Homework # 3

1. a) $u(x,t) = U(x) \longrightarrow \quad 0 = \frac{d^2U}{dx^2} \quad 0<x<1, \ U(0) = 1/2, \ U(1) = 2$

$\rightarrow U(x) = \frac{3x+1}{2}$

b) $u(x,t) = U(x) + v(x,t)$ solves the problem, then $v(x,t)$ must satisfy

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} & 0<x<1, \ t>0 \\ v(0,t) = 0 = v(1,t) \\ v(x,0) = \sin(2\pi x) \end{cases}$$

by separation of variables, this is a problem done in class: $v(x,t) = T(t)\varphi(x) \longrightarrow \begin{cases} dT/dt = -\lambda T \\ \frac{d^2\varphi}{dx^2} + \lambda\varphi = 0 & 0<x<1 \\ \varphi(0) = 0 = \varphi(1) \end{cases}$

Hence, $\lambda = \lambda_n = n^2\pi^2 \quad n = 1, 2, \ldots$

$\varphi = \varphi_n(x) = \sin(n\pi x)$, so

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

Now $v(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) = \sin(2\pi x) \rightarrow b_n = \begin{cases} 0 & n \neq 2 \\ 1 & n = 2 \end{cases}$

$$\rightarrow u(x,t) = \frac{3x+1}{2} + e^{-4\pi^2 t}\sin(2\pi x)$$

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2. $u(x,t) = T(t)\varphi(x) \rightarrow \dot{T}/T = \varphi''/\varphi = -\lambda$ (as above)

$\rightarrow \begin{cases} \varphi'' + \lambda\varphi = 0 & 0<x<1 \\ \varphi'(0) = 0 = \varphi(1) \end{cases}$ and $\dot{T} = -\lambda T$

$\varphi(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x \rightarrow \varphi'(x) = \sqrt{\lambda}\{-A\sin\sqrt{\lambda}\,x + B\cos\sqrt{\lambda}\,x\}$

$\rightarrow \varphi'(0) = \sqrt{\lambda}B = 0$. If $\lambda = 0$, $\varphi(x) = Ax + B$, $\varphi'(0) = A = 0$

so $\varphi(x) = B$; but with $\varphi(1) = B = 0 \rightarrow \varphi \equiv 0 \rightarrow \lambda \neq$ eigenvalue.

Hence $B = 0 \rightarrow \varphi(x) = \cos\sqrt{\lambda}\,x$. Now $\varphi(1) = 0 = \cos\sqrt{\lambda} \rightarrow$

$\sqrt{\lambda} = \sqrt{\lambda_n} = (n + 1/2)\pi \quad n = 0, 1, 2, \ldots$, $\varphi_n(x) = \cos[(n+1/2)\pi x]$.

Also $T(t) = T_n(t) = e^{-(n+1/2)^2\pi^2 t} \rightarrow u(x,t) = \sum_{n=0}^{\infty} a_n e^{-(n+1/2)^2\pi^2 t}\cos[(n+\frac{1}{2})\pi x]$

Letting $t \to 0$,

$$f(x) = \sum_{0}^{\infty} a_n \cos[(n+1/2)\pi x] \rightarrow a_n = 2\int_0^1 f(x)\cos[(n+\tfrac{1}{2})\pi x]\,dx$$

3. a) $u(x,t) = T(t)\varphi(x) \longrightarrow \dot{T}/T = \varphi''/\varphi = -\lambda$

$\rightarrow \dot{T} = -\lambda T$ and (1) $\begin{cases} \varphi'' + \lambda\varphi = 0 & 0<x<1 \\ \varphi(0) = 0 \end{cases}$

Now for the boundary condition at $x=1$,

$u_t(1,t) + a\,u_x(1,t) = \dot{T}(t)\varphi(1) + aT(t)\varphi'(1) = 0$,

or $-\lambda T(t)\varphi(1) + aT(t)\varphi'(1) = 0$ using the $T(t)$ equation.

Since $T(t) \not\equiv 0$, then (2) $a\varphi'(1) - \lambda\varphi(1) = 0$

That is (1), (2) make up the EVP.

b) (1) implies $\varphi(x) = \sin\sqrt{\lambda}\,x$, and substitution into (2)

gives $a\sqrt{\lambda}\cos\sqrt{\lambda} - \lambda\sin\sqrt{\lambda} = \sqrt{\lambda}\left\{a\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda}\right\} = 0$

If $\lambda = 0$ then $\varphi(x) = Ax + B$; $\varphi(0) = 0 \rightarrow B = 0$ and (2) would imply $aA = 0 \rightarrow A = 0 \rightarrow \varphi \equiv 0$, so $\lambda \neq$ eigenvalue. Hence

$a\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda} = 0$, or $\dfrac{a}{\sqrt{\lambda}} = \tan\sqrt{\lambda}$

With $a = 1$, let $r = \sqrt{\lambda}$, then we must find the solutions to $\frac{1}{r} = \tan(r)$. Now you have to find your favorite rootfinder to compute $r_1, r_2, r_3, \ldots$. Then $\lambda_n = r_n^2$. In this case, $\lambda_1 \simeq 11.735$, $\lambda_2 \simeq 41.439$, $\lambda_3 \simeq 90.808$, $\lambda_4 \simeq 159.903$, etc.

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4. $u(x,t) = T(t)\varphi(x) \longrightarrow t\dot{T}/T = (\varphi'' + 2\varphi)/\varphi = -\lambda$

$\rightarrow \begin{cases} \varphi'' + (\lambda+2)\varphi = 0 & 0<x<\pi \\ \varphi(0) = 0 = \varphi(\pi) \end{cases}$ and $t\dot{T} = -\lambda T$, $t>0$

From the EVP, $\lambda = \lambda_n = -2 + n^2$ and $\varphi = \varphi_n = \sin(nx)$ $n = 1,2,3,4$

For $T(t)$, $t\dot{T} + \lambda_n T = 0$ which has integrating factor t^{λ_n}; i.e. $(t^{\lambda_n}T)^{\cdot} = 0 \rightarrow T(t) = T_n(t) = t^{-\lambda_n} = t^{2-n^2}$

(up to a multiplicative constant). Hence

$u(x,t) = \sum_{n=1}^{\infty} a_n t^{2-n^2}\sin(nx) = a_1 t\sin(x) + a_2 t^{-2}\sin(2x) + a_3 t^{-7}\sin(3x) + \ldots$

With $u(x,0) = 0$ if $t\rightarrow 0$, $u(x,t) \Rightarrow 0$ only if $a_j = 0$ $j \geq 2$, (but a_1 is) arbitrary $\rightarrow$ have a 1 param. set of solutions