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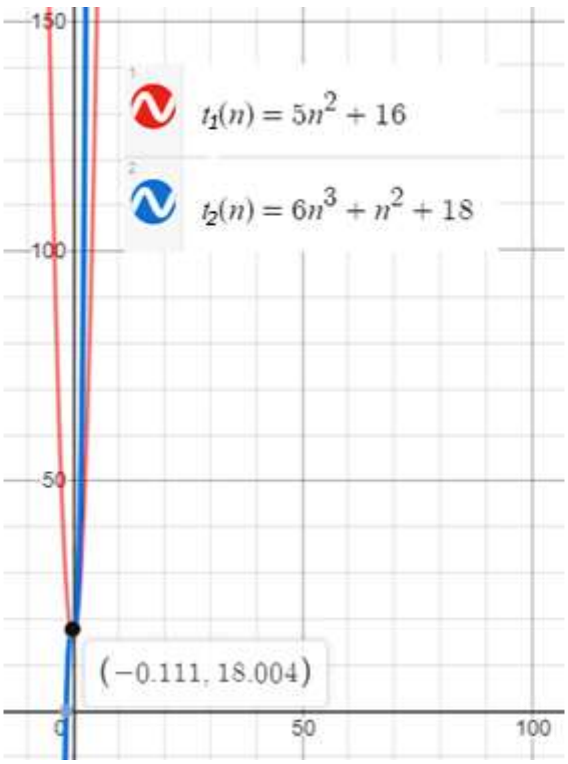
Homework - CS 2020 Problem Sheet #2

Problem 2.1

$t_1(n) = 5n^2 + 16$ $t_2(n) = 6n^3 + n^2 + 18$

// We say $f(n) = O(g(n))$ if there exist positive constants c and n_0 such that $f(n) \leq c*g(n)$ for all $n \geq n_0$ //



a) $t_1(n) \leq c*n$
$c = 20$ $n_0 = 1$
for n>1,
$5n^2 + 16 \leq 20n^2$
Therefore:
$t_1 \in O(n^2)$

$t_2(n) \leq c*n$
$c = 40$ $n_0 = 0$
for n>0,
$6n^3 + n^2 + 18 \leq 40n^3$
Therefore:
$t_2 \in O(n^3)$

b) $t_1 + t_2 = 5n^2 + 16 + 6n^3 + n^2 + 18$
$t_3 = 6n^3 + 6n^2 + 34$

$k = 40$ $n_0 = 1$
for $n > 1$, $6n^3 + 6n^2 + 34 \leq 40n^3$
Therefore:
$t_3 \in O(n^3)$

c) Prove that if $f_1 \in O(g_1)$ and $f_2 \in O(g_2)$, then $(f_1 + f_2) \in O(\max\{g_1, g_2\})$.

$f_1 \leq c_1g_1$ for $n \geq n_1$ and $f_2 \leq c_2g_2$ for $n \geq n_2$

$$\begin{array}{lll} & f_1 \leq c_1g_1 & \text{for } n \geq n_1 \\ + & f_2 \leq c_2g_2 & \text{for } n \geq n_2 \\ \hline \end{array}$$

$f_1 + f_2 \leq c_1g_1 + c_2g_2$ for $n \geq n_1+n_2$
$f_1 + f_2 \leq c_1 \max(g_1, g_2) + c_2 \max(g_1, g_2)$ for $n \geq n_1+n_2$
$f_1 + f_2 \leq (c_1 + c_2) \max(g_1, g_2)$ for $n \geq n_1+n_2$

Therefore: $f_1 + f_2 \in O(\max(g_1, g_2))$

Problem 2.2

$1^2 + 3^2 + 5^2 + \ldots (2n-1)^2 = \sum_{k=1}^{n} (2k-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$

Mathematical Induction

a) Test the statement for the first possible value, n=1:

$$(2\times 1-1)^2 = \frac{2\times 1(2\times 1-1)(2\times 1+1)}{6}$$

$$1^2 = \frac{6}{6}$$

$$1 = 1$$

b) Assume the statement is true for n=k:

$$1^2 + 3^2 + 5^2 + \ldots (2k-1)^2 = \frac{2k(2k-1)(2k+1)}{6}$$

c) Prove that the statement is true for n = k + 1:

$$\underbrace{1^2 + 3^2 + 5^2 + \ldots (2k-1)^2}_{\text{By assumption}} + [2(k+1)-1]^2 = \frac{2(k+1)(2(k+1)-1)(2(k+1)+1)}{6}$$

LHS (left hand side) RHS (right hand side)

LHS:

$$\frac{2k(2k-1)(2k+1)}{6} + (2k+2-1)^2 =$$

$$\frac{2k(2k-1)(2k+1) + 6(2k+1)^2}{6} =$$

$$\frac{2k(4k^2-1)+6(4k^2+4k+1)}{6} =$$

$$\frac{8k^3-2k+24k^2+24k+6}{6} =$$

$$\frac{8k^3+24k^2+22k+6}{6} =$$

(we factorize)

$$\frac{2(4k^3+12k^2+11k+3)}{6} =$$

$$\frac{2(k+1)(2k+1)(k+3)}{6} =$$

$$\frac{2(k+1)(2k+2-1)(2k+2+1)}{6} =$$

$$\frac{\mathbf{2(k+1)(2(k+1)-1)(2(k+1)+1)}}{\mathbf{6}} = \textbf{RHS}$$

Since n = 1 is true and then we assume that n = k is true, n= k+1 is true whenever n=k is accepted to be true for all n ≥ 1.

(By mathematical induction the statement holds for every positive integer *n*)