Jona Bako
j.bako@jacobs-university.de

Due: April 20, 2021
Assignment 7

Calculus and Linear Algebra II

1. Consider the function

$$\boldsymbol{f(x) = \begin{cases} 0 & if \ -\pi < x < 0 \\ \sin x & if \quad 0 < x < \pi \end{cases}}$$

Find the Fourier series for f.

- $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} \sin x\ dx\right) = \frac{1}{2\pi}\left((-\cos x)|_0^{\pi}\right) = \frac{1}{2\pi}(2) = \boldsymbol{\frac{1}{\pi}}$

- $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \mathrm{dx} = \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx\ \mathrm{dx}$
  $*\sin(x)\cos(y) = \frac{1}{2}(\sin(y+x) - \sin(y-x))* \to a_n = \frac{1}{\pi}\int_0^{\pi}\frac{(\sin((n+1)x) - \sin((n-1)x))}{2}\ \mathrm{dx}$
  $a_n = \frac{1}{\pi}\left(\frac{1}{2}\int(\sin((n+1)x)\ \mathrm{dx} - \frac{1}{2}\int(\sin((n-1)x)\ \mathrm{dx}\right)\Big|_0^{\pi} \to *u = (n\pm1)x \to \frac{du}{dx} = n\pm1 \to dx = \frac{1}{n\pm1}du*$
  $a_n = \frac{1}{\pi}\left(\frac{1}{2n+2}\int(\sin(u)\ \mathrm{du} - \frac{1}{2n-2}\int(\sin(u)\ \mathrm{du}\right)\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{1}{2n+2}\int(\sin(u)\ \mathrm{du} - \frac{1}{2n-2}\int(\sin(u)\ \mathrm{du}\right)\Big|_0^{\pi}$
  $a_n = \frac{1}{\pi}\left(\frac{\cos((n-1)x)}{2n-2} - \frac{\cos((n+1)x)}{2n+2}\right)\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{n\cos((n-1)x)+\cos((n-1)x)-n\cos((n+1)x)+\cos((n+1)x)}{2(n^2-1)}\right)\Big|_0^{\pi}$
  $*\ \sin(x)\sin(y) = \frac{1}{2}(\cos(y-x) - \cos(y+x))\ *\ and\ *\ \cos(x)\cos(y) = \frac{1}{2}(\cos(y+x) + \cos(y-x))\ *$
  $a_n = \frac{1}{\pi}\left(\frac{n(\cos((n-1)x)-\cos((n+1)x))}{2(n^2-1)} + \frac{\cos((n+1)x)+\cos((n-1)x)}{2(n^2-1)}\right)\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{n\sin x\sin nx + \cos x\cos nx}{(n^2-1)}\right)\Big|_0^{\pi} \to$
  $a_n = \frac{1}{\pi}\left(\frac{n\sin\pi\sin n\pi + \cos\pi\cos n\pi}{(n^2-1)} - \frac{n\sin 0\sin 0 + \cos 0\cos 0}{(n^2-1)}\right) = \frac{1}{\pi}\left(\frac{-\cos n\pi - 1}{(n^2-1)}\right) = -\boldsymbol{\frac{\cos n\pi + 1}{\pi(n^2-1)}}$

- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \mathrm{dx} = \frac{1}{\pi}\int_0^{\pi}\sin x\sin nx\ \mathrm{dx}$
  $*\ \sin(x)\sin(y) = \frac{1}{2}(\cos(y-x) - \cos(y+x))\ * \to b_n = \frac{1}{\pi}\int_0^{\pi}\frac{(\cos((n-1)x) - \cos((n+1)x))}{2}\ \mathrm{dx}$
  $b_n = \frac{1}{\pi}\left(\frac{1}{2}\int(\cos((n-1)x)\ \mathrm{dx} - \frac{1}{2}\int(\cos((n+1)x)\ \mathrm{dx}\right)\Big|_0^{\pi} \to *u = (n\pm1)x \to \frac{du}{dx} = n\pm1 \to dx = \frac{1}{n\pm1}du*$
  $b_n = \frac{1}{\pi}\left(\frac{1}{2n-2}\int(\sin(u)\ \mathrm{du} - \frac{1}{2n+2}\int(\sin(u)\ \mathrm{du}\right)\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{1}{2n-2}\int(\cos(u)\ \mathrm{du} - \frac{1}{2n+2}\int(\cos(u)\ \mathrm{du}\right)\Big|_0^{\pi}$
  $b_n = \frac{1}{\pi}\left(\frac{\sin((n-1)x)}{2n-2} - \frac{\sin((n+1)x)}{2n+2}\right)\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{n\sin((n-1)x)+\sin((n-1)x)-n\sin((n+1)x)+\sin((n+1)x)}{2(n^2-1)}\right)\Big|_0^{\pi}$
  $*\ \sin(x)\cos(y) = \frac{1}{2}(\sin(y+x) - \sin(y-x))*\ and\ *\ \cos(x)\sin(y) = \frac{1}{2}(sin(y+x) + \sin(y-x))\ *$
  $b_n = \frac{1}{\pi}\left(\frac{-n(\sin((n+1)x)-\sin((n-1)x))}{2(n^2-1)} + \frac{\sin((n+1)x)+\sin((n-1)x)}{2(n^2-1)}\right)\Big|_0^{\pi} = \frac{1}{\pi}\left(\frac{\cos x\sin nx - n\sin x\cos nx}{(n^2-1)}\right)\Big|_0^{\pi} \to$
  $b_n = \frac{1}{\pi}\left(\frac{\cos\pi\sin n\pi - n\sin\pi\cos n\pi}{(n^2-1)} - \frac{\cos 0\sin 0 - n\sin 0\cos 0}{(n^2-1)}\right) = \frac{1}{\pi}\left(\frac{-\sin n\pi}{(n^2-1)}\right) = -\boldsymbol{\frac{-\sin n\pi}{\pi(n^2-1)}}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \to \mathbf{f(x)} = \boldsymbol{\frac{1}{\pi} + \sum_{n=1}^{\infty}\frac{\cos n\pi\,(1-\cos n\pi\,)}{\pi(n^2-1)} + \sum_{n=1}^{\infty}\frac{\sin^2 n\pi}{\pi(n^2-1)}}$$

---

2. Use the formula $e^{ix} = \cos x + i \sin x$ to show that

$$\cos 3x = 4\cos^3 x - 3\cos x$$

Proof:

$$e^{ix} = \cos x + i\sin x \rightarrow e^{3ix} = (\cos x + i\sin x)^3 \rightarrow \cos 3x + i\sin 3x = (\cos x + i\sin x)^3$$

$$(\cos x + i\sin x)^3 = \cos^3 x + 3i\sin x\cos^2 x - 3\sin^2 x\cos x - i\sin^3 x$$

$$\cos 3x + (\sin 3x)i = \cos^3 x - 3\sin^2 x\cos x + (3\sin x\cos^2 x - \sin^3 x)i$$

$$\cos 3x = \cos^3 x - 3\sin^2 x\cos x = \cos^3 x - 3\cos x\,(1 - \cos^2 x) = \cos^3 x - 3\cos x + 3\cos^3 x$$

$$\rightarrow \mathbf{\cos 3x = 4\cos^3 x - 3\cos x}$$

---

3. Suppose f is an even function, i.e. f(-x) = f(x). Show that in the Fourier expansion of f

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

we have $b_n = 0$ for all $n \geq 1$.

Proof:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ \mathrm{dx} \rightarrow \sin nx \ is\ an\ odd\ function, and\ f(x)\ is\ an\ even\ function.$$

$$The\ product\ of\ an\ even\ and\ odd\ function, is\ going\ to\ be\ odd\ function, and$$

$$the\ integral\ of\ an\ odd\ function\ over\ a\ symmetric\ interval, is\ going\ to\ be\ 0.$$

$$Therefore: b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \ \mathrm{dx} = 0$$

---

4. Determine the sign of each one of the following permutations. Show the steps in your computation.

(a) $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix} \rightarrow \underset{2\ flips}{\begin{pmatrix} 3 & 1 & 2 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}} \rightarrow \underset{2\ flips}{\begin{pmatrix} 3 & 1 & 5 & 2 & 4 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}} \rightarrow \underset{1\ flip}{\begin{pmatrix} 3 & 1 & 5 & 4 & 2 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}} \rightarrow \begin{matrix} total = 5\ flips \\ sgn\ \sigma_1 = -1\ (odd) \end{matrix}$

(b) $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 4 & 2 \end{pmatrix} \rightarrow \underset{4\ flips}{\begin{pmatrix} 5 & 1 & 2 & 3 & 4 & 6 \\ 5 & 3 & 1 & 6 & 4 & 2 \end{pmatrix}} \rightarrow \underset{2\ flips}{\begin{pmatrix} 5 & 3 & 1 & 2 & 4 & 6 \\ 5 & 3 & 1 & 6 & 4 & 2 \end{pmatrix}} \rightarrow \underset{2\ flips}{\begin{pmatrix} 5 & 3 & 1 & 6 & 2 & 4 \\ 5 & 3 & 1 & 6 & 4 & 2 \end{pmatrix}} \rightarrow$

$\underset{1\ flip}{\begin{pmatrix} 5 & 3 & 1 & 6 & 4 & 2 \\ 5 & 3 & 1 & 6 & 4 & 2 \end{pmatrix}} \rightarrow \begin{matrix} total = 9\ flips \\ sgn\ \sigma_1 = -1\ (odd) \end{matrix}$

---

5. Determine the value of following determinants. Show the steps in your computation.

(a) $\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 3 \\ 1 & 5 & 8 \end{vmatrix} = \underset{\substack{\downarrow \\ 19}}{(1 \times 2 \times 8 + 0 \times 5 \times 3 + 1 \times 3 \times 1)} - \underset{\substack{\downarrow \\ 21}}{(3 \times 2 \times 1 + 3 \times 5 \times 1 - 1 \times 0 \times 8)} = -2$

(b) $\begin{vmatrix} 1 & 1 & 3 \\ 0 & 4 & 6 \\ 1 & 5 & 9 \end{vmatrix} = \underset{\substack{\downarrow \\ 42}}{(1 \times 4 \times 9 + 0 \times 5 \times 3 + 1 \times 6 \times 1)} - \underset{\substack{\downarrow \\ 42}}{(3 \times 4 \times 1 + 6 \times 5 \times 1 - 1 \times 0 \times 9)} = 0$

(c) $\begin{vmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{vmatrix} = 0 \times \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} - a \times \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & c \\ d & 0 & 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ d & 0 & 0 \end{bmatrix} - 0 \times \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ d & 0 & 0 \end{bmatrix}$

$\begin{vmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{vmatrix} = -a(\underset{\substack{\downarrow \\ bcd}}{(0 \times 0 \times 0 + 0 \times 0 \times 0 + b \times c \times d)} - \underset{\substack{\downarrow \\ 0}}{(0 \times 0 \times d + c \times 0 \times 0 - 0 \times b \times 0)}) = -abcd$