Homework_5

1. $u(x,0) = f(x) = e^x$, $\frac{\partial u}{\partial t}(x,0) = g(x) = \sin x$, so by d'Alembert's formula, $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

$$= e^x\left(\frac{e^{ct}+e^{-ct}}{2}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct}\sin(s)\,ds$$

$$= e^x\cosh(ct) + \frac{1}{2c}[\cos(x-ct) - \cos(x+ct)] \text{, or}$$

$$= e^x\cosh(ct) + \frac{1}{c}\sin(x)\sin(ct)$$

2. (a) $u_{tt} = c^2\{u_{rr} + \frac{2}{r}u_r\}$ $\quad u(r,0) = \phi(r)$, $u_t(r,0) = \psi(r)$

$v = ru \rightarrow u_{tt} = \frac{1}{r}v_{tt}$, $u_r = \frac{1}{r}v_r - \frac{1}{r^2}v \rightarrow u_{rr} = \frac{1}{r}v_{rr} - \frac{2}{r^2}v_r + \frac{2}{r^3}v$

so

$$\frac{1}{r}v_{tt} = c^2\left\{\frac{1}{r}v_{rr} - \frac{2}{r^2}v_r + \frac{2}{r^3}v + \frac{2}{r}\left[\frac{1}{r}v_r - \frac{1}{r^2}v\right]\right\}$$

$$= c^2\frac{1}{r}v_{rr} \quad \text{or} \quad v_{tt} = c^2 v_{rr}$$

(b) Thus, $v(r,t)$ is of the form $v(r,t) = F(r-ct) + G(r+ct)$

$\rightarrow u(r,t) = \frac{1}{r}\{F(r-ct) + G(r+ct)\}$ where F, G are arbitrary differentiable functions

(c) If $v(r,0) = \phi^*(r)$, $v_t(r,0) = \psi^*(r)$ then d'Alembert's solution is $v(r,t) = \frac{1}{2}[\phi^*(r-ct) + \phi^*(r+ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct}\psi^*(s)\,ds$

But because $\phi^*(r) = r\phi(r)$, $\psi^*(r) = r\psi(r)$, then

$$u(r,t) = \frac{1}{2r}[(r-ct)\phi(r-ct) + (r+ct)\phi(r+ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)\,ds$$

3. $u(x,t) = T(t)\varphi(x) \rightarrow (\rho_0\ddot{T} + \beta\dot{T})/\tau_0 T = \varphi''/\varphi = -\lambda$

$$\rightarrow \text{EVP: } \left\{\begin{array}{ll}\varphi'' + \lambda\varphi = 0 & 0 < x < l \\ \varphi(0) = \varphi(l) = 0 & \end{array}\right\} \rightarrow \begin{array}{l}\lambda = \lambda_n = (n\pi/l)^2, n = 1,2,\ldots \\ \varphi = \varphi_n(x) = \sin\left(\frac{n\pi x}{l}\right)\end{array}$$

and

$$\rho_0\ddot{T} + \beta\dot{T} + \lambda_n\tau_0 T = 0 \quad n = 1,2,\ldots$$

(This is basically the damped spring equation)

$T(t) = e^{rt} \rightarrow e^{rt}\{\rho_0 r^2 + \beta r + \lambda_n\tau_0\} = 0$

so r must satisfy

$$\rho_0 r^2 + \beta r + \lambda_n\tau_0 = 0$$

Now $~~r~~ = r_n^{\pm} = \frac{1}{2\rho_0}\left\{-\beta \pm \sqrt{\beta^2 - 4\rho_0\tau_0\lambda_n}\right\}$

Given that $\beta^2 - 4\rho_0\tau_0\lambda_1 < 0$, and since $\lambda_1 < \lambda_2 < \cdots$ then $\beta^2 - 4\rho_0\tau_0\lambda_n < 0$ for all $n \geq 1$. For convenience let $\omega_n^2 \doteq 4\rho_0\tau_0\lambda_n - \beta^2 > 0$; then $r_n^{\pm} = -\frac{\beta}{2\rho_0} \pm i\frac{\omega_n}{2\rho_0}$, so a fundamental set of solutions for $T(t) = T_n(t)$ is $\left\{ e^{-\beta t/2\rho_0}\cos\left(\frac{\omega_n t}{2\rho_0}\right),\ e^{-\beta t/2\rho_0}\sin\left(\frac{\omega_n t}{2\rho_0}\right)\right\}$. Hence

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\beta t/2\rho_0}\left\{a_n\cos\left(\frac{\omega_n t}{2\rho_0}\right) + b_n\sin\left(\frac{\omega_n t}{2\rho_0}\right)\right\}\sin\left(\frac{n\pi x}{l}\right)$$

As $t \to 0$ $u \to f(x)$, $u_t \to g(x)$, so

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right) \quad \longrightarrow \quad a_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$u_t(x,t) = -\frac{\beta}{2\rho_0}e^{-\beta t/2\rho_0}\sum_1^{\infty}\left\{a_n\cos\left(\frac{\omega_n t}{2\rho_0}\right) + b_n\sin\left(\frac{\omega_n t}{2\rho_0}\right)\right\}\sin\left(\frac{n\pi x}{l}\right)$$

$$+\ e^{-\beta t/2\rho_0}\sum_1^{\infty}\frac{\omega_n}{2\rho_0}\left\{-a_n\sin\left(\frac{\omega_n t}{2\rho_0}\right) + b_n\cos\left(\frac{\omega_n t}{2\rho_0}\right)\right\}\sin\left(\frac{n\pi x}{l}\right)$$

so, $t \to 0$, we obtain

$$g(x) = -\frac{\beta}{2\rho_0}\sum_1^{\infty} a_n\sin\left(\frac{n\pi x}{l}\right) + \frac{1}{2\rho_0}\sum_1^{\infty}\omega_n b_n\sin\left(\frac{n\pi x}{l}\right)$$

Since the first series on the right side is known, we can compute b_n's in the usual way by using the orthogonality of the sines. Hence

$$b_m = \frac{4\rho_0}{\omega_m\rho_0 l}\int_0^l\left[g(x) + \frac{\beta}{2\rho_0}f(x)\right]\sin\left(\frac{m\pi x}{l}\right)dx.$$