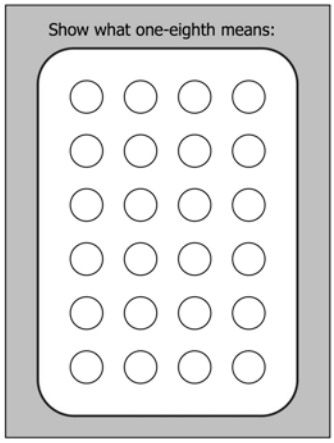
Math Notes: Analysis of the Fraction-of-a-Set Task

Description of the task:

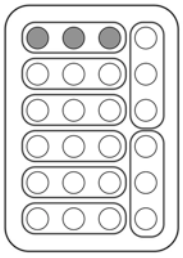
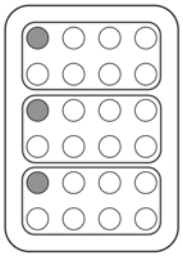
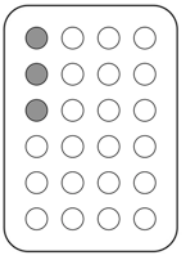


This task concerns showing a fraction-of-a-set of objects. It illustrates a part/whole interpretation of fractions using a set representation. The whole consists of 24 circles and the task is to show $\frac{1}{8}$ of the whole.

What approaches can be used to show one-eighth of the set?

In the context of a set representation, a fraction refers to the number of objects in the whole that have a given attribute (e.g., shaded gray) compared to the total number of objects in the whole. Objects in the whole can also be grouped into subsets of equal size. There are three common types of approaches that can be used to show one-eighth of the set. The approaches are described below and are illustrated in the figures that follow.

- *Approach #1* involves dividing the set of circles (the whole) into 8 equal subsets of 3 (24 ÷ 8 = 3). Each of these subsets consists of 3 circles. Since the fraction has a numerator of 1, only one of these subsets (or three circles) needs to be shaded.
- *Approach #2* involves partitioning the whole into equal subsets, containing a multiple of 8 circles, and then making a set composed of $\frac{1}{8}$ of each of the subsets. In the example below, there are 3 groups of 8 in 24 (24 ÷ 8 = 3). $\frac{1}{8}$ of each subset is 1 (in this case $\frac{1}{8}$ of $\frac{1}{3}$ which equals $\frac{1}{24}$). Combining these subsets yields $\frac{3}{24}$ which is equivalent to $\frac{1}{8}$.
- *Approach #3.* Other approaches can involve computations to find the number of circles that should be shaded. For examples, the task can be thought of as $\frac{1}{8}$ of 24. Multiplying $\frac{1}{8} \times 24$ yields 3. Alternatively, dividing 24 by 8 also yields 3. This results in shading 3 circles.

Approach #1	Approach #2	Approach #3
		

What do equivalent fractions mean in the context of set representations?

In a part-whole interpretation of fractions, two fractions are equivalent if they refer to the same-sized whole[1] and represent quantities of the same size. In a set representation, this means that two fractions

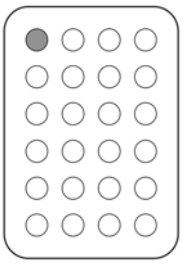
[1] It is possible to use wholes that vary in size to show that two fractions are equivalent using a set model. This involves using a ratio interpretation of fractions, which is outside the scope of the module.

are equivalent if, for each fraction, the wholes contain the same number of objects and the fractions are represented by same number of such objects. For example, $\frac{1}{8}$ and $\frac{3}{24}$ can be shown to be equivalent by using a set of 24 objects as the whole. In the case of $\frac{1}{8}$, the whole (a set of 24 objects) is divided into 8 subsets containing 3 objects each, and the objects in one of these subsets are shaded (i.e., 3 objects total are shaded). The representation produced is like the one shown in Approach #1. In the case of $\frac{3}{24}$, three of the objects in the set of 24 objects are shaded. The representation produced is like the one shown in Approach #3. Because the whole is the same and the same number of objects are shaded in both cases, we can conclude that $\frac{1}{8}$ and $\frac{3}{24}$ are equivalent fractions.

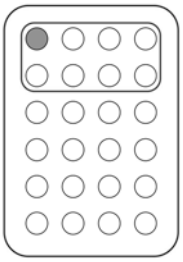
What are common ways that students may misinterpret a fraction-of-a-set task?

In addition to the three approaches described for showing one-eighth of the set, there are three patterns of reasoning that often lead students to create a representation that does not show one-eighth of 24 circles. These patterns of reasoning are described below.

- *Numerator represents the number of circles to shade.* In early experiences with set representations, students often see sets containing the same number of objects as the denominator of the fraction they are asked to show (e.g., show $\frac{1}{8}$ of 8 circles). When the number of objects in the set is equal to the denominator of the fraction being represented, the numerator represents the number of objects to shade. Students often over generalize this rule to sets where the number of objects is not equal to the denominator of the fraction. In the case of showing $\frac{1}{8}$ of 24 circles, a student using this approach would shade 1 circle.



- *Redefining the whole.* Related to treating the numerator as the number of circles to shade, students may redefine the whole (the set of circles) so that the number of circles in the whole is the same as the denominator of the fraction being represented. Then, numerator represents the number of circles to shade. Thus, in the case of showing $\frac{1}{8}$ of 24 circles, the student would treat 8 circles as the whole and shade one of those circles. The representation shows $\frac{1}{8}$ of 8 circles rather than $\frac{1}{8}$ of 24 circles.



- *Denominator indicates the number of circles to shade (the number in a group)*. When showing $\frac{1}{8}$ of 24 circles, a student may divide 24 by 8 and get 3. Then, the student might divide the set of circles into 3 equal subsets of 8 and shade all the circles in one of those subsets. The resulting representation shows $\frac{1}{3}$ of 24 circles.