PHY293—Waves & Modern Physics

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Contents

1 Harmonic Oscillators

- Think of simple harmonic motion (SHM) as circular motion projected into one dimension. (a wave is rotation in the complex plane)
- One of the simplest system is a mass on the spring, with known force $F = -k\Delta x$.
- For **SHM**, the motion must be periodic, and the force must be proportional to displacement.
- Using Newton's 2nd law on the spring force,

$$m\ddot{x} = -kx \tag{1.1}$$
$$\ddot{x} + \frac{k}{m}x = 0 \tag{1.2}$$

- For a mass on the vertical spring, the equilibrium position will be lower due to gravity.

$$k(y_1 - y_0) - mg = 0 \tag{1.3}$$
$$y_1 = y_0 + \frac{mg}{k} \tag{1.4}$$

  y_1 is the new equilibrium position. SHM will still occur if the system is disturbed.
- Especially when dealing with energy, it is a good idea to have the origin at $y = y_1$.

1.1 The Differential Equation

- The solution to (1.2) can be represented as

$$x = A\cos(\omega t + \varphi_0) \tag{1.5}$$

- A represents the amplitude
- φ_0 is the phase angle (initial phase)
- $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency.

- The velocity and acceleration can be easily calculated by taking derivatives.

$$\dot{x} = -A\omega \sin(\omega t + \varphi_0) \tag{1.6}$$
$$\ddot{x} = -A\omega^2 \cos(\omega t + \varphi_0) \tag{1.7}$$

- Another way to represent the full solution is

$$x = a\cos(\omega t) + b\sin(\omega t) \tag{1.8}$$

where $a = A\cos\varphi_0, b = -A\sin\varphi_0$.

Example 1 ()

Determine the amplitude and phase constant of a pendulum moving with a motion described by a sum of two functions, $x_1(t) = 0.25\cos\omega t$ and $x_2(t) = -0.5\sin\omega t$.

Solution 1 ():

$$0.25 = A\cos\phi_0 \tag{1.9}$$
$$-0.50 = -A\sin\phi_0 \tag{1.10}$$

Figure out phi_0 from the ratio of eq.(1.10)/(1.9)

1.2 Energy of SHM

- A mass has kinetic energy of $T = \frac{1}{2}mv^2$.
- The potential energy is related to the restoring force and by definition,

$$\Delta U = -\int F \cdot \mathrm{d}x = -\int_{x_i}^{x^f} (-kx')\mathrm{d}x' = \frac{1}{2}k(x_f^2 - x_i^2) \tag{1.11}$$

- The conservation of energy here follows from newton's 2nd law.

$$m\ddot{x} = -kx \tag{1.12}$$

- Looking at the potential energy

$$x = A\cos(\omega t + \varphi_0) \tag{1.13}$$
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t) \tag{1.14}$$

- Looking at the kinetic energy,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t) \tag{1.15}$$

- The total energy is

$$E = T + \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}mv_{MAX}^2 \tag{1.16}$$

1.3 Physics of Small Vibrations

- Most system will oscillate with SHM when the amplitude is small. Recall the taylor expansion of an analytic function

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2 f''(a) + \ldots \tag{1.17}$$

If a is a minima, $f'(a) = 0$. For $x \approx a$, the higher order terms

- The potential energy of a pendulum is
$$U = mgy = mgL(1-\cos\theta) \tag{1.18}$$

- **For this course, an angle $\theta < 10°$, it will be considered small.**

- For a simple pendulum,
$$-mg\sin\theta = ma \tag{1.19}$$
$$-mg\sin\theta = m\ddot{s}, s = L\theta, \mathrm{d}s = L\mathrm{d}theta \tag{1.20}$$
$$-g\sin\theta = L\ddot{\theta} \tag{1.21}$$

  For small angles, $\theta \approx \sin\theta$
$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{1.22}$$

  This is the equation for SHM

- The energy of the pendulum is
$$E = T + U\frac{1}{2}mv^2 + mg\left(\frac{x^2}{2L}\right) \tag{1.23}$$