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Assignment 4

Calculus and Linear Algebra II

1. Use the method of Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraints:

$(a)\ f(x,y) = x^2 + y^2,\ xy = 4$

$$\begin{Bmatrix} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ xy = 4 \end{Bmatrix} \to \begin{Bmatrix} 2x = \lambda y \\ 2y = \lambda x \\ xy = 4 \end{Bmatrix} \to \begin{Bmatrix} \lambda = \frac{2x}{y} \\ \lambda = \frac{2y}{x} \\ xy = 4 \end{Bmatrix} \to \begin{Bmatrix} \frac{2x}{y} = \frac{2y}{x} \\ xy = 4 \end{Bmatrix} \to \begin{matrix} x^2 = y^2 \\ xy = 4 \end{matrix} \to \begin{matrix} x = y \text{ or } x = -y \\ xy = 4 \end{matrix} \to$$

$$\begin{matrix} x^2 = 4 \to x = y = \pm 2 \\ -x^2 = 4 \to \text{complex solution} \end{matrix} \to \begin{matrix} \mathbf{P_1(2,2,8)} \\ \mathbf{P_2(-2,-2,8)} \end{matrix} \to \begin{matrix} f(2,2) = 8 \\ f(-2,-2) = 8 \end{matrix} \to \boldsymbol{minimums}$$

$(b) f(x,y) = e^{xy},\ x^3 + y^3 = 54$

$$\begin{Bmatrix} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ xy = 4 \end{Bmatrix} \to \begin{Bmatrix} xe^{xy} = 3\lambda y^2 \\ ye^{xy} = 3\lambda x^2 \\ x^3 + y^3 = 54 \end{Bmatrix} \to \begin{Bmatrix} xye^{xy} = 3\lambda y^3 \\ xye^{xy} = 3\lambda x^3 \\ x^3 + y^3 = 54 \end{Bmatrix} \to \begin{Bmatrix} 3\lambda y^3 = 3\lambda x^3 \\ x^3 + y^3 = 54 \end{Bmatrix} \to \begin{Bmatrix} y^3 = x^3 \\ x^3 + y^3 = 54 \end{Bmatrix} \to$$

$$\{2x^3 = 54\} \to \{x^3 = 27\} \to \begin{matrix} x = 3 \\ y = 3 \end{matrix} \to \boldsymbol{P(3,3,e^9) \to maximum}$$

$(c) f(x,y,z) = x^2 + y^2 + z^2,\ x^4 + y^4 + z^4 = 1$

$$\begin{Bmatrix} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \\ x^4 + y^4 z^4 = 1 \end{Bmatrix} \to \begin{Bmatrix} 2x = 4\lambda x^3 \\ 2y = 4\lambda y^3 \\ 2z = 4\lambda z^3 \\ x^4 + y^4 + z^4 = 1 \end{Bmatrix} \to \begin{Bmatrix} 1 = 2\lambda x^2 \\ 1 = 2\lambda y^2 \\ 1 = 2\lambda z^2 \\ x^4 + y^4 + z^4 = 1 \end{Bmatrix} \to \begin{Bmatrix} \frac{1}{2x^2} = \frac{1}{2y^2} = \frac{1}{2z^2} \\ x^4 + y^4 + z^4 = 1 \end{Bmatrix} \to$$

$$\begin{Bmatrix} x^2 = y^2 = z^2 \\ 3x^4 = 1 \end{Bmatrix} \to \left\{x^4 = \frac{1}{3}\right\} \to \left\{x = y = z = \pm\frac{\sqrt[4]{3}}{3}\right\} \to 9 \text{ possible points:}$$

$\mathbf{P_1}\left(\frac{\sqrt[4]{3}}{3}, \frac{\sqrt[4]{3}}{3}, \frac{\sqrt[4]{3}}{3}, \sqrt{3}\right)$ $\mathbf{P_2}\left(-\frac{\sqrt[4]{3}}{3}, \frac{\sqrt[4]{3}}{3}, \frac{\sqrt[4]{3}}{3}, \sqrt{3}\right)$ $\mathbf{P_3}\left(\frac{\sqrt[4]{3}}{3}, -\frac{\sqrt[4]{3}}{3}, \frac{\sqrt[4]{3}}{3}, \sqrt{3}\right)$ $\mathbf{P_4}\left(\frac{\sqrt[4]{3}}{3}, \frac{\sqrt[4]{3}}{3}, -\frac{\sqrt[4]{3}}{3}, \sqrt{3}\right)$

$\mathbf{P_5}\left(-\frac{\sqrt[4]{3}}{3}, -\frac{\sqrt[4]{3}}{3}, \frac{\sqrt[4]{3}}{3}, \sqrt{3}\right)$ $\mathbf{P_6}\left(-\frac{\sqrt[4]{3}}{3}, \frac{\sqrt[4]{3}}{3}, -\frac{\sqrt[4]{3}}{3}, \sqrt{3}\right)$ $\mathbf{P_7}\left(\frac{\sqrt[4]{3}}{3}, -\frac{\sqrt[4]{3}}{3}, -\frac{\sqrt[4]{3}}{3}, \sqrt{3}\right)$ $\mathbf{P_8}\left(-\frac{\sqrt[4]{3}}{3}, -\frac{\sqrt[4]{3}}{3}, -\frac{\sqrt[4]{3}}{3}, \sqrt{3}\right)$

$f(P) = 3x^2 = \sqrt{3} \to$ ***maximums***

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2. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest and farthest from the origin.

$g_1(x) \to x + y + 2z = 2 \qquad\qquad g_2(x) \to x^2 + y^2 - z = 0$

$(x, y, z) \to$ *a point that satisfies both of the constraints*

$distance\ from\ origin = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$\to by\ minimizing\ f(x, y, z) = x^2 + y^2 + z^2$

$\nabla f = (2x, 2y, 2z) \qquad \nabla g_1 = (1, 1, 2)\ \nabla g_1 = (2x, 2y, -1)$

$$\left.\begin{matrix} 2x = \lambda_1 + 2x\lambda_2 \\ 2y = \lambda_1 + 2y\lambda_2 \\ 2z = 2\lambda_1 - \lambda_2 \\ x + y + 2z = 2 \\ x^2 + y^2 - z = 0 \end{matrix}\right\} \to \left.\begin{matrix} 2x = \frac{\lambda_1}{1-\lambda_2} \\ 2y = \frac{\lambda_1}{1-\lambda_2} \\ 2z = 2\lambda_1 - \lambda_2 \\ x + y + 2z = 2 \\ x^2 + y^2 - z = 0 \end{matrix}\right\} \to \left.\begin{matrix} x = y \\ x + z = 1 \\ 2x^2 - z = 0 \end{matrix}\right\} \to \left.\begin{matrix} x = y \\ 2x^2 + x - 1 = 0 \end{matrix}\right\} \to \left.\begin{matrix} x = y \\ (2x-1)(x+1) = 0 \end{matrix}\right\} \to$$

$$\left.\begin{matrix} x_1 = y_1 = \frac{1}{2} \\ x_2 = y_2 = -1 \end{matrix}\right\} \to \left.\begin{matrix} z_1 = 1 - x_1 = \frac{1}{2} \\ z_2 = 1 - x_2 = 2 \end{matrix}\right\} \to \left.\begin{matrix} f(x_1, y_1, z_1) = \frac{3}{4} \\ f(x_2, y_2, z_2) = 6 \end{matrix}\right\} \to \begin{matrix} \boldsymbol{P_1\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right) \to nearest\ point\ to\ origin} \\ \boldsymbol{P_2(-1, -1, 2, 6) = farthest\ point\ to\ origin} \end{matrix}$$

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3. Each one of the differential equations or initial value problems below is either separable or linear. Identify the type and apply the methods discussed in the class and find the general solution. You do not have to express the dependent variable explicitly as a function of the independent variable.

$(a) y' - y = 2te^{2t} \to \boldsymbol{linear}$

$\mu = e^{\int(-1)dt} = e^{-t} \to y = \frac{1}{e^{-t}}\int(2te^{2t}e^{-t})dt \to \mathbf{y = e^t \int(2te^t)dt} \to \int(2te^t)dt = \mathbf{2\int(te^t)dt}$

$\to u = t, dv = e^t, du = 1, v = e^t \to \int udv = uv - \int duv \to \int(te^t)dt = \mathbf{te^t} - \boldsymbol{\int(e^t)dt} \to$

$\mathbf{2(te^t - e^t) + C} = 2te^t - 2e^t + C \to y = e^t(2te^t - 2e^t + C) = 2te^{2t} - 2e^{2t} + e^tC = \boldsymbol{2e^{2t}(t-1) + e^tC}$

$(b) t^3y' + 4t^2y = e^{-t},\ \ y(-1) = 0,\ \ \ t < 0 \to y' + \frac{4}{t}y = \frac{1}{e^tt^3} \to \boldsymbol{linear}$

$\mu = e^{\int\left(\frac{4}{t}\right)dt} = e^{4\ln t} = \left(e^{\ln(t)}\right)^4 = t^4 \to y = \frac{1}{t^4}\int\left(\frac{t^4}{e^tt^3}\right)dt = \boldsymbol{\frac{1}{t^4}\int(e^{-t}t)dt} \to$

$u = t, dv = e^{-t}, du = 1, v = -e^{-t} \to \int udv = uv - \int duv \to \int udv = -te^{-t} - \int(-e^{-t})dt = \boldsymbol{-te^{-t} - e^{-t} - C} \to$

$y = \frac{1}{t^4}(-te^{-t} - e^{-t} + C) \to \boldsymbol{y = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{C}{t^4}}$

$0 = -\frac{e^2}{(-2)^3} - \frac{e^2}{(-2)^4} + \frac{C}{(-2)^4} \to 0 = 2e^2 - e^2 + C \to C = \pm\sqrt{e} \to \boldsymbol{y = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4} \pm \frac{\sqrt{e}}{t^4}}$

$(c)\ 2y' + y = 3t^2 \to y' + \frac{1}{2}y = \frac{3t^2}{2} \to \boldsymbol{linear}$

$\mu = e^{\int\left(\frac{1}{2}\right)dt} = e^{\frac{t}{2}} \to y' = \boldsymbol{\frac{1}{e^{\frac{t}{2}}}\int\left(\frac{3t^2e^{\frac{t}{2}}}{2}\right)dt} \to \int\left(\frac{3t^2e^{\frac{t}{2}}}{2}\right)dt = \boldsymbol{\frac{3}{2}\int\left(t^2e^{\frac{t}{2}}\right)dt} \to u = t^2, dv = e^{\frac{t}{2}}, du = 2t, v = 2e^{\frac{t}{2}} \to$

$\int udv = uv - \int duv \rightarrow \int \left(t^2 e^{\frac{t}{2}}\right) dt = 2e^{\frac{t^3}{2}} - \int \left(4e^{\frac{t^2}{2}}\right) dt = 2t^2 e^{\frac{t}{2}} - 4\int \left(2te^{\frac{t}{2}}\right) dt = 2t^2 e^{\frac{t}{2}} - 8\int \left(te^{\frac{t}{2}}\right) dt \rightarrow$

$\frac{3}{2}\left(\mathbf{2t^2 e^{\frac{t}{2}} - 8\int \left(te^{\frac{t}{2}}\right) dt}\right) \rightarrow u = t, dv = e^{\frac{t}{2}}, du = 1, v = 2e^{\frac{t}{2}} \rightarrow \int \left(te^{\frac{t}{2}}\right) dt = 2te^{\frac{t}{2}} - \int 2e^{\frac{t}{2}} = \mathbf{2te^{\frac{t}{2}} - 4e^{\frac{t}{2}}} \rightarrow$

$\frac{1}{e^{\frac{t}{2}}}\left(\frac{3}{2}\left(2t^2 e^{\frac{t}{2}} - 8\left(2te^{\frac{t}{2}} - 4e^{\frac{t}{2}}\right)\right) + C\right) = \frac{1}{e^{\frac{t}{2}}}\left(\frac{3}{2}\left(2t^2 e^{\frac{t}{2}} - 16te^{\frac{t}{2}} + 32e^{\frac{t}{2}}\right) + C\right) = \frac{1}{e^{\frac{t}{2}}}\left(3t^2 e^{\frac{t}{2}} - 24te^{\frac{t}{2}} + 48e^{\frac{t}{2}} + C\right) =$
$3t^2 - 24t + 48 + \frac{C}{e^{\frac{t}{2}}} = 3(t^2 - 8t + 16) + \frac{C}{e^{\frac{t}{2}}} \rightarrow \boldsymbol{y = 3(t^2 - 8(t-2) + \frac{C}{e^{\frac{t}{2}}}}$

$(d)\ y' = (1-2x)y^2, \quad y(0) = -\frac{1}{6} \rightarrow \boldsymbol{separable}$

$\int \left(\frac{1}{y^2}\right) dy = \int (1-2x)dx \rightarrow -\frac{1}{y} = x - x^2 + C \rightarrow y = -\frac{1}{x - x^2 + C} \rightarrow$

$y(0) = -\frac{1}{0 - 0^2 + C} \rightarrow -\frac{1}{6} = -\frac{1}{C} \rightarrow C = -6 \rightarrow \boldsymbol{y(t) = -\frac{1}{x - x^2 + 6}}$

$(e)\ y' = \frac{2x}{1+2y} \rightarrow y', \quad y(2) = 0 \rightarrow 2x\frac{1}{1+2y} \rightarrow \boldsymbol{separable}$

$\int (1+2y)dy = \int (2x)dx \rightarrow y + y^2 = x^2 + C \rightarrow 0 + 0^2 = 2^2 + C \rightarrow C = -4$

$y = \frac{-1 \pm \sqrt{1 + 4x^2 + 4C}}{2} \rightarrow 0 = \frac{-1 \pm \sqrt{1 + 4 \cdot 2^2 + 4C}}{2} \rightarrow 1 = \pm\sqrt{17 + 4C} \rightarrow 1 = \pm(17 + 4C) \rightarrow$ $\boldsymbol{-4 \rightarrow solution}$, $-\frac{9}{2} \rightarrow \sqrt{\Delta} < 0$

$\boldsymbol{y + y^2 = x^2 - 4}$

$(f)\ y' = \frac{e^{-x} - e^{x}}{3+4y}, \quad y(0) = 1 \rightarrow (e^{-x} - e^{x})\frac{1}{2+4y} \rightarrow \boldsymbol{separable}$

$\int (3+4y)dy = \int (e^{-x} - e^{x})dx \rightarrow 3y + 2y^2 = -e^{-x} - e^{x} + C \rightarrow 3 + 2 = -e^{0} - e^{0} + C \rightarrow C = 7$

$y = \frac{-3 \pm \sqrt{9 - 8(e^{-x} + e^{x} - C)}}{4} \rightarrow 1 = \frac{-3 \pm \sqrt{9 - 8(e^{0} + e^{0} - C)}}{4} \rightarrow 7 = \pm\sqrt{-7 + 8C} \rightarrow 49 = \pm(-7 + 8C) \rightarrow$ $\boldsymbol{7 \rightarrow solution}$, $\boldsymbol{-\frac{21}{4}} \rightarrow \sqrt{\Delta} < 0$

$\boldsymbol{3y + 2y^2 = -e^{-x} - e^{x} + 7}$

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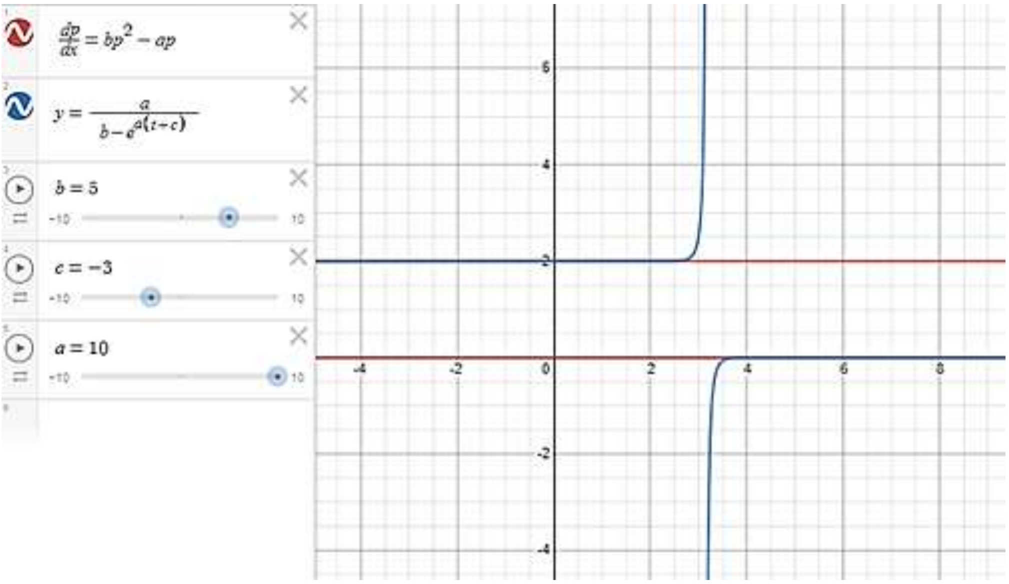
4. Consider an organism in which each member of the population requires a partner for reproduction, and that each member relies on chance encounters for meeting a mate. For such a population it is reasonable to assume that the birth rate at time t is proportional to $p(t)^2$, where p(t) denotes the population at time t, while the death rate is proportional to $p(t)$. Consequently, the population size p(t) satisfies the differential equation

$$\frac{dp}{dt} = bp^2 - ap, \quad a, b > 0. \qquad (1)$$

(a) Find all the equilibria and classify them.



$$\frac{dp}{dt} = bp^2 - ap = 0$$

$$\frac{dp}{dt} = p(bp - a) = 0$$

Equilibria:

$p_1 = 0 \to stable$

$p_2 = \frac{a}{b} \to unstable$

(b) Find the general solution for this differential equation.

$p' = bp^2 - ap \to p' + ap = bp^2$

$$\boldsymbol{y' = p(x)y = q(x)y^n} \to y' = p', p(x) = a, q(x) = b, y^n = p^2 \to \boldsymbol{n = 2}$$

$$\mathrm{v} = \mathrm{y}^{1-\mathrm{n}} = p^{1-2} = \boldsymbol{p^{-1}} \to \boldsymbol{v} = \frac{\boldsymbol{1}}{\boldsymbol{y}}\, and\, \frac{1}{1-n}v' + p(x)v = q(x) \to \boldsymbol{-v' + av = b}$$

Solve $-v' + av = b \to -\frac{dv}{dt} = b - av \to \left(-\frac{1}{b-av}\right)dv = (1)dt$:

$$\int\left(-\frac{1}{b-av}\right)dv = \int(1)dt \to \int(1)dt = \boldsymbol{t} \to \int\left(-\frac{1}{b-av}\right)dv \to$$

$$u = b - av \to \frac{du}{dv} = -a \to dv = -\frac{1}{a}du \to -\frac{1}{a}\int\left(-\frac{1}{u}\right)du = \frac{\ln u}{a} = \frac{\boldsymbol{\ln b - av}}{\boldsymbol{a}} \to$$

$$\frac{\boldsymbol{\ln b - av}}{\boldsymbol{a}} = \boldsymbol{t + C} \to \ln(b - av) = at + aC \to e^{(at+aC)} = b - av \to \boldsymbol{v} = \frac{\boldsymbol{b - e^{(at+aC)}}}{\boldsymbol{a}}$$

$$Substitute\ v \to \frac{1}{p} = \frac{b - e^{(at+aC)}}{a} \to p = \frac{a}{b - e^{(at+aC)}} \to \boldsymbol{p(t)} = \frac{\boldsymbol{a}}{\boldsymbol{b - e^{a(t+C)}}}$$

(c) Solve the initial value problem with (1) and the initial condition $p(0) = p_0$.

$$p_0 = \frac{a}{b - e^{a(0+C)}} \to p_0 = \frac{a}{b - e^{aC}} \to e^{aC} = b - \frac{a}{p_0} \to \ln e^{aC} = \ln\left(b - \frac{a}{p_0}\right) \to \boldsymbol{C} = \frac{\boldsymbol{\ln\left(b - \frac{a}{p_0}\right)}}{\boldsymbol{a}}$$

$$p(t) = \frac{a}{b - e^{a\left(t + \frac{\ln\left(b - \frac{a}{p_0}\right)}{a}\right)}} = \frac{a}{b - e^{at}e^{\ln\left(b - \frac{a}{p_0}\right)}} = \frac{\boldsymbol{a}}{\boldsymbol{b - e^{at}\left(b - \frac{a}{p_0}\right)}}$$

(d) Assume that $p_0 < \frac{a}{b}$. Show that the as $t \to 1,\ p(t)$ approaches zero.

$$\lim_{t\to 1} p(t) = \lim_{t\to 1} \frac{a}{b - e^{at}\left(b - \frac{a}{p_0}\right)} = \frac{a}{b - e^{a}\left(b - \frac{a}{p_0}\right)} \to since\ p_0 < \frac{a}{b}\ then\ b < \frac{a}{p_0}, therefore \left(b - \frac{a}{p_0}\right) < 0 \to$$

$$b - e^{a}\left(b - \frac{a}{p_0}\right) > 0 \to \frac{a}{b-e^{a}\left(b-\frac{a}{p_0}\right)} \to the\ denumerator\ is\ positive\ multiple\ of\ an\ exponential, therefore$$

$$a\ divided\ by\ this\ comparitively\ bigger\ denumerator\ will\ come\ close\ to\ 0 \to \mathbf{\lim_{t\to 1} \frac{a}{b-e^{at}\left(b-\frac{a}{p_0}\right)} = 0}$$

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5. Consider the Lotka-Volterra equation given by the system of equations

$$\frac{dx}{dt} = a_1 x - b_1 xy$$

$$\frac{dy}{dt} = -a_2 y + b_2 xy$$

Consider the quantity:

$$V(t) = b_2 x(t) + b_1 y(t) - a_2 \log x(t) - a_1 \log y(t).$$

- Show that V(t) is a constant of motion, that is, $V'(t) = 0$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \to \frac{dy}{dx} = \frac{y(b_2x - a_2)}{x(a_1 - b_1y)} \to \frac{dy(a_1 - b_1y)}{y} = \frac{dx(b_2x - a_2)}{x} \to$$

$$a_1 \int \frac{1}{y} dy - b_1 \int \frac{y}{y} dy = b_2 \int \frac{x}{x} dx - a_2 \int \frac{1}{x} dx \to$$

$$a_1 \ln y - b_1 y = b_2 x - a_2 \ln x \to V(t) = b_2 x + b_1 y - a_2 \ln x - a_1 \ln y$$

$$\frac{dV}{dt} = b_2 \frac{dx}{dt} + b_1 \frac{dy}{dt} - a_2 \frac{d(\ln x)}{dt}\frac{dx}{dt} - a_1 \frac{d(\ln y)}{dt}\frac{dy}{dt} \to$$

$$\frac{dV}{dt} = b_2(a_1x - b_1xy) + b_1(-a_2y + b_2xy) - \frac{a_2}{x}(a_1x - b_1xy) - \frac{a_1}{y}(-a_2y + b_2xy) \to$$

$$\frac{dV}{dt} = a_1b_2x - b_1b_2xy - a_2b_1y + b_1b_2xy - a_1a_2 + a_2b_2y + a_1a_2 - a_1b_2x \to$$

$$\frac{dV}{dt} = \cancel{a_1b_2x} - \cancel{a_1b_2x} - \cancel{b_1b_2xy} + \cancel{b_1b_2xy} - \cancel{a_2b_1y} + \cancel{a_2b_2y} - \cancel{a_1a_2} + \cancel{a_1a_2} \to$$

$$\mathbf{\frac{dV}{dt} = V' = 0}$$

- Show that the minimum possible value of V is attained at the equilibrium $(x, y) = \left(\frac{a2}{b2}, \frac{a1}{b1}\right)$.

$We\ have\ equilibrum\ when:$ $\begin{cases} \frac{dx}{dt} = 0 \to a_1 = b_1 y \to y = \frac{a_1}{b_1} \\ \frac{dy}{dt} = 0 \to a_2 = b_2 x \to x = \frac{a_2}{b_2} \end{cases}$

$$V(t) = b_2 \frac{a_2}{b_2} + b_1 \frac{a_1}{b_1} - a_2 ln \frac{a_2}{b_2} - a_1 ln \frac{a_1}{b_1} \to$$

$$V(t) = a_2 - a_2 ln \frac{a_2}{b_2} + a_1 - a_1 ln \frac{a_1}{b_1} \to$$

$$V(t) = a_2 \left(1 - \ln \frac{a_2}{b_2}\right) + a_1 \left(1 - ln \frac{a_1}{b_1}\right) \to$$

$$V(t) = a_2(1 - \ln a_2 + \ln b_2) + a_1(1 - \ln a_1 + \ln b_1)$$

$We\ have\ minimum:$

$$For \frac{dx}{dt}, let\ y = 2\frac{a_1}{b_1}$$

$$\frac{dx}{dt} = a_1 x - b_1 xy$$

$$\frac{dx}{dt} = a_1 \frac{a_2}{b_2} - b_1 \frac{a_2}{b_2} 2 \frac{a_1}{b_1} = -\frac{a_1 a_2}{b_2}$$

$$for\ y = \frac{1}{2}\frac{a_1}{b_1} \to$$

$$\frac{dx}{dt} = a_1 \frac{a_2}{b_2} - b_1 \frac{a_2}{b_2} \frac{1}{2} \frac{a_1}{b_1} = \frac{a_1 a_2}{2 b_2}$$

$$For \frac{dy}{dt}, let\ x = 2\frac{a_2}{b_2}$$

$$\frac{dy}{dt} = -a_2 y + b_2 xy$$

$$\frac{dy}{dt} = -a_2 \frac{a_1}{b_1} + b_2 2 \frac{a_2}{b_2} \frac{a_1}{b_1} = \frac{a_1 a_2}{b_1}$$

$$for\ x = \frac{1}{2}\frac{a_2}{b_2} \to$$

$$\frac{dy}{dt} = -a_2 \frac{a_1}{b_1} + b_2 \frac{1}{2} \frac{a_2}{b_2} \frac{a_1}{b_1} = -\frac{a_1 a_2}{b_1}$$

$$Since\ V(t)\ is\ result\ of\ an\ integration, dx(t) depends\ on \frac{dy}{dx}, and\ since \frac{dy}{dx} approaches$$
$$the\ point \left(\frac{a2}{b2}, \frac{a1}{b1}\right), it\ means\ that\ this\ point\ is\ a\ minimum.$$
