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Assignment 2

Calculus and Linear Algebra II

1. Use the chain rule to find $\frac{dz}{dt}$ *for*

(a) $z = x^2 + 2y^2, \quad x = 1 + t^3, y = 2 - t$

$$\frac{dz}{dt} = 2x \times \frac{d}{dt}[x] + 4y \times \frac{d}{dt}[y] = 2(1 + t^3) \times 3t^2 + 4(2 - t) \times (-1)$$
$$\frac{dz}{dt} = 6t^5 + 6t^2 + 4t - 8$$

(b) $z = \sin x \sin y, \quad x = 2t^2, y = \sqrt{t}$

$$\frac{dz}{dt} = \frac{d}{dt}[\sin x] \times \sin y + \sin x \times \frac{d}{dt}[\sin y]$$
$$\frac{dz}{dt} = \cos x \times \frac{d}{dt}[x] \times \sin y + \sin x \times \cos y \times \frac{d}{dt}[y]$$
$$\frac{dz}{dt} = \cos(2t^2) \times 4t \times \sin(\sqrt{t}) + \sin(2t^2) \times \cos(\sqrt{t}) \times + \frac{1}{2\sqrt{t}}$$
$$\frac{dz}{dt} = 4t \sin(\sqrt{t}) \cos(2t^2) + \frac{\sin(2t^2)\cos(\sqrt{t})}{2\sqrt{t}}$$

(c) $x = e^{xyz}, \quad x = \sin t, y = \cos t, z = t$

$$\ln x = \ln e^{xyz} => \ln x = xyz => z = \frac{\ln x}{xy} => \frac{dz}{dt} = \frac{d}{dt}\left[\frac{\ln x}{xy}\right]$$
$$\frac{dz}{dt} = \frac{\frac{d}{dt}[\ln x] \times xy - \ln x \times \frac{d}{dt}[xy]}{x^2y^2} = \frac{\frac{1}{x} \times \frac{d}{dt}[x] \times xy - \ln x \times \left(\frac{d}{dt}[x] \times y + x \times \frac{d}{dt}[y]\right)}{x^2y^2}$$
$$\frac{dz}{dt} = \frac{\frac{\cos t \times \sin t \times \cos t}{\sin t} - \ln(\sin t) \times (\cos t \times \cos t + \sin t \times (-\sin t))}{\sin^2 t \cos^2 t}$$
$$\frac{dz}{dt} = \frac{\cos^2 t - \ln(\sin t)\cos^2 t + \ln(\sin t)\sin^2 t}{\sin^2 t \cos^2 t} = \frac{1 - \ln(\sin t)}{\sin^2 t} + \frac{\ln(\sin t)}{\cos^2 t}$$

2. Use the chain rule to find $\frac{\partial z}{\partial s}$ *and* $\frac{\partial z}{\partial t}$

(a) $z = x^2 + y^2, \quad x = s\cos t, \ y = s \sin t$

- $\frac{\partial z}{\partial s} = 2x\frac{\partial}{\partial s}x + 2y\frac{\partial}{\partial s}y = 2s\cos^2 t + 2s\sin^2 t = 2s(\cos^2 t + \sin^2 t) = 2s$
- $\frac{\partial z}{\partial t} = 2x\frac{\partial}{\partial t}x + 2y\frac{\partial}{\partial t}y = -2s^2 \sin t \cos t + 2s^2 \sin t \cos t = 0$

(b) $z = \sin\emptyset \cos\theta, \quad \emptyset = st^2, \theta = s^2t$

- $\frac{\partial z}{\partial s} = \cos\emptyset\cos\theta\frac{\partial}{\partial s}[\emptyset] - \sin\emptyset\sin\theta\frac{\partial}{\partial s}[\theta] = t^2\cos(st^2)\cos(s^2t) - 2st\sin(st^2)\sin(s^2t)$
- $\frac{\partial z}{\partial t} = \cos\emptyset\cos\theta\frac{\partial}{\partial t}[\emptyset] - \sin\emptyset\sin\theta\frac{\partial}{\partial t}[\theta] = 2st\cos(st^2)\cos(s^2t) - s^2\sin(st^2)\sin(s^2t)$

(c) $z = e^r \sin\theta$, $r = st, \theta = \sqrt{s^2+t^2}$

- $\frac{\partial z}{\partial s} = e^r \sin\theta \frac{\partial}{\partial s}[r] + e^r \cos\theta \frac{\partial}{\partial s}[\theta] = te^{st}\sin\left(\sqrt{s^2+t^2}\right) + \frac{se^{st}\cos\left(\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}$
- $\frac{\partial z}{\partial t} = e^r \sin\theta \frac{\partial}{\partial s}[r] + e^r \cos\theta \frac{\partial}{\partial s}[\theta] = se^{r}\sin\left(\sqrt{s^2+t^2}\right) + \frac{te^{st}\cos\left(\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}$

3. Use the implicit differentiation formula to find $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$ *where*

(a) $x^2 + y^2 + z^2 = 3xyz$

- $\frac{\partial}{\partial x}[x^2+y^2+z^2] = \frac{\partial}{\partial x}[3xyz] \Rightarrow 2x + 2z\frac{\partial z}{\partial x} = 3yz + 3xy\frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \frac{3yz-2x}{2z-3xy}$
- $\frac{\partial}{\partial y}[x^2+y^2+z^2] = \frac{\partial}{\partial y}[3xyz] \Rightarrow 2y + 2z\frac{\partial z}{\partial y} = 3xz + 3xy\frac{\partial z}{\partial y} \Rightarrow \frac{\partial z}{\partial y} = \frac{3xz-2y}{2z-3xy}$

(b) $yz = \log(x+z)$

- $\frac{\partial}{\partial x}[yz] = \frac{\partial}{\partial x}[\log(x+z)] \Rightarrow y\frac{\partial z}{\partial x} = \frac{1+\frac{\partial z}{\partial x}}{\ln 10(x+z)} \Rightarrow \frac{\partial z}{\partial x} = \frac{1}{y\ln 10(x+z)-1}$
- $\frac{\partial}{\partial y}[yz] = \frac{\partial}{\partial y}[\log(x+z)] \Rightarrow z + y\frac{\partial z}{\partial y} = \frac{\frac{\partial z}{\partial y}}{\ln 10(x+z)} \Rightarrow \frac{\partial z}{\partial y} = -\frac{z\ln 10(x+z)}{y\ln 10(x+z)-1}$

(c) $xyz = x + y + z$

- $\frac{\partial}{\partial x}[xyz] = \frac{\partial}{\partial x}[x+y+z] \Rightarrow yz + xy\frac{\partial z}{\partial x} = 1 + \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \frac{1-yz}{xy-1}$
- $\frac{\partial}{\partial y}[xyz] = \frac{\partial}{\partial y}[x+y+z] \Rightarrow xz + xy\frac{\partial z}{\partial x} = 1 + \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial y} = \frac{1-xz}{xy-1}$

4. In each one of the following problems, determine ∇f *, the direction derivative of f at the given point a in the direction of u and the unit vector with the maximum rate of change:*

(a) $f(x,y) = 1 + x\sqrt{y}$, $a = (2,4), u = (2,1)$

- $\nabla f(2,4) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \sqrt{y} \\ \frac{x}{2\sqrt{y}} \end{bmatrix} = \begin{bmatrix} \sqrt{4} \\ \frac{2}{2\sqrt{4}} \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$
- $D_u f(x,y) = \nabla f \cdot u = 2 \times 2 + \frac{1}{2} \times 1 = 4.5$
- $u = \frac{1}{|\nabla f(2,4)|} \times \nabla f(2,4) = \frac{1}{\sqrt{2^2+\left(\frac{1}{2}\right)^2}} \times \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{\sqrt{\frac{17}{4}}} \times \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} = \frac{2\sqrt{17}}{17} \times \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{4\sqrt{17}}{17} \\ \frac{\sqrt{17}}{17} \end{bmatrix}$

(b) $f(x,y,z) = xe^y + ye^z - ze^x, \quad a = (0,0,0), u = (1,2,1)$

- $\nabla f(0,0,0) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} e^y - ze^x \\ xe^y + e^z \\ ye^z - e^x \end{bmatrix} = \begin{bmatrix} e^0 - 0 \times e^0 \\ 0 \times e^0 + e^0 \\ 0 \times e^0 - e^0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
- $D_u f(x,y,z) = \nabla f \cdot u = 1 \times 1 + 1 \times 2 - 1 \times 1 = 2$
- $u = \frac{1}{|\nabla f(0,0,0)|} \times \nabla f(0,0,0) = \frac{1}{\sqrt{3}} \times \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix}$

(c) $f(x,y) = \log(x^3 + y^3), \quad a = (1,1), u = (1,0)$

- $\nabla f(1,1) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{3x^2}{(x^3+y^3)\ln 10} \\ \frac{3y^2}{(x^3+y^3)\ln 10} \end{bmatrix} = \begin{bmatrix} \frac{3\times 1^2}{(1^3+1^3)\ln 10} \\ \frac{3\times 1^2}{(1^3+1^3)\ln 10} \end{bmatrix} = \begin{bmatrix} \frac{3}{2\ln 10} \\ \frac{3}{2\ln 10} \end{bmatrix}$
- $D_u f(x,y) = \nabla f \cdot u = \frac{3}{2\ln 10} \times 1 + \frac{3}{2\ln 10} \times 0 = \frac{3}{2\ln 10}$
- $u = \frac{1}{|\nabla f(1,1)|} \times \nabla f(1,1) = \frac{1}{\sqrt{\frac{9}{2\ln^2 10}}} \times \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} = \frac{\ln 10\sqrt{2}}{3} \times \begin{bmatrix} \frac{3}{2\ln 10} \\ \frac{3}{2\ln 10} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

5. Find the equation of the tangent plan to the following level surfaces:

(a) $x^2 + y^2 - z^2 = 1, \quad (1,1,1)$

$$x^2 + y^2 - z^2 - 1 = 0$$

- $\nabla f(1,1,1) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ -2z \end{bmatrix} = \begin{bmatrix} 2 \times 1 \\ 2 \times 1 \\ -2 \times 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$
- $2(x-1) + 2(y-1) - 2(z-1) = 0$

(b) $yz = \log(x+z) \Rightarrow yz - \log(x+z) = 0, \quad (0,0,1)$

- $\nabla f(0,0,1) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} -\frac{1}{(x+z)\ln 10} \\ z \\ y - \frac{1}{(x+z)\ln 10} \end{bmatrix} = \begin{bmatrix} -\frac{1}{(0+1)\ln 10} \\ 1 \\ 0 - \frac{1}{(0+1)\ln 10} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\ln 10} \\ 1 \\ -\frac{1}{\ln 10} \end{bmatrix}$
- $-\frac{x}{\ln 10} + y - \frac{(z-1)}{\ln 10} = 0$

(c) $x^2y + y^2z + z^2x = 3, \quad (1,1,1)$

- $\nabla f(1,1,1) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2xy + z^2 \\ x^2 + 2yz \\ y^2 + 2xz \end{bmatrix} = \begin{bmatrix} 2 \times 1 \times 1 + 1^2 \\ 1^2 + 2 \times 1 \times 1 \\ 1^2 + 2 \times 1 \times 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$
- $3(x-1) + 3(y-1) - 3(z-1) = 0$

6. (Bonus) A function f is called homogenous of degree n if for all x, y, t we have

$$f(tx, ty) = t^n f(x, y)$$

(a) Verify that $f(x,y) = Ax^2 + Bxy + Cy^2$ is homogenous of order 2.

$$f(tx, ty) = A(tx)^2 + B(tx)(ty) + C(ty)^2$$

$$f(tx, ty) = At^2x^2 + Bt^2xy + Ct^2y^2$$

$$f(tx, ty) = t^2(Ax^2 + Bxy + Cy^2)$$

$$f(tx, ty) = t^2 f(x, y) \Rightarrow homogenous\ of\ order\ 2$$

(b) If f is homogenous of degree n show that:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

<u>Proof:</u>

Let u = tx and v = ty:

$$\frac{\partial}{\partial x} f(u, v) = \frac{\partial}{\partial x}(t^n f(x, y))$$

$$\boldsymbol{\frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} = nt^{n-1}f(x, y)}$$

Get the derivative with respect to x:

$$\frac{\partial}{\partial x}\big(f(u,v)\big) = \frac{\partial}{\partial x}\big(t^n f(x,y)\big)$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = t^n \frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial u} t = t^n \frac{\partial f}{\partial x}$$

$$\boldsymbol{\frac{\partial f}{\partial u} = t^{n-1} \frac{\partial f}{\partial x}}$$

Get the derivative with respect to y:

$$\frac{\partial}{\partial y}\big(f(u,v)\big) = \frac{\partial}{\partial y}\big(t^n f(x,y)\big)$$

$$\frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = t^n \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} t = t^n \frac{\partial f}{\partial y}$$

$$\boldsymbol{\frac{\partial f}{\partial v} = t^{n-1} \frac{\partial f}{\partial y}}$$

Substitute the 2 results into $\frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} = nt^{n-1} f(x,y)$:

$$\cancel{t^{n-1}} \frac{\partial f}{\partial x} x + \cancel{t^{n-1}} \frac{\partial f}{\partial y} y = n\cancel{t^{n-1}} f(x,y)$$

$$=> \boldsymbol{\frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y = n f(x,y)}$$