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Definitions and Defining in Mathematics and Mathematics Teaching

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What is a mathematical definition? We tend to think of the definition of a *word*, and assume that the definition tells us what that word means, often with several possible meanings in common usage[1]. Note that there is already a kind of paradox at work here, since we define a word using other words, so it must be assumed that we already know the meanings of the words used in the definition. But how then does this all get started in mathematics? We won't go there just yet; read on.

Concept image and concept definition

An important question in mathematics is where a definition comes from or, more specifically, how it gets made, by whom, and why. Note the fundamental shift in perspective in this formulation. A definition is something made by people, and made with a purpose. Defining is something people do, not something delivered from some sanctified authority or out of the mists of history. And the definition is not of a word, but of an idea or concept. The word is just the name given to the idea.

So what comes first is the concept or idea, and only once it is conceived might it have need for a name. Not all ideas need specific names; instead it is sometimes possible to refer to them using words already in use. Because we don't give names and definitions to everything, another important question is by what criteria do we decide what is appropriate to be defined and named, and when?

From this viewpoint, definitions are the product of an evolutionary process. First comes the concept, possibly labeled with a provisional name. You may well ask, "How can the concept be known without a definition?" The answer is that it can have a kind of partial, experiential, intuitive, or perceptual existence, perhaps with many, even archetypal examples, something which some authors[2] call the *concept image*. For example, we all have a strong image of the concept of "tree," but we might be challenged to write down a clear definition. For mathematical examples, take the concepts of "rectangle," or "curve," or "even number." An important historical example is the notion of a "continuous function" that evolved over time from attempts to mathematically describe the motion of an object moving continuously through space. But the concept image still lacks a clear articulation, i.e. criteria for how to exactly characterize examples and non-examples. In this context, a proposal of a definition is not a definitive act of conventional agreement; rather it is tentative, an experiment to see whether it leads to a correct and helpful discrimination of the idea. Stable consensual definitions are typically the result of many rounds of such experimentation, some even stretching over long periods of history[3]. For a teacher to create for students an experience with this kind of evolution, she could try to have an upper elementary or middle school class develop a precise and accurate definition of the concept of "rectangle." For secondary students or preservice teachers, such an activity with the definition of "polygon" could be similarly instructive.

In general terms, these issues may seem to be of mainly philosophical or linguistic interest. But in mathematics, they are central to the culture of the discipline, to its need for precise language, to the generation of new knowledge, and to the teaching and learning of mathematics.

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[1] For example, dictionary.reference.com lists 28 meanings of the word "even" (also used in mathematics), used variously as adjective, adverb, verb, idiom, and proper noun.

[2] Tall, D., & Vinner, S. (1981), Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169

[3] The notion of continuous function is such an example. Others are the notions of area and volume: For what regions of the plane is it "meaningful" to measure its area? Similar questions arise for volumes of regions of space, and these lead to apparent paradoxes. These are subtle questions, resolved only in 20th century mathematics after long struggles with the ideas.

Good mathematical definitions

The core requirements of a good mathematical definition are:

- Precision: The definition names and unambiguously specifies a certain concept[4] and is logically consistent.
- Accuracy: The concept so specified is in agreement with the concept image, and generally understood meaning.
- Usability: The terms of the definition are well understood by the intended users of the definition.

For example, consider the definition: An *odd number* is a number of the form 2k + 1, where k is a whole number. This definition is precise (it unambiguously defines a certain class of numbers), but it is inaccurate (it leaves out negative odd integers), and it is usable only by persons familiar with algebraic notation, and therefore not generally usable in the elementary grades. This definition can be made both precise and accurate by replacing "whole number" with "integer." Notice that the first two criteria are absolute, but the third is relative to who the users are, and what can be assumed about their prior knowledge. Or consider an effort to define "*super numbers*" as those that are prime and also a sum of three consecutive whole numbers. This definition is not precise because, although it is clearly stated, the conditions are contradictory and so no such numbers exist.

These ideas are not new. In his 1914 essay about mathematical definitions and their use in education, Henri Poincaré [5] called attention to these same requirements of precision, accuracy, and usability.

In addition to the core requirements of mathematical definitions, there are other characteristics that are considered to be desirable on more aesthetic grounds. One such quality is efficiency. A definition is considered better if it is not logically redundant. In other words, it should avoid including a condition that is already consequent of other conditions in the definition. For example, when children define rectangles, they may include "four right angles," "four corners," "four sides," which are redundant, but overlook "closed." Or, one could define an isosceles triangle as being a triangle with two equal sides or as a triangle with two equal angles but the definition should not include both. One reason for efficiency, beyond the virtue of honoring simplicity, is a practical one: Being efficient reduces the burden of checking whether an example satisfies the conditions of the definition. A second valued quality of definitions is that they should identify objects or ideas that are mathematically interesting, significant, or useful to name and define. One would probably not honor with a definition those numbers whose decimal representation involves no 3s, for example. On the other hand, we call attention to prime numbers p such that p+2 is also prime, calling (p,p+2) "twin primes." For example: (3,5), (5,7), (11,13), (17,19). It is still not known if there are infinitely many of them.

Multiple, and partial, definitions

Note that we speak above of "a," not "the," good definition. A good definition must precisely and accurately characterize the concept being defined, as well as be accessible. But there can, and often do, exist many possible good definitions for a concept. When this occurs, there arises the problem of choosing one as the definition, and then reconciling this mathematically with the other alternatives that we do not wish to forsake.

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[4] By "concept" here, we refer to a broad range of possibilities, such as a particular object or idea (the unit square, a standard algorithm), a class of objects (prime numbers, pyramids), a process (multiplication of integers, rotations), etc. Furthermore, the definition has a (sometimes tacit) "universe of application." For example, in defining "even numbers," it is often understood by the context, but unsaid, that "numbers" here are meant to be integers. Specification (implicit or otherwise) of this universe is an essential part of the precision of a definition.
[5] Poincaré, H. (1952). Mathematical definitions and education. In *Science and Method* (F. Maitland, Trans.) New York: Dover. (Original work published in 1908).

The familiar example of even numbers is a good case in point. With young children whose number universe is the whole numbers, there are several potential definitions of *even number*.

(1) (Share) A number that can be divided into two equal parts with none left over.

(2) (Pair) A number that can be divided into groups of two with none left over.

(3) (Alternating) Starting with zero, the even and odd numbers alternate on the number line: even, odd, even, odd, even, odd, …

(4) (Unit digit) The digit in the ones place is 0, 2, 4, 6, or 8.

One can reasonably say that, within the whole number universe, each of these options qualifies as a precise, accurate, and usable (by 8-year olds) definition of even number. So which, if any, might one choose, and why? And what, then, of the others?

Since the concept image of even number for children is rooted (at least in part) in the notion of equal sharing, one might argue that options (1) and (2) come closest to capturing this idea. Both of them express important but distinct characteristics of even numbers, so whichever one is chosen as the definition, one would want to know that the other defines the same class of numbers. This mathematical equivalence of (1) and (2) is a special case of commutativity of multiplication by 2:

(1)	$2\times N$	=	$N + N$	is the same as
(2)	$N\times 2$	=	$2 + 2 + 2 + \ldots + 2$	(a sum of N twos)

This could be seen by children, for example, using a 2 x N rectangular array. Option (3) can be considered a "recursive" definition of even numbers. They are the numbers that can be reached from zero by skip counting by twos. In this way, we can discern the equivalence of (2) and (3), for example. But counting by twos would not be a very convenient way to test the evenness of very large numbers. From the point of view of utility, (4) is easily the most convenient; with it one can instantly test the evenness of arbitrarily large numbers. But it can reasonably be argued that (4) is <u>not</u> an appropriate (or "natural") definition of even number, since it is so remote from the concept image of evenness; it is based on an artifact of base ten place value representation, rather than an intrinsic property of the number. So the more appropriate status of (4) is not as a definition, but as a kind of "theorem" about even numbers, one whose proof is accessible once students know more about place value and divisibility.

While even numbers involve integers, the concept of even number arises already in the whole number world, prior to children's exposure to integers. So how, in this whole number universe, can one define the notion of even number and satisfy the requirement of accuracy?[6] One option is to give a partial definition: "A whole number is *even* if…." Or to give a more specialized definition: "An *even whole number* is…"

When mathematical need, not concept image, precipitates definitions

It is important to point out that the framework we have presented – the [concept image → concept definition] evolution – is not adequate to account for all of the ways that definitions arise in mathematics or in teaching. Sometimes a mathematical concept arises not from a concept image or intuition, but

[6] This is similar to the problem of discussing subtraction when saying "we can't take a big number from a small one" is destined to become false, and possibly provoke confusion. There are other intellectually honest ways of saying this; for example, "With the numbers we have now, we can't …" See Ball, D., & Bass, H. (2009, March). *With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures*. Paper prepared based on keynote address at the 43rd Jahrestagung für Didaktik der Mathematik, Oldenburg, Germany.

rather from some mathematical "imperative" to create the concept. A task of teaching then is to provide and support intuitive images or models to give the concept more vivid meaning.

What are these "mathematical imperatives?" Consider this: In the learners' present mathematical environment, one can pose problems that may not have solutions in that environment. For example, if second graders try to subtract 7 from 2, there is no number in the whole number universe that is an answer to this problem. So then there is a kind of "mathematical pressure" to invent a larger mathematical environment in which such a problem does have a solution. Several of the expansions of our number systems are of this nature:

Whole numbers → integers:

> Here the "pressure" arises from trying to solve problems such as 3 – 5, where no answer exists in the whole numbers; expanding to the integers allows a solution –– -2.

Whole numbers → rational numbers:

> In this case, students encounter a need for new numbers when they try to solve problems such as 20 ÷ 8; expanding to fractions allows a solution ––2½.

Real numbers → complex numbers:

> Here the imperative for expansion can arise from efforts to find a square root for -1: Can one find/invent/define such a number?

Each of these expansions involve definitions – of what the new numbers are and of what the arithmetic operations mean with the new numbers. Similar issues of definition arise when extending exponential notation from whole number exponents to integers, or to fractional exponents.

Fractions present a useful example. Young students begin to encounter situations where numbers do not divide evenly. For example, how many cookies does each person get if 4 people share 6 cookies? Each person can have one cookie, but children will instinctively seek to share out all the cookies, and divide the two leftover cookies each in two pieces. The problem for them is then how to name this solution. Many young children will solve the problem correctly as 1½ cookies per person but name the number as 2, not discriminating between whole cookies and half-cookies. They are ready to learn about a new class of numbers, born from division problems. From there we can go on to construct various intuitive concepts of fraction, like part-whole, and eventually develop a mathematical definition and with it a basis for placement of fractions on the number line. Here, development of the concept image and concept definition progress hand-in-hand.

Mathematical definitions are not just helpful; they are necessary for reasoning

Just as in ordinary language, mathematical definitions facilitate the flow of discourse, giving names – sometimes symbolic and sometimes verbal – and characterization to the important elements in a mathematical discourse. Formulation of definitions helps assure that all participants have a shared sense of the meanings for the terms that are used. This condition is a prerequisite for rational and disciplined discourse in any field, but the demand for and the nature of definitions in mathematics is unique.

Mathematical definitions involve the process of "compression" of ideas, a main value and characteristic of mathematics. Mathematical ideas that are complex, multi-layered, and developed over time can be distilled and named by definitions so that they can be assimilated and enter efficiently into our vocabulary and thinking. For simple expressions – like "prime number," "circle," "pi," "$5^{-3/4}$," and "polynomial" – that we utter fluently, imagine the complexity of explaining their meanings from first principles. Since mathematics is partly a hierarchical subject, such compression is essential so that we do not sink under the weight of having to explain from scratch every concept with which we are concerned. Language, expressed carefully with definitions, is one medium for such compression.

Finally, the most fundamental and essential role for definitions – in both mathematics and mathematics education – is as a foundation for mathematical reasoning. In mathematical reasoning, definitions can function like axioms. We have made this point elsewhere:[7]

> Mathematical language is the foundation of mathematical reasoning that is complementary to the base of publicly shared knowledge. *Language* is used here expansively, comprising the entire linguistic infrastructure that supports mathematical communication with its requirements for precision, clarity, and economy of expression. Language is essential for mathematical reasoning and for communicating about mathematical ideas, claims, explanations, and proofs. Language is a medium in which mathematics is enacted, used, and created.
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> Language includes the nature and role of definitions in mathematics; the nature of, and rules for, manipulating symbolic notation; and the conceptual compression afforded by timely use of such notation. Definitions and terms play a crucial role: Not simply delivered names to be memorized, definitions and terms originate in, and emerge from, new ideas and concepts and develop through active investigation and reflection. Definitions and terms facilitate reasoning about those new ideas by naming and specification. Decisions about what to name, when to name it, and how to specify that which is being named are important components of mathematical sensibility and discrimination central to the construction of mathematical knowledge.
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> Some disagreements stem from divergent or unreconciled uses of terminology, whereas others are rooted in substantive and conflicting mathematical claims. The ability to distinguish between issues of terminology and issues of mathematical claims requires sensitivity to the nature and role of language in mathematics.

And Magdalene Lampert has written:[8]

> Mathematical discourse is about figuring out what is true, once the members of the discourse community agree on their definitions and assumptions. These definitions and assumptions are not given, but are negotiated in the process of figuring out what is true.

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[7] Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, and D. Schifter (Eds.), A Research Companion to Principles and Standards for School Mathematics, (pp. 27-44). Reston, VA: National Council of Teachers of Mathematics.

[8] Lampert, M. (1990). When the problem is not the question and the answer is not the solution. *American Educational Research Journal, 27,* 29–63.