Homework # 7

1. Consider constants $p, \sigma > 0$ and the problem

$$p\frac{d^2\psi}{dx^2} + \mu\sigma\psi = 0 \qquad 0 < x < 1$$
$$\psi(0) = 0 = \psi(1)$$

Then $\psi(x) = \sin\sqrt{\mu\sigma/p}\, x$; so if $0 = \psi(1) = \sin\sqrt{\mu\sigma/p}$ we have $\mu_n = \frac{p}{\sigma} n^2\pi^2 \quad n = 1, 2, \ldots$. We want $\mu_1 \le \lambda_1$. By the Monotonicity theorem, we need $p \le 1 + x^2$ on $[0,1]$ and $\sigma \ge 1 + x^2$ on $[0,1]$. Choose $p = 1, \sigma = 2$, then $\frac{1}{2}\pi^2 \le \lambda_1$. For an upper bound, we want $p \ge 1 + x^2 \ge \sigma$ on $[0,1]$ so let $p = 2$, $\sigma = 1$. Therefore $\frac{1}{2}\pi^2 \le \lambda_1 \le 2\pi^2$ (or, approximately $4.93 < \lambda_1 < 19.74$ — not a very tight bound!)

2. If we let $u(x,t) = A(t)(\pi - x)/\pi + v(x,t)$ then v satisfies

$$\begin{cases} v_t = v_{xx} + F(x,t) & 0 < x < \pi \\ v(0,t) = 0 = v(\pi,t) \\ v(x,0) = 0 \end{cases}$$

, where $F(x,t) \doteq e^{-t}\sin 3x - \dot{A}(t)(1 - x/\pi)$.

Now we have a problem with homogeneous b.c.'s. The eigenfunctions found from the homogeneous problem are $\{\sin(nx)\}_{n\ge 1}$, so let $v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(nx)$

$$F(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin(nx)$$

Then, upon substitution into the equation, gives

$$f_n(t) = \frac{2}{\pi}\int_0^{\pi} F(x,t)\sin(nx)\,dx = e^{-t}\delta_{3n} - \frac{2}{n\pi}\dot{A}(t) \quad \left(\delta_{3n} = \begin{cases} 1 & n = 3 \\ 0 & n \ne 0 \end{cases}\right)$$

$$\left.\begin{array}{l} \dot{a}_n + n^2 a_n = f_n(t) \\ a_n(0) = 0 \end{array}\right\} \rightarrow a_n(t) = \int_0^t f_n(\tau)\, e^{-n^2(t-\tau)}\,d\tau$$

$$= \frac{1}{8}(e^{-t} - e^{-9t})\delta_{3n} - \frac{2}{n\pi}A(t)$$
$$+ \frac{2n}{\pi}\int_0^t e^{-n^2(t-\tau)}A(\tau)\,d\tau$$

3. This problem is the damped-wave equation version of a problem I did in class. With eigenfunctions $\{\sin(\frac{n\pi x}{\ell})\}$, $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x)$. Substitution gives

$$(*)\quad \frac{d^2a_n}{dt^2} + \beta\frac{da_n}{dt} + n^2\pi^2 a_n = f_n(t) = 2\int_0^1 q_0 \sin(n\pi x)dx \cos(\omega t)$$

$$= \begin{cases} 0 & n = \text{even} \\ \frac{4q_0}{n\pi}\cos\omega t & n = \text{odd} \end{cases}$$

Define $\omega_n \doteq \sqrt{n^2\pi^2 - \beta^2/4} > 0$ for all $n = 1, 2, \ldots$, then a fundamental set of solutions for the problem is $\{e^{-\beta t/2}\cos(\omega_n t), e^{-\beta t/2}\sin(\omega_n t)\}$. From the zero initial conditions, $a_n(0) = 0$, $\frac{da_n}{dt}(0) = 0$, so for $n =$ even, $a_n(t) \equiv 0$. From now on, $n =$ odd. A particular solution of the form $a(t) = A\cos(\omega t) + B\sin(\omega t)$, substituted into (*) gives the following algebraic system for the coefficients:

$$\begin{bmatrix} n^2\pi^2 - \omega^2 & \beta\omega \\ -\beta\omega & n^2\pi^2 - \omega^2 \end{bmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 4q_0/n\pi \\ 0 \end{pmatrix}. \text{ Thus}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{4q_0/n\pi}{(n^2\pi^2 - \omega^2)^2 + \beta^2\omega^2}\begin{pmatrix} n^2\pi^2 - \omega^2 \\ \beta\omega \end{pmatrix}.$$

So writing $a_n(t)$ as the sum of the particular solution and a linear combination of the fundamental set of solutions, then applying the initial conditions, gives

$$a_n(t) = \begin{cases} A\left\{\cos\omega t - e^{-\beta t/2}\cos(\omega_n t) - \frac{\beta}{2\omega_n}e^{-\beta t/2}\sin(\omega_n t)\right\} \\ \quad + B\left\{\sin(\omega t) - \frac{\omega}{\omega_n}e^{-\beta t/2}\sin(\omega_n t)\right\} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$