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Assignment 10

Calculus and Linear Algebra II

1. (a) Suppose U is a unitary matrix. Prove that $|det\ U| = 1$.

$$* \det(AB) = \det(A)\det(B) * and * UU^* = I *$$

$$\det(U)\det(U^*) = \det(UU^*) = \det(I) = 1$$

$$Let \det(U) = a + bi, then \ \det(U^*) = a - bi$$

$$(a + bi)(a - bi) = 1 \rightarrow a^2 + b^2 = 1$$

$$\det(U) = a + bi \rightarrow |\det(U)| = |a + bi| = \sqrt{a^2 + b^2} = \sqrt{1} = 1$$

(b) Show that if λ is a complex number with $|\lambda| = 1$ then the matrix

$$U_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$$

is unitary.

$$* Conjugate\ of\ the\ quotient\ of\ two\ complex\ numbers\ z1\ and\ z2\ is\ the\ quotient\ of\ their\ conjugates *$$

$$U_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \rightarrow U_\lambda^* = \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & \overline{\frac{1}{\lambda}} \end{pmatrix} = \begin{pmatrix} \bar{\lambda} & 0 \\ 0 & \frac{\bar{1}}{\bar{\lambda}} \end{pmatrix}$$

$$* The\ multiplication\ of\ a\ complex\ number\ z = a + bi\ and\ its\ conjugate\ \bar{z} = a - bi\ gives\ a^2 + b^2\ or\ \sqrt{|z|} *$$

$$U_\lambda U_\lambda^* = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}\begin{pmatrix} \bar{\lambda} & 0 \\ 0 & \frac{\bar{1}}{\bar{\lambda}} \end{pmatrix} = \begin{pmatrix} \lambda\bar{\lambda} & 0 \\ 0 & \frac{\bar{1}}{\lambda\bar{\lambda}} \end{pmatrix} = \begin{pmatrix} \sqrt{1} & 0 \\ 0 & \frac{1}{\sqrt{1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \rightarrow U_\lambda\ is\ unitary$$

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2. Let A be the matrix

$$A = \begin{pmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{pmatrix}$$

(a) Show that A is Hermitian.

$$\left.\begin{array}{l} A^T = \begin{pmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{pmatrix} = A \to symmetric \\ A^* = \begin{pmatrix} \bar{2} & \overline{\sqrt{2}} & \bar{0} \\ \overline{\sqrt{2}} & \bar{2} & \overline{\sqrt{2}} \\ \bar{0} & \overline{\sqrt{2}} & \bar{2} \end{pmatrix} = \begin{pmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{pmatrix} = A \end{array}\right\} \to A \to Hermitian$$

(b) Find the eigenvalues of A.

$$\begin{vmatrix} t-2 & -\sqrt{2} & 0 \\ -\sqrt{2} & t-2 & -\sqrt{2} \\ 0 & -\sqrt{2} & t-2 \end{vmatrix} = (t-2)(t^2-4t+2) - 2(t-2) = (t-2)(t^2-4t) = t(t-2)(t-4)$$

$$\boldsymbol{f(t) = -t^3 + 6t^2 - 8t = t(t-2)(t-4) \to eigenvalues\colon \lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 4}$$

(c) Find an orthonormal basis for R^3 consisting of eigenvectors of A.

$$\begin{pmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} 2x_1 + \sqrt{2}x_2 = 0 \\ \sqrt{2}x_1 + 2x_2 + \sqrt{2}x_3 = 0 \\ \sqrt{2}x_2 + 2x_3 = 0 \end{cases} \to \begin{cases} x_1 = -\frac{\sqrt{2}}{2}x_2 = x_3 \\ x_2 = -\sqrt{2}x_1 = -\sqrt{2}x_3 \\ x_3 = -\frac{\sqrt{2}}{2}x_2 = x_1 \end{cases} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = x_2\begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = x_3\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} \sqrt{2}x_2 = 0 \\ \sqrt{2}x_1 + \sqrt{2}x_3 = 0 \\ \sqrt{2}x_2 = 0 \end{cases} \to \begin{cases} x_1 = -x_3 \\ x_2 = 0 \\ x_3 = -x_1 \end{cases} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & \sqrt{2} & 0 \\ \sqrt{2} & -2 & \sqrt{2} \\ 0 & \sqrt{2} & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} -2x_1 + \sqrt{2}x_2 = 0 \\ \sqrt{2}x_1 - 2x_2 + \sqrt{2}x_3 = 0 \\ \sqrt{2}x_2 - 2x_3 = 0 \end{cases} \to \begin{cases} x_1 = \frac{\sqrt{2}}{2}x_2 = x_3 \\ x_2 = \sqrt{2}x_1 = \sqrt{2}x_3 \\ x_3 = \frac{\sqrt{2}}{2}x_2 = x_1 \end{cases} \to \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = x_2\begin{pmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix} = x_3\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\boldsymbol{eigenvectors\colon v_1 = x_1\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = x_2\begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 1 \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = x_3\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, v_2 = x_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_3 = x_1\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = x_2\begin{pmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{pmatrix} = x_3\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}}$$

$$\textbf{orthonormal basis} \to \begin{pmatrix} \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{1} \\ \boldsymbol{-\sqrt{2}} & \boldsymbol{0} & \boldsymbol{\sqrt{2}} \\ \boldsymbol{1} & \boldsymbol{-1} & \boldsymbol{1} \end{pmatrix}$$

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3. Suppose A is a matrix which is both Hermitian and unitary. What are possible eigenvalues of A?

Give an example of infinitely many such matrices.

$$Possible\ eigenvalues\ of\ A\ are\ those\ that\ are\ on\ the\ unit\ circle\ and\ are\ real\ numbers, such\ as\ \lambda = \pm 1$$

$$\boldsymbol{For\ example\colon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} or \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}}$$

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4. Show that the matrix

$$A_t = \begin{pmatrix} \mathbf{1} & \boldsymbol{t} \\ \boldsymbol{t} & \mathbf{4} \end{pmatrix}$$

is Hermitian. Find all values of t for which this matrix is positive definite.

$$\left.\begin{array}{l} A^T = \begin{pmatrix} 1 & t \\ t & 4 \end{pmatrix} = A \rightarrow symmetric \\ A^* = \begin{pmatrix} \overline{1} & \overline{t} \\ \overline{t} & \overline{4} \end{pmatrix} = \begin{pmatrix} 1 & t \\ t & 4 \end{pmatrix} = A \end{array}\right\} \rightarrow A \rightarrow Hermitian$$

$$\boldsymbol{Positive\ definite \rightarrow eigenvalues\ are\ positive}$$

$$\begin{vmatrix} \lambda - 1 & -t \\ -t & \lambda - 4 \end{vmatrix} = (\lambda - 1)(\lambda - 4) - t^2 = \boldsymbol{\lambda^2 - 5\lambda + 4 - t^2}$$

$$\boldsymbol{f(t) = \lambda^2 - 5\lambda + 4 - t^2 \rightarrow eigenvalues: \lambda_{1,2} = \frac{5 \pm \sqrt{25 - 4(4 - t^2)}}{2} = \frac{5 \pm \sqrt{9 + 4t^2}}{2}}$$

$$\boldsymbol{\frac{5 \pm \sqrt{9 + 4t^2}}{2} > 0 \rightarrow \begin{cases} \frac{5 + \sqrt{9 + 4t^2}}{2} > 0 \rightarrow \sqrt{9 + 4t^2} > -5 \rightarrow t^n, n\ is\ even \rightarrow t \in]-\infty, \infty[\\ \frac{5 - \sqrt{9 + 4t^2}}{2} > 0 \rightarrow \sqrt{9 + 4t^2} < 5 \rightarrow 9 + 4t^2 < 25 \rightarrow t^2 < 4 \rightarrow t \in]-2, 2[\end{cases}}$$

$$\boldsymbol{t \in]-2, 2[\rightarrow (for\ only\ positive\ eigenvalues)}$$

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5. Let A be the matrix given by

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{-3} & \mathbf{4} & \mathbf{0} \\ \mathbf{-3} & \mathbf{4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{pmatrix}$$

Compute the singular values of A.

$$\boldsymbol{A^T A} = \begin{pmatrix} \mathbf{-3} & \mathbf{-3} & \mathbf{0} \\ \mathbf{4} & \mathbf{4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{pmatrix} \begin{pmatrix} \mathbf{-3} & \mathbf{4} & \mathbf{0} \\ \mathbf{-3} & \mathbf{4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{18} & \mathbf{-24} & \mathbf{0} \\ \mathbf{-24} & \mathbf{32} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

$$\begin{vmatrix} t - 18 & 24 & 0 \\ 24 & t - 32 & 0 \\ 0 & 0 & t - 1 \end{vmatrix} = (t - 18)(t - 32)(t - 1) - 576(t - 1) = \boldsymbol{t^3 - 51t^2 + 50t}$$

$$\boldsymbol{f(t) = t^3 - 51t^2 + 50t = t(t - 1)(t - 50) \rightarrow eigenvalues: \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 50}$$

$$The\ singular\ values\ in\ S\ are\ square\ roots\ of\ eigenvalues\ from\ AA^T\ or\ A^T A.$$

$$They\ are\ the\ diagonal\ entries\ of\ the\ S\ matrix\ and\ are\ arranged\ in\ descending\ order.$$

$$\boldsymbol{Thus, singular\ values\ of\ A: \sigma_1 = 0, \sigma_2 = 1, \sigma_3 = 5\sqrt{2}}$$
