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Homework - CS 2020 Problem Sheet #4

Problem 4.1

Let Σ be a finite set (called an alphabet) and let Σ^* be the set of all words that can be created out the symbols in the alphabet Σ. (Σ^* is the Kleene closure of Σ, which includes the empty word ϵ.) A word $p \in \Sigma^*$ is called a prefix of a word $w \in \Sigma^*$ if there is a word $q \in \Sigma^*$ such that $w = pq$. A prefix p is called a proper prefix if $p \neq w$.

a) Let $\preccurlyeq \subseteq \Sigma^* \times \Sigma^*$ be a relation such that $p \preccurlyeq w$ for $p,w \in \Sigma^*$ if p is a prefix of w. Show that $\preccurlyeq$ is a partial order.
For a relation to be partial order, we need to proove that is is reflective, antisymetric and transitive.We proove for each condition as such:
- Reflexive: $\forall\, p \in \Sigma^*, (p, p) \in \preccurlyeq$ (meaning $q = \epsilon$)
- Antisymmetric: $\forall\, p \in \Sigma^*, [(p, w) \in \Sigma^* \wedge (w, p) \in \Sigma^*] => p=w$
  ($w = pq \wedge p = wq => q = \epsilon$)
- Trasitive: $\forall\, p, w, z \in \Sigma^*, [(p, w) \in \Sigma^* \wedge (w, z) \in \Sigma^*] => (p, z)^* \in \Sigma^*$
  ($w = pq \wedge z = wq => z = \mathbf{p}qq$, therefore p is also prefix of z)

b) Let$\prec \subset \Sigma^* \times \Sigma^*$ be a relation such that for $p \prec w$ for $p,w \in \Sigma^*$ if p is a proper prefix of w. Show that $\prec$ is a strict partial order.
- Irreflexive: $\forall\, p \in \Sigma^*, (p, p) \notin \preccurlyeq$ (not proper prefix)
- Asymmetric: $\forall\, p \in \Sigma^*, (p, w) \in \Sigma^* => (w, p) \notin \Sigma^*$ (not proper prefix)
- Trasitive: $\forall\, p, w, z \in \Sigma^*, (p, w) \in \Sigma^* \wedge (w, z) \in \Sigma^* => (p, z) \in \Sigma^*$
  ($w = pq \wedge z = wq => z = \mathbf{p}qq$, therefore p is also prefix of z)

c) Are the two order relations $\preccurlyeq$ and $\prec$ total?
Neither relation is total. The $\prec$ relation isn't partial order, so it doesn't even fulfill the first condition to be total, while the $\preccurlyeq$ relations fails on the second condition, as not every two elements of the poset $(\Sigma^*, \preccurlyeq)$ are comparable, i.e. if in the poset (p, w), w is an empty word, and p is a proper prefix. (equivalence relation in itself cannot be comparable).

Problem 4.2

Let A,B and C be sets and let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions.

a) Prove the following statement: If $g \circ f$ is bijective, then f is injective and g is surjective.
Since $g \circ f$ is bijective, then it is both injective and surjective.

- Let f: A - > B and g: B - > C. Proove that if $g \circ f$: A - > C is injective, then so is f.
  Suppose $f(x) = f(y)$, for some $x,y \in A$ (domain of f). (Take g in both sides of equation)

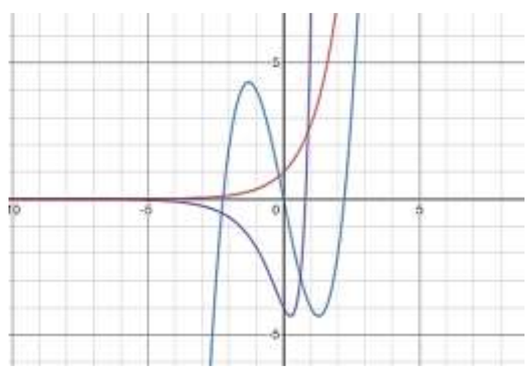
  Then $g(f(x))=g(f(y))$, i.e.

  $(g \circ f)(x)=(g \circ f)(y)$, so since $y \circ f$ is injective, we have $x=y$.

  x and y were arbitrary so this proof holds $\forall x,y$**, thus this shows f is injective.**
- Let f: A - > B and g: B - > C. Prove that if $g \circ f$: A - > C is surjective, then so is g.
  Take any $y \in C$ (domain of g).
  Since $g \circ f$ is surjective, there exists some $a \in A$ such that $(g \circ f)(a)=y$.
  This can be writen as $g(f(a))=y$. Set $b = f(a) \in B$.
  Then $g(b)=g(f(a))=y$ **Therefore g is surjective.**

b) Find an example demonstrating that $g \circ f$ is not bijective even though f is injective and g is surjective.
Take as functions f and g:
$f(x) = e^x$ (injective)
$g(x) = x^3 - 5x$ (surjective)
$f(g(x)) = e^{3x} - 5e^x$ (not bijective)

not bijective –> demolishes proposition



c) Find an example demonstrating that $g \circ f$ is bijective even though f is not surjective and g is not injective.
Take as functions f and g:
$g(x) = x^{1/2}$ (not surjective)
$f(x) = x^2$ (not injective)
$f(g(x)) = x$ (bijective)

bijective –> demonstrates proposition