$\boxed{\text{Homework \# 10}}$

1. $u(x,y) = X(x)Y(y) \rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\mu \rightarrow$

$$\left.\begin{array}{l} \frac{d^2X}{dx^2} + \mu X = 0 \\ X(0) = 0 = X(\pi) \end{array}\right\} \rightarrow \mu = \mu_n = n^2,\ X = X_n(x) = \sin(nx),\ n = 1, 2, \ldots$$

Also, $\frac{d^2Y}{dy^2} - n^2 Y = 0 \rightarrow Y(y) = Ae^{-ny} + Be^{ny}$, but to preserve boundedness as $y \to \infty$, $B = 0$. Hence, $u(x,y) = \sum_{n=1}^{\infty} A_n e^{-ny}\sin(nx) \rightarrow h(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$. The Fourier coefficients of h are given by $A_n = \frac{2}{\pi}\int_0^{\pi} h(x)\sin(nx)dx$

2. Since $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 2\left(-2x + x^2 + 1 - y^2,\ 2y(-1+x)\right)/(1-2x+x^2+y^2)^2$, then $\frac{\partial u}{\partial y} = 0 \Leftrightarrow$ either $y = 0$ or $x = 1$, and $\frac{\partial u}{\partial x} = 0 \Leftrightarrow$ either $y = x - 1$ or $y = 1 - x$. Thus, there is no intersection of these zero sets in the unit disk, but only on the boundary, so there can't be any minima or maxima in $\{x^2 + y^2 < 1\}$.

I found this example rather interesting because it initially looked like a harmonic function, but failed to have a mean value property ($u(0,0) = 1$, but $u = 0$ almost everywhere on the boundary). The denominator of u is $(x-1)^2 + y^2$, so we have a singularity at $(x,y) = (1,0)$. (To see this let $x = 1 - \varepsilon$, $y = \varepsilon$, then $u = (1-\varepsilon)/\varepsilon$ which blows up as $\varepsilon \to 0$). In an appendix to this answer set I give a picture of the surface, and also a symbolic check of the harmonic nature of the function.

3. (a) A steady state temperature with no dependence on ϕ, θ in the boundary conditions means u only depends on the radial coordinate r. So we only have the radial part of the Laplacian in 3D. Thus

$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)=0 \rightarrow r^2\frac{du}{dr}=c_1 \rightarrow u(r)=-\frac{c_1}{r}+c_2$. After applying the b.c.s, $u(r)=\frac{4\gamma}{r}+100-4\gamma$.

(b) $\frac{du}{dr}=-\frac{4\gamma}{r^2}<0$, so hottest temperature is at $r=1$, namely $u=100°C$, and the coolest temperature is at $r=2$, namely $u(2)=100-2\gamma$ °C.

(c) $u(2)=100-2\gamma=20° \rightarrow \gamma=40$.

4. (a) $\mathcal{F}(e^{-|x|})=\int_{-\infty}^{\infty}e^{-ikx-|x|}dx=\int_{-\infty}^{0}e^{-ikx+x}dx+\int_{0}^{\infty}e^{-ikx-x}dx$

$$=\frac{1}{1-ik}+\frac{1}{1+ik}=\frac{2}{1+k^2}$$

(b) Because $-a(x^2+ikx/a)=-a(x+ik/2a)^2-k^2/4a$

$$\mathcal{F}(e^{-ax^2})=\int_{-\infty}^{\infty}e^{-ikx-ax^2}dx=e^{-k^2/4a}\int_{-\infty}^{\infty}e^{-a(x+ik/2a)^2}dx$$

Let $y=x+ik/2a$, then

$$\mathcal{F}(e^{-ax^2})=e^{-k^2/4a}\int_{-\infty}^{\infty}e^{-ay^2}dy \underset{z=y\sqrt{a}}{\uparrow =} \frac{2e^{-k^2/4a}}{\sqrt{a}}\int_0^{\infty}e^{-z^2}dz$$

$$=\sqrt{\frac{\pi}{a}}\,e^{-k^2/4a}$$

5. $\hat{u}(k,t)=\int_{-\infty}^{\infty}e^{-ikx}u(x,t)dx \rightarrow \begin{cases}\hat{u}_t=(-ik)^2\hat{u}-t\hat{u}=-(k^2+t)\hat{u}\\ \hat{u}(k,0)=\hat{f}(k)\end{cases}$

so $\frac{d}{dt}\left(e^{k^2t+t^2/2}\hat{u}\right)=0 \rightarrow \hat{u}(k,t)=\hat{f}(k)e^{-k^2t-t^2/2}$

Therefore, $u=\mathcal{F}^{-1}(\hat{f}\hat{g})=f*g$, where

$\hat{g}(k,t)=e^{-k^2t-t^2/2}$, so $g(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-k^2t-t^2/2}dk$

Thus,

$$2\pi\, e^{t^2/2} g(x,t) = \int_{-\infty}^{\infty} e^{-t(k^2 - ixk/t)}\, dk$$

$$= e^{-x^2/4t} \int_{-\infty}^{\infty} e^{-t l^2}\, dl$$

$$\left\{ \begin{array}{l} k^2 - ixk/t = \\ (k - ix/2t)^2 + x^2/4t^2 \\ \text{then let} \\ l = k - ix/2t \end{array} \right.$$

$$= \frac{2e^{-x^2/4t}}{\sqrt{t}} \int_{0}^{\infty} e^{-\sigma^2}\, d\sigma \qquad \sigma = l\sqrt{t}$$

$$= \sqrt{\frac{\pi}{t}}\; e^{-x^2/4t}$$

$$\rightarrow g(x,t) = \frac{e^{-x^2/4t - t^2/2}}{2\sqrt{\pi t}}$$

$$\rightarrow u(x,t) = \frac{e^{t^2/2}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(y)\, e^{-(x-y)^2/4t}\, dy$$

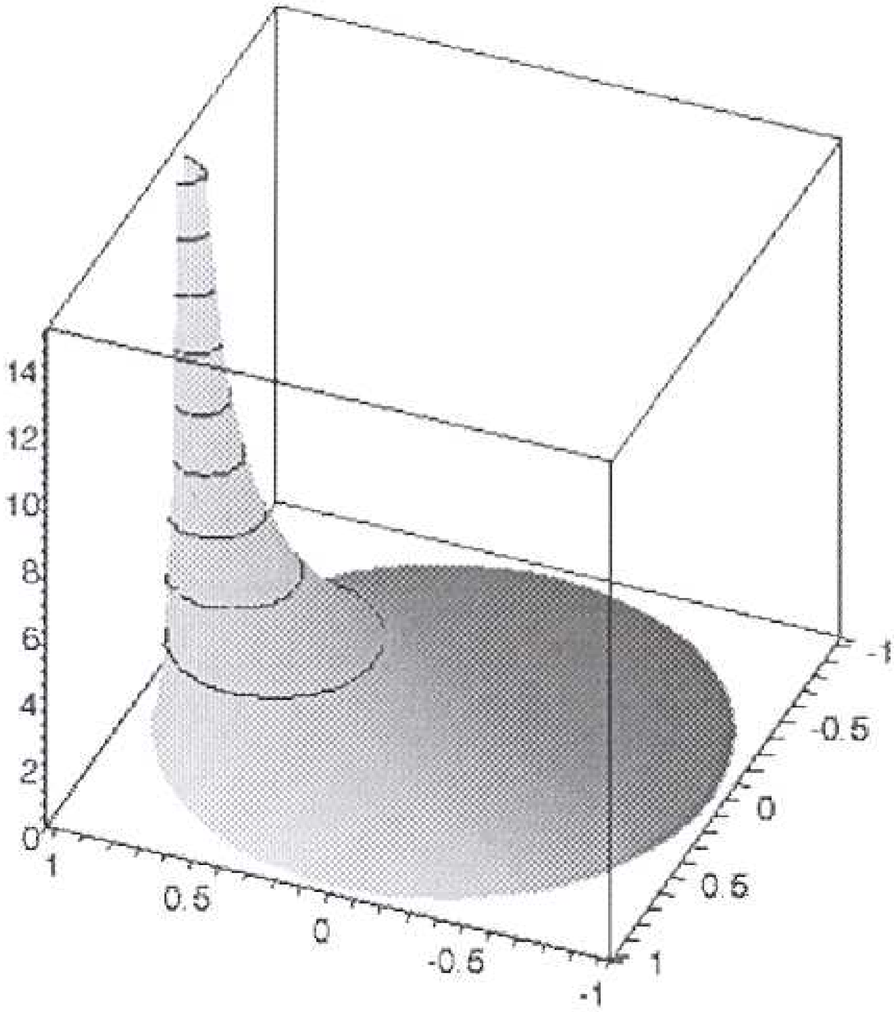
Appendix: graph of u(x,y) from problem 2

```
> u:= (x,y)->simplify((1-x^2-y^2)/(1-2*x+x^2+y^2));
```

$$u := (x,y) \to simplify\left(\frac{1-x^2-y^2}{1-2x+x^2+y^2}\right)$$

```
> x:= r*cos(t): y:= r*sin(t):
```

```
>
plot3d([x,y,u(x,y)],r=0..1,t=0..2*Pi,axes=boxed,grid=[30,65],view=0..15,st
```

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>
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Appendix: u(x,y) from problem 2 is harmonic

```
> u:= (x,y)-> (1 - x^2 - y^2)/(1 - 2*x + x^2 + y^2);
```

$$u := (x, y) \to \frac{1 - x^2 - y^2}{1 - 2x + x^2 + y^2}$$

```
> diff(u(x,y),x,x)+diff(u(x,y),y,y);
```

$$-\frac{4}{1 - 2x + x^2 + y^2} + \frac{4x(-2 + 2x)}{(1 - 2x + x^2 + y^2)^2} + \frac{2(1 - x^2 - y^2)(-2 + 2x)^2}{(1 - 2x + x^2 + y^2)^3} - \frac{4(1 - x^2 - y^2)}{(1 - 2x + x^2 + y^2)^2} + \frac{8y^2}{(1 - 2x + x^2 + y^2)^2}$$
$$+ \frac{8(1 - x^2 - y^2)y^2}{(1 - 2x + x^2 + y^2)^3}$$

```
> simplify(%);
```

$$0$$

```
> U:= (r,t)-> (1 - r^2)/(1 - 2*r*cos(t) + r^2);
```

$$U := (r, t) \to \frac{1 - r^2}{1 - 2r\cos(t) + r^2}$$

```
> r*diff(r*diff(U(r,t),r),r) + diff(U(r,t),t,t);
```

$$r\left(-\frac{2r}{1 - 2r\cos(t) + r^2} - \frac{(1 - r^2)(-2\cos(t) + 2r)}{(1 - 2r\cos(t) + r^2)^2}\right.$$
$$\left. + r\left(-\frac{2}{1 - 2r\cos(t) + r^2} + \frac{4r(-2\cos(t) + 2r)}{(1 - 2r\cos(t) + r^2)^2} + \frac{2(1 - r^2)(-2\cos(t) + 2r)^2}{(1 - 2r\cos(t) + r^2)^3} - \frac{2(1 - r^2)}{(1 - 2r\cos(t) + r^2)^2}\right)\right)$$
$$+ \frac{8(1 - r^2)r^2\sin(t)^2}{(1 - 2r\cos(t) + r^2)^3} - \frac{2(1 - r^2)r\cos(t)}{(1 - 2r\cos(t) + r^2)^2}$$

```
> simplify(%);
```

$$0$$

```
>
```