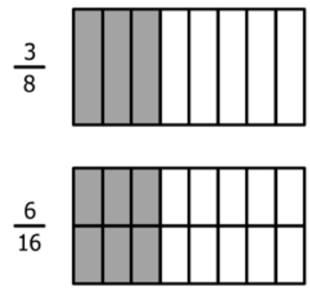
Math Notes: Methods for Generating and Explaining Equivalent Fractions

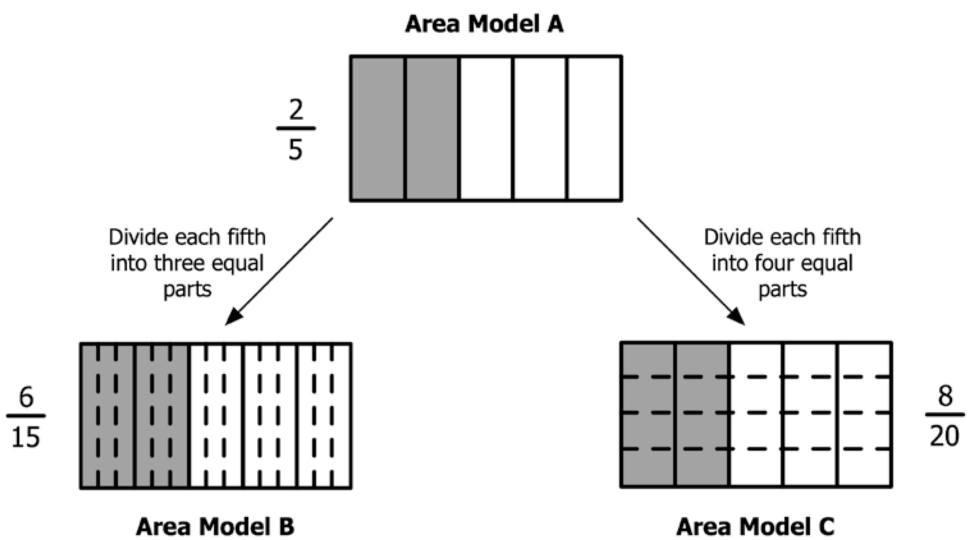
This document discusses equivalent fractions, in particular, the meaning and generation of equivalent fractions using area models, number lines, set models, and arithmetic approaches. The document closes with some challenges for students in reasoning about equivalent fractions.

Reasoning about equivalent fractions using an area model



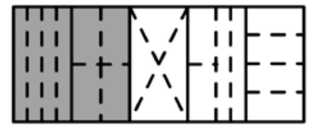
The meaning of equivalent fractions in the context of an area model. In a part-whole interpretation of fractions, two fractions are equivalent if they refer to the same-sized whole and represent the same size portion of that whole. When representing fractions using an area model, two fractions are equivalent if their areas are the same with respect to the same-sized whole. For example, the diagrams to the right show that $\frac{3}{8}$ and $\frac{6}{16}$ are equivalent because the fractions correspond to equal areas of the same-sized whole.

Generating equivalent fractions using an area model. One way to generate equivalent fractions using an area model is to divide each of the parts in the whole into the same number of equal parts. This can be seen in the examples below:



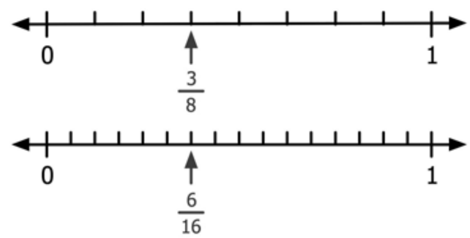
Area Model A represents $\frac{2}{5}$: the whole is divided into five equal parts and two of those parts are shaded. In Area Model B, each of the five equal parts was divided into three equal parts to generate fifteenths, of which six are shaded. This shows that $\frac{6}{15}$ is equivalent to $\frac{2}{5}$. In Area Model C, each of the five equal parts was divided into four equal parts to generate twentieths, showing that $\frac{8}{20}$ is also equivalent to $\frac{2}{5}$. Thus, the area model has been used to show that $\frac{2}{5}$, $\frac{6}{15}$, and $\frac{8}{20}$ are equivalent fractions.

As seen in the examples above, the partitions can be made in any direction, as long as each of the original parts is divided into the same number of equal parts (i.e., parts with equal area). In fact, the subdivisions do not even have to be made in the same way for each part. For example, the diagram to the right shows another way to divide each of the original 5 parts into 4 equal parts. However, when the divisions do not result in congruent pieces, further explanation might be needed to justify that they do in fact correspond to the same area. In addition, even though the divisions can be made in many different ways, there are sometimes mathematical or pedagogical reasons for making them in the same manner.

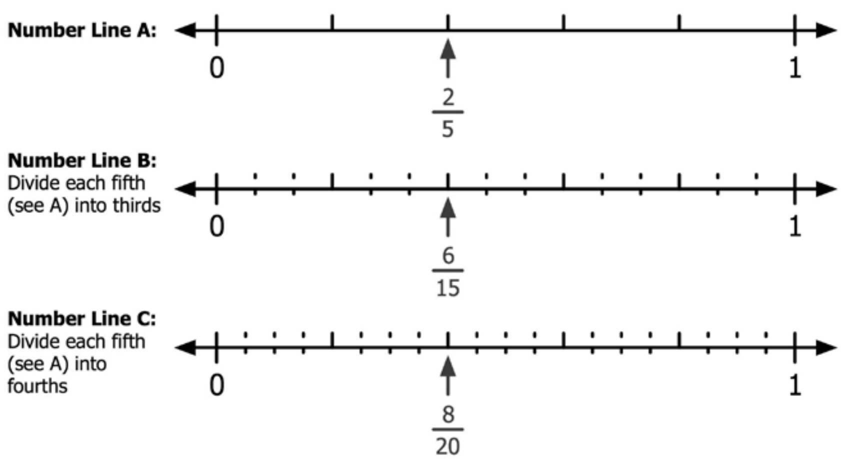


Reasoning about equivalent fractions using a number line

The meaning of equivalent fractions in the context of a number line. When representing fractions using a linear model (such as a number line or paper strip), two fractions are equivalent if their directed distances from 0 are the same with respect to a same-sized unit distance. For example, the number lines to the right show that that $\frac{3}{8}$ and $\frac{6}{16}$ are equivalent because the fractions correspond to the same directed distance from 0, with the unit distance being the distance from 0 to 1. In other words, $\frac{3}{8}$ and $\frac{6}{16}$ are two names for the same point on the number line. In general, *two fractions are equivalent if and only if they represent the same point on the number line,* relative to the unit interval as whole.



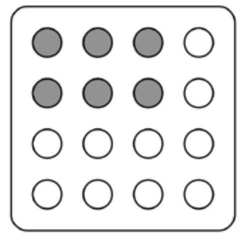
Generating equivalent fractions using a number line. One way to generate equivalent fractions using a linear model (such as a number line) is to represent the given fraction and then further divide each of the parts in the unit interval (the whole) into the same number of equal parts. This can be seen in the examples below:



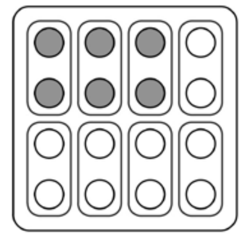
On Number Line A, the unit interval (the whole) is divided into five equal parts and the point is located at the end of the second part to the right of 0 is labeled $\frac{2}{5}$. On Number Line B, each of the five equal parts in the representation of $\frac{2}{5}$ was divided into three equal parts to generate fifteenths, showing that $\frac{6}{15}$ is equivalent to $\frac{2}{5}$. On Number Line C, each of the five parts in the representation of $\frac{2}{5}$ was divided into four equal parts to generate twentieths, showing that $\frac{8}{20}$ is also equivalent to $\frac{2}{5}$.

Reasoning about equivalent fractions using a set model

The meaning of equivalent fractions in the context of a set model. In a part-whole interpretation of fractions, two fractions are equivalent if they refer to the same-sized whole[1] and represent the same size portion of that whole. In a set model, this means that two fractions are equivalent if, for each fraction, the wholes contain the same number of objects and the fractions are represented by same number of such objects. For example, the diagrams to the right show that $\frac{3}{8}$ and $\frac{6}{16}$ are equivalent fractions because the wholes are the same (16 objects in both representations) and the same number of objects are shaded (i.e., 6 objects are shaded in both cases).

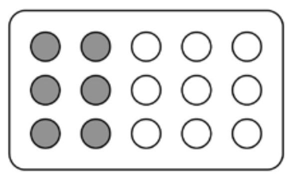


$\frac{6}{16}$

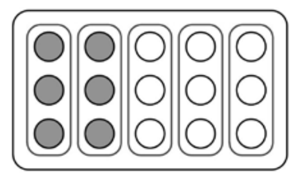


$\frac{3}{8}$

Generating equivalent fractions using a set model. Equivalent fractions can be generated using a set model by representing a fraction and then dividing the whole into subsets containing equal numbers of objects. This can be seen in the diagrams to right where the set of 15 objects was divided into 5 sets each containing 3 objects (with 2 of those subsets being completely shaded). $\frac{6}{15}$ is equivalent to $\frac{2}{5}$ because the wholes are the same (16 objects) and the same number of objects are shaded (6 objects are shaded). Thus, the area model has been used to show that $\frac{6}{15}$ and $\frac{2}{5}$ are equivalent fractions.



$\frac{6}{15}$



$\frac{2}{5}$

Reasoning about equivalent fractions using arithmetic approaches

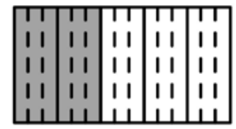
The meaning of equivalent fractions in an arithmetic context. Two fractions are equivalent if a number *m* exists such that multiplying (or dividing) one fraction by $\frac{m}{m}$ produces the second fraction. For example, $\frac{3}{8}$ and $\frac{6}{16}$ are equivalent fractions because multiplying the numerator and denominator of $\frac{3}{8}$ by 2 generates $\frac{6}{16}$. Another example, $\frac{30}{100}$ and $\frac{6}{20}$ are equivalent fractions because dividing the numerator and denominator of $\frac{30}{100}$ by 5 generates $\frac{6}{20}$.

Generating equivalent fractions using an arithmetic procedure. A common arithmetic procedure for generating equivalent fractions is to multiply (or divide) the numerator and denominator of a fraction by the same non-zero number. For example, multiplying both the numerator and denominator of $\frac{2}{5}$ by 3 generates the equivalent fraction $\frac{6}{15}$: $\frac{2}{5} = \frac{2\times 3}{5\times 3} = \frac{6}{15}$. Multiplying the numerator and

[1] It is possible to use wholes that vary in size to show that two fractions are equivalent using a set model. This involves using a ratio interpretation of fractions, which is outside the scope of the module.

denominator by the same non-zero number, *m*, is equivalent to multiplying the original fraction by $\frac{m}{m}$, which is equivalent to 1. For example, in this case, $\frac{2 \times 3}{5 \times 3} = \frac{2}{5} \times \frac{3}{3} = \frac{2}{5} \times 1$ because $\frac{3}{3}$ is equivalent to 1.

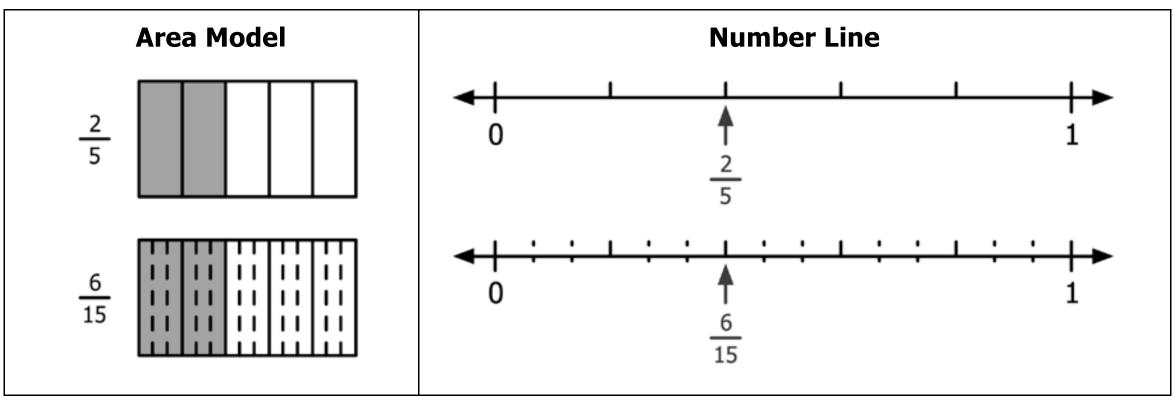
This procedure can be explained using an area model. The fraction $\frac{6}{15}$ is represented in the diagram on the right. Each of the original 5 parts in $\frac{2}{5}$ was divided into 3 equal parts. This corresponds to multiplying the 5 in the denominator of $\frac{2}{5}$ by 3. At the same time, each of the original 2 shaded parts was also divided into 3 equal parts. This corresponds to multiplying the numerator, 2, by 3.



Common patterns in students' reasoning about equivalent fractions.

There are several known challenges for students in reasoning about equivalent fractions.

1. *Two fractions that "look" different may in fact be equivalent.* A challenge for students in reasoning about equivalent fractions is understanding that two fractions that look different in terms of the number of parts, subdivisions, or objects in the set, may in fact be equivalent. For example, in the figure below, $\frac{3}{8}$ and $\frac{6}{16}$ are equivalent even though $\frac{3}{8}$ has 3 shaded parts and $\frac{6}{16}$ has 6 shaded parts. Similar challenges arise with linear models such as number lines.



2. **Partial understanding of procedures for generating equivalent fractions arithmetically.** Students may misremember rules for creating equivalent fractions arithmetically (e.g., thinking that <u>adding</u> two to both the numerator and denominator produces an equivalent fraction), which points to the importance of supporting students in understanding the relationship between an arithmetic procedure and what the procedure means in the context of an area model, number line, or set model.