MAT257—Analysis

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September 17, 2021

Contents

1 Course Overview

- $\mathbb{R} \to \mathbb{R}^n$
- Linear Algebra
- Continuity
- Differentiability
- Integration
- Key theorem of this class is **Stokes' Theorem**

$$\int_C \mathrm{d}\omega = \int_{\partial C} \omega \tag{1.1}$$

Generalizes the fundamental theorem of calculus:

$$\int_{[a,b]} F'(t)\mathrm{d}t = F(b) - F(a) = \int_{\partial[a,b]} F \tag{1.2}$$

Note that $\partial[a,b] = \{b+, a-\}$.

2 Continuity

- Roughly speaking, continuity from $\mathbb{R} \to \mathbb{R}$ means if two points are near, their images should be near also.
- Thus, in $\mathbb{R}^n$, the intuitive meaning should be similar.

2.1 Norms and Inner Product

Note there are 2 conventions for $\mathbb{R}^n$

1. The set of all n-dimensional real column vectors.
2. The set of all n-dimensional real row vectors.

In this class, the distinction is not very important.

Definition: For $x, y \in \mathbb{R}^n$, "The standard (or euclidian) inner product of x and y, denoted

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \tag{2.1}$$

The norm-squared of x is

$$|x|^2 = \langle x, x\rangle = \sum x_i^2 \tag{2.2}$$

and the norm of x is

$$|x| = \sqrt{|x|^2} = \sqrt{\sum x_i^2} \tag{2.3}$$

Proposition: If $x, y, z \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, then

0. The inner product is bilinear & the norm is "semi-linear".

$$\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle \tag{2.4}$$

$$\langle z, ax + by\rangle = \dots \tag{2.5}$$

$$|ax| = |a||x| \tag{2.6}$$

Aside:

$$1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1 \tag{2.7}$$

1. $$|x| \geq 0 \& |x| = 0 \iff x = 0 \tag{2.8}$$

2. $$\langle x, y\rangle = \langle y, x\rangle \tag{2.9}$$

3. *Cauchy-Schwarz inequality*

$$|\langle x, y\rangle| \leq |x||y| \tag{2.10}$$

with equality if $x \& y$ are dependent.

4. *Triangle inequality*

$$|x + y| \leq |x| + |y| \tag{2.11}$$

5. *Polarization identity*

$$\langle x, y\rangle = \frac{|x + y|^2 - |x - y|^2}{4} \tag{2.12}$$

Proof:

1. $|x| = \sqrt{\sum x_i^2}$ $|x| = 0 \implies \sum x_i^2 = 0 \implies \forall i, x_i^2 = 0 \implies \forall i, x_i = 0 \implies x = 0$
2. For $s, t \in \mathbb{R}^n$

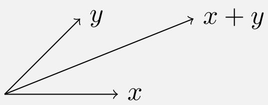
$$|s+t|^2 = |s|^2 + |t|^2 + 2\langle s, t\rangle \tag{2.13}$$

Look at

$$0 \leq \Big||y|^2 x - \langle x, y\rangle y\Big|^2 = |y|^4|x| + \langle x, y\rangle^2 |y|^2 - 2|y|^2\langle x, y\rangle^2 \tag{2.14}$$

$$= |y|^2\left(|y|^2|x|^2 - \langle x, y\rangle^2\right) \tag{2.15}$$

This is equal to zero only if $|y|^2 x - \langle x, y\rangle y = 0$. If we have equality, that implies $x \& y$ are dependent. **Why, what does this mean?**

3. 

As both sides of the triangle inequality are ≥ 0, square both sides.

$$|x+y|^2 \stackrel{?}{\leq} (|x| + |y|)^2 \tag{2.16}$$

$$\langle x+y, x+y\rangle \stackrel{?}{\leq} |x|^2 + |y|^2 + 2|x||y| \tag{2.17}$$

$$\langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \stackrel{?}{\leq} |x|^2 + |y|^2 + 2|x||y| \tag{2.18}$$

$$|x|^2 + |y|^2 + 2\langle x, y\rangle \stackrel{?}{\leq} |x|^2 + |y|^2 + 2|x||y| \tag{2.19}$$

$$\langle x, y\rangle \stackrel{?}{\leq} |x||y| \tag{2.20}$$

(2.20) is true by cauchy-schwarz.

4. The proof is trivial because you can expand the right hand side.
Note: The inner product and the norm are not independent. If you know how to compute one, you can compute the other.

2.2 Distance Functions

Definition: If $x, y \in \mathbb{R}^n$,define the distance between $x \& y$

$$d(x, y) = |x - y| \tag{2.21}$$

Theorem:

1. d is symmetric: $d(x, y) = d(y, x)$
2. d is positive definite: $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

The significance of this theorem is that this is all we need to know about distances to comment on continuity.
Aside: Later, this theorem will become a definition for a distance function or a metric.

Proof:

1.
$$d(x,y) = |x-y| = |-(y-x)| = |-1||y-x| = |y-x| = d(y,x) \tag{2.22}$$

2.
$$d(x,y) = 0 \iff |x-y| = 0 \iff x-y = 0 \iff x = y \tag{2.23}$$

3.
$$|x-z| \overset{?}{\leq} |x-y| + |y-z| \tag{2.24}$$

   This is true by the previous triangle inequality, $|p|+|q| \geq |p+q|$. Letting $p = x-y, q = y-z \implies p+q = x-z$.

There are other norms and distance functions that we will rarely use.

- The euclidian norm which we use is $|x|_{L^2} = \sqrt{\sum x_i^2}$.
- There is a L1 norm $|x|_{L^1} = \sum |x_i|$.
- The L-infinity norm is $|x|_{L^\infty} = \max |x_i|$.

The distance functions for these norms also satisfys these three properties.

- There is a bijection between linear maps from $\mathbb{R}^n \to \mathbb{R}^m$ and the set of $m \times n$ matrices with real coefficients. This bijection can be realized by choosing a basis.
- In $\mathbb{R}^n$ or $\mathbb{R}^m$, there is a natural basis (the standard basis) $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i-$th position
- by $A \in M_{m\times n} \to L_A(x) = Ax$, where $x \in \mathbb{R}^n$.
- If T is a linear transformation, $M_T = \Bigg(Te_1 | Te_2 | \ldots | Te_n \Bigg)$

Definition: Homomorphism: A map that preserve the structure.

Theorem:

0. Bijective
$$L_{(M_T)} = T, M_{(L_A)} = A \tag{2.25}$$

1.
   - $A \to L_A$ is linear: $L_{aA+bB} = aL_A + bL_b$
   - $T \to M_T$ is linear: $M_{aT+bS} = aM_T + bM_S$

2. Given $T : \mathbb{R}^n \to \mathbb{R}^m, S : \mathbb{R}^m \to \mathbb{R}^p$, and $S \circ T \equiv T//S : \mathbb{R}^n \to \mathbb{R}^p$.
   Then, $M_S M_T = M_{S\circ T}$.

End of the review.

3 Rectangles

- It is common to use intervals in $\mathbb{R}$. In $\mathbb{R}^n$, we use rectangles.
- To specify a rectangle, we must bound the each of the n coordinates.

Definition: Given $a_i \leq b_i$, where $i = 1, \ldots, n$,

- The closed rectangle corresponding to a_i, b_i is defined as

$$R = \prod_{i=1}^{n} [a_i, b_i] = \{x \in \mathbb{R}^n : \forall i \;\; a_i \leq x_i \leq b_i\} \tag{3.1}$$

- The opened rectangle defined by a_i, b_i is defined as

$$R = \prod_{i=1}^{n} (a_i, b_i) = \{x \in \mathbb{R}^n : \forall i \;\; a_i < x_i < b_i\} \tag{3.2}$$

- If $X \& Y$ are sets, we define (from set theory) the cartesian product $X \times Y = \{(x, y) : x \in X, y \in Y\}$
- Given 3 sets, the cartesian product is strictly speaking not associative as $((x, y), z) \neq (x, (y, z))$. However, for convinence we agree that $((x, y), z) = (x, y, z) = (x, (y, z))$. Thus, the cartesian product is associative.
- $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$.

Definition:

- $A \subset \mathbb{R}^n$ is called an open set if $\forall a \in A \exists$ an open rectangle $R : x \in R \subset A$.
- $A \subset \mathbb{R}^n$ is called an open set if $\forall a \in A \exists$ an open ball $B : x \in B \subset A$. An open ball $B = B_r(x_0) = \{y \in \mathbb{R}^n : |x_0 - y| < r\}$ Note an open ball can be defined with any norm.

Theorem: Defining "open" using rectangles is equivalent to define "open" using balls.

> **Proof**: $\implies$ Every open rectangle is open using the ball definition.
> $\impliedby$ Every open ball is open using the rectangle definition.

Definition: A set B is "closed" if $\mathbb{R}^n \;\, B = B^C$ is open.

Proposition: [**De-Morgan's Laws**]
If Y_α is any collection of subsets of some universe U,

$$\left(\bigcup Y_\alpha\right)^C = \bigcap Y_\alpha^C \tag{3.3}$$

$$\left(\bigcap Y_\alpha\right)^C = \bigcup Y_\alpha^C \tag{3.4}$$

Theorem:

1. $\emptyset, \mathbb{R}^n$ are clopen.
2. Any union of open sets is open. Any intersection of closed sets is closed.
3. A finite intersection of open sets is open. A finite union of closed sets is closed.

Proof:

1. $\mathbb{R}^n$ is open. $x \in \prod(x_i - 1, x_i + 1) \subset \mathbb{R}^n$. $\implies$ $\emptyset$ is closed. The empty set has no points, thus the condition holds. "Every horse in an empty set of horses has horns." $\implies$ $\mathbb{R}^n$ is closed.

2. Suppose $\{A_\alpha\}_{\alpha \in I}$, where I is an arbiturary indexing set, is a collection of open sets.

$$A = \bigcup_{\alpha \in I} A_\alpha = \{x : \exists \alpha \in I \;\; x \in A_\alpha\} \tag{3.5}$$

   Let $x \in A$, find α such that $s \in A_\alpha$. Find an open rectangle R such that $x \in R \subset A_\alpha \subset A$

   Suppose $\{B_\alpha\}_{\alpha \in I}$ is a collection of closed sets, show $\cap B_\alpha$ is closed. $\left(\bigcap B_\alpha\right)^C = \bigcup B_\alpha^C$ is open $\implies$ $\bigcap B_\alpha^C$ is closed.

3. > **Lemma 1:** The intersection of two open rectangles, if non-empty, is an open rectangle.

   Suppose A_1 and A_2 are open. Pick $x \in A_1 \cap A_2$. By openness of A_1, $x \in A_1 \implies \exists R_1 : x \in \mathbb{R}_1 \subset A_1$. Similarly, by openness of A_2, $x \in A_2 \implies \exists R_2 : x \in \mathbb{R}_2 \subset A_2$. Then, $x \in R_1 \cap R_2 \equiv R \subset A_1 \cap A_2$.

   Suppose A_i, $i = 1, \ldots, n$ are open.

$$\bigcap_{i=1}^{n} A_i = \left(\bigcap_{i=1}^{n-1} A_i\right) \bigcap A_n \tag{3.6}$$

   By induction hypothesis, $\left(\bigcap_{i=1}^{n-1} A_i\right)$ is an open set. The intersection of two open sets are open $\implies$ the intersection of n open sets are open.

   Suppose B_i, $i = 1, \ldots, n$ is closed,

$$\left(\bigcup_{i=1}^{n} B_i\right)^C = \bigcup_{i=1}^{n} B_i^C \tag{3.7}$$

- Consider the following example,

$$\bigcap_{n>0} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0, 1] \tag{3.8}$$

  This infinite intersection of open sets is not an open set due to the pointS 0 and 1.

Clopen Sets: Suppose $A \subset \mathbb{R}^n$ is clopen $\implies$ A^C is clopen. Suppose neither is empty.

Consider the line segment $l_{xy}(t) = ty + (1 - t)x$.

$$l_{xy}(0) = x \in A \tag{3.9}$$
$$l_{xy}(1) = y \in A^C \tag{3.10}$$
$$t_0 = \sup_{t \in [0,1]} \{l_{xy}(t) \in A\} \tag{3.11}$$
$$l_{xy}(t_0) = z \tag{3.12}$$

if $z \in A$, the rectangele containing $z \cap l_{xy}$ includes $l(t_0 + \epsilon) \in A^C$ for some ϵ.

Similarly if $z \in A^C$ $\implies$ one of A and A^C is not clopen so the other one isn't clopen either.

Thus, the only clopen sets is $\emptyset$ and $\mathbb{R}^n$