Homework #1

1. $\frac{dI}{dt} = \beta I (N-I) \rightarrow \beta\, dt = \frac{dI}{I(N-I)} = \frac{dI}{N}\left\{\frac{1}{I} + \frac{1}{N-I}\right\}$

$\rightarrow \beta N t + \text{constant} = \ln(I) - \ln(I-N) = \ln\left(\frac{I}{I-N}\right) \rightarrow$

$\frac{I}{I-N} = C e^{\beta N t}$ $\quad$ (so $C = \frac{I_0}{I_0 - N}$ if $I(0) = I_0$)

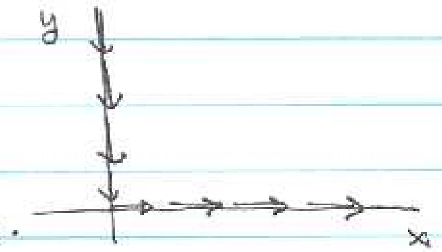
Hence $\quad I = \frac{C N e^{\beta N t}}{C e^{\beta N t} - 1} = \frac{I_0 N e^{\beta N t}}{I_0 e^{\beta N t} - (I_0 - N)} = \frac{I_0 e^{\beta N t}}{1 - (I_0/N)(1 - e^{\beta N t})}$

Also $\quad N - I = \frac{N - I_0}{1 - (I_0/N)(1 - e^{\beta N t})} \rightarrow \beta I (N - I) = \frac{\beta I_0 (N - I_0) e^{\beta N t}}{\left[1 - (I_0/N)(1 - e^{\beta N t})\right]^2}$

$$\beta I(N-I) = \frac{dI}{dt}$$

2. a) $\begin{cases} \frac{dx}{dt} = x\{a - g(y)\} = ax - x g(y) \\ \frac{dy}{dt} = y\{bx - c\} = bxy - cy \end{cases}$

b) along the y-axis ($y > 0$), $x = 0 \rightarrow \frac{dx}{dt} = 0$; i.e. if we found ourselves on the y-axis, x can not increase or decrease. Since trajectories cannot intersect, except at equilibrium points (steady states), with $\frac{dy}{dt} = -cy < 0$, all we can do is run down the axis.



That is, the y axis acts as a barrier to any trajectories leaving the positive quadrant. Similarly, on the x axis, $dx/dt = ax > 0$, $dy/dt = 0$, so it too does not allow trajectories to escape the positive quadrant. So if we are in the positive quadrant, we must stay in it. The model would have a critical failing if this was not so.

c) The Jacobian of the system is

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} a - g(y) & -x g'(y) \\ by & bx - c \end{bmatrix}$$

where I am thinking of the model as $\begin{cases} \frac{dx}{dt} = F(x,y) \\ \frac{dy}{dt} = G(x,y) \end{cases}$

For critical points $\quad 0 = x\{a - g(y)\}$

$\qquad\qquad\qquad\qquad\quad 0 = y\{bx - c\}$

If $y = 0 \rightarrow x = 0$ i.e. $(x,y) = (0,0)$ is a critical point (extinction)

If $y \neq 0 \rightarrow x = c/b \rightarrow a - g(y) = 0$. But g is monotone increasing, so g^{-1} exists. Hence, $(x,y) = (c/b, g^{-1}(a))$ is another critical point.

Now

$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$ so its eigenvalues are

$\lambda = a, -c \rightarrow (0,0)$ is a saddle point (unstable)

For

$$J(c/b, g^{-1}(a)) = \begin{bmatrix} 0 & -cg'(g^{-1}(a))/b \\ bg^{-1}(a) & 0 \end{bmatrix}$$

since $g'(g^{-1}(a)) > 0$, $g^{-1}(a) > 0$ then the eigenvalues are

$\lambda = \pm i\sqrt{cg^{-1}(a)g'(g^{-1}(a))}$ so $(c/b, g^{-1}(a))$ is a center. This is a degenerate case for the full (nonlinear) system, but this is all that is asked for in this problem.

d) One can not really measure the total viral population. It could range over orders of magnitude. Besides, knowledge of this number is not really interesting. It is the distribution of the infecting agent over the host population that determines the percentage of people that get sick.