Jona Bako
j.bako@jacobs-university.de

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Assignment 5

Calculus and Linear Algebra II

1. For each pair of vectors u and subspace W determine $proj_W$u and $orth_W$u. In parts (b) and (c) justify your answer.

(a) $u = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, W is the one-dimensional space spanned by $w = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

$$proj_W u = \frac{u \cdot w}{||w||^2} w = \frac{2}{9}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ \frac{2}{9} \end{pmatrix}$$

$$orth_W u = u - proj_W u = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ \frac{2}{9} \end{pmatrix} = \begin{pmatrix} \frac{5}{9} \\ -\frac{13}{9} \\ \frac{16}{9} \end{pmatrix}$$

(b) u is an arbitrary non-zero vector and W is the one-dimensional subspace spanned by u.

$proj_W u = u \rightarrow \textit{the projection is on the same line, thus it is also } u$
$orth_W u = 0 \rightarrow \textit{there is no orthogonal projection, } u - proj_W u = u - u = 0$

(c) u is an arbitrary non-zero vector in an inner product space V and W = V.

$proj_W u = u \rightarrow \textit{the projection is on the same space, thus it is also } u$
$orth_W u = 0 \rightarrow \textit{there is no orthogonal projection, } u - proj_W u = u - u = 0$

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2. Suppose u and v are vectors in an inner product space V.

(a) Show that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle$$

Proof by expanding left hand side:

$$\|u + v\|^2 = (u + v) \cdot (u + v) = \sum_{i=0}^{n} (u_i + v_i)^2$$
$$\|u + v\|^2 = \sum_{i=0}^{n} (u_i)^2 + 2u_i v_i + (v_i)^2$$
$$\|u + v\|^2 = \sum_{i=0}^{n} (u_i)^2 + 2 \times \sum_{i=0}^{n} u_i v_i + \sum_{i=0}^{n} (v_i)^2$$
$$\|u + v\|^2 = u \cdot u + +2 \times (u \cdot v) + v \cdot v$$
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle$$

(b) Use (a) to prove the following identity:

$$\|u-v\|^2 + \|u+v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Proof by expanding left hand:

$$\|u-v\|^2 + \|u+v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

$$\|u\|^2 + \|v\|^2 - 2\langle u, v\rangle + \|u\|^2 + \|v\|^2 + 2\langle u, v\rangle = 2(\|u\|^2 + \|v\|^2)$$

$$2\|u\|^2 + 2\|v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

$$2(\|u\|^2 + \|v\|^2) = 2(\|u\|^2 + \|v\|^2)$$

(note: for expanding $\|u-v\|^2$, the same proof as above can be applied)

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3. Consider the polynomials $L_0(x) = 1,\ L_1(x) = x,\ L_2(x) = x^2 - \frac{1}{3}$ viewed as elements of the space of continuous functions on [-1, 1] equipped with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) Show that $\{L_0, L_1, L_2\}$ is an orthonormal set, i.e.

$$\langle L_i, L_j\rangle = \begin{cases} 1 & if\ \ i = j \\ 0 & if\ \ i \neq j \end{cases}$$

$$\left.\begin{array}{r}
i \neq j: \\
\langle L_0, L_1\rangle = \int_{-1}^{1}(x)dx = \left.\frac{x^2}{2}\right|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0 \\
\langle L_0, L_2\rangle = \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)\,dx = \left.\frac{x^3 - x}{3}\right|_{-1}^{1} = \frac{1-1}{3} - \frac{-1+1}{3} = 0 \\
\langle L_1, L_2\rangle = \int_{-1}^{1}\left(x\left(x^2 - \frac{1}{3}\right)\right)\,dx = \int_{-1}^{1}\left(x^3 - \frac{x}{3}\right)\,dx = \left.\frac{3x^4 - 2x^2}{12}\right|_{-1}^{1} = \frac{3-2}{12} - \frac{3-2}{12} = 0 \\
i = j: \\
\langle L_0, L_0\rangle = \int_{-1}^{1}(1)dx = x|_{-1}^{1} = 1 + 1 = 2 \neq 1 \\
\langle L_1, L_1\rangle = \int_{-1}^{1}(x^2)\,dx = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \neq 1 \\
\langle L_2, L_2\rangle = \int_{-1}^{1}\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)\,dx = \left.\frac{9x^5 - 10x^3 + 5x}{45}\right|_{-1}^{1} = \frac{9-10+5}{45} + \frac{9-10+5}{45} = \frac{8}{45} \neq 1
\end{array}\right\} \rightarrow \begin{array}{c} \{L_0, L_1, L_2\} \\ is\ not\ an \\ orthonormal \\ set \end{array}$$

(b) Express $f(x) = (1\ +\ x)^2$ as a linear combination of L_0, L_1, L_2.

$$f(x) = (1\ +\ x)^2 = 1 + 2x + x^2 = c_0L_0 + c_1L_1 + c_2L_2$$

$$\boldsymbol{f(x) = L_0 + 2L_1 + L_2 + \frac{1}{3}} \rightarrow c_0 = 1, c_1 = 2, c_2 = 1$$

(c) Find the projection of $g(x) = x^4$ on the space spanned by L_0, L_1.

$W \rightarrow space\ spanned\ by\ L_0\ and\ L_1 \quad g(x) = x^4 \rightarrow u$

We normalize the orthogonal functions to orthonormal bases: $v_0 = \frac{L_0}{\|L_0\|} = \frac{1}{\sqrt{2}}, v_1 = \frac{L_1}{\|L_1\|} = \frac{\sqrt{3}}{\sqrt{2}}x$

$$proj_W u = \sum_{i=0}^{k} \langle u, v_i \rangle\, v_i = \left(\int_{-1}^{1} \left(\frac{x^4}{\sqrt{2}}\right) dx\right) \frac{1}{\sqrt{2}} + \left(\int_{-1}^{1} \left(\frac{\sqrt{3}x^5}{\sqrt{2}}\right) dx\right) \frac{\sqrt{3}}{\sqrt{2}} x$$

$$proj_W u = \frac{2}{5\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 \times \frac{\sqrt{3}}{\sqrt{2}} x = \mathbf{\frac{1}{5}}$$

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4. Vectors v, u, w are given by

$$v = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

(a) Compute $orth_v u$.

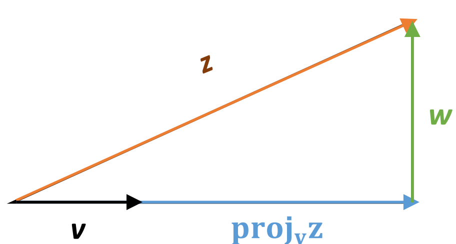
$orth_v u = u - proj_v u$

$$orth_v u = u - \frac{u \cdot v}{\|v\|^2} v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{7}{21} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{3} \\ 1 - \frac{2}{3} \\ 1 - \frac{4}{3} \end{pmatrix} = \begin{pmatrix} \mathbf{\frac{2}{3}} \\ \mathbf{\frac{1}{3}} \\ -\mathbf{\frac{1}{3}} \end{pmatrix}$$

(b) Verify that $\langle v, w \rangle = 0$ and give a description of all vectors z with $orth_v z = w$.

$$\langle v, w \rangle = 2 + 2 - 4 = 0$$

$w\ is\ the\ orthogonal\ projection\ of\ vector\ v\ on\ vector\ z.$
$We\ can\ also\ say\ that\ all\ vectors\ z\ will$
$have\ a\ magnitude\ of\ \sqrt{proj_v z^2 + w^2}$



$A\ mathematical\ description\ of\ any\ vector\ z\ would\ be:$

$$z = c \times v + w\ , where\ c \in R$$

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5. Apply Gram-Schmidt process to the vectors v_1, v_2, v_3 below to find an orthonormal basis e_1, e_2, e_3 for the space W they span.

$$v_1 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ -6 \\ 8 \end{pmatrix}$$

Solution:

$$\mathbf{e_1} = \frac{\mathbf{v_1}}{\|\mathbf{v_1}\|} = \frac{\mathbf{1}}{\mathbf{3}}\begin{pmatrix}\mathbf{3}\\\mathbf{0}\\\mathbf{0}\end{pmatrix} = \begin{pmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{pmatrix}$$

$$\widetilde{\mathrm{e}_2} = \mathrm{v}_2 - \langle \mathrm{v}_2, \mathrm{e}_1\rangle \mathrm{e}_1 = \begin{pmatrix}2\\8\\6\end{pmatrix} - 2\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\8\\6\end{pmatrix} \ -> \ \mathbf{e_2} = \frac{\widetilde{\mathbf{e_2}}}{\|\widetilde{\mathbf{e_2}}\|} = \frac{\mathbf{1}}{\mathbf{10}}\begin{pmatrix}\mathbf{0}\\\mathbf{8}\\\mathbf{6}\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\\frac{\mathbf{4}}{\mathbf{5}}\\\frac{\mathbf{3}}{\mathbf{5}}\end{pmatrix}$$

$$\widetilde{\mathrm{e}_3} = \mathrm{v}_3 - \langle \mathrm{v}_3, \mathrm{e}_1\rangle \mathrm{e}_1 - \langle \mathrm{v}_3, \mathrm{e}_2\rangle \mathrm{e}_2 = \begin{pmatrix}2\\-6\\8\end{pmatrix} - 2\begin{pmatrix}1\\0\\0\end{pmatrix} - 0\begin{pmatrix}0\\\frac{4}{5}\\\frac{3}{5}\end{pmatrix} = \begin{pmatrix}0\\-6\\8\end{pmatrix} \ -> \mathbf{e_3} = \frac{\widetilde{\mathbf{e_3}}}{\|\widetilde{\mathbf{e_3}}\|} = \frac{\mathbf{1}}{\mathbf{10}}\begin{pmatrix}\mathbf{0}\\\mathbf{-6}\\\mathbf{8}\end{pmatrix} = \begin{pmatrix}\mathbf{0}\\-\frac{\mathbf{3}}{\mathbf{5}}\\\frac{\mathbf{4}}{\mathbf{5}}\end{pmatrix}$$
