G: Uniform Convergence of Fourier Series

From previous work on the prototypical problem (and other problems)

$$\begin{cases} u_t = Du_{xx} & 0 < x < l \ , \ t > 0 \\ u(0,t) = 0 = u(l,t) & t > 0 \\ u(x,0) = f(x) & 0 < x < l \end{cases} \tag{1}$$

we developed a (formal) series solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 Dt/l^2} \sin(\frac{n\pi x}{l}) \ , \tag{2}$$

with $b_n = \frac{2}{l}\int_0^l f(y)\sin(\frac{n\pi y}{l})dy$. These are the Fourier sine coefficients for the initial data function $f(x)$ on $[0,l]$. We have no real way to check that the series representation (2) is a solution to (1) because we do not know we can interchange differentiation and infinite summation. We have only assumed that up to now. In actuality, (2) makes sense as a solution to (1) if the series is uniformly convergent on $[0,l]$ (and its derivatives also converges uniformly[1]). So we first discuss conditions for an infinite series to be differentiated (and integrated) term-by-term. This can be done if the infinite series and its derivatives converge uniformly. We list some results here that will establish this, but you should consult Appendix B on calculus facts, and review definitions of convergence of a series of numbers, absolute convergence of such a series, and uniform convergence of sequences and series of functions. Proofs of the following results can be found in any reasonable real analysis or advanced calculus textbook.

0.1 Differentiation and integration of infinite series

Let $\mathcal{I} = [a,b]$ be any real interval.

Theorem 1 (term-by-term differentiation): If, on the interval $\mathcal{I}$,
a) $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly;
b) $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly;

[1] We can get away with less restrictive conditions, but the analysis is beyond the level of these Notes.

c) $f_n'(x)$ is continuous.
Then the series $\sum_{n=1}^{\infty} f_n(x)$ can be differentiated term-by-term; that is, $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$.

Theorem 2 (term-by-term integration): If, on the interval $\mathcal{I}$,
a) $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly;
b) each $f_n'(x)$ is integrable.
Then the series $\sum_{n=1}^{\infty} f_n(x)$ can be integrated term-by-term; that is, $\int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx$.

Remark: Note the weaker conditions needed for integration versus differentiation.

Theorem 3 (Weirstrass M-test for uniform convergence of an infinite series of functions):
Suppose $\{f_n\}_{n\geq 1}$ is a sequence of functions defined on $\mathcal{I}$, and suppose $|f_n(x)| \leq M_n$ for all $x \in \mathcal{I}$, $n \geq 1$. Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly in $\mathcal{I}$ if the series of numbers $\sum_{n=1}^{\infty} M_n$ converges absolutely.

Theorem 4 (Ratio test):
A series of numbers $\sum_{n=1}^{\infty} M_n$ converges absolutely if, for some $N \geq 1$, and some constant $0 < \rho < 1$, $|M_{n+1}/M_n| \leq \rho < 1$, for all $n \geq N$.

Now fix $t_0 > 0$ and consider series (2) for all $t \geq t_0$, $0 \leq x \leq l$, and also assume f is absolutely integrable (i.e. $\int_0^l |f(x)|dx < \infty$); f being piecewise continuous on $0 \leq x \leq l$ is more than sufficient. Then we have

$$|u_n(x,t)| = |b_n|e^{-n^2\pi^2 Dt/l^2}|\sin(\frac{n\pi x}{l})| \leq |b_n|e^{-n^2\pi^2 Dt_0/l^2} \ ,$$

where $|b_n| \leq (2/l)\int_0^l |f(y)|dy$. Define $M_n := ((2/l)\int_0^l |f(y)|dy)e^{-n^2\pi^2 Dt_0/l^2}$. Then, for all $t \geq t_0$, $|u_n(x,t)| \leq M_n$. Also,

$$\frac{M_{n+1}}{M_n} = \frac{e^{-(n+1)^2\pi^2 Dt_0/l^2}}{e^{-n^2\pi^2 Dt_0/l^2}} = e^{[n^2-(n+1)^2]\pi^2 Dt_0/l^2} = e^{-(2n+1)\pi^2 Dt_0/l^2} \leq e^{-\pi^2 Dt_0/l^2} < 1 \ .$$

Thus, with $\rho := e^{-\pi^2 D t_0 / l^2}$, $\sum_{n=1}^{\infty} M_n$ converges absolutely, by Theorem 4, and by Theorem 3, $\sum_{n=1}^{\infty} u_n$ converges uniformly on $[0, l]$, $t \geq t_0 > 0$. A similar argument holds for the convergence of derivatives u_t, u_{xx}. So, for $t \geq t_0$, the series (2) for u can be differentiated term-by-term. Since each term, $u_n(x, t)$, satisfies the pde and b.c.s, so does u. After our discussion of the properties of the Fourier series, and the uniform convergence result on the Fourier series, the convergence of u holds all the way down to $t = 0$ (given the appropriate conditions on $u(x, 0) = f(x)$).

For the latter argument we turn to the uniform convergence of the trig series.

0.2 Uniform convergence of classical Fourier series

Let[2] f be piecewise smooth on $(-1, 1)$, continuous on $[-1, 1]$, with $f(-1) = f(1)$. Thus, when f is considered extended to the whole real line, it is continuous everywhere, and is a 2-periodic function on $\mathbb{R}$. Then its Fourier series converges everywhere (pointwise) to f. For purposes below we will also assume f' is square-integrable; that is, $\int_{-1}^{1} (f'(x))^2 dx < \infty$. Let

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(n\pi x) + b_n \sin(n\pi x)\} \ ,$$

where $\{a_n, b_n\}$ are the Fourier coefficients for f :

$$\begin{bmatrix} a_n \\ \\ b_n \end{bmatrix} = \int_{-1}^{1} f(x) \begin{bmatrix} \cos(n\pi x) \\ \\ \sin(n\pi x) \end{bmatrix} dx \ \ .$$

The purpose here is to show the sequence of partial sums $\{S_N\}_{N \geq 1}$ is a Cauchy sequence, which is sufficient for the Fourier series for f to converge uniformly to f. Recall that $\{S_N\}_{N \geq 1}$ is a Cauchy sequence if, for any $\varepsilon > 0$, there is an N^* such that for any $N, M > N^*$, $|S_N(x) - S_M(x)| < \varepsilon$ for all $x \in (-1, 1)$.

Assume without loss of generality that $N > M$; from the definition of S_N,

$$S_N(x) - S_M(x) = \sum_{n=M+1}^{N} \{a_n \cos(n\pi x) + b_n \sin(n\pi x)\} \ . \tag{3}$$

[2]For convenience only we have scaled the interval to have semi-length unity by redefining $\tilde{x} = x/l$, then dropping the tilde notation.

The essential "trick" is to write the right side of (3) as a dot product of two vectors:

$$S_N(x) - S_M(x) = \tag{4}$$
$$(\pi(M+1)a_{M+1}, \pi(M+1)b_{M+1}, \ldots, \pi N a_{M+1}, \pi N a_{M+1})\cdot$$
$$(\frac{\cos((M+1)\pi x)}{\pi(M+1)}, \frac{\sin((M+1)\pi x)}{\pi(M+1)}, \ldots, \frac{\cos(N\pi x)}{\pi N}, \frac{\sin(N\pi x)}{\pi N})\ .$$

Now we need three pieces of information, namely,

A. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

B. (Schwarz's inequality): given sequences $\{c_n\}_{n=1}^N, \{d_n\}_{n=1}^N$, then

$$|\sum_{n=1}^{N} c_n d_n| \leq (\sum_{n=1}^{N} c_n^2)^{1/2} (\sum_{n=1}^{N} d_n^2)^{1/2} \ .$$

C. (Bessel's inequality): if $\{a_n, b_n\}$ are the Fourier coefficients of f, where f' is square-integrable on $(-1,1)$, then

$$\pi^2 \sum_{n=1}^{\infty} n^2 (a_n^2 + b_n^2) \leq \int_{-1}^{1} (f'(x))^2 dx \ .$$

Remark: In the last section of this appendix are proofs of these three statements.

Now apply Schwarz's inequality B to (4):

$$\begin{aligned} |S_N(x) - S_M(x)| &\leq (\sum_{n=M+1}^{N} (\pi n)^2 (a_n^2 + b_n^2))^{1/2} (\sum_{n=M+1}^{N} \frac{1}{(n\pi)^2} [\cos^2(n\pi x) + \sin^2(n\pi x)])^{1/2} \\ &= (\sum_{n=M+1}^{N} n^2 (a_n^2 + b_n^2))^{1/2} (\sum_{n=M+1}^{N} \frac{1}{n^2})^{1/2} \\ &\leq (\frac{1}{\pi^2} \int_{-1}^{1} |f'(x)|^2 dx)^{1/2} (\sum_{n=M+1}^{N} \frac{1}{n^2})^{1/2} \end{aligned}$$

using Bessel's inequality C in the last step. Since, from fact A,

$$0 < \frac{\pi^2}{6} - \sum_{n=1}^{N} \frac{1}{n^2} = \frac{\pi^2}{6} - \sum_{n=1}^{M} \frac{1}{n^2} - \sum_{n=M+1}^{N} \frac{1}{n^2} \ ,$$

then

$$|S_N(x) - S_M(x)| \le \left(\frac{1}{\pi^2}\int_{-1}^{1}|f'(x)|^2dx\right)^{1/2}\left(\frac{\pi^2}{6} - \sum_{n=1}^{M}\frac{1}{n^2}\right)^{1/2} , \qquad (5)$$

where the first factor on the right side of the inequality is just a constant, and the second factor can be made arbitrarily small if M is taken sufficient large. Hence, $\{S_N\}$ is a Cauchy sequence.

Remark: We can let $N \to \infty$ in (5) to obtain

$$|f(x) - S_M(x)| \le \left(\frac{1}{\pi^2}\int_{-1}^{1}|f'(x)|^2dx\right)^{1/2}\left(\frac{\pi^2}{6} - \sum_{n=1}^{M}\frac{1}{n^2}\right)^{1/2} .$$

Example: $f(x) = x^2$ on $[-1, 1]$; computing the Fourier coefficients, we have

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x) .$$

(Note: the equality is fully justified in this case.) Now,

$$\int_{-1}^{1}(f'(x))^2dx = \frac{8}{3} , \quad \text{so} \quad \left(\frac{1}{\pi^2}\int_{-1}^{1}(f'(x))^2dx\right)^{1/2} \cong 0.519798 .$$

Then for

M	$(0.519798)(\frac{\pi^2}{6} - \sum_{n=1}^{M}\frac{1}{n^2})^{1/2}$
2	0.32667
10	0.16035
50	0.07314
100	0.05185
1000	0.01643
100000	0.00164

Remark: In actuality the theory gives very conservative estimates. You only need a small number of terms usually to get a high order accurate approximation for series that converge uniformly.

Exercise: Compute estimates of $|f(x) - S_M(x)|$ for $f(x) = 1 - |x|$ on $[-1, 1]$, for $M = 10, 100, 1000$.

0.3 Proofs of statements A, B, and C

1. Showing A

For $f(x) = x^2$ defined on $(-1, 1)$ and extended to the real line, its Fourier series is $f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x)$, which converges uniformly to f, so

$$1 = f(1) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{since } (-1)^n \cos(n\pi) = (-1)^n(-1)^n = 1\ .$$

Solving for $\sum \frac{1}{n^2}$ gives $\pi^2/6$.

2. Showing B

First, for two arbitrary real numbers c, d, and for any $\alpha > 0$,

$$0 \leq (\alpha|c| - \frac{1}{\alpha}|d|)^2 = \alpha^2 c^2 - 2|cd| + \frac{1}{\alpha^2} d^2 \tag{6}$$

so $|cd| \leq \frac{1}{2}\{\alpha^2 c^2 + d^2/\alpha^2\}$.

Second, for real vectors $(c_1, c_2, \dots, c_N), (d_1, d_2, \dots, d_N)$, and real number α,

$$|(c_1, c_2, \dots, c_N) \cdot (d_1, d_2, \dots, d_N)| = |\sum_n^N c_n d_n| \leq \sum_1^N |c_n d_n| \leq \frac{1}{2} \sum_1^N \{\alpha^2 c_n^2 + \frac{1}{\alpha^2} d_n^2\}$$

where we have applied (6) to each term in the sum. If $\sum_1^N c_n^2 \neq 0$, define

$\alpha^2 := (\sum_1^N d_n^2 / \sum_1^N c_n^2)^{1/2}$. Then

$$
\begin{aligned}
|\sum_1^N c_n d_n| &\leq \frac{1}{2}\alpha^2 \sum_1^N c_n^2 + \frac{1}{2\alpha^2}\sum_1^N d_n^2 \\
&= \frac{1}{2}(\frac{\sum_1^N d_n^2}{\sum_1^N c_n^2})^{1/2} \sum_1^N c_n^2 + \frac{1}{2}(\frac{\sum_1^N c_n^2}{\sum_1^N d_n^2})^{1/2} \sum_1^N d_n^2 \\
&= \frac{1}{2}(\sum_1^N d_n^2)^{1/2}(\sum_1^N c_n^2)^{1/2} + \frac{1}{2}(\sum_1^N c_n^2)^{1/2}(\sum_1^N d_n^2)^{1/2} \\
&= (\sum_1^N c_n^2)^{1/2}(\sum_1^N d_n^2)^{1/2} \ , \qquad (7)
\end{aligned}
$$

which is B. (Note: if $\sum_1^N c_n^2 = 0$, then $c_n = 0$ for all n, which implies (7) automatically.)

3. Showing C

We first want the Fourier coefficients for $f'(x)$, given

$$
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n \cos(n\pi x) + b_n \sin(n\pi x)\} \ .
$$

By integration-by-parts,

$$
\int_{-1}^{1} f'(x)\cos(n\pi x)dx = f(x)\cos(n\pi x)|_{-1}^{1} + n\pi \int_{-1}^{1} f(x)\sin(n\pi x)dx \ . \qquad (8)
$$

Since $f(-1) = f(1)$, there is no contribution from the first term on the right side of (8). With $\{a_n, b_n\}$ being the Fourier coefficients of f, let $\{a_n^{(1)}, b_n^{(1)}\}$ be the Fourier coefficients for $f'(x)$. Then (8) just states that $a_n^{(1)} = n\pi b_n$. Similarly, integrating-by-parts the integral $\int_{-1}^{1} f'(x)\sin(n\pi x)dx$ yields $b_n^{(1)} = -n\pi a_n$. Therefore, the Fourier series for $f'(x)$ is

$$
\begin{aligned}
f'(x) &\sim \sum_{n=1}^{\infty}\{a_n^{(1)}\cos(n\pi x) + b_n^{(1)}\sin(n\pi x)\} \\
&= \pi\sum_{n=1}^{\infty} n\{b_n\cos(n\pi x) - a_n\sin(n\pi x)\} \ .
\end{aligned}
$$

Let

$$S_N^{(1)}(x) = \pi \sum_{n=1}^{N} n\{b_n \cos(n\pi x) - a_n \sin(n\pi x)\} \ .$$

Now

$$
\begin{aligned}
0 &\le \int_{-1}^{1} (f'(x) - S_N^{(1)}(x))^2 dx \\
&= \int_{-1}^{1} (f'(x))^2 dx - 2\int_{-1}^{1} f'(x) S_N^{(1)}(x) dx + \int_{-1}^{1} (S_N^{(1)}(x))^2 dx \\
&= \int_{-1}^{1} (f'(x))^2 dx - 2\pi \sum_{n=1}^{N} n b_n \int_{-1}^{1} f'(x)\cos(n\pi x) dx + 2\pi \sum_{n=1}^{N} \int_{-1}^{1} f'(x)\sin(n\pi x) dx \\
&+ \pi^2 \sum_{n=1}^{N}\sum_{m=1}^{N} \int_{-1}^{1} nm\{b_n b_m \cos(n\pi x)\cos(m\pi x) - b_m a_n \cos(m\pi x)\sin(n\pi x) \\
&- a_m b_n \cos(n\pi x)\sin(m\pi x) + a_m a_n \sin(n\pi x)\sin(m\pi x)\} dx \\
&= \int_{-1}^{1} (f'(x))^2 dx - 2\pi^2 \sum_{n=1}^{N} n^2(a_n^2 + b_n^2) + \pi^2 \sum_{n=1}^{N} n^2(a_n^2 + b_n^2) \\
&= \int_{-1}^{1} (f'(x))^2 dx - \pi^2 \sum_{n=1}^{N} n^2(a_n^2 + b_n^2) \ \ ,
\end{aligned}
$$

so that

$$\pi^2 \sum_{n=1}^{N} n^2(a_n^2 + b_n^2) \le \int_{-1}^{1} (f'(x))^2 dx \ .$$

Since this holds for all $N \ge 1$, then this gives C on page 4. We have made use of the fact that the single summations combine and the double summation collapses because of the orthogonality property of $\{\sin(n\pi x), \cos(n\pi x)\}$ on $(-1, 1)$.