**Some Calculus background you should be familiar with, or review, for Math 404**

It will be, for the most part, assumed you have at your fingertips the basics of (multivariable) functions, calculus, and elementary differential equations. If there has been too much of a gap since you took those courses, you must spend time reviewing that material or you will not be successful in learning partial differential equation techniques as given in Math 404. Below is a brief guide to some needed calculus material, but it is by no means a complete representation of al relevant material.

# Integration and Differentiation

*Leibniz Theorem*: A handy result we will apply over and over, at least in restrictive cases to the statement here concerns the integral $I(t) = \int_{a(t)}^{b(t)} f(x,t)dx$.

> **Theorem**: if *f(x,t)* and $\partial f / \partial t$ are continuous on the rectangle [A,B] x [c,d], where [A,B] contains the union of all the intervals [a(t),b(t)], and if a(t) and b(t) are differentiable on [c,d], then
> $$\frac{\partial I}{\partial t} = \frac{\partial}{\partial t}\int_{a(t)}^{b(t)} f(x,t)dx = f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)dx \ .$$

*Exercises*:

1. Let $f(t) = \int_t^{t^2} \sin(s)ds$ . First, use the Leibniz theorem to compute *df/dt*. Second, integrate the integral directly, then take the derivative to obtain the same result.
2. Define the two-variable function $u(x,t) = \int_{x-t}^{x+t} g(y)dy$ for an arbitrary integrable function *g(y),* and show that *u(x,t)* is a solution to the partial differential equation $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ .$$

If $f = f(x,y,z)$ is a scalar function, and $F = (F_1, F_2, F_3)$ is a vector function, then the notation for the *gradient* of *f* is given by $\nabla f = grad(f) = (f_x, f_y, f_z)$, where $f_x = \partial f / \partial x$, etc. IN Cartesian coordinates this is sometimes written as $\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$, where $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors in the x, y, and z directions, respectively. Here $f$ has domain in 3-space (that is, in $\Re^3$), but we have the analogous formulae in the plane or in n-space. The *directional derivative* of *f* at the (vector) point *a* in the direction of the vector *v* is $\lim_{t\to 0} \frac{f(a+tv) - f(a)}{t} = v \cdot \nabla f(a)$.

It follows that the rate of change of a quantity *f(x)* seen by a moving particle *x(t)* is $(d/dt)f(x) = \nabla f \cdot (dx/dt)$.

The *divergence* of the vector function $F = (F_1, F_2, F_3)$ is given by

$$divF = \nabla \bullet F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad .$$

Therefore, the Laplacian of $u$ is

$$\Delta u(x, y, z) = \nabla^2 u = div(grad(u)) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad .$$

Also, $|\nabla u|^2 = |grad(u)|^2 = (u_x)^2 + (u_y)^2 + (u_z)^2$ .

The *curl* of the vector function $F = (F_1, F_2, F_3)$ is given by

$$curlF = \nabla \times F = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

**Theorem**: Let *F, G* be two vector functions such that *curl(F), curl(G), div(F),div(G)* exist, and let *f* be a scalar function such that *grad(f)* exists. Then

a) *curl(F + G) = curl(F) + curl(G)*
b) *div(F + G) = div(F) + div(G)*
c) *curl( fF ) = f curl(F) + grad(f)*$\times$ *F*
d) *div( fF ) = f div(F) + grad(f)*$\bullet$ *F*
e) *curl(grad(f)) = 0*
f) *div(curl(F)) = 0*
g) *curl(curl(F)) = grad(div(F)) -* $\nabla^2 F$

If an equation involves a solution needing partial derivatives of second order, then we are interested in a continuous function on a domain D that has *continuous* partial derivatives up to order two, and we say the function is of class $C^2(D)$. If the function only needs continuous partial derivatives of first order, then the function is of class $C^1(D)$. For example, if we are interested in solving the one-dimensional heat equation for *u* on the domain $a < x < b, 0 < t < T$, then *u* should be of class $C^2((a,b))$ in *x*, $C^1((0,T))$ in *t*. (The equation holds in the open domain, i.e. the region not consisting of its boundary because we can not assume the derivatives of the function hold on the domain's boundary.)

Now let D be an open, bounded, simply-connected set in the plane (or in 3-space) with smooth boundary. (By simply-connected we mean D has no holes. By smooth boundary, which is bounded here, we mean there is a unit normal vector at each boundary point, i.e., at each point on the boundary, there is a vector that is perpendicular to the tangent plane at the point, of unit length.) Call the unit (outward pointing) normal vector $\hat{n}$ .

**Divergence Theorem**: Let g be any continuous scalar function on D and its boundary S, and continuously differentiable on D, and let F be a continuously differentiable vector function on D. Then

$$\iint_D \{g\,div(F) + F \bullet grad(g)\}dx = \int_S (gF)\bullet \hat{n}ds \ .$$

*Exercises*:

1. Use the Divergence theorem, with letting $F = \nabla u = grad(u)$ to obtain $\iint_D (g\nabla^2 u + \nabla u \bullet \nabla g)dx = \int_S g(\nabla u \bullet \hat{n})ds$. This expression is often called *Green's First Identity*.
2. Take the expression in part 1, interchange g and u, and subtract the two expressions to obtain $\iint_D (u\nabla^2 g - g\nabla^2 u)dx = \int_S (u\nabla g - g\nabla u)\bullet \hat{n}ds$. This is *Green's Second Identity*.

I have written the divergence theorem as if it were a planar theorem, but in fact it holds in higher dimensions. A two-dimension theorem to keep in mind is

**Green's Theorem**: Let D be a bounded planar domain with piecewise $C^1$ boundary curve S. (sometimes S is denoted $\partial D$). (Piecewise $C^1$ mean continuously differentiable except at a finite number of points.) Consider S parameterized such that it is traversed once with D on the left (traversed counterclockwise). Let p(x,y) and q(x,y) be any $C^1$ functions defined on the closure of D (D + S, i.e. the union of the two sets, i.e. cl(D)). Then $\iint_D (q_x - p_y)dxdy = \int_S pdx + qdy$ .

A completely equivalent formulation of Green's theorem is obtained by substituting p = -g and q = +f. If F = (f,g) is any $C^1$ vector field in cl(D), then $\iint_D (f_x + g_y)dxdy = \int_S (-gdx + fdy)$. If $\hat{n}$ is the unit outward-pointing normal vector on S, then $\hat{n}$ = (+dy/ds, -dx/ds). Hence, Green's theorem takes the form $\iint_D \nabla \bullet Fdxdy = \int_S F\cdot nds$, where $\nabla \bullet F = div(F) = f_x + g_y$ is the divergence of F.

# Trigonometric and Hyperbolic Functions

*Addition formulas (trig):*

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

So, for instance, cos(2x) = $\cos^2$(x) – $\sin^2$(x) = 1 – 2$\sin^2$(x) = 2$\cos^2$(x) – 1; thus, for example, $\sin^2$(x) = (1 – cos(2x))/2 .

*Exercises*: using the addition formulas, show the following integrals are true:

1. For arbitrary positive integers n and m,

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases}$$

$$\int_0^1 \cos(m\pi x)\sin(n\pi x)dx = 0 \quad \text{for all } n \text{ and } m \ .$$

2. Sketch a graph of tan(x) for x > 0 and superimpose on the graph the graph of the function x/2. Numerically approximate the first 5 solutions to the transcendental equation tan(x) = x/2.
3. Show that $\sin(11\pi x)\cos(10\pi x) = \frac{1}{2}\{\sin(21\pi x) + \sin(\pi x)\}$

*Hyperbolic functions*: sinh(x) = $(e^x - e^{-x})/2$, cosh(x) = $(e^x + e^{-x})/2$, tanh(x) = sinh(x)/cosh(x), , etc.

$$\frac{d}{dx}\sinh(x) = \cosh(x), \quad \frac{d}{dx}\cosh(x) = \sinh(x), \text{ etc.}$$
$$\sinh(x \pm y) = \sinh(x)\cosh(y) \pm \cosh(x)\sinh(y)$$
$$\cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$

*Exercises*

1. Show that sinh(x), tanh(x) are odd functions and sketch a graph of each. Then show cosh(x) and sech(x) are even functions and sketch a graph of each.
2. Show that sinh(ax) and cosh(ax) form a fundamental set of solutions for the differential equation $\frac{d^2u}{dx^2} - a^2u = 0$ .

# Sequences and Series of Functions

We are going to be dealing with series of functions (Fourier series), so you should recall a few things about sequences and series.

*Definition*: **convergence of a series** of (real) numbers: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$ converges if the tail end can be made arbitrarily small; i.e. given any tolerance ε > 0, there is an M > 1 such that for m > M, $|\sum_{n=m}^{\infty} a_n| < \varepsilon$ .

*Definition*: **absolute convergence** of a series: $\sum_1^{\infty} a_n$ converges absolutely if $\sum_1^{\infty} |a_n|$ converges.

*Remark*: the **Comparison Test**: If $|a_n| \le b_n$ for all *n*, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely. The contrapositive necessarily follows: If $\sum_{n=1}^{\infty} |a_n|$ diverges, so does $\sum_{n=1}^{\infty} b_n$ . The limit comparison test states that if $a_n \ge 0, b_n \ge 0$, if $\lim_{n\to\infty} a_n / b_n = L$, where $0 \le L < \infty$, and if $\sum_{n=1}^{\infty} b_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$ .

*Remark*: the **ratio test**: the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\frac{|a_{n+1}|}{|a_n|} \leq \rho < 1$ for some constant $\rho$, and for $n \geq N \geq 1$. (We don't care if the inequality is not met for the first N terms.)

*Examples*: For $\sum_{n=1}^{\infty} (\frac{1}{2})^n$, so $a_n = 1/2^n$, hence $\frac{|a_{n+1}|}{|a_n|} = \frac{1}{2} = \rho$, so series is absolutely convergent. For $\sum_1^{\infty} \frac{1}{n}$, $\frac{|a_{n+1}|}{|a_n|} = \frac{n}{n+1} \to 1$. Hence, there is no upper bound less than 1, so the ratio test fails, i.e. give no information. This series actually diverges. The ratio test also fails for the series $\sum_1^{\infty} \frac{1}{n^2}$, but the series converges to $\pi^2/6$. In fact, the p-series $\sum_1^{\infty} 1/n^p$ converges for p > 1, and diverges (is infinite) for $p \leq 1$.

*Definition*: **uniform convergence of sequence of functions**: assume the sequence $\{f_n(x)\}_{n=1,2,\ldots}$ of functions is defined on an interval $I$ of the real numbers. Then $\{f_n\}$ converges uniformly on $I$ to f(x) if for any tolerance $\varepsilon > 0$, there is an M such that for $m > M$, $|f_m(x) - f(x)| < \varepsilon$ for all x in $I$.

*Definition*: **uniform convergence of a series of functions**: with $f_n$'s defined on interval $I$, $\sum_1^{\infty} f_n(x)$ converges *uniformly* on $I$ to $f(x)$ if the sequence of partial sums $\{s_N\}_{N=1,2,\ldots}$, $s_N = \sum_{n=1}^{N} f_n(x)$, converges uniformly to $f(x)$ on $I$.

**Comparison Test**: If $|f_n(x)| \leq c_n$ for all n and for all $a \leq x \leq b$, where the $c_n$'s are constants, and if $\sum_1^{\infty} c_n$ converges, then $\sum_1^{\infty} f_n(x)$ converges *uniformly* in the interval [a,b], as well as absolutely.

**Convergence Theorem**: If $\sum_1^{\infty} f_n(x)$ converges uniformly to $f(x)$ in [a,b] and if all the functions $f_n(x)$ are continuous in [a,b], then the sum $f(x)$ is also continuous in [a,b] and $\sum_1^{\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

The last statement is called term-by-term integration.

**Convergence of Derivatives**: If all the functions $f_n(x)$ are differentiable in [a,b] and if the series $\sum_1^{\infty} f_n(c)$ converges for some $c$, and if the series of derivatives $\sum_1^{\infty} f'_n(x)$ converges *uniformly* in [a,b], then $\sum_1^{\infty} f_n(x)$ converges uniformly to a function $f(x)$ and $\sum_1^{\infty} f'_n(x) = f'(x)$.