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Assignment 3

Calculus and Linear Algebra II

1. Find all the critical points of the following functions and classify them:

(a) $f(x, y) = x^4 + y^4 - 4xy + 2$

$\frac{\partial f}{\partial x} = 4x^3 - 4y = 0 \quad \frac{\partial f}{\partial y} = 4y^3 - 4x = 0 \quad \rightarrow \quad y = x^3$

$4x(x^8 - 1) = 0 \rightarrow \boldsymbol{x_1 = 0},\ \boldsymbol{x_2 = 1},\ \boldsymbol{x_3 = -1}$ *and* $\boldsymbol{y_1 = 0},\ \boldsymbol{y_2 = 1},\ \boldsymbol{y_3 = -1}$

$\frac{\partial^2 f}{\partial x^2}(x, y) = 12x^2 \quad \frac{\partial^2 f}{\partial y^2}(x, y) = 12y^2 \quad \frac{\partial^2 f}{\partial x \partial y}(x, y)^2 = -4$

$\boldsymbol{P_1(0, 0)} \rightarrow D_2 = (0)(0) - (-4)^2 = -16 < 0 \rightarrow \boldsymbol{saddle\ point}$

$\boldsymbol{P_2(1, 1)} \rightarrow D_2 = (12)(12) - (-4)^2 = 128 > 0\ and\ \frac{\partial^2 f}{\partial x^2} > 0 => \boldsymbol{local\ minimum}$

$\boldsymbol{P_2(-1, -1)} \rightarrow D_2 = (12)(12) - (-4)^2 = 128 > 0\ and\ \frac{\partial^2 f}{\partial x^2} > 0 => \boldsymbol{local\ minimum}$

(b) $f(x, y) = (1 - xy)(x + y) = x + y - x^2y - xy^2$

$\frac{\partial f}{\partial x} = 1 - 2xy - y^2 = 0 \quad \frac{\partial f}{\partial y} = 1 - x^2 - 2xy = 0 \quad \rightarrow \quad 1 - x^2 = 1 - y^2 \rightarrow x^2 = y^2$

2 *cases*:
- $x = y \rightarrow 1 - 2y^2 - y^2 = 0 \rightarrow y = \sqrt{\frac{1}{3}} \rightarrow \boldsymbol{x_1 = y_1 = \frac{1}{\sqrt{3}}}$ *and* $\boldsymbol{x_2 = y_2 = -\frac{1}{\sqrt{3}}}$
- $x \neq y \rightarrow complex\ solutions$

$\frac{\partial^2 f}{\partial x^2}(x, y) = -2y \quad \frac{\partial^2 f}{\partial y^2}(x, y) = -2x \quad \frac{\partial^2 f}{\partial x \partial y}(x, y)^2 = -2x - 2y$

$\boldsymbol{P_1}\left(\sqrt{\boldsymbol{\frac{1}{3}}}, \sqrt{\boldsymbol{\frac{1}{3}}}\right) \rightarrow D_1 = \left(-2\sqrt{\frac{1}{3}}\right)\left(-2\sqrt{\frac{1}{3}}\right) - \left(-2\sqrt{\frac{1}{3}} - 2\sqrt{\frac{1}{3}}\right)^2 = \left(-\frac{12}{3}\right) < 0 \rightarrow \boldsymbol{saddle\ point}$

$\boldsymbol{P_2}\left(-\sqrt{\boldsymbol{\frac{1}{3}}}, -\sqrt{\boldsymbol{\frac{1}{3}}}\right) \rightarrow D_2 = \left(2\sqrt{\frac{1}{3}}\right)\left(2\sqrt{\frac{1}{3}}\right) - \left(2\sqrt{\frac{1}{3}} + 2\sqrt{\frac{1}{3}}\right)^2 = \left(-\frac{12}{3}\right) < 0 \rightarrow \boldsymbol{saddle\ point}$

(c) $f(x, y) = e^y(x^2 - y^2) = x^2e^y - y^2e^y$

$\frac{\partial f}{\partial x} = 2xe^y = 0 \rightarrow \boldsymbol{x_1 = x_2 = 0} \quad \frac{\partial f}{\partial y} = x^2e^y - 2ye^y - y^2e^y = 0 \rightarrow -2ye^y - y^2e^y = 0$

$-ye^y(2 + y) = 0 \rightarrow \boldsymbol{y_1 = 0},\ \boldsymbol{y_2 = -2}$

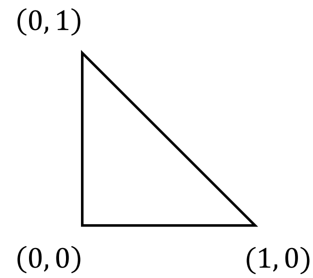
$\frac{\partial^2 f}{\partial x^2}(x, y) = 2e^y \quad \frac{\partial^2 f}{\partial y^2}(x, y) = x^2e^y - 2e^y - 4ye^y - y^2e^y \quad \frac{\partial^2 f}{\partial x \partial y}(x, y)^2 = 2xe^y$

$\boldsymbol{P_1(0, 0)} \rightarrow D_1 = (2)(-2) - (0)^2 = (-4) < 0 \rightarrow \boldsymbol{saddle\ point}$

$\boldsymbol{P_2(0, -2)} \rightarrow D_2 = \left(\frac{2}{e^2}\right)\left(-\frac{2}{e^2} + \frac{8}{e^2} - \frac{4}{e^2}\right) - (0)^2 = \frac{4}{e^4} > 0\ and\ \frac{\partial^2 f}{\partial x^2} > 0 => \boldsymbol{local\ minimum}$

2. Determine the absolute maximum and minimum value of f on the set D for

(a) $f(x,y) = 1 + 3x + 2y$, D is the closed triangular region with vertices $(0,0), (1,0), (0,1)$.



$\frac{\partial f}{\partial x} = 3 \neq 0 \qquad \frac{\partial f}{\partial y} = 2 \neq 0 \qquad \rightarrow\ no\ critical\ points$

$f(0{,}0) = 1 \qquad f(1{,}0) = 1 + 3 = 4 \qquad f(0{,}1) = 1 + 2 = 3$

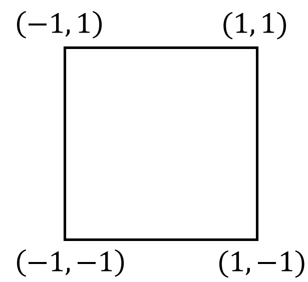
$y = 0 \rightarrow 3x + 1,\ 0 \leq x \leq 1 \rightarrow max = \mathbf{4}, min = \mathbf{1}$

$x = 0 \rightarrow 2y + 1,\ 0 \leq y \leq 1 \rightarrow max = \mathbf{3}, min = \mathbf{1}$

$y = -x + 1 \rightarrow x + 3,\ 0 \leq x \leq 1 \rightarrow max = \mathbf{4}, min = \mathbf{3}$

$Absolute\ maximum\ and\ minimum\ value\ of\ f\ on\ the\ set\ D\ are\ respectively\ 4\ and\ 1$

(b) $f(x,y) = x^2 + y^2 + 1$, $D = \{(x,y) :\ |x| \leq 1, |y| \leq 1\} \rightarrow -1 \leq x \leq 1, -1 \leq y \leq 1$



$\frac{\partial f}{\partial x} = 2x = 0 \qquad \frac{\partial f}{\partial y} = 2y = 0 \qquad \rightarrow\ critical\ point \rightarrow (0{,}0)$

$f(0{,}0) = \mathbf{1} \quad f(-1,-1) = 3 \quad f(-1{,}1) = 3 \quad f(1,-1) = 3 \quad f(1{,}1) = 3$

$x = -1 \rightarrow y^2 + 2, -1 \leq y \leq 1 \rightarrow max/min = \mathbf{3}$

$x = 1 \rightarrow y^2 + 2, -1 \leq y \leq 1 \rightarrow max/min = \mathbf{3}$

$y = -1 \rightarrow x^2 + 2, -1 \leq x \leq 1 \rightarrow max/min = \mathbf{3}$

$y = 1 \rightarrow x^2 + 2, -1 \leq x \leq 1 \rightarrow max/min = \mathbf{3}$

$Absolute\ maximum\ and\ minimum\ value\ of\ f\ on\ the\ set\ D\ are\ respectively\ 3\ and\ 1$

(c) $f(x,y) = x^3 - 3xy^2, D = \{(x,y): x^2 + y^2 \leq 1\} \rightarrow g(x) = x^2 + y^2 - 1 = 0.$

$\frac{\partial f}{\partial x} = 3x^2 - 3y^2 = 0 \ \rightarrow\ (\mathrm{x} + \mathrm{y})(\mathrm{x} - \mathrm{y}) = 0 \qquad \frac{\partial f}{\partial y} = -6xy = 0$

$\begin{array}{l} (\mathrm{x} + \mathrm{y}) = 0 \rightarrow \mathrm{x} = -\mathrm{y} \rightarrow 6\mathrm{y}\,2 = 0 \rightarrow \mathrm{y} = \mathrm{x} = 0 \\ (\mathrm{x} - \mathrm{y}) = 0 \rightarrow \mathrm{x} = \mathrm{y} \rightarrow -6\mathrm{y}\,2 = 0 \rightarrow \mathrm{y} = \mathrm{x} = 0 \end{array} \rightarrow \textbf{critical point} \ \rightarrow\ (\mathbf{0}, \mathbf{0}) \rightarrow f(0{,}0) = \mathbf{0}$

$\left.\begin{array}{ll} \frac{\partial f}{\partial x} = 3x^2 - 3y^2 & \frac{\partial g}{\partial x} = 2x \\ \frac{\partial f}{\partial y} = -6xy & \frac{\partial g}{\partial x} = 2y \end{array}\right\} \rightarrow \begin{array}{c} 3x^2 - 3y^2 = \lambda \times 2x \\ -6xy = \lambda \times 2y \end{array} \rightarrow \begin{array}{l} \lambda = \frac{3x^2 - 3y^2}{2x} \\ \lambda = \frac{-6xy}{2y} \end{array} \rightarrow \begin{array}{c} \lambda = \frac{3x}{2} - \frac{3y^2}{2x} \\ \lambda = -3x \end{array} \rightarrow \frac{3x}{2} - \frac{3y^2}{2x} = -3x$

$3x^2 - 3y^2 = -6x^2 \rightarrow y^2 = 3x^2 \quad \rightarrow \quad x^2 + y^2 - 1 = 0 \rightarrow x^2 + 3x^2 - 1 = 0 \rightarrow x^2 = \frac{1}{4} \rightarrow$

$\boldsymbol{x_{\frac{1}{2}} = \pm\frac{1}{2} \quad y_{\frac{1}{2}} = \pm\frac{\sqrt{3}}{2}} \quad \rightarrow \quad \left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right), \left(-\frac{\mathbf{1}}{\mathbf{2}}, \frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right), \left(\frac{\mathbf{1}}{\mathbf{2}}, -\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right)\ \boldsymbol{and}\ \left(-\frac{\mathbf{1}}{\mathbf{2}}, -\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right)$

$f\left(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right) = \frac{1}{8} - \frac{9}{8} = -\mathbf{1} \qquad f\left(-\frac{\mathbf{1}}{\mathbf{2}}, \frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right) = -\frac{1}{8} + \frac{9}{8} = \mathbf{1}$

$f\left(\frac{\mathbf{1}}{\mathbf{2}}, -\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right) = \frac{1}{8} - \frac{9}{8} = -\mathbf{1} \qquad f\left(-\frac{\mathbf{1}}{\mathbf{2}}, -\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\right) = -\frac{1}{8} + \frac{9}{8} = \mathbf{1}$

$Absolute\ maximum\ and\ minimum\ value\ of\ f\ on\ the\ set\ D\ are\ respectively\ 1\ and\ -1$

3. Determine all critical points of the function

$$f(x,y) = x^2 - 2xy + y^2 + 1$$

Show that there are infinitely many of them and D = 0 at each one. Without using the second derivative test (which cannot be applied) show that f has a minimum at each critical point.

$$\frac{\partial f}{\partial x} = 2x - 2y = 0 \qquad \frac{\partial f}{\partial y} = 2y - 2x = 0 \qquad \rightarrow \qquad x = y \rightarrow (\boldsymbol{infinite\ values}) \rightarrow$$
$$(infinite\ critical\ points\ as\ long\ as\ the\ x\ and\ y\ values\ are\ equal)$$

$$f(x,x) = f(y,y) = x^2 - 2x^2 + x^2 + 1 = y^2 - 2y^2 + y^2 + 1 = \mathbf{1} \rightarrow (\boldsymbol{constant\ value})$$
$$\rightarrow (the\ infinite\ critical\ points\ have\ the\ same\ extreme\ value)$$

- $This\ is\ a\ minimum, because\ it\ is\ the\ lowest\ value\ our\ function\ can\ take - Proof:$

$$Factorize\ the\ function \rightarrow f(x,y) = (x-y)^2 + 1 \rightarrow (x-y)^2\ is\ always \geq 0,$$

$$therefore\ if\ we\ substitute\ the\ lowest\ value\ 0\ in\ (x-y)^2 \rightarrow f(x,y) = 0 + 1 = \mathbf{1}$$

$$=> Thus, the\ function\ has\ a\ minimum\ (f = 1) at\ each\ critical\ point\ (infinte\ of\ them)$$

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4. Find the point on the surface defined by

$$z^2 = 9 + xy$$

that is closest to the origin (0, 0, 0).

$$distance = \sqrt{(x-0)^2 + (y-0)^2 + \left(\sqrt{9+xy} - 0\right)^2}$$

$$\Delta(x,y) = x^2 + y^2 + 9 + xy$$

$$\left\{\begin{matrix} \dfrac{\partial \Delta}{\partial x} = 2x + y = 0 \\ \dfrac{\partial \Delta}{\partial y} = 2y + x = 0 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} y = -2x \\ -4x + x = 0 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} y = 0 \\ x = 0 \end{matrix}\right\} \rightarrow z^2 = 9 \rightarrow z = \pm 3$$

$$Points\ closest\ to\ origin \rightarrow \boldsymbol{P_1(0,0,3)}\ \boldsymbol{and}\ \boldsymbol{P_1(0,0,-3)}$$

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5. This problem deals with a counter-intuitive phenomenon for functions of several variables.

(a) Suppose $f : R \rightarrow R$ is differentiable and has only one critical point at c and $f''(c) > 0$, hence c is a local minimum for f . Show that c is indeed a global minimum, that is, for every $x \in R$, we have $f(x) \geq f(c)$.

- Since the function has only one critical point ($f'(c) = 0$) which is a minimum, it means $f'(x)$ for every other $x \in R$ will be ≥ 0. If the point at c is not a global minimum, it means there has to be a second point where $f(x) \leq f(c)$. For this to happen the value of the function has to decrease, meaning $f'(x) < 0$. However as this doesn't happen, (at both sides of the point at c the first derivative is positive), f(c) is the lowest value the function can take, meaning the point at c is a global minimum.

(b) Consider the function $f : R^2 \rightarrow R$ defined by

$$f(x,y) = x^2 + y^2(1-x)^3 = x^2 + y^2 - 3xy^2 + 3x^2y^2 - x^3y^2$$

Show that f has only one critical point at (0, 0).

$$\frac{\partial f}{\partial x} = 2x - 3y^2 + 6xy^2 - 3x^2y^2 = 0 \qquad \frac{\partial f}{\partial y} = 2y - 6xy + 6x^2y - 2x^3y = -2y(x-1)^3 = 0$$

$$\begin{matrix} 2x - 3y^2 + 6xy^2 - 3x^2y^2 = 0 \\ -2y(x-1)^3 = 0 \end{matrix} \rightarrow \begin{matrix} y = 0 \rightarrow 3y^2 - 6y^2 + 3y^2 = -2 \rightarrow x = \emptyset \rightarrow no\ solution \\ x = 1 \rightarrow -3y^2 = 0 \rightarrow y = 0 \rightarrow only\ critical\ point\ (0,0) \end{matrix}$$

(c) Show that (0, 0) is a local minimum for f.

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2 + 6y^2(1-x) \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = -2(x-1)^3 \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y)^2 = -6y(x-1)^2$$

$$D(0,0) = (2)(2) - (0)^2 = 4 > 0\ and \frac{\partial^2 f}{\partial x^2} > 0 => local\ minimum$$

(d) Show that $(0,0)$ is not a global (or absolute) minimum for f by finding values of (x,y) such that $f(x,y) < f(0,0)$. It's a nice exercise to try to imagine how this can happen.

$Take\ a\ point\ with\ coordinates\ (x,y) = (-6,0)\ and\ substitute\ to\ calculate\ value\ of\ function$:

$$f(0,0) = 0 \quad f(4,1) = 4^2 + 1^2(1-4)^3 = -11 < 0 \rightarrow f(4,1) < f(0,0)$$