rules in Boxes 3-1 to 3-5 all hold. In both systems, all arithmetic is determined by these rules.

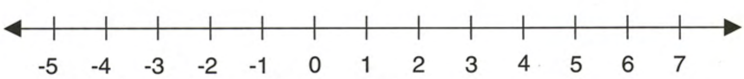
Finally, the two procedures actually produce the same system. The end result is essentially the same, whether one first annexes the negatives and then the fractions, or the other way around. The hard part is making sure that you can actually do it—that there really is a system in which you can add, subtract, multiply, and divide, and where all the rules work in harmony to tell you how to do it. Mathematicians call this system the *rational numbers*.

Arithmetic into Geometry—The Number Line

The rational numbers are harder to visualize than the whole numbers or even the integers, but there is a picture that lets you think about rational numbers geometrically. It lets you interpret whole numbers, negative numbers, and fractions all as part of one overall system. Furthermore, it provides a uniform way to extend the rational number system to include numbers such as π and $\sqrt{2}$ that are not rational;[9] it provides a link between arithmetic and geometry; and it paves the way for analytic geometry, which connects algebra and geometry. This conceptual tool is called the *number line*. It can be seen in a rudimentary way in many classrooms, but its potential for organizing thinking about number and making connections with geometry seems not to have been fully exploited. Finding out how to realize this potential might be a profitable line of research in mathematics education.

The potential for organizing thinking about number and making connections with geometry seems not to have been fully exploited.

The number line is simply a line, but its points are labeled by numbers. One point on the line is chosen as the origin. It is labeled 0. Then a positive direction (usually to the right) is chosen for the line. This choice amounts to specifying which side of the origin will be the positive half of the line; the other side is then the negative half. Finally, a unit of length is chosen. Any point on the line is labeled by its (directed) distance from the origin measured according to this unit length. The point is labeled positive if it is on the positive half of the line and as negative if it is on the negative half. The integers, then, are the points that are a whole number of units to the left or the right of the origin. Part of the number line is illustrated below, with some points labeled.[10]



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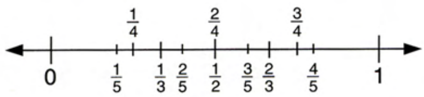
Rational numbers fit into this scheme by dividing up the intervals between the integers. For example, $\frac{1}{2}$ goes midway between 0 and 1, and $\frac{3}{2}$ goes midway between 1 and 2. The numbers $\frac{1}{3}$ and $\frac{2}{3}$ divide the interval from 0 to 1 into three parts of equal length, and the numbers $\frac{7}{3} = 2\frac{1}{3}$ and $\frac{8}{3} = 2\frac{2}{3}$ divide the interval between 2 and 3 similarly. If you locate fractions with different denominators on the line, they may appear to be arranged somewhat irregularly.



However, if you fix a denominator, and label all points by numbers with that fixed denominator, then you get an evenly spaced set, with each unit interval divided up into the same number of subintervals. Thus all rational numbers, whatever their denominators, have well-defined places on the number line. In particular, decimals with one digit to the right of the decimal point partition each unit interval on the number line into subintervals of length $\frac{1}{10}$, and decimals with two digits to the right of the decimal point refine this to intervals of length $\frac{1}{100}$, with 10 of these fitting into each interval of length $\frac{1}{10}$. See Box 3-6.

Box 3-6

The Number System of Finite Decimals

Although they are not usually singled out explicitly, the finite decimals, such as 3, -104, 21.6, 0.333, 0.0125, and 3.14159, form a number system in the sense that you can add them and multiply them and get finite decimals. You can also subtract finite decimals, but you cannot always divide them. For example, $\frac{1}{3}$ cannot be exactly represented as a finite decimal, although it can be approximated by 0.333. The finite decimal system is intermediate between the integers and the rational numbers.

The advantage of working with finite decimals rather than all the rational numbers is that the usual arithmetic for integers extends almost without change. The only complication is that one must keep track of the decimal point. (This seemingly small complication is actually a large conceptual leap.) For example,

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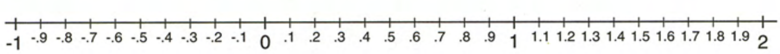
$$\begin{array}{r} 3.14159 \\ +\ .0125 \\ \hline 3.15409 \end{array} \qquad \begin{array}{r} 104 \\ \times\ .333 \\ \hline 312 \\ 312 \\ 312 \\ \hline 34.632 \end{array}$$

The finite decimal system does allow division by 10 (and by its divisors, 2 and 5), and it may be characterized as the smallest number system containing the integers and allowing division by 10. Indeed, another way of representing finite decimals is as rational numbers with denominators that are powers of 10. For example, 21.6 = 216/10 and 0.0125 = 125/10,000.

It may not seem a huge gain to be able to divide by 10. What is the point of enlarging the system of integers to the system of finite decimals? It is that arithmetic can remain procedurally similar to the arithmetic of whole numbers, and yet finite decimals can be arbitrarily small and, as a consequence, can approximate any number as closely as you wish. This process is best illustrated by using the number line.

The integers occupy a discrete set of points on the number line, each separated from its neighbors on either side by one unit distance:

The finite decimals with at most one digit to the right of the decimal point label the positions between the integers at the division points:



If you allow two digits to the right of the decimal point, these tenths are further subdivided into hundredths.

As you can see, space between these numbers is already rather small. It would be very difficult to draw a picture of the next division, defined by decimals with three digits to the right of the decimal point. Nonetheless, you can imagine this subdivision process continuing on and on, giving finer and finer partitions of the line.

continued

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Box 3-6 Continued

Geometrically, the digits in a decimal representation can be viewed as being parts of an "address" of the number, with each successive digit locating it more and more accurately. Thus if you have the decimal 1.41421356237, the integer part tells you that the number is between 1 and 2. The first decimal place tells you that the number is between 1.4 and 1.5. The next place says that the number is between 1.41 and 1.42. The first decimal place specifies the number to within an interval of $\frac{1}{10}$. The second decimal place specifies the number to within an interval of length $\frac{1}{100}$, and so on.

If you think of it in this way, you can imagine applying this "address system" to any number, not just finite decimals. For finite decimals the procedure would effectively stop, with all digits beyond a given point being zero. With a number that is not a finite decimal, the process would go on forever, with each successive digit giving the number 10 times more precision. Thus, the finite decimals give you a systematic method for approximating *any* number to *any* desired accuracy. In particular, although the reciprocal of an integer will not usually be a finite decimal, you can approximate it by a finite decimal. Thus, $\frac{1}{3}$ is first located between 0 and 1, then between 0.3 and 0.4, then between 0.33 and 0.34, and so on.

But once you have started allowing approximation, there is no need or reason to restrict yourself to rational numbers. All numbers on the number line—even those that are not rational—can be approximated by finite decimals. For example, the number $\sqrt{2}$ is approximately 1.41421. Expanding the rational number system to include all numbers on the number line brings you to the *real number system*. Finite decimals give you access to arbitrarily accurate approximate arithmetic for all real numbers. That is one reason for their ubiquitous use in calculators.

NOTE: The finite decimals, also called decimal fractions, were first discussed by Stevin, 1585/1959.

The potential of the number line does not stop at providing a simple way to picture all rational numbers geometrically. It also lets you form geometric models for the operations of arithmetic. These models are at the same time more visual and more sophisticated than most interpretations. Consider addition. We have already mentioned that one way to interpret addition of whole numbers is in terms of joining line segments. Now you can refine that interpretation by taking a standard segment of a given (positive) length to be the segment of that length with its left endpoint at the origin. Then the right

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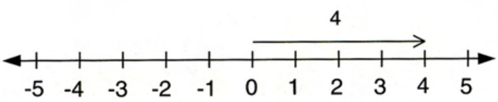
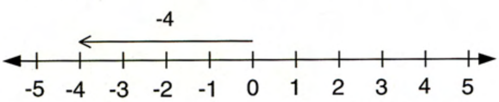
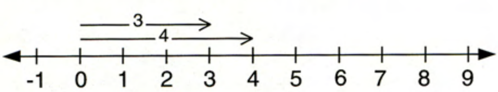
endpoint will lie at the point labeled by the length of the segment. To encompass negative numbers, you must give your segments more structure. You must provide them with an *orientation*—a beginning and an end, a head and a tail. These oriented segments may be represented as arrows. The positive numbers are then represented by arrows that begin at the origin and end at the positive number that gives their length. Negative numbers are represented by arrows that begin at the origin and end at the negative number. That way, 4 and -4, for example, have the same length but opposite orientation. (*Note:* For clarity, arrows are shown above rather than on the number line.)

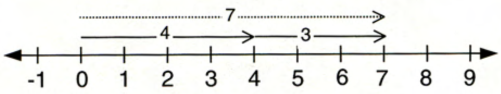




Suppose I want to compute 4 + 3 on the number line. It is difficult to add the arrows when they both begin at the origin:



But the arrows may be moved left or right, as needed, as long as they maintain the same length and orientation. To add the arrows, I move the second arrow so that it begins at the end of the first arrow.



The result of the addition is an arrow that extends from the beginning of the first arrow to the end of the second arrow.

This geometric approach is quite general: It works for negative integers and rational numbers, although in the latter case it is hard to interpret the answer in simple form without dividing the intervals according to a common denominator.

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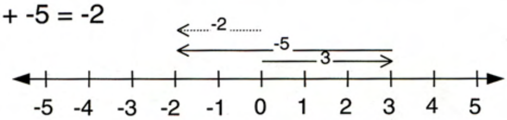
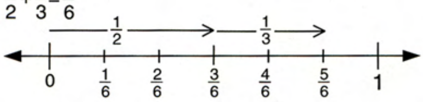
$3 + -5 = -2$

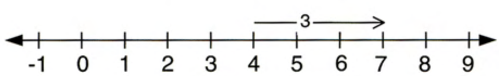


$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

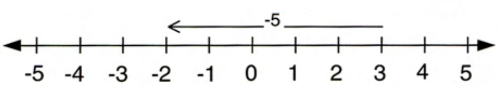


Another method (see below) for illustrating addition on the number line is simpler because it uses only one arrow. The method is more subtle, however, because it requires that some numbers be interpreted as points and others as arrows.

$4 + 3 = 7$



$3 + -5 = -2$



Interpret the first number as a point and the second number as an arrow. Position the beginning of the arrow at the point. The result of the addition is given by the point at the end of the arrow.

Numbers on the number line have a dual nature: They are simultaneously points and oriented segments (which we represent as arrows). A deep understanding of number and operations on the number line requires flexibility in using each interpretation. A principal advantage to this shorthand method for addition is that it supports the idea that adding 3, for example, amounts to moving the line (translating) three units to the right. By similar reasoning, adding -5 amounts to translating five units to the left. In general, adding any number may be interpreted as a translation of the line. The size of the translation depends on the size of the number, and the direction of the translation depends on its sign (i.e., positive or negative).

Multiplication on the number line is subtler than addition. Multiplication by whole numbers, however, may be interpreted as repeated addition:

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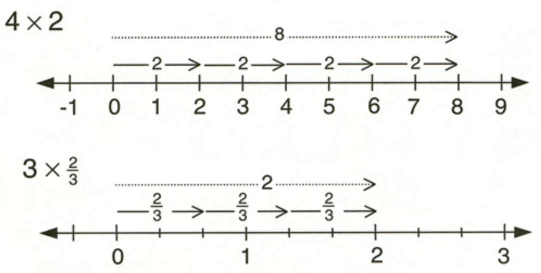

In what way does multiplication transform the line? Multiplication by 4, for example, stretches the line so that all points are four times as far from the origin as they previously were, given a constant unit. Division by 4 (or multiplication by $\frac{1}{4}$) reverses this process, thereby shrinking the line. Then multiplication by $\frac{3}{5}$, for example, may be interpreted as stretching by a factor of 3 and then shrinking by a factor of 5. Multiplication by -1 takes positive numbers to their negative counterparts and vice versa, which amounts to flipping the line about the origin.

These geometric interpretations of addition and multiplication as transformations of the line are quite sophisticated despite their pictorial nature. Nonetheless, these interpretations are important because they provide a way to picture the differences between addition and multiplication. Furthermore, the interpretations provide links between number, algebra, geometry, and higher mathematics.

Nested Systems of Numbers

While the number line gives a faithful geometric picture of the real number system, it does not make it easy to see geometrically the expansion of the number systems from whole numbers to integers to rationals, with each system contained in the next. The schematic picture in Box 3-7 illustrates how the number systems are related as sets. In the center is zero, surrounded on the right by the positive whole numbers and on the left by their negative counterparts. Together they form the integers. In the next larger circle are the rationals, which include the integers as a subset. In elementary school, children begin with the right half of the innermost circle (the whole numbers) and then learn about the right half of the next larger circle (nonnegative rationals). In the middle grades, the two circles are completed with the introduction of integers and negative rationals. In the late middle grades or high school, rationals are augmented to form real numbers.

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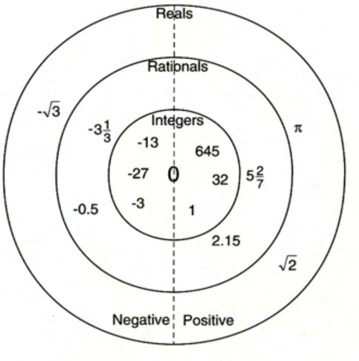
Box 3-7

The Real Number System and Its Subsystems



The number systems that have emerged over the centuries can be seen as being built on one another, with each new system subsuming an old one. This remarkable consistency helps unify arithmetic. In school, however, each number system is introduced with distinct symbolic notations: negation signs, fractions, decimal points, radical signs, and so on. These multiple representations can obscure the fact that the numbers used in grades pre-K through 8 all reside in a very coherent and unified mathematical structure—the number line.

Representations

In this chapter we are concerned primarily with the physical representations for number, such as symbols, words, pictures, objects, and actions.[11] Physical representations serve as tools for mathematical communication, thought, and calculation, allowing personal mathematical ideas to be externalized, shared, and preserved.[12] They help clarify ideas in ways that support reasoning and build understanding. These representations also support the development of efficient algorithms for the basic operations.[13]

Mathematics requires representations. In fact, because of the abstract nature of mathematics, people have access to mathematical ideas only through the representations of those ideas.[14] Although on its surface school math-

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