Numerical Conformal Maps

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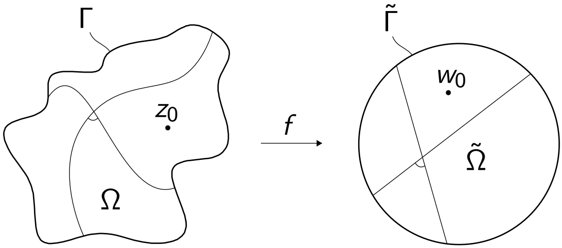
Motivation

Maps to and from the unit disk ($\tilde{\Omega} = \mathbb{D}$)

The general case

Conclusion

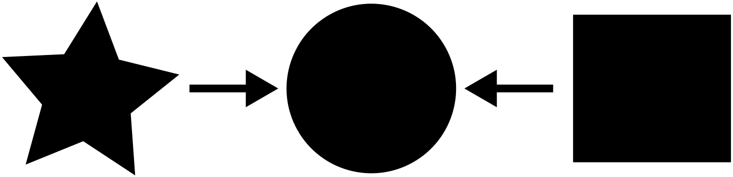
Conformal maps



- holomorphic function $f : x + \mathrm{i}y = z \mapsto w = u + \mathrm{i}v$:
- bijective: $f'(z) \neq 0$
- conjugate harmonic components:
  - $\Delta u(x, y) = \Delta v(x, y) = 0$
  - **and**: $\nabla u \perp \nabla v$
- existence by Riemann's theorem

Why do we care?

- Riemannian submanifold of shape space
- tracking of textured regions
- canonization



Related work

- spectral methods (Bergman, 1970):
  - $f'(z)$ from Bergman/Szegö kernel
  - contour integration
  - little intuitive
- polygonal approximations (Driscoll and Trefethen, 2002):
  - Schwarz-Christoffel formula
  - parameter problem
  - contour integration
- **boundary integral methods** (Symm, 1966)
- specialized algorithms (Wegmann, 2005)

Symm's ansatz

1. Observation: $f(z) = (z - z_0)\exp(\hat{f}(z))$ holomorphic
2. Let $\hat{f}(z) = u(z) + \mathrm{i}v(z)$,
   - u adjusts the modulus of $(z - z_0)$,
   - and v the argument.
3. On $\Gamma = \{z \subset \mathbb{C} \mid z = z(t)\}$, set $u = -\log|z - z_0|$.
4. In $\overline{\Omega}$, we have single layer potential representation:

$$u(z) = -\frac{1}{2\pi}\int \log|z - z(t)|\sigma(t)dt$$

5. Potential σ from *Symm's equation*

$$\frac{1}{2\pi}\int \log|z(s) - z(t)|\sigma(t)|z'(t)|dt = \log|z(s) - z_0|$$

Discretization

- NURBS approximation of Γ **and** σ:

$$z(t) = \sum_{i=1}^{n} z_j B_j(t), \quad \sigma(t) = \sum_{i=1}^{n} \sigma_j B_j(t)$$

- semi-discrete equation:

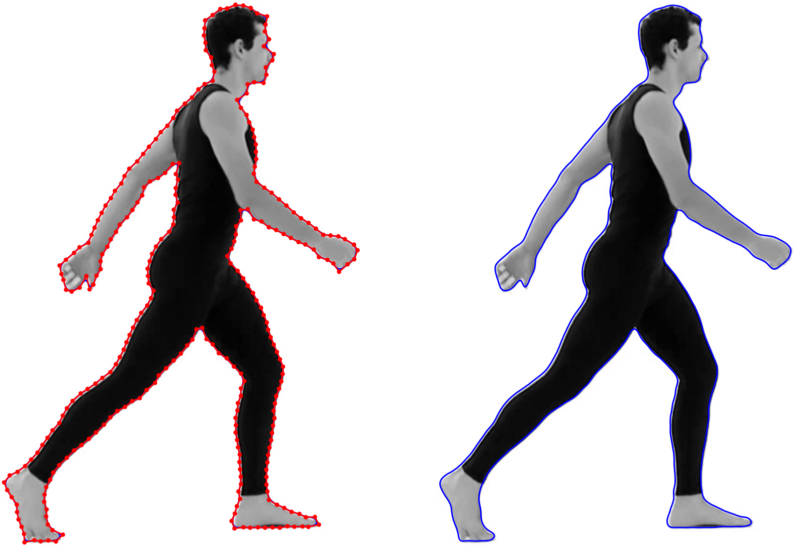
$$\sum_{i}^{n} \sigma_j \frac{1}{2\pi} \int \log|z(s) - z(t)| B_j(t) |z'(t)| dt = \log|z(s) - z_0|$$

- collocation at s_i, $i = 1, \ldots, n$:

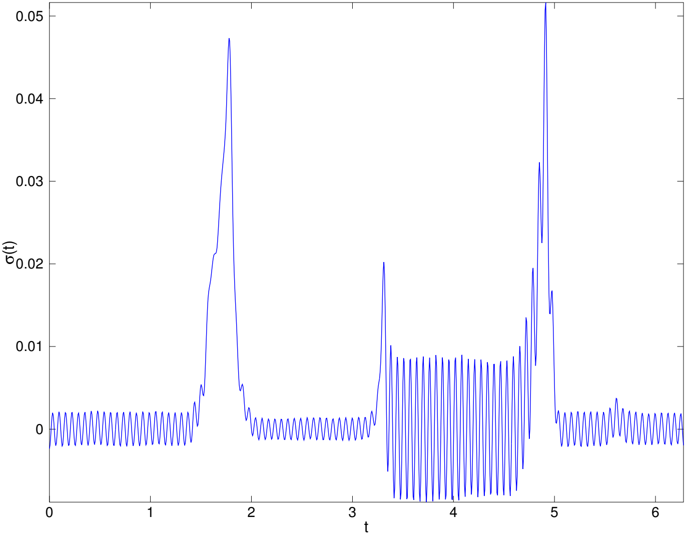
$$a_{ij} = \frac{1}{2\pi} \int \log|z(s_i) - z(t)| B_j(t) |z'(t)| dt, \quad b_i = \log|z(s_i) - z_0|$$

- adaptive Gauss-Kronod quadrature

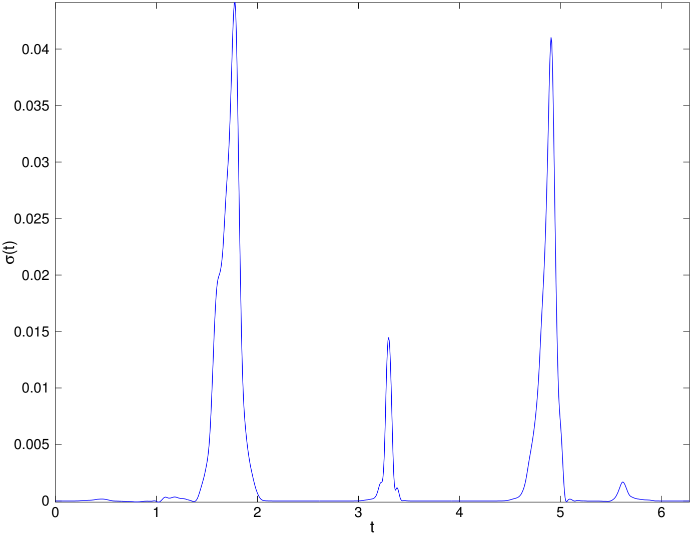
Geometric approximation



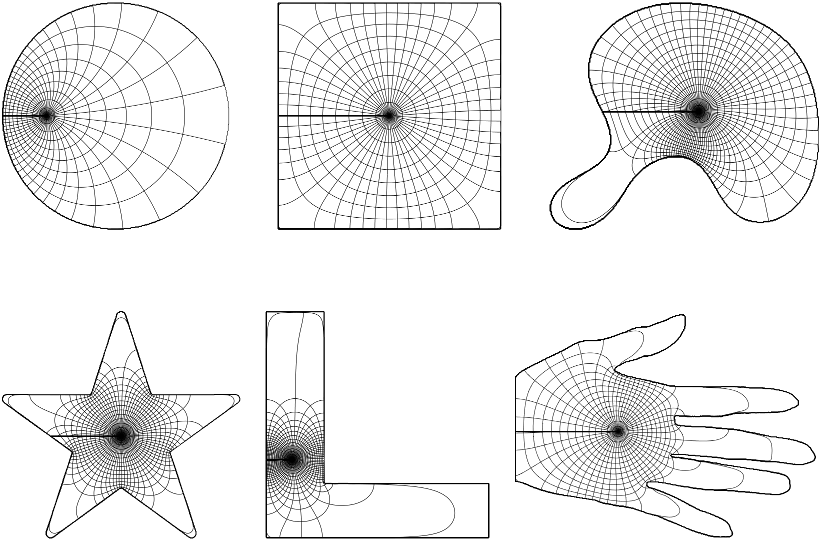
Symm's equation is first-kind Fredholm!



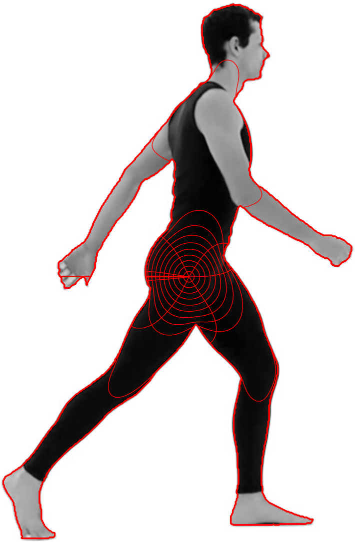
Tikhonov regularization

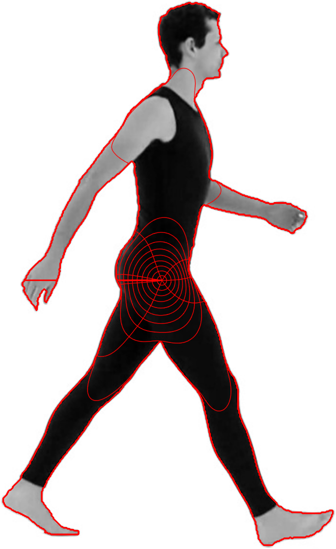


Results I



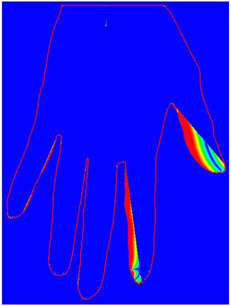
Results II

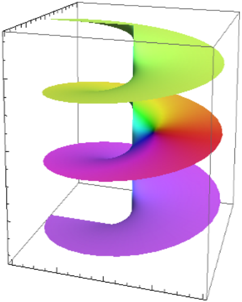


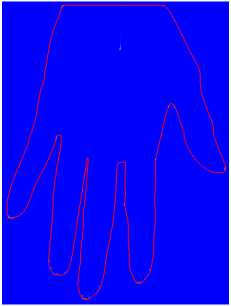


Making `atan2(y,x)` continuous

$$\arg(f(z)) = \arg(z - z_0) - \frac{1}{2\pi}\int \arg(z - z(t))|z'(t)|\sigma(t)dt$$







The inverse mapping f^{-1}

- $\theta(t) = \arg(f(z(t)))$: boundary correspondence function

- By Cauchy's formula:

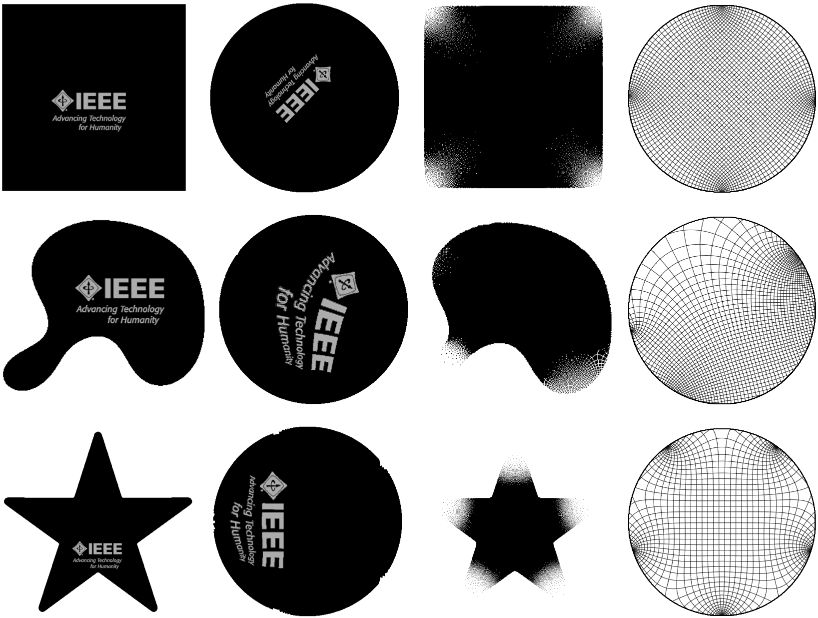
$$z = f^{-1}(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f^{-1}(\mathrm{e}^{\mathrm{i}\tau})\mathrm{e}^{\mathrm{i}\tau}}{\mathrm{e}^{\mathrm{i}\tau} - w} d\tau$$

- Using $\tau = \theta(t)$:

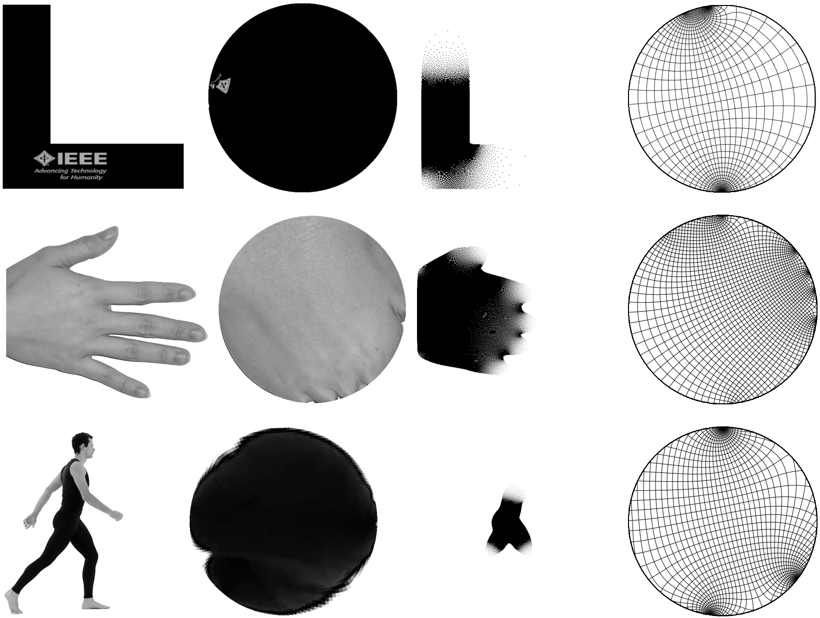
$$z = \frac{1}{2\pi} \int_0^{2\pi} \frac{z(t)\mathrm{e}^{\mathrm{i}\theta(t)}}{\mathrm{e}^{\mathrm{i}\theta(t)} - w} \theta'(t) dt$$

- Note that f^{-1} is never evaluated explicitly!

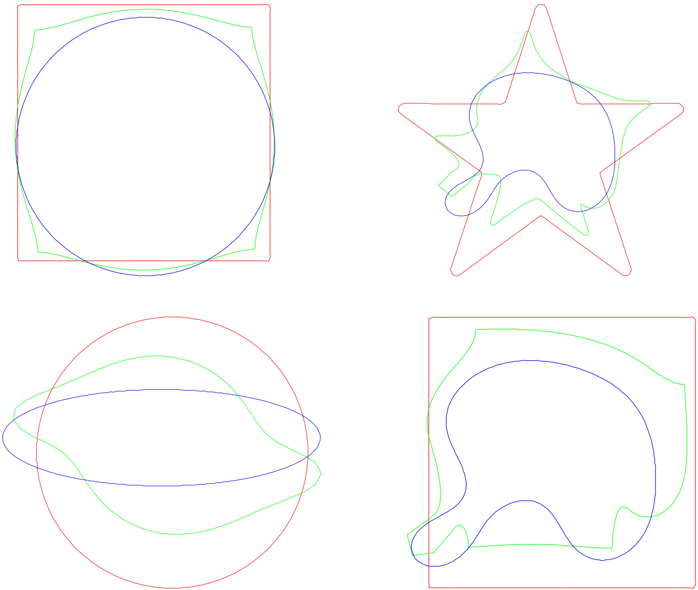
Results I



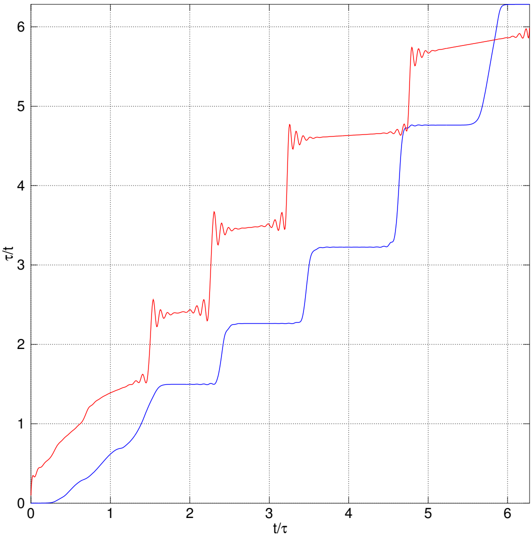
Results II



Interpolation



Inversion of θ



Work in progress

- Error metric on the boundary:

$$E(f) = \int_{\Gamma} u'(z)v'(z) + \partial_n u(z)\partial_n v(z)dz$$

- Everything in the interior is known by $\Delta u = \Delta v = 0$.

- Dirichlet-to-Neumann map $u(z) \mapsto \partial_n u(z)$ from

$$\int_{\Gamma} \partial_n u(z)\,\Gamma(z,\bar{z})dz = u(\bar{z}) + \int_{\Gamma} u(z)\,\partial_n \Gamma(z,\bar{z})dz$$

- $E(f) = 0$ by fixed-point scheme

To-do list

- direct mapping
- application: matching, tracking, etc.
- local h/p-refinement and coarsening
- parameter selection (Mozorov?)
- comparative studies

References

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