Smallest Enclosed Balls (SEB) - Algorithms

Jônatas O. L. da Silva

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1. Introduction

The Smallest Enclosed Balls (SEB) problem consistis in covering a finite set of points in the plane with the smallest possible circle. The problem was originally proposed in [Sylvester 1857].

In this document was be presented two approaches to solve the problem. The first one is a recursive approach proposed by [Welzl 1991] and the second is a reformulation of the original problem as a quadratic programming problem. Here will not be discussed about the mathematical aspects of the second approach since it can be found in [Elzinga and Hearn 1972]. Instead it I will explain the details about both algorithm and some important aspects find out during the written of the code and during the tests.

Some results to both both algorithms are also presented.

2. Welzl Agorithm

The first algorithm is based in [Welzl 1991] and uses a recursive approach to solve the SEB problem. The algorithm selects one point p randomly and uniformly from P (the set of points), and recursively finds the minimal circle containing P – p, i.e. all of the other points in P except p. If the returned circle also encloses p, it is the minimal circle for the whole of P and is returned.

Otherwise, point p must lie on the boundary of the result circle. It recurses, but with the set R of points known to be on the boundary as an additional parameter.

The recursion terminates when P is empty, and a solution can be found from the points in R: for 0 or 1 points the solution is trivial, for 2 points the minimal circle has its center at the midpoint between the two points, and for 3 points the circle is the circumcircle of the triangle described by the points.

3. Hearn Agorithm

The second approach is based in [Elzinga and Hearn 1972] and reformulate the original problem as a squared euclidean distance and then maximize the objective function to find the points on the boundary of a circle. The main aspect of the algorithm consists in the point that will be used in the optimization problem. The author suggests using only four points. So, we decompose the main problem in small subproblems. After find the optimal values of each subproblem, a varification is made to verify if all points is inside of the circle. If it, the algorithm stops we the solution, otherwise, one point is removed of the subproblem, and another point is inserted to generate a new subproblem. The algoritm iterates until find the best solution.

4. Difficulties of both algorithms

With the Welzl Agorithm, the difficulty come from big problems, which in some cases it could not be solved. It occurs because the limit of recursive operations is reached. This limit can vary with the computer. In my personal laptop (Macbook IOs 8gb RAM, processor 1.8 Ghz Dual-core) the limit was about 500 points.

With the Hearn Agorithm was find difficulties to solve problems with negative values in the points. To solve it a manipulation of the points was made. All of negative points was moved to positive side of the Cartesian's plane, thus, was the points needed also be moved. The algorithm solves the problem with the new points and then the optimal center point of this new problem is manipulated again to reflect the original problem. An geometry example of this manipulation can be seen in the Figures 4, 4.

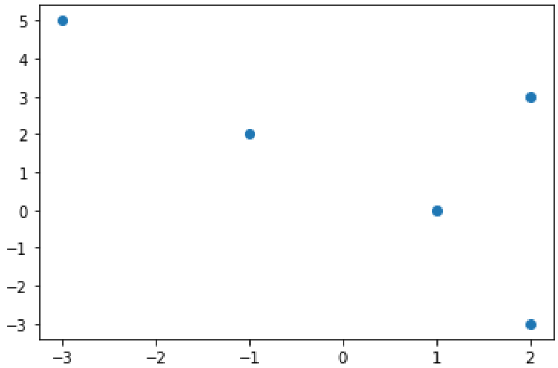

Figure 1. Original data points

Firstly, in Figure 4 we have the original points (-1,2), (2,3), (-3,-5), (1,0) and (2,3). After transform the data points, we have a new set of points as in Figure 4.

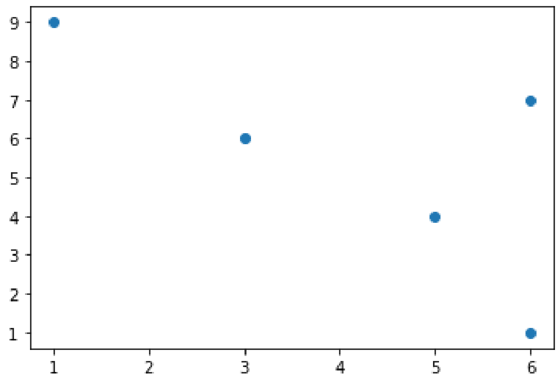

Figure 2. Manipulated data points

The formula to obtain the new manipulated data points is:

$$\begin{array}{c} x_i + \alpha \quad \forall i = 1, 2, ..., n \\ y_i + \beta \quad \forall i = 1, 2, ..., n \end{array} \tag{1}$$

.where x and y are the first and second index of all data points tuple. α é the most negative value in vector x, and β is the most negative value in vector y. After achieve the result in

Table 1. Results to n=250 and n=500

n	Welzl Algorithm	Hearn Algorithm
500	0.745819	0.730309

Table 2. Results to Hearn Algorithm in a greater scale

n	Hearn Algorithm
1000	0.733296
5000	1.239857
10000	2.005075

the manipulated data, we can easy subtract the center manipulated point with the vector (α, β) to find the original center point. The radius continues the same.

Figure 4 shows the circle to the original data points.

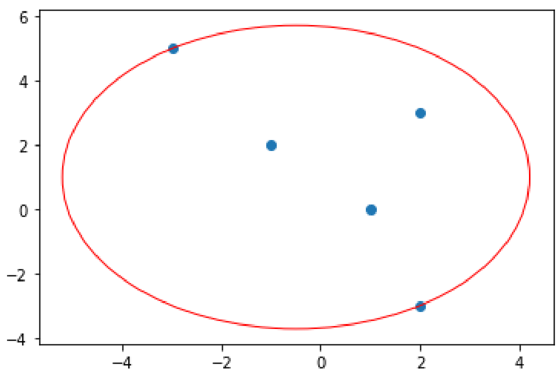

Figure 3. Circle to original data points

5. Experiments

In experiments was used a Laptop IOs 8gb Ram, dual-core i5 1.8 Ghz.

The maximum size to Welzl Algorithm was defined as 500 data points. More the it the algorithm crashes because the maximum limit of recursive operations was achieved.

The Table 5 shows the time result in seconds to n=500 for both algorithms. The results presents a similar computation time for both algorithms.

In the Table 5 the result to the Hearn Algorithm is presented to n=1000, n=5000 and n=10000.

6. Conclusion

The goal in this first project was understood the problem and implement the first version of the algorithm. Two algorithms was implemented based in two popular algorithms dedicated to SEB problem. Some aspects needed to be improved and the a list with the next steps can me made as following:

- Implements a heuristic in Welzl Algorithm to reduce the number of recursion calls and be able to solve larger problems;

- Improve the data structure in Simplex Method to solve the subproblems in Hearn Algorithm;
- Implements a heuristic to improve the choice of the point the leave and entry in the new subproblem at the Hearn Algorithm;
- Find out new ways to solve the problem based in [Xu et al. 2003] which contemplate an excellent list of approaches dedicated to solve the SEB problem.

References

Elzinga, D. J. and Hearn, D. W. (1972). The minimum covering sphere problem. *Management science*, 19(1):96–104.

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Xu, S., Freund, R. M., and Sun, J. (2003). Solution methodologies for the smallest enclosing circle problem. *Computational Optimization and Applications*, 25(1):283–292.