Some Review of ODEs

Because of Principle #1, you should review your course notes from elementary ordinary differential equations and keep your old textbook handy. The comments here are not meant to be a substitute.

For our purposes we are only concerned with solving linear first order and second order odes. In your first course you probably only discussed initial value problems, that is time-dependent problems defined for time $t > t_0$. The new topic in odes we need for this pde course is to solve (two-point) boundary value problems. After reviewing some specific aspects of initial value problems, we'll start the discussion of a very special class of boundary value problems.

Unlike pdes, odes can be classified more or less by order. The order of an ode is the order of the highest derivative appearing in the equation.

Linear first-order ODEs

These have the form $A\frac{dy}{dt} + By + C = 0$, where A, B, C can depend, at most, on the variable t, and we assume $A \neq 0$ either as a constant, or as a function of t. For precise conditions to have the existence of a global solution, consult any ODE book. If we let $p = p(t) \doteq B/A$, $q = q(t) \doteq -C/A$, then we can write the equation in the form

(1) $$\frac{dy}{dt} + p(t)y = q(t)$$

examples: (a), (b), (c) are linear, (d) and (e) are not

(a) $\frac{dy}{dt} = \sin(2t-1)$ (b) $(1+t^2)\frac{dy}{dt} + \ln(t)\,y = e^{t}$

(c) $\sin(2t)\,dy + \cos(2t)\,dt = 0$ (d) $4\frac{dy}{dt} + 3y^2 = \tanh(t)$

(e) $y\frac{dy}{dt} + ty + t^2 = 0$

Remark: given a 1^{st} order equation, think that it takes an integration to solve it, so the general solution to a 1^{st} order equation involves a constant of integration. That is, without any extra information about the solution function besides the equation, the solution is a 1-parameter family of functions

example: $\frac{dy}{dt} = 1 \longrightarrow y(t) = t + C$

$\frac{dy}{dt} + \sin(t)\, y = 0 \longrightarrow y(t) = C e^{\cos(t)}$

Thus, a well-posed problem includes the equation, the domain it holds over, and a value for the solution to have at a given point.

example: $\frac{dy}{dt} + \sin(t)\, y = 0$, $t > 0$, with $y(0) = 1$

the solution to this problem is $y(t) = e^{-1+\cos(t)}$.

Note: the equation need not hold at initial point $t=0$ because we do not require you to have a derivative there (though in this example the derivative does exist and is continuous)

First order linear equations should be solved by the method of integrating factors. Given (1), integrate $p(t)$, then exponential the result. That expression is the integrating factor (IF). For example, in the example above, $e^{-\cos(t)}$ is the IF. Multiply the equation by the IF and combine the two terms on the left hand side of the equation into an exact derivative.

example: $\frac{dy}{dt} + \sin(t)\, y = 0$ $\int \sin(t)\, dt \to -\cos(t)$ up to a constant of integration, so $IF = e^{-\cos(t)} \longrightarrow$

$$e^{-\cos(t)} \frac{dy}{dt} + e^{-\cos(t)} \sin(t)\, y = \frac{d}{dt}\left[e^{-\cos(t)} y(t) \right] = 0 \longrightarrow$$

$e^{-\cos(t)} y(t) = C$ = constant of integration. Thus, $y(t) = Ce^{\cos(t)}$.

If $y(0) = 1$ then $y(0) = Ce^{1} \to C = e^{-1} \to y(t) = e^{-1}e^{\cos(t)} = e^{-1+\cos(t)}$.

example: $\frac{dy}{dt} + py = q$ p, q are constants

IF $= e^{pt} \to e^{pt}\frac{dy}{dt} + pe^{pt}y = qe^{pt}$ or $\frac{d}{dt}[e^{pt}y] = qe^{pt}$

$\to e^{pt}y(t) = \frac{q}{p}e^{pt} + C \to y(t) = \frac{q}{p} + Ce^{-pt}$

Comment: if you know the domain of the problem it is best to use that information but performing definite integration rather than indefinite integration. For example, if in the last example, we have it hold for $t>0$, and $y(0)=2$, then write $\frac{d}{dt}[e^{pt}y] = qe^{pt} \to e^{pt}y(t) - y(0) = \int_0^t qe^{pt'}dt'$

or $e^{pt}y(t) - 2 = \frac{q}{p}e^{pt'}\Big|_0^t = \frac{q}{p}(e^{pt}-1) \to y(t) = 2e^{-pt} + \frac{q}{p}(1-e^{-pt})$

Then one clearly sees $y(0)=2$ and $y(t) \to q/p$ as $t \to \infty$.
The q/p is y's steady state, or equilibrium value.

example: $\frac{dy}{dt} - 2ty = 1$, $y(0) = 3$, to hold for $t>0$

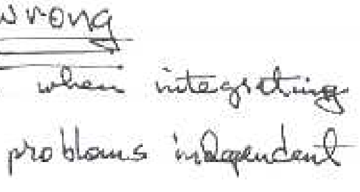
the IF is $e^{-t^2} \to \frac{d}{dt}[e^{-t^2}y] = e^{-t^2} \to$

$$e^{-t^2}y(t) - y(0) = \int_0^t e^{-\bar{t}^2}d\bar{t} \to y(t) = 3e^{t^2} + e^{t^2}\int_0^t e^{-\bar{t}^2}d\bar{t}$$

Two remarks: note that when I integrated I used "dummy" variable of integration, here denoted $\bar{t}$ arbitrarily.
That is $e^{-t^2}y(t) - 3 = \int_0^t e^{-t^2}dt$ is wrong
Always use a dummy variable of integration when integrating where the integration range depends on the problems independent variable. You will be penalized if you do not.

Secondly, a special function that comes up often in pdes (particularly in heat and mass transfer), is the error function

$$\mathrm{erf}(z) \doteq \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$$



Note the following properties:

i) $\mathrm{erf}(0) = 0$ ii) $z>0 \to \mathrm{erf}(z) > 0$

iii) $\frac{d}{dz}\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}e^{-z^2} > 0$, so erf is monotone increasing

iv) $z<0$ $\mathrm{erf}(z) = -\mathrm{erf}(-z)$, i.e. erf is odd. It's graph looks like

v) $\int_0^\infty e^{-s^2}ds = \frac{\sqrt{\pi}}{2}$ so $\mathrm{erf}(z) \to 1$ as $z \to +\infty$.

Therefore, the solution to the last example could be written as $y(t) = 3e^{t^2} + \frac{\sqrt{\pi}}{2} e^{t^2} \operatorname{erf}(t)$

You will see $\operatorname{erf}(x)$ again in the pde course

exercises (practice)

1) $\frac{dy}{dt} + 3y = 0$ ans: $y(t) = Ce^{-3t} = y(0)e^{-3t}$

2) $t\frac{dy}{dt} + 4y = t^2$ on any interval not containing $t=0$
ans: $y(t) = \frac{t^2}{6} + C/t^4$

3) $(\sin(t))\frac{dy}{dt} + \cos(t)\, y = \cos(t)$, $\pi/2 < t < \pi$ ans: $y(t) = 1 + C\csc(t)$

4) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = 2$
hint: substitute $v = \frac{dy}{dt}$ to get a 1st order equation in v
but being a 2nd order equation, ans: $y(t) = \frac{1}{2}t + C_1e^{-4t} + C_2$
there will be 2 constants of integration.

Linear Second-order ODEs

These have the form $A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy + D = 0$ ($A \neq 0$), but we most commonly write the equation in the form

(2) $a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)\, y = f(t)$

where the coefficients are continuous on the interval (2) is supposed to hold on, and $a(t)$ is not 0 in the interval (but can vanish at the initial, i.e. endpoint, of the interval). The f term might be a sum of terms, but it does not depend on y or any of its derivatives. If (2) is such that f is identically zero ($f \equiv 0$), the equation is <u>homogeneous</u>. Otherwise, $f \neq 0$, the equation is <u>nonhomogeneous</u>. If (2) has variable coefficients a, b, c, there is no general technique for obtaining an explicit solution. If we have a solution, we can obtain another, linearly independent solution, by <u>reduction of order</u> method you can look up in your ODE textbook.

So the strategy, given (2), with coefficients defined and continuous on some interval, say $t>0$, is to find two linearly independent solutions, say $y_1(t)$ & $y_2(t)$. Then the general solution is given by their linear combination: $y(t) = C_1 y_1(t) + C_2 y_2(t)$. The two constants C_1, C_2 are found by applying the two initial conditions.

Lemma: Let $y_1(t), y_2(t)$ be two solutions to (2) for $t>0$. Define the Wronskian $W = W(f(t), g(t)) = f(t)g'(t) - g(t)f'(t)$

$$= \det \begin{bmatrix} f(t) & g(t) \\ \frac{df}{dt}(t) & \frac{dg}{dt}(t) \end{bmatrix}$$, for any differentiable functions f & g.

Then either $W(y_1(t), y_2(t)) \equiv 0$ (y_1 and y_2 are linearly dependent), or else $W(y_1(t), y_2(t)) \neq 0$ for any t in the domain (y_1 & y_2 are linearly independent on the domain). In the latter case, y_1, y_2 are called a fundamental set of solutions for (2).

example: Consider $\frac{d^2y}{dt^2} - 4y = 0$ on the real line. $y_1(t) = e^{2t}, y_2(t) = e^{-2t}$ form a fundamental set of solutions for the equation. $(W(e^{2t}, e^{-2t}) = -2e^{2t}e^{-2t} - 2e^{-2t}e^{2t} = -4$ for all $t)$

But so is $y_1(t) = \sinh(2t)$, $y_2(t) = \cosh(2t)$.

We'll just mention two classes of homogeneous, 2^{nd} order (linear) odes, the constant coefficient ones, and the Cauchy-Euler equations. (These are actually equivalent; there is a transcendental transformation that can convert one type to the other type.) Consider

(3) $$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \qquad a, b, c \text{ are constant}$$

Thus, the equation holds on the whole real line. Assume a solution that is a simple exponential:

(4) $$y(t) = e^{rt}$$

Substitute (4) into (3) $= e^{rt}\{ar^2 + br + c\} = 0$.

Since e^{rt} is never zero, this holds and (4) is a solution to (3) if and only if

(5) $$ar^2 + br + c = 0 \rightarrow r = r_{1,2} = \frac{1}{2a}\{-b \pm \sqrt{b^2 - 4ac}\}$$

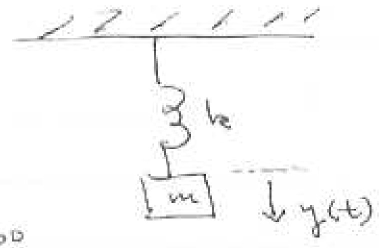
(5) is called the <u>characteristic</u> <u>equation</u> for (3). There are 3 non-degenerate possibilities regarding (5).

(i) roots r_1, r_2 are real, unequal: then a fundamental set of solutions is $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$ and the general solution is given by $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

(ii) roots are real, equal: then a fundamental set of solution is given by $y_1 = e^{rt}$, $y_2 = te^{rt}$ and the general solution is $y = (C_1 + C_2 t) e^{rt}$.

(iii) roots are complex conjugates, say $r_{1,2} = a \pm bi$, with a, b real ($i = \sqrt{-1}$). Then a fundamental set of solutions is given by $y_1 = e^{at}\cos(bt)$, $y_2 = e^{at}\sin(bt)$, and the general solution is $y = e^{at}\{C_1 \cos(bt) + C_2 \sin(bt)\}$

<u>example</u>: spring-mass model



$$m\frac{d^2y}{dt^2} + ky = 0$$

Let $\omega \doteq \sqrt{k/m} \implies \frac{d^2y}{dt^2} + \omega^2 y = 0$

Comment: ω has units of 1/time, so it is called the natural frequency of the system.

$y = e^{rt} \to$ the characteristic equation is $r^2 + \omega^2 = 0 \to$ $r = \pm i\omega \to y_1 = \cos(\omega t)$, $y_2 = \sin(\omega t)$ form a fundament set, so a general solution is $y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$.

By use of the <u>addition formulas</u> for the trig functions to match coefficients, the general solution here can be written equivalently as $y(t) = a\cos(\omega t - \phi)$. a is the <u>amplitude</u> of the motion, which is oscillatory, and ϕ is the <u>phase</u> of the motion. These two numbers uniquely characterize the motion if we are given initial conditions.

<u>example</u>: spring-mass-dashpot system (so we include frictional forces here):

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0 \qquad t > 0$$

$$y(0) = 2, \quad \frac{dy}{dt}(0) = 0$$

Thus, we pull the mass down 2 units of length and release it.

Now, substituting $y = e^{rt}$, the characteristic equation is $mr^2 + cr + k = 0$. Think of mass m and spring constant k being fixed, but constant c, the damping coefficient, can take on different values. The possibilities are

(i) the roots are real and negative (discriminant is $c^2 - 4km > 0$)
thus, $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} = e^{-ct/2m}\{C_1 e^{t\sqrt{c^2-4km}/2m} + C_2 e^{-t\sqrt{c^2-4km}/2m}\}$
Now apply the initial conditions: $y(0) = 2 = C_1 + C_2$;
$\frac{dy}{dt}(0) = r_1 C_1 + r_2 C_2 = r_1 C_1 + r_2(2 - C_1) = 0$. Solving the two equations for the two unknowns gives
$y(t) = 2\left[\frac{r_1}{r_1 - r_2} e^{r_1 t} - \frac{r_2}{r_1 - r_2} e^{r_2 t}\right]$

(ii) the roots are real, equal (and negative; i.e. $c^2 - 4km = 0$ and $r_1 = r_2 = -c/2m$). Then $y(t) = e^{-ct/2m}\{C_1 + C_2 t\}$.
Now $y(0) = 2 = C_1$, $\frac{dy}{dt}(0) = 0 = -\frac{c}{m} + C_2 \rightarrow y(t) = e^{-ct/2m}\{2 + \frac{ct}{m}\}$

(iii) the roots are complex conjugates (so $c^2 - 4km < 0$, so let $\sqrt{4km - c^2}/2m \doteq \beta$; then $r_{1,2} = -\frac{c}{2m} \pm i\beta$). Now
$y(t) = e^{-ct/2m}\{C_1 \cos\beta t + C_2 \sin\beta t\}$, and $y(0) = C_1 = 2$,
$\frac{dy}{dt}(0) = \beta C_2 - \frac{c}{2m} C_1 = 0 \rightarrow y(t) = e^{-ct/2m}\{2\cos\beta t + \frac{c}{m\beta}\sin\beta t\}$

$$= e^{-ct/2m}\left\{2\cos\beta t + \frac{2c}{\sqrt{4km - c^2}}\sin\beta t\right\}$$

$$= 4\sqrt{\frac{km}{4km - c^2}}\cos(\beta t - \varphi)$$

where $\varphi = \arctan\left(\frac{c}{2m\beta}\right)$. In case (i) both terms decay exponentially and this is the overdamped case (decay, but no oscillations; c is considered "large"). Case (ii) is the damped oscillations (c is "small") and is called the underdamped case. Case (ii) is the intermediate, or critically damped case.

Practice exercises. Find the general solution to the following

1. $\frac{d^2y}{dt^2} + 4y = 0$ — ans: $y = C_1\cos(2t) + C_2\sin(2t)$
2. $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} = 0$ — ans: $y = C_1 + C_2 e^{-3t}$
3. $\frac{d^2y}{dt^2} = 0$, $t > 0$, $y(0) = 0$, $\frac{dy}{dt}(0) = 4$ — ans: $y = 4$
4. $3\frac{d^2y}{dt^2} - 5\frac{dy}{dt} - 2y = 0$ — ans: $y = C_1 e^{2t} + C_2 e^{-t/3}$
5. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} = 2$, $y(0) = 0$, $\frac{dy}{dt}(0) = 4$ — ans: $y = \frac{7}{8}(1 - e^{-4t}) + \frac{1}{2}t$

Example: Cauchy-Euler equations

These are of the type

(6) $$A t^2 \frac{d^2y}{dt^2} + Bt\frac{dy}{dt} + Cy = 0$$

or

$$A(x-x_0)^2\frac{d^2y}{dx^2} + B(x-x_0)\frac{dy}{dx} + Cy = 0$$

where A, B, C are constants. (The latter equation can be transformed into the first equation by making a change of variable $t = x - x_0$.) It is assumed that the first equation (respectively, the second equation) is defined on an interval not containing $t=0$ ($x = x_0$). Then let $y(t) = t^r$ (respectively, $y(x) = (x-x_0)^r$), substitute into the equation to derive

$$A t^2 (r(r-1)t^{r-2}) + Bt(rt^{r-1}) + Ct^r = t^r\{Ar^2 + (B-A)r + C\} = 0$$

(7) so $Ar^2 + (B-A)r + C = 0$ is the characteristic equation for the Cauchy-Euler equation (6). Again we have the 3 cases:

Case 1: roots r_1, r_2 to (7) are real, unequal. Then a fundamental set of solutions is t^{r_1}, t^{r_2}, and the general solution to (6) is $y(t) = C_1 t^{r_1} + C_2 t^{r_2}$.

Case 2: roots r_1, r_2 are real and equal. Let $r = r_1 = r_2$, then a fundamental set of solutions is $t^r, t^r\ln(t)$ and the general solution to (6) can be written as $y(t) = t^r\{C_1 + C_2\ln(t)\}$

Comment: we assume here the domain is an interval in the positive reals. If it is an interval in the negative real numbers, then we could use $\ln(|t|)$, or make a change of independent variables.

Case 3: $r_{1,2} = a \pm ib$, $b \neq 0$. Then a fundamental set of solutions is $t^a\cos(b\ln(t))$, $t^a\sin(b\ln(t))$ and the general solution to (6) would be $y(t) = t^a\{C_1\cos(b\ln(t)) + C_2\sin(b\ln(t))\}$.

example: $(x+1)^2\frac{d^2y}{dx^2} - \frac{1}{2}y = 0 \qquad x > 0$.

$y(x) = (x+1)^r \longrightarrow r^2 - r - \frac{1}{2} = 0 \rightarrow r = \frac{1}{2}(1 \pm \sqrt{3}) \rightarrow$ a general solution is $y(x) = \sqrt{x+1}\,\{C_1(x+1)^{\sqrt{3}/2} + C_2(x+1)^{-\sqrt{3}/2}\}$.

<u>Nonhomogeneous 2nd order linear equations</u>

Given the general equation

(8) $$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t) \qquad t > a$$

with $f(t) \not\equiv 0$, then if (8) has 2 solutions, $y_1(t)$ & $y_2(t)$, then $u(t) = y_1(t) - y_2(t)$ satisfies the homogeneous version of (8) by simply writing (8) with y_1 and y_2, and subtracting; so $\frac{d^2u}{dt^2} + p(t)\frac{du}{dt} + q(t)u = 0$. Since this equation has a fundamental set of solutions, call them $\varphi_1(t), \varphi_2(t)$, then $u(t) = C_1\varphi_1(t) + C_2\varphi_2(t)$, and $y_1(t) = C_1\varphi_1(t) + C_2\varphi_2(t) + y_2(t)$. This is a solution to (8) with 2 free parameters to satisfy initial conditions (or 2 boundary conditions), so it represents the general solution to (8). The general solution has the form of the general solution to the homogeneous equation plus a particular solution to the nonhomogeneous equation. So, given (8) along with initial conditions (or boundary conditions if (8) is defined on a bounded interval), the strategy is

1. find a fundamental set of solutions φ_1, φ_2 to the homogeneous equation;
2. find a particular solution, $y_p(t)$, to the nonhomogeneous equation;
3. write the general solution $y(t) = y_p(t) + C_1\varphi_1(x) + C_2\varphi_2(x)$, and apply the initial conditions (boundary conditions) to determine C_1 and C_2, and hence the unique solution to the problem.

Note: the order is important here because to enact step 2 you need information about fundamental solutions to the homogeneous equation in step 1.

There are two main approaches to obtaining a particular solution. They are

a) method of ~~un~~determined coefficients

b) variation of parameters method

Most of the ODEs encountered in this course are constant coefficient equations (i.e. p & q are constants, but not f in (8)), so the undetermined coefficients method is most useful for us. But it is restrictive; namely, you <u>must</u> have the coefficients on the left side to be constant, and f to be of the form

$$e^{at}P(t)\begin{cases}\cos(bt)\\ \sin(bt)\end{cases} \text{ or}$$

or sums of such terms. (Note — this means a simple exponential is valid by letting $b=0$ and the polynomial $P(t)$ be the constant polynomial. A simple sin or cos. would be to have $a=0$, $P(t)=$ constant, etc.) If these conditions are not met, you must use variation of parameters.

<u>example</u>: $\frac{d^2y}{dt^2}+\omega^2 y = 3e^{-t}\cos(2t)$, $y(0)=0$, $\frac{dy}{dt}(0)=1$

For $\frac{d^2y}{dt^2}+\omega^2 y=0$ we have $\varphi_1(t)=\sin(\omega t)$, $\varphi_2(t)=\cos(\omega t)$.

For $f(t)=3e^{-t}\cos(2t)$, let $y_p(t)=Ae^{-t}\cos(2t)+Be^{-t}\sin(2t)$.

Then ask: is any term here a solution to the homogeneous equation? If the answer is yes then multiply the right hand side by t and ask the question again. If the answers is still yes, you have to multiply by t again, then proceed. In our case here the answer is no, so substitute y_p into the homogeneous equation and determine A, B. Thus,

$$e^{-t}\underbrace{\{(\omega^2-3)A-4B\}}_{=3}\cos(2t)+e^{-t}\underbrace{\{(\omega^2-3)B+4A\}}_{=0}\cos(2t)=3e^{-t}\cos(2t)$$

so

$$\Rightarrow A=\frac{3(\omega^2-3)}{(\omega^2-3)^2+16},\quad B=-\frac{12}{(\omega^2-3)^2+16} \quad \text{and so}$$

$$y(t)=C_1\sin(\omega t)+C_2\cos(\omega t)+Ae^{-t}\cos(2t)+Be^{-t}\sin(2t)$$

$$y(0)=C_2+A=0 \qquad \frac{dy}{dt}(0)=\omega C_1-A+2B=1 \rightarrow C_1=\frac{1}{\omega}(1-2B+A)$$

$$\rightarrow y(t)=\frac{1}{\omega}(1-2B+A)\sin(\omega t)-A\cos(\omega t)+Ae^{-t}\cos(2t)+Be^{-t}\sin(2t)$$

Please pick up your ODE and review the method and do some problems to re-familiarize yourself with the method.

<u>example</u> : $\frac{d^2y}{dt^2} + y = 2\sec(t) \qquad 0 < t < \pi/2$

Here the right hand side of the equation is not in a form where undetermined coefficients method can be used, so apply variation of parameters method. Here, for the homogeneous equation, a fundamental set of is $\varphi_1(t) = \cos t$, $\varphi_2(t) = \sin t$, so write $y_p(t) = u_1(t)\varphi_1(t) + u_2(t)\varphi_2(t)$

$$= u_1(t)\cos(t) + u_2(t)\sin(t)$$

where u_1, u_2 are to be determined from <u>first-order</u> equations.

Differentiating ; $\frac{dy_p}{dt} = \frac{du_1}{dt}\cos t + \frac{du_2}{dt}\sin t - u_1 \sin t + u_2 \cos t$

Set $\frac{du_1}{dt}\cos t + \frac{du_2}{dt}\sin t = 0$ (so we don't have to deal with second derivatives of 2 unknown functions), then

$\frac{d^2y_p}{dt^2} = -\frac{du_1}{dt}\sin t - u_1 \cos t + \frac{du_2}{dt}\cos t - u_2 \sin t$; now add y_p:

$$\begin{array}{lllll} y_p = & & + u_1 \cos t & & + u_2 \sin t \\ \hline \frac{d^2y_p}{dt^2} + y_p = & -\frac{du_1}{dt}\sin t + \frac{du_2}{dt}\cos t & = & 2\sec(t) & \end{array}$$

or

$-\left(-\frac{du_2}{dt}\tan t\right)\sin t + \frac{du_2}{dt}\cos t = 2\sec(t)$

so multiply by $\cos t$ to obtain $\frac{du_2}{dt}(\sin^2 t + \cos^2 t) = 2$

so $\frac{du_2}{dt} = 2 \rightarrow u_2 = 2t$ and $\frac{du_1}{dt} = -2\tan t \rightarrow u_1 = -\ln|\cos t|$
$= -\ln(\cos t)$

so $y_p(t) = -\cos(t)\ln(\cos t) + 2t \sin t$ and the general solution is

$y(t) = C_1 \sin t + C_2 \cos t - \cos(t)\ln(\cos t) + 2t \sin t \quad , \quad 0 < t < \pi/2.$

Again, return to your ODE book to review the method.

<u>Eigenvalue Problems</u>

We will deal will a new class of ODEs in this course, which I'll briefly introduce here through an example

<u>example</u> : $\frac{d^2\varphi}{dx^2} + \lambda \varphi = 0 \qquad 0 < x < 2$

with $\varphi(0)=0$, $\varphi(2)=0$

First of, these problems will come from spatial problems in this course, and so the equations will be defined on finite intervals. Hence, there will be a condition for φ at the left boundary point and a condition for φ at the right boundary point, so these eigenvalue problems (EVPs) are 2-point boundary value problems, as opposed to the initial-value problems you studied in your elementary ODE class. Secondly, the parameter λ is unknown a priori, so must be determined along with $\varphi(x)$.

Note that $\varphi(x) \equiv 0$ is a solution, and this gives no information about λ, so it is of no interest. The question asked of EVPs is: for what values of λ does the problem have a non-trivial solution? If $\lambda = \lambda_1$ is such a value, with $\varphi = \varphi_1(x) \not\equiv 0$, then λ_1 is called an eigenvalue for the problem, and $\varphi_1(x)$ is called the eigenfunction associated with λ_1.

For the specific EVP above, $\varphi(x) = e^{rx}$ gives the characteristic polynomial $r^2+\lambda=0 \rightarrow r = \pm i\sqrt{\lambda}$.
Comment: when we study these type problems in the course, we'll determine that for λ to be an eigenvalue, it will be real and nonnegative. A fundamental set of solutions is $e^{ix\sqrt{\lambda}}$, $e^{-ix\sqrt{\lambda}}$, or $\sin(\sqrt{\lambda}x)$, $\cos(\sqrt{\lambda}x)$.
Since our ODE problems are always real, always use sines and cosines. Thus $\varphi(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$, but $\varphi(0) = 0 = C_1$; so $\varphi(x) = C_2\sin\sqrt{\lambda}x$. Now $\varphi(1)=0=C_2\sin\sqrt{\lambda}$, and since we can not have $C_2=0$, then $\sin\sqrt{\lambda}=0$. This can happen at values $\sqrt{\lambda} = \pm\pi, \pm 2\pi, \pm 3\pi, \dots$. We don't want $\lambda=0$ (or else $\varphi(x)=0$) and we can absorb the sign into multiplicative coefficients, so for this problem

the eigenvalues are

$$\lambda = \lambda_n = n^2\pi^2 \qquad n = 1, 2, 3, \ldots$$

and the eigenfunctions associated with each eigenvalue λ_n is $\varphi = \varphi_n(x) = \sin(n\pi x) \qquad n = 1, 2, \ldots$.

Remark: the terminology comes from the algebraic eigenvalue case,

$$A\underline{v} = \lambda \underline{v}$$

where one looks for all λ ~~for~~ which there is a solution vector $\underline{v} \neq \underline{0}$ i.e. $\underline{v}$ is not the $\underline{0}$ vector. We could, of course, write the equation like

$$-\frac{d^2}{dx^2}\varphi = \lambda \varphi \qquad \text{defined on } (0, l)$$

where the matrix A is replaced by a differential operator $\frac{d^2}{dx^2}$.
There are also pde analogues. For example if Ω is some open connected set in 3 space $\mathbb{R}^3$, with boundary denoted by $\partial\Omega$, then a typical pde EVP is

$$-\nabla^2\varphi = \lambda\varphi \qquad \text{in } \Omega$$

$$\varphi(\underline{x}) = 0 \qquad \text{for all } \underline{x} \text{ in } \partial\Omega$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Here are a few practice problems

(1) $t\frac{d^2y}{dt^2} + \frac{dy}{dt} = -1, \quad t > 1$ ans: $y = 1 - t + \ln(t)$

$y(1) = 0 \quad \frac{dy}{dt}(1) = 0$

(2) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 3e^{-x}$ ans: $y = C_1 e^{-x} + C_2 x e^{-x} + \frac{3}{2}x^2 e^{-x}$

(3) if $u_1(x) = 1 + x$, $u_2(x) = e^x$ form a fundamental set of solutions to the homogeneous equation, what is the general solution to

$$x\frac{d^2y}{dx^2} - (1+x)\frac{dy}{dx} + y = x^2 e^{2x} \qquad x > 0$$

ans: $y = C_1(1+x) + C_2 e^x + \frac{1}{2}e^{2x}(x-1)$

To summarize, you must know how to

1 - solve a first order, linear ode by method of integrating factors

2 - solve second order, linear, homogeneous odes by characteristic equation method

3 - for a linear, nonhomogeneous ode, you must be able to obtain a particular solution by undetermined coefficients method and/or variation of parameters method, and be able to know when you use one versus the other method.

4 - know what Cauchy-Euler equations are and how to solve them.

Extra problems

1. Define or characterize the following
   a) error and complementary error functions
   b) fundamental set of solutions
   c) Wronskian of two functions; what can you say about the Wronskian of two solutions to a linear, 2^{nd} order ode on interval $[a,b]$?
   d) Cauchy-Euler equations

2. What type of function is a solution to $\frac{dT}{dt}(t) = p\,T(t)$?

3. What is the characteristic equation for
   a) $2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$
   b) $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} - 16y = 0$

   Write a fundamental set of solutions for each equation
   (ans: a) $e^{x/2}\cos(x/2)$, $e^{x/2}\sin(x/2)$; b) $(1+x)^4$, $(1+x)^{-4}$)

4. $u_1(x) = e^{2x}$, $u_2(x) = xe^{2x}$ are two solutions to the equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$. What is the Wronskian of these two functions and what does it say about the nature of these solutions?