Morphological Metrics: An Introduction to a Theory of Formal Distances

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ABSTRACT

An introduction to the classification and analysis of morphological metrics, or distance functions on "shapes". The importance of such functions is explained, followed by the types of questions that the investigation of these metrics should address. A simple classification of some basic metric types is given, along with notes about their perceptual and musical ramifications. Finally, the problems of metric and dimensional integration are raised, with preliminary suggestions for future work in this area.

1) Introduction

1.1) Definition of Problem

1.1.1) Scope of Paper

This paper formulates some of the primary questions involved in constructing a theory of morphological metrics, or "formal distances". In simplest terms, this involves the way that morphologies (see definitions) of all sorts (melodies, phrases, abstract shapes, larger forms) may be "judged" in terms of their distances from some other morphology, by either the listener, the theorist, the composer, or the analyst.

The measurement of morphological distance is inextricably related (and arguably, equivalent) to the fundamental issues in music theory of recognition, creation, and analysis of parametric variation and transformation. Musical parameters can include pitch, duration, harmonic relations, timbral values, or any other sonic parameter to which one could assign values, and which might serve as a formative parameter for musical perception.

This paper is not a complete classification, but a prolegomenon to a theory of morphological metrics. Its primary purpose is to describe the questions that need to be investigated, and to present a useful and non-stylistically biased language for the investigation.

Certain important topics are only briefly mentioned here, and might be dealt with at greater length elsewhere. These topics include: the problem of metrics on morphologies of differing length, metrics on non-discrete morphologies, and the significance of metrics on difference functions on morphologies (for example, metrics on the derivatives of continuous functions).

1.1.2) Holarchy in Musical Form

In *META + HODOS*, James Tenney (1986) made an important distinction, drawn from gestalt psychology, between *statistical* and *morphological* features of what he called *temporal gestalt units (TGU)* in music. In general, statistical features of a given *TGU* are the "global" properties, like the mean and range values for a parameter. These values are not in general dependent on the sequence in which each of the component events occurs in time. Morphological features of a *TGU*, on the other hand, are described by the "profile" of parametric values in that *TGU*. The *order* of events is clearly the distinguishing feature between a statistical measure of a *TGU* and a morphological one.

An important aspect of Tenney's theory is that statistical measures of parametric profiles become parametric values themselves at a "higher level," in what he referred to as *hierarchical temporal gestalt formation*. Recently, Tenney (Belet, 1987) has proposed the substitution of the word *holarchy*, since the formal theory proposed does not necessarily imply any precedence of given forms (Abraham, 1987).

The concept of a morphological metric assumes that parametric values can be somehow measured, and further assumes that morphologies may exist on any holarchical level. The discussion that follows is intended to apply to "lower" levels (like melodies and rhythmic sequences), and "higher" levels of musical form (like a morphology consisting of the mean pitch of several phrases).

1.1.3) Some Historical Notes

Much of traditional music theory has concerned itself with aspects of recognition and invariance of morphological units. Explicit reference to methodical morphological variation of higher holarchical forms is rather rare. At the melodic and rhythmic level, one could say that, aside from harmony, morphological transformation has been the single most important theoretical concern in western music. In other world musics, like classical central Javanese music, morphology is an even more important determinant of musical perception and creation than harmony, or at least an equal partner (harmony here, is not limited to "vertical" relations, but include all systems of pitch class usage, like Javanese pathet, western scales and modes, etc.).

An ancient and rather illustrative example is the elaborate system of the Classical Masoretic Hebrew tropes, used for the cantillation of the Tanach (Old Testament). These tropes are twenty-seven simple and archetypal melodic figures notated in the text as neumes (Binder, 1959; Rosowsky, 1957). The realization of these neumes in terms of harmony, melody, rhythm and articulation is highly diverse between any two singers. They are even more highly diversified between different cultures — so much so that one can usually identify a singer's geographical origin by the manner in which the tropes are sung. Even within the context of one singer's performance, or one culture, there are in fact proscribed multiple interpretations of the same sign — used for different texts. For example, in the Ashkenazic tradition, there are six different realizations, that are used to sing different books or at different times of the year.

The pertinence of this tradition to the problem at hand is the clear morphological invariance of these tropes which exists at a deeper level than their variation. Tropes sung by a singer from one tradition may easily be recognized by a singer from another tradition. All of the realizations tend to retain certain invariants of the tropes — "up and down, curvature, accentual pattern" — while freely transforming other parameters. It might be said, in the terminology given below, that certain important metrics (like a *directional* metric, which only measures "contour") taken on two cultures' realizations of the same trope would in general yield zero or near zero values. In fact, a study of morphological metrics in this cantillation tradition might reveal much about the basic factors of morphological invariance in human perception.

In western European art music, the compositional techniques of morpholgical transformation have been well documented. Every student learns of inversion (invariant under a *magnitude* metric), expansion/contraction and transposition (in general, invariant under *intervallic* metrics), and even the retrograde (in general, invariant under *unordered* metrics).

In the twentieth century, many composers and theorists have shifted their primary musical focus from harmony to morphology. Schoenberg, Hauer and other serialists discovered and defined their own transformations (inversion, transposition, retrograde, and retrograde inversion), which may be seen as an important experimental canon from which to proceed. Evidence from perceptual research and compositional development seems to suggest continued evolution of this hypothetical canon. For example, if one had to pick four transformations so as to preserve melodic recognition, even the earliest data from experimental psychology seems to suggest that retrograde would probably not make the list (White, 1960). Recognition of "distorted" melodic forms was, of course, not the only criteria Schoenberg used for selecting these transformations. It does seem, however, that Schoenberg was trying to initiate an era of motivic experimentation that might significantly evolve those four primitive functions, while still focusing on the perceptual, compositional, and formal goals of the motivic transformation.

> "Tonality and rhythm provide for *coherence* in *music*; variation delivers all that is grammatically necessary. I define variation as changing a number of a unit's features, while preserving others." (Schoenberg, "*Connection of Musical Ideas*," from *Style and Idea*)

The classical serial transformations provides a good "jumping off point" for a theory of morphological metrics. While Schoenberg's notion of inversion "preserves" in all cases the size of successive intervals in a given melody (that unit's "features"), there are more general cases of morphological metrics sensitive to interval size in any parameter, proportionally or absolutely, or, more relevantly, linearly (as in the 12-tone row) or combinatorially. Retrograde may be seen as a specific instance of permutation of a set of values. A metric intended to reflect invariance under retrograde (or permutation in general) may measure only absolute values in given parametric profiles, but not the differences, or intervals, between values (nor even transposition!).

1.2) Definitions of terms

Definition: A *morphology* is a function M on the set of integers greater than 0, called I, into into any other set E, such that M(i) = e, where e is in E, and i is in I.

Comment: A *morphology* is an n-dimensional set of values with an explicit order. That is, every n-dimensional point in a morphology M, has an associated value i, which is unique in that morphology. In general, the values of I are the integers 1 ... n. However, there is no restriction made as to whether or not a morphology is finite. For convenience in comparing morphologies, however, the range of I is usually considered to be from 1 (the "beginning" of the morphology) to n (the "end" of the morphology).

Definition: An *element* is one point e_i in the set E such that $M(i) = e_i$, for some morphology M.

Comment: An *element* is a list of values associated with one unique order value (i) for a morphology. That is, an element is one n-dimensional point in a morphology. We may denote an element as e_i.

Definition: The *length* (L) of a morphology M is the number of elements in that morphology.

Definition: A *morphological metric* is a distance function of two morphologies, D(M1,M2) = n, where:

1) n >= 0
2) D(M1,M2) = D(M2,M1)
3) D(M1,M2) = 0 iff M1 = M2
4) D(M1,M2) <= D(M1,M3) + D(M3,M2)

Comment: This is the standard definition of a metric where 1) signifies that all distances are positive, 2) that the distance function is transitive, 3) that distances of zero imply equality[1], and 4) that the "triangle inequality" is satisfied (in very simple terms, "the shortest distance between two points is a straight line").

Definition: An *Interval* is the difference, $\Delta(ei,ej)$, under some function Δ (not necessarily, but usually, a metric) between two elements in a morphology.

Comment: Most morphological metrics require calculations of distances within morphologies themselves. Morphological metrics then measure the distance of one set of ordered distances to another. For example, it is usually more useful to consider ordered sets of pitch intervals than the absolute pitches themselves in comparing two morphologies. The computation of *interval* is often quite simple, such as an absolute value difference, or a ratio. However, *interval* calculation can be more complex, as in, for example, multi-dimensional weighted differences of values in a morphology with many parameters (Tenney, Polansky; 1979). Interval calculations are, in general, higly specific to the parameter under consideration. They are, in a sense, independent of the morphological metric in use. For example, pitch, usually represented by some logarithmic notation (like 12-tone equal temperament) would usually be computed by arithmetic differences, frequencies and durations most naturally by ratiometric interval measures. Both of these types of interval calculations are interchangeable within most of the metrics described below.

This paper, in its attempt to discuss a more abstract formulation of morphological metrics, will use either simple interval calculations (like the absolute value of arithmetic distance between parametric values), or the generalized notation $\Delta(e_i,e_j)$. It is assumed that in the application of these ideas to actual compositional and analytical procedures, interval calculation methods will be customized for specific cases.

1.3) Perceptual Criteria

A metric on a set of morphologies will organize that set according to certain definite criteria, defining a metric space with morphologies as the points in that space. In this way, morphologies will be distinguishable along certain axes, and those axes will represent the distinguishing morphological features measured by the metric. For example, one axis might represent difference in absolute pitch values (the axis would then be a transpostion scale factor), and another might represent the difference in durational contour (coincidence of up or down in duration of morphological elements). Morphologies may then be plotted in that space from a fixed morphology at the origin, and distances in that space can then be computed in any number of ways (like a simple Euclidean metric).

Metrics may be heuristically derived. The distinguishing features might be simply related to known perceptual criteria for morphological recognition and classification. A metric which, unlike the above example, did not consider absolute values of pitches, but only pitch intervals, would create a morphological metric space in which all transpositions of a morphology were the same point.

Metrics may be experimentally derived. The distinguishing features measured might be intended for more complex experiments in reorganization and/or evolution of our perceptual criteria for morphological recognition and classification. For example, a metric function in which the function itself is random or stochastic in some way would create an unusual metric space where the location of a given point would not be fully determinate.

In general, however, metrics may be thought to be more or less "sensitive" to perceptual distance judgements. Criteria already established in psychoacoustics and perceptual research, like contour, common fate, intervallic similarity and proportionality can be closely approximated by simple metrics (some are defined below). Experimental psychological evidence (e.g. Dowling, 1978), suggests that the perceptual criteria used in recognition and judgement of morphologies are highly context dependent. It is likely that in even the simplest musical context, complex shifts in the perceiver's use of morphological metrics occur quite rapidly.

This paper describes some of the simplest metrics which seem to reflect perceptual criteria in common use. The specification of precise and simple metrics is an important first step towards studying the more complex mutability of the perception of morphological spaces. From a "generative," or compositional point of view, it is perhaps even more important to isolate and precisely define those metrics which are needed to produce desired perceptual domains.

2) Morphological Metrics

The following is a preliminary classification of a few simple metrics, specifically those that map some known basic perceptual attributes. The metric features discussed here are: *order*; *linearity and combinatoriality*; *directionality*; *and magnitude (absolute value and intervallic value).*

These classifications are not exclusive. A metric may be be ordered in one way and unordered in another, or magnitudinal in one way, and directional in another. In addition, degrees of order and non-order, linearity and combinatoriality can often be measured. This classification of metric types is intended as a useful taxonomy for describing primitives of morphological distance. It is preliminary in the sense that there are other descriptors that might be added, and many (in fact infinitely many) other possible metrics, and many other possible refinements and extensions of the metrics themselves.

2.1) Ordered and Unordered Metrics

Definition: An *Ordered* metric is one in which the sequence of values in the function is retained in the calculation of distance. *Unordered* metrics do not use this sequence of values.

Comment: In some sense, unordered metrics operate on non-morphological data, in that the value of the metric would not change whether or not the sets were morphological or not. If at some level of the calculation, the values of the morphologies can be scrambled with respect to i without affecting the metric, the metric is unordered at that level.

Metrics may be ordered and unordered at the same time, depending on the criteria under consideration. Often, sequence of values is retained at one stage of the calculation, but discarded at the next (in what might be called a statistical data reduction). For example, if we are considering a metric on two sets of intervals, at the lowest level we clearly need to know morphological sequence in order to even calculate the intervals, but we may or may not consider the sequence of intervals themselves in the distance calculation.

Example 1 illustrates how a metric may be unordered in some respects, and ordered in others. For two morphologies M1 and M2, E_{1i} and E_{2i} are the elements of M1 and M2 respectively. L is the length of M1 and M2, which, for the sake of simplicity in this example, is assumed to be the same:

$$D(M1,M2) = \frac{\sum_{i=2}^{L} |e_{1i} - e_{1i-1}|}{(L-1)} - \frac{\sum_{i=2}^{L} |e_{2i} - e_{2i-1}|}{(L-1)}$$

This metric, called an *Unordered linear magnitude (ULM)* metric, is the difference in the absolute value of the mean interval of two morphologies. It is ordered, in a simple sense, because the morphological sequence is preserved in the calculation of successive intervals. On a higher, and more important level, however, it is unordered, because once the intervals are computed, the sequence of their occurrence in the two morphologies is "thrown out" in the statistical average.

Example 2 is an ordered version of the above:

$$D(M1,M2) = \frac{\left| \sum_{i=2}^{L} |e_{1i} - e_{1i-1}| - |e_{2i} - e_{2i-1}| \right|}{(L-1)}$$

In this metric, called an *Ordered linear magnitude (OLM)* metric, the mean of the differences of corresponding intervals in the two morphologies is considered, retaining a high degree of morphological order.

Perceptually, ordered metrics tend to reflect a context in which the perciever is able to retain sequential information about morphologies, and unordered metrics, a perceptual context in which the perceiver in some way "reduces" or disregards sequential information. [2]

2.1.1) Linear and Combinatorial Metrics

Another type of order is the degree to which a given element in a morphology is "compared with" other elements in the same morphology in the calculation of a metric.

Definition: A metric is *linear* if in the calculation of distance, the number of intervals used within either of the morphologies in the calculation of a metric is less than or equal to L-1 (where L is the length of the metric). A metric is *combinatorial* if within one of the two morphologies, more than L-1 intervals are computed.

Comment: The degree of *order* of a metric is determined by the type of calculation done *between* the two morphologies. The degree of *linearity* or *combinatoriality* of a metric is determined by the type of calculation done *within* each morphology.

There is a wide range of combinatoriality possible for all metrics of two morphologies, roughly corresponding to how many cells of a matrix of all possible intervals are used in the calculation of distance. A measure of combinatoriality (C) of a metric might be given informally as $C = I/L_m$ where I is the number of intervals used in the calculation, and L_m is the number of possible intervals. For any given morphology M of length L, there are L-1 possible linear intervals, and $L_m = (L*(L+1)/2) - L$ possible (non-redundant) combinatorial intervals, constituting "half" the matrix L x L (minus the diagonal).[3] For example, for lengths 2-10 the number of intervals are 1, 3, 6, 10, 15, 21, 28, 36, 45. In general, L_m is the second binomial coefficient of L. Most linear metrics only consider intervals between the cells of the matrix (e_1,e_2), (e_2,e_3), (e_3,e_4) etc. As such, their measure of combinatoriality is $(L-1)/L_m$. By the above suggested measure of a metric's combinatoriality (C), the measure of combinatoriality of strictly linear metrics of morphologies of length 2-10 would be: 1, 1, .50, .40, .33, .29, .25, .22, .20. Obviously, the longer the morphology, the less accurately a linear metric may measure its internal structure.

The simplest combinatorial metrics are very similar to the linear metrics in the examples above, but instead of computing the means of successive linear intervals, or corresponding linear intervals, the means computed are from each interval to all other intervals. **Example 3** is an *Unordered combinatorial magnitude (UCM)* metric, based on Example 1:

$$D(M1,M2) = \left| \frac{\left(\sum_{j=1}^{L-1}\left(\sum_{i=1}^{L-j}\Delta(e_{1i}, e_{1i+j})\right)\right)}{L_m} - \frac{\left(\sum_{j=1}^{L-1}\left(\sum_{i=1}^{L-j}\Delta(e_{2i}, e_{2i+j})\right)\right)}{L_m} \right|$$

This example takes the difference of the means of differences between all non-redundant cells of each morphology's matrix. Note that in this example, the more general notation for interval distance function is used — which might represent absolute value of arithmetic differences, ratio, or any other such function. [4]

Example 4 is an *Ordered combinatorial magnitude (OCM)* metric. In this metric, the mean of the differences between corresponding cells of the two matrices is considered:

$$D(M1,M2) = \frac{\sum_{j=1}^{L-1}\left(\sum_{i=1}^{L-j}\left| \Delta(e_{1i}, e_{1i+j}) - \Delta(e_{2i}, e_{2i+j}) \right|\right)}{L_m}$$

The degree of order of a metric, and its linearity or combinatoriality, reflects the importance of sequence in the judgement of morphological distance. In some cases order is not desired, as in, for example, purely statistical measures. Similarly, there are situations in which a strictly linear metric can yield more useful results, as in most traditional comparisons of melodic variation. The more combinatorial the metric, however, the more finely it measures the differences in the internal structure of two morphologies.

2.2) Directional and Magnitudinal Metrics

Definition: A metric is *directional* if it considers the sign of an interval in a morphology, and *magnitudinal* if it considers the value of that interval.

Comment: Directional metrics reduce a morphology to one of three values — "down, same, up"— usually represented as -1, 0, and 1. Magnitudinal metrics measure actual parametric differences in morphologies. Directional metrics measure contour, or direction of change. Magnitudinal metrics measure the amount of change. These two types of metrics may easily be combined.

A simple and useful function for directional interval calculation is $\Delta(e_i,e_j) = sgn(e_i,e_j)$ where:

$$sgn(e_i,e_j) = 1,\ e_i > e_j;\ \ 0,\ e_i = e_j;\ \ -1,\ e_i < e_j$$

Note that this interval distance function is not strictly a metric.

Example 5 is an *Ordered linear directional (OLD)* metric:

$$D(M1,M2) = \frac{\sum_{i=2}^{L} diff(sgn(e_{1i},e_{1i-1}), sgn(e_{2i},e_{2i-1}))}{(L-1)}$$

where $diff(x,y) = 1, x \neq y ; 0, x = y$ -and-

$sgn(e_i,e_j) = 1, e_i > e_j ;\ 0, e_i = e_j ; -1, e_i < e_j$; or $sgn(e_i,e_j) = ((e_i - e_j) / | e_i - e_j |) / 1$

In this example, only the sign of the interval is considered (directional), only intervallic direction between successive elements is used (linear), and corresponding intervals are compared between the two morphologies (ordered). This metric, though surprisingly simple, will quite accurately measure a perceptual notion of contour of two morphologies along a given parameter. Examples 1-4 are all magnitudinal.

2.2.1) Absolute and Intervallic Magnitudes

Magnitudinal metrics can further be categorized by the values used in the calculation: *intervallic* or *absolute* . In general, absolute metrics do not make interval calculations within a morphology. **Example 6**, the simplest metric so far, is an *Ordered linear absolute magnitude (OLAM)* metric:

$$D(M1, M2) = \frac{\sum_{i=2}^{L} | e_{1i} - e_{2i} |}{L}$$

This is the only metric thus far described which will not recognize transposition as an invariant.

2.3) List of Morphological Metrics

The following is a list of 10 primitive metrics based on the above taxonomy, with simple equations for each.[5] All equations assume equal length morphologies (thus L = L1 = L2). For notational convenience, the word *intervallic* is assumed (thus OLM = OLIM).

• Ordered linear direction (OLD)

$$\frac{\sum_{i=2}^{L} \mathrm{diff}\,(\mathrm{sgn}\,(\Delta(e_{1i}, e_{1i-1})), \mathrm{sgn}\,(\Delta(e_{2i}, e_{2i-1})))}{(L-1)}$$

• Unordered Linear Direction (ULD)

$$\frac{\sum_{v=-1}^{1} |\#e_1{}^{v} - \#e_2{}^{v}|}{L-1}$$

where $\#e_n{}^{v}$ = number of intervals in Mn s.t. $\mathrm{sgn}(e_i, e_j) = v$; $v = \{-1, 0, 1\}$

• Unordered Combinatorial Direction (UCD)

$$\frac{\sum_{v=-1}^{1} | \#e_1{}^{v} - \#e_2{}^{v} |}{L_m}$$

where $\#e_n{}^{v}$ = # of intervals in Mn s.t. $\mathrm{sgn}(e_{ni}, e_{ni+j}) = v$; where $v = \{-1, 0, 1\}$; $j = \{1, ..., L-1\}$; $i = \{1, ..., L-j\}$

• Ordered Combinatorial Direction (OCD)

$$\frac{\sum_{j=1}^{L-1} \left(\sum_{i=1}^{L-j} \mathrm{diff}\,(\mathrm{sgn}\,(\Delta(e_{1i}, e_{1i+j})), \mathrm{sgn}\,(\Delta(e_{2i}, e_{2i+j})))\right)}{L_m}$$

• Ordered Linear Magnitude (OLM)

$$\frac{\sum_{i=2}^{L} | \Delta(e_{1i}, e_{1i-1}) - \Delta(e_{2i}, e_{2i-1}) |}{(L-1)}$$

• Unordered Linear Magnitude (ULM)

$$\left| \frac{\sum_{i=2}^{L} \Delta(e_{1i}, e_{1i-1})}{(L-1)} - \frac{\sum_{i=2}^{L} \Delta(e_{2i}, e_{2i-1})}{(L-1)} \right|$$

• Ordered Linear Absolute Magnitude (OLAM)

$$\frac{\sum_{i=1}^{L} \Delta(e_{1i}, e_{2i})}{L}$$

• Ordered Combinatorial Magnitude (OCM) [6]

$$\frac{\sum_{j=1}^{L-1} \sum_{i=1}^{L-j} | \Delta(e_{1i}, e_{1i+j}) - \Delta(e_{2i}, e_{2i+j}) |}{L_m}$$

• Unordered Combinatorial Magnitude (UCM) [6]

$$\left| \frac{\sum_{j=1}^{L-1} \left(\sum_{i=1}^{L-j} \Delta(e_{1i}, e_{1i+j}) \right)}{L_m} - \frac{\sum_{j=1}^{L-1} \left(\sum_{i=1}^{L-j} \Delta(e_{2i}, e_{2i+j}) \right)}{L_m} \right|$$

• Unordered Linear Absolute Magnitude (ULAM)[7]

$$\left| \frac{\sum_{i=1}^{L} e_{1i}}{L} - \frac{\sum_{i=1}^{L} e_{2i}}{L} \right|$$

3) Additional Topics: Metric combinations and morphological length

3.1) Combination across dimensions

A simple method for metrics on multi-dimensional morphologies is for the *interval* calculation itself to be multi-dimensional. In several of the above examples, a Euclidean metric (in one dimension) has been used, and this can easily be extended to n-dimensions, with the proper weighting and scaling of parametric values (Tenney, Polansky; 1979). The metric itself may also be multi-dimensional, but this will usually result in a certain loss of generality of the metric.

3.2) Combination across metrics

Perceptual spaces may be represented as one quadrant (positive values for all axes) of a multi-dimensional real-number space (R^n), where the origin is some "source" morphology, and the axes represent distance values of other morphologies to that source under given metrics. The distance of a morphology from the source is some n-dimensional metric in the real numbers, and distances from any morphology to another may be computed as well.[8] Note that the space is not necessarily Euclidean, since the metric used for distance measurements in that space itself may not be the Euclidean metric (for example, the coarser "city-block" might be used). Multi-dimensional metric spaces provide a useful, and "finely tunable" tool for precisely measuring morphological distance, and may be made highly sensitive to almost any set of perceptual criteria.

3.3) Metrics and Morphological Length

One of the most important problems in the description of morphological metrics is the measurement of distance between morphologies of varying lengths. Since this question is somewhat independent of the above discussion, it will only be briefly noted here.

One of the simplest and most effective techniques is suggested by the information theoretic work of Gregory Chaitin (1979), and independently referred to by Tenney, Rosenboom (personal correspondences), and others as the measure of the "intersection over the union." In this technique, the metric calculation uses the maximum of the two morphological lengths, and intervals or values in the shorter morphology are defined to be zero for $i > L_{min}$ (where L_{min} is the shorter length, L_{max} the longer). Perceptually, this technique assumes that values in a longer morphology are "completely different" than values which do not exist in a shorter. This technique is simple, but effective, and at least to this author, most often gives results that are in reasonable accord with expectation. This metric is called *non-normalized*, because the difference of the two lengths is preserved in the metric. **Example 8** shows a simple use of this technique:

$$D(M1,M2) = \frac{\sum_{i=2}^{L_{max}} | \Delta(e_{1i}, e_{1i-1}) - \Delta(e_{2i}, e_{2i-1}) |}{L_{max}}$$

A second class of techniques are *normalized* metrics, which first alter the morphologies themselves so that the two lengths are equal. To normalize two morphologies of varying lengths, either the shorter must be made longer, or the longer made shorter. Even in the restricted case here of discrete morphologies, there are many such normalization methods (for example, "sampling" the longer by interpolating a new set of points using the ratio of the two lengths as a "sampling width"). These normalizations will necessarily be rather complex, and will have distinct perceptual ramifications. Normalization technique is an important area for further research.

4) Conclusions

This paper, almost deliberately, avoids certain restrictive formalizations of morphological metrics. The subject is hybrid in nature, combining perception, musical form, and mathematics, and a complete formalization of this topic needs to be the result of considerable work in each of the related disciplines. In addition, any such formalism needs to clarify whether the principles stated are "formative" or "descriptive." That is, to what extent does the theory describe or analyze known phenomena, and to what extent does it generate a field of new, and possibly unknown, phenomena.

The intention here is to do a little of both — begin with certain primitives of perception and composition and then attempt to formulate a language and set of techniques that will allow for future experiment. It is likely that many of the ideas stated here will find more elegant statements at a later point, since generalizations and abstractions tend to become more obvious when a theory is brought more and more to application.

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6) Notes

1) Equality, in this case, means invariance. Morphologies may be equal under one metric, and very different under another. In general, more highly ordered metrics will produce finer distinction between morphologies.

2) In general, unordered metrics will produce "smaller" distance measures than ordered metrics. One illustrative, if imprecise, way to state the distinction between these types of metrics is to say that "the difference of means" is less than the "difference of the means of differences."
3) In most cases, this non-redundancy is a result of the interval calculation being a metric, since d(a,b) = d(b,a). In some cases, such as directional metrics, where the interval calculation is not transitive, "half the matrix and the diagonal" is still "redundant," since the diagonal is always 0, and the one diagonal half of the matrix will be a simple sign inversion of the other.
4) The difference function between means in the metric itself may also be generalized, and a more general, albeit abstract, notation for morphological metrics can be developed. I have not taken this further step here in order to give the reader more concrete examples. Such a general notation needs to preserve the disintinction between magnitudinal and directional metrics.
5) Several obvious but "non-primitive" metrics are not mentioned. One example relevant to the OLD metric is the *Ordered linear interval direction (OLID)* metric (or the corresponding OCID metric), which would measure the sign of "intervals between intervals" in a morphology. This metric would then measure the contour of the first order difference function on a morphology, or the contour of the "rate of change".
6) Note that, by definition, UCAM and OCAM are not very useful or meaningful measures.
7) This metric is more truly a statistical measure, but is "generated" by this taxonomy of morphological metrics.
8) This latter statement must be interpreted with care, since all points in that space are specifically referenced to the source morphology, and distances between other points may not always be "expected" values!