Homework #2

1. $u_x - u_y + u = 1 \qquad u(x,0) = \sin x \qquad \to \Gamma = \{(x,y) = (s,0) = (\phi(s), \psi(s)) \mid |s| < \infty\}$

$\frac{dx}{dt} = 1 \to x = t + x_0(s) \quad ; \quad x|_{t=0} = x_0(s) = s$

$\frac{dy}{dt} = -1 \to y = -t + y_0(s) \quad ; \quad y|_{t=0} = y_0(s) = 0 \to x = t+s, \ y = -t$

Hence, $t = -y$, $s = x+y$

$\frac{du}{dt} = -u + 1 \to u = 1 + e^{-t} u_0(s) \quad ; \quad u|_{t=0} = 1 + u_0(s) = \sin(s)$

$\therefore \quad u = 1 + e^{-t}(-1 + \sin(s)) \to u = 1 + e^{y}(-1 + \sin(x+y))$.

2. $x u_x + y u_y = 3 \qquad u(1,y) = \ln y \qquad \Gamma = \{(x,y) = (1,y) = (1,s) = (\phi(s), \psi(s))\}$

$\frac{dx}{dt} = x \to x = x_0(s) e^{t} \quad ; \quad x|_{t=0} = 1 = x_0(s) \to x = e^{t}$

$\frac{dy}{dt} = y \to y = y_0(s) e^{t} \quad ; \quad y|_{t=0} = s = y_0(s) \to y = s e^{t}$

Thus, $s = \frac{y}{x}$ and $t = \ln x$

$\frac{du}{dt} = 3 \to u = 3t + u_0(s) \quad ; \quad u|_{t=0} = u_0(s) = \ln s$

$\to u = 3t + \ln s \quad ;$ so $u(x,y) = 3\ln x + \ln(y/x)$

$= 2\ln x + \ln y = \ln(x^2 y)$

3. $x^2 u_x - y^2 u_y = 0$, $u(1,y) = F(y)$

$\to \Gamma = \{(x,y) = (1,s) = (\phi(s), \psi(s))\}$

$\frac{dx}{dt} = x^2 \to x^{-2}dx = dt \to x = \frac{1}{c-t} \quad ; \quad x|_{t=0} = 1 = \frac{1}{c} \to x = \frac{1}{1-t}$

$\frac{dy}{dt} = -y^2 \to y = \frac{1}{t - c_1} \quad ; \quad y|_{t=0} = -\frac{1}{c_1} = s \to y = \frac{1}{t + 1/s}$

Thus $t = \frac{x-1}{x}$, $s = \frac{xy}{x(1-y)+y}$.

$\frac{du}{dt} = 0 \to u = u_0(s) = u|_{t=0} = F(s)$ i.e. $u(t,s) = F(s)$, or

$u(x,y) = F\left(\frac{xy}{x(1-y)+y}\right)$