75 points

Math 404 NAME (Print Last, First)): Key

Midterm Exam

March 2006

Game Rules: No calculators, notes, or other aids permitted. Show all work. Answer what is asked in the space provided. If you need extra space, use the back of the page, or spare blank sheet attached on the back of the exam. Look over the whole exam and work on the parts you feel most comfortable with first. Enjoy…

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1. Consider the problem $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \qquad 0 < x < \pi, \quad t > 0$

$$u(0,t) = 0 = \frac{\partial u}{\partial x}(\pi,t)$$

$$u(x,0) = 2\sin(3x/2) + \sin(7x/2)$$

a) If $u(x,t) = T(t)\varphi(x)$, derive the equation for T and the eigenvalue problem for $\varphi(x)$ (equation and boundary conditions)

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$u(x,t) = T(t)\varphi(x) \rightarrow \dot{T}/T = \varphi''/\varphi = -\lambda \rightarrow \boxed{\begin{array}{l} \dot{T} = -\lambda T \\ \begin{cases} \varphi'' + \lambda\varphi = 0 \\ \varphi(0) = 0 = \varphi'(\pi) \end{cases} \end{array}}$

b) Solve the eigenvalue problem to obtain the eigenvalues and associated eigenfunctions.

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$\varphi'' + \lambda\varphi = 0$ on $0 < x < \pi$, $\varphi(0) = 0 \rightarrow \varphi(x) = \sin\sqrt{\lambda}\,x$

$\varphi'(x) = \sqrt{\lambda}\cos\sqrt{\lambda}\,x \rightarrow 0 = \sqrt{\lambda}\cos\sqrt{\lambda}\,\pi$

1) $\sqrt{\lambda} = 0 \rightarrow \varphi = Ax + B$; $\varphi(0) = B = 0$, $\varphi'(\pi) = A = 0 \rightarrow \varphi \equiv 0$

so $\lambda = 0$ is <u>not</u> an eigenvalue

2) $\cos\sqrt{\lambda}\,\pi = 0 \rightarrow \sqrt{\lambda}\,\pi = (n+1/2)\pi$ $\boxed{n = 0, 1, 2, \ldots}$

or $\boxed{\lambda = \lambda_n = (n+1/2)^2}$

$\rightarrow \boxed{\varphi = \varphi_n(x) = \sin(n+1/2)x}$

Also $T(t) = T_n(t) = e^{-(n+1/2)^2 t}$

c) Use the initial data to determine the solution $u(x, t)$. By Superposition principle,

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$$u(x,t) = \sum_{n=0}^{\infty} b_n e^{-(n+1/2)^2 t} \sin(n+\tfrac{1}{2})x$$

$b_0 \sin(\tfrac{1}{2}x) + b_1 \sin(\tfrac{3}{2}x) + b_2 \sin(\tfrac{5}{2}x) + \cdots$

$$\rightarrow u(x,0) = 2\sin(\tfrac{3}{2}x) + \sin(\tfrac{7}{2}x) = \sum_{n=0}^{\infty} b_n \sin(n+\tfrac{1}{2})x \rightarrow b_n = \begin{cases} 2 & n=1 \\ 1 & n=3 \\ 0 & \text{otherwise} \end{cases}$$

$$\rightarrow \boxed{u(x,t) = 2e^{-9t/4}\sin(3x/2) + e^{-49t/4}\sin(7x/2)}$$

(25) 2. Let $f(x) = \dfrac{\pi^2 - 2\pi|x|}{8} \quad -\pi \le x \le \pi$

a) Compute the Fourier cosine series for f; i.e. $f(x) \sim \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$. (So

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determine the coefficients. Note that the function is an even function. Also, you can use the fact that $\cos(n\pi) = (-1)^n$, and

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \begin{cases} \pi & n = m \neq 0 \\ 0 & n \neq m \end{cases}$$

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f(x) is continuous, piecewise smooth, so the Fourier cosine series converges everywhere to the extended function $\tilde{f}(x)$.

$$0 = \int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \sum_{n\ge 1} a_n \underbrace{\cancel{\int_{-\pi}^{\pi} \cos nx\, dx}}_{0} = a_0\pi \rightarrow a_0 = 0$$

For $m = 1, 2, \ldots$

$$\int_{-\pi}^{\pi} f(x) \cos mx\, dx = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \pi a_m$$

$$\shortparallel$$

$$\int_{-\pi}^{\pi} \frac{\pi^2 - 2\pi|x|}{8} \cos mx\, dx = 2\int_0^{\pi} \frac{\pi^2 - 2\pi x}{8} \cos mx\, dx = \frac{\pi^2}{4}\Big\{ \cancel{\int_0^{\pi} \cos mx dx} \Big\}^{0} - \frac{\pi}{2}\Big\{\int_0^{\pi} x \cos mx dx\Big\}$$

So $$a_m = -\frac{1}{2}\int_0^{\pi} x \cos mx\, dx = -\frac{1}{2}\Big\{ \underbrace{\frac{x}{m} \sin mx \Big|_0^{\pi}}_{=0} - \frac{1}{m}\int_0^{\pi} \sin mx\, dx \Big\}$$

$$= -\frac{1}{2}\Big\{ \frac{\cos mx}{m^2} \Big|_0^{\pi} \Big\} = \frac{1}{2m^2}\{1 - (-1)^m\} = \begin{cases} 0 & m = \text{even} \\ 1/m^2 & m = \text{odd} \end{cases}$$

$$\rightarrow \boxed{f(x) \sim \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{\cos nx}{n^2} \underset{n=2k+1}{=} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)x]}{(2k+1)^2}}$$

b) Sketch a graph of the Fourier series in part a) on the whole real line — f is continuous, piecewise smooth, so the F.C.S. converges everywhere to f(x). That is, the graph of the F.C. Series is the graph of f.

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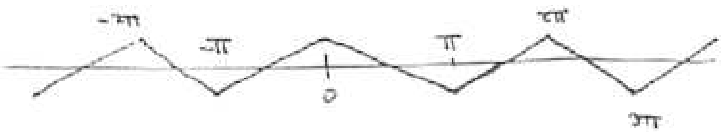

(20) 3. Determine the solution to the following Cauchy problem:

$$2\frac{\partial u}{\partial t} + \cos(t)\frac{\partial u}{\partial x} = 0 \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = e^{-x} \qquad -\infty < x < \infty$$

$$\Gamma = \{(x,t) = (s,0) \mid \ |s| < \infty\}$$

$$\frac{dt}{d\tau} = 2 \rightarrow t = 2\tau + t_0(s); \quad t\big|_{\tau=0} = t_0 = 0 \rightarrow t = 2\tau$$

$$\frac{dx}{d\tau} = \cos(2\tau) \rightarrow x = \frac{1}{2}\sin(2\tau) + x_0(s); \quad x\big|_{\tau=0} = x_0 = s$$

$$\rightarrow x = \frac{1}{2}\sin(2\tau) + s$$

$$\frac{du}{d\tau} = 0 \rightarrow u = u_0(s) = e^{-s} \rightarrow \boxed{u(x,t) = e^{-x + \frac{1}{2}\sin(t)}}$$