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Assignment 8

Calculus and Linear Algebra II

1. Find the value of

$$\begin{vmatrix} \mathbf{1+a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ \mathbf{a} & \mathbf{1+b} & \mathbf{c} & \mathbf{d} \\ \mathbf{a} & \mathbf{b} & \mathbf{1+c} & \mathbf{d} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{1+d} \end{vmatrix} \rightarrow$$

$$\rightarrow (1+a)\begin{vmatrix} 1+b & c & d \\ b & 1+c & d \\ b & c & 1+d \end{vmatrix} - b \times \begin{vmatrix} a & c & d \\ a & 1+c & d \\ a & c & 1+d \end{vmatrix} + c \times \begin{vmatrix} a & 1+b & d \\ a & b & d \\ a & b & 1+d \end{vmatrix} - d \times \begin{vmatrix} a & 1+b & c \\ a & b & 1+c \\ a & b & c \end{vmatrix}$$

- $\begin{vmatrix} 1+b & c & d \\ b & 1+c & d \\ b & c & 1+d \end{vmatrix} = \begin{matrix} (1+b)(1+c+d+cd-cd) - c(b+bd-bd) + d(bc-b-bc) \\ \downarrow \\ 1+c+d+b+bd+bc-bc-bd \end{matrix} = 1+b+c+d$
- $\begin{vmatrix} a & c & d \\ a & 1+c & d \\ a & c & 1+d \end{vmatrix} = \begin{matrix} a(1+c+d+cd-cd) - c(a+ad-ad) + d(ac-a-ac) \\ \downarrow \\ a+ac+ad-ac-ad \end{matrix} = a$
- $\begin{vmatrix} a & 1+b & d \\ a & b & d \\ a & b & 1+d \end{vmatrix} = \begin{matrix} a(b+bd-bd) - (1+b)(a+ad-ad) + d(ab-ab) \\ \downarrow \\ ab-a-ab \end{matrix} = -a$
- $\begin{vmatrix} a & 1+b & c \\ a & b & 1+c \\ a & b & c \end{vmatrix} = \begin{matrix} a(bc-b-bc) - (1+b)(ac-a-ac) + c(ab-ab) \\ \downarrow \\ -ab+a+ab \end{matrix} = a$

$$\det = (1+a)(1+b+c+d) - b \times a + c \times (-a) - d \times a = 1+b+c+d+a+ab+ac+ad-ab-ac-ad$$

$$\begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix} = 1+a+b+c+d$$

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2. Find the values of x for which the following matrix is invertible

$$\boldsymbol{A_x} = \begin{pmatrix} \mathbf{1} & \boldsymbol{x} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \boldsymbol{x} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} \end{pmatrix} \rightarrow detA = 1 \times \begin{vmatrix} 1 & x & 2 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} - x \times \begin{vmatrix} 0 & x & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} + 2 \times \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} - 1 \times \begin{vmatrix} 0 & 1 & x \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & x & 2 & 1 \\ 0 & 1 & x & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{vmatrix} = \begin{matrix} 1(2-4x+2) - x(-2x+2) + 2(-2+2) - 1(-1+x) \\ \downarrow \\ 2-4x+2+2x^2-2x-4+4+1-x \end{matrix} = 2x^2-7x+5$$

$$\begin{vmatrix} 1 & x & 2 & 1 \\ 0 & 1 & x & 2 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{vmatrix} \neq 0 \rightarrow 2x^2-7x+5 \neq 0 \rightarrow (x-1)(2x-5) \neq 0 \rightarrow x_1 \neq 1, x_2 \neq \frac{5}{2}$$

$$For\ all\ x \in R \backslash \left\{1, \frac{5}{2}\right\}, the\ matrix\ A_x\ is\ invertible.$$

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3. For each matrix compute adjA, detA, and when A is invertible its inverse A^{-1}.

(a) $A = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{4} & \mathbf{1} \end{pmatrix}$

- $adjA = \begin{pmatrix} \begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} \\ \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}^T = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 3 \end{pmatrix}$
- $detA = 1 \times 3 - 2 \times 0 + 0 \times 0 = 3$
- $A^{-1} = \frac{1}{detA} adjA = \frac{1}{3}\begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{4}{3} & 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$

- $adjA = \begin{pmatrix} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} \\ \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} \\ \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{pmatrix}^T = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{pmatrix}$
- $detA = 2 \times 3 - (-1) \times (-2) + 0 \times 1 = 4$
- $A^{-1} = \frac{1}{detA} adjA = \frac{1}{4}\begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix}$

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4. Determine whether the following states are true or false. If they are true provide a justification.

If they are false, provide counterexamples.

(a) If A and B are 2×2 matrices then det(A + B) = detA + det B. **False**

counterexample:

$$\left| \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} \right| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{vmatrix} = 0 + (-3)$$

$$-12 \neq -3$$

(b) If A is a 3×3 matrix whose all 9 entries are different. Then A is invertible. **False**

$$counterexample: \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \to \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(45-48) - 2(36-42) + 3(32-35) = -3 + 12 - 9 = 0$$

(c) If A is a 3×3 matrix whose all 9 entries with 7 entries equal to zero, then $\det A = 0$. **True**

$$justification: \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg) = \boldsymbol{aei - afh - bdi + bfg + cdh - ceg} \to$$

$$Since\ the\ determinant\ of\ a\ 3 \times 3\ matrix\ depends\ on\ \boldsymbol{the\ sum\ of\ products\ of\ 3\ different\ entries},$$

$$we\ need\ at\ least\ 3\ non-zero\ entries\ for\ the\ determinant\ to\ have\ a\ value\ different\ from\ zero.$$

$$Since\ in\ matrix\ A\ 7\ out\ of\ 9\ entries\ equal\ to\ 0, then\ we\ have\ only\ two\ non-zero\ entries, thus\ detA = 0.$$

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5. Find the characteristic polynomial of the following matrices. Use the characteristic polynomial to compute the eigenvalues and the corresponding eigenvectors.

(a) $\begin{pmatrix} 6 & 6 \\ 3 & -1 \end{pmatrix} \to f(t) = \begin{vmatrix} t-6 & -6 \\ -3 & t+1 \end{vmatrix} = (t-6)(t+1) - (-6)(-3) = t^2 - 5t - 24$

$$\boldsymbol{f(t) = t^2 - 5t - 24} = (t+3)(t-8) \to \boldsymbol{eigenvalues}: \boldsymbol{\lambda_1 = -3, \lambda_2 = 8}$$

$$\begin{pmatrix} 6-(-3) & 6 \\ 3 & -1-(-3) \end{pmatrix} v_1 = 0 \to \begin{pmatrix} 9 & 6 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} 9x_1 + 6y_1 = 0 \\ 3x_1 + 2y_1 = 0 \end{cases} \to \begin{cases} x_1 = -\frac{2}{3}y_1 \\ y_1 = -\frac{3}{2}x_1 \end{cases} \to \boldsymbol{v_1} = \begin{pmatrix} \boldsymbol{x_1} \\ -\boldsymbol{\frac{3}{2}x_1} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\frac{2}{3}y_1} \\ \boldsymbol{y_1} \end{pmatrix}$$

$$\begin{pmatrix} 6-8 & 6 \\ 3 & -1-8 \end{pmatrix} v_2 = 0 \to \begin{pmatrix} -2 & 6 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{cases} -2x_2 + 6y_2 = 0 \\ 3x_2 - 9y_2 = 0 \end{cases} \to \begin{cases} x_2 = 3y_2 \\ y_2 = \frac{1}{3}x_2 \end{cases} \to \boldsymbol{v_2} = \begin{pmatrix} \boldsymbol{x_2} \\ \boldsymbol{\frac{1}{3}x_2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{3y_2} \\ \boldsymbol{y_2} \end{pmatrix}$$

$$\boldsymbol{eigenvectors}: \boldsymbol{v_1 = x_1}\begin{pmatrix} \boldsymbol{1} \\ -\boldsymbol{\frac{3}{2}} \end{pmatrix} = \boldsymbol{y_1}\begin{pmatrix} -\boldsymbol{\frac{2}{3}} \\ \boldsymbol{1} \end{pmatrix}, \boldsymbol{v_2 = x_2}\begin{pmatrix} \boldsymbol{1} \\ \boldsymbol{\frac{1}{3}} \end{pmatrix} = \boldsymbol{y_2}\begin{pmatrix} \boldsymbol{3} \\ \boldsymbol{1} \end{pmatrix}$$

(b) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \to f(t) = \begin{vmatrix} t-1 & -1 & 0 \\ -2 & t & 0 \\ 0 & 0 & t-3 \end{vmatrix} = (t-1)(t^2 - 3t) - 2t + 6 = t^3 - 4t^2 + t + 6$

$$\boldsymbol{f(t) = t^3 - 4t^2 + t + 6} = (t+1)(t-2)(t-3) \to \boldsymbol{eigenvalues}: \boldsymbol{\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 3}$$

$$\begin{pmatrix} 1+1 & 1 & 0 \\ 2 & 0+1 & 0 \\ 0 & 0 & 3+1 \end{pmatrix} v_1 = 0 \to \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{cases} 2x_1 + y_1 = 0 \\ 2x_1 + y_1 = 0 \\ 4z_1 = 0 \end{cases} \to \begin{cases} x_1 = -\frac{1}{2}y_1 \\ y_1 = -2x_1 \\ z_1 = 0 \end{cases} \to \boldsymbol{v_1} = \begin{pmatrix} \boldsymbol{x_1} \\ -\boldsymbol{2x_1} \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\frac{1}{2}y_1} \\ \boldsymbol{y_1} \\ \boldsymbol{0} \end{pmatrix}$$

$$\begin{pmatrix}1-2 & 1 & 0\\ 2 & 0-2 & 0\\ 0 & 0 & 3-2\end{pmatrix}v_2=0\rightarrow\begin{pmatrix}-1 & 1 & 0\\ 2 & -2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x_2\\ y_2\\ z_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\rightarrow\begin{cases}-x_2+y_2=0\\ 2x_2-2y_2=0\\ z_2=0\end{cases}\rightarrow\begin{cases}x_2=y_2\\ y_2=x_2\\ z_2=0\end{cases}\rightarrow\boldsymbol{v_2}=\begin{pmatrix}\boldsymbol{x_2}\\ \boldsymbol{x_2}\\ \mathbf{0}\end{pmatrix}=\begin{pmatrix}\boldsymbol{y_2}\\ \boldsymbol{y_2}\\ \mathbf{0}\end{pmatrix}$$

$$\begin{pmatrix}1-3 & 1 & 0\\ 2 & 0-3 & 0\\ 0 & 0 & 3-3\end{pmatrix}v_3=0\rightarrow\begin{pmatrix}-2 & 1 & 0\\ 2 & -3 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_3\\ y_3\\ z_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\rightarrow\begin{cases}-2x_3+y_3=0\\ 2x_3-3y_3=0\\ 0=0\end{cases}\rightarrow\begin{cases}x_3=0\\ y_3=0\\ z_3\in R\end{cases}\rightarrow\boldsymbol{v_3}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \boldsymbol{z_3}\end{pmatrix}$$

$$\boldsymbol{eigenvectors}:\boldsymbol{v_1}=\boldsymbol{x_1}\begin{pmatrix}\mathbf{1}\\ \mathbf{-2}\\ \mathbf{0}\end{pmatrix}=\boldsymbol{y_1}\begin{pmatrix}-\frac{\mathbf{1}}{\mathbf{2}}\\ \mathbf{1}\\ \mathbf{0}\end{pmatrix},\boldsymbol{v_2}=\boldsymbol{x_2}\begin{pmatrix}\mathbf{1}\\ \mathbf{1}\\ \mathbf{0}\end{pmatrix}=\boldsymbol{y_2}\begin{pmatrix}\mathbf{1}\\ \mathbf{1}\\ \mathbf{0}\end{pmatrix},\boldsymbol{v_3}=\boldsymbol{z_3}\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{1}\end{pmatrix}$$

(c) $\begin{pmatrix}0&0&0&0\\0&0&2&0\\0&2&0&0\\1&0&0&0\end{pmatrix}\rightarrow f(t)=\begin{vmatrix}t&0&0&0\\0&t&-2&0\\0&-2&t&0\\-1&0&0&t\end{vmatrix}=t(t(t^2)+2(-2t))=t^4-4t^2$

$$\boldsymbol{f(t)}=\boldsymbol{t^4}-\boldsymbol{4t^2}=t^2(t^2-4)\rightarrow\boldsymbol{eigenvalues}:\boldsymbol{\lambda_1}=\mathbf{0},\boldsymbol{\lambda_2}=\mathbf{2},\boldsymbol{\lambda_3}=\mathbf{-2}$$

$$\begin{pmatrix}0&0&0&0\\0&0&2&0\\0&2&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\y_1\\z_1\\u_1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\rightarrow\begin{cases}0=0\\2z_1=0\\2y_1=0\\x_1=0\end{cases}\rightarrow\begin{cases}x_1=0\\y_1=0\\z_1=0\\u_1\in R\end{cases}\rightarrow\boldsymbol{v_1}=\begin{pmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\boldsymbol{u_1}\end{pmatrix}$$

$$\begin{pmatrix}-2&0&0&0\\0&-2&2&0\\0&2&-2&0\\1&0&0&-2\end{pmatrix}\begin{pmatrix}x_2\\y_2\\z_2\\u_2\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\rightarrow\begin{cases}-2x_2=0\\-2y_2+2z_2=0\\2y_2-2z_2=0\\x_2-2u_2=0\end{cases}\rightarrow\begin{cases}x_2=0\\y_2=z_2\\z_2=y_2\\u_2=0\end{cases}\rightarrow\boldsymbol{v_2}=\begin{pmatrix}\mathbf{0}\\\boldsymbol{y_2}\\\boldsymbol{y_2}\\\mathbf{0}\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\\boldsymbol{z_2}\\\boldsymbol{z_2}\\\mathbf{0}\end{pmatrix}$$

$$\begin{pmatrix}2&0&0&0\\0&2&2&0\\0&2&2&0\\1&0&0&2\end{pmatrix}\begin{pmatrix}x_3\\y_3\\z_3\\u_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\rightarrow\begin{cases}x_3=0\\y_3+z_3=0\\y_3+z_3=0\\x_3+2u_3=0\end{cases}\rightarrow\begin{cases}x_3=0\\y_3=-z_3\\z_3=-y_3\\u_3=0\end{cases}\rightarrow\boldsymbol{v_3}=\begin{pmatrix}\mathbf{0}\\\boldsymbol{y_3}\\-\boldsymbol{y_3}\\\mathbf{0}\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\-\boldsymbol{z_3}\\\boldsymbol{z_3}\\\mathbf{0}\end{pmatrix}$$

$$\boldsymbol{eigenvectors}:\boldsymbol{v_1}=\boldsymbol{u_1}\begin{pmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{1}\end{pmatrix},\boldsymbol{v_2}=\boldsymbol{y_2}\begin{pmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{1}\\\mathbf{0}\end{pmatrix}=\boldsymbol{z_2}\begin{pmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{1}\\\mathbf{0}\end{pmatrix},\boldsymbol{v_3}=\boldsymbol{y_3}\begin{pmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{-1}\\\mathbf{0}\end{pmatrix}=\boldsymbol{z_3}\begin{pmatrix}\mathbf{0}\\\mathbf{-1}\\\mathbf{1}\\\mathbf{0}\end{pmatrix}$$
