Homework # 6

1. $E(t) = \frac{1}{2}\int_0^1 \left\{\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right\} dx \rightarrow \frac{dE}{dt}(t) = \int_0^1 \left\{\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right\} dx$

Substituting for $\partial^2 u/\partial t^2$ and integrating-by parts gives

$$\frac{dE}{dt} = \int_0^1 \left\{\frac{\partial u}{\partial t}\left[\frac{\partial^2 u}{\partial x^2} - r\frac{\partial u}{\partial t}\right]\right\} dx + \cancel{\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^1}^{\,0} - \int_0^1 \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t}\, dx = -\int_0^1 \left(\frac{\partial u}{\partial t}\right)^2 dx \leq 0$$

If $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$, not both 0, then $E(0) = \frac{1}{2}\int_0^1 \{g^2 + (f')^2\} dx > 0$ and $dE/dt < 0$.

2. $u(x,t) = T(t)\varphi(x) \rightarrow \frac{d^2T}{dt^2} = -\lambda T$, $\begin{cases}(1+x)^2\varphi'' + \lambda\varphi = 0 & 0 < x < l \\ \varphi(0) = 0 = \varphi(l)\end{cases}$

This is an Euler equation, so $\varphi(x) = (x+1)^r \rightarrow (x+1)^r\{r(r-1)+\lambda\} = 0$. Thus $r^2 - r + \lambda = 0 \rightarrow r = \frac{1}{2}\{1 \pm \sqrt{1-4\lambda}\}$. As in class, assume $\lambda > 1/4$ and write $\omega \stackrel{\circ}{=} \sqrt{\lambda - 1/4}$ then $r = \frac{1}{2} \pm i\omega$. Hence, a fundamental set of solutions is $\{(x+1)^{\frac{1}{2}\pm i\omega} = (x+1)^{1/2}(x+1)^{\pm i\omega}\}$. But since $(x+1)^{\pm i\omega}$ can be written in terms of $\cos[\omega \ln(1+x)]$, $\sin[\omega\ln(1+x)]$, we prefer the fundamental set $\{(x+1)^{1/2}\cos(\omega\ln(1+x)),\ (x+1)^{1/2}\sin(\omega\ln(1+x))\}$. Thus, $\varphi(x) = \sqrt{x+1}\,\{A\cos(\omega\ln(x+1)) + B\sin(\omega\ln(x+1))\}$.

$\varphi(0) = A = 0$, $0 = \varphi(l) = \sqrt{l+1}\, B\sin[\omega\ln(l+1)] \longrightarrow$

$\omega\ln(l+1) = n\pi \rightarrow \sqrt{\lambda - 1/4} = \frac{n\pi}{\ln(l+1)} \Rightarrow \lambda = \lambda_n = \frac{1}{4} + \left(\frac{n\pi}{\ln(l+1)}\right)^2$

$n = 1, 2, 3, \ldots$

(note: $\lambda_n > 1/4$ for all $n \geq 1$.) Also, $\varphi(x) = \varphi_n(x) = \sqrt{1+x}\,\sin\left[\frac{n\pi\ln(1+x)}{\ln(1+l)}\right]$

Now

$T_n(t) = a_n\cos(\sqrt{\lambda_n}\,t) + b_n\sin(\sqrt{\lambda_n}\,t)$. Therefore,

$$u(x,t) = \sqrt{1+x}\sum_{n=1}^{\infty}\{a_n\cos(\sqrt{\lambda_n}\,t) + b_n\sin(\sqrt{\lambda_n}\,t)\}\sin\left[\frac{n\pi\ln(1+x)}{\ln(1+l)}\right].$$

3. (a) $\varphi'' + \lambda x^{-2}\varphi = 0 \longrightarrow p \equiv 1, q \equiv 0, \sigma = x^{-2}$

(b) $(\sin(x)\varphi')' + \lambda\sin(x)\varphi = 0 \longrightarrow p(x) = \sigma(x) = \sin(x)$, $q \equiv 0$

(c) $(x\varphi')' - x^{-2}\varphi + \lambda\varphi = 0 \longrightarrow p(x) = x$, $q(x) = x^{-2}$, $\sigma \equiv 1$

(d) multiply by p: $p\varphi'' - xp\varphi' + \lambda p\varphi = 0$ which has to be of the form $(p\varphi')' + \lambda p\varphi = 0$ so $p' = -xp \rightarrow p(x) = e^{-x^2/2}$.

Also $\sigma(x) = e^{-x^2/2}$ while $q \equiv 0$.

(e) note that $e^x\left(e^{-x}[\varphi'' - \varphi']\right) + \lambda\varphi = e^x[e^{-x}\varphi']' + \lambda\varphi = 0$

so $p(x) = \sigma(x) = e^{-x}$, $q \equiv 0$.

(f) $(x\varphi)'' + \lambda x\varphi = x\varphi'' + 2\varphi' + \lambda x\varphi = 0 \rightarrow$

$x^2\varphi'' + 2x\varphi' + \lambda x^2\varphi = (x^2\varphi')' + \lambda x^2\varphi = 0 \rightarrow p(x) = \sigma(x) = x^2$, $q \equiv 0$.

4. $\varphi(x) = v(z)$, $z = \ln(x)$ so

$$\frac{d\varphi}{dx} = \frac{1}{x}\frac{dv}{dz} \rightarrow \frac{d^2\varphi}{dx^2} = -\frac{1}{x^2}\frac{dv}{dz} + \frac{1}{x^2}\frac{d^2v}{dz^2} \rightarrow \frac{d^2v}{dz^2} + \lambda v = 0$$

$v(0) = 0 = v(\ln(2)) \rightarrow v(z) = \sin(\sqrt{\lambda}\, z)$ such that $\sin(\sqrt{\lambda}\ln 2) = 0$

$\rightarrow \lambda = \lambda_n = \left(\frac{n\pi}{\ln(2)}\right)^2$ $n = 1, 2, 3, \ldots$ and $v(z) = v_n(z) = \sin\left(\frac{n\pi z}{\ln(2)}\right)$

$\rightarrow \varphi_n(x) = \sin\left(\frac{n\pi \ln(x)}{\ln(2)}\right)$.