Math Notes: Analysis of Fraction Comparison Tasks

This document discusses the fraction comparison problem in Session 5, Part 6. Comparing two fractions asks which one is greater. The fractions must be in reference to the same (or same-sized) whole for this question to be meaningful. Below we unpack the mathematics, and also share relevant information about student thinking and relative difficulty, of each comparison problem. After discussing these particular examples, some general comments about what makes pairs of fractions more or less difficult to compare are discussed.

a) $\frac{1}{8}$ or $\frac{1}{5}$

- A fraction with a 1 in the numerator is called a *unit fraction*. Unit fractions are important because they can be thought of as building blocks for other fractions. For example $\frac{7}{8}$ can be interpreted as seven one-eighths. This idea is reflected in the working definition of fractions, which defines $\frac{n}{d}$ as n of $\frac{1}{d}$.

- Unit fractions can be compared by reasoning about the size of the equal parts into which each whole has been partitioned. When partitioning a whole into equal parts, the size of each part becomes smaller as the number of parts becomes greater. With respect to this problem, if a whole is partitioned into 8 equal parts, each of those parts will be smaller than if that same whole were partitioned into 5 equal parts. Therefore, $\frac{1}{8}$ is less than $\frac{1}{5}$ because, although each fraction corresponds to one part of the whole, eighths are smaller than fifths. Unit fractions are the simplest case of the "common numerator" strategy.

- When first learning about fractions, students often think that $\frac{1}{8}$ is greater than $\frac{1}{5}$ because 8 is greater than 5 or because eighths have more pieces than fifths.

- Fifths can be challenging for students to represent on a number line or an area model because it is more difficult to equally partition a whole into an odd number of equal parts. Eighths, however, are easy to make because repeated halving can be used.

b) $\frac{6}{10}$ or $\frac{7}{10}$

- These two fractions have a *common denominator*, which means that they can be compared by simply comparing their numerators: 7 is greater than 6, which means that $\frac{7}{10}$ is greater than $\frac{6}{10}$. This works because fractions with a common denominator have the same-sized parts. The fraction with the greater numerator has more of these same-sized parts, making it the greater fraction.

- Because the denominator is 10, this problem also provides opportunities to make easy connections to decimals.

c) $\frac{5}{6}$ or $\frac{3}{4}$

- These two fractions are both "missing" one part from the whole. They can be compared by reasoning about which fraction is "missing less"—in other words, which fraction is closer to 1. Using this strategy involves reasoning about the size of the part of each fraction, which is another reason that being able to reason about unit fractions is so fundamental. For example, $\frac{5}{6}$ is missing $\frac{1}{6}$ from the whole (i.e., $\frac{5}{6}$ is $\frac{1}{6}$ less than 1), and $\frac{3}{4}$ is missing $\frac{1}{4}$ from the whole (i.e., $\frac{3}{4}$ is $\frac{1}{4}$ less than 1). Because $\frac{1}{4}$ is greater than $\frac{1}{6}$, $\frac{3}{4}$ is further away from 1, which means that $\frac{5}{6}$ is greater than $\frac{3}{4}$.
- Comparing these two fractions is tricky because the fraction with the smaller number in the denominator is farther away from the benchmark (one whole) because it has bigger pieces. This makes it the smaller fraction.
- When first learning about fractions, students may think that these fractions are equal because there are both "missing" one part from the whole.

d) $\frac{5}{6}$ or $\frac{16}{15}$

- These two fractions lie on opposite sides of a common benchmark, one whole. The fractions can be compared by reasoning that $\frac{5}{6}$ is less than 1 and $\frac{16}{15}$ is more than 1.
- While any two fractions that represent distinct numbers can be compared to a benchmark, this strategy works particularly well when one fraction is more than a common benchmark such as $\frac{1}{2}$ or 1 and the other fraction is less than this benchmark.

e) $\frac{3}{3}$ or $\frac{5}{5}$

- These two fractions are equivalent; in fact, both are equivalent to 1. Students learn early on that fractions of the form $\frac{d}{d}$ ($d \neq 0$) are equivalent to 1.
- Representing these fractions on a number line provides an opportunity to discuss the idea that points on the number line can be named in multiple ways. And it is fairly accessible for students to understand that there are (infinitely) many fractions that are equivalent to 1.

Some factors that can impact the difficulty of a fraction comparison problem:

- *Common denominators*. If the two fractions have a common denominator, they can be compared directly and are, therefore, among the easiest types of comparison problems. Comparing fractions with common denominators tends to be easier for students than comparing fractions with common numerators (see below) because, with common denominators, the greater numerator corresponds to the greater fraction, and thus matches the familiar reasoning used with whole numbers.

- *Unit fractions and common numerators.* Unit fractions are usually easy for students to compare because students can reason about the size of the parts (see (a) above). Unit fractions are a special case of "common numerators." Students can apply the same reasoning in (a) to non-unit fractions with a common numerator, because a common numerator implies that the fractions have the same number of parts. Therefore, as with common denominators, fractions with common numerators can be compared directly. However, common numerators require a little more reasoning than common denominators because when the numerators are the same, the fraction with the greater denominator is actually the lesser fraction.

- *Denominator's ease for partitioning*. A fraction's denominator is a main factor in how difficult the fraction is to represent. The easiest denominator is 2, since halving is often easy. More generally, if the denominator is even, one can begin by halving, followed by partitioning each half into a smaller number of equal parts. Repeating this "uses up" all of the factors of 2 in the denominator perhaps leaving an odd number. For example, when partitioning a whole into twelfths, the whole can first be partitioned into halves, each half partitioned in half again to make fourths, and then each fourth is partitioned into thirds. Large denominators can be difficult to accurately partition because the parts become too small to distinguish.

- *Familiarity and number sense*. Some fractions—for example, halves, thirds, and fourths—are more familiar to students and thus, are easier to work with, perhaps because students have a sense of their value. Unit fractions and fractions that are equivalent to one-half or one whole are fractions that students also tend to have developed more number sense about, which can support their comparisons.

- *When one denominator is a factor of the other.* Pairs of fractions in which one denominator is a factor of the other tend to be easier comparison problems. They are easier to compare using an area model or number line because the partitions made for the lesser denominator can be further subdivided to make the partitions for the greater denominator. In addition, it is easy to find a common denominator for such pairs of fractions because the greater denominator can be used as the common denominator.

- *When one fraction is greater than one (or one-half) and the other is less.* Comparison problems in which one fraction is greater than one and the other is less than one are easy for students because it is straightforward to tell whether a fraction is greater than or less than one. Though not as easy as comparing to one, one-half can be used in a similar way. A fraction is greater than one-half if its numerator is greater than half of its denominator, and this is often simple for students to quickly determine. One-half serves as an easy benchmark, for example, when one of the fractions is obviously closer to 0 and the other closer to 1 (e.g., $\frac{1}{8}$ and $\frac{13}{14}$).

- *Pairs that are each "missing" the same number of pieces from the whole (or from one-half)*. Pairs that are each missing one piece from the whole are fairly easy for students to compare using the strategy discussed with (c) and (d) above. However, this same type of reasoning is more difficult to apply (or to see that it can be applied) when the number of missing parts is greater than one (e.g., $\frac{4}{7}$ and $\frac{8}{11}$, which are both missing 3 pieces from the whole).

- *Closeness of the fractions*. When the fractions are close together it can be more difficult for students to compare them using a representation because inaccuracies in the drawing can lead to the incorrect conclusion about which fraction is greater.