Phys 111 Computational Project

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1 Introduction

Our chosen system consists of two charged particles, with charges q_1 and q_2 and masses m_1 and m_2 respectively, placed in a constant magnetic field, for simplicity's sake chosen to be $\mathbf{B} = B_0\,\hat{z}$.

In a system with just a single charged particle and a constant magnetic field along $\hat{z}$, the particle undergoes gyration in the x-y plane about an axis parallel to $\hat{z}$ and approaches some constant speed in the $\hat{z}$ direction, as depicted in Fig. 1. If we take $|q| = |m| = 1$, then this terminal z-velocity is determined by B_0, and the radius of gyration is determined by both B_0 and the particle's initial velocity in the x-y plane.

Figure 1: A single charged particle gyrating in the constant magnetic field $\mathbf{B} = B_0\,\hat{z}$.

By introducing a second particle, there will now be a Coulombic interaction between the two particles. Due to the Coulomb potential being proportional to $1/r$, we hope to see strong sensitivity to initial conditions and chaotic behavior as a result of introducing this second particle.

2 Equations of Motion

2.1 Generalized Coordinates

Since our system consists of two particles in three dimensions, this system has a maximum of six degrees of freedom. If we made certain assumptions about the mass or charge of the particle (such as the particles having equal charge-mass ratios), we might be able to reduce the number of degrees of freedom, but since we are not making any of these assumption, we are left with the initial six degrees of freedom, with generalized coordinates given by:

$$(x_1, y_1, z_1) \quad (x_2, y_2, z_2).$$

2.2 Lagrangian(s)

The Lagrangian of a single point charge q in an electric and magnetic field is given by

$$\mathscr{L}(\mathbf{r}, t) = \frac{1}{2} m \dot{\mathbf{r}}^2 - q\,\phi(\mathbf{r}, t) + q\,\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t) \tag{1}$$

where $\phi(\mathbf{r}, t)$ is the scalar potential of the electric field and $\mathbf{A}(\mathbf{r}, t)$ is the vector potential of the magnetic field. Our system has a constant magnetic field of strength B_0 in the $\hat{z}$ direction, which has a vector potential of

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A} = -B_0 y\,\hat{x}. \tag{2}$$

Since we have only two charged particles, the only electric field felt by one particle is the electric field of other particle, so the electric scalar potential of each particle is given by

$$\phi_i(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|} = \frac{1}{4\pi\epsilon_0} \frac{q_j}{d} \tag{3}$$

where we have defined $d = |\mathbf{r}_i - \mathbf{r}_j|$.

Substituting these potentials and our generalized coordinates into Eq. (1), we get the Lagrangians of our two particles:

$$\mathscr{L}_1(x_1, y_1, z_1) = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) - \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{d} - q_1 B_0 y_1 \dot{x}_1, \tag{4a}$$

$$\mathscr{L}_2(x_2, y_2, z_2) = \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) - \frac{1}{4\pi\epsilon_0}\frac{q_2 q_1}{d} - q_2 B_0 y_2 \dot{x}_2, \tag{4b}$$

where d is given by $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

2.3 Euler-Lagrange Equations

Taking the Euler-Lagrange equations of these two Lagrangians, we get the following six differential equations:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}_1}\right) = \frac{\partial \mathscr{L}}{\partial x_1} \implies \ddot{x}_1 = \frac{q_1 B_0}{m_1}\dot{y}_1 + \frac{q_1 q_2}{4\pi\varepsilon_0 m_1}\frac{x_2 - x_1}{d^3} \tag{5a}$$

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{y}_1}\right) = \frac{\partial \mathscr{L}}{\partial y_1} \implies \ddot{y}_1 = -\frac{q_1 B_0}{m_1}\dot{x}_1 + \frac{q_1 q_2}{4\pi\varepsilon_0 m_1}\frac{y_2 - y_1}{d^3} \tag{5b}$$

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{z}_1}\right) = \frac{\partial \mathscr{L}}{\partial z_1} \implies \ddot{z}_1 = \frac{q_1 q_2}{4\pi\varepsilon_0 m_1}\frac{z_2 - z_1}{d^3} \tag{5c}$$

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{x}_2}\right) = \frac{\partial \mathscr{L}}{\partial x_2} \implies \ddot{x}_2 = \frac{q_2 B_0}{m_2}\dot{y}_2 - \frac{q_1 q_2}{4\pi\varepsilon_0 m_2}\frac{x_2 - x_1}{d^3} \tag{6a}$$

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{y}_2}\right) = \frac{\partial \mathscr{L}}{\partial y_2} \implies \ddot{y}_2 = -\frac{q_2 B_0}{m_2}\dot{x}_2 - \frac{q_1 q_2}{4\pi\varepsilon_0 m_2}\frac{y_2 - y_1}{d^3} \tag{6b}$$

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{z}_2}\right) = \frac{\partial \mathscr{L}}{\partial z_2} \implies \ddot{z}_2 = -\frac{q_1 q_2}{4\pi\varepsilon_0 m_2}\frac{z_2 - z_1}{d^3} \tag{6c}$$

2.4 Non-dimensionalization

To non-dimensionalize our equations of motion, we can combine the constants we have in our equations (e.g., q, m, ε_0, B_0) to construct non-dimensionalized lengths and time:

$$\left[\left(\frac{B_0^2\varepsilon_0}{m}\right)^{1/3}\right] = \mathrm{m}^{-1} \qquad \left[\frac{qB_0}{m}\right] = \mathrm{s}^{-1}.$$

Using these to write our generalized coordinates and their derivatives in terms of dimensionless variables, we get

$$t = \tilde{t}\frac{m}{qB_0}, \qquad l = \tilde{l}\left(\frac{m}{B_0^2\varepsilon_0}\right)^{1/3},$$

$$\frac{dl}{dt} = \left(\frac{qB_0}{m}\right)\left(\frac{m}{B_0^2\varepsilon_0}\right)^{1/3}\frac{d^2\tilde{l}}{d\tilde{t}^2}, \qquad \frac{dl^2}{dt^2} = \left(\frac{qB_0}{m}\right)^2\left(\frac{m}{B_0^2\varepsilon_0}\right)^{1/3}\frac{d^2\tilde{l}}{d\tilde{t}^2}$$

where the remaining derivatives all have the same non-dimensionalization constants.

For simplicity, we pick q_1 and m_1 to replace the q and m in these constants (picking q_2 and m_2 would be equally valid, but we need to be consistent). Substituting these non-dimensional variables into our equations of motion in Eq. (5) and Eq. (6), we arrive at

$$\ddot{\tilde{x}}_1 = \dot{\tilde{y}} + \frac{1}{4\pi}\frac{q_2}{q_1}\frac{\tilde{x}_1 - \tilde{x}_2}{\tilde{d}^3} \tag{7a}$$

$$\ddot{\tilde{y}}_1 = -\dot{\tilde{x}} + \frac{1}{4\pi}\frac{q_2}{q_1}\frac{\tilde{y}_1 - \tilde{y}_2}{\tilde{d}^3} \tag{7b}$$

$$\ddot{\tilde{z}}_1 = \frac{1}{4\pi}\frac{q_2}{q_1}\frac{\tilde{z}_1 - \tilde{z}_2}{\tilde{d}^3} \tag{7c}$$

$$\ddot{\tilde{x}}_2 = \frac{q_2 m_1}{q_1 m_2}\dot{\tilde{y}}_2 + \frac{1}{4\pi}\frac{q_2 m_1}{q_1 m_2}\frac{\tilde{x}_1 - \tilde{x}_2}{\tilde{d}^3} \tag{8a}$$

$$\ddot{\tilde{y}}_2 = -\frac{q_2 m_1}{q_1 m_2}\dot{\tilde{x}}_2 + \frac{1}{4\pi}\frac{q_2 m_1}{q_1 m_2}\frac{\tilde{y}_1 - \tilde{y}_2}{\tilde{d}^3} \tag{8b}$$

$$\ddot{\tilde{z}}_2 = \frac{1}{4\pi}\frac{q_2 m_1}{q_1 m_2}\frac{\tilde{z}_1 - \tilde{z}_2}{\tilde{d}^3} \tag{8c}$$

These q and m factor on our equations for the second particle just specify its behavior differs first particle's in terms of their charge and mass ratios.

TODO: look into getting the ratio terms in both sets of DEs. Currently, with our non-dimensionalizing, we define the length and time scale based on particle 1 to have a length/time of "1", and define particle 2 relative to that. Instead, could we define the midpoint of their scales to be one, so that reciprocal coeffs get us the correct scales?

3 Dynamics

4 Conclusion