Determining a Physical Distributed Parameter for a Neuronal Cable Model on a Metric Tree Graph

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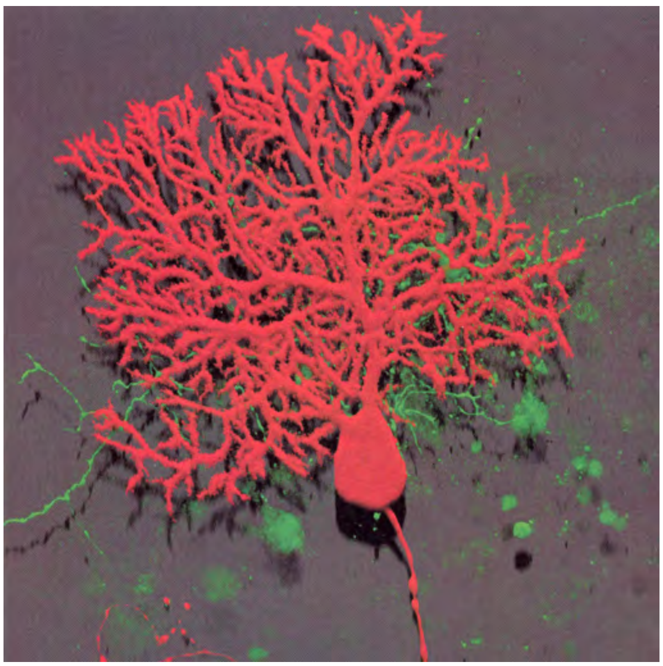
(Technology Review, 2009)

Motivation:

Current flow in dendrites of CNS neurons

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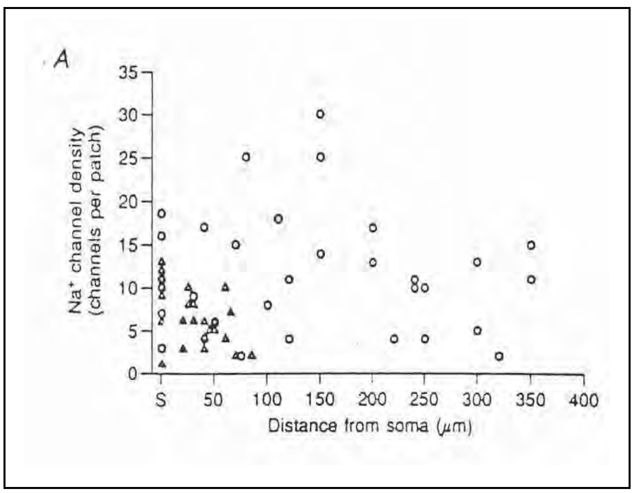
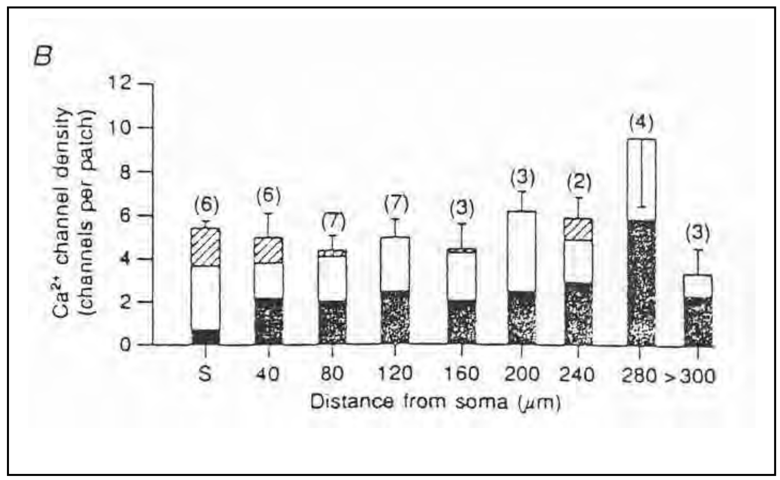
Neuronal Cable Model (single branch, linear I-v)

$$C_m \frac{\partial v}{\partial t} + g(v - E) = \frac{a}{2R_i} \frac{\partial^2 v}{\partial x^2}$$

Conductance, g , can vary on branch due, e.g. to spatial distribution of ionic channels





(MaGee and Johnston, 1989)

Non-dimensionalized Cable Model to be imposed on a tree graph:

$$\frac{\partial v}{\partial t} + q(x)v = \frac{\partial^2 v}{\partial x^2} \qquad\qquad q(x) \propto g$$

The Set-up:

Domain: $\Omega = E \oplus V = \{e_1, e_2, \ldots, e_N\} \oplus \{v_1, v_2, \ldots, v_M\}$

$$\partial\Omega = \{v \in V \mid id(v) = 1\} = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$$

$\Omega =$ *metric graph* if every edge $e_j \in E$ is identified with an interval of the real line with positive length l_j.

$\Omega =$ metric **tree** graph

Problem: $\dfrac{\partial v}{\partial t} + q(x)v = \dfrac{\partial^2 v}{\partial x^2}$ in $\{\Omega \setminus V\} \times (0,T)$ (1)

(**KN**) $\sum_{e_j \sim \nu} \partial v_j(\nu, t) = 0$ for $\nu \in V \setminus \partial\Omega$, and $t \in [0,T]$ (2)

$v(\cdot, t)$ is continuous at each vertex, for all $t \in [0,T]$

$\partial v = f$ on $\partial\Omega \times [0,T]$ $\quad f \in F^T := L^2([0,T], R^m)$ (3)

$v|_{t=0} = 0$ in Ω (4)

$\partial v_j(\nu, t)$ denotes the derivative of v at the vertex ν taken along the edge e_j in the direction toward the vertex.

The sum in (2) is taken over all edges incident to vertex ν.

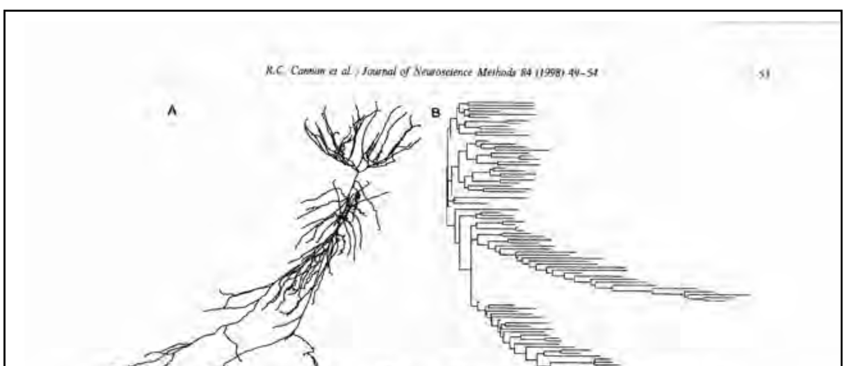
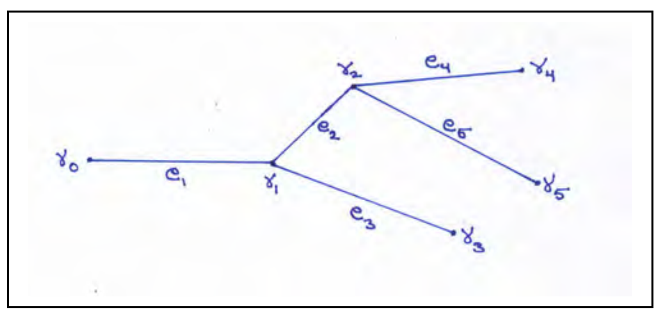


Fig. 3. A sample of representations available with the morphology editor as applied to the CA1 cell n184. A: a simple skeleton projection. B: the appropriate lengths and connections of all branches without direction information, in the form of a dendrogram.



Response operator for system: $R^T = \{R_{ij}^T\}_{i,j=1}^m$ defined by

$$\left(R^T f\right)(t) = v^f(\cdot,t)_{|\partial\Omega} \qquad f \in F^T := L^2([0,T],R^m)$$

Spectral Data(SD): $\{\lambda_n, \varphi_n \mid \partial\Omega\}$

Let $\varphi = \varphi^f(x,\lambda)$ be the solution to

$$L\varphi = -\frac{d^2\varphi}{dx^2} + q(x)\varphi = \lambda\varphi \qquad \text{in} \quad \{\Omega \setminus V\}$$

φ satisfies KN condition at $V \setminus \partial\Omega$, $\partial\varphi(\gamma_j,\lambda) = f_j$, $\gamma_j \in \partial\Omega \rightarrow$

Spectrum $\{\lambda_n\}$ is real, discrete, $\{\varphi_n\}$=ON basis in $L^2(\Omega)$

Titchmarsh-Weyl (TW) matrix function: $\varphi^f \mid_{\partial\Omega} = M(\lambda)f$

$$\varphi^f(x,\lambda) = \sum_{n=1}^{\infty} \frac{< f, \varphi_n \mid_{\partial\Omega} >}{\lambda_n - \lambda}\varphi_n(x) \quad \rightarrow$$

$$M(\lambda) = \left(M_{ij}(\lambda)\right), \quad M_{ij}(\lambda) = \sum_{n\geq 1} \frac{\varphi_n(\gamma_i)\varphi_n(\gamma_j)}{\lambda_n - \lambda}$$

The Response operator and TW-function are connected with each other by Fourier-Laplace transform, so knowledge of one implies knowledge of the other.

Spectral Controllability Lemma: For any T > 0, for any n ≥ 1, there exists a control $f = f_n \in H_0^1(0,T;R^m)$ such that $v^f(\cdot,T) = \varphi_n$ in Ω.

So **SD** can be found using the Response operator R^T.

Companion wave equation problem:

$$\frac{\partial^2 w}{\partial t^2} + q(x)w = \frac{\partial^2 w}{\partial x^2} \quad \text{in} \quad \{\Omega \setminus V\} \times (0,T) \tag{5}$$

(**KN**) $\sum_{e_j \sim v} \partial w_j(v,t) = 0$ for $v \in V \setminus \partial\Omega$, and $t \in [0,T]$ (6)

$w(\cdot,t)$ is continuous at each vertex, for all $t \in [0,T]$

$$\partial w = f \quad \text{on} \quad \partial\Omega \times [0,T] \qquad f \in F^T := L^2([0,T],R^m) \tag{7}$$

$$v|_{t=0} = \frac{\partial w}{\partial t}|_{t=0} = 0 \quad \text{in} \quad \Omega \tag{8}$$

→ $w = w^f \in C([0,T];H^1) \cap C^1([0,T];H)$

Response operator for system: : $R^T = \{R_{ij}^T\}_{i,j=1}^m$ defined by

$$(R^T f)(t) = w_t^f(\cdot,t)_{|\partial\Omega}$$

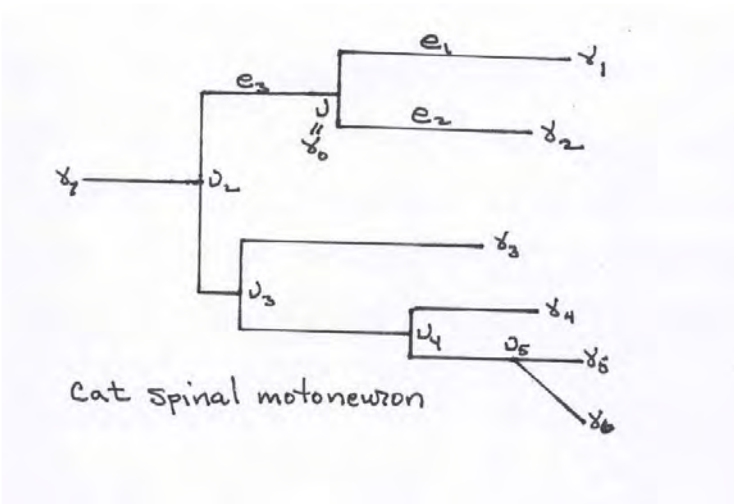
Again,

R^T, $\forall T > 0$ ↔ Fourier-Laplace transform ↔ $M(\lambda)$

Tree Graph Algorithm: Using $M(\lambda)$ and R^T

Idea (Avdonin and Kurasov, 2008): Use boundary data TW-function to determine *q* on boundary edges, then prune tree to smaller tree, recomputed $M(\lambda)$ for the smaller tree, and continue to "prune" edges until have IP on single interval.

(Single edge case theory in Avdonin, Bell, 2012)



Suppose conductance already found on e_1, e_2

Denote $\tilde{M}(\lambda)$ the *M* matrix for reduced graph $\tilde{\Omega} = \Omega \setminus \{e_1, e_2\}$

Rename $v = \gamma_0$ the "new" boundary vertex for $\tilde{\Omega}$

$\tilde{M}_{i0}$, $\tilde{M}_{0i}$, $\tilde{M}_{00}$ are matrix entries related to $v = \gamma_0$

Other entries $\tilde{M}_{ij}$ are the same as the corresponding M_{ij} of the original matrix *M* (for Ω).

For γ_1 fixed we have Cauchy problems

$$\begin{cases} -\varphi''+q(x)\varphi=\lambda\varphi & x\in e_1 \\ \partial\varphi(\gamma_1,\lambda)=1, & \varphi(\gamma_1,\lambda)=M_{11}(\lambda) \end{cases}$$ and

$$\begin{cases} -\varphi''+q(x)\varphi=\lambda\varphi & x\in e_2 \\ \partial\varphi(\gamma_2,\lambda)=0, & \varphi(\gamma_2,\lambda)=M_{12}(\lambda) \end{cases}$$

Solve these, use KN condition at $v=\gamma_0$ to recover

$$\varphi(\gamma_0,\lambda),\quad \partial\varphi(\gamma_0,\lambda)\rightarrow \tilde{M}_{00}(\lambda)=\frac{\varphi(\gamma_0,\lambda)}{\partial\varphi(\gamma_0,\lambda)},\ \tilde{M}_{0i}(\lambda)=\frac{M_{1i}(\lambda)}{\partial\varphi(\gamma_0,\lambda)},\ i=3,\ldots,m$$

To find $\tilde{M}_{i0}(\lambda), i=3,\ldots,m$, fix γ_i consider $\psi(x,\lambda)$, the solution to EVP with b.c. $\partial\psi(\gamma_i,\lambda)=1, \partial\psi(\gamma_j,\lambda)=0,\quad j\neq i$. Then function ψ solves Cauchy problems on edges e_1, e_2

$$\begin{cases} -\psi''+q(x)\psi=\lambda\psi & x\in e_j,\ j=1,2 \\ \partial\psi(\gamma_j,\lambda)=0, & \psi(\gamma_j,\lambda)=M_{ij}(\lambda) \end{cases}$$

Now consider the linear combination

$\tilde{\phi}(x,\lambda)=\psi(x,\lambda)-\dfrac{\partial\psi(\gamma_0,\lambda)}{\partial\varphi(\gamma_0,\lambda)}\varphi(x,\lambda)$. On subgraph $\tilde{\Omega}$ this $\tilde{\phi}$ satisfies b.c.s $\partial\tilde{\phi}(\gamma_i,\lambda)=1, \partial\tilde{\phi}(\gamma_j,\lambda)=0, j\neq i$. Thus, we obtain $\tilde{M}_{i0}(\lambda)=\psi(\gamma_0,\lambda)-\partial\psi(\gamma_0,\lambda)\tilde{M}_{00}(\lambda)$. To recover all elements of the reduced TW function, the procedure needs to be applied for all $i=3,\ldots,m$. We also put $\tilde{M}_{ij}=M_{ij}$ for $i,j=3,\ldots,m$.

Other Related Inverse Problems

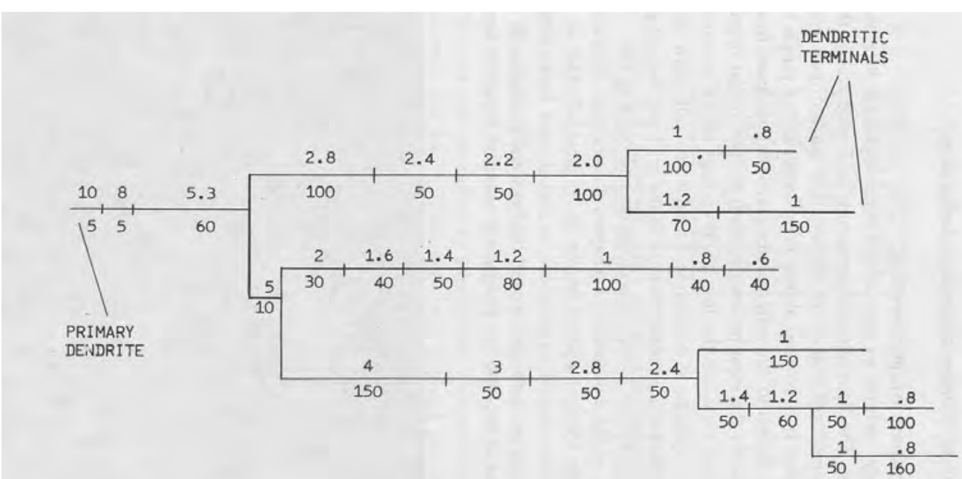
1. Large Changes in Ion Channel Density: *N(x)*

$$C_m \frac{\partial v}{\partial t} + g(v - E) = \frac{\partial^2 v}{\partial x^2} \;,\; C_m = C_0 + C_1 N, \quad g = g_0 N$$

$$\rightarrow \begin{cases} (1+q(x))v_t + q(x)v = v_{xx} & 0 < x < l, & t > 0 \\ v_x(0,t) = f(t), & v_x(l,t) = 0, & t > 0 \\ v(x,0) = 0 & & 0 < x < l \end{cases}$$

1. Recovering other physical parameters:

$$C_m \frac{\partial v}{\partial t} + g(v - E) = \frac{1}{a(x)} \frac{\partial}{\partial x}\left\{ a(x)^2 \frac{\partial v}{\partial x} \right\} \text{ on } \{\Omega \setminus V\}$$



(J. Barrett, 1988)