

ICS 2020 Problem Sheet #4

Problem 4.1: prefix order relations

(2+2+1 = 5 points)

Let Σ be a finite set (called an alphabet) and let Σ^* be the set of all words that can be created out of the symbols in the alphabet Σ . (Σ^* is the Kleene closure of Σ , which includes the empty word ϵ .) A word $p \in \Sigma^*$ is called a prefix of a word $w \in \Sigma^*$ if there is a word $q \in \Sigma^*$ such that $w = pq$. A prefix p is called a proper prefix if $p \neq w$.

- a) Let $\preceq \subseteq \Sigma^* \times \Sigma^*$ be a relation such that $p \preceq w$ for $p, w \in \Sigma^*$ if p is a prefix of w . Show that \preceq is a partial order.
- b) Let $\prec \subset \Sigma^* \times \Sigma^*$ be a relation such that for $p \prec w$ for $p, w \in \Sigma^*$ if p is a proper prefix of w . Show that \prec is a strict partial order.
- c) Are the two order relations \preceq and \prec total?

Make sure you write complete proofs for the properties of the order relations. Do not assume something is 'obvious' or 'trivial' — always reason with the definition of the order relation.

Problem 4.2: function composition

(2+1+1 = 4 points)

Let A, B and C be sets and let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions.

- a) Prove the following statement: If $g \circ f$ is bijective, then f is injective and g is surjective.
- b) Find an example demonstrating that $g \circ f$ is not bijective even though f is injective and g is surjective.
- c) Find an example demonstrating that $g \circ f$ is bijective even though f is not surjective and g is not injective.

Problem 4.3: prime numbers with a fixed prime gap (haskell)

(1 point)

We call the difference between two successive prime numbers their *prime gap*. Prime numbers with a prime gap of 2 are called *twin primes* while prime numbers with a prime gap of 4 are called *cousin primes*. Prime numbers with a prime gap of 6 are called *sexy primes*.

The predicate `isPrime` shown below determines whether a number is a prime number or not.

```
1 isPrime :: Integer -> Bool
2 isPrime n = null [ x | x <- [2..n `div` 2], n `mod` x == 0 ]
```

Implement a function `primes` that takes two arguments `a` and `b` and returns the list of all prime numbers in the interval $[a, b]$. With that, implement a function `gappies` that receives three arguments, a prime gap `g`, a lower interval bound `a`, and an upper interval bound `b`. The function returns all prime number pairs with the prime gap `g` in the interval $[a, b]$. By “currying” the first argument of `gappies`, we easily obtain the functions `twins`, `cousins`, and `sexies`.

Some sample results so that you can test your implementation:

```
> twins 1 100
[(3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73)]
> cousins 1 100
```

```
[(3,7),(7,11),(13,17),(19,23),(37,41),(43,47),(67,71),(79,83)]
> sexies 100 150
[(101,107),(103,109),(107,113),(131,137)]
```

Below is a template that may serve as a starting point.

```
1  isPrime :: Integer -> Bool
2  isPrime n = null [ x | x <- [2..n `div` 2], n `mod` x == 0]
3
4  primes :: Integer -> Integer -> [Integer]
5  primes a b = undefined
6
7  gappies :: Integer -> Integer -> Integer -> [(Integer,Integer)]
8  gappies g a b = undefined
9
10 twins   = gappies 2
11 cousins = gappies 4
12 sexies  = gappies 6
```