PHY293—Waves & Modern Physics

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Contents

1 Harmonic Oscillators

- Think of simple harmonic motion (SHM) as circular motion projected into one dimension. (a wave is rotation in the complex plane)
- One of the simplest system is a mass on the spring, with known force $F = -k\Delta x$.
- For **SHM**, the motion must be periodic, and the force must be proportional to displacement.
- Using Newton's 2nd law on the spring force,

$$m\ddot{x} = -kx \tag{1.1}$$
$$\ddot{x} + \frac{k}{m}x = 0 \tag{1.2}$$

- For a mass on the vertical spring, the equilibrium position will be lower due to gravity.

$$k(y_1 - y_0) - mg = 0 \tag{1.3}$$
$$y_1 = y_0 + \frac{mg}{k} \tag{1.4}$$

  y_1 is the new equilibrium position. SHM will still occur if the system is disturbed.
- Especially when dealing with energy, it is a good idea to have the origin at $y = y_1$.

1.1 The Differential Equation

- In general, the equation for simple harmonic motion is

$$\ddot{x} + \omega^2 x = 0 \tag{1.5}$$

- The solution to (1.2) can be represented as

$$x = A\cos(\omega t + \varphi_0) \tag{1.6}$$

- A represents the amplitude
- φ_0 is the phase angle (initial phase)
- $\omega = \sqrt{\dfrac{k}{m}}$ is the angular frequency.
- The velocity and acceleration can be easily calculated by taking derivatives.

$$\dot{x} = -A\omega\sin(\omega t + \varphi_0) \tag{1.7}$$

$$\ddot{x} = -A\omega^2\cos(\omega t + \varphi_0) \tag{1.8}$$

- Another way to represent the full solution is

$$x = a\cos(\omega t) + b\sin(\omega t) \tag{1.9}$$

where $a = A\cos\varphi_0, b = -A\sin\varphi_0$.

Example 1

Determine the amplitude and phase constant of a pendulum moving with a motion described by a sum of two functions, $x_1(t) = 0.25\cos\omega t$ and $x_2(t) = -0.5\sin\omega t$.

Solution 1 ():

$$0.25 = A\cos\phi_0 \tag{1.10}$$

$$-0.50 = -A\sin\phi_0 \tag{1.11}$$

Figure out phi_0 from the ratio of eq.(1.11)/(1.10)

1.2 Energy of SHM

- A mass has kinetic energy of $T = \dfrac{1}{2}mv^2$.
- The potential energy is related to the restoring force and by definition,

$$\Delta U = -\int F \cdot \mathrm{d}x = -\int_{x_i}^{x^f} (-kx')\mathrm{d}x' = \frac{1}{2}k(x_f^2 - x_i^2) \tag{1.12}$$

- The conservation of energy here follows from newton's 2nd law.

$$m\ddot{x} = -kx \tag{1.13}$$

- Looking at the potential energy

$$x = A\cos(\omega t + \varphi_0) \tag{1.14}$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t) \tag{1.15}$$

- Looking at the kinetic energy,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t) \tag{1.16}$$

- The total energy is

$$E = T + \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}mv_{MAX}^2 \tag{1.17}$$

1.3 Physics of Small Vibrations

- Most system will oscillate with SHM when the amplitude is small. Recall the taylor expansion of an analytic function

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2 f''(a) + \dots \tag{1.18}$$

  If a is a minima, $f'(a) = 0$. For $x \approx a$, the higher order terms

- The potential energy of a pendulum is

$$U = mgy = mgL(1-\cos\theta) \tag{1.19}$$

- **For this course, an angle $\theta < 10°$, it will be considered small.**
- For a simple pendulum,

$$-mg\sin\theta = ma \tag{1.20}$$
$$-mg\sin\theta = m\ddot{s}, s = L\theta, \mathrm{d}s = L\mathrm{d}theta \tag{1.21}$$
$$-g\sin\theta = L\ddot{\theta} \tag{1.22}$$

  For small angles, $\theta \approx \sin\theta$

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{1.23}$$

  This is the equation for SHM

- The energy of the pendulum is

$$E = T + U = \frac{1}{2}mv^2 + mg\left(\frac{x^2}{2L}\right) \tag{1.24}$$

- For physical pendulum,

$$\tau = I\alpha = I\ddot{\theta} \tag{1.25}$$
$$-mgd\sin\theta = I\ddot{\theta} \tag{1.26}$$

  For small $\theta : \sin\theta \approx \theta$

$$\ddot{\theta} + \frac{mgd}{I}\theta = 0 \tag{1.27}$$

- For LC circuits,

$$\ddot{i} + \frac{1}{LC}i = 0 \tag{1.28}$$

  When a resistor is connected, energy is lost as heat and the circuit behaves like a damped oscilator.

2 Damped Oscillations

The damped oscillator involves the addition to the drag force that is proportional to $-v$ to the simple harmonic oscillator.

Define

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0 \tag{2.1}$$
$$x = A\exp\left(\left(\sqrt{\frac{\gamma^2}{4} - \omega_0^2} - \frac{\gamma}{2}\right)t\right) + B\exp\left(-\left(\sqrt{\frac{\gamma^2}{4} - \omega_0^2} - \frac{\gamma}{2}\right)t\right) \tag{2.2}$$

For $\omega_0^2 \neq \dfrac{\gamma^2}{4}$.

Critical damping gives the minimum time for the system to return to equilibrium when $\omega_0^2 \neq \dfrac{\gamma^2}{4}$.

$$x = (Ax + B)\exp(-\omega_0 t) \tag{2.3}$$

Example 2

A mass $m = 3$ is attached to a spring with a value of $k = 600$

1. Determine the value of the damping constant b that would produce critical damping.
2. Determine the value of damping constant b that would decrease the angular frequency by 10%/
3. A mass recieve an impulse that ives it a initial velocity $v = 2$. What is the maximum resultant displacement and the time when it occurs.

Solution 2 ():

1.

$$\omega_0 = \frac{\gamma}{2} \tag{2.4}$$

$$x = (A + Bt)\exp\left(-\frac{\gamma t}{2}\right) \tag{2.5}$$
$$\dot{x} = \exp\left(-\frac{\gamma t}{2}\right)\left(B - \frac{\gamma B t}{2} - A\frac{\gamma}{2}\right) \tag{2.6}$$
$$\tag{2.7}$$

$x(0) = 0, \dot{x}(0) = v_i$

$$x = v_i t \exp\left(-\frac{\gamma t}{2}\right) \tag{2.8}$$
$$\dot{x} = v_i \exp\left(-\frac{\gamma t}{2}\right)\left(1 - \frac{\gamma t}{2}\right) = 0 \tag{2.9}$$
$$t = \frac{2}{\gamma} = \frac{2m}{b} \tag{2.10}$$
$$x\left(\frac{2}{\gamma}\right) = v_i t \exp\left(-\frac{\gamma t}{2}\right) = \frac{2v_i}{\gamma e} \tag{2.11}$$

2.1 Energy

For underdamped oscillator, assume $\omega \approx \omega_0$.

$$E = T + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{2.12}$$
$$x = A_0 \exp\Big((i\omega_0 - \frac{\gamma}{2})t\Big) \tag{2.13}$$
$$\dot{x} = -A_0 \exp\left(t\left(i\omega_0 - \frac{\gamma}{2}\right)\right) = A_0\left(i\omega_0 - \frac{\gamma}{2}\right)\exp\left(t\left(i\omega_0 - \frac{\gamma}{2}\right)\right) \tag{2.14}$$
$$E = \frac{1}{2}kA_0^2\exp(-\gamma t) \tag{2.15}$$
$$\tag{2.16}$$

The rate of change of energy is

$$\dot{E} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = \dot{x}(m\ddot{x} + k\dot{x}) \tag{2.17}$$

Recall $m\ddot{x} = -kx - b\dot{x}$

$$\dot{E} = -b\dot{x}^2 = -\gamma E \tag{2.18}$$

Example 3

The energy of a simple harmonic oscillator is ovserved to reduce by a factor of two after 10 complete cycles.

1. How amny cycles will it take to reduce it by a factor of 8?
2. By what factor would it be reduced after 100 cycles?

Solution 3 ():

1.

$$\frac{E}{E_0} = e^{-\gamma t} \tag{2.19}$$

$$\frac{1}{2} = e^{-\gamma 10T} \tag{2.20}$$

$$\left(\frac{1}{2}\right)^3 = \left(e^{-\gamma 10T}\right)^3 = e^{-\gamma 30T} \tag{2.21}$$

2.2 Quality Factor

Definition: The **quality factor** is a convenient measurement on how good an oscillator is (how many oscillations it can make before its amplitude would decrease by a certain rate) is defined as

$$Q = \frac{\omega_0}{\gamma} = \frac{\sqrt{km}}{b} \tag{2.22}$$

where $\gamma = 2\beta = \dfrac{b}{m}$.

- If $Q = \dfrac{1}{2}$, the system is critically damped.
- Energy $E = E_0 \exp(-\gamma t)$.
- At two different times t_1 and t_2 seperated by period T,

$$\frac{E(t+T)}{E(t)} = \exp(-\gamma T) \tag{2.23}$$

  Which leads to

$$\frac{E(t) - E(t+T)}{E(t_1)} = 1 - e^{-\gamma T} \approx -\gamma T \approx \frac{2\pi\gamma}{\omega} = \frac{2\pi}{Q} \tag{2.24}$$

  Thus,

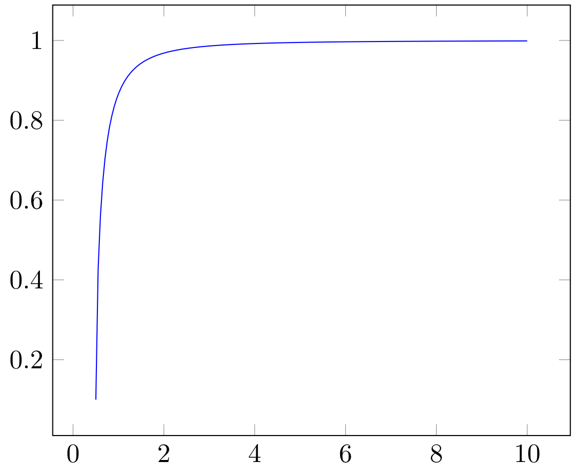
$$Q = \frac{\text{Energy stored in the oscillator}}{\text{Energy dissipated per radian}} \tag{2.25}$$

- We can rewrite the equation of the damped oscillator

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \tag{2.26}$$

$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x = 0 \tag{2.27}$$

  As $\gamma = \dfrac{\omega_0}{Q}, \omega = \omega_0\sqrt{1 - \dfrac{1}{4Q^2}}$.



Example 4

WHen an electron in H is moved from $n = 2$ to $n = 3$ states, the atom behaves like a damped oscillator when the lights of frequency 4.57×10^{14} Hz is emmited. The lifetime of the excited atom is approximately 10 ns. What is the value of the quality factor?

Solution 4 (): As $\gamma = \frac{1}{\tau}, \omega_0 = 2\pi f$,

$$Q = \frac{\omega_0}{\gamma} = 2\pi f \tau = 2.87 \times 10^7 \tag{2.28}$$

- RLC circuit:

$$RI + L\dot{I} + \frac{q}{C} = 0 \tag{2.29}$$
$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0 \tag{2.30}$$
$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0 \tag{2.31}$$

3 Driven Oscillations

In the undamped case,

$$m\ddot{x} + kx = F_0 e^{i\omega t} \tag{3.1}$$
$$m\ddot{x} + kx = kae^{i\omega t} \tag{3.2}$$
$$\tag{3.3}$$

The particular solution is

$$x = A(\omega)e^{i(\omega t - \delta)} \tag{3.4}$$

where

- $\tan\delta = 0$ and $A(\omega) = \frac{a}{1 - (\omega/\omega_0)^2}$ when $\omega < \omega_0$
- $\tan\delta = \pi$ and $A(\omega) = -\frac{a}{1 - (\omega/\omega_0)^2}$ when $\omega > \omega_0$

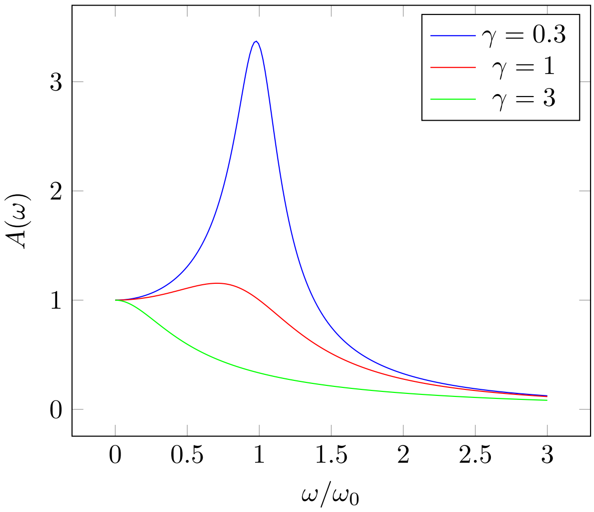
3.1 With damping

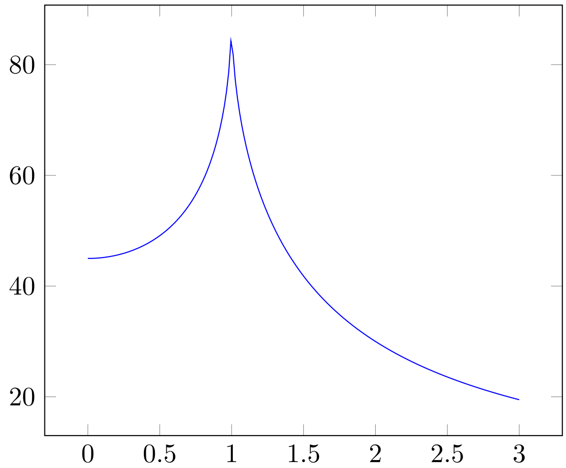
$$\ddot{x} + \gamma\dot{x} + \omega_0^2 = ae^{i\omega t} \tag{3.5}$$

Solving for x, the particular solution is

$$x = A(\omega)e^{i(\omega t - \delta)} \tag{3.6}$$

with $\tan\delta = \dfrac{\omega\gamma}{\omega_0^2 + \omega^2}$, $A(\omega) = \dfrac{a\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}}$.





If it is driven at the resonance frequency,

$$A = \frac{a\omega_0}{\gamma} \tag{3.7}$$

$$\delta = \frac{\pi}{2} \tag{3.8}$$

For large damping, $\omega_{max} \neq \omega_0$.

3.2 Power

If the motion of a driven oscillator is

$$x = A(\omega)e^{i(\omega t - \delta)} \tag{3.9}$$

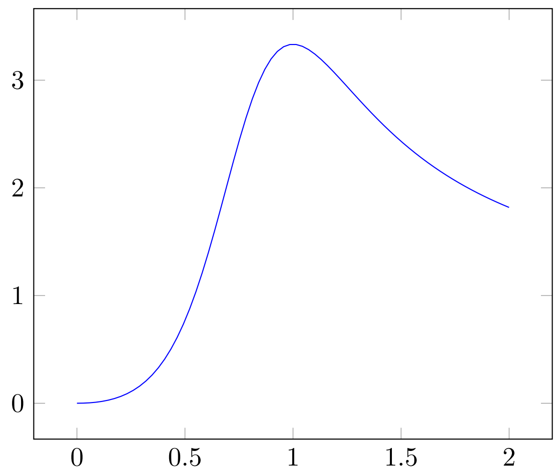
The velocity is

$$\dot{x} = iA(\omega)\omega e^{i(\omega t - \delta)} \tag{3.10}$$

Note that power lost to damping is

$$P = b\dot{x}^2 \tag{3.11}$$

The full width at half height of the period-averaged power is $\gamma/\omega_0 = \dfrac{1}{Q}$



If the driving frequency is close to the natural frequency,

$$(\omega^2 - \omega_0^2) = (\omega - \omega_0)(\omega + \omega_0) \approx -2\omega_0 \Delta\omega \tag{3.12}$$

Then,

$$\overline{P}(\omega) = \frac{F_0^2}{2m\gamma[\frac{4\Delta\omega^2}{\gamma^2} + 1]} \tag{3.13}$$

3.3 Reasonance in RLC circuits

$$RI + L\dot{I} + \frac{q}{C} = \varepsilon_0 \cos(\omega t) \tag{3.14}$$

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = \varepsilon_0 \cos(\omega t) \tag{3.15}$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{\varepsilon_0}{L} \cos(\omega t) \tag{3.16}$$

$$\omega_0^2 = \frac{1}{LC}, \gamma = \frac{R}{L}, Q = \frac{\omega_0}{\gamma} = \sqrt{\frac{L^2}{RC}}$$

$$q(t) = q_0(\omega) e^{i(\omega t - \delta)} \tag{3.17}$$