MAT257—Analysis

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Contents

1 Course Overview

- $\mathbb{R} \to \mathbb{R}^n$
- Linear Algebra
- Continuity
- Differentiability
- Integration
- Key theorem of this class is **Stokes' Theorem**

$$\int_C \mathrm{d}\omega = \int_{\partial C} \omega \tag{1.1}$$

Generalizes the fundamental theorem of calculus:

$$\int_{[a,b]} F'(t)\mathrm{d}t = F(b) - F(a) = \int_{\partial[a,b]} F \tag{1.2}$$

Note that $\partial[a, b] = \{b+, a-\}$.

2 Continuity

- Roughly speaking, continuity from $\mathbb{R} \to \mathbb{R}$ means if two points are near, their images should be near also.
- Thus, in $\mathbb{R}^n$, the intuitive meaning should be similar.

2.1 Norms and Inner Product

Note there are 2 conventions for $\mathbb{R}^n$

1. The set of all n-dimensional real column vectors.
2. The set of all n-dimensional real row vectors.

In this class, the distinction is not very important.

Definition: For $x, y \in \mathbb{R}^n$, "The standard (or euclidian) inner product of x and y, denoted

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \tag{2.1}$$

The norm-squared of x is

$$|x|^2 = \langle x, x\rangle = \sum x_i^2 \tag{2.2}$$

and the norm of x is

$$|x| = \sqrt{|x|^2} = \sqrt{\sum x_i^2} \tag{2.3}$$

Proposition: If $x, y, z \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, then

1. The inner product is bilinear & the norm is "semi-linear".

$$\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle \tag{2.4}$$

$$\langle z, ax + by\rangle = \dots \tag{2.5}$$

$$|ax| = |a||x| \tag{2.6}$$

Aside:

$$1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1 \tag{2.7}$$

2.

$$|x| \geq 0 \& |x| = 0 \iff x = 0 \tag{2.8}$$

3.

$$\langle x, y\rangle = \langle y, x\rangle \tag{2.9}$$

4. *Cauchy-Schwarz inequality*

$$|\langle x, y\rangle| \leq |x||y| \tag{2.10}$$

with equality if $x \& y$ are dependent.

5. *Triangle inequality*

$$|x + y| \leq |x| + |y| \tag{2.11}$$

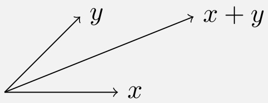
6. *Polarization identity*

$$\langle x, y\rangle = \frac{|x + y|^2 - |x - y|^2}{4} \tag{2.12}$$

Proof:

1. $|x| = \sqrt{\sum x_i^2}$ $|x| = 0 \implies \sum x_i^2 = 0 \implies \forall i, x_i^2 = 0 \implies \forall i, x_i = 0 \implies x = 0$

2. For $s, t \in \mathbb{R}^n$
$$|s+t|^2 = |s|^2 + |t|^2 + 2\langle s, t\rangle \tag{2.13}$$
Look at
$$0 \le \Big| |y|^2 x - \langle x, y\rangle y \Big|^2 = |y|^4|x| + \langle x, y\rangle^2 |y|^2 - 2|y|^2\langle x, y\rangle^2 \tag{2.14}$$
$$= |y|^2\left(|y|^2|x|^2 - \langle x, y\rangle^2\right) \tag{2.15}$$
This is equal to zero only if $|y|^2 x - \langle x, y\rangle y = 0$. If we have equality, that implies $x \& y$ are dependent.
Why, what does this mean?

3. 

As both sides of the triangle inequality are ≥ 0, square both sides.
$$|x+y|^2 \overset{?}{\le} (|x| + |y|)^2 \tag{2.16}$$
$$\langle x+y, x+y\rangle \overset{?}{\le} |x|^2 + |y|^2 + 2|x||y| \tag{2.17}$$
$$\langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \overset{?}{\le} |x|^2 + |y|^2 + 2|x||y| \tag{2.18}$$
$$|x|^2 + |y|^2 + 2\langle x, y\rangle \overset{?}{\le} |x|^2 + |y|^2 + 2|x||y| \tag{2.19}$$
$$\langle x, y\rangle \overset{?}{\le} |x||y| \tag{2.20}$$
(2.20) is true by cauchy-schwarz.

4. The proof is trivial because you can expand the right hand side.
Note: The inner product and the norm are not independent. If you know how to compute one, you can compute the other.

Definition: If $x, y \in \mathbb{R}^n$,define the distance between $x \& y$
$$d(x, y) = |x - y| \tag{2.21}$$

Theorem:

1. d is symmetric: $d(x, y) = d(y, x)$
2. d is positive definite: $d(x, y) \ge 0$ and $d(x, y) = 0 \iff x = y$
3. Triangle inequality: $d(x, z) \le d(x, y) + d(y, z)$

The significance of this theorem is that this is all we need to know about distances to comment on continuity.
Aside: Later, this theorem will become a definition for a distance function or a metric.

Proof:

1.

$$d(x, y) = |x - y| = |-(y - x)| = |-1||y - x| = |y - x| = d(y, x) \tag{2.22}$$

2.

$$d(x, y) = 0 \iff |x - y| = 0 \iff x - y = 0 \iff x = y \tag{2.23}$$

3.

$$|x - z| \overset{?}{\leq} |x - y| + |y - z| \tag{2.24}$$

This is true by the previous triangle inequality, $|p| + |q| \geq |p + q|$. Letting $p = x - y, q = y - z \implies p + q = x - z$.

There are other norms and distance functions that we will rarely use.

- The euclidian norm which we use is $|x|_{L^2} = \sqrt{\sum x_i^2}$.
- There is a L1 norm $|x|_{L^1} = \sum |x_i|$.
- The L-infinity norm is $|x|_{L^\infty} = \max |x_i|$.

The distance functions for these norms also satisfys these three properties.