MAT257—Analysis 2

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1 Course Overview

- $\mathbb{R} \to \mathbb{R}^n$
- Linear Algebra
- Continuity
- Differentiability
- Integration
- Key theorem of this class is **Stokes' Theorem**

$$\int_C \mathrm{d}\omega = \int_{\partial C} \omega \tag{1.1}$$

Generalizes the fundamental theorem of calculus:

$$\int_{[a,b]} F'(t)\mathrm{d}t = F(b) - F(a) = \int_{\partial[a,b]} F \tag{1.2}$$

Note that $\partial[a,b] = \{b+, a-\}$.

2 Distances

- Roughly speaking, continuity from $\mathbb{R} \to \mathbb{R}$ means if two points are near, their images should be near also.
- Thus, in $\mathbb{R}^n$, the intuitive meaning should be similar.

2.1 Norms and Inner Product

Note there are 2 conventions for $\mathbb{R}^n$

1. The set of all n-dimensional real column vectors.
2. The set of all n-dimensional real row vectors.

In this class, the distinction is not very important.

Definition 1: For $x, y \in \mathbb{R}^n$, "The standard (or euclidian) inner product of x and y, denoted

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \tag{2.1}$$

The norm-squared of x is

$$|x|^2 = \langle x, x\rangle = \sum x_i^2 \tag{2.2}$$

and the norm of x is

$$|x| = \sqrt{|x|^2} = \sqrt{\sum x_i^2} \tag{2.3}$$

Proposition 1: If $x, y, z \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$, then

0. The inner product is bilinear & the norm is "semi-linear".

$$\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle \tag{2.4}$$

$$\langle z, ax + by\rangle = \dots \tag{2.5}$$

$$|ax| = |a||x| \tag{2.6}$$

Aside:

$$1 = \sqrt{1} = \sqrt{-1 \cdot -1} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1 \tag{2.7}$$

1. $$|x| \geq 0 \& |x| = 0 \iff x = 0 \tag{2.8}$$

2. $$\langle x, y\rangle = \langle y, x\rangle \tag{2.9}$$

3. *Cauchy-Schwarz inequality*

$$|\langle x, y\rangle| \leq |x||y| \tag{2.10}$$

with equality if $x \& y$ are dependent.

4. *Triangle inequality*

$$|x + y| \leq |x| + |y| \tag{2.11}$$

5. *Polarization identity*

$$\langle x, y\rangle = \frac{|x+y|^2 - |x-y|^2}{4} \tag{2.12}$$

Proof. 1. $|x| = \sqrt{\sum x_i^2}$ $|x| = 0 \implies \sum x_i^2 = 0 \implies \forall i, x_i^2 = 0 \implies \forall i, x_i = 0 \implies x = 0$

2. For $s, t \in \mathbb{R}^n$

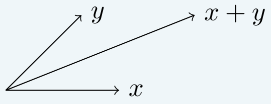
$$|s + t|^2 = |s|^2 + |t|^2 + 2\langle s, t\rangle \tag{2.13}$$

Look at

$$0 \le \Big||y|^2x - \langle x,y\rangle y\Big|^2 = |y|^4|x| + \langle x,y\rangle^2|y|^2 - 2|y|^2\langle x,y\rangle^2 \tag{2.14}$$

$$= |y|^2\left(|y|^2|x|^2 - \langle x,y\rangle^2\right) \tag{2.15}$$

This is equal to zero only if $|y|^2x - \langle x,y\rangle y = 0$. If we have equality, that implies $x\&y$ are dependent.
Why, what does this mean?

3. 

As both sides of the triangle inequality are ≥ 0, square both sides.

$$|x+y|^2 \overset{?}{\le} (|x|+|y|)^2 \tag{2.16}$$

$$\langle x+y, x+y\rangle \overset{?}{\le} |x|^2 + |y|^2 + 2|x||y| \tag{2.17}$$

$$\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \overset{?}{\le} |x|^2 + |y|^2 + 2|x||y| \tag{2.18}$$

$$|x|^2 + |y|^2 + 2\langle x,y\rangle \overset{?}{\le} |x|^2 + |y|^2 + 2|x||y| \tag{2.19}$$

$$\langle x,y\rangle \overset{?}{\le} |x||y| \tag{2.20}$$

(2.20) is true by cauchy-schwarz.

4. The proof is trivial because you can expand the right hand side.
Note: The inner product and the norm are not independent. If you know how to compute one, you can compute the other.

□

2.2 Distance Functions

Definition 2: If $x, y \in \mathbb{R}^n$,define the distance between $x\&y$

$$d(x,y) = |x-y| \tag{2.21}$$

Theorem 1:

1. d is symmetric: $d(x,y) = d(y,x)$
2. d is positive definite: $d(x,y) \ge 0$ and $d(x,y) = 0 \iff x = y$
3. Triangle inequality: $d(x,z) \le d(x,y) + d(y,z)$

The significance of this theorem is that this is all we need to know about distances to comment on continuity.
Aside: Later, this theorem will become a definition for a distance function or a metric.

Proof. 1.

$$d(x,y) = |x-y| = |-(y-x)| = |-1||y-x| = |y-x| = d(y,x) \tag{2.22}$$

2.

$$d(x,y) = 0 \iff |x-y| = 0 \iff x-y = 0 \iff x = y \tag{2.23}$$

3.

$$|x-z| \overset{?}{\le} |x-y| + |y-z| \tag{2.24}$$

This is true by the previous triangle inequality, $|p|+|q| \ge |p+q|$. Letting $p = x-y, q = y-z \implies p+q = x-z$.

□

There are other norms and distance functions that we will rarely use.

- The euclidian norm which we use is $|x|_{L^2} = \sqrt{\sum x_i^2}$.
- There is a L1 norm $|x|_{L^1} = \sum |x_i|$.
- The L-infinity norm is $|x|_{L^\infty} = \max |x_i|$.

The distance functions for these norms also satisfys these three properties.

- There is a bijection between linear maps from $\mathbb{R}^n \to \mathbb{R}^m$ and the set of $m \times n$ matrices with real coefficients. This bijection can be realized by choosing a basis.
- In $\mathbb{R}^n$ or $\mathbb{R}^m$, there is a natural basis (the standard basis) $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i-$th position
- by $A \in M_{m\times n} \to L_A(x) = Ax$, where $x \in \mathbb{R}^n$.
- If T is a linear transformation, $M_T = \left(Te_1 | Te_2 | \dots | Te_n \right)$

Definition 3: Homomorphism: A map that preserve the structure.

Theorem 2:

0. Bijective

$$L_{(M_T)} = T, M_{(L_A)} = A \tag{2.25}$$

1.
   - $A \to L_A$ is linear: $L_{aA+bB} = aL_A + bL_b$
   - $T \to M_T$ is linear: $M_{aT+bS} = aM_T + bM_S$
2. Given $T : \mathbb{R}^n \to \mathbb{R}^m, S : \mathbb{R}^m \to \mathbb{R}^p$, and $S \circ T \equiv T//S : \mathbb{R}^n \to \mathbb{R}^p$.
   Then, $M_S M_T = M_{S\circ T}$.

End of the review.

3 Rectangles

- It is common to use intervals in $\mathbb{R}$. In $\mathbb{R}^n$, we use rectangles.
- To specify a rectangle, we must bound the each of the n coordinates.

Definition 4: Given $a_i \leq b_i$, where $i = 1, \dots, n$,

- The closed rectangle corresponding to a_i, b_i is defined as

$$R = \prod_{i=1}^{n} [a_i, b_i] = \{x \in \mathbb{R}^n : \forall i \;\; a_i \leq x_i \leq b_i\} \tag{3.1}$$

- The opened rectangle defined by a_i, b_i is defined as

$$R = \prod_{i=1}^{n} (a_i, b_i) = \{x \in \mathbb{R}^n : \forall i \;\; a_i < x_i < b_i\} \tag{3.2}$$

- If $X \& Y$ are sets, we define (from set theory) the cartesian product $X \times Y = \{(x, y) : x \in X, y \in Y\}$

- Given 3 sets, the cartesian product is strictly speaking not associative as $((x, y), z) \neq (x, (y, z))$. However, for convinence we agree that $((x, y), z) = (x, y, z) = (x, (y, z))$. Thus, the cartesian product is associative.
- $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$.

Definition 5:

- $A \subset \mathbb{R}^n$ is called an open set if $\forall a \in A \exists$ an open rectangle $R : x \in R \subset A$.
- $A \subset \mathbb{R}^n$ is called an open set if $\forall a \in A \exists$ an open ball $B : x \in B \subset A$. An open ball $B = B_r(x_0) = \{y \in \mathbb{R}^n : |x_0 - y| < r\}$ Note an open ball can be defined with any norm.

Theorem 3: Defining "open" using rectangles is equivalent to define "open" using balls.

Proof. $\Longrightarrow$ Every open rectangle is open using the ball definition.
$\Longleftarrow$ Every open ball is open using the rectangle definition. □

Definition 6: A set B is "closed" if $\mathbb{R}^n\ B = B^C$ is open.

Proposition 2:
If Y_α is any collection of subsets of some universe U,

$$\left(\bigcup Y_\alpha\right)^C = \bigcap Y_\alpha^C \tag{3.3}$$

$$\left(\bigcap Y_\alpha\right)^C = \bigcup Y_\alpha^C \tag{3.4}$$

Theorem 4:

1. $\emptyset, \mathbb{R}^n$ are clopen.
2. Any union of open sets is open. Any intersection of closed sets is closed.
3. A finite intersection of open sets is open. A finite union of closed sets is closed.

Proof. 1. $\mathbb{R}^n$ is open. $x \in \prod (x_i - 1, x_i + 1) \subset \mathbb{R}^n$. $\Longrightarrow$ $\emptyset$ is closed. The empty set has no points, thus the condition holds. "Every horse in an empty set of horses has horns." $\Longrightarrow$ $\mathbb{R}^n$ is closed.

2. Suppose $\{A_\alpha\}_{\alpha \in I}$, where I is an arbiturary indexing set, is a collection of open sets.

$$A = \bigcup_{\alpha \in I} A_\alpha = \{x : \exists \alpha \in I \ \ x \in A_\alpha\} \tag{3.5}$$

Let $x \in A$, find α such that $s \in A_\alpha$. Find an open rectangle R such that $x \in R \subset A_\alpha \subset A$

Suppose $\{B_\alpha\}_{\alpha \in I}$ is a collection of closed sets, show $\cap B_\alpha$ is closed. $\left(\bigcap B_\alpha\right)^C = \bigcup B_\alpha^C$ is open $\Longrightarrow$ $\bigcap B_\alpha^C$ is closed.

3. **Lemma 1**: The intersection of two open rectangles, if non-empty, is an open rectangle.

Suppose A_1 and A_2 are open. Pick $x \in A_1 \cap A_2$. By openness of A_1, $x \in A_1 \Longrightarrow \exists R_1 : x \in \mathbb{R}_1 \subset A_1$. Similarly, by openness of A_2, $x \in A_2 \Longrightarrow \exists R_2 : x \in \mathbb{R}_2 \subset A_2$. Then, $x \in R_1 \cap R_2 \equiv R \subset A_1 \cap A_2$.
Suppose A_i, $i = 1, \ldots, n$ are open.

$$\bigcap_{i=1}^{n} A_i = \left(\bigcap_{i=1}^{n-1} A_i\right) \bigcap A_n \tag{3.6}$$

By induction hypothesis, $\left(\bigcap_{i=1}^{n-1} A_i\right)$ is an open set. The intersection of two open sets are open $\Longrightarrow$ the intersection of n open sets are open.
Suppose B_i, $i = 1, \ldots, n$ is closed,

$$\left(\bigcup_{i=1}^{n} B_i\right)^C = \bigcup_{i=1}^{n} B_i^C \tag{3.7}$$

□

Definition 7: Clopen Sets: Suppose $A \subset \mathbb{R}^n$ is clopen $\implies A^C$ is clopen. Suppose neither is empty. Consider the line segment $l_{xy}(t) = ty + (1-t)x$.

$$l_{xy}(0) = x \in A \tag{3.8}$$

$$l_{xy}(1) = y \in A^C \tag{3.9}$$

$$t_0 = \sup_{t\in[0,1]} \{l_{xy}(t) \in A\} \tag{3.10}$$

$$l_{xy}(t_0) = z \tag{3.11}$$

if $z \in A$, the rectangele containing $z \cap l_{xy}$ includes $l(t_0 + \epsilon) \in A^C$ for some ϵ.
Similarly if $z \in A^C \implies$ one of A and A^C is not clopen so the other one isn't clopen either.
Thus, the only clopen sets is $\emptyset$ and $\mathbb{R}^n$

- Consider the following example,

$$\bigcap_{n>0} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) = [0,1] \tag{3.12}$$

  This infinite intersection of open sets is not an open set due to the points 0 and 1.

Definition 8: Given $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, there is a tricotomy (**exactly** one of the following is true)

1. x belongs to the *interior* of A: $\exists$ open rectangle R such that $x \in R \subset A$.
2. x belongs to the *exterior* of A: $\exists$ open rectangle R such that $x \in R \subset A^C$.
3. x belongs to the *boundary* or A: Every open rectangle R such that $x \in R$ has $R \cap A^C \neq \emptyset$ AND $R \cap A \neq \emptyset$.

- The closure of A is the complement of the exterior. $\overline{A} = (\text{ext}A)^C$. It will satisfy either condition 1 or 3.
- **Claims:**
  1. $\overline{A} \ni x$ iff. every open rectangle $R \ni x$ satisfies $R \cap A \neq \emptyset$.
  2. $\text{int}A \cup \text{ext}A \cup \text{Bd}A = \mathbb{R}^n$
  3. $\text{cl} = A \cup \text{Bd}A$
  4. $\text{int}A = A \setminus \text{Bd}A$.
  5. $\text{int}S$ is the largest open set in S, $\text{int}S = \bigcup_{U\subset S} U$
  6. $\overline{S}$ is the smallest closed set containing S, $\overline{S} = \bigcap_{C\supset S} C$.

Example 1

$A = [0,1) \subset \mathbb{R}$

- $\text{int}A = (0,1)$
- $\text{ext}A = (-\infty, 0) \cup (1, \infty)$
- $\text{Bd}A = \{0,1\}$
- $\text{cl}A = [0,1]$

4 Compactness

Definition 9: An **open cover** of a set A is a collection $\{U_\alpha\}$ of open sets in $\mathbb{R}^n$ such that

$$\bigcup_{\alpha\in I} U_\alpha \supset A \tag{4.1}$$

A **subcover** of $\{U_\alpha\}_{\alpha \in I}$ is a collection $\{U_\alpha\}_{\alpha \in I'}$ where $I' \subset I$ such that

$$\bigcup_{\alpha \in I'} U_\alpha \supset A \tag{4.2}$$

Definition 10: A set A is called **compact** if **EVERY** open cover of A has a finite sub-cover.

- Note: Showing one finite open cover with a finite subcover is not sufficient.

Examples:

1. If $F \subset \mathbb{R}^n$ is finite, then it is compact.
2. $\mathbb{R}$ is not compact. Take $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n-1, n+1) = \bigcup_{n \in \mathbb{Z}} (-n, n)$. These open covers does not have a finite subcover.

4.1 Finding all compact subsets of $\mathbb{R}^n$

Theorem 5: $[a, b]$ is compact.

Proof. Let $\{U_\alpha\}_{\alpha \in J}$ be an open cover of $[a, b]$. We will first show there's a subcover from a to $g > a$.

Define $G = \{g \in [a, b] : \exists J' \subset J\}$ such that J' is a finite subcover of $[a, g]$.

To show $b \in G$ will prove the theorem. Set $\gamma = \sup G$. For G to have a supremum, it must be bounded ($G \subset [a, b]$) and non-empty ($a \in G$).

Claim: $\gamma = b$. Suppose $\gamma < b$, as $\gamma \in [a, b]$, $\exists \beta \in J$ such that $\gamma \in U_\beta$.

As U_β is open, $\exists (g', g'') : \gamma \in (g', g'') \subset [g', g''] \subset U_\beta$.

$[a, g''] = [a, g'] \cup [g', g'']$.
As $g' < \gamma$, $[0, g']$ has a finite subcover. $[g', g'']$ is covered by a single set U_β. Thus, $g'' \in G$ and this is a contradiction as $g'' > \gamma$.
Next, we show $b = \gamma \in G$.
If b is covered by $\{U_\alpha\}_{\alpha \in J}$, hence some interval (b^-, b^+) is covered by one set U_α. As $\sup G = b > b^-, \exists g' \in G : b^- < g' < b$.

$$[a, b] = [a, g'] \cup [b^-, b] \tag{4.3}$$

□

Theorem 6: If $A \subset \mathbb{R}^n$ is compact and $b \subset \mathbb{R}^m$ is compact. Then, $A \times B \subset \mathbb{R}^{n+m}$ is compact.

Proof. Suppose $U = \{U_\alpha\}$ is an open cover of $A \times B$.

WLOG, each U_α is itself an open rectangle.

> **Lemma 2**: For every $x \in A$, we can find an open set $N_x \ni x : N_x \times B$ can be covered with finitely many of the U_αs.
>
> *Proof.* Write $U_\alpha = V_\alpha \times W_\alpha$, where V_α, W_α are open rectangles in $\mathbb{R}^n, \mathbb{R}^m$ respectively.
> Consider that $\{W_\alpha : x \in V_\alpha\}$ covers B which is compact. So find $\alpha_1, \ldots, \alpha_p : \{W_{\alpha_1}, \ldots, W_{\alpha_p}\}$ cover B. So, $\{U_{\alpha_1}, \ldots, U_{\alpha_p}\}$ cover $\{x\} \times B$.
>
> Let $N_x = \bigcap_{i=1}^{p} V_{\alpha_i} \subset V_{\alpha_i} \subset V_{\alpha_i} \forall i$.
>
> Now, $N_x \times B \subset \bigcup_{i=1}^{p} N_x \times W_{\alpha_i} \subset \bigcup_{i=1}^{p} V_{\alpha_i} \times W_{\alpha_i} = \bigcup_{i=1}^{p} U_{\alpha_i}$. □

□

Now, $\{N_x\}_{x\in A}$ is an open cover of A. By compactness of A, find $x_1,\dots,x_q : \bigcup_{j=1}^{q} N_{x_j} \supset A$. i.e. $\bigcup_{j=1}^{q} N_{x_j} \times B \supset A \times B$.

For each $j = 1,\dots,q$ find U_{ji} which are rectangles in U, $i = 1,\dots,p_j : \bigcup_{i=1}^{p_j} U_{ji} \supset N_{x_j} \times B$.

Now, $\bigcup_{j=1}^{p} \bigcup_{i=1}^{p_j} U_{ji} \supset A \times B$.

Corollary 1: Closed rectangles $R = \prod_{i=1}^{n} [a_i, b_i]$ are compact.

Proposition 3: A closed subset of a compact set is compact.

Corollary 2: Every closed and bounded subset of $\mathbb{R}^n$ is compact.

Theorem 7: Every compact set in $\mathbb{R}^n$ is closed and bounded.

Proof. Construct a cover for S with open balls of radius R. Given S is compact, it is covered by finitely many elements. Thus, S is bounded.
Let $x \in S^C, y \in S$, Let $B_y = B(y, \frac{1}{3}|x-y|), C_y = B(x, \frac{1}{3}|x-y|)$ □

If $X \subset \mathbb{R}^n$ is compact,

- Every open cover has a finite subcover
- Closed and bounded
- Every sequence $(x_n)_n \in X$ has a converging subsequences that converge in X.

Continuity:

- $\epsilon - \delta$
- f^{-1}(open) is open
- If x_n converges to x, then $f(x_n)$ converges to $f(x)$.

5 Continuity

Definition 11: Given $F : \mathbb{R}^n \to \mathbb{R}^m$,

- $C \subset \mathbb{R}^n$, the image of C is $F(C) := \{F(\gamma) : \gamma \in C\}$
- $D \subset \mathbb{R}^m$, the preimage of D is $F^{-1}(D) := \{\gamma \in \mathbb{R}^n : F(\gamma) \in D\}$

Note the image behaves better on points, but preimage behaves better on sets, as,

$$F^{-1}(D_1 \cup D_2) = F^{-1}(D_1) \cup F^{-1}(D_2) \tag{5.1}$$
$$F^{-1}(D_1 \cap D_2) = F^{-1}(D_1) \cap F^{-1}(D_2) \tag{5.2}$$
$$F^{-1}(D^C) = F^{-1}(D)^C \tag{5.3}$$

$$F(C_1 \cup C_2) = F(C_1) \cup F(C_2) \tag{5.4}$$
$$F(C_1 \cap C_2) \subset F(C_1) \cap F(C_2) \tag{5.5}$$
$$F(C^C) \neq F(C)^C \tag{5.6}$$

Definition 12: $\pi_i : \mathbb{R}^n \to \mathbb{R}$

$$\pi_i \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_i \tag{5.7}$$

Definition 13: For $F : \mathbb{R}^n \to \mathbb{R}^m$, or

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix} \tag{5.8}$$

Where $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, m$ are the coordinate functions of f. $f_i = \pi_i \circ F$

Definition 14: Given $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^p$, and $h = g \circ f : \mathbb{R}^n \to \mathbb{R}^p$

$$h(x) = g(f(x)) = (g \circ f)(x) \tag{5.9}$$

Definition 15: A function $f : \mathbb{R} \to \mathbb{R}$, the graph of f is

$$\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \tag{5.10}$$

A function $f : \mathbb{R}^n \to \mathbb{R}^m$, the graph of f is

$$\Gamma_f = \{x, f(x) : x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m} \tag{5.11}$$

Definition 16: Suppose $f : A \subset \mathbb{R}^n \to \mathbb{R}^m ; a \in \overline{A}$

$$\lim_{x \to a} f(x) = b \text{ means } \forall \varepsilon > 0 \exists \delta > 0 : x \in (B_\delta(a) \setminus \{x\}) \cap A \implies f(x) \in B_\epsilon(b) \tag{5.12}$$

- If the limit exists, it is unique.

Definition 17: $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in A$ if $\lim_{x \to a} f(x) = f(a)$.
f is continuous on $A \iff f$ is cont. at every $a \in A$.

$$\iff \forall a \, \forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in A : |x - a| < \delta \implies |f(x) - f(a)| < \epsilon \tag{5.13}$$

Definition 18: $B \subset A$ is *open in* A if $\exists U$ open in $\mathbb{R}^n$ such that $B = U \cap A$.

Theorem 8:
$f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is cont. iff whenever $U \subset \mathbb{R}^m$ is open, $f^{-1}(U)$ is open in A. (i.e. $\exists V \subset \mathbb{R}^n$ which is open and s.t. $f^{-1}(U) = V \cap A$.)

Proof in the case where $A = \mathbb{R}^n$. $\implies$ Assume $U \subset \mathbb{R}^m$ is open, NTS $f^{-1}(U)$ is open.

Pick $a \in f^{-1}(U)$, then $f(a) \in U$ so pick $\epsilon > 0$ s.t. $B_\epsilon(f(a)) \subset U$, by continuity, find $\delta > 0$ s.t. $f(B_\delta(a)) \subset B_\epsilon(f(a)) \subset U$.

So, $a \in B_\delta(a) \subset f^{-1}(U)$. So, $f^{-1}(U)$ is open.

$\impliedby$ Given $a \in \mathbb{R}^n$ and $\epsilon > 0$, consider $B_\epsilon(f(a))$ it is open. So, $a \in f^{-1}(B_\epsilon(f(a)))$ is open.

So $\exists \delta > 0 : B_\delta(a) \subset f^{-1}(B_\epsilon(f(a)))$. $\square$

Theorem 9: $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^p$, f, g cont. $\implies g \circ f$ is continuous.

Proof. Given $U \in \mathbb{R}^p$ open, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. By the continuity of g, $g^{-1}(U)$ is open. By the continuity of f, $f^{-1}(g^{-1}(U))$ is open. Thus, $g \circ f$ is continuous. □

Theorem 10: If $f : \mathbb{R}^n \to \mathbb{R}^m$ cont and $C \subset \mathbb{R}^n$ is compact. Then, $f(C)$ is compact.
"A cont. image of a compact is compact"

Sketch of Proof. Given an open cover $\{U_\alpha\}$ of $f(C)$, $\{f^{-1}(U\alpha)\}$ is an open cover of C. Hence, it has a finite subcover. Which in itself corresponds to a finite subcover for $f(C)$ □

Corollary 3: A cont. function on a compact set is bounded.

Theorem 11: f cont $\iff$ (U open $\implies f^{-1}(U)$ open). $\iff$ (D closed $\implies f^{-1}(D)$ closed)
True because $f^{-1}(D^C) = (f^{-1})^C$

6 Integration

Definition 19: $f : \mathbb{R}^n \to \mathbb{R}, A \subset \mathbb{R}^n$, an Oscillation on A is

$$O(f, A) = \sup_{x \in A} f(x) - \inf_{x \in A} f(x) \tag{6.1}$$

Oscillation of f at $a \in \mathbb{R}^n$ is

$$O(f, a) = \lim_{r \to 0} O(f, B_r(a)) \tag{6.2}$$

Informally, this is the "jump" of f at a. So, we claim that f is continuous at a iff $O(f, a) = 0$.

Theorem 12: Continuous functions are integrable.

Proof. Assume f is continuous on a rectangle R. A function is continuous if its oscillation at every point is equal to 0. Pick a number δ, for every $a \in R$ find a ball $B(a)$ s.t. $O(f, B(a)) < \delta$. For each a find an open rectangle R_a containing a and s.t. $\overline{R_a} \subset B(a)$. So, $O(f, \overline{R_a}) < \delta$.
The collection $\{R_a\}$ covers R. By compactness, we can find $a_1, \dots, a_p$ s.t. $R_{a_1}, \dots, R_{a_p}$ cover R.
Find a partition P whose cut points (t_{ij})s are all of the endpoints of all of the R_{a_i}s.
Now, if $S \in P$, then $S \subset R_{a_i}$ for some i, so, $O(f, S) \leq O(f, R_{a_i}) < \delta$. Now,

$$U(f, P) - L(f, P) = \sum_{S \in P} \mathrm{vol}(S)(M_S(f) - m_S(f)) = \sum_{S \in P} \mathrm{vol}(S) O(f, s) \leq \delta \sum \mathrm{vol}(s) = \delta \mathrm{vol}(R) \tag{6.3}$$

Hence, choose $\delta = \dfrac{\varepsilon}{v(R)}$ that proves the theorem. □

Our goal now is to prove a theorem of the following form:

- f is integrable $\iff$ f is continuous except on a ~~tiny set~~ set of measure 0.

Definition 20: A set $A \subset \mathbb{R}^n$ is measure zero in $\mathbb{R}^n$ if $\forall \varepsilon > 0$ you can find a sequence R_i of open or closed rectangles $(i = 1, 2, 3, \dots)$ such that $A \subset \bigcup R_i$ and $\sum \mathrm{vol} R_i < \epsilon$
Examples:

1. A finite set is of measure 0. (use rectangles small enough)
2. An infinite sequence of points $\{x_i\}$ is of measure 0. (use rectangles with volumes less than a geometric sequence)
3. A set in $\mathbb{R}^m$ is of measure 0 in $\mathbb{R}^n$ where $m < n$. (use a single rectangle that's very thin) **warning: the notion of "measure zero" is dependent of dimension**.

Definition 21: A set X is countable if there is a sequence x_i, $i = 1, 2, 3, \ldots$ s.t. the $\{x_i\} = X$. Or, $\exists f : \mathbb{N} \to X$ s.t. $f(\mathbb{N}) = X$. **Claims:**

1. Finite sets are countable.
2. A subset of a countable set is countable.

Definition 22: A set $A \in \mathbb{R}^n$ is said to be content zero if $\forall \varepsilon > 0$ it is contained in a finite union of rectangles whose $\sum \text{vols} < \varepsilon$.
Examples:

1. $Z \subset \mathbb{R}$ is measure zero, but not content zero.

Proposition 4: Compact set A of measure zero is of content zero.

Proof. Suppose $\varepsilon > 0$, cover A with countably many open rectangle whose sum of volumes is less than ε. By compactness, finitely many of those who already cover A, and the sum of their volumes is still less than ε. □

Proposition 5: $R = \prod[a_i, b_i]$, $vol(R) > 0$ is not of content zero $\implies$ R is not measure 0. (This also shows $[0, 1]$ is not countable)

Proof. Suppose $(R_i)_{i=1}^N$ are rectangles that cover R. We will show that $\sum_{i=1}^{N} v(R_i) \geq v(R) > 0$.
WLOG, $R_i \subset R$.
Find a partition P of R s.t. if $S \in P$ then $S \subset R_i$. Then,

$$\text{vol}(R) = \sum_{S \in P} \text{vol}(S) \leq \sum_{i=1}^{N} \sum_{S \in P \wedge S \subset R_i} \text{vol}(S) = \sum_{i=1}^{N} \text{vol}(R_i) \tag{6.4}$$

□

Theorem 13: For f bounded on a rectangle $R \subset \mathbb{R}^n$, $f : R \to \mathbb{R}$ is integrable $\iff$ f is continuous except for a set of measure zero.

Proof. Assume f is continuous except on a set E (evil set) of measure 0. Let $\varepsilon > 0$. As E is measure zero, find rectangles B_i s.t. $\bigcup_{i=1}^{\infty} B_i \supset E$, and $\sum_{i=1}^{\infty} \text{vol}(B_i) < \delta_1 = \dfrac{\varepsilon}{4M}$.
Now for every $y \in R \setminus E, f$ is continuous at y, so $O(f, y) = 0$, so find a rectangle A_y s.t. $y \in \text{int}(A_y)$ and $O(f, A_y) < \delta_2 = \dfrac{\varepsilon}{2\text{vol}(R)}$.

$$\bigcup_{y \in R \setminus E} \text{int}(A_y) \cup \bigcup_{i=1}^{\infty} \text{int}(B_i) \supset R \tag{6.5}$$

By compactness, there is a finite subcover, $A_{y1}, \ldots, A_{yp}, B_{i1}, \ldots, B_{yq}$. Let P be the partition of R s.t. if $S \in P$ then $S \subset A_{ij}$ or $S \subset B_{ij}$ for some j.
Because f is bounded on R so $|f| \leq M$

$$U(f, P) - L(f, P) = \sum_{S \in P} \text{vol}(S) O(f, S) \tag{6.6}$$

$$\leq \sum_{S \in P \wedge \exists j : S \subset A_{yi}} \text{vol}(S) O(f, s) + \sum_{S \in P \wedge \exists j : S \subset B_{yj}} \tag{6.7}$$

$$\leq \sum \text{vol}(S) \delta_2 + \sum \text{vol}(S) \cdot 2m \tag{6.8}$$

$$\leq \text{vol}(R) \delta_2 + 2M \delta_1 \tag{6.9}$$

Assume f is integrable, let $E = \text{disc}(f) = \{x : f \text{ isn't continuous at x}\} = \{x : O(f, x) > 0\} = U_n\{x : O(f, x) > \dfrac{1}{n}\}$.
Our goal is to show that each E_n is measure zero, but we will show that each E_n is content zero.

Fix some n and fix $\varepsilon > 0$, as f is integrable, find a partition P s.t. $U(f,P) - L(f,P) < \delta$ where $\delta = \frac{\varepsilon}{2n}$.

$$\delta > \sum_{S \in P} \mathrm{vol}(S)O(f,S) \geq \sum_{s \in P \wedge \mathrm{int} S \cap E_n \neq \emptyset} \mathrm{vol}(S)O(f,S) > \sum_{s \in P \wedge \mathrm{int} S \cap E_n \neq \emptyset} \implies \sum_{s \in P \wedge \mathrm{int} S \cap E_n \neq \emptyset} \mathrm{vol}(S) < \frac{n\delta}{2} = \frac{\varepsilon}{2} \tag{6.10}$$

Now, $\{S \in P : \int(S) \cap E_n\}$ covers E_n except perhaps $E_n \cap G$ where $G = \bigcup_{S \in P} \mathrm{bd}S$. But, G itself is of content zero so $E_n \cap G$ can be covered with further rectangles, whose total volume is $\frac{\varepsilon}{2}$.
Now, all rectangles taken together cover E_n and have total volume less than ε. □

Warning: this theorem often makes people think measure zero sets can be ignored. This is not true.

Example: $f : [0,1] \to \mathbb{R}$. Let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

$f(x) = 0$ except on $\mathbb{Q} \cap [0,1]$ which is measure zero. However, f is not integrable because the discontinuities are of measure one.

Corollary 4: If g is integrable and f differs from g on a set of content zero. Then, f is integrable too and $\int f = \int g$.

1. Changing finitely many points keeps $\int$.
2. Changing g on a closed set of measure zero keeps $\int$.

Sketch of Proof. $g = f$ except on B of content zero. Cover B with finitely many rectangles.
Take a partition that is good for g. Refine the partition with the rectangles which cover B. As B is content zero, the volumes of the rectangles that intersect B can be made arbiturarily small. □

Definition 23: C is Jordan measurable if C is bounded and $\mathrm{bd}C$ is of content zero.

Definition 24: Given a set $C \subset \mathbb{R}^n$. Define

$$\chi_c(x) = 1_c(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases} \tag{6.11}$$

$$f\chi_c(x) = \begin{cases} f(x) & x \in C \\ 0 & x \notin C \end{cases} \tag{6.12}$$

Define

$$\mathrm{vol}(C) = R \supset C\chi_c \tag{6.13}$$

when C is bounded and $\mathrm{bd}C$ is of content zero.
Suppose f is Jordan-measurable set $C \in \mathbb{R}^n$, then

$$\int_C f = \int_R f \cdot \chi_C \tag{6.14}$$

where R is any rectangle containing C.

- We have defined the integral, but currently we can integrate close to nothing, except for functions in one dimension.
- To integrate functions, we need Fubini's theorem but we must be careful when we state it. Consider these mishaps:
- Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be rectangles, set $R = A \times B \subset \mathbb{R}^n \times \mathbb{R}^m$. Let $f : R \to \mathbb{R}$ be an integrable function, and let $g(x) = \int_B f(x,y)\mathrm{d}y$. Then,

$$\int_R f = \int_A g \mathrm{d}x \tag{6.15}$$

This is wrong. The function f is integrable with respect to x, y but not necesarily with respect to y alone. Consider the function $f:[0,1]\times[0,1]\to\mathbb{R}$ such that $f(x,y)=\begin{cases}1 & x\in\mathbb{Q}\wedge y=0.5\\ 0 & \text{otherwise}\end{cases}$ This set of discontinuities is of measure 0 in $\mathbb{R}^2$, but is of measure 1 in $\mathbb{R}$

Now consider another function

$$f(x,y)=\begin{cases}\frac{1}{q} & x,y\in\mathbb{Q}, x=\frac{p}{q}\\ 0 & \text{otherwise}\end{cases} \tag{6.16}$$

And another function

$$h(x)=\begin{cases}1 & x=\frac{p}{q}\\ 0 & \text{otherwise}\end{cases} \tag{6.17}$$

The set of discontinuities of h is $\mathbb{Q}$, which is of measure zero $\implies$ h is integrable and $\int_0^1 h(x)=0$. Because for any $\varepsilon>0$, there is only a content zero set in which $h>\varepsilon$. Hence, the integral of h is less than ε.

If we try to use

$$g(x)=\begin{cases}\int_B f(x,y)\mathrm{d}y & f(x,\_)\text{is integrable}\\ 17 & \text{otherwise}\end{cases} \tag{6.18}$$

Now, $g(x)=\begin{cases}0 & x\notin\mathbb{Q}\\ 17 & x\in\mathbb{Q}\end{cases}$, which is not integrable.

- Note that if f is continuous, all that is not an issue.
- Likewise, if $f(x,-)$ is integrable except for finitely many x's.

Theorem 14: $A\subset\mathbb{R}^n$ and $B\subset\mathbb{R}^m$ be rectangles, set $R=A\times B\subset\mathbb{R}^n\times\mathbb{R}^m$. Let $f:R\to\mathbb{R}$ be an integrable function, and let $\underline{g}(x)=\int_B f(x,y)\mathrm{d}y$.

$$\underline{g}(x)=\mathbb{L}\int_B f(x,y)\mathrm{d}y=L(f(x,-))=\sup L(f(x,-),P) \tag{6.19}$$
$$\bar{g}(x)=\mathbb{U}\int_B f(x,y)\mathrm{d}y=U(f(x,))=\inf U(f(x,),P) \tag{6.20}$$
$$\tag{6.21}$$

Then,

$$\int_R f=\int_A \underline{g}=\int_A \bar{g} \tag{6.22}$$

Now consider the same example, $\bar{g}(x)=\begin{cases}0 & x\notin Q\\ \frac{1}{q} & x\in Q\end{cases}=h(x)$, which is integrable.

Note that if f is continuous, $\underline{g}(x)=\bar{g}(x)=\int_B f(x,y)\mathrm{d}y$.

Proof. As f is integrable, there is a partition P of R where $U(f,P)-L(f,P)<\varepsilon$. Given a partition P of R, we can write it as $P_A\times P_B$ where P_A and P_B are partitions of A and B. Similarly, given an element of $S\in P$, we can write $S=S_A\times S_B$ where $S_A\in P_A$ and $S_B\in P_B$.

Lemma 3: Given a sequence of functions $h_k : X \to \mathbb{R}$. Then,

$$\sum_k \inf_{x\in X} h_k(x) = \inf_{x\in X} \sum_k h_k(x) \tag{6.23}$$

Proof.

$$\inf_{x\in X} h_k(x) \le h_k(y) \text{ for all y} \tag{6.24}$$

$$\sum_k \inf_P h_k(x) \le \sum_k \inf_P \sum_k h_k(x) \tag{6.25}$$

$$\sum_k \inf_P \sum_k h_k(x) = \sum_k \inf_P \sum_k h_k(x) \text{ for all } k \tag{6.26}$$

$$\tag{6.27}$$

□

Given a partition $P = P_A \times P_B$ of R,

$$\begin{aligned}
L(f,P) &= \sum_{S\in P} \text{vol}(S) \inf_{(x,y)\in S} f(x,y) && (6.28)\\
&= \sum_{S_A\in P_A \wedge S_B \in S_B} \text{vol}(S_A)\text{vol}(S_B) \inf_{x\in S_A} \inf_{y\in S_B} f(x,y) && (6.29)\\
&= \sum_{S_A\in P_A} \text{vol}(S_A) \sum_{S_B\in P_B} \text{vol}(S_B \inf_{x\in S_A} \inf_{y\in S_B} f(x,y)) && (6.30)\\
&\le \sum_{S_A\in P_A} \text{vol}(S_A) \inf x \in S_A \sum_{S_B\in P_B} \text{vol}(S_B) \inf_{y\in S_B} f(x,y) && (6.31)\\
&\le \sum_{S_A\in P_A} \text{vol}(S_A) \inf x \in S_A \underline{g}(x) && (6.32)\\
&= L(\underline{g}, P_A) && (6.33)
\end{aligned}$$

We have shown that $L(f,P) \le L(\underline{g}, P_A)$. By similar reasoning, we can show $U(\bar{g}, P_A) \le U(f,P)$. Now we show $L(\underline{g}, P_A) \le U(\bar{g}, P_A)$. This can be done by two ways. we know $L(\bar{g}, P_A)$ and $U(\underline{g}, P_A)$ are both less than $U(\bar{g}, P_A)$ and greater than $L(\underline{g}, P_A)$.
Now, assume $\varepsilon > 0$ and P was chosen such that $U(f,P) - L(f,P) < \varepsilon$ as f is integrable. Then, $U(\underline{g}, P_A) - L(\underline{g}, P_A) < \varepsilon$ and $U(\bar{g}, P_A) - L(\bar{g}, P_A) < \varepsilon$. So $\underline{g}$ and $\bar{g}$ are both integrable on A. Also, $\int_A \bar{g}$ and $\int_A \underline{g}$ are between $L(f,P)$ and $U(f,P)$ for any P. Taking the infimum over all P, for $U(f,P)$ and the supremum over all P for $L(f,P)$ we get

$$\int f \le \int \bar{g} \le \int f \tag{6.34}$$

$$\int f \le \int \underline{g} \le \int f \tag{6.35}$$

Thus,

$$\int f = \int \bar{g} = \int \underline{g} \tag{6.36}$$

□

Theorem 15: Given any $A \in \mathbb{R}^n$, and U is an open cover of A, $\exists$ open $W \supset A$ and a countable collection of functions $\Phi = \{\varphi : W \to [0,1]\} \subset C^\infty$ such that

1. Locally finite:

$$\forall x \in W \exists \text{ neighborhood } V \ni x \, |\{i : \text{supp}\phi \cap V \ne \phi\}| < \infty \tag{6.37}$$

2. Sum = 1

$$\forall x \in A \tag{6.38}$$

Lemma 4: Given a compact $C \subset U$ open set U, $\exists f \in C^\infty(\mathbb{R}^n)$ s.t. $f|_C = 1$, $\mathrm{supp} f \subset U$.

Proof. Step 1: $\exists$ smooth 1D seashore: $\sigma \in C^\infty(\mathbb{R})$ where

$$\begin{cases} \sigma(x) = 0 & x \leq 0 \\ \sigma(x) > 0 & x > 0 \end{cases} \tag{6.39}$$

One such σ is

$$\sigma(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x} & x > 0 \end{cases} \tag{6.40}$$

Step 2: $\exists$ smooth 1D bumps $\beta_\epsilon \in C^\infty(\mathbb{R})$. Set $\beta_\epsilon(x) = \sigma(\epsilon + x)\sigma(\epsilon - x)$
Step 3: $\exists$ smooth nD bumps: given $a \in \mathbb{R}^n, \epsilon > 0 \exists \beta \in C^\infty, \beta \geq 0, \beta(a) > 0, \beta(x) = 0 \impliedby |x - a| < \epsilon$

$$\beta(x) = \beta_{\epsilon^2}(|x - a|^2) \tag{6.41}$$

Step 4: $\exists$ smooth step functions $\Theta : \mathbb{R} \to [0, 1] \in C^\infty(\mathbb{R})$ such that $\Theta(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 1 \end{cases}$.

$$\Theta(x) = \frac{\int_0^x \beta_{1/2}(t - \frac{1}{2})\mathrm{d}x}{\int_0^1 \beta_{1/2}(t - \frac{1}{2})\mathrm{d}x} \tag{6.42}$$

For each $x \in C$ find $\epsilon_x > 0$ such that $\overline{B_{\epsilon_x}(x)} \subset U$ because U is open. $\square$