

MAT292—ODE

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1 Newton's Law of Cooling

- Newton's law of cooling states that

$$\frac{du}{dt} = -k(u - T_0) \quad (1.1)$$

- Note that there is one trivial solution, the equilibrium solution is $u(t) = T_0$. The meaning of this solution is the temperature of an object doesn't change when it is already at the equilibrium temperature.

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$$\frac{\frac{du}{dt}}{u - T_0} = -k \quad (1.2)$$

$$\frac{d}{dt} \log(u - T_0) = -k \quad (1.3)$$

$$\log(u - T_0) = -kt + c_1 \quad (1.4)$$

$$u = T_0 + \exp(c_1) \exp(-kt) = T_0 + c_2 \exp(-kt) \quad (1.5)$$

Note that $c_2 = \exp(c_1) > 0$. However, this is not a complete solution as it cannot describe the solutions with $u < T_0$.

- **Warning:** note that the integral of $\frac{1}{x}$ is $\log|x|$, **NOT** $\log(x)$. This is what caused the solution to be incomplete.
- Hence, $c_2 = \pm \exp(c_1)$. Note that $c_1 = \pm\infty$ is allowed thus so c_2 can take any value.
- Below are the integral curves.

2 Classifications

Definition: A differential equation is an equation containing one or more unknown functions of one or more independent variables.

- The order of a differential equation is the order of the highest derivative
- The most general n-th order ODE:

$$F[t, y, y', y'', \dots, y^{(n)}] = 0 \quad (2.1)$$

- Linear ODE.
- Autonomous ODE is an ODE which does not explicitly depend on the independent variable, like $y' = y$. $y' = ty$ is not autonomous.
- Seperable first order ODE is a first order ODE that can be written as $y' = p(t)q(y)$.
- Newton's Law of cooling is first order, linear, autonomous, and seperable.

3 Systems of Differential Equations

- Think of a zombie apocalypse. You need to find a good time to find food.
- Let x be the number of people, and y be the number of zombies. This can be modelled by the *Lotka–Volterra* or Predator–Prey equations.

$$\frac{dx}{dt} = \alpha x - \beta xy \quad (3.1)$$

$$\frac{dy}{dt} = -\gamma y + \delta xy \quad (3.2)$$

- The term αx is inspired by short term population growth.
- The term $-\beta xy$ is inspired by the fact that zombies are eating people.
- The term $-\gamma y$ is inspired by the fact zombie die.
- The δxy term is inspired by the fact that people can be converted to zombies.
- Note that this is **not** a linear equation. The term xy is nonlinear. Let the dependent variable be $z = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$xy = z^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

- The most general quadratic form for vector (system of) equations is

$$z^T A z + b^T z + c \quad (3.3)$$

- Now take two twitter accounts, with each account telling its followers to unfollow the other account, with rates $m > 0, n > 0$ respectively. The accounts will naturally grow by word of mouth, with rates $k > 0, l > 0$ respectively. Note these constraints are important.

$$p' = kp - mo \quad (3.4)$$

$$o' = lo - np \quad (3.5)$$

- There are oversimplification for this model. It ignores the fact that when somebody unfollows they cannot unfollow again.

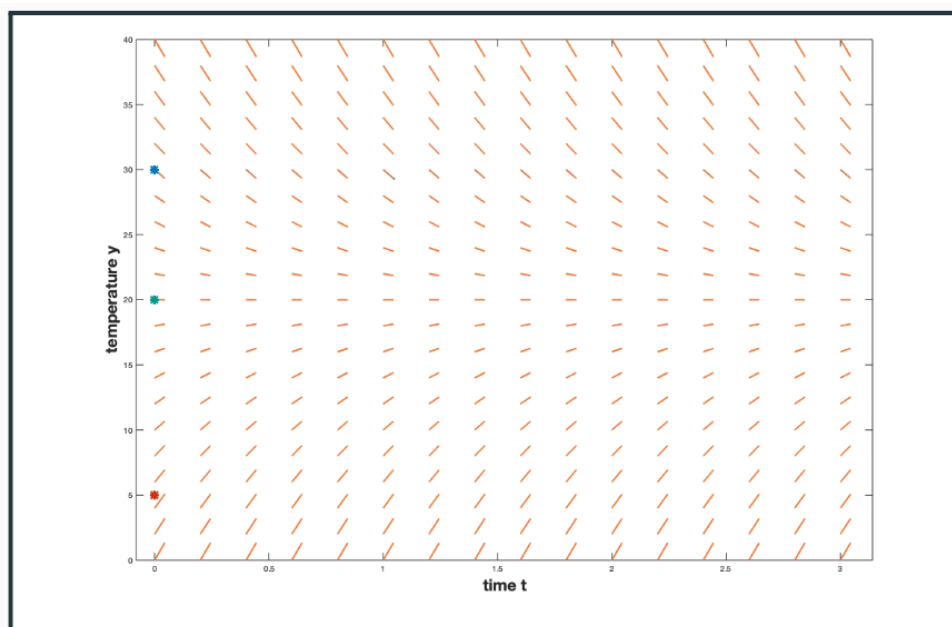
4 Qualitative Methods: Direction Fields and Phase Lines

Definition: Consider the ODE $y' = f(t, y)$. We can draw a **direction field** as follows:

- Draw a $t - y$ -coordinate system.
- Evaluate $f(t, y)$ over a rectangular grid of points.
- Draw a line at each point (t, y) of the grid with slope $f(t, y)$

- Let's look again at Newton's law of cooling:

$$y' = -1.5(y - 20) \quad (4.1)$$



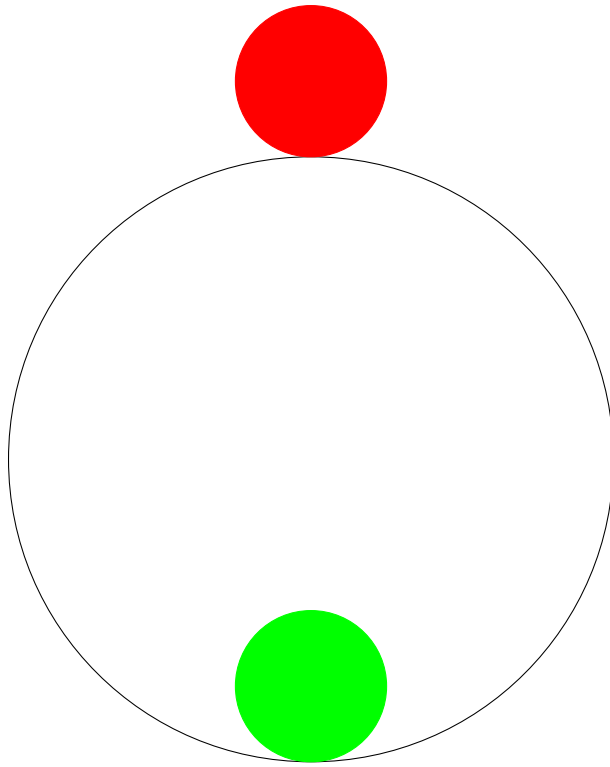
- Based on the initial conditions, we can draw the approximate solution by following direction field.
- A lot of the behavior of the differential equation are visible from the slope field.

Definition: Consider an autonomous first-order ODE $y' = f(y)$.

If $f(c) = 0$ for a specific value c , we call c an **equilibrium** of the ODE.

We say it is

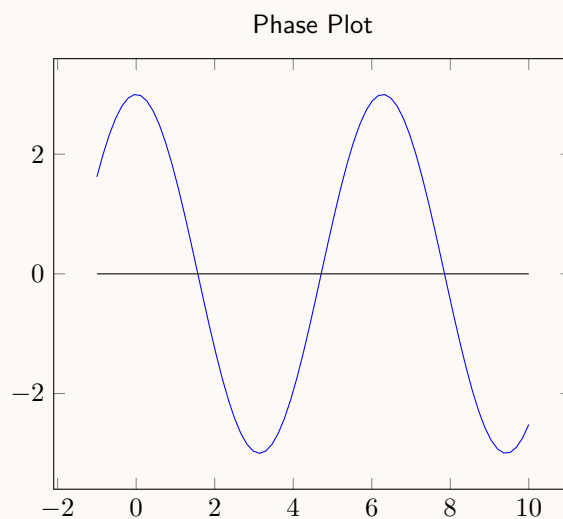
1. a **stable equilibrium**, if a solution starting at a value close to c approaches $y = c$ as $t \rightarrow \infty$.
2. an **unstable equilibrium**, if a solution starting at a value close to c moves away from $y = c$ as $t \rightarrow \infty$.
3. a **semistable equilibrium**, if we observe either behavior, depending on if the solution starts just above or below c .



- The red circle is in unstable equilibrium. The green circle is in stable equilibrium.
- Something resting on the saddle point of $y = x^3$ will be in semistable equilibrium.

Example 1

Find and classify the equilibria of the ODE $y' = 3 \cos y$



Solution 1 (): To find equilibrium, set $y' = 0$.

$$y' = 3 \cos y = 0 \quad (4.2)$$

$$y = \frac{\pi}{2} + n\pi, n \in \mathbb{Z} \quad (4.3)$$

At the equilibrium at $y = \frac{\pi}{2}$, In the phase diagram, anything below or above it s

5 Linear Equations: Method of Integrating Factors

- No general method for finding analytic solutions to first order differential equations.
- There exist classes of equations for which we know a corresponding solution method.

Definition: Standard form for a first order linear differential equation:

$$\frac{dy}{dt} + p(t)y = g(t) \quad (5.1)$$

- For newton's law of cooling, the standard form is

$$\frac{du}{dt} + ku = kT_0 \quad (5.2)$$

Derivation:

$$\frac{dy}{dt} + p(t)y = g(t) \quad (5.3)$$

$$\mu(t) \frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \quad (5.4)$$

$$\mu(t)p(t) = \mu'(t) \quad (5.5)$$

$$p(t) = \frac{\mu'(t)}{\mu(t)} = \frac{d}{dt} \log(\mu(t)) \quad (5.6)$$

$$\mu(t) = \exp\left(\int p(t)dt\right) \quad (5.7)$$

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)g(t) \quad (5.8)$$

$$\mu(t)y(t) = \int \mu(t')g(t')dt' \quad (5.9)$$

$$y(t) = \frac{\int \mu(t')g(t')dt' + C}{\mu(t)} \quad (5.10)$$

Note that the C is in the numerator and **must** be divided by $\mu(t)$.

Example 2

$$\frac{du}{dt} = -k(u - T_0 - A \sin(\omega t)) \quad (5.11)$$

In standard form,

$$\frac{du}{dt} + ku = kT_0 + kA \sin(\omega t) \quad (5.12)$$

Calculate the integrating factor

$$\mu(t) = \exp\left(\int k dt\right) = \exp(kt) \quad (5.13)$$

Calculate the general solution

$$y = \exp(-kt) \int \exp(kt')k(T_0 + A \sin(\omega t'))dt' \quad (5.14)$$

$$= T_0 + \frac{kA}{k^2 + \omega^2} (k \sin(\omega t) - \omega \cos(\omega t)) + C \exp(-kt) \quad (5.15)$$

6 Existence and Uniqueness of Solutions

- Uniqueness is the question if a model can only follow one process or not.

Theorem: [Existence and uniqueness for linear first-order ODEs] Consider the IVP $y' + p(t)y = g(t)$ with initial value $y(t_0) = y_0$ and an interval $I = (\alpha, \beta)$. If

1. $t_0 \in I$
2. $p(t)$ continuous on I
3. $g(t)$ continuous on I

Then,

1. This IVP has a solution and this solution is unique.
2. This solution exists for all time $t \in I$.
3. The ODE has a general solution that depends on one free parameter.

Proof. Integrating factor method constructs the unique solution. □

Example 3

$ty' + 2y = 4t^2$, $y(1) = 2$. As $t \neq 0$, $t' + 2\frac{y}{t} = 4t$.

Pick $I = (0, \infty)$. $p(t), g(t)$ are continuous on I . Thus, this IVP has a unique solution.

The general solution is $y = t^2 + \frac{C}{t^2}$.

Theorem: [Picard–Lindelöf] Consider the IVP $y' = f(t, y)$ with initial value $y(t_0) = y_0$. Consider furthermore an open rectangle $R = (\alpha, \beta) \times (\gamma, \delta)$ in the $t - y$ plane. If

1. $(t_0, y_0) \in R$
2. f is continuous in R .
3. $\frac{\partial f}{\partial y}$ is continuous in R .

Then the IVP has a unique solution. ~~The solution exists for $t \in (\alpha, \beta)$.~~ The solution exist for some interval $(t_0 - h, t_0 + h)$ where $(t_0 - h, t_0 + h) \subset (\alpha, \beta)$.

Remarks:

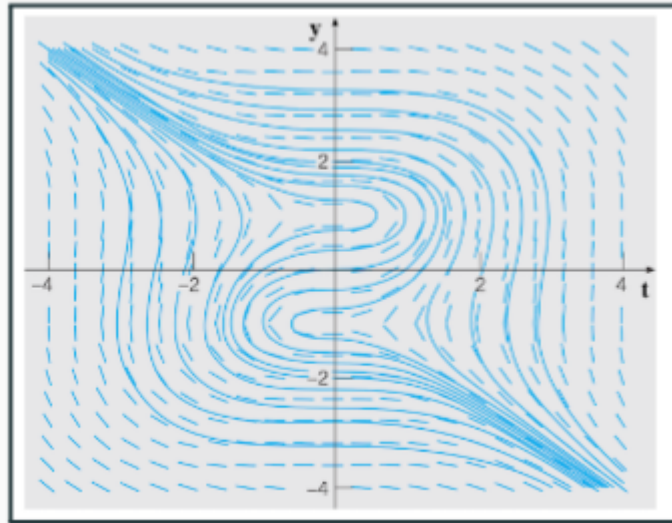
1. Non-linear ODEs don't necessarily have a general solution.
2. The solution might be implicit. e.g. $\sqrt{y^2 + \log y} = 5t$

Clarifications:

- We need f continuous in the rectangle to get existence.
- We need $\frac{\partial f}{\partial y}$ continuous in the rectangle to get uniqueness.

Proof. The proof for this theorem is beyond the scope of this class. □

- Note that these theorems are **sufficient** conditions for existence and uniqueness. Even if the hypotheses are not satisfied, it may be possible that existence and uniqueness holds. However, they are not guaranteed by the theorem.
- If existence and uniqueness holds, solution curves **cannot** cross each other.
- For the equation $y' = \frac{t^2}{1 - y^2}$, Picard–Lindelöf guarantees local existence and uniqueness, however, the existence and uniqueness at given initial conditions only lasts for a finite interval.



7 Autonomous Equations and Population Dynamics

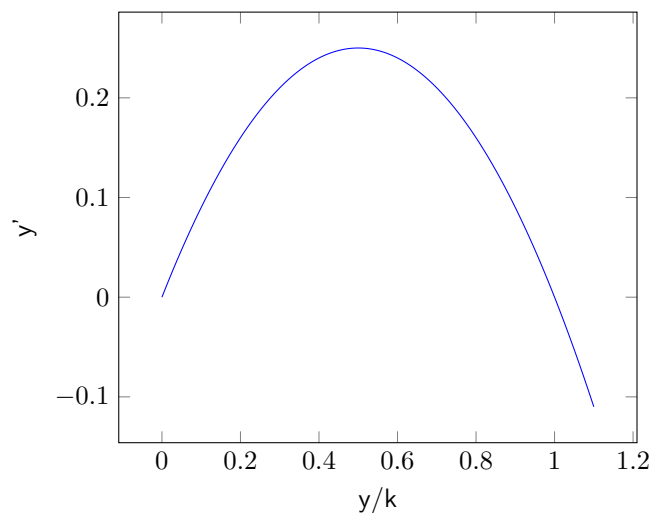
- Exponential Growth $y' = ky$.
- Logistic Growth:
 - If uninhibited, we assume exponential growth.
 - However, in the long run, population is limited to k . Consider the model

$$y' = ryh(y) \quad (7.1)$$

We would want several properties about $h(y)$.

- * $h(y) \approx 1$ if $y \ll k$.
- * $0 < h(y) < 1$ if $y < k$.
- * $h(y) = 0$ if $y = k$.
- * $h(y) < 0$ if $y > k$.
- To keep the model as simple as possible, we will fulfill all the conditions with a linear function $h(y) = 1 - \frac{y}{k}$.
- We arrive at the ODE

$$y' = r \left(1 - \frac{y}{k}\right) y \quad (7.2)$$
- As this is an autonomous ODE, we can draw a phase plot.



- There are two fixed points. $y/k = 0$ is unstable and $y/k = 1$ is stable.
- If $y/k < 1/2$, then the solution will have an inflection point.