J: Brief Introduction to Green's Functions: ODEs

0.1 Self-Adjoint Form of ODEs

Consider the second order linear equation

$$A(x)\frac{d^2u}{dx^2} + B(x)\frac{du}{dx} + C(x)u = F(x) \qquad \text{on } a < x < b \tag{1}$$

where $A, dA/dx, B, C$ and F are continuous functions on $a < x < b$. Assuming $A(x)$ does not vanish in the interval (otherwise we would have to deal with a *singular point* that we will not consider here), divide the equation by $A(x)$, then we have $d^2u/dx^2 + (B/A)du/dx +$ the rest; so multiply by

$$p(x) := e^{\int_a^x B(\xi)/A(\xi)\, d\xi}$$

then we have

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + \text{ the rest .}$$

That is, let $q(x) := C(x)p(x)/A(x)$, $f(x) := F(x)p(x)/A(x)$, then (1) is transformed to

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x) \qquad \text{in } a < x < b \tag{2}$$

This is called the **self-adjoint** form of the equation, and it has several advantages (for theoretical purposes) over (1). (The left-hand side of (2) is similar to the operator we introduced in the Sturm-Liouville theory of EVPs, but here we are not interested in eigenvalues or the sign of the coefficients, so we will stick with the plus sign in front of $q(x)$ and we do not care if q changes sign on the interval.) But (2) suggests introducing an operator notation $\mathcal{L}_x$ by

$$\mathcal{L}_x u := \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u \text{ .}$$

So $\mathcal{L}_x$ is a *differential operator*, and (2) can be written as

$$\mathcal{L}_x u = f \qquad \text{in } a < x < b \text{ .}$$

0.2 Initial Value Problems (IVPs)

Consider (2) for $x > a$. For the homogeneous case ($f \equiv 0$) let $u_1(x), u_2(x)$ be a fundamental set of solutions, i.e. two linearly independent solutions to the homogeneous equation. Then, by variation of parameters method for finding a particular solution to (2), we write $u(x) = w(x)u_1(x) + z(x)u_2(x)$ and go through the procedure of Appendix A of determining w and z. The result is

$$\begin{aligned} u(x) &= u_2(x)\int_a^x \frac{u_1(\xi)f(\xi)}{p(\xi)W(\xi)}\,d\xi - u_1(x)\int_a^x \frac{u_2(\xi)f(\xi)}{p(\xi)W(\xi)}\,d\xi \\ &= \int_a^x f(\xi)\left\{\frac{u_1(\xi)u_2(x) - u_2(\xi)u_1(x)}{p(\xi)W(\xi)}\right\}\,d\xi \\ &= \int_a^x f(\xi)R(x,\xi)\,d\xi \end{aligned}$$

where $W(\xi) := u_1(\xi)u_2'(\xi) - u_2(\xi)u_1'(\xi) =$ Wronskian of u_1, u_2. Since u_1 and u_2 are linearly independent on $x > a$, the Wronskian never vanishes on $\xi > a$. Actually, by differentiation, it is straight forward to show that $p(\xi)W(\xi) =$ a constant, so define $K := p\{u_1'u_2 - u_2'u_1\}$. As long as du_1/dx and du_2/dx remain bounded as $x \to a$, we also have $u(a) = 0$ and $\frac{du}{dx}(a) = 0$.

Defining $R(x,\xi)$ above, for fixed $\xi > a$, R solves the problem

$$\begin{cases} \frac{d}{dx}\left(p\frac{dR}{dx}\right) + qR = 0 \quad x > \xi \\ \\ R(\xi,\xi) = 0 \\ \\ \frac{dR}{dx}(\xi,\xi) = 1/p(\xi) \end{cases}$$

Also, $R(x,\xi) = -R(\xi,x)$, and $u(x) = \int_a^x R(x,\xi)f(\xi)d\xi$ solves

$$\mathcal{L}_x u = f(x) \qquad x > a, \quad u(a) = 0 = \frac{du}{dx}(a)\ .$$

Because the denominator is constant, R is just a linear combination of u_1 and u_2. Hence, by linearity, we would just attach a general solution to the homogeneous equation, namely $C_1u_1 + C_2u_2$, to this u to have a solution that satisfies non-zero initial conditions after evaluating the two constants C_1, C_2.

Example 1: Consider

$$\begin{cases} u'' + u = f(x) \qquad x > 0 \quad \text{so } p \equiv 1, q \equiv 1 \\ \\ u(0) = u'(0) = 0 \end{cases}$$

So $R'' + R = 0$ for $x > \xi$, $R = 0$ and $R' = 1$ at $x = \xi$, hence $R(x,\xi) = \sin(x-\xi)$. Therefore, $u(x) = \int_0^x f(\xi)\sin(x-\xi)\, d\xi$ satisfies the IVP, which is what you would obtain by the variation of parameters method, or the Laplace transform method.

Example 2: Consider

$$\begin{cases} u'' + u = f(x) & x > 0 \quad \text{so } p \equiv 1, q \equiv 1 \\ \\ u(0) = 1,\ u'(0) = -1 \end{cases}$$

From the first example, $u(x) = \int_0^x f(\xi)\sin(x-\xi)\, d\xi + C_1\sin(x) + C_2\cos(x)$, so applying the initial conditions gives $C_2 = 1$, $C_1 = -1$, that is,

$$u(x) = \int_0^x f(\xi)\sin(x-\xi)\, d\xi - \sin(x) + \cos(x) \ .$$

0.3 Boundary Value Problems (BVPs)

Now consider (2) again, that is
Example 3:

$$\begin{cases} (pu')' + qu = -f(x) \quad \text{on } a < x < b \\ \\ u(a) = u(b) = 0 \ . \end{cases} \tag{3}$$

(The negative sign on the right is for convenience in applications.) First ignore the second boundary condition and write

$$u(x) = -\int_a^x R(x,\xi)f(\xi)d\xi + C_1u_1(x) + C_2u_2(x)$$

from our developments in the previous subsection. Now employ the boundary conditions:

$$\begin{aligned} C_1u_1(a) + C_2u_2(a) &= 0 \\ C_1u_1(b) + C_2u_2(b) &= \int_a^b R(b,\xi)f(\xi)d\xi \end{aligned}$$

or, in matrix form

$$\begin{bmatrix} u_1(a) & u_2(a) \\ \\ u_1(b) & u_2(b) \end{bmatrix} \begin{bmatrix} C_1 \\ \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \int_a^b R(b,\xi)f(\xi)d\xi \end{bmatrix}$$

In order to solve for C_1, C_2 uniquely, we need the determinent of the matrix to be nonzero, that is, $D := u_1(a)u_2(b) - u_2(a)u_1(b) \neq 0$. Assume this, and recall K from the previous subsection above. Then after some straightforward, but tedious calculations in finding the C_1, C_2 and manipulating the result, we obtain

$$u(x) = \int_a^x \left\{ \frac{1}{KD}(u_1(\xi)u_2(a) - u_2(\xi)u_1(a))(u_1(x)u_2(b) - u_1(b)u_2(x)) \right\} f(\xi)d\xi$$

$$+ \int_x^b \left\{ \frac{1}{KD}(u_1(x)u_2(a) - u_2(x)u_1(a))(u_1(\xi)u_2(b) - u_2(\xi)u_1(b)) \right\} f(\xi)d\xi$$

Now define $G = G(x, \xi)$ by

$$G(x,\xi) = \frac{1}{KD} \begin{cases} (u_1(x)u_2(a) - u_2(x)u_1(a))(u_1(\xi)u_2(b) - u_2(\xi)u_1(b)) & x \leq \xi \\ (u_1(\xi)u_2(a) - u_2(\xi)u_1(a))(u_1(x)u_2(b) - u_1(b)u_2(x)) & x \geq \xi \end{cases} . \tag{4}$$

Then we can write

$$u(x) = \int_a^b G(x,\xi)f(\xi)d\xi \ ,$$

where G is called the **Green's function** for the BVP (3).

0.4 Properties of the Green's Function

The point here is that, given an equation (or $\mathcal{L}_x$) and boundary conditions, we only have to compute a Green's function once. Then we have a solution formula for $u(x)$ for any $f(x)$ we want to utilize. But we should like to not go through all the computations above to get the Green's function representation. Our strategy here is to identify properties that G satisfy, and derive a formula for it from the properties, just like we did for IVPs.

0. First, since the problem (2), and (3), was written in self-adjoint form, we have $G(x, \xi) = G(\xi, x)$; that is, G is symmetric [1].

[1] In computing the Green's function it is easy to make algebraic mistakes; so it is best to start with the equation in self-adjoint form, and checking your computed G to see if it is symmetric. If it is not, you have an incorrect form.

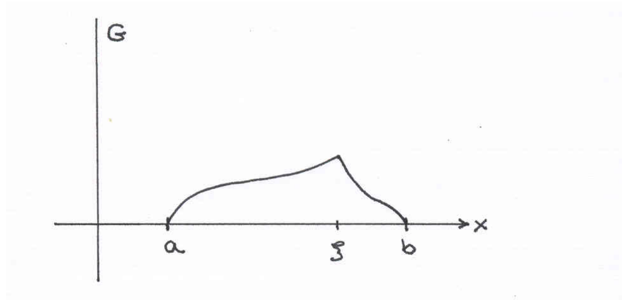

Figure 1: A way of picturing Green's function $G(x, \xi)$, $\xi \in (a, b)$ fixed.

i. Now, staring at the expression (4) above, each component is just a constant times a linear combination of the fundamental set of solutions; therefore, for fixed $\xi \in (a, b)$, G as a function of x satisfies

$$\frac{d}{dx}\left(p\frac{dG}{dx}\right) + qG = 0 \qquad \text{for } x \neq \xi \tag{5}$$

ii. Also

$$G(a, \xi) = 0 = G(b, \xi) \tag{6}$$

Remember, we are dealing with BVP (3). If we change the boundary conditions in (3), the associated G will satisfy the appropriate boundary conditions of the problem (as a function of x).

iii. G is continuous at $x = \xi$; that is,

$$G_{|x=\xi+} = G_{|x=\xi-} \tag{7}$$

iv. Also, dG/dx experience a specific jump discontinuity at $x = \xi$, namely

$$\frac{dG}{dx}_{|x=\xi+} - \frac{dG}{dx}_{|x=\xi-} = -\frac{1}{p(\xi)} \tag{8}$$

Another way of expressing these last two conditions is to write $[[G]]_{|\xi} = 0$ and $[[G_x]]_{|\xi} = -1/p(\xi)$, respectively.

Hence, instead of all the previous calculations to get expressions like (4), we use properties (i)-(iv) directly to construct $G(x, \xi)$. The idea is to write down general homogeneous solutions to (i) for $x < \xi$ and for $x > \xi$, ξ being fixed in the interval. There are four constants to determine. Apply these representations to (ii). This eliminates two constants. Applying the continuity and jump conditions (iii) and (iv) eliminates the other two constants, giving us a unique G. Figure 1 gives a way of picturing the construction of G.

Remark: One interpretation of G can be made thinking of u as 'displacement', f as a 'force per unit length'. Then $G(x, \xi)$ is the displacement at location x due to a unit force at location ξ. Property (0) above gives us a **reciprocity law**: the displacement at location ξ due to a unit force at location x is the same as the displacement at location x due to a unit force at location ξ. So $G(x, \xi)f(\xi)$ is the displacement at location x due to a force of magnitude $f(\xi)$ at location ξ. Since $f(x)$ is a 'distributed force', the total contribution to the displacement at x due to the distributed force over the whole interval requires us to sum up all the contributions. Hence,

$$u(x) = \int_a^b G(x, \xi)f(\xi)\, d\xi \quad .$$

Example:

$$\begin{cases} u'' = -f(x) & 0 < x < 1 \\ \\ u(0) = u(1) = 0 \end{cases}$$

By (i), $G(\cdot, \xi)$, for fixed $\xi \in (0, 1)$ must satisfy $G_{xx} = 0$, $x \neq \xi$, so on both parts of the interval partitioned by ξ,

$$G(x, \xi) = \begin{cases} Ax + B & x < \xi \\ \\ Cx + D & x > \xi \end{cases}$$

From (ii), $G_{|_{x=0}} = 0 = G_{|_{x=1}}$ implies $B = 0, C + D = 0$, so that

$$G(x, \xi) = \begin{cases} Ax & x < \xi \\ \\ C(x-1) & x > \xi \end{cases}$$

Now (iii) implies $C(\xi - 1) = A\xi = (A - 1)(\xi - 1) = A\xi - A - \xi + 1$ since (iv) implies $C = A - 1$. Hence, $A = 1 - \xi$ and $C = -\xi$, so

$$G(x, \xi) = \begin{cases} x(1-\xi) & x < \xi \\ \\ \xi(1-x) & x > \xi \quad . \end{cases}$$

(Note the symmetry in G.) Therefore, the solution to the problem is

$$u(x) = \int_0^1 f(\xi)G(x,\xi)\, d\xi = (1-x)\int_0^x f(\xi)\xi\, d\xi + x\int_x^1 f(\xi)(1-\xi)\, d\xi\ .$$

For example, if $f \equiv 1$, then this gives the easily checked $u(x) = \frac{x}{2}(1-x)$.

Example:

$$\begin{cases} u'' - u = -f(x) & 0 < x < 1 \\ \\ u'(0) = u'(1) = 0 \end{cases}$$

For $\xi \in (0,1)$ fixed, $G_{xx} - G = 0$, for $x \neq \xi$, has fundamental set of solutions $\sinh(x), \cosh(x)$, so we have

$$G(x,\xi) = \begin{cases} A\sinh(x) + B\cosh(x) & x < \xi \\ \\ C\sinh(x) + D\cosh(x) & x > \xi\ . \end{cases}$$

Applying the boundary conditions gives

$$G_x(0,\xi) = A = 0 \ \text{ and } \ G_x(1,\xi) = 0 \ \ \Rightarrow \ \ C = -D\tanh(1)$$

so

$$G(x,\xi) = \begin{cases} B\cosh(x) & x < \xi \\ \\ D\{\cosh(x) - \tanh(1)\sinh(x)\} & x > \xi\ . \end{cases}$$

Now $[[G]]_\xi = 0$ implies $B\cosh(\xi) = D[\cosh(\xi) - \tanh(1)\sinh(\xi)]$, and $[[G_x]]_\xi = -1$ implies $D[\sinh(\xi) - \tanh(1)\cosh(\xi)] - B\sinh(\xi) = -1$. solving these two equations in the two unknowns gives $D = \coth(1)\cosh(\xi)$ and $B = \coth(1)\cosh(\xi) - \sinh(\xi)$. Therefore

$$G(x,\xi) = \begin{cases} \cosh(x)[\coth(1)\cosh(\xi) - \sinh(\xi)] & x < \xi \\ \\ \cosh(\xi)[\coth(1)\cosh(x) - \sinh(x)] & x > \xi\ . \end{cases}$$

Example:

$$\begin{cases} xu'' + u' = -f(x) & 0 < x < 1 \\ \\ u(1) = 0,\ u(0) = \text{ finite} \end{cases}$$

For $\xi \in (0,1)$ fixed, $xG_{xx} + G_x = (xG_x)_x = 0$, $x \neq \xi$, gives

$$G(x,\xi) = \begin{cases} A\ln(x) + B & x < \xi \\ \\ C\ln(x) + D & x > \xi \end{cases} .$$

$G_{|x=1} = 0 = D$ and $G_{|x\to 0}$ remaining finite implies $A = 0$, so

$$G(x,\xi) = \begin{cases} B & x < \xi \\ \\ C\ln(x) & x > \xi \end{cases} .$$

Now $[[G]]_\xi = 0$ if $C\ln(\xi) = B$, and $[[G_x]]_\xi = -1/\xi$ implies $C/\xi - 0 = -1/\xi$, so $C = -1$ and $B = -\ln(\xi)$, so that

$$G(x,\xi) = \begin{cases} -\ln(\xi) & x < \xi \\ \\ -\ln(x) & x > \xi \end{cases} .$$

or put another way, $G(x,\xi) = -\ln[\max(x,\xi)]$. Hence

$$u(x) = -\ln(x)\int_0^x f(\xi)\,d\xi - \int_x^1 f(\xi)\ln(\xi)\,d\xi \quad .$$

Remark: Look back at the last three examples and note that the only solution to the homogeneous problem is $u \equiv 0$. *The existence of a Green's function is equivalent to the existence of a unique solution of the homogeneous BVP with its homogeneous boundary conditions.* Let us state the situation in a couple more ways: under the given homogeneous boundary conditions, either $\mathcal{L}_x u = -f$ has a uniquely determined solution $u(x)$ for every given continuous $f(x)$, or the homogeneous problem $\mathcal{L}_x u = 0$ (with its boundary conditions) has a nontrivial solution. Another way of stating this is: $\mathcal{L}_x u = 0$ plus boundary conditions has $u \equiv 0$ as its only solution, in which case there exists a Green's function for the BVP, or $\mathcal{L}_x u = 0$ plus boundary conditions has a nontrivial solution, in which case there is no Green's function.

But all is not lost in the second situation. If, for any nonzero solution, $v(x)$, to the *homogeneous* problem, $f(x)$ is such that f is orthogonal to v, that is, $\int_a^b vf\,dx = 0$, then there is a function $\mathbb{G}(x,\xi)$ called the **generalized**

Green's function because it satisfies the properties of the Green's function plus an extra property, such that we can still write

$$u(x) = \int_a^b \mathbb{G}(x,\xi) f(\xi)\, d\xi$$

for the solution of the BVP $\mathcal{L}_x u = -f$ in $a < x < b$, with homogeneous boundary conditions at $x = a, b$. This either-or statement is called the **Fredholm Alternative**, and it and its generalizations form very important results in applied mathematics. We will not delve into problems in this course where we need to build generalized Green's functions, but *you should always check to see if the homogeneous problem has nontrivial solutions* before trying to construct a Green's function.

Let us briefly make one last comment here. The homogeneous equation $\mathcal{L}_x u = (pu')' + qu = 0$ looks like a Sturm-Liouville problem with $\lambda = 0$. In fact the Fredholm Alternative is wrapped up into the case of whether $\lambda = 0$ is an eigenvalue or not.

Exercises:

1. What is the Green's function for the BVP $u'' = -f(x)$, $0 < x < 1$, $u(0) = u'(1) = 0$. Use your Green's function to find the solution $u(x)$ when $f(x) = e^{2x}$.

2. Find the Green's function for the BVP

$$\begin{cases} \frac{d^2u}{dx^2} = -f(x) & -1 < x < 1 \\ u(\pm 1) = 0 \end{cases}$$

   Again find $u(x)$ when $f(x) = e^x$.

3. Find the Green's function for the BVP

$$\begin{cases} \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) = -f(x) & 0 < x < 1 \\ u(0) = 0 = \frac{du}{dx}(1) \end{cases}$$

   where $p(x) > 0$ on $[0, 1]$ and is continuously differentiable on $(0, 1)$.

4. Consider the BVP $\frac{d^2u}{dx^2} = -f(x)$, $0 < x < a$, $u(0) = 0, u(a) = \frac{du}{dx}(a)$.

(a) If $a \neq 1$, show the homogeneous problem has only the zero solution, and construct the Green's function for the problem.

(b) If $a = 1$, show the homogeneous problem has a nonzero solution, so there is no Green's function. In this case what is the condition on f so that there is a generalized Green's function $\mathbb{G}$ so that the solution can be written in the form $u(x) = \int_0^1 \mathbb{G}(x,\xi) f(\xi) d\xi$.

Remark: Knowledge of a Green's function can be helpful for *nonlinear* 2-point boundary-value problems too, which are not solvable in closed form.

Example: Consider the nonlinear problem

$$\left\{ \begin{array}{l} \frac{d^2u}{dx^2} = -F(x, u(x)) \quad 0 < x < 1 \\ u(0) = 0 = u(1) \;\; . \end{array} \right. \tag{9}$$

One specific example of this is the equation $u'' + \mu \sin(u) = 0$, which is associated with the undamped pendulum problem and the inextensible elastica problem (see Figure 2). The Green's function associated with (9) is

$$G(x,\xi) = \left\{ \begin{array}{ll} x(1-\xi) & x < \xi \\ \xi(1-x) & x > \xi \end{array} \right.$$

so we can write the solution $u(x)$ as

$$u(x) = \int_0^1 G(x,\xi) F(\xi, u(\xi)) \, d\xi \;\; .$$

This is a nonlinear integral equation (of Fredholm type). This doesn't look any more promising an approach than attacking the nonlinear differential equation, but remember that integration is a "smoothing" operation that allows theoretical and numerical approaches. For example, if we make an initial guess of the solution, say $u^{(0)}(x)$, then for $k = 0, 1, 2, \ldots$, write

$$u^{(k+1)}(x) = \int_0^1 G(x,\xi) F(\xi, u^{(k)}(\xi)) \, d\xi = \int_0^1 G(x,\xi) f_k(\xi) \, d\xi$$

so at each iteration we know $f_k(\xi) = F(\xi, u^{(k)}(\xi))$. If $u^{(k)}(x) \to$ some function $v(x)$ as $k \to \infty$ for all $x \in [0,1]$, then $v(x)$ is a solution to the integral equation (and hence the BVP); so we need to know when such convergence

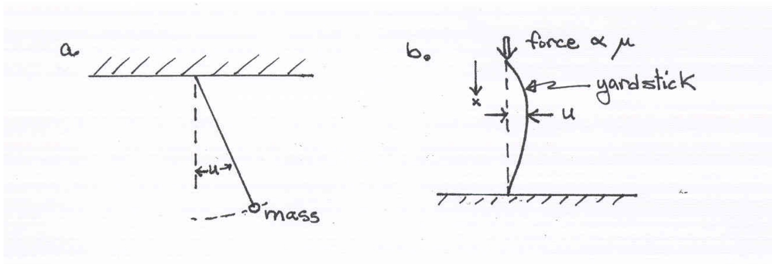

Figure 2: (a) u represents the angle a pendulum makes, where μ is proportional to gravity constant. (b) u is displacement from the vertical, with μ proportional to applied force.

takes place. (One convergence result is to consider $F(\xi, z)$ as a function of two variables defined on $[0,1] \times \mathbb{R}$, and for F to be continuous in ξ for all z, and $\partial F/\partial z$ to be continuous and bounded. A numerical scheme can be implemented based on the above iteration scheme [2].)

Exercises: Consider the problem

$$\begin{cases} u_t = u_{xx} + q(x) & 0 < x < 1\ ,\ t > 0 \\ u(0,t) = 0\ ,\ u_x(1,t) = 0 & \end{cases} \tag{10}$$

1. First consider the steady state problem for (10), namely

$$0 = U'' + q(x) \quad ,\ U(0) = 0 = U'(1)$$

   and show that $U(x) = \int_0^x \int_{x'}^1 q(s)\, ds dx'$.

2. Next show that the Green's function for the steady state problem is $G(x, \xi) = \min(x, \xi)$. Therefore, we can write the steady state solution as $U(x) = \int_0^x \xi q(\xi)\, d\xi + x \int_x^1 q(\xi)\, d\xi$.

3. How do we resolve the equivalence of these two representations? (Note the uniqueness of the steady state solution. Suppose U_1, U_2 are two steady state solutions. Then $U = U_1 - U_2$ satisfies $U'' = 0$ on $(0,1)$, $U(0) = U'(1) = 0$, which implies $U(x) \equiv 0$.)

[2] A rather old, but nice, reference to this subject is Keller's *Numerical Methods for Two-Point Boundary-Value Problems.*