

sorting report

January 22, 2026

0.0.1 Assignment 1 b

```
[1]: import numpy as np
from matplotlib import pyplot as plt
import time
```

```
[6]: # Insertion Sort
def insertion_sort(arr, metadata=None):
    metadata = metadata or {}
    metadata['insertion_comparisons'] = metadata.get('insertion_comparisons', 0)
    metadata['insertion_swaps'] = metadata.get('insertion_swaps', 0)
    start = time.perf_counter_ns()
    for i in range(1, len(arr)):
        key = arr[i]
        j = i - 1
        while j >= 0:
            metadata['insertion_comparisons'] += 1
            if arr[j] <= key:
                break
            arr[j + 1] = arr[j]
            metadata['insertion_swaps'] += 1
            j -= 1
        arr[j + 1] = key
    end = time.perf_counter_ns()
    metadata["insertion_us"] = (end - start) / 1_000
    return arr, metadata

# Merge Sort
def merge_sort(arr, metadata=None):
    metadata = metadata or {}
    metadata['merge_comparisons'] = metadata.get('merge_comparisons', 0)
    metadata['merge_swaps'] = metadata.get('merge_swaps', 0)
    start = time.perf_counter_ns()
    if len(arr) > 1:
        mid = len(arr) // 2
        L = arr[:mid]
        R = arr[mid:]
```

```

merge_sort(L, metadata)
merge_sort(R, metadata)

i = j = k = 0

while i < len(L) and j < len(R):
    metadata['merge_comparisons'] += 1
    if L[i] < R[j]:
        arr[k] = L[i]
        metadata['merge_swaps'] += 1
        i += 1
    else:
        arr[k] = R[j]
        metadata['merge_swaps'] += 1
        j += 1
    k += 1

while i < len(L):
    arr[k] = L[i]
    metadata['merge_swaps'] += 1
    i += 1
    k += 1

while j < len(R):
    arr[k] = R[j]
    metadata['merge_swaps'] += 1
    j += 1
    k += 1

end = time.perf_counter_ns()
metadata["merge_us"] = (end - start) / 1_000
return arr, metadata

# Heap Sort
def heapify(arr, n, i, metadata=None):
    metadata = metadata or {}
    metadata['heap_comparisons'] = metadata.get('heap_comparisons', 0)
    metadata['heap_swaps'] = metadata.get('heap_swaps', 0)
    largest = i
    l = 2 * i + 1
    r = 2 * i + 2

    metadata['heap_comparisons'] += 1
    if l < n and arr[l] > arr[largest]:
        largest = l

    metadata['heap_comparisons'] += 1

```

```

    if r < n and arr[r] > arr[largest]:
        largest = r

    if largest != i:
        arr[i], arr[largest] = arr[largest], arr[i]
        metadata['heap_swaps'] += 1
        heapify(arr, n, largest, metadata)

def heap_sort(arr, metadata=None):
    metadata = metadata or {}
    metadata['heap_comparisons'] = metadata.get('heap_comparisons', 0)
    metadata['heap_swaps'] = metadata.get('heap_swaps', 0)
    start = time.perf_counter_ns()
    n = len(arr)

    for i in range(n // 2 - 1, -1, -1):
        heapify(arr, n, i, metadata)

    for i in range(n - 1, 0, -1):
        arr[i], arr[0] = arr[0], arr[i]
        metadata['heap_swaps'] += 1
        heapify(arr, i, 0, metadata)

    end = time.perf_counter_ns()
    metadata["heap_us"] = (end - start) / 1_000
    return arr, metadata

# Quick Sort
def quick_sort(arr, metadata=None):
    IS_FIRST_CALL = metadata is None
    metadata = metadata or {}
    metadata['quick_comparisons'] = metadata.get('quick_comparisons', 0)
    metadata['quick_swaps'] = metadata.get('quick_swaps', 0)
    start = time.perf_counter_ns()
    if len(arr) <= 1:
        return arr, None
    else:
        # Pivot at first element for worst case scenario
        pivot = arr[0] #arr[len(arr) // 2]
        idxFromLeft = None
        idxFromRight = None
        while idxFromLeft is None or idxFromLeft < idxFromRight:
            idxFromLeft = 1
            while idxFromLeft < len(arr):
                metadata['quick_comparisons'] += 1
                if arr[idxFromLeft] >= pivot:
                    break

```

```

        idxFromLeft += 1
        idxFromRight = len(arr) - 1
        while idxFromRight > 0:
            metadata['quick_comparisons'] += 1
            if arr[idxFromRight] <= pivot:
                break
            idxFromRight -= 1
        if idxFromLeft < idxFromRight:
            arr[idxFromLeft], arr[idxFromRight] = arr[idxFromRight], arr[idxFromLeft]
            metadata['quick_swaps'] += 1
            arr[0], arr[idxFromRight] = arr[idxFromRight], arr[0]
            metadata['quick_swaps'] += 1
            left, _ = quick_sort(arr[:idxFromRight], metadata)
            right, _ = quick_sort(arr[idxFromRight + 1:], metadata)
            if IS_FIRST_CALL:
                end = time.perf_counter_ns()
                metadata["quick_us"] = (end - start) / 1_000

        return left + [arr[idxFromRight]] + right, metadata

# Run the sorting algorithms and collect data
N = list(range(2,200))

results_us = {
    'insertion': [],
    'merge': [],
    'heap': [],
    'quick': []
}

results_comp = {
    'insertion': [],
    'merge': [],
    'heap': [],
    'quick': []
}

for n in N:
    data = list(range(n))
    np.random.shuffle(data)
    data.sort(reverse=True) # worst case

    for sort in [insertion_sort, merge_sort, heap_sort, quick_sort]:
        sorted_arr, metadata = sort(data.copy())

        key = list(metadata.keys())[0].split('_')[0]

```

```

        results_us[key].append(metadata[f'{key}_us'])
        results_comp[key].append(metadata[f'{key}_comparisons'])

fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(16, 7), sharex=True)

# ---- Left: Runtime (s) ----
for key in results_us:
    ax1.plot(N, results_us[key], label=key.capitalize())

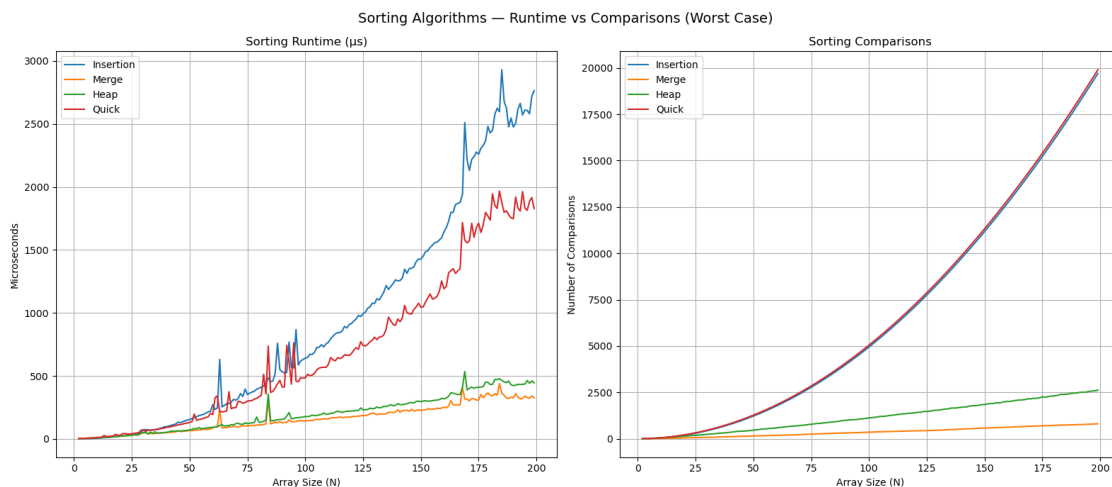
ax1.set_title("Sorting Runtime (s)")
ax1.set_xlabel("Array Size (N)")
ax1.set_ylabel("Microseconds")
ax1.legend()
ax1.grid(True)

# ---- Right: Comparisons ----
for key in results_comp:
    ax2.plot(N, results_comp[key], label=key.capitalize())

ax2.set_title("Sorting Comparisons")
ax2.set_xlabel("Array Size (N)")
ax2.set_ylabel("Number of Comparisons")
ax2.legend()
ax2.grid(True)

plt.suptitle("Sorting Algorithms - Runtime vs Comparisons (Worst Case)",
             ↪fontsize=14)
plt.tight_layout()
plt.show()

```



We see that Insertion sort is slow, Merge sort is fast. The execution time of Insertion sort approaches 3000 microseconds at $n = 200$.

```
[10]: def estimate_complexity(N:list, ops:list):
    common_complexities = {
        'T(n)': lambda n: n,
        'T(n log n)': lambda n: n * np.log2(n),
        'T(n^2)': lambda n: n**2,
        'T(log n)': lambda n: np.log2(n),
    }
    estimates = {}
    for name, func in common_complexities.items():
        estimated_ops = [func(n) for n in N]
        scaling_factor = np.mean([a / b for a, b in zip(ops, estimated_ops)])
        estimated_ops = [scaling_factor * e for e in estimated_ops]
        error = np.mean([(a - b) ** 2 for a, b in zip(ops, estimated_ops)])
        estimates[name] = error
    estimates = {k: round(np.log(v), 1) for k, v in estimates.items()}
    best_fit = min(estimates, key=estimates.get)
    return best_fit, estimates

for key, values in results_comp.items():
    best_fit, estimates = estimate_complexity(N, values)
    print(f'{key.capitalize()} Sort is {best_fit}')
```

Insertion Sort is $T(n^2)$

Merge Sort is $T(n \log n)$

Heap Sort is $T(n \log n)$

Quick Sort is $T(n^2)$

```
[17]: import pandas as pd
print("# DotNet results:\n")
df = pd.read_csv('DotnetComparison/sorting_microseconds.csv')

plt.figure(figsize=(12, 8))
for key in df.columns:
    if key == 'N':
        continue
    # Skipping the first row since the initialization time messes up the results
    N = df.loc[1:, 'N']
    values = df.loc[1:, key]

    best_fit, estimates = estimate_complexity(N, values)
    print(f'{key.capitalize()} Sort is {best_fit}\n- errors: {estimates}\n')
```

```
plt.plot(N, values, label=f'{key.capitalize()} Sort Comparisons')
plt.legend()
plt.xlabel('Array Size (N)')
plt.ylabel('Number of Comparisons')
plt.title('Sorting Algorithm Comparisons')
```

DotNet results:

Insertion Sort is $T(n^2)$

- errors: {'T(n)': np.float64(5.8), 'T(n log n)': np.float64(5.6), 'T(n²)': np.float64(3.8), 'T(log n)': np.float64(6.4)}

Merge Sort is $T(n \log n)$

- errors: {'T(n)': np.float64(3.2), 'T(n log n)': np.float64(2.5), 'T(n²)': np.float64(6.5), 'T(log n)': np.float64(4.8)}

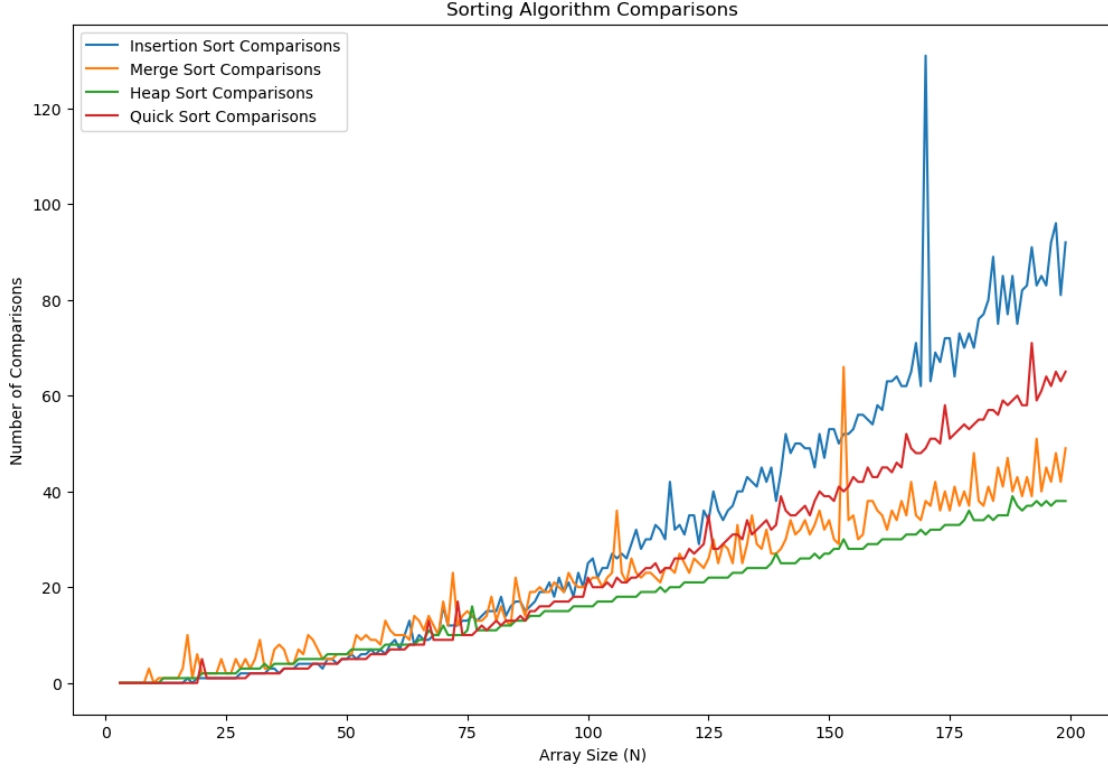
Heap Sort is $T(n \log n)$

- errors: {'T(n)': np.float64(2.9), 'T(n log n)': np.float64(1.6), 'T(n²)': np.float64(5.0), 'T(log n)': np.float64(4.4)}

Quick Sort is $T(n^2)$

- errors: {'T(n)': np.float64(4.9), 'T(n log n)': np.float64(4.6), 'T(n²)': np.float64(1.3), 'T(log n)': np.float64(5.7)}

[17]: Text(0.5, 1.0, 'Sorting Algorithm Comparisons')



0.0.2 Task 3

0.1 Small-o proof: $o(g(n))$

0.2 Intuition: strictly upper bound

We have:

$$f(n) = o(g(n)) \iff f(n) = o(g(n)),$$

$$f(n) = o(g(n)) \iff \forall c > 0, \exists n_0 \geq 1 \text{ such that } f(n) < c \cdot g(n) \text{ for all } n \geq n_0,$$

$$\iff \lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \infty,$$

$$\iff \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0.$$

We want to show that $\frac{n^2}{\log n} = o(n^2)$ as $n \rightarrow \infty$.

By definition of little-o, $f(n) = o(g(n))$ means $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$.

Here, $f(n) = \frac{n^2}{\log n}$ and $g(n) = n^2$, so:

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{\frac{n^2}{\log n}}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{\log n} = 0.$$

This proves that $\frac{n^2}{\log n} = o(n^2)$.

0.2.1 Task 3b

We want to check if $n^2 = o(n^2)$.

Here, let $f(n) = n^2$ and $g(n) = n^2$. Then:

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2} = \lim_{n \rightarrow \infty} 1 = 1 \neq 0$$

Since the limit is **not 0**, we conclude that:

$$n^2 \neq o(n^2)$$

0.2.2 Task 4a Masters theorem

$$*R1 \ T(n) = 16T\left(\frac{n}{4}\right) + n$$

constants: $a = 16$, $b = 4$

$$\text{Case 1: } F(n) = n^{\log_4 16 - c} = n^{2-c}$$

$$f(n) = n^2$$

$$T(n) = \Theta(n^2)$$

0.2.3 Task 4b

$$T(n) = 4T\left(\frac{n}{2}\right) + n \text{ recursive relation to be solved with substitution method}$$

Inductive hypothesis to test:

1. $T(n) \leq cn^2$, where $c \geq 0$
2. $T(n) \geq cn^2$, where $c \geq 0$
3. $T(n) \leq (cn^2 - bn)$, where $c \geq 0$ and $b \geq 0$.

0.2.4 Test for hypothesis 1

With substitution method we go from $T(n)$ to $T(\frac{n}{2})$ which is the recursive term in the equation.

1. $T(\frac{n}{2}) \leq c(\frac{n}{2})^2 = \frac{cn^2}{4}$
2. $T(n) \leq 4 \cdot \frac{cn^2}{4} + n$
3. $T(n) = cn^2 + n$

0.2.5 Hypothesis check

4. $cn^2 + n \leq cn^2$ This hypothesis does not stand

0.2.6 Test for hypothesis 2

Based on the answer from hypothesis 1. We can confirm hypothesis 2 to be true, because:

$$cn^2 + n \geq cn^2$$

0.2.7 Test for hypothesis 3

1. $T(\frac{n}{2}) = c \cdot (\frac{n}{2})^2 - b \cdot (\frac{n}{2})$
2. $T(n) = 4 \cdot T(\frac{n}{2}) + n$
3. $T(n) \leq 4 \cdot (\frac{cn^2}{4} - \frac{bn}{2}) + n$
4. $T(n) \leq cn^2 - 2bn + n$
5. $cn^2 - 2bn + n \leq cn^2 - bn$

0.2.8 Inequality term subtracting cn^2 on both sides.

7. $-2bn + n \leq -bn$
8. $n \leq -bn + 2bn$
9. $n \leq bn$
10. $1 \leq b$

This hypothesis works if $b \geq 1$