MAT292—ODE

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1 Newton's Law of Cooling

- Newtons law of cooling states that
$$\frac{\mathrm{d}u}{\mathrm{d}t} = -k(u - T_0) \tag{1.1}$$
- Note that there is one trivial solution, the equilibrium solution is $u(t) = T_0$. The meaning of this solution is the temperature of an object doesn't change when it is already at the equilibrium temperature.
-
$$\frac{\frac{\mathrm{d}u}{\mathrm{d}t}}{u - T_0} = -k \tag{1.2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\log(u - T_0) = -k \tag{1.3}$$
$$\log(u - T_0) = -kt + c_1 \tag{1.4}$$
$$u = T_0 + \exp(c_1)\exp(-kt) = T_0 + c_2\exp(-kt) \tag{1.5}$$
Note that $c_2 = \exp(c_1) > 0$. However, this is not a complete solution as it cannot describe the solutions with $u < T_0$.
- **Warning:** note that the integral of $\frac{1}{x}$ is $\log|x|$, **NOT** $\log(x)$. This is what caused the solution to be incomplete.
- Hence, $c_2 = \pm\exp(c_1)$. Note that $c_1 = \pm\infty$ is allowed thus so c_2 can take any value.
- Below are the integral curves.

2 Classifications

Definition 1: A differential equation is an equation containing one or more unknown functions of one or more independent variables.

- The order of a differential equation is the order of the highest derivative
- The most general n-th order ODE:
$$F[t, y, y', y'', \ldots, y^{(n)}] = 0 \tag{2.1}$$
- Linear ODE.
- Autonomous ODE is an ODE which does not explicitly depend on the independent variable, like $y' = y$. $y' = ty$ is not autonomous.
- Seperable first order ODE is a first order ODE that can be written as $y' = p(t)q(y)$.
- Newton's Law of cooling is first order, linear, autonomous, and seperable.

3 Systems of Differential Equations

- Think of a zombie apocalypse. You need to find a good time to find food.
- Let x be the number of people, and y be the number of zombles. This can be modelled by the *Lotka–Volterra* or Predator–Prey equations.
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x - \beta xy \tag{3.1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\gamma y + \delta xy \tag{3.2}$$
- The term αx is inspired by short term population growth.
- The term $-\beta xy$ is inspired by the fact that zombies are eating people.
- The term $-\gamma y$ is inspired by the fact zombie die.
- The δxy term is inspired by the fact that people can be converted to zombies.
- Note that this is **not** a linear equation. The term xy is nonlinear. Let the dependent variable be $z = \begin{pmatrix} x \\ y \end{pmatrix}$. Then $xy = z^T \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$
- The most general quadratic form for vector (system of) equations is
$$z^T Az + b^T z + c \tag{3.3}$$
- Now take two twitter accounts, with each account telling its followers to unfollow the other account, with rates $m > 0, n > 0$ respectively. The accounts will naturally grow by word of mouth, with rates $k > 0, l > 0$ respectively. Note these constraints are important.
$$p' = kp - mo \tag{3.4}$$
$$o' = lo - np \tag{3.5}$$
- There are oversimplification for this model. It ignores the fact that when somebody unfollows they cannot unfollow again.

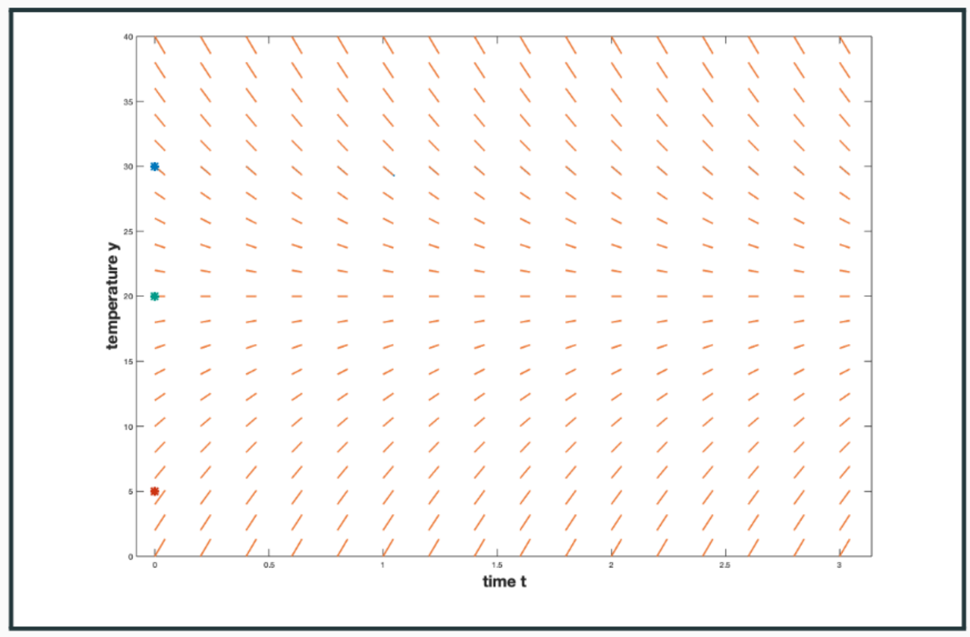
4 Qualitative Methods: Direction Fields and Phase Lines

Definition 2: Consider the ODE $y' = f(t, y)$. We can draw a **direction field** as follows:

- Draw a $t - y-$coordinate system.
- Evaluate $f(t, y)$ over a rectangular grid of points.
- Draw a line at each point (t, y) of the grid with slope $f(t, y)$

- Let's look again at Newton's law of cooling:

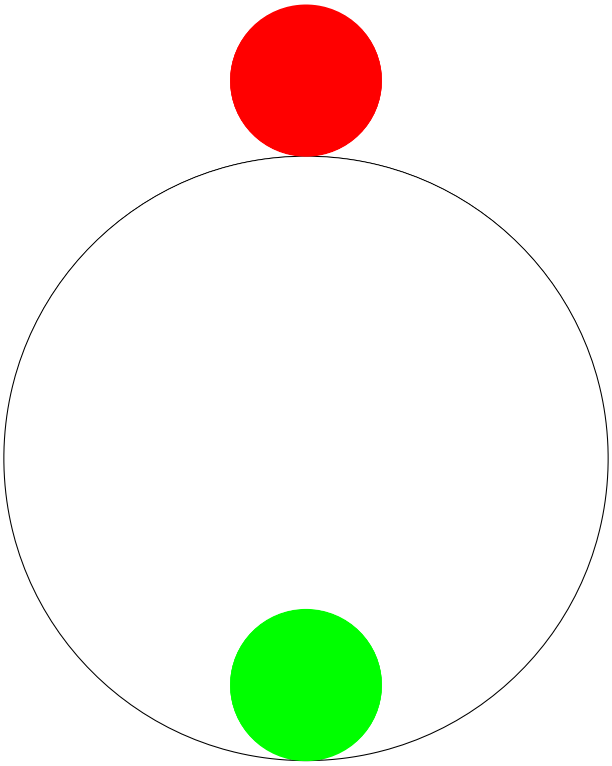
$$y' = -1.5(y - 20) \tag{4.1}$$



- Based on the initial conditions, we can draw the approximate solution by following direction field.
- A lot of the behavior of the differential equation are visible from the slope field.

Definition 3: Consider an autonomous first-order ODE $y' = f(y)$.
If $f(c) = 0$ for a specific value c, we call c an **equilibrium** of the ODE.
We say it is

1. a **stable equilibrium**, if a solution starting at a value close to c approaches $y = c$ as $t \to \infty$.
2. an **unstable equilibrium**, if a solution starting at a value close to c moves away from $y = c$ as $t \to \infty$.
3. a **semistable equilibrium**, if we observe either behavior, depending on if the solution starts just above or below c.

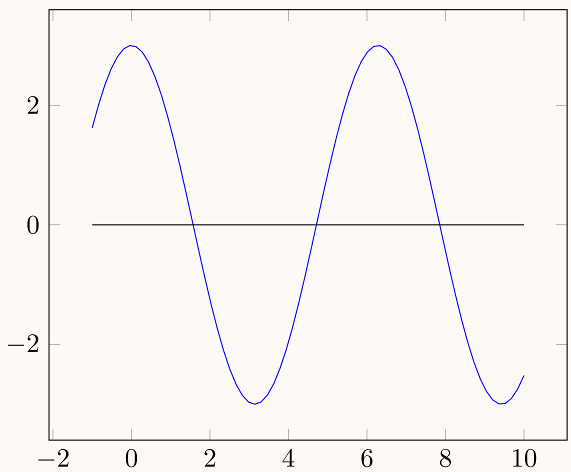


- The red circle is in unstable equilibrium. The green circle is in stable equilibrium.
- Something resting on the saddle point of $y = x^3$ will be in semistable equilibrium.

Example 1

Find and classify the equilibria of the ODE $y' = 3\cos y$

Phase Plot



Solution 1 (): To find equilibrium, set $y' = 0$.

$$y' = 3\cos y = 0 \tag{4.2}$$

$$y = \frac{\pi}{2} + n\pi, n \in \mathbb{Z} \tag{4.3}$$

At the equilibrium at $y = \frac{\pi}{2}$, In the phase diagram, anything below or above it s

5 Linear Equations: Method of Integrating Factors

- No general method for finding analytic solutions to first order differential equations.
- There exist classes of equations for which we know a corresponding solution method.

Definition 4: Standard form for a first order linear differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = p(t)y = g(t) \tag{5.1}$$

- For newton's law of cooling, the standard form is

$$\frac{\mathrm{d}u}{\mathrm{d}t} + ku = kT_0 \tag{5.2}$$

Derivation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} - p(t)y = g(t) \tag{5.3}$$

$$\mu(t)\frac{\mathrm{d}y}{\mathrm{d}t} - \mu(t)p(t)y = \mu(t)g(t) \tag{5.4}$$

$$\mu(t)p(t) = \mu'(t) \tag{5.5}$$

$$p(t) = \frac{\mu'(t)}{\mu(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\log(\mu(t)) \tag{5.6}$$

$$\mu(t) = \exp\left(\int p(t)\mathrm{d}t\right) \tag{5.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)y(t)) = \mu(t)g(t) \tag{5.8}$$

$$\mu(t)y(t) = \int \mu(t')g(t')\mathrm{d}t' \tag{5.9}$$

$$y(t) = \frac{\int \mu(t')g(t')\mathrm{d}t' + C}{\mu(t)} \tag{5.10}$$

Note that the C is in the numerator and **must** be divided by $\mu(t)$.

Example 2

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -k(u - T_0 - A\sin(\omega t)) \tag{5.11}$$

In standard form,

$$\frac{\mathrm{d}u}{\mathrm{d}t} + ku = kT_0 + kA\sin(\omega t) \tag{5.12}$$

Calculate the integrating factor

$$\mu(t) = \exp\left(\int k\mathrm{d}t\right) = \exp(kt) \tag{5.13}$$

Calculate the general solution

$$\begin{aligned} y &= \exp(-kt)\int \exp(kt')k(T_0 + A\sin(\omega t'))\mathrm{d}t & (5.14)\\ &= T_0 + \frac{kA}{k^2+\omega^2}(k\sin(\omega t) - \omega\cos(\omega t)) + C\exp(-kt) & (5.15)\end{aligned}$$

6 Existance and Uniqueness of Solutions

- Uniqueness is the question if a model can only follow one process or not.

Theorem 1: Consider the IVP $y' + p(t)y = g(t)$ with initial value $y(t_0) = y_0$ and an interval $I = (\alpha, \beta)$. If

1. $t_0 \in I$
2. $p(t)$ continuous on I
3. $g(t)$ continuous on I

Then,

1. This IVP has a solution and this solution is unique.
2. This solution exists for all time $t \in I$.
3. The ODE has a general solution that depends on one free parameter.

Proof. Integrating factor method constructs the unique solution. □

Example 3

$ty' + 2y = 4t^2$, $y(1) = 2$. As $t \neq 0$, $t' + 2\frac{y}{t} = 4t$.
Pick $I = (0, \infty)$. $p(t), g(t)$ are continous on I. Thus, this IVP has a unique solution.
The general solution is $y = t^2 + \frac{C}{t^2}$.

Theorem 2: Consider the IVP $y' = f(t, y)$ with initial value $y(t_0) = y_0$. Consider furthermore an open rectangle $R = (\alpha, \beta) \times (\gamma, \delta)$ in the $t - y$ plane. If

1. $(t_0, y_0) \in R$
2. f is continuous in R.
3. $\frac{\partial f}{\partial y}$ is continuous in R.

Then the IVP has a unique solution. ~~The soluton exists for $t \in (\alpha, \beta)$.~~ The solution exist for some interval $(t_0 - h, t_0 + h)$ where $(t_0 - h, t_0 + h) \subset (\alpha, \beta)$.

Remarks:

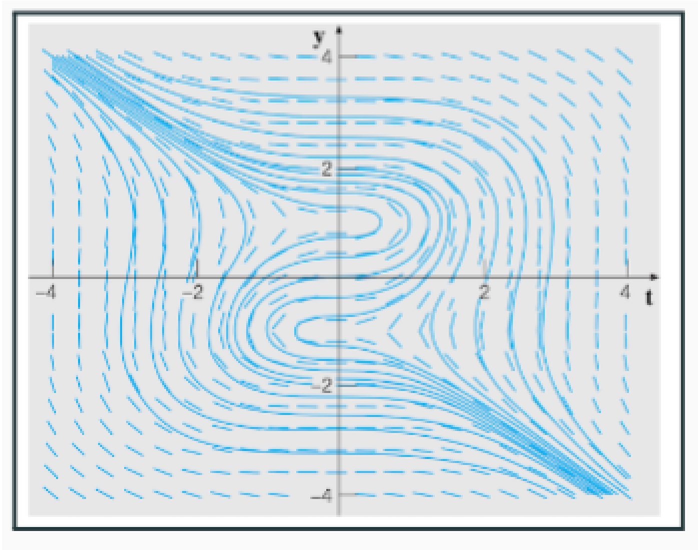
1. Non-linear ODEs don't necessarily have a general solution.
2. The solution might be implicit. e.g. $\sqrt{y^2 + \log y} = 5t$

Clarifications:

- We need f continuous in the rectangle to get existance.
- We need $\frac{\partial f}{\partial y}$ continuous in the rectangle to get uniqueness.

Proof. The proof for this theorem is beyond the scope of this class. □

- Note that these theorems are **sufficient** conditions for existance and uniqueness. Even if the hypotheses are not satisfied, it may be possible that existance and uniqueness holds. However, they are not guarenteed by the theorem.
- If existance and uniqueness holds, solution curves **cannot** cross each other.
- For the equation $y' = \frac{t^2}{1 - y^2}$, Picard–Lindelöf guarentees local existance and uniqueness, however, the existance and uniqueness at given initial conditions only lasts for a finite interval.



7 Autonomous Equations and Population Dynamics

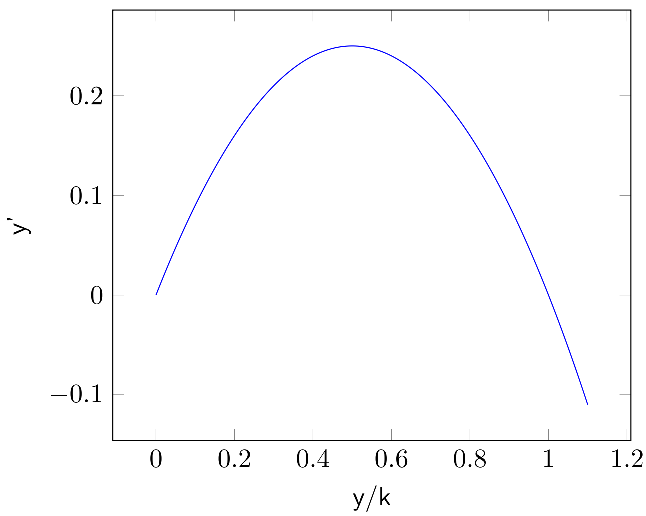
- Exponential Growth $y' = ky$.
- Logistic Growth:
  - If uninhibited, we assume exponential growth.
  - However, in the long run, population is limited to k. Consider the model

$$y' = ryh(y) \tag{7.1}$$

  We would want several properties about $h(y)$.
    * $h(y) \approx 1$ if $y << k$.
    * $0 < h(y) < 1$ if $y < k$.
    * $h(y) = 0$ if $y = k$.
    * $h(y) < 0$ if $y > k$.
  - To keep the model as simple as possible, we will fulfill all the conditions with a linear function $h(y) = 1 - \frac{y}{k}$.
  - We arrive at the ODE

$$y' = r\left(1 - \frac{y}{k}\right)y \tag{7.2}$$

  - As this is an autonomous ODE, we can draw a phase plot.



  - There are two fixed points. $y/k = 0$ is unstable and $y/k = 1$ is stable.
  - If $y/k < 1/2$, then the solution will have an inflection point.

8 Variation of Parameters

Consider the 2nd order nonhomogeneous ODE $y'' + p(t) + q(t)y = g(t)$. The general solution of this ODE can be written as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) \tag{8.1}$$

where $Y(t)$ is a particular solution of the 2nd order ODE.

- Assume we have already found the homogenous solution $y(t) = c_1 y_1(t) + c_2 y_2(t)$.
- To get a particular solution of the try non-constant parameters

$$Y(t) = u_1(t) y_1(t) + u_2(t) y_2(t) \tag{8.2}$$

Now consider a system of ODES $x' = P(t)x + g(t)$. If given the homogenous solution, we can find the fundamental matrix $X(t)$ such that $X' = P(t)X$.

To find the particular solution, $x_p = X(t)U(t)$. Plugging this into the ODE we get

$$(X(t)U(t))' = P(t)X(t)U(t) + g(t) \tag{8.3}$$
$$X'(t)U(t) + X(t)U'(t) = P(t)X(t)U(t) + g(t) \tag{8.4}$$
$$X(t)U'(t) = g(t) \tag{8.5}$$
$$U'(t) = X^{-1} g(t) \tag{8.6}$$

- For a second order equation, use $X = \begin{pmatrix} y \\ y' \end{pmatrix}$. Then, the particular solution is given by

$$y_p = y_1 \int \frac{-y_2 g}{y_1 y_2' + y_1' y_2} dt + y_2 \int \frac{y_1 g}{y_1 y_2' + y_1' y_2} dt \tag{8.7}$$