D: Dirac Delta Distributions

You have probably been introduced to the Dirac delta distribution (commonly mis-named the Dirac delta function) in your ODE course while discussing the Laplace transform technique, but probably not in the way we discuss it below.

From the forced string problem

$$\rho u_{tt} = T u_{xx} + f_0(x) \qquad (\rho = \text{density/length}, \ T = \text{tension})$$

in the equilibrium case we write

$$\frac{d^2 u}{dx^2} = -f(x) \qquad (\text{with } f := f_0(x)/T)$$

and consider f a distributed force per unit length. We can imagine a point force too (at $x = a$; see figure 1):

$$\frac{d^2 u}{dx^2} = 0 \quad 0 < x < l \ , \ x \neq a \in (0, l) \ .$$

Can we represent such a point force within the differential equation?

Approximate the point force by a sequence of functions $f_\epsilon(x)$ that have an ever shrinking support [1]. One example would be

$$f_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon} & -\epsilon + a < x < a + \epsilon \\ \\ 0 & \text{otherwise} \end{cases}$$

We see that $\int f_\epsilon(x) \ dx = 1$ for any $\epsilon > 0$, over any interval containing support $(a - \epsilon, a + \epsilon)$, and for fixed $x \neq a$, $f_\epsilon(x) \to 0$ as $\epsilon \to 0$. Here

$$\int_{-\infty}^{x} f_\epsilon(x') \ dx' \to \begin{cases} 1 & x > a \\ 0 & x < a \end{cases} \quad \text{as } \epsilon \to 0 \ .$$

Also, for any $\phi(x)$ that is continuous in an interval about a, $\int f_\epsilon(x)\phi(x) \ dx \to \phi(a)$ as $\epsilon \to 0$; that is, all the mass of f_ϵ, hence $f_\epsilon(x)\phi(x)$ becomes concentrated at $x = a$.

If in an informal sense we write "$lim_{\epsilon \to 0} f_\epsilon(x)$" $= \delta(x - a)$, then we expect this limit to satisfy

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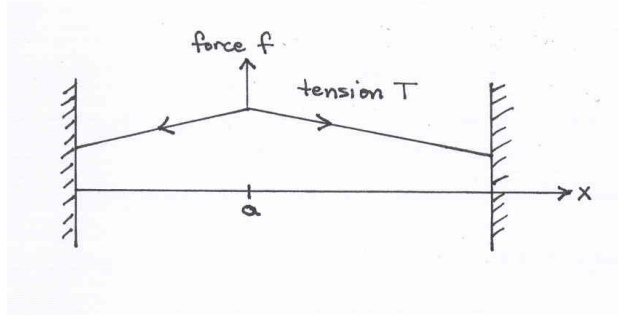[1] *Support* of a function is the set of values x where the function is nonzero.

Figure 1: Steady point force applied to an inextensible string.

i. $\int_I \delta(x)\, dx = 1$, I is any interval about $x = a$

ii. $\delta(x - a) = 0$ for $x \neq a$

iii. $\int_{-\infty}^{x} \delta(x' - a)\, dx' = H(x - a)$, $\quad H(\cdot)$ being the Heaviside function

iv. for smooth ϕ in interval I, $\int_I \delta(x - a)\phi(x)\, dx = \phi(a)$.

So δ is the **Dirac delta distribution** if it is defined on $\mathbb{R}$ and satisfies (i)-(iv).

For the 2D version of the Dirac delta distribution (readily generalizble to higher dimensions), we write $\delta(\mathbf{x} - \mathbf{x'})$, $\mathbf{x} = (x, y)$, $\mathbf{x'} = (\xi, \eta)$, so we can also write $\delta(x - \xi, y - \eta)$, which we can interpret as a product of 1D delta distributions: $\delta(x - \xi, y - \eta) = \delta(x - \xi)\delta(y - \eta)$.

Of course, to make analytic sense of all these claims in this appendix would require us to go into distribution theory, which is beyond the level of this course. That said, the formal notion of the Dirac delta distribution is a very handy concept in an applied sense.

Remark: We can expect that the fundamental solution to the heat equation, $S(x, \epsilon) = e^{-x^2/4\epsilon}/\sqrt{4\pi\epsilon}$ would act like the f_ϵ above; that is, formally $S(x, \epsilon) \to \delta(x)$ as $\epsilon \to 0$. So the family of Gaussian ("bell" curves) converge to a unit point force as the time ϵ shrinks to zero. Then, again informally,

$$\int_{-\infty}^{\infty} S(x - y, \epsilon) f(y)\, dy \quad \to \quad \int_{-\infty}^{\infty} \delta(x - y) f(y)\, dy = f(x)$$

as $\epsilon \to 0$.