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Assignment 6

Calculus and Linear Algebra II

1. Find the QR decomposition for the following matrices

(a) $A = \begin{pmatrix} 6 & 2 & 2 \\ 0 & 8 & -6 \\ 0 & 6 & 8 \end{pmatrix} \rightarrow u_1 = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}, u_3 = \begin{pmatrix} 2 \\ -6 \\ 8 \end{pmatrix}$

$$\widetilde{e_1} = \mathrm{u}_1 = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \rightarrow \mathbf{e_1} = \frac{\mathbf{u_1}}{\|\mathbf{u_1}\|} = \frac{\mathbf{1}}{\mathbf{6}}\begin{pmatrix} \mathbf{6} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$$\widetilde{e_2} = \mathrm{u}_2 - \langle \mathrm{u}_2, \mathrm{e}_1 \rangle \mathrm{e}_1 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} \rightarrow \mathbf{e_2} = \frac{\widetilde{\mathbf{e_2}}}{\|\widetilde{\mathbf{e_2}}\|} = \frac{\mathbf{1}}{\mathbf{10}}\begin{pmatrix} \mathbf{0} \\ \mathbf{8} \\ \mathbf{6} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \frac{\mathbf{4}}{\mathbf{5}} \\ \frac{\mathbf{3}}{\mathbf{5}} \end{pmatrix}$$

$$\widetilde{e_3} = \mathrm{u}_3 - \langle \mathrm{u}_3, \mathrm{e}_1 \rangle \mathrm{e}_1 - \langle \mathrm{u}_3, \mathrm{e}_2 \rangle \mathrm{e}_2 = \begin{pmatrix} 2 \\ -6 \\ 8 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 0\begin{pmatrix} 0 \\ \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 8 \end{pmatrix} \rightarrow \mathbf{e_3} = \frac{\widetilde{\mathbf{e_3}}}{\|\widetilde{\mathbf{e_3}}\|} = \frac{\mathbf{1}}{\mathbf{10}}\begin{pmatrix} \mathbf{0} \\ \mathbf{-6} \\ \mathbf{8} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -\frac{\mathbf{3}}{\mathbf{5}} \\ \frac{\mathbf{4}}{\mathbf{5}} \end{pmatrix}$$

$\boldsymbol{v_1} = \langle \mathrm{u}_1, \mathrm{e}_1 \rangle \mathrm{e}_1 + \langle \mathrm{u}_1, \mathrm{e}_2 \rangle \mathrm{e}_2 + \langle \mathrm{u}_1, \mathrm{e}_3 \rangle \mathrm{e}_3 = \mathbf{6}e_1$
$\boldsymbol{v_2} = \langle \mathrm{u}_2, \mathrm{e}_1 \rangle \mathrm{e}_1 + \langle \mathrm{u}_2, \mathrm{e}_2 \rangle \mathrm{e}_2 + \langle \mathrm{u}_2, \mathrm{e}_3 \rangle \mathrm{e}_3 = \mathbf{2}e_1 + \mathbf{10}e_2$
$\boldsymbol{v_2} = \langle \mathrm{u}_3, \mathrm{e}_1 \rangle \mathrm{e}_1 + \langle \mathrm{u}_3, \mathrm{e}_2 \rangle \mathrm{e}_2 + \langle \mathrm{u}_3, \mathrm{e}_3 \rangle \mathrm{e}_3 = \mathbf{2}e_1 + \mathbf{10}e_3$

$$\boldsymbol{A = QR} \rightarrow \begin{pmatrix} \mathbf{6} & \mathbf{2} & \mathbf{2} \\ \mathbf{0} & \mathbf{6} & \mathbf{-6} \\ \mathbf{0} & \mathbf{6} & \mathbf{8} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\mathbf{4}}{\mathbf{5}} & -\frac{\mathbf{3}}{\mathbf{5}} \\ \mathbf{0} & \frac{\mathbf{3}}{\mathbf{5}} & \frac{\mathbf{4}}{\mathbf{5}} \end{pmatrix} \begin{pmatrix} \mathbf{6} & \mathbf{2} & \mathbf{2} \\ \mathbf{0} & \mathbf{10} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{10} \end{pmatrix}$$

(b) $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix} \rightarrow u_1 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\widetilde{e_1} = \mathrm{u}_1 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \rightarrow \mathbf{e_1} = \frac{\mathbf{u_1}}{\|\mathbf{u_1}\|} = \frac{\mathbf{1}}{\mathbf{3}}\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{3} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

$$\widetilde{e_2} = \mathrm{u}_2 - \langle \mathrm{u}_2, \mathrm{e}_1 \rangle \mathrm{e}_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \rightarrow \mathbf{e_2} = \frac{\widetilde{\mathbf{e_2}}}{\|\widetilde{\mathbf{e_2}}\|} = \frac{\mathbf{1}}{\mathbf{2}}\begin{pmatrix} \mathbf{0} \\ \mathbf{2} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}$$

$$\widetilde{e_3} = \mathrm{u}_3 - \langle \mathrm{u}_3, \mathrm{e}_1 \rangle \mathrm{e}_1 - \langle \mathrm{u}_3, \mathrm{e}_2 \rangle \mathrm{e}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 0\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \mathbf{e_3} = \frac{\widetilde{\mathbf{e_3}}}{\|\widetilde{\mathbf{e_3}}\|} = \mathbf{1}\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

$\boldsymbol{v_1} = \langle \mathrm{u}_1, \mathrm{e}_1 \rangle \mathrm{e}_1 + \langle \mathrm{u}_1, \mathrm{e}_2 \rangle \mathrm{e}_2 + \langle \mathrm{u}_1, \mathrm{e}_3 \rangle \mathrm{e}_3 = \mathbf{3}e_1$

$\boldsymbol{v_2} = \langle \mathrm{u}_2, \mathrm{e}_1 \rangle \mathrm{e}_1 + \langle \mathrm{u}_2, \mathrm{e}_2 \rangle \mathrm{e}_2 + \langle \mathrm{u}_2, \mathrm{e}_3 \rangle \mathrm{e}_3 = \mathbf{2}e_2$

$\boldsymbol{v_2} = \langle \mathrm{u}_3, \mathrm{e}_1 \rangle \mathrm{e}_1 + \langle \mathrm{u}_3, \mathrm{e}_2 \rangle \mathrm{e}_2 + \langle \mathrm{u}_3, \mathrm{e}_3 \rangle \mathrm{e}_3 = e_3$

$$\boldsymbol{B = QR} \rightarrow \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{3} & \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

2. Suppose Q_1 and Q_2 are orthogonal matrices (of size n × n).

(a) Show that Q_1Q_2 is also an orthogonal matrix.

$$\text{Since Q1 and Q2 are orthogonal matrices} \rightarrow {Q_1}^T Q_1 = \mathrm{I_n} \text{ and } {Q_2}^T Q_2 = \mathrm{I_n} \rightarrow \text{then}$$

$$(Q_1Q_2)^{\mathrm{T}}(Q_1Q_2) = \left({Q_2}^T {Q_1}^T\right)Q_1Q_2 \;=\; {Q_2}^T\left({Q_1}^T Q_1\right)Q_2 = {Q_2}^T Q_2 = \mathrm{I_n}$$

(b) Show that Q_1^{-1} is also an orthogonal matrix.

$$\textit{The inverse of an orthogonal matrix equal to its transpose} \rightarrow \mathrm{Q_1}^{-1} = \mathrm{Q_1}^T$$

$$\text{Since } \mathrm{Q_1} \text{ is orthogonal} \rightarrow \;\mathrm{Q_1}^T = \mathrm{I} = \;\mathrm{Q_1}^T\mathrm{Q_1}, \text{then}$$

$$\mathrm{Q_1}^T\;\mathrm{Q_1}^{TT} = \;\mathrm{Q_1}^T\mathrm{Q_1} = \mathrm{I} = \mathrm{Q_1}\;\mathrm{Q_1}^T = \;\mathrm{Q_1}^{TT}\;\mathrm{Q_1}^T$$

$$\text{So, }\; \mathrm{Q_1}^T \text{ is orthogonal, thus }\; \mathrm{Q_1}^{-1}\; \textit{is also orthogonal}$$

(c) Prove that $det\ Q \;=\; \pm 1$ and give examples of $n \times n$ orthogonal matrices where the determinant is equal to 1 and 1.

$$\textit{Since } Q \textit{ is orthogonal, then } QQ^T = I = Q^TQ \textit{ by definition}.$$

$$\textit{Using the fact that } det(AB) = \det(A)\det(B), \textit{we have}$$

$$\det(I) = 1 = \det(QQ^T) = \det(Q)\det(Q^T) = \det(Q)\det(Q) = \det(Q)^2$$

$$\textit{Since we have } det(Q)^2 = 1, \textit{then } det(Q) = \sqrt{1} = \pm 1.$$

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3. If Q is an orthogonal $\boldsymbol{n} \times \boldsymbol{n}$ matrix, and $\boldsymbol{v} \in \boldsymbol{R}^{\boldsymbol{n}}$ show that

$$\|Qv\| \;=\; \|v\|.$$

$$\textit{Let } I_n \textit{ denote an } n \times n \textit{ identity matrix}, Q \textit{ is } n \times n \textit{ orthogonal matrix}$$

$$\textit{and } v \textit{ is an } n - \textit{dimensional vector}. \rightarrow$$

$$\|Qv\|^2 \;= Qv \cdot Qv = (Qv)^TQv = v^TQ^TQv = v^T(Q^TQ)v \rightarrow$$

$$\|Qv\|^2 = v^TIv = v^Tv = v \cdot v = \|v\|^2$$

$$\textit{Since we have } \|Qv\|^2 \;=\; \|v\|^2, \textit{then } \|Qv\| \;=\; \|v\|.$$

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4. For each pair of functions $\boldsymbol{f}, \boldsymbol{g}$ as continuous functions on $[\mathbf{0}, \mathbf{2\pi}]$ determine $\langle \boldsymbol{f}, \boldsymbol{g} \rangle, \|\boldsymbol{f}\|, \|\boldsymbol{g}\|$.

(a) $f(x) = 2\sin x\,,\ \ g(x) = \cos x$

$$\langle f, g\rangle = \int_0^{2\pi} 2\sin x\cos x\ dx = \int_0^{2\pi} 2u\,du = \sin^2 x\big|_0^{2\pi} = 0 - 0 \to \langle \boldsymbol{f}, \boldsymbol{g}\rangle = \mathbf{0}$$

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_0^{2\pi} 4\sin^2 x\,dx} \to 4\int \sin^2 x\,dx = 4\left(\frac{2-1}{2}\int \sin^{2-2} x\,dx + \frac{\sin^{2-1} x\cos x}{2}\right) =$$

$$4\left(\frac{1}{2}\int 1\,dx + \frac{\sin x\cos x}{2}\right) \to \sqrt{2x + 2\sin x\cos x\big|_0^{2\pi}} = \sqrt{4\pi} \to \|\boldsymbol{f}\| = \mathbf{2}\sqrt{\boldsymbol{\pi}}$$

$$\|g\| = \sqrt{\langle g, g\rangle} = \sqrt{\int_0^{2\pi} \cos^2 x\,dx} \to \int \cos^2 x\,dx = \frac{2-1}{2}\int \cos^{2-2} x\,dx + \frac{\sin x\cos^{2-1} x}{2} =$$

$$\frac{1}{2}\int 1\,dx + \frac{\sin x\cos x}{2} \to \sqrt{\frac{x + \sin x\cos x}{2}\bigg|_0^{2\pi}} = \sqrt{\frac{2\pi}{2} - \frac{0}{2}} \to \|\boldsymbol{g}\| = \sqrt{\boldsymbol{\pi}}$$

(b) $f(x) = x,\ \ g(x) = x^2 - 1.$

$$\langle f, g\rangle = \int_0^{2\pi} (x^3 - x)\,dx = \frac{x^4}{4} - \frac{x^2}{2}\bigg|_0^{2\pi} = \frac{16\pi^4}{4} - \frac{4\pi^2}{2} - \frac{0^4}{4} + \frac{0^2}{2} = 4\pi^4 - 2\pi^2 = 2\pi^2(2\pi^2 - 1)$$

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_0^{2\pi} x^2\,dx} = \sqrt{\frac{x^3}{3}\bigg|_0^{2\pi}} = \sqrt{\frac{8\pi^3}{3}} \to \|\boldsymbol{f}\| = \mathbf{2}\boldsymbol{\pi}\sqrt{\frac{\mathbf{2}\boldsymbol{\pi}}{\mathbf{3}}}$$

$$\|g\| = \sqrt{\langle g, g\rangle} = \sqrt{\int_0^{2\pi} (x^4 - 2x^2 + 1)dx} = \sqrt{\frac{x^5}{5} - \frac{2x^3}{3} + x\bigg|_0^{2\pi}} = \sqrt{\frac{32\pi^5}{5} - \frac{16\pi^3}{3} + 2\pi} \to$$

$$\|\boldsymbol{g}\| = \sqrt{\frac{\mathbf{96}\boldsymbol{\pi}^{\mathbf{5}} - \mathbf{80}\boldsymbol{\pi}^{\mathbf{3}} + \mathbf{30}\boldsymbol{\pi}}{\mathbf{15}}}$$

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5. Let T denote the vector space spanned by the function $\mathbf{1}, \cos \boldsymbol{nx}, \sin \boldsymbol{nx}$ for $\boldsymbol{n} \geq \mathbf{1}$. That is

$$T = \left\{ c_0 + \sum_{k=1}^{N} a_k \cos kx + \sum_{k=1}^{N} b_k \sin kx : a_k, b_k \in R, N \geq 1 \right\}.$$

(a) Show that $\sin x$ and $\cos x$ are linearly independent.

Suppose that $\cos x = c \sin x$ *for all* x. *Substituting* $x = 0$ *we obtain* $\cos 0 = c \sin 0 = 0$, *which is a contradiction. Hence sin x and cos x are linearly independent.*

(b) Show that the function $\cos^2 x$ and $\cos^3 x$ both belong to T by writing them as above.

Note that

$$\cos^2 x = \frac{1 + \cos 2x}{2} = \frac{1}{2} + \frac{1}{2} \cos 2x$$

which corresponds to $c_0 = a_2 = 1\,2$ *and other coefficients equal to zero.*

Similarly

$$\cos^3 x = \cos x \cos^2 x = \frac{1}{2} \cos x (1 + \cos 2x) = \frac{1}{2} \cos x + \frac{1}{2} 2 \cos x \cos 2x.$$

Using the identity

$$\cos A \cos B = \cos(A + B) + \cos(A - B)$$

we have

$$\cos x \cos 2x = \frac{1}{2} (\cos 3x + \cos x).$$

Hence

$$\cos^3 x = \frac{1}{2} \cos x + \frac{1}{4} (\cos 3x + \cos x) = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x.$$

This corresponds to $a_1 = \frac{3}{4}, a_3 = \frac{1}{4}$ *and all other coefficients equal to* 0.

(c) Explain why the function $|\sin x|$ is not in T.

Suppose that

$$|\sin x| = c_0 + \sum_{k=1}^{N} a_k \cos kx + \sum_{k=1}^{N} b_k \sin kx .$$

Note that the right − hand side is a linear combination of differentiable functions and is hence differentiable. On the other hand, one can see that the function $|\sin x|$ *is not differentiable at* $x = 0$. *This shows that such an equality is not possible.*
