Some Calculus background you should be familiar with, or review, for Math 404

- Integration and differentiation of common functions (trig, exponential, logs, etc.).

  A handy result concerning the integral $I(t) = \int_{a(t)}^{b(t)} f(x,t)dx$.

  Theorem: if *f(x,t)* and $\partial f / \partial t$ are continuous on the rectangle [A,B] x [c,d], where [A,B] contains the union of all the intervals [a(t),b(t)], and if a(t) and b(t) are differentiable on [c,d], then

  $$\frac{\partial I}{\partial t} = \frac{\partial}{\partial t}\int_{a(t)}^{b(t)} f(x,t)dx = f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)dx \ .$$

- Understanding partial derivatives, the chain rule, etc.
- Integration-by-parts
- Convergence of series, uniform convergence if terms depend on a variable
- **Addition formulas** (trig):

  $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

  $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

  So, for instance, $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$; thus, for example, $\sin^2(x) = (1 - \cos(2x))/2$.

- **Hyperbolic functions**: $\sinh(x) = (e^x - e^{-x})/2$, $\cosh(x) = (e^x + e^{-x})/2$, tanh(x), etc.

  $\frac{d}{dx}\sinh(x) = \cosh(x), \quad \frac{d}{dx}\cosh(x) = \sinh(x),$ etc.

  $\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$

  $\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$

- If $f = f(x,y,z)$ is a scalar function, and $F = (F_1, F_2, F_3)$ is a vector function, then the notation for the *gradient* of *f* is given by $\nabla f = grad(f) = (f_x, f_y, f_z)$, where $f_x = \partial f / \partial x$, etc. Here *f* has domain in 3-space (that is, in $\Re^3$), but we have the analogous formulae in the plane or in n-space. The *directional derivative* of *f* at the (vector) point *a* in the direction of the vector *v* is

  $$\lim_{t \to 0} \frac{f(a + tv) - f(a)}{t} = v \cdot \nabla f(a) \ .$$

  It follows that the rate of change of a quantity *f(x)* seen by a moving particle *x(t)* is $(d/dt)f(x) = \nabla f \cdot (dx/dt)$.

  The *divergence* of the vector function *F* is given by

  $divF = \nabla \bullet F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$. Therefore, the Laplacian of u is

$$\Delta u(x, y, z) = \nabla^2 u = divgrad(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad .$$

Also, $|\nabla u|^2 = |grad(u)|^2 = (u_x)^2 + (u_y)^2 + (u_z)^2$.

If an equation involves a solution needing partial derivatives of second order, then we are interested in a continuous function on a domain D that has *continuous* partial derivatives up to order two, and we say the function is of class C^2(D). If the function only needs continuous partial derivatives of first order, then the function is of class C^1(D). For example, if we are interested in solving the one-dimensional heat equation for u on the domain $a < x < b$, $0 < t < T$, then u should be of class $C^2((a,b))$ in x, $C^1((0,T))$ in t.

Green's Theorem: Let D be a bounded planar domain with piecewise C^1 boundary curve C. (sometimes C is denoted ∂D). Consider C parameterized such that it is traversed once with D on the left (traversed counterclockwise). Let p(x,y) and q(x,y) be any C^1 functions defined on the closure of D (D + C, i.e. the union of the two sets, i.e. cl(D)). Then $\iint_D (q_x - p_y)dxdy = \int_C pdx + qdy$.

A completely equivalent formulation of Green's theorem is obtained by substituting p = -g and q = +f. If F = (f,g) is any C^1 vector field in cl(D), then $\iint_D (f_x + g_y)dxdy = \int_C (-gdx + fdy)$. If n is the unit outward-pointing normal vector on C, then n = (+dy/ds, -dx/ds). Hence, Green's theorem takes the form $\iint_D \nabla \bullet Fdxdy = \int_C F \cdot nds$, where $\nabla \bullet F = f_x + g_y$ denotes the divergence of F.

Series: we are going to be dealing with series of functions (Fourier series), so you should recall a few things about sequences and series.

Def'n: **convergence of a series** of (real) numbers: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$ converges if the tail end can be made arbitrarily small; i.e. given any tolerance ε > 0, there is an M > 1 such that for m > M, $|\sum_{n=m}^{\infty} a_n| < \varepsilon$.

Def'n: **absolute convergence** of a series: $\sum_1^{\infty} a_n$ converges absolutely if $\sum_1^{\infty} |a_n|$ converges.

Remark: the **Comparison Test**: If $|a_n| \le b_n$ for all n, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely. The contrapositive necessarily follows: If $\sum_{n=1}^{\infty} |a_n|$ diverges, so does $\sum_{n=1}^{\infty} b_n$. The limit comparison test states that if $a_n \ge 0, b_n \ge 0$, if $\lim_{n\to\infty} a_n / b_n = L$, where $0 \le L < \infty$, and if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

Remark: the **ratio test**: the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\frac{|a_{n+1}|}{|a_n|} \le \rho < 1$ for some constant ρ, and for $n \ge N \ge 1$. (We don't care if the inequality is not met for the first N terms.)

Examples: For $\sum_{n=1}^{\infty} (\frac{1}{2})^n$, so $a_n = 1/2^n$, hence $\frac{|a_{n+1}|}{|a_n|} = \frac{1}{2} = \rho$, so series is absolutely convergent. For $\sum_1^{\infty} \frac{1}{n}$, $\frac{|a_{n+1}|}{|a_n|} = \frac{n}{n+1} \to 1$. Hence, there is no upper bound less than 1, so the ratio test fails, i.e. give no information. This series actually diverges. The ratio test also fails for the series $\sum_1^{\infty} \frac{1}{n^2}$, but the series converges to $\pi^2/6$. In fact, the p-series $\sum_1^{\infty} 1/n^p$ converges for p > 1, and diverges (is infinite) for $p \le 1$.

Def'n: **uniform convergence of sequence of functions**: assume the sequence $\{f_n(x)\}_{n=1,2,\ldots}$ of functions is defined on an interval *I* of the real numbers. Then $\{f_n\}$ converges uniformly on *I* to f(x) if for any tolerance $\varepsilon > 0$, there is an M such that for $m > M$, $|f_m(x) - f(x)| < \varepsilon$ for all x in *I*.

Def'n: **uniform convergence of a series of functions**: with f_n 's defined on interval I, $\sum_1^{\infty} f_n(x)$ converges *uniformly* on I to *f(x)* if the sequence of partial sums $\{s_N\}_{N=1,2,\ldots}$, $s_N = \sum_{n=1}^{N} f_n(x)$, converges uniformly to *f(x)* on *I*.

Comparison Test: If $|f_n(x)| \le c_n$ for all n and for all $a \le x \le b$, where the c_n*'s* are constants, and if $\sum_1^{\infty} c_n$ converges, then $\sum_1^{\infty} f_n(x)$ converges *uniformly* in the interval [a,b], as well as absolutely.

Convergence Theorem: If $\sum_1^{\infty} f_n(x)$ converges uniformly to *f(x)* in [a,b] and if all the functions $f_n(x)$ are continuous in [a,b], then the sum *f(x)* is also continuous in [a,b] and $\sum_1^{\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

The last statement is called term-by-term integration.

Convergence of Derivatives: If all the functions $f_n(x)$ are differentiable in [a,b] and if the series $\sum_1^{\infty} f_n(c)$ converges for some *c*, and if the series of derivatives $\sum_1^{\infty} f'_n(x)$ converges *uniformly* in [a,b], then $\sum_1^{\infty} f_n(x)$ converges uniformly to a function *f(x)* and $\sum_1^{\infty} f'_n(x) = f'(x)$.

Supplementary Problems from Calculus

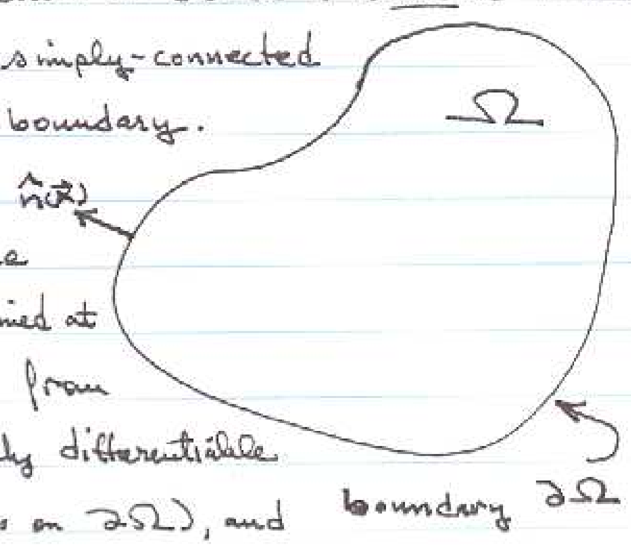
1. Given $\int_{a(t)}^{b(t)} g(y)\,dy$, then a special form of the Leibniz rule is $\frac{d}{dt}\int_{a(t)}^{b(t)} g(y)\,dy = g(b(t))\frac{db}{dt}(t) - g(a(t))\frac{da}{dt}(t)$, where it is assumed g is piecewise continuous and a & b are differentiable in t. (The full statement of Leibniz rule is my 1^{st} theorem in my writeup "needed Calculus background.pdf")
Use this rule to verify that $u(x,t) = \int_{x-t}^{x+t} g(y)dy$ is a solution to $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$.

2. Use the addition formulas to show that
a) $\sin(11\pi x)\cos(10\pi x) = \frac{1}{2}\sin(21\pi x) + \frac{1}{2}\sin(\pi x)$
b) $\sin(5x)\sin(2x) = \frac{1}{2}\cos(3x) - \frac{1}{2}\cos(7x)$, and thus
$$\int_0^{\pi} \sin(5x)\sin(2x)\,dx = 0$$

3. a) define $\cosh(2x)$ and graph the function $u(x) = \cosh(2x) - 1$.
b) define $\tanh(x)$ and sketch a graph of it on $\mathbb{R} = (-\infty,\infty)$.

4. Suppose $\varphi = \varphi(x,y)$ is twice continuously differentiable in the xy plane and $\underline{u} = \underline{u}(x,y)$ is the vector grad φ. (If we think of $\underline{u}$ as a velocity vector, φ is called the velocity potential.) If div $\underline{u} = 0$, show this means $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$ in the plane. (such a function is called a harmonic function)

5. Let Ω be a open, bounded, simply-connected set in the plane with smooth boundary.
(simply-connected means no holes; Smooth boundary means there is a unit ~~outer~~ normal vector $\hat{n}$ defined at every point on the boundary). Recall from calculus that G is a continuously differentiable function defined Ω (continuous on $\partial\Omega$), and $\underline{F}$ is a vector-valued function defined on Ω and its boundary, and



is continuously differentiable in Ω, then

$$(*) \quad \iint_{\Omega} \{ G \operatorname{div} \underline{F} + \underline{F} \cdot \operatorname{grad} G \} d\underline{x} = \int_{\partial\Omega} (G\underline{F}) \cdot \hat{n} \, ds$$

If $\underline{F} = \operatorname{grad} u$ for some smooth scaler function $u(x,y)$ defined on Ω and its boundary, then use (*) to show

$$(a) \quad \iint_{\Omega} \{ G \nabla^2 u + \operatorname{grad} u \cdot \operatorname{grad} G \} d\underline{x} = \int_{\partial\Omega} G \, (\hat{n} \cdot \operatorname{grad} u) \, ds$$

($\nabla^2 u$ = Laplacian of u = $\operatorname{div}(\operatorname{grad} u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$)

and

$$(b) \quad \iint_{\Omega} \{ u \nabla^2 G - G \nabla^2 u \} d\underline{x} = \int_{\partial\Omega} \{ (u \operatorname{grad} G - G \operatorname{grad} u) \cdot \hat{n} \} ds$$

((a) and (b) here are called Green's First and Second Identity, resp.)

6. Sketch a graph of $\tan(x)$ for $x > 0$ and superimpose on the graph $x/2$. Numerically approximate the first 5 solutions to the transcendental equation

$$\tan(x) = x/2 \, .$$