Math Notes: The Strategies Underlying Students' Errors When Identifying Points on the Number Line [1]

Tasks that ask students to label rational number points on the number line are common across curricula in the upper elementary grades. A correct response to these tasks requires coordinating the directed distance of the marked point from zero in relation to a unit distance (the distance from zero to one or its equivalent) on the line. Then, the relationship is represented using one of two notational systems for rational number – fraction notation or decimal notation. For instance, the marked point shown below in Figure 1 can be labeled $\frac{1}{5}$ because the interval from zero to one (the whole) is divided into five parts of equal length (fifths) and the point is located at the end (farthest from 0) of the first of those parts. These types of tasks are also commonly seen on state assessments (e.g., California Department of Education, 2009; Massachusetts Department of Elementary & Secondary Education, 2008). Such tasks target foundational rational number concepts – that a fraction (or a decimal) is a number with a specific location on a number line. These concepts are also emphasized and described as core content in the Common Core State Standards (2010).

What fraction should be written at the point?

Figure 1. A typical number line task in the upper elementary grades

When tasks like the one shown above are used with students to support their understanding of these foundational fraction concepts, many upper elementary school teachers report that their students produce incorrect answers such as $\frac{5}{1}$, $\frac{2}{6}$, and $\frac{1}{1}$. What do these types of answers reveal about the nature of students' understanding? The purpose of this paper is to examine the patterns of reasoning underlying students' incorrect answers on these types of tasks.

Understanding Patterns' of Reasoning Underlying Incorrect Answers

To explore the nature of students' patterns of reasoning when labeling rational number points on the number line, I interviewed thirty-one fifth grade students in an urban school district in Northern California. I designed four number line tasks that asked students to label marked points on the number line as fractions (see number lines in Figure 2). For two of the number lines (A and B), the intervals between consecutive integers were partitioned into parts of equal length. For the other two number lines (C and D), the intervals between consecutive integers were not partitioned into parts of equal length, in order to problematize counts of parts on the number line. For each number line, the student was asked, "what fraction name would you call this point?" For all tasks students were instructed to write down the fraction and to explain their answer verbally[2].

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[1] This paper is an earlier version of the following article:
Shaughnessy, M. M. (2011). Identify fractions and decimals on a number line. *Teaching Children Mathematics, 17*(7), 428-434.

[2] For a complete description of the methods used in this study, see Shaughnessy (2009).

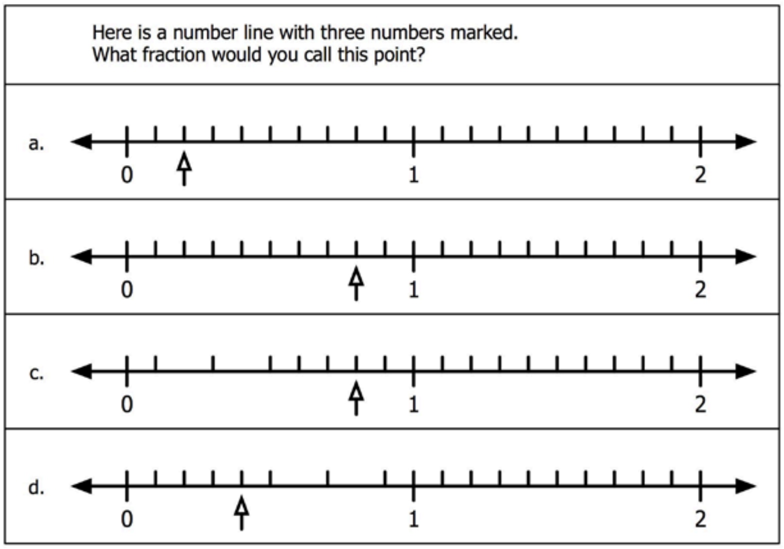

Figure 2. Interview Tasks

More students appropriately labeled points when the interval between zero and one was equally partitioned than when the interval was unequally partitioned. This difference was statistically significant (Shaughnessy, 2009). An analysis of students' incorrect answers and their verbal reasoning revealed four common patterns of reasoning underlying incorrect answers:

1. Using unconventional notation
2. Redefining the unit
3. A two-count strategy focusing on discrete tick marks (or parts) rather than distances
4. A one-count strategy focusing on discrete tick marks (or parts) rather than distances.

Use of Unconventional Notation

Labeling a marked point on the number line using fraction notation requires an understanding of conventions. Although a student may coordinate the directed distance of the marked point from zero in relation to a unit distance, the representation used by the student to convey the relationship may not draw upon standard conventions. While students in this study did not appear to use unconventional notation when asked to represent points on the line as fractions, prior research has indicated students' understanding of part-whole relations (specifically area models) and their use of conventional notation for fractions are independent (Saxe, Taylor, McIntosh, & Gearhart, 2005). It is plausible that younger students might have produced responses such as $\frac{10}{8}$ when presented with the task shown in Figure 3. A response of $\frac{10}{8}$ likely indicates a coordination of the directed distance of the marked point from zero in relation to a unit distance; however, the response indicates a limited understanding of conventional notation because the student writes the denominator and numerator in opposite spots.

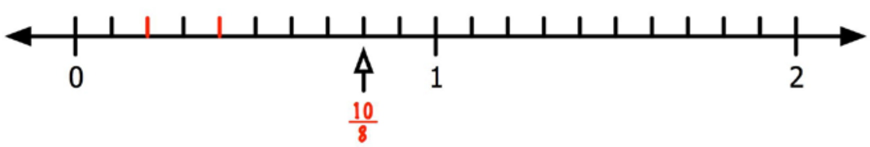

Figure 3. An example of the use of unconventional notation

Redefining the Unit

During initial rational number instruction in the elementary grades, students often see number lines marked from zero to one, like the ones shown on the first page, and thus the shown interval is the unit interval. When presented with number lines in which the shown interval is not the unit interval, such as the tasks used in this interview, students may redefine the whole on the number line and treat the entire interval shown as the whole rather than the interval from zero to one. For example, as shown in Figure 4, one student described the marked point as $\frac{8}{20}$. The student added two tick marks to divide the interval between zero and one into ten parts of equal length and explained her answer in this way:

> *It's ten from zero to one so it has to be another ten from one to two so that would make twenty, then I counted from zero to the arrow to get eight.*

Thus, while this student was attuned to the need to determine the number of parts of equal length that constituted the whole to determine the denominator, the entire interval of the number line shown was treated as the whole. This reflects losing track of the convention that the unit interval is always taken to be the whole on the number line.

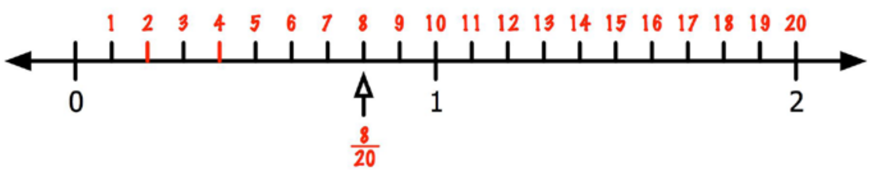

Figure 4. An example of redefining the unit

A Two-Count Strategy Focusing on Number of Tick Marks (or Parts) Rather Than Distances

Students often produce incorrect answers on number line tasks by focusing on discrete marks and/or parts without considering the directed distance of the part from zero in relation to the unit distance. Students may use a two-count strategy for labeling a rational number point on the line as a fraction, one count, the denominator, is the count of the number of tick marks (or parts) displayed in the unit interval. The second count, the numerator, is the number of tick marks (or parts) from zero to the target point. Importantly, distance is not being considered. For instance, as shown in Figure 5, one student represented the marked point as $\frac{6}{8}$ because the interval between zero and one was divided into eight parts and the point is at the end of the sixth part. Thus, this student did not coordinate the distance of the marked point from zero with the unit distance. Of course this same (flawed) reasoning produces correct answers on a number line in which intervals are equally partitioned for the student.

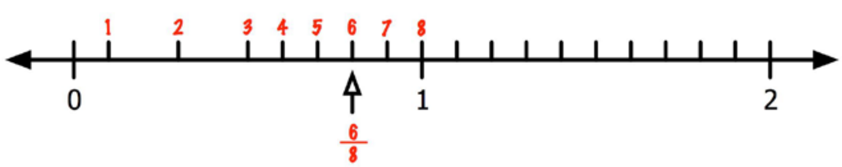

Figure 5. An example of a two-count strategy focusing on number of tick marks (or parts)

Dividing the unit interval into parts of equal length however, is not necessarily an indicator that students are coordinating the directed distance of the marked point from zero with the unit distance. Students may partition the unit interval into parts of equal length but then count the number of tick marks rather than focusing on the number of parts. For instance, as shown in Figure 6, one student partitioned the unit interval into equal parts. The student then counted the number of tick marks starting with zero and ending on the one (eleven tick marks) to determine the denominator. Then, the student counted the number of tick marks starting with 0 and ending with the target point (nine) to determine the numerator.

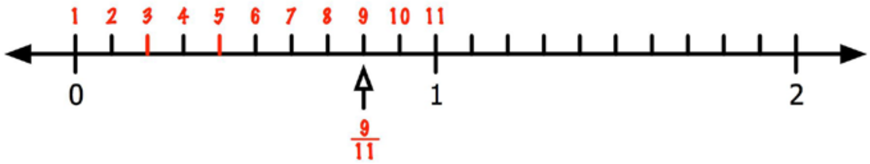

Figure 6. A second example of a two-count strategy focusing on number of tick marks (or parts)

While these types of responses might be particularly visible on tasks in which number lines are unequally partitioned, this type of reasoning can also emerge on standard number line tasks like the one shown in Figure 1. For instance, labeling the marked point in Figure 1 as $\frac{2}{6}$ because there are six tick marks and the arrow is pointed to the second one.

A One-Count Strategy Focusing on Numbers of Tick Marks (or Parts) Rather than Distances

Students may also count discrete quantities in ways that do not attempt to coordinate the number of discrete quantities from zero to the target point with the number of discrete quantities from zero to one. Students may use a one-count strategy focusing only on the number of tick marks or parts from zero to the target point. In the case of fractions, the number of tick marks (or parts) (regardless of their length) from zero to the target point may be treated as the denominator. For example, as shown in Figure 7, one student labeled the target point as $\frac{1}{6}$ and explained:

> *One two three four five six [pointing to each tick mark to the right of zero and up to the target point] and then I saw the one [referring to the one marked on the number line] and put it there.*

Thus, this particular student focused on the number of tick marks up to the target point – six – and treated this number as the denominator and then 'one' is identified as the numerator because 'one' is marked on the number line to the right of the target point. Another instance of this type of error is when students treat the number of tick marks up to the target point as the numerator and zero as the denominator because they are in the "zero space" (the space between zero and one).

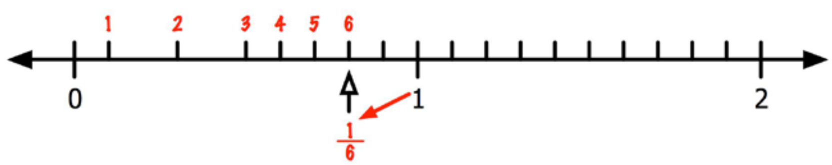

Figure 7. An example of a one-count strategy focusing on number of tick marks (or parts)

Implications for Instruction

Several implications for instruction arise out of the findings of this study. First, understanding these patterns of reasoning can help teachers make sense of the errors produced by students on standard number line tasks. For instance, when a student responds "$\frac{2}{6}$", in response to the task shown in Figure 1, it indicates that the student is likely using a two-count strategy focusing on the number of tick marks, rather than distances on the number line. These types of patterns of reasoning are not specific to the tasks shown in this document, they extend to other number line tasks in which students are asked to label rational number points on the number line. Secondly, the types of errors and tasks described in this document may be useful to discuss with students. By asking students to reflect upon common misunderstanding, teachers can engage students in deepening their own understanding of the number line. Thirdly, it is important for students to have experiences with a variety of number line tasks – number lines that are pre-partitioned and others not partitioned, tasks in which students are given a fraction and asked to locate it on the number line, and tasks in which students are asked to label a marked point using fraction notation. This is important because it allows teachers to see the ways in which students are reasoning, and it supports students in reasoning about the ways in which the number line works, including notational conventions connected with the number line.

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