Supplementary Problems

1. Solve
$$\begin{cases} u_t = u_{xx} & 0 < x < \pi, \quad t > 0 \\ u(0,t) = 2+t = u(\pi,t) \\ u(x,0) = 0 \end{cases}$$

2. (a) Solve
$$\begin{cases} u_t = k\, u_{xx} & 0 < x < L, \quad t > 0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = 1 \end{cases}$$

(b) Find the average temperature in the rod at time t — call it $U(t)$.

(c) Given that $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \pi^2/8$, show that
$$\frac{8}{\pi^2} e^{-(\pi^2/L^2)kt} \le U(t) \le e^{-(\pi^2/L^2)kt}$$

3. Solve
$$\begin{cases} u_t = u_{xx} & 0 < x < 1 \quad t > 0 \\ u_x(0,t) = 0 = u(1,t) \\ u(x,0) = 1-x \end{cases}$$

4. A rod of homogeneous radioactive material lies along $0 \le x \le L$. The neutron density $n(x,t)$ is affected by two processes, fission (with fission constant $k > 0$), and diffusion (diffusion constant $D > 0$). Conservation of neutrons leads to
$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} + k n$$
Let $n = 0$ at the rod ends. Obtain the series solution for $n(x,t)$ and show the rod will explode ($n \to \infty$ as $t \to \infty$) if and only if $k > \pi^2 D/L^2$.

5. A string with fixed ends is struck ~~with~~ a sudden blow with a hammer at $t=0$. As a result we have $u(x,0) = 0$ $(0 \le x \le L)$, and $u_t(x,0) = -V$ $L/4 \le x \le L/2$, 0 otherwise. Find the formula for the motion.

6. Consider $\frac{\partial^2 u}{\partial t^2} + 3\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (x+1)\frac{\partial u}{\partial x} \qquad 0 < x < \pi,\ t > 0$

$$u(0,t) = 0 = \frac{\partial u}{\partial x}(\pi, t) \qquad t > 0$$

a) Is this equation hyperbolic, parabolic, or elliptic? Why?

b) If $u(x,t) = T(t)\varphi(x)$, What is the eigenvalue problem for this pde problem? (Do not solve it)

c) Solve the Steady State problem, $u(x,t) = U(x)$, to this pde problem.

7. Sketch the Fourier sine series of f, and determine its Fourier coefficients given

$$f(x) = \begin{cases} 1 & 0 < x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$$

8. If you extend the function f given in #7 as an even function on $(-L, L)$, so you have the Fourier cosine series $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$; what are the coefficients?

9. Sketch the Fourier sine series of $f(x) = \cos(\pi x/L)$, $0<x<L$, and determine its Fourier coefficients.

<u>Some</u> <u>answers</u>

1. $u = 2 + t + \sum_{n=1}^{\infty} a_n(t)\sin(nx)$, $a_n(t) = -\frac{2}{n\pi}(1-(-1)^n)\left\{2e^{-n^2 t} + \frac{1-e^{-n^2 t}}{n^2}\right\}$

2. $u(x,t) = \frac{4}{\pi}\sum \frac{e^{-\lambda_{2k+1} kt}}{2k+1}\sin\left[\frac{(2k+1)\pi x}{L}\right] \qquad \lambda_{2k+1} = \left(\frac{(2k+1)\pi}{L}\right)^2$

from which you should do the rest.

3. $u(x,t) = \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{e^{-(n+1/2)^2\pi^2 t}}{(n+1/2)^2} \cos\left[(n+\tfrac{1}{2})\pi x\right]$

4. $n(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n^2\pi^2 D/L^2 - k)t} \sin\left(\frac{n\pi x}{L}\right)$, so we get blowup if $\frac{n^2\pi^2 D}{L^2} < k$ for any n, and in particular $n=1$.

5. $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi c t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$, $b_n = \frac{2}{\pi c} \int_0^L u_t(x,0) \sin\left(\frac{n\pi x}{L}\right) dx$

$$= \frac{2LV}{nc\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right]$$

6. hyperbolic ; leading Terms are $u_{tt} - u_{xx}$ + lower order terms. Apply the classification definition.

$$\left.\begin{aligned} &\varphi'' + (x+1)\varphi' + \lambda\varphi = 0 \\ &\varphi(0) = 0 = \varphi'(\pi) \end{aligned}\right\}$$ This is the eigenvalue problem.

The steady state solution must be $U(x) \equiv 0$.

7. $f(x) \sim \sum_1^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ $\quad b_n = \frac{2}{n\pi}\left[1 + 2\cos\left(\frac{n\pi}{6}\right) - 3\cos\left(\frac{n\pi}{2}\right)\right]$

8. $f(x) \sim 7/6 + \frac{2}{\pi} \sum_1^{\infty} \frac{1}{n}\left[3\sin\left(\frac{n\pi}{2}\right) - 2\sin\left(\frac{n\pi}{6}\right)\right] \cos\left(\frac{n\pi x}{L}\right)$

9. $f(x) \sim \sum_1^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ (you will want to use the addition formulas on the form $\sin A \cos B$),

$$b_n = \begin{cases} 0 & n = \text{odd} \\ \frac{1}{\pi}\frac{4n}{n^2-1} & n = \text{even} \end{cases}$$