Predator-mediated coexistence with chemorepulsion

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Model: Two competing prey, a common predator

$$
\begin{aligned}
u_t &= d\triangle u + u(1-u-a_1 v - b_1 w)\\
v_t &= d\triangle v + v(r(1-v) - a_2 u - b_2 w) \quad \text{in } \Omega\times(0,T)\\
w_t &= \nabla\cdot(\nabla w + \chi_0 w \nabla z) + w(-\mu + b_1 u + b_2 v)\\
\tau z_t &= d_1 \triangle z - \mu_z z + \delta u
\end{aligned}
$$

Reduction ("direct interaction"): $z = (\delta/\mu_z)u$

$$
\begin{aligned}
u_t &= d\triangle u + u(1-u-a_1 v - b_1 w)\\
v_t &= d\triangle v + v(r(1-v) - a_2 u - b_2 w) \quad \text{in } \Omega\times(0,T)\\
w_t &= \nabla\cdot(\nabla w + \chi w \nabla u) + w(-\mu + b_1 u + b_2 v)
\end{aligned}
\tag{1}
$$

Along with homogeneous Neumann b.c.s and initial conditions

Some Theory

$\Omega \subset \mathbb{R}^n$ connected, bounded, $\partial\Omega$ smooth, $p > n = 2$
$W^{1,p}_{bdy} \doteq \{y \in W^{1,p}(\Omega; \mathbb{R}^2) | \nu \cdot \nabla y_{\partial\Omega} = 0\}$
$X \doteq \{y = (u, v, w) \in (W^{1,p}_{bdy})^3 | y(\overline{\Omega}) \in \mathcal{H}\}$; Assume
$(u_0, v_0, w_0) \in X$

Theorem: $\exists\ T = T(u_0, v_0, w_0)$ s.t. i) there is a unique maximal classical solution (u, v, w) on $\Omega^T \doteq \Omega \times (0, T)$; ii) $(u_0, v_0, w_0) \geq 0$ implies $(u, v, w) \geq 0$ on Ω^T ; iii) if $||(u(\cdot, t), v(\cdot, t), w(\cdot, t))||_{L^\infty}$ is bounded away from boundary of $\mathcal{H}$ for all $t \in (0, T)$, then $T = +\infty$: i.e. (u, v, w) is global in time.

Proof similar to arguments in Wrzosek, 2004, Wang and Hillen, 2007, and relies on Amann's theory, 1990, 1993.

Constant Steady State (u_s, v_s, w_s)

Non-spatial (ODE) case: For competitive system (no predator),
$a_1 > 1, a_2 < r$, v "wins", u "loses" ($(u,v) \to (0,1)$ as $t \to \infty$)
$a_1 < 1, a_2 > r$, u "wins", v "loses" ($(u,v) \to (1,0)$ as $t \to \infty$).

Full ODE system: there is a non-empty (admissible parameter set) $\mathcal{A}$ such that

Proposition (predator-mediated coexistence):
$(a_1, a_2, b_1, b_2, r, \mu) \in \mathcal{A}$, then $(u_s, v_s, w_s) \in (\mathbb{R}^+)^3$ and is globally stable positive solution.

Assume parameters always in $\mathcal{A}$.

Question: Can a (chemotactic) prey defense destabilize the situation? Is there pattern formation? How do the two cases above differ?

Convergence to the Constant Steady State

Let $(a_1, a_2, b_1, b_2, r, \mu) \in \mathcal{A}$.

$$\overline{V} = \int_{u_s}^{u} \frac{u'-u_s}{u'} du' + \frac{v'-v_s}{v'} dv' + \frac{w'-w_s}{w'} dw'$$
$$V(t) = V(u,v,w) = \int_\Omega \overline{V}(u,v,w) dx$$
$$\mathcal{H}_L = \{(u,v,w) | V(u,v,w) \leq L\}$$

L can be large enough for $\mathcal{H}_L$ to be positive invariant.

Proposition: Let $(u_0, v_0, w_0) \in \mathcal{H}_L$. If $r > (a_1 + a_2)^2/4$ and $\chi^2 < 4du_s/(w_s u_{max}^2)$, then $V(t) \to 0$ as $t \to \infty$; i.e., (u_s, v_s, w_s) is globally asymptotically stable for system (1).

Proof: $V(t)$ is a strong Lyapunov functional

Pattern Formation

Setup: linearize about the constant steady state

$$\frac{\partial}{\partial t}\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ \chi w_s & 0 & 1 \end{bmatrix} \triangle \begin{bmatrix} U \\ V \\ W \end{bmatrix} + \begin{bmatrix} -u_s & -a_1 u_s & -b_1 u_s \\ -a_2 v_s & -r v_s & -b_2 v_s \\ b_1 w_s & b_2 w_s & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

Consider $(U, V, W) = e^{\lambda t + i\vec{k}\cdot\vec{x}}\vec{\psi}$, $k = |\vec{k}|$:

$$\mathcal{J} = \begin{bmatrix} dk^2 + u_s + \lambda & a_1 u_s & b_1 u_s \\ a_2 v_s & dk^2 + r v_s + \lambda & b_2 v_s \\ \chi w_s k^2 - b_1 w_s & -b_2 w_s & k^2 + \lambda \end{bmatrix}$$

Dispersion Relation

The $det(\mathcal{J}) = 0$ gives the dispersion relation (DR):

$$\lambda^3 + a(k^2)\lambda^2 + b(k^2, \chi)\lambda + c(k^2, \chi) = 0$$

DR gives relation between wave number k and temporal behavior (frequency $\sim 1/\lambda$)

instability if either $c(k^2, \chi) < 0$ or $a(k^2)b(k^2, \chi) - c(k^2, \chi) < 0$

Under some technical conditions on parameters, there exist a $\chi_c(k^2, d) > 0$ such that

$$c(k^2, \chi) < 0 \quad \text{iff} \quad \chi > \chi_c(k^2, d)$$

Chemotaxis-induced instability

$\chi > \chi_c$ does **not** guarantee pattern formation Ω is bounded.

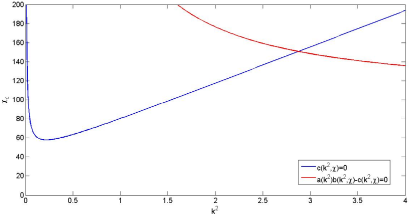


Figure: A plot of χ versus k^2. The blue graph is χ_c as a function of k^2.

Assume $(a_1, a_2, b_1, b_2, r, \mu) \in \mathcal{A}$, $b_1 d k^2 > v_s(a_1 b_2 - r b_1)$

Theorem: Let $\Omega = (0, L)$. Let $\chi_m > \chi_c$ be the first value of χ such that either $k_1 L/\pi$ or $k_2 L/\pi$ is an integer. Then χ_m is a bifurcation parameter and $\chi > \chi_m$ is a necessary and sufficient condition for pattern formation of system (1).

Diffusion-induced instability

$c(k^2, \chi)$ can be written in form

$c(k^2, \chi) = k^6 d^2 + A(k^2, \chi) d + B(k^2, \chi)$

$A(k^2, \chi) = b_1 u_s w_s k^4 \{\chi_{c_1}(k^2, r) - \chi\}$
$B(k^2, \chi) = u_s v_s w_s (r b_1 - a_1 b_2) k^2 \{\chi_{c_2}(k^2, r) - \chi\}$

$\chi_{c_1}(k^2, r)$ is always positive, decreasing in k^2 to a positive limit.

$\chi_{c_2}(k^2, r)$ is dependent on signs of $r b_1 - a_1 b_2, r - a_1 a_2$, etc.

Example: $r b_1 - a_1 b_2 > 0$, $r - a_1 a_2 > 0$. Then $\chi_{c_2} > 0$, decreasing to a positive limit. Assume parameters s.t. there exists $\hat{k}^2 > 0$ ($\chi_{c_2} > \chi_{c_1}$ iff $k^2 > \hat{k}^2$), and $\hat{k}^2 > 0$ large enough so $(0, \hat{k}^2)$ contains a mode

Diffusion-induced instability continued

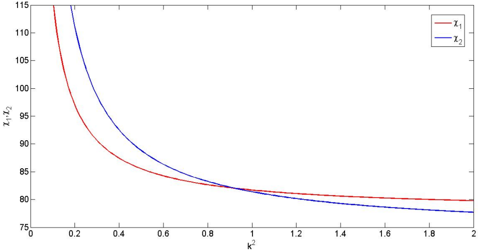
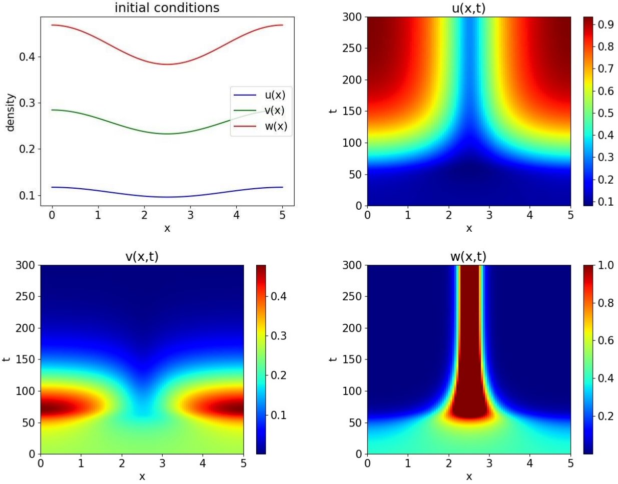


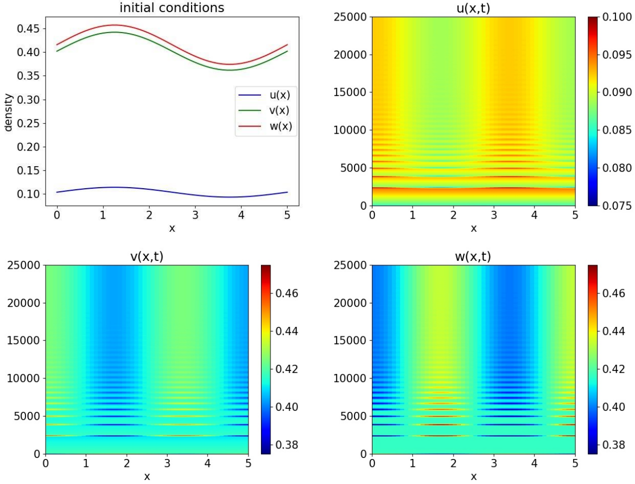
Figure: A plot of χ_{c_1} (red), χ_{c_2} (green) versus k^2.

(k^2, χ) in I, II: (u_s, v_s, w_s) unstable for $0 < d < d_+$;
in III: (u_s, v_s, w_s) unstable for $0 < d\_ < d < d_+$;
in IV: (u_s, v_s, w_s) always locally stable

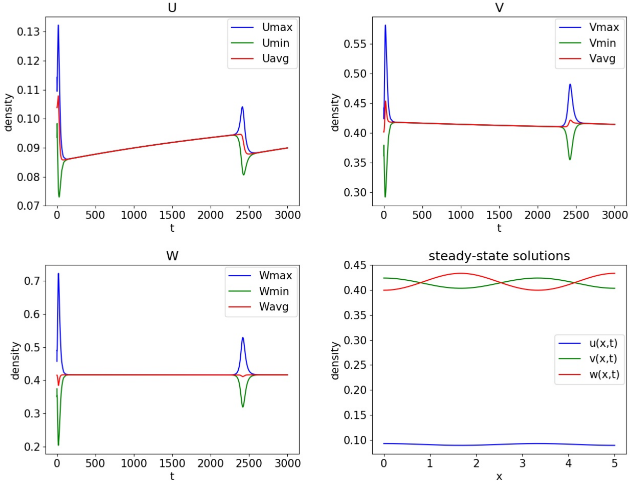
Case 1: u wins


initial conditions
density
0.4
0.3
0.2
0.1
0
1
2
3
4
5
x
u(x)
v(x)
w(x)
u(x,t)
t
300
250
200
150
100
50
0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
v(x,t)
0.4
0.3
0.2
0.1
w(x,t)
1.0
0.8
0.6
0.4
0.2

Case 2: v wins



Case 2: Early epoch time vs steady state



Summary

- Consistency across finite difference, continuous FE, discontinuous FE methods, and ranges of parameters
- Here graphs with "slow" prey ($d = 0.1$); for $d \approx 1$, predator "bumps" are wider
- Prey are "out of phase" with predator, v tends to hide behind u
- "Win, don't loose" dynamics

Thank you for your attention

Figure: This is AfterMath, my cruising home