Predicate Derivation Rules

The structure of a predicate argument is identical a propositional argument except that the wffs are predicate wffs. Moreover, predicate arguments can only be proven by using derivation rules. But recall that predicates are prepositional statements whenever its variables are constants or, ultimately, when they are not quantified. Therefore, the derivation rules we have already encountered will still be implemented to prove an argument is valid. Hence, predicate logic only provides a few additional derivation rules that deals with quantifiers. The first rule is a predicate equivalence rule. It is called ***negation***. It is

$$\left[\big(\forall x\big)P(x)\right]' \leftrightarrow \big(\exists x\big)[P(x)]'$$
$$\left[\big(\exists x\big)P(x)\right]' \leftrightarrow \big(\forall x\big)[P(x)]'$$

This is evident from the definition of universal and existential quantifiers. For example,

$$\begin{aligned} \left[\big(\forall x\big)P(x)\right]' &= \left[\bigwedge_{x_i \in D_x} P(x_i)\right]' \\ &= \bigvee_{x_i \in D_x} [P(x_i)]' \quad \text{by De Morgan's Law} \\ &= \big(\exists x\big)[P(x)]' \quad \text{by Existential Quantifier Definition} \end{aligned}$$

A similar process can be used to prove negation of the existential quantifier.

The remaining derivation rules of predicates we will discuss here are inference rules; Unlike propositional inference rules, these rules come with restrictions. The first two rules are used to eliminate quantifiers called ***universal instantiation*** and ***existential instantiation***. The universal instantiation states

$$\big(\forall x\big)P(x) \to P(c)$$

where c can be a variable or constant, but it must not already be an existing variable of the predicate or a quantified variable in its scope. That is, if the predicate has more than one (1) variable, then none of the other variables can be c as well. For instance, let $P(x, y) \Rightarrow x < y$, then if you use the universal instantiation rule on the wff $\big(\forall x\big)P(x, y)$ and transform x to y to get $P(y, y)$, which is now invalid.

The next inference rule, existential instantiation, states

$$\big(\exists x\big)P(x) \to P(c)$$

where c is a new constant in the wff. In other words, no variable or constant of any predicate in the entire wff can already be c. For example, let $P(x) = (x \text{ is even})$ and $Q(x) = (x \text{ is odd})$, then if you used the existential instantiation rule twice on the wff $\big(\exists x\big)P(c) \wedge \big(\exists x\big)Q(x)$ and transform both xs to cs to get $P(c) \wedge Q(c)$, the wff, although originally true, becomes false. When eliminating quantifiers, you should implement this rule first.

Once the quanifiers are eliminated, your predicate argument becomes a propositional argument (implies you can use the old rules); however, the consequent of a predicate argument may include quantifiers which means you will need the two (2) remaining rules called ***universal generalization*** and ***existential generalization***. These rules add quanifiers. The first of these rules, universal generalization, states

$$P(c) \to \big(\forall x\big)P(x)$$

where c is not a free variable or derived from an existential instantiation, and x does not already exist in the predicate. To elaborate consider saying "1 is odd", and then generalizing that "all integers are odd". This would be irrational; hence, going from $P(x)$ (where x is a free variable) or $\big(\exists x\big)P(x)$ to $\big(\forall x\big)P(x)$ would be invalid. The last rule, existential generalization, states

$$P(c) \rightarrow \big(\exists x\big)P(x)$$

where x is a variable that does not already exist as a variable of the predicate. The reasoning for this restriction is the same as universal instantiation's reason.

The process for proving a predicate argument is divided into three steps which are

1. **Remove quantifiers using instantiation rules.**
2. **Use propositional derivation rules to structure the antecedent into the consequent.**
3. **Add quantifiers using generalization rules if neccessary.**

To clarify, we will prove the below predicate argument is a tautology.

$$\big(\forall x\big)\big(A(x) \rightarrow B(x)'\big) \wedge \big(\exists x\big)\big(C(x) \wedge A(x)\big) \rightarrow \big(\exists x\big)\big(C(x) \wedge B(x)'\big)$$

Proof. The proof sequence is as follows;

Step	**Reason**
1 $\big(\forall x\big)\big(A(x) \rightarrow B(x)'\big)$	Hypothesis
2 $\big(\exists x\big)\big(C(x) \wedge A(x)\big)$	Hypothesis
3 $\big(C(a) \wedge A(a)\big)$	Existential Instantiation Rule on 2
4 $\big(A(a) \rightarrow B(a)'\big)$	Universal Instantiation Rule on 1
5 $B(a)'$	Modus Ponens on $3, 4$
6 $C(a)$	Simplification Rule on 3
7 $\big(C(a) \wedge B(a)'\big)$	Conjunction Rule on $5, 6$
8 $\big(\exists x\big)\big(C(x) \wedge B(x)'\big)$	Existential Generalization Rule on 7

$\square$

The last thing to note is that if instances of a predicate have different values, they represent different propositional statements; i.e, $P(a) \neq P(b)$.