Homework # 8

1. $\varphi = R(r)\,\Theta(\theta) \rightarrow r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\frac{1}{R} + \lambda r^2 = -\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \mu$

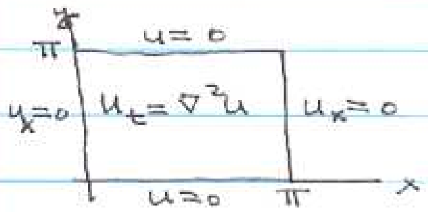
$$\rightarrow \left.\begin{array}{l} \frac{d^2\Theta}{d\theta^2} + \mu\Theta \\ \Theta(0) = \Theta(\pi/2) = 0 \end{array}\right\} \rightarrow \begin{array}{l} \mu = \mu_m = 4m^2 \qquad m = 1,2,3,\ldots \\ \Theta = \Theta_m(\theta) = \sin(2m\theta) \end{array}$$

Also $R(r)$ satisfies

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda r^2 - 4m^2)R = 0$$
$$R(1) = 0$$

$\rightarrow R(r) = J_{2m}(\sqrt{\lambda}\, r)$. Thus, from the initial condition $J_{2m}(\sqrt{\lambda}) = 0 \rightarrow \{\lambda_{2m,k}\}_{k=1,2,\ldots}$ is the family of eigenvalues. For the smallest, $k=1$, $m=1$, so compute $J_2(s_1) = 0 \rightarrow s_1 \simeq 5.1356$; $\lambda_{2,1} = s_1^2 \simeq 26.3746$

2.



$u = T(t)\varphi(x,y) \rightarrow \dot{T}/T = \nabla^2\varphi/\varphi = -\lambda$

$\rightarrow \dot{T} = -\lambda T \rightarrow T(t) = e^{-\lambda t}$

$\nabla^2\varphi + \lambda\varphi = 0$ with the given b.c.s

$\varphi = XY \rightarrow$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \lambda = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \mu \rightarrow \begin{cases} \frac{d^2Y}{dy^2} + \mu Y = 0, & 0<y<\pi \\ Y(0) = 0 = Y(\pi) \end{cases}$$

$\rightarrow \mu = \mu_m = m^2$, $Y = Y_m(y) = \sin(my)$ $m = 1,2,3,\ldots$. So,

$$\left.\begin{array}{l} \frac{d^2X}{dx^2} + (\lambda - m^2)X = 0, \quad 0<x<\pi \\ \frac{dX}{dx}(0) = 0 = \frac{dX}{dx}(\pi) \end{array}\right\} \begin{array}{l} X = A\sin(\sqrt{\lambda-m^2}\,x) + B\cos(\sqrt{\lambda-m^2}\,x) \\ \downarrow \\ X'(0) = \sqrt{\lambda-m^2}\,A = 0 \end{array}$$

case 1) $\lambda = m^2 \rightarrow X =$ constant

case 2) $\lambda - m^2 > 0$, $A = 0 \rightarrow X'(\pi) = 0 = -B\sin(\sqrt{\lambda-m^2}\,\pi) \rightarrow$

$\lambda = \lambda_{nm} = n^2 + m^2$ $\quad(\pi\sqrt{\lambda-m^2} = n\pi \rightarrow \lambda - m^2 = n^2)$ $\quad n = 1,2,\ldots$

$\rightarrow X = X_n = \cos(nx)$

so we have components $\varphi_{0m} = \sin(my)$ and

$\varphi_{nm} = \cos(nx)\sin(my)$; therefore

$$u(x,y,t) = \sum_{m=1}^{\infty} A_{0m} e^{-\lambda_{0m}t} \sin(my) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} e^{-\lambda_{nm}t} \cos(nx)\sin(my)$$

Let $t \to 0 \Rightarrow$

$$f(x,y) = \sum_{m=1}^{\infty} A_{0m} \sin(my) + \sum_{n,m=1} A_{nm} \cos(nx)\sin(my)$$

For $k \geq 1$, $l \geq 1$

$$\int_0^{\pi}\int_0^{\pi} f(x,y)\cos(kx)\sin(ly)\,dx\,dy = \sum_{m=1}^{\infty} A_{0m}\left(\int_0^{\pi}\sin(my)\sin(ly)\,dy\right)\cdot \underbrace{\cancel{\left(\int_0^{\pi}\cos(kx)\,dx\right)}}_{0} + \sum_{n,m=1}^{\infty} A_{nm}\left(\int_0^{\pi}\sin(my)\sin(ly)\,dy\right)\left(\int_0^{\pi}\cos(kx)\cos(nx)\,dx\right)$$

$$= A_{kl}\left(\frac{\pi}{2}\right)^2$$

Similarly, for $l \geq 1$,

$$\int_0^{\pi}\int_0^{\pi} f(x,y)\sin(ly)\,dy\,dx = \sum_{m=1}^{\infty} A_{0m}\left(\int_0^{\pi}\sin(my)\sin(ly)\,dy\right)\left(\int_0^{\pi}dx\right) + \sum_{n,m=1}^{\infty} A_{nm}\left(\int_0^{\pi}\sin(my)\sin(ly)\,dy\right)\underbrace{\cancel{\left(\int_0^{\pi}\cos(nx)\,dx\right)}}_{0}$$

$$= A_{0l}\,\frac{\pi^2}{2}$$

3. $$\frac{d^2h}{dt^2} fg = c^2\left\{\frac{d^2f}{dx^2} gh + \frac{d^2g}{dy^2} fh\right\} - \beta fg\frac{dh}{dt} \Rightarrow$$

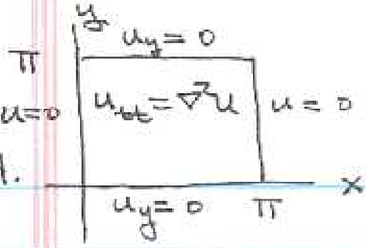
$$\frac{1}{h}\left\{\frac{d^2h}{dt^2} + \beta\frac{dh}{dt}\right\} = \frac{c^2}{f}\frac{d^2f}{dx^2} + \frac{c^2}{g}\frac{d^2g}{dy^2} = -\lambda \quad \text{for some } \lambda.$$

Hence, $\frac{d^2h}{dt^2} + \beta\frac{dh}{dt} + \lambda h = 0 \quad t > 0$. Also

$$\frac{1}{f}\frac{d^2f}{dx^2} = -\frac{1}{g}\frac{d^2g}{dy^2} - \frac{\lambda}{c^2} = -\mu \Rightarrow \frac{d^2f}{dx^2} + \mu f = 0,$$

and $$\frac{d^2g}{dy^2} + \left(\frac{\lambda}{c^2} - \mu\right)g = 0$$

4.



$u = T\varphi \to \ddot{T}/T = \nabla^2\varphi/\varphi = -\lambda \Rightarrow$

$\nabla^2\varphi + \lambda\varphi = 0$ ⊕ given b.c.s, and

$\ddot{T} + \lambda T = 0 \to T(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)$

$$\varphi = XY \to \frac{d^2X}{dx^2}\frac{1}{X} + \frac{d^2Y}{dy^2}\frac{1}{Y} + \lambda = 0 \Rightarrow$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \lambda = -\frac{1}{X}\frac{d^2X}{dx^2} = \mu \Rightarrow \frac{d^2X}{dx^2} + \mu X = 0, \quad X(0) = 0 = X(\pi)$$

$$\Rightarrow X = X_n(x) = \sin(nx), \quad \mu = \mu_n = n^2, \quad n = 1, 2, 3, \dots \Rightarrow$$

$$\frac{d^2Y}{dy^2} + (\lambda - \mu_n)Y = 0, \quad \frac{dY}{dy}(0) = 0 = \frac{dY}{dy}(\pi)$$

$Y(y) = \cos(\sqrt{\lambda - \mu_n}\, y)$ solves the equation and first b.c.

$$\frac{dY}{dy}(\pi) = 0 = -\sqrt{\lambda-\mu_n}\sin(\sqrt{\lambda-\mu_n}\, y)\Big|_{y=\pi}$$

case 1) $\lambda = \mu_n = n^2 \to Y = Ay + B \to \frac{dY}{dy} = A = 0 \Rightarrow Y = \text{const.}$

case 2) $\lambda - \mu_n > 0 \to \sqrt{\lambda - \mu_n}\,\pi = m\pi \to \lambda = \lambda_{nm} = n^2 + m^2$

$Y = Y_m(y) = \cos(my) \qquad n = 1, 2, \dots, \quad m = 0, 1, \dots$

Thus,

$$u(x,y,t) = \sum_{n=1}^{\infty}\left\{A_{n0}\cos(\sqrt{\lambda_{n0}}\,t) + B_{n0}\sin(\sqrt{\lambda_{n0}}\,t)\right\}\sin(nx)$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{A_{nm}\cos(\sqrt{\lambda_{nm}}\,t) + B_{nm}\sin(\sqrt{\lambda_{nm}}\,t)\right\}\sin(nx)\cos(my)$$

at $t=0$

$$u(x,y,0) = 0 = \sum_{n=1} A_{n0}\sin(nx) + \sum_{n,m=1} A_{nm}\sin(nx)\cos(my)$$

$$\Rightarrow A_{nm} = 0 \qquad n \geq 1, \ m \geq 0$$

$$u_t(x,y,0) = g(x,y) = \sum_{n=1}^{\infty}\sqrt{\lambda_{n0}}\,B_{n0}\sin(nx) + \sum_{n,m=1}^{\infty}\sqrt{\lambda_{nm}}\,B_{nm}\sin(nx)\cos(my)$$

As in problem 2, multiply by $\sin(kx)\cos(ly)$, $k, l > 0$, and integrate to obtain

$$B_{kl} = \frac{4}{\pi^2\sqrt{\lambda_{kl}}}\int_0^{\pi}\int_0^{\pi} g(x,y)\sin(kx)\cos(ly)\,dx\,dy$$

Similarly,

$$B_{k0} = \frac{2}{\pi^2\sqrt{\lambda_{k0}}}\int_0^{\pi}\int_0^{\pi} g(x,y)\sin(kx)\,dx\,dy$$