Homework # 11

1. $\mathcal{F}_c(e^{-ax}) = \int_0^\infty e^{-ax}\cos(kx)\,dx$

If you do two integrations-by-parts on the integral $I = \int e^{-ax}\cos(kx)\,dx$, you get

$$I = \frac{e^{-ax}(k\sin kx - a\cos kx)}{a^2+k^2}$$

So $\mathcal{F}_c(e^{-ax}) = I\Big|_{x=0}^{x=\infty} = \dfrac{a}{a^2+k^2}$

2. $\mathcal{F}_s(u(x,y)) = \hat{u}(k,y) = \int_0^\infty u(x,y)\sin kx\,dx \rightarrow u(x,y) = \mathcal{F}_s^{-1}(\hat{u})$

$$= \frac{2}{\pi}\int_0^\infty \hat{u}(k,y)\sin kx\,dk$$

and $\mathcal{F}_s(u_{xx}) = -k^2\hat{u} + k\,u(0,y)$, so application to pde $\rightarrow$

$-k^2\hat{u} + k + \hat{u}_{yy} = 0 \quad y>0$

$\hat{u}(k,0) = 0$, $\hat{u}$ remains bounded as $y \to \infty$

$\rightarrow \hat{u} = Ae^{-ky} + \underbrace{B}_{0}e^{ky} + \frac{1}{k} = \frac{1}{k}(1-e^{-ky})$

$\quad$ 0 boundedness condition

$\rightarrow u(x,y) = \dfrac{2}{\pi}\int_0^\infty (1-e^{-ky})\dfrac{\sin(kx)}{k}\,dk = 1 - \dfrac{2}{\pi}\int_0^\infty e^{-ky}\dfrac{\sin kx}{k}\,dk$

remark: see Note after #5 below

because $\int_0^\infty \frac{\sin kx}{k}\,dk = \frac{\pi}{2}$ $(x>0)$ (brought up in class).

Note: It is easy to check that u is harmonic in the positive quadrant, and satisfies the boundary conditions.

3. (a) $F(s) = \int_0^\infty e^{-st}f(t)\,dt$, so $\mathcal{L}(-tf(t)) = \int_0^\infty f(t)(-te^{-st})\,dt =$

$$\int_0^\infty f(t)\frac{d}{ds}(e^{-st})\,dt = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \frac{dF}{ds}(s)$$

(b) $\mathcal{L}(H(t-b)f(t-b)) = \int_0^\infty e^{-st}H(t-b)f(t-b)\,dt$

$$= \int_b^\infty e^{-st}f(t-b)\,dt$$

$$\underset{\tau=t-b}{=} \int_0^\infty e^{-s(\tau+b)}f(\tau)\,d\tau$$

$$= e^{-sb}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau = e^{-bs}F(s)$$

4. We can write f as $f(t) = t^2[H(t-5) - H(t-8)]$. What we have to work with is the shift formula of exercise 3(b). Thus, for example, we need to change t^2 into a polynomial in $t-5$ when being multiplied by $H(t-5)$. Therefore,

$$H(t-5)t^2 - H(t-8)t^2 = H(t-5)\{(t-5)^2 + 10(t-5) + 25\}$$
$$- H(t-8)\{(t-8)^2 + 16(t-8) + 64\}$$

$$\to \mathcal{L}(f(t)) = e^{-5s}\mathcal{L}(t^2 + 10t + 25) - e^{-8s}\mathcal{L}(t^2 + 16t + 64)$$
$$= e^{-5s}\left(\frac{2}{s^3} + \frac{10}{t^2} + \frac{25}{s}\right) - e^{-8s}\left(\frac{2}{s^3} + \frac{16}{s^2} + \frac{64}{s}\right)$$

5. $U(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt \to s^2 U - s u(x,0) - u_t(x,0) = c^2 U_{xx}$

or $U_{xx} - \frac{s^2}{c^2} U = -\frac{s}{c^2}\sin x$. A particular solution

has the form $A \sin x$, so substituting gives

$$-A\sin x - \frac{s^2}{c^2} A \sin x = -\frac{s}{c^2}\sin x \to A = \frac{s}{s^2 + c^2}.$$

Thus, $U(x,s) = A_1 e^{xs/c} + A_2 e^{-xs/c} + \frac{s}{s^2+c^2}\sin x$

$$= \frac{s}{s^2 + c^2}\sin x$$

because we require boundedness in u, hence U, as $x \to \pm\infty$. Therefore,

$$u(x,t) = (\sin x)\mathcal{L}^{-1}\left(\frac{s}{s^2+c^2}\right) = \sin(x)\cos(ct).$$

Note on #2: On first inspection, since $x>0$ & $y>0$ the question might be raised why not take the Fourier sine transform in y: $U(x,k) = \int_0^\infty u(x,y)\sin(ky)\,dy$. From taking the transform of the equation you would obtain $U_{xx} - kU + k\overset{0}{\cancel{u(x,0)}} = 0 \Rightarrow$ $U = Ae^{-kx} + Be^{kx}$. But because of the boundedness requirement, $B = 0$ ($k>0$ here). But then you need $U(0,k) = \int_0^\infty \underbrace{u(0,y)}_{1} \sin ky\,dy$, but this integral does not exist!