Pressure vessel optimization with SymPy, SciPy

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1 Problem Setup

One of the most famous Aerostructures is here under study and depicted in the next figure. The vessel consists of:

- Cylindrical Section (Length L, Diameter D)
- Two Hemispherical End Caps (each with radius $D/2$)
- A bound constraint for the ratio L/D constrained in the range $[0.1, 10]$

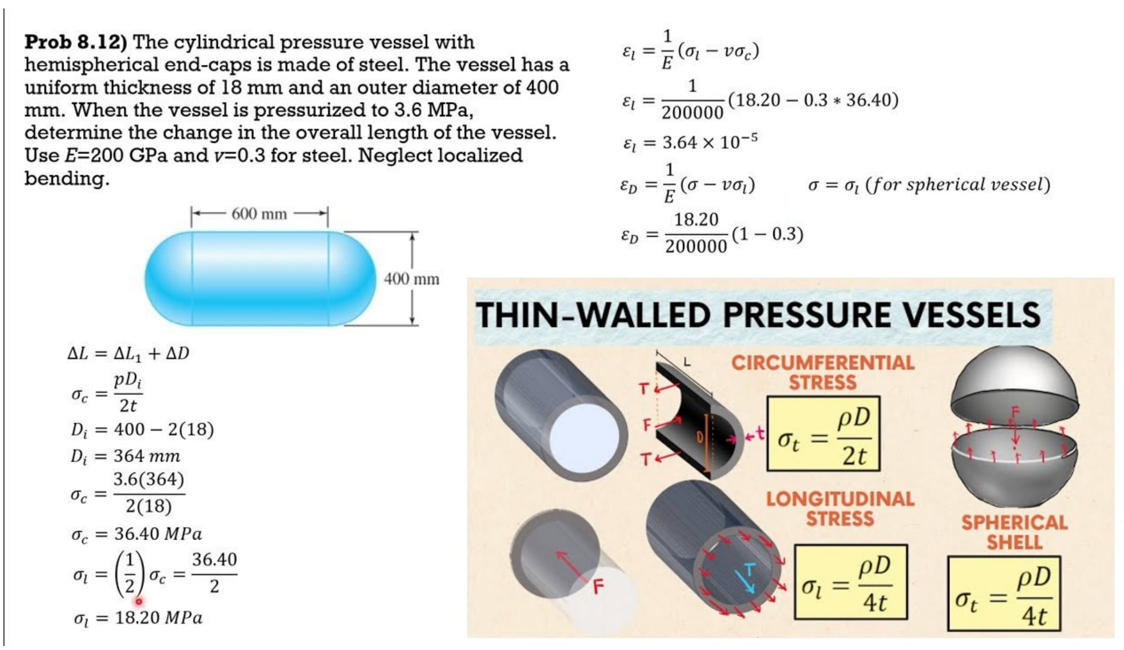

Figure 1: In this cylindrical pressure vessel, L is 600 mm, and D= 400 mm

We need to minimize the mass M of the vessel, given:

- Internal pressure p,
- Density ρ,
- Yield strength σ_y,
- the two Design Variables (DV or Unknowns) which are cylinder diameter D and length L that describes the internal volume V

2 Theoretical solution

2.1 Expressions for Volume

The internal volume is given by:

$$V = \frac{\pi}{4} D^2 L + \frac{2}{3} \pi \left(\frac{D}{2} \right)^3$$

$$V = \frac{\pi}{4} D^2 L + \frac{\pi}{6} D^3$$

2.2 Expressions for thicknesses

From thin-wall pressure vessel theory, the required thicknesses are:

- Cylindrical section (Hoop stress governs):

$$t_{\mathrm{c}} = \frac{pD}{2\sigma_y}$$

- Hemispherical caps (Membrane stress governs):

$$t_h = \frac{pD}{4\sigma_y}$$

2.3 Mass Expression

The total mass consists of the mass of the cylindrical shell and the two hemispherical end caps. First the Mass of the cylindrical shell is given by:

$$m_c = \rho \cdot Volume = \rho \left(\pi D L t_c \right)$$

Substituting $t_c = \frac{\rho D}{2\sigma_s}$ it gives:

$$m_c = \rho \pi D L \frac{pD}{2\sigma_y} = \frac{\rho \pi p D^2 L}{2\sigma_y}$$

The Mass of the two hemispherical end caps are given hereafter:

$$m_h = \rho \cdot Volume = \rho \left(2 \cdot 2\pi \left(\frac{D}{2} \right)^2 t_h \right)$$

Substituting $t_h = \frac{pD}{4\sigma_y}$ it leads to:

$$m_h = \rho \cdot 2 \cdot 2\pi \frac{D^2 pD}{4} \frac{p\sigma_y}{4} = \frac{\rho\pi p D^3}{4\sigma_y}$$

Thus, the total mass is:

$$m_{total} = M = m_c + m_h = \frac{\rho\pi p D^2 L}{2\sigma_y} + \frac{\rho\pi p D^3}{4\sigma_y}$$

- Step 1: Express the Dimensionless Mass M^*

  We derived earlier that the total mass of the vessel is:

  $$M = \frac{\rho\pi p D^2 L}{2\sigma_y} + \frac{\rho\pi p D^3}{4\sigma_y}$$

  Dividing by the characteristic mass $V\rho p/\sigma_y$, the dimensionless mass is:

  $$M^* = \frac{M}{V\rho p/\sigma_y}$$

  Using the volume equation:

  $$V = \frac{\pi}{4} D^2 L + \frac{\pi}{6} D^3$$

  we express M^* in terms of $L/D = \lambda$.

- Step 2: Differentiate M w.r.t. λ^*

  We take:

  $$\frac{dM^*}{d\lambda} = 0$$

  to find the critical points.

- Step 3: Solve for λ_{opt} From previous numerical results, we expect L/D very small (i.e. a sphere) to be the optimal shape. This makes sense because: - A sphere minimizes surface area for a given volume. - A spherical pressure vessel distributes stress evenly, reducing material requirements.

3 Possible extension

- Use sciPy minimize with more constraints and real properties from the figure 1
- Add a reserve factor on yield stress.
- Material selection for mass and CO2 minimization

4 Simply with symPy/sciPy

Please check the notebook to start!!

5 Solution

Step 1: Express Volume V in Terms of $L/D = \lambda$
We use the internal volume equation:

$$V = \frac{\pi}{4} D^2 L + \frac{\pi}{6} D^3$$

Since $\lambda = L/D$, we substitute $L = \lambda D$.

$$V = \frac{\pi}{4} D^2 (\lambda D) + \frac{\pi}{6} D^3$$

$$V = \frac{\pi}{4} \lambda D^3 + \frac{\pi}{6} D^3$$

Factor out D^3 :

$$V = \frac{\pi D^3}{4} \left(\lambda + \frac{2}{3} \right)$$

Solving for D :

$$D = \left(\frac{4V}{\pi \left(\lambda + \frac{2}{3} \right)} \right)^{\frac{1}{3}}$$

And for L :

$$L = \lambda D = \lambda \left(\frac{4V}{\pi \left(\lambda + \frac{2}{3} \right)} \right)^{\frac{1}{3}}$$

Step 2: Express Dimensionless Mass M^*
The total mass is:

$$M = \rho \pi D L t_c + 2\rho \cdot 2\pi \left(\frac{D}{2} \right)^2 t_h$$

Using the thin-wall thickness equations:

$$t_c = \frac{pD}{2\sigma_y}, \quad t_h = \frac{pD}{4\sigma_y}$$

$$M = \rho \pi D L \frac{pD}{2\sigma_y} + 2\rho \cdot 2\pi \left(\frac{D}{2} \right)^2 \frac{pD}{4\sigma_y}$$

$$M = \frac{\rho \pi p D^2 L}{2\sigma_y} + \frac{\rho \pi p D^3}{4\sigma_y}$$

Dividing by the characteristic mass $V\rho p/\sigma_y$:

$$M^* = \frac{M}{V\rho p/\sigma_y} = \frac{\frac{\pi p D^2 L}{2\sigma_y} + \frac{\pi p D^3}{4\sigma_y}}{V p/\sigma_y}$$

Cancel p/σ_y:

$$M^* = \frac{\pi D^2 L}{2V} + \frac{\pi D^3}{4V}$$

Substituting D and L :

$$M^* = \frac{\pi\left(\frac{4V}{\pi\left(\lambda+\frac{2}{3}\right)}\right)^{\frac{2}{3}} \cdot \lambda\left(\frac{4V}{\pi\left(\lambda+\frac{2}{3}\right)}\right)^{\frac{1}{3}}}{2V} + \frac{\pi\left(\frac{4V}{\pi\left(\lambda+\frac{2}{3}\right)}\right)}{4V}$$

Simplifying:

$$M^* = \frac{\pi\lambda\left(\frac{4V}{\pi\left(\lambda+\frac{2}{3}\right)}\right)}{2V} + \frac{\pi\left(\frac{4V}{\pi\left(\lambda+\frac{2}{3}\right)}\right)}{4V}$$

Canceling V:

$$M^* = \frac{\pi\lambda}{2} \cdot \frac{4}{\pi\left(\lambda+\frac{2}{3}\right)} + \frac{\pi}{4} \cdot \frac{4}{\pi\left(\lambda+\frac{2}{3}\right)}$$

$$M^* = \frac{2\lambda}{\lambda+\frac{2}{3}} + \frac{1}{\lambda+\frac{2}{3}}$$

$$M^* = \frac{2\lambda+1}{\lambda+\frac{2}{3}}$$

Step 3: Differentiate and Solve for λ_{opt}

Taking the derivative:

$$\begin{aligned}
\frac{dM^*}{d\lambda} &= \frac{(2)\left(\lambda+\frac{2}{3}\right) - (2\lambda+1)(1)}{\left(\lambda+\frac{2}{3}\right)^2} \\
&= \frac{2\lambda+\frac{4}{3}-2\lambda-1}{\left(\lambda+\frac{2}{3}\right)^2} \\
&= \frac{\frac{1}{3}}{\left(\lambda+\frac{2}{3}\right)^2}
\end{aligned}$$

Setting $dM^*/d\lambda = 0$:

$$\frac{\frac{1}{3}}{\left(\lambda+\frac{2}{3}\right)^2} = 0$$

Since the numerator is a constant (1/3), this equation has no real solution where $dM^*/d\lambda = 0$. This means the function is monotonically decreasing for small λ and increasing for large λ.

Step 4: Identify the Minimum

Since M^* has no critical points, we check the boundary values:
Substituting:

- For $\lambda = 0.1$ (sphere as L largely inferior to D):

$$M^* = \frac{2(0.1)+1}{0.1+\frac{2}{3}} = 1.56$$

- For $\lambda = 10$ (long cylinder):

$$M^* = \frac{2(10)+1}{10+\frac{2}{3}} = 1.96$$

Since $M*$ is lowest at $L/D = 0.1$, the optimal shape is indeed a sphere!

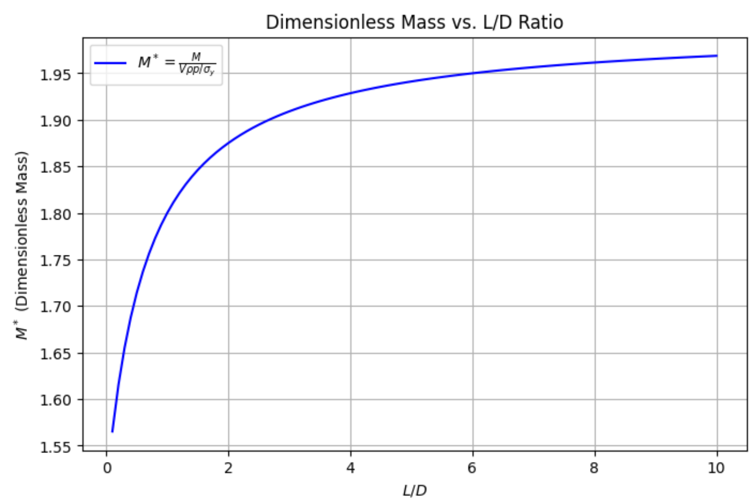

Figure 2: Do you know why in spacecraft the vessel is spherical ? Check for the minimum