Morphological metrics

Larry Polansky

Morphological Metrics

Larry Polansky

ABSTRACT

This article describes a number of methods for measuring morphological similarity in music. A variety of *metric equations* are described which attempt to understand certain kinds of musical and perceptual distances between pairs of shapes in any musical parameter. Different techniques for applying these metrics, such as weightings, variants on each class of metric, and multidimensional and multimetric combinations are also described.

INTRODUCTION

How are two melodies more similar to each other than they are to a third? What are the principles and mechanisms by which composers and listeners create and perceive formal similarity, not just in sets of pitches, but in duration series, large scale forms, and perhaps even timbral data?

This article introduces techniques, called *metrics*, for measuring distances between musical, perceptual, and compositional *morphologies*.[1] These functions measure musical similarity. They suggest ways that morphologies of all sorts (melodies, phrases, abstract shapes, tuning systems, duration series, sectional statistics, larger forms) may be organized in terms of their distance from each other.

The measurement of musical morphological distance involves music-theoretic ideas of creation, recognition, and the analysis of parametric variation and transformation. Morphological parameters in music can include sets of pitches, durations, harmonic relations, and sequences of timbral values (like spectra, temporal values for attack and decay transients, and so on). Elements of a morphology may consist of any parameter to which ordered values may be assigned, or any formative parameters for musical composition, analysis, or cognition.

The results of morphological measurement might be used in a variety of musical disciplines such as theory, analysis, and cognition, and especially composition. Many of the functions described below are intended to be generative: rather than reflecting conventional theories of music organization, they can suggest new musical forms.

Each metric is based on certain primitives of morphological perception: contour, magnitude, pairwise relationships. The metrics are also derived from simple mathematical assumptions — fundamental interval functions, mean values, normalization procedures — that provide formal descriptions of musical morphology as well as techniques for using morphological ideas in composition.

MORPHOLOGY

In *Meta + Hodos*, James Tenney (1961) distinguished between *statistical* and *morphological* features of *temporal gestalt units (TGs)* in music. In general, statistical features of a TG are global, order-independent properties, like the mean, range and standard deviation values for a parameter. These values are generally not dependent on their order within the TG. Morphological features of a TG, on the other hand, are described by the morphological "profile" of parametric values in that TG. The order of events is the distinguishing feature between morphological and statistical ones, yet it is difficult to measure morphology except in relation to other morphologies.

Statistical measures like mean (μ) and standard deviation (σ) are not affected by the reordering of values in a sample (or TG). Measures of dependence like the covariance and correlation coefficient do produce measures of morphological similarity *between* two populations by considering corresponding relationships of individual values to the mean, or standard deviation.[2] Morphology tends to be more relational, and to some extent more taxonomical, in our experience. Although we can say the pitch mean of a TG is equal to n, in describing aspects of morphology not only do we generally include more dimensions, but are more likely to do so in a relational way. It is often difficult to absolutely measure aspects of a TG's morphology. Our descriptions are often something like: "the morphology of TG *M* is closer to the morphology of TG *N* than it is to the morphology of TG *O*".[3]

In Tenney's theory, statistical measures of parametric profiles become parametric values themselves at a "higher level" in what he refers to as hierarchical temporal gestalt formation.[4] *Shape* is a result of parametric state differences at the next lower hierarchical level. By measuring differences in "shape" at some level through functions like the metrics proposed here, state differences at the next hierarchical level may also result (this is certainly the case at the acoustical level). The morphological metrics described here assume not only that parametric elements may be distinguished, and that their values may somehow be measured, but that morphs exist on any hier/hol/heterarchical level. These metrics are meant to apply to "lower levels" (like melodies and rhythmic sequences) as well as "higher levels" (such as morphs consisting of the μs or δs of some parameter). They may apply to what might be called micro- (timbre) and macro- (stylistic) levels as well.

Music theory has often considered the recognition and invariance of morphological units, though explicit reference to methodical morphological variation of higher holarchical forms is rare. Aside from harmony, morphology (as in the case of melodic and rhythmic variation) is one of the most important theoretical, compositional and perceptual focuses of western music. In other musical cultures, such as central Javanese court music, morphology plays an equally important role, and could be said to often supersede harmonic relations. Many musical traditions (including our own) recognize morphological invariance as an important structural element. Melodies are recognizable under modal or tonal transposition, but also when expanded or compressed in pitch range in many ways. However, only a few aspects of morphological similarity have been formally described.

DEFINITION OF MORPHOLOGY

A *morphology* (*morph*) is an ordered set M. The *elements* of M are identified as M_i, where i goes from 1 to L. L is the *length* of M. When distinguishing between the lengths of two different morphs, L is notated as M_L or N_L. For the purposes of this article, L is assumed to be ≥ 2.[5]

Morphs are ordered shapes, such as melodies, duration series, harmonic orderings, spectra, or statistical measures of formal segregation, like the succession of mean pitches of sections of a piece (Tenney 1961; Tenney & Polansky 1980). In most of the metrics defined here, it is crucial to be able to say that "M_i comes before M_j", or more simply, $i \leq j$. This is not necessarily temporal order, it may be a ranking of any kind.[6] Elements of a morphology may be of any dimensionality, for example, $M = \{x_1 y_1 z_1, x_2 y_2 z_2, x_3 y_3 z_3, ... x_L y_L z_L\}$. In this case, the M_{ith} element $= \{x_i y_i z_i\}$.[7]

MORPHOLOGICAL INVARIANCE

In this century composers have focussed a new attention on formal aspects of morphology. Schoenberg and other serialists codified transformations (T, I, R, and combinations) to form an experimental canon. Forte, Babbitt, Rahn, Morris, Lewin and others have made extraordinary contributions to revealing the underlying morphological complexity of the 12-tone equal tempered system.

Schoenberg's fundamental transformations (inversion and transposition) seem to correspond (in large part) with human musical experience. Composers and researchers (either in music theory or music cognition), however, do not always share the same goals or motivations. For example, to state that in general the retrograde of a melody is not considered more recognizable than a random

permutation of a melody (e.g., White 1960), does not necessarily lessen its function as a compositional or theoretical tool.

Ideas of morphological invariance and similarity often emanate from cognitive processes such as transposition and contour-preserving transformations. Other morphological transformations, like retrogrades, may not. Schoenberg characterizes his approach to *morphological invariance* in this way:

> Tonality and rhythm provide for coherence in music; variation delivers all that is grammatically necessary. I define variation as changing a number of a unit's feature, while preserving others. (from "Connection of Musical Ideas", (Schoenberg 1975)).[8]

Morphology, most often in the melodic context, has been an important area for research in music cognition, theory and ethnomusicology. In recognition, recall, and cognitive processing of melodic (and morphological) information, different parameters are utilized in different ways. Length, register, loudness, timbre, and articulation play significant roles in similarity judgements. Early studies by Dowling (1982, 1978, 1972, 1971; Dowling & Fujitani 1981; Dowling & Hollombe 1977) discussed the different effects of "contour and scale" in melodic perception, along with memory and recognition of various transformations and distortions of melodic forms. These studies and others have demonstrated that these two parameters are to some extent perceptually (and certainly compositionally) isolatable, and that they are individually quite complex. Other studies in music cognition, experimental psychology (such as Cuddy, Cohen & Miller 1979; Edworthy 1983, 1985; Monahan & Carterette 1985), and ethnomusicology (such as Adams 1976; Kolinski 1965a, b; Becker 1980; Seeger 1960) have considered melodic transformation, judgements of contour and melodic similarity, and taxonomies of contour and melodic form. Deliege (1987) tries to integrate contour similarity into Lerdahl and Jackendoff's (1983) "grouping preference rules".

Recently, music theorists have begun to formally investigate contour and its role in musical invariance. Morris (1989, 1992) has described the formal aspects of contour, and others have made important contributions in the areas of contour invariance and inclusion criteria (Marvin & Laprade 1987; Marvin 1991; Freidman 1987, 1985; for an excellent survey, see Marvin 1990). Some of this author's work in contour and morphology is represented in (Polansky & Bassein 1992; Polansky 1992a, 1987; Polansky & McKinney 1991).

The literature of atonal theory often deals with morphological similarity, even though the fundamental element, the pc-set, usually imposes a particular order upon a collection of actual pitches (as well as eliminating duplications).[9] For example, consider Morris' (1979–80) *SIM* function on two interval vectors and the corresponding *absolute similarity* (*ASIM*) for different length (cardinality) pc-sets:

$$sim(R,S) = \sum_{n=1}^{6} |a_n - b_n| \qquad \textit{(Morris SIM function)}$$

$$ASIM(R,S) = \frac{SIM(R,S)}{\#V(R) + \#V(S)} \qquad \textit{(Morris ASIM function)}$$

R, S are two pc-set-classes; a_n is the number of occurrences of n in the interval vector of S; and $\#V(R)$, $\#V(S)$ are the *cardinalities* of R and S respectively. These functions resemble, in both philosophy and technique, the *ordered magnitude metrics* suggested below, even including the way in which "different length" morphs are handled.[10] Like the magnitude metrics below, Morris' function(s) find higher values for dissimilar set-classes, and low values for similar ones. Like most of the metrics in this article, *ASIM* is scaled by definition from [0,1].

Hermann's excellent survey and analytical study of similarity functions, (1994) discusses "variants" on Morris' *SIM* function, including Teitelbaum's earliest study (1965), and those of Lord (1981) and Isaacson (1990). Rahn (1979–80) provides a comprehensive discussion of set-class similarity relationships under transposition and inversion, including Lewin's powerful mathematical generalization of set-class similarity (1977, 1979–80). In his discussion of Regener's "common-note function" (Regener 1974), Rahn clarifies important issues of scaling and similarity functions on sets of different length. Like Morris' *ASIM* function (which Rahn discusses), the scaling techniques Rahn mentions are similar to ones proposed here. Like Rahn was, I am "... less interested in the numerical values of these functions than in the concepts they embody", but of course both the techniques and results are important.

Other theorists have developed similarity measures within the context of atonal music, some of which are on ordered pc-segments.[11] The use of *order*, nonassumption of *pitch classes*, and the avoidance of the concept of *inclusion* and (T-, I-, R-related) equivalence classes are important functional distinctions between the techniques in this article and similarity functions described by the authors mentioned above.

Morphological metrics may measure any quantifiable parameter (including time- and spectral-domain representations of sound). When they *do* describe pitch (I most often use melodic examples because they are easiest to visualize) pitch-class invariance operators (*T*, *I*, *R* etc.) and stylistically based assumptions of harmony are not invoked. These metrics are intended to be as adaptable to the measurements of different scales as to the comparison of rhythmic sequences; as useful in comparing melodic contour as in comparing spectra.

DISTANCE, SIMILARITY AND METRICS

The concept and measurement of *distance* is fundamental to formal theories of perception (e.g., Attneave 1950; Shepard 1963, 1987; Krumhansl 1978), and to our sense of the world. Without this idea, it is difficult to develop intuitive ideas of similarity, movement and transformation. The simplest distance function, "equal to or not equal to",[12] implies the ability to make a distinction between two objects or gestalts. The perception that melody A is closer to melody B than it is to melody C implicitly assumes that not only is similarity defined, but also *degree* of similarity. Informally, similarity may be defined as the "inverse" of distance, since a distance of zero usually means equality (or equivalence).

Every distance function depends on a deeper concept: *distinction*. Fundamental to the notion of distance is the ability to say that melody *A* is separate and distinguishable from melody *B*. In order to say that two entities, objects, temporal gestalts, events, points, or things are a certain distance apart, these things must first be distinguishable. It will be assumed that all morphs discussed here are in some way distinct entities. However, there is a difference between distinguishable and *equivalent* points: two different melodies which are transpositions of each other are equivalent under most similarity measures. This concept is central to serial and atonal theory, and is also important in the development of metrics (see the comments on the *identity* criteria, below).

METRICS

The words "distance" and "similarity", useful in many contexts, are less well defined than the mathematical notion of a metric, a relation on two points with certain conditions. Not all distance functions are metrics, but all metrics are essentially distance functions.

The usual mathematical definition of a metric is a real-numbered function on a set S of the form $d(a,b) = a$, where:

1) $a \geq 0$ *(non-negativity)*
2) $d(a,b) = d(b,a)$ *(symmetry)*
3) $d(a,b) = 0$ *iff* $a = b$ *(identity)*
4) $d(a,b) \leq d(a,c) + d(c,b)$ *(triangle inequality)*

for all *a, b, c* elements of *S*.

1) *Non-negativity* is intuitively necessary to preclude any form of directionalized distance or "vector". Non-negativity is not a necessary assumption, it can be derived

from conditions 2–4. The simplest definition of pitch interval — C up to G is the same as G down to C (a fifth) — satisfies this condition. This measure between two pitches is in fact a metric, like the standard absolute value metric on the integers.

2) *Symmetry*, or commutivity, says that the distance between *a* and *b* is the same as the distance between *b* and *a*; the order of terms is inconsequential. There are many situations in which it does not obtain, especially in distance functions which model perception and the physical world. For example, consider the metric which is the amount of energy it takes to walk between any two places in San Francisco: symmetry is confounded by gravity.[13] Inclusion metrics are often asymmetrical: for example, if the spectra of timbre *A* is included in the spectra of timbre *B*, $d(A,B)$ = 0, but $d(B,A) \neq 0$ because *B* is not included in *A*.

3) *Identity*, sometimes called "nondegeneracy" (Schreider 1974) says that $\alpha = 0$ for $d(a,b)$ if and only if *a* and *b* are the "same point". It provides a useful definition for "same point".[14] This is also a musically meaningful definition for *invariance*: two morphs can be said to be the "same morph" under a given metric.[15] In a metric which only considers ordered intra-element absolute value differences, two morphs related by inversion or transposition are the same. Schoenberg's 48-row forms, or Forte's (unordered) set-classes are examples. Each may be defined as collections of set-classes which are the same collection under some group of metrics (in serialism, those which are zero between a row and its inversion, transposition or retrograde). Invariance means that some aspect of two morphs (absolute intervals, signed intervals) is the same.

The 48-row forms are not, however, the same rows. They are distinct points *{a, b, c,...}* for which $d(a,b) = 0$, but $a \neq b$, seeming to violate the identity criteria. But the concept of equality $(a = b)$ is more primitive than that of metric equality $(d(a,b) = 0)$, the former referring to logical equality in the set upon which the metric is imposed.[16] Metrics often "redefine" the underlying set to create equivalence classes. Mathematically, it is not quite correct to say that for two row classes, the "serial" metric represents transposition as a zero distance on the set of all rows. That metric should first be defined on the sets of equivalence classes (or mathematically, quotient spaces), that we are, from a practical standpoint, inter-ested in representing. In this case, it might be said that the difference between two rows is some distance function "modulo transposition, inversion, and retrograde".[17]

Not all musical distance functions are metrics. For example, the signed difference between two pitches or pitch classes, ("ordered pitch-class interval" or "directed interval", e.g., *i<6,9>* = 3, *i<9,6>* = 9) is not a metric because of the sign. The "unordered pitch interval" (e.g., *ip{3,7}* = *ip{7,3}* = 4) is a metric, producing an absolute difference between the two pitches or pitch classes irrespective of order (Morris 1991).

In the metrics below, these two steps are often conflated: taking metrics on *representations* of morphology which form equivalence classes, and to some extent, not conforming precisely to the mathematical sense of the identity criteria. For example, the two morphs:

{5, 3, 7, 1} and {6, 5, 9, 2}

may be described as: Down, Up, Down. A metric on that description (the *OLD*, described below) would find the two morphs equal, but the two morphs are not *equal* in the underlying set of all possible 4-element morphs. They are members of an equivalence class recognized by a zero metric value, described by the direction representation.

As another example, take the usual metric on the real numbers, where decimal expansions are rounded to their "limits" (e.g., .99999... = 1). This is not equivalent to saying that decimal expansions are equal (or indistinguishable) in the underlying space. The set of points in a metric space is not inextricably linked to the metric itself. By some other metric (defining a different metric space) they may not all be equal.

4) The *triangle inequality*, is the "strongest" condition. In proving that a function is a metric, the triangle inequality is usually the most difficult condition to verify. It says, in simple language, that "the shortest distance between two points is a straight line". The triangle inequality ensures a kind of regularity to the *metric space* without which behavior in the space would not correspond to intuitive notions of distance.[18]

A metric and a set of points define a *metric space*: "a set which possesses a sensible notion of 'distance'" (Lederman & Vajda 1982, p. 505). There are many common metrics on the integers, reals, and other sets, including the familiar absolute value function on the integers and the Euclidean metric on the reals. Particular metrics are often chosen in order to define what distance in some given space truly means, in other words, what the space "looks like". For example, the "city-block" or "taxi-cab" metric on two-dimensional space (or higher dimension) describes a space in which movement occurs along one dimension at a time:

$$d(a,b) = \left| x_1 - y_1 \right| + \left| x_2 - y_2 \right| \qquad \qquad (city\text{-}block\ metric)$$

where $a = \{x_1 x_2\}$ and $b = \{y_1 y_2\}$. For example, on a violin from middle C at *pp* to high E at *ff* we have to "move" (perceptually, or as a performer) along the two axes of pitch and amplitude, each of which may be done independently.

The Euclidean metric (on two dimensions):

$$d(a,b)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \qquad \textit{(Euclidean metric)}$$

integrates dimensions, reflecting situations in which the natural movement is "across" dimensions, as we envision physical space. The Euclidean and city-block metrics are both examples of Minkowski metrics, and are written more generally as:

$$d(a,b)=(|x_1-y_1|^n+|x_2-y_2|^n)^{\frac{1}{n}} \qquad \textit{(Minkowski metric)}$$

These two fundamental forms, which might be called the *sum of absolute values* and the *square root of the sum of squares* are two different ways of assuring positive values for distances between two points in 2-space. The Euclidean is used more often in statistical measures like standard deviation and correlation coefficient.

The relationship of the two metrics is (Shreider 1974):

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq |x_1-x_2| + |y_1-y_2| \qquad \textit{(Euclidean vs. city-block metric)}$$

or more simply (by setting x_2 and y_2 equal to zero):

$$\sqrt{x^2+y^2} \leq |x| + |y| \qquad \textit{(root of sum squares vs. sum of absolute values)}$$

The hypotenuse is shorter than the sum of the sides: the triangle inequality. The Euclidean metric is often called the L_2 metric, the city-block the L_1, referring to their relationship to the plane and the line, respectively.

The squared and absolute value forms have several important distinctions. The squared form is often mathematically preferred for its algebraic manipulability. The two sides of the plus sign may be written out in long form, making it easier to derive and transform equations.[19] Algebraic expansion is more difficult for the absolute value form. The squared form is also advantageous for continuous functions: x_i may be written as $f(i)$, and squared and "rooted" without reference to specific values for the function (also not the case with the absolute value form). The two forms produce highly correlated results. The squared form may be viewed as a "circle" around a point, the absolute value form as a "90-degree rotated" square inscribed within that circle (Kaplansky 1972). The absolute value is easier to use in the case of simple, discrete metrics because it can be computed without taking square roots. The squared form is useful in computer calculations, and in

describing the relationship of magnitude metrics with the standard deviation and correlation coefficient.[20,21]

MORPHOLOGICAL METRICS

Morphological metrics **are metrics** *d(M, N)* **on** *morphologies (morphs)*

Morphological metrics generally utilize the distinguishing feature of morphs, that they are ordered. Most of the metrics discussed below do not yield the same value if the elements of one of the morphs are rearranged. This seems at first obvious, but commonly used measures and relationships, especially in atonal and serialist theory, do not necessarily reflect order[22] (like the Z-relationship, SC-inclusion, and interval vectors). PC-sets are, in general, unordered, or more precisely, transformed into a normal order.[23] Order may refer to the temporal occurrence of a morph's elements or a ranking. For example, the pitches of a morph may be ordered by their durations.

Morphological metrics may take advantage of some powerful properties of metrics, with immediate compositional implications. For example, in designing "mutation functions" between morphs (Polansky, Burk & Rosenboom 1987; Polansky 1991) straight lines of "mutant morphs" between two morphs *M* and *N* under a given metric can be given by the equation:

$$d(M,N) = (d(M,O) + d(O,N))$$

where *O* represents all morphs which satisfy the equality, and where generally, depending on the length and "grain" (or resolution) of the metric, some small difference is allowed between the two sides of the equation.

Common metrics on real-valued functions are useful models for morphological metrics. For example, given two real-valued continuous functions *f(t)* and *g(t)*, over the range *[m, n]*, two intuitive magnitude metrics are:

$$d(x,y) = \max \{|x(t)-y(t)|\} \qquad \qquad \text{(max or sup metric)}$$

and

$$\int_n^m |f(t)-g(t)|\, dt \qquad \qquad \text{(magnitude metric}^{24}\text{)}$$

or a scaled version

$$\frac{\int_n^m |f(t) - g(t)| \, dt}{|m-n|} \qquad \qquad \textit{(magnitude metric scaled)}$$

The *max metric* measures the maximum difference between two functions, and the *magnitude metrics* measure the average of point-by-point differences in the two functions (Bryant 1985).[25] If the denominator of the *magnitude metric scaled* function is multiplied by the result of the *max metric* on the same two functions, the metric is "normalized" to the maximum range of the functions (as suggested below for the discrete magnitude metrics). This metric might also be scaled by the mean, standard deviation, or other measure of either/or both of the two functions.

Musically, it is often important to measure difference between the "change" in two functions, which are generally discrete, not continuous. In the context of atonal set theory, this would be analogous to studying interval vectors rather than common tone theorems (though they can of course be related). By replacing *f(t)* and *g(t)* with their derivatives of any order, metrics are obtained which measure the average magnitude difference of the corresponding rate of change of two functions (again, for the 1st, 2nd, or nth derivative). For discrete functions (morphs) the derivative is replaced by what may be called the first (2nd,... nth) order difference function. For two morphs *M, N*, of length *L* the "(absolute) magnitude metric" above becomes:

$$\sum_{i=1}^{L} |M_i - N_i| \quad \text{or, scaled by length:} \quad \frac{\sum_{i=1}^{L} |M_1 - N_i|}{L}$$

Taking this metric on successive intra-morphological differences produces the *OLM* (*ordered linear magnitude*) metric described below. This idea may be generalized by the averaging of metrics on successive orders of the derivatives a simplified version of the *Sobalev norm* (Lederman & Vajda 1982, p. 771), informally given as:[26]

$$d(f,g) = \sum_{i=0}^{n} \left[\sqrt[2]{\int (f(t)^i - g(t)^i)^2} \right] \qquad \textit{(simplified "Sobalev" metric)}$$

where *i* indicates the order of the derivative of each function (*f⁰* is the function itself). This is the L_2 version. The L_1 version, averaged over the number of derivatives, might be:

$$d(f,g) = \frac{\sum_{i=1}^{n}\left[\int\left|f(t)^i - g(t)^i\right|\right]}{n} \qquad \text{(city-block, averaged "Sobalev" metric)}$$

In the discrete form, weighting the difference functions, the following version of the *OLM* is obtained:

$$d(M,N) = \frac{\sum_{i=0}^{n}\left[\alpha(i) * \frac{\sum_{j=1}^{(L-i)}\left|M^i - N^i\right|}{L-i}\right]}{\sum \alpha(i)} \qquad \text{("Sobalev-OLM")}$$

where i is the order of difference function on M, N (by any notion of "difference"). The length of M, N decreases by 1 each time. If i begins at 1, rather than 0, the elements of M, N are excluded from the average (transpositions of melodies, for example, will have a distance of zero). $\alpha(i)$ is any weighting function indexed by order of the difference function. $N \le L-1$, where L is the length of M, N. The weighting function allows for greater or lesser sensitivity to activity in higher order difference functions. In simpler terms, it allows for a degree of preference in whether the metric reflects "rippling" or overall morphological shape.

INTERVAL

Because it is usually more important to consider a morph as a set of relative rather than absolute values, most morphological metrics require some inner distance function, or *interval*. The specific interval used can be left general in the metric's definition in order to allow for variation.[27]

For example, the *OLM* (Polansky 1987; described below), measures some version of average absolute magnitude change between corresponding adjacent elements of two morphs. *Interval* may be defined in the very specific sense of measuring arithmetic change in a given parameter:

$$\sum_{i=1}^{L-1}\frac{\left|\left|M_i - M_{i+1}\right| - \left|N_i - N_{i+1}\right|\right|}{L-1} \qquad \text{(OLM, no }\Delta)\text{[28]}$$

or the *root of the difference of squares* form:

$$\sum_{i=1}^{L-1} \frac{\sqrt{(M_i-M_{i+1})^2 - (N_i-N_{i+1})^2}}{L-1} \qquad (OLM,\ no\ \Delta,\ squared)^{29}$$

where N and M are two morphs of length L, and N_i and M_i are the *ith* elements of M and N. In general, I will discuss the absolute value forms of the magnitude metrics, rather than the root of the difference of squared form (which I often refer to as the *squared form* of the metric). I call both of these metrics "no Δ" because in what follows specific interval calculations like those used above (linear differences of adjacent elements), will tend to be replaced with a generalized Δ *function*.

The following shows the forms of the *OLM* on two short morphs, M, N, and their first-[30] and second-order (absolute value) difference functions:


```
(1st order absolute interval)      (2nd order absolute interval)

M = (0,2,4,1,0)   [2,2,3,1]        [0,1,2]
N = (2,3,0,4,1)   [1,3,4,3]        [2,1,1]

OLM, no Δ(M,N) (1st order)   = (1+1+1+2)/4          = 1.25
OLM, no Δ(M,N) (2nd order)   = (2+0+1/3)            = 1.00
OLM, no Δ² (1st order)       = (√3 + √5 + √7 + √8)/4 = 2.360
OLM, no Δ² (2nd order)       = (√2 + 1)/3           = 0.804
```
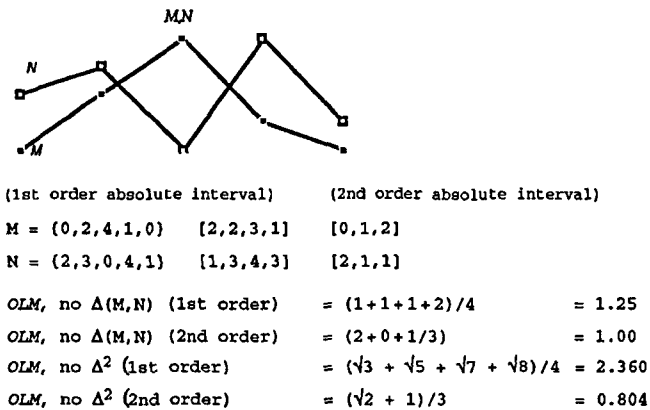
Fig. 1.   Morph example.

This example is similar to the correlation coefficient of two sets, but holds inversion invariant, resulting in positive values rather than [-1, 1]. With this interval, the *OLM* reflects a "dependence" in local linear movement rather than *dependence* or *independence* about the means. Metrics on lower order difference functions tend to be greater (more distant) than higher order ones, just as successive derivatives of many functions tend to "flatten out".

Absolute value (or equivalently, squared difference) intervals within morphs are often inappropriate. The specific interval used does not characterize or define the metric form. For example, in comparing two duration or frequency series, ratiometric intervals would likely be used (both within morphs and between corresponding values of the two morphs). Specific interval functions can be chosen

by the theorist or composer to measure specific types of musical change. I will generally use Δ to signify a generalized interval function, as in the following version of the *OLM*:

$$\sum_{i=1}^{L-1} \frac{\left| \Delta (M_i, M_{i+1}) - \Delta (N_i, N_{i+1}) \right|}{L-1} \qquad\qquad (OLM,\ general)$$

The *(OLM, no Δ)* might represent an *OLM* metric using unordered pitch interval, while the *(OLM, general)*, the *interval class* (Morris 1991) if Δ is defined as:

$$\Delta = \left| min \left[\{a-b\},\ \{b-a\} \right] (mod\ 12) \right| \qquad\qquad (interval\ class)$$

where a,b are two pitch classes defined in the usual way (mod 12). In this form, the *(OLM, general)* results in values between [0,6], averaging the difference in interval classes (0–6) between corresponding adjacent elements of the morphs. Multiplying the denominator by 6 normalizes the function to [0,1] (various normalization and scaling procedures will be described below).

The "interval class (IC)" metric may be rewritten as:

$$\sum_{i=1}^{L-1} \frac{\left| min\{(M_i-M_{i-1})\,mod\,12, (M_{i-1}-M_i)\,mod\,12\} - min\{(N_i-N_{i-1})\,mod\,12, (N_{i-1}-N_i)\,mod\,12\} \right|}{(L-1)*6}$$

$$(IC\ metric)$$

a variant on Morris' *SIM* function (on non-normal form, unordered sets), generalizable to any modulus and, in fact, any notion of intervallic equivalence (not just around the tritone).

Intervals may be characterized as *signed* or *unsigned*, *arithmetic* or *ratiometric*, *directional* or *magnitudinal*. Signed intervals, not metrics themselves, reflect a "greater than/less than" relationship, unsigned intervals do not (and are usually metrics themselves). Directional intervals *only* measure "greater than/less than/equal to" relationships, and magnitudinal ones, *only* the amount of change.[31] Ratiometric intervals (which are not metrics) are generally more appropriate for acoustical parameters (like frequency, amplitude, time) which have not already been converted to psychoacoustic scales (pitch, loudness, rhythmic values). Psychoacoustic descriptions of acoustical phenomena tend to be the results of ratiometric Δs themselves, though of course not always the ones we might want.

Many other kinds of intervals are possible, including *harmonic* ones which determine harmonic distance between two pitches (Chalmers 1993; Tenney 1987; Barlow 1987). I have used harmonic interval functions in two pieces: *Two Children's Songs* for two bass winds (Polansky 1992b) and *Roads to Chimacum* for string or mandolin quartet (Polansky 1993b). These pieces use harmonic mutation functions[32] which make use of harmonic lookup tables, and calculate intervallic distance by "dereferencing" a particular interval from the desired distance (Polansky 1992a). This table may be dynamically changed by the user, and could be filled with values derived from standard harmonic distance functions. Other composers, such as Barlow (1980, 1987), Tenney (1984, 1987), and Scholz (1994) have recently made interesting use of harmonic metrics in their work.

Interval (Δ) can be a complicated or simple function, and obviously need not be a metric itself (as in *min*). The choice of Δ depends on the given musical parameter and the type of inter-element change being measured.

META-INTERVAL

The concept of generalized interval may be extended further to the metrics themselves. For example, in ordered linear metrics, the general form

$$|\Delta M_i - \Delta N_i| \qquad\qquad (meta\text{-}interval\ absolute\ value\ form)$$

has been used most often. This is a simplified representation of the *OLM*. The minus and absolute value signs are arbitrary, depending on musical context. Variations on this form are seen below in various statistical versions of magnitude metrics. Ratiometric, maxima, and other interval calculations are as possible in the comparison of inter-morphological intervals as they are in the calculation of intra-morphological ones (although of course not all intervals will result in metrics).

Δ may be generalized to what might be called meta-interval, or ψ. ψ may be any metric, such as ratio, max, and root of the difference of squares. This allows for greater generality in inter-morphological interval calculation. The following shows the use of ψ in simple metric equations: the *OLM* and *ULM*. The *ULM*, or unordered linear magnitude metric is described in detail below. While the *OLM* may be thought of as a generalized "mean of differences", the *ULM* may be thought of as the "difference of means":

$$\frac{\sum_{i=1}^{L-1} \psi\big(\Delta(M_i), \Delta(N_i)\big)}{(L-1)*(maxint)} \qquad\qquad (OLM,\ meta\text{-}interval\ form)$$

where *maxint* is the maximum Δ in M and N^{33}

$$\psi \left(\frac{\displaystyle\sum_{i=1}^{M_L-1} \Delta(M_i)}{M_L-1}, \frac{\displaystyle\sum_{i=1}^{N_L-1} \Delta(N_i)}{N_L-1} \right) \qquad\qquad (ULM, \text{ meta-interval form})$$

An example of a "*ULM*-style" metric from atonal set theory would be the absolute value of the difference of the means of two interval vectors. The meta-interval form of the *ULM* is similar to various forms of the standard deviation (see below, under "Statistical Variations of Magnitude Metrics"). Meta-interval forms are possible in all of the metrics described below, including direction metrics.

INTERVAL CALCULATION INDICES

The notation M_i allows for generalization of elements between which intervals are taken; M_i and some other index in M. Adjacent intervals are one form of calculation between M_i and M_{i+1} or M_{i-1}. Several other types of indices may also be used. In statistics, intervals are often taken to the mean (μ) of a sample. In the equations below for linear metrics, indices of the form $\Delta(M_i, M_{i+1})$ are generally used. However, $\Delta(M_i, M_1)$, $\Delta(M_i, M_{i-n})$, $\Delta(M_i, M_\mu)$, $\Delta(M_i, M_{max})$, and $\Delta(M_i, M_L)$ are all useful interval index forms.

The *adjacency interval (AI)* of a metric may be any value $< L$. If $AI = 2$, intervals are taken between every other value. In *combinatorial metrics* (metrics which measure a greater number of relationships than purely adjacent ones) this may be further extended to both adjacency column interval and adjacency row interval, which need not be the same. *AI* may change within a metric (for example, shrinking towards the end).[34]

Fundamental indexing means that intervals are taken to some value f which may or may not be an element of the morph. If $f = M_i$ for some i, than i is called a *fundamental index*. If $f \neq M_i$ for some i, than f is called a *fundamental value* or, informally, *phantom fundamental*. A standard technique is to set the *mean of the intervals* in a morph to be the phantom fundamental, relating magnitude metrics to the correlation coefficient and standard deviation.

As an example of generalized indexing, the *(OLM, general)* can be rewritten from $\Delta(M_i, M_{i+1})$ form ($AI = 1$), to $\Delta(M_i, M_f)$ and $\Delta(M, f)$, the former using a fundamental interval, the latter using a phantom fundamental. This might be appropriate, for example, if all pitches in M and N are viewed in harmonic relationship to a specific pitch in M (M_i), not an element of N.

Different indexing systems may be used for each morph in a metric. The notation may be generalized further by using M_{in_l}. For example, the *(OLM, meta-interval form)* may be rewritten as follows:

$$\frac{\displaystyle\sum_{i=1}^{L} \psi\!\left(\Delta(M_{in_l}), \Delta(N_{in_l})\right)}{(L-1)*(maxint)} \qquad\qquad (OLM,\ generalized\ interval)$$

a mathematically useful form for expressing a canonical template for the *OLM*.[35]

Different interval indices, particularly fundamental indices, have been useful in my own composition, especially in the designing of *mutation functions* which transform one morph into another by a prescribed distance (like an inverse metric) (Polansky 1992a; Polansky & McKinney 1991; Polansky, Burk & Rosenboom 1987). In the program *Soundhack*, which implements spectral mutation functions, the user may take all intervals between adjacent spectral frames, or to a user-definable "fundamental" absolute amplitude value for a spectral frequency-bin (Erbe 1994; Polansky & Erbe 1996). In my piece *51 Melodies...* for two guitars and rock band (Polansky 1991), the pitch mutations take all intervals to a fundamental of E, which may or may not appear in the melody. However, this ensures that the melodies themselves all "relate" around that tonic center (Fig. 2).

For combinatorial metrics where interval calculations on all or some subsets of the possible pairwise relationships of a morph are required, a conceptually similar standard indexing system may be used. Columns of combinatorial matrices are treated as "nested inner loops" of the "outer loops" of rows (see Fig. 3).

The top row contains the differences between the first and all succeeding values. The "inner diagonal" (not the identity, but one diagonal higher) of the matrix contains the adjacent linear intervals. The choice of which index to use for "row and column" is arbitrary, but confusing if not kept consistent.[36] For the combinatorial metrics below, j is used for inner loops and i for the outer ones in the following general form: ·

$$\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \Delta(M_i, M_j) \qquad\qquad (general\ combinatorial\ form)$$

COMBINATORIAL AND LINEAR METRICS

Combinatorial and *linear* metrics differ in the number of intra-interval calculations.[37] The number of possible pairwise intervals or relations in a morph of length L is:

excerpt from *51 Melodies...*

Fig. 2.   Two pages from the score for *51 Melodies...* (beginning with the Target, and continuing on for several mutations).

Fig. 2. Continued.

$$L_m = \frac{L^2-L}{2} \qquad \text{(number of pairwise relationships)}$$

often called the second-order binomial coefficient of L.[38] The *degree of combinatoriality* #L of a metric is the number of intra-morphological intervals used in its calculations (#I) scaled by the number of possible intervals, L_m:

$$\begin{array}{llll} \Delta(M_i,M_{i+1}) & \Delta(M_i,M_{i+2}) & ... & \Delta(M_i,M_L) \\ & \Delta(M_{i+1},M_{i+2})... & & \Delta(M_{i+1},M_L) \\ & ... & & \\ & & ... & \\ & & ... & \Delta(M_{L-1},M_L) \end{array}$$

Fig. 3.   Combinatorial metric matrix.

$$\#I/L_m = \#L$$

#L typically ranges from *2/L* to 1, *#I* from *L-1* to L_m. I have called *linear* those metrics where $\#I \le L$. Generally, for phantom fundamental or fundamental indices, $\#L = L/L_m$ or $\#L = 2/(L-1)$ for linear metrics. Similarly, *#L= $L-1/L_m$ = 2/L* when adjacency intervals are used.[39]

A metric is *combinatorial* when $L < \#I \le L_m$ (or *#L > 2/L*). There is a wide range of possibilities for *#I* between *L-1* and L_m. For example, only even-numbered rows might be considered important in a morph's interval matrix, or intervals might be calculated between elements which are not more than two elements "away". On long morphs such as waveforms, some form of stochastic interval sampling might be used. For simplicity's sake, most of the equations in this article use *L*, *L-1*, or L_m intervals. I have not attempted to generalize the indexing notation to represent all these possibilities. However, this is easily done (in software, and mathematically) by leaving the concepts of index, adjacency, and fundamental as variables.[40]

A standard example from mathematical analysis, of what I am calling a combinatorial metric is given by:

$$d(M,N) = \max_{i,j} | M_{ij} - N_{ij} | \qquad \text{(*"matrix max-metric"*)}$$

where *M,N* are two matrices (Lederman & Vajda 1982).

The *grain* of a metric may be defined as one over the number of possible values that a metric may return (in terms of *L*). The grain of combinatorial metrics is smaller (more sensitive) than linear ones. Compositionally, the grain of a metric might be used to define what is meant by "continuity" in a metric space. That is, two morphs are as "close as they can be without being the same" in a given morphological metric space if they are different by 1 unit of the metric's grain.

ORDERED AND UNORDERED METRICS

Ordered metrics make use of the *corresponding* order of elements between two morphs. That is, they compare some notion of $\Delta(M_i)$ and $\Delta(N_i)$ for some specific i, or function of i, which might be as simple as:

$$f(i) = i+1 \ (mod\, N_L)$$

displacing the "corresponding" interval by one. Ordered metrics are sensitive to corresponding differences and similarities between morphs, and as such an ordered form of a metric is generally greater than its corresponding unordered form (the morphs are "more different"). The correlation coefficient of two samples is ordered (but not a metric), since it uses the covariance of two samples, which multiplies the *corresponding differences* of the *i*th values with the mean of each sample (Wonnacott & Wonnacott 1979, p. 99).

Unordered metrics do not use corresponding intervals, but preserve in some way, order within morphs. The difference in average values of two morphs is not truly morphological: a morph has a zero distance from all its permutations. The same idea might be generalized to a metric on standard deviation of values, average or standard deviation of intervals, maxima or range of intervals or values.

Unordered metrics, while less discriminating than their ordered counterparts, have several advantages. They are less sensitive to localized differences, and tend to discern more general shape similarities. More practically, they can be used on morphs of unequal length without any further techniques, since they are essentially statistical measures.

Another feature of unordered metrics is that, being more statistical, they are better than their ordered equivalents at recognizing displaced patterns. The seemingly obvious but formally difficult problem of recognizing the similarity between the two morphs in Fig. 4 necessitates at least some combination of ordered and unordered metrics. Unordered metrics will yield values which indicate almost total similarity (except for two points), ordered metrics will yield values which indicate that these two morphs are extremely dissimilar (which they are, in terms of their corresponding morphology).

Fig. 4. Morph example.

METRIC EQUATIONS

The four fundamental forms for magnitude and direction metrics are:

	Magnitude Metrics		**Direction Metrics**	
	Linear	Combinatorial	Linear	Combinatorial
ordered	OLM	OCM	OLD	OCD
unordered	ULM	UCM	ULD	UCD

Each form has different versions, including various scaling techniques (absolute, relative, and unscaled), interval functions and indices (intervals to the mean or phantom fundamental, adjacency intervals), *orders* (i.e., metrics on *nth*-order difference functions of the morphs), and other variations. Magnitude metrics can generally be rewritten from the absolute value form to the root of the difference of squares form, and direction metrics can be generalized to *n-ary* contours (Polansky & Bassein 1992) by extending the *sgn* function to reflect the grain of contour. In other words, these basic forms can be extended to reflect a variety of musical ideas.

In the functions below, M and N are generally assumed to be of equal length. This condition is necessary for the ordered metrics and unnecessary for the unordered ones.[41] Δ notation is used to make the equations more general, usually in the magnitude metrics. In most of the magnitude metrics, for $\Delta(M_i, M_{i+1})$, one might substitute the example intervals:

1) $|M_i - M_{i+1}|$ (*absolute value*)
2) M_i / M_{i+1} (*ratio*)
3) $max\{\Delta(M_i, M_{i+1})\}$ (*max of some function*)
4) $(M_i - M_{i+1})^2$ (*squared difference*)
5) $\sqrt{(M_i - M_{i+1})^2}$ (*root of squared difference*)

or any version of these where M_μ replaces M_{i+1} (i.e., intervals to some mean interval). Any metric may be used for Δ. In general, any interval function which results in the larger equation being a metric is valid for Δ. Note that all Δs above yield positive values, all except the ratio interval (which is not even commutative) are metrics. 1) and 5) are the city-block and Euclidean metrics themselves, in one dimension.

The basic metric equations use the simplest possible interval indexing (adjacency for the linear metrics, "row then column" indexing for the combinatorial ones). Substituting different indexing schemes is quite simple and often useful. For example $\Delta(M_i, f)$ can be used instead of $\Delta(M_i, M_{i+1})$, where f might typically be

M_μ, or $\Delta(M_i, M_f)$ where M_f is some fixed index in M. Note that in this case, $\#I = L$, instead of $L\text{-}1$ (the denominator of the metric should be adjusted). The linear equations assume $AI = 1$, and run from $i = 0$ to $i = L\text{-}1$, taking intervals between elements M_i and M_{i+1}. However, they could all be rewritten with $AI = 2$, for instance, running from $i = 0$ to $i = L\text{-}2$, taking intervals between M_i and M_{i+2}.

NORMALIZATION

Metric values are usually *normalized* to the closed real-valued interval [0,1]. A value of 0 signifies that the two morphs are the same point in the metric space (by definition); a value of 1 means they are "as far apart as possible". Normalization may not always be desirable. In magnitude metrics, unscaled, unbounded results may be appropriate for certain situations. Direction metrics are normalized by definition (see below). Several scaling and normalization techniques are suggested below for magnitude metrics.

DIRECTION METRICS

Direction metrics measure *contour* differences between morphs. A morph and some "contour preserving" distortion are the same under direction metrics. They are listed here from least to most sensitive. All use the contour function:[42]

$sgn(\Delta(M_i, M_j)) =$
                1 or +, where $M_i > M_j$ ("goes down")
                0, where $M_i = M_j$ ("stays same")
                -1 or $-$, where $M_i < M_j$ ("goes up")

The assignments of -1 (is less than), 0 (is equal to), and 1 (is greater than) are arbitrary.[43] Any three values or symbols may be used to represent the ternary relationship (Polansky & Bassein 1992). Direction metrics are by definition scaled to [0,1], since they measure a percentage of total values which are different in contour between M, N.

ULD (UNORDERED LINEAR DIRECTION)

The simplest form of a direction metric measures the differences in average "up-ness, down-ness and equal-ness" between two morphs:

$$\frac{\sum\limits_{v=(-1,0,1)} |\#_v M - \#_v N|}{(L-1)*2}$$

$\#_v$ is the number of intervals in M where $sgn(\Delta(M_{i+1}, M_i)) = v$; $v = \{-1, 0, 1\}$ (see the definition of the sgn function). The ULD measures the difference in the linear contour "vectors" of two morphs, where the three values in the vector are the number of occurrences of $\{-1,0,1\}$.[44] For example, a linear contour vector for a morph where $L = 4$ could be [003], indicating three 1s, or all ascending intervals (e.g., $M = \{2,3,4,5\}$)[45] and another [300], indicating all descending (e.g., $N = \{5,4,3,2\}$). The difference is 6, or twice the number of intervals. The sum of the values in the vector equals $L-1$. Scaling the denominator by 2 is necessitated by the maximum number of possible different contours.[46]
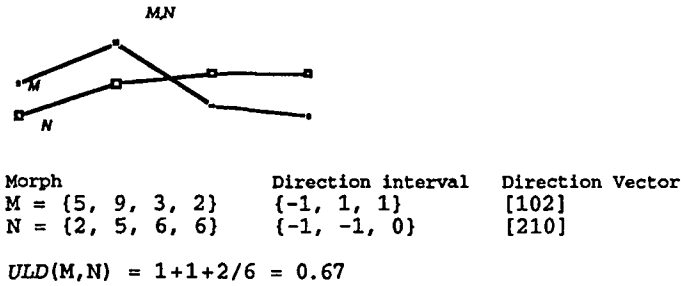


```
Morph                  Direction interval    Direction Vector
M = {5, 9, 3, 2}       {-1, 1, 1}            [102]
N = {2, 5, 6, 6}       {-1, -1, 0}           [210]

ULD(M,N)  = 1+1+2/6 = 0.67
```

Fig. 5.   Morph example.

The ULD is a statistical comparison of linear interval contour, independent of the *corresponding* respective intra-morphological orders of two morphs. Like the ULM (below), it is only trivially a morphological metric, leading naturally to the OLM and OCD. It is still a useful measure, reflecting a salient perceptual relationship between morphs. Under the ULD, morphs which "go up a lot" (linearly) will be closer to others that "go up a lot", even if they do not go up in the same places. ULD values range from [0,1], with a grain of $1/((L-1)*2)$.

OLD (ORDERED LINEAR DIRECTION)

$$\frac{\sum\limits_{i=1}^{L-1} \text{diff}\,(sgn(\Delta(M_i, M_{i+1})), sgn(\Delta(N_i, N_{i+1})))}{L-1} \qquad (OLD)$$

where:

$$diff \quad = 0 \text{ iff } sgn(M_{int}) = sgn(N_{int})$$
$$= 1 \text{ iff } sgn(M_{int}) \neq sgn(N_{int})^{47}$$

The *OLD* measures the percentage of different contour values between corresponding linear intervals. That is, if ΔM_I "goes up" where ΔN_I "goes down" or stays the same, the sum of direction dissimilarities is incremented. For morphs with the same linear contour, *OLD* = 0. If two morphs differ in every place (linearly), *OLD* = 1.

The *ULD* and *OLD* are somewhat independent. Even with *OLD* = *1*, it is entirely possible that *ULD* = *0*, as the following example shows:

Morph	Direction interval	Vector
O = {5, 3, 6, 1, 4}	[1, -1, 1, -1]	[202]
P = {3, 6, 1, 4, 2}	[-1, 1, -1, 1]	[202]

Fig. 6.   Morph examples.

ULD(O,P) = 0, or *O,P* are the "same point" in *ULD*-space. *OLD(O,P)* = *1*, or "they are as different as they can be" in *OLD*-space, because each corresponding interval has a different contour. Even though the *ULD* has a finer grain, in general, *OLD* ≥ *ULD* since the *OLD* is "more discriminating". However, in this case, the ULD recognizes a "contour inversion" and the *OLD* does not. *OLD* = *ULD* for *strictly* monotonic morphologies (that is, where the < or > relationship holds, not ≤ or ≥).

OLD values range from [0,1] with a grain of *1/L-1*. Note that the *OLD* is simply an *L-1* dimensional discrete metric, so is by definition a metric itself.

OCD (ORDERED COMBINATORIAL DIRECTION)

$$\frac{\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} diff\,(sgn(\Delta(M_i, M_j)),\ sgn(\Delta(N_i, N_j)))}{L_m} \qquad (OCD)$$

The *OCD* is the combinatorial version of the *OLD*. It is the most discriminating of the direction metrics. The *OCD* measures the complete, cell-by-cell network of contour similarity between two morphs. The *OCD* closely reflects melodic perception, tracking the difference between the combinatorial contour of two melodies.[48]

The *OLD*, *ULD* and *OCD* are generally independent (though the *OLD* is "included" as the diagonal for the *OCD*). For the two morphs (above) M = {5,9,3,2}, N = {2,5,6,6} with combinatorial contour matrices:[49]

```
  − + +              − − −
    + +     and        − −
      +                   0
```

the *diff* value between corresponding cells is 5, so $OCD(M,N) = .83$.[50] If O = {5,3,6,1,4} and P = {3,6,1,4,2} (see the above comparison of the *ULD* and *OLD*), with contour matrices:

```
  + − + +            − + − +
    − + −     and      + + +
      + +                − −
        −                  +
```

the *diff* value is 8, so $OCD(O,P) = .8$ (again, very different), where the $OLD(O,P) = 1$ (completely different "diagonals") and $ULD(O,P) = 0$ (half up, half down for each).

OCD values range from [0,1] with a grain of $1/Lm$.[51]

UCD (UNORDERED COMBINATORIAL DIRECTION)

The combinatorial version of the *ULD* (or the unordered version of the *OCD*) is:

$$\frac{\sum_{v=(-1,0,1)} |\#_v M - \#_v N|}{(L_M * 2)} \qquad (UCD)$$

where intra-morphological intervals in *M, N* are calculated as in the *OCD* above.

The *UCD* compares the statistics of combinatorial "up/equal/down-ness" of each morph. It does not discern similarities in *corresponding* intervals. In general, *OCD* $\geq$ *UCD*, since the *OCD* is more sensitive.

In the *UCD*, the difference vector is comprised of L_m total values, expressing the number of combinatorial ternary contour intervals.[52] For *M, N* above, the

combinatorial contour vectors are [105] and [510] which are the sums of the ternary contours in the contour matrices.

UCD(M,N) = (4+1+5)/12 = .83 (in this case the same as the value for the *ULD*). For O = {5,3,6,1,4} and P = {3,6,1,4,2}, with the same combinatorial contour vector [406], *UCD(O,P) = 0*, where it was quite high, .8, under the *OCD*. Thus, in terms of statistical contour, *O* and *P* are the same morph, but not in terms of ordered contour. The *UCD* and *ULD* metrics, in their simplest forms, are equivalent to applying Morris' (1979–80) *SIM* function to the combinatorial and linear contour vectors.

The following example shows the *UCD* taken on three morphs, *Q, R, S*:

$$Q = \{5,3,7,6\} \quad R = \{2,1,2,1\} \quad S = \{8,3,5,4\}$$
$$[402] \qquad\qquad [123] \qquad\qquad [204]$$
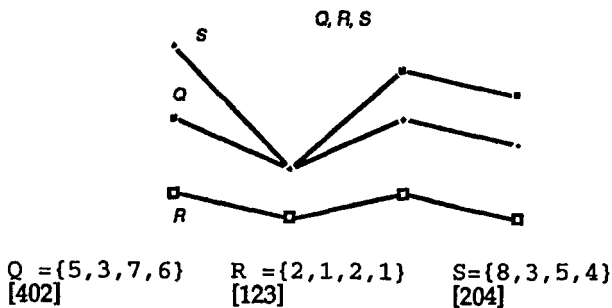
Fig. 7.   Morph examples.

$(L_m = 12)$
```
UCD (Q, R) = 3+2+1/(12) = .5
UCD (R, S) = 1+2+1/(12) = .33
UCD (Q, S) = 2+0+2/(12) = .33
```

The least similar morphs under the *UCD* are *Q* and *R*, though because of their magnitudes, they visually appear very similar. This example emphasizes that each metric measures similarity of some morphological feature, not a kind of overall perceptual similarity.

As with the *OLD* and *ULD*, it is possible to have a low value for the *OCD* and a high one for the *UCD*, and vice versa. The *UCD* like the *ULM*, is only trivially a morphological metric, leading naturally to the *OCD*. It is more discriminating than the *ULD* (in general, *UCD* ≥ *ULD*). *UCD* = *OCD* for monotonic morphologies.

The *ULD* and *UCD* do not require equal length morphs. Their more general equations for morphs of unequal length are:

$$\frac{\sum_{v=(-1,0,1)} \left| \frac{\#_v M}{M_L - 1} - \frac{\#_v N}{N_L - 1} \right|}{2} \qquad \text{(ULD, unequal length form)}$$

$$\frac{\sum_{v=(-1,0,1)} \left| \frac{\#_v M}{L_m} - \frac{\#_v N}{L_n} \right|}{2} \qquad \text{(UCD, unequal length form)}$$

Note that $\#_v$ is different for the *ULD* and the *UCD*. Using *M, N, O, P*, where the linear contour vectors are [102], [210], and [202] (for *M, N* and *O = P* respectively) and combinatorial contour vectors [105], [510], and [406] (for *M, N* and *O = P* respectively), *ULD(M|N,O)* = *ULD(M|N,P)*, and similarly for the *UCD*):

$ULD(M,O) =.17^{53}$ $ULD(N, O) =.5$
$UCD(M,O) =.23^{54}$ $UCD(N, O) =.585$

M, with its higher curvature, is more similar to *O* and *P* than the "flatter" *N*.
   UCD values range from [0,1] with a grain of $1/(L_m * 2)$.

SOME COMMENTS ON DIRECTION METRICS

Unordered direction metrics (*ULD* and *UCD*) measure similarity of general curvature, while ordered direction metrics (*OLD* and *OCD*) measure similarity of specific curvature. Since there can be several combinatorial direction matrices for a given set of linear direction intervals (Polansky & Bassein 1992), the *OLD* and *OCD* measure distinct morphological features. Consider the linear contour series:

$\{+, -, +\}$ $(e_1 > e_2, \ e_2 < e_3, \ e_3 > e_4)$
                linear contour vector = [102]

representing any of the following 4-element morphs *M, N*, and *O* below, each with different combinatorial contour matrices[55] (Fig. 8).
   The combinatorial and linear contour vectors for M, N, and O are:

M	N	O
[102] [402]	[102] [123]	[102] [204]
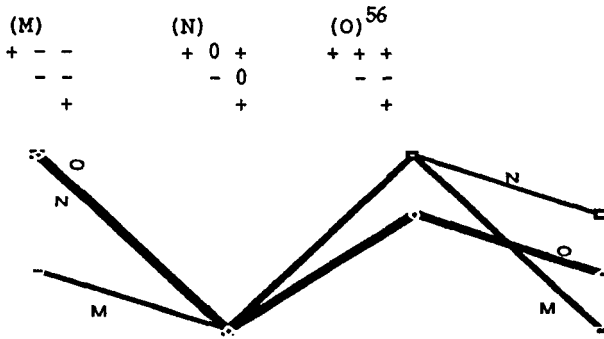
Fig. 8. Morph examples.

Note that:

$$OLD = ULD = 0$$

for all {M, N, O}. Although the morphs are different, they have the same linear contour vectors (*ULD*), and the same combinatorial contour matrix *diagonals* (used by the *OLD*). These three morphs are the same in ordered and unordered linear contour space. However, for the combinatorial metrics:

```
OCD(M,N) =.5   OCD(M,O) =.33   OCD(N,O) =.33
UCD(M,N) =.5   UCD(M,O) =.17   UCD(N,O) =.33
```

The closest relationship is between *M* and *O* in *UCD*-space, the most distant, between *M* and *N* in *OCD*- and *UCD*-space.

MAGNITUDE METRICS

Magnitude metrics measure the difference in intervallic *magnitude* between two morphs, regardless of direction. In general, a morph and its inversion are the same according to these metrics. Morphs which are "almost inversions" of another morph are similar under magnitude metrics. For example, two morphs with their last values inverted:

```
M = {1,3,7,2,5}    N = {1,3,7,12,9}
```

will cause "perturbations" in linear magnitude metrics, and greater "perturbations" in combinatorial ones.

Linear magnitude metrics often measure, in the case where adjacency intervals are used, the difference between sequential values. Combinatorial magnitude

metrics operate on morphological magnitude matrices, similar to the interval vector in atonal theory. As in the atonal theory interval vector, the combinatorial magnitude matrix for a given morph and its inversion are the same.


```
M = {5,3,2,6,9}, N = {5,7,8,4,1}
M', N' = {2,1,4,3}
Combinatorial magnitude matrix M, N =
                                2314
                                136
                                47
                                3
```
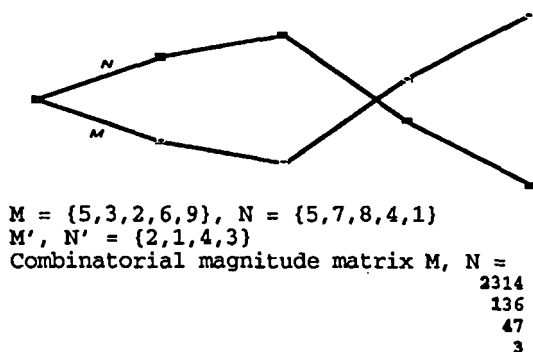
Fig. 9.   Morph examples.

The choice of Δ is important in magnitude metrics, since *interval magnitude* may not be generally defined. Different functions will work better with different parameters. In pc- or p-space, an absolute magnitude function might be appropriate, since intervals are reduced by pitch-class and inversional equivalence. Harmonic distance functions are also possible, the simplest example being a kind of 12-ET "lookup-table" of harmonic ranking (the ranking is arbitrary, a composer may assign her own):

```
d(a,b) = 0 (octave); 1 (fifth); 2 (major third);
         3 (minor third); ... 11 (minor second)
```

In frequency or duration space, a ratio function is more appropriate. *Max, min* and various statistical measures are also useful.

OLM (ORDERED LINEAR MAGNITUDE)

The *OLM* measures the average difference between corresponding intervals in two morphs:

$$\sum_{i=1}^{L-1} \frac{|\Delta(M_i, M_{i+1}) - \Delta(N_i, N_{i+1})|}{L-1} \qquad (OLM)$$

where Δ is some interval function such as:

$$|M_i - M_{i+1}| \quad \text{or} \quad \sqrt{(M_i - M_{i+1})^2} \quad \text{or} \quad M_i / M_{i+1}$$

The *OLM* depends on the particular definition of Δ. It generalizes the discrete version of the *magnitude metric* (see above) to the derivatives of real-valued functions. Simplifying the *OLM* equation makes this clear:

$$\frac{\sum_{i=1}^{L-1} |\Delta(M) - \Delta(N)|}{(L-1)} \qquad\qquad (\textit{simplified OLM})$$

where $\Delta(M)$ is some first-order difference function on M. The *OLM* may be generalized to nth-order difference functions, suggesting a variety of musical uses.[57]

Unlike direction metrics, magnitude metrics are, by definition, unscaled. Since intervals are not bounded, the *OLM* in its unscaled form yields indefinitely large values. A simple scaling technique for the *OLM*, generally useful for magnitude metrics, is:

$$\frac{\sum_{i=1}^{L-1} |\Delta(M_i, M_{i+1}) - \Delta(N_i, N_{i+1})|}{(L-1) * (\textit{maxint})} \qquad\qquad (\textit{scaled OLM})$$

where *maxint* is the maximum interval occurring in M, N. Normalized in this way, the *OLM* assumes values between [0,1], assuming a value of 1 only if 1) M, N are completely comprised of intervals *maxint* and 0, and 2) whenever $\Delta M = 0$, $\Delta N = maxint$, and vice versa. A variety of other scaling techniques are possible (see below for the *ULM*).

Another form of the *OLM* is the root of the difference of squares:

$$\frac{\sum_{i=1}^{L-1} \sqrt{(\Delta(M_i, M_{i+1})^2 - (\Delta(N_i, N_{i+1}))^2}}{(L-1) * (\textit{maxint})} \qquad (\textit{OLM, root of difference of squares})$$

Since the *OLM* is an average of metrics (for example, absolute value differences), it is itself a metric, since the sum of two metrics on a set is also a metric.[58]

ULM (UNORDERED LINEAR MAGNITUDE)

The *ULM* is derived from a rearrangement of the unscaled *OLM*:

$$\left| \frac{\sum_{i=1}^{M_L-1} \Delta(M_i, M_{i+1})}{M_L-1} - \frac{\sum_{i=1}^{N_L-1} \Delta(N_i, N_{i+1})}{N_L-1} \right| \qquad (ULM)^{59}$$

By analogy with the *ULD*, the *ULM* measures the *difference of the average intervals* of two morphs, whereas the *OLM* measures the *average difference of corresponding intervals* of two morphs.

Where $M_L = N_L$, the *ULM* looks suspiciously like the *OLM*, since in general:

$$\sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Rewriting M_L as L, we get a simplified version of the *ULM*:

$$\frac{|\sum \Delta M| - |\sum \Delta N|}{L} \qquad (simplified\ ULM)$$

However, the *OLM* (the average of the differences) and *ULM* (the difference of the averages) yield different values since:

$$\left| \sum a \right| - \left| \sum b \right| \neq \sum |a-b|$$

as Fig. 10 (unscaled) shows.

A simpler, more conventionally statistical notation for the *ULM* is:

$$\left| \Delta M_\mu - \Delta N_\mu \right| \qquad (ULM,\ difference\ of\ mean\ intervals)$$

where ΔM_μ is the mean interval in M. Similarly, the *OLM* may be rewritten as:

$$\left| \Delta M_i - \Delta N_i \right|_\mu \qquad (OLM,\ mean\ difference\ of\ intervals)$$

The *ULM* is not sensitive to intervallic order, generating a space in which morphs

```
M = {1,6,2,5,11}              [5,4,3,6]
N = {3,15,13,2,9}             [12,2,11,7]

OLM(M,N) = (7+2+8+1)/4   =    4.5
ULM(M,N) = |4.5 - 8|     =    3.5
```
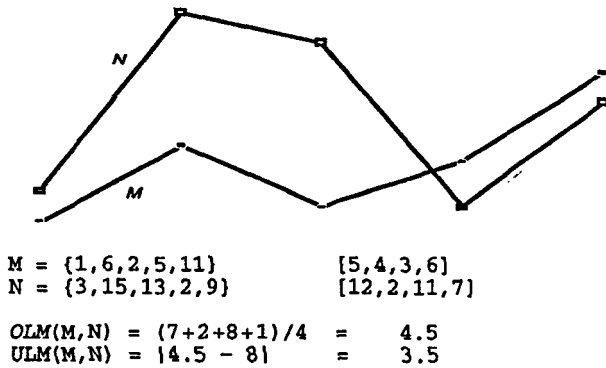
Fig. 10.   Morph examples.

are "closer" to each other than in *OLM*-space. In general, $OLM \geq ULM$. Since M_L need not equal N_L, the *ULM* does not require equal length morphs.

ABSOLUTE AND RELATIVE SCALING[60]

Two different types of scaling are shown below for the *ULM*. The first, *absolute scaling*, uses *maxint(M,N)* for both morphs, normalizing the two morphs to the larger of the two ranges. Absolute scaling measures "contour" to some extent; the intervals of the smaller morph are "stretched" in the calculation. The second, *relative scaling*, uses individual values for *maxint(M)* and *maxint(N)* to normalize each morph. It preserves the absolute ranges of each morph in the metric.

$$\frac{\left| \dfrac{\displaystyle\sum_{i=1}^{M_{L-1}} \Delta(M_i, M_{i+1})}{M_L - 1} - \dfrac{\displaystyle\sum_{i=1}^{N_{L-1}} \Delta(N_i, N_{i+1})}{N_L - 1} \right|}{maxint(M,N)}$$

(ULM; absolute scaling)

For this to equal 0, the numerator must be 0, so $M_\mu = N_\mu$ (where M_μ and N_μ are the means of the intervals). To equal 1, $N_\mu = maxint$ and $M_\mu = 0$ for all i (or vice versa), implying that both morphs are "straight lines", explicitly not allowed. Therefore, the range of the *ULM* with absolute scaling is [0,1).[61]

These scaling techniques work best if Δ is a metric itself. If it is, *maxint(M)* is similar to what is called the *diameter* of a metric space. *Maxint(M,N)* should be used only when Δ is the same for *M, N*.

$$\left| \frac{\sum\limits_{i=1}^{M_L-1} \Delta(M_i, M_{i+1})}{(M_L-1) * maxint(M)} - \frac{\sum\limits_{i=1}^{N_L-1} \Delta(N_i, N_{i+1})}{(N_L-1) * maxint(N)} \right| \qquad (ULM,\ relative\ scaling)$$

Assuming $M_L = N_L$, the *relative scaled ULM* may be rewritten as:

$$\sum_{i=1}^{L-1} \frac{\text{"scaled average"}}{L-1} \qquad (ULM,\ scaled\ average\ form)$$

where "scaled average" is:

$$\frac{(\Delta M * maxint(N)) - (\Delta N * maxint(M))}{(maxint(M) * maxint(N))} \qquad (\text{"scaled average"})$$

The relative scaled *ULM* assumes values between [0,1), similar to the absolute scaled form, since each side of the minus sign in the summation is the average interval divided by the maximum interval. Even the unscaled version of this metric, since it is a statistical measure, is "scaled" to the means of the two morphs (see the example below).

The relative scaled *OLM* is derived from the relative scaled *ULM*:

$$\frac{\sum\limits_{i=1}^{L-1} \left| \dfrac{\Delta(M_i, M_j)}{maxint(M)} - \dfrac{\Delta(N_i, N_j)}{maxint(N)} \right|}{L-1} \qquad (relative\ scaled\ OLM)$$

or

$$\frac{\sum\limits_{i=1}^{L-1} \dfrac{\left| (\Delta M * maxint(N)) - (\Delta N * maxint(M)) \right|}{maxint(M) * maxint(N)}}{L-1} \qquad (rewritten\ relative\ scaled\ OLM)$$

A final form of scaling, especially useful for squared forms of the OLM and OCM, is the *max* of the root of the squared differences of corresponding intervals between two morphs, or:

$$maxint = max \sqrt{\left(M_{int_i} - N_{int_j}\right)^2} \qquad \text{(\textit{maxint, squared form})}$$

This form of *maxint* stays smaller than either absolute or relative *maxint*, and can often be a more effective form of normalization.[62]

OCM (ORDERED COMBINATORIAL MAGNITUDE)

The *OCM* is the combinatorial version of the *OLM*. Its three principle forms are:

$$\frac{\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \left| \Delta(M_i,M_j) - \Delta(N_i,N_j) \right|}{L_m} \qquad \text{(\textit{OCM})}$$

$$\frac{\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \left| \Delta(M_i,M_j) - \Delta(N_i,N_j) \right|}{L_m * maxint(M,N)} \qquad \text{(\textit{absolute scaled OCM})}$$

$$\frac{\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \sqrt{\left(\Delta(M_i,M_j) - \Delta(N_i,N_j)\right)^2}}{L_m * maxint(M,N)} \qquad \text{(\textit{absolute scaled OCM, squared form})}$$

where *maxint* is the maximum combinatorial interval in the two morphs.

$$\frac{\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \left| \frac{\Delta(M_i,M_j)}{maxint(M)} - \frac{\Delta(N_i,N_j)}{maxint(N)} \right|}{L_m} \qquad \text{(\textit{relative scaled OCM})}$$

The *OCM* measures the average cell-by-cell difference between two absolute magnitude matrices of equal length morphs. For example, for *M,N* above, with $\Delta = |a-b|$:

```
        M = {1,  6,  2,  5,  11}        N = {3,  15,  13,  2,  9}
                    5   1   4  10                   12  10   1   6
    matrix(M) =         4   1   5       matrix(N) =      2  13   6
                            3   9                           11   4
                                6                                7
```

Difference Matrix =

$$
\begin{array}{cccc}
\mathbf{7} & 9 & 3 & 4 \\
 & \mathbf{2} & 12 & 1 \\
 & & \mathbf{8} & 5 \qquad \text{(sum = 52)} \\
 & & & \mathbf{1}
\end{array}
$$

$$OCM(M,N) = 52/10 = 5.2 \qquad (OLM(M,N) = 4.5)$$

As in the *OCD*, adjacent differences are shown in the main diagonals of the matrices. The rows are differences between the first element and all succeeding, the second element and all succeeding, and so on. The highlighted diagonal shows the linear intervals used in the *OLM* above. These matrices are alternative representations of the standard interval vector, except that values are generalized by Δ. The difference matrix is also generalizable by ψ. The *OCM* has a smaller grain and greater sensitivity than the *OLM*, as shown by this example.

The *OCM* is strongly related to the *OLM*, including it as its diagonal differences. $L\text{-}1/L_m$ values of the computation are the same between the two metrics. For a given linear absolute difference magnitude vector, several combinatorial matrices are possible. For example, take the vector:

[2,3,2,1]

which may correspond to the morphs:

{5,7,10,8,9} or {5,3,6,8,7}

and the two different absolute difference magnitude matrices:

$$
\begin{array}{cccc}
\mathbf{2} & 5 & 3 & 4 \\
 & \mathbf{3} & 1 & 2 \\
 & & \mathbf{2} & 1 \\
 & & & \mathbf{1}
\end{array}
\qquad\qquad
\begin{array}{cccc}
\mathbf{2} & 1 & 3 & 2 \\
 & \mathbf{3} & 5 & 4 \\
 & & \mathbf{2} & 1 \\
 & & & \mathbf{1}
\end{array}
$$

Therefore, $OLM(M, N) = 0$ does not imply that $OCM(M, N) = 0$. However, note that the values in one matrix are simply rearranged into the other (the interval vectors are the same), implying that $OLM(M,N) = 0 \Rightarrow UCM(M,N) = 0$ and $ULM(M,N) = 0$. This is an "artifact" of the fact that while partial inversions will distinguish the *OLM* and *OCM*, they do not affect the unordered magnitude metrics.

When scaled, combinatorial *maxints* may of course be different than linear ones. If row and column intervals are weighted in the *OCM* (see the section below on weightings), the *OCM* yields musical results which the *OLM* can not.

Two morphs with the same combinatorial magnitude matrix are related by transposition and/or inversion. That is, every matrix is associated with a set of morphs, all related by the standard serial operations. This magnitude matrix is a slightly stronger condition than for example, the Z-relation (i.e., it includes less members in a given "equivalence class" created by the relation), because two interval vectors (in the sense of atonal theory) might be the same, but the intervals might occur in different places, as in the example above. In other words, the combinatorial matrix described here is "more" morphological than the atonal interval vector. The *UCM* (described below) and *ULM* metrics would find two Z-related morphs to be the same ($d = 0$). The *OCM* and *OLM*, however, typically distinguish between Z-related morphs. The *OCM* is thus closely related to but distinct from Morris' *SIM* function. It is more distinct when weightings and different scaling functions are used.

UCM (UNORDERED COMBINATORIAL MAGNITUDE)

The *UCM* is the combinatorial version of the *ULM*, the difference between average combinatorial intervals.

$$\left| \frac{\sum_{i=1}^{L-1_M} \sum_{j=i+1}^{L_M} |\Delta(M_i, M_j)|}{L_M} - \frac{\sum_{i=1}^{L-1_N} \sum_{j=i+1}^{L_N} |\Delta(N_i, N_j)|}{L_N} \right|$$ (*UCM, unscaled*)

The *UCM* is a useful statistical measure. Like the *ULM*, it does not require that $L_m = L_n$. The relative and absolute scaled forms are easily derived and similar to those of the *ULM*, as is the simple notation for the difference of mean combinatorial intervals.

For $M = \{1, 6, 2, 5, 11\}$ and $N = \{3, 15, 13, 2, 9\}$, the unscaled *OCM*, *OLM* and *ULM* respectively, equalled 5.2, 4.5 and 3.5, The unscaled *UCM* = 2.4, the smallest of the four unscaled forms (least sensitive, since it averages each magnitude matrix).

The *UCM* is closely related to Morris' *SIM* function. An atonal set theory version of the *UCM* would be the weighted difference in average interval between two interval vectors (weighted by the number of entries in a given place). This function (which might be called SIM_{avg}) reflects the difference in "width" or skew of two pc-sets, for example, for set classes 5–19 and 5–34, with interval vectors [212122] and [032221], this metric is:

$SIM_{avg}(5-19, 5-34)$[63]

$= |\,((2^* 1) + (1^*2) \,\ldots\, (2^*6))/10$
$\quad\; - \;((0^* 1) + (3^*2) \,\ldots\, (1^*6))/10\,|$
    (each set scaled by its own # of intervals)
$= |\,(2+2+6+4+10+12/10) \quad - \quad (0+3+4+6+10+6)/10\,|$
$= |\,3.6 - 2.9\,| \quad$ (5—19 is "wider" than 5—34)
$= .7$

As a second example, using Morris' terminology and *ASIM* function for pc-sets of different cardinality (Morris 1979—1980) SIM_{avg} of sets $R = (3,4,5)$ and $S = (8,9,1,3)$, with $V(R) = [210000]$ and $V(S) = [110121]$:

$SIM_{avg}(R,S) =$
$\quad\quad |\,(2+1/3) - (1+2+4+10+6/6)\,| = |1 - 3.83\,| = 2.83$
$\quad\quad 2.83/6 = .471$
$ASIM(R,S) = .555$

where R has an average interval of 1 semitone, S almost 4 semitones. Morris' function is nicely scaled to [0,1], while SIM_{avg} will range from [0, 6], so in the above example it is scaled in the same way to make the values comparable.

STATISTICAL VARIATIONS OF MAGNITUDE METRICS

The *OLM* and *ULM* suggest a wide variety of statistical comparisons between two morphs, including correlation coefficient (Uno 1991; Morris 1989; Hermann 1994), differences in standard deviation, statistical distribution, range, and so on.[64] Magnitude metrics are similar in principle to the correlation coefficient (*CC*):

$$r = \frac{\sum (x_i - \mu_x)(y_i - \mu_x)}{\sqrt{\sum (x_i - \mu_x)^2 \sum (y_i - \mu_x)^2}} \qquad \textit{(correlation coefficient)}$$

The *CC* is not a metric since it yields negative and positive values [−1, 1]. −1 and 1 indicate negative and positive dependence and 0 indicates that the two variables x and y are independent (e.g., Wonnacott & Wonnacott 1972). To illustrate the difference between the *CC* and some of the metrics, consider M = {50,10,1} and its retrograde N = {1,10,50}. The *CC* of *M, N* is −.76, the *OLM* (relative or absolute scaled) is 1, but the *ULM* and *OLD* are 0. If the morph

were "symmetrical", the CC would be -1, which would "correspond" to the highest possible value of 1 for the OLD. The CC is the covariance over the product of the standard deviations of two variables. However, the covariance (the numerator) can be rewritten in its scaled form:

$$\frac{\sum (x_i - \mu_x)(y_i - \mu_x)}{L_{xy}} \qquad \textit{(covariance, scaled)}$$

where L_{xy} is a translation into my own terminology of the more common $p(x,y)$, the probability of (x,y) in the sample (in the probabilistic form of this equation). In other words, the intervals around the mean are weighted by the number of their occurrence in the sample. In the metrics, this weight is usually 1, so $L_{xy} = L\text{-}1$ (in general).

The covariance is similar to the OLM when the mean interval is used as a phantom fundamental, and the denominator of the correlation (the product of the standard deviations) is a scalar, similar to *maxint*. In this way, the CC is a version of the *relative scaled OLM* (or more humbly, vice versa), but the OLM is a metric and the CC is not.[65] The CC assumes negative values (the individual terms in the covariance are negative or positive: "above or below" the mean), combining aspects of the OLM and OLD.

The *max-ULM* is an example of extending magnitude metrics to other statistical features. This measures the difference in interval ranges between two morphologies, in which M_L and N_L may be different:

$$\left| \max\left(\Delta(M_i, M_{i+1})\right) - \max\left(\Delta(N_i, Ni+1)\right)\right| \qquad \textit{(max-ULM)}^{66}$$

An ordered version, the *max-OLM*, is derived from the standard *max* metric on real-valued functions (Bryant 1985; Copson 1968 p. 59, for a two-dimensional version):

$$d(x,y) = max\{|x(t) - y(t)| : t \text{ element of } [a,b]\} \qquad \textit{(standard max metric)}$$

$$max\left|\left(\Delta(M_i, M_{i+1})\right) - \left(\Delta(N_i, N_{i+1})\right)\right| \qquad \textit{(max-OLM)}$$

where i goes from 1 to $L\text{-}1$. This is generalized to the *max*-OCM and *max*-UCM:

$$max \left| \sum_{i=1}^{L-1} \sum_{j=i+1}^{L} \Delta\left(M_{i}, M_{j}\right) - \Delta\left(N_{i}, N_{j}\right) \right| \qquad (max\text{-}OCM)$$

$$\left| max\left(\sum_{i=1}^{L_M-1} \sum_{j=i+1}^{L_M} \Delta\left(M_{i}, M_{j}\right) \right) - max\left(\sum_{i=1}^{L_N-1} \sum_{j=i+1}^{L_N} \Delta\left(N_{i}, N_{j}\right) \right) \right| \qquad (max\text{-}UCM)$$

The example below illustrates the difference between the *max-ULM* and the *max-OLM*:

morphs   absolute magnitude intervals
M = {30,35,45,43}   [5,10,2]
N = {1,25,3,9}   [24,22,6]
O = {30,31,28,33}   [1,3,5]

max-ulm(M,N) = |24-10| = 14   max-OLM(M,N) = 19
max-ulm(M,O) = |5-10| = 5   max-OLM(M,O) = 7
max-ulm(N,O) = |24-5| = 19   max-OLM(N,O) = 23
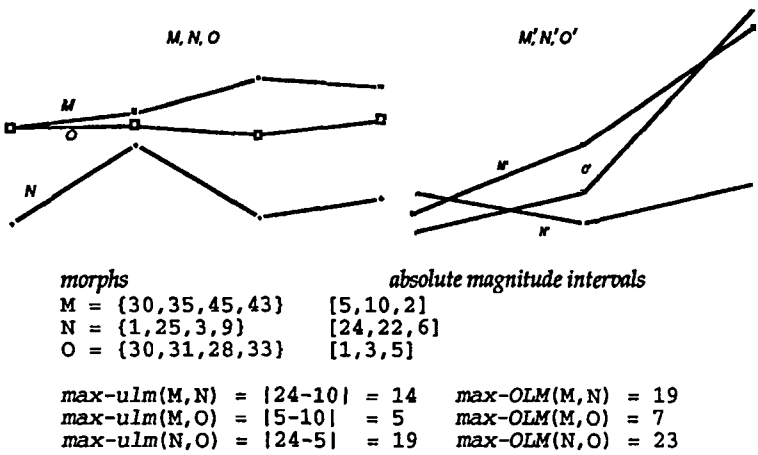
Fig. 11.   Morph example.

Note that the *max-OLM* ≥ *max-ULM*. By these two metrics, *M* and *O* are "closer" than either *M, N* or *N, O*. The similarity ranking (from close to distant) is:

[(*M*,*O*), (*M, N*), (*N*,*O*)]

which in this case places the three morphs unambiguously on a line with *O* and *N* as endpoints. This one-dimensional ordering is not possible for greater numbers of morphological comparisons, or more complicated relationships where techniques such as multidimensional scaling become useful in "placing" a group of morphs in an n-dimensional space induced by a metric (see below, and Polansky 1993a).

The slightly more complex σ-*ULM* measures the differences in standard deviation of interval size between two morphs:

$$\left| \sqrt{V_m} - \sqrt{V_n} \right| \qquad (\sigma\text{-}ULM)$$

where V_m is the variance of Ms intervals, defined as:[67]

$$\sum_{i=1}^{L-1} \frac{\left(\Delta M_i - \left(\Delta M_\mu\right)\right)^2}{L-1} \qquad \text{(interval variance of M)}$$

ΔM_μ is the average (or median, or any function of) the intervals in M (written as ΔM_i), or:

$$\Delta M_u = \frac{\sum \Delta M_i}{L-1}$$

M_L need not equal N_L. To show how this "constructed metric" corresponds to the metric forms outlined in this article, the σ-ULM may be considered as the ordinary ULM with a phantom fundamental ΔM_μ, and with the Δ function:

$$\Delta = \left(\Delta'(M_i, M_j) - \Delta'(M_\mu)\right)^2 \qquad \text{(Δ function for σ-ULM)}$$

where Δ' is some lower level interval function on elements of M, and a ψ function of:

$$\varphi = \left| \sqrt{a} - \sqrt{b} \right| \qquad \text{(ψ function for σ-ULM)}$$

scaled by $\#I$ (the number of intervals).

The σ-ULM can be compared to the absolute value form of the unscaled ULM with intervals taken to the mean, given below just for the first of the two morphs:

$$\sum_{i=1}^{L-1} \frac{\left| \Delta M_i - \Delta M_\mu \right|}{L} \qquad \text{(ULM, mean interval fundamental from)}$$

The difference between this form of the ULM and the σ-ULM is the difference between (shortening the notation a bit):

$$\sum \left| x_i - x_\mu \right| \quad \text{and} \quad \sqrt{\sum \left(x_i - x_\mu \right)^2} \qquad \text{("numerators", ULM, standard deviation)}$$

similar to the difference between the average city-block and Euclidean metrics from the mean.[68] In other words, the ULM, rewritten as the root of the *sum* of squared differences, and with the mean as phantom fundamental, is the standard deviation (while the OLM is strongly related to the CC).

The following compares the σ-*ULM* and the *ULM* on *M*, *N*, and *O* above. The *ULM*, in this example, uses the absolute value of differences between each absolute value interval and the mean interval of each morph.

```
M = {30,35,45,43}              [5,10,2]
N = {1,25,3,9}                 [24,22,6]
O = {30,31,28,33}              [1,3,5]
```

$$\Delta M_\mu = 5.67 \qquad \Delta N_\mu = 17.33 \qquad \Delta O_\mu = 3$$
$$VM = (.67^2 + 4.33^2 + 3.67^2)/3 \qquad = 11.22$$
$$VN = (6.67^2 + 4.67^2 + 11.33^2)/3 \qquad = 64.89$$
$$VO = (0 + 2^2 + 2^2)/3 \qquad = 2.67$$

$$\sqrt{VM} = 3.35 \qquad \sqrt{VN} = 8.06 \qquad \sqrt{VO} = 1.63$$

e.g., $ULM_M = (.67 + 4.33 + 3.67)/3$ = 2.89
ULM_N = 7.56
ULM_O = 1.33

σ-*ULM (M,N)* = 4.71   *ULM (M,N)* = 4.67
σ-*ULM (M, O)* = 1.72   *ULM (M, O)* = 1.56
σ-*ULM (N,O)* = 6.43   *ULM (N,O)* = 6.23

For these particular forms, $ULM \leq \sigma\text{-}ULM$. Both preserve the "ordering" of the *max-OLM* and *max-ULM* above, while more sensitively measuring the same morphological feature. Since these variance and standard deviation values are not scaled in any way, they should not be "directly compared" with values obtained by other metrics.

These examples suggest a range of mathematical and musical applications.[69] For example, the σ-*ULM* might be applied to the elements of a morph or any order difference function. It might be generalized to both the *OLM* (where M_L must equal N_L) and combinatorial metrics (by taking the variance of all combinatorial intervals over L_M) for different results, and also applied to parameters such as harmonic distance, duration, and so on. Variance and other statistical metrics, while not respecting individual order, are useful in comparing important morphological features, and can be used in conjunction with truly morphological metrics (like the *OCD*) to precisely measure various aspects of morphological similarity.

EXAMPLES OF MAGNITUDE METRICS

The following is a detailed example of the computation of simple versions of the four primitive magnitude metrics (*ULM, OLM, UCM, OCM*) on two morphs, *M*, *N* (with *M'* and *N'* the first-order difference functions):

M = {1,5,12,2,9,6}   N = {7,6,4,9,8,1}
M'= {4,7,10,7,3}   N'= {1,2,5,1,7}
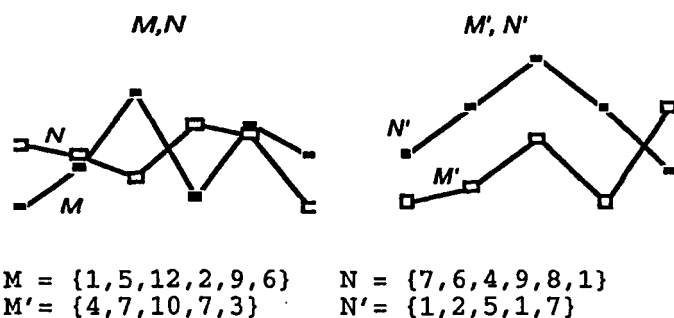
Fig. 12.   Morph examples.

M and *N* might be viewed as short pitch sequences:

Fig. 13.   Morph examples as pitch sequences.

or as duration sequences, or even the harmonic structures seen in Fig. 14 where harmonic values are taken to the note C, in the following "harmonic distance" order:[70] unison (1), fifth (2), fourth (3), major third (4), minor third (5), major sixth (6), minor sixth (7), minor seventh (8), major second (9), tritone (10), major seventh (11), and minor second (12).

Fig. 14.   Morph examples as harmonic sequences.

The absolute magnitude matrices of these two morphs are:

```
M-matrix =  4 11 1 8 5        N-matrix =  1 3 2 1 6
               7 3 4 1                       2 3 2 5
              10 3 6                          5 4 3
                 7 4                            1 8
                 3                              7
```

```
Magnitude difference-matrix =  3 8 1 7 1
                                 5 0 2 4
                                 5 1 3
                                 6 4
                                 4
```

(See Figs. 15 and 16)

It is obvious from the example that unordered metrics and relative scaling (except for the *OCM* above) tend to find greater degrees of similarity than ordered metrics and absolute scaling, respectively.

M_{maxint} and N_{maxint} are different for the combinatorial and linear metrics; they represent the maximum combinatorial and linear intervals, respectively. Form 3) < 2), because both use maximum interval calculations, but only the larger of the maximum intervals is used in the denominator of 2). In other words, 2) "expands" the smaller range morph to be the same as the larger. 3) reflects the actual total range of the two morphs, while 2) may be more useful in compounding two morphs with very different ranges.

7) –9), 10) –12), the versions of the *ULM* and the *UCM*, are all, by definition, "scaled" to some extent, since both sides of the inter-morphological interval are means in and of themselves, and thus "relatively" scaled. 8) and 9), and 11) and

OLM
1) Unscaled OLM
        = 3+5+5+6+4 = 23/5   = 4.6
2) Absolute scaled OLM
        = 23/(5*10)   = 0.46
3) Relative scaled OLM
        = |4/10 - 1/7| + |7/10 -2/7| _ /5
        = (|.4 - 0.142| + |.7 -0.285) _ /5   = 0.44

OCM
4) Unscaled OCM
        = 54/15   = 3.6
5) Absolute scaled OCM
        = 54/(15*11)   = 0.327
6) Relative scaled OCM
        = |4/11 - 1/8| + |1 - 3/8| _/15   = 0.366

ULM
7) Unscaled ULM
        = |$M_{intmean}$ - $N_{intmean}$|
        = |6.2 - 3.2| = 3.0
8) Absolute scaled ULM
        = |$M_{intmean}$ - $N_{intmean}$|/maxint
        = |6.2 - 3.2|/10   = 0.3
9) Relative scaled ULM
        = |($M_{intmean}$/M_{maxint}) - ($N_{intmean}$/N_{maxint})|
        = |.62 - .457 | = 0.1628

UCM
10) Unscaled UCM
        = |$M_{intmean}$ - $N_{intmean}$|
        = |5.133   - 3.53| = 1.6
11) Absolute scaled UCM
        = |$M_{intmean}$ - $N_{intmean}$|/maxint
        = |5.133   - 3.53|/11   = 0.14545
12) Relative scaled UCM
        = |($M_{intmean}$/M_{maxint}) - ($N_{intmean}$/N_{maxint})|
        = |(6.2/10) -(3.2/8)|
    |.62 - 0.457| = 0.025

Fig. 15.   Numerical comparison of different metrics.
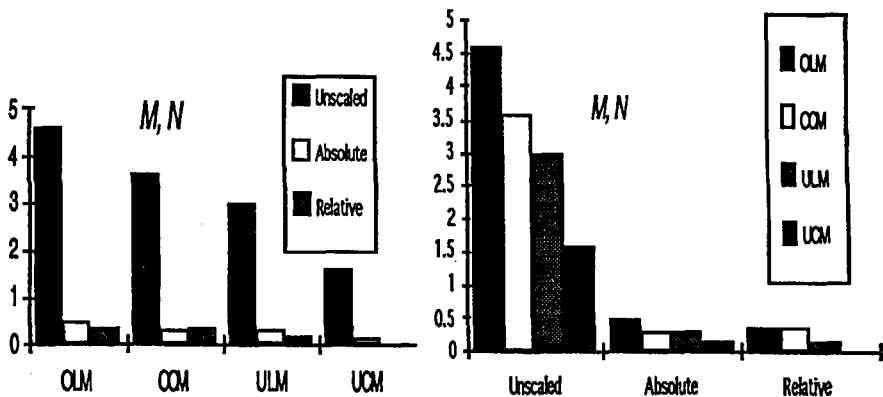
Fig. 16.   Graphic comparison of different metrics.

12) scale the metric to between [0,1). In 10) − 12), interval means are the mean combinatorial interval of each morph. Among the many other relationships between these forms, notice that 11) is simply 10) divided by *maxint*.

The chart below shows the values for all of the above metrics on these two morphs, with some additional metric forms and statistical variants:

Table 1.   Comparison of metric forms.

Values	Magnitude metrics	Values	Direction metrics
0.300000	*ULM*	0.4^{71}	*ULD*
0.162857	relative scaled *ULM*	0.2	*OLD*
3.000000	unscaled *ULM*	0.266667	*UCD*
		0.666667	*OCD*
0.460000	*OLM*		
0.649884	*OLM* squared		
0.442857	relative scaled *OLM*		
4.600000	unscaled *OLM*		
6.498840	unscaled *OLM* squared		
0.327273	*OCM*		
0.479978	*OCM* squared		
0.365909	relative scaled *OCM*		
3.600000	unscaled *OCM*		
5.279755	unscaled *OCM* squared		
0.145455	*UCM*		
0.025000	relative scaled *UCM*		
1.600000	unscaled *UCM*		
−.379865	cc		
−.10744	cc′		
−.20362	combinatorial cc		

The canonical *OLM*, *ULM*, *OCM*, and *UCM* above are the absolute scaled versions, which I tend to use most often. The "squared" versions of the *OCM* and *OLM* (absolute and relative squared) are as above. The corresponding versions for the *ULM* and *UCM* are not given. *CC* and combinatorial *CC* are the standard correlation functions on both the linear absolute magnitude values and the combinatorial magnitude intervals of the two morphs. *CC′* is the correlation function on the first-order difference functions of the two morphs.

As a second example, take the short morphs

M	= {1,2,4}		N	= {1,5,9}	
M′	= {1,2}		N′	= {4,4}	
M_{matrix}	=	1 3	N_{matrix}	=	4 8
		2			4

Values for the 12 magnitude metric forms are given below in slightly different format, pointing out the three forms of the (non-squared) magnitude metrics:

	Relative scaled	**Absolute scaled**	**Unscaled**
UCM	0	.416	3.33
OCM	.111	.416	3.33
OLM	.25	.625	2.5
ULM	.25	.625	2.5

The following table shows the more complete set of metrics as above, again with the absolute scaled forms considered to be canonical (shortest names):

Table 2.   Comparison of metric forms.

Value	Metric
0.625	*ULM*
0.25	relative scaled *ULM*
2.5	unscaled *ULM*
0.625	*OLM*
0.917136	*OLM* squared
0.25	relative scaled *OLM*
2.5	unscaled *OLM*
3.668542	unscaled *OLM* squared
0.416667	*OCM*
0.61472	*OCM* squared
0.111111	relative scaled *OCM*
3.333333	unscaled *OCM*
4.917761	unscaled *OCM* squared
0.416667	*UCM*
0	relative scaled *UCM*
3.333333	unscaled *UCM*

Note again that the *OLM* ≥ *ULM* and *OCM* ≥ *UCM*. However, the same relationships do not hold for the *OLM* and *OCM* that hold for the *OLD* and *OCD* (in this example, *OLM* > *OCM*). Unweighted metrics do not distinguish well between ordered and unordered forms for short morphs. All direction metrics are zero for these morphs; they are the same in all four "contour spaces". The relative scaled *UCM* finds the two morphs to be equal; their average intervals in relationship to their maximum (combinatorial) intervals are the same, though *N* spreads the interval size more equally. The relationships *unscaled* ≥ *absolute scaled* ≥ *relative scaled* relationships are evidenced in this example.

WEIGHTINGS[72]

Internal weighting functions on intra-morphological intervals help distinguish the relative importance of specific intervals or sets of intervals in similarity calculations. A perceptual motivation for weighting functions can be seen in the case of the *OLD*. Often, in recognizing similarity, the first interval's contour will be more important than the second, the second interval more important than the third, and so on. To model this effect of "memory decay", the *OLD* (or by extension, the *OLM*) can be rewritten as:

$$\frac{\sum_{i=1}^{L-1} f(i) * diff\left(sgn\left(\Delta(M_i,M_j)\right), sgn\left(\Delta(N_i,N_j)\right)\right)}{\sum f(i)} \qquad (weighted\ OLD)$$

If $f(i) = 1$ (no weighting), the metric's denominator is $L-1$, L, or L_m. The use of a weighting function $f(i)$ is similar to the use of a probability function $p(i)$ in the computation of expected value, mean, and standard deviations (see for example, Neter, Wasserman & Whitmore 1978, p. 118).

$f(i)$ can be any function or set of discrete weights. For example, to "zero out" the similarity effect of all but a few intervals in two morphs, $f(i) = 1$ for those indices, and zero for all the others. This is the same as describing a frequency or probability distribution where the denominator is the total number of intervals and $f(i) * int_i$ is the number of occurrences of a given interval. As such, any standard statistical or probability distributions may be used.

More typically some monotonically decreasing function such as $1/i$ or $1/i^2$ is used. Tenney's (1987) "half-cosine interpolation", a function which starts and ends slowly, smoothly interpolating in the middle, has also proved useful (Fig. 17).

To emphasize perceptual weighting of events occurring at the beginning and end of morphs some triangular function, or approximation of an "inverse Gaussian" or "inverse normal" may be used. In practice, especially for short morphs, a simple triangular form is adequate:

Fig. 17.   Half-cosine curve.

$$\frac{|L/2 - i|}{L/2} \qquad\qquad\qquad (inverse\ triangular\ weighting)$$

where i is the index (either linear, row, or column, see below) and L is the number of intervals.[74]

In combinatorial metrics, *row* and *column* weights may be used. Row weights, in the form of the matrices used here, bias intervals according to the number of the morph element (intervals taken to the first element, to the second element, and so on). Column weights adjust by *adjacency* of an interval (one away, two away, and so on). Because each column weight starts further into the matrix row, they also incorporate something of the row weighting. For this reason, it is often simpler to specify only a column weight function.[75]

$$\frac{\displaystyle\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} (f(i) * g(j)) |\Delta(M_i, M_j) - \Delta(N_i, N_j)|}{\left(\sum f(i) * \sum g(j)\right) * maxint}$$

(row and column weighted, absolute scaled UCM)

where $f(i)$ and $g(j)$ are column and row weight functions. The unordered combinatorial metrics have a slightly different weighting equation:

$$\frac{\left| \left(\dfrac{\displaystyle\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} (f_m(i) * g_m(j)) \Delta(M_i, M_j)}{\left(\sum f_m(i) + \sum g_m(j)\right)} \right) - \left(\dfrac{\displaystyle\sum_{i=1}^{L-1} \sum_{j=i+1}^{L} (f_n(i) * g_n(j)) \Delta(N_i, N_j)}{\left(\sum f_n(i) + \sum g_n(j)\right)} \right) \right|}{maxint\,(N,M)}$$

where $g_m(j)$ and $f_m(i)$ are row and column weighting functions on M, *maxint (M,N)* as above.

In this more complex equation, separate weighting functions can be used for the two morphs, which may be of different lengths. The weighted equations for the other metrics (*OCM, ULM*, relative and unscaled forms) are easily derived from these examples.

The following example shows the differences between weighted and unweighted versions of the *OLM, OCM, UCM, OLD,* and *UCD* in various forms. $f(i) = 1/i$, where i is the order number in the morph, or the respective row or column.

M = {1,5,12,2,9,6}   N = {7,6,4,9,8,1}
M′ = {4,7,10,7,3}   N′ = {1,2,5,1,7}

Table 3.   Comparison of weighted and unweighted metrics.

Metric	Values	Weighted Values (1/i)
OLM	0.46	0.414599
OLM squared	0.649884	0.574196
relative scaled *OLM*	0.442857	0.367362
unscaled *OLM*	4.6	4.145991
unscaled *OLM* squared	6.49884	5.741964
OCM	0.327273	0.478788
OCM squared	0.479978	0.672608
relative scaled *OCM*	0.365909	0.478532
unscaled *OCM*	3.6	5.266663
unscaled *OCM* squared	5.279755	7.398692
UCM	0.145455	0.342639
relative scaled *UCM*	0.025	0.213554
unscaled *UCM*	1.6	3.769034
OLD	0.8	0.91241
UCD	0.266667	0.266667

Differences in metric values with this weighting function show similarity of early morph elements more heavily weighted, and also by the relative adjacency of intervals in combinatorial metrics (both row and column weightings are *1/i*). This is seen in the *OLD*, which measures the two difference vectors:

[− − + − +] and [+ + − + +]

and emphasizes the differences in the first four places, nearly negating the only similarity (index 5).

Weightings may be complex, and may be used to fine tune metrics and produce a wide variety of results using the same morphs and metric functions. The use of weighting is an important area for further research.

METRICS ON DIFFERENT LENGTH MORPHS

Morphs of unequal length are a fundamental problem in the design of *ordered* metrics. Various authors have approached this issue in discipline- and context-specific ways (Morris 1979–80; Rahn 1979–80; Pennycook & Stammen 1993;

Rabiner et al. 1978). For the ordered metrics several different techniques of length normalization are possible, including (Polansky 1987):

1) *Down-sampling*, or *decimating* the shorter morph to equal the length of the longer.
2) *Up-sampling* (*interpolating*) the longer to equal the length of the shorter.[76]
3) *Zero-padding* the shorter to equal the length of the longer.
4) *Truncating* the longer to equal the length of the shorter.
5) *Windowing* and *averaging* the shorter through the longer.
6) *Fixed dimensional* (*time-point*) sampling of morphs at equal sampling lengths.

In *down-sampling*, problems occur when the morphs are of non-integer related lengths. Salient information may be "accidentally" missed when "out-of-phase" with the "sampling rate", obfuscating magnitude and direction similarities between M, N:

$$M = \{3,5,7,9\} \qquad N = \{1,3,1,5,1,7,1,9\}$$
$$N_{decimated} = \{1,1,1,1\}$$

Up-sampling is more reliable because interpolated information is based upon patterns in the shorter morph:

$$M_{interpolated} = \{3,4,5,6,7,8,9,10\}$$

In many "real-world" judgements of morphological similarity, we are adept at cognitive similarity judgements using interpolation, sampling, and filtering for distortions of morphology regardless of length and/or size. For example see Fig. 18.

    Zero-padding and *Truncating*, are common in signal processing comparisons and transformations of sounds and sound files.[77] Truncation implies that we "stop paying attention" to a morph when it exceeds the length of the morph under comparison. Zero-padding assumes the opposite, that we "wait" until the length of the longer morph has been equalled by some kind of "dead-space after" the former. Although these descriptions are temporal, the techniques may be used in other dimensions as well.

    Zero-padding may be combined with a monotonically decreasing weighting, which "cross-fades" or de-emphasizes the problem of having no values to compare
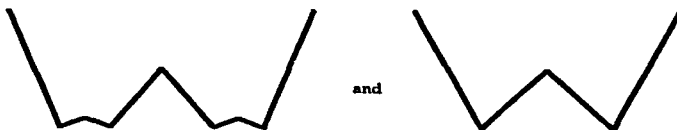
Fig. 18.   Similar morphs.

at the end of the longer morph. A steeply decreasing weighting function, reaching zero before the end of the shorter morph, is equivalent to truncation.

Windowing, perhaps the most interesting technique for further exploration, functions to some extent like convolution.[78] In windowing, the shorter morph "moves" through the longer one, taking a metric each time:
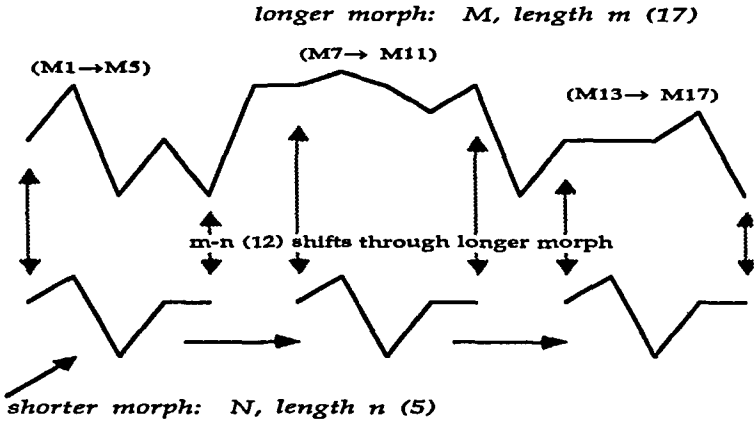
Fig. 19.   Windowing comparison of different length morphs.

A weighted average is taken as the final step

$$d(M,N) = \frac{\displaystyle\sum_{i=0}^{M_L - N_L} \alpha_i\,(d,(M,N))}{\displaystyle\sum \alpha_i} \qquad \textit{(weighted metric on different length morphs)}$$

where M is the longer morph, and α_i is some weighting function (as in the above equations for weighted morphs). In other words if $M_L = 100$ and $N_L = 10$, 90 metrics are taken, comparing N to $(M_1 - M_{10})$, $(M_2 - M_{11})$, ... $(M_{91} - M_{10})$. The weighted average of those metrics is $d(M,N)$. The choice of a weighting function is crucial. A simple approach is to weight metric values highest toward the beginning, middle and end, as in the following linear function.[79]

$$\alpha_i = \frac{abs \mid i \bmod (L_{diff}/2) - (L_{diff}/4)\mid}{L_{diff}/4}$$

(linear weighting function for metrics on morphs of different lengths)

where L_{diff} is the positive difference in length between the morphs. This function weights the beginning, middle and end of the shorter morph with the longer,

attempting to relate salient features. A differently shaped "w", constructed by changing the two constants, or one with more "ripples", tailors the technique to reflect different points, or a different "frequency" of common saliency. This technique is also useful for weighting intra-morphological intervals.
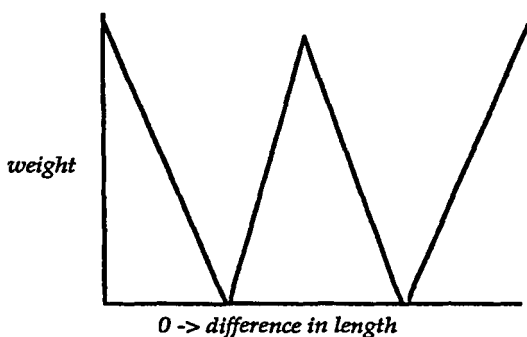
Fig. 20.   "W" weighting function.

The final technique, *fixed dimensional sampling*, reduces two morphs to a common length by sampling through another dimension, typically, time. For example, the similarity of two pitch/duration morphs may be measured by taking a metric on elements occurring at times *(a, b, c,...)* for some finite number of time points. Using this technique, similar to decimation in one dimension, both morphs are ordered by a common perceptual axis, like time.[80]

MULTIMETRICS AND MULTIDIMENSIONAL METRICS

Often what is needed is a metric on more than one dimension of a morph (*multidimensional metric*), or a metric which is the result of several metrics on the same pair of morphs (*multimetric*).

A simple multimetric is the standard *max metric*, treating two (or more) metrics as different points in a two- (or more) dimensional space. For example:

$d(M,N) = max \ (OCD(M,N),OLD(M,N))$ *(max multimetric)*

Simple, two-dimensional Euclidean forms may be used to combine metrics:

$d(M,N) = \sqrt{OCD(M,N)^2 + ULD(M,N)^2}$ *(Euclidean multimetric)*

resulting in a space such as the following:
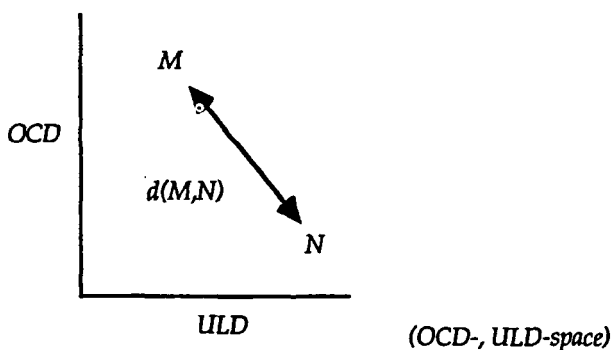
Fig. 21.   Two-dimensional morphological metric space.

Different metrics may be arithmetically weighted together into a multimetric. For example:

$$d(M,N) = \frac{\alpha(OCM(M,N) + \beta(OCD(M,N))}{\alpha+\beta} \qquad \text{(multimetric OCM, OCD)[81]}$$

returns a value between [0,1], the weighted average of two different metrics (each of which has many possible forms). Any number of metrics may be used in a multimetric, with weights and forms reflecting specific musical and perceptual similarities. Interesting results may be produced by combining, for example, a *ULM* and an *OLM*; the *OLM* might be weighted low, producing a metric which measures the difference of average magnitudes, with a slight "hint" of order. Similarly, by weighting magnitude and direction metrics differently, one can "tune" the metrics to measure attenuated or emphasized effects of inversion and contour. Multimetrics may also consist of the same metric with different Δs; for example, two versions of the *OLM*, the first with arithmetic absolute value intervals, the second ratiometric intervals. Metrics which measure range may be combined with those which measure average interval, linear with combinatorial (especially interesting if the *OLD* is combined with the *OCD*), and so on, creating a wide variety of morphological metric spaces, and suggesting an interesting area for future experimentation.

  Multidimensional metrics are more problematic since they try to integrate the effects of varying musical parameters, which necessitate different interval functions, into one measure (Tenney & Polansky 1980). A simple approach mirrors that of multimetrics; individual metrics for individual parameters which are weighted and averaged into a composite metric (see below).

EXAMPLE OF MULTIMETRICS:
RUTH CRAWFORD'S *PIANO STUDY IN MIXED ACCENTS*

The following examples show the use of different metrics to envision certain morphological ideas in Ruth Crawford's *Piano Study in Mixed Accents*. This work consists of 111 phrases of varying length. The first note of each phrase (all sixteenth notes) is accented, suggesting a (composer specified) set of morphs on which metrics may be taken (Fig. 22).

The first example takes four different unweighted metrics on adjacent morphs for the entire piece. For ordered metrics on morphs of different lengths, windowing was used with the "w" weighting function described above. Note that although values vary greatly between the contour and magnitude metrics, all four of the metrics often follow somewhat similar trajectories, implying certain deep connections between contour and magnitude (among other things) in the work, at least in the momentary transitions between one morph and the next (Fig. 23).

Viewing the individual metric "functions" (i.e., a metric over time) for this piece shows the morphological structure more clearly, and also demonstrates some of the properties of the metrics (Fig. 24).

Fig. 22.   Section 1 of Ruth Crawford's *Piano Study in Mixed Accents* (©1932 Theodore Presser Company. Used with permission).
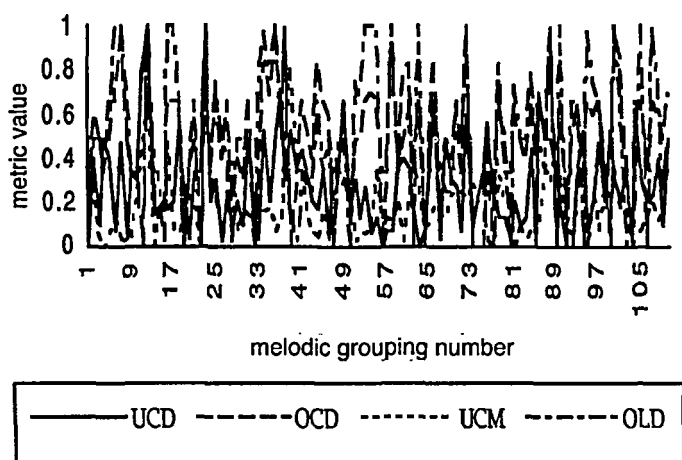
Fig. 23.   Four different metrics, adjacent morphologies, entire piece, Ruth Crawford's *Piano Study in Mixed Accents*.
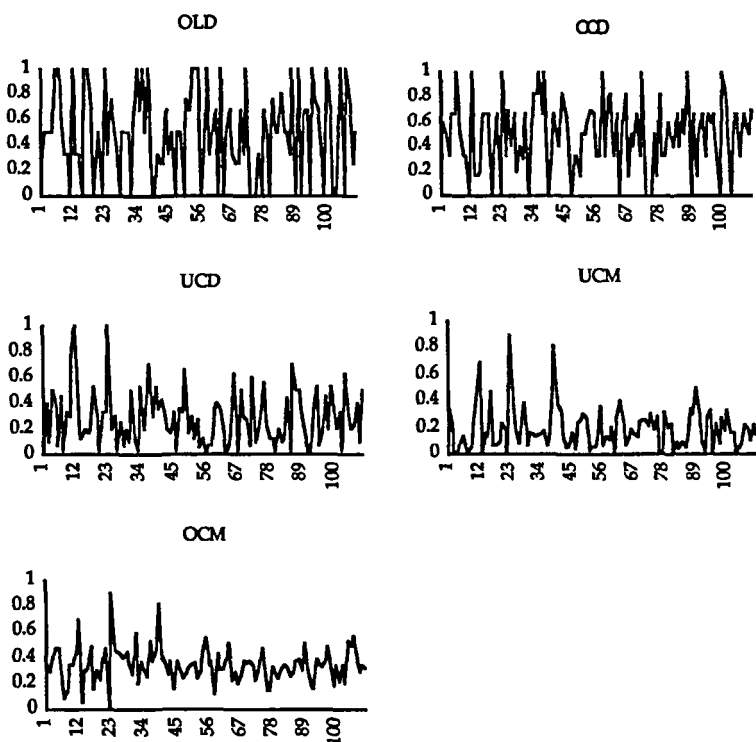
Fig. 24.   Five different metrics as functions for the entire piece, *Piano Study in Mixed Accents*, adjacent morphologies.

The *OCD* and *OLD* have the greatest range (showing a great deal of adjacent contour variation in the work). The magnitude metrics tend to be more "damped", which may be a secondary result of Ruth Crawford's compositional concern with interval class, not interval magnitude. New sections of the work begin at morphs 13, 28, 85, and 100; most of the metrics find some sort of peak at or around those values (this might be coincidental).

The next example shows the correlation coefficients on the 10 pairs of the five metrics above, for the whole work.
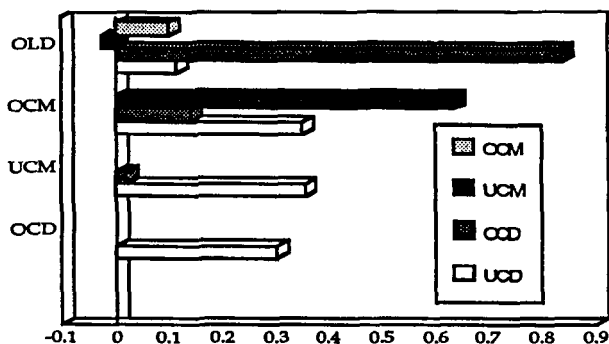
Fig. 25. Correlation coefficients of different metrics over the course of *Piano Study in Mixed Accents*.

The *OLD* and *OCD* are, expectedly, the most highly correlated, with the *OCM* and *UCM* next. There are also positive correlations for the *OCM* and *UCD*, *UCM* and *UCD*, and *OCD* and *UCD* (the last not surprising). The *UCM* and *OCD* are almost completely independent, as is the *OLD* and *UCM* (actually, a small negative correlation). Small positive correlations exist between the *OLD* and *OCM* and *UCD*, and the *OCM* and *OCD*.

Taking metrics on adjacent morphs is one possible single-dimensional solution to illustrating morphological structure. Metrics in the above example could have been taken to the first morph, the first morph in a current section, the last morph, some kind of average of the metrics between the first and last morphs, or many other possibilities. Since metric spaces need an origin or zero value, the choice of a given morph, or set of morphs to assign to that origin is musically and theoretically important.

The next example combines the *OCD* and *OCM* as *(OCM+OCD)/2*, using the graphs for the *OCM* and *OCD* above (adjacent morphs, entire piece). As in the example above, the "w-shaped" weighting function is used to window the shorter grouping through the longer in morphs of unequal length. This example illustrates a simple but somewhat effective way of integrating and envisioning metric values

on different morphological features. With regard to the incipients of new sections, note the spike at morph 13 (.8), a smaller one at 28 (.57), a slight hump at 85 (.45) and an anomalous "dip" at 100 (.09).

OCM+OCD/2
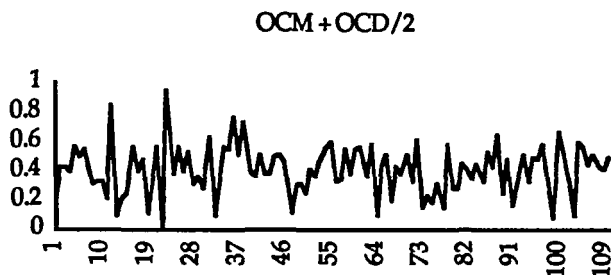
Fig. 26.   *(OCD+OCM)/2* metric, adjacent groups: Ruth Crawford's *Piano Study in Mixed Accents.*

Metrics on adjacent morphs measure local, moment-by-moment (or morph-by-morph) morphological movement. The following two examples illustrate the use of a non-temporal metric space visualization. Each of these graphs shows morphological similarity in section 3 of the work (the longest section). The x-axis is the *OCD* or *OCM* value to the first morph in the section, the y-value the *OCD* or *OCM* value to the last morph. Gray lines are drawn in for purely visual reasons, emphasizing the "shape" of this two-dimensional metric space. They mean nothing in terms of *distance* in the space; metric values between for example, the 3rd and 7th morph in the section are not contained in this graph in any way (see the section below on multidimensional scaling for more on this concept) (Fig. 27).

EXAMPLE OF MULTIMETRICS, MULTIDIMENSIONAL METRICS, AND ENVISIONING METRIC SPACES

The following is an extended example of multimetrics, multidimensional metrics, and what might be called *multimetric spaces*, using five equal length melodic jazz tune excerpts (Fig. 28).

The pitch and duration[82] morphs are shown in Fig. 29. There is no common y-axis here, each morph in one dimension is simply graphed below the previous one.

Each of the four duration functions rises characteristically near the beginning (a rest at the end of a smaller phrase) and at the end. For each of four different metrics, the *OLM*, *OLD*, *ULM*, and *OCD* there are three different half matrices of metric values between each of the five melodies: duration, pitch, and the average of pitch and duration[83] (see Table 4).
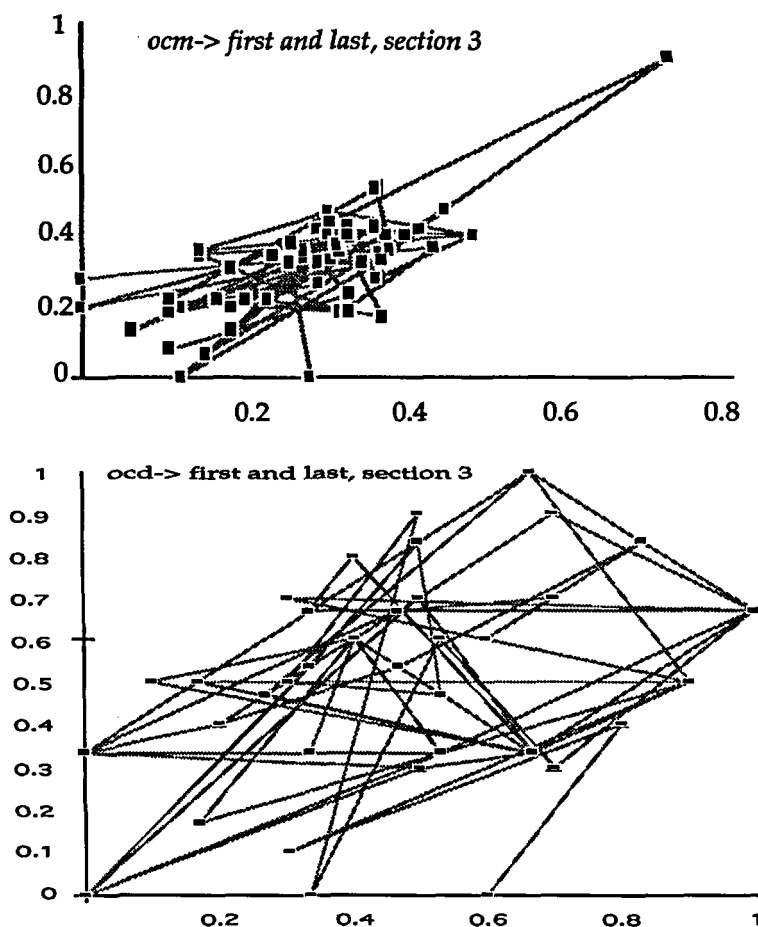
Fig. 27.   OCM and OCD metrics to first and last melodic groups, third section: Ruth
            Crawford's *Piano Study in Mixed Accents*.

The two examples in Fig. 30 show combinations of the four metric-spaces, in
the pitch and duration dimensions. The origin of the space is (melody 1, melody
5).[84] The choice of these two melodies as the origin, for the purpose of this
example, is arbitrary, illustrating only how the matrix of metric values can be
plotted in two-dimensional spaces, in the absence of more sophisticated multi-
dimensional reduction techniques like multidimensional scaling (see below). Note
that these pictures of the similarity spaces imply nothing about the similarity
between melodies 2, 3, and 4, except in the *similarity of their similarities* to
melodies 1 and 5.

The example shown in Fig. 31 reduces the information in Fig. 30 by first
averaging the four different metric values for pitch and duration (multimetrics), and

Fig. 28.   Musical examples for multimetric comparisons.
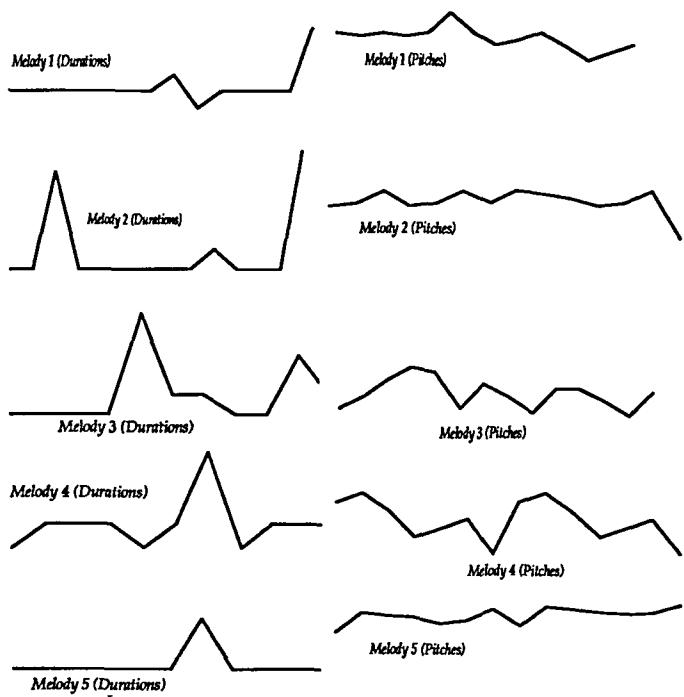
Fig. 29.   Visual representation of the musical examples above.

next averaging those two values for each melody. In other words, the pitch/dura-
tion values below show an extremely "information-dense" (if completely un-

Table 4. Metric values for melodies.

Pitch				Duration				Average			
(OLD)											
.46	.62	.46	.38	.38	.46	.69	.31	.42	.54	.58	.35
	.77	.15	.54		.46	.92	.46		.62	.54	.50
		.92	.54			.62	.46			.77	.50
			.38				.46				.42
(OCD)											
.59	.48	.57	.67	.44	.63	.59	.36	.52	.55	.58	.52
	.71	.40	.65		.51	.55	.37		.61	.47	.51
		.73	.56			.55	.48			.64	.52
			.65				.35				.50
(ULM)											
.04	.18	.01	.18	.13	.03	.15	.04	.09	.10	.08	.11
	.09	.05	.13		.11	.03	.15		.10	.04	.14
		.19	.04			.12	.11			.16	.08
			.19				.19				.19
(OLM)											
.16	.25	.22	.29	.21	.22	.27	.15	.18	.23	.24	.22
	.22	.13	.33		.26	.36	.26		.23	.23	.29
		.29	.33			.32	.25			.30	.34
			.22				.19				.21

weighted) value for the similarities of each melody to melody 1 and melody 5. Because successive means of different data sets are taken, this set of distance values will tend to "flatten out", blurring similarity distinctions. Weighting the different features (metrics, dimensions) in diverse ways will have a great effect on this space, and will most likely become a study in and of itself.

The following examples show another way of envisioning morphological metric space. Considering any single morph as a source, different metrics are represented as dimensions or axes of the space. Each other melody has a single, visually apparent distance from the source (Euclidean, city-block, or any other). Euclidean distances from melody 1 are shown by the example arrows.

In the first example (Fig. 32), melodies 1—4 are plotted in *OLM*-, *OCD*-space (x, y), a multimetric space which separates contour and magnitude along the two dimensions. Note that melody 5 is clearly the most dissimilar from melody 1 by both the *OLM* and *OCD*, somewhat justifying the use of those melodies as the "origin" for the examples above (the opposite is the case in terms of duration!).

The second example shows the same metric space, with melody 5 as the origin. Also shown in both graphs are the means of the metrics for pitch and duration, and a distance measure with four attributes. In the first example those attributes are: pitch and duration ordered combinatorial contour, and pitch and duration ordered linear magnitude. In the second example: pitch and duration linear ordered contour, and pitch and duration linear unordered magnitude. Since the *ULM* is generally less
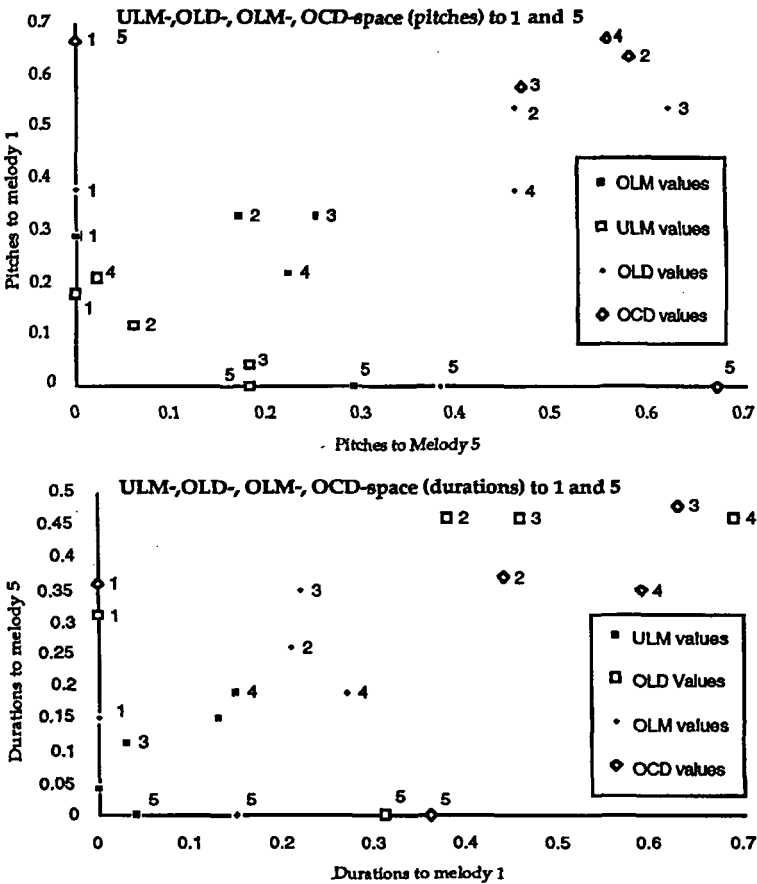
Fig. 30.   Multimetric space for pitches and durations of melodies.
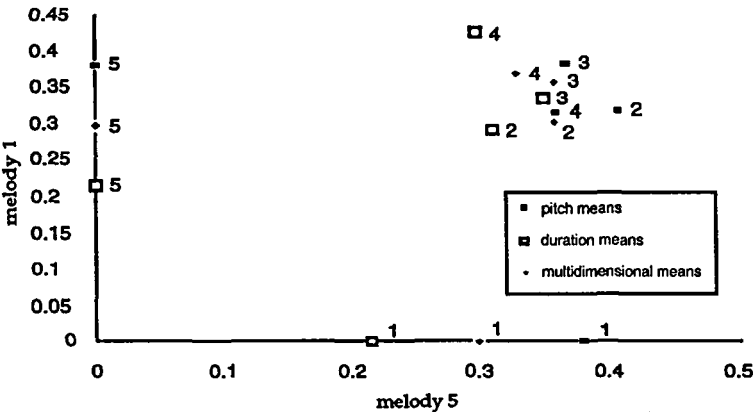

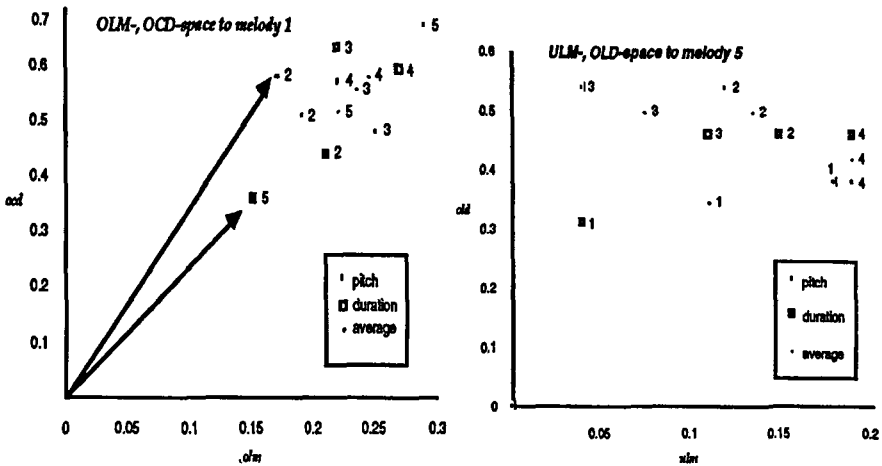Fig. 31.   Pitch, duration, and multidimensional metric means.

Fig. 32. Multimetric space with a fixed melody as the origin.

discriminating than the *OLM*, and similarly for the *OLD* vs. the *OCD*, the first graph is more "tightly packed", the second more diffuse.

The colinearity of the three metric values for each melody is a simple result of the average, which must fall on the line defined by the two other points.

The corresponding Euclidean metrics (shown by the length of the arrows in the first chart above) for the 12 points in each of these spaces are:

To Melody 1					To Melody 5			
mel	pitch	dur	∂		mel	pitch	dur	∂
2	.61	.49	.78		1	.69	.31	.76
3	.54	.66	.86		2	.66	.49	.82
4	.61	.65	.89		3	.56	.47	.74
5	.73	.39	.83		4	.67	.5	.84

where ∂ is the Euclidean distance value of the pitch and duration (Euclidean distance) values. In other words, ∂ is some "absolute" distance in one dimension of each melody from either 1 or 5, combining all four metrics in the pitch and duration dimensions.

By a long chain of measures, any of which could have been "tuned" in a number of ways, the table above says that melody 4 (from "Wolverine Blues" by Fred Morton and John Spikes) is the most dissimilar to both melodies 1 ("Never Never Land", by Mark Charlap, from *Peter Pan*) and 5 ("When You Wish Upon a Star", by Leigh Harline, Paul J. Smith, and Ned Washington, from *Pinocchio*), and that melodies 2 ("Stormy Weather" by Harold Arlen) and 3 ("Round Midnight" by Thelonius Monk) are most similar to melodies 1 and 5, respectively.

Also, the most similar of the set are 3 and 5, the most dissimilar 4 and 1. Using these "dense metrics", the visualization of the space can be reduced even further:
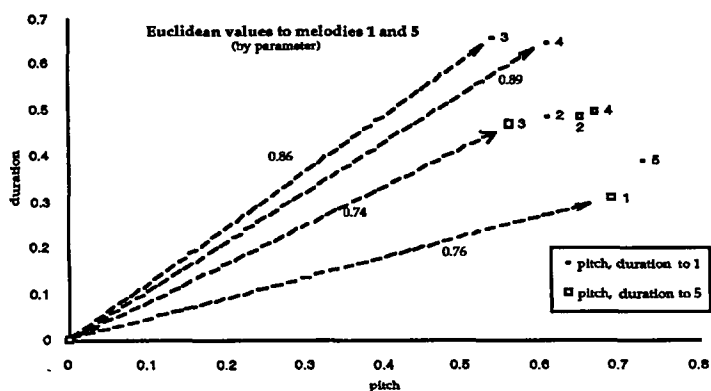
Fig. 33.   Multimetric space for pitches and durations of melodies.

In the above, only a few of the combined Euclidean distances in pitch- and duration-space are indicated by arrows and the distance values themselves. Recall that the "source" metric spaces for each of the melodies are different: *ULM-*, *OLD-*space to melody 5, *OLM-*, *OCD-*space to melody 1. Note also that physical distances between points (except to the origin, and to the axes) are illusory; this graph does not imply anything about distances *between* melodies 3, 4, and 5.

   Alternatively, distances to melodies 1 and 5 may be used as the axes (leaving out melodies 1 and 5 themselves), resulting in a closely packed (the heavy square is added) set of large distances from 1 and 5 (see Fig. 34).
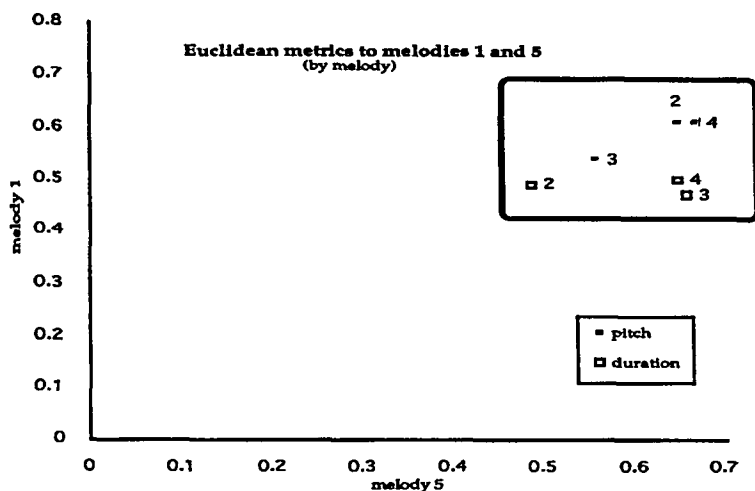
Fig. 34.   Combination of a number of metrics to melody 1 and 5.

This type of visualization allows for an interesting compositional application. By creating morphs (perhaps stochastically) which lie within balls of small radii around points in this space, one can design a set of "melodies" with a desired set of similarities to two (or more, simply by adding more "source melody axes") pre-existing melodies. Interpolations through this kind of space are possible, as are many other operations (like inversion, compression, and so on) suggesting many compositional applications.[85]

VISUALIZATION OF METRICS BY MULTIDIMENSIONAL SCALING

A standard technique for visualizing large numbers of pairwise relationships in a small number of dimensions is *multidimensional scaling*.

"This approach represents the similarity relations between objects in terms of a geometric model that consists of a set of points embedded in a dimensionally organized metric space, where the points correspond to the objects under consideration. The central assumption of this type of model is that the similarity data can be related by a linear or monotonic decreasing function to the interpoint distances in the metric space, that is, the larger the measure of similarity between two objects, the smaller the distance between the corresponding points in the metric space". (Krumhansl 1978)

The following examples illustrate the use of MDS[86] in visualizing morphological metric relationships in the piece discussed earlier, Ruth Crawford's *Piano Study in Mixed Accents*. The first graph (not multidimensional scaling) shows five different metrics on the first section of the piece. This graph is a "detail" from the larger graph above of the whole piece, but contains an additional metric, the *OCM*. Note the common trajectories, like the "hump" in the metric differences between morphs 4 and 5 (except for the *OLD*), and 11 and 12.

A set of morphs can be considered in a more general similarity context by envisioning their pairwise relationship matrix in a space of small dimensionality. The various clusterings, singularities, and overall shapes which result from the set of paired distances may be seen at a glance. The following five examples are multidimensional scalings of the pairwise relationship matrices of different metrics on three of the sections of the Ruth Crawford piece.[87]

I intend these examples to be an illustration of the use of a technique like MDS combined with the metric functions, more than as an actual "analysis" of the piece. MDS is more usually used with experimental data to structure a set of pairwise similarity judgements by subjects, and to try and suggest "dimensions of similarity"
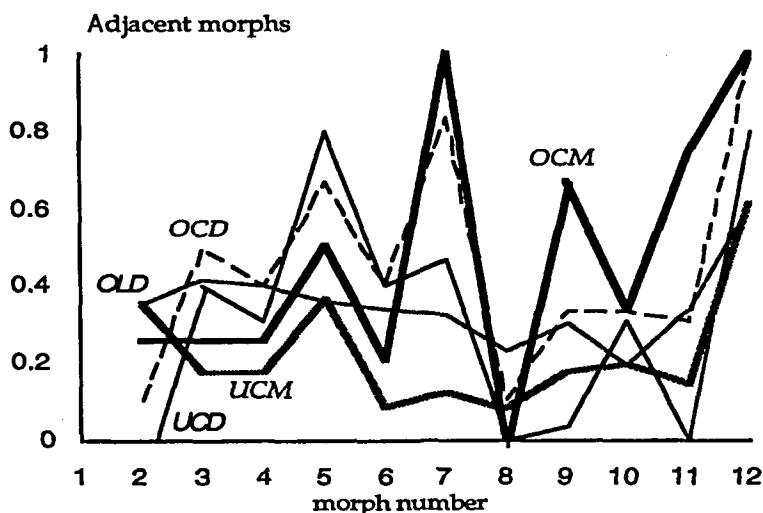
Fig. 35.   Five different metrics, compared on section 1, *Piano Study in Mixed Accents*.

for a multidimensional, or more precisely, multi-attribute data set, especially when it is unclear what those attributes are. When using MDS, one must be careful not to assume meanings for given dimensions, although they are often suggested when intuitively considering the data set. In applying MDS to the metrics, the criteria for "similarity judgements" are known (the metrics themselves), although the dimensions of the constructed MDS space are still difficult to "name".[88] In these examples, the "w" windowing function was used on the different length morphs for both (relative scaled) *OCM* and *OCD*.

The first two MDS plots show two different metrics on the twelve morphs of section 1 (see the score example above). Clusterings show similarity (3,5; 1,2,4,6,8), and the *OCD* shows an interesting clustering of "odd/even" morphs; every other morph is more or less placed into one of two areas in *OCD*-space. Since morph 12 is short (2 notes, a descending major seventh) its singularity is not particularly relevant here.

The next example (Fig. 38) plots the *OCD* in the final section (score shown below). The piece is a retrograde of itself, with symmetry around the middle section, but the composer adds two last notes at the end (C♯ down to A♯, after the final F down to D, imitating the minor third at the half-step from what would be expected to be the final notes).[89] For this and other reasons, the number of notes in each morph differs between sections 5 and 1. However, the *OCD* still shows symmetry around the "x-axis" for the two sections. Morph 1 is now the short morph, and thus spaced widely from the others. Some, but not all, of the even/odd clustering disappears. Most importantly, a new morphological similarity space emerges because of the way that the same notes "phase" through different
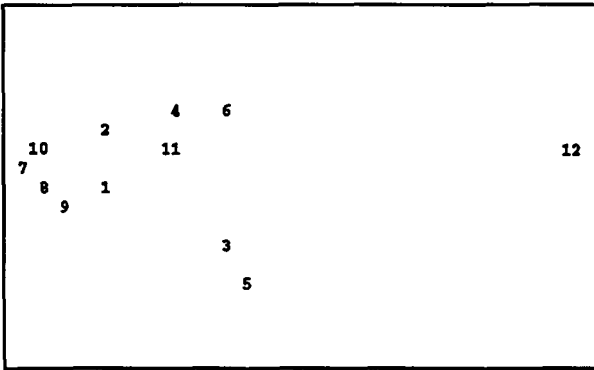
Fig. 36.   Multidimensional scaling coordinates in two dimensions: *OCM* metric (section 1).
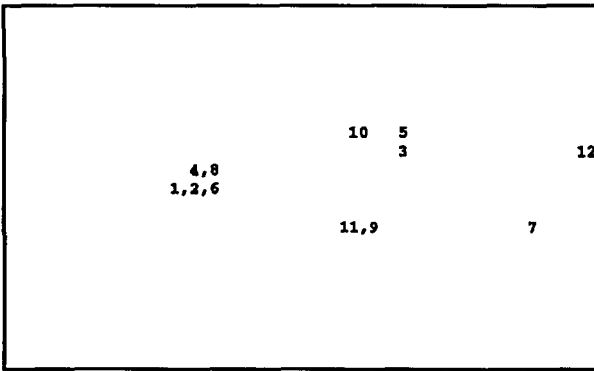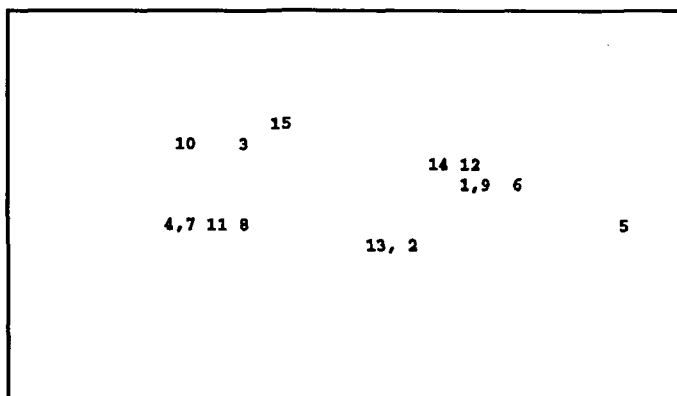

Fig. 37.   Multidimensional scaling coordinates in two dimensions: *OCD* metric (section 1).

morphological groupings as a result of the added two notes at the end of the piece (Fig. 39).

The final two examples (Figs. 40 and 41) show the effect of different scalings for one metric on the same section of the piece, section 2. Note that the *1/i* scaling technique for the *OCD* produces small, but interesting clustering changes. In all cases, MDS-induced two-dimensional similarities between morphs are reduced.

CONCLUSIONS, FURTHER IDEAS

In this article I have presented possibilities and techniques for approaches to the consideration of morphological similarity. I hope that others will extend, refine, and explore these ideas in a number of ways that I may not be able to foresee.

Fig. 38.   Last section of *Piano Study in Mixed Accents* (©1932 Theodore Presser Company. Used with permission).
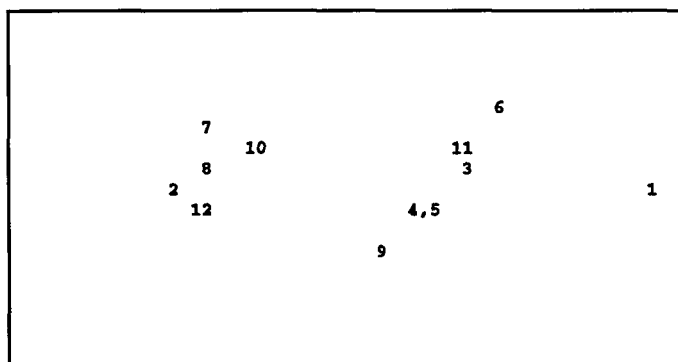
Fig. 39.   Multidimensional scaling coordinates in two dimensions: *OCD* metric (section 5), unity scaling.

The examples in the last section of the article illustrate a variety of partial solutions to the problem of visualizing and making use of the nontransitive, multidimensional nature of the pairwise metric values of a set of morphs. The usual difficulty of dealing with the $(N^2 - N)/2$ pairwise relationship matrix is compounded in this case by the number of musical parameters considered, and the number of different metric spaces, each a kind of "morphological feature" space.

The use of morphological similarity spaces can be an interesting and fertile compositional and analytical tool. It is tempting to search for a kind of single, unified metric which will in some way reflect innate perceptual similarity criteria. For composers, however, the variety of different measures and envisioning

Fig. 40.   Multidimensional scaling coordinates in two dimensions: *OCD* metric (section 2), unity scaling.
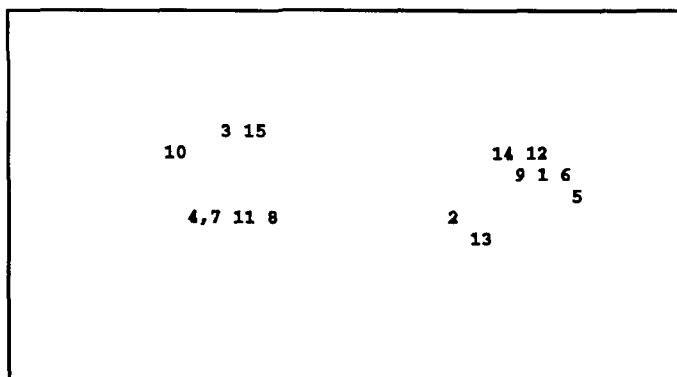
Fig. 41.   Multidimensional scaling coordinates in two dimensions: *OCD* metric (section 2), *1/i* scaling.

techniques is an advantage, facilitating experimentation, and with it, unexpected musical results.

Although I have, for simplicity's sake, exemplified the metrics via simple melodies and duration series, they have interesting applications in other domains. For example, spectra may be considered as morphs, and the comparison of spectra by these metric functions could be an interesting topic for new synthesis and timbral analysis techniques.[90] Scales and tuning systems are another class of morphs where similarity, and questions of "between-ness" suggest interesting musical possibilities. The metric functions may be used generatively as well as analytically; a new set of morphs may be "fitted" to a predetermined or computed morphological metric trajectory or space configuration. New scales and spectra (as well as melodies, forms, and so on) may be generated more or less continuously

between pre-existing ones, providing a formal structure for morphogenesis (the creation of new morphs) and continuous morphological transformation.

ACKNOWLEDGEMENTS

NOTES

1. Many of the ideas in this present article supersede and greatly expand upon a brief previous article (Polansky 1987). This current article contains numerous developments and revisions incorporating ideas drawn from recent composition, teaching, and software.
2. For an example of the use of the correlation coefficient on musical morphology in relation to Tenney's ideas, see Uno (1991).
3. Even what appears to be purely "quantitative" often depends on relation, and usually on metrics. A pitch mean of "60.7" means little if there is no standard for comparison. Our number systems themselves depend on such a metric; without being able to judge the relative distances of the numbers $\{0, 1, 3,$ and $1,000,000\}$ in some reasonable way, we would neither have an arithmetic nor a useful number system. In other words, though numbers often serve as purely descriptive or taxonomical entities (the concept of one apple may be irrespective of a metric in our experience), most useful taxonomies are ordered, and incorporate a concept of distance. There are many examples of "partially ordered taxonomies" in music, such as Forte's list of set-classes with its various relations.
4. Tenney (Belet 1987) later suggested substituting the word *holarchical* for *hierarchical* (also see Abraham 1987), since his theory implies no precedence of forms. Polansky, Burk and Rosenboom (1990) have further suggested the term *heterarchical*, because of the various ways that forms at lower levels may become forms at higher ones, and vice versa.
5. In Polansky and Bassein (1992, p. 275), a morphology is defined with more mathematical precision, for the purposes of a specific proof, as "a finite sequence of (not necessarily distinct) elements chosen from an ordered set".
6. The definition of a morphology is similar to the concept of a "time-series" in statistics.

7. In this article I have not dealt with the possibility that elements in a morph may be of different dimensionalities.

8. It is interesting to contrast Schoenberg's idea of "variation" with that of modern statistics. Tufte (1990, p. 22) points out the shift in this century towards studying and displaying standard deviations rather than means: "Measured assessments of variability are at the heart of quantitative reasoning".

9. "In both p- and pc-space, the order of a set's members is not defined" (Morris 1991, p. 20). Generally, the metrics in this article are assumed to operate in the pitch domain, on what Morris calls (ordered) *psegs* (Morris 1991, p. 5) — a list of pitches. This does not mean, of course, that the metrics cannot be easily adapted to transformations of that data, as I have tried to show often in this article.

10. Hermann (1994) refers to this as "the cardinality problem", in his exhaustive examination of how atonal theorists have constructed similarity functions. Isaacson (1990) also addresses this in his *ICvSIM* function, which satisfies his criteria of "be(ing) useful for sets of any size".

11. Morris (1989) proposes several interesting similarity functions for pc-segments, which although different in philosophy from the ones presented here, clearly share a common goal. Of special interest is Morris' use of the correlation coefficient (p. 120–121), *CC*, as a measure.

12. Called the discrete metric: $d(x,y) = 1$ if $x \neq y$; 0 if $x = y$.

13. Shreider (1974) provides some similar and amusing examples of nonsymmetrical distance functions in relation to travelling around Moscow.

14. Topological spaces which do not satisfy identity in "both directions", or $x = y \Rightarrow d(x,y) = 0$ but not vice versa, are called *psuedometric* spaces. Spaces which satisfy identity and symmetry, but not the triangle inequality, are called *semimetric*. Spaces which satisfy identity and the triangle inequality, but not symmetry are called *quasimetric*. (Sim 1976). The application of these spaces to music and to musical morphology in particular is an interesting area for further exploration.

15. Nick Didkovsky (personal communication) has suggested a special notation for two morphs which are "equal under a given metric", for example: *Equal(N, M, OLD)*, rather than "*M = N* under the *OLD* metric".

16. I am grateful to my colleague Dennis Healy, of the Dartmouth College Math and Computer Science Department, for suggesting this way of explaining equivalence classes and the difference between metric and point equality, and for some of the examples in this section.

17. Roeder (1987) describes several interesting metrics, including toroidal and other geometric ones on set classes and pc-space. He renotates the Morris *SIM* function, and refers to it as a metric, stressing that it is a metric on ICs (or Vs, as Morris calls them). However, Morris' *sim(S,R)* violates the identity condition on pc-sets in the way that I have described: two different set classes may have the same interval vector. Morris does not use the term "metric" in the original article. "All pc [pitch-class] sets within a single SC [Set Class] have the same [interval-class]-content. Thus, each SC is associated with one V [interval-class content, or 'interval vector']. However, the converse is not the case; some Vs are associated with two distinct SCs. In other words, two pc-sets may have the same V but not be related under T_n or T_nI [transposition, or transposition and inversion]". (Morris 1991). This, of course, is the Z-relation. ("Z-related sets are not related by IT". (Forte 1965).) Hermann (1994) analyses similarity functions in terms of whether or not they solve this "z-related sc problem".

18. Again, taking Morris' *sim* function as an example, it clearly satisfies the first three conditions. The triangle inequality is slightly more difficult to prove, yet intuitively obvious. One only has to try the function on a few interval vectors to be convinced. Since each individual difference is a metric, and the *sim* function is the sum of those differences, the *sim* function is also a metric.

19. For an excellent example in the derivation of a simple, computable form of the standard deviation, see Wyatt and Bridges 1967, p. 29.

20. The Morris *SIM* and *ASIM* function and the Lord *sf* (similarity function) are examples of

L_1 functions. Teitelbaum's *s.i.* (similarity index) and Isaacson's *ICvSIM* relation are examples of L_2 functions.

21. Shepard (1987) gives an excellent description of the difference between the L_1 and L_2 forms in psychological measures.

22. Although order is discussed frequently in this context ("Presumably, a refined measure of similarity would take order into account" (Forte 1965)), certain theorists have dealt with it in various ways different from the ones I propose here. For an example of a discussion of ordered pc-sets, see (Morris 1989, pp. 107–109).

23. Forte (1973, p. 3): "If however, the two sets [[0,2,3] and [2,3,0]] are regarded as distinct, it is evident that they are distinct on the basis of difference in order, in which case they are called *ordered sets*". This is an excellent way of describing what I mean by morphological.

24. Apel (1993) presents an interesting application of this in comparing MQ spectral analyses.

25. The magnitude metric may also be defined as follows:

$$d(x,y) = \sqrt{\int_a^b |f(t) - g(t)|^2}$$

the root of the integral of the square of the differences, or the Euclidean measure rather than the "city-block" measure. There is no corresponding *min* metric, which would violate the triangle inequality.

26. Thanks again to Dennis Healy for introducing me to this. The musical interpretation and rewriting of the more formal mathematical representation of the Sobalev norm, which is a similar concept to a metric, is of course my own.

27. "... both Forte's interval vector and my interval function count 'intervals' in some traditional sense, that is, 'distances' associated with pairs, whether nondirected or directed, of pitch-classes (x,y) or x to y. In each case, the notion of 'interval' is implicitly conceptualized as expressing some basic relation between the two pitch-classes involved". (Lewin 1977, p. 227).

28. Formally, this is similar to taking the average difference in corresponding elements of the *INT* function of two pc-segs (Morris 1991, p. 43). The Teitelbaum, Lord and Isaacson similarity relations on interval vectors are roughly equivalent to this form, each with slight variations. Teitelbaum's is the L_2 form, Lord's the L_1 form divided by 2, and Isaacson the L_2 "partially scaled" by the standard deviation of the vector value differences. Hermann, (1994, p. 111), shows how to scale the Isaacson *ICvSIM* measure to [0,1], using the conventional method of normalizing a standard deviation.

29. An alternate form is the square root of the sum itself, as in the standard deviation or Isaacson's *ICvSIM*.

30. That is: $|M_i - M_{i-1}|$ ..., or the "discrete first order derivative". In statistical analysis of time series, this is called a "lag", and a number of lags may be used to analyse a series (*i-1, i-2, ...*) up to the total number of pairwise relationships.

31. For a more rigorous description of direction and magnitude, see (Polansky & Bassein 1987); or Morris' *COM* function (Morris 1989, p. 28).

32. These mutations are coded in HMSL as special cases of the metric functions; everything described in this article is to some extent characteristic of the mutations as well.

33. For the purpose of this article, it will be assumed that at least one of the morphs in any given metric comparison has at least one "nonzero" interval by the Δ used, preventing *maxint* and similar scalars from producing a zero denominator. This is similar to saying that at least one of the morphs is not a straight line.

34. Like *adjacency interval*, many of these more abstract ideas are proposed as areas for further development. I have implemented these notions in a general software environment, but in my own composition, I have not made use of *all* possible metrics, indexing schemes, and interval calculations!

35. The *OLM* is coded like this in my HMSL implementation of these metrics.

36. Alternately:

        (i, i+1), (i+1, i+2), ...

               (i, i+2), (i+1, i+3) ...

                       (i, i+3), (i+1, i+4) ...

           ...   .

but the form used in this article seems more common (e.g., Morris 1991, p. 22).

37. Cuddy and Cohen (1976) investigate certain aspects of perception in the recognition of melodies in which adjacent and nonadjacent or combinatorial, and intervals are altered.

38. Also expressed as:

$$\sum_{i=1}^{L} i$$

see (Polansky & Bassein 1992) for more on this function, which is also fundamental in atonal set theory as the total number of values in the interval vector of a set class of cardinality L.

39. These measures are only meaningful for $L > 2$.

40. In a metric, two morphs may use a different number of interval calculations, roughly having the same effect as taking a metric on morphs of different length.

41. Length equivalence is often assumed in measures of statistical dependency like correlation coefficient as well.

42. Since the *sgn* function is three-valued, these direction metrics measure the differences between what I have called *ternary contour* (Polansky & Bassein 1992). To extend their definition to measure *n-ary* contours, the *sgn* function must be changed to represent *n-ary* values. If *n* gets large for *n-ary* contours, the results approach those of magnitude metrics.

43. These are the opposite of Morris' *COM* function (Morris 1989, p. 28), also a ternary function, in which $COM(a,b) = 1$ if $b > a$. In (Polansky & Bassein 1992) we call attention to Knuth's clarifying suggestion of using a "balanced number system" rather than positive and negative values. Computationally it is often simplest to use the equation:

$$sgn\,(M_i, M_j) = \frac{(M_i - M_j)}{|M_i - M_j|}$$

44. For *n-ary* contour functions like those suggested in Polansky and Bassein (1991), these linear contour vectors (as well as the corresponding combinatorial contour vectors) would be *n-1* places long. See Friedman (1987, 1985), Morris (1989), and Marvin and Laprade (1987) for related uses of direction vectors.

45. Note that my notation for morphs is different from the frequent use in atonal set theory of "curly brackets" for unordered sets.

46. Morris' (1979–80) formula for the "number of ICs in common between" two sets R and S uses the same principle:

$$k = \frac{\#V(R) + \#V(S) - SIM\,(R,S)}{2}$$

as does Lord's *sf* (Lord 1981), which may simply be written as *SIM/2* (for sets of the same cardinality). Morris's *k* is an example of "union minus (over) the intersection", an important concept in the information theoretic measure of different sized messages (e.g., Chaitin 1979, and also suggested to me by both David Rosenboom and James Tenney, personal correspondences). Hermann's *SR* (set relate) function is another example of this general principle, though more elaborate. The "union over the intersection" is a fundamental para-

digm for many of the ideas in this paper for the measurement of similarity between two sets.

47. The *diff* function is commonly used in coding theory for metrics on binary strings. Grimaldi (1989, pp. 647–648) provides a formal description of its use in the Hamming Metric.

48. For more on combinatorial contour, see (Polansky 1987; Polansky & Bassein 1992), Morris (1989), or (Marvin & Laprade 1987; where their more or less equivalent *CSIM* function is described). I have used the *OCD* as an important part of many pieces since 1986, including *3 Studies,* and *17 Simple Melodies of the Same Length* (Polansky 1994b).

49. These are the same as Morris' *COM*-matrix:

$$\text{sgn}(M_i - M_{i+1}) \quad \text{sgn}(M_i - M_{i+2}) \quad \dots \quad \text{sgn}(M_i - M_L)$$
$$\text{sgn}(M_{i+1} - M_{i+2}) \quad \dots \quad \text{sgn}(M_{i+1} - M_L)$$
$$\dots \quad \dots \quad \dots$$
$$\text{sgn}(M_{L-1} - M_L)$$

Morris (1989, p. 28) points out also that "Com(parison) matrices are not uniquely associated with one contour. The contours <4 1 5 9> and <6 2 7 8> will also generate the [same] comparison matrix... If the graphic representations of contours that have the same comparison matrices are compared, all the contours will have the same visual (and aural) pattern". Under the *OCD*, these two morphs are the same.

50. Coincidentally the same as the *ULD* value for the two morphs. When $L = 4$, the binomial coefficient $= 6 = 3*2$, resulting in the same *grain* as the *ULD*.

51. For an interesting example of the *OCD*, see Langmead (1995, 1995a), where it is used to measure differences between morphological representations of different cognitively based timbral features.

52. Alternate notations for the combinatorial contour vector each having their own advantages, are Polansky and Bassein's (1992) ternary numbers for combinatorial contour, and Morris' (1989) rank order form for contour space.

53. $(|1/3 - 2/4| + |0/3 - 0/4| + |2/3 - 2/4|)/2 = (.17 + .17)/2 = .17$

54. $(|1/6 - 4/10| + |0/6 - 0/10| + |5/6 - 6/10|)/2 = (.23 + .23)/2 = .23$

55. An equivalent notation for the contour of these three morphologies, using Morris' "ranking" method, is $M = \{2143\}$, $N = \{2121\}$ and $O = \{4132\}$. In this method, each element in the morphology is represented in order by the ranking vector by a number representing its "rank" from least to greatest. If all values are equal, the vector consists of 1s. In a strictly monotonically increasing morphology, the vector goes from 1 to L.

56. Note that changing one matrix value, for example 2nd row, 2nd column of M, to – rather than +, results in what Polansky and Bassein (1992) called "impossible melodies": contours which are easily described but which violate transitivity. That is, one may have arbitrary linear contours, but not arbitrary combinatorial ones, since *most* are impossible.

57. In fact all of the metrics — magnitude and direction, ordered and unordered — may be generalized in this way. If Δ in the above equation is the *sgn* function, and the absolute value is changed to *diff*, this is the *OLD*.

58. See Copson (1968, p. 93), and also note that the *max* of two metrics is a metric (p. 61).

59. Most of the unordered magnitude metrics may be renotated to take into account the fact that $|a-b| = |a| - |b|$.

60. This section of the paper benefitted greatly from conversations with Steve Berkley who implemented some of these ideas in his program *corrMorph* (Berkley 1993).

61. The parentheses indicates the "half-open" interval $0 \rightarrow 1$.

62. My appreciation to Chris Langmead for first suggesting this form.

63. I have not seen this particular atonal similarity measure, but I am not seriously proposing it since it reduces information too severely to be of much musical use. I am only trying to point out how morphological metrics may be transformed into their atonal similarity function cousins.

64. See (Ames 1992, 1991) for a thorough analysis of musical statistical distributions in music. There are standard mathematical metrics upon *distributions*, which offers an interesting direction for further development.

65. As such, it is difficult to average a set of CCs.

66. An even simpler version would be the difference in absolute value ranges of M, N.

67. This is only one version of the equation for variance which may be thought of as the average difference between members of a population and the mean.

68. Treating $x_i - x_\mu$ as a, and $x_j - x_\mu$ as b, and the summation as one addition, we have $|a| + |b|$ compared to $\sqrt{(a^2 + b^2)}$ and squaring both sides, the inequality: $(|a| + |b|)^2 \geq a^2 + b^2$.

69. Isaacson's (1990) *ICvSIM* is another good example; the standard deviation (unscaled) of the difference of two interval vectors.

70. This is the simple ranking of harmonic distance as described earlier. More complex and sensitive ones may be used, especially for non-12-tone-ET pitch space. A simple adjustment to this ranking would be to make the harmonic distance equal for an interval and its inversion. I am not proposing any specific ranking, only using the one above as an example of how a morph might be comprised of harmonic distances.

71. The linear contour vectors of M, N respectively are [302] and [104], and the combinatorial contour vectors, [10 0 5] and [609]. The linear contour series (adjacent intervals) are [—+—+] and [++—++]. The combinatorial contour matrices are:

```
 - - - - -        + + - - +
  - + - -          + - - +
   + + +            - - +
    - -              + +
     +                +
```

72. I am indebted to Steve Berkley for implementing the first version of weighted metrics in his program *corrMorph* (Berkley 1993). In doing so, he contributed and helped clarify some of these ideas.

73. For real-time applications, I have often used the average of a series of uniform random distributions to approximate the Gaussian function.

74. All of these weighting equations are directly applicable to windowing metrics of different lengths, as described in the section on windowing below.

75. If both are used, column weights can be rescaled to "eliminate" the influence of row (an artifact of the way these matrices are written). In other words, "visualize" the half-matrix as in B rather than A below:

A: Column#						B: Column#				
1	2	3	4	5		1	2	3	4	5
	a	b	c	d		a	b	c	d	e
		e	f	g		e	f	g		
			h	i		h	i			
				j		j				

76. In the program *Soundhack*, by Tom Erbe (Erbe 1994), functions similar to these metrics are used on sound spectra. The problem of unequal length sounds is solved by offering the choice of truncating to the shorter file, or time-stretching the shorter to the longer. Carter Scholz also suggested (personal communication) padding the shorter by other values, such as the mean.

77. Zero-padding is often used in FFTs and DSP algorithms to ensure a power-of-two number of samples.

78. DTW, or Dynamic Time Warping, a technique from speech recognition, has been explored in real-time by Pennycook and Stammen (1993) as a similar solution to the problem of comparing musical sequences of different lengths.

79. I have used a simple linear form of an exponential or trigonometric equation of the same "shape". In weighting these discrete, short morphs, I have found linear functions acceptable and faster to compute in real time. To apply these metrics to the spectral domain, on sounds themselves, more sophisticated windowing or weighting functions would be needed, like

those used in FFT applications (Hamming, Hanning, and other "raised cosine" window functions).

80. Ideas similar to this have been a common technique in later serialist works, and (Rahn 1977) offers some interesting formal comments on the notion of "timepoint similarity in pitch, in which several different timepoints are associated with the same pitch".

81. This form was used in my *17 Simple Melodies of the Same Length* (Polansky 1994a, 1988).

82. The rest in melody 4 is added to the duration of the previous note.

83. Simple metric forms are used for these examples. All metrics are unweighted, and the morphs are of equal length. For the *OLM* and *ULM*, the absolute scaled forms are used. Durations and pitch intervals are simple arithmetic differences (unsigned in the magnitude metrics, only signs in the direction metrics).

84. The metric values in these charts are the top row and final column of the matrices above. The "top" corner is thus a redundant value (1 is to 5 as 5 is to 1). The values lying on the axis themselves (1 and 5) are symmetrical.

85. I used this idea in my piece *The Casten Variation*, for solo piano or for ensemble, which is based on Ruth Crawford's *Piano Study in Mixed Accents*. (Polansky 1994b).

86. For another example of MDS using the metric functions described here, see (Berkley 1991).

87. I used the commercially available program SysStat for these MDS plots. For more on multidimensional scaling and the geometrical representation of similarity see Shepard (1962a, 1962b, 1964, 1974), Torgerson (1965), Tversky (1977) and Krumhansl (1978). As examples of musical applications of the technique, see Wessel (1979), Grey (1975) and Vaughan (1991).

88. Since this is not a study in MDS, I have not listed the usual technical information (stress, type of scaling and so on). In all cases I used Kruskal scalings and Euclidean metrics. Stress values were all within standard limits for the program used.

89. For more on the "palindromic" details of this piece, see (Nelson 1986).

90. The author and Tom Erbe have explored some of these in what we call "spectral mutation" in the program *Soundhack* (Polansky & Erbe 1994; Erbe 1994; Polansky 1992a).

REFERENCES

Abraham, R. (1986). Mathematics and evolution: A manifesto. *International Synergy Journal*, 1(3), 14–28.

Adams, C. (1976). Melodic contour typology. *Ethnomusicology*, 20(2), 179–215.

Ames, C. (1992). A catalog of sequence generators. *Leonardo Music Journal*, 1(2), 55–72.

Ames, C. (1991). A catalog of statistical distributions: Techniques for transforming random, determinate and chaotic sequences. *Leonardo Music Journal*, 1(1), 55–70.

Apel, T. (1993). A metric for determining the distance between MQ frames. Paper written for Systems of Musical Thought, Graduate Seminar, Dartmouth College.

Ash, R. (1970). *Basic Probability Theory*. New York: John Wiley and Sons.

Attneave, F. (1950). Dimensions of similarity. *American Journal of Psychology*, 63, 516–556.

Barlow, C. (1987). Two essays on theory. *Computer Music Journal*, 11(1), 44–60.

Barlow, C. (1980). *Bus journey to Parametron (all about Çogluotobüsisletmesi)*. Feedback Papers 21–23. Köln: Feedback-Studio-Verlag.

Becker, J (1980). *Traditional Music in Modern Java*. Honolulu: University Press of Hawaii.

Belet, B. (1987). An interview with James Tenney. *Perspectives of New Music*, 25(1,2), 459–466.

Berkley, S. (1993). corrMorph: Extending Morphological Metrics with Convolution Operators. Paper written for Systems of Musical Thought, Graduate Seminar, Dartmouth College.

Bryant, V. (1985). *Metric Spaces: Iteration and Application*, Cambridge: Cambridge University Press.

Chaitin, G. (1979). Towards a mathematical definition of life. In R.D. Levine, M. Tribus (eds.), *The Maximum Entropy Formulation*. Cambridge: MIT Press.

Chalmers, J. (1993). *Divisions of the Tetrachord*. Lebanon: Frog Peak Music.

Copson, E.T. (1968). *Metric Spaces*. Cambridge: Cambridge University Press.

Crawford, R. (1931). *Piano Study in Mixed Accents*. Bryn Mawr: Theodore Presser Company.

Cuddy, L. & Cohen, A.J. (1976). Recognition of transposed melodic sequences. *Quarterly Journal of Experimental Psychology*, 28, 255—270.

Cuddy, L., Cohen, A.J. & Miller, J. (1979). Melody recognition: The experimental application of musical rules. *Canadian Journal of Psychology*, 33, 148—157.

Deliege, I. (1987). Grouping conditions in listening to music: An approach to Lerdahl and Jackendoff's grouping preference rules. *Music Perception*, Summer, 4(4), 325—360.

Dowling, W.J. (1982). Melodic information processing and its development. In D. Deutsch, (ed.), *The Psychology of Music*. New York: Academic Press.

Dowling, W.J. (1978). Scale and contour: Two components of memory for melodies. *Psychological Review*, 85(4), 341—354.

Dowling, W.J. (1972). Recognition of inversions of melodies and melodic contours. *Perception and Psychophysics*, 9, 348—349.

Dowling, W.J. (1971). Recognition of melodic transformations: Inversion, retrograde, and retrograde inversion. *Perception and Psychophysics*, 12, 417—421.

Dowling, W.J. & Fujitani, D.S. (1971). Contour, interval and pitch recognition in memory for melodies. *Journal of the Acoustical Society of America*, 49, 524—531.

Dowling, W.J. & Hollombe, A.W. (1977). The perception of melodies distorted by splitting into several octaves: Effects of increasing proximity and melodic contour. *Perception and Psychophysics*, 21, 60—64.

Dugundgi, J. (1966). *Topology*. Boston: Allyn and Bacon.

Edworthy, J. (1985). Interval and contour in melody processing. *Music Perception*, 2(3), 375—388.

Edworthy, J. (1983). Melodic contour and musical structure. In W.J. Dowling, D. Harwood (eds.), *Musical Structure and Cognition*. New York: Academic Press.

Erbe, T. (with Polansky, L.). (1994). *Soundhack User Manual*. (Manual section on Spectral Mutation written by L. Polansky). Available via FTP from: http://music.calarts.edu/~tre.

Forte, A. (1973). *The Structure of Atonal Music*. New Haven: Yale University Press.

Forte, A. (1965). The domain and relations of set-complex theory. *Journal of Music Theory*, 9(1), 173—180.

Friedmann, M. (1987). A response: My contour, their contour. *Journal of Music Theory*, Fall, 268—274.

Friedmann, M. (1985). A methodology for the discussion of contour: Its application to Schoenberg's music. *Journal of Music Theory*, 29, 223—248.

Grey, J.M. (1975). An exploration of musical timbre. CCRMA Report No. STAN-M-2.

Grimaldi, R.P. (1989). *Discrete and Combinatorial Mathematics*. New York: Addison Wesley Publishing.

Hermann, R. (1994). *A General Measurement for Similarity Relations: A Heuristic for Constructing or Evaluating Aspects of Possible Musical Grammars*. Ph.D Dissertation. Eastman School of Music.

Isaacson, E. (1990). Similarity of interval-class content between pitch-class sets. *Journal of Music Theory*, 34(1), 1—28.

Kaplansky, I. (1972). *Set Theory and Metric Spaces*. Boston: Allyn and Bacon.

Kolinski, M. (1965a). The general direction of melodic movement. *Ethnomusicology*, 9, 240—264.

Kolinski, M. (1965b). The structure of melodic movement — A new method of analysis (revised version). In *Studies in Ethnomusicology*, 2, (pp. 96—120). New York: Oak Publications.

Knuth, D. (1973). *The Art of Computer Programming. Volume 1: Fundamental Algorithms*. Reading: Addison Wesley.

Krumhansl, C. (1978). Concerning the applicability of geometric models to similarity data: The interrelationship between similarity and spatial density. *Psychological Review*, 85(5), 445—463.

Langmead, C. (1995). A Theoretical Model of Timbre Perception Based on Morphological Representations of Time-Varying Spectra. M.A. Thesis in Electro-Acoustic Music. Dartmouth College.

Langmead, C. (1995a). *PAST* (software), available from http://music.dartmouth.edu/~langmead

Lederman, W. & Vajda, S. (eds.). (1982). *Handbook of Applicable Mathematics. Volume IV: Analysis.* New York: John Wiley and Sons.

Lerdahl, F. & Jackendoff, R. (1983). *A Generative Theory of Tonal Music.* Cambridge: MIT Press.

Lewin, D. (1979–80). A response to a response: On PC set relatedness. *Perspectives of New Music,* 13(2), 498–502.

Lewin, D. (1977). Forte's interval vector, my interval function, and Regener's common-note function. *Journal of Music Theory,* Fall, 21(2), 194–237.

Lord, C.H. (1981). Intervallic similarity relations in atonal music. *Journal of Music Theory,* Spring, 25(1), 91–111.

Marvin, E. (1991). The perception of rhythm in non-tonal music: Rhythmic contours in the music of Edgard Varèse. *Music Theory Spectrum,* 13(1), 61–78.

Marvin, E. (1990). A generalization of contour theory to diverse musical spaces: Analytical applications to the music of Dallapiccola and Stockhausen. Paper presented at the Society for Music Theory. Oakland, California.

Marvin, E. & Laprade, P. (1987). Relating music contours: Extensions of a theory for contour. *Journal of Music Theory,* 31(2), 225–267.

Monahan, C. & Carterette, E. (1985). Pitch and duration as determinants of musical space. *Music Perception,* Fall, 3(1), 1–32.

Morris, R. (1992). New directions in the theory and analysis of musical contours. Paper presented at the Society for Music Theory Meeting. Kansas City.

Morris, R. (1991). *Class Notes for Atonal Music Theory.* Lebanon: Frog Peak Music.

Morris, R. (1989). *Composition with Pitch Classes: A Theory of Compositional Design.* New Haven: Yale University Press.

Morris, R. (1979–80). A similarity index for pitch-class sets. *Perspectives of New Music,* 18(2), 445–660.

Nelson, M. (1986). In pursuit of Charles Seeger's heterophonic ideal: Three palindromic works by Ruth Crawford. *Musical Quarterly,* 72(4), 458–475.

Neter, J., Wasserman, W. & Whitmore, G.A. (1978). *Applied Statistics.* Boston: Allyn and Bacon.

Pennycook, B. & Stammen, D. (1993). Real-time recognition of melodic fragments using the dynamic timewarp algorithm. In *Proceedings of the International Computer Music Conference 1993.* (pp. 232–235). Tokyo: International Computer Music Association.

Polansky, L. (1994a). *The Casten Variation.* Score. For solo piano, or ensemble. Lebanon: Frog Peak Music.

Polansky, L. (1994b). Live interactive computer music in HMSL: 1984–1992. *Computer Music Journal,* 18(2), 59–77.

Polansky, L. (1993a). Envisioning Ruth Crawford Seeger's *Piano Study in Mixed Accents.* Response presented at Society for Music Theory. Montreal. Extended version of paper given at West Coast Society for Music Theory, (1994). Oakland, California.

Polansky, L. (1993b). *Roads to Chimacum.* Score. For string or mandolin quartet. Lebanon: Frog Peak Music.

Polansky, L. (1992a). More on morphological mutation functions: Recent techniques and developments. In *Proceedings of the International Computer Music Conference 1992* (pp. 57–60). San Jose: International Computer Music Association.

Polansky, L. (1992b). *Two Childrens' Songs.* Score. For two bass winds. Lebanon: Frog Peak Music.

Polansky, L. (1991). *51 Melodies...* Score. For two electric guitars and rock band. Lebanon: Frog Peak Music.

Polansky, L. (1988). *17 Simple Melodies of the Same Length,* for solo performer and computer. Software. Frog Peak Music. Available on cassette, *Works for Performers and Live Interactive Computer.* Lebanon: Frog Peak Music.

Polansky, L. (1987). Morphological metrics: An introduction to a theory of formal distances. In *Proceedings of the International Computer Music Conference 1987* (pp. 197–205). San Francisco: International Computer Music Association.

Polansky, L. (1987a). Distance Musics I-VI, For Any Number of Programmer/Performers and Live, Programmable Computer Music Systems. *Perspectives of New Music*, 25(1,2), 537—544.

Polansky, L., Burk, P. & Rosenboom, D. (1990). HMSL (Hierarchical Music Specification Language): A theoretical overview. *Perspectives of New Music*, 28(2), 136—178.

Polansky, L., Burk, P. & Rosenboom, D. (1987). HMSL: Overview (Version 3.1) and Notes on Intelligent Instrument Design. In *Proceedings of the International Computer Music Conference 1987* (pp. 220—227). Urbana: International Computer Music Association.

Polansky, L. & Bassein, R. (1992). Possible and impossible melody: Some formal aspects of contour. *Journal of Music Theory*, 36(2), 284. (Paper delivered at the Society for Music Theory. Oakland, California).

Polansky, L. & Erbe, T. (1996). Spectral mutation in Soundhack.

Polansky, L. & McKinney, M. (1991). Morphological mutation functions: Applications to motivic transformation and a new class of cross-synthesis techniques. In *Proceedings of the International Computer Music Conference 1991* (pp. 234—241). Montreal: International Computer Music Association.

Rabiner, L.R., Rosenberg, A.E. & Levinson, S.E. (1978). Considerations in dynamic time warping algorithms for discrete word recognition. *IEEE Transactions on Acoustics, Speech, Signal Processing*, ASSP-25, 575—582.

Rahn, J. (1979—80). Relating sets. *Perspectives of New Music*, 13(2), 483—498.

Rahn, J. (1975). On pitch or rhythm: Interpretations of orderings of and in pitch and time. *Perspectives of New Music*, 13(2), 182—203.

Regener, E. (1974). On Allen Forte's theory of chords. *Perspectives of New Music*, 13(1), 191—212.

Roeder, J. (1987). A geometric representation of pitch-class series. *Perspectives of New Music*, 25(1,2), 362—409.

Schoenberg, A. (1975). *Style and Idea*. London: Faber and Faber.

Scholz, C. (1994). *Lattice*. Piece on the CD *Hallways: 11 Musicians and HMSL*. Lebanon: Frog Peak Music.

Seeger, C. (1960). On the moods of a music logic. *Journal of the American Musicological Society*, 13, 224—261.

Shepard, R.N. (1987). Towards a universal law of generalization for psychological space. *Science*, September 11, 1317—1323.

Shepard, R.N. (1974). Representation of structure in similarity data: Problems and prospects. *Psychometrika*, 39, 373—421.

Shepard, R.N. (1964). Attention and the metric structure of the stimulus space. *Journal of Mathematical Psychology*, 1, 54—87.

Shepard, R.N. (1963). Analysis of proximities as a technique for the study of information processing in man. *Human Factors*, 1, 54—87.

Shepard, R.N. (1962a). The analysis of proximities: Multidimensional scaling with an unknown distance function. Part I. *Psychometrika*, 27, 125—140.

Shepard, R.N. (1962b). The analysis of proximities: Multidimensional scaling with an unknown distance function. Part II. *Psychometrika*, 27, 219—246.

Shreider, Y.A. (1974). *What is Distance?* Translated and adapted from the Russian by L. Cohn & H. Edelberg. Chicago: University of Chicago Press.

Sim, B. (1976). *Fundamentals of Topology*. New York: MacMillan Publishing.

Teitelbaum, R. (1965). Intervallic relations in atonal music. *Journal of Music Theory*, Fall, 24(2), 72—127.

Tenney, J. (1987). About "changes": 64 studies for 6 harps. *Perspectives of New Music*, 25(1,2), 64—87.

Tenney, J. (1984). *Bridge (for two pianos, eight hands)*. Score. Baltimore: Smith Publications.

Tenney, J. (1961). *Meta + Hodos*. M.A. Thesis, University of Illinois at Champaign-Urbana. Published by Frog Peak Music (1987).

Tenney, J. & Polansky, L. (1980). Temporal gestalt perception in music: A metric space model. *Journal of Music Theory*, 24, 205—241.

Torgerson, W.S. (1965). Multidimensional scaling of similarity. *Psychometrika*, 30, 379–393.

Tufte, E. (1990). *Envisioning Information*. Cheshire, CT: Graphics Press.

Tversky, A. (1977). Features of similarity. *Psychological Review*, 84, 327–352.

Uno, Y. (1991). Examining the roles of musical dimensions in Messiaen's *Mode de valeurs et d'intensités*: contour, entropy and hierarchical segmentation. Paper presented at the Society for Music Theory. Cincinatti.

Vaughan, K. (1992). Experimental ethnomusicology: A perceptual basis for Jairazbhoy's circle of thàt. *The World of Music*, 34(3), 99–119.

Wessel, D. (1979). Timbre space and musical structure. *Computer Music Journal*, 3(2), 45–52.

White, B. (1960). The recognition of distorted melodies. *American Journal of Psychology*, 73, 100–107.

Wonnacott, T. & Wonnacott, R. (1972). *Introductory Statistics*. New York: John Wiley and Sons.

Wyatt, W. & Bridges, C. (1974). *Statistics for the Behavioral Sciences* Boston: D.C. Heath and Company.

Photo by Jody Diamond

Larry Polansky
Music Department
Bregman Electronic Music Studio
Dartmouth College
Hanover, NH 03755
USA
Tel.: +1-603-646-2139
E-mail: larry.polansky@dartmouth.edu
http://music.dartmouth.edu

Larry Polansky is a composer, theorist, performer, programmer, editor and teacher living in Lebanon, New Hampshire, USA. He is an Associate Professor of Music at Dartmouth College, and is the co-founder and director of Frog Peak Music (A Composers' Collective).