Homework # 9

1. $\nabla^2 u = 0$ in Ω, $u(x,y) = X(x)Y(y) \to \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \mu$

$$\to \frac{d^2X}{dx^2} + \mu X = 0 \qquad \frac{d^2Y}{dy^2} - \mu Y = 0$$
$$\frac{dX}{dx}(0) = 0 = \frac{dX}{dx}(L) \qquad Y(0) = 0$$

$X = \cos\sqrt{\mu}\,x \to X' = -\sqrt{\mu}\sin\sqrt{\mu}\,x \to \sqrt{\mu}\sin\sqrt{\mu}\,L = 0$

If $\mu = 0$, then X = constant. If $\sin\sqrt{\mu}\,L = 0$, then $\mu = \mu_n = \left(\frac{n\pi}{L}\right)^2$, $X = X_n = \cos\left(\frac{n\pi x}{L}\right)$

For the Y equation, if $\mu = 0$, $Y(y) = A_0 y$ (since $Y(0) = 0$); otherwise $Y(y) = Y_n(y) = \sinh\left(\frac{n\pi y}{L}\right)$ $n = 1, 2 \ldots$

Hence, $u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$.

Finally, at $y = K$

$$u(x,K) = 6\cos\left(\frac{8\pi x}{L}\right) = A_0 K + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi K}{L}\right)\cos\left(\frac{n\pi x}{L}\right),$$

so $A_0 = 0$, $A_8 \sinh\left(\frac{8\pi K}{L}\right) = 6$, $A_n = 0$ $n \neq 6$; thus

$$u(x,y) = \frac{6\sinh\left(\frac{8\pi y}{L}\right)}{\sinh\left(\frac{8\pi K}{L}\right)}\cos\left(\frac{8\pi x}{L}\right).$$

2. $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$ in $\Omega = \{(r,\theta) \mid r > a\}$,

so $u(r,\theta) = R(r)\Theta(\theta) \to$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \mu \to \frac{d^2\Theta}{d\theta^2} + \mu\Theta = 0$$
$$0 \le \theta < 2\pi$$

Because we need 2π-periodicity, $\mu = \mu_n = n^2$ $n = 0, 1, \ldots$ and $\Theta = A\cos(n\theta) + B\sin(n\theta)$. If $\mu = \mu_0 = 0$, then $R = c\ln(r) + d$ but because we want bounded solutions, $c = 0$. For $\mu_n = n^2 > 0$, $r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2 R = 0$ is an Euler equation, so letting $R = r^\alpha$ gives the characteristic equation $\alpha^2 - n^2 = 0 \to \alpha = \pm n \to$ $R = ar^n + br^{-n}$. Again, for boundedness, $a = 0$.

Putting these results together,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n} \{A_n \cos(n\theta) + B_n \sin(n\theta)\}$$

Therefore,

$$u(a,\theta) = \ln(2) + 4\cos(3\theta) = A_0 + \sum_{n=1}^{\infty} \{a^{-n} A_n \cos(n\theta) + a^{-n} B_n \sin(n\theta)\}$$

so that $A_0 = \ln(2)$, $B_n = 0$ for all $n \geq 1$, and $A_3 = 4a^3$, $A_n = 0$ for $n \neq 3$

$$\rightarrow \quad u(r,\theta) = \ln(2) + 4a^3 r^{-3} \cos(3\theta)$$

3. $$\int_{\Omega} f \, d\Omega \underset{\text{by equation}}{=} \int_{\Omega} \nabla^2 u \, d\Omega \underset{\text{divergence theorem}}{=} \int_{\partial\Omega} \hat{n} \cdot \nabla u \, ds \underset{\text{def'n of } \partial/\partial n}{=} \int_{\partial\Omega} \frac{\partial u}{\partial n} \, ds \overset{\text{b.c.}}{=} \int_{\partial\Omega} g \, ds$$

4. (a) u depending only on r means

$$0 = \nabla^2 u = \frac{d^2 u}{dr^2} + \frac{2}{r}\frac{du}{dr} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right)$$

$$\rightarrow r^2 \frac{du}{dr} = C \rightarrow \frac{du}{dr} = Cr^{-2} \rightarrow u(r) = C_1 r^{-1} + C_2$$

(b) Similarly $\nabla^2 u = k^2 u \rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) - k^2 u = 0$

$$u = v/r \rightarrow \frac{du}{dr} = \frac{1}{r}\frac{dv}{dr} - \frac{1}{r^2} v \rightarrow$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r \frac{dv}{dr} - v\right) - k^2 \frac{v}{r} = 0 \rightarrow \frac{d^2 v}{dr^2} - k^2 v = 0 \rightarrow$$

$$\rightarrow \quad v = A \cosh(kr) + B \sinh(kr)$$

$$\rightarrow \quad u = A \frac{\cosh(kr)}{r} + B \frac{\sinh(kr)}{r}$$