

MAT257—Analysis 2

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1 Course Overview

- $\mathbb{R} \rightarrow \mathbb{R}^n$
- Linear Algebra
- Continuity
- Differentiability
- Integration
- Key theorem of this class is **Stokes' Theorem**

$$\int_C d\omega = \int_{\partial C} \omega \quad (1.1)$$

Generalizes the fundamental theorem of calculus:

$$\int_{[a,b]} F'(t) dt = F(b) - F(a) = \int_{\partial[a,b]} F \quad (1.2)$$

Note that $\partial[a, b] = \{b+, a-\}$.

2 Continuity

- Roughly speaking, continuity from $\mathbb{R} \rightarrow \mathbb{R}$ means if two points are near, their images should be near also.
- Thus, in \mathbb{R}^n , the intuitive meaning should be similar.

2.1 Norms and Inner Product

Note there are 2 conventions for \mathbb{R}^n

1. The set of all n-dimensional real column vectors.
2. The set of all n-dimensional real row vectors.

In this class, the distinction is not very important.

Definition: For $x, y \in \mathbb{R}^n$, "The standard (or euclidian) inner product of x and y , denoted

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i \quad (2.1)$$

The norm-squared of x is

$$|x|^2 = \langle x, x \rangle = \sum x_i^2 \quad (2.2)$$

and the norm of x is

$$|x| = \sqrt{|x|^2} = \sqrt{\sum x_i^2} \quad (2.3)$$

Proposition: If $x, y, z \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, then

0. The inner product is bilinear & the norm is "semi-linear".

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle \quad (2.4)$$

$$\langle z, ax + by \rangle = \dots \quad (2.5)$$

$$|ax| = |a||x| \quad (2.6)$$

Aside:

$$1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1 \quad (2.7)$$

- 1.

$$|x| \geq 0 \& |x| = 0 \iff x = 0 \quad (2.8)$$

- 2.

$$\langle x, y \rangle = \langle y, x \rangle \quad (2.9)$$

3. *Cauchy-Schwarz inequality*

$$|\langle x, y \rangle| \leq |x||y| \quad (2.10)$$

with equality if $x \& y$ are dependent.

4. *Triangle inequality*

$$|x + y| \leq |x| + |y| \quad (2.11)$$

5. *Polarization identity*

$$\langle x, y \rangle = \frac{|x + y|^2 - |x - y|^2}{4} \quad (2.12)$$

Proof. 1. $|x| = \sqrt{\sum x_i^2} \quad |x| = 0 \implies \sum x_i^2 = 0 \implies \forall i, x_i^2 = 0 \implies \forall i, x_i = 0 \implies x = 0$

2. For $s, t \in \mathbb{R}^n$

$$|s + t|^2 = |s|^2 + |t|^2 + 2\langle s, t \rangle \quad (2.13)$$

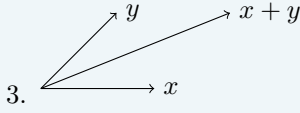
Look at

$$0 \leq \left| |y|^2 x - \langle x, y \rangle y \right|^2 = |y|^4 |x| + \langle x, y \rangle^2 |y|^2 - 2|y|^2 \langle x, y \rangle^2 \quad (2.14)$$

$$= |y|^2 (|y|^2 |x|^2 - \langle x, y \rangle^2) \quad (2.15)$$

This is equal to zero only if $|y|^2 x - \langle x, y \rangle y = 0$. If we have equality, that implies $x \& y$ are dependent.

Why, what does this mean?



As both sides of the triangle inequality are ≥ 0 , square both sides.

$$|x + y|^2 \stackrel{?}{\leq} (|x| + |y|)^2 \quad (2.16)$$

$$\langle x + y, x + y \rangle \stackrel{?}{\leq} |x|^2 + |y|^2 + 2|x||y| \quad (2.17)$$

$$\langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \stackrel{?}{\leq} |x|^2 + |y|^2 + 2|x||y| \quad (2.18)$$

$$|x|^2 + |y|^2 + 2\langle x, y \rangle \stackrel{?}{\leq} |x|^2 + |y|^2 + 2|x||y| \quad (2.19)$$

$$\langle x, y \rangle \stackrel{?}{\leq} |x||y| \quad (2.20)$$

(2.20) is true by cauchy-schwarz.

4. The proof is trivial because you can expand the right hand side.

Note: The inner product and the norm are not independent. If you know how to compute one, you can compute the other.

□

2.2 Distance Functions

Definition: If $x, y \in \mathbb{R}^n$, define the distance between x & y

$$d(x, y) = |x - y| \quad (2.21)$$

Theorem:

1. d is symmetric: $d(x, y) = d(y, x)$
2. d is positive definite: $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

The significance of this theorem is that this is all we need to know about distances to comment on continuity.

Aside: Later, this theorem will become a definition for a distance function or a metric.

Proof. 1.

$$d(x, y) = |x - y| = |-(y - x)| = |-1||y - x| = |y - x| = d(y, x) \quad (2.22)$$

2.

$$d(x, y) = 0 \iff |x - y| = 0 \iff x - y = 0 \iff x = y \quad (2.23)$$

3.

$$|x - z| \stackrel{?}{\leq} |x - y| + |y - z| \quad (2.24)$$

This is true by the previous triangle inequality, $|p| + |q| \geq |p + q|$. Letting $p = x - y, q = y - z \implies p + q = x - z$.

□

There are other norms and distance functions that we will rarely use.

- The euclidian norm which we use is $|x|_{L^2} = \sqrt{\sum x_i^2}$.
- There is a L1 norm $|x|_{L^1} = \sum |x_i|$.
- The L-infinity norm is $|x|_{L^\infty} = \max |x_i|$.

The distance functions for these norms also satisfy these three properties.

- There is a bijection between linear maps from $\mathbb{R}^n \rightarrow \mathbb{R}^m$ and the set of $m \times n$ matrices with real coefficients. This bijection can be realized by choosing a basis.

- In \mathbb{R}^n or \mathbb{R}^m , there is a natural basis (the standard basis) $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th position}$

- by $A \in M_{m \times n} \rightarrow L_A(x) = Ax$, where $x \in \mathbb{R}^n$.

- If T is a linear transformation, $M_T = \begin{pmatrix} Te_1 | Te_2 | \dots | Te_n \end{pmatrix}$

Definition: Homomorphism: A map that preserve the structure.

Theorem:

0. Bijective

$$L_{(M_T)} = T, M_{(L_A)} = A \quad (2.25)$$

- $A \rightarrow L_A$ is linear: $L_{aA+bB} = aL_A + bL_B$
 • $T \rightarrow M_T$ is linear: $M_{aT+bS} = aM_T + bM_S$
- Given $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $S : \mathbb{R}^m \rightarrow \mathbb{R}^p$, and $S \circ T \equiv T/S : \mathbb{R}^n \rightarrow \mathbb{R}^p$.
 Then, $M_S M_T = M_{S \circ T}$.

End of the review.

3 Rectangles

- It is common to use intervals in \mathbb{R} . In \mathbb{R}^n , we use rectangles.
- To specify a rectangle, we must bound the each of the n coordinates.

Definition: Given $a_i \leq b_i$, where $i = 1, \dots, n$,

- The closed rectangle corresponding to a_i, b_i is defined as

$$R = \prod_{i=1}^n [a_i, b_i] = \{x \in \mathbb{R}^n : \forall i \ a_i \leq x_i \leq b_i\} \quad (3.1)$$

- The opened rectangle defined by a_i, b_i is defined as

$$R = \prod_{i=1}^n (a_i, b_i) = \{x \in \mathbb{R}^n : \forall i \ a_i < x_i < b_i\} \quad (3.2)$$

- If X & Y are sets, we define (from set theory) the cartesian product $X \times Y = \{(x, y) : x \in X, y \in Y\}$
- Given 3 sets, the cartesian product is strictly speaking not associative as $((x, y), z) \neq (x, (y, z))$. However, for convinence we agree that $((x, y), z) = (x, y, z) = (x, (y, z))$. Thus, the cartesian product is associative.
- $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$.

Definition:

- $A \subset \mathbb{R}^n$ is called an open set if $\forall a \in A \exists$ an open rectangle $R : x \in R \subset A$.
- $A \subset \mathbb{R}^n$ is called an open set if $\forall a \in A \exists$ an open ball $B : x \in B \subset A$. An open ball $B = B_r(x_0) = \{y \in \mathbb{R}^n : |x_0 - y| < r\}$. Note an open ball can be defined with any norm.

Theorem: Defining “open” using rectangles is equivalent to define “open” using balls.

Proof. \implies Every open rectangle is open using the ball definition.

\impliedby Every open ball is open using the rectangle definition. □

Definition: A set B is “closed” if $\mathbb{R}^n \setminus B = B^C$ is open.

Proposition: [De-Morgan’s Laws]

If Y_α is any collection of subsets of some universe U ,

$$\left(\bigcup Y_\alpha\right)^C = \bigcap Y_\alpha^C \quad (3.3)$$

$$\left(\bigcap Y_\alpha\right)^C = \bigcup Y_\alpha^C \quad (3.4)$$

Theorem:

1. \emptyset, \mathbb{R}^n are clopen.
2. Any union of open sets is open. Any intersection of closed sets is closed.
3. A finite intersection of open sets is open. A finite union of closed sets is closed.

Proof. 1. \mathbb{R}^n is open. $x \in \prod (x_i - 1, x_i + 1) \subset \mathbb{R}^n. \implies \emptyset$ is closed. The empty set has no points, thus the condition holds. “Every horse in an empty set of horses has horns.” $\implies \mathbb{R}^n$ is closed.

2. Suppose $\{A_\alpha\}_{\alpha \in I}$, where I is an arbitrary indexing set, is a collection of open sets.

$$A = \bigcup_{\alpha \in I} A_\alpha = \{x : \exists \alpha \in I \ x \in A_\alpha\} \quad (3.5)$$

Let $x \in A$, find α such that $x \in A_\alpha$. Find an open rectangle R such that $x \in R \subset A_\alpha \subset A$

Suppose $\{B_\alpha\}_{\alpha \in I}$ is a collection of closed sets, show $\bigcap B_\alpha$ is closed. $\left(\bigcap B_\alpha\right)^C = \bigcup B_\alpha^C$ is open $\implies \bigcap B_\alpha$ is closed.

Lemma 1: The intersection of two open rectangles, if non-empty, is an open rectangle.

3.

Suppose A_1 and A_2 are open. Pick $x \in A_1 \cap A_2$. By openness of A_1 , $x \in A_1 \implies \exists R_1 : x \in R_1 \subset A_1$. Similarly, by openness of A_2 , $x \in A_2 \implies \exists R_2 : x \in R_2 \subset A_2$. Then, $x \in R_1 \cap R_2 \equiv R \subset A_1 \cap A_2$.

Suppose $A_i, i = 1, \dots, n$ are open.

$$\bigcap_{i=1}^n A_i = \left(\bigcap_{i=1}^{n-1} A_i\right) \cap A_n \quad (3.6)$$

By induction hypothesis, $\left(\bigcap_{i=1}^{n-1} A_i\right)$ is an open set. The intersection of two open sets are open \implies the intersection of n open sets are open.

Suppose $B_i, i = 1, \dots, n$ is closed,

$$\left(\bigcup_{i=1}^n B_i\right)^C = \bigcap_{i=1}^n B_i^C \quad (3.7)$$

□

Definition: Clopen Sets: Suppose $A \subset \mathbb{R}^n$ is clopen $\implies A^C$ is clopen. Suppose neither is empty. Consider the line segment $l_{xy}(t) = ty + (1-t)x$.

$$l_{xy}(0) = x \in A \quad (3.8)$$

$$l_{xy}(1) = y \in A^C \quad (3.9)$$

$$t_0 = \sup_{t \in [0,1]} \{l_{xy}(t) \in A\} \quad (3.10)$$

$$l_{xy}(t_0) = z \quad (3.11)$$

if $z \in A$, the rectangle containing $z \cap l_{xy}$ includes $l(t_0 + \epsilon) \in A^C$ for some ϵ .
Similarly if $z \in A^C \implies$ one of A and A^C is not clopen so the other one isn't clopen either.
Thus, the only clopen sets is \emptyset and \mathbb{R}^n

- Consider the following example,

$$\bigcap_{n>0} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0, 1] \quad (3.12)$$

This infinite intersection of open sets is not an open set due to the points 0 and 1.

Definition: Given $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, there is a tricotomy (**exactly** one of the following is true)

1. x belongs to the *interior* of A : \exists open rectangle R such that $x \in R \subset A$.
2. x belongs to the *exterior* of A : \exists open rectangle R such that $x \in R \subset A^C$.
3. x belongs to the *boundary* of A : Every open rectangle R such that $x \in R$ has $R \cap A^C \neq \emptyset$ AND $R \cap A \neq \emptyset$.

- The closure of A is the complement of the exterior. $\bar{A} = (\text{ext}A)^C$. It will satisfy either condition 1 or 3.

• **Claims:**

1. $\bar{A} \ni x$ iff. every open rectangle $R \ni x$ satisfies $R \cap A \neq \emptyset$.
2. $\text{int}A \cup \text{ext}A \cup \text{Bd}A = \mathbb{R}^n$
3. $\text{cl}A = A \cup \text{Bd}A$
4. $\text{int}A = A \setminus \text{Bd}A$

Example 1 ()

$$A = [0, 1) \subset \mathbb{R}$$

- $\text{int}A = (0, 1)$
- $\text{ext}A = (-\infty, 0) \cup (1, \infty)$
- $\text{Bd}A = \{0, 1\}$
- $\text{cl}A = [0, 1]$

4 Compactness

Definition: An **open cover** of a set A is a collection $\{U_\alpha\}$ of open sets in \mathbb{R}^n such that

$$\bigcup_{\alpha \in I} U_\alpha \supset A \quad (4.1)$$

A **subcover** of $\{U_\alpha\}_{\alpha \in I}$ is a collection $\{U_\alpha\}_{\alpha \in I'}$ where $I' \subset I$ such that

$$\bigcup_{\alpha \in I'} U_\alpha \supset A \quad (4.2)$$

Definition: A set A is called **compact** if **EVERY** open cover of A has a finite sub-cover.

- Note: Showing one finite open cover with a finite subcover is not sufficient.

Examples:

1. If $F \subset \mathbb{R}^n$ is finite, then it is compact.
2. \mathbb{R} is not compact. Take $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n-1, n+1) = \bigcup_{n \in \mathbb{Z}} (-n, n)$. These open covers does not have a finite subcover.

4.1 Finding all compact subsets of \mathbb{R}^n

Theorem: [Heine-Borel] $[0, 1]$ is compact.

Proof. Let $\{U_\alpha\}_{\alpha \in J}$ be an open cover of $I = [0, 1]$. We will first show there's a subcover from 0 to $g > 0$. Define $G = \{g \in [0, 1] : \exists J' \subset J \text{ such that } J' \text{ is a finite subcover of } [0, g]\}$. To show $1 \in G$ will prove the theorem. Set $\gamma = \sup G$. For G to have a supremum, it must be bounded ($G \in [0, 1]$) and non-empty ($0 \in G$). Claim: $\gamma = 1$. Suppose $\gamma < 1$, as $\gamma \in I$, $\exists \beta \in J$ such that $\gamma \in U_\beta$. As U_β is open, $\exists (g', g'') : \gamma \in (g', g'') \subset [g', g''] \subset U_\beta$. $[0, g''] = [0, g'] \cup [g', g'']$. As $g' < \gamma$, $[0, g']$ has a finite subcover. $[g', g'']$ is covered by a single set U_β . Thus, $g'' \in G$ and this is a contradiction. \square