2 *The Nature of Classroom Tasks*

One of the most important points that we will make is that students develop mathematical understanding as they invent and examine methods for solving mathematical problems. This is quite different than the usual claim which says that students acquire understanding as they listen to clear explanations by the teacher and watch the teacher demonstrate how to solve problems. In this chapter, we will explain what we mean when we say that students should be encouraged to invent and examine methods for solving problems, and we will show why this is essential for building important mathematical understandings.

Why Are Tasks Important?

Students learn from the kind of work they do during class, and the tasks they are asked to complete determines the kind of work they do (Doyle 1983, 1988). If they spend most of their time practicing paper-and-pencil skills on sets of worksheet exercises, they are likely to become faster at executing these skills. If they spend most of their time watching the teacher demonstrate methods for solving special kinds of problems, they are likely to become better at imitating these methods on similar problems. If they spend most of their time reflecting on the way things work, on how various ideas and procedures are the same or different, on how what they already know relates to the situations they encounter, they are likely to build new relationships. That is, they are likely to construct new understandings. How they spend their time is determined by the tasks that they are asked to complete. The tasks make all the difference.

Students also form their perceptions of what a subject is all about from the kinds of tasks they do. If they are asked in history class only to memorize the names, dates, and locations of historical events, they will think that history is about remembering facts from the past. If students are asked in mathematics class only to practice prescribed procedures by completing

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sets of exercises, they will think that mathematics is about following directions to move symbols around as quickly as possible. If we want students to think that doing mathematics means solving problems, they will need to spend most of their time solving problems. Students' perceptions of the subject are built from the kind of work they do, not from the exhortations of the teacher. These perceptions guide their expectations for what they will do in mathematics class and influence their inclination to participate in the kind of classroom community we are describing in this book (see Chapter 4). Once again, it starts with the tasks. The tasks are critical.

What Kinds of Tasks Are Important?

What kinds of tasks should teachers use if they want their students to build important mathematical understandings? As we argued in Chapter 1, students build mathematical understandings by reflecting and communicating, so the tasks must allow and encourage these processes. This requires several things: First, the tasks must allow the students to treat the situations as problematic, as something they need to think about rather than as a prescription they need to follow. Second, what is problematic about the task should be the mathematics rather than other aspects of the situation. Finally, in order for students to work seriously on the task, it must offer students the chance to use skills and knowledge they already possess. Tasks that fit these criteria are tasks that can leave behind something of mathematical value for students. We will explore these criteria for selecting and designing tasks in the next section.

Tasks Should Encourage Reflection and Communication

Reflecting and communicating are the processes through which understanding develops. One of the simplest principles we can suggest is that if you would like students to understand, then be sure they are reflecting on what they are doing and communicating about it to others. Tasks are the key. They provide the context in which students can reflect on and communicate about mathematics.

Reflecting means turning something over in your head, thinking again about it, trying to relate it to something else you know. If a task encourages you to reflect on something, you do not rush through it as quickly as you can. Tasks that encourage reflection take time. Communicating means talking and listening. It means sharing the method you developed to solve a problem and responding to questions about your method. It means listening to others share their methods and asking questions to make sure you understand.

In order for students to reflect on mathematics and communicate their experience, they must see that there is something intriguing on

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which to reflect and something worthwhile to communicate. They must sense a difficulty that they would like to resolve and discuss. In order for the task to meet these needs, two things are essential: one is that students must make the task their own. Students must set the goal of solving the problem. The second is that the intriguing or perplexing part of the situation should be the *mathematics*. The task could, of course, be interesting in lots of ways, but if students are to build *mathematical* understandings, then it should be interesting in a mathematical way.

For something to be a problem for a student, he or she must see it as a challenge and must want to know the answer. The student must set a goal of resolving the problem. The goal might come from the student, or be adopted by the student after listening to peers or the teacher. The important thing is that the student makes the goal his or her own.

Goals come in many shapes and sizes: A goal might be to find the answer to a question posed by the teacher or by peers; it might be to increase the efficiency of a computation procedure; it might be to develop a strategy to solve a large, messy problem; it might be to generate a problem for others to solve. All of these goals define problems for students. They set intellectual challenges that create the need for resolution. The goals might set short-term problems, solvable in a few minutes, or they might set long-term, large-scale problems, solvable only after days or weeks.

Students will work to achieve goals only if they believe the goals are worth the effort. The reasons for perceiving worth may include the student's personal values (remember that all students are naturally curious) or values emerging in the culture of the classroom (for example, students may wish to participate in the class discussion and have something to contribute; see Chapters 3 and 4). It is important that the student attaches worth to the goal beyond that of immediate external rewards (Hatano 1988). If students are working only toward an external reward, such as leaving for recess early, this can work against reflecting thoughtfully about what they are doing.

Some tasks suggest interesting problematic situations that are not very mathematical. For example, suppose a group of sixth graders was given a budget of $100 and asked to plan a class party. This kind of task has become a favorite mathematical activity. However, the task may be resolved with little mathematics, especially of a kind that would challenge sixth graders. The situation still might be problematic, but the problems raised and resolved might be social or political ones.

The reverse can also occur. Some situations might look mathematical but would not be very problematic. For example, suppose a student, say Joanne, wanted to memorize the multiplication facts, perhaps to please her parents, or her teacher, or even herself. She borrowed a set of flash

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cards from her teacher and drilled herself for a number of days until she knew them all. Although we might applaud Joanne's motivation and discipline, and agree that knowing multiplication facts is important, we would not say that she engaged in solving a problem. The activity might even be called mathematical, but it did not make mathematics problematic. There was no need to search for and develop a method to solve the problem, and there was no need to reflect on what was happening. This is important because it means that it is unlikely that the activity facilitated the development of Joanne's mathematical understanding. We are not saying the activity was wasteful or unimportant: The point is that understanding develops only as students reflect on and communicate about situations that are mathematically problematic.

Here is an example of a task that is mathematically problematic: Suppose students are presented with the task of developing a method for adding $\frac{1}{3} + \frac{1}{4}$. If students have not yet added fractions with unlike denominators, this could be a task that allows the mathematics to become problematic for students. They would need to rely on their past experiences and then extend their knowledge to generate a solution. They would need to reflect on what they know about fractions, perhaps use resources such as diagrams or fraction pieces, and continually think about whether the methods they are developing are consistent with methods for related problems and whether the answer that is produced is reasonable. All of this involves intensive reflection. If students are asked to work on the task in small groups or to present and defend their method of solution, then communication becomes an integral part of the activity. Communication increases the likelihood that students will think again about their own method, and hear about other methods that would work just as well or better. It is not hard to see that understanding would be a natural outcome of this kind of task.

Tasks Should Allow Students to Use Tools

Tasks that encourage reflection and communication are tasks that link up with students' thinking. One way to describe this is to say that students should see ways in which they can use the tools they possess to begin the task. We define tools broadly to include things the student already knows and materials that can be used to solve problems. Tools are resources or *learning supports,* as we will call them in Chapter 5, and include skills that have been acquired (e.g., counting and adding single-digit numbers can be used as tools to add multidigit numbers), physical materials (e.g., fraction pieces can be used to add $\frac{1}{3} + \frac{1}{4}$), written symbols (often used as records for things that have been figured out), and verbal language (often used to communicate with others about the task). We will describe

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the role of tools more completely in Chapter 5, so we will make only a few points here, in relation to tasks.

Using tools to work on mathematical tasks can be thought of like using tools to complete tasks around the home. Tools are very handy, and we use many of them without even thinking. We use our reading skill to study the directions for how to open the new aspirin bottle; we use water, detergent, and a dish cloth to wash the dishes; and we use our fingers to flip the latch on the window. How did we learn to use these tools so well? We learned to use them because we were given time to explore the tools and time to practice using them in different ways. Of course, it is likely that we also had a bit of instruction in how to use them, but we did not learn by sitting back and watching someone else use them. One does not learn to use a hammer skillfully by watching someone else hammer nails. Tools are used effectively when their owners can practice using them on a variety of tasks. It is the same way with mathematical tools. Students need to have time to explore them, try them out, and use them in a variety of situations.

A second thing about tools is that they are used when there is a need to use them, when they can help to solve a problem or complete a task. Tools are used for a purpose. It is likely that you did not practice using a dish cloth just so you could get good at it. There were dirty dishes that needed to be washed. The same is true for mathematical tools. Students get good at using mathematical tools by using them to solve problems. Usually there is little point in practicing with tools just to be practicing.

It is important to note that tools are used when the *user* sees a need for using them. This means two things: One is that the user chooses the tool to use and finds out if it was a good choice by using it. Choosing a sledge hammer rather than a nail hammer to pound in the tomato stake may not be the best choice (the sledge hammer may be too heavy for the thin stake) but the user will learn about sledge hammers and nail hammers and tomato stakes by trying it out. Something similar is true for mathematical tools. Counting by ones may not be the best tool to find 45 + 38, but the tool user will find out something valuable about counting by ones and about 45 + 38 by trying it out, reflecting on the process, and communicating with others about it.

Another implication of using tools when the user sees a need is that the tasks need to be suitable for the tools that are available. It would be inappropriate to ask someone to build an intricate piece of furniture if they had never used a saw or chisel. Building a wood crate for storing toys might be a better first task. Similarly, it would be inappropriate to ask students to solve $\frac{1}{4} + \frac{1}{3}$ if they did not yet know the meaning of $\frac{1}{4}$ and $\frac{1}{3}$. This does not mean that tasks should be easy for students, or that students should know how to complete them before they start. Rather, it

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means that students should already have some tools available that allow them to begin thinking about the problem and trying out methods that might work. After students have talked about fractions such as $\frac{1}{4}$ and $\frac{1}{3}$, and perhaps represented them with fraction pieces, then they have some tools they can use to begin solving $\frac{1}{4} + \frac{1}{3}$. Tasks should be challenges for students, but they should link up with where students are and with what they already know and can do.

Tasks Should Leave Behind Important Residue

William Brownell (1946) pointed out a number of years ago that it is better to think of understanding as that which comes naturally while students solve mathematical problems rather than as something we should teach directly. More recently, Davis (1992) suggested that we have too long been designing our curriculum and instruction on the idea that we should first teach students skills and then have students apply them to solve problems. Davis argued that it is better to *begin* with problems, allow students to develop methods for solving them, and recognize that what students take away from this experience is what they have learned. Such learning is likely to be deep and lasting. Davis referred to the learning that students take with them from solving problems as "residue."

The point that both Brownell and Davis were making is that we build understandings or relationships by discovering them and hearing about them and using them as we solve problems. Teachers can point out relationships, but they become meaningful as students use them for solving problems. For example, teachers can point out that 38 means 38 ones, or 3 tens and 8 ones, or 2 tens and 18 ones, and so on. But these relationships only become meaningful for students when they use them to solve problems. For example, if students solve 45 + 38 by adding 3 tens and 4 tens to get 70, and 8 ones and 5 ones to get 13 ones, and then combine these to get 83, then the relationships between tens and ones become significant.

Thinking of understandings as outcomes of solving problems rather than as concepts that we teach directly requires a fundamental change in our perceptions of teaching. Many of us have been brought up to think that the best way to teach mathematics is to teach important concepts, like place value or common denominators, by explaining them clearly and demonstrating how to use them and then having students practice them. Our recommendation is that we change our way of thinking and teaching so that students are allowed to develop concepts, such as place value and common denominators, in the context of solving problems. This means that when selecting tasks or problems, we need to think ahead about the kinds of relationships that students might take with them from the experience.

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We cannot provide a list of all the residues that are important because there is no one correct list, and if there were it would be very long. There are many kinds of understandings that are important, and different students are likely to build different ones. We can, however, identify two types of residue that are essential and that can provide useful guides for selecting tasks. One type can be called insights into the structure of mathematics, and the second type is the strategies or methods for solving problems.

Mathematical systems are filled with relationships. Take the base-ten number system for an example. The simple looking numeral 328 is loaded with relationships that can be constructed by students—relationships between the values of the digits, between the units represented by the different positions, and so on. Tasks that invite students to explore relationships of this kind, while they are solving problems, are likely to leave behind insights into the structure of this mathematical system (Cobb et al. 1991; Fuson and Briars 1990; Hiebert and Wearne 1993).

Tasks that are likely to focus students' attention on mathematical relationships are tasks such as: developing several different methods for solving 28×17 and discussing the efficiency of the methods; finding how many triangles can be drawn inside a rectangle, pentagon, hexagon, and so on, using the vertices of the polygon, and looking for a pattern; and deciding whether it is possible to find a fraction between any two fractions and explaining why or why not. Tasks like this provide opportunities for students to get inside mathematical systems and discover how they work. In general, tasks that encourage students to reflect on mathematical relationships are likely to leave behind insights into structure.

If tasks are problematic for students, and if students are allowed to work out methods to complete the tasks, then they also are likely to take with them strategies for solving problems. Two kinds of strategies will be left as residue. One kind of strategy is a specific technique for completing specific kinds of tasks. Two quite different examples will help to illustrate this process. First, consider a routine-looking computation problem. Suppose students had not yet added decimal fractions and the task involved adding $1.34 + 2.5$. After students developed methods for completing the task they would likely take with them specific strategies that could be used to add similar decimal fractions in the future.

A second example comes from a larger scale real-life situation. The day we were writing the first draft of this chapter, Cal Ripken Jr. broke Lou Gehrig's record for consecutive games played in Major League baseball. Mr. Gehrig's record was 2130; Mr. Ripken was playing in his 2131st consecutive game. The Baltimore Orioles had especially large crowds

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during the weeks leading up to this event. On this day, they set up 260 extra seats for the game and charged $5000 for each seat, with the proceeds used to help find treatments and a cure for Lou Gehrig's disease. There are many questions that could be asked about this situation, including statistical comparisons of Mr. Gehrig's and Mr. Ripken's baseball careers, estimates on the number of people that have seen, in person, each of them play, and percentage of revenue from today's game that was contributed to fight Lou Gehrig's disease. Of course, to answer the questions students would need to do some additional research. Tasks like this provide experiences in finding, organizing, and manipulating lots of information. Students are likely to take with them a variety of specific techniques for organizing and manipulating numbers.

But it is likely that students will take with them another kind of strategy from solving both kinds of problems that is even more important than the specific techniques they acquire. As students develop their own methods for solving problems, they develop general approaches for inventing specific procedures or adapting ones they already know to fit new problems. In other words, they learn how to construct their own methods (Fennema et al. 1993; Hiebert and Wearne 1993; Kamii and Joseph 1989; Wearne and Hiebert 1989).

This kind of residue is extremely valuable because it enables students to solve a variety of problems without having to memorize different procedures for each new problem. Although students can acquire specific strategies for specific tasks through more traditional forms of instruction, we believe that they acquire general approaches for developing their own procedures only if they are allowed to treat tasks as problematic. In other words, students learn how to construct methods to solve problems if they allowed to do just that.

A major advantage of thinking about learning as the residue that gets left behind when solving problems is that it provides a way of dealing with a very common difficulty. Many students have trouble connecting the concepts they are learning with the procedures they are practicing (Hiebert 1986). They often end up memorizing and practicing procedures that they do not understand (e.g., adding fractions with unlike denominators). This has damaging consequences, such as forgetting procedures, learning slightly flawed procedures without knowing it, or applying them rigidly without adjusting them for slightly different problems (National Assessment of Educational Progress 1983). In general, if students separate their conceptual understandings from their procedures it means that they cannot solve problems very well.

We believe that the reason so many students separate concepts and procedures, and acquire many procedures they do not understand, is that traditional instruction encourages this separation. By trying to teach

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concepts and procedures directly, we artificially separate them. Although we may try to get students to hook them back together, this is more difficult than we think and most students are not successful. They learn procedures by imitating and practicing rather than by understanding them, and it is hard to go back and try to understand a procedure after you have practiced it many times (Hatano 1988; Resnick and Omanson 1986; Wearne and Hiebert 1988a). Without understanding, it is easy to forget procedures and distort them. And it is hard to adjust them to solve different kinds of problems.

An alternative is to begin with problems. If students are encouraged to develop their own procedures for solving problems, then they must use what they already know, including the understandings they have already constructed. There is no other way to do it. Understandings and procedures remain tightly connected because procedures are built on understandings. The methods students first develop may not be the most efficient ones, but they will be methods students understand. This is exactly what we are finding in classrooms that treat arithmetic in this way (Carpenter et al. 1989; Hiebert and Wearne 1992, 1993, in press; Kamii and Joseph 1989; Murray et al. 1992).

We believe that if we want students to understand mathematics, it is more helpful to think of understanding as something that results from solving problems, rather than something we can teach directly. In particular, we believe that teaching concepts and procedures separately is potentially damaging. It is more appropriate to engage students in solving problems because it is only through solving problems that their concepts and procedures develop together and remain connected in a natural and productive way.

What Changes Should We Make in Our Current Curriculum?

In this chapter we have described the kinds of tasks that fit into the system of instruction we outlined in Chapter 1. Most of our discussion says more about *how* the content should be treated than *what* content should be included. The system of instruction that we recommend is an approach to treating content, not a prescription for the selection of content.

Nevertheless, we can say a few things about content: First, we believe that much of the content in current curricula, as presented in popular textbooks, is appropriate *as long as* students are allowed to make the mathematics problematic. The system of instruction we describe does not mean a wholesale replacement of the curriculum. In fact, there may be few content changes that are required.

A second point is that the reason for including particular topics may

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be different now than in the past. Arithmetic computation, for example, has occupied the lion's share of the curriculum in elementary school because of the importance that has been attached to rapid paper-and-pencil calculation skills. This is being challenged by the reform documents which point out that these skills are rapidly declining in importance (NCTM 1989). We agree that students do not need to become high-speed paper-and-pencil calculators; electronic calculators do that job better. The great amount of time spent practicing fast execution of paper-and-pencil procedures is better spent elsewhere. But we believe that computation is still an important topic (Hiebert 1990). It provides a rich site for students to develop methods for solving problems and to gain important understandings about the number systems and about operations within number systems. Studying computation serves as a vehicle for building mathematical understandings. Of course, it still is useful to possess some computation skills, but these develop alongside the insights into how numbers work as students develop their own methods and examine them carefully (Carpenter et al. 1989; Fennema et al. 1996; Hiebert and Wearne 1992, 1993, in press).

A third point about content is that the criteria identified earlier can be used to decide whether classroom tasks contain appropriate content. The task should allow and encourage students to problematize the mathematics of the situation, and it should invite students to use the tools they already possess to solve the problem. Such tasks are likely to leave behind something of mathematical value. These criteria cannot be used to select topics or to say that one topic is more important than another. But they do say that tasks should be selected for the mathematics of the situation, rather than other extraneous features and that, as one completes the task and looks back, the mathematics of the situation should be the most salient residue. Mathematics should be the focal point, both going into the task and coming out of the task.

Using these three criteria, it is easy to see that much of the content in the current curricula could be framed into tasks that would be appropriate. On the other hand, some tasks that are being proposed as innovative and reform-minded would be inappropriate. Simple computation problems, such as $38 + 45$ and $\frac{1}{4} + \frac{1}{3}$ can be mathematically problematic for students if they are introduced at the right time and treated appropriately, and they can leave behind important residue. In contrast, planning parties with $100 budgets might look interesting and engaging, but might have few mathematical goals going in and leave little mathematical residue coming out. When deciding whether a task is appropriate, it is helpful to look at the way in the which the goals students set will shape the task and the kind of mathematical understandings that are likely to be left behind.

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Tasks Form the Foundation for Instruction

The system of instruction we outlined in Chapter 1 is an interrelated ensemble of five dimensions. Instruction depends on all five working together, and the nature of the tasks is only one of the five. Still, the tasks provide a foundation for instruction that is critical. The underlying processes of reflection and communication are possible only when the tasks are appropriately problematic. The entire system of instruction we are describing depends on tasks that allow and encourage students to treat mathematics as problematic. The way in which the other classroom dimensions build on these kinds of tasks will become clear in the next several chapters.

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