MAT257—Analysis

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Contents

1 Course Overview

- $\mathbb{R} \to \mathbb{R}^n$
- Linear Algebra
- Continuity
- Differentiability
- Integration
- Key theorem of this class is **Stokes' Theorem**

$$\int_C \mathrm{d}\omega = \int_{\partial C} \omega \tag{1.1}$$

  Generalizes the fundamental theorem of calculus:

$$\int_{[a,b]} F'(t)\mathrm{d}t = F(b) - F(a) = \int_{\partial[a,b]} F \tag{1.2}$$

  Note that $\partial[a,b] = \{b+, a-\}$.

2 Review

2.1 Continuity

- Roughly speaking, continuity from $\mathbb{R} \to \mathbb{R}$ means if two points are near, their images should be near also.
- Thus, in $\mathbb{R}^n$, the intuitive meaning should be similar.

Note there are 2 conventions for $\mathbb{R}^n$

1. The set of all n-dimensional real column vectors.
2. The set of all n-dimensional real row vectors.

In this class, the distinction is not very important.

Definition: For $x, y \in \mathbb{R}^n$, "The standard (or euclidian) inner product of x and y, denoted

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \tag{2.1}$$

The norm-squared of x is

$$|x|^2 = \langle x, x \rangle = \sum x_i^2 \tag{2.2}$$

and the norm of x is

$$|x| = \sqrt{|x|^2} = \sqrt{\sum x_i^2} \tag{2.3}$$

If $x, y, z \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, then

1. The inner product is bilinear & the norm is "semi-linear".

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle \tag{2.4}$$

$$\langle z, ax + by \rangle = \dots \tag{2.5}$$

$$|ax| = |a||x| \tag{2.6}$$

Aside:

$$1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1 \tag{2.7}$$

2.

$$|x| \geq 0 \& |x| = 0 \iff x = 0 \tag{2.8}$$

3.

$$\langle x, y \rangle = \langle y, x \rangle \tag{2.9}$$

4. *Cauchy-Schwarz inequality*

$$|\langle x, y \rangle| \leq |x||y| \tag{2.10}$$

with equality if $x \& y$ are dependent.

5. *Triangle inequality*

$$|x + y| \leq |x| + |y| \tag{2.11}$$

6. *Polarization identity*

$$\langle x, y \rangle = \frac{|x + y|^2 - |x - y|^2}{4} \tag{2.12}$$

Proof. 1. $|x| = \sqrt{\sum x_i^2}$ $|x| = 0 \implies \sum x_i^2 = 0 \implies \forall i, x_i^2 = 0 \implies \forall i, x_i = 0 \implies x = 0$

2. For $s, t \in \mathbb{R}^n$

$$|s + t|^2 = |s|^2 + |t|^2 + 2\langle s, t \rangle \tag{2.13}$$

Look at

$$0 \leq \Big| |y|^2 x - \langle x, y \rangle y \Big|^2 = |y|^4 |x| + \langle x, y \rangle^2 |y|^2 - 2|y|^2 \langle x, y \rangle^2 \tag{2.14}$$

$$= |y|^2 \left(|y|^2 |x|^2 - \langle x, y \rangle^2 \right) \tag{2.15}$$

This is equal to zero only if $|y|^2 x - \langle x, y \rangle y = 0$. If we have equality, that implies $x \& y$ are dependent. **Why, what does this mean?**

$\square$