Abstract

Since antiquity, the spectrum from consonance to dissonance has been central to harmonic theory in the West. Many music theorists in recent centuries have relied upon the results of psychoacoustic experimentation to inform and/or 'verify' certain hypotheses on the nature of consonance and of musical perception in general. The author is apprehensive about taking the psychoacoustic literature as axiomatic as such perceptual experimentation is subjective and perhaps unverifiable: in any human subject, perceptions are shaped by the subject's personal history. This text presents a speculative model of harmonic relationships that avoids the issue of consonance/dissonance altogether in an attempt to avoid this reliance on subjective methods. Instead, we begin by exploring a harmonic space model in which intervallic relationships between tones are represented as vectors in a multidimensional lattice. By inspecting and cataloguing unique collections of tones in this model, a retinue of attributes are developed that serve to classify these collections both in terms of their internal structures and in relation to one another. These novel attributes can be deployed both as parameters and concepts in the composition and analysis of musical systems at micro and macro organizational scales. Furthermore, we build upon these concepts to outline possible approaches to motion between tones, and among collections of tones.

Introduction and Literature Review

[I've given this a few tries, but I'm having trouble knowing how to contextualize this all, without getting lost in the weeds. Need to give it another go …]

Theory

1.1 Tone

Given a set, S, of sinusoids whose frequencies are a subset of a harmonic series with fundamental f, when observed together throughout some temporal duration, if S unambiguously coheres to form a stable impression of an individual pitch corresponding to f, we shall refer to S in aggregate as a **tone** with a frequency of f.

1.2 Interval Ratio

One of the ways in which a tone can be differentiated from another tone is by observing the **interval ratio** between their fundamental frequencies, reduced such that they are coprime integers.

$$IR(t_a,\ t_b) = \frac{f(t_a)}{GCF(f(t_a), f(t_b))} : \frac{f(t_b)}{GCF(f(t_a), f(t_b))}$$

where t_a and t_b are tones, $f(t)$ is the frequency of t, and $GCF(a, b)$ is the greatest common factor of a and b.

1.3 Harmonic Space

Intervallic relationships between tones can be presented in a multi-dimensional model with discrete steps known as a **harmonic space lattice.** In a harmonic space lattice**,** each dimension is associated with a unique prime number, p, such that a change in value, Δ_v, along an axis corresponds to a multiplication of p raised to the exponent Δ_v. Figure 1 plots twelve tones in a harmonic space lattice with prime dimensions of two, three, and five.[1]

[1] Theoretically, *Harmonic Space* has infinite dimensions, one for every prime number, though it is unlikely that primes higher than 19 are of much practical use in tuning theory, as it is thought that there is a natural tendency for humans to 'misperceive' intervals as being in as simple a relationship as possible, within some range of tolerance. For example, the interval 41:20 is said to be perceived and understood as a 2:1 octave.The software accompanying this study allows for the following primes: [2, 3, 5, 7, 11, 13, 17, 19]

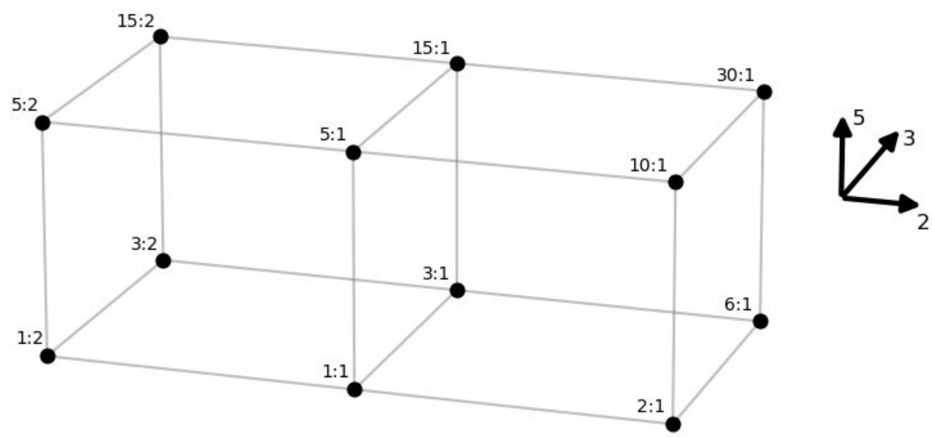


Figure 1. Twelve tones represented as black dots in a harmonic space lattice with dimensions of 2, 3, and 5.

1.4 Harmonic Space Vector

In harmonic space, the position of one tone in relation to another is found by decomposing the interval ratio between the two tones into its prime components—by first vectorizing the exponents of the prime factorization of both sides of the ratio, and subtracting the vector associated with the consequent from the vector associated with the antecedent. This will be referred to as a **harmonic space vector**:

$$HSV(ir) = pe(ir_a) - pe(ir_c),$$

$$\text{such that } x = \prod_{i=0}^{\infty} k_i^{pe_i(x)}$$

where k is the list of prime numbers, *PE(x)* is the list of exponents of the prime factorization of x, ir is an interval ratio, ir_a is the antecedent of the interval ratio, and ir_c is the consequent of the interval ratio.

In figure 2, we show representations of various collections of pitches in Western notation, as interval ratios, as harmonic space vectors, and as plotted in a three-dimensional harmonic space lattice.

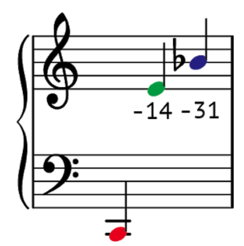
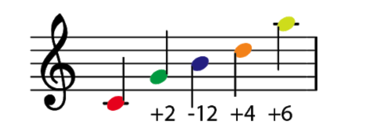
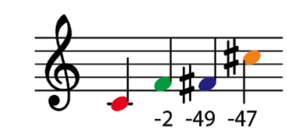
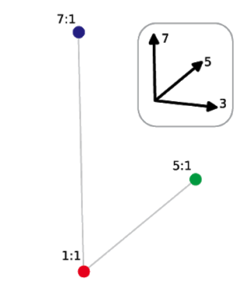
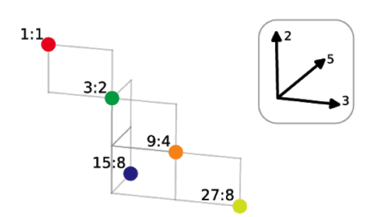
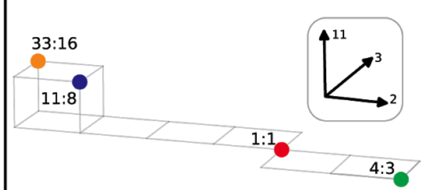
Western Notation			
Frequency (hz.)	65, 327, 458	262, 393, 491, 590, 884	262, 349, 360, 540
Interval Ratio	1:1, 5:1, 7:1	1:1, 3:2, 15:8, 9:4, 27:8	1:1, 4:3, 11:8, 33:16
Prime Factors / Harmonic Space Vector	$\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} 0 & -1 & -3 & -2 & -3 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 0 & 2 & -3 & -4 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$
Harmonic Space Lattice			

Figure 2. Three collections of pitches represented in Western Notation, as a list of frequencies, as a list of interval ratios, as a combination of prime factors and harmonic space vectors, and as plotted in a harmonic space lattice.

1.5 Octave Equivalency

Many theorists have struggled to explain the near-universal acceptance of the notion of octave equivalency—that tones with frequencies that differ by a power of two are equivalent in some sense, and belong, therefore, to a shared *chroma*, or *pitch-class*. Why would it be the case that multiples of two have a special and unique status, while, say, multiples of three do not? Is it a difference of kind, something unique about the perception of “two-ness”, or a difference of degree, that two is just the simplest prime, and stands out as a consequence of that simplicity?

I don't believe this is a question to which there is a knowable answer, so I won't venture my guess. I'll just note axiomatically that in this text, I accept and make use of the concept of octave equivalency, though in a slightly non-traditional way. In harmonic theory texts that makes use of the "language of ratios"[2], it is traditional to octave-generalize all intervallic ratios such that they are situated in the octave between 1:1 and 2:1. For example, 17:4 would be generalized to 17:16, and 3:7 would be generalized to 12:7. This kind of generalization is sometimes applied to harmonic space lattices. Figure three shows this approach, which can be useful in representing more complex intervallic relationships in a 3D rendering, as you can do away with the dimension associated with the number two altogether. However, some key aspects of the structure of the intervallic relationships are lost in this process.

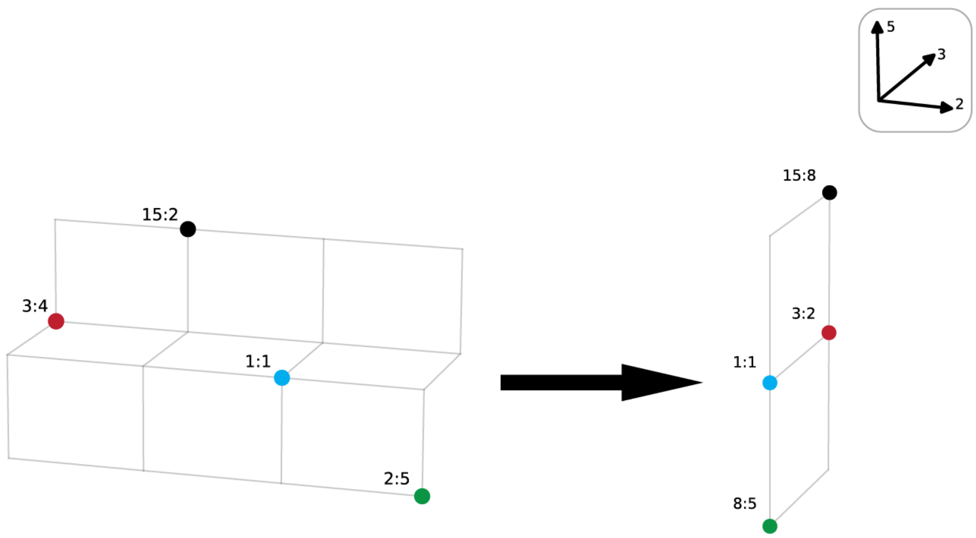


Figure 3. A collection of tones represented in standard (left) and octave-generalized (right) harmonic space.

I am proposing another approach to adjusting registration in a harmonic space context: assign a degree of octave shift to each dimension, such that successive steps along an axis in harmonic space represent consistent intervallic relationships. Figure 4 shows how different octave-shifts applied to a collection of tones in a harmonic space lattice affects their registration,

[2] Partch, Harry. Genesis of a Music. [FIND QUOTE]

presented in standard western notation with cent deviations.

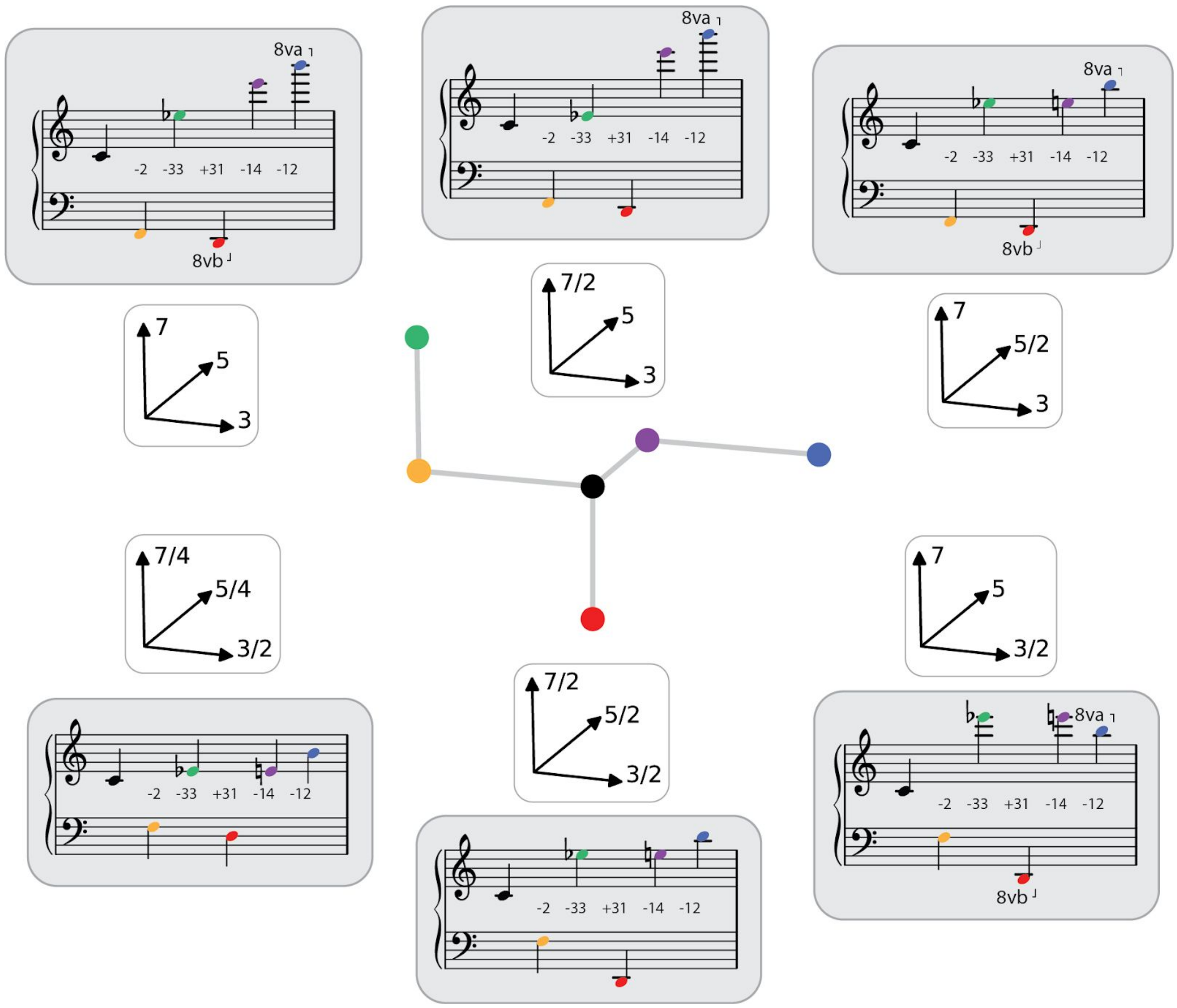


Figure 4. Standard western notations for of a single shape in harmonic space with a fixed prime dimensionality, but with six different octave shifts.

1.6 Containment

Given a harmonic series *A* and a harmonic series *B* whose fundamental aligns with one of the partials of *A*, all of *B*'s partials are contained within *A*. (see Figure 5) Therefore, given *tones A* and *B*, we can say that *A contains B* if the frequency of *B* is a multiple of the frequency of *A*. Similar to music-theoretical conceptions of *polarity* and *rootedness*, **containment** is useful in accounting for harmonic relationships as it provides a clear directionality to harmonic space.

Motion from *A* to *B* presents no new information to an observer—all of *B*'s partials are already included in the harmonic series implied by *A*—while motion from *B* to *A* provides new information in the form of the partials in harmonic series *A* that are less than the frequency of *B*.

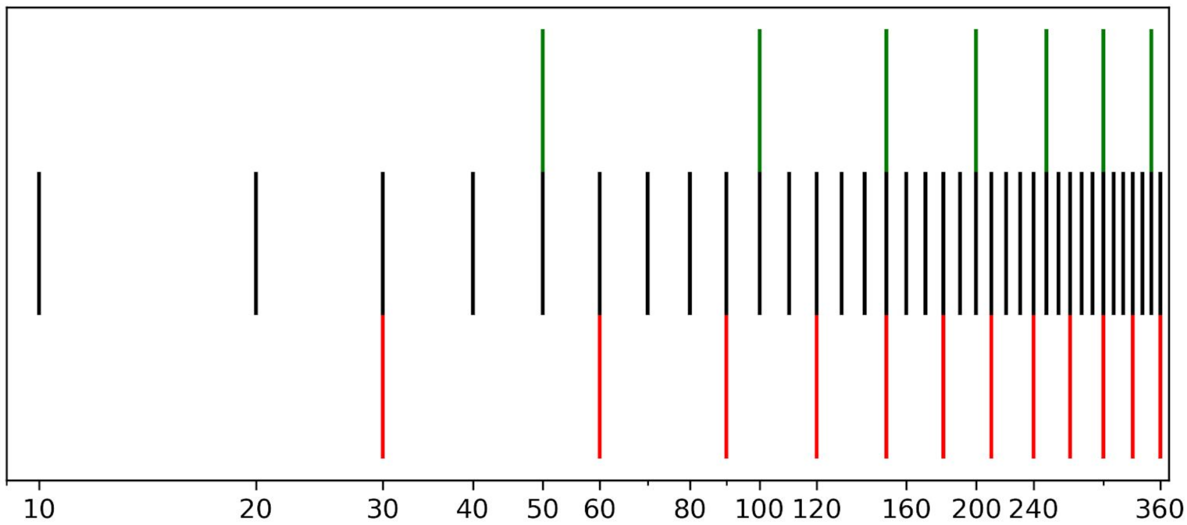


Figure 5. Harmonic series with fundamentals at 10 hz. (black), 30 hz. (red), and 50 hz. (green). All of the red and green partials align with partials in the black harmonic series; a *tone* with a frequency at 10 is thus said to *contain* tones at 30 hz, 50 hz, and any other frequency aligned with one of its partials.

In keeping with our axiomatic acceptance of octave equivalency, we extend the definition of containment to allow for octave shifts in the following way: tone *A* will be said to contain tone *B* if F_B is a multiple of some frequency *z* and F_A is the product of *z* and some power of two.

$$f(T_b) = xz \wedge f(T_a) = z2^y \rightarrow T_a \; contains \; T_b$$

where *x* and *y* are positive integers, *z* is a positive real number, and T_a and T_b are tones.

$$\frac{f(T_a)}{f(T_b)} = \frac{2^y}{x} \rightarrow T_a \; contains \; T_b$$

Where x and y are positive integers and T_a and T_b are tones.[3]

I should note that not all intervals have a containment relationship; in many cases, pitches are ambiguous to one another with regard to containment.

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[3] I think this is equivalent to the above equation, but not 100% confident.

2.0 Collection

The harmonic content of any collection of tones can be explicitly specified using the terms and concepts defined above: (1) a list of harmonic space vectors, (2) the primes associated with each dimensions, (3) the octave shift associated with each dimension, and (4) the frequency associated with the origin of the harmonic space.

2.1 Ordinal position

In order to compare the structures of such collections to one another, and to make a catalog of unique structures, collections must be cast to ***ordinal position***—that is, their simplest possible non-negative form with their axes arranged in order of (1) decreasing range size, then (2) increasing mean value for all tones along a given axis. Figure 6 shows the process of recasting a shape to ordinal position.

$$\left[\begin{array}{c|c|c|c|c|c|c} -1 & -1 & -1 & -1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & -1 & -2 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{c|c|c|c|c|c|c} 1 & 1 & 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 2 & 2 & 2 \end{array}\right] \quad \left[\begin{array}{c|c|c|c|c|c|c} 2 & 3 & 1 & 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

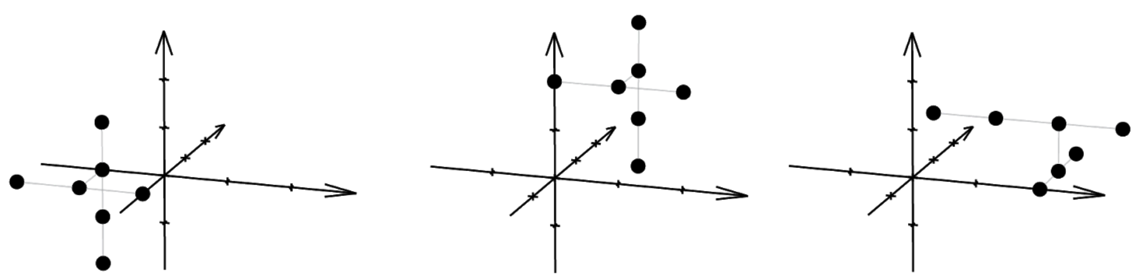


Figure 6. From left to right, recasting a shape to ordinal position. First, translating to the simplest non-negative form (center), before transposing such that the dimensions are arranged in order of decreasing range size.

2.2 Chord

In this text, the term **chord** refers to any collection of tones in a harmonic space lattice in which each tone is no further than one step from its nearest neighbor. In other words, a **chord** is a collection represented by a set of points that are all connected via an unbroken chain of interval ratios that correspond to the ratios associated with the dimensionality of the space. As in Figure 4, each chord can be realized in an infinite number of ways by setting different primes and octave-shifts for each dimension, and assigning different frequencies for the origin of the harmonic space lattice. For the purposes of analyzing the structure of chords in harmonic space though, all of these infinite variations are equivalent.

2.3 Root

Every single-step connection between points in a harmonic space lattice represents an interval with a containment relationship. For a connection between two points along a given axis, the point with the lower coordinate value for that axis contains the other point. Each point in a chord thus either (1) contains at least one other point, (2) is contained by at least one other point, or (3) both. Any point in a chord that is *not* contained by any other points shall be considered a **root** of that chord. All chords have at least one root, though it is possible for a chord to have many roots. The total possible number of roots for a chord with a specified number of tones and number of dimensions is given by the following formula:

[Need to figure this one out! Tricky math problem, I am finding!]

Figure 7 displays three different chords with roots labeled in red.

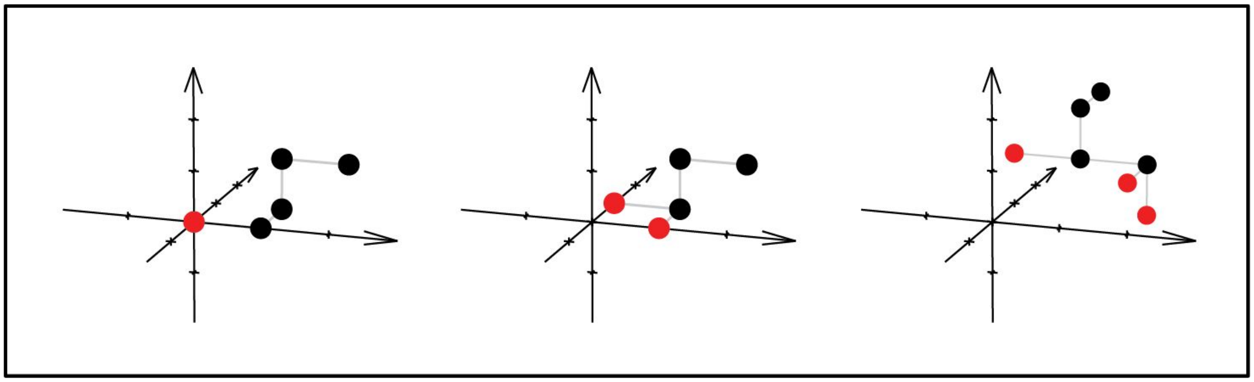


Figure 7. Three chords with all roots colored red.

2.4 Root Breakpoint

For each pair of roots in a chord, there may be at least one tone that is contained by both roots. The simplest such tone—that is, the one that is closest to the origin—is known as a **root breakpoint**. For chords with more than one root, each root has at least one root breakpoint associated with it, and there may be as many breakpoints as pairwise combinations of roots. However, breakpoints between different sets of roots may be located in the same place. Figure 8 displays a few chords with roots and root breakpoints labeled. Remember that containment extends beyond direct orthogonal connections: tones may contain tones to which they are not directly connected, as in the fourth example in figure 8.

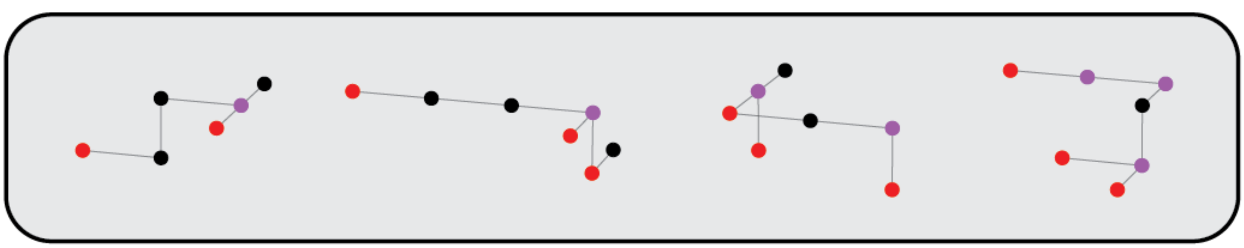


Figure 8. Four chords with all roots labeled red and root breakpoints labeled purple.

2.5 Extremity

An **extremity** is, in a sense, the opposite of a root: any tone in a chord that does not contain any other tones. Every chord has at least one extremity, and perhaps many extremities.

The maximum number of extremities is given by the same formula as for the maximum number of roots in a chord.

2.6 Extremity Breakpoint

The **extremity breakpoint** between any two extremities in a chord is the chord tone that contains both extremities and is maximally far from the origin. As with root breakpoints, for chords with more than one extremity, there is at least one breakpoint for every root. There may be as many total breakpoints in a chord as there are pairwise combinations of extremities.

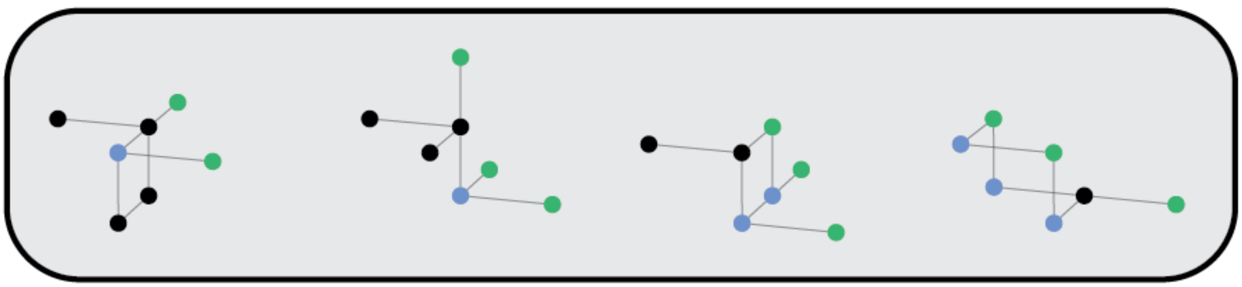


Figure 9. Four chords with all extremities labeled green and extremity breakpoints labeled blue.

2.7 Tree Diagram

It may also be useful to simplify the visualization of the root, extremity, and breakpoint statuses of the points in a chord by notating them as tree diagrams, as in figure 10. Tree diagrams allow for the plotting of some aspects of the structure of chords that have more than three dimensions.

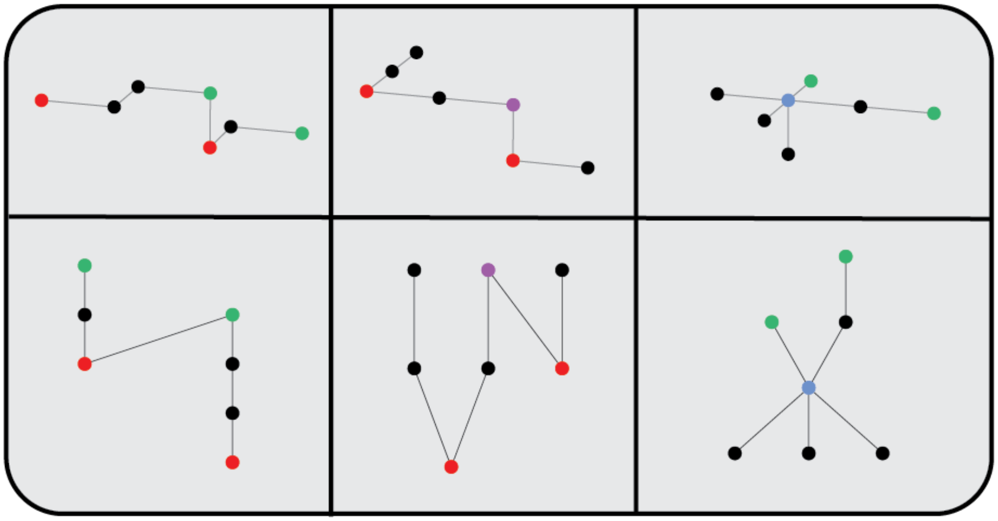


Figure 10. Three chords represented as harmonic space lattices (above) and as tree diagrams (below). From the left, each chord contains colored labels for roots (red) with extremities (green), roots (red) with root breakpoints (purple), and extremities (green) with extremity breakpoints (blue).

2.8 Branch

The term **branch** is used in this text to denote any chord with only one root. Branches are similar to harmonic series aggregates: the frequencies of all of the tones in a branch lie upon a harmonic series whose fundamental shares the pitch chroma of that branch's root. The root of any branch in ordinal position is located at the origin of the harmonic space.

2.5 Transposition

From ordinal position, branches can be cast to a number of different configurations via the **transposition** of dimensions. A branch in a two-dimensional harmonic space has two transpositions depending on which dimension is mapped to each axis. A branch in a three-dimensional harmonic space can be transposed into six configurations according to the six permutations of its axes. Figure 11 shows all transpositions of a branch in a three-dimensional harmonic space. The number of possible transpositions in higher-dimensional spaces is given by the number of permutations of its dimensions:

$$P(n) = n!\text{, where n is the number of dimensions.}$$

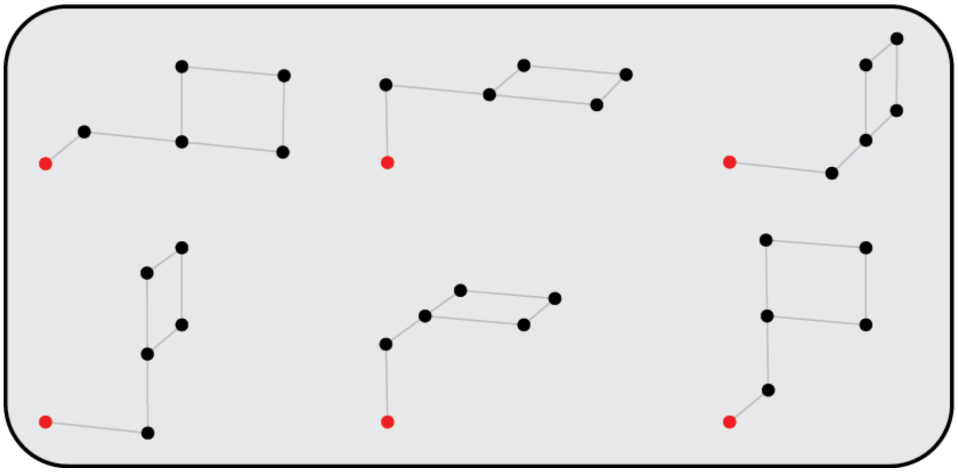


Figure 11. All of the six possible transpositions of a branch in a three-dimensional harmonic space.

2.6 Transposition Shell

A **transposition shell** can be formed by collecting all of the unique interval vectors across all transpositions of a branch. Analysis of the structure of such a transposition shell on its own and in comparison to the branch from which it is generated can lead to a number of quantifiable attributes that are potentially useful in characterizing and classifying said branch. Transposition shells can also serve to classify groups of branches by similarity: branches that share a common transposition shell can be classified into groups according to this affinity. Figure 12 displays a series of branches that would generate a common transposition shell.

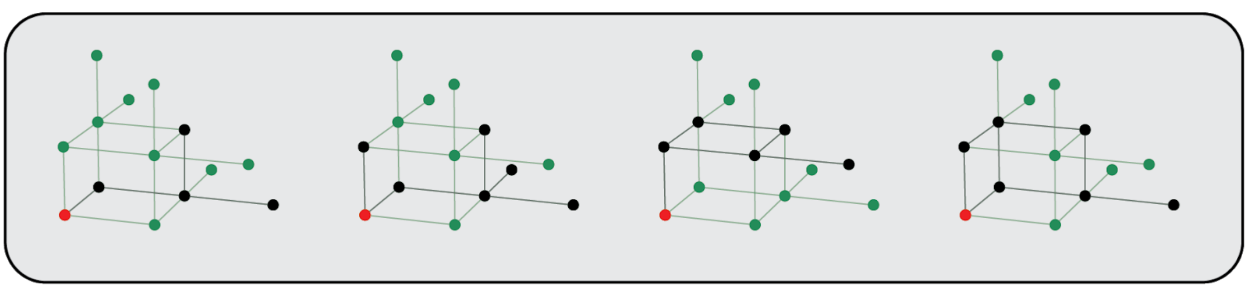


Figure 12. Four branches (in black with root in red) superimposed atop their shared transposition shell (in green).

For chords with more than one root, there is a different transposition shell for each root. Each shell is formed by collecting transformations after re-orienting the harmonic space such that the origin matches the particular root.

2.7 Paths

A path is an unbroken chain of orthogonal connections between two locations in harmonic space in which the directionality of the containment relationship between successive connections is consistent. That is to say, for a set of four locations in harmonic space, *A*, *B*, *C*, and *D*, orthogonally connected in sequence, for this grouping to be classified as a path, either *A* contains *B*, *B* contains *C*, and *C* contains *D*, or *D* contains *C*, *C* contains *B*, and *B* contains *A*. There may be multiple possible paths between the two tones, as seen in Figure 13.

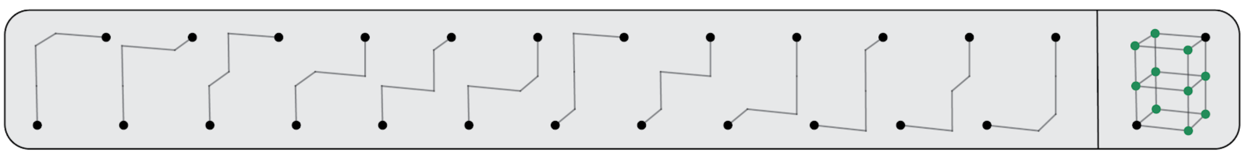


Figure 13. All possible paths from the origin to [1, 1, 2] (left), and their combined multipath (right).

2.8 Multipath

The **multipath** of an interval refers to the union of the points crossed by all of the paths between its two tones (see figure 13). The multipath of any interval forms a hyperrectangle with the same dimensionality as the harmonic space vector associated with that interval.

2.7 Multipath Shell

The **multipath shell** of a branch is a collection formed by a union of all of the multipaths between the origin and the other points in the branch. Similar to a transposition shell, there may be multiple branches that share a common complement shell, and branches can be grouped by this commonality. Figure 14 displays a series of branches that share a common complement shell.

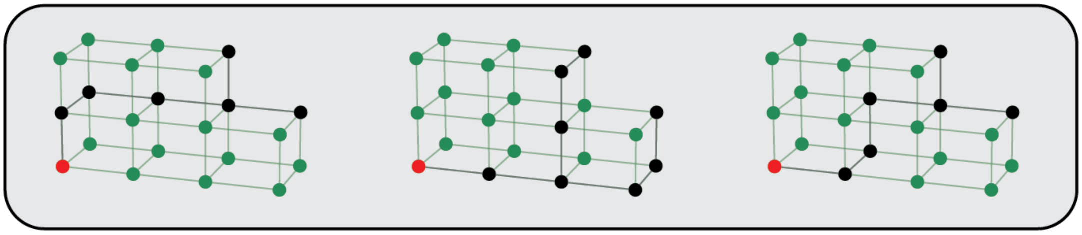


Figure 14. Three branches (in black, with root in red) that share a common multipath shell (in green).

2.8 Root Layout

Any chord can also be classified in terms of the structure of its roots. Chords with only one root, aka *branches*, have a root situated at the origin. A hypothetical tone that would contain all roots of a multi-root chord and is minimally distant from each root is situated at the origin. **[This is basically Terhardt's *Virtual Pitch*, or Rameau's bass/root, excepting the possibility of octave shift. Should I just use that term, Virtual Pitch?]** A chord's **root layout** is the list of harmonic space vectors of all of its roots. Similar to the treatment of chords, for the purposes of cataloging, a root layout is equivalent to all of its transformations. In order to assess this equivalency, the ordinal position of a root layout must be assessed separately from the ordinal position of the chord altogether, as they may differ.

3 Classifiers

3.0 Branch Decomposition

Chords—as well as branches—can be decomposed and examined in terms of the number of branches they envelop. Figure 15 shows one such **branch decomposition**. A chord's **unique branch decomposition** can be found by casting this subset to ordinal position and removing duplicates. The total number of branches as well as the number of unique branches in each of these subsets as well as the ratio between these values can serve as quantifiable measures in characterizing the structure of a chord.

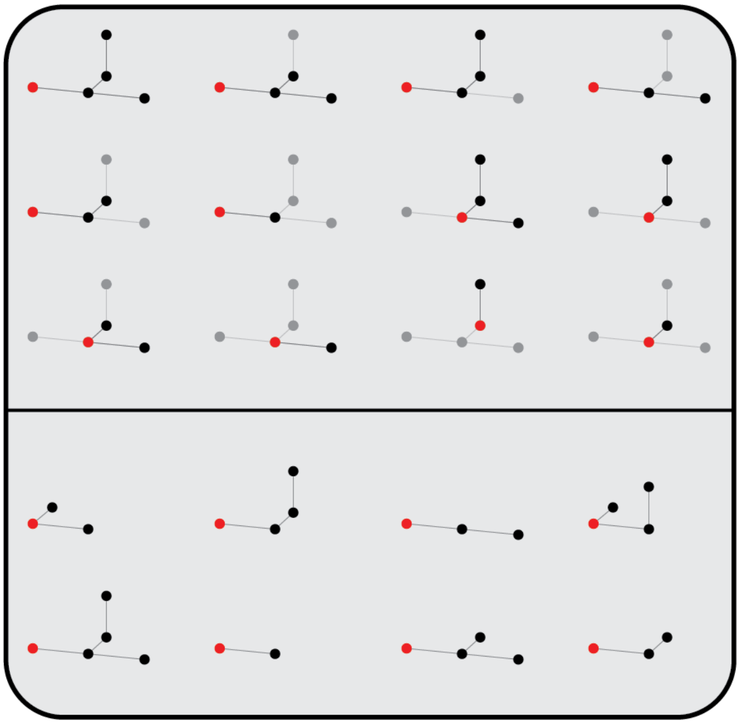


Figure 15. Top: A branch (top left) and its branch decomposition (all). Bottom: The unique branch decomposition of the same branch, in ordinal position.

3.2 Containment Size and Containment Index

A chord's **containment size** is defined as the number of pairwise combinations of tones in a chord in which one of the tones contains the other. The **containment index** is defined as the containment size of a chord divided by the total number of pairwise combinations of tones in a chord.

3.3 Asymmetry

The asymmetry of a chord in relation to one of its roots can be assessed by comparing all of its transformations. For example, given a three-dimensional chord in a three dimensional harmonic space, of its six transformations about a given root, some transformations may be identical to one another. In other words, among those transformations, there will either be six, three, or one unique shape(s). Formally, the **asymmetry** of a chord in relation to a root is defined as the number of unique shapes among its transformations divided by its total possible transformations, as shown in figure 16. This is a measure that spans from 0 to 1, with higher values indicating a higher degree of asymmetry.[4]

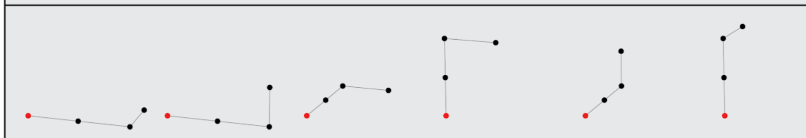
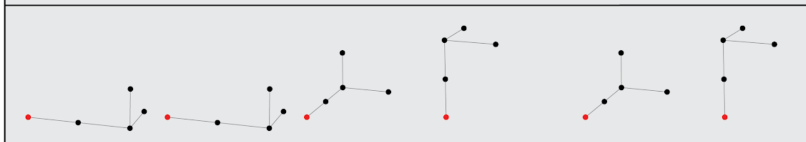
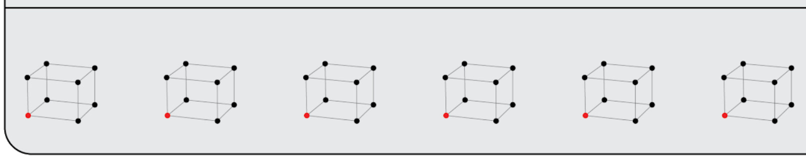
	Total Transforms	Unique Transforms	Asymmetry
	6	6	1
	6	3	0.5
	6	1	0.17

Figure 16. All six transformations of three different branches in three dimensional harmonic space, with associated measures of asymmetry.

3.4 Stability

The **stability** of a chord about a given root can be assessed by taking the average of the number of occurrences of each harmonic space vector among all of the chord's transformations.

$$s(c) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{m} [f_j = i]\,;$$

where $[P]$ is defined to be 1 if *P* is true, and 0 if it is false; *c* is a list of harmonic space vectors representing the tones of a chord; and *f* is a concatenated list of all transformations of *c*.

[4] Why measure asymmetry, rather than symmetry? It may be useful when comparing chords with differing numbers of dimensions to fix the value associated with the least symmetrical situation in a given dimensionality to some constant—in this case, 1. This way, there is a uniform scale when using asymmetry to analyze collections of chords with mixed numbers of dimensions.

Stability is a more finely graded way of quantifying the self-similarity among the transformations of a chord. Figure 17 shows all forty-four of the five-tone branches ranked inj order of their stability.

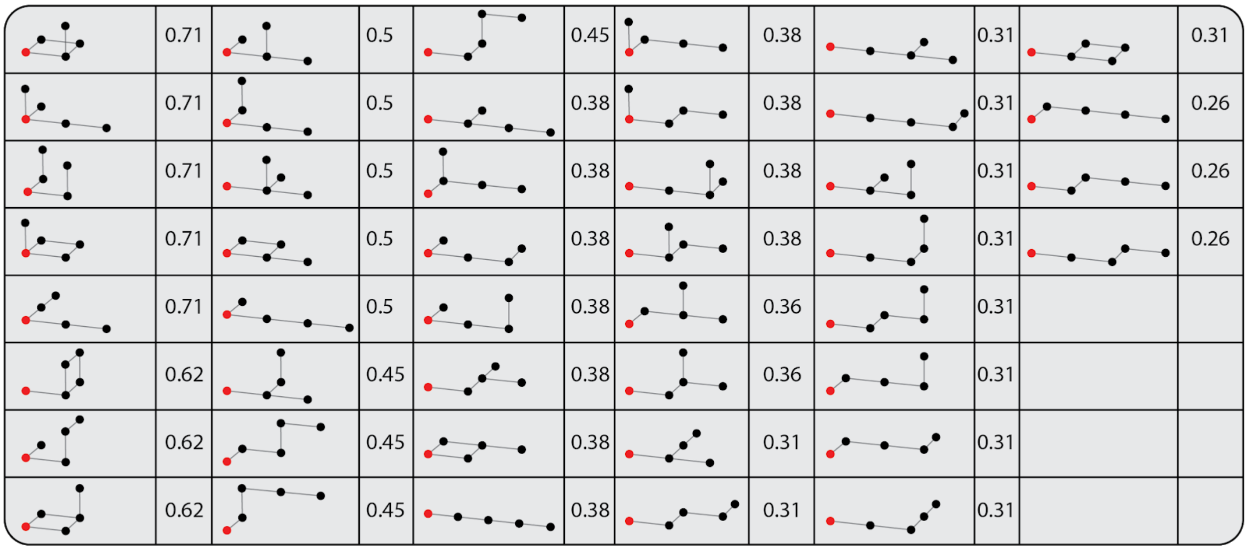


Figure 17. The forty-four unique five-tone branches (in three dimensional harmonic space), arranged in order of decreasing stability.

3.5 Loop

A **loop** is any collection of four tones connected to each other in harmonic space such that they form a square. Chords and branches can be characterized in terms of the number of loops they include.

3.6 Route

A **route** consists of any subset of tones in a chord that are connected in an uninterrupted chain bounded by two endpoints. Each non endpoint tone in the chain is orthogonally connected to two tones, while endpoints of a route are orthogonally connected to either more or less than two tones. Chords can be characterized by the number of routes they include. Figure 18 shows a series of six-tone chords each of which have a different number of routes.

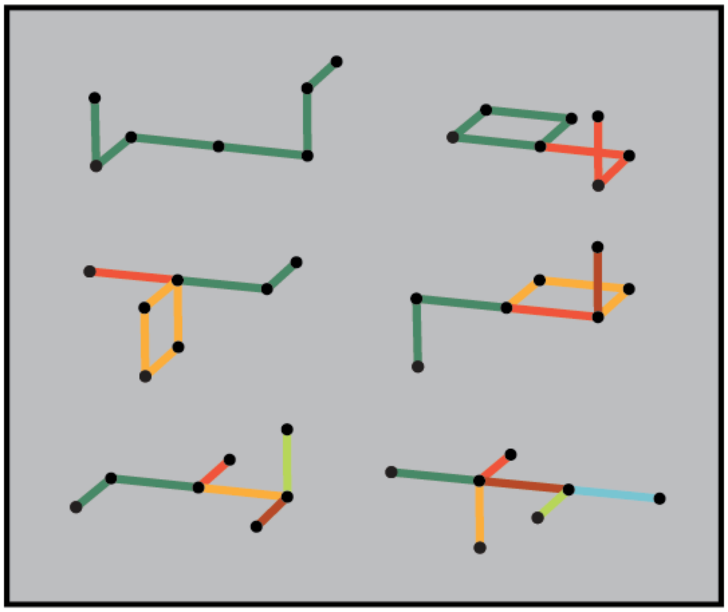


Figure 18. Six seven-tone chords each with differing numbers of routes, with each route visualized in a unique color.

3.7 Transposition Shell Size and Transposition Shell Proportion

The **size** of a chord's transposition shell is given by the number of unique tones in the shell. The **proportion** of a chord's transposition shell is defined as the number of tones in a chord divided by the size of the shell.

3.8 Multipath Shell Size and Multipath Shell Proportion

Similar to transposition shells, the **size** of a branch's multipath shell is given by the number of unique tones in the shell. The **proportion** of a branch's multipath shell is given by the number of tones in a branch divided by the size of the shell. (Multipath shells are not a concept that makes sense for chords with multiple roots, since multipaths are all about drawing all possible paths from an origin to a positive harmonic series vector).

3.9 Mean Root Distance

In order to quantify the root layout, it may be useful to gather the average of the Manhattan distances between the origin and each root. Formally, the **mean root distance** is defined as follows:

$$MRD(X) = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{j,i}$$

$$X = \left[\begin{array}{c|c|c|c} X_{1,1} & X_{2,1} & \ldots & X_{n,1} \\ X_{1,2} & X_{2,2} & \ldots & X_{n,2} \\ \ldots & \ldots & \ldots & \ldots \\ X_{1,m} & X_{2,m} & \ldots & X_{n,m} \end{array} \right]$$

where *X* is the list of harmonic space vectors of the roots of a chord, in root ordinal position, *m* is the dimensionality of the harmonic space, and *n* is the number of roots in the chord.

[Speculation: It stands to reason that chords whose root layout distance is relatively larger would have less of a tendency to imply the toneness associated with the virtual pitch situated at the origin.]

3.10 Mean Root Angle

The mean root angle is defined as the average of the angles between each pairwise combination of vectors in a root layout.

$$MRA(X) = \frac{2(n-2)!}{n!} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} cos^{-1}(X_i \cdot X_j)$$

where *X* is the list of harmonic space vectors of the roots of a chord, and *n* is the number of roots in the chord.

4.0 Catalogues

[I'm at a loss for how to present the catalogues of unique branches, chords, transposition shells, multipath shells, and root layouts. I have written some software that is essentially a sortable spreadsheet with all of the statistics for each unique chord, that lets you visualize and sonify each chord, altering the timbres, playback modes, primes,

octave shifts, transformation, root fundamental frequency, for all 3d chords up to seven tones. Should I just include the tables as an appendix? Inspect the structures of catalogues of unique branches, unique chords, transposition shell group members, and multipath shell group members via graphing each statistic in the group? Try to cluster them?].

5 Trajectories Within a Chord

6 Trajectories in Free Space

7 Trajectories between chords