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Assignment 1

Calculus and Linear Algebra II

1. Compute the partial derivatives $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *for:*

(a) f (x, y) = $x^2 + xy + y^2$

- $\frac{\partial f}{\partial x} = 2x + y$
- $\frac{\partial f}{\partial y} = x + 2y$

(b) f (x, y) = $\frac{x+y}{x-y}$

- $\frac{\partial f}{\partial x} = \frac{(x-y)\frac{\partial}{\partial x}(x+y)-(x+y)\frac{\partial}{\partial x}(x-y)}{(x-y)^2} = \frac{x-y-x-y}{(x-y)^2} = \frac{-2y}{x^2-2xy+y^2}$
- $\frac{\partial f}{\partial y} = \frac{(x-y)\frac{\partial}{\partial y}(x+y)-(x+y)\frac{\partial}{\partial y}(x-y)}{(x-y)^2} = \frac{x-y+x+y}{(x-y)^2} = \frac{2x}{x^2-2xy+y^2}$

(c) f (x, y) = $x \log y$

- $\frac{\partial f}{\partial x} = \log y$
- $\frac{\partial f}{\partial y} = \frac{x}{\mathrm{y}\ \ln 10}$

(d) f (x, y) = $arctan\left(\frac{y}{x}\right)$

- $\frac{\partial f}{\partial x} = -\frac{y}{x^2\left(1+\frac{y^2}{x^2}\right)} = -\frac{y}{x^2+y^2}$
- $\frac{\partial f}{\partial y} = \frac{1}{x\left(1+\frac{y^2}{x^2}\right)} = \frac{1}{x+\frac{y^2}{x}} = \frac{x}{x^2+y^2}$

(e) f (x, y) = $\sin(x^2 + y^2 - xy)$

- $\frac{\partial f}{\partial x} = (2x - y)\cos(x^2 + y^2 - xy)$
- $\frac{\partial f}{\partial y} = (2y - x)\cos(x^2 + y^2 - xy)$

2. Use implicit differentiation to compute $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$ *for the following implicitly defined functions:*

(a) $x^2 + y^2 + z^3 = 2xyz$

- $\frac{\partial}{\partial x}[x^2 + y^2 + z^3] = \frac{\partial}{\partial x}[2xyz] => 2x + 3z^2\frac{\partial z}{\partial x} = 2yz + 2xy\frac{\partial z}{\partial x} =>$

  $3z^2\frac{\partial z}{\partial x} - 2xy\frac{\partial z}{\partial x} = 2yz - 2x => \frac{\partial z}{\partial x} = \frac{2yz - 2x}{3z^2 - 2xy}$

- $\frac{\partial}{\partial y}[x^2 + y^2 + z^3] = \frac{\partial}{\partial y}[2xyz] => 2y + 3z^2\frac{\partial z}{\partial y} = 2xz + 2xy\frac{\partial z}{\partial y} =>$

  $3z^2\frac{\partial z}{\partial y} - 2xy\frac{\partial z}{\partial y} = 2xz - 2y => \frac{\partial z}{\partial x} = \frac{2xz - 2y}{3z^2 - 2xy}$

(b) $yz = \log(x + z)$

- $\frac{\partial}{\partial x}[yz] = \frac{\partial}{\partial x}[\log(x + z)] => y\frac{\partial z}{\partial x} = \frac{1+\frac{\partial z}{\partial x}}{(x+z)\ln 10} => y\frac{\partial z}{\partial x} = \frac{1}{(x+z)\ln 10} + \frac{1}{(x+z)\ln 10}\frac{\partial z}{\partial x} =>$

  $\frac{\partial z}{\partial x} = \frac{1}{(x + z)\ln 10} \times \frac{(x + z)\ln 10}{xy\ln 10 + yz\ln 10 - 1} = \frac{1}{y\ln 10\,(x + z) - 1}$

- $\frac{\partial}{\partial y}[yz] = \frac{\partial}{\partial y}[\log(x + z)] => z + y\frac{\partial z}{\partial y} = \frac{1}{(x+z)\ln 10}\ \frac{\partial z}{\partial y} => \frac{\partial z}{\partial y} = \frac{z\ln 10(x+z)}{1-y\ln 10(x+z)}$

(c) $z - x = e^{yz}$

- $\frac{\partial}{\partial x}[z - x] = \frac{\partial}{\partial x}[e^{yz}] => \frac{\partial z}{\partial x} - 1 = ye^{yz}\frac{\partial z}{\partial x} => \frac{\partial z}{\partial x} = \frac{1}{1-ye^{yz}}$
- $\frac{\partial}{\partial y}[z - x] = \frac{\partial}{\partial y}[e^{yz}] => \frac{\partial z}{\partial y} = \left(z + y\frac{\partial z}{\partial y}\right)e^{yz} => \frac{\partial z}{\partial y} - ye^{yz}\frac{\partial z}{\partial y} = ze^{yz} =>$
  $\frac{\partial z}{\partial y} = \frac{ze^{yz}}{1-ye^{yz}}$

(d) $xyz = 1$

- $\frac{\partial}{\partial x}[xyz] = \frac{\partial}{\partial x}[1] => yz + yx\frac{\partial z}{\partial x} = 0 => \frac{\partial z}{\partial x} = -\frac{z}{x}$
- $\frac{\partial}{\partial y}[xyz] = \frac{\partial}{\partial y}[1] => xz + xy\frac{\partial z}{\partial y} = 0 => \frac{\partial z}{\partial y} = -\frac{z}{y}$

(e) $x^2y + y^2z + z^2x = 1$

- $\frac{\partial}{\partial x}[x^2y + y^2z + z^2x] = \frac{\partial}{\partial x}[1] => 2xy + y^2\frac{\partial z}{\partial x} + z^2 + 2xz\frac{\partial z}{\partial x} = 0 =>$
  $\frac{\partial z}{\partial x}(y^2 + 2xz) = -z^2 - 2xy => \frac{\partial z}{\partial x} = \frac{-z^2 - 2xy}{y^2 + 2xz} = -\frac{z^2 + 2xy}{y^2 + 2xz}$
- $\frac{\partial}{\partial y}[x^2y + y^2z + z^2x] = \frac{\partial}{\partial y}[1] => x^2 + 2yz + y^2\frac{\partial z}{\partial y} + 2xz\frac{\partial z}{\partial y} = 0 =>$
  $\frac{\partial z}{\partial y}(y^2 + 2xz) = -x^2 - 2yz => \frac{\partial z}{\partial y} = \frac{-x^2 - 2yz}{y^2 + 2xz} = -\frac{x^2 + 2yz}{y^2 + 2xz}$

3. Compute the second order partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ *and* $\frac{\partial^2 f}{\partial x^2}$ *and* $\frac{\partial^2 f}{\partial y^2}$ *for:*

(a) $f(x,y) = x^2 - y^2$

- $\frac{\partial^2}{\partial x \partial y}[f(x,y)] = \frac{\partial^2}{\partial x \partial y}[x^2 - y^2]$

1) $\frac{\partial}{\partial y}[f(x,y)] = \frac{\partial}{\partial y}[x^2 - y^2] = -2y$

2) $\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}[f(x,y)]\right] = \frac{\partial}{\partial x}[-2y] = 0$

- $\frac{\partial^2}{\partial x^2}[f(x,y)] = \frac{\partial^2}{\partial x^2}[x^2 - y^2]$

1) $\frac{\partial}{\partial x}[f(x,y)] = \frac{\partial}{\partial x}[x^2 - y^2] = 2x$

2) $\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}[f(x,y)]\right] = \frac{\partial}{\partial x}[2x] = 2$

- $\frac{\partial^2}{\partial y^2}[f(x,y)] = \frac{\partial^2}{\partial y^2}[x^2 - y^2]$

1) $\frac{\partial}{\partial y}[f(x,y)] = \frac{\partial}{\partial y}[x^2 - y^2] = -2y$

2) $\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}[f(x,y)]\right] = \frac{\partial}{\partial y}[-2y] = -2$

(b) $f(x,y) = \mathrm{x}\log(y) + \mathrm{y}\log(x)$

- $\frac{\partial^2}{\partial x \partial y}[f(x,y)] = \frac{\partial^2}{\partial x \partial y}[\mathrm{x}\log(y) + \mathrm{y}\log(x)]$

1) $\frac{\partial}{\partial y}[f(x,y)] = \frac{\partial}{\partial y}[\mathrm{x}\log(y) + \mathrm{y}\log(x)] = \frac{x}{y\ln 10} + \log(x)$

2) $\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}[f(x,y)]\right] = \frac{\partial}{\partial x}\left[\frac{x}{y\ln 10} + \log(x)\right] = \frac{1}{y\ln 10} + \frac{1}{x\ln 10}$

- $\frac{\partial^2}{\partial x^2}[f(x,y)] = \frac{\partial^2}{\partial x^2}[\mathrm{x}\log(y) + \mathrm{y}\log(x)]$

1) $\frac{\partial}{\partial x}[f(x,y)] = \frac{\partial}{\partial x}[\mathrm{x}\log(y) + \mathrm{y}\log(x)] = \log(y) + \frac{y}{x\ln 10}$

2) $\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}[f(x,y)]\right] = \frac{\partial}{\partial x}\left[\log(y) + \frac{y}{x\ln 10}\right] = -\frac{y}{x^2\ln 10}$

- $\frac{\partial^2}{\partial y^2}[f(x,y)] = \frac{\partial^2}{\partial y^2}[\mathrm{x}\log(y) + \mathrm{y}\log(x)]$

1) $\frac{\partial}{\partial y}[f(x,y)] = \frac{\partial}{\partial y}[\mathrm{x}\log(y) + \mathrm{y}\log(x)] = \frac{x}{y\ln 10} + \log(x)$

2) $\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}[f(x,y)]\right] = \frac{\partial}{\partial y}\left[\frac{x}{y\ln 10} + \log(x)\right] = -\frac{x}{y^2\ln 10}$

4. Compute the gradient vector ∇f for:

(a) $f(x, y) = x^2 + x\log y$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x + \log y \\ \frac{x}{y\ln 10} \end{bmatrix}$$

(b) $f(x, y, z) = xyz(1 + x + y + z) = xyz + x^2yz + xy^2z + xyz^2$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} yz + 2xyz + y^2z + yz^2 \\ xz + x^2z + 2xyz + xz^2 \\ xy + x^2y + xy^2 + 2xyz \end{bmatrix}$$

(c) $f(x, y) = x^2 - y^2$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$$

(d) $f(x, y) = \sin x + \sin y$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos x \\ \cos y \end{bmatrix}$$

(e) $f(x, y) = \frac{y^2}{x}$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{y^2}{x^2} \\ \frac{2y}{x} \end{bmatrix}$$

5. Consider the function:

$$f(x, y) = x^2 - xy + y^2$$

(a) Find the equation of the tangent plane to the graph of the function at the point corresponding to (x, y) = (1, 2).

Equation of the tangent plane to the surface given by z = $f(x, y)$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$z = x^2 - xy + y^2$$
$$f(x_0, y_0, z_0) = x^2 - xy + y^2 - z$$

$x = 1 \qquad y_0 = 2 \qquad z_0 = f(1, 2) = 1^2 - 1 \times 2 + 2^2 = 3$

$$a = \frac{\partial f}{\partial x}(1, 2, 3) = 2x - y = 0$$
$$b = \frac{\partial f}{\partial y}(1, 2, 3) = -x + 2y = 3$$
$$c = \frac{\partial f}{\partial z}(1, 2, 3) = -1$$

$3(y - 2) - 1(z - 3) = 0 => 3y - 6 - z + 3 = 0 => z = 3y - 3$

$$P = 3y - 3$$

(b) Determine all the points on the graph of this function where the tangent plane is parallel to the xy – plane.

$$f(x, y, z) = x^2 - xy + y^2 - z$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x - y \\ -x + 2y \\ -1 \end{bmatrix}$$

The tangent plane at (x_0, y_0, z_0) is parallel to the xy – plane if ∇f = k $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for some k.

$$\begin{bmatrix} 2x - y \\ -x + 2y \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} => \begin{matrix} k = -1 \\ z = 0 \end{matrix} => \begin{matrix} 2x - y = 0 \\ -x + 2y = 0 \end{matrix} => -x + 4x = 0 => x = 0 => y = 0$$

The only point on the graph of this function where the tangent plane is parallel to the xy – plane is: (x, y, z) = (0, 0, 0)