MIE407 Nuclear Reactor Theory & Design

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Nuclear Stability:

- Stability of nucleus is the result of the balance between the strong nuclear force and the electromagnetic force.
- Range of the strong nuclear force is about $1\,\mathrm{fm}$ (femtometre, $10^{-13}\,\mathrm{cm}$)
- The strong nuclear force acts equally between protons and neutrons, but neutrons also reduce repulsions between protons by pushing them further apart from each other.
- Fermi approx: $r = R_0 A^{1/3}$ where $R_0 = 1.2\,\mathrm{fm}$
- The strong nuclear force is repulsive at very small distances, which contributes to the incompressibility of the nucleus.
- The strong nuclear force is insignificant at larger than around 4 proton diameters.

Nuclear Binding Energy:

- The binding energy of a nucleus can be found by calculating the mass defect of the nucleus compared to the mass of the unbound nucleons.

$$E_B = [Zm_p + (A-Z)m_n - m]c^2 \equiv \Delta M \times c^2 \tag{1}$$

- The binding energy is usually expressed as the average binding energy per nucleon $\varepsilon = E_B/A$.
- $1\,\mathrm{u}{=}931.5\,\mathrm{MeV/c^2}$

Compound nucleus decay modes:

- Neutron capture (n,γ): Nucleus decays to a lower energy state by emitting some gamma rays. $n +^A X \rightarrow^{A+1} X^* \rightarrow^{A+1} X + \gamma$
- Elastic scattering (n,n'): Neutron is re-emitted after leaving the nucleus in the ground state.
- Inelastic scattering $(n,n'\gamma)$: Neutron (usually high energy) is re-emitted at a lower energy, leaving the nucleus in an excited state. Then, the nucleus emits some gamma rays to return to the ground state.
- Fission (n,f): Nucleus splits into two fragments with approx. $2:3$ mass ratio.
- Particle emission $(n,\alpha),(n,p),(n,kn)$: A particle other than a neutron, or multiple neutrons are emitted. Only occurs with very high energy neutrons. Hence, only a small number of $(n,2n)$ reactions occur in nuclear reactors.

Nuclear Fission

- When a thermal neutron ($\leq 1\,\mathrm{eV}$) is absorbed by a U-235 nucleus, it forms a compound nucleus that fissions with $\approx 84\%$ probability.
- Fission is a threshold reaction. In cases for U-235, the threshold energy is lower than the energy gained by binding the extra neutron. Hence, fissions may occur by neutrons with any energy. However, for U-238, the threshold energy is about 1 MeV greater than the energy gained by binding the extra neutron. Hence, fissions only occur by neutrons with kinetic energy greater than 1 MeV, which rarely occurs in even a fast reactor.
- Under thermal conditions ($0.0253\,\mathrm{eV}$ or $2200\,\mathrm{m/s}$) the distribution of fission products is a strong bimodal distribution (Peaks at $A = 96, 135$).
- Most neutrons (prompt neutrons) emitted in fission are released at the instant of fission.
- A small number of fission fragments also emit neutrons (delayed neutrons). These compose of $< 1\%$ of the total neutrons, but they are significant in the transient behavior of the reactor.
- Delayed neutrons result from high nuclear excitation of the daughter (when the excess energy exceed the nuclear binding energy of the neutron)
- Between 0 to 7 prompt neutrons may be emitted by fission, on average around 2.4. This is very important.
- The fission neutron spectrum is approximately $\chi(E) = 0.484 \sinh\left(\sqrt{2E}\right) e^{-E} \mathrm{MeV}^{-1}$. Integrate this over a interval of E to obtain the probability of energy in that interval. (Average $2\,\mathrm{MeV}$)
- Delayed neutrons are much lower energy ($400\,\mathrm{keV}$)
- Reaction Rate stuff
  - Intensity $I = nv$ $(1/\mathrm{cm}^2\mathrm{s})$
  - Flux ϕ $(1/\mathrm{cm}^2\mathrm{s})$
  - Number density $N = \frac{\rho N_A}{M}$ $(1/\mathrm{cm}^3)$
  - Microscopic cross section σ $(1/\mathrm{cm}^2)$
  - Macroscopic cross section $\Sigma = N\sigma$ $(1/\mathrm{cm})$
  - Reaction rate $R = \Sigma\phi$ $(1/\mathrm{cm}^3\mathrm{s})$
- *Passive* or non-multiplying media are characterized by the scattering (σ_s) and capture (σ_c) cross sections.
- *Multiplying* media contain at least one fissile or fissionable (fissile at high energy only) isotope and are further characterized by fission cross-section σ_f and ν.

1 Propogation of Neutrons in a Passive Medium

- Due to interactions with the medium, the initial intensity I_0 decreases to $I(x)$ at depth x.
- The rate of decrease is the reaction rate $I' = -R = \Sigma I$, with solution $I(x) = I_0 e^{-\Sigma x}$. Σ is the total cross-section.
- Understood in probability terms, $e^{-\Sigma x}$ is the probability that a neutron survive to depth x. Also, for very small Δx, $\Sigma \Delta x$ is the probability that a neutron will interact in Δx.
- Finally, $\Sigma e^{-\Sigma x}\Delta x$ is the probability that a neutron will interact between distance x and $x + \Delta x$.
- The **mean free path** is defined as the average distance a neutron travels before interacting

  $$\lambda = \int_0^\infty x p(x) \mathrm{d}x = \int_0^\infty x \Sigma e^{-\Sigma x} \mathrm{d}x = \frac{1}{\Sigma} \tag{2}$$

  Note that the mean free path can be used for a single reaction type also, e.g. $\lambda_s, \lambda_c, \lambda_f$.
- Point neutron source. The intensity of the source is $\dot{S}$.

- In empty space, $\phi(r) = \frac{\dot{S}}{4\pi r^2}$.
- In medium, $\phi(r) = \frac{\dot{S}}{4\pi r^2} e^{-\Sigma r}$
- If there is scattering, these equations will underestimate the flux.

2 Neutron Flux and Current

- In analyzing a nuclear reactor, we are interested in the neutron population at any position/time, and the rates of all nuclear reaction at any position/time.
- The neutron density is $n(\mathbf{r}, \mathbf{v}, t)$. It is useful to express the velocity vector as the speed v and the unit direction $\mathbf{\Omega}$. We also use kinetic energy (which contains the same information as the speed $E = \frac{1}{2}mv^2$) i,e, $n(\mathbf{r}, E, \mathbf{\Omega}, t)$.
- $n(\mathbf{r}, E, \mathbf{\Omega}, t)$ is a density function with units $1/\mathrm{cm}^3\mathrm{eVsr}$. $n(\mathbf{r}, E, \mathbf{\Omega}, t)\mathrm{d}V\mathrm{d}E\mathrm{d}\Omega$.
- With $\mu := \cos\theta$, we have $\mathrm{d}\Omega = -\mathrm{d}\varphi\mathrm{d}\mu/4\pi$, when we integrate over the all solid angles, we can integrate φ from 0 to 2π and μ from 1 to -1.
- Here, we define $\mathrm{d}\Omega = \mathrm{d}S/4\pi = -\mathrm{d}\varphi\,\mathrm{d}(\cos\theta)/4\pi$. Note that $\int_{4\pi}\mathrm{d}\Omega = \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\varphi\int_{-1}^{1}\mathrm{d}\mu = 1$
- The **angular neutron density** $n(\mathbf{r}, E, \mathbf{\Omega})$ is an important reactor parameter used in the definition of neutron flux and current.
- The integral of the angular neutron density over all solid angles is the **neutron density** in $\mathrm{cm}^{-3}\mathrm{eV}^{-1}$

$$n(\mathbf{r}, E) = \int_{4\pi} n(\mathbf{r}, E, \mathbf{\Omega})\mathrm{d}\Omega. \tag{3}$$

- If some quantity q with density ρ can flow by a velocity field $\mathbf{v}$, we can define the **vector flux**

$$\mathbf{j} = \rho\mathbf{v}. \tag{4}$$

  We can define the **neutron flux density** as

$$\phi(\mathbf{r}, E, \mathbf{\Omega}) = n(\mathbf{r}, E, \mathbf{\Omega})\mathbf{v}(\mathbf{r}, E, \mathbf{\Omega}). \tag{5}$$

- The magnitude of vector flux density is the combined distance traveled by these neutrons (**this is important for determining reaction rate**) is defined as the **scalar neutron flux**,

$$\Phi(\mathbf{r}, E) = \int_{4\pi}\phi(\mathbf{r}, E, \mathbf{\Omega})\mathrm{d}\Omega = v\int_{4\pi} n(\mathbf{r}, E, \mathbf{\Omega})\mathrm{d}\Omega. \tag{6}$$

- The **neutron (net) current density** (**this is important for determining neutron transport**) is

$$\mathbf{J}(\mathbf{r}, E) = \int_{4\pi}\mathbf{v}n(\mathbf{r}, E, \mathbf{\Omega})\mathrm{d}\Omega = v\int_{4\pi}\hat{\mathbf{\Omega}}\,n(\mathbf{r}, E, \mathbf{\Omega})\mathrm{d}\Omega \tag{7}$$

- Since the integral in (7) is performed over all direction, which yields a vector that points in a particular direction. The direction of $\mathbf{J}$ represents the direction of **net** neutron flow.
- We can specify a unit vector $\hat{n}$ that is normal to $\mathrm{d}A$, then $\mathbf{J}\cdot\hat{n}\mathrm{d}A$ is the net number of neutrons crossing the surface $\mathrm{d}A$ per unit energy and unit time.

3 Neutron Diffusion Equation

- We will first try to understand how neutrons flows through an volume element $\Delta V = \Delta x \Delta y \Delta z$ within a passive medium. In a passive medium, there are two types of interactions (absorption and scattering).
- For sake of simplicity, we will assume a partial current $\mathbf{J} = J_x \hat{x}$. Due to interactions, we have

$$\frac{J_x(x+\Delta x, y, z) - J_x(x, y, z)}{\Delta x} \rightarrow \frac{\partial J_x}{\partial x} \mathrm{d}V \tag{8}$$

  We can repeat this for the J_y and J_z to obtain the **leakage** form

$$\mathrm{d}\mathcal{L} = \operatorname{div} \mathbf{J} \mathrm{d}V. \tag{9}$$

- At steady state, the net leakage (9) represents a mismatch between neutron gain and loss.
- The overall neutron balance in $\mathrm{d}V$ becomes

$$\mathrm{d}\mathcal{L} = (Q - R_a)\mathrm{d}V \tag{10}$$

  where Q is the source density in $\mathrm{cm}^{-3}\mathrm{s}^{-1}$ and $R_a = \Sigma_a \phi$ is the rate of absorption per unit volume.

$$\operatorname{div} \mathbf{J} + \Sigma_a \Phi = Q. \tag{11}$$

  equation (11) is the **neutron balance equation**.
- We need an equation to relate $\mathbf{J}$ and Φ. The most common approximation is the **diffusion approximation**, which assumes the neutron flux (**not neutron density**) behaves like a fluid

$$\mathbf{J}(\mathbf{r}) = -\mathbf{D}(\mathbf{r}) \nabla \Phi(\mathbf{r}) \tag{12}$$

- Here, the diffusion coefficient $\mathbf{D}$ has the dimensions of length. This depends on spatial position because the material in the reactor is not homogeneous. The diffusion coefficient is related to the cross-sections of the medium

$$D = \frac{\lambda_{tr}}{3} = \frac{1}{\Sigma_{tr}}, \tag{13}$$

  where λ_{tr} is the transport mean free path in cm.

3.1 Transport Cross-Section

- The macroscopic transport cross-section is determined by the density of the medium,

$$\Sigma_{tr} = N \sigma_{tr} \tag{14}$$

- The microscopic transport cross-section accounts for both absorption and inelastic scattering

$$\sigma_{tr} = \sigma_a + (1 - \bar{\mu})\sigma_s, \tag{15}$$

  where $\bar{\mu} \approx \frac{2}{3A}$ is the average cosine of the scattering angle θ.
- In the center of mass frame, the scattering is isotropic, but in the lab frame it is not due to conservation of momentum. $0 \leq \theta < 180°$ or $0 \leq \bar{\theta} < 90°$. As $\mu = \cos\theta, 1 \geq \mu > 0$.
- Typically, $\bar{\mu} << 1$ for heavier atoms $A >> 1$, so $\sigma_{tr} = \sigma_a + \sigma_s = \sigma_t$. In this course, **make this approximation unless otherwise told**.

3.2 Limitations of the Diffusion Approximation

- Underlying diffusion theory is an approximation that expresses the scattering kernel as an expression in Legendre polynomials, in which the first order term are used

$$\Sigma_s(z, \mathbf{\Omega} \to \mathbf{\Omega}') = \Sigma_s(z, \Omega \cdot \Omega') = \frac{1}{2\pi} \sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(\mu) \Sigma_s(z) \tag{16}$$

where $\mu = \cos(\mathbf{\Omega} \cdot \mathbf{\Omega}')$ and P_l is the Legendre polynomial of order l.

- This approximation is good **except**:
  - Near a strong source or absorber
  - At the boundary between two media with dissimilar neutron diffusion characteristics

3.3 Equation and Boundary Conditions

- Combining the diffusion approximation with the neutron balance equation, we obtain the **neutron diffusion equation**

$$(\mathrm{div}(D\nabla) + \Sigma_a)\Phi = Q \tag{17}$$

- A reactor core is modelled composed of numerous homogeneous regions in which this dependence on $\mathbf{D}$ vanishes. Then, this reduces to

$$(-D\Delta + \Sigma_a)\Phi = Q \tag{18}$$

- The net current at a plane $x = b$ inside a homogeneous material region in a slab reactor consists of two contributions

$$J_+(b) = \frac{\Phi(b)}{4} - \frac{D}{2} \frac{\mathrm{d}\Phi}{\mathrm{d}x} \tag{19}$$

$$J_-(b) = \frac{\Phi(b)}{4} + \frac{D}{2} \frac{\mathrm{d}\Phi}{\mathrm{d}x} \tag{20}$$

- These equations yield the net current at $x = b$ which agrees with Fick's law.
- Since neutrons leaving one region must enter the other, the additional boundary conditions are

$$J_+^A(b) = J_+^B(b) \tag{21}$$

$$J_-^A(b) = J_-^B(b) \tag{22}$$

Then substituting the values for J, we obtain

$$\frac{\Phi_A(b)}{4} - \frac{D_A}{2} \frac{\mathrm{d}\Phi_A}{\mathrm{d}x} = \frac{\Phi_B(b)}{4} - \frac{D_B}{2} \frac{\mathrm{d}\Phi_B}{\mathrm{d}x} \tag{23}$$

$$\frac{\Phi_A(b)}{4} + \frac{D_A}{2} \frac{\mathrm{d}\Phi_A}{\mathrm{d}x} = \frac{\Phi_B(b)}{4} + \frac{D_B}{2} \frac{\mathrm{d}\Phi_B}{\mathrm{d}x} \tag{24}$$

We can add and subtract these equations and obtain

$$\Phi_A(b) = \Phi_B(b) \tag{25}$$

$$D_A \frac{\mathrm{d}\Phi_A}{\mathrm{d}x} = D_B \frac{\mathrm{d}\Phi_B}{\mathrm{d}x} \tag{26}$$

Thus, we know from Fick's law that

$$J_A(b) = J_B(b) \tag{27}$$

$$\tag{28}$$

- When region B is vacuum, there is no return current. Hence,

$$0 = J_{-}^{A}(b) = \frac{\Phi_A(b)}{4} + \frac{D_A}{2}\frac{\mathrm{d}\Phi_A}{\mathrm{d}x} \tag{29}$$

$$\frac{\mathrm{d}\Phi_A}{\mathrm{d}x} = -\frac{1}{2D_A}\Phi_A(b) = -\frac{3}{2}\frac{\mathrm{d}\Phi_A}{\mathrm{d}\lambda_{tr}^{A}} \tag{30}$$

- We approximate beyond the boundary the neutron current is linear thus,

$$\delta_{ex} = -\frac{\Delta_A(b)}{\frac{\mathrm{d}\Phi_A}{\mathrm{d}x}} = \frac{2}{3}\frac{\lambda_{tr}^{A}}{\Phi_A(b)}\Phi_A(b) = \frac{2}{3}\lambda_{tr}^{A} \tag{31}$$

With neutron transport theory, we get a better approximation for $\delta_{ex} = 0.71\lambda_{tr}^{A}$. This value δ_{ex} is called the **extrapolation length**.

3.4 Solution of the Neutron Diffusion Equation

- Consider a point isotropic source $\dot{S}$ in an infinite homogeneous and isotropic medium. The problem has spherical symmetry and has no source apart from $r = 0$, so the neutron diffusion equation reduces to

$$-D\Delta\Phi + \Sigma_a\Phi = 0 \tag{32}$$

$$-\frac{D}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) + \Sigma_a\Phi = 0 \tag{33}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) - \frac{\Sigma_a}{D}\Phi = 0 \tag{34}$$

We will define the **diffusion area** as

$$L^2 = \frac{D}{\Sigma_a} = \frac{1}{3\Sigma_{tr}\Sigma_a} \tag{35}$$

Therefore,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) - \frac{\Phi}{L^2} = 0 \tag{36}$$

- We can think of the diffusion area as
- Using the substitution $\Phi(r) = R(r)/r$, the differential equation becomes

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2} - \frac{R}{L^2} = 0 \tag{37}$$

So the general solution is

$$R(r) = Ae^{-r/L} + Be^{r/L}. \tag{38}$$

- We can determine A and B using the boundary conditions at $r = 0$ and $r = \infty$. With the condition $\Phi(\infty) = \lim_{r\to\infty} R(r)/r = 0$, we have $B = 0$. For the boundary condition at $r = 0$, we have $\Phi(0) = \dot{S} = A/r$