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Assignment 9

Calculus and Linear Algebra II

1. Find the characteristic polynomial, eigenvalues, and the corresponding eigenvectors of the following matrices. In each case, determine if the matrix A is diagonalizable or not. If it is the case, find an invertible matrix P such that $P^{-1}AP = D$ is diagonal and determine D.

(a) $\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix} \to \begin{vmatrix} t-1 & -2 & -1 \\ 0 & t-1 & -2 \\ 0 & 0 & t-1 \end{vmatrix} = (t-1)(t-1)(t-1) = \boldsymbol{t^3 - 3t^2 + 3t - 1}$

$$\boldsymbol{f(t) = t^3 - 3t^2 + 3t - 1} = (t-1)(t-1)(t-1) \to \boldsymbol{eigenvalues: \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1}$$

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} 2y + z = 0 \\ z = 0 \\ 0 = 0 \end{cases} \to \begin{cases} x = x \\ y = 0 \\ z = 0 \end{cases} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{eigenvectors: v_1 = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$$\boldsymbol{The\ 3 \times 3\ matrix\ is\ not\ diagonizable, as\ it\ doesn't\ have\ 3\ distinct\ eigenvalues.}$$

(b) $\begin{pmatrix} \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{4} & \mathbf{1} \\ \mathbf{0} & \mathbf{8} & \mathbf{2} \end{pmatrix} \to \begin{vmatrix} t-2 & 0 & -1 \\ 0 & t-4 & -1 \\ 0 & -8 & t-2 \end{vmatrix} = (t-2)\big((t-2)(t-4) - 8\big) = \boldsymbol{t^3 - 8t^2 + 12t}$

$$\boldsymbol{f(t) = t^3 - 8t^2 + 12t} = t(t-2)(t-6) \to \boldsymbol{eigenvalues: \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 6}$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} 2x + z = 0 \\ 4y + z = 0 \\ 8y + 2z = 0 \end{cases} \to \begin{cases} z = -2x \\ z = -4y \\ z = -4y \end{cases} \to \begin{cases} x = 2y = -\frac{1}{2}z \\ y = \frac{1}{2}x = -\frac{1}{4}z \\ z = -2x = -4y \end{cases} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ \frac{1}{2} \\ -2 \end{pmatrix} = y\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = z\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 8 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} z = 0 \\ 2y + z = 0 \\ 8y = 0 \end{cases} \to \begin{cases} x = x \\ y = 0 \\ z = 0 \end{cases} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & 8 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} -4x + z = 0 \\ -2y + z = 0 \\ 8y - 4z = 0 \end{cases} \to \begin{cases} z = 4x \\ z = 2y \\ z = 2y \end{cases} \to \begin{cases} x = \frac{1}{2}y = \frac{1}{4}z \\ y = 2x = \frac{1}{2}z \\ z = 4x = 2y \end{cases} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = y\begin{pmatrix} \frac{1}{2} \\ 1 \\ 2 \end{pmatrix} = z\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

$$\boldsymbol{eigenvectors: v_1 = x_1\begin{pmatrix} 1 \\ \frac{1}{2} \\ -2 \end{pmatrix} = y_1\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} = z_1\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 1 \end{pmatrix}, v_2 = x_2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = x_3\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = y_3\begin{pmatrix} \frac{1}{2} \\ 1 \\ 2 \end{pmatrix} = z_3\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix}}$$

$$\boldsymbol{The\ 3 \times 3\ matrix\ is\ diagonizable, as\ it\ has\ 3\ distinct\ eigenvalues.}$$

$$\boldsymbol{Assume\ x_{1,2,3} = 1 \rightarrow P = \begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & 0 & 2 \\ -2 & 0 & 4 \end{pmatrix} and\ P^{-1} = \begin{pmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -1 & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{12} \end{pmatrix}}$$

$$\boldsymbol{P^{-1}AP = \begin{pmatrix} 0 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -1 & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{12} \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & 8 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & 0 & 2 \\ -2 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & -2 & \frac{1}{2} \\ 0 & 2 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & 0 & 2 \\ -2 & 0 & 4 \end{pmatrix} = D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}}$$

(c) $\boldsymbol{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} \rightarrow \begin{vmatrix} t & -1 & 0 & 0 \\ -1 & t & 0 & 0 \\ 0 & 0 & t-2 & 0 \\ 0 & 0 & -1 & t-1 \end{vmatrix} = (t+1)(t-2)(t-1)(t-1) = \boldsymbol{t^4 - 3t^3 + t^2 + 3t - 2}$

$$\boldsymbol{f(t) = t^4 - 3t^3 + t^2 + 3t - 2} = (t+1)(t-2)(t-1)(t-1) \rightarrow \boldsymbol{eigenvalues\colon \lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 2}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} x + y = 0 \\ x + y = 0 \\ 3z = 0 \\ z + 2u = 0 \end{cases} \rightarrow \begin{cases} x = -y \\ y = -x \\ z = 0 \\ u = 0 \end{cases} \rightarrow \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = x\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -x + y = 0 \\ x - y = 0 \\ z = 0 \\ z = 0 \end{cases} \rightarrow \begin{cases} x = y \\ y = x \\ z = 0 \\ u = u \end{cases} \rightarrow \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{cases} -2x + y = 0 \\ x - 2y = 0 \\ 0 = 0 \\ z - u = 0 \end{cases} \rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = u \\ u = z \end{cases} \rightarrow \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = z\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = u\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{eigenvectors\colon v_1 = x_1\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = y_1\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = x_2\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = y_2\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = u_3\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, v_4 = z_4\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = u_4\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}$$

$$\boldsymbol{The\ 4 \times 4\ matrix\ is\ diagonizable, as\ it\ has\ 4\ distinct\ eigenvectors.}$$

$$\boldsymbol{Assume\ x_{1,2} = u_{3,4} = 1 \rightarrow P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} and\ P^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}$$

$$\boldsymbol{P^{-1}AP = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}}$$

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2. Suppose

$$\boldsymbol{A}=\begin{pmatrix}\mathbf{3} & \mathbf{-1}\\ \mathbf{-2} & \mathbf{2}\end{pmatrix}$$

(a) Find eigenvalues and corresponding eigenvectors of $\boldsymbol{A}$.

$$\begin{vmatrix}t-3 & 1\\ 2 & t-2\end{vmatrix}=(t-3)(t-2)-2=\boldsymbol{t^2-5t+4}$$

$$\boldsymbol{f(t)=t^2-5t+4}=(t-1)(t-4)\to \boldsymbol{eigenvalues}:\boldsymbol{\lambda_1=1,\lambda_2=4}$$

$$\begin{pmatrix}2 & -1\\ -2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}2x-y=0\\ -2x+y=0\end{cases}\to\begin{cases}x=\frac{1}{2}y\\ y=2x\end{cases}\to\begin{pmatrix}x\\ y\end{pmatrix}=x\begin{pmatrix}1\\ 2\end{pmatrix}=y\begin{pmatrix}\frac{1}{2}\\ 1\end{pmatrix}$$

$$\begin{pmatrix}-1 & -1\\ -2 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}-x-y=0\\ -2x-2y=0\end{cases}\to\begin{cases}x=-y\\ y=-x\end{cases}\to\begin{pmatrix}x\\ y\end{pmatrix}=x\begin{pmatrix}1\\ -1\end{pmatrix}=y\begin{pmatrix}-1\\ 1\end{pmatrix}$$

$$\boldsymbol{eigenvectors}:\boldsymbol{v_1=x_1\begin{pmatrix}1\\ 2\end{pmatrix}=y_1\begin{pmatrix}\frac{1}{2}\\ 1\end{pmatrix},v_2=x_2\begin{pmatrix}1\\ -1\end{pmatrix}=y_2\begin{pmatrix}-1\\ 1\end{pmatrix}}$$

(b) Find a formula for $\boldsymbol{A^n}$ for $\boldsymbol{n\geq 1}$.

$$Assume\ x_{1,2}=1\to P=\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix},D=P^{-1}AP=\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix},P^{-1}=\begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{pmatrix}$$

$$A^n=PD^nP^{-1}=\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 4^n\end{pmatrix}\begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{pmatrix}=\begin{pmatrix}1 & 4^n\\ 2 & -4^n\end{pmatrix}\begin{pmatrix}\frac{1}{3} & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{pmatrix}\to\boldsymbol{A^n}=\begin{pmatrix}\frac{\mathbf{1+2^{2n+1}}}{\mathbf{3}} & \frac{\mathbf{1-4^n}}{\mathbf{3}}\\ \frac{\mathbf{2(1-4^n)}}{\mathbf{3}} & \frac{\mathbf{2+4^n}}{\mathbf{3}}\end{pmatrix}$$

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3. Find the general solution for the initial value problems

(a) $\boldsymbol{x'}=\begin{pmatrix}\mathbf{1} & \mathbf{-1} & \mathbf{4}\\ \mathbf{3} & \mathbf{2} & \mathbf{-1}\\ \mathbf{2} & \mathbf{1} & \mathbf{-1}\end{pmatrix}\boldsymbol{x},\quad \boldsymbol{x(0)}=\begin{pmatrix}\mathbf{1}\\ \mathbf{1}\\ \mathbf{1}\end{pmatrix}$

$$\begin{vmatrix}t-1 & 1 & -4\\ -3 & t-2 & 1\\ -2 & -1 & t+1\end{vmatrix}=(t-1)\big((t-2)(t+1)+1\big)-(-3t-1)-4(2t-1)=\boldsymbol{t^3-2t^2-5t+6}$$

$$\boldsymbol{f(t)=t^3-2t^2-5t+6}=(t-1)(t+2)(t-3)\to\boldsymbol{eigenvalues}:\boldsymbol{\lambda_1=-2,\lambda_2=1,\lambda_3=3}$$

$$\begin{pmatrix}3 & -1 & 4\\ 3 & 4 & -1\\ 2 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\to\begin{cases}3x-y+4z=0\\ 3x+4y-z=0\\ 2x+y+z=0\end{cases}\to\begin{cases}x=-y=-z\\ y=-x=z\\ z=-x=y\end{cases}\to\begin{pmatrix}x\\ y\\ z\end{pmatrix}=x\begin{pmatrix}1\\ -1\\ -1\end{pmatrix}=y\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}=z\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}$$

$$\begin{pmatrix}0 & -1 & 4\\ 3 & 1 & -1\\ 2 & 1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\to\begin{cases}-y+4z=0\\ 3x+y-z=0\\ 2x+y-2z=0\end{cases}\to\begin{cases}x=-\frac{1}{4}y=-z\\ y=-4x=4z\\ z=-x=\frac{1}{4}y\end{cases}\to\begin{pmatrix}x\\ y\\ z\end{pmatrix}=x\begin{pmatrix}1\\ -4\\ -1\end{pmatrix},y\begin{pmatrix}-\frac{1}{4}\\ 1\\ \frac{1}{4}\end{pmatrix},z\begin{pmatrix}-1\\ 4\\ 1\end{pmatrix}$$

$$\begin{pmatrix}-2 & -1 & 4\\ 3 & -1 & -1\\ 2 & 1 & -4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\to\begin{cases}-2x-y+4z=0\\ 3x-y-z=0\\ 2x+y-4z=0\end{cases}\to\begin{cases}x=\frac{1}{2}y=z\\ y=2x=2z\\ z=x=\frac{1}{2}y\end{cases}\to\begin{pmatrix}x\\ y\\ z\end{pmatrix}=x\begin{pmatrix}1\\ 2\\ 1\end{pmatrix},y\begin{pmatrix}\frac{1}{2}\\ 1\\ \frac{1}{2}\end{pmatrix},z\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}$$

$$\boldsymbol{eigenvectors}:\boldsymbol{v_1=x_1\begin{pmatrix}1\\ -1\\ -1\end{pmatrix}=y_1\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}=z_1\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix},v_2=x_2\begin{pmatrix}1\\ -4\\ -1\end{pmatrix},y_2\begin{pmatrix}-\frac{1}{4}\\ 1\\ \frac{1}{4}\end{pmatrix},z_2\begin{pmatrix}-1\\ 4\\ 1\end{pmatrix},v_3=x_3\begin{pmatrix}1\\ 2\\ 1\end{pmatrix},y_3\begin{pmatrix}\frac{1}{2}\\ 1\\ \frac{1}{2}\end{pmatrix},z_3\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}}$$

$$x(t)=c_1e^{\lambda_1 t}v_1+c_2e^{\lambda_2 t}v_2+c_3e^{\lambda_3 t}v_3\to Assume\ x_{1,2,3}=1\to c_1e^{-2t}\begin{pmatrix}1\\ -1\\ -1\end{pmatrix}+c_2e^{t}\begin{pmatrix}1\\ -4\\ -1\end{pmatrix}+c_3e^{3t}\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}$$

$$x(0)=\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}\to\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}=c_1e^0\begin{pmatrix}1\\ -1\\ -1\end{pmatrix}+c_2e^0\begin{pmatrix}1\\ -4\\ -1\end{pmatrix}+c_3e^0\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}\to\begin{pmatrix}c_1\\ -c_1\\ -c_1\end{pmatrix}+\begin{pmatrix}c_2\\ -4c_2\\ -c_2\end{pmatrix}+\begin{pmatrix}c_3\\ 2c_3\\ c_3\end{pmatrix}=\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}\to$$

$$\begin{cases}c_1+c_2+c_3=1\\ -c_1-4c_2+2c_3=1\\ -c_1-c_2+c_3=1\end{cases}\to\begin{cases}c_1=-\frac{1}{3}\\ c_2=\frac{1}{3}\\ c_3=1\end{cases}\to\boldsymbol{general\ solution}\to\boldsymbol{x(t)=\frac{2}{3}e^{-2t}\begin{pmatrix}1\\ -1\\ -1\end{pmatrix}-\frac{1}{6}e^{t}\begin{pmatrix}1\\ -4\\ -1\end{pmatrix}+\frac{1}{2}e^{3t}\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}}$$

(b) $\boldsymbol{x'=\begin{pmatrix}2 & 2\\ 2 & 5\end{pmatrix}x,\quad x(0)=\begin{pmatrix}3\\ 2\end{pmatrix}}$

$$\begin{vmatrix}t-2 & -2\\ -2 & t-5\end{vmatrix}=(t-2)(t-5)-4=\boldsymbol{t^2-7t+6}$$

$$\boldsymbol{f(t)=t^2-7t+6}=(t-1)(t-6)\to\boldsymbol{eigenvalues}:\boldsymbol{\lambda_1=1,\lambda_2=6}$$

$$\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}x+2y=0\\ 2x+4y=0\end{cases}\to\begin{cases}x=-2y\\ y=-\frac{1}{2}x\end{cases}\to\begin{pmatrix}x\\ y\end{pmatrix}=x\begin{pmatrix}1\\ -\frac{1}{2}\end{pmatrix}=y\begin{pmatrix}-2\\ 1\end{pmatrix}$$

$$\begin{pmatrix}-4 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\to\begin{cases}-4x+2y=0\\ 2x-y=0\end{cases}\to\begin{cases}x=\frac{1}{2}y\\ y=2x\end{cases}\to\begin{pmatrix}x\\ y\end{pmatrix}=x\begin{pmatrix}1\\ 2\end{pmatrix}=y\begin{pmatrix}\frac{1}{2}\\ 1\end{pmatrix}$$

$$\boldsymbol{eigenvectors}:\boldsymbol{v_1=x_1\begin{pmatrix}1\\ -\frac{1}{2}\end{pmatrix}=y_1\begin{pmatrix}-2\\ 1\end{pmatrix},v_2=x_2\begin{pmatrix}1\\ 2\end{pmatrix}=y_2\begin{pmatrix}\frac{1}{2}\\ 1\end{pmatrix}}$$

$$x(t)=c_1e^{\lambda_1 t}v_1+c_2e^{\lambda_2 t}v_2\to Assume\ x_{1,2}=1\to x(t)=c_1e^{t}\begin{pmatrix}1\\ -\frac{1}{2}\end{pmatrix}+c_2e^{6t}\begin{pmatrix}1\\ 2\end{pmatrix}$$

$$x(0)=\begin{pmatrix}3\\ 2\end{pmatrix}\to\begin{pmatrix}3\\ 2\end{pmatrix}=c_1e^0\begin{pmatrix}1\\ -\frac{1}{2}\end{pmatrix}+c_2e^0\begin{pmatrix}1\\ 2\end{pmatrix}\to\begin{pmatrix}c_1\\ -\frac{1}{2}c_1\end{pmatrix}+\begin{pmatrix}c_2\\ 2c_2\end{pmatrix}=\begin{pmatrix}3\\ 2\end{pmatrix}\to$$

$$\begin{cases}c_1+c_2=3\\ -\frac{1}{2}c_1+2c_2=2\end{cases}\to\begin{cases}2c_1+2c_2=6\\ \frac{1}{2}c_1-2c_2-2\end{cases}\to\frac{5c_1}{2}=4\to\boldsymbol{c_1=\frac{8}{5}}\to c_2=3-\frac{8}{5}\to\boldsymbol{c_2=\frac{7}{5}}$$

$$\boldsymbol{general\ solution}\to\boldsymbol{x(t)=\frac{8}{5}e^{t}\begin{pmatrix}1\\ -\frac{1}{2}\end{pmatrix}+\frac{7}{5}e^{6t}\begin{pmatrix}1\\ 2\end{pmatrix}}$$

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4. A radioactive substance R_1 having decay rate $k_1 = 0.2\ g/h$ disintegrates into a second radioactive substance R_2. Substance R_2 disintegrates into R_3 with the decay rate $k_2 = 0.1\ g/h$. We assume that R_3 is stable. If $m_i(t)$ represents the mass of substance R_i at time t for $i = 1, 2, 3$ the applicable equations are:

$$\frac{dm_1}{dt} = -k_1 m_1$$
$$\frac{dm_2}{dt} = k_1 m_1 - k_2 m_2$$
$$\frac{dm_3}{dt} = k_2 m_2$$

$where\ k_1 and\ k_2 are\ the\ constant\ given\ above. Use\ eigenvalue\ method\ to\ solve\ the\ above\ system\ under\ the\ conditions$

$$m_1(0) = 4g, m_2(0) = 0, m_3(0) = 0.$$

$$\begin{cases} m_1' = -0.2m_1 \\ m_2' = 0.2m_1 - 0.1m_2 \\ m_3' = 0.1m_2 \end{cases} \to A = \begin{pmatrix} -0.2 & 0 & 0 \\ 0.2 & -0.1 & 0 \\ 0 & 0.1 & 0 \end{pmatrix} \to eigenvalues: \begin{cases} \lambda_1 = -0.2 \\ \lambda_2 = -0.1 \\ \lambda_3 = 0 \end{cases}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0.2 & 0.1 & 0 \\ 0 & 0.1 & 0.2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} 0 = 0 \\ 0.2x + 0.1y = 0 \\ 0.1y + 0.2z = 0 \end{cases} \to \begin{cases} x = -\frac{1}{2}y = z \\ y = -2x = -2z \\ z = x = -\frac{1}{2}y \end{cases} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = y\begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix} = z\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -0.1 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0 & 0.1 & 0.1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} -0.1x = 0 \\ 0.2x = 0 \\ 0.1y + 0.1z = 0 \end{cases} \to \begin{cases} x = 0 \\ y = -z \\ z = -y \end{cases} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = z\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -0.2 & 0 & 0 \\ 0.2 & -0.1 & 0 \\ 0 & 0.1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} -0.2x = 0 \\ 0.2x - 0.1y = 0 \\ 0.1y = 0 \end{cases} \to \begin{cases} x = 0 \\ y = 0 \\ z = z \end{cases} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{eigenvectors: v_1 = x_1\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = y_1\begin{pmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix} = z_1\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, v_2 = y_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = z_2\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, v_3 = z_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

$$Asume\ x_1, y_2, z_3 = 1 \to m_i(t) = c_1 e^{-0.2t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 e^{-0.1t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3 e^{0.1t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} m_1(t) \\ m_2(t) \\ m_3(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-0.2t} \\ -2c_1 e^{-0.2t} \\ c_1 e^{-0.2t} \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 e^{-0.1t} \\ -c_2 e^{-0.1t} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c_3 e^{0.1t} \end{pmatrix}$$

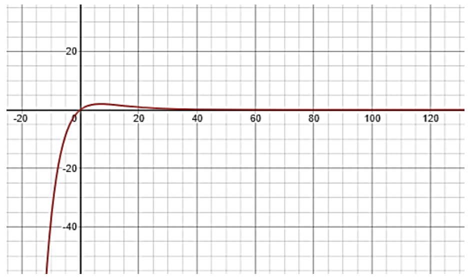
$$\begin{pmatrix} m_1(0) \\ m_2(0) \\ m_3(0) \end{pmatrix} = \begin{pmatrix} c_1 \\ -2c_1 + c_2 \\ c_1 - c_2 + c_3 \end{pmatrix} \to \begin{cases} c_1 = 4 \\ -2c_1 + c_2 = 0 \\ c_1 - c_2 + c_3 = 0 \end{cases} \to \begin{cases} \boldsymbol{c_1 = 4} \\ \boldsymbol{c_2 = 8} \\ \boldsymbol{c_3 = 4} \end{cases}$$

$$\boldsymbol{general\ solution\ of\ system \to m_i(t) = 4e^{-0.2t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + 8e^{-0.1t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + 4e^{0.1t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4e^{-0.2t} \\ -8e^{-0.2t} + 8e^{-0.1t} \\ 4e^{-0.2t} - 8e^{-0.1t} + 4e^{0.1t} \end{pmatrix}}$$

$\boldsymbol{Find\ the\ limit\ of\ m_i(t)\ as\ t \to \infty, for\ i = 1, 2, 3.}$

$$\boldsymbol{i = 1} \qquad \lim_{t\to\infty}\left(4e^{-0.2t}\right) = 0$$

$$\boldsymbol{i = 2} \qquad \lim_{t\to\infty}\left(-8e^{-0.2t} + 8e^{-0.1t}\right) = 0$$



$$\boldsymbol{i = 3} \qquad \lim_{t\to\infty}\left(4e^{-0.2t} - 8e^{-0.1t} + 4e^{0.1t}\right) = -\infty$$

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5. Suppose that the matrices A and B are given by:

$$\boldsymbol{A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$$

(a) Compute e^A and e^B using the definition of the exponential function.

$$As\ A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\ and\ B^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$higher\ powers\ will\ result\ in\ only\ null\ matrices. Thus:$$

$$e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ldots = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

$$e^B = \sum_{n=0}^{\infty}\frac{A^n}{n!} = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ldots = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$$

(b) Let $\boldsymbol{C = A + B}$. Compute $\boldsymbol{C^2, C^3}$ and guess a formula for $\boldsymbol{C^n}$. Use this formula to compute $\boldsymbol{e^C}$.

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \boldsymbol{C^2 = I}$$

$$C^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \boldsymbol{C^3 = C}$$

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, D = P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, P^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$A^n = PD^nP^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} (-1)^n & 1 \\ (-1)^{n+1} & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{(-1)^n + 1}{2} & \frac{(-1)^{n+1} + 1}{2} \\ \frac{(-1)^{n+1} + 1}{2} & \frac{(-1)^n + 1}{2} \end{pmatrix}$$

$$C^n = I \; for\; even\; n, and\; C^n = C \; for\; odd\; n, thus\; \boldsymbol{C^n} = \begin{pmatrix} \dfrac{(-1)^n+1}{2} & \dfrac{(-1)^{n+1}+1}{2} \\ \dfrac{(-1)^{n+1}+1}{2} & \dfrac{(-1)^n+1}{2} \end{pmatrix}$$

$$e^C = Pe^D P^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{e} & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{e} & e \\ -\frac{1}{e} & e \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \rightarrow \boldsymbol{e^C} = \begin{pmatrix} \dfrac{e^2+1}{2\,e} & \dfrac{e^2-1}{2\,e} \\ \dfrac{e^2-1}{2\,e} & \dfrac{e^2+1}{2\,e} \end{pmatrix}$$

(c) Show that $e^C \neq e^A e^B$. This shows that the formula $e^{x+y} = e^x e^y$ does not in general hold/ for matrices.

$$e^A e^B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} \dfrac{e^2+1}{2\,e} & \dfrac{e^2-1}{2\,e} \\ \dfrac{e^2-1}{2\,e} & \dfrac{e^2+1}{2\,e} \end{pmatrix} \neq e^C$$