# Optional Homework Assignment #12

1. $g(t) = H(T-t)$ for $T>0$ fixed

$$u(x,t) = \frac{x}{2\sqrt{\pi D}}\int_0^t g(t-\tau)\frac{e^{-x^2/4D\tau}}{\tau^{3/2}}d\tau \underset{t'=t-\tau}{=} \frac{x}{2\sqrt{\pi D}}\int_0^t g(t')\frac{e^{-x^2/4D(t-t')}}{(t-t')^{3/2}}dt'$$

$$= \frac{x}{2\sqrt{\pi D}}\int_0^t H(T-t)\frac{e^{-x^2/4D(t-t')}}{(t-t')^{3/2}}dt' \underset{\tau=t-t'}{=} \begin{cases} \frac{x}{2\sqrt{\pi D}}\int_0^t \frac{e^{-x^2/4D\tau}}{\tau^{3/2}}d\tau & \text{if } t<T \\ \frac{x}{2\sqrt{\pi D}}\int_0^T \frac{e^{-x^2/4D\tau}}{\tau^{3/2}} & \text{if } t\geq T \end{cases}$$

Now

$$\frac{x}{2\sqrt{\pi D}}\int_0^t \frac{e^{-x^2/4D\tau}}{\tau^{3/2}}d\tau \underset{r=\frac{x}{2\sqrt{D\tau}}}{=} \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{Dt}}}^\infty e^{-r^2}dr = \operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}\right)$$

$$-\frac{4\sqrt{D}}{x}dr = \frac{d\tau}{\tau^{3/2}}$$

so

$$u(x,t) = \begin{cases} \operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}\right) & t<T \\ \operatorname{erfc}\left(\frac{x}{2\sqrt{DT}}\right) & t\geq T \end{cases}$$

2. $U(r,s) = \int_0^\infty e^{-st}u(r,t)\,dt \rightarrow sU - \underbrace{u(r,0)}_{0} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) \rightarrow$

$r^2\frac{\partial^2 U}{\partial r^2} + r\frac{\partial U}{\partial r} - \frac{s}{D}r^2 U = 0$, and $U(a,s) = \int_0^\infty e^{-st}dt = 1/s$

Just like in the class example, $U$ can be written as a linear combination of modified Bessel functions $K_0, I_0$ but because of the demand for boundedness, $U(r,s) = A K_0(r\sqrt{s/D})$.

Letting $r\to a$, $1/s = AK_0(a\sqrt{s/D}) \rightarrow$

$$U(r,s) = \frac{K_0(r\sqrt{s/D})}{sK_0(a\sqrt{s/D})}$$

Now for $s \gg 1$,

$$U \sim \frac{\sqrt{\frac{\pi}{2r\sqrt{s/D}}}\,e^{-r\sqrt{s/D}}}{s\sqrt{\frac{\pi}{2a\sqrt{s/D}}}\,e^{-a\sqrt{s/D}}} = \sqrt{\frac{a}{r}}\,\frac{1}{s}e^{-\sqrt{s}(r-a)/\sqrt{D}}$$

Now $\mathcal{L}^{-1}\left(\frac{1}{s}e^{-b\sqrt{s}}\right) = \operatorname{erfc}\left(\frac{b}{2\sqrt{t}}\right)$; here $b = \frac{r-a}{\sqrt{D}}$, so

for small $t$ $\quad u(r,t) \sim \sqrt{\frac{a}{r}}\operatorname{erfc}\left(\frac{r-a}{2\sqrt{Dt}}\right)$.