MAT257—Analysis 2

Jonah Chen

September 24, 2021

Contents

1 Course Overview

- $\mathbb{R} \to \mathbb{R}^n$
- Linear Algebra
- Continuity
- Differentiability
- Integration
- Key theorem of this class is **Stokes' Theorem**

$$\int_C \mathrm{d}\omega = \int_{\partial C} \omega \tag{1.1}$$

  Generalizes the fundamental theorem of calculus:

$$\int_{[a,b]} F'(t)\mathrm{d}t = F(b) - F(a) = \int_{\partial[a,b]} F \tag{1.2}$$

  Note that $\partial[a, b] = \{b+, a-\}$.

2 Distances

- Roughly speaking, continuity from $\mathbb{R} \to \mathbb{R}$ means if two points are near, their images should be near also.
- Thus, in $\mathbb{R}^n$, the intuitive meaning should be similar.

2.1 Norms and Inner Product

Note there are 2 conventions for $\mathbb{R}^n$

1. The set of all n-dimensional real column vectors.
2. The set of all n-dimensional real row vectors.

In this class, the distinction is not very important.

Definition: For $x, y \in \mathbb{R}^n$, "The standard (or euclidian) inner product of x and y, denoted

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \tag{2.1}$$

The norm-squared of x is

$$|x|^2 = \langle x, x \rangle = \sum x_i^2 \tag{2.2}$$

and the norm of x is

$$|x| = \sqrt{|x|^2} = \sqrt{\sum x_i^2} \tag{2.3}$$

Proposition: If $x, y, z \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, then

0. The inner product is bilinear & the norm is "semi-linear".

$$\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle \tag{2.4}$$

$$\langle z, ax + by \rangle = \dots \tag{2.5}$$

$$|ax| = |a||x| \tag{2.6}$$

Aside:

$$1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1 \tag{2.7}$$

1.

$$|x| \geq 0 \& |x| = 0 \iff x = 0 \tag{2.8}$$

2.

$$\langle x, y \rangle = \langle y, x \rangle \tag{2.9}$$

3. *Cauchy-Schwarz inequality*

$$|\langle x, y \rangle| \leq |x||y| \tag{2.10}$$

with equality if $x \& y$ are dependent.

4. *Triangle inequality*

$$|x + y| \leq |x| + |y| \tag{2.11}$$

5. *Polarization identity*

$$\langle x, y \rangle = \frac{|x + y|^2 - |x - y|^2}{4} \tag{2.12}$$

Proof. 1. $|x| = \sqrt{\sum x_i^2}$ $|x| = 0 \implies \sum x_i^2 = 0 \implies \forall i, x_i^2 = 0 \implies \forall i, x_i = 0 \implies x = 0$

2. For $s, t \in \mathbb{R}^n$

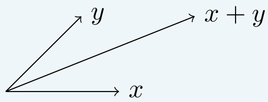
$$|s + t|^2 = |s|^2 + |t|^2 + 2\langle s, t \rangle \tag{2.13}$$

Look at

$$0 \leq \Big||y|^2 x - \langle x, y \rangle y\Big|^2 = |y|^4 |x| + \langle x, y \rangle^2 |y|^2 - 2|y|^2 \langle x, y \rangle^2 \tag{2.14}$$

$$= |y|^2 \left(|y|^2 |x|^2 - \langle x, y \rangle^2\right) \tag{2.15}$$

This is equal to zero only if $|y|^2 x - \langle x, y \rangle y = 0$. If we have equality, that implies $x \& y$ are dependent. **Why, what does this mean?**

3. 

As both sides of the triangle inequality are ≥ 0, square both sides.

$$|x+y|^2 \overset{?}{\leq} (|x|+|y|)^2 \tag{2.16}$$
$$\langle x+y, x+y\rangle \overset{?}{\leq} |x|^2+|y|^2+2|x||y| \tag{2.17}$$
$$\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \overset{?}{\leq} |x|^2+|y|^2+2|x||y| \tag{2.18}$$
$$|x|^2+|y|^2+2\langle x,y\rangle \overset{?}{\leq} |x|^2+|y|^2+2|x||y| \tag{2.19}$$
$$\langle x,y\rangle \overset{?}{\leq} |x||y| \tag{2.20}$$

(2.20) is true by cauchy-schwarz.

4. The proof is trivial because you can expand the right hand side.
Note: The inner product and the norm are not independent. If you know how to compute one, you can compute the other.

□

2.2 Distance Functions

Definition: If $x, y \in \mathbb{R}^n$,define the distance between $x\&y$

$$d(x,y) = |x-y| \tag{2.21}$$

Theorem:

1. d is symmetric: $d(x,y) = d(y,x)$
2. d is positive definite: $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$
3. Triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$

The significance of this theorem is that this is all we need to know about distances to comment on continuity.
Aside: Later, this theorem will become a definition for a distance function or a metric.

Proof. 1.

$$d(x,y) = |x-y| = |-(y-x)| = |-1||y-x| = |y-x| = d(y,x) \tag{2.22}$$

2.

$$d(x,y) = 0 \iff |x-y| = 0 \iff x-y = 0 \iff x = y \tag{2.23}$$

3.

$$|x-z| \overset{?}{\leq} |x-y| + |y-z| \tag{2.24}$$

This is true by the previous triangle inequality, $|p|+|q| \geq |p+q|$. Letting $p = x-y, q = y-z \implies p+q = x-z$.

□

There are other norms and distance functions that we will rarely use.

- The euclidian norm which we use is $|x|_{L^2} = \sqrt{\sum x_i^2}$.
- There is a L1 norm $|x|_{L^1} = \sum |x_i|$.
- The L-infinity norm is $|x|_{L^\infty} = \max |x_i|$.

The distance functions for these norms also satisfys these three properties.

- There is a bijection between linear maps from $\mathbb{R}^n \to \mathbb{R}^m$ and the set of $m \times n$ matrices with real coefficients. This bijection can be realized by choosing a basis.

- In $\mathbb{R}^n$ or $\mathbb{R}^m$, there is a natural basis (the standard basis) $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i-$th position

- by $A \in M_{m\times n} \to L_A(x) = Ax$, where $x \in \mathbb{R}^n$.

- If T is a linear transformation, $M_T = \left(Te_1 | Te_2 | \dots | Te_n \right)$

Definition: Homomorphism: A map that preserve the structure.

Theorem:

0. Bijective

$$L_{(M_T)} = T, M_{(L_A)} = A \tag{2.25}$$

1.
   - $A \to L_A$ is linear: $L_{aA+bB} = aL_A + bL_b$
   - $T \to M_T$ is linear: $M_{aT+bS} = aM_T + bM_S$

2. Given $T : \mathbb{R}^n \to \mathbb{R}^m, S : \mathbb{R}^m \to \mathbb{R}^p$, and $S \circ T \equiv T//S : \mathbb{R}^n \to \mathbb{R}^p$.
   Then, $M_S M_T = M_{S\circ T}$.

End of the review.

3 Rectangles

- It is common to use intervals in $\mathbb{R}$. In $\mathbb{R}^n$, we use rectangles.

- To specify a rectangle, we must bound the each of the n coordinates.

Definition: Given $a_i \leq b_i$, where $i = 1, \dots, n$,

- The closed rectangle corresponding to a_i, b_i is defined as

$$R = \prod_{i=1}^{n} [a_i, b_i] = \{x \in \mathbb{R}^n : \forall i \;\; a_i \leq x_i \leq b_i\} \tag{3.1}$$

- The opened rectangle defined by a_i, b_i is defined as

$$R = \prod_{i=1}^{n} (a_i, b_i) = \{x \in \mathbb{R}^n : \forall i \;\; a_i < x_i < b_i\} \tag{3.2}$$

- If $X\&Y$ are sets, we define (from set theory) the cartesian product $X \times Y = \{(x, y) : x \in X, y \in Y\}$

- Given 3 sets, the cartesian product is strictly speaking not associative as $((x, y), z) \neq (x, (y, z))$. However, for convinence we agree that $((x, y), z) = (x, y, z) = (x, (y, z))$. Thus, the cartesian product is associative.

- $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$.

Definition:

- $A \subset \mathbb{R}^n$ is called an open set if $\forall a \in A \exists$ an open rectangle $R : x \in R \subset A$.
- $A \subset \mathbb{R}^n$ is called an open set if $\forall a \in A \exists$ an open ball $B : x \in B \subset A$. An open ball $B = B_r(x_0) = \{y \in \mathbb{R}^n : |x_0 - y| < r\}$ Note an open ball can be defined with any norm.

Theorem: Defining "open" using rectangles is equivalent to define "open" using balls.

Proof. $\implies$ Every open rectangle is open using the ball definition.
$\impliedby$ Every open ball is open using the rectangle definition. □

Definition: A set B is "closed" if $\mathbb{R}^n\ B = B^C$ is open.

Proposition: **[De-Morgan's Laws]**
If Y_α is any collection of subsets of some universe U,

$$\left(\bigcup Y_\alpha\right)^C = \bigcap Y_\alpha^C \tag{3.3}$$

$$\left(\bigcap Y_\alpha\right)^C = \bigcup Y_\alpha^C \tag{3.4}$$

Theorem:

1. $\emptyset, \mathbb{R}^n$ are clopen.
2. Any union of open sets is open. Any intersection of closed sets is closed.
3. A finite intersection of open sets is open. A finite union of closed sets is closed.

Proof. 1. $\mathbb{R}^n$ is open. $x \in \prod(x_i - 1, x_i + 1) \subset \mathbb{R}^n$. $\implies \emptyset$ is closed. The empty set has no points, thus the condition holds. "Every horse in an empty set of horses has horns." $\implies \mathbb{R}^n$ is closed.

2. Suppose $\{A_\alpha\}_{\alpha \in I}$, where I is an arbiturary indexing set, is a collection of open sets.

$$A = \bigcup_{\alpha \in I} A_\alpha = \{x : \exists \alpha \in I \ \ x \in A_\alpha\} \tag{3.5}$$

Let $x \in A$, find α such that $s \in A_\alpha$. Find an open rectangle R such that $x \in R \subset A_\alpha \subset A$

Suppose $\{B_\alpha\}_{\alpha \in I}$ is a collection of closed sets, show $\cap B_\alpha$ is closed. $\left(\bigcap B_\alpha\right)^C = \bigcup B_\alpha^C$ is open $\implies \bigcap B_\alpha^C$ is closed.

3. **Lemma 1:** The intersection of two open rectangles, if non-empty, is an open rectangle.

Suppose A_1 and A_2 are open. Pick $x \in A_1 \cap A_2$. By openness of A_1, $x \in A_1 \implies \exists R_1 : x \in \mathbb{R}_1 \subset A_1$. Similarly, by openness of A_2, $x \in A_2 \implies \exists R_2 : x \in \mathbb{R}_2 \subset A_2$. Then, $x \in R_1 \cap R_2 \equiv R \subset A_1 \cap A_2$.
Suppose A_i, $i = 1, \ldots, n$ are open.

$$\bigcap_{i=1}^{n} A_i = \left(\bigcap_{i=1}^{n-1} A_i\right) \bigcap A_n \tag{3.6}$$

By induction hypothesis, $\left(\bigcap_{i=1}^{n-1} A_i\right)$ is an open set. The intersection of two open sets are open $\implies$ the intersection of n open sets are open.
Suppose B_i, $i = 1, \ldots, n$ is closed,

$$\left(\bigcup_{i=1}^{n} B_i\right)^C = \bigcup_{i=1}^{n} B_i^C \tag{3.7}$$

□

Definition: Clopen Sets: Suppose $A \subset \mathbb{R}^n$ is clopen $\implies$ A^C is clopen. Suppose neither is empty. Consider the line segment $l_{xy}(t) = ty + (1-t)x$.

$$l_{xy}(0) = x \in A \tag{3.8}$$
$$l_{xy}(1) = y \in A^C \tag{3.9}$$
$$t_0 = \sup_{t \in [0,1]} \{l_{xy}(t) \in A\} \tag{3.10}$$
$$l_{xy}(t_0) = z \tag{3.11}$$

if $z \in A$, the rectangele containing $z \cap l_{xy}$ includes $l(t_0 + \epsilon) \in A^C$ for some ϵ.
Similarly if $z \in A^C$ $\implies$ one of A and A^C is not clopen so the other one isn't clopen either.
Thus, the only clopen sets is $\emptyset$ and $\mathbb{R}^n$

- Consider the following example,
$$\bigcap_{n>0} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0,1] \tag{3.12}$$
This infinite intersection of open sets is not an open set due to the points 0 and 1.

Definition: Given $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, there is a tricotomy (**exactly** one of the following is true)

1. x belongs to the *interior* of A: $\exists$ open rectangle R such that $x \in R \subset A$.
2. x belongs to the *exterior* of A: $\exists$ open rectangle R such that $x \in R \subset A^C$.
3. x belongs to the *boundary* or A: Every open rectangle R such that $x \in R$ has $R \cap A^C \neq \emptyset$ AND $R \cap A \neq \emptyset$.

- The closure of A is the complement of the exterior. $\overline{A} = (\text{ext}A)^C$. It will satisfy either condition 1 or 3.
- **Claims:**
  1. $\overline{A} \ni x$ iff. every open rectangle $R \ni x$ satisfies $R \cap A \neq \emptyset$.
  2. $\text{int}A \cup \text{ext}A \cup \text{Bd}A = \mathbb{R}^n$
  3. $\text{cl} = A \cup \text{Bd}A$
  4. $\text{int}A = A \setminus \text{Bd}A$.
  5. $\text{int}S$ is the largest open set in S, $\text{int}S = \bigcup_{U \subset S} U$
  6. $\overline{S}$ is the smallest closed set containing S, $\overline{S} = \bigcap_{C \supset S} C$.

Example 1

$A = [0,1) \subset \mathbb{R}$

- $\text{int}A = (0,1)$
- $\text{ext}A = (-\infty, 0) \cup (1, \infty)$
- $\text{Bd}A = \{0, 1\}$
- $\text{cl}A = [0,1]$

4 Compactness

Definition: An **open cover** of a set A is a collection $\{U_\alpha\}$ of open sets in $\mathbb{R}^n$ such that

$$\bigcup_{\alpha \in I} U_\alpha \supset A \tag{4.1}$$

A **subcover** of $\{U_\alpha\}_{\alpha \in I}$ is a collection $\{U_\alpha\}_{\alpha \in I'}$ where $I' \subset I$ such that

$$\bigcup_{\alpha \in I'} U_\alpha \supset A \tag{4.2}$$

Definition: A set A is called **compact** if **EVERY** open cover of A has a finite sub-cover.

- Note: Showing one finite open cover with a finite subcover is not sufficient.

Examples:

1. If $F \subset \mathbb{R}^n$ is finite, then it is compact.
2. $\mathbb{R}$ is not compact. Take $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n-1, n+1) = \bigcup_{n \in \mathbb{Z}} (-n, n)$. These open covers does not have a finite subcover.

4.1 Finding all compact subsets of $\mathbb{R}^n$

Theorem: [Heine-Borel] $[a, b]$ is compact.

Proof. Let $\{U_\alpha\}_{\alpha \in J}$ be an open cover of $[a, b]$. We will first show there's a subcover from a to $g > a$.

Define $G = \{g \in [a, b] : \exists J' \subset J\}$ such that J' is a finite subcover of $[a, g]$.

To show $b \in G$ will prove the theorem. Set $\gamma = \sup G$. For G to have a supremum, it must be bounded ($G \subset [a, b]$) and non-empty ($a \in G$).

Claim: $\gamma = b$. Suppose $\gamma < b$, as $\gamma \in [a, b]$, $\exists \beta \in J$ such that $\gamma \in U_\beta$.

As U_β is open, $\exists (g', g'') : \gamma \in (g', g'') \subset [g', g''] \subset U_\beta$.

$[a, g''] = [a, g'] \cup [g', g'']$.
As $g' < \gamma$, $[0, g']$ has a finite subcover. $[g', g'']$ is covered by a single set U_β. Thus, $g'' \in G$ and this is a contradiction as $g'' > \gamma$.
Next, we show $b = \gamma \in G$.
If b is covered by $\{U_\alpha\}_{\alpha \in J}$, hence some interval (b^-, b^+) is covered by one set U_α. As $\sup G = b > b^-, \exists g' \in G : b^- < g' < b$.

$$[a, b] = [a, g'] \cup [b^-, b] \tag{4.3}$$

□

Theorem: [] If $A \subset \mathbb{R}^n$ is compact and $b \subset \mathbb{R}^m$ is compact. Then, $A \times B \subset \mathbb{R}^{n+m}$ is compact.

Proof. Suppose $U = \{U_\alpha\}$ is an open cover of $A \times B$.

WLOG, each U_α is itself an open rectangle.

Lemma 2: For every $x \in A$, we can find an open set $N_x \ni x : N_x \times B$ can be covered with finitely many of the U_αs.

Proof. Write $U_\alpha = V_\alpha \times W_\alpha$, where V_α, W_α are open rectangles in $\mathbb{R}^n, \mathbb{R}^m$ respectively.
Consider that $\{W_\alpha : x \in V_\alpha\}$ covers B which is compact. So find $\alpha_1, \ldots, \alpha_p : \{W_{\alpha_1}, \ldots, W_{\alpha_p}\}$ cover B. So, $\{U_{\alpha_1}, \ldots, U_{\alpha_p}\}$ cover $\{x\} \times B$.

Let $N_x = \bigcap_{i=1}^{p} V_{\alpha_i} \subset V_{\alpha_i} \subset V_{\alpha_i} \forall i$.

Now, $N_x \times B \subset \bigcup_{i=1}^{p} N_x \times W_{\alpha_i} \subset \bigcup_{i=1}^{p} V_{\alpha_i} \times W_{\alpha_i} = \bigcup_{i=1}^{p} U_{\alpha_i}$. □

□

Now, $\{N_x\}_{x \in A}$ is an open cover of A. By compactness of A, find $x_1, \ldots, x_q : \bigcup_{j=1}^{q} N_{x_j} \supset A$. i.e. $\bigcup_{j=1}^{q} N_{x_j} \times B \supset A \times B$.

For each $j = 1, \ldots, q$ find U_{ji} which are rectangles in U, $i = 1, \ldots, p_j : \bigcup_{i=1}^{p_j} U_{ji} \supset N_{x_j} \times B$.

Now, $\bigcup_{j=1}^{p} \bigcup_{i=1}^{p_j} U_{ji} \supset A \times B$.

Corollary: Closed rectangles $R = \prod_{i=1}^{n} [a_i, b_i]$ are compact.

Proposition: A closed subset of a compact set is compact.

Corollary: Every closed and bounded subset of $\mathbb{R}^n$ is compact.

Theorem: Every compact set in $\mathbb{R}^n$ is closed and bounded.

Proof. Construct a cover for S with open balls of radius R. Given S is compact, it is covered by finitely many elements. Thus, S is bounded.
Let $x \in S^C, y \in S$, Let $B_y = B(y, \frac{1}{3}|x - y|), C_y = B(x, \frac{1}{3}|x - y|)$ □

If $X \subset \mathbb{R}^n$ is compact,

- Every open cover has a finite subcover
- Closed and bounded
- Every sequence $(x_n)_n \in X$ has a converging subsequences that converge in X.

Continuity:

- $\epsilon - \delta$
- f^{-1}(open) is open
- If x_n converges to x, then $f(x_n)$ converges to $f(x)$.

5 Continuity

Definition: [Image and Preimage] Given $F : \mathbb{R}^n \to \mathbb{R}^m$,

- $C \subset \mathbb{R}^n$, the image of C is $F(C) := \{F(\gamma) : \gamma \in C\}$
- $D \subset \mathbb{R}^m$, the preimage of D is $F^{-1}(D) := \{\gamma \in \mathbb{R}^n : F(\gamma) \in D\}$

Note the image behaves better on points, but preimage behaves better on sets, as,

$$F^{-1}(D_1 \cup D_2) = F^{-1}(D_1) \cup F^{-1}(D_2) \tag{5.1}$$
$$F^{-1}(D_1 \cap D_2) = F^{-1}(D_1) \cap F^{-1}(D_2) \tag{5.2}$$
$$F^{-1}(D^C) = F^{-1}(D)^C \tag{5.3}$$

$$F(C_1 \cup C_2) = F(C_1) \cup F(C_2) \tag{5.4}$$
$$F(C_1 \cap C_2) \subset F(C_1) \cap F(C_2) \tag{5.5}$$
$$F(C^C) \neq F(C)^C \tag{5.6}$$

Definition: [Projection] $\pi_i : \mathbb{R}^n \to \mathbb{R}$

$$\pi_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i \tag{5.7}$$

Definition: [Coordinate Functions] For $F : \mathbb{R}^n \to \mathbb{R}^m$, or

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix} \tag{5.8}$$

Where $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, m$ are the coordinate functions of f. $f_i = \pi_i \circ F$

Definition: [Composition] Given $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^p$, and $h = g \circ f : \mathbb{R}^n \to \mathbb{R}^p$

$$h(x) = g(f(x)) = (g \circ f)(x) \tag{5.9}$$

Definition: [Graph] A function $f : \mathbb{R} \to \mathbb{R}$, the graph of f is

$$\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \tag{5.10}$$

A function $f : \mathbb{R}^n \to \mathbb{R}^m$, the graph of f is

$$\Gamma_f = \{x, f(x) : x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m} \tag{5.11}$$

Definition: [Limit] Suppose $f : A \subset \mathbb{R}^n \to \mathbb{R}^m ; a \in \overline{A}$

$$\lim_{x \to a} f(x) = b \text{ means } \forall \varepsilon > 0 \exists \delta > 0 : x \in (B_\delta(a) \setminus \{x\}) \cap A \implies f(x) \in B_\epsilon(b) \tag{5.12}$$

- If the limit exists, it is unique.

Definition: [Continuity] $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in A$ if $\lim_{x \to a} f(x) = f(a)$.

f is continuous on $A \iff f$ is cont. at every $a \in A$.

$$\iff \forall a \, \forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in A : |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \tag{5.13}$$

Definition: $B \subset A$ is *open in* A if $\exists U$ open in $\mathbb{R}^n$ such that $B = U \cap A$.

Theorem:

1. $f : \mathbb{R}^n \to \mathbb{R}^m$ is cont. iff for every open set $V \subset \mathbb{R}^m, f^{-1}(V)$ is also open.
2. $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is cont. iff for every open set $V \subset \mathbb{R}^m, f^{-1}(V)$ is open in A.