

# Mathematics 41C - 43C

Mathematics Department  
Phillips Exeter Academy  
Exeter, NH  
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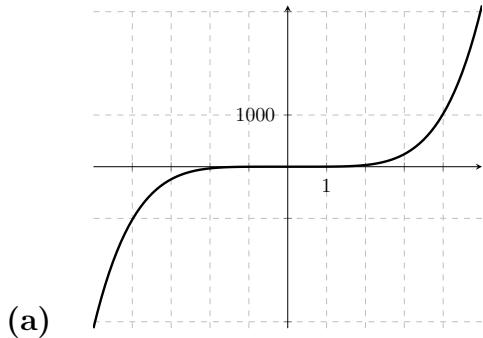
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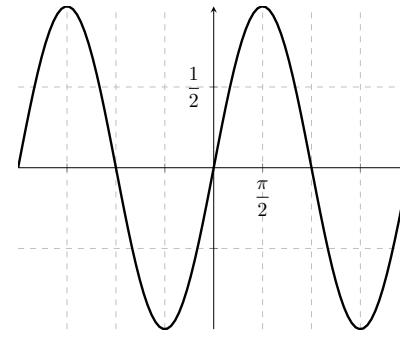




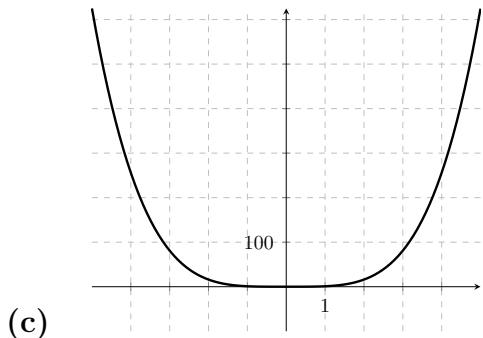
**L0.1** Without a graphing tool, choose from the following equations  $y = x^2$ ,  $y = x^3$ ,  $y = x^5$ ,  $y = x^4$ ,  $y = -x$ ,  $y = x$ ,  $y = \sqrt{x}$ ,  $y = \frac{1}{x}$ ,  $y = \sin(x)$ ,  $y = \cos(x)$ ,  $y = \tan(x)$ ,  $y = x^{1/3}$  and  $y = \log(x)$  to describe each of the graphs below:



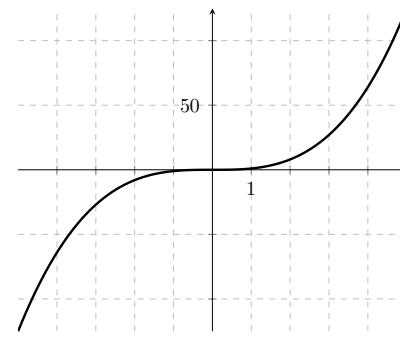
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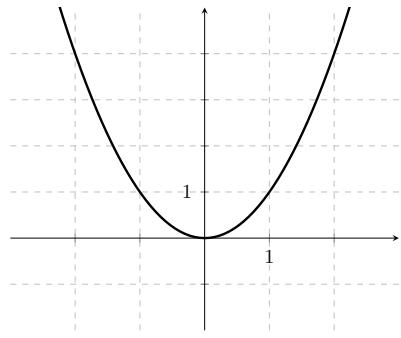
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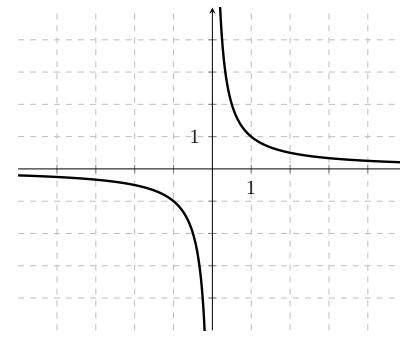
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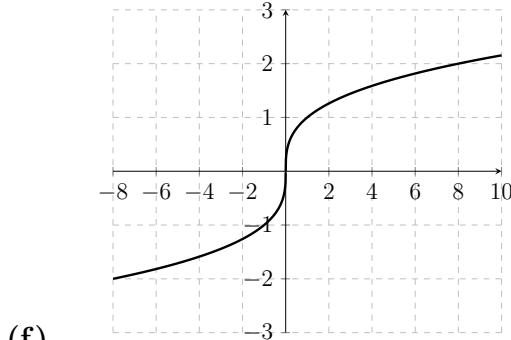
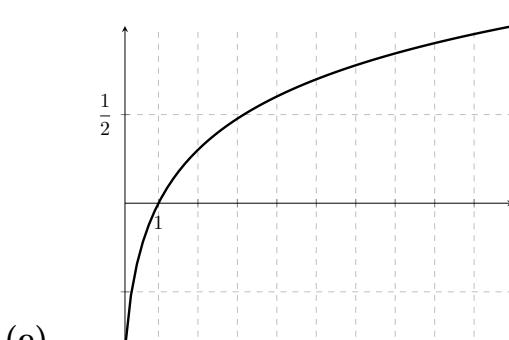
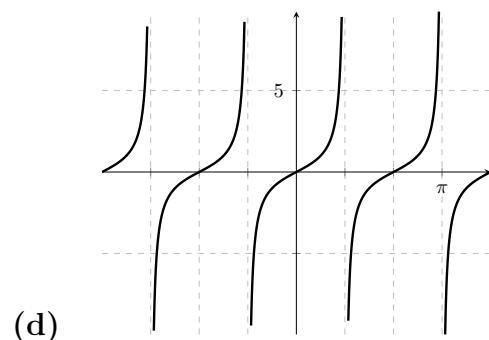
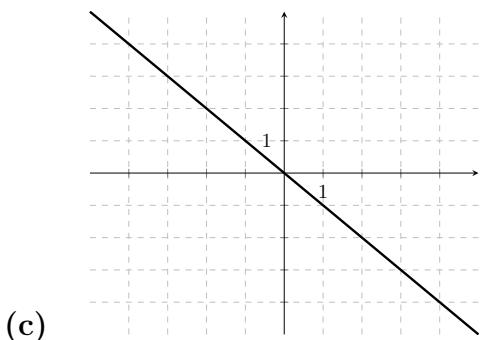
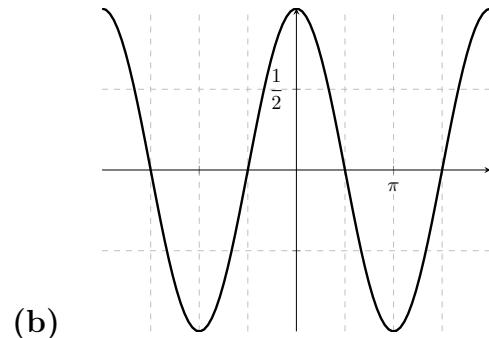
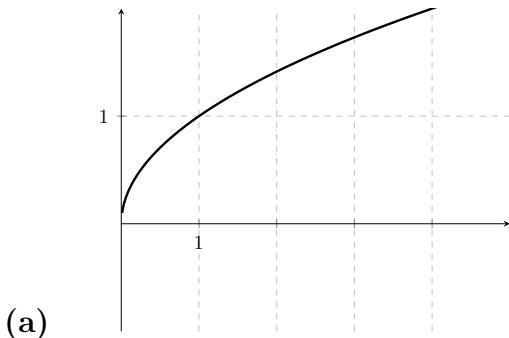


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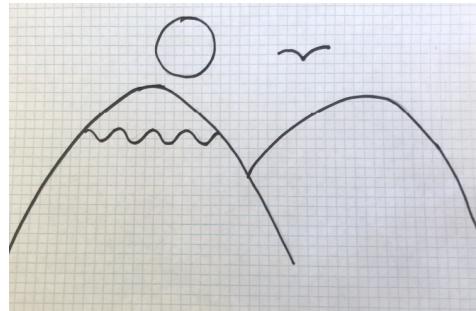


(f)

**L0.2** Choose from the following equations:  $y = x^2$ ,  $y = x^3$ ,  $y = x^5$ ,  $y = x^4$ ,  $y = -x$ ,  $y = x$ ,  $y = \sqrt{x}$ ,  $y = \frac{1}{x}$ ,  $y = \sin(x)$ ,  $y = \cos(x)$ ,  $y = \tan(x)$ ,  $y = x^{1/3}$ , and  $y = \log(x)$  to describe each of the graphs below:



**L0.3** Using Desmos, see if you can draw the picture shown:



**L0.4** Sketch a picture of a face by hand, and then see if you can draw the same picture in Desmos with functions.

**L0.5** The functions you have worked with in this lab are all important in mathematics, and you will work with them in Math 41C, 42C and 43C. Which are your favorites? Summarize how you used transformations to draw the pictures in this lab.

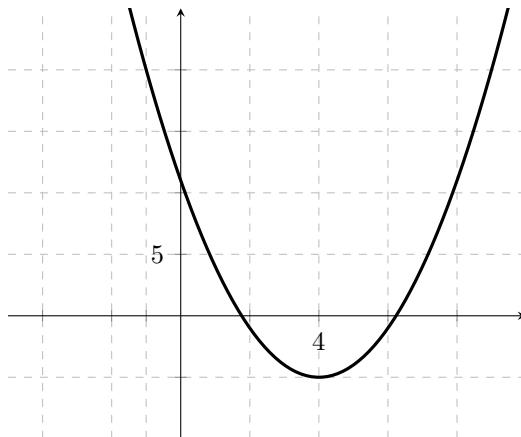
**P1.1** Big Red Taxi charges a fixed amount of \$2.40 plus \$3.10 per mile. Big Blue Taxi only charges for mileage at a rate of \$3.50 per mile. Write equations expressing the amount of the fare as a function of the distance traveled for each of the cabs, and sketch graphs of the functions on the same set of axes. What is common to these two functions? What is different? What can you say about the rate of change of each fare function?

**P1.2** A college savings account grows at an annual rate of 4.5%. Write an equation expressing the amount in the account  $t$  years after an initial deposit of \$5000.00, and obtain a graph of the function. Compare this function with the functions in #P1.1. How are they similar? How are they different? [Hint: consider what is constant in each function.]

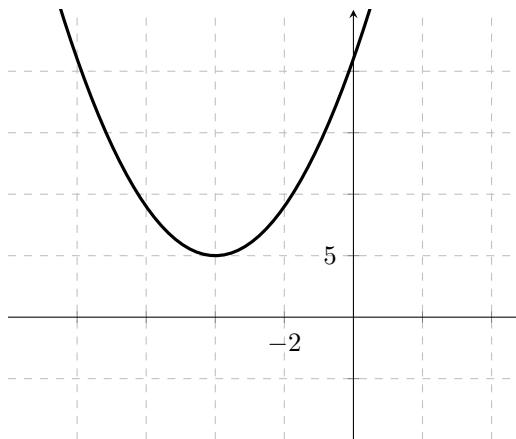
**P1.3** A function  $f$  is called *even* if it has the property that  $f(-x) = f(x)$  for all  $x$ -values in the domain. What does this property tell us about the appearance of the graph of  $y = f(x)$ ? Show that  $C(x) = \frac{2^x + 2^{-x}}{2}$  is an even function. Give other examples of even functions.

**P1.4** A function  $f$  is called *odd* if it has the property that  $f(-x) = -f(x)$  for all  $x$ -values in the domain. What does this property tell us about the appearance of the graph  $y = f(x)$ ? Show that  $C(x) = \frac{2^x - 2^{-x}}{2}$  is an odd function. Give other examples of odd functions.

**P1.5** The graph of  $y = g(x)$  is shown below. Knowing that  $g(x)$  is a translation of  $f(x) = x^2$ , write an equation for  $y = g(x)$  in terms of  $f(x)$ .



**P1.6** (Continuation) Write an equation for the graph shown below in terms of  $g(x)$ .



**P1.7** After being dropped from the top of a tall building, the height of an object is described by  $h(t) = 400 - 16t^2$ , where  $h$  is in feet and  $t$  is the time in seconds.

- Sketch a graph that shows the height on the vertical axis and time on the horizontal axis. Paying attention to the scale, label the  $h$ -intercept with an  $A$  and the  $t$ -intercept with a  $B$ .
- Draw line segment  $AB$  and find its slope. What does the slope of  $AB$  tell you about the falling object?
- Let  $C$  be the point when  $t = 2$  and  $D$  be the point when  $t = 2.1$ , and draw the line segment  $CD$ . What does the slope of  $CD$  tell you about the falling object?
- If the height of the object dropped from the tall building were given by  $H(t) = 450 - 16t^2$  instead of  $h(t) = 400 - 16t^2$  how would your answers to (b) and (c) change, if at all?

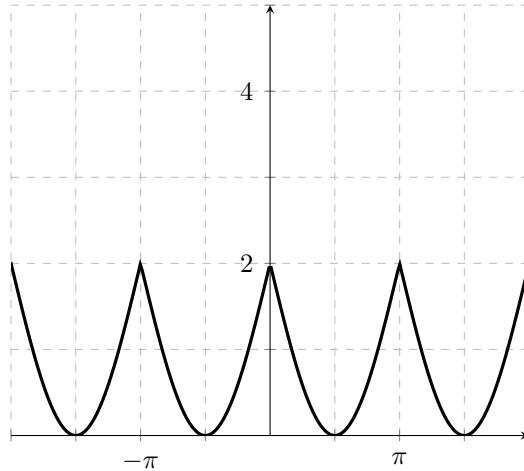
**P1.8** Given  $f(x) = 3x + 4x^2$ .

- Write expressions for  $f(x-1)$  and  $f(2x)$ . Simplify if you think it is informative, and sketch all three curves.
- Find  $x$  if  $f(x) = 7$  and compare your approach with your classmates.
- Find  $x$  if  $f(x-1) = 7$  and  $f(2x) = 7$ , and again compare approaches.

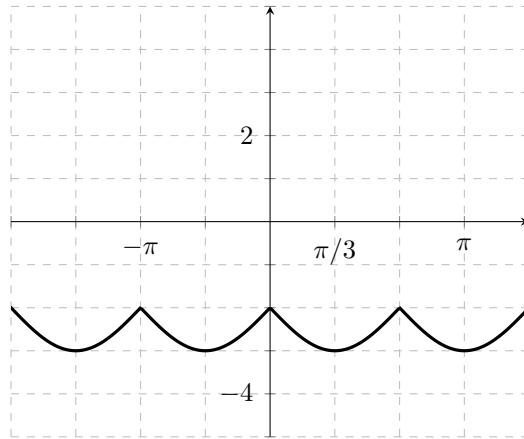
**P1.9** Given  $g(x) = x + \frac{3}{2-x}$

- Calculate  $g(x) = 0$ . Is there a way to rewrite  $g(x)$  that you find helpful?
- Use your result from  $g(x) = 0$  to solve for  $g(x+1) = 2$ .

**P1.10** The graph of  $y = g(x)$  is shown below. Write an equation for  $g(x)$  in terms of  $f(x) = -2|\sin(x)| + 4$ .



**P1.11** (Continuation) Write an equation for the graph shown below in terms of  $f(x)$



**P1.12** How many solutions are there to the equation  $x^2 = 2^x$ ? Find all of them.

**P1.13** (Continuation) From the solutions, which  $x$ -value is largest? Which is largest for  $x^2 = 1.5^x$ ? What about  $x^2 = 1.1^x$ ?

**P1.14** Suppose we compare the graphs of  $y = x^2$  and  $y = 1.01^x$ . The first is a parabola, and the second is an exponential function with a growth rate of 1%. Which function is greater for  $x = 100$ ?  $x = 1000$ ?  $x = 10,000$ ? Based on these observations, which function grows faster in the long run?

**P1.15** (Continuation) Make a conjecture about which is the faster growing function: a power function  $y = x^n$ , where  $n$  is a positive integer, or  $y = b^x$  where  $b > 1$ .

**P1.16** Which is best, to have money in a bank that pays 9 percent annual interest, one that pays 9/12 percent monthly interest, or one that pays 9/365 percent daily interest? A bank is said to *compound* its annual interest when it applies a fraction of its annual interest to a fraction of a year.

**P1.17** (Continuation) Inflation in the country of Sylvania has reached alarming levels. Many banks are paying 100 percent annual interest, some banks are paying 100/12 percent monthly interest, a few are paying 100/365 percent daily interest, and so forth. Trying to make sense of all of these promotions, Milou decides to graph the function  $E$ , given by  $E(x) = \left(1 + \frac{1}{x}\right)^x$ . What does this graph reveal about the sequence  $v_n = \left(1 + \frac{1}{n}\right)^n$ , where  $n$  is a positive integer? Calculate the specific values:  $v_1$ ,  $v_{12}$ ,  $v_{365}$  and  $v_{31536000}$ .

**P1.18** ((Continuation) Numerical and graphical evidence from #P1.17 suggests that this sequence has a limiting value. This example is so important that a special letter is reserved for the limiting value (as is done for  $\pi$ ). We define  $e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$ , which means that the sequence  $v_n = \left(1 + \frac{1}{n}\right)^n$  approaches  $e$  as  $n$  approaches infinity. This limit is an example of an *indeterminate form*. For some additional limit practice, use a calculator to evaluate  $\lim_{n \rightarrow \infty} \left(1 + \frac{0.09}{n}\right)^n$ , which is greater than 1. Make up a story to go with this question.

**P1.19** Without a calculator and on the same set of axes, graph both  $f(x) = x^3$  and  $y = f(x - 1) + 2$ .

**P1.20** Without a calculator and on the same set of axes, graph  $f(x) = x$  and  $y = 1/f(x)$ .

**L1.1** Please use a computer-based spreadsheet during this lab when doing so makes the computations less tedious.

- (a) Make a table of values where column 1 is  $x = 0, 1, 2, \dots$ , and  $y = x^2$  by filling in the second column in Table 1.

Table 1: Differences

$x$	$y = x^2$	Diff <sub>1</sub>	Diff <sub>2</sub>
0	0	1	2
1	1	3	
2	4		
3			
4			
5			
6			

- (b) Fill in the third column in the table by recording the difference between successive  $y$ -values. Is there a pattern to the column of differences? Do the values in this column fit a linear function? Explain. As a check, fill in the fourth column that shows the differences of the differences, also known as second differences. How does this column relate to the linear function you found?

**L1.2** As a side note to what you have observed, can you write simple formula for the sum of the first  $n$  odd numbers?

**L1.3** Using the triples of coefficients  $a$ ,  $b$  and  $c$  that your teacher assigns you from Table 2, carry out the same calculations as in number #L1.1, but replacing  $y = x^2$  with the quadratic equation  $y = ax^2 + bx + c$ . Compute the first and second differences using Table 3 as a model for your results. All of the class results should be entered in the Diff<sub>2</sub> column of Table 2.

Table 2: Class Data

$a$	$b$	$c$	Diff <sub>2</sub>
1	1	0	
1	1	1	
2	0	0	
2	1	0	
2	2	2	
3	0	0	
3	1	0	
3	3	2	
4	0	0	
4	1	0	
4	1	1	
4	$\pi$	$\sqrt{2}$	

Table 3: Your Data

$x$	$y = ax^2 + bx + c$	Diff <sub>1</sub>	Diff <sub>2</sub>
0			
1			
2			
3			
4			
5			
6			

**L1.4** How does the number in  $\text{Diff}_2$  relate to the coefficient of  $x^2$ ?

**L1.5** Now, try a cubic function,  $y = ax^3 + bx^2 + cx + d$ . Using the values for  $a$ ,  $b$ ,  $c$  and  $d$  that your teacher assigns you from Table 4, compute the differences of your cubic polynomial. How many difference columns are needed this time to get a constant difference? Share your results with the rest of the class in the table created by your teacher, and fill in the results below.

Table 4: Class Data

$a$	$b$	$c$	$d$	$\text{Diff}_3$
1	0	0	0	
1	1	0	0	
1	1	1	0	
1	1	1	1	
2	0	0	0	
2	1	0	0	
2	1	1	0	
2	1	1	1	
3	1	0	0	
3	1	1	0	
3	1	1	1	
$\pi$	1	1	1	

**L1.6** How does this constant relate to the coefficient of  $x^3$ ?

**L1.7** Extend the previous results by speculating on what will happen with a fourth-degree polynomial. Try one and see if you were correct.

**L1.8** Write a paragraph summarizing your observations about differences for polynomial functions. You should include a generalization about the constant differences for a polynomial relative to its degree. You should also discuss what the first differences are for any degree polynomial.

**P2.1** Using a strategy like the one you used in Lab 1, create a table of differences for the function  $y = 2^x$ . You should see that the difference of consecutive  $y$ -values never becomes 0, but does behave in a predictable way. Repeat for  $y = 3^x$  and  $y = 1.2^x$ . Describe how differences for an exponential function behave compared to differences for a polynomial function.

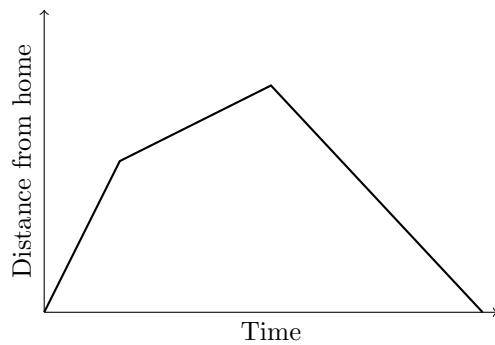
**P2.2** Given a function  $f$ , each solution of the equation  $f(x) = 0$  is called a *zero* of  $f$ . Without using a calculator, find the zeros of each function.

(a)  $s(x) = \sin(3x)$       (b)  $L(x) = \log_5(x - 3)$

**P2.3** The point  $(4, 16)$  is on the graph of  $f(x) = x^2$ . Treating  $(x, x^2)$  as an arbitrary point on the graph of  $f(x) = x^2$ , the fraction  $\frac{x^2 - 16}{x - 4}$  represents the slope between two points on the graph. Find the value of the slope when  $x$  is close to 4. How does the slope behave as  $x$  gets closer and closer to 4? Repeat for the point  $(-4, 16)$ .

**P2.4** Which description matches the graph below?

- (a) Casey walked from home to the park. Starting off slowly, Casey's pace gradually increased. At the park Casey turned around and walked slowly back home.
- (b) From home Casey took a bike ride heading east going up a steep hill. After a while the slope eased off. At the top Casey raced down the other side.
- (c) From home Casey went for a jog. At the end of the street Casey bumped into a friend and slowed down. After Casey's friend left, Casey walked quickly back home.



**P2.5** Sketch a graph of the rate-of-change function for Casey's distance from home given the graph in #P2.4.

**P2.6** The first few terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, ..., and each successive term is the sum of the two previous terms. Make a table of differences for this sequence. What patterns do you notice in the first differences? The second differences? And so on?

**P2.7** The half-life of ibuprofen in the bloodstream is about 2 hours. This means that after 2 hours of being consumed, only 100 mg of the 200 mg of ibuprofen taken will remain in the bloodstream. After another 2 hours (4 hours total) only 50 mg will remain in the bloodstream. A patient is prescribed 200 mg of ibuprofen to be taken every 4 hours. Fill in the following table, which records the amount of ibuprofen in the patient's body hours after the initial dose is consumed. What is the long-term stable concentration?

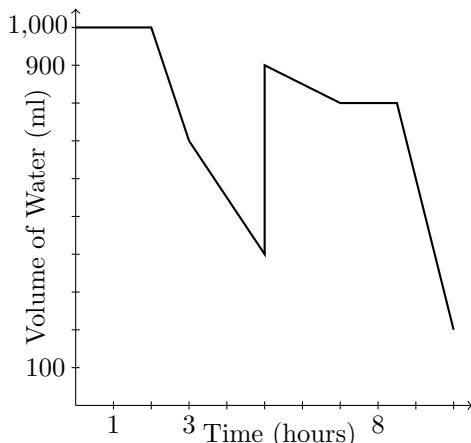
Elapsed time in hours	0	2	4	6	8	10	12	14	16
Mg of ibuprofen in the body	200	100	250						

**P2.8** The zeros of the function  $Q$  are  $-4$  and  $5$ . Find the zeros of the given functions.

- (a)  $f(x) = Q(4x)$       (b)  $t(x) = Q(x - 3)$       (c)  $j(x) = 3Q(x + 2)$

**P2.9** The point  $A(0, 1)$  is on the graph of  $f(x) = 2^x$ . If  $B(x, 2^x)$  is any other point on the graph of  $f(x) = 2^x$ , the fraction  $\frac{2^x - 1}{x - 0}$  represents the slope between the two points  $A$  and  $B$  on the graph. Find the value of the slope when  $x$  is close to 0. How does the slope behave as  $x$  gets closer and closer to 0?

**P2.10** Use the graph shown to tell the story of Jo's water consumption.



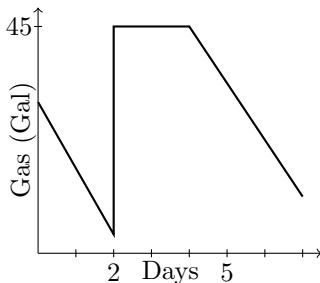
**P2.11** Suppose a herd of goats currently number 25. Consider two possible rates of growth for the herd and compare the size of the herd over time for these two growth scenarios.

- (a) An increase of 10 goats per year.  
 (b) An increase of 20 % per year.

**P2.12** Suppose we know the following about the function  $f(x)$ . The domain is  $-10 \leq x \leq 15$ , the range is  $-20 \leq y \leq 5$ , the  $x$ -intercepts are  $(-1, 0)$  and  $(7, 0)$ , the  $y$ -intercept is at  $y = -10$ . For each function, identify the domain, the range, and  $x$ -intercepts and the  $y$ -intercept, if possible. If there is not enough information to identify any feature, explain why this is the case.

(a)  $g(x) = -2f(2x)$       (b)  $m(x) = f(x + 5) - 1$

**P2.13** Use the graph to tell the story of Cam's gas use over the week.



**P2.14** We have now discussed problems with differences and slopes for various functions. These problems are actually about rates of change, specifically how a function changes with respect to its independent variable. The change of a function can be thought of as another function, which for now, we will call the *rate-of-change function*. Use your experience and intuition to respond to the following:

- (a) Describe the rate of change for  $y = 3x+1$ . Describe the rate-of-change function for any linear function.
- (b) Describe the rate-of-change function for a parabola.
- (c) Describe the rate-of-change function for an exponential function.

**P2.15** Sketch a graph of distance (in miles) from Exeter versus time (in hours) for a car trip you take on a Sunday afternoon from PEA to Deerfield Academy, which is about 120 miles away. Some of the time you will be driving in towns, most of the time you will be on highways, and you will want to make a stop to eat some lunch about midway through your trip. Next, sketch a graph of the rate-of-change function that corresponds to your distance vs. time graph.

**P2.16** Sketch the graph of the function that has a positive and increasing rate of change.

**P2.17** The graph of  $y = f(2x - 5)$  is obtained by applying first a horizontal translation and then a horizontal compression to the graph of  $y = f(x)$ . Explain. Is it possible to achieve the same result by applying first a horizontal compression and then a horizontal translation to the graph of  $y = f(x)$ ?

**P2.18** A driver was overheard saying, "My trip to New York City was made at 80 kilometers per hour." Do you think the driver was referring to an instantaneous speed or an average speed? What is the difference between these two concepts?

**P2.19** (Continuation) Let  $R(t)$  denote the speed of the car after  $t$  hours of driving. Assuming that the trip to New York City took exactly five hours, draw a careful graph of a plausible speed function  $R$ . It is customary to use the horizontal axis for  $t$  and the vertical axis for  $R$ . Each point on your graph represents information about the trip; be ready to explain the story behind your graph. In particular, the graph should display reasonable maximum and minimum speeds.

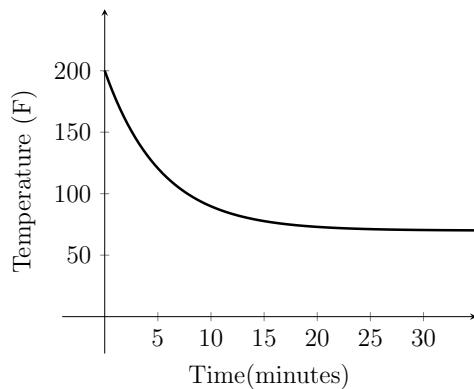
**L2.1** Complete as many of the 3 activities as possible during the lab period:

- (a) A ball is tossed softly from one person to another, tracing out a curved path that rises to a maximum height in the middle of the path. There are two components to the path –one is the height of the ball above the ground, and the other is the horizontal distance from the person tossing the ball.
- (b) A pendulum starts at a angle of about 60 deg from vertical. There are two components to the path of the pendulum – one is the height of the pendulum bob above the horizontal line through the equilibrium position, and the other is the horizontal displacement from equilibrium. (The equilibrium position is the position of the pendulum bob at rest.)
- (c) A weight is attached to the end of a spring that is suspended from a hook. When the weight is pulled down from its equilibrium position and then released, it oscillates up and down. There is only one component to this path.

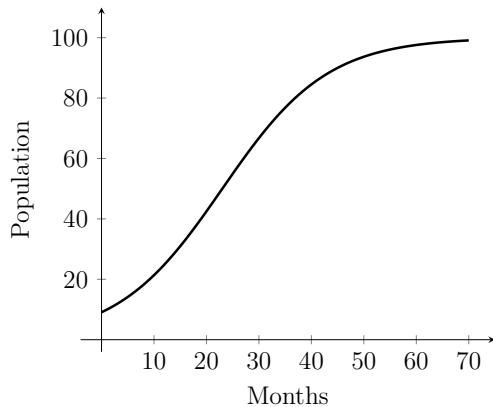
Choose one of scenarios, and think about the shape that a distance versus time graph should have. If you chose a scenario that involves two motion components, consider the shape of each. Then use the Video Physics app to obtain a graph of the phenomenon described. If you chose a scenario that involves two motion components, graph both. In addition, obtain a picture of the rate-of-change graph for each motion.

**L2.2** (Continued) Examine the graphs you obtained above and explain the shapes of both the motion graphs and the associated rate-of-change graphs.

**L2.3** The following graph illustrates the temperature of a fresh cup of coffee as it cools. Sketch a graph of the rate-of-change curve of this *cooling curve*.



**L2.4** The population of chipmunks living around PEA has been growing in recent years towards what is considered by experts to be the maximum possible PEA chipmunk population of 100, as shown in the graph below. Sketch a graph of the rate-of-change function for this *constrained* population.



**L2.5** For each of the following functions, sketch the graph of the function and use this graph to sketch a graph of its rate-of-change. Be sure to label and show an appropriate scale.

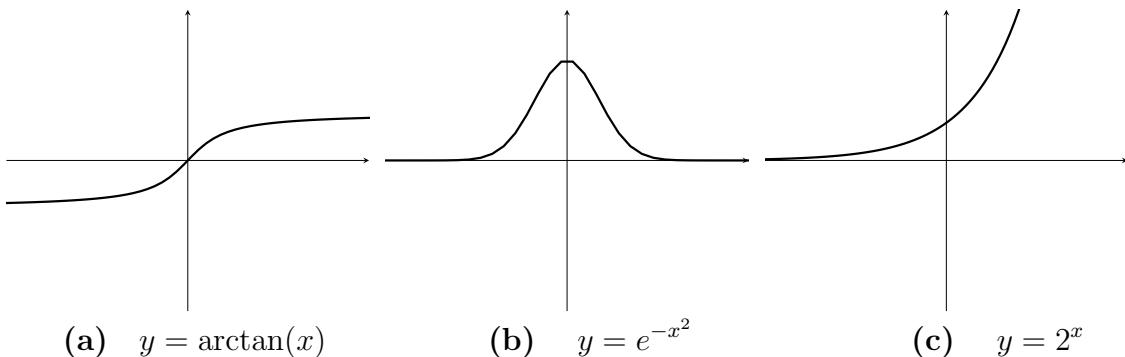
(a)  $y = x^2$       (b)  $y = \sin(x)$       (c)  $y = e^x$       (d)  $y = \frac{1}{x}$       (e)  $y = \sqrt{x}$

**L2.6** Now try the same for this next set of functions:

(a)  $y = x^3$       (b)  $y = \cos(x)$       (c)  $y = \frac{1}{x^2}$       (d)  $y = \log(x)$

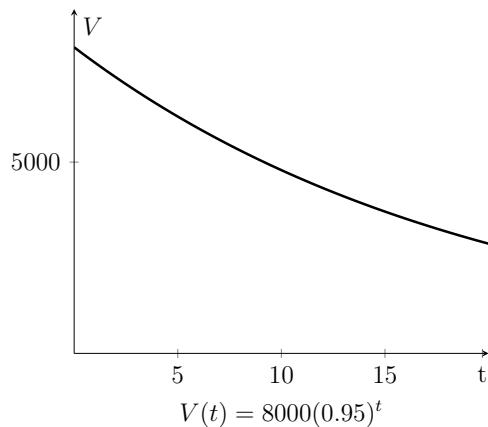
**L2.7** Compile your results into a report. In addition to your graphs, your report should contain a paragraph describing the reasoning you used to draw the rate-of-change graphs. Be sure to address how this lab relates to Laboratory 1.

**P3.1** Using a technique similar to what you used in Lab 2, sketch the rate of change function for each of the following.

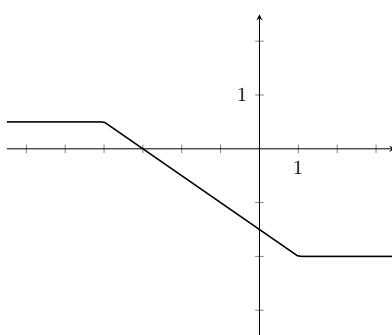


**P3.2** The equation  $V(t) = 8000(0.95)^t$  models the volume in cubic centimeters of a shrinking balloon that is losing 5 percent of its helium each day.

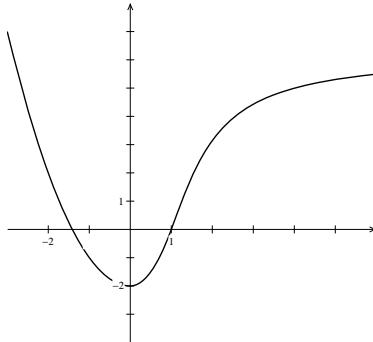
- Calculate  $V(0)$ . What does the value tell you about the graph? What does the value tell you about the balloon?
- Calculate  $V(10)$ . What does the value tell you about the graph? What does the value tell you about the balloon?
- Find  $t$  so that  $V(t) = 5000$ . Describe the balloon at this moment.
- Calculate  $\frac{V(14) - V(12)}{14 - 12}$ . What does the value of the fraction tell you about the graph? What does this value tell you about the balloon?
- Calculate  $\frac{V(14) - V(13)}{14 - 13}$ . What does this value tell you about the balloon?



- P3.3** Two different graphs for  $y = f(x)$  are shown. For each, sketch a graph of the reciprocal function  $y = \frac{1}{f(x)}$ .



Function A

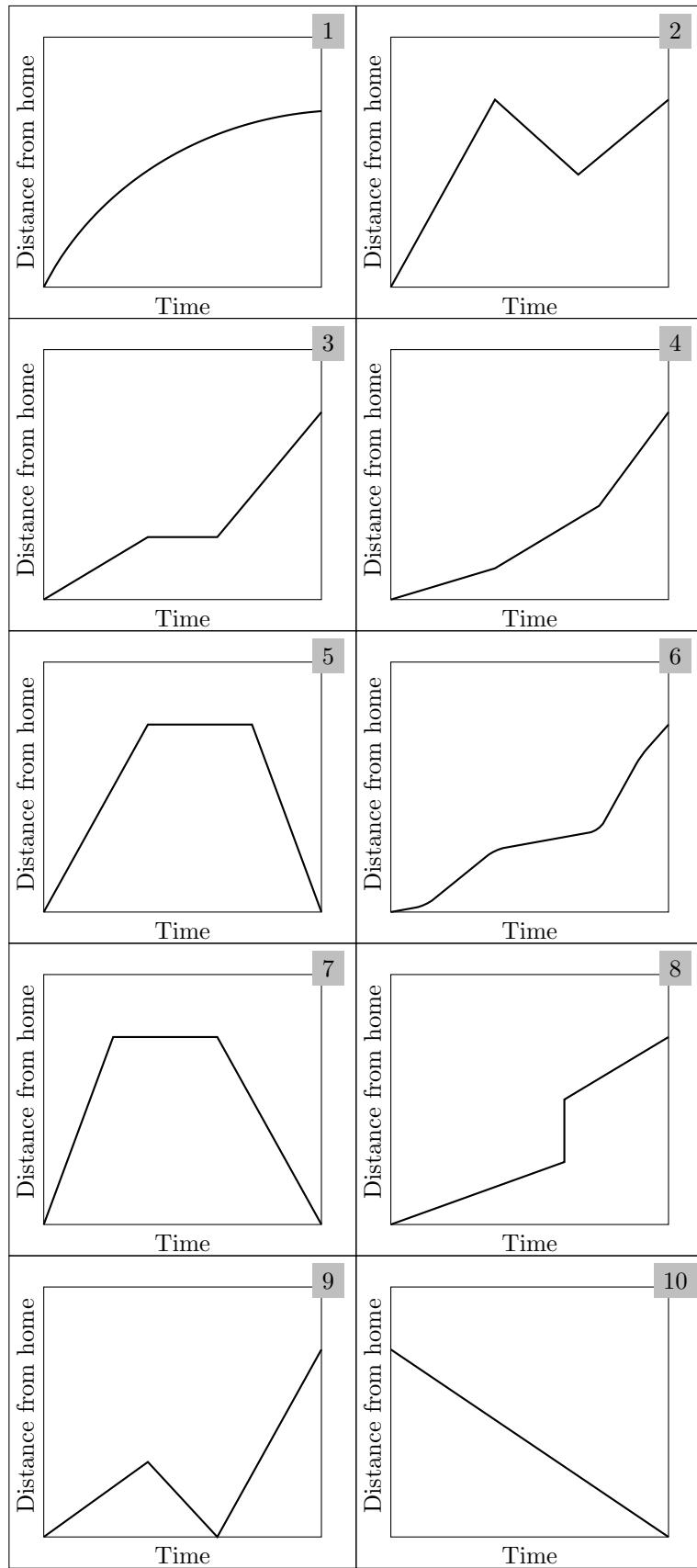


Function B

- P3.4** The function  $g(x) = \sec(x)$  is the secant function and is defined as the reciprocal of the cosine function. Similarly,  $h(x) = \csc(x)$  is the cosecant function and is defined as the reciprocal of the sine function. Without using technology and working in radians, draw accurate graphs of these two reciprocal functions on the interval  $-2\pi \leq x \leq 2\pi$ . It might help to sketch  $y = \sin(x)$  and  $y = \cos(x)$  first.

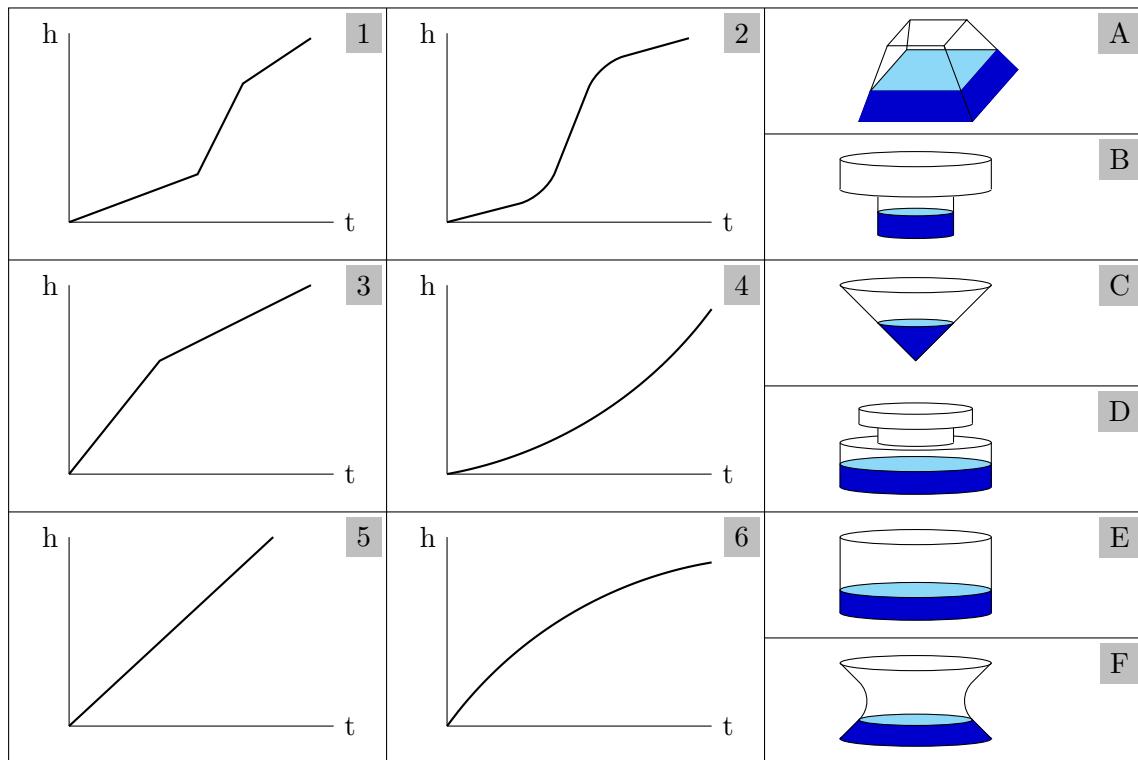
- P3.5** Match the following descriptions with the 10 distance vs. time graphs on the following page.

- (a) Rory ran from home to the bus stop and waited. Rory realized the bus had already left so Rory walked home.
- (b) Opposite Rory's home is a hill. Rory climbed slowly up the hill, walked across the top, and then ran quickly down the other side.
- (c) Rory skateboarded from home, gradually building up speed. Rory slowed down to avoid some rough ground, but then speeded up again.
- (d) Rory walked slowly along the road and stopped to check the time. Seeing how late it was Rory then started running.
- (e) Rory left home for a run. Rory soon got tired and gradually came to a stop!
- (f) Rory walked to the store at the end of the street, bought a newspaper and then ran all the way back.
- (g) Rory went out for a walk with some friends. After noticing Rory's wallet was at home, Rory ran home to get it and then had to run to catch up with the others.
- (h) This graph is just plain wrong. How can Rory be in two places at once?
- (i) After the party, Rory walked slowly all the way home.
- (j) Make up your own story!



**P3.6** What single word describes a function  $f$  that has the property  $f(x) = f(x + 60)$  for all values of  $x$ ?

**P3.7** Water is flowing into each container at a constant rate. As the volume increases the height of the water also increases. Match each graph of height versus time with its container.



**P3.8** When logarithms are calculated using  $e$  as the base, they are called *natural*, and written  $y = \ln(x)$  instead of  $y = \log_e(x)$ . Graph  $y = e^x$  and  $y = \ln(x)$  on the same set of axes. Recall that exponential functions and logarithmic functions with the same base are inverse functions.

- (a) How are the graphs of these two functions related? Compare the domains and ranges of the functions.
- (b) What are the axis intercepts for the two curves? Estimate the slope of the tangent lines to the curves at the intercepts. The slope of the tangent line to a curve at a point is the *slope of the curve at that point*.
- (c) Can you find a relationship between the slopes you found in the previous part?

**P3.9** Sketch the graph of a function that has the property that its rate-of-change is linear. How does your graph compare with the graphs of your classmates?

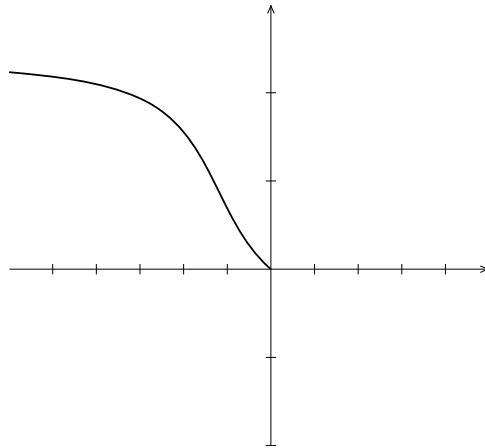
**P3.10** Sketch the graph of a function whose rate of change is:



**P3.11** For which of the following functions can it be said  $f(a + b) = f(a) + f(b)$  for all  $a$  and  $b$  in the domain of the given function?

- (a)  $f(x) = 3x$ , (b)  $f(x) = -x + 5$ , (c)  $f(x) = \sqrt{x}$ , (d)  $f(x) = 2^x$ .

**P3.12** Part of the graph of  $y = f(x)$  is shown below. Draw the rest of the graph given



**P3.13** Find a function for which  $f(x + a) = f(x)f(a)$  for all numbers  $x$  and  $a$ .

**P3.14** On the graph of  $y = f(x)$ , it is given that  $(-2, 5)$  is the highest point,  $(2, -7)$  is the lowest point, and  $x = -4$ ,  $x = 1$  and  $x = 3$  are the  $x$ -intercepts. Find the highest and lowest points on the graph as well as the  $x$ -intercepts of the curves.



**P3.15** The point  $(0, 1)$  is the  $y$ -intercept on the graph of  $y = b^x$  for  $b > 0$ . The expression  $\frac{b^x - 1}{x - 0}$  approximates the slope of  $y = b^x$  at its  $y$ -intercept, when  $x$  is close to 0. Fill in the missing entries in the following table.

b	Slope of $y = b^x$ at its $y$ -intercept
0.075	
0.5	
1.5	
2	
3	
5	
8	
15	

**P3.16** (Continuation) Make a scatterplot with values of  $b$  on the horizontal axis and the value of the slope of  $y = b^x$  on the vertical axis. The shape of the scatterplot suggests a simple relationship between the slope and the value of  $b$ . What familiar function describes this relationship?

**P3.17** (Continuation) What is the slope of  $y = e^x$  at  $(0, 1)$ ?

**P3.18** Is the graph of  $k(x) = x - x^2$  an even function, and odd function, or neither? How do you know?

**P3.19** Is the graph of  $t(x) = x^3 + x$  an even function, an odd function, or neither? How do you know?

**P3.20** For which of the following functions can it be said  $f(a + b) = f(a) + f(b)$  for all  $a$  and  $b$  in the domain of the given function?

- (a)  $f(x) = -\frac{x}{2}$ ,      (b)  $f(x) = x^2$ ,      (c)  $f(x) = \frac{1}{x}$ .

**P3.21** After being dropped from the top of a tall building, the height of an object is described by  $h(t) = 400 - 16t^2$ , where  $h(t)$  is measured in feet and  $t$  is measured in seconds.

- (a) How many seconds did it take for the object to reach the ground? Use a graphing app to obtain a graph of height vs. time in a suitable window.
- (b) What is the height of the object when  $t = 2$ ? What is the height of the object when  $t = 4$ ? Use algebra to obtain an equation for the line that goes through the two points you just calculated. Add your line to the graph of height vs. time. A line that is determined by two points on a curve is known as a *secant line*.
- (c) What is the average rate at which the height of the object is changing between 2 and 4 seconds after it is dropped? How does this rate relate to the slope of the secant line?

**P3.22** (Continuation)

- (a) How fast is the height of the object changing at the instant 2 seconds after being dropped?
- (b) Explain why this rate of change is given by the *difference quotient*  $\frac{h(t) - h(2)}{t - 2}$ , where  $t$  is a time very close to 2 seconds.
- (c) Verify that the closer  $t$  is to 2, the closer the rate of change is to  $-64$ .

We can summarize this situation by writing  $\lim_{t \rightarrow 2} \frac{h(t) - h(2)}{t - 2} = -64$ . The notation  $\lim_{t \rightarrow 2}$  is read as “the limit as  $t$  approaches 2”. This limit is known as the instantaneous rate of change when  $t = 2$ .

**P3.23** (Continuation)

- (a) What is an equation for the tangent line to the curve at the point with  $t = 2$ ?
- (b) How is the slope of the tangent related to the instantaneous rate of change at that point?
- (c) Use a graphing app to graph the curve  $h(t) = 400 - 16t^2$  together with the tangent line at  $t = 2$ .
- (d) Gradually zoom in on the point with  $t = 2$ . What do you notice about the curve and the tangent line as you zoom in? Because the curve and the tangent line are indistinguishable on a small interval containing  $t = 2$ , this curve is *locally linear* at  $t = 2$ .

**P3.24** (Continuation) Suppose we now drop the object from a taller building, so that the initial height is 600 ft, thus  $h(t) = 600 - 16t^2$ .

- (a) What is the height of the object at time  $t = 2$ ?
- (b) Find the instantaneous rate of change of the height at  $t = 2$ .
- (c) Find the equation of the tangent line to the curve at the point where  $t = 2$ .
- (d) How do your answers compare with your results for the problem with initial height 400 ft?

**L3.1** Use a graphing app to view the function  $f(x) = \sin(x)$  over an appropriate domain, where  $x$  is in radians. (You many change your mind about what is *appropriate* as you work through this lab.)

**L3.2** Create a new function  $g(x)$  where

$$g(x) = \frac{f(x + 0.001) - f(x)}{0.001}.$$

Notice that for a given  $x$ -value,  $g(x)$  describes the average rate-of-change of  $f(x)$  on the interval  $[x, x + 0.001]$ . Obtain a graph of  $g(x)$ . This average rate-of-change function is an approximation for the instantaneous rate-of-change function, which we call the *derivative*. This average rate-of-change function can help us both understand and find an equation for the derivative.

**L3.3** (Continuation) Have you seen a function that looks like  $g(x)$  before? Make a guess as to which well-known function is the derivative of  $\sin(x)$ . Verify or debunk your guess by graphing it on the same axes as the approximation function  $g(x)$ .

**L3.4** Repeat steps 1 through 3 above for each function below:

- (a)  $y = x^2$     (b)  $y = x^3$     (c)  $y = \cos(x)$     (d)  $y = e^x$   
(e)  $y = \frac{1}{x}$     (f)  $y = \frac{1}{x^2}$     (g)  $y = \ln(x)$     (h)  $y = \sqrt{x}$

Make sure to keep a record of your results including a sketch of each function and its derivative along with their equations.

**L3.5** Write a brief report summarizing what you have learned in this lab. Include a table listing all of the function equations and your guess for each derivative. Note any patterns you see, generalizations you might be willing to make and any result that surprised you.

**P4.1** Use the results of Lab 3 to find a point on the graph of  $y = \ln(x)$  where the slope is equal to (a) 1, (b) 2, (c)  $\frac{1}{3}$ .

**P4.2** Choose a value of  $b$ ,  $b \neq 0$ , and graph the function  $y = \frac{b^x - 5}{b^x + 3}$ .

- (a) What happens to the  $y$ -values as  $x$  increases without bound,  $x \rightarrow \infty$ ?
- (b) What happens to the  $y$ -values as  $x$  decreases without bound,  $x \rightarrow -\infty$ ?

**P4.3** Use the results of Lab 3 to find three points on the graph of  $y = \sin(x)$  where the slope is equal to  $\frac{1}{2}$ .

**P4.4** Use the results of Lab 3 to find the slope of the function  $f(x) = x^2$  at  $x = -\frac{3}{2}$ , and then write the equation of the line tangent to  $f(x) = x^2$  at  $x = -\frac{3}{2}$ . Use a graphing app to graph  $y = f(x)$  and this tangent line on the same axes.

**P4.5** Without technology, make accurate sketches of  $f(x) = \sqrt{x}$  and  $g(x) = \frac{1}{\sqrt{x}}$ . Next sketch  $h(x) = \frac{1}{2\sqrt{x}}$  on the same axes. Compare  $h(x)$  to the derivative of  $y = f(x)$ .

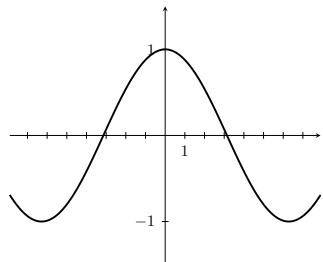
**P4.6** Match the graphs with the descriptions:

	1		2	Moving at a steady constant pace. <span style="float: right;">A</span>
	3		4	Moving at a fast pace, changing gradually to a slower pace. <span style="float: right;">B</span>
	5		6	Moving at a fast steady pace. <span style="float: right;">C</span>
				Moving fast, then slowing slightly, then going faster again. <span style="float: right;">D</span>
				Moving at a steady pace and then stops for a period of time. <span style="float: right;">E</span>
				Moving at a slow pace and then rapidly increases. <span style="float: right;">F</span>

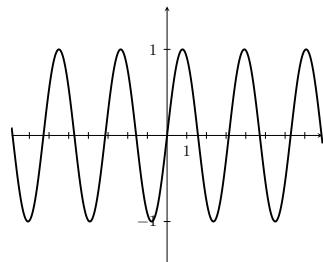
**P4.7** Use the results of Lab 3 to find the slope of the function  $f(x) = \frac{1}{x}$  at  $x = \frac{5}{2}$ , and then write the equation of the line tangent to  $f(x) = \frac{1}{x}$  at  $x = \frac{5}{2}$ . Use a graphing app to graph  $y = f(x)$  and its tangent line on the same axes.

**P4.8** Decide which of the following equations are graphed below:

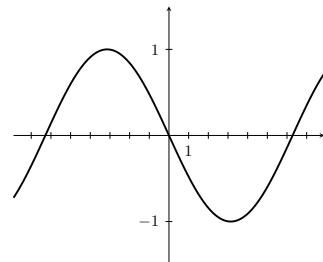
$$y = \sin(2x), \quad y = \cos(0.5x), \quad y = \sin(-0.5x), \quad y = \cos(-2x) \quad y = \sin(3x)$$



(a)



(b)



(c)

**P4.9** The expression

$$\frac{\sin(t+h) - \sin(t)}{h} \text{ can be written as } \frac{\sin(t+\Delta t) - \sin(t)}{\Delta t},$$

in which  $h$  is replaced by  $\Delta t$ . The symbol  $\Delta$  (the Greek letter capital *delta*) is chosen to represent the word *difference*. It is customary to refer to  $\Delta t$  as the *change in t*. The corresponding change in  $\sin(t)$  is  $\sin(t+\Delta t) - \sin(t)$ , which can be abbreviated  $\Delta \sin(t)$ . Notice that  $\Delta \sin(t)$  depends on both  $t$  and  $\Delta t$ . The ratio

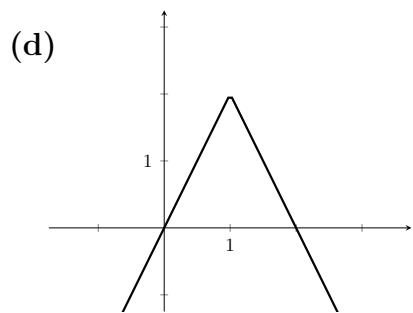
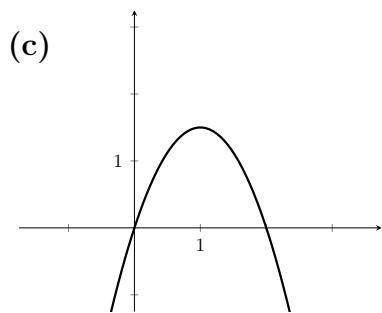
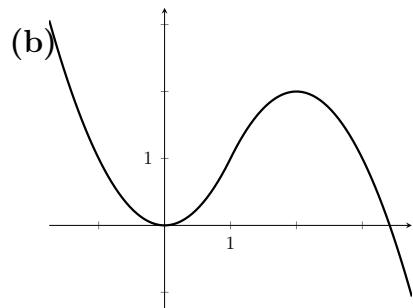
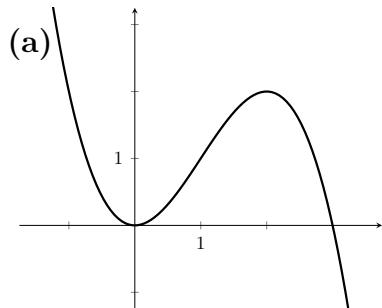
$$\frac{\sin(t+\Delta t) - \sin(t)}{\Delta t} = \frac{\Delta \sin(t)}{\Delta t}$$

is called a *difference quotient*. Working in radian mode, calculate  $\Delta \sin t$  and  $\frac{\Delta \sin(t)}{\Delta t}$  for  $t = 0.48$  and the following values of  $\Delta t$ :

- (a)  $\Delta t = 0.1$       (b)  $\Delta t = 0.01$       (c)  $\Delta t = 0.001$ .

What do you notice about the values of these difference quotients?

**P4.10** The four graphs below belong to  $f$ ,  $g$ ,  $f'$  and  $g'$ . Figure out which is which.



**P4.11** Use the difference quotient technique of Lab 3 to obtain a graph of an approximation to the derivative of  $f(x) = \tan x$ . What function is shown in the approximation of the derivative?

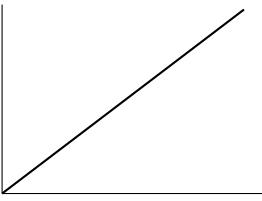
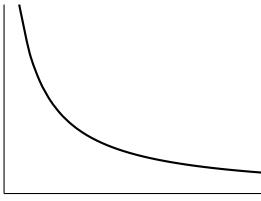
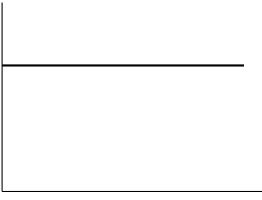
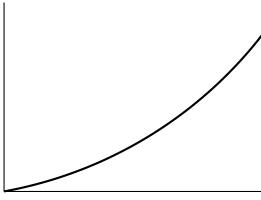
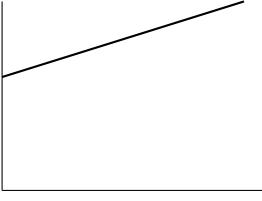
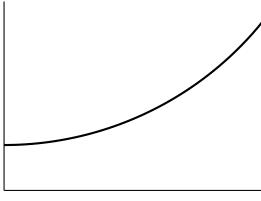
**P4.12** Given  $f(x) = \sqrt{x}$  and  $g(x) = x^2 + 9$ , find:

- (a)  $f(x - 1)$ ,
- (b)  $g(2x)$ ,
- (c)  $f(g(5))$ ,
- (d)  $g(f(5))$ ,
- (e)  $f(g(x))$ ,
- (f)  $g(f(x))$ .

**P4.13** The function  $p$  defined by  $p(t) = 3960(1.02)^t$  describes the population of Dilcue, North Dakota,  $t$  years after it was founded.

- (a) Find the founding population.
- (b) At what rate was the population growing ten years after the founding?
- (c) At what annual rate has the population of Dilcue been growing?

**P4.14** Match the graphs with the descriptions:

	1		2	The cost of hiring an electrician per hour, including a fixed call out fee.
	3		4	The connection between the length and width of a rectangle of fixed area.
	5		6	Speed against time for a car traveling at constant speed.
				The area of a circle as the radius increases.
				The width of a square as the length of the square increases.
				The 1954 population increased slowly at first, then increased more quickly.

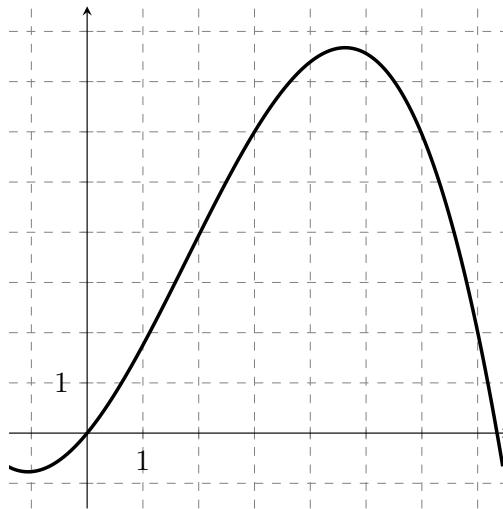
**P4.15** You have seen that the expression  $\left(1 + \frac{1}{n}\right)^n$  gets closer and closer to “e” as  $n$  approaches  $\infty$ , which is expressed formally as  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ . The limit notation of this equation implies that the difference between  $e$  and  $\left(1 + \frac{1}{n}\right)^n$  can be made small by choosing a large value of  $n$ . How large does  $n$  need to be so that the difference between  $\left(1 + \frac{1}{n}\right)^n$  and  $e$  is less than 0.01? 0.0001? 0.000001?

**P4.16** Working in radians, evaluate  $\frac{\sin(1 + \Delta t) - \sin(1)}{\Delta t}$  for  $\Delta t = 0.1$ ,  $\Delta t = -0.1$ ,  $\Delta t = 0.01$ , and  $\Delta t = -0.01$ . Based on your results, what would you say is the value of  $\lim_{\Delta t \rightarrow 0} \frac{\sin(1 + \Delta t) - \sin(1)}{\Delta t}$ ?

**P4.17** Find a linear approximation for  $y = \sin x$  around  $(0, 0)$ . Compare the values of the linear approximation with the actual values of  $\sin x$  at  $x = \frac{\pi}{4}$  and  $x = \frac{\pi}{2}$ . What is the percentage error of the approximation in each case?

**P4.18** On the graph  $y = f(x)$  shown below, draw lines whose slopes are

$$(a) \frac{f(7) - f(3)}{7 - 3}, \quad (b) \lim_{h \rightarrow 0} \frac{f(6 + h) - f(6)}{h}, \quad (c) \frac{f(7)}{7}, \quad (d) \lim_{h \rightarrow 0} \frac{f(h)}{h}.$$



**P4.19** (Continuation) On the graph  $y = f(x)$  shown, mark points where the  $x$ -coordinate has the following properties (a different point for each equation):

$$(a) \frac{f(x) - f(2)}{x - 2} = \frac{1}{2}, \quad (b) \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = -1,$$

$$(c) \frac{f(x)}{x} = \frac{1}{2}, \quad (d) \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = 0.$$

**P4.20** (Continuation) On a separate system of axes, graph the derivative function  $f'$ .

**P4.21** What does each limit say about the graphs of  $f(x) = \tan^{-1}(x)$  and  $g(x) = \frac{x^2 - 1}{x^2 + 1}$ ?

$$(a) \lim_{x \rightarrow \infty} \tan^{-1} x, \quad (b) \lim_{x \rightarrow -\infty} \tan^{-1} x, \quad (c) \lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^2 + 1}, \quad (d) \lim_{x \rightarrow -\infty} \frac{x^2 - 1}{x^2 + 1}.$$

**P4.22** The tangent line to a function can be thought of as a linear approximation for the function at a point. Find the equation of the tangent line to  $y = \sqrt{x}$  at  $x = 9$  and graph both  $y = \sqrt{x}$  and the linear approximation function. What value does this linear approximation give for  $\sqrt{8.9}$ ?  $\sqrt{9.1}$ ?  $\sqrt{10}$ ? Use a calculator to compare the approximate values with the actual values. Interpret your results. Be sure to look at a graph of the curve and the line.

**P4.23** Write a paragraph reflecting upon the concepts average rate of change, instantaneous rate of change, slope of secant, slope of tangent, and local linearity.

## Part 1

**L4.1** Using a driver on the 8<sup>th</sup> tee, which is situated on a plateau 10 yards above the level fairway, Sami hits a fine shot. The flight of the golf ball is described by the parabolic trajectory with equation  $y(x) = 10 + 0.5x - 0.002x^2$ .

- (a) Construct a graph of the path of the golf ball in a suitable window.
  - (b) Find how far down the fairway the ball lands.
  - (c) Determine the coordinates of the highest point of the trajectory.
  - (d) Make a guess as to the derivative of this function at the point  $(50, 30)$ .

**L4.2** (Continuation) Use the guess that you made above to write an equation for the tangent line  $l(x)$  at the point  $(50, 30)$ , which is a linear approximation. Add the line  $l(x)$  to the graph of the trajectory and adjust your guess for the slope as needed. Where does the actual trajectory lie in relation to this line?

**L4.3** Over what interval of  $x$ -values does  $l(x)$  appear to approximate the trajectory well? Is there an  $x$ -value after which the linear approximation stops begin useful? Fill in the table below, and then use the data you have collected to determine the interval of  $x$ -values for which the difference between the linear approximation and the actual value of  $f(x)$  is less than 10% of  $f(x)$ .

**L4.4** Determine the linear approximation for the trajectory with the starting point  $(0, 10)$  as the point of tangency. Graph this line along with the quadratic trajectory. What is the difference between the maximum height of the actual trajectory and the value of this linear approximation using the same  $x$ -value?

**L4.5** Determine the linear approximation *centered* at the point  $(200, 30)$ . Using the word *centered* is another way of indicating the point of tangency. Once again, you should graph this line and the quadratic trajectory of the ball. Where will the ball land according to this linear approximation? Compare with the actual value found in #L4.1.

Part 2

**L4.6** Referring to the golf ball trajectory  $y(x) = 10 + 0.5x - 0.002x^2$ , suppose that the laws of physics (in particular, those regarding the force of gravity) were suddenly suspended at the moment when the ball reaches its highest point. What path would the ball follow from that point forward? Write a new function, which will be a *piecewise* function, that describes this entire trajectory. Graph your result.

**L4.7** Does your new function have the property of local linearity at all points on the curve? Did you check the point that corresponds to the moment that gravity ceased to exist? Make sure to confirm your answer by zooming in on your graph.

**L4.8** Change the second half of your piecewise function to the linear function  $y(x) = -0.1x + 53.75$ . This new piecewise function is parabolic when  $x < 125$  and linear when  $x > 125$ . Does it possess the property of local linearity at all points? Did you check  $x = 125$ ? Which of the two piecewise functions you created, the one in Problem L4.6 or the one in Problem L4.8, has a well-defined linear approximation centered at  $x = 125$ ? Why?

Part 3

**L4.9** Summarize what you have learned in this lab about linear approximations. You should be sure to describe what a linear approximation is, how you create a linear approximation (including a formula if you found one), how to visually and numerically determine an interval where you are confident in that approximation, and in what situations a linear approximation does not exist.

**P5.1** Consider the function  $f(x) = x^2$ .

- (a) Find the slope of the line between the points  $(3, f(3))$  and  $(3 + \Delta x, f(3 + \Delta x))$  for  $\Delta x = 0.5$ ,  $\Delta x = 0.1$ ,  $\Delta x = 0.05$ , and  $\Delta x = 0.01$ . What happens to the slope as  $\Delta x$  approaches zero? Use a limit to write this result.
- (b) Obtain an expression for  $\frac{f(3 + \Delta x) - f(3)}{\Delta x}$  by substitution using  $f(x) = x^2$ , then simplify this expression using algebra. Use the result to evaluate the limit expression  $\lim_{\Delta x \rightarrow 0} \frac{f(3 + \Delta x) - f(3)}{\Delta x}$ .

**P5.2** For  $f(x) = x^2$ , simplify the expression  $\frac{f(x + \Delta x) - f(x)}{\Delta x}$  and use the result to evaluate  $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ . What does this limit tell you about the derivative of  $f$ ?

**P5.3** For  $f(x) = x^2 - 3x$ , simplify the expression  $\frac{f(x + \Delta x) - f(x)}{\Delta x}$  and use the result to evaluate  $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ . What does this tell you about the derivative of  $f$ ?

**P5.4** In the quadratic functions of the three previous problems, why did you need to simplify the expressions before evaluating the limits?

**P5.5** Evaluate the following limits by zooming in on a graph showing both the numerator function and the denominator function.

$$(a) \lim_{x \rightarrow 0} \frac{\sin 3x}{x} \quad (b) \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

**P5.6** (Continuation)

- (a) Use algebra to evaluate  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$  by first simplifying the ratio  $\frac{x^2 - 1}{x - 1}$ .
- (b) Compare the graphs of  $f(x) = \frac{x^2 - 1}{x - 1}$  and  $g(x) = x + 1$ , and the domains of each function.

**P5.7** Use difference quotients with small intervals to graph an approximation for the slope of the graph of  $y = e^x$ . Compare the slope graph to the original graph. What does this tell you about the value of the slope of  $y = e^x$  at any point on the curve?

**P5.8** (Continuation) Find the function for the slope at any point on the graph of:

$$(a) y = 3e^x \quad (b) y = e^{x-3}$$

**P5.9** You have discovered the derivatives for the following functions. Fill in the table with the derivatives.

$f(x)$	$f'(x)$
$x^2$	
$x^3$	
$\sin x$	
$\cos x$	
$e^x$	
$\frac{1}{x}$	
$\frac{1}{x^2}$	
$\ln x$	
$\sqrt{x}$	

**P5.10** In our previous work we have often used a difference quotient to represent the rate of change of a function. Write a paragraph explaining your understanding of the limit of a difference quotient as an expression of the derivative of  $f(x)$ :

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The expression on the right side of the equation above is known as the *limit definition of the derivative*.

**P5.11** A *linear function* has the form  $L(x) = mx + b$  where  $m$  and  $b$  are constants (that is, the values of  $m$  and  $b$  do not depend on the value of  $x$ ). What is the derivative  $L'(x)$ ? Use algebra and the limit definition of the derivative to show how to go step-by-step from  $L(x)$  to  $L'(x)$ .

**P5.12** Consider the absolute value function  $y = |x|$ , which can be defined as

$$y = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

- (a) When a function is piecewise defined, we need to consider the derivative of each piece separately. What is the derivative of each “half” of the function? What is the derivative at  $x = 0$ ? How do you know?
- (b) To gain insight into part (a), zoom in on the graph of  $y = |x|$  around  $(0, 0)$ . Do you see local linearity? Explain how the picture relates to the derivative or lack of a derivative for this function.

**P5.13** Tell how the slope of the curve  $y = 3^x$  at its  $y$ -intercept compares to the slope of the curve  $y = 2 \cdot 3^x$  at its  $y$ -intercept. What can you say about the slopes at other pairs of points on these curves that have the same  $x$ -values?

**P5.14 (a)** If your last name begins with A through M, draw the graph of  $y = \sin x$  for  $-2\pi \leq x \leq 2\pi$ , working in radians. If your last name begins with N through Z, draw the graph of  $y = \sin x$  for  $-360^\circ \leq x \leq 360^\circ$ , working in degrees. Now find the slope of your curve at the origin. Is your answer for the slope consistent with what you know is the derivative of  $\sin x$  at  $x = 0$ ?

**(b)** Compare your answers with the your classmates' answers. Does it make a difference if you are working in radians as opposed to degrees? [Your results will suggest why we use radian measure for angles when we are using the tools of calculus.]

**P5.15** (Continuation) Working in radians, evaluate  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ . Interpret your answer.

**P5.16** Consider the piecewise-defined function  $f(x) = \begin{cases} -x + 4 & \text{if } x < 0, \\ x^2 + 4 & \text{if } x \geq 0. \end{cases}$

- (a)** Is this function differentiable everywhere? Explain. When a function fails to have a derivative at a point, it is said to be *nondifferentiable* at that point.
- (b)** Adjust the linear part of  $f$  so that the answer to (a) is the opposite of what you found.
- (c)** Now adjust the quadratic part of  $f$  instead so that the answer to (a) is the opposite of what you found.

**P5.17** Why does  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$  represent the same value as  $\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$ ?

**P5.18** How do the slopes of the curves  $y = m \sin x$  and  $y = \sin x$  compare at the origin?

Working in radians, also compare the slope of the curve  $y = m \sin x$  at  $(\pi, 0)$  with the slope of  $y = \sin x$  at  $(\pi, 0)$ . Is it possible to compare slopes for other points on these two curves?

**P5.19** The IRS tax formula for married couples is a piecewise-linear function, which in 2014 was as follows:

$$T(x) = \begin{cases} 0.10x & \text{for } 0 \leq x \leq 18\,150 \\ 1815 + 0.15(x - 18\,150) & \text{for } 18\,150 < x \leq 73\,800 \\ 10\,162.50 + 0.25(x - 73\,800) & \text{for } 73\,800 < x \leq 148\,850 \\ 28\,925 + 0.28(x - 148\,850) & \text{for } 148\,850 < x \leq 226\,850 \\ 50\,765 + 0.33(x - 226\,850) & \text{for } 226\,850 < x \leq 405\,100 \\ 109\,587.50 + 0.35(x - 405\,100) & \text{for } 405\,100 < x \leq 457\,600 \\ 127\,962.50 + 0.396(x - 457\,600) & \text{for } 457\,600 < x \end{cases}$$

This function indicates the tax  $T(x)$  for each nonnegative taxable income  $x$ .

- (a) What is the tax for a couple whose taxable income is \$50 000? \$100 000? \$300 000?
- (b) What is the meaning of “I’m in the 25% tax bracket”? What about the 33% bracket? Why does each piece (except for the first) of  $T$  begin with a number (1815 for example)?
- (c) Draw a rough sketch of the graph of  $T$  using pencil and paper. After making your drawing, open the Desmos file provided by your teacher which contains a definition and graph of the function.
- (d) Explain why  $T'(x)$  makes sense for all but six nonnegative values of  $x$  for these six values, why is  $T$  nondifferentiable?
- (e) Graph the derivative function  $T'$ . How many distinct values are there in the range of  $T'$ ?

**P5.20** Find the average rate of change for  $y = x^2$  between  $x = 0$  and  $x = 2$ . Is there a point with the  $x$ -value in the interval  $(0, 2)$  where the instantaneous rate of change is the same as the average rate of change over the interval? Explain and find the point. Verify your calculations with a graph.

**P5.21** (Continuation) How would you explain your answer to the previous problem with slopes and lines?

**P5.22** Find an approximate value for  $F(2.3)$  given only the information  $F(2.0) = 5.0$  and  $F'(2.0) = 0.6$ . Explain how this problem uses a linear approximation.

**P5.23** An investment account is set up with an initial deposit of \$1000, and it grows at an annual rate of 5%.

- (a) Write an equation for the amount  $A$  in the account as a function of time  $t$  in years.
- (b) Find the *doubling time* for this account, the time it takes for the amount in the account to double.
- (c) If you solved part (b) with algebra, you had to solve the equation  $2 = (1.05)^t$ . Explain. Since you are solving for an exponent, you would need to use logarithms since a logarithm is the inverse of an exponential; thus, the logarithm of an exponential expression yields an exponent. For this particular equation,  $\ln 2 = t \ln 1.05$  (Why?) Complete the work of finding the doubling time  $t$  from this equation.
- (d) In part (c) we happened to use a logarithm with base  $e$ , but we could have used any base logarithm. We often use base  $e$  or base 10 logarithms because those functions are readily available on calculators. What if we had used a base 1.05 logarithm applied to both sides of the equation  $2 = (1.05)^t$ ? What equation would we get for  $t$ ?
- (e) The solutions in parts (c) and (d) suggest the equality  $\frac{\ln 2}{\ln 1.05} = \log_{1.05} 2$ . Verify that this is true.

**P5.24** (Continuation) The equation  $a = b^x$  can be solved for  $x$  using the two approaches shown in the previous problem.

- (a) Write out two possible solutions, one using a logarithm with a base  $c$ , the other using a logarithm with a base  $b$ .
- (b) Equate the two expressions for  $x$  in part (a) to yield the *change of base* formula:

$$\log_b a = \frac{\log_c a}{\log_c b}.$$

**L5.1** *Vertical shift.* Graph the function  $f(x) = x^2$  and the tangent line to  $f(x)$  at  $x = 1$ . Now choose three values for a constant  $k$  and graph the functions  $f_k(x) = x^2 + k$  on the same axes as  $f(x)$ . Notice that the new functions are vertical shifts of the origin function. Consider how a vertical shift by  $k$  units affects the equation of the tangent line at  $x = 1$ , and write equations for the tangent lines at  $x = 1$  for your new functions  $f_k(x)$ . Check your conclusions by adding the lines to your graph.

**L5.2** (Continuation) What do all of the tangent lines have in common? How does your answer relate to the derivative of  $f_k(x) = x^2 + k$  at  $x = 1$ ?

- (a) What is the derivative of  $f_k(x) = x^2 + k$  at  $x = 1$ ?
- (b) In general, how do the derivatives of a function and a vertical shift of that function, obtained by adding a constant to the function, relate to each other?
- (c) Find the derivatives of the following functions.
  - i.  $y = x^2 + 5$
  - ii.  $y = -2 + \cos(x)$
  - iii.  $g(x) = f(x) + k$

**L5.3** *Vertical stretch/shrink.* Graph the function  $f(x) = \sin(x)$  and the tangent line to  $f(x)$  at  $x = \pi/4$ . Choose three values for a constant  $k$ , and on the same set of axes, graph the functions  $f_k(x) = k \sin(x)$ . Notice that the new functions are vertical stretches (or shrinks) of the original function. Consider how a vertical stretch by  $k$  units affects the equation of the tangent line at  $x = \pi/4$ , and write equations for the tangent lines at  $x = \pi/4$  for your new functions  $f_k(x)$ . Check your conclusions by adding the lines to your graph.

**L5.4** (Continuation) What do all of the tangent lines have in common? How does your answer relate to the derivative of  $f_k(x) = k \sin(x)$  at  $x = \pi/4$ ?

- (a) What is the derivative of  $f_k(x) = k \sin(x)$  at  $x = \pi/4$ ?
- (b) In general, how do the derivatives of a function and a vertical stretch or shrink of that function, obtained by multiplying the function by a constant, relate to each other?
- (c) Find the derivatives of the following functions.
  - i.  $y = 5x^2$
  - ii.  $y = -2 \cos(x)$
  - iii.  $g(x) = k \cdot f(x)$

**L5.5 Horizontal shift.** Graph the function  $f(x) = \sqrt{x}$  and the tangent line to  $f(x)$  at  $x = 4$ . Choose three values for a constant  $k$ , and on the same set of axes, graph the functions  $f_k(x) = \sqrt{x - k}$ . Notice that the new functions are horizontal shifts of the original function. Consider how a horizontal shift by  $k$  units affects the equation of the tangent line at  $x = 4$ , and write equations for the tangent lines at  $x = 4$  for your new functions  $f_k(x)$ . Check your conclusions by adding the lines to your graph.

**L5.6** (Continuation) What do all of the tangent lines have in common? How does your answer relate to the derivative of  $f_k(x) = \sqrt{x - k}$  at  $x = 4$ ?

- (a) What is the derivative of  $f_k(x) = \sqrt{x - k}$  at  $x = 4$ ?
- (b) In general, how do the derivatives of a function and a horizontal shift of that function, obtained by subtracting a constant from the  $x$  value, relate to each other?
- (c) Find the derivatives of the following functions.
  - i.  $y = (x + 5)^2$
  - ii.  $y = \cos\left(x - \frac{\pi}{4}\right)$
  - iii.  $g(x) = f(x - k)$

**L5.7 Horizontal stretch/shrink.** Graph the function  $f(x) = \cos x$  and the tangent line to  $f(x)$  at  $x = \frac{\pi}{2}$ , which is the smallest positive  $x$ -intercept of the graph. Choose three values for a constant  $k$ , and on the same set of axes, graph the functions  $f_k(x) = \cos(kx)$ . Notice that the new functions are horizontal stretches (shrinks) of the original function. Consider how a horizontal shift by  $k$  units affects the location of the smallest positive  $x$ -intercept of the function, and write equations for the tangent lines at these intercepts. Check your conclusions by adding the lines to your graph.

**L5.8** (Continuation) How does the  $x$ -intercept shift as  $k$  changes? What do all of the tangent lines have in common? How does your answer relate to the derivative of  $f_k(x) = \cos(kx)$  at  $x = \frac{\pi}{2}$ ?

- (a) What is the derivative of  $f_k(x) = \cos kx$  at  $x = 4$ ?
- (b) In general, how do the derivatives of a function and a horizontal stretch of that function, obtained by multiplying  $x$  by a constant, relate to each other?
- (c) Find the derivatives of the following functions.
  - i.  $y = \sqrt{5x}$
  - ii.  $y = \sin(2x)$
  - iii.  $g(x) = f(kx)$

**L5.9** Write a paragraph which summarizes your understanding of the relationship, which you have explored in this lab, between the derivatives of function and a simple transformation of that same function.

**P6.1** Find the values of the expressions  $\lim_{h \rightarrow 0} \frac{e^h - 1}{h}$  and  $\lim_{k \rightarrow 1} \frac{\ln k}{k - 1}$ . Show that each value can be interpreted as a slope, and thus as a derivative.

**P6.2** Each of the following represents a derivative. Use this information to evaluate each limit.

$$(a) \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \quad (b) \lim_{h \rightarrow 0} \frac{1}{h} \left( \sin\left(\frac{\pi}{6} + h\right) - \sin\left(\frac{\pi}{6}\right) \right) \quad (c) \lim_{x \rightarrow a} \frac{e^x - e^a}{x - a}$$

**P6.3** In previous labs and problem sets, we learned that the derivative of  $y = \sqrt{x}$  is  $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ . In this problem we will see how to obtain this derivative from the limit definition of the derivative.

(a) Explain why  $\sqrt{x}$  has the following limit expression as its derivative.

$$\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

- (b) Re-express the limit formula in part (a) by using the technique of multiplying the numerator and denominator by the same expression  $\sqrt{x+h} + \sqrt{x}$ . Simplify and cancel common factors.
- (c) Taking the limit at  $h$  approaches 0 of the expression in part (b), obtain the expression for the derivative of  $\sqrt{x}$ .

**P6.4** Find the approximate values for both  $g(2.1)$  and  $g(1.85)$ , given that  $g(2.0) = -3.5$  and (a)  $g'(2.0) = 10.0$  (b)  $g'(2.0) = -4.2$ .

**P6.5** What is the formula for the linear approximation for a function  $f(x)$  at  $x = a$ ? Write out in words what you think the linear approximation means. Include the following terms in your paragraph: slope of linear approximation, slope of tangent line, instantaneous rate of change, and derivative.

**P6.6** Find the derivatives of the following functions that have undergone a combination of transformations. Your answers should look like “ $f'(x) = \dots$ ”, or “ $y' = \dots$ ”, or a similar form.

(a) $y = e^{2x-2}$	(d) $y = 1 + 3\sqrt{x-4}$
(b) $p = 3960 \cdot e^{0.15t} + 5280$	(e) $y = -1 + \cos(3x + \pi)$
(c) $g(x) = \frac{5}{x+2}$	(f) $h(x) = (5x+1)^3$

**P6.7** After being dropped from the top of a tall building, the height of an object is described by  $y = 400 - 16t^2$ , where  $y$  is measured in feet and  $t$  is measured in seconds. Find a formula for the rate of descent (in feet per second) for this object. Your answer will depend on  $t$ . How fast is the object falling after 2 seconds?

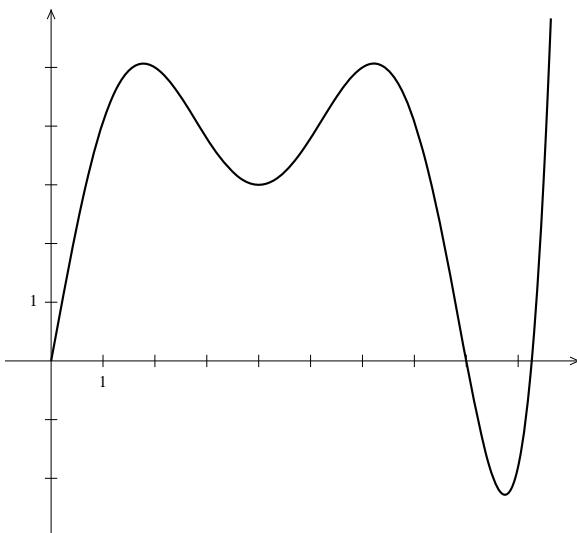
**P6.8** Calculate derivatives for  $A(r) = \pi r^2$  and  $V(r) = \frac{4}{3}\pi r^3$ . The resulting functions  $A'$  and  $V'$  should look familiar. Could you have anticipated their appearance? Explain.

**P6.9** (Continuation) If  $a$  is a number, then  $f'(a)$  is the derivative of  $f(x)$  evaluated at  $a$ . This means that to find  $f'(a)$ , you first determine a formula for  $f'(x)$  and then substitute the value of  $a$  for  $x$ .

- (a) Find the values of  $A'(2)$  and  $V'(2)$ .
- (b) Explain the geometric meaning of the quantities found in part (a).

**P6.10** Interpret the diagram as a *velocity-time* graph for an object that is moving along a number line. The horizontal axis represents time (in seconds) and the vertical axis represents velocity (in meters per second).

- (a) The point  $(8.8, -2.2)$  is on the graph. Find it in the diagram and describe what is going on around that point. In particular, what is the significance of the sign? Choose three other conspicuous points on the graph and interpret them.
- (b) Suppose the object starts its journey when  $t = 0$  at a definite point  $P$  on the number line. Use the graph to estimate the position of the object (in relation to  $P$ ) two seconds later.



**P6.11** (Continuation) On a separate set of axes, sketch the derivative of the function whose graph appears above. Interpret your graph in this context.

**P6.12** You have found the derivatives of certain power functions. The derivative of  $f(x) = x^2$  is  $f'(x) = 2x$ , and the derivative of  $g(x) = x^3$  is  $g'(x) = 3x^2$ .

- (a) What do you think is a formula for the derivative of a power function  $y = x^n$ ?
- (b) Look at the graphs of some other power functions (such as  $x^4, x^{-2}, \sqrt{x}$ , and so on) and their derivatives. [Desmos is an app that allows you to graph the derivative when the operator  $\frac{d}{dx}$  is applied to a function of  $x$ .] See if your guess from part (a) is correct, and if not, modify your formula.

**P6.13** The *linear approximation* for  $(1 + x)^k$ :

- (a) Find the linear approximation for  $(1+x)^2$  centered at  $a = 0$ . Verify graphically that you have indeed found the tangent line approximation for this function.
- (b) Find the linear approximation for  $(1+x)^3$  centered at  $a = 0$ . Verify graphically.
- (c) Find the linear approximation for  $\sqrt{1+x}$  centered at  $a = 0$ . Verify graphically.
- (d) Justify the linear approximation formula  $(1 + x)^k \approx 1 + kx$ .

**P6.14** Illuminated by the parallel rays of the setting Sun, Andy rides alone on a merry-go-round, casting a shadow that moves back and forth on a wall. The merry-go-round takes 9 seconds to make one complete revolution, Andy is 24 feet from its center, and the Sun's rays are perpendicular to the wall. Let  $N$  be the point on the wall that is closest to the merry-go-round.

- (a) Interpreted in radian mode,  $f(t) = 24 \sin\left(\frac{2\pi}{9}t\right)$  describes the position of the shadow relative to  $N$ . Explain.
- (b) Calculate the speed (in feet per second) of Andy's shadow when it passes  $N$ , and the speed of the shadow when it is 12 feet from  $N$ .

**P6.15** Find the derivative of  $\ln(3x)$ . Explain why this function can be viewed as a vertical shift or a horizontal shrink of the function  $\ln x$ . Show that both approaches lead to the same derivative.

**P6.16** Find the derivative of  $e^{x+5}$ . Explain why this function can be viewed as a horizontal shift or a vertical stretch of the function  $e^x$ . Show that both approaches lead to the same derivative.

**P6.17** Recall the change-of-base formula for logarithms:  $\log_b a = \frac{\log_c a}{\log_c b}$ . Apply this formula to rewrite the following logarithms in terms of natural logarithms.

- (a)  $\log_2 x$ ,
- (b)  $\log_{10}(x + 3)$ ,
- (c)  $\log_3(x^2)$ .

**P6.18** (Continuation) Use the strategy in the previous number to find the derivative formula for any base logarithm function by answering parts (a)-(c) below.

- (a) What is the derivative of  $\ln x$ ? What is the derivative of  $\frac{\ln x}{\ln 3}$ ? What is the derivative of  $\log_3 x$ ?
- (b) Rewrite  $\log_b x$  in terms of  $\ln x$  by using the change-of-base formula.
- (c) Find the derivative of  $\log_b x$  by using the expression found in part (b).
- (d) Find the derivatives of the expressions given in the previous problem:  $\log_2 x$ ,  $\log_{10}(x + 3)$ , and  $\log_3(x^2)$ .

**P6.19** By now you know that the derivative of  $y = e^x$  is  $y' = e^x$ . To find the derivative of other exponential functions, such as  $2^x$ , it is helpful to rewrite the exponential as a base  $e$  exponential. This can be accomplished by recognizing that 2 is equal to some power of  $e$  or in symbols,  $2 = e^k$ .

- (a) Find the exact solution for  $k$  in the equation  $2 = e^k$  using natural logarithms.
- (b) Using your previous answer, show how to rewrite  $2^x$  as  $e^{kx}$ .
- (c) Find the derivative of  $2^x$  by finding the derivative of the equivalent expression  $e^{kx}$ .
- (d) Write a general formula for the derivative of  $b^x$ , for any base  $b > 0$ .

**P6.20** (Continuation) Find the derivatives of the following functions.

(a)  $y = 2^{x+5}$ ,      (b)  $A = 1000 \cdot (1.05)^t$ ,      (c)  $T = 68 + 132 \cdot (0.9)^t$ .

**P6.21** *Simple harmonic motion.* An object is suspended from a spring, 40 cm above a laboratory table. At time  $t = 0$  seconds, the object is pulled 24 cm below its equilibrium position and released. The object bobs up and down thereafter. Its height  $y$  above the laboratory table is described, with  $t$  measured in radians, by  $y = 40 - 24 \cos(2\pi t)$ .

- (a) What is the period of the resulting motion?
- (b) Find the average velocity of the object during the first 0.50 second of motion.
- (c) Find the instantaneous velocity of the object when  $t = 0.25$  second. Find a way of convincing yourself that the object never moves any faster than it does at this instant.

In the previous lab we investigated what happens to a tangent line to the graph of a function when we apply basic transformations to the function, the tangent line being the linear approximation for the curve near the point of tangency. The concept of local linearity played a large role in Lab 6, and we will again use this concept to gain insight into the derivative of the sum of two functions and the derivative of the product of two functions.

### Part 1: The Addition Rule for Derivatives

**L6.1** Find an equation for  $l(x)$ , the linear approximation for  $f(x) = x^2 + 1$  at  $P = (1, 2)$ .

Find an equation for  $m(x)$ , the linear approximation for  $g(x) = \frac{1}{x}$  at  $Q = (1, 1)$ . Graph these four functions in a window that includes  $P$  and  $Q$ .

**L6.2** Now graph the sum functions  $s(x) = f(x) + g(x)$  and  $r(x) = l(x) + m(x)$ . What do you notice about  $s(x)$  and  $r(x)$  near  $x = 1$ ? What does this suggest about the linear approximation for  $s(x)$  near the point with  $x = 1$ ?

**L6.3** Choose another  $x$ -value, find  $P$  and  $Q$ , and consider the linear approximations to  $f(x)$  and  $g(x)$  in the same way that you did in problems 1 and 2 above.

**L6.4** Consider the following related question: If you have two linear functions, how does the slope of the sum function relate to the slopes of the linear functions?

**L6.5** We know from our previous work that the general equations for the linear approximations for two functions  $f$  and  $g$  centered at  $x = a$  are  $f(x) \approx f'(a)(x - a) + f(a)$  and  $g(x) \approx g'(a)(x - a) + g(a)$ . How can these two equations be combined to find a linear approximation for the function  $s(x) = f(x) + g(x)$  centered at the point with  $x = a$ ? What is the slope of this linear approximation for  $s(x)$ ? How does this slope relate to the derivatives of  $f(x)$  and  $g(x)$  at  $x = a$ ?

**L6.6** Write a statement for the *Addition Rule* relating the derivative of  $s(x)$  to the derivatives of  $f(x)$  and  $g(x)$ .

**L6.7** Apply the Addition Rule to find the derivatives of the following functions.

$$(a) \quad y = x^2 + \frac{1}{x} \qquad (b) \quad y = x + \sin x$$

### Part 2: The Product Rule for Derivatives

**L6.8** Find an equation for  $l(x)$ , the linear approximation for  $f(x) = e^{-x}$  at  $P = (1, e^{-1})$ .

Find an equation for  $m(x)$ , the linear approximation for  $g(x) = x^2$  at  $Q = (1, 1)$ . Graph these four functions in a window that includes  $P$  and  $Q$ .

**L6.9** Now graph the product functions  $p(x) = f(x) \cdot g(x)$  and  $q(x) = l(x) \cdot m(x)$ . What do you notice about  $p(x)$  and  $q(x)$  near  $x = 1$ ? What does this suggest about the linear approximation for  $p(x)$  near the point with  $x = 1$ ? What type of function is  $q(x)$ ? How does this product compare with  $p(x)$  near  $x = 1$ ? Explain.

**L6.10** What is the exact value of  $p(1)$ ? Use your graph to find the approximate value of the slope of the curve  $p(x)$  at  $x = 1$ . Use this point and the approximate slope to write an equation for the tangent line to  $p(x)$  at the point with  $x = 1$ . Name this tangent line the function  $L(x)$ .

**L6.11** The exact value of the slope of the linear approximation for  $p(x)$  is  $1 \cdot (-e^{-1}) + e^{-1} \cdot 2$ . How close is this number to your guess? Verify that your slope in number 3 is approximately this value. How does this slope relate to  $f$  and  $g$  and their derivatives at  $x = 1$ ? Now make a conjecture about how the derivative of  $p(x)$  relates to  $f(x)$ ,  $g(x)$ ,  $f'(x)$ , and  $g'(x)$ . Whereas you do not yet know how to find the derivative of the product function  $p(x)$  (and thus the slope  $p'(1)$ ), you can find the derivative of  $q(x)$ , which approximates  $p(x)$  near  $x = 1$ . This value  $q'(1)$  gives you the exact value of the slope of  $L(x)$ .

**L6.12** Work through the following steps to investigate your conjecture about the derivative of the product of two functions.

- (a) Write the formulas for the linear approximations of the functions  $f$  and  $g$  centered at the points with  $x = a$ .
- (b) Multiply the linear approximations of the functions  $f$  and  $g$  from part (a). Notice that this is a quadratic function since it is the product of two linear functions.
- (c) Verify that the derivative of the quadratic in part (b) is the linear expression  $2f'(a)g'(a)(x - a) + f(a)g'(a) + f'(a)g(a)$ . What is the value of this expression at the point of tangency  $x = a$ ? Explain why this is also the value of  $p'(a)$ .
- (d) Write a general formula for the derivative of  $p(x)$ , where  $p$  is the product of  $f$  and  $g$ . Compare your formula with the *Product Rule* for derivatives (which you can look up). Adjust your formula as necessary.

**L6.13** Investigate the derivative of  $p(x)$  algebraically and graphically.

- (a) Use the Product Rule to find the derivative of  $p(x) = x^2e^{-x}$ .
- (b) Graph both  $p(x)$  and its derivative on the same set of axes.
- (c) Does your graph of the derivative seem reasonable compared to the shape of  $p(x)$ ? If not, modify your derivative formula. Explain how the shape of  $p'(x)$  makes sense.

**L6.14** Apply the Product Rule to find derivatives of the following functions.

$$(a) y = x \sin x \quad (b) m(x) = (x + 1)e^x$$

### Part 3: Summary

**L6.15** Summarize in a paragraph what you have learned in this lab. You should reflect upon the Addition Rule and the Product Rule, and how you understand those procedures. You should also consider how the range of derivatives you can calculate has been expanded by these two rules.

**P7.1** If  $y = \sin x$ , then the derivative equation  $y' = \cos x$  can also be expressed with the notation  $\frac{dy}{dx} = \cos x$ . Show where this notation comes from by writing a limit that relates  $\frac{dy}{dx}$  and  $\frac{\Delta y}{\Delta x}$ . Is  $\frac{dy}{dx}$  a ratio of two numbers in the usual sense? The form  $\frac{dy}{dx}$  for the derivative is known as *Leibniz notation*.

**P7.2** *Oscillations about a line.*

- (a) Graph the functions  $f(x) = x + \sin x$  and  $g(x) = x$ .
- (b) Find the derivative of  $f$ .
- (c) What is the slope of the graph of  $f$  at the points where  $f(x) = g(x)$ ?
- (d) Which of the following characteristics apply to the graph of  $f$ ? (1) periodic; (2) always increasing; (3) alternately increasing and decreasing; (4) non-decreasing; (5) rotated sinusoidal; (6) horizontal tangent at  $x$  values equal to odd multiples of  $\pi$

**P7.3** Find the derivative of each of the following functions.

- (a)  $f(x) = x + \ln x$
- (b)  $g(t) = 3t - 5 \sin t$
- (c)  $L(x) = \sqrt{4 - x}$
- (d)  $P(t) = 12 + 4 \cos(\pi t)$

**P7.4** A quadratic function  $F$  is defined by  $F(x) = ax^2 + bx + c$ , where  $a$ ,  $b$ , and  $c$  are constants and  $a$  is nonzero. Find the derivative of  $F$ , and then find the value of  $x$  that makes  $F'(x) = 0$ . The corresponding point on the graph  $y = F(x)$  is special. Why?

**P7.5** Find a function that fits the description  $f'(t) = -0.42f(t)$ . There are many from which to choose.

**P7.6** Assuming  $A$  and  $C$  are constants, find the derivatives of the following functions:

- (a)  $y = A \cdot e^x$ ; (b)  $y = e^x + C$ . For which of these functions is it true that  $\frac{dy}{dx} = y$ ?

**P7.7** Given that  $f$  is a differentiable function and that the value of  $c$  does not depend on  $x$ , explain the following differentiation properties:

- (a) If  $g(x) = f(x - c)$ , then  $g'(x) = f'(x - c)$ .
- (b) If  $g(x) = c \cdot f(x)$ , then  $g'(x) = c \cdot f'(x)$ .
- (c) If  $g(x) = f(cx)$ , then  $g'(x) = c \cdot f'(cx)$ .

**P7.8** The PEA Ski Club is planning a ski trip over a long weekend. They have 40 skiers signed up to go, and the ski resort is charging \$180 per person. The resort manager offers to reduce the group rate of \$180 per person by \$3 for each additional registrant as long as revenue continues to increase.

- (a) Calculate how much money (revenue) the resort will receive if no extra students sign up beyond the original 40. How much is the revenue if 5 extra students sign up?
- (b) Let  $x$  be the number of additional registrations beyond the original 40. In terms of  $x$ , write expressions for  $p(x)$ , the total number of people going, and for  $q(x)$ , the cost per person. Revenue is the product of the number of people and the cost per person, so the revenue function is  $r(x) = p(x) \cdot q(x)$ .
- (c) Use the product rule to find the derivative  $r'(x)$  in terms of  $p(x)$ ,  $p'(x)$ ,  $q(x)$ , and  $q'(x)$ .
- (d) Find the derivative  $r'(x)$  by first multiplying out the product of  $p(x)$  and  $q(x)$ , then using the Addition Rule. Compare your answer with the answer from part (c). What value of  $x$  yields the maximum revenue.

**P7.9** Find the derivative of each of the following functions.

(a)  $f(x) = x^2 + x^{-2}$ ,    (b)  $g(t) = \sqrt{5t}$ ,    (c)  $y = (1+x)^n$ .

**P7.10** A roller coaster descends from a height of 100 ft on a track that is in the shape of the parabola  $y = 100 - \frac{1}{4}x^2$ , where  $y$  is the height in feet and  $x$  is the horizontal distance in feet from the point of maximum height.

- (a) What is the slope of the track at any point  $(x, y)$ ? Where will the track run into the ground and at what angle?
- (b) To allow the ride to transition smoothly to the ground, another piece of track in the shape of the parabola  $y = a(x - 40)^2$  is joined to the previous track and a horizontal track at  $y = 0$ . What is the slope of this track assuming  $a$  is a constant?
- (c) Using a graphing app, find an approximate value for  $a$  that fits the tracks together to give a smooth ride from the top to the ground.
- (d) Use your knowledge of calculus to find the exact value of  $a$  that smoothly fits the tracks together.

**P7.11** An object moves along the  $x$ -axis according to the equation  $x(t) = 4t - t^2$ .

- (a) Obtain a graph of  $x(t)$  versus  $t$ . Explain what the height of any point above the horizontal axis on the graph represents.
- (b) Employ *differentiation* (which is the name of the process for finding a derivative) to find a formula for the velocity of the object.
- (c) Use this derivative to find the velocity and speed of the object each time it passes the point  $x = 0$ .

**P7.12** Find the derivative of each of the following functions.

- (a)  $y = \sin x \cos x$
- (b)  $A(x) = x \cdot e^x$
- (c)  $P(u) = (u + 2)^5$

**P7.13** Consider the equations  $y'(t) = 0.12$  and  $p'(t) = 0.12 \cdot p(t)$ . They say similar but different things about the functions whose rates of change they are describing.

- (a) For each equation, find a function with the given derivative. There are many possible answers for each one.
- (b) For each equation, find the particular function that has a value of 36 when  $t = 0$ .

**P7.14** The population of Halania is increasing at a rate of 1.3% per year while per capita energy consumption is increasing at a constant rate of  $8 \times 10^6$  BTUs per year.

- (a) Explain why the functions  $P(t) = 23(1.013)^t$  and  $E(t) = 8t + 150$  are reasonable models for population and per capita energy consumption, where  $P$  is in millions of people,  $E$  is in millions of BTUs, and  $t$  is in years since 2010. What are the meanings of the constants 23, 1.013, 8, and 150 in these models?
- (b) Write an expression for total energy consumption  $T(t)$ , which is the product of population and per capita energy consumption. Use the product rule to find the derivative of this function.
- (c) How fast will total energy consumption be changing at the beginning of 2020? What is the percent change at that time?
- (d) Obtain a graph of the percent change in total energy consumption. Comment on the shape of this graph.

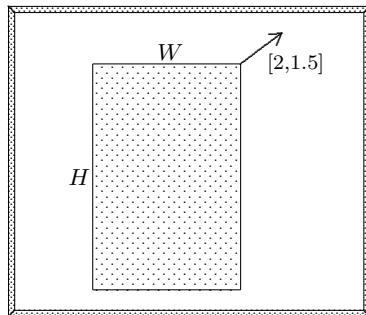
**P7.15** Kelly is using a mouse to enlarge a rectangular frame on a computer screen.

As shown below, Kelly is dragging the upper right corner at 2 cm per second horizontally and 1.5 cm per second vertically. Because the width and height of the rectangle are increasing, the enclosed area is also increasing. At a certain instant, the rectangle is 11 cm wide and 17 cm tall. By how much does the area increase during the next 0.1 second? Make calculations to show that most of the additional area comes from two sources – a contribution due solely to increased width, and a contribution due solely to increased height. Your calculations should also show that the rest of the increase is insignificant – amounting to less than 1%.

**P7.16** (Continuation) Repeat the calculations, using a time increment of 0.001 second.

As above, part of the increase in area is due solely to increased width, and part is due solely to increased height. What fractional part of the change is not due solely to either effect?

**P7.17** (Continuation) Let  $A(t) = W(t) \cdot H(t)$ , where  $A$ ,  $W$ , and  $H$  stand for area, width, and height, respectively. The previous examples illustrate the validity of the equation  $\Delta A = W \cdot \Delta H + H \cdot \Delta W + \Delta W \cdot \Delta H$ , in which the term  $\Delta W \cdot \Delta H$  plays an insignificant role as  $\Delta t \rightarrow 0$ . Divide both sides of this equation by  $\Delta t$  and find limits as  $\Delta t \rightarrow 0$ . This will yield an equation involving derivatives that shows that the functions  $\frac{dA}{dt}$ ,  $W$ ,  $H$ ,  $\frac{dW}{dt}$ , and  $\frac{dH}{dt}$  are related in special way. What rule does this demonstrate?



**L7.1** Let  $y = x^3 - x$ . Find the derivative  $\frac{dy}{dx}$ . Graph both functions on the same set of axes.

- (a) What do you notice about the graph of  $y$  for those  $x$ -values where  $\frac{dy}{dx} = 0$ ?
- (b) What do you notice about the graph of  $y$  for those  $x$ -values where  $\frac{dy}{dx} > 0$ ?
- (c) What do you notice about the graph of  $y$  for those  $x$ -values where  $\frac{dy}{dx} < 0$ ?

**L7.2** Repeat #L7.1 for the following functions.

- (a)  $y = \sin x$
- (b)  $y = x^4 + 2x^3 - 7x^2 - 8x + 12$
- (c)  $y = x + \frac{1}{x}$

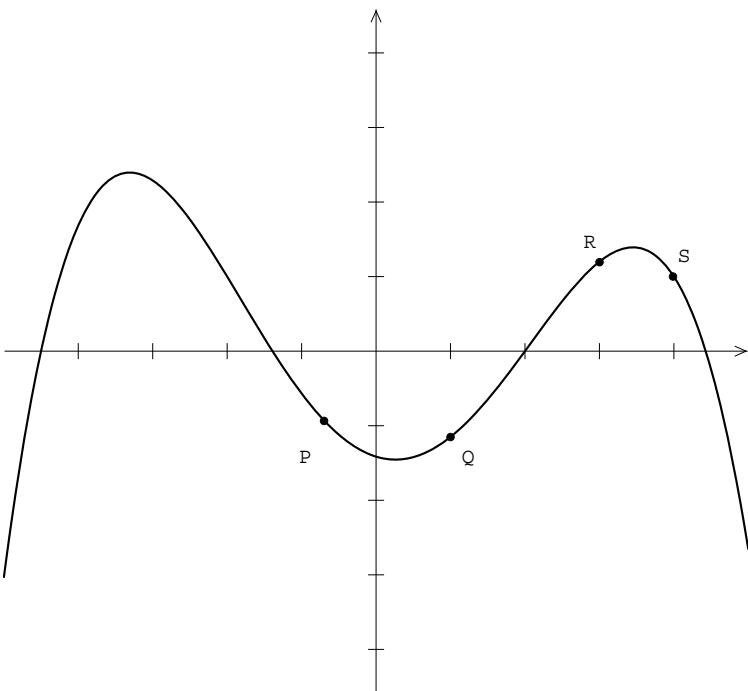
**L7.3** An  $8 \times 15$  rectangular sheet of metal can be transformed into a rectangular box by cutting four congruent squares from the corners and folding up the sides. The volume  $V(x)$  of such a box depends on  $x$ , the length of the sides of the square cutouts.

- (a) Use algebra to find an expression for the volume  $V(x)$ . For what values of  $x$  does  $V(x)$  make sense?
- (b) Use  $V'(x)$  to find the largest value of  $V(x)$  and the  $x$ -value that produces it. Explain your reasoning. Use a graph of  $V(x)$  to confirm your answer.

**L7.4** Write a paragraph summarizing what the values of the first derivative tell you about the graph of a function. You should include in your summary the technical terms increasing, decreasing, local maximum, and local minimum.

**P8.1** The figure below shows the graph of  $y = f(x)$ , where  $f$  is a *differentiable* function. The points  $P$ ,  $Q$ ,  $R$ , and  $S$  are on the graph. At each of these points, determine which of the following statements applies:

- |                       |                        |                        |
|-----------------------|------------------------|------------------------|
| (a) $f'$ is positive  | (b) $f'$ is negative   | (c) $f$ is increasing  |
| (d) $f$ is decreasing | (e) $f'$ is increasing | (f) $f'$ is decreasing |



The graph  $y = f(x)$  is called *concave up* at points  $P$  and  $Q$ , and *concave down* at points  $R$  and  $S$ .

**P8.2** Use a derivative to confirm that  $(0, 0)$  and  $(\frac{2}{3}, \frac{4}{27})$  are *locally extreme* points on the graph of  $y = x^2 - x^3$ . Explain this terminology. Then consider the section of the curve that joins these extreme points, and find coordinates for the point on it where the slope is steepest.

**P8.3** Find the extreme points on  $y = x + \frac{1}{x}$  by using a derivative. Confirm your answer graphically.

**P8.4** (Continuation) Find the extreme points on  $y = x + \frac{c}{x}$ , where  $c$  is a positive constant. Can you confirm your answer graphically? Explain.

**P8.5** Suppose the revenue and cost functions (in thousands of dollars) for a manufacturer are  $r(x) = 9x$  and  $c(x) = x^3 - 6x^2 + 15x$ , where  $x$  is in thousands of units.

- (a) What amount should be produced to generate the most profit? [Note: Profit equals revenue minus cost.]
- (b) Compare the slopes of the revenue and cost functions at the  $x$ -value found in (a).

**P8.6** (Continuation) Graph the revenue and cost functions from the previous problem. The location of the maximum profit, which is the greatest difference between  $r(x)$  and  $c(x)$ , appears to occur where the slopes of the two functions are equal. Use the derivative of profit  $p(x) = r(x) - c(x)$  to show that this is true in general.

**P8.7** Use what you know about derivatives of power functions (functions of the form  $y = x^n$ ) to find the derivative of  $Q(x) = 60 - \frac{12}{x\sqrt{x}}$

**P8.8** An object travels counterclockwise at 1 unit per second around the unit circle starting at the point  $(1, 0)$ .

- (a) Write parametric equations for the position of the object in terms of  $t$ , the time in seconds. Use these equations to represent the position of the object as a vector with  $x$ - and  $y$ -components given as functions of time  $t$  in seconds. You can think of the position of the object as the tip of a vector with its tail at the origin.
- (b) Find the *velocity vector* at the instant when the object is at each of the following points:  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 0)$ ,  $(0, -1)$ . How is the velocity vector oriented relative to the position vector?

**P8.9** (Continuation) Find the velocity vector in terms of  $t$  for the object by taking derivatives of the  $x$ - and  $y$ -components found in part (a). Obtain a sketch of the position and velocity vectors at the following points:  $(1, 0)$ ,  $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ , and  $(-0.6, 0.8)$ . Confirm that the velocity vector has a constant orientation relative to the position vector.

**P8.10** (Continuation) Suppose the object is actually tethered to a string of unit length as it is spun around a circle. What would happen to the object if the string suddenly broke when the object reached  $(-0.6, 0.8)$ ?

**P8.11** *Oscillations between envelope lines.*

- (a) Graph the functions  $f(x) = x \sin x$ ,  $g(x) = x$ , and  $h(x) = -x$ . How is the oscillating curve of  $f$  related to the *envelope* lines  $g$  and  $h$ ?
- (b) Find the derivative of  $f$ .
- (c) At what values of  $x$  does the graph of  $f$  touch the envelope lines? What are the  $x$ -values for the turning points of  $f$ ? Are these two sets of points the same?

**P8.12** Suppose that  $f'(a) = 0$  and that  $f'(x)$  changes from positive to negative at  $x = a$ . What does this tell you about the point  $(a, f(a))$  on the graph of  $y = f(x)$ ?

**P8.13** Suppose that  $f'(a) = 0$  and that  $f'(x)$  changes from negative to positive at  $x = a$ . What does this tell you about the point  $(a, f(a))$  on the graph of  $y = f(x)$ ?

**P8.14** Suppose that  $f'(a) = 0$  but that  $f'(x)$  does not change sign at  $x = a$ . What does this tell you about the point  $(a, f(a))$  on the graph of  $y = f(x)$ ? Show a graph that has this property.

**P8.15** In most cases, a derivative,  $f'$ , has its own derivative,  $f''$ , which is useful in many contexts. This function is called the *second derivative of  $f$* . If  $y = f(x)$ , then we can also write the second derivative as  $\frac{d^2y}{dx^2}$ . For each of the following, calculate the first and second derivatives. It may help to write (c) as a product.

$$(a) \ y = \sin x, \quad (b) \ s(t) = 400 - 16t^2, \quad (c) \ g(x) = \frac{e^x}{x}.$$

**P8.16** Obtain a graph of  $y = \ln(1 - x)$ . What is the linear approximation centered at  $x = 0$ ?

**P8.17** Calculate the derivative of each of the following functions:

$$(a) \ f(t) = t^2 e^{-t}, \quad (b) \ g(u) = u^2 \sqrt{u}, \quad (c) \ y = x \ln x.$$

**P8.18** The linear equation  $y - 5 = m(x - 2)$  represents a family of lines.

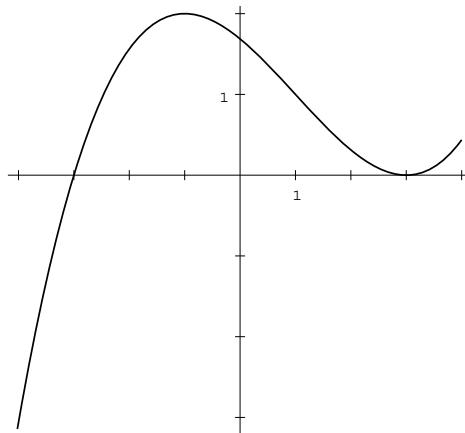
- (a) Describe the family of lines.
- (b) What values of  $m$  will form a right triangle in the first quadrant bounded by the axes and the line?
- (c) Find expressions for the  $x$ - and  $y$ -intercepts in terms of  $m$ .
- (d) Solve for the value of  $m$  that minimizes the area of the triangle in (b).

**P8.19** Just as the second derivative is the derivative of the first derivative, the third derivative is the derivative of the second derivative, and so on. For the following functions, find the first and second derivatives, and continue finding higher derivatives until you are able to deduce a formula for the  $n$ th derivative.

$$(a) \ y = t \cdot e^t, \quad (b) \ f(x) = x^n.$$

**P8.20** Suppose that  $f(x)$  is defined for  $-4 \leq x \leq 4$ , and that its derivative  $f'$  is shown below. Use the information in this graph and the additional fact  $f(-1) = 3$  to answer the following:

- (a) Is it possible that  $f(3) \leq 3$ ? Explain.
- (b) Is it possible that  $11 \leq f(3)$ ? Explain.
- (c) For what  $x$  does  $f(x)$  reach its maximum value?
- (d) For what  $x$  does  $f(x)$  reach its minimum value?
- (e) Estimate the minimum value and make a sketch of  $y = f(x)$  for  $-4 \leq x \leq 4$ .





The Tilt-a-Whirl is a popular carnival ride in which riders sit in carts that can be spun by the rider in circles, while the carts themselves are going around in a larger circle. The distance from the center of the large circle to the center of a cart's small circle is 5 meters. The cart's small circle has a one meter radius from center to seat. It takes 12 seconds to go once around the large circle. The small circles are actually controlled by the riders, but suppose that a rider is spinning at a constant rate of once every three seconds.

We can describe the location  $(x, y)$  of a rider by setting up our coordinate system with the center of the large circle at the origin. We want to know the location of a rider as a function of time, and we will use parametric equations where  $x$  and  $y$  are functions of time  $t$ . Recall that in Math 3-4 we used parametric equations to describe the location of a point on a rotating wheel (a Ferris wheel, for example), and we modeled the location of a point on a wheel that rolls along the ground. Both situations are helpful references for modeling the Tilt-a-Whirl, which is like a rotating wheel — at least the large circle is — but with the addition of a small circle that spins as it rotates around the large circle.

### Part 1

**L8.1** The Tilt-a-Whirl starts up and you are in one of the carts. Sketch a graph of what you think your path looks like as the large circle turns around its center (which is the origin of your coordinate system), and your cart is simultaneously spinning around its small circle.

**L8.2** Write parametric equations for  $Q(x, y)$ , where  $Q$  is the center of a cart's small circle as the large circle turns around the origin. Use the relevant values given in the first paragraph.

**L8.3** Write parametric equations for the motion of  $R(x, y)$ , where  $R$  is a point on a cart's small circle, and the motion is expressed relative to the center of the small circle. Equivalently, find the equations for  $R$  assuming the large circle is not moving. Use the relevant values given in the first paragraph of the introduction.

**L8.4** Explain why the parametric equations for  $P(x, y)$ , where  $P$  represents the location of a cart as it spins around the small circle while also rotating around the large circle, can be found by adding  $Q$  and  $R$ , as in  $P(x, y) = Q(x, y) + R(x, y)$ .

**L8.5** Obtain a graph of  $P(x, y)$  using graphing technology. How does your graph compare with the graph you sketched in number 1? Adjust your equations as necessary to get a reasonable result.

Part 2

**L8.6** You can create a position vector  $\mathbf{s}$  from the origin to the point  $P$ , the magnitude of which gives you the rider's distance from the origin. The components of  $\mathbf{s}$  are the parametric functions  $x(t)$  and  $y(t)$  that you found in Part 1 when you graphed  $P(x, y)$ . Write an equation in terms of  $x(t)$  and  $y(t)$  for the magnitude of  $\mathbf{s}$ , and graph the magnitude versus time for one 12-second revolution,  $0 \leq t \leq 12$ .

**L8.7** The velocity can also be represented as a vector, which we will call  $\mathbf{v}$ . The velocity has a magnitude and a direction, and the components of  $\mathbf{v}$  are the derivatives of the components of  $\mathbf{s}$ . Use calculus to find the velocity vector  $\mathbf{v} = [x'(t), y'(t)]$ .

**L8.8** The magnitude of the velocity vector is the speed of the rider, which is given by  $|\mathbf{v}| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ . Graph the speed over the 12-second interval of a single revolution.

**L8.9** Now do the same thing for acceleration: find the acceleration vector using derivatives and graph its magnitude as a function of time.

**L8.10** You now have three graphs that you can align as functions of time, or you can plot all three on the same set of axes. Based on these graphs, as well as the graph you generated in Part 1, at what points on the Tilt-a-Whirl do you think riders have the most fun? Explain your reasoning.

Part 3

**L8.11** Open the accompanying Geogebra file and run the animations in the graphing windows. Compare the graph in the window at the bottom of the screen, which shows the magnitudes of the position/velocity/acceleration vectors, to the graphs you found in Part 2.

**L8.12** In the top window, you will notice a point moving around the Tilt-a-Whirl curve, as well as vectors for position, velocity, and acceleration. The position vector has its tail at the origin, the tail of the velocity vector is anchored to the tip of the position vector, and likewise the acceleration vector is attached to the velocity vector. Study the animation in conjunction with the graph of the magnitudes of the vectors. Now reconsider an earlier question: At what points on the Tilt-a-Whirl do you think riders have the most fun?

**L8.13** Write a summary of what you have learned in this lab, especially regarding parametric equations, vectors, and their derivatives.

Extension

As an additional challenge, investigate what happens if you speed up the rotation of the cart. One way you can do this in the Geogebra simulation is by introducing a slider parameter to the vector equations in place of the number 3 that corresponds to the spinning rate of the cart. If  $k$  is the parameter, then the cart spins once every  $k$  seconds instead of once every 3 seconds.





**L9.1** Let  $y = x^4 - x^2$ . Find the first derivative  $\frac{dy}{dx}$  and the second derivative  $\frac{d^2y}{dx^2}$ . Graph all three functions on the same set of axes in a suitable sized window centered at the origin.

- (a) What do the values of  $\frac{d^2y}{dx^2}$  tell you about the graph of  $\frac{dy}{dx}$ ? This is an application of the concepts of Laboratory 7. You should include in your explanation words such as increasing, decreasing, turning points, and so on.
- (b) What do the values of  $\frac{d^2y}{dx^2}$  tell you about the graph of  $y$ ? As with the previous lab, consider where the second derivative is positive, where it is negative, and where it equals zero. The terms *concave up* and *concave down* may be useful in your description.

**L9.2** Repeat the questions #L9.1 for the following functions.

(a)  $y = \sin x$       (b)  $g(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$       (c)  $f(x) = x + \frac{1}{x}$

**L9.3** Fill in the blanks in the following statements based on your observations in #L9.1 and #L9.2.

- (a) If the second derivative is positive, then the first derivative is \_\_\_\_\_ and the graph of the function is \_\_\_\_\_.
- (b) If the second derivative is negative, then the first derivative is \_\_\_\_\_ and the graph of the function is \_\_\_\_\_.
- (c) If the second derivative changes sign (goes from positive to negative or negative to positive), then the first derivative has a local \_\_\_\_\_ and the graph of the function changes \_\_\_\_\_. This location on the function is called a *point of inflection*.

**L9.4** Write a lab report summarizing what the values of the second derivative tell you about the graph of the first derivative, and hence the graph of the function.

**P10.1** What can be said about the derivative of



**P10.2** Suppose that  $f''(a) = 0$  and that  $f''(x)$  changes from negative to positive at  $x = a$ . What does this tell you about the point  $(a, f(a))$  on the graph of  $y = f(x)$ ?

**P10.3** Use derivatives to find coordinates for the inflection points on the graph of  $y = xe^{-x}$ . Examine a graph to confirm your answers.

**P10.4** Suppose that  $f''(a) = 0$  and that  $f''(x)$  changes from positive to negative at  $x = a$ . What does this tell you about the point  $(a, f(a))$  on the graph of  $y = f(x)$ ?

**P10.5 Composition of functions.** The function  $h$  defined by  $h(x) = f(g(x))$  is a composition of functions in which  $f$  is composed with  $g$ . Let  $f(x) = \sin x$  and  $g(x) = 2\pi x$ . Find expressions for each of the following functions and their derivatives:

- $$(a) \ h(x) = f(g(x)) \quad (b) \ u(x) = g(f(x))$$

**P10.6** Graph the function  $y = |x^2 - 4|$  over the interval  $-4 \leq x \leq 4$ .

- (a) What is the equation of the tangent line at the point with  $x = 0$ ? If your definition of tangent line is based upon the phrase “only one point of intersection,” then is there another line that intersects the curve only once at the point with  $x = 0$ ?

(b) Come up with a definition of tangent line that results in a unique line for  $x$ -values on the curve such that  $-4 \leq x \leq 4$ . You will want to consider how “local linearity” can be used in your definition.

(c) Notice that this graph is not “smooth” at the points where  $x = \pm 2$ . Can you find a unique tangent line to the curve at either of those points? Based on your definition in part (b), can you say that a tangent line exists at either of those points? How does the existence of a tangent line at a point relate to *differentiability*, the existence of the derivative at that point?

**P10.7** The second derivative of  $f(x) = x^4$  is 0 when  $x = 0$ . Does that mean the origin is an inflection point on the graph of  $f$ ? Explain.

**P10.8** A plastic box has a square base and rectangular sides, but no top. The volume is 256 cc. What is the smallest amount of plastic that can be used to make this box? What are the dimensions of the box?

**P10.9** (Continuation) The previous problem can be solved graphically without using calculus techniques, and this strategy is often used in Book 3 problems. So why do we need calculus? Suppose you want to solve the problem in general for any volume  $V$ , not just  $V = 256 \text{ cc}$ . Solve this new problem using the techniques of calculus. Your minimum surface area and the accompanying dimensions will be given in terms of  $V$ .

**P10.10** At an inflection point, the tangent line does something unusual, which tangent lines drawn at non-inflection points do not do. What is this unusual behavior?

**P10.11** Given the functions  $f(x) = e^x$  and  $g(x) = 2x - 2$ , differentiate the following compositions of these functions.

- (a)  $y = f(g(x))$       (b)  $y = g(f(x))$

**P10.12** An object moves along the  $x$ -axis with its position at time  $t$  given by  $x = t + 2 \sin t$ , for  $0 \leq t \leq 2\pi$ .

- (a) What is the velocity  $\frac{dx}{dt}$  of the object at time  $t$ ?  
 (b) For what values of  $t$  is the object moving in the positive direction?  
 (c) For what values of  $t$  is the object moving in the negative direction?  
 (d) When and where does the object reverse its direction?

**P10.13** A windmill extracts energy from a stream of air according to the power function  $P(x) = 2kx^2(V - x)$ , where  $k$  is a constant,  $V$  is the velocity of the wind, and  $x$  is the average of the wind speeds in front and behind the windmill.

- (a) Find the derivative of  $P$ .  
 (b) Use the derivative to find the value of  $x$  that maximizes the power  $P$ . Your answer will be in terms of the parameters  $k$  and  $V$ . What is the maximum power?  
 (c) According to Betz's Law, the maximum power that can be extracted from the wind is  $16/27$  times the power in the wind stream, which is  $\frac{k}{2} \cdot V^3$ . Verify that the value for  $P$  found in (b) is indeed  $\frac{16}{27} \cdot \frac{k}{2} \cdot V^3$ .

*The following questions refer to the article from The Economist that follows.*

**P10.14** In the second paragraph, the author states: "Nobody cares much about inflation; only whether it is going up or down." How can this statement be translated into the language of derivatives?

**P10.15** "National debt" refers to how much the government has borrowed throughout the years of its history. A "budget deficit" refers to how much a government spends in a year beyond its annual revenues (which come mainly from taxes).

- (a) How is one of these related to the other by differentiation?  
 (b) Using the language of calculus, explain what happens to national debt if budget deficits are growing each year. What if the budget deficits are declining? What would have to occur with budget deficits for the national debt to decrease?

**P10.16** Find three other examples in the article that involve the way things change. Translate your examples into the language of derivatives.

**P10.17** What does an increase in the amount of change in a quantity say about the particular phenomenon? What about a decrease in the amount of change?

## THE TYRANNY OF DIFFERENTIAL CALCULUS

$$\frac{d^2 P}{dT^2} > 0 = \text{misery}$$

*from The Economist, 6 April 1991*

**Rates of change and the pace of decay**

*"The pace of change slows,"* said a headline on the *Financial Times's* survey of world paints and coatings last week. Growth has been slowing in various countries — slowing quite quickly in some cases. Employers were invited recently to a conference on Techniques for Improving Performance Enhancement. It's not enough to enhance your performance, Jones, you must improve your enhancement.

Suddenly, everywhere, it is not the rate of change of things that matters, it is the rate of change of rates of change. Nobody cares much about inflation; only whether it is going up or down. Or rather, whether it is going up fast or down fast. *"Inflation drops by a disappointing two points,"* cries the billboard. Which, roughly translated, means that prices are still rising, but less fast than they were, though not quite as much less fast as everybody had hoped.

No respectable American budget director has discussed reducing the national debt for decades; all talk sternly about the need to reduce the budget deficit, which is, after all, roughly the rate at which the national debt is increasing. Indeed, in recent years it is not the absolute size of the deficit that has mattered so much as the trend: is the rate of change of the rate of change of the national debt positive or negative?

Blame Leibniz, who invented calculus (yes, so did Newton, but he called it fluxions, and did his best to make it incomprehensible). Rates of change of rates of change are what mathematicians call second-order differentials or second derivatives. Or blame Herr Daimler. Until the motor car came along, what mattered was speed, a first-order differential. The railway age was an era of speed. With the car came the era of acceleration, a second-order differential: the fact that a car can do 0-60mph in eight seconds is a far more important criterion

for the buyer than that it can do 110mph. Acceleration limits are what designers of jet fighters, rockets and racing cars are chiefly bound by.

Politicians, too are infected by ever higher orders of political calculus. No longer is it necessary to have a view on abortion or poll taxes. Far better to commission an opinion poll to find out what the electorate's view is, then adopt that view. Political commentators, and smart-ass journalists, make a living out of the second-order differential: predicting or interpreting what politicians think the electorate thinks. Even ethics has become infected (*"You did nothing wrong, Senator, and even the appearance of it does not stink, but people might think it will appear to stink"*).

**Bring back the integral**

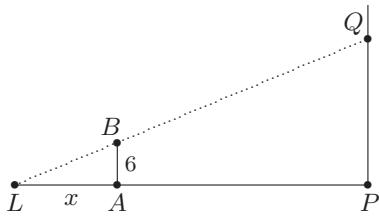
It boils down to biology. There is virtually nothing in the human brain or the nerves that measures a steady state. Everything responds to change. Try putting one hand in hot water and the other in cold water. After a minute put both hands in tepid water. It will feel hot to one, cold to the other. The skin's heat sensors measure changes in temperature.

The visual system of the from is a blank screen on which on things that move (delicious flies or dangerous enemies) show up, like shooting stars in a night sky. Human eyes have "edge detectors" to catch second-order differentials. They respond to the places where the rate of change of the rate of change of light hits zero — these mark the edges of things.

Soon second-order differentials will be passé, and the third order will be all the rage, with headlines reading *"inflation's rate of increase is leveling off"*, or *"growth is slowing quite quickly"*. Frenzy will then be a steady state. It will be high time to reverse the slide into perpetual differentiation. Workers of the world, integrate!

Part 1

**L10.1** The diagram below shows the shadow  $PQ$  that is cast onto a wall by a six-foot person, represented by  $AB$ , who is illuminated by a spotlight on the ground at  $L$ . The distance from the light to the wall is  $LP = 50$  feet, and the distance from the light to the person is  $LA = x$ , a variable quantity. The length of the shadow depends on  $x$ , so call it  $S(x)$ .



- (a) Use geometry to find a formula for  $S(x)$ .
- (b) For what values of  $x$  does  $S(x)$  make sense? In other words, find the *domain* of  $S$ .
- (c) Explain why  $S(x)$  is a decreasing function of  $x$ .

**L10.2** Find  $\Delta S/\Delta x$  when  $x = 12$  and  $\Delta x = 0.4$ . What does this ratio represent in terms of rate of change of the length of the shadow? Recall that the symbol  $\Delta$  (the Greek letter capital *delta*) is chosen to represent the word *difference*.

- (a) Find  $\Delta S/\Delta x$  when  $x = 12$  and  $\Delta x = 0.04$ , and again when  $\Delta x = 0.004$ . What do you observe about the value of  $\Delta S/\Delta x$  when  $\Delta x$  is close to 0?
- (b) What is the limiting value of  $\Delta S/\Delta x$  as  $\Delta x$  approaches 0?

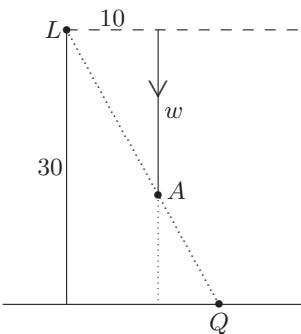
**L10.3** Suppose that  $x$  is a function of time, specifically  $x(t) = 4t$ , which means that the person is walking towards the wall at 4 ft/sec. This implies that the length of the shadow is also a function of time, so we can write a new function  $y(t) = S(x(t))$  as a composition of two functions  $S(x)$  and  $x(t)$ .

- (a) Write out in words the meaning of the functions  $S(x)$ ,  $x(t)$ , and  $y(t)$ .
- (b) Calculate  $\Delta x/\Delta t$  and  $\Delta y/\Delta t$  when  $t = 3$  and  $\Delta t = 0.1$ ,  $\Delta t = 0.01$ , and  $\Delta t = 0.001$ . What do you notice about the trend in these values?
- (c) Your calculations suggest that  $\frac{dx}{dt}$ ,  $\frac{dy}{dx}$ , and  $\frac{dy}{dt}$  should be related in a simple way. Explain.
- (d) Write an equation that connects the rates  $x'(3)$ ,  $S'(12)$ , and  $y'(3)$ . Notice that the primes in this list of derivatives *do not all mean the same thing*. Explain.

**L10.4** Suppose that the person runs toward the wall at 20 ft/sec. At what rate, in ft/sec, is the shadow length decreasing at the instant when  $x = 12$ ?

Part 2

**L10.5** The diagram below shows a falling object  $A$ , which is illuminated by a streetlight  $L$  that is 30 feet above the ground. The object is 10 feet from the lamppost, and  $w$  feet below the light, as shown. Let  $S$  be the distance from the base of the lamppost to the shadow  $Q$ .



- (a) Confirm that  $S = \frac{300}{w}$  and  $\frac{dS}{dw} = -\frac{300}{w^2}$ . What is the significance of the minus sign in the derivative?
- (b) Find  $\Delta S$  when  $w = 9$  and  $\Delta w = 0.24$ . Explain why  $\Delta S$  is approximately equal to  $\frac{dS}{dw} \cdot \Delta w$ .

**L10.6** Suppose the object was dropped from 30 feet above ground. From the laws of physics we know that the distance fallen is  $w = 16t^2$ . Calculate  $w$  and  $\frac{dw}{dt}$  when  $t = 0.75$  sec. Calculate  $\Delta w$  when  $t = 0.75$  and  $\Delta t = 0.01$ , and notice that  $\frac{dw}{dt} \cdot \Delta t \approx \Delta w$ .

**L10.7** The distance  $S$  from  $Q$  to the lamppost is also a function of time, so it makes sense to calculate  $\frac{dS}{dt}$ . Calculate this velocity when  $t = 0.75$  sec. Confirm that  $\frac{dS}{dt}$  is in fact equal to the product of  $\frac{dS}{dw}$  and  $\frac{dw}{dt}$  for all values of  $t$  in the domain.

Part 3

**L10.8** The function defined by  $h(x) = (x^2 + 5)^2$  is a composition of two functions. One of the functions is  $g(x) = x^2 + 5$  and the other is  $f(x) = x^2$ . Confirm that  $h(x) = f(g(x))$ . You can find  $h'(x)$  by first multiplying  $x^2 + 5$  by itself, collecting like terms, and then differentiating term-by-term. Do so, and then confirm that  $h'(x) = f'(g(x))g'(x)$ . This is an example of the *Chain Rule* for derivatives. In what form did you encounter the Chain Rule in Parts 1 and 2?

**L10.9** Another way of understanding the Chain Rule is through linear approximation. We begin with the composite function  $h(x) = f(g(x))$ .

- (a) Using the linear approximation for  $g(x)$  near  $x = a$ , show that  $h(x) \approx f(g'(a)(x - a) + g(a))$ .
- (b) Confirm that  $\frac{d}{dx}[f(mx + b)] = f'(mx + b) \cdot m$  by referring to your previous work with derivatives of functions composed with linear functions.
- (c) Hence show that  $h'(x) \approx f'(g'(a)(x - a) + g(a)) \cdot g'(a)$ , and thus explain the following equation:  $h'(a) \approx f'(g(a)) \cdot g'(a)$ .
- (d) How does (c) establish the Chain Rule  $h'(x) = f'(g(x))g'(x)$ ?

**L10.10** Use the Chain Rule to find the derivative of the function  $h(x) = \sqrt{1 - x^2}$ .

**L10.11** Write a paragraph summarizing your understanding of the Chain Rule. Be sure to include what types of functions you can now find the derivatives of, which you were unable to find previously.

**P11.1** If  $y = f(u)$  and  $u = g(x)$ , then the Chain Rule can also be written as  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ .

- (a) How does this notation compare with  $h'(x) = f'(g(x))g'(x)$ ? Explain how the two forms are the same.
- (b) For the function  $y = e^{\sin x}$ , identify the functions  $f$  and  $g$  that are composed with  $f$  being the “outer” function and  $g$  being the “inner” function.
- (c) Find  $\frac{dy}{dx}$  using the Chain Rule.

**P11.2** The function  $y = e^{-x^2}$  has a graph shaped like a *bell curve*. This function is widely used in statistics by using various basic transformations of the graph.

- (a) Confirm that the graph has a bell shape. What are the main features of the graph?
- (b) Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$ .
- (c) Describe the concavity of the graph, and find the coordinates of any points where the concavity changes, the so-called points of inflection.

**P11.3** The annual government expenditures  $E$  in Geerland can be modeled as a function of population  $P$  by  $E(P) = 10\sqrt{P}$ , with units of thousands of dollars. The population is growing and is modeled by the function  $P(t) = 100e^{0.03t}$ , where  $t$  is in years.

- (a) Express  $E$  as a function of time.
- (b) Find the derivative of  $E$  with respect to time.
- (c) Comment on the graphs of  $E(t)$  and  $E'(t)$ .

**P11.4** (Continuation) Suppose now the population of Geerland grows according to the function  $P(t) = 10000 - 9900e^{-0.03t}$ .

- (a) Graph  $P$  over a suitable domain. Why do you think this model is called *constrained growth*? What is the constraint, or upper limit, on the population?
- (b) Find the derivative of  $E$  with respect to time using the constrained growth model for population.
- (c) Comment on the graphs of  $E$  and the derivative of  $E$ .

**P11.5** Find the derivatives of the following functions.

<b>(a)</b> $H(u) = (\sin u)^3$	<b>(b)</b> $C(t) = \frac{e^t + e^{-t}}{2}$	<b>(c)</b> $v(x) = \sqrt{r^2 - x^2}$
<b>(d)</b> $y = \frac{1}{\cos x}$	<b>(e)</b> $u = (t+1)^2 \ln t$	

**P11.6** A particle moves along a number line according to  $x = t^4 - 4t^3 + 3$ , during the time interval  $-1 \leq t \leq 4$ . Calculate the velocity function  $\frac{dx}{dt}$  and the acceleration function  $\frac{d^2x}{dt^2}$ . Use them to help you give a detailed description of the position of the particle according to the following questions.

- (a) At what times is the particle (instantaneously) at rest, and where does this happen?
- (b) During what time intervals is the position  $x$  increasing? When is  $x$  decreasing?
- (c) At what times is the acceleration of the particle zero? What does this signify?
- (d) What is the complete range of positions of the particle?
- (e) What is the complete range of velocities of the particle?

**P11.7** For each of the following description of  $\frac{dy}{dx}$ , with constants  $c$  and  $k$ , what type of function is  $y$ ? For example, your answer in (a) could be something like this: *The derivative is a constant, therefore the function must be \_\_\_\_\_.*

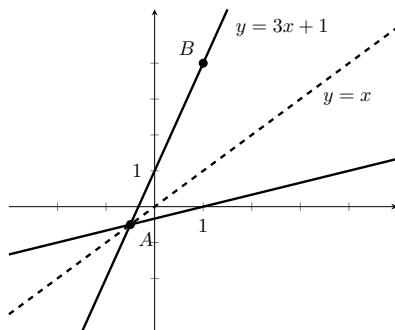
(a)  $\frac{dy}{dx} = c$       (b)  $\frac{dy}{dx} = ky$       (c)  $\frac{dy}{dx} = \cos x$

**P11.8** (Continuation) Equations involving derivatives, of which these are examples are known as *differential equations*. Give a particular solution of  $y$  as a function of  $x$  for each one. Why are there many solutions for each equation?

**P11.9** If we ignore the effects of air resistance, a falling object has a constant acceleration of 9.8 m/sec<sup>2</sup>.

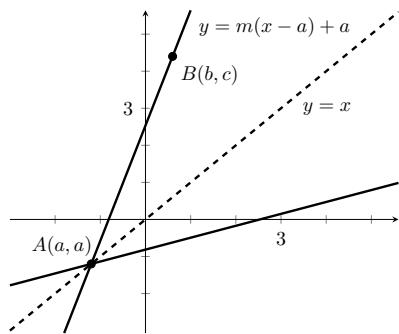
- (a) If the object falls from an initial height of 100 meters with initial velocity of 0, find equations for the velocity and height as functions of time (until the object hits the ground). With what speed does it strike the ground?
- (b) If the object falls from an initial height of 100 meters and it is initially projected upward with a velocity of 20 m/sec, find equations for the velocity and height as functions of time (until the object hits the ground). What is the maximum height and when does that occur? With what speed does it strike the ground?

**P11.10** The graph shows a line and its reflection across the line  $y = x$ . Find the images of points  $A$  and  $B$ , then find the equation of the image line. What is the relationship between the slope of the original line and the slope of its reflection?



**P11.11** (Continuation) The graph shows  $y = m(x - a) + a$  and its reflection over  $y = x$ .

- (a) Explain why the equation  $y = m(x - a) + a$  fits the line shown.
- (b) Find the images of  $A$  and  $B$ , and the equation of the reflected line.
- (c) What is the relationship between the slope of a line  $l$  and the slope of the reflection of  $l$  over  $y = x$ ?



**P11.12** Given the graph of a set of points  $(a, b)$ , the set of points  $(b, a)$  is a reflection over what line?

**P11.13** (Continuation) If the set of points  $(x, y)$  that satisfy the equation  $y = x^2$  is reflected through the line  $y = x$ , then the resulting set of points satisfies which of the following equations?

- (a)  $x = y^2$
- (b)  $y = \sqrt{x}$

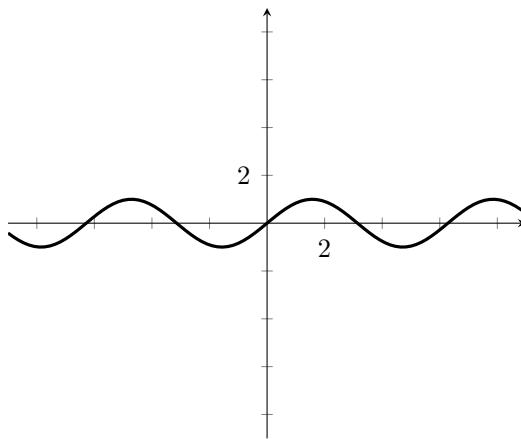
**P11.14** *Implicit Differentiation.* Find the derivative of both sides of the equation for the unit circle equation  $x^2 + y^2 = 1$ . Term-by-term differentiation is straightforward for two of the three terms of this equation, specifically  $\frac{d}{dx}(x^2) = 2x$  and  $\frac{d}{dx}(1) = 0$ .

- (a) Explain how the chain rule is used in the differentiation  $\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$ . This derivative assumes that  $y$  is an *implicitly defined function of  $x$* .
- (b) Put all the pieces together and solve for  $\frac{dy}{dx}$  from the circle equation. How else could you have found this derivative?

**P11.15** Find the derivatives of the following functions.

(a)  $y = \frac{1}{\sin x}$       (b)  $u = (t - 1)^3 \ln t$       (c)  $f(x) = e^{\cos x}$

**P11.16** Sketch on the axes below the graph of the reflection of  $y = \sin x$  over  $y = x$ .



**P11.17** The function  $\sin^{-1} x$  defines an angle whose sine ratio is  $x$ . You may recall from Math 3-4 that for this operation to give only one angle, the range of angles has to be limited.

- (a) What is the range of the function  $y = \sin^{-1} x$ ? If you are unsure, you could try in radian mode various sine ratios on your calculator (which are necessarily between -1 and 1, inclusive) and see what range of angles result.
- (b) Why is the range of  $y = \sin^{-1} x$  limited?
- (c) If you start with an angle measuring 1 radian and take the sine of this angle, you get approximately 0.8415. If you then take the inverse sine of 0.8415, you get back to the 1 radian angle you started with (or nearly so depending on how you rounded). In contrast, if you start with an angle measuring 2 radians and take the sine of this angle, you get approximately 0.9093. If you then take the inverse sine of 0.9093, you get approximately 1.142, which is not close to 2 radians. What accounts for the different results in these two examples?

**P11.18** (Continuation) What is the range of the function  $y = \cos^{-1} x$ ? This is equivalent to asking how the domain of  $y = \cos x$  is restricted so that the reflection of the graph across  $y = x$  is also a function.

**P11.19** In radian mode, the parametric equation  $(x, y) = (5 \cos t, 15 \sin t)$  traces an ellipse as  $t$  varies from 0 to  $2\pi$ .

- (a) Confirm by graphing and by algebra that this ellipse also has equation  $9x^2 + y^2 = 225$ .
- (b) Find the  $t$ -value that corresponds to the point  $(4, 9)$ .
- (c) Find the components of velocity  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ , and find their values at  $(4, 9)$ .
- (d) On a graph of the ellipse, show the vector  $\left[ \frac{dx}{dt}, \frac{dy}{dt} \right]$  with its tail at  $(4, 9)$  using the values of the components found in (c). Use these components of velocity to find the slope of the line tangent to the curve at  $(4, 9)$ .

**P11.20** Consider the function  $y = \tan x$  and its inverse  $y = \tan^{-1} x$ .

- (a) Graph  $y = \tan x$  over the domain  $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$ . This will result in three branches of the function, each branch situated between vertical *asymptotes*. What are the equations of the asymptotes?
- (b) Now enter a graph of  $y = \tan^{-1} x$  on the same set of axes. Notice that only one branch of this new function is allowed. How do you account for the appearance of  $y = \tan^{-1} x$  relative to the graph of  $y = \tan x$ ? Your explanation should include a reflection across a line, horizontal asymptotes, and limits on the domain and range of the two functions.

**P11.21** You previously discovered that  $\frac{d}{dx}(\ln x) = \frac{1}{x}$ . This can also be established through the derivative of  $e^x$ , which is the inverse of  $\ln x$ :

- (a) starting with  $y = \ln x$ , write  $x$  as a function of  $y$ , so that  $e^y = x$ ;
- (b) use implicit differentiation to show that  $e^y \frac{dy}{dx} = 1$ ;
- (c) solve for  $\frac{dy}{dx}$  and substitute for  $y$  in terms of  $x$  to show that  $\frac{dy}{dx} = \frac{1}{x}$ .

**P11.22** Explain why the graphs of inverse functions are related by a reflection across the line  $y = x$ .

- (a) On the same system of coordinate axes, and using the same scale on both axes, make careful graphs of both  $y = \ln x$  and  $y = e^x$ .
- (b) Let  $P = (a, b)$  be a point on the graph of  $y = \ln x$ , and let  $Q = (b, a)$  be the corresponding point on the graph of  $y = e^x$ . How are the slopes of the curves related at these two points? Explain.
- (c) Find the slopes at  $P$  and  $Q$  by using the derivatives of  $\ln x$  and  $e^x$ . Confirm the relationship between the slopes of corresponding points on the graphs of these inverse functions by letting  $a = 0.5, 1$ , and  $1.5$ .

**P11.23** (a) Given  $\sin \theta = 0.352$ , show how to find  $\cos \theta$  without finding the value of  $\theta$ .

- (b) If  $\sin \theta = x$ , then what is  $\cos \theta$  in terms of  $x$ ?
- (c) Explain the equivalence of  $\cos(\arcsin x)$  and  $\sqrt{1 - x^2}$ .
- (d) Use implicit differentiation to find the derivative of  $y = \sin^{-1} x$  by first rewriting the equation as  $\sin y = x$ . Your answer should simplify to the formula  $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$ .

**L11.1** A ball moves along a straight path according to the linear equation  $y = 3t$ , where the position  $y$  from the starting point is measured in feet and time  $t$  is measured in seconds.

- (a) What are the initial positions and velocity of the ball? What are the position and velocity after the ball has rolled for 1 second?
- (b) Write an equation for the velocity  $\frac{dy}{dt}$  at time  $t$  in the form  $\frac{dy}{dt} = \underline{\hspace{2cm}}$ .
- (c) Determine another linear relationship between an object's position and time, in which the velocity equation at time  $t$  is the same as in (b). How have you transformed the graph of  $y = 3t$ ?
- (d) The equation in (b) represents all linear functions whose rate of change is a constant 3 ft/sec. This is an example of a *differential equation*. Now write a differential equation for the rate of change of a general linear function  $y = mx + b$ . How many functions does your differential equation represent?
- (e) Solve the differential equation  $\frac{dy}{dx} = 10$  for the particular solution that contains the point  $(0, 5)$ .

**L11.2** The height of an object dropped from a building is described by the quadratic equation  $h = 400 - 16t^2$ , where  $h$  is measured in feet and  $t$  is measured in seconds.

- (a) What are the initial height and velocity of the object? What are these values 2 seconds later?
- (b) Write a differential equation for the velocity at time  $t$ .
- (c) What is another quadratic equation relating an object's height and time in which the velocity equation is the same as in (b)? How have you transformed the graph of  $h = 400 - 16t^2$ ?
- (d) Write a differential equation for the rate of change of a general quadratic function  $y = ax^2 + bx + c$ .
- (e) Solve the differential equation  $\frac{dh}{dt} = -10t$  for the particular solution that has an initial height  $h = 150$  meters.

**L11.3** The population of a town is given by exponential equation  $P = 4123e^{0.02t}$ , where  $t$  is measured in years since the beginning of 2010.

- (a) Determine the initial population (in 2010, which is at time  $t = 0$ ) and the initial instantaneous growth rate.
- (b) Write a differential equation for the growth rate of  $P$  at any time  $t$ . Notice that within this differential equation you will find  $P$  multiplied by some constant. Rewrite the differential equation in terms of  $P$ .
- (c) Determine another exponential relationship in which the differential equation, in terms of  $P$ , is the same as in (b). How have you transformed the graph of  $P = 4123e^{0.02t}$ ?

- (d) Write a differential equation for the rate of change of a general exponential function with base  $e$ , initial value  $A$ , and growth rate  $k$ .

- (e) Solve the differential equation  $\frac{dP}{dt} = 0.15P$  for the particular solution with an initial value of  $P = 225$ .

**L11.4** The height of a weight on a spring is given by the sinusoidal  $h = 12 \sin(2\pi t)$ . The position of the weight when it is hanging at rest on the spring,  $h = 0$ , is called the equilibrium position.

- (a) Graph this function and describe the motion of the weight. How long does it take for one complete oscillation? How far does the weight move?

- (b) Write a differential equation for the velocity at time  $t$ . Graph the velocity on the same set of axes as the position. How are the two curves related?

- (c) How can you transform the equation for  $h$  in a way that the velocity stays the same for all values of  $t$ ? What does this transformation mean in the context of the problem?

- (d) Solve the differential equation  $\frac{dh}{dt} = 12\pi \cos(4\pi t)$  for the particular solution with an initial height of  $h = 10$ . What is the range of values for  $h$  in your solution?

**L11.5** In #L11.1 through #L11.4, you have considered linear, quadratic, exponential, and sinusoidal functions. For which of these equations is the rate of change at time  $t$  constant? For which of these equations is the percent rate of change at time  $t$  constant?

**L11.6** Write a paragraph summarizing what you have learned in this lab about differential equations. Be sure to include observations about *families of functions* that have the same differential equation. You will also want to incorporate the concepts of constant change, linear change, constant percentage change, and sinusoidal change in your write-up.

**P12.1** Find  $\frac{dy}{dx}$  for each of the following functions:

(a)  $y = x\sqrt{1-x^2}$

(b)  $y = \sin(x^2)$

(c)  $y = 9 - \cos^2(90x)$

**P12.2** The core temperature of a potato that has been baking in a  $375^\circ$  oven for  $t$  minutes is modeled by the equation  $C = 375 - 300(0.96)^t$ . Find a formula for the rate of change of the temperature in degrees per minute for this potato.

**P12.3** The equations  $y = 2^x$  and  $y = \log_2 x$  represent inverse functions. Find the derivatives of each function, and explain how they are related.

**P12.4** A can with a volume of 1000 cc consists of a cylinder with a top and a bottom. The height of the can is  $h$  cm, and the radius of the top and bottom is  $r$  cm.

- (a) What is the surface area of the side? What is the surface area of the top and bottom? Write an equation for the total surface area  $A$  in terms of  $r$  and  $h$ .
- (b) Explain why  $\pi r^2 h = 1000$ . Use this equation to express surface area  $A$  as a function of  $r$  alone.
- (c) Use a derivative to find the value of  $r$  that gives the minimum value of  $A$ . Confirm your answer graphically. What is the value of  $h$  that corresponds to this  $r$ -value? How would you describe the shape of this can of minimum surface area?

**P12.5** (a) On the same system of coordinate axes, and using the same scale on both axes, make careful graphs of both  $y = \sin x$  and  $y = \sin^{-1} x$ . These graphs should intersect only at the origin. Be sure that your angles are measured in radians. Why is it necessary to restrict the domain of  $y = \sin x$  to  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ , and thus to restrict the range of  $y = \sin^{-1} x$  to  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ ?

- (b) Let  $P = (a, b)$  be a point on the graph of  $y = \sin x$ , where  $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$ , and let  $Q = (b, a)$  be the corresponding point on the graph of  $y = \sin^{-1} x$ . How are the slopes of the curves related at these two points? Explain.
- (c) Find the slopes at  $P$  and  $Q$  by using the derivatives of  $\sin x$  and  $\sin^{-1} x$ . Confirm the relationship between the slopes of corresponding points on the graphs of these inverse functions by letting  $a = 0.5, 1$ , and  $1.5$ .

**P12.6** Find the derivatives of the following functions.

(a)  $L(x) = \cos^{-1} x$  (Hint: implicit differentiation)

(b)  $y = x \sin^{-1} x$  (Hint: product rule)

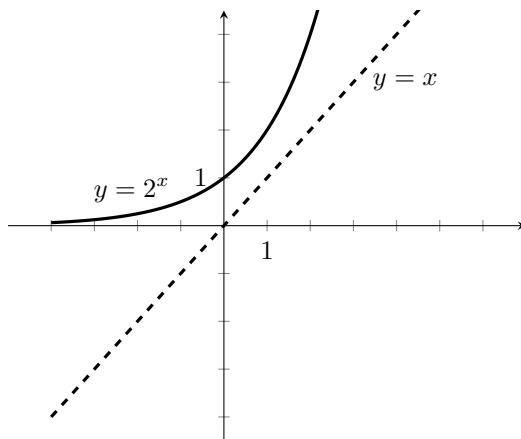
**P12.7** With the help of the Chain Rule and the Power Rule, write out the derivative of  $f(x) = (\sin x)^{1/2}$ . Then find a formula for the general example of this type, which has the form  $f(x) = (g(x))^n$ .

**P12.8** Using the Chain Rule and Power Rule, show that the derivative of  $R(x) = \frac{1}{g(x)}$  is  $R'(t) = -\frac{g'(t)}{[g(t)]^2}$ . (Hint: first write  $R$  as a power function  $R(x) = [g(x)]^{-1}$ ).

**P12.9** (Continuation) *Quotient Rule.* A function  $Q$  defined as a ratio of two functions  $Q(x) = \frac{f(x)}{g(x)}$  can be differentiated by first rewriting  $Q$  as a product  $Q(x) = f(x) \cdot \frac{1}{g(x)}$ . Show that  $Q'(x)$  can be written as:

$$\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

**P12.10** Sketch the graph of the reflection of  $y = 2^x$  over  $y = x$  on the axes below.



**P12.11** Graph the function  $f(x) = x - \sin x$  and its inverse. Although one cannot find an explicit formula for  $f^{-1}(x)$ , one can find its derivative at certain points.

- (a) Find the derivative of  $f^{-1}(x)$  at the point  $(\pi, \pi)$ .
- (b) Find the derivative of  $f^{-1}(x)$  at the point  $(\frac{\pi}{2} - 1, \frac{\pi}{2})$ .

**P12.12** Write a paragraph summarizing what you have learned in this lab about the derivative of the inverse of a function. Be sure to include the graphical relationship between a function and its inverse, techniques you have learned for finding the derivative of an inverse function, and the connection between the slope at a point on a function's graph and the slope at the corresponding point on the inverse function's graph.

**P12.13** Use implicit differentiation to find the slope of the ellipse  $x^2 + 9y^2 = 225$  at  $(9, 4)$ . In other words, find  $\frac{dy}{dx}$  without first solving for  $y$  as an explicit function of  $x$ .

**P12.14** Let  $f(x) = x^3$ ,  $g(x) = x^4$ , and  $k(x) = x^7$ . Notice that  $k(x) = f(x) \cdot g(x)$ . Is it also true that  $k'(x) = f'(x) \cdot g'(x)$ ? Explain.

**P12.15** Use a derivative to calculate the slope of the curve  $y = x^3$  at the point  $(2, 8)$ .

Now find the slope of the curve  $x = y^3$  at the point  $(8, 2)$ . How could you have found the second slope from the first slope without finding another derivative?

**P12.16** After first writing  $\tan x = \frac{\sin x}{\cos x}$ , use the Quotient Rule to show that  $\frac{d}{dx}(\tan x) = \sec^2 x$ . Recall that  $\sec x = \frac{1}{\cos x}$ .

**P12.17** Find the derivatives of the following:

(a)  $f(x) = x^3 + 3^x$       (b)  $M(\theta) = 8 \tan(3\theta)$       (c)  $H(u) = (\sin u)^3$

**P12.18** Find  $\frac{dy}{dx}$  for each of the following:

(a)  $y = \frac{x-1}{x+1}$       (b)  $y = \frac{1}{\cos x}$       (c)  $y = \sin^{-1} 2x$       (d)  $y = \frac{x}{x^2 + 1}$

**P12.19** In radians, the parametric equation  $(x, y) = (4 \tan t, 3 \sec t)$  traces both branches of a hyperbola as  $t$  varies from  $0$  to  $2\pi$ .

(a) Confirm by graphing and by algebra that this hyperbola also has equation  $16y^2 - 9x^2 = 144$ . You will need the identity  $\tan^2 t + 1 = \sec^2 t$ .

(b) Find the  $t$ -value that corresponds to the point  $\left(\frac{16}{3}, 5\right)$ .

(c) Find the components of velocity  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ , and find their values at  $\left(\frac{16}{3}, 5\right)$ .

(d) On a graph of the hyperbola, show the vector  $\left[\frac{dx}{dt}, \frac{dy}{dt}\right]$  with its tail at  $\left(\frac{16}{3}, 5\right)$  using the values of the components found in (c). Use these components of velocity to find the slope of the line tangent to the curve at  $\left(\frac{16}{3}, 5\right)$ .

**P12.20** The equation  $A(t) = 6.5 - \frac{20.4t}{t^2 + 36}$  models the pH of saliva in the mouth  $t$  minutes after eating candy. The lower the pH, the more acidic is the saliva. A pH of 7 is neutral.

(a) Using a graph of this function, what is the normal pH in the mouth? How long after eating candy does it take for the pH to return to within 0.1 of normal?

(b) After eating the candy, when is the mouth becoming more acidic? When is the acidity decreasing?

(c) At what time is there a point of inflection on the graph? What interpretation can you give to this point in the context of the problem?

**P12.21** Consider the curve, in radian mode, defined by  $(x, y) = (\cos t - 3 \sin t, 2 \cos t \sin t)$ .

(a) Graph this curve for  $0 \leq t \leq 2\pi$ . In which direction is the curve swept out as  $t$  increases?

- (b) For what points on the curve is  $\frac{dy}{dt} = 0$ ? What are the equations of the tangent lines at those points?
- (c) For what points on the curve is  $\frac{dx}{dt} = 0$ ? What are the equations of the tangent lines at those points?
- (d) Find the slope of the curve at any point in terms of the parameter  $t$ . Use this expression to find the equation of the tangent line at the point with  $t = 1$ .

**P12.22** The acceleration due to gravity on the Moon is about  $1.6 \text{ m/sec}^2$ .

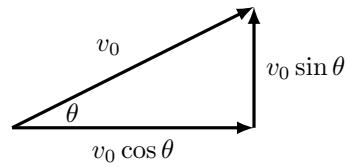
- (a) Represent the acceleration with a differential equation using the derivative of the velocity  $v$  of a falling object on the Moon.
- (b) Use the differential equation from (a) to find an equation for the velocity of an object that is propelled upward with an initial velocity of  $25 \text{ m/sec}$ .
- (c) Since  $v = \frac{dh}{dt}$ , where  $h$  is the height of the object, you can use the result of (b) to write a differential equation for the height of the object. Solve this for  $h$  as a function of  $t$  given that the initial height of the object is 3 meters.

**P12.23** Suppose the acceleration due to gravity is a constant  $g \text{ m/sec}^2$ , the initial velocity is  $v_0 \text{ m/sec}$ , and the initial height is  $h_0$  meters. As in the previous problem, write a differential equation for acceleration, then use this to find the differential equation for velocity. Finally, solve it for velocity to get an equation for  $h$  as a function of  $t$ . You should obtain this well-known formula from physics:  $h = -\frac{g}{2}t^2 + v_0t + h_0$ .

**L12.1** Open the Geogebra simulation in the file “Projectile motion.ggb”, which contains a catapult and a castle. Your catapult always launches the projectile with an initial velocity of 50 m/sec, but you can adjust the angle of the launch.

- (a) For which angle can you reach the furthest point on the plain? Ignore the castle for now.
- (b) For which angle will the projectile land on top of the closest tower? Is there another, much greater, angle that will also work? Make sure your answer references the location and height of the tower.

**L12.2** You have seen that trigonometry helps decompose a vector into its vertical and horizontal components. The diagram shows how to decompose a projectile vector,  $v_0$ , that makes an angle of  $\theta$  with the ground.



- (a) Explain how the three vectors are related. How are their lengths related?
- (b) If we assume that gravity acts with a force of  $g$ , that the projectile is launched from an initial height of  $h_0$  and additionally that there is no air resistance, we can use the component vectors to build parametric equations that describe the motion of the projectile. Convince yourself that the following parametric equations describe such a setup.

$$x(t) = (v_0 \cos \theta) t \quad y(t) = -\frac{g}{2}t^2 + (v_0 \sin \theta) t + h_0$$

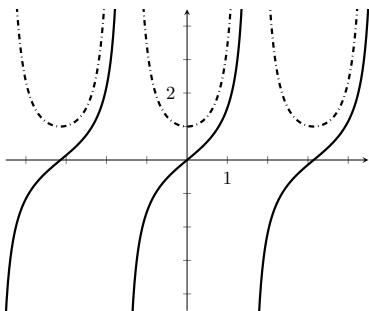
- (c) Throughout this lab we will use the acceleration due to gravity  $g = 9.81 \text{ m/sec}^2$ . Suppose we let  $\theta = 40^\circ$ , the initial velocity  $v_0 = 50 \text{ m/sec}$ , and the initial height  $h_0 = 10 \text{ m}$ . Write out the equations for  $x(t)$  and  $y(t)$ .
- (d) Graph the path of the projectile.
- (e) How long does it take for the projectile to hit the ground? Where does it hit the ground? What is the speed at the moment of impact?

**L12.3** Return to the Geogebra simulation mentioned above to answer the following.

- (a) What are the speeds with which the projectile strikes the tower in the two solutions found #L12.1(b)? Is there an advantage to one launch angle over the other?
- (b) Compare the slopes of the trajectories found in #L12.1(b) at the moment of impact on the tower. Use the slopes to find the angles of impact.

**L12.4** Compile a report summarizing your results and the techniques of calculus you have used in this lab. Be sure to discuss the main concepts of this lab and what new problems you can now solve. Among other things, you should be sure to note the distinction between the slope of the path and the speed of the projectile.

**P13.1** Two functions are shown on the graph below, one of which is the derivative of the other. Which is the derivative function? Write the equations for the two functions.



**P13.2** Find a formula for the derivative of  $y = \tan^{-1} x$  by using the technique of implicit differentiation with  $x = \tan y$ . Your final answer should be in terms of  $x$ , and without any trig functions.

**P13.3** Solve the following *antiderivative* questions. In other words, given their derivatives below, find  $W$ ,  $F$ , and  $g$ . How can you check your answers? Could there be more than one answer for each?

$$(a) \frac{dW}{dx} = 10x^9 \quad (b) F'(u) = 120e^{0.25u} \quad (c) g'(t) = -10\pi \sin 2\pi t$$

**P13.4** What are the dimensions of the largest cylinder that can be inscribed in a sphere of radius 1? You solved a problem like this in precalculus by writing the volume of the cylinder in terms of its radius alone (or its height), then finding the maximum point on the volume graph. Now solve the problem using the techniques of calculus. Verify your answer graphically.

**P13.5** (Continuation) What are the dimensions of the largest cylinder that can be inscribed in a sphere of radius  $R$ ? Your answer should be in terms of the parameter  $R$ . This parameter makes the problem difficult to solve graphically, hence techniques of calculus are useful.

**P13.6** A population grows according to the exponential function  $P(t) = 100e^{0.07t}$ .

- (a) What is the initial population? What is the population after one year? What is the average percent growth over the first year?
- (b) What is  $P'(t)$ , the instantaneous growth rate? What is the initial instantaneous growth rate? How does this compare with the average percent growth over the first year? Explain why there is a difference between these two values.
- (c) The function  $P(t)$  can also be expressed as  $100(1+r)^t$  by substituting  $1+r$  for  $e^{0.07}$ . What is the value of  $r$ ? How is this related to (a) and (b)?

**P13.7** The slope of the curve  $y = \tan x$  is equal to 2 at exactly one point  $P$  whose  $x$ -coordinate is between 0 and  $\frac{\pi}{2}$ . Let  $Q$  be the point where the curve  $y = \tan^{-1} x$  has slope  $\frac{1}{2}$ . Find coordinates for both  $P$  and  $Q$ . How are these coordinate pairs related?

**P13.8** The following expressions are derivatives of which inverse trig functions?

$$(a) \frac{1}{\sqrt{1-x^2}} \quad (b) \frac{1}{1+x^2} \quad (c) -\frac{1}{\sqrt{1-x^2}}$$

**P13.9** A potato, initially at room temperature ( $70^\circ\text{F}$ ), is placed in a hot oven ( $350^\circ\text{F}$ ) for 30 minutes. After being taken out of the oven, the potato sits undisturbed for 30 more minutes on a plate in the same room ( $70^\circ\text{F}$ ). Let  $T(t)$  be the temperature of the potato at time  $t$  during the 60-minute interval  $0 \leq t \leq 60$ . Draw plausible graphs of both  $T(t)$  and  $T'(t)$ . Other than  $T(0)$ , you are not expected to know any specific values of  $T$ .

**P13.10** Use implicit differentiation to find the slope of the curve  $e^y + x^2 = 2$  at the point  $(-1, 0)$ . Graph the curve and the tangent line at this point.

**P13.11** Alex is in the desert in a jeep, 10 km from the nearest point  $N$  on a long, straight road. On the road, the jeep can do 50 kph, but in the desert sands, it can manage only 30 kph. Alex is very thirsty and knows that there is a gas station that has ice-cold Pepsi, located at point  $P$  that is 20 km down the road from  $N$ . Alex decides to drive there by following a straight path through the desert to a point that is between  $N$  and  $P$ , and  $x$  km from  $N$ . The total time  $T(x)$  for the drive to the gas station is a function of this quantity  $x$ . Find an explicit expression for  $T(x)$ , then calculate  $T'(x)$ . Use algebra to find the minimum value of  $T(x)$  and the value of  $x$  that produces it.

**P13.12** Find the derivatives of the following functions.

$$(a) y = \tan^{-1} x^2 \quad (b) u = \frac{\cos t}{t} \quad (c) H(x) = \frac{2x-1}{x^3}$$

**P13.13** The slope of the curve  $y = \sin x$  is equal to  $\frac{1}{2}$  at exactly one point  $P$  whose  $x$ -coordinate is between 0 and  $\frac{\pi}{2}$ . Let  $Q$  be the point where the curve  $y = \sin^{-1} x$  has slope 2. Find coordinates for both  $P$  and  $Q$ .

**P13.14** Given the function  $f(x) = \frac{\ln x}{x}$ , use first and second derivatives to find the coordinates of all maximum and minimum points, and any points of inflection. Use a graph to verify your answers.

**P13.15** If the slope of the graph  $y = f(x)$  is  $m$  at the point  $(a, b)$ , then what is the slope of  $x = f(y)$  at the point  $(b, a)$ ?

**P13.16** Find the derivatives of the following functions.

$$(a) v = e^{-x} \sin x \quad (b) g(t) = \frac{t}{1+t} \quad (c) y = \ln \sin x$$

**P13.17** In the Assembly Hall one day, Tyler spends some time trying to figure which row gives the best view of the screen, which is 18 feet tall with its bottom edge 6 feet above eye level. Tyler finds that sitting 36 feet from the plane of the screen is not satisfactory, for the screen is far away and subtends only a 24.2-degree viewing angle. (Verify this.) Sitting 4 feet from the screen is just as bad because the screen subtends the same 24.2-degree angle from this position. (Verify this also.) Find the optimal viewing distance – the distance that makes the screen seem the largest – and the angular size of the screen at this distance. (The angle subtended by the screen has its vertex at the eye of the viewer and angle sides formed by the top and bottom of the screen.)

**P13.18** Find the value of  $\lim_{h \rightarrow 0} \frac{\ln(a+h) - \ln(a)}{h}$  without using a calculator.

**P13.19** The differential equation  $\frac{dy}{dt} = -\frac{1}{3}y^2$  is given.

- (a) Show that  $y = \frac{3}{t}$  is a solution, but ...
- (b) ... that neither  $y = \frac{3}{t} + 2$  nor  $y = \frac{2}{t}$  is a solution.
- (c) Find another solution to the differential equation.

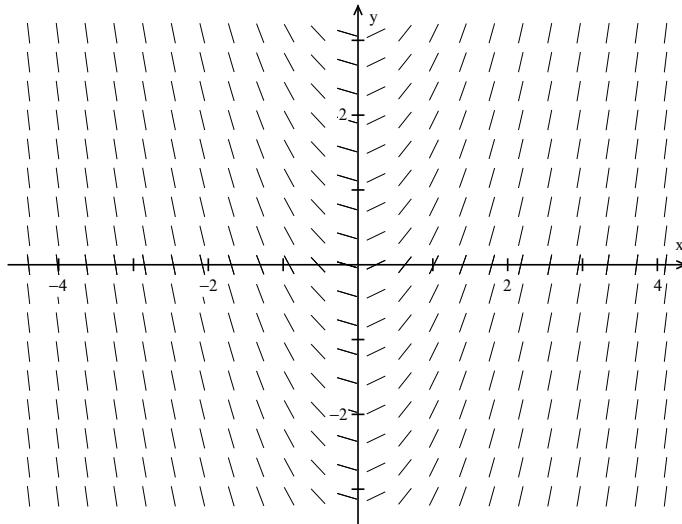
**P13.20** After investigating the derivative of  $x^n$  when  $n$  is a positive integer and determining that  $\frac{d}{dx}(x^n) = nx^{n-1}$ , we extended this Power Rule to all powers. We now have the tools to examine this extension more critically.

- (a) (Negative powers) If the power is a negative integer, we can rewrite the function as  $y = \frac{1}{x^n}$ , where  $n$  is a positive integer. Use the Quotient Rule to find  $\frac{dy}{dx}$ , then use the Power Rule to find the derivative of  $x^{-n}$ . Recall that  $x^{-n} = \frac{1}{x^n}$ . Your answers should agree.
- (b) (Rational powers) Suppose the power is a rational number, so that  $y = x^{a/b}$ , where  $a$  and  $b$  are positive integers. We can rewrite the equation as  $y^b = x^a$ , where  $y$  is now an implicit function of  $x$ . Use implicit differentiation to solve for  $\frac{dy}{dx}$ , then use the Power Rule to find the derivative of  $x^{a/b}$ . Your answers should agree.

**P13.21** Identify each of the following rules for derivatives.

(a)  $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$     (b)  $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$     (c)  $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

The figure below shows the *slope field* for the differential equation  $\frac{dy}{dx} = 2x$ . Each point is assigned a slope equal to the value of the derivative at that point. That slope is represented as a line segment. For example, the point  $(1, 1)$  has a segment with slope 2 because  $\frac{dy}{dx}$  is  $2(1) = 2$  at  $x = 1$ , and similarly the point  $(-2, -2)$  has a segment with slope  $-4$ . Many other points are represented in the slope field, which allows you to visualize the behavior of the solutions to the differential equation. Since the derivative depends only on  $x$ , all of the slopes along vertical lines are the same. A solution curve can be drawn on this graph by choosing a point on the curve and then sketching the curve by following the direction of the slope field, which is why it is also called a *direction field*.



**L13.1** Slope fields can be drawn using Geogebra with the command  $SlopeField \left[ \frac{dy}{dx}, 20 \right]$ .

The number 20 here represents the number of line segments shown, this can be adjusted up or down to get a better picture.

- Produce the slope field for  $\frac{dy}{dx} = 2x$  by entering  $SlopeField [2x, 20]$ . Observe how the slope field changes as you change the graphing window.
- Discuss the solution curve that contains the point  $(0, 0)$ . Use Geogebra to draw the solution curve with the command  $SolveODE [2x, (0, 0)]$ . Is it what you expected?
- Discuss the solution curve that contains the point  $(-2, -2)$  and draw the curve with Geogebra.
- Add 2 more solution curves to your graphing window. What characteristics are shared by all of the solution curves? Do all the curves have the same general behavior?
- Use your knowledge of derivatives to write a function  $y = f(x)$  for the general solution to the differential equation  $\frac{dy}{dx} = 2x$ . How does the general solution compare with the curves drawn in the slope field?

**L13.2** Consider the differential equation  $\frac{dy}{dx} = y$ .

- (a) Create a slope field for this differential equation. Discuss the appearance of the slope field. In what direction can you move without changing the slopes? Why?
- (b) Draw two different solution curves for the points  $(0, 1)$  and  $(0, -2)$ . How do the behaviors of these curves differ? Is there any other type of behavior that a solution curve can have?
- (c) Solve the differential equation for an explicit function by using your knowledge of derivatives. How does your symbolic solution give you insight into the behavior of solution curves drawn in the slope field?

**L13.3** Consider the differential equation  $\frac{dy}{dx} = \sin(x)$ . Create a graph of the slope field, draw some solution curves, and discuss the different behaviors that the solution curves can possess. You should also solve the differential equation and compare your symbolic solution with your observations about the slope field.

**L13.4** One version of Newton's law of cooling states that a hot liquid will cool at a rate that is proportional to the difference between the temperature of the liquid and the ambient temperature of the surroundings (such as room temperature, or the temperature inside a refrigerator). This description of change can be written as the differential equation  $\frac{dT}{dt} = k(T - A)$ , where  $T$  is the temperature of the liquid,  $t$  is time,  $k$  is a constant of proportionality, and  $A$  is the ambient temperature.

- (a) Let  $A = 68^\circ$  and  $k = -0.05$ . Create a graph of the slope field. Be sure that your window spans the horizontal line  $y = 68$ . Why?
- (b) Draw the solution curve with an initial temperature (at time  $t = 0$ ) of  $200^\circ$ , then compare this curve with the solution curve for an initial temperature of  $0^\circ$ . If necessary change your window. What happens if the initial temperature is  $68^\circ$ ?
- (c) This differential equation is not one that we can solve symbolically at this time; however, you can learn a lot about the solution through the visualization provided by the slope field. Describe the three qualitatively different solutions and how they depend upon the initial temperature.

**L13.5** Summarize your results in a lab report, being sure to explain how a slope field helps you to visualize the qualitatively different behaviors of the solutions to a differential equation.

**P14.1** There are many functions for which  $f(3) = 4$  and  $f'(3) = -2$ . A quadratic example is  $f(x) = x^2 - 8x + 19$ . Find an example of such an exponential function  $f(x) = a \cdot b^x$ .

**P14.2** Use implicit differentiation to find the slope of  $9x^2 + y^2 = 225$  at  $(4, 9)$ . In other words, find  $\frac{dy}{dx}$  without first solving for  $y$  as an explicit function of  $x$ .

**P14.3** Given that  $P'(t) = k \cdot P(t)$  and  $P(0) = 2718$ , find  $P(t)$ .

**P14.4** Find  $\frac{dy}{dx}$  for each of the following curves.

$$(a) \ y = x \cdot 2^x \quad (b) \ y = \sin\left(\frac{1}{x}\right) \quad (c) \ y = \frac{2x}{x^2 - 1} \quad (d) \ y = \frac{e^x - e^{-x}}{2}.$$

**P14.5** For the function  $y = \ln(1 - x)$ , explain why  $y$  is virtually the same as  $-x$  when  $x$  is close to zero.

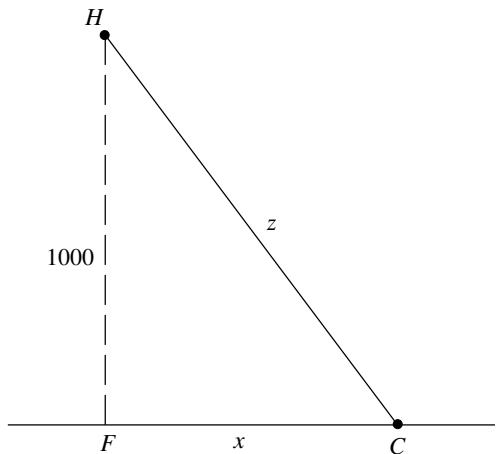
**P14.6** Give an example of a function for each of the following scenarios. Compare the graphs of the two functions.

- (a) The instantaneous growth rate is a constant 12%.
- (b) The annual growth rate is a constant 12%.

**P14.7** Solve the following antiderivative questions. In other words, find  $F$ ,  $g$ , and  $S$ .

$$(a) \ F'(x) = 6x^5 \quad (b) \ g'(t) = 10 \cos(5t) \quad (c) \ \frac{dS}{du} = \frac{1}{2}e^u + \frac{1}{2}e^{-u}$$

**P14.8** A police helicopter is hovering 1000 feet above a highway, using radar to check the speed of a car  $C$  below. The radar shows that the distance  $HC$  is 1250 feet and increasing at 66 ft/sec. Is the car exceeding the speed limit of 65 mph? To figure out the speed of the car, let  $x = FC$ , where  $F$  is the point on the highway that is directly beneath  $H$ , and let  $z = HC$ . Notice that  $x$  and  $z$  are both functions  $t$  and that  $z(t)^2 = 1000^2 + x(t)^2$ . Differentiate this equation with respect to  $t$ . The new equation involves  $\frac{dx}{dt}$  and  $\frac{dz}{dt}$ , as well as  $x$  and  $z$ . The information from the radar means that  $\frac{dz}{dt} = 66$  when  $z = 1250$ . Use this data to calculate the value  $\frac{dx}{dt}$ .

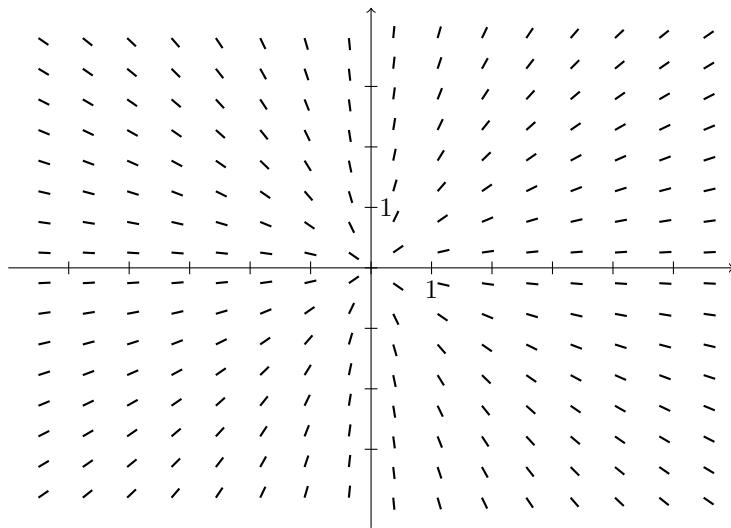


**P14.9** Find a function whose graph  $y = f(x)$ : (a) has negative slope, which increases as  $x$  increases; (b) has positive slope, which decreases as  $x$  increases.

**P14.10** If  $0 < f''(a)$ , then the graph of  $y = f(x)$  is said to be concave up (or have *positive curvature*) at the point  $(a, f(a))$ . Explain this terminology based on what is happening with the first derivative at  $(a, f(a))$ . If  $f''(a) < 0$ , then the graph of  $y = f(x)$  is said to be concave down (or have *negative curvature*) at the point  $(a, f(a))$ . Explain this terminology as well.

**P14.11** Examine the following slopefield for a certain differential equation.

- (a) Sketch solution curves through the following points:  $(1, 1)$ ,  $(-4, 2)$ ,  $(3, -5)$ .
- (b) What do all the solutions of this differential equation seem to have in common?



**P14.12** Consider the function  $f(x) = \begin{cases} 4 - 2x, & \text{for } x < 1 \\ ax^2 + bx, & \text{for } x \geq 1 \end{cases}$ . What values of  $a$  and  $b$  guarantee that  $f$  is both continuous and differentiable?

**P14.13** Find the first and second derivative of each of the following functions.

- (a)  $f(z) = z \ln(z)$
- (b)  $f(u) = \tan^{-1} u$
- (c)  $f(t) = e^{-t^2}$

**P14.14** A function  $g$  is continuous on the interval  $[-2, 4]$ , with  $g(-2) = 5$  and  $g(4) = 2$ . Its derivatives have properties summarized in the table.

$x$	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x < 4$
$g'(x)$	positive	undefined	negative	0	negative
$g''(x)$	positive	undefined	positive	0	negative

- (a) Find the  $x$ -coordinates of all globally extreme points for  $g$ . Justify your answer.
- (b) Find the  $x$ -coordinates of all inflection points for  $g$ . Justify your answer.
- (c) Make a sketch of  $y = g(x)$  that is consistent with the given information.

**P14.15** The volume of a cube is increasing at 120 cc per minute at the instant when its edges are 8 cm long. At what rate are the edge lengths increasing at that instant? Consider differentiating both sides of the equation  $V = x^3$ , with respect to  $t$ , to get  $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$ , and then use the given values.

**P14.16** A model for the net result of the attractive and repulsive forces that exist between two atoms is given by  $F(x) = \frac{c}{x^2} + \frac{d}{x^3}$  where  $x$  is the separation distance between the atoms, and  $c$  and  $d$  are constants.

- (a) Determine values for  $c$  and  $d$  which give a maximum attraction of 1 at  $x = 1$ .
- (b) Graph the force function.
- (c) Determine where the graph has an inflection point. How is the force changing at the inflection point?

**P14.17** A particle moves along a number line according to  $x = t^4 - 4t^3 + 3$  during the time interval  $-1 \leq t \leq 4$ .

- (a) Calculate the velocity function,  $\frac{dx}{dt}$ , and the acceleration function,  $\frac{d^2x}{dt^2}$ .
- (b) At what times is the particle instantaneously at rest? Where does this happen?
- (c) At what times is the particle's position  $x$  increasing? When is  $x$  decreasing?
- (d) At what times is the acceleration zero? What does this signify?
- (e) What is the complete range of positions of the particle?
- (f) What is the complete range of velocities of the particle?

**P14.18** A rumor spreads among the students at PEA according to the differential equation  $\frac{dR}{dt} = 0.08R(1 - R)$ , where  $R$  is the proportion of the student population who have heard the rumor.

- (a) Obtain a slopefield and a solution with initial value  $R(0) = 0.03$ . If  $t$  is in minutes, how long does it take for 99% of the students to hear the rumor?
- (b) Explain the shape of the solution curve and what in the differential equation causes this characteristic shape of the *logistic curve*.
- (c) Restricting  $R$  to the range of 0 to 1, how many different types of solution curves can you perceive by examining the slopefield?
- (d) Allowing  $R$  to go outside the range of 0 to 1, what other families of solution curves can you perceive by examining the slopefield?

Given the derivative  $\frac{dy}{dx}$ , we have had a lot of practice finding the linear approximation for  $y$ . We have noticed, however, that the linear approximation is generally less accurate as you increase the distance from the center of the approximation. One method for achieving a more accurate approximation is a recursive technique known as Euler's method.

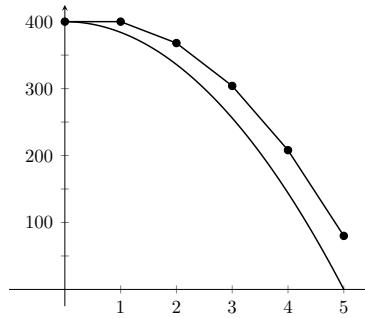
**L14.1** The velocity of an object falling from the top of a tall building is  $y' = -32t$ . The height of the building is 400 ft and the initial velocity is 0 ft/sec. We thus have the initial point  $(t_0, y_0) = (0, 400)$ . The linear approximation centered at this point is  $y = y'(0) \cdot (t - 0) + 400$ , where  $y'(0)$  is the derivative at the point  $(0, 400)$  and thus the slope of the linear approximation at that point.

- (a) Confirm that the value of the linear approximation one second later, where  $t = 1$ , is  $y = 400$ .
- (b) If we want to approximate the function at  $t = 2$ , we could substitute  $t = 2$  into the linear approximation formula. Instead, however, we can use a two-step approximation method by making a new line through the point  $(1, 400)$  that is parallel to the tangent to the curve at the point with  $t = 1$ . This new line has equation  $y = y'(1) \cdot (t - 1) + 400$ . Confirm that the approximation from this line at  $t = 2$  is  $y = 368$ .
- (c) Continue the process of recursively taking the result after each step and substituting this point back into the linear approximation formula. Fill in the remaining  $y$ -values in the table below.

$t$	0	1	2	3	4	5
$y$	400	400	368			

**L14.2** How do the approximation values in the table compare to the actual  $y$ -values? We can find out by solving the differential equation  $y' = -32t$  for the curve that contains the point  $(0, 400)$ .

- (a) Explain why the general solution is  $y = -16t^2 + C$ , and the particular solution is  $y = -16t^2 + 400$ .
- (b) Compare the approximation with the actual value at  $t = 5$ . Explain how all this relates to the graph below.
- (c) In the graph shown, the stepsize for each line segment was  $\Delta t = 1$ . How would the graph above look if you changed the stepsize to  $\Delta t = 0.5$ ? Use  $\Delta t = 0.5$  to find the Euler's method approximation using the spreadsheet option in Geogebra, and examine the accuracy of the approximation after 10 steps, at  $t = 5$ .



**L14.3** A population is changing at a rate given by the differential equation  $\frac{dP}{dt} = 0.05P$ . The initial population at time  $t = 0$  is  $P_0 = 3140$ .

- (a) Find the Euler's method approximation for the population at  $t = 10$  using a stepsize of  $\Delta t = 0.5$ .
- (b) Solve the differential equation  $\frac{dP}{dt} = 0.05P$  for the particular function  $P(t)$  with  $P_0 = 3140$ . Compare the Euler's method approximation with the actual value of  $P$  at  $t = 10$ .
- (c) Recompute the Euler's method approximation for the population at  $t = 10$  using a stepsize of  $\Delta t = 0.1$ . Graph on the same set of axes the function  $P(t)$  and the sequence of points generated by Euler's method. Discuss the graph.

**L14.4** To make soft drinks a large 100-gallon vat is used to hold the inflow and outflow of sugar water. Measured in tablespoons per minute, the change in the amount of sugar can be modeled by  $\frac{dS}{dt} = \frac{2000 - 3S}{100}$ , measured in tablespoons per minute.

- (a) Compute the Euler's method approximation using a stepsize of  $\Delta t = 1$  for the amount of sugar in the vat after 20 minutes. The initial amount of sugar in the vat is  $S(0) = 350$ .
- (b) Graph the points found in #L14.4a. What does this graph represent?
- (c) What happens to the sugar concentration in the long run? You may want to compute and graph the Euler's method approximation out to an hour or more.
- (d) The differential equation given in this part is not one that we can reasonably expect to solve at this time. How do we know that the Euler's method approximation gives a fairly accurate picture of the behavior of the solution? Is our choice of  $\Delta t = 1$  a small enough stepsize?
- (e) What do you notice if you superimpose a slopefield on the graph of the Euler's method points?

**L14.5** Write a summary explaining how Euler's method works and how it can be used to generate a numerical approximation for the solution to a differential equation. Be sure to discuss the relative certainty of your estimation with regard to stepsize. You will want to include in your summary the general recursive pattern to Euler's method, what the points generated by it represent, and how all this relates to the prior lab on slopefields.

**P15.1** The function  $f(t) = (1 + r)^t$  can be rewritten as  $f(t) = e^{\alpha t}$  for some number  $\alpha$ .

- (a) Letting  $r = 0.05$ , calculate  $\alpha$ . Given that  $t$  is in years, what do  $r$  and  $\alpha$  represent?
- (b) Using this value of  $\alpha$ , sketch the graphs of  $y = e^{\alpha t}$  and  $y = 1 + \alpha t$  on the same set of axes. How are these graphs related?
- (c) The linear model is attractively simple to work with, but the exponential model slowly diverges from it. What is the largest interval of  $t$ -values for which the linear  $y$ -value is within 99% of the exponential  $y$ -value? In other words, how long does it take for the linear model to diverge more than 1% from the exponential model?

**P15.2** At noon, a blue sports car was 15 miles south of an intersection, heading due north along a straight highway at 40 mph. Also at noon, a red sports car was 20 miles west of the same intersection, heading due east along a straight highway at 80 mph.

- (a) At noon, when the cars were 25 miles apart, at what rate was their separation decreasing?
- (b) At 1 pm, when the cars were 65 miles apart, at what rate was their separation increasing?
- (c) At what time of day were the cars closest, and how far apart were they?

**P15.3** A population is changing at a rate given by the differential equation  $\frac{dP}{dt} = 0.12P$ . The initial population at  $t = 0$  is  $P_0 = 144$ .

- (a) Find the Euler's Method approximation for the population at  $t = 6$  using stepsize of  $\Delta t = .02$ .
- (b) Solve the differential equation  $\frac{dP}{dt} = 0.12P$  for the particular function  $P(t)$  with  $P_0 = 144$ . Compare the Euler's Method approximation with the actual value of  $P$  at  $t = 6$ .

**P15.4** *Separation of variables.* The circle  $x^2 + y^2 = 25$  is a solution curve for the differential equation  $\frac{dy}{dx} = -\frac{x}{y}$ .

- (a) Confirm that this is true by differentiating  $x^2 + y^2 = 25$ .
- (b) You can solve the differential equation  $\frac{dy}{dx} = -\frac{x}{y}$  by putting all the  $x$ 's on one side of the equation and all the  $y$ 's on the other side. The resulting equation  $y \frac{dy}{dx} = -x$  can be solved by guessing an *antiderivative* function for each side of the equation. Think about the Chain Rule in reverse to find a function with derivative  $y \frac{dy}{dx}$ . What function has a derivative  $-x$ ?

- (c) Explain why the solution curves  $x^2 + y^2 = C$ , where  $C$  is a constant, are all solutions for the differential equation in this problem. Verify that the slope field reveals the solution curves.

**P15.5** (Continuation) Consider the differential equation  $\frac{dy}{dx} = \frac{x}{y}$ .

- (a) Use a slope field to make a conjecture about the solutions to this differential equation.
- (b) Solve the differential equation by the method of separation of variables. In other words, multiply both sides by  $y$  and then find the antiderivative functions for each side of the equation. To find the general solution, you will want to include a constant in each antiderivative. What can you do when the two constants are added or subtracted?
- (c) Give a general equation for the solution curves for this differential equation.

**P15.6** A running track is made up of two parallel straightaways attached to semi-circles on each end. The straightaways have length  $l$  and the semi-circles have radius  $r$ .

- (a) Find an expression for the total length of the track in terms of  $l$  and  $r$ . 
- (b) Find an expression for the total area enclosed by the track.
- (c) If the length of the track is 400 meters, what dimensions will make it so the track encloses the largest possible area?

**P15.7** Write (but do not try to solve) differential equations given the following descriptions of change.

- (a) The temperature  $T$  of a cup of water placed in a freezer set to 20 degrees decreases at a rate that is proportional to the difference between  $T$  and 20.
- (b) The velocity of a car going downhill increases linearly as a function of time.
- (c) The rate at which the thickness  $y$  of ice increases is inversely proportional to the thickness of the ice.
- (d) A population grows at a continuous annual rate of 2%, and immigrants arrive continuously at a rate of 150 per year.

**P15.8** For each equation, find  $\frac{dy}{dx}$ .

(a)  $y = \ln(10 - x)$       (b)  $\ln y = 3 \ln x$       (c)  $2\sqrt{y} = 8 - 0.4x$

**P15.9** The parabolic arc  $y = 7 - x - x^2$  defined on the interval  $-3.5 \leq x \leq 1$  joins smoothly at  $(1, 5)$  to another parabolic arc  $y = a(x - c)^2$  for the interval  $1 \leq x \leq 6$ . Find the values of  $a$  and  $c$ . Verify your answer graphically and with calculus techniques.

**P15.10** A population is constrained by an upper limit 100 and is changing at the rate  $\frac{dP}{dt} = 0.04(100 - P)$ . The initial population at  $t = 0$  is  $P_0 = 8$ .

- (a) Use Euler's Method to generate values for the population from  $t = 0$  to  $t = 100$  using a stepsize  $\Delta t = 1$ .
- (b) Graph the ordered pairs  $(t, P)$  generated with Euler's Method. Is this what you expected for the shape of the graph?
- (c) For an extra challenge, solve the original differential equation and graph the solution on the same set of axes as the ordered pairs from Euler's Method.

**P15.11** You have solved some separable differential equations by writing them in the form  $f(y)\frac{dy}{dx} = g(x)$ . Show that  $\frac{dy}{dx} = 2xy^2$  is also of this type by writing the equation in this form. Find the solution curve that goes through the point  $(5, 1)$ . Confirm your answer with Geogebra using slopefields and SolveODE.

**P15.12** Boyle's Law states that for a gas at constant temperature, the pressure times the volume of the gas is constant, or  $PV = c$ .

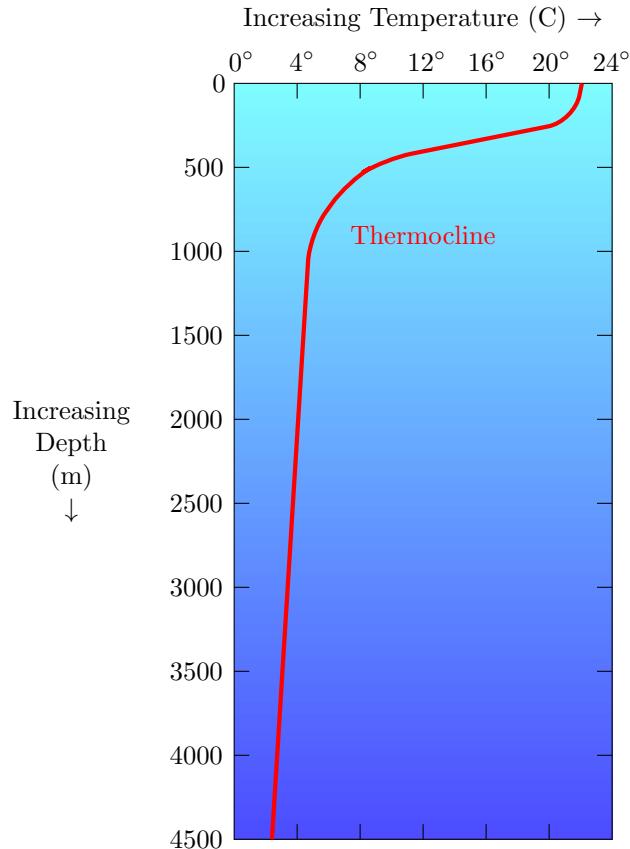
- (a) Turn this into a rate problem by taking derivatives with respect to time. In other words, apply the operator  $\frac{d}{dt}$  to both sides of the equation.
- (b) If the volume is decreasing at a rate of  $10 \text{ cm}^3/\text{sec}$ , how fast is the pressure increasing when the pressure is  $100 \text{ N/cm}^2$  and the volume is  $20 \text{ cm}^3$ ?

**P15.13** *Constrained population growth.* A population  $P$  (in millions) grows in proportion to how much more it has left to grow before it reaches its upper limit  $L$ .

- (a) For this population, complete the differential equation:  $\frac{dP}{dt} = k(\quad)$ .
- (b) Let  $k = 0.05$  and  $L = 100$  (million). Obtain a slopefield for this differential equation and confirm that the solution curves level off at 100.
- (c) Solve this differential equation for the curve that contains the point  $(0, 50)$ . You can separate variables by dividing both sides of the differential equation by the factor that you placed in the blank.

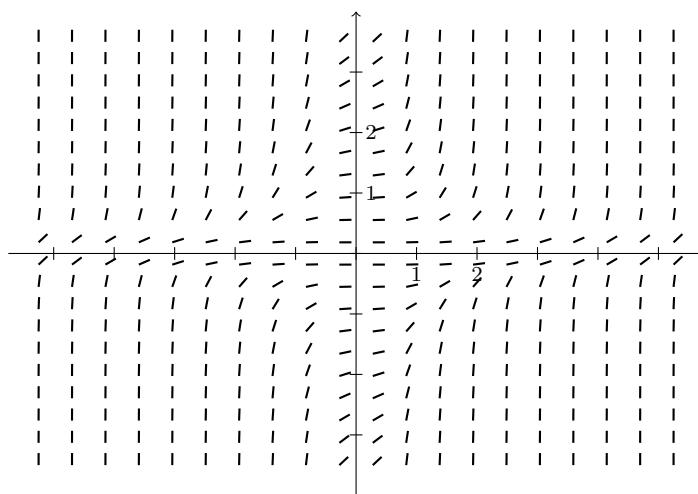
**P15.14** Solve the differential equation  $\frac{dy}{dx} = \frac{y}{x}$  by separation of variables. You will need to divide both sides by  $y$ . Find the particular solution that contains  $(4, 2)$ .

**P15.15** The graphic displays the temperature of the ocean water versus depth. Describe, in words and with a graph, the rate of change of the temperature as a function of depth.



**P15.16** Examine the slopefield shown below.

- Sketch solution curves through the following points:  $(-4, -1)$ ,  $(2, 2)$ ,  $(4, -2)$ .
- What general behaviors of the solution curves can you deduce from the slopefield?



In Laboratory 12 we simplified differential equations for a falling object by ignoring air resistance. We begin with constant acceleration due to gravity, which is represented by the differential equation

$$\frac{dv}{dt} = -g.$$

Confirm and explain the equations for velocity and height:

$$v(t) = -gt + v_0, \quad h(t) = -\frac{g}{2}t^2 + v_0t + h_0.$$

To model the descent of a parachutist, however, we need air resistance in our model. The force of air resistance on a parachute is in the upward direction, which opposes the downward force of gravity. Air resistance increases as speed increases (up to a point) and in this lab we will assume it is proportional to the velocity squared. The equation for acceleration is thus

$$\frac{dv}{dt} = -g + kv^2.$$

We now have a differential equation that we cannot solve symbolically; however, we can use Euler's method to generate a numerical solution. The constant of proportionality is  $k = \frac{C\rho A}{2m}$ , where  $C$  is the coefficient of air resistance,  $\rho$  is the density of air,  $A$  is the surface area of the skydiver before opening the parachute or the parachute after opening, and  $m$  is the mass of the skydiver. Assume that the skydiver jumps from a plane so the initial downward velocity is 0, and the constants have the values  $g = 9.8 \text{ m/sec}^2$ ,  $C = 0.57$ ,  $\rho = 1.3 \text{ kg/m}^3$ ,  $m = 75 \text{ kg}$ , and  $A = 0.7 \text{ m}^2$ .

**L15.1** Use Euler's method with a stepsize of  $\Delta t = 0.5$  to generate approximate values for the velocity during the first 25 seconds of freefall. What are the units of velocity?

**L15.2** Discuss the shape of the graph of velocity vs. time. What is the *terminal velocity* of the skydiver in freefall? What is this velocity in km/hr?

**L15.3** Assume the parachute is opened after reaching terminal velocity, and that the surface area of the parachute is  $30 \text{ m}^2$ . Generate a new set of velocities using the freefall terminal velocity as your new initial velocity and a smaller stepsize  $\Delta t = 0.1$ . Why is it necessary to use a smaller  $\Delta t$ ? (Hint: What happens if you use  $\Delta t = 0.5$ ?) What is the new terminal velocity of the skydiver? Express your answer in km/hr. What is the landing speed for the skydiver? Is this reasonably “safe”?

**L15.4** Now try to put all of the velocities together in one graph by opening the parachute after 20 seconds. In your worksheet, you can start with  $A = 0.7$  and  $\Delta t = 0.5$  in your formulas and change to  $A = 30$  and  $\Delta t = 0.1$  after 20 seconds.

**L15.5** You can now use the sequence of velocities to approximate the height of the skydiver at any time. Since  $\frac{dh}{dt} = v$ , confirm that you can use Euler's method to approximate the heights with the recursive equation  $h_n = h_{n-1} + v_{n-1} \cdot \Delta t$ . How is this work an example of numerical integration?

**L15.6** Generate a graph of the height vs. time of a skydiver jumping out of a plane at a height of 1000 meters and opening the parachute after 20 seconds. Extend the calculations until the skydiver lands on the ground. Explain the shape of the height vs. time graph in the context of this problem.

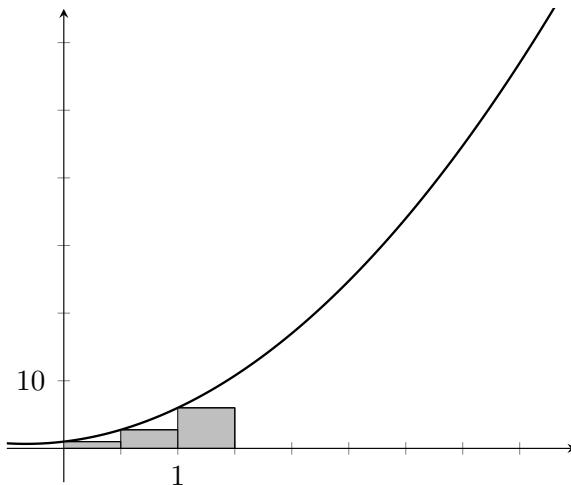
**L15.7** Write a lab report that includes an analysis of your work in the context of the problem, an explanation of how you utilized Euler's method, the recursive formulas you used, what you learned about Euler's method, and relevant graphs. As always, extensions, surprises, and "a-ha" moments are welcome.

*The Fundamental Theorem of Calculus.* Complete #P16.1–#P16.11 as a separate activity in class and for homework.

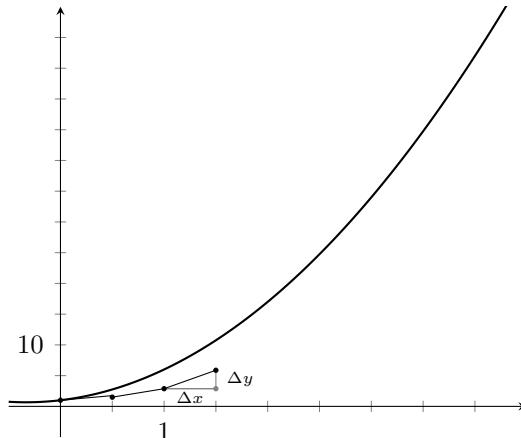
- P16.1** Use Euler's method to approximate the function  $y = f(x)$  that has derivative  $\frac{dy}{dx} = 3x^2 + 2x + 1$  and initial value  $(0, 1)$ . Use a stepsize of  $\Delta x = 0.5$ , and generate a sequence of values from  $x_0 = 0$  to  $x_n = 4$ . Fill in the table of values below.

$x_{n-1}$	$y_{n-1}$	$\frac{dy}{dx}\Big _{n-1}$	$y_n = \frac{dy}{dx}\Big _{n-1} \cdot \Delta x + y_{n-1}$
0	1	1	1.5
0.5	1.5	2.75	2.875
1.0	2.875		
1.5			
2.0			
2.5			
3.0			
3.5			
4.0			

- P16.2** A graph of  $\frac{dy}{dx}$  is shown below. A sequence of rectangles with a base on the  $x$ -axis, a width of  $\Delta x$ , and a corner on the  $\frac{dy}{dx}$  curve is shown. Draw the additional rectangles up to where  $x = 4$ . Which column in the table above gives the heights of these rectangles? What product gives the area of each rectangle?



**P16.3** On the set of axes below, plot the points from #P16.1. The first few points are shown. The result is an approximation for the solution curve for  $\frac{dy}{dx} = 3x^2 + 2x + 1$ . What quantities in the graph from #P16.2 correspond to the  $\Delta y$  values below? A representative  $\Delta y$  is shown. What is the equation for the solution curve shown? How can you adjust Euler's method to make the approximation more accurate?



**P16.4** We can generalize the previous work by letting the initial value be at the point with  $x_0 = a$  and the last value be at  $x_n = b$ . Complete the expressions for the first two y-value approximations:

$$y_1 = y_0 + \left. \frac{dy}{dx} \right|_0 \cdot \Delta x \quad \text{and} \quad y_2 = y_1 + \underline{\hspace{2cm}}.$$

**P16.5** Show that  $y_2$  can be rewritten as:

$$y_2 = y_0 + \left. \frac{dy}{dx} \right|_0 \cdot \Delta x + \left. \frac{dy}{dx} \right|_1 \cdot \Delta x.$$

Continue the substitution pattern in the sequence of equations below.

$$y_3 = y_0 + \left. \frac{dy}{dx} \right|_0 \cdot \Delta x + \left. \frac{dy}{dx} \right|_1 \cdot \Delta x + \underline{\hspace{2cm}},$$

$$y_4 = y_0 + \left. \frac{dy}{dx} \right|_0 \cdot \Delta x + \left. \frac{dy}{dx} \right|_1 \cdot \Delta x + \underline{\hspace{2cm}} + \underline{\hspace{2cm}},$$

$$y_5 = y_0 + \underline{\hspace{2cm}},$$

⋮

$$y_n = y_0 + \left. \frac{dy}{dx} \right|_0 \cdot \Delta x + \left. \frac{dy}{dx} \right|_1 \cdot \Delta x + \cdots + \left. \frac{dy}{dx} \right|_{n-1} \cdot \Delta x.$$

Relate the last equation to the graph in #P16.3.

**P16.6** Verify that:  $y_n - y_0 = \sum_{k=0}^{n-1} \frac{dy}{dx} \Big|_k \cdot \Delta x$

**P16.7** To improve the Euler's method approximation above, we should make  $\Delta x$  smaller. What happens to  $n$  as  $\Delta x$  is reduced? What is the limit of the Euler's method approximation as  $\Delta x$  approaches 0?

**P16.8** If we start with a differential equation  $\frac{dy}{dx} = f(x)$ , and we call the solution function  $F(x)$  (so that  $F(x)$  is an antiderivative of  $f(x)$ ), then  $y_0 = F(a)$  and  $y_n \approx F(b)$ . Why? Explain how we get the following equation:

$$F(b) - F(a) = \lim_{\Delta x \rightarrow 0} \sum_{k=0}^{n-1} \frac{dy}{dx} \Big|_k \cdot \Delta x$$

The expression on the right side of the equation above defines the definite integral

$$\int_a^b f(x) dx = \lim_{\Delta x \rightarrow 0} \sum_{k=0}^{n-1} f(x_k) \cdot \Delta x$$

where  $f(x)$  has been substituted for  $\frac{dy}{dx}$ .

**P16.9** *The Fundamental Theorem of Calculus.* The two equations in #P16.8 can be combined to form the equation

$$\int_a^b f(x) dx = F(b) - F(a)$$

What is the relationship between  $f(x)$  and  $F(x)$ ?

**P16.10** Evaluate the integral

$$\int_0^4 (3x^2 + 2x + 1) dx,$$

which is the integral of the function that we started with at the beginning of this problem set. How does the numerical value of this integral relate to antiderivatives? How is it connected to Euler's method? What is the relationship between this integral and the graphs shown in #P16.2 and #P16.3?

**P16.11** Write in symbols, with explanation, the Fundamental Theorem of Calculus. How does this theorem link derivatives and antiderivatives? (These two "halves" of calculus are often called *Differential Calculus* and *Integral Calculus*.)

**P16.12** Solve the following antiderivative questions and thus find  $F$ ,  $h$ , and  $A$ .

(a)  $\frac{dF}{du} = \frac{1}{u^2}$       (b)  $h'(t) = \sin 2t$       (c)  $\frac{dA}{dx} = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

**P16.13** Find  $\frac{dy}{dx}$  for the function defined implicitly by  $y = xy^2 + 1$ .

**P16.14** Find at least two different functions  $S$  for which  $S'(x) = x^2$ .

**P16.15** Graph  $y = \cos\left(\frac{1}{x}\right)$  and find and graph  $\frac{dy}{dx}$ . How does the graph of the derivative explain the behavior of the graph of  $y$ ?

**P16.16** Find antiderivatives for the following:

(a)  $\frac{1}{y} \frac{dy}{dx}$       (b)  $(\sin x)^2 \cos x$       (c)  $e^u \cdot u'$

**P16.17** Find the derivatives of the reciprocal functions of  $\cos$ ,  $\sin$ , and  $\tan$ , namely the  $\sec$  (secant),  $\csc$  (cosecant), and  $\cot$  (cotangent) functions.

**P16.18** What is the derivative of: (a)  $x^\pi$ , (b)  $\pi^x$ , (c)  $\pi^\pi$ ?

**P16.19** Find functions with the following derivatives:

(a)  $10x^9$       (b)  $x^{99}$       (c)  $\frac{1}{2\sqrt{x}}$       (d)  $\sqrt{x}$

**P16.20** (Continuation) Given that  $n$  is a constant, find the antiderivatives.

(a)  $nx^{n-1}$       (b)  $x^n$

**P16.21** Graph  $y = \sin x^2$  and find and graph  $\frac{dy}{dx}$ . How does the graph of the derivative explain the behavior of the graph of  $y$ ?



**P17.1** Find the values of the following integrals by using antiderivatives

$$(a) \int_0^3 (x^2 + 2x + 1) dx \quad (b) \int_{-1}^1 e^x dx \quad (c) \int_1^{10} \frac{1}{x} dx$$

**P17.2** Some of the particular solutions of the differential equation  $\frac{dL}{dt} = k(M - L)$  are called *learning curves*, where  $L$  is the amount of learning of a body of facts or of a particular skill.

- (a) Finish the following statement: The rate of learning is proportional to ...
- (b) In a learning curve, the initial value of  $L$  is less than  $M$ . What is the shape of the curve? What is the significance of the parameter  $M$ ?

**P17.3** Determine antiderivatives of the following:

$$(a) \frac{1}{3x} \quad (b) \frac{1}{5+x} \quad (c) \frac{1}{10-0.2x}$$

**P17.4** The following derivatives relate to the inverse trig functions  $\sin^{-1} x$ ,  $\cos^{-1} x$ , and  $\tan^{-1} x$ ? Find the antiderivatives writing  $y$  as a function of  $x$  in each case.

$$(a) \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad (b) \frac{dy}{dx} = \frac{1}{1+x^2} \quad (c) \frac{dy}{dx} = \frac{-2}{\sqrt{1-4x^2}}$$

**P17.5** The differential equation  $\frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right) \left(\frac{P}{m} - 1\right)$  is called the *threshold model* for population growth. The parameter  $k$  is a constant of proportionality,  $M$  is the maximum attainable population, and  $m$  is called the threshold value of the population ( $m < M$ ). Refer to this differential equation to answer the following questions.

- (a) Explain why the population will decline if the value of  $P$  exceeds  $M$ .
- (b) Why does the population have to be greater than  $m$  in order to grow? What happens to the population if  $P < m$ ?
- (c) Sketch a typical solution curve given an initial population  $P(0) = P_0$  such that  $m < P_0 < M$ .

**P17.6** To solve the following separable differential equation for the particular solution with initial value  $y_0 = 1000$ , it is tempting to add  $0.5y$  to both sides of the equation, but the left side will not be easy to antidifferentiate. Instead, if you divide both sides by  $(20 - 0.5y)$ , then the left side will be a derivative of an  $\ln$  function.

$$\frac{dy}{dx} = 20 - 0.5y$$

**P17.7** How can you use calculus to determine the distance traveled by a vehicle over an interval of time if you are given an explicit function  $f(t)$  for the speed of the vehicle at any time  $t$ ?

**P17.8** The volume of a cube is increasing at 120 cc per minute at the instant when its edges are 8 cm long. At what rate are the edge lengths increasing at that instant? (Hint: Write a formula for volume as a function of the edge length  $x$ , then differentiate both sides with respect to time  $t$ .)

**P17.9** *Falling object with no air resistance.* An object is thrown upward from an initial height  $h_0$  with initial velocity  $v_0$ , where upward motion is given a positive sign and downward motion has a negative sign. Ignore air resistance, thus acceleration is constant and equal to the acceleration  $g$  due to gravity.

- (a) Confirm the differential equation for acceleration  $\frac{dv}{dt} = -g$ , and solve this equation for  $v(t)$ .
- (b) Since  $v = \frac{dh}{dt}$ , you can now write a differential equation for  $\frac{dh}{dt}$ . Solve this equation for  $h(t)$ . The answer is a quadratic function commonly found in elementary physics textbooks.

**P17.10** Find antiderivatives of the following:

- (a)  $\sqrt{y(t)} y'(t)$
- (b)  $f'(t) \sin f(t)$
- (c)  $\frac{1}{1+w^2} \frac{dw}{dt}$

**P17.11** Not wanting to be caught exceeding the speed limit, the driver of a red sports car suddenly decides to slow down a bit. The table below shows how the speed of the car (in feet per second) changes second by second. Estimate the distance traveled during this 6-second interval.

Time	0	1	2	3	4	5	6
Speed	110.0	99.8	90.9	83.2	76.4	70.4	65.1

**P17.12** (Continuation) The red sports car's speed during the time interval  $0 \leq t \leq 6$  is actually described by the function  $f(t) = \frac{44000}{(t+20)^2}$ . Use this function to calculate the exact distance traveled by the sports car.

**P17.13** Find the antiderivative of

- (a)  $e^{3x}$
- (b)  $e^{-\frac{x}{10}}$
- (c)  $Ae^{kx}$ , where  $A$  and  $k$  are constants

**P17.14** In years past many countries disposed of radioactive waste by sealing it in barrels and dumping it in the ocean. The acceleration of the barrel is the sum of the downward force of gravity, the upward force of the buoyancy of the barrel, and the drag force of the water that is directed opposite to the motion. The forces of gravity and buoyancy are constant, and for a 55 gallon barrel filled with 527 pounds of waste, the acceleration due to gravity minus buoyancy is a constant  $3.49 \text{ ft/sec}^2$ . The acceleration due to the drag force is proportional to the velocity with a constant of proportionality 0.0049. Thus the differential equation for the velocity of the barrel is

$$\frac{dv}{dt} = 3.49 - 0.0049v.$$

- (a) Explain the significance of the terms in this differential equation.
- (b) Solve this separable equation to find the velocity as an explicit function of  $t$ .
- (c) Velocity is the derivative of position, so  $v = \frac{dh}{dt}$ , where  $h$  is the height of the object above the ocean floor. Do one more antiderivative to find  $h$  as a function of  $t$ .
- (d) A barrel is dumped in an area with an ocean depth of 300 feet. How long will it take for the barrel to reach the ocean floor?
- (e) Tests have shown that the barrels tend to rupture if they land with a velocity greater than 40 ft/sec. Will the barrels remain intact when dropped a depth of 300 feet?

Note: Since 1993, ocean dumping of radioactive waste has been banned by international treaty.

**P17.15** The speed in ft/sec of a red sports car is given by  $f(t) = \frac{44000}{(t+20)^2}$  for  $0 \leq t \leq 6$ .

- (a) Explain why the sums  $\sum_{k=0}^5 1 \cdot f(k)$  and  $\sum_{k=1}^6 1 \cdot f(k)$  are reasonable approximations to the distance traveled by the car during this 6-second interval. Why is the true distance between these two estimates?
- (b) Explain why  $\sum_{k=0}^{11} 0.5 \cdot f(0.5k)$  is a better approximation than  $\sum_{k=0}^5 1 \cdot f(k)$ .
- (c) Explain why  $\sum_{k=0}^{119} 0.05 \cdot f(0.05k)$  is an even better approximation.
- (d) How is the sum in part (c) related to the Euler's method of approximation?

**P17.16** (Continuation) The speed of a red sports car is given by  $f(t) = \frac{44000}{(t+20)^2}$  for  $0 \leq t \leq 6$ .

- (a) Given a large positive integer  $n$ , the sum  $\sum_{k=1}^n \frac{6}{n} \cdot f\left(\frac{6k}{n}\right)$  is a reasonable estimate of the distance traveled by the sports car during this 6-second interval. Explain why.
- (b) Another reasonable estimate is  $\sum_{k=0}^{n-1} \frac{6}{n} \cdot f\left(\frac{6k}{n}\right)$ . Compare it to the preceding.
- (c) What is the significance of the expression  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{6}{n} \cdot f\left(\frac{6k}{n}\right)$ ? What is the value of this expression? (You have found this value in an earlier exercise.)

**P17.17** Let  $f(x) = \cos(\ln x)$  and  $g(x) = \ln(\cos x)$ .

- (a) Describe the domain and range of  $f$  and  $g$ . Graphs may be helpful here.
- (b) Find  $f'(x)$  and  $g'(x)$ .

**P17.18** Find the first and second derivatives of  $A(t) = e^{-t^2}$ , then find all points where  $A'$  is negative and  $A''$  is positive. What does this tell you about the graph of  $y = A(t)$ ?

**P17.19** Problems #P17.15 and #P17.16 were about accumulating velocity to calculate the distance traveled by a sports car. The distance is approximated to any degree of accuracy by a *sum of products*, specifically rate times time. The precise answer is a *limit* of such sums. For  $0 \leq t \leq 6$  and  $\Delta t = 6/n$ , the distance traveled is  $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} f(k\Delta t)\Delta t = \int_0^6 f(t)dt$ . The expression  $\int_0^6 f(t)dt$  for the exact distance traveled is read “the *integral* of  $f(t)$  from  $t = 0$  to  $t = 6$ ”. The integration symbol is an elongated S - the initial letter of *sum*. The symbol  $dt$ , although it does not stand for a specific value, represents the  $\Delta t$  in the approximating sum, and it identifies the *integration variable*.

- (a) Since  $f(t)$  is velocity, the distance traveled, namely  $\int_0^6 f(t)dt$ , is the antiderivative of  $f(t)$ . Thus, the accumulation of velocities (which is an Euler's method problem) can be calculated by using an antiderivative. Find the antiderivative of  $f(t)$ , which we will call  $F(t)$ , then take the difference  $F(6) - F(0)$  to find the distance traveled.
- (b) Explain how this accumulation problem is also an *area* problem, specifically the area under the curve  $y = f(t)$  between  $t = 0$  and  $t = 6$ .
- (c) Explain how this one problem is simultaneously a problem about accumulating change over an interval to find the net change in one function *and* a problem about accumulating area to find the area under a curve defined by another function.

**P17.20** (Continuation) Suppose the speed of a car in ft/sec is given by  $f(t) = 90e^{-0.1t}$ .

- (a) Write an integral for the distance traveled from time  $t = 0$  to  $t = 5$  seconds.
- (b) Write an integral for the distance traveled from time  $t = 5$  to  $t = 10$  seconds.
- (c) Evaluate each of the integrals in (a) and (b).

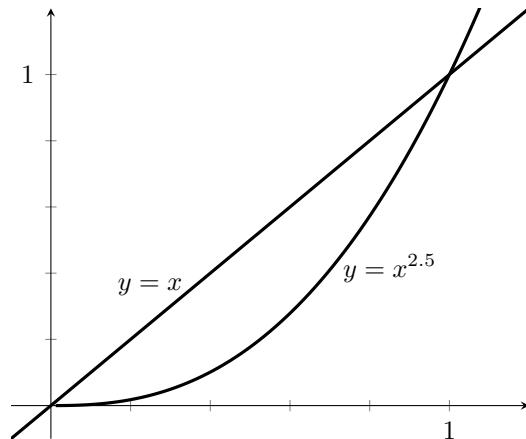
**P17.21** Find the values of the following integrals

(a)  $\int_0^1 x dx$       (b)  $\int_0^1 x^2 dx$       (c)  $\int_0^1 x^3 dx$

**P17.22** Find the area under the curve  $y = f(x)$  between  $x = 0$  and  $x = 1$  for

(a)  $f(x) = x$  (b)  $f(x) = x^2$  (c)  $f(x) = x^3$

**P17.23** Evaluate an integral to find the area of the region bounded by  $y = x$  and  $y = x^{2.5}$ .



Perceived inequity in the distribution of income in our society is of ongoing concern; hence, the “Occupy” movement that began a few years ago. In this lab you will investigate how to quantify inequity in a society’s income distribution using the *Gini Index*.

**L17.1** Suppose our society has 25 family units with incomes listed below.

197 800	177 600	14 200	17 600	28 000
61 800	67 600	104 600	79 600	12 800
15 400	207 200	186 600	111 400	58 600
63 400	96 200	88 600	167 200	16 400
18,000	65 000	58 000	38 000	48 400

Your first task is to order the population by income, and then determine the percent of total income that is earned by each fifth of the population. Since there are 25 incomes, each fifth will be the sum of five incomes. The lowest fifth will consist of the lowest five incomes, the second fifth will have the next five lowest incomes, and so on. Fill in the chart below with the sum of the incomes in each fifth in column 2, then convert these numbers to percent of total income in column 3.

Fifth of population	Aggregate Income	Percent Aggregate Income
Lowest fifth		
Second fifth		
Third fifth	316 400	15.8
Fourth fifth		
Highest fifth		
<b>Total</b>	2 000 000	100

**L17.2** Use the column for percent aggregate income to fill in the chart below with the cumulative percent of income in the lowest fifth, the lowest two fifths, and so on.

Fifths of population	Cumulative Percent Aggregate Income
Lowest one-fifth	
Lowest two-fifths	13.3
Lowest three-fifths	
Lowest four-fifths	
<b>All fifths</b>	100.0

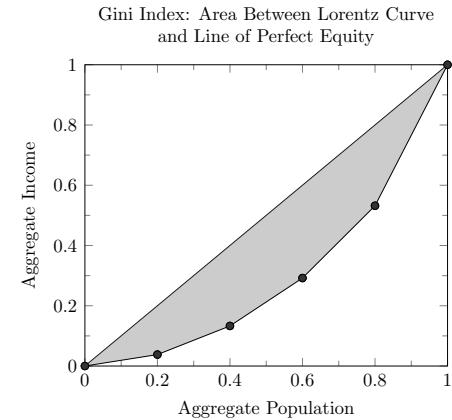
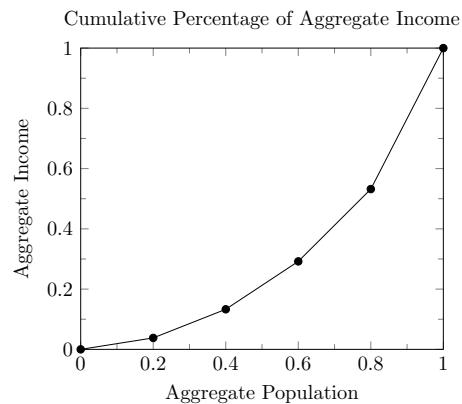
**L17.3** The data in this chart is plotted as ordered pairs in the following graph with proportion of population on the horizontal axis and proportion of cumulative percent aggregate income on the vertical axis. A curve that fits the points of a cumulative percent aggregate income distribution is known as a *Lorenz curve*. Why must a Lorenz curve be increasing?

**L17.4** Equity/Inequity

- (a) Sketch the Lorenz curve for a society with perfect *equity* of income distribution, meaning income is distributed equally throughout the population.
- (b) On the same axes, sketch the Lorenz curve for a society with perfect *inequity* of income distribution, which corresponds to one household earning all the income and everyone else earning nothing.

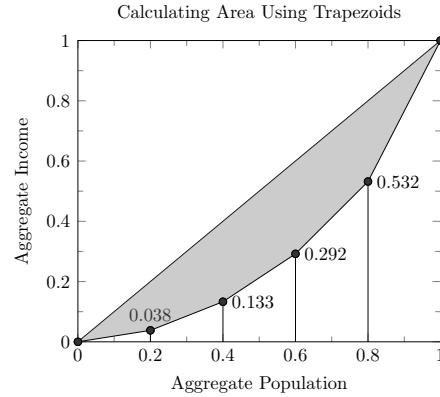
**L17.5** A typical Lorenz curve lies between the extremes sketched in the previous exercise. Corrado Gini first defined the Gini index (or Gini coefficient) in 1912, and it has been widely used by economists since then. The Gini index is based upon the area of the region bounded by the line of perfect equity and the Lorenz curve, as shown in the figure below. To scale the Gini index on the interval from 0 to 1, this area is then divided by the area bounded by the extremes of perfect equity and perfect inequity.

- (a) What is the area bounded by the extremes of perfect equity and perfect inequity? Thus, what must the area bounded by the Lorenz curve be multiplied by?
- (b) What is the Gini index for a society with perfect equity of income distribution? Perfect inequity of income distribution?
- (c) Estimate the Gini index for the income distribution given above.



**L17.6** We can find the area needed to calculate the Gini index by finding the area below the Lorenz curve (that is, bounded by the Lorenz curve and the x-axis) and subtracting it from the area of the triangle highlighted in the figure above. You can estimate the area under the Lorenz curve without actually finding an equation for the curve. One method is to join the successive data points with line segments and draw trapezoids as shown in the figure below.

- (a) Find the areas of the five trapezoids shown in the figure below, the first one actually being a triangle.
- (b) Use the total area of the trapezoids to determine the area of the polygonal region bounded by the line of perfect equity and the line graph of the data points.
- (c) Use this area to calculate the Gini index.



**L17.7** The relevant area for the Gini index can also be estimated by evaluating an integral. To do so, however, requires that we have an explicit formula  $L(x)$  for the Lorenz curve.

- (a) Confirm that the area between the line of perfect equity and the Lorenz curve is given by
- $$\int_0^1 x \, dx - \int_0^1 L(x) \, dx$$
- (b) A reasonable, simple expression for  $L(x)$  is the power function  $x^n$ . Explain.
  - (c) Find a value of  $n$  that gives a good fit of  $x^n$  to the data points for the cumulative aggregate income distribution. You now have an estimate for  $L(x)$ .
  - (d) Use integrals to find the area in (a), hence calculate a value for the Gini index.

**L17.8** The following table contains information on the U.S. income distribution, which is published each year by the U.S. Census Bureau. In your class, assign a selection of years spread throughout the given years from 1967 to 2012. For each year assigned, estimate the value of the Gini index for that year. Share your results with your classmates and consolidate all of the results into a graph of Gini index versus year. What trend do you notice in the historical results?

**L17.9** Write a lab report with a summary of your results, observations, and interpretations.

Year	Shares of aggregate income				
	Lowest fifth	Second fifth	Third fifth	Fourth fifth	Highest fifth
2014	3.1	8.2	14.3	23.2	51.2
2013	3.2	8.4	14.4	23.0	51.0
2012	3.2	8.3	14.4	23	51
2011	3.2	8.4	14.3	23	51.1
2010	3.3	8.5	14.6	23.4	50.2
2009	3.4	8.6	14.6	23.2	50.3
2008	3.4	8.6	14.7	23.3	50
2007	3.4	8.7	14.8	23.4	49.7
2006	3.4	8.6	14.5	22.9	50.5
2005	3.4	8.6	14.6	23	50.4
2004	3.4	8.7	14.7	23.2	50.1
2003	3.4	8.7	14.8	23.4	49.8
2002	3.5	8.8	14.8	23.3	49.7
2001	3.5	8.7	14.6	23	50.1
2000	3.6	8.9	14.8	23	49.8
1999	3.6	8.9	14.9	23.2	49.4
1998	3.6	9	15	23.2	49.2
1997	3.6	8.9	15	23.2	49.4
1996	3.6	9	15.1	23.3	49
1995	3.7	9.1	15.2	23.3	48.7
1994	3.6	8.9	15	23.4	49.1
1993	3.6	9	15.1	23.5	48.9
1992	3.8	9.4	15.8	24.2	46.9
1991	3.8	9.6	15.9	24.2	46.5
1990	3.8	9.6	15.9	24	46.6
1989	3.8	9.5	15.8	24	46.8
1988	3.8	9.6	16	24.2	46.3
1987	3.8	9.6	16.1	24.3	46.2
1986	3.8	9.7	16.2	24.3	46.1
1985	3.9	9.8	16.2	24.4	45.6
1984	4	9.9	16.3	24.6	45.2
1983	4	9.9	16.4	24.6	45.1
1982	4	10	16.5	24.5	45
1981	4.1	10.1	16.7	24.8	44.3
1980	4.2	10.2	16.8	24.7	44.1
1979	4.1	10.2	16.8	24.6	44.2
1978	4.2	10.2	16.8	24.7	44.1
1977	4.2	10.2	16.9	24.7	44
1976	4.3	10.3	17	24.7	43.7
1975	4.3	10.4	17	24.7	43.6
1974	4.3	10.6	17	24.6	43.5
1973	4.2	10.4	17	24.5	43.9
1972	4.1	10.4	17	24.5	43.9
1971	4.1	10.6	17.3	24.5	43.5
1970	4.1	10.8	17.4	24.5	43.3
1969	4.1	10.9	17.5	24.5	43
1968	4.2	11.1	17.6	24.5	42.6
1967	4	10.8	17.3	24.2	43.6

**P18.1** Sterling, a student of Calculus, was given an assignment to find the derivative with respect to  $x$  of five functions. Below are Sterling's five answers. For each, reconstruct the expression that Sterling differentiated, and write your answers in terms of  $y$  and  $x$ . Can you be sure that your answers agree with the questions on Sterling's assignment?

- (a)  $2.54 + \frac{dy}{dx}$
- (b)  $\frac{dy}{dx} \sec^2 y$
- (c)  $\sqrt{5y} \frac{dy}{dx}$
- (d)  $\frac{7}{y^2} \frac{dy}{dx}$
- (e)  $(y - \cos x) \left( \frac{dy}{dx} + \sin x \right)$

**P18.2** *Displacement* is the difference between the initial position and the final position of an object. It is related to velocity by the process of antiderivatiation. Let  $x(t)$  be the position of an object moving along a number line. Suppose that the velocity of the object is  $\frac{dx}{dt} = 4 - 3 \cos 0.5t$  for all  $t$ . Calculate and compare

- (a) the displacement of the object during the interval  $0 \leq t \leq 2\pi$ ;
- (b) the distance traveled by the object during the interval  $0 \leq t \leq 2\pi$ .

**P18.3** A jet is flying at its cruising altitude of 6 miles. Its path carries it directly over Anjali, who is observing it and making calculations. At the moment when the elevation angle is  $60^\circ$ , Anjali finds that this angle is increasing at  $72^\circ/\text{minute}$ . Use this information to calculate the speed of the jet. Is your answer reasonable?

**P18.4** Because of the close connection between integrals and derivatives, integral signs are often used to denote antiderivatives. Thus statements such as  $\int x^2 dx = \frac{1}{3}x^3 + C$  are common. The term *indefinite integral* often appears as a synonym for *antiderivative*, and an integral such as  $\int_0^1 x^2 dx$  is often called *definite*. Evaluate the following indefinite integrals.

- (a)  $\int \cos 2x dx$
- (b)  $\int e^{-x} dx$
- (c)  $\int 110(0.83)^x dx$

**P18.5** The density of air that is  $x$  km above sea level can be approximated by the function  $f(x) = 1.225(0.903)^x \text{ kg/m}^3$ .

- (a) Approximate the number of kg of air in a column that is one meter square, five km tall, and based at sea level, by breaking the column into blocks that are one km tall.
- (b) Make another approximation by breaking the column into blocks that are only one meter tall. Recall that your calculator can be used to sum a series.
- (c) Which answer is a better approximation?
- (d) By the way, you are calculating the mass of the column of air. To test the plausibility of your answer, re-calculate the mass assuming that its density is constant, using both the sea-level value and the 5-km value.

**P18.6** (Continuation) It is possible to find an exact answer for the mass of the column of air by using an integral. Calculate this exact value, assuming that the base of the 5-km tall column of air is: (a) at sea level and (b) 5 km above sea level.

**P18.7** *Immigration model for population growth.* The population of Gravesend grows in proportion to its current population with a constant of proportionality 0.03. It also grows due to a constant immigration rate of 100 000 people per year. The current population is 10 million.

- (a) Explain why the differential equation  $\frac{dP}{dt} = 0.03P + 0.1$  can be used to describe the population growth in Gravesend.
- (b) Use Euler's method to estimate the population for the next 30 years. How long will it take for the population to double its current size? When will it reach 100 million?
- (c) How does your Euler's method estimate relate to the value of the definite integral  $\int_0^{30} 0.03P(t) + 0.1 dt$ ?
- (d) You can actually find a symbolic formula for  $P(t)$  by solving the given differential equation. You solved a differential equation similar to this in #P15.10. Solve the differential equation, then compare the exact values from your solution with the estimates in part (b).

**P18.8** You have now worked through a few accumulation problems, such as finding the distance traveled by a car, finding the population of a country, and finding the mass of a column of air. In each case the desired quantity could be approximated by a sum of products — speed times time, area times height, etc. The precise answer was a limit of such sums. For example, if the cross-sectional area of a water reservoir is  $A(x)$ , where  $x$  is the depth of the cross section, for  $0 \leq x \leq b$ , then if  $\Delta x = b/n$  the volume of the reservoir is

$$\int_0^b A(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n A(k\Delta x)\Delta x.$$

- (a) Explain the relationship between the approximating sum and the integral. Be sure to include the role of each of the symbols in the equation.
- (b) Write the integral for the distance traveled by a car whose speed is  $f(t)$  for  $0 \leq t \leq 6$ .
- (c) Write an equivalent integral for the following expressions:
  - (i)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \sin\left(\frac{k\pi}{n}\right) \frac{\pi}{n}$
  - (ii)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{2k}{n}\right)^2 \cdot \frac{2}{n}$
- (d) Evaluate and interpret the two integrals from (c).

**P18.9** Apply the Fundamental Theorem of Calculus by using an antiderivative to find the area of the first quadrant region that is enclosed by the coordinate axes and the parabola  $y = 9 - x^2$ .

**P18.10** (Continuation) Write an approximating sum, or Riemann sum for  $\int_0^3 9 - x^2 dx$ .

Draw a graphical representation that shows how the limit of a Riemann sum approaches the value of a definite integral.

**P18.11** A standard six-sided die is rolled 5 times. What is the probability that

- (a) The first roll is an ace?
- (b) The first two rolls are aces?
- (c) None of the rolls is an ace?
- (d) At least one of the rolls is an ace?

**P18.12** “Common integration is only the memory of differentiation.” (Augustus De Morgan) Explain what you think is the meaning of this quote.

**P18.13** Evaluate: (a)  $\int_0^\pi \cos x dx$ ; (b)  $\int_0^\pi \sin x dx$ .

**P18.14** Being successful when using antidifferentiation to evaluate an integral often depends upon recognizing the *form of the integrand*. To illustrate this remark, the following definite integrals have a common form. Evaluate each of them.

(a) $\int_6^{15} 2x\sqrt{64 + x^2} dx$	(b) $\int_0^{\pi/2} \cos x \sqrt{\sin x} dx$
(c) $\int_0^1 e^x \sqrt{3 + e^x} dx$	(d) $\int_1^e \frac{1}{x} \sqrt{\ln x} dx$

**P18.15** When two six-sided dice both land showing ones, it is called *snake-eyes*. What is the probability of this happening? What is the most likely sum of two dice, and what is its probability?

**P18.16** The water in a cylindrical tank is draining through a small hole in the bottom of the tank. The depth of water in the tank is governed by *Torricelli's Law*:  $\frac{dy}{dt} = -0.4\sqrt{y}$ , where  $y$  is measured in cm and  $t$  is in seconds. The rate constant 0.4 depends upon the size of the cylinder and the size of the hole.

- (a) Restate Torricelli's Law in words and without symbols.
- (b) Draw a sketch of the rate of change of the depth with respect to time. Now draw a sketch of the depth of the water versus time using a starting depth of 64 cm.
- (c) This equation states that at the moment the water is 49 cm deep, the water level is dropping at a rate of 2.8 cm/sec. Suppose the water is 64 cm deep; estimate the water level a half second later.

**P18.17** (Continuation) By writing Torricelli's Law in the form  $\frac{1}{\sqrt{y}} \frac{dy}{dt} = -0.4$ , solve this differential equation for  $y$  as a function of  $t$ . Use the information that the starting depth is 64 cm to obtain the correct *particular* solution.

- (a) Compare  $y(0.5)$  with your estimate in the preceding questions. Was your estimate too high or too low? How could you have predicted this?
- (b) What is the minimum time needed to empty the tank completely?

**P18.18** A coin with a 2 cm diameter is dropped onto a sheet of paper ruled by parallel lines that are 3 cm apart. Which is more likely, that the coin will land on a line, or that it will not? Calculate the probability that the coin will land on a line.

We have seen in our previous work how the accumulation of an Euler's method approximation and the area under a curve are two views of the same problem. We have also noted that an integral calculated using an antiderivative gives an exact value for the limit of a sum as the step size shrinks towards zero. If an antiderivative is not known, numerical integration can be used, as with some of the problems in this lab.

- L18.1** The speed in mph of a red sports car as the driver slows to be under the speed limit is described by  $s(t) = 80(0.97)^t$  for  $0 \leq t \leq 5$ .

- (a) Complete the table below.

$t$	1	2	3	4	5
$s(t)$					

- (b) Use a graphing application to construct a speed versus time graph.  
 (c) Use the points in (a) to estimate the distance traveled during the entire 5 seconds. Discuss the various strategies that you and your classmates come up with. How does your estimate relate to the graph?

- L18.2** A conical reservoir is 12 meters deep and 8 meters in diameter.

- (a) Confirm that the surface area of the water is described by  $A(h) = \frac{\pi h^2}{9}$ , given that the water is  $h$  meters deep.  
 (b) Complete the table below in a graphing app and construct a graph of surface area versus depth of water.

Depth ( $h$ )	0	3	6	9	12
Surface Area ( $A$ )					

- (c) Estimate the volume of the reservoir using the points in the table in part (b) to *accumulate* surface areas for various depths. You may want to consider that the cone can be approximated with a stack of cylinders. How can you get a better estimate? Discuss your strategy.  
 (d) There is a well-known formula for the volume of a cone, so you can get an exact volume to compare with your estimates. What do your estimates, as well as the exact value, have to do with the graph of surface area versus depth?

**L18.3** From midnight to 9 am, snow accumulates on a driveway at a rate modeled by the function  $s(t) = 7t e^{\cos t}$ , where  $t$  is in hours and  $s$  is in cubic feet per hour.

- (a) Graph of  $s(t)$ . At about what time is the snow accumulating the fastest?
- (b) Use your graph to come up with a rough estimate of the average rate of snow accumulation, and thus estimate the total amount of snow that accumulates by 9 am. Explain how this method relates to the graph of  $s$  versus  $t$ .
- (c) Refine your estimate of the total amount of snow by using the rate of accumulation at each hour (1 am, 2 am, and so on) to estimate the amount of snow that accumulates during each hour. Your estimate of the total will be a sum of these nine numbers.
- (d) Further refine your estimate by using half-hour intervals of accumulation.
- (e) If you continue to use smaller intervals—say minute intervals, then seconds, etc.—to calculate the accumulation of snow, is there a theoretical limit to your sum? How does this theoretical limit relate to the graph of  $s$  versus  $t$ ?

**L18.4** Can any of these three accumulation problems be solved with an antiderivative? If so, use the techniques of antiderivatives to find the exact value of an appropriate integral.

**L18.5** Write a lab report including your results, analysis of your work, and any new insights and techniques that you have used here.

**P19.1** Two real numbers are chosen at random between 0 and 10. What is the probability that the sum is less than 5? greater than 10? This is a two-dimensional problem, so you will want to investigate the sum  $x + y$  in the region where  $0 \leq x \leq 10$  and  $0 \leq y \leq 10$ . The solution can be modeled using a ratio of areas, a technique known as Geometric Probability.

**P19.2** The derivative of a step function has only one value, and yet it could be misleading to simply say that the derivative is a “constant function.” Why?

**P19.3** There is no general method that will find an explicit solution for every differential equation, but the antiderivative approach can be applied to separable equations. For example, you have seen how  $\frac{dy}{dt} = -0.4\sqrt{y}$  can be solved by rewriting it as  $\frac{1}{\sqrt{y}} \frac{dy}{dt} = -0.4$ , then antiderentiating both sides to obtain  $2\sqrt{y} = -0.4t + C$ . Notice that there are infinitely many solutions, thanks to the *antiderivative constant*  $C$ . Another separable example that occurs often is illustrated by  $\frac{dy}{dt} = -0.4y$ . You already know that a solution to this equation is an exponential function (which one?), but we can also use the separable technique. We can rewrite the equation as  $\frac{1}{y} \frac{dy}{dt} = -0.4$ , then antiderentiante both sides to obtain  $\ln y = -0.4t + C$ , which is equivalent to  $y = ke^{-0.4t}$ .

- (a) Solve  $\frac{dy}{dt} = y^2 \sin t$  by the separable technique.
- (b) Solve  $\frac{dy}{dt} = y + 2$  by the separable technique.
- (c) Show that  $\frac{dy}{dt} = y + 2t$  is not a separable equation.

**P19.4** (Continuation) Is it necessary to place an antiderivative constant on both sides of an antiderivative equation? Explain.

**P19.5** Two real numbers between  $-2$  and  $2$  are chosen at random. What is the probability that the sum of their squares is greater than  $1$ ?

**P19.6** A *cycloid* is traced out by the parametric equation  $(x, y) = (t - \sin t, 1 - \cos t)$ .

- (a) Confirm graphically that one arch of this curve is traced out by using  $t$ -values from  $0$  to  $2\pi$ .
- (b) Estimate the area under one arch of the cycloid by three different methods:
  - i. the average of the areas of a rectangle and a triangle;
  - ii. the sum of the areas of five rectangles with heights on the curve;
  - iii. using an antiderivative to find the area under a parabola that contains the peak and the  $x$ -intercepts of the arch.
- (c) Which of the estimates in (b) do you think is closest to the true area?

**P19.7** The integral  $\int_0^\pi x \sin x \, dx$  is not one we can compute using the techniques we know now by finding an antiderivative. We can, however, estimate the area under the curve  $y = x \sin x$  from  $x = 0$  to  $x = \pi$  by a variety of methods, a few of which are outlined below.

- (a) First, divide the interval from  $x = 0$  to  $x = \pi$  into five subintervals using the sequence of points  $0, \pi/5, 2\pi/5, 3\pi/5, 4\pi/5, \pi$ . The first subinterval is  $[0, \pi/5]$ , the second subinterval is  $[\pi/5, 2\pi/5]$ , and so on up to the fifth subinterval  $[4\pi/5, \pi]$ .
- (b) *Left-hand endpoints.* Approximate the area by summing up the five rectangles with heights determined by the value of  $y = x \sin x$  at the left-hand endpoints of each subinterval. Draw a picture of the curve and the rectangles. How well do you think the sum approximates the actual area under the curve?
- (c) *Right-hand endpoints.* Approximate the area by summing up the five rectangles with heights determined by the value of  $y = x \sin x$  at the right-hand endpoints of each subinterval. Draw a picture of the curve and the rectangles. How well do you think the sum approximates the actual area under the curve?
- (d) *Midpoints.* Approximate the area by summing up the five rectangles with heights determined by the value of  $y = x \sin x$  at the midpoints of each subinterval. Draw a picture of the curve and the rectangles. How well do you think the sum approximates the actual area under the curve?

**P19.8** (Continuation) Evaluate the integral  $\int_0^\pi x \sin x \, dx$  using a numerical integrator on a calculator or some other technology. Now discuss the accuracy of the approximations from the previous problem.

**P19.9** (Continuation) The area approximation with rectangles improves as the number of rectangles increases, which means the width of each rectangle decreases. In the limit as the number of rectangles increases without bound, the sum of the areas of the rectangles is exactly equal to the area. Which endpoint technique is represented by each of the following limits? Explain.

- (a)  $\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} (x_k \sin x_k) \left(\frac{\pi}{n}\right)$ , where  $x_k = k \cdot \frac{\pi}{n}$
- (b)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n (x_k \sin x_k) \left(\frac{\pi}{n}\right)$ , where  $x_k = k \cdot \frac{\pi}{n}$

**P19.10** Two real numbers between 0 and 4 are chosen at random. What is the probability that the sum of the two numbers is greater than the product of the two numbers?

**P19.11** The acceleration due to gravity  $g$  is not constant — it is a function of the distance  $r$  from the center of the Earth, whose radius is  $R = 6.378 \times 10^6$  m, and whose mass is  $M = 5.974 \times 10^{24}$  kg. You learn in physics that

$$g = \begin{cases} GMrR^{-3} & \text{for } 0 \leq r < R \\ GMr^{-2} & \text{for } R \leq r \end{cases}$$

where  $G = 6.673 \times 10^{-11}$  Nm $^2$ /kg $^2$  is the gravitational constant.

- (a) Sketch a graph of  $g$  as a function of  $r$ .
- (b) Is  $g$  a continuous function of  $r$ ?
- (c) Is  $g$  a differentiable function of  $r$ ?
- (d) What familiar value of  $g$  is found at the surface of the earth, where  $r = R$ ?

**P19.12** Show that  $\int_1^2 \frac{1}{x} dx$  and  $\int_5^{10} \frac{1}{x} dx$  have the same value.

**P19.13** (Continuation) Find the exact value of  $\int_b^{2b} \frac{1}{x} dx$ .

**P19.14** Find the area of the region bounded by  $y = \frac{1}{1+x^2}$ , the  $x$ -axis, and the lines  $x = 1$  and  $x = -1$ .

**P19.15** Find the area of the “triangular” region enclosed by  $y = \cos x$ ,  $y = \sin x$ , and  $x = 0$ .

**P19.16** Recall that the Gini index is defined as twice the area of the region between the line  $y = x$  and the Lorenz curve from  $x = 0$  to  $x = 1$ . Suppose the equation of the Lorenz curve for the U.S. income distribution in 1990 is  $L(x) = x^{2.36}$ . Use  $L(x)$  to calculate the value of the U.S. Gini index for 1990.

**P19.17** The following definite integrals have something significant in common which will allow you to evaluate each of them by finding an antiderivative.

(a)  $\int_0^3 \frac{2x}{1+x^2} dx$

(b)  $\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} dx$

(c)  $\int_{-1}^0 \frac{e^x}{3+e^x} dx$

(d)  $\int_4^5 \frac{2x+3}{x^2+3x-4} dx$

**P19.18** Find the area of the region enclosed by the line  $y = x$  and the parabola  $y = 2-x^2$ . You will need to find the points of intersection of the line and the parabola.

In this lab we will investigate probability questions that are answered by calculating the ratios of areas. Some of the areas are of geometric figures with well-known area formulas. Other regions of interest will require techniques of calculus to determine the areas.

**L19.1** A coin of radius 1 cm is tossed onto a plane surface that has been tessellated by rectangles, all of which measure 5 cm by 12 cm. What is the probability that the coin lands entirely within one of the rectangles?

**L19.2** Two numbers  $x$  and  $y$  between 0 and 1 are chosen at random.

- (a) Sketch the region of the  $xy$ -plane that symbolizes the pair of numbers chosen. What is the area of this region?
- (b) What is the probability that the sum is less than  $\frac{1}{2}$ ? This can be written in symbols as  $P\left(x + y < \frac{1}{2}\right)$ . Be sure to indicate a subset of the region in part (a) that has an area equal to the probability you wish to calculate.
- (c) What is the probability that the product is less than  $\frac{1}{2}$ ? This can be written in symbols as  $P\left(xy < \frac{1}{2}\right)$ . Again, be sure to indicate a subset of the region in part (a) that has an area equal to the probability you wish to calculate.

**L19.3** *Random Quotient.* If two numbers  $x$  and  $y$  between 0 and 1 are chosen at random, determine the probability of each of the following events.

$$(a) \frac{x}{y} < 1 \quad (b) \frac{x}{y} < 0.25 \quad (c) \frac{x}{y} < 2 \quad (d) 10 < \frac{x}{y} \quad (e) 0.2 < \frac{x}{y} < 0.3$$

**L19.4** *Random Product.* If two numbers  $x$  and  $y$  between 0 and 1 are chosen at random, determine the probability of each of the following events.

$$(a) xy < 0.9 \quad (b) xy > 0.5 \quad (c) xy < 0.1 \quad (d) 0.2 < xy < 0.3$$

**L19.5** *Buffon's needle problem (1777).* Suppose a flat surface is marked off with parallel lines spaced 5 cm apart. If a needle of length 5 cm is dropped at a random place on the surface with a random orientation, what is the probability that the needle will lie across one of the lines? Similar to the previous problems, each drop of the needle can be represented by two random numbers, the distance of the needle from a line (you'll need to decide which point of the needle to use and which line) and the angle of orientation of the needle.

**L19.6** After you have worked out your answers to the problems in this lab, write a report with your results and a summary of the main ideas common to the problems. Be sure to emphasize how the techniques and concepts of calculus are used in your calculations.

**P20.1** The Fundamental Theorem of Calculus states that an integral  $\int_a^b f(t)dt$  can be evaluated by the formula  $F(b) - F(a)$ , where  $F$  is an antiderivative for  $f$ . One of the issues we have glossed over is the assumption that an antiderivative of  $f$  actually exists. What must be true of the function  $f$  in order to apply the Fundamental Theorem? Can  $F$  be any antiderivative of  $f$ ? Explain.

**P20.2** (Continuation) Because of this close connection between integration and differentiation, integral signs are often used to denote antiderivatives. Thus statements such as  $\int x^2 dx = \frac{1}{3}x^3 + C$  are common, where  $C$  is called the constant of integration. The notation shown is a helpful way to order the steps in calculating a definite integral:

$$\begin{aligned}\int_5^8 x^2 dx &= \frac{1}{3}x^3 \Big|_5^8 \\ &= \frac{1}{3} \cdot 8^3 - \frac{1}{3} \cdot 5^3 \\ &= 129\end{aligned}$$

- (a) Why is there no need to insert a constant of integration in the antiderivative?
- (b) Evaluate  $\int e^{2x} dx$  and  $\int_{-1}^1 e^{2x} dx$ .

**P20.3** Suppose a projectile is propelled vertically into the air such that the height (in meters) as a function of time (in seconds) is given by  $h(t) = -5t^2 + 1000t$ .

- (a) How long does it take for the projectile to return to the ground?
- (b) What is the displacement of the projectile during the time it is in the air?
- (c) What is the total distance traveled by the projectile?

**P20.4** (Continuation) Compare the displacement to the distance traveled for an object moving with velocity  $v(t) = \sin t$  from  $t = 0$  to  $t = 2\pi$ .

**P20.5** Evaluate each definite integral

$$(a) \int_1^e \frac{1}{x} dx \quad (b) \int_{-2}^2 |x| dx \quad (c) \int_0^{2\pi} \cos x dx \quad (d) \int_{-1}^1 (e^x + e^{-x}) dx$$

**P20.6** Find the area of the region enclosed by the line  $y = x + 2$  and the parabola  $y = 4 - x^2$ .

**P20.7** Determine what property of a function  $f$  guarantees each of the following and graph a typical function in each case.

- (a)  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$  holds for all  $a$ ?
- (b)  $\int_{-a}^a f(x) dx = 0$  holds for all  $a$ ?

**P20.8** *S-I-R model.* The spread of an infectious disease can be modeled by differential equations that give the rate of change with respect to time of three segments of a population:  $S$  is the number of susceptibles who are not yet infected but could become infected;  $I$  is the number of people who are currently infected; and,  $R$  is the number of recovered people who have been infected but can no longer infect others or be reinfected. Consider the following set of coupled differential equations where  $a$  and  $b$  are positive constants:

$$\frac{dS}{dt} = -aSI \quad \frac{dI}{dt} = aSI - bI \quad \frac{dR}{dt} = bI$$

- (a) Why is the sum of these three equations equal to 0?
- (b) Explain each term in the equations and why they make sense.
- (c) Suppose the starting values of each segment of the population are  $S_0 = 1000$ ,  $I_0 = 10$ , and  $R_0 = 0$ . On the same set of axes, sketch plausible graphs of  $S$ ,  $I$ , and  $R$  versus time.

**P20.9** Evaluate each of the indefinite integrals by finding an antiderivative.

$$(a) \int \frac{2x+5}{x^2+5x-6} dx \quad (b) \int \cos 3t dt \quad (c) \int \frac{1}{\sqrt{1-u^2}} du \quad (d) \int \sec^2 2\theta d\theta$$

**P20.10** Check that  $F(t) = \frac{1}{2}(t - \sin t \cos t)$  is an antiderivative for  $f(t) = \sin^2 t$ . Then find an antiderivative for  $g(t) = \cos^2 t$ .

**P20.11** Jordan is jogging along a straight path with velocity given by a differentiable function  $v$  for  $0 \leq t \leq 40$ . The table below shows certain values of  $v(t)$ , where  $t$  is in minutes and  $v(t)$  is measured in meters per minute. As always, you should have a visual representation before you as you solve this problem.

$t$ (minutes)	0	12	20	24	40
$v$ (m/min)	0	200	240	-220	150

- (a) Explain the meaning of the definite integral  $\int_0^{40} |v(t)| dt$  in this context.
- (b) Find an approximate value of  $\int_0^{40} |v(t)| dt$  using the given data.

**P20.12** Give two reasons why  $\int_a^b f(x) dx = - \int_b^a f(x) dx$  should be true.

**P20.13** The area of the unit circle is given by the integral  $2 \int_{-1}^1 \sqrt{1-x^2} dx$ . Explain. Use a geometric interpretation to find the value of this integral.

**P20.14** (Continuation) The previous integral is difficult to evaluate by finding an antiderivative; however, parameterizing the circle leads to a more accessible antiderivative.

- (a) Confirm that a set of parametric equations  $x(t)$  and  $y(t)$  for a circle with radius  $r$  and center at  $(0, 0)$  is  $(x(t), y(t)) = (r \cos t, r \sin t)$ .
- (b) In the non-parametric version, the area of the circle is  $2 \int_{-r}^r y \, dx$ , where  $y = \sqrt{r^2 - x^2}$ . In the parametric version, as  $x$  goes from  $-r$  to  $r$ , the value of the angle  $t$  goes from  $\pi$  to 0. In the integrand,  $y \, dx$  changes to  $y(t) \frac{dx}{dt} dt$ . Explain this transformation of the integrand.
- (c) Explain and evaluate the parametric version of the area:  $2 \int_{\pi}^0 r \sin t (-r \sin t) dt$ .

**P20.15** For the differential equation  $\frac{dy}{dx} = x - y$ ,

- (a) obtain a graph of the slope field;
- (b) add to the slope field a graph of the solution curve containing the point  $(0, 1)$ ;
- (c) use Euler's method with 10 steps from  $(0, 1)$  to find the value of the solution at the point with  $x = 2$ ;
- (d) describe the behavior of the various types of solution curves as revealed by the slope field.

**P20.16** The velocity of an object moving on the  $x$ -axis is  $v(t) = \frac{1}{2}t^3 - 3t^2 + 10$ .

- (a) Find the displacement of the object from  $t = 0$  to  $t = 6$ . How is this value related to area?
- (b) Write a single integral that represents the total distance traveled from  $t = 0$  to  $t = 6$ . Find the value of this integral, for which you may want to use a numerical integrator.
- (c) What must be true of the graph of the velocity if the displacement over an interval is exactly 0?

**P20.17** Evaluate an integral to find a formula, in terms of  $a$  and  $b$ , for the area enclosed by the ellipse  $(x, y) = (a \cos t, b \sin t)$ . Work parametrically as with the area of the circle problem above.

**P20.18** Show that the derivative with respect to  $x$  of  $\ln|x|$  equals  $\frac{1}{x}$ , i.e.  $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ . Thus show that  $\int \frac{1}{x} \, dx = \ln|x| + C$  is a true statement.

**P20.19** Write a paragraph, along with helpful graphics, explaining the relationships between Euler's method, accumulating rate of change, a definite integral, and the sum of the areas of rectangles.

**L20.1** This lab is centered on flipping a fair coin 100 times and counting the number of heads, which we will simulate with technology. First you should open the Geogebra file named “Lab 20 simulation”. This file simulates  $a$  trials of 100 coin flips, where the number of heads in each trial is recorded. The value of  $a$  is controlled by a slider, which is initially set to 100. This means there are 100 trials of 100 coin tosses. The accompanying frequency histogram displays the number of times each number of heads occurred. For example, the bin centered on 45 has a height equal to the number of times 45 heads occurred in 100 trials of 100 coin flips. Since heads is as equally likely as tails, you might think that every trial would result in 50 heads. As the histogram shows, this is not the case. Why? Discuss the shape of the histogram.

**L20.2** Observe what happens to the frequency histogram as you increase the value of  $a$ , the number of trials. Does the center of the histogram shift? What about the heights of the rectangles? The  $y$ -axis is fixed to the edge of the graphing window to make it easy to adjust the height of the window without needing to shift the window.

**L20.3** It becomes rather tedious to continually adjust the graphing window as you change the number of trials of 100 coin flips. The alternative is a probability histogram, which divides the individual frequencies, which are the rectangle heights, by the total number of trials. This causes the entire area of the histogram to remain at 1 no matter how you set the slider for the number of trials. To see this, you should highlight the probability histogram and un-highlight the frequency histogram. Now gradually adjust the slider from 100 to 5000 and describe the changing shape of the probability histogram.

**L20.4** As the number of trials increases, the shape of the probability histogram becomes more like the so-called *bell curve*, or *normal curve*. The equation of the standard normal curve is the familiar function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The factor  $\frac{1}{\sqrt{2\pi}}$  makes the area under this curve equal to 1. Graph this function along with the probability histogram, then adjust the function with a horizontal shift so that the peak of the curve aligns with the highest values of the histogram.

**L20.5** At this point you will notice that the normal curve is too tall and narrow compared with the histogram. You should adjust the function  $f$  with a horizontal stretch by changing  $\frac{(x - 50)^2}{2}$  to  $\frac{1}{2} \left( \frac{x - 50}{s} \right)^2$ . Why? The value of  $s$  can be set by creating a slider with a range of 1 to 10. Now as you increase  $s$  you will see the spread of the normal curve widen as well.

**L20.6** You will also need to decrease the height of the normal curve, which can be accomplished by changing the factor  $\frac{1}{\sqrt{2\pi}}$  to  $\frac{1}{s\sqrt{2\pi}}$ . This also achieves the goal of keeping the area under the curve at 1, just as with the probability histogram. Why does the introduction of  $s$  in two places in the formula for  $f$  keep the area under the curve at 1?

**L20.7** Write down the function  $f$  that best fits the probability histogram. Note the location of the center of the normal curve and the value of  $s$ . In statistics, the center is called the *mean* and the spread  $s$  is called the *standard deviation*.

**L20.8** What is the probability that the number of heads in 100 coin flips is between 45 and 55, inclusive? One way to answer this question is by summing the areas of the histogram bars from 45 to 55. On the other hand, you can find the area under the normal curve from 44.5 to 55.5, which is the value of the integral

$$\int_{44.5}^{55.5} \frac{1}{s\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-50}{s}\right)^2} dx.$$

We are not able to calculate this integral using an anti-derivative, but we can find the value using numerical integration, which Geogebra implements with the command **integral[f,44.5,55.5]**. Notice that we have replaced a discrete problem (the area of part of the histogram) with a continuous problem (the area under the normal curve).

**L20.9** Use numerical integration to answer the following questions: What is the probability that the number of heads  $x$  in 100 tosses is such that

- (a)  $45 \leq x \leq 55$ ?    (b)  $40 \leq x \leq 60$ ?    (c)  $x > 57$ ?    (d)  $x < 36$ ?

**L20.10** Write a report summarizing your work in this lab and explaining your understanding of the various concepts introduced here.

**P21.1** Water is being pumped into a pool at a rate  $P(t)$  cubic feet per hour. The table below gives values for  $P(t)$  for selected values of  $t$ .

$t$	0	2	4	6	8	10	12
$P(t)$	0	46	53	57	60	62	63

- (a) Draw a graph of the ordered pairs  $(t, P(t))$ . Use the areas of trapezoids to estimate the amount of water that was pumped into the pool during these 12 hours.
- (b) Over these same 12 hours, water has been leaking from the pool at the rate of  $L(t)$  cubic feet per hour, where  $L(t) = 25e^{-0.05t}$ . How much water has leaked out during the time interval  $0 \leq t \leq 12$ ?
- (c) Use your answers to parts (a) and (b) to estimate the amount of water in the pool at time  $t = 12$  hours.

**P21.2** A pyramid that is 12 inches tall has a base whose area is 80 square inches. The pyramid is sliced by a plane that is  $y$  inches from the vertex and parallel to the base. Let  $A(y)$  be the area of the *cross-section*. Find a formula for  $A(y)$ , then evaluate  $\int_0^{12} A(y) dy$ . Explain the significance of the answer.

**P21.3** You have recently (in Problem Set 18) evaluated a number of integrals of the form  $\int_a^b \sqrt{u} du$ . For example,  $\int_1^e \frac{1}{x} \sqrt{\ln x} dx$  is equal to  $\int_0^1 \sqrt{u} du$ , where  $u = \ln x$ . Notice that  $u = 1$  corresponds to  $x = e$  and  $u = 0$  corresponds to  $x = 1$ , which is why the limits on the two integrals are different. Make up an example that is equivalent to  $\int_4^9 \sqrt{u} du$ , by replacing  $u$  by an expression  $f(x)$  of your choice. Choose corresponding integration limits for your  $dx$  integral, then find its value. This technique of transforming one integral into another that has the same value is known as *integration by substitution*.

**P21.4** Use integration by substitution to find the values of the following integrals.

$$(a) \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} dx \quad (b) \int_{\frac{2}{3\pi}}^{\frac{2}{\pi}} \frac{\cos \frac{1}{x}}{x^2} dx$$

**P21.5** (Continuation) What area problem is represented by the integral  $\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} dx$ ?

**P21.6** The arc  $y = \sqrt{x}$  for  $0 \leq x \leq 4$  is revolved around the  $x$ -axis, which generates a surface known as a *paraboloid*. Note that the cross-sections perpendicular to the  $x$ -axis are circles.

- (a) What is the area of the cross-section at  $x = 1$ ? At  $x = 4$ ? At  $x = r$ ?
- (b) Obtain and evaluate an integral for the volume of this paraboloid.

**P21.7** Find the exact area bounded by the x-axis and one arch of the cycloid traced by the parametric equation  $(x, y) = (t - \sin t, 1 - \cos t)$ .

**P21.8** Find the value of  $\lim_{n \rightarrow \infty} \frac{1}{n} (e^{1/n} + e^{2/n} + e^{3/n} + \dots + e^{n/n})$ . There are at least two ways to proceed.

**P21.9** Find the area bounded by the curves with equations  $y = x^2 - 6x + 11$  and  $y = -\frac{1}{2}x^2 + 2x + 4$ .

**P21.10** *The Mean-Value Theorem.* Suppose that  $f$  is continuous on the interval  $a \leq x \leq b$ , and that  $f$  is differentiable on the interval  $a < x < b$ . Draw the graph of  $f$ , and then draw the segment that joins  $P = (a, f(a))$  to  $Q = (b, f(b))$ . Now consider all the lines that can be drawn tangent to the curve  $y = f(x)$  for  $a < x < b$ . It is certain that at least one of these lines bears a special relationship to segment  $PQ$ . What is this relationship?

**P21.11** The radius of a spherical container is  $r$  cm, and the water in it is  $h$  cm deep. Use an integral  $\int_0^h A(z) dz$  to find a formula for the volume of water in the container. Check your formula for the special cases  $h = 0$ ,  $h = r$ , and  $h = 2r$ .

**P21.12** Explain the geometric significance of the result  $\int_0^R 4\pi x^2 dx = \frac{4}{3}\pi R^3$ . In particular, contrast the way in which volume was accumulated in the preceding exercise with the way in which volume is being accumulated in this example.

**P21.13** An American roulette wheel consists of slots numbered 1 to 36, half of which are red and half of which are black, plus two green slots numbered 0 and 00. The game is played by rolling a ball into the spinning wheel, where it comes to rest in one of the 38 slots.

- (a) If you place a \$1 bet on red, and the ball ends up in a red slot, you win \$1 (meaning you get back your \$1 bet plus \$1). What is the probability of winning when betting on red?
- (b) If you play red 1000 times, how much will you expect to win (or lose)? What is the average amount you win or lose per bet? This is known as the expected value of the game.
- (c) Show that the expected value of playing red is the sum of two terms, each of which is the product of a probability and an amount won or lost.
- (d) Why is it that casino games generally have a negative expected value from the perspective of the player?

**P21.14** The conclusion of the Mean-Value Theorem does not necessarily follow if  $f$  is not known to be a differentiable function. Provide an example that illustrates this remark.

**P21.15** The special case of the Mean-Value Theorem that occurs when  $f(a) = 0 = f(b)$  is called *Rolle's Theorem*. Write a careful statement of this result. Does “0” play a significant role?

**P21.16** Sketch a graph and shade the indicated region for each of the following integrals. Find the value of each integral, and give an explanation of why some of the answers are positive and some are negative.

(a)  $\int_0^2 (x^2 + 1) \, dx$

(b)  $\int_{-2}^0 (x^2 + 1) \, dx$

(c)  $\int_0^{-2} (x^2 + 1) \, dx$

(d)  $\int_0^1 (x^2 - 1) \, dx$

**P21.17** Find the value of  $\int_{-2}^2 \sin x \, dx$ , and explain the answer.

**P21.18** Given a function  $f$ , about which you know only  $f(2) = 1$  and  $f(5) = 4$ , can you be sure that there is an  $x$  between 2 and 5 for which  $f(x) = 3$ ? If not, what additional information about  $f$  would allow you to conclude that  $f$  has this *intermediate-value property*?

**P21.19** In the carnival game “Chuck-A-Luck” you roll three fair dice. If you roll three 6’s you win \$5. If you roll two 6’s you win \$3. If you roll one 6 you win \$1, and if you roll no 6’s you lose \$1.

(a) On a single play of the game, how many ways can you win \$5? How many ways can you win \$3? How many ways can you win \$1? How many ways can you lose \$1?

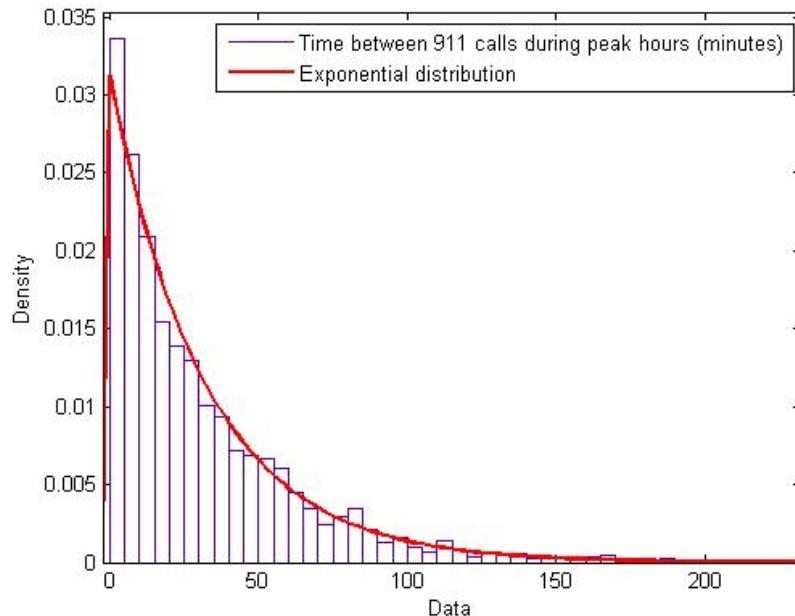
(b) Show that the expected value of this game is \$0.

**P21.20** Obtain a graph of the region enclosed by the positive coordinate axes, the curve  $y = e^{-x}$ , and the line  $x = a$ , where  $a > 0$ .

(a) Write an expression for the area of this region. For what value of  $a$  is the area equal to 0.9? equal to 0.999? equal to 1.001?

(b) The expression  $\int_0^\infty e^{-x} \, dx$ , which is defined as  $\lim_{b \rightarrow \infty} \int_0^b e^{-x} \, dx$ , is an example of an *improper integral*. What is its numerical value, and how is this number to be interpreted?

The following graph displays a histogram for the time (in minutes) between 911 calls during peak hours in a certain city. The width of each histogram rectangle is 5 minutes, and the height has been converted from frequency to probability by dividing by the total number of calls.



<http://punkrockor.com/2011/02/10/isanythingreallyexponentiallydistributed/>

**L21.1** Discuss the meaning in context for the values represented by the probability histogram. Why is the total area of the histogram rectangles equal to 1?

**L21.2** You will be fitting an exponential curve to the histogram, as shown in the graphic, which has an equation of the form  $y = Ae^{-kt}$ . For this model to represent probability, the area under the curve in the first quadrant must equal 1, which is indicated by the equation  $\int_0^{\infty} Ae^{-kt} dt = 1$ . Evaluate the integral and determine what this tells you about how the values of  $A$  and  $k$  are related.

**L21.3** The histogram data has been converted to a set of ordered pairs found in the spreadsheet view of the Geogebra file “Lab 21 data”. Graph this data as a set of ordered pairs (midpoint of each rectangle interval, height of the rectangle).

**L21.4** After highlighting both columns of data, select the menu option for analyzing two-variable data. Be sure it shows the variables in the correct order. If not, there is an option to flip the ordered pairs. Choose the option for exponential least squares, which will open a window with a plot of the ordered pairs, the least-squares curve, and the equation of the curve. Record the equation of the curve that is shown.

**L21.5** Your exponential model may not fit the area requirement derived in number 2; adjust the values of the parameters so that the area under the curve equals 1.

**L21.6** You are now able to answer probability questions about the time  $t$  between 911 calls using an integral to represent area under the curve. Calculate the probability that the time in minutes,  $t$ , is: (a)  $t < 10$ ; (b)  $t > 30$ ; and (c)  $t < 60$ .

**L21.7** Estimate from the graph the average time between 911 calls. This is *not* the average height of the histogram.

**L21.8** Use an integral with your exponential probability model to calculate the average time between 911 calls. Recall that average (or expected) value in this context can be found by summing up the products of  $t$  times the probability of  $t$  occurring for all possible values of  $t$ . Since our probability model is continuous, this results in an infinite sum that can be represented by an integral of the form

$$\int_0^\infty t \cdot P(t) dt.$$

Use the numerical integration command in Geogebra to calculate the average time between 911 calls. Compare your result with your estimate from #L21.7. How does the average compare with the parameter of your model?

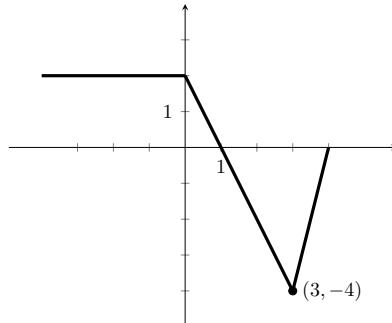
**L21.9** Suppose  $x$  represents the life span (in hours) of an electrical component, and the probability that a component fails is distributed exponentially, as described by

$$P(x) = \frac{1}{50}e^{-x/50}.$$

- (a) Determine the probability that a randomly selected component fails within the first 10 hours.
- (b) Suppose you start with 1000 components. How many will fail within the first 10 hours? What proportion is this? Of the remaining components, what proportion will fail in an additional 10 hours? This result illustrates the memoryless property of an exponential probability distribution: The probability that a component lasts at least  $t$  hours is the same as the probability that the component lasts at least  $t$  additional hours given that it has already lasted  $h$  hours.
- (c) Determine the average life span of a component.

**L21.10** Write a report summarizing your work in this lab and explaining your understanding of the various concepts introduced here. In particular, be sure to discuss why it is that area under the exponential curve is our quantity of interest and not the particular value at a point.

- P22.1** Let  $f(t)$  be defined by the graph shown below. Let  $g(x) = \int_0^x f(t) dt$  define an area function.



- (a) Rewrite the following integrals in terms of  $g$ :

i.  $\int_0^1 f(t) dt =$       ii.  $\int_0^{-2} f(t) dt =$       iii.  $\int_1^4 f(t) dt =$

- (b) Define the area functions  $h(x) = \int_{-3}^x f(t) dt$  and  $k(x) = \int_1^x f(t) dt$ . Fill in the chart below.

$x$	-3	-1	0	1	3	4
$g(x)$	-6					
$h(x)$		4				
$k(x)$			-1			

- (c) Use the chart to sketch graphs of the three area functions on the same set of axes. What do you notice about the three graphs? Express the functions  $h(x)$  and  $k(x)$  in terms of  $g(x)$ .
- (d) We previously learned that a function has infinitely many antiderivatives, each differing by a constant. How does this relate to the results in part (c)?

- P22.2** Referring to your previous work with the normal curve, and using numerical integration, what are the values of the following improper integrals?

(a)  $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$       (a)  $\int_{-\infty}^{\infty} \frac{1}{15\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-100}{15})^2} dx$

- P22.3** (Continuation) When you use numerical integration to calculate the previous integrals, what is the smallest interval you need to integrate over to get 99.8% of the area? Round off to the nearest unit.

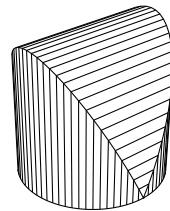
- P22.4** Kelly completed a 250-mile drive in exactly 5 hours — an average speed of 50 mph. The trip was not actually made at a constant speed of 50 mph, of course, for there were traffic lights, slow-moving trucks in the way, etc. Nevertheless, there must have been at least one instant during the trip when Kelly's speedometer showed exactly 50 mph. Give two explanations for why this assertion is true — one using a distance-versus-time graph, and the other using a speed-versus-time graph. Make your graphs consistent with each other.

**P22.5** (Continuation) A student drew the line that joins  $(0, 0)$  to  $(5, 250)$ , and remarked that any actual distance-versus-time graph has to have points that lie above this line and points that lie below it. What do you think of this remark, and why?

**P22.6** (Continuation) Another student was thinking that the area between the distance-versus-time graph and the time axis was a significant number. What do you think of this idea, and why?

**P22.7** Sasha took a wooden cylinder and created an interesting sculpture from it. The finished object is 6 inches tall and 6 inches in diameter. It has square cross-sections perpendicular to the circular base of the cylinder. Draw the top view and the two simplest side views of this sculpture. The volume of a thin slice of the object is approximately  $A(x)\Delta x$ , where  $-3 \leq x \leq 3$ . Explain.

- (a) What is an equation for the base if you center it at the origin of the  $xy$ -plane? What is an equation for  $A(x)$ , the area of a square cross section?
- (b) What is the volume of the object?



**P22.8** *Integration by Parts* is the name given to the Product Rule for derivatives when it is used to solve integration problems. The first step is to convert  $(f \cdot g)' = f \cdot g' + g \cdot f'$  into the form  $f(x) \cdot g(x) \Big|_{x=a}^{x=b} = \int_a^b f(x)g'(x) dx + \int_a^b g(x)f'(x) dx$ . Explain this reasoning, then notice an interesting consequence of this equation: If either of the integrals can be evaluated, then the other can be too. Apply this insight to obtain the value of

$$\int_0^\pi x \cos x dx.$$

**P22.9** (Continuation) Use integration by parts to verify the equation

$$\int_0^\infty x \cdot ae^{-ax} dx = \frac{1}{a},$$

which is the expected value of the exponential distribution

**P22.10** Let  $f(x) = x^2 - 2$ . Investigate the area functions  $A(x) = \int_0^x t^2 - 2 dt$ .

- (a) Evaluate integrals to fill in the following table.

$x$	-2.0	-1.0	-0.5	0.0	0.5	1.0	2.0
$A(x)$							

- (b) Plot the ordered pairs  $(x, A(x))$ .

- (c) At what points do the minimum and maximum values of  $A(x)$  occur on the interval  $-2 \leq x \leq 2$ ? Where does  $f(x)$  equal 0?  
 (d) What can you conclude about the functions  $A(x)$  and  $f(x)$ ?

**P22.11** (Continuation) Use an antiderivative to show that the derivative of  $\int_1^x t^2 dt$  is  $x^2$ , which can be written symbolically as  $\frac{d}{dx} \int_1^x t^2 dt = x^2$ .

**P22.12** (Continuation) Show  $\frac{d}{dx} \int_a^x f(t) dt = f(x)$ . Hint: Assume a function  $F$  is an antiderivative of  $f$ . This result is often called the *Second Fundamental Theorem of Calculus*.

**P22.13** (Continuation) We now have two Fundamental Theorems of Calculus. Explain how they are related.

**P22.14** Let  $R_k$  be the region enclosed by the positive coordinate axes, the curve  $y = \sec^2 x$ , and the line  $x = k$ , where  $k$  is a number between 0 and  $\frac{\pi}{2}$ . Find the area of  $R_k$ , and notice what happens to this area as  $k \rightarrow \frac{\pi}{2}$ . Explain why the expression  $\int_0^{\pi/2} \sec^2 x dx$  is called an improper integral. Does it have a numerical value?

**P22.15** Use integration by parts to find the following indefinite integrals.

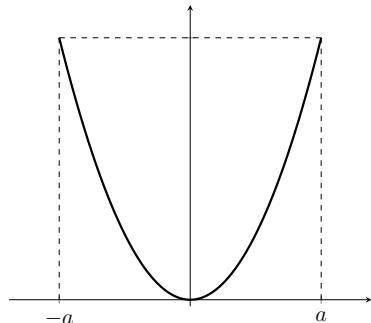
$$(a) \int x \sin x dx \quad (b) \int te^t dt \quad (c) \int x \ln x dx$$

**P22.16** Evaluate the following improper integrals.

$$(a) \int_1^\infty \frac{1}{x} dx \quad (b) \int_1^\infty \frac{1}{x^2} dx \quad (c) \int_1^\infty \frac{1}{\sqrt{x}} dx$$

**P22.17** (Continuation) For what values of  $p$  does the integral  $\int_1^\infty \frac{1}{x^p} dx$  have a finite value? Explain and justify your answer.

**P22.18** The diagram shows the parabolic arc  $y = x^2$  inscribed in the rectangle  $-a \leq x \leq a$ ,  $0 \leq y \leq a^2$ . This curve separates the rectangle into two regions. Find the ratio of their areas, and show that it does not depend on the value of  $a$ .



Part I: Heating Curve

- L22.1** The following table shows temperature in degrees F versus time in minutes for a cold drink that is taken out of the refrigerator at time 0.

Time (min)	Temperature (°F)
0	35.0
1	38.8
2	42.4
3	45.7
5	51.5
6	54.1
8	58.6
11	64.3

We know from previous work that a model for this data is an exponential curve that levels off at the ambient temperature. Enter this data in the spreadsheet of a Geogebra file, and then obtain a graph. You will notice that we do not have enough data to easily determine the ambient temperature directly from the graph.

- L22.2** Rather than trying to guess the ambient temperature, we can use our calculus knowledge to estimate parameters of an appropriate model for the data. A differential equation for the heating phenomenon is

$$\frac{dT}{dt} = k(A - T),$$

where  $A$ , the ambient temperature, and  $k$ , the growth rate, are the parameters of the model. Explain a rationale for this equation.

- L22.3** The differential equation above is linear in terms of the temperature  $T$ , which implies that the ordered pairs  $\left(T, \frac{dT}{dt}\right)$  have a linear trend. We are not given values of  $\frac{dT}{dt}$ , but we can approximate each of the values with a difference quotient  $\frac{\Delta T}{\Delta t}$ . You should now add to your spreadsheet a column with the difference quotients.

- L22.4** Each difference quotient approximates the derivative over an interval, so each difference quotient should be paired with the average  $T$  on each interval (rather than the left-hand or right-hand  $T$ ). Make a column on the spreadsheet with the average  $T$  on each interval.

**L22.5** Now you are ready to make a plot of the ordered pairs that approximate  $\left(T, \frac{dT}{dt}\right)$  and fit a line to the data. Highlight the two columns with average  $T$  and with the difference quotients, then select the menu option for analyzing two-variable data. Be sure it shows the variables in the correct order. If not, there is an option to flip the ordered pairs. Choose the linear least-squares option, which will open a window with a plot of the ordered pairs, the least-squares line, and the equation of the line.

**L22.6** The linear least-squares line has an equation of the form  $y = mx + b$ . Equate your least-squares equation with the differential equation above, and confirm that  $m = -k$  and  $b = kA$ .

**L22.7** Use the slope and intercept from the least-squares equation to solve for  $k$  and  $A$ .

**L22.8** Now we are ready to solve the differential equation to find a model for the data. Show that the solution is  $T(t) = A - Ce^{-kt}$ , where  $C$  is a constant of integration.

**L22.9** Substitute your values for  $A$  and  $k$  into the model  $T(t)$ . The value of  $C$  can be estimated from the initial value of the temperature. Graph your model along with the original data for time and temperature.

**L22.10** Discuss the various characteristics of your model. What is the ambient temperature? What is the meaning of  $k$ ? How well does your model fit the data?

## Part II: The Logistic Model for Growth

**L22.11** In this section you will fit a logistic model to the data for the total number of Starbucks stores open at the end of each year from 1987 through 2008. Open the Geogebra file named “Starbucks data” to find the data in the spreadsheet view.<sup>1</sup> To make the data more manageable, insert a column for “Years since 1987” by subtracting 1987 from the Year column. Now obtain a plot of the set of ordered pairs (Years since 1987; Stores).

**L22.12** Discuss why a logistic model is a reasonable model for this data.

**L22.13** One of the parameters for a logistic model is the maximum number of stores. The graph of the given data does not show this growth limit, but as in Part I, we can use a differential equation to estimate this parameter. Our differential equation for logistic growth is

$$\frac{dS}{dt} = kS \left(1 - \frac{S}{M}\right),$$

where  $S$  is the number of stores as a function of time  $t$ ,  $k$  is a growth rate, and  $M$  is the upper limit on the number of stores. We can rewrite this equation as

$$\frac{1}{S} \frac{dS}{dt} = k \left(1 - \frac{S}{M}\right),$$

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<sup>1</sup><http://www.starbucks.com/about-us/company-information/starbucks-company-timeline>

which is a linear equation with ordered pairs  $\left(S, \frac{1}{S} \frac{dS}{dt}\right)$ . Create a column in your spreadsheet for the average  $S$  on each interval (why?) and then a column for the difference quotients  $\frac{\Delta S}{\Delta t}$  divided by the average  $S$ .

**L22.14** Make a plot of the ordered pairs that approximate  $\left(S, \frac{1}{S} \frac{dS}{dt}\right)$  and fit a line to the data using the menu options as in Part I. You may need to go to the menu Options, Rounding, and change the rounding default to five significant figures so that the value of the slope is displayed.

**L22.15** Equate your least-squares equation with the differential equation above, and confirm that  $b = k$  and  $m = -\frac{k}{M}$ . Use the slope  $m$  and intercept  $b$  from the least-squares equation to solve for  $k$  and  $M$ .

**L22.16** The solution to the logistic differential equation is

$$S(t) = \frac{Ce^{kt}}{1 + \frac{C}{M}e^{kt}},$$

where  $C$  is a constant of integration. (You do not need to confirm this solution.) Substitute your values for  $M$  and  $k$  into this model. The value of  $C$  can be approximated by the initial number of stores. Graph your model along with the original data for the ordered pairs (Years since 1987, number of stores).

**L22.17** Discuss various characteristics of your model. What is maximum number of stores predicted by your model? How well does your model fit the data?

**L22.18** Add the following data points to your graph, and discuss how well your prediction based upon data through 2008 fits the data that comes after 2008.

Year	2009	2010	2011	2012	2013	2014
Stores	16 635	16 858	17 003	18 066	19 767	21 366

### Part III: Atmospheric Carbon Dioxide

**L22.19** Open the Geogebra file “CO2 data” for the annual average atmospheric CO<sub>2</sub> concentration (in parts per million) since 1950. Obtain a graph of the ordered pairs (Years since 1950, CO<sub>2</sub>).<sup>2</sup>

**L22.20** An exponential model looks promising for this data, so try to fit an exponential least squares model to the ordered pairs (Years since 1950, CO<sub>2</sub>). You will undoubtedly notice that it does not give a good result. This is because an exponential least squares model can be used only if the data has a horizontal asymptote at 0. This data has a horizontal asymptote well above 0, which represents the ambient CO<sub>2</sub>

<sup>2</sup><http://www.esrl.noaa.gov/gmd/ccgg/trends/data.html>

concentration from before the industrial revolution. As with Parts I and II, we will use a differential equation model to estimate the unknown parameter for ambient CO<sub>2</sub> concentration.

- L22.21** Explain why the following differential equation is a reasonable model for exponential growth above an ambient level  $A$ , where  $k$  is a growth constant and  $C$  is the atmospheric CO<sub>2</sub> concentration at time  $t$ :

$$\frac{dC}{dt} = k(C - A).$$

- L22.22** This differential equation is linear with respect to  $C$ , so using the techniques learned in this lab, find the least squares line that fits the ordered pairs  $\left(C, \frac{dC}{dt}\right)$ .

- L22.23** Use the slope and intercept from the least squares line to estimate values for  $k$  and  $A$ .

- L22.24** Confirm that the solution to the differential equation is  $C(t) = A + Be^{kt}$ , where  $B$  is a constant of integration, and  $t$  is years since 1950.

- L22.25** Substitute values for  $k$  and  $A$  into your model, and use  $C(0)$  to estimate a value for  $B$ . Graph your model together with the CO<sub>2</sub> data.

- L22.26** Discuss and critique your model. What does your model suggest about future levels of atmospheric CO<sub>2</sub>? Is it reasonable to extrapolate in this way?

### Summary

- L22.27** Discuss the techniques introduced in this lab, especially what the three parts have in common. What problems are you able to solve with data analysis and calculus that you were unable to solve prior to this lab?

**P23.1** The area function  $A(x) = \int_1^x \frac{1}{t} dt$  defines a well-known function. What is it?

What value of  $x$  gives an area of 1, which is to say  $A(x) = 1$ ? For what  $x$  is  $A(x) = -1$ ?

**P23.2** (Continuation) Explain graphically and algebraically why  $A(b) - A(a) = \int_a^b \frac{1}{t} dt$ .

**P23.3** (Continuation) Explain why  $A(km) - A(m) = \ln k$  for any nonzero  $k$  and  $m$ .

**P23.4** The temperature for a typical May 15<sup>th</sup> in Exeter can be modeled by the function  $T(t) = -12.1 \cos\left(\frac{\pi}{12}t\right) + 55.5$ , where  $t$  is the hours after sunrise, and  $0 \leq t \leq 15$ . How do we find the average temperature during the nearly 15 hours of sunlight on this day?

- (a) Suppose we use the temperature at the beginning of each hour to calculate the average. Find the value of  $\frac{1}{15} \sum_{i=0}^{14} T(t_i)$ , where  $t_0 = 0, t_1 = 1, \dots, t_{14} = 14$ .
- (b) You can obtain a more accurate average by using the temperatures at half-hour intervals. Calculate the value of  $\frac{1}{30} \sum_{i=0}^{29} T(t_i)$ , where  $t_i = 0.5i$ . What would this summation look like if we used the temperature at the end of each half hour interval?
- (c) Confirm that the average temperature using  $n$  equally spaced times starting at  $t_0 = 0$  is given by  $\frac{1}{n} \sum_{i=0}^{n-1} T(t_i)$ .
- (d) The time interval used in the averaging process is  $\Delta t = \frac{15}{n}$ . Confirm that introducing  $\Delta t$  into the previous summation leads to  $\frac{1}{15} \sum_{i=0}^{n-1} T(t_i) \Delta t$ .
- (e) The previous summation has the familiar form of a Riemann sum and an area using left-hand endpoint rectangles. As we let  $n$  increase without bound, the summation approaches an integral. Write out the definite integral represented by  $\lim_{n \rightarrow \infty} \frac{1}{15} \sum_{i=0}^{n-1} T(t_i) \Delta t$ .
- (f) Calculate the value of this integral.

**P23.5** (Continuation) *Average value.* Explain why the average value of a function over an interval  $a \leq x \leq b$  can be defined by the limit of a Riemann sum

$$\lim_{n \rightarrow \infty} \frac{1}{b-a} \sum_{i=0}^{n-1} f(x_i) \Delta x, \quad \text{or equivalently} \quad \frac{1}{b-a} \int_a^b f(x) dx.$$

**P23.6** The driver of a red sports car, which is rolling along at 110 feet per second, suddenly steps on the brake, producing a steady deceleration of 25 feet per second per second. How many feet does the red sports car travel while coming to a stop?

**P23.7** To evaluate  $\int_0^{\pi/2} e^{\sin x} \cos x dx$ , Steph tried integration by parts. Was this a good idea?

**P23.8** Find the average value of  $\sin x$  over one arch of the curve,  $0 \leq x \leq \pi$ . Compare your answer to the average of the maximum and minimum values on this interval.

**P23.9** The standard method for approximating an integral  $\int_a^b f(x) dx$  is to use a Riemann sum. Assuming that  $f(x)$  is defined for all  $x$  in the interval  $[a, b]$ , let  $a = x_0 < x_1 < x_2 < \dots < x_n = b$  be a partition of the interval  $[a, b]$ . For each subinterval  $[x_{i-1}, x_i]$ , with  $i = 1, 2, \dots, n$ , choose an  $x_i^*$  in that subinterval such that  $x_{i-1} \leq x_i^* \leq x_i$ , and denote  $\Delta x_i = x_i - x_{i-1}$ . The sum  $\sum_{i=1}^n f(x_i^*) \Delta x_i$  is called a Riemann sum of  $f(x)$  on  $[a, b]$  with respect to the partition  $a = x_0 < x_1 < x_2 < \dots < x_n = b$ . There are different schemes for choosing the evaluation points  $x_i^*$ , and the corresponding sums are named accordingly. What are the schemes that correspond to *left-hand sum*, *right-hand sum*, *lower sum*, *upper sum*, and *midpoint sum*?

**P23.10** The *trapezoidal sum* is the arithmetic mean of the left and right-hand sums. Explain the name, and show that it fits the definition of a Riemann sum when  $f$  is continuous

**P23.11** The *Mean-Value Theorem for Integrals* says: If  $f$  is a function that is continuous for  $a \leq x \leq b$ , then there is a number  $c$  between  $a$  and  $b$  for which  $f(c) \cdot (b-a) = \int_a^b f(x) dx$ . Interpret this statement graphically, and relate it to the average value of  $f(x)$  on the interval  $[a, b]$ .

**P23.12** (Continuation) Use the First Fundamental Theorem of Calculus to show how the Mean-Value Theorem for Integrals is related to the Mean-Value Theorem for Derivatives.

**P23.13** Apply the trapezoidal method with  $\Delta x = 0.5$  to approximate  $\int_1^3 \frac{1}{x} dx$ . Your answer will be slightly larger than  $\ln 3$ . How could this have been anticipated?

**P23.14** For what type of velocity function will the average value of velocity, which can be written as  $\frac{1}{b-a} \int_a^b v(t) dt$ , equal the average of the initial and final velocities  $\frac{1}{2}(v(a) + v(b))$ ? In other words, under what conditions will the following equation be true?

$$\frac{1}{b-a} \int_a^b v(t) dt = \frac{1}{2}(v(a) + v(b))$$

**P23.15** Show that the expected value of the standard normal distribution is the center of the distribution. In other words, confirm the following equation:

$$\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 0.$$

**P23.16** What is the average value of an odd function over the interval  $[-a, a]$ ?

**P23.17** Explain and show with a diagram the differences between the average value of a function and the expected value of a probability distribution.

**P23.18** A model for the pH level of the saliva in the mouth  $t$  minutes after eating a piece of candy is given by  $f(t) = 6.5 - \frac{20t}{t^2 + 36}$ .

- (a) What is the average pH in the mouth during the first 10 minutes after eating a piece of candy?
- (b) During the first hour?

**P23.19** *Predator/Prey Model.* In a population model for lynx and hares, the lynx (*predator*) feeds on the hare (*prey*). The changes in the size of each population depend on both populations because of the predator/prey interaction. The differential equation for the hare population is

$$\frac{dh}{dt} = 0.05h - 0.001hl,$$

and the differential equation for the lynx population is

$$\frac{dl}{dt} = -0.03l + 0.0002hl$$

- (a) Write the Euler's method equations that approximate the values of the two populations,  $h_n$  and  $l_n$ , in terms of  $h_{n-1}$ ,  $l_{n-1}$ , and the stepsize  $\Delta t$ . You do not need to compute any values.
- (b) Explain the effect of each term in the two differential equations. In particular, consider which terms give the natural growth/decline rates in the absence of the other species and which terms are due to interaction between the species.
- (c) Make a conjecture about how you think the growth and decline of one population depends on the other population.

In a population model in which a predator feeds on a prey as the primary food source, changes in the size of each population depend jointly on the number of individuals in each species. A system of differential equations for lynx (predator) and hares (prey) is:

$$\frac{dh}{dt} = a \cdot h - b \cdot h \cdot l \quad \frac{dl}{dt} = -c \cdot l + d \cdot h \cdot l$$

where  $l$  represents the number of lynx,  $h$  represents the number of hares, and  $a$ ,  $b$ ,  $c$ , and  $d$  are constants. Solving this system symbolically is beyond this course, but we can generate approximate values for each population by using Euler's method.

**L23.1** Write the recursive equations that use Euler's method to approximate the values of the two populations,  $h_n$  and  $l_n$ , in terms of  $h_{n-1}$ ,  $l_{n-1}$ , the constants  $a$ ,  $b$ ,  $c$ , and  $d$ , and the stepsize  $\Delta t$ .

**L23.2** Let the initial size of the hare population be 200 and the initial lynx population be 50. Use the following values for the four constants:  $a = 0.05$ ,  $b = 0.001$ ,  $c = 0.03$ ,  $d = 0.0002$ . Examine numerically and graphically the populations versus time for 200 months with a stepsize of 1 month. Since the equations both involve values of  $h_{n-1}$  and  $l_{n-1}$ , you will need to generate the values of the two populations in sync, both populations one step at a time.

**L23.3** Discuss the relative shapes of the two graphs for lynx and hares. It is helpful to graph both populations on the same set of axes. Are the graphs periodic? Are they sinusoidal? What is the relationship between the peaks and valleys of the two graphs? What seem to be the values about which each population oscillates?

**L23.4** Examine the effect of small changes in the parameter values. For example, change  $a$  by a small amount and see what happens to the populations. How does the value of each parameter affect the graphs? What happens if you change the starting value of a population?

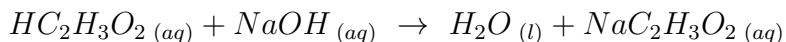
**L23.5** The rate of change  $\frac{dl}{dh}$  of the lynx with respect to the hares can be modeled by the ratio of  $\frac{dl}{dt}$  to  $\frac{dh}{dt}$ , so that  $\frac{dl}{dh} = \frac{\frac{dl}{dt}}{\frac{dh}{dt}}$ . Using the values of the parameters from #L23.2, obtain a graph of the slope field for  $\frac{dl}{dh}$ . What does the slope field tell you about the relationship between  $l$  and  $h$ ? Draw a particular solution curve in the slope field using the initial values of  $l_0 = 50$  and  $h_0 = 200$ .

**L23.6** Each of the differential equations in this system can be factored to find ordered pairs  $(l, h)$  that make the differential equations equal to 0. One such ordered pair is  $(0, 0)$ . What is another ordered pair with this property? These points are called stable points since the populations will remain constant if set to those values. How does this stable point that is not  $(0, 0)$  relate to the graphs of the lynx and hares versus time, and also the slope field and solution curve for lynx verse hares?

**L23.7** Assemble your work into a report that highlights with graphs and explanations the key concepts of this lab. Be sure to include how the techniques of calculus were used to investigate the predator/prey model.

**Goal:** The goal of this activity is to practice applying your differentiation skills to data collected in the chemistry lab (most specifically to **determine an equivalency point**). It is assumed that you have no prior knowledge of chemistry and you are not responsible for any of the chemistry content beyond being prepared for class. The idea is that you will have a better sense of differentiation by applying your skills to data you physically collect?data you watch changing in real time.

**Background:** For the purposes of this lab, an acid can be thought of as a chemical substance that increases the concentration of H<sup>+</sup> in solution. In our lab we will add sodium hydroxide (NaOH: a strong base) to vinegar (HC<sub>2</sub>H<sub>3</sub>O<sub>2</sub>: a common weak acid). The OH<sup>-</sup> from the base will combine with the H<sup>+</sup> from the acid to form water:



The above reaction is known as a neutralization (water is neutral) and the physical act of slowly adding the base to the acid is an example of a titration.

As you are adding your base to the acid you will measure the pH of the solution. The function “p” is defined such that p(x) = −log(x). In our specific case pH = −log[H<sup>+</sup>]. A less acidic solution has less H<sup>+</sup> and a more acidic solution has more H<sup>+</sup>. A glass of wine might have a small bit of H<sup>+</sup> such that [H<sup>+</sup>] = 10<sup>−4</sup> and pH = 4. Lemon juice (more acidic) might have enough H<sup>+</sup> such that [H<sup>+</sup>] = 10<sup>−2</sup> and pH = 2. Acids have pH values less than 7 and bases pH values greater than 7.

As you add the base (NaOH) to the acid (HC<sub>2</sub>H<sub>3</sub>O<sub>2</sub>) the solution becomes less and less acidic. When the number of OH<sup>-</sup> ions added exactly matches the number of HC<sub>2</sub>H<sub>3</sub>O<sub>2</sub> molecules originally in the solution you have reached the equivalency point. This is an important point from a chemist’s perspective and you should take a chemistry class if you wish to know more. The equivalency point will correspond to a maximum in the first derivative curve of pH versus volume — it is worth thinking about why this is the case both during and after the lab activity (it is a little early to worry about at this point).

### Pre-Lab Questions:

1. As you add more and more base to your acid, will the pH increase or decrease? Explain.
2. A student measures the pH of human gastric acid (stomach acid) to be 2.5. What is the H<sup>+</sup> concentration, [H<sup>+</sup>], in the gastric acid?
3. Human blood typically maintains an H<sup>+</sup> concentration of  $[4.0 \times 10^{-8}]$ . What is the pH of human blood? Is human blood acidic or basic?
4. May require research: What is numerical differentiation and more specifically what is Newton’s difference quotient? Can Newton’s difference quotient be used to approximate a second derivative? Explain.

**Procedure:**

- Use a volumetric pipette to measure out 25.0 mL of  $\text{HC}_2\text{H}_3\text{O}_2$  (aq) and transfer to a 150 mL beaker.
- Add a few drops of the indicator phenolphthalein. Phenolphthalein turns pink at the end-point of the titration.
- On the data table below, record the concentration of the sodium hydroxide written on the bottle.
- Use a volumetric pipette to measure out 25.0 mL of  $\text{HC}_2\text{H}_3\text{O}_2$  (aq) and transfer to a 150 mL beaker.
- Add a few drops of the indicator phenolphthalein. Phenolphthalein turns pink at the end-point of the titration.
- On the data table below, record the concentration of the sodium hydroxide written on the bottle.
- With a waste beaker under the tip, pour some NaOH into the buret, allow some to drain out, and make sure there are no air bubbles in the tip of the buret. Fill the burette with the NaOH solution up past the 0 mL mark and then drain down to the 0.0 mL mark.
- Open up Logger Pro and make sure the Y-axis reads pH and the X-axis reads volume — see Mr. M if this is not the case.
- You may need to calibrate the pH probe — see any instructions on the overhead.
- Set the beaker containing the acetic acid solution on the stir plate under the buret, add the stir bar to the beaker, and place the pH probe in the beaker — when moving the pH probe from one solution to another or back to the storage solution, thoroughly rinse it with distilled or deionized water and gently pat-dry before final transfer.
- Turn on the stir plate and arrange everything so that the stir bar won't hit the pH probe. The pH meter can be set against one side of the beaker and out of the way of the stir bar (see picture) by hanging it over the edge using the storage solution cap and a rubber band (a partner can hold it there until you are sure it is steady or you can clamp it in place). Add enough distilled water to be sure the pH probe is covered.
- Begin the LoggerPro program by pressing **COLLECT**. BEFORE YOU BEGIN THE TITRATION . . . plot the initial pH of your acetic acid solution by pressing **keep** and then writing "0" when prompted for the volume. Note this initial pH in the data section below.
- Add 1.0 mL of NaOH and plot the new pH, again by pressing **keep** and then writing "1.0" as the program prompts you for a volume. Add successive 1.0 mL volumes of the NaOH and record the pH and **total volume** of NaOH added after each addition. After adding approximately 15ml **start adding only** 0.50 ml each time. After adding



approximately 20ml **start adding only** 0.25 ml (or as close as you can) each time until you clear the equivalency point. Keep adding; you can return to 1.0 ml volumes when you have reached a high pH that has clearly leveled out (is not significantly changing).

- After you have added a lot of base (past equivalency), add the NaOH in one more 5.0 mL volume, keep that data point and hit **STOP** at a total volume of at least 25 mL NaOH added.
- When the titration is complete, hit **STOP** and then **save your data**.
- Now **export** your data: In logger pro select file, export data, CSV format. Save this file on a flash drive or on the cloud or email it to yourself — you need to be able to open it in an Excel spreadsheet on your own computer.
- See Mr. M **before** you print the graph (he will move the x axis **up** to create a “negative” region). Use file, “print graph” **not** print. Print four copies of your pH plot (2 per partner).

**Data:**

- Concentration of standardized NaOH used in the titration.  $[\text{NaOH}] = \underline{\hspace{10cm}}$
- The pH of the acetic acid solution before the titration began.  $\text{pH}_{\text{HC}_2\text{H}_3\text{O}_2} = \underline{\hspace{10cm}}$

**Analysis of the Data:** (Complete the following in **Excel**)

- Using Newton's difference quotient and your original data, create a new data table representing a numerical approximation of the **first** derivative. Generate a plot of this data.
- Using Newton's difference quotient create another new data table representing a numerical approximation of the **second** derivative. Generate a plot of this data.
- **Print** a spreadsheet showing the original data, the first derivative data, and the second derivative data.

**Questions and Analysis:** (Complete the data analysis above in **Excel** before proceeding)

- (1) What volume of added NaOH corresponds to the “equivalency point” in this titration? Explain your reasoning and all the evidence that points to this conclusion. Use calculus vocabulary whenever possible.
- (2) On one of the original titration plots printed in class, graphically show a representation of Newton's difference quotient for any two consecutive points around  $x$  (volume NaOH added) = 10ml (or anywhere there is a nice curve). Explain how this differs from the “true” derivative? How could it be improved upon?
- (3) On the other copy of the original titration plot printed in class, sketch a plot of an idealized first derivative. In a different color, sketch the plot of an idealized second derivative. Colored pencils are probably best to keep everything legible as you draw one sketch atop another.
- (4) Referencing a rate, what does the maximum in the first derivative curve physically represent? (Be sure you are referencing a rate).

- (5) Explain any points of interest in the second derivative curve.
- (6) Titrations are usually only examined using numerical methods. Why do you think this is the case? Explain. Feel free to research.

**What to turn in:** (Do not type anything!)

- (1) The answers to the pre-lab questions.
- (2) The two titration plots you printed in class, with the appropriate markings as required in the “Questions and Analysis” section above.
- (3) A printed spreadsheet showing the original data, the first derivative data, and the second derivative data.
- (4) The answers to the each numbered item in the “Questions and Analysis” section above.

**Extra Credit:** The early part of our titration curve can be modeled using:

$$pH = pK_a + \log \left( \frac{x}{\text{original moles acid} \times 0.025L - x} \right)$$

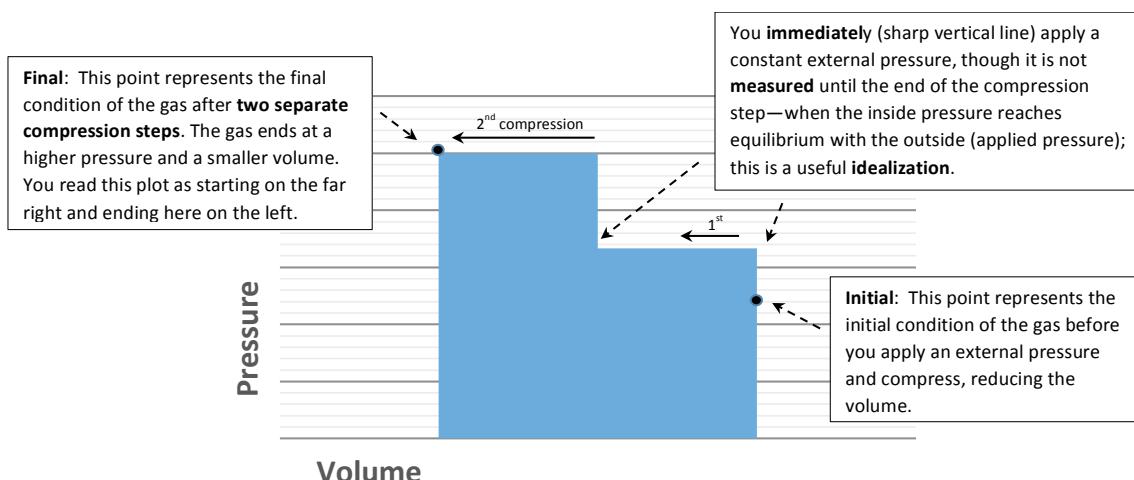
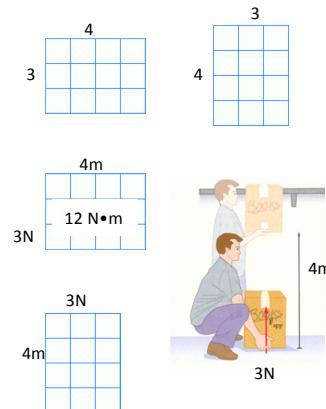
Where x is the number of moles of OH<sup>-</sup> added (not ml). Using the second derivative of the above function find an inflection point in the early part of the curve. Convert this mole number to volume (ml) using the molarity of base solution. What do you notice about this value?

**Goal:** The goal of this activity is to **determine the amount of work done to compress an ideal gas** in 1-step, 3-steps, 5-steps, and an infinite number of steps (a hypothetical). It is assumed that you have limited knowledge of thermodynamics (a field that branches across chemistry and physics) and you are not responsible for any of the chemistry or physics content beyond being prepared for class. The idea is to have a better sense of integration by applying your knowledge to data you physically collect in real time.

**Background:** Geometry is often useful in explaining and visualizing arithmetic. The product of two multiplied numbers, for example, can be easily visualized as the area of a rectangle. The diagrams to the right can be thought of in similar ways ( $3 \cdot 4 = 12$ ).

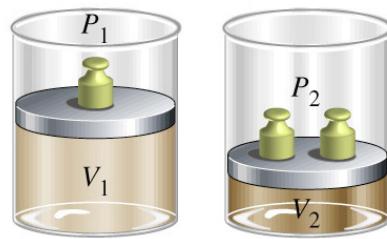
Let's consider another example that involves the calculation of work. The work associated with a constant force ( $F$ ) applied in a straight line that changes an object's position ( $x$ ) follows a simple equation:  $w = F \cdot \Delta x$ . To the right we see 3 Newtons of force constantly applied to move an object from  $x = 0$  meters to  $x = 4$  meters. The work done, as shown to the right, equals 12 Newton-meters.

In the lab you will be looking at the work associated with the compression of a gas. Assuming a constant pressure is applied, the work done to compress the gas is the product of the pressure applied (closely related to force) and the change in volume (closely related to a change in position):  $w = -P_{ex} \cdot \Delta V$ . (See footnote.<sup>3</sup>) The area diagramed in the plot below represents the work associated with a two-step compression. **Carefully study the plot** (there is a lot more going on here than just areas, and this diagram is the key to the entire lab) and think about how much work, in total, was required to compress the gas (**also pay close attention to all the notes**).



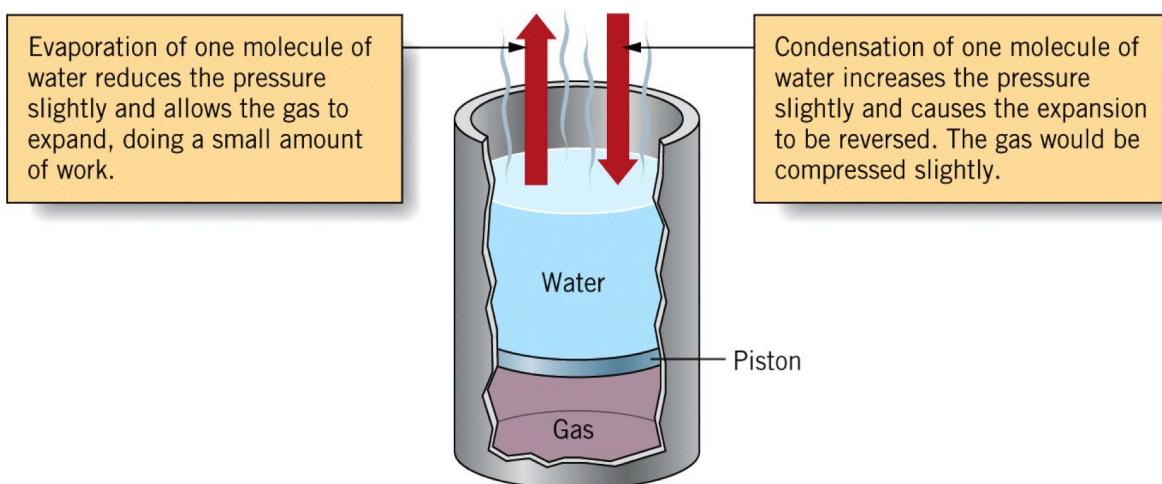
<sup>3</sup>The negative sign – in  $w = -P_{ex} \cdot \Delta V$  helps account for the negative change in volume ( $V_f - V_i$ ) when you compress a gas. Therefore the work will remain positive; you do work to compress a gas.

It will be important to consider the pressure you apply to the gas with each step as an approximately constant value (though it may not feel this way as you push down). Ideally you would simply place a particular weight on the compression cylinder for each step. A ten-pound weight would compress the gas a certain volume and an additional ten pounds would further compress the gas. Visualizing adding weights helps understand the idealization where the **higher pressure is reached immediately** and then **remains constant throughout each step** in your plots of pressure vs. volume. This is an idealization and exaggerates the simplifications in  $w = -P_{ex} \cdot \Delta V$  as applied to a real process.



### Pre-Lab Questions (may well require research):

1. The work associated with a constant force ( $F$ ) applied in a straight line that changes an object's position ( $x$ ) along the direction of the force follows a simple equation:  $w = F \cdot \Delta x$ . If Pressure equals Force per unit Area ( $F/A$ ), show that to compress a gas  $w = -P_{ex} \cdot \Delta V$ , where  $\Delta V$  is a change in volume.
2. What is the ideal gas law? Label each variable and provide units.
3. Is air, the substance you will compress in the lab tomorrow, an ideal gas? Explain.
4. What is an integral? What is the difference between a definite and an indefinite integral?
5. Within the context of thermodynamics, what is a reversible process? How might the concept of a thermodynamically reversible process relate to calculus? See example below.



**Procedure:**

**Total Cylinder Volume = 236ml +/- 2ml if the top position is 15.5cm**

**Total Cylinder Volume = 228ml +/- 2ml if the top position is 15.0cm**

**Volume per Centimeter = 15ml +/- 1ml**

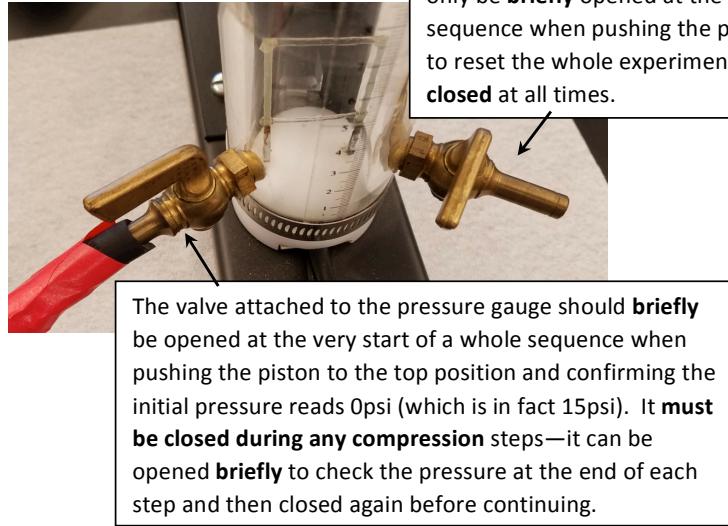
**Add 15 psi to all pressure measurements: 0.0 psi on the guage = 15 psi**

**It is fine that your units of work will likely be psi·mL. 1 psi·mL = 0.00689 Joules.**

(1) Spend a little time just “playing” with the gas compression chambers (but be gentle and careful!). Get a sense of how they work. Make sure you can open the turn valves and reset the system pressure and position. Notice that the pressure gauge is reset with a push-button and will “stick” at the highest pressure recorded during a fast compression/relaxation. Be sure to check the total volume of your specific cylinder using the data in the procedure box at the bottom of the previous page!

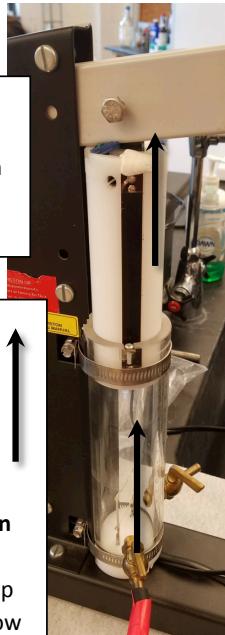
(2) Press “collect” in Logger Pro and then “stop” so that you can see how to track temperature changes (especially max temperatures) for individual compression steps throughout the day.

(3) Spend a little time working with the valve system.



The valve that connects to the atmosphere should only be **briefly** opened at the very start of a whole sequence when pushing the piston to the top position to reset the whole experiment. Otherwise it **must be closed at all times**.

**Before** starting an entire sequence, always push the piston to the top start position and briefly open both valves to reset. **Then close both valves.** Take note of the top position so you know the **volume of your specific gas chamber**.

**One-Step Compression**

(4) Ask a partner to completely compress the gas in a single, fast step. Record the maximum measured pressure (again this should be thought of as approximately a constant pressure applied throughout the step — can be thought of as one really heavy weight dropped on the compression chamber). Record the final position of the compression

plunger and convert this to a final volume. Record the maximum temperature reached during the compression.

(5) On your graph paper draw a diagram analogous to the one in the background information, but for a 1-step compression. Make it large, and especially broad, so that you can later draw 3- and 5-step diagrams with similar scales. The  $x$  and  $y$  axis do not need to start at  $(0, 0)$ ; and only the area of interest needs to be to scale.

(6) Calculate the area under your plot. What does this value represent?

(7) Integrate the function  $dw = -P_{ex} \cdot \Delta V$  where  $P_{ex}$  is a constant (numerically equal to the maximum pressure you recorded after adding 15psi to the gauge reading). Then show how the result is evaluated over the volume range you recorded showing all the steps used in the standard notation applied to definite integrals. In what way is applying the original function,  $dw = -P_{ex} \cdot \Delta V$ , to these procedural steps an awkward use of the differential notation “d” (as in  $dw$  and  $dV$ )? Why can we get away with it?

### Three-Step Compression (read all of 8-10 before proceeding)

(8) Ask a partner to completely compress the gas in three separate steps, stopping after each step and holding the plunger steady. These do not have to be exactly equal steps, just do your best.

(9) Record the maximum measured pressure after the first step. Record the position of the compression plunger after the first step. Repeat these recordings for each subsequent step, so that you end with three sets of pressure and position data, and maximum temperatures.

(10) Convert all your position data to final volumes.

(11) Now draw a diagram on your graph paper describing the 3-step compression. Make sure the scale matches that of the 1-step diagram you drew earlier. You can always adjust and redraw these as needed.

(12) Calculate the work required to compress the gas in 3 steps.

### Five-Step Compression

(13) Repeat steps 7-11, but for a 5-step compression.

### Finishing Up

(14) Make sure you have finished all your graph paper plots and answered any questions included in the procedure below each plot before you leave class today.

### Lab Questions (again, may require research):

(1) Draw a diagram representing a compression via an infinite number of steps. Roughly sketch this diagram, it does not need to be perfect. Just imagine how the plot will change as the number of steps increases a whole lot! Try to do this on your own before ?finding? the answer.

(2) What is an isothermal process? Think about the temperature readings during the various compressions. Which type of compression 1-step, 3-step, or 5-step was the closest

to an isothermal process?

(3) If you completed an essentially infinite number of steps and waited a long time after each step, you could approximate an isothermal compression. Explain why this makes sense.

(4) One way to think about the diagram drawn in response to question #1 at the top of the page is to imagine that we apply a new constant pressure an infinite number of times (drop infinitesimally small weights on the compression chamber again and again). Fortunately, we can also imagine that the pressure changes according to the ideal gas law;  $P_{\text{external}} = P_{\text{internal}}$  if the steps are infinitesimally small. Rewrite  $w = -P_{\text{ex}} \cdot \Delta V$  in terms of differential notation “d” (as in dw and dV). Now, **using the ideal gas law**, rewrite the equation such that  $P$  is in terms of **only** volume and  $nRT$  (each of these latter values can be thought of as essentially constant).

(5) Determine the antiderivative of the final function derived in #5 above. Remember that  $nRT$  can all be thought of as constant under the slow, infinite step, compression.

(6) Show how the result in #5 above is evaluated over the volume range you used in the lab. The appropriate  $R$  value in  $J/(mole \cdot K) = 8.314J/(mole \cdot K)$ . To compare this with your lab data recall that  $1\text{psi} \cdot \text{mL} = 0.00689$  Joules. Room temperature ( $T = 298\text{K}$ ). Show your work using the **standard notation** applied to **definite** integrals.  $n = 0.00965$  moles of gas if the top position of your cylinder is 15.5cm;  $n = 0.00932$  moles of gas if the top position of your cylinder is 15.0cm.

(7) Does the “+C” constant of integration serve a purpose in our lab? Explain.

(8) Your calculations indicate that you did a lot more work in the 1-step than in the 5-step, and certainly more than in the hypothetical, infinite step compression. **Calculate** the difference in work required to compress the air in 1-step versus an infinite number of steps. Comment on the fact that work is a “**path dependent**” function.

(9) The fact that energy is conserved and that the final energy of the system (the compressed gas) is the same regardless of whether it was compressed in 1-step or an infinite number of steps, leads to an interesting question: Where did all the extra work you did to compress the gas in 1-step go? Comment on that question.

**What to turn in:** (Do not type anything!)

(1) The answers to all the pre-lab questions.

(2) All the diagramed plots completed during the lab procedure (or redrawn as needed to clear any errors). These should be on graph paper.

(3) The answers to any questions posed in the procedure. These answers should be written out below the respective diagram plots with which they are associated, on the exact same graph paper pages.

(4) The answers to the final set of lab questions.