

PHY293—Waves & Modern Physics

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September 21, 2021

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1 Harmonic Oscillators

- Think of simple harmonic motion (SHM) as circular motion projected into one dimension. (a wave is rotation in the complex plane)
- One of the simplest system is a mass on the spring, with known force $F = -k\Delta x$.
- For **SHM**, the motion must be periodic, and the force must be proportional to displacement.
- Using Newton's 2nd law on the spring force,

$$m\ddot{x} = -kx \quad (1.1)$$

$$\ddot{x} + \frac{k}{m}x = 0 \quad (1.2)$$

- For a mass on the vertical spring, the equilibrium position will be lower due to gravity.

$$k(y_1 - y_0) - mg = 0 \quad (1.3)$$

$$y_1 = y_0 + \frac{mg}{k} \quad (1.4)$$

y_1 is the new equilibrium position. SHM will still occur if the system is disturbed.

- Especially when dealing with energy, it is a good idea to have the origin at $y = y_1$.

1.1 The Differential Equation

- In general, the equation for simple harmonic motion is

$$\ddot{x} + \omega^2 x = 0 \quad (1.5)$$

- The solution to (1.2) can be represented as

$$x = A \cos(\omega t + \varphi_0) \quad (1.6)$$

- A represents the amplitude
- φ_0 is the phase angle (initial phase)
- $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency.

- The velocity and acceleration can be easily calculated by taking derivatives.

$$\dot{x} = -A\omega \sin(\omega t + \varphi_0) \quad (1.7)$$

$$\ddot{x} = -A\omega^2 \cos(\omega t + \varphi_0) \quad (1.8)$$

- Another way to represent the full solution is

$$x = a \cos(\omega t) + b \sin(\omega t) \quad (1.9)$$

where $a = A \cos \varphi_0$, $b = -A \sin \varphi_0$.

Example 1 ()

Determine the amplitude and phase constant of a pendulum moving with a motion described by a sum of two functions, $x_1(t) = 0.25 \cos \omega t$ and $x_2(t) = -0.5 \sin \omega t$.

Solution 1 ():

$$0.25 = A \cos \phi_0 \quad (1.10)$$

$$-0.50 = -A \sin \phi_0 \quad (1.11)$$

Figure out ϕ_0 from the ratio of eq.(1.11)/(1.10)

1.2 Energy of SHM

- A mass has kinetic energy of $T = \frac{1}{2}mv^2$.
- The potential energy is related to the restoring force and by definition,

$$\Delta U = - \int F \cdot dx = - \int_{x_i}^{x_f} (-kx') dx' = \frac{1}{2}k(x_f^2 - x_i^2) \quad (1.12)$$

- The conservation of energy here follows from newton's 2nd law.

$$m\ddot{x} = -kx \quad (1.13)$$

- Looking at the potential energy

$$x = A \cos(\omega t + \varphi_0) \quad (1.14)$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t) \quad (1.15)$$

- Looking at the kinetic energy,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t) \quad (1.16)$$

- The total energy is

$$E = T + \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}mv_{MAX}^2 \quad (1.17)$$

1.3 Physics of Small Vibrations

- Most system will oscillate with SHM when the amplitude is small. Recall the Taylor expansion of an analytic function

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2 f''(a) + \dots \quad (1.18)$$

If a is a minima, $f'(a) = 0$. For $x \approx a$, the higher order terms

- The potential energy of a pendulum is

$$U = mgy = mgL(1 - \cos \theta) \quad (1.19)$$

- **For this course, an angle $\theta < 10^\circ$, it will be considered small.**
- For a simple pendulum,

$$-mg \sin \theta = ma \quad (1.20)$$

$$-mg \sin \theta = m\ddot{s}, s = L\theta, ds = Ld\theta \quad (1.21)$$

$$-g \sin \theta = L\ddot{\theta} \quad (1.22)$$

For small angles, $\theta \approx \sin \theta$

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \quad (1.23)$$

This is the equation for SHM

- The energy of the pendulum is

$$E = T + U = \frac{1}{2}mv^2 + mg\left(\frac{x^2}{2L}\right) \quad (1.24)$$

- For physical pendulum,

$$\tau = I\alpha = I\ddot{\theta} \quad (1.25)$$

$$-mgd \sin \theta = I\ddot{\theta} \quad (1.26)$$

For small θ : $\sin \theta \approx \theta$

$$\ddot{\theta} + \frac{mgd}{I}\theta = 0 \quad (1.27)$$

- For LC circuits,

$$\ddot{i} + \frac{1}{LC}i = 0 \quad (1.28)$$

When a resistor is connected, energy is lost as heat and the circuit behaves like a damped oscillator.

2 Damped Oscillations

The damped oscillator involves the addition to the drag force that is proportional to $-v$ to the simple harmonic oscillator.

Define

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0 \quad (2.1)$$

$$x = A \exp\left(\left(\sqrt{\frac{\gamma^2}{4} - \omega_0^2} - \frac{\gamma}{2}\right)t\right) + B \exp\left(-\left(\sqrt{\frac{\gamma^2}{4} - \omega_0^2} - \frac{\gamma}{2}\right)t\right) \quad (2.2)$$

For $\omega_0^2 \neq \frac{\gamma^2}{4}$.

Critical damping gives the minimum time for the system to return to equilibrium when $\omega_0^2 = \frac{\gamma^2}{4}$.

$$x = (Ax + B) \exp(-\omega_0 t) \quad (2.3)$$

Example 2 ()

A mass $m = 3$ is attached to a spring with a value of $k = 600$

1. Determine the value of the damping constant b that would produce critical damping.
2. Determine the value of damping constant b that would decrease the angular frequency by 10%/
3. A mass recieve an impulse that ives it a initial velocity $v = 2$. What is the maximum resultant displacement and the time when it occurs.

Solution 2 ():

1.

$$\omega_0 = \frac{\gamma}{2} \quad (2.4)$$

$$x = (A + Bt) \exp\left(-\frac{\gamma t}{2}\right) \quad (2.5)$$

$$\dot{x} = \exp\left(-\frac{\gamma t}{2}\right) \left(B - \frac{\gamma B t}{2} - A \frac{\gamma}{2}\right) \quad (2.6)$$

$$(2.7)$$

$$x(0) = 0, \dot{x}(0) = v_i$$

$$x = v_i t \exp\left(-\frac{\gamma t}{2}\right) \quad (2.8)$$

$$\dot{x} = v_i \exp\left(-\frac{\gamma t}{2}\right) \left(1 - \frac{\gamma t}{2}\right) = 0 \quad (2.9)$$

$$t = \frac{2}{\gamma} = \frac{2m}{b} \quad (2.10)$$

$$x\left(\frac{2}{\gamma}\right) = v_i t \exp\left(-\frac{\gamma t}{2}\right) = \frac{2v_i}{\gamma e} \quad (2.11)$$

2.1 Energy

For underdamped oscillator, assume $\omega \approx \omega_0$.

$$E = T + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \quad (2.12)$$

$$x = A_0 \exp\left((i\omega_0 - \frac{\gamma}{2})t\right) \quad (2.13)$$

$$\dot{x} = -A_0 \exp\left(t\left(i\omega_0 - \frac{\gamma}{2}\right)\right) = A_0 \left(i\omega_0 - \frac{\gamma}{2}\right) \exp\left(t\left(i\omega_0 - \frac{\gamma}{2}\right)\right) \quad (2.14)$$

$$E = \frac{1}{2}kA_0^2 \exp(-\gamma t) \quad (2.15)$$

$$(2.16)$$

The rate of change of energy is

$$\dot{E} = \frac{d}{dt} \left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2 \right) = \dot{x}(m\ddot{x} + kx) \quad (2.17)$$

Recall $m\ddot{x} = -kx - b\dot{x}$

$$\dot{E} = -b\dot{x}^2 = -\gamma E \quad (2.18)$$

Example 3 ()

The energy of a simple harmonic oscillator is observed to reduce by a factor of two after 10 complete cycles.

1. How many cycles will it take to reduce it by a factor of 8?
2. By what factor would it be reduced after 100 cycles?

Solution 3 ():

1.

$$\frac{E}{E_0} = e^{-\gamma t} \quad (2.19)$$

$$\frac{1}{2} = e^{-\gamma 10T} \quad (2.20)$$

$$\left(\frac{1}{2}\right)^3 = (e^{-\gamma 10T})^3 = e^{-\gamma 30T} \quad (2.21)$$

2.2 Quality Factor

Definition: The **quality factor** is a convenient measurement on how good an oscillator is (how many oscillations it can make before its amplitude would decrease by a certain rate) is defined as

$$Q = \frac{\omega_0}{\gamma} = \frac{\sqrt{km}}{b} \quad (2.22)$$

where $\gamma = 2\beta = \frac{b}{m}$.

- If $Q = \frac{1}{2}$, the system is critically damped.
- Energy $E = E_0 \exp(-\gamma t)$.
- At two different times t_1 and t_2 separated by period T ,

$$\frac{E(t+T)}{E(t)} = \exp(-\gamma T) \quad (2.23)$$

Which leads to

$$\frac{E(t) - E(t+T)}{E(t)} = 1 - e^{-\gamma T} \approx -\gamma T \approx \frac{2\pi\gamma}{\omega} = \frac{2\pi}{Q} \quad (2.24)$$

Thus,

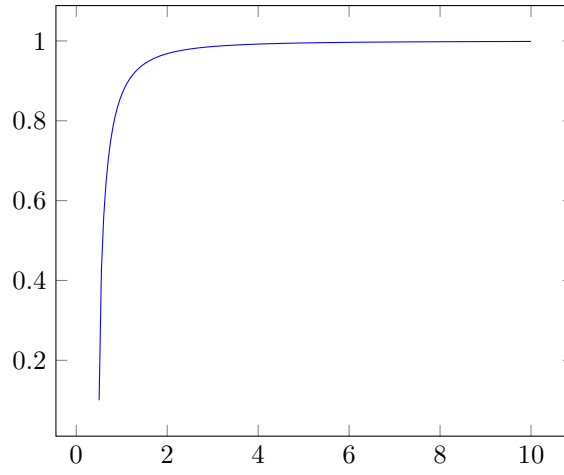
$$Q = \frac{\text{Energy stored in the oscillator}}{\text{Energy dissipated per radian}} \quad (2.25)$$

- We can rewrite the equation of the damped oscillator

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0 \quad (2.26)$$

$$\ddot{x} + \frac{\omega_0}{Q} \dot{x} + \omega_0^2 x = 0 \quad (2.27)$$

$$\text{As } \gamma = \frac{\omega_0}{Q}, \omega = \omega_0 \sqrt{1 - \frac{1}{4Q^2}}.$$



Example 4 ()

When an electron in H is moved from $n = 2$ to $n = 3$ states, the atom behaves like a damped oscillator when the lights of frequency 4.57×10^{14} Hz is emitted. The lifetime of the excited atom is approximately 10 ns. What is the value of the quality factor?

Solution 4 (): As $\gamma = \frac{1}{\tau}$, $\omega_0 = 2\pi f$,

$$Q = \frac{\omega_0}{\gamma} = 2\pi f\tau = 2.87 \times 10^7 \quad (2.28)$$

- RLC circuit:

$$RI + L\dot{I} + \frac{q}{C} = 0 \quad (2.29)$$

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0 \quad (2.30)$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0 \quad (2.31)$$

3 Driven Oscillations

In the undamped case,

$$m\ddot{x} + kx = F_0 e^{i\omega t} \quad (3.1)$$

$$m\ddot{x} + kx = ka e^{i\omega t} \quad (3.2)$$

$$(3.3)$$

The particular solution is

$$x = A(\omega) e^{i(\omega t - \delta)} \quad (3.4)$$

where

- $\tan \delta = 0$ and $A(\omega) = \frac{a}{1 - (\omega/\omega_0)^2}$ when $\omega < \omega_0$
- $\tan \delta = \pi$ and $A(\omega) = -\frac{a}{1 - (\omega/\omega_0)^2}$ when $\omega > \omega_0$