PHY293—Waves & Modern Physics

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Contents

1 Harmonic Oscillators

- Think of simple harmonic motion (SHM) as circular motion projected into one dimension. (a wave is rotation in the complex plane)
- One of the simplest system is a mass on the spring, with known force $F = -k\Delta x$.
- For **SHM**, the motion must be periodic, and the force must be proportional to displacement.
- Using Newton's 2nd law on the spring force,

$$m\ddot{x} = -kx \tag{1.1}$$
$$\ddot{x} + \frac{k}{m}x = 0 \tag{1.2}$$

- For a mass on the vertical spring, the equilibrium position will be lower due to gravity.

$$k(y_1 - y_0) - mg = 0 \tag{1.3}$$
$$y_1 = y_0 + \frac{mg}{k} \tag{1.4}$$

y_1 is the new equilibrium position. SHM will still occur if the system is disturbed.

- Especially when dealing with energy, it is a good idea to have the origin at $y = y_1$.

1.1 The Differential Equation

- In general, the equation for simple harmonic motion is

$$\ddot{x} + \omega^2 x = 0 \tag{1.5}$$

- The solution to (1.2) can be represented as

$$x = A\cos(\omega t + \varphi_0) \tag{1.6}$$

- A represents the amplitude
- φ_0 is the phase angle (initial phase)
- $\omega = \sqrt{\dfrac{k}{m}}$ is the angular frequency.
- The velocity and acceleration can be easily calculated by taking derivatives.

$$\dot{x} = -A\omega\sin(\omega t + \varphi_0) \tag{1.7}$$
$$\ddot{x} = -A\omega^2\cos(\omega t + \varphi_0) \tag{1.8}$$

- Another way to represent the full solution is

$$x = a\cos(\omega t) + b\sin(\omega t) \tag{1.9}$$

where $a = A\cos\varphi_0, b = -A\sin\varphi_0$.

Example 1

Determine the amplitude and phase constant of a pendulum moving with a motion described by a sum of two functions, $x_1(t) = 0.25\cos\omega t$ and $x_2(t) = -0.5\sin\omega t$.

Solution 1 ():

$$0.25 = A\cos\phi_0 \tag{1.10}$$
$$-0.50 = -A\sin\phi_0 \tag{1.11}$$

Figure out phi_0 from the ratio of eq.(1.11)/(1.10)

1.2 Energy of SHM

- A mass has kinetic energy of $T = \dfrac{1}{2}mv^2$.
- The potential energy is related to the restoring force and by definition,

$$\Delta U = -\int F\cdot \mathrm{d}x = -\int_{x_i}^{x^f}(-kx')\mathrm{d}x' = \frac{1}{2}k(x_f^2 - x_i^2) \tag{1.12}$$

- The conservation of energy here follows from newton's 2nd law.

$$m\ddot{x} = -kx \tag{1.13}$$

- Looking at the potential energy

$$x = A\cos(\omega t + \varphi_0) \tag{1.14}$$
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t) \tag{1.15}$$

- Looking at the kinetic energy,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t) \tag{1.16}$$

- The total energy is

$$E = T + \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}mv_{MAX}^2 \tag{1.17}$$

1.3 Physics of Small Vibrations

- Most system will oscillate with SHM when the amplitude is small. Recall the taylor expansion of an analytic function

$$f(x) = f(a) + (x-a)f'(a) + (x-a)^2 f''(a) + \dots \tag{1.18}$$

  If a is a minima, $f'(a) = 0$. For $x \approx a$, the higher order terms

- The potential energy of a pendulum is

$$U = mgy = mgL(1-\cos\theta) \tag{1.19}$$

- **For this course, an angle $\theta < 10°$, it will be considered small.**
- For a simple pendulum,

$$-mg\sin\theta = ma \tag{1.20}$$

$$-mg\sin\theta = m\ddot{s}, s = L\theta, \mathrm{d}s = L\mathrm{d}theta \tag{1.21}$$

$$-g\sin\theta = L\ddot{\theta} \tag{1.22}$$

  For small angles, $\theta \approx \sin\theta$

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \tag{1.23}$$

  This is the equation for SHM

- The energy of the pendulum is

$$E = T + U = \frac{1}{2}mv^2 + mg\left(\frac{x^2}{2L}\right) \tag{1.24}$$

- For physical pendulum,

$$\tau = I\alpha = I\ddot{\theta} \tag{1.25}$$

$$-mgd\sin\theta = I\ddot{\theta} \tag{1.26}$$

  For small $\theta : \sin\theta \approx \theta$

$$\ddot{\theta} + \frac{mgd}{I}\theta = 0 \tag{1.27}$$

- For LC circuits,

$$\ddot{i} + \frac{1}{LC}i = 0 \tag{1.28}$$

  When a resistor is connected, energy is lost as heat and the circuit behaves like a damped oscilator.

2 Damped Oscillations

The damped oscillator involves the addition to the drag force that is proportional to $-v$ to the simple harmonic oscillator.

Define

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0 \tag{2.1}$$

$$x = A\exp\left(\left(\sqrt{\frac{\gamma^2}{4} - \omega_0^2} - \frac{\gamma}{2}\right)t\right) + B\exp\left(-\left(\sqrt{\frac{\gamma^2}{4} - \omega_0^2} - \frac{\gamma}{2}\right)t\right) \tag{2.2}$$

For $\omega_0^2 \neq \frac{\gamma^2}{4}$.

Critical damping gives the minimum time for the system to return to equilibrium when $\omega_0^2 \neq \frac{\gamma^2}{4}$.

$$x = (Ax + B)\exp(-\omega_0 t) \tag{2.3}$$

Example 2

A mass $m = 3$ is attached to a spring with a value of $k = 600$

1. Determine the value of the damping constant b that would produce critical damping.
2. Determine the value of damping constant b that would decrease the angular frequency by 10%/
3. A mass recieve an impulse that ives it a initial velocity $v = 2$. What is the maximum resultant displacement and the time when it occurs.

Solution 2 ():

1.

$$\omega_0 = \frac{\gamma}{2} \tag{2.4}$$

$$x = (A + Bt)\exp\left(-\frac{\gamma t}{2}\right) \tag{2.5}$$

$$\dot{x} = \exp\left(-\frac{\gamma t}{2}\right)\left(B - \frac{\gamma B t}{2} - A\frac{\gamma}{2}\right) \tag{2.6}$$

$$\tag{2.7}$$

$x(0) = 0, \dot{x}(0) = v_i$

$$x = v_i t \exp\left(-\frac{\gamma t}{2}\right) \tag{2.8}$$

$$\dot{x} = v_i \exp\left(-\frac{\gamma t}{2}\right)\left(1 - \frac{\gamma t}{2}\right) = 0 \tag{2.9}$$

$$t = \frac{2}{\gamma} = \frac{2m}{b} \tag{2.10}$$

$$x\left(\frac{2}{\gamma}\right) = v_i t \exp\left(-\frac{\gamma t}{2}\right) = \frac{2v_i}{\gamma e} \tag{2.11}$$

2.1 Energy

For underdamped oscillator, assume $\omega \approx \omega_0$.

$$E = T + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{2.12}$$

$$x = A_0 \exp\Big((i\omega_0 - \frac{\gamma}{2})t\Big) \tag{2.13}$$

$$\dot{x} = -A_0 \exp\Big(t\left(i\omega_0 - \frac{\gamma}{2}\right)\Big) = A_0\left(i\omega_0 - \frac{\gamma}{2}\right)\exp\Big(t\left(i\omega_0 - \frac{\gamma}{2}\right)\Big) \tag{2.14}$$

$$E = \frac{1}{2}kA_0^2\exp(-\gamma t) \tag{2.15}$$

$$\tag{2.16}$$

The rate of change of energy is

$$\dot{E} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = \dot{x}(m\ddot{x} + k\dot{x}) \tag{2.17}$$

Recall $m\ddot{x} = -kx - b\dot{x}$

$$\dot{E} = -b\dot{x}^2 = -\gamma E \tag{2.18}$$

Example 3

The energy of a simple harmonic oscillator is ovserved to reduce by a factor of two after 10 complete cycles.

1. How amny cycles will it take to reduce it by a factor of 8?
2. By what factor would it be reduced after 100 cycles?

Solution 3 ():

1.

$$\frac{E}{E_0} = e^{-\gamma t} \tag{2.19}$$
$$\frac{1}{2} = e^{-\gamma 10T} \tag{2.20}$$
$$\left(\frac{1}{2}\right)^3 = \left(e^{-\gamma 10T}\right)^3 = e^{-\gamma 30T} \tag{2.21}$$

2.2 Quality Factor

Definition 1: The **quality factor** is a convenient measurement on how good an oscillator is (how many oscillations it can make before its amplitude would decrease by a certain rate) is defined as

$$Q = \frac{\omega_0}{\gamma} = \frac{\sqrt{km}}{b} \tag{2.22}$$

where $\gamma = 2\beta = \dfrac{b}{m}$.

- If $Q = \dfrac{1}{2}$, the system is critically damped.
- Energy $E = E_0 \exp(-\gamma t)$.
- At two different times t_1 and t_2 seperated by period T,

$$\frac{E(t+T)}{E(t)} = \exp(-\gamma T) \tag{2.23}$$

Which leads to

$$\frac{E(t) - E(t+T)}{E(t_1)} = 1 - e^{-\gamma T} \approx -\gamma T \approx \frac{2\pi\gamma}{\omega} = \frac{2\pi}{Q} \tag{2.24}$$

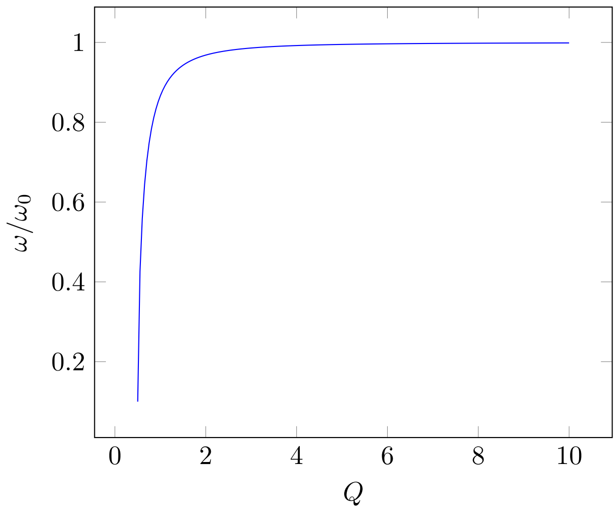
Thus,

$$Q = \frac{\text{Energy stored in the oscillator}}{\text{Energy dissipated per radian}} \tag{2.25}$$

- We can rewrite the equation of the damped oscillator

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \tag{2.26}$$
$$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x = 0 \tag{2.27}$$

As $\gamma = \dfrac{\omega_0}{Q}, \omega = \omega_0\sqrt{1 - \dfrac{1}{4Q^2}}$.



Example 4

WHen an electron in H is moved from $n = 2$ to $n = 3$ states, the atom behaves like a damped oscillator when the lights of frequency 4.57×10^{14} Hz is emmited. The lifetime of the excited atom is approximately 10 ns. What is the value of the quality factor?

Solution 4 (): As $\gamma = \frac{1}{\tau}, \omega_0 = 2\pi f$,

$$Q = \frac{\omega_0}{\gamma} = 2\pi f \tau = 2.87 \times 10^7 \tag{2.28}$$

- RLC circuit:

$$RI + L\dot{I} + \frac{q}{C} = 0 \tag{2.29}$$

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0 \tag{2.30}$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0 \tag{2.31}$$

3 Driven Oscillations

In the undamped case,

$$m\ddot{x} + kx = F_0 e^{i\omega t} \tag{3.1}$$

$$m\ddot{x} + kx = kae^{i\omega t} \tag{3.2}$$

$$\tag{3.3}$$

The particular solution is

$$x = A(\omega)e^{i(\omega t - \delta)} \tag{3.4}$$

where

- $\tan\delta = 0$ and $A(\omega) = \frac{a}{1 - (\omega/\omega_0)^2}$ when $\omega < \omega_0$
- $\tan\delta = \pi$ and $A(\omega) = -\frac{a}{1 - (\omega/\omega_0)^2}$ when $\omega > \omega_0$

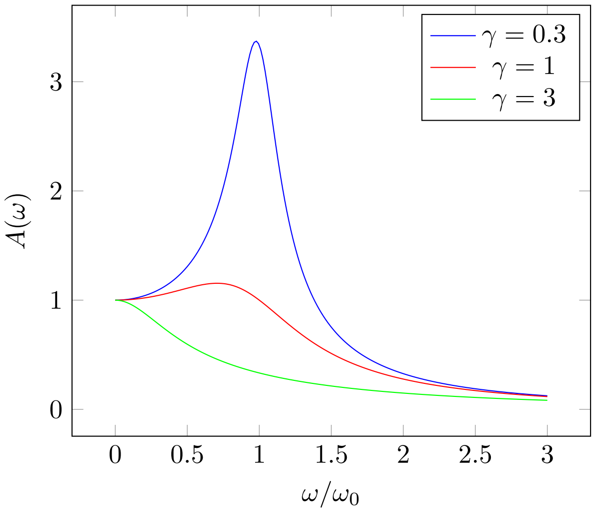
3.1 With damping

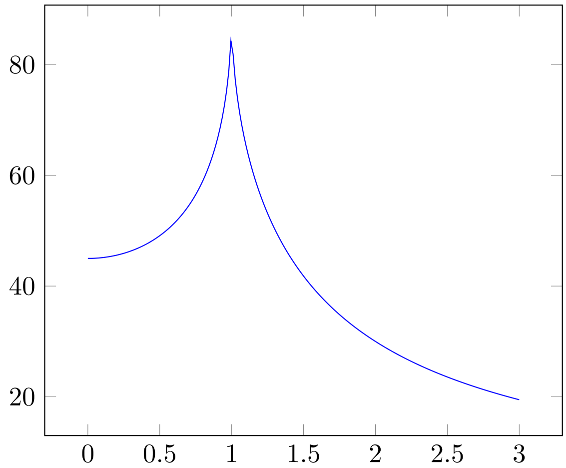
$$\ddot{x} + \gamma\dot{x} + \omega_0^2 = ae^{i\omega t} \tag{3.5}$$

Solving for x, the particular solution is

$$x = A(\omega)e^{i(\omega t-\delta)} \tag{3.6}$$

with $\tan\delta = \dfrac{\omega\gamma}{\omega_0^2+\omega^2}$, $A(\omega) = \dfrac{a\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}}$.





If it is driven at the resonance frequency,

$$A = \frac{a\omega_0}{\gamma} \tag{3.7}$$

$$\delta = \frac{\pi}{2} \tag{3.8}$$

For large damping, $\omega_{max} \neq \omega_0$.

3.2 Power

If the motion of a driven oscillator is

$$x = A(\omega)e^{i(\omega t-\delta)} \tag{3.9}$$

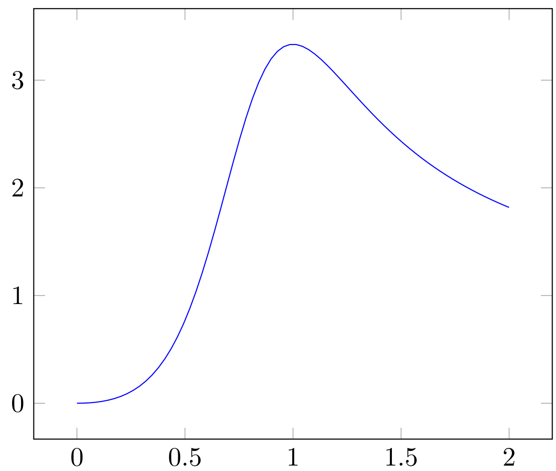
The velocity is

$$\dot{x} = iA(\omega)\omega e^{i(\omega t-\delta)} \tag{3.10}$$

Note that power lost to damping is

$$P = b\dot{x}^2 \tag{3.11}$$

The full width at half height of the period-averaged power is $\gamma/\omega_0 = \dfrac{1}{Q}$



If the driving frequency is close to the natural frequency,

$$(\omega^2 - \omega_0^2) = (\omega - \omega_0)(\omega + \omega_0) \approx -2\omega_0 \Delta\omega \tag{3.12}$$

Then,

$$\overline{P}(\omega) = \frac{F_0^2}{2m\gamma[\frac{4\Delta\omega^2}{\gamma^2} + 1]} \tag{3.13}$$

3.3 Reasonance in RLC circuits

$$RI + L\dot{I} + \frac{q}{C} = \varepsilon_0 \cos(\omega t) \tag{3.14}$$

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = \varepsilon_0 \cos(\omega t) \tag{3.15}$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{\varepsilon_0}{L}\cos(\omega t) \tag{3.16}$$

$$\omega_0^2 = \frac{1}{LC}, \gamma = \frac{R}{L}, Q = \frac{\omega_0}{\gamma} = \sqrt{\frac{L^2}{RC}}$$

$$q(t) = q_0(\omega) e^{i(\omega t - \delta)} \tag{3.17}$$

$$q_0(\omega) = \frac{\varepsilon_0}{L\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{R\omega}{L}\right)^2}} = \frac{\varepsilon_0}{\omega\sqrt{\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2}} \tag{3.18}$$

3.4 Transient Phenomena

When the driving force is first applied and the system is distributed from equilibium the system will be inclined to oscillate at its natural frequency, especially if the system is delicate.This is shown by the homogenous solution of the differential equation.

4 Coupled Oscillator

- For a system of coupled oscillators can be described by equations

$$m_1\ddot{x}_1 = k_{11}x_1 + \cdots + k_{1n}x_n \tag{4.1}$$

$$\vdots \tag{4.2}$$

$$m_n\ddot{x}_n = k_{n1}x_1 + \cdots + k_{nn}x_n \tag{4.3}$$

Thus,

$$M\ddot{x} = Kx \tag{4.4}$$