CHE260—Thermodynamics & Heat Transfer

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1 Basics

Fourier's law relates heat flux to temperature

$$\dot{q} = -k\nabla T \tag{1.1}$$

Newton's Law of Cooling is the fundamental law of convection.

$$\dot{q} = h(T - T_\infty) \tag{1.2}$$

Thermal radiation is

$$\dot{q} = \sigma A(T^4 - T_\infty^4) \tag{1.3}$$

2 Conduction

2.1 Heat Equation

The heat equation is a statement of the first law (conservation of energy), which describes conduction.

$$(\partial_t - \alpha\nabla^2)T = 0 \tag{2.1}$$

2.2 Steady State

In steady state, $\partial_t T = 0$, so the heat equation reduces to Laplace's equation for temperature $\nabla^2 T = 0$. Energy Balance for something:

$$\rho c_p \partial_t T = -\frac{1}{A}\partial_x(\dot{q}A) \tag{2.2}$$

1-dimensional Heat Equation

$$(\partial_t - \alpha\partial_x^2)T = 0 \tag{2.3}$$

where $\alpha := \frac{k}{\rho c_p}$, with units of m^2/s.

- High k — Material conducts heat well.
- High ρc_p — Material stores energy well.

In cylinderical coordinates (in the r direction),

$$(\partial_t - \frac{\alpha}{r}\partial_r(r\partial_r))T = 0 \tag{2.4}$$

In spherical coordinates (in the r direction),

$$(\partial_t - \frac{\alpha}{r^2}\partial_r(r^2\partial_r))T = 0 \tag{2.5}$$

Common Solutions: Steady state solution for cartesian coordinates is

$$T = mx + b \tag{2.6}$$

2.3 Thermal Resistance

When dealing with steady state behavior for multiple connected mediums, it is useful to define a quantity called thermal resistance with the property that

$$\dot{Q} = \frac{T_1 - T_2}{R} \tag{2.7}$$

The calculated values for resistances are

- For conduction, $R = \frac{L}{kA}$.
- For convection, $R = \frac{1}{hA}$.
- For radiation, $R = \frac{1}{\varepsilon\sigma A(T^2 - T_\infty^2)(T - T_\infty)}$.

For two mediums connected in series,

$$R_T = R_1 + R_2 \tag{2.8}$$

For two mediums connected in parallel,

$$R_T = \frac{R_1 R_2}{R_1 + R_2} \tag{2.9}$$

It is also useful to define an overall heat transfer coefficient $U = \frac{1}{R_T A}$

$$\dot{Q} = UA(T_1 - T_2) \tag{2.10}$$

The rvalue is L/k, hence

$$\dot{Q} = \frac{\Delta T}{R_{value}} \times A \tag{2.11}$$

which is in english units. L is in feet and k is in BTU per h square feet Fahernhit.

The critical radius of insulation is when the resistance is stationary. When increasing the radius, we are increacing conductive resistance but decreasing convective resistance.

$$r_{crit} = \frac{k}{h} \tag{2.12}$$

For a very long fin with perimeter P, conduction and convection coefficients k and h in an enviroment at T_∞, (we are also assuming steady state somehow, but not sure how)

$$(k\partial_x(A\partial_x) - hP)T = -hPT_\infty \tag{2.13}$$

Assume A, k, P are constant. Define $\Theta = T - T_\infty$ and $a^2 = \frac{hP}{kA}$. Then,

$$(\partial_x^2 - a^2)\Theta = 0 \tag{2.14}$$

Using boundary conditions of $\Theta(0) = \Theta_b$ and $\Theta(\infty) = 0$ (from the definition of T_∞), the unique solution is

$$\Theta = \Theta_b e^{-ax} \tag{2.15}$$

The rate of heat loss is just equal to the heat entering the base of the fin. Using fourier's law,

$$\dot{Q} = \sqrt{hPkA}(T_b - T_\infty) \tag{2.16}$$

For a finitely long fin, assume the heat transfer to the air at the end is negligiable (adiabatic). Hence, $\partial_x \Theta|_{x=L} = 0$.

$$\Theta = \Theta_b \frac{\cosh(a(L-x))}{\cosh(aL)} \tag{2.17}$$

By similar means, the **rate of heat loss** for this case is

$$\dot{Q} = \dot{Q}_\infty 2 \tag{2.18}$$

Where $\dot{Q}_\infty$ is the rate of heat loss assuming the fin is infinitely long.

If there is some heat lost to the air, we can correct the length $L_C = L + \frac{A}{P}$

To define the **fin efficiency**, we first define the "ideal" fin to be a fin with infinite heat conductivity, and

$$\eta_{fin} := \frac{\dot{Q}_{fin}}{\dot{Q}_{fin,max}} = \frac{\tanh(aL)}{aL} \tag{2.19}$$

For $L >> a$, $\tanh(aL)$ is approximately 1.

We define the **fin effectiveness** as

$$\varepsilon_{fin} := \frac{\dot{Q}_{fin}}{\dot{Q}_{no\,fin}} = \sqrt{\frac{kP}{hA}} \tag{2.20}$$

To maximize effectiveness,

- Maximize P/A, use a lot of small fins.
- Maximize k, use copper or something similar.
- Don't use fin for liquids because h is big.
- Generally, we use fins if $\epsilon \gtrapprox 2$.

2.4 Lump Bodies

The lump capacitance is defined as $L = \nabla^2 T$. For a lump at uniform temperature, $L = 0$ and the heat equation reduces to

$$\partial_t T = 0 \tag{2.21}$$

This assumption is valud when the object is small or have high thermal conductivity.

The stored thermal energy of the lump is being lost to the surrounding through convection. So $\dot{E}_{store} + \dot{Q}_{conv} = 0$. Defining a time constant $\tau = \frac{\rho v c_p}{hA}$ and a dimensionless temperature $\Theta = \frac{T - T_\infty}{T_0 - T_\infty}$,

$$\Theta = e^{-t/\tau} \tag{2.22}$$

The dimensionless constant hL/k is called Biot's number

$$\frac{T_1 - T_2}{T_2 - T_\infty} = B \tag{2.23}$$

where k is thermal conductivity of the **solid**. If Bi $>> 1$, then the temperature drop within the body is much greater than temperature drop outside the body. If Bi $<< 1$, then the temperature drop within the body is much smaller than temperature drop outside the body. Hence, the body can be considered as a lump.