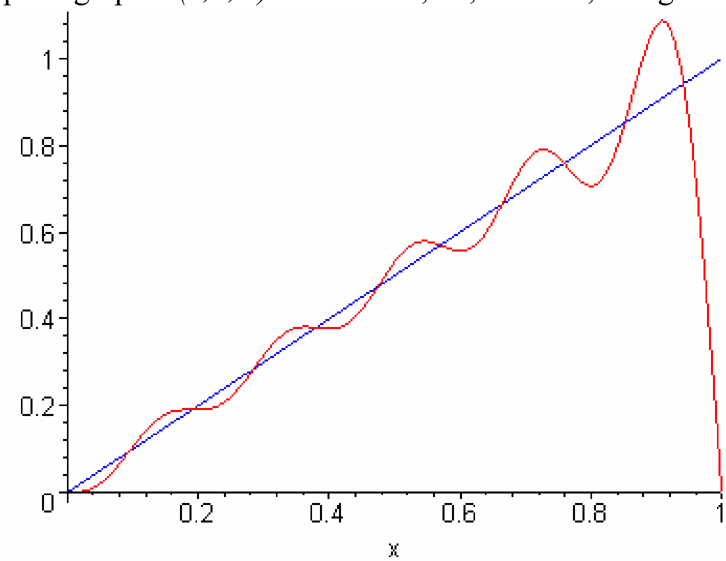
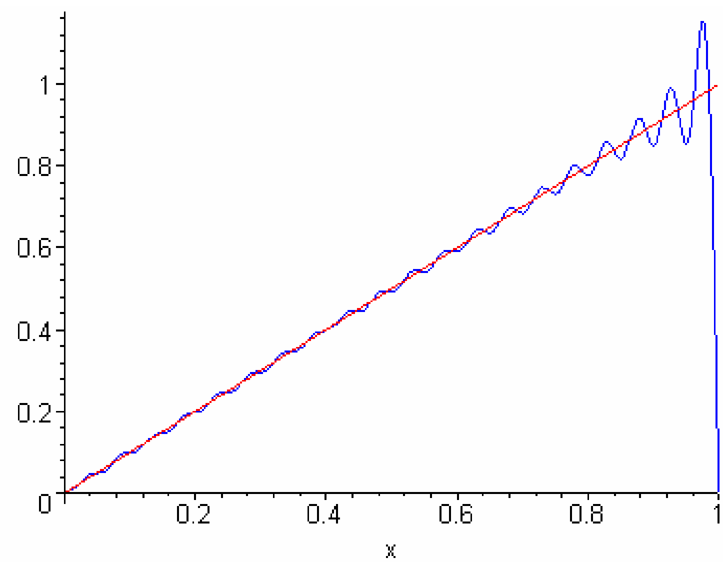
Example: graphs on the solution to the heat equation problem

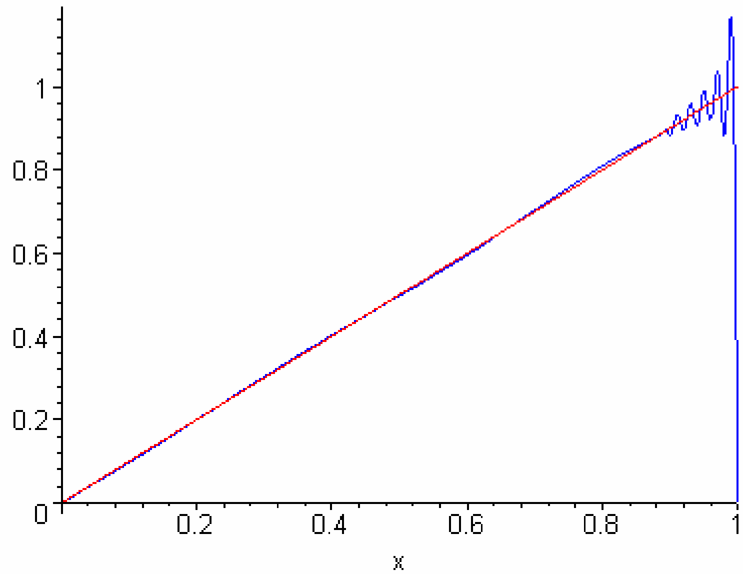
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1, \quad t > 0$$
$$u(x,0) = x$$
$$u(0,t) = 0 = u(1,t)$$

By separation of variables, and Fourier methods, the solution is given by the series
$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \exp(-n^2\pi^2 t)\sin(n\pi x)$$

Let *u(x,t;N)* be the notation for the series truncated to the first N terms. The following plots graph *u(x,0;N)* for N = 10, 40, and 100, along with *f(x) = x*:







This certainly shows the series converging for *t = 0* to the initial condition as $N \to \infty$, except near *x = 1*, where *u(1, t) = 0*, but *f(1-) = 1*. What is happening in a neighborhood of *x = 1* is **Gibbs' phenomena**. Joseph Williard Gibbs, American (Yale) physicist, observed the overshoot behavior of Fourier series at jump discontinuities in the function being approximated, no matter how many terms were taken in the series, and wrote about it to Michelson in 1899. However, the phenomenon was observed and analyzed by Henry Wilbraham, a Cambridge mathematician, in 1848. I'll say more about it when we discuss Fourier series in more detail.

Finally, note that $\lim_{t\to\infty} u(x,t) = 0$ on [0, 1] (which is the steady state solution *U(x)* for the problem). So the next graph plots *f(x) = x = u(x, 0),* and for N = 50, also plots *u(x,0.01;50), u(x, 0.1; 50), u(x, 0.2; 50)* so you can see the (uniform) convergence of u toward $U(x) \equiv 0$.