Homework 2

Due Monday 9/12.

1. Evaluate the following limits. Justify your answers. Some of the limits may not exist.

(a) $\lim\limits_{(x,y)\to(2,3)} \frac{x^3+1}{3-4y}$

(b) $\lim\limits_{(x,y)\to(-2,5)} \frac{x-y+7}{x+y-3}$

(c) $\lim\limits_{(x,y)\to(0,0)} y^2 \sin\left(\frac{1}{3x^2-y^3}\right)$

(d) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^2}$

2. Evaluate the limits by converting to polar coordinates.

(a) $\lim\limits_{(x,y)\to(0,0)} e^{-\frac{1}{x^2+y^2}}$

(b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$

3. Show that the limit of $f(x,y) = \frac{\sin(xy)}{x^2+y^2}$ as (x,y) approaches $(0,0)$ along a line $y = mx$ is $\frac{m}{1+m^2}$ (use L'Hôpital's rule). Conclude that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

4. Let $f(x,y) = \frac{\sin(xy)}{x+y}$.

(a) Show that along any line $y = mx$ (with $m \neq -1$) the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ is zero.

(b) Show that the limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist by looking along the path $y = -\sin x$. (*Hint:* after substituting $y = -\sin x$, use L'Hôpital twice, and you get something where the numerator approaches 2 and the denominator approaches 0).

5. Let

$$f(x,y) = \begin{cases} \frac{x|y|}{y} \text{ if } y \neq 0, \\ 0 \text{ if } y = 0. \end{cases}$$

Show that f is continuous at the origin, but is discontinuous at any other point along the x-axis.

6. Show that

$$f(x,y) = \begin{cases} \frac{\sqrt{xy}-1}{xy-1} \text{ if } xy \neq 1, \\ \frac{1}{2} \text{ if } xy = 1. \end{cases}$$

is continuous everywhere.

7. Give an example of a function $f(x,y)$ such that $f(0,y)$ and $f(x,0)$ are continuous functions of one variable, but $f(x,y)$ is not continuous at $(0,0)$. Your function should be defined everywhere (you can make a piecewise defined function to extend it to $(0,0)$ if need be). *There are a few examples already on this page.*