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Homework - ICS 2020 Problem Sheet #6

Problem 6.1

Prove that the two elementary Boolean functions → (implication) and ¬ (negation) are universal, i.e., they are sufficient to express all possible Boolean functions.

To prove that the two elementary Boolean functions, → and ¬ are universal, we need to show that they are functionally complete. This means that all formulas in propositional logic can be rewritten to an equivalent that solely uses the set {→, ¬}. table for logical implication:

Table for negation:

P	¬ P
1	0
0	1

Table for logical implication:

P	Q	P → Q
1	1	1
1	0	0
0	1	1
0	0	1

Let's find a form using the two functions that will equal the AND operator, i.e.:

$$P \land Q = (P \rightarrow \neg Q) \rightarrow \neg(P \rightarrow \neg Q)$$

P	Q	¬Q	P → ¬Q	¬(P → ¬Q)	(P → ¬Q) → ¬(P → ¬Q)	P ∧ Q
0	0	1	1	0	**0**	**0**
0	1	0	1	0	**0**	**0**
1	0	1	1	0	**0**	**0**
1	1	0	0	1	**1**	**1**

We proved that the formula is equal to the AND logical operator. From the above table we can also see that (P → ¬Q) is the contrary to P ∧ Q, meaning it's equivalent to the NAND operator. Let's now find a form using the two functions that will equal the OR operator, i.e.:

$$P \lor Q = (P \rightarrow \neg P) \rightarrow \neg(Q \rightarrow \neg Q)$$

P	Q	¬P	¬Q	P → ¬P	Q → ¬Q	¬(Q → ¬Q)	(P → ¬Q) → ¬(P → ¬Q)	P ∨ Q
0	0	1	1	1	1	0	**0**	**0**
0	1	1	0	1	0	1	**1**	**1**
1	0	0	1	0	1	0	**1**	**1**
1	1	0	0	0	0	1	**1**	**1**

We now proved that the formula is equal to the OR logical operator.
Lastly, we write that the NOT operator ¬ is equivalent to itself:
¬P = ¬P / ¬Q = ¬Q

P	¬P	Q	¬Q
0	1	0	1
1	0	1	0

Therefore, the two Boolean functions are universal.

Problem 6.2

Simplify the following Boolean formulas by repeatedly applying Boolean equivalence laws. (The simplified formulas contain at most one $\wedge$ or $\vee$.) Indicate in every step which law you have applied. You obtain points for the derivation, not for the result alone.

a) $\phi(A, B) = (\neg \mathbf{A} \vee \neg B) \wedge (\neg \mathbf{A} \vee B) \wedge (A \vee \neg B)$
   Apply absorbtion law:
   $\phi(A, B) = \mathbf{\neg A \wedge (A \vee \neg B)}$
   Apply distributivity law:
   $\phi(A, B) = \mathbf{\neg A \wedge A} \vee \neg A \wedge \neg B)$
   $\phi(A, B) = \mathbf{0 \vee \neg A \wedge \neg B}$
   Apply identity law:
   $\phi(A, B) = \neg A \wedge \neg B$
b) $\phi(A, B, C) = (\mathbf{A \wedge \neg B}) \vee (\mathbf{A \wedge \neg B} \wedge C)$
   Apply absorbtion law:
   $\phi(A, B, C) = A \wedge \neg B$
c) $\phi(A, B, C, D) = (A \vee \mathbf{\neg(B \wedge A)}) \wedge (C \vee (D \vee C))$
   Apply de Morgan's laws
   $\phi(A, B, C, D) = \mathbf{(A \vee (\neg B \vee \neg A))} \wedge (\mathbf{C \vee (D \vee C))}$
   Apply associativity law:
   $\phi(A, B, C, D) = ((A \vee \neg A) \vee \neg B)) \wedge ((C \vee C) \vee D)$
   $\phi(A, B, C, D) = (\mathbf{1 \vee \neg B}) \wedge (C \vee D)$
   Apply identity law:
   $\mathbf{1 \wedge (C \vee D)}$
   Apply identity law:
   $\phi(A, B, C, D) = C \vee D$
d) $\phi(A, B, C) = (\mathbf{\neg(A \wedge B)} \vee \neg C) \wedge (\neg A \vee B \vee \neg C)$
   Apply de Morgan's law:
   $\phi(A, B, C) = \mathbf{(\,(\neg A} \vee \mathbf{\neg B) \vee \neg C)} \wedge (\mathbf{\neg A \vee B \vee \neg C})$
   Apply associativity law:
   $\phi(A, B, C) = \mathbf{(\,(\neg A \vee \neg C)} \vee \neg B) \wedge \mathbf{(\neg A \vee \neg C} \vee B)$
   Apply absorbtion law:
   $\phi(A, B, C) = \neg A \vee \neg C$
e) $\phi(A, B) = (A \vee \mathbf{B}) \wedge (\neg A \vee \mathbf{B}) \wedge (A \vee \mathbf{\neg B}) \wedge (\neg A \vee \mathbf{\neg B})$
   Apply absorbtion law:
   $\phi(A, B) = B \wedge \neg B$
   $\phi(A, B) = 0$

Problem 6.3

Consider the following boolean formula:

$\phi(P, Q, R, S) = (\neg P \lor Q) \land (\neg Q \lor R) \land (\neg R \lor S) \land (\neg S \lor P)$

a) How many interpretations of the variables P, Q, R, S satisfy ϕ? Provide a proof for your answer (e.g., by providing a truth table).

P	Q	R	S	$\neg P \lor Q$	$\land$	$\neg Q \lor R$	$\neg R \lor S$	$\land$	$\neg S \lor P$	$\varphi(P, Q, R, S)$
0	0	0	0	1	**1**	1	1	**1**	1	1
0	0	0	1	1	**1**	1	1	**0**	0	0
0	0	1	0	1	**1**	1	0	**0**	1	0
0	0	1	1	1	**1**	1	1	**0**	0	0
0	1	0	0	1	**0**	0	1	**1**	1	0
0	1	0	1	1	**0**	0	1	**0**	0	0
0	1	1	0	1	**1**	1	0	**0**	1	0
0	1	1	1	1	**1**	1	1	**0**	0	0
1	0	0	0	0	**0**	1	1	**1**	1	0
1	0	0	1	0	**0**	1	1	**1**	1	0
1	0	1	0	0	**0**	1	0	**0**	1	0
1	0	1	1	0	**0**	1	1	**1**	1	0
1	1	0	0	1	**0**	0	1	**1**	1	0
1	1	0	1	1	**0**	0	1	**1**	1	0
1	1	1	0	1	**1**	1	0	**0**	1	0
1	1	1	1	1	**1**	1	1	**1**	1	1

There are only 2 interpretations that satisfy ϕ:

- First Interpretation: P = 0, Q = 0, R = 0, S = 0
- Second Interpretation: P = 1, Q = 1, R = 1, S = 1

b) Given the interpretations that satisfy ϕ, write the formula for ϕ in disjunctive normal form (DNF).
According to the table only the first and last row have 1 as their result . We logically combine them with the OR operator, and the AND operator in between the variables, negating when equal to 0:

DNF (ϕ) -> $\gamma = (\neg P \land \neg Q \land \neg R \land \neg S) \lor (P \land Q \land R \land S)$