MAT354 Complex Analysis

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1 Rational Functions

1.1 Classification of rational functions of order 2

(up to fractional linear transformations of the source and target):

1. One double pole β
2. Two distinct poles a, b

In case 1: Make a fractional linear transformation to move β to ∞

$$z = \beta + \frac{1}{\zeta} \tag{1}$$

We set a rational function with double pole at ∞, i.e. a polynomial of degree 2

$$\begin{align} w &= az^2 + bz + c \tag{2} \\ &= a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \tag{3} \\ \tag{4} \end{align}$$

Making a change of coordinates in the source and the target

$$\begin{align} w_1 &= w + \frac{b^2}{4a} - c \tag{5} \\ z_1 &= z + \frac{b}{2a} \tag{6} \\ \tag{7} \end{align}$$

so we have $w_1 = z_1^2$

In case 2: Make a fractional linear transformation to move a, b to $0, \infty$.

$$w = \frac{z-b}{z-a} \tag{8}$$

Rational function of order 2 with poles at $0, \infty$ can be written $w = Az + B + \frac{C}{z}$. Make the coefficients of z and $1/z$ equal by $z_1 = \sqrt{\frac{A}{C}}z$ and $w_1 = \frac{1}{A}(w - B)$ then $w = z + \frac{1}{z}$.

1.2 Rational functions of order 1

Fractional linear transformation

$$w = S(z) = \frac{az+b}{cz+d}, ad - bc \neq 0 \tag{9}$$

Note that $S(\infty) = a/c$ and $S(-d/c) = \infty$.

We want to show that all fractional linear transformations can be written as a composition of translation, inversion, homothety

For $c = 0$, $w = az + b$ which is a translation, homothety.

For $c \neq 0$,

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}(z+d/c) + b + \frac{bc-ad}{c^2}}{z+d/c} = \frac{a}{c} + \frac{bc-ad}{c^2}\frac{1}{z+d/c} \tag{10}$$

This is a composition of

1. translation: $z_1 = z + d/c$
2. inversion: $z_2 = 1/z_1$
3. homethety: $z_3 = \frac{bc-ad}{c^2} \cdot z_2$
4. translation: $z_4 = z_3 + a/c$

Theorem–: Given any 3 distinct points z_2, z_3, z_4, $\exists!$ fractional linear transformation $S : z_2, z_3, z_4 \mapsto 1, 0, \infty$

Proof.

$$S(z) = \begin{cases} \frac{z-z_3}{z-z_4} \Big/ \frac{z_2-z_3}{z_2-z_4} & \text{otherwise} \\ \frac{z-z_3}{z-z_4} & \text{if } z_2 = \infty \\ \frac{z_2-z_4}{z-z_4} & \text{if } z_3 = \infty \\ \frac{z-z_3}{z_2-z_3} & \text{if } z_4 = \infty \end{cases} \tag{11}$$

Suppose also $T : z_2, z_3, z_4 \mapsto 1, 0, \infty$. Consider $ST^{-1} : 1, 0, \infty \mapsto 1, 0, \infty$. ST^{-1} is also a fractional linear transformation $\frac{az_b}{cz+d}$
Given any pair of circles/lines □

Definition–Cross ratio:

$$(z_1 : z_2 : z_3 : z_4) = S(z_1) \tag{12}$$

is the cross ratio of z_1, z_2, z_3, z_4.

Theorem–:

1. If z_1, z_2, z_3, z_4 are distinct points, and T is a fractional linear transformation, then
$$(z_1 : z_2 : z_3 : z_4) = (Tz_1 : Tz_2 : Tz_3 : Tz_4) \tag{13}$$
2. $(z_1 : z_2 : z_3 : z_4)$ is real if and only if z_1, z_2, z_3, z_4 lie on a circle or a line.

Proof.

1. Let $Sz = (z : z_2 : z_3 : z_4)$. Then, $ST^{-1} : Tz_2, Tz_3, Tz_4 \mapsto 1, 0, \infty$. Then, $(Tz_1 : Tz_2 : Tz_3 : Tz_4)$ is by definition equal to Tz_1 under the fractional linear transformation that takes Tz_2, Tz_3, Tz_4 to $1, 0, \infty$, which is precisely ST^{-1}. So, $(Tz_1 : Tz_2 : Tz_3 : Tz_4) = ST^{-1}(Tz_1) = Sz_1 = (z_1 : z_2 : z_3 : z_4)$.
2. First, we show the image of the real axis under fractional linear transformation T^{-1} is either a circle of line.

   $w = T^{-1}(z)$ for $z \in \mathbb{R}$, we want to see that w satisfies the equation of a circle or line.

   We are interested in all w such that $z = Tw = \frac{aw+b}{cw+d}$ is real. If $z \in \mathbb{R}$, then $Tw = \overline{Tw}$ and

$$\frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}} \tag{14}$$
$$(aw+b)(\bar{c}\bar{w}+\bar{d}) = (cw+d)(\bar{a}\bar{w}+\bar{b}) \tag{15}$$
$$\underbrace{(a\bar{c}-\bar{a}c)}_{\text{imaginary}}|w|^2 + \underbrace{(a\bar{d}-\bar{b}c)w + (b\bar{c}-\bar{a}d)}_{\text{imaginary}} + \underbrace{b\bar{d}-\bar{b}d}_{\text{imaginary}} = 0 \tag{16}$$

   If $a\bar{c}-\bar{a}c \neq 0$, then this is an equation of a circle. If $a\bar{c}-\bar{a}c = 0$, then this is an equation of a line.

   Next, $Sz = (z : z_2 : z_3 : z_4)$ is real on the image of the real axis under S^{-1} and nowhere else. $S^{-1} : 1, 0, \infty \mapsto z_2, z_3, z_4$

□

Fractional linear transformations T takes the set of all circles and lines in the complex plane to itself.

Given any pair of circles/lines, there is a fractional linear transformation taking one to the other.

Example 1 ()

Fractional linear transformation that takes the upper half plane H^+ to the unit disk D and the real axis to the unit circle.
We will take i to 0, so the numerator should be $z - i$. $w = \frac{z-i}{z+i} : i \mapsto 0, 0 \mapsto -1, \infty \mapsto 1, 1 \mapsto -i$

2 Holomorphic Functions

- $f(z)$ complex valued functions in an open set $\Omega \subset \mathbb{C}$ or $\Omega \subset \mathbb{C} \cup \{\infty\}$
- f is holomorphic if $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists. i.e. for some $c \in \mathbb{C}, f(z+h) - f(z) = ch + \varphi(h)h$ where $\varphi(h) \in o(h)$.
- This is similar to the definition of the derivative from an open set in the plane to an open set in the plane. (writing $z = x + iy, f(z) = u + iv, c = a + ib, h = \xi + i\eta$ and $f : (x, y) \mapsto (u, v)$)
- The derivative at z takes

$$h \mapsto ch \tag{17}$$
$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{18}$$

The matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$

- For a function to be holomorphic, it requires an additional constraint than being simply differentiable. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. This is the Cauchy-Riemann equations. Or, $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$.
- The derivative at z is a linear transformation $h \mapsto ch$.
- The jacobian determinant is $a^2 + b^2 = |f'(z)|^2$.
- Consider $f(x, y)$ differentiable, but complex valued. The differential $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$. For example, $z = x + iy$ or $\bar{z} = x - iy$. Then, $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - i\mathrm{d}y$.
- Then we have $\mathrm{d}x = \frac{1}{2}(\mathrm{d}z + \mathrm{d}\bar{z})$ and $\mathrm{d}y = \frac{1}{2i}(\mathrm{d}z - \mathrm{d}\bar{z})$. Then,

$$\mathrm{d}f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\mathrm{d}z + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\mathrm{d}\bar{z} \tag{19}$$

- So, we **define**

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \tag{20}$$
$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{21}$$

- Thus, we can write the 1-form $\mathrm{d}f = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \bar{z}}\mathrm{d}\bar{z}$.
- $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are defined as the dual basis for $\mathrm{d}z, \mathrm{d}\bar{z}$.
- We can rewrite the Cauchy-Riemann equations as $\frac{\partial f}{\partial \bar{z}} = 0$. This means for holomorphic functions, it's **only** a function of z, **not** $\bar{z}$.

Definition–Harmonic Function: $f(x, y)$ is a **harmonic function** if $f \in C^2$ and $\Delta f = 0$, or $\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$. (laplace equation)

- We will see that holomorphic functions are harmonic. (but we need to first show we can differentiate holomorphic functions twice) So, the real and imaginary parts of holomorphic functions are also harmonic.
- Remark: $\frac{\partial f}{\partial \bar{z}} = 0 \iff \frac{\partial \bar{f}}{\partial z} = 0$. Why? Consider $f = u + iv, \bar{f} = u - iv$.

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{22}$$

$$\overline{\frac{\partial f}{\partial \bar{z}}} = \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x} - i\frac{\partial \bar{f}}{\partial y}\right) = \frac{\partial \bar{f}}{\partial z} \tag{23}$$

$$\tag{24}$$

Lemma 1: If $f(z)$ is holomorphic in a connected open set Ω and $f'(z) = 0$ in Ω, then f is constant.

Proof.

$$\mathrm{d}f = \underbrace{\frac{\partial f}{\partial z}}_{0} \mathrm{d}z + \underbrace{\frac{\partial f}{\partial \bar{z}}}_{0 \text{ holomorphic}} \mathrm{d}\bar{z} = 0 \tag{25}$$

□

Proposition: Given $f(z)$ is holomorphic in a connected open set Ω, then

1. If $|f(z)|$ is constant, then $f(z)$ is constant.
2. If $\mathrm{Re}(f(z))$ is constant, then $f(z)$ is real.

Proof. 1. $|f(z)|^2 = f(z)\overline{f(z)}$ is constant, so

$$0 = \frac{\partial |f|^2}{\partial z} = \frac{\partial f}{\partial z}\bar{f} + f\cancel{\frac{\partial \bar{f}}{\partial z}} = \frac{\partial f}{\partial z}\bar{z} \tag{26}$$

so either $\bar{f} = 0$ so $f = 0$ thus f is constant or $\frac{\partial f}{\partial z} = 0$ so f is constant.

2. $\mathrm{Re}(f) = f + \bar{f}$ is constant, so

$$0 = \frac{\partial (f + \bar{f})}{\partial z} = \frac{\partial f}{\partial z} + \cancel{\frac{\partial \bar{f}}{\partial z}} = \frac{\partial f}{\partial z} \tag{27}$$

so $\frac{\partial f}{\partial z} = 0$ and f is constant.

□

2.1 Mapping Properties

Suppose f is holomorphic at some point z_0. The **tangent mapping** of f at z_0 is

$$w = f(z_0) + f'(z_0)(z - z_0), \tag{28}$$

if $f'(z) \neq 0$, then the tangent mapping preserves angles and their orientation.

Definition–Conformal Mapping: A mapping f is **conformal** if f is holomorphic and $f'(z_0) \neq 0$. i.e. if f preserves angles and orientation.

Lemma 2: A $\mathbb{R}$-linear transformation $\mathbb{C} \to \mathbb{C}$ which preserves angles is of the form either $w = cz$ or $w = c\bar{z}$.

Consider $w = f(z)$ in a connected open set Ω. If f is treated as a function from $\mathbb{R}^2 \to \mathbb{R}^2$ has $\det f' \neq 0$ in Ω.

If f preserves angles at every point in Ω, then $\frac{\partial f}{\partial z} = 0$ or $\frac{\partial f}{\partial \bar{z}} = 0$. They cannot be both zero at the same point, as otherwise $\det f' = 0$ at that point. As $f \in C^1$, the partial derivatives are continuous. This means $\{z \in \Omega | \frac{\partial f}{\partial z} = 0\}, \{z \in \Omega | \frac{\partial f}{\partial \bar{z}} = 0\}$ are disjoint sets, and their union is Ω. Since Ω is connected, one of them must be empty.

So, either $\frac{\partial f}{\partial \bar{z}} = 0$ throughout $\Omega \implies f$ is holomorphic, or $\frac{\partial f}{\partial z} = 0$ throughout $\Omega \implies f$ is anti-holomorphic.

Theorem–: f preserves angles at every point in $\Omega \iff f$ is either holomorphic or anti-holomorphic in Ω.

Theorem–Inverse Function: Suppose f is holomorphic in a neighborhood of z_0 and $f'(z_0) \neq 0$. Then there are neighborhoods U of z_0 and V of $w_0 = f(z_0)$ such that f maps U **onto** V, with an inverse $z = g(w)$ which is holomorphic in V. And,

$$g'(w) = \frac{1}{f'(z)}. \tag{29}$$

Proof (to be completed later). We will use the fact that partial derivatives of holomorphic functions are continuous, which we will prove later.
If $f'(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and g' is the inverse, then $g'(w) = \frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, so g satisfies the cauchy riemann equations and g is holomorphic. $\square$

3 Power Series

- A complex power series $f(w) = \sum_{n=0}^{\infty} a_n w^n$. Note that w is not a complex number, it's just a symbol. Complex power series means $a_n \in \mathbb{C}$.
- Suppose we have another power series $g(z) = \sum_{p=0}^{\infty} b_p z^p$. We want to compose
$$(f \circ g)(z) = a_0 + a_1(b_0 + b_1 z + \cdots) + a_2(b_0 + b_1 z + \cdots)^2 + \cdots \tag{30}$$
- First we need to ask if this even make sense? The answer is yes if $b_0 = 0$. However, in calculus every formal power series is the taylor series of some C^∞ functions, which can be composed. So why do we have this restriction?
- Consider taylor series of $f(g(z))$ at $z = z_0$. Let $w_0 = g(z_0)$ and the taylor series at w_0 is
$$f(w) = \sum_{n=0}^{\infty} a_n (w - w_0)^n. \tag{31}$$
Then we replace w with the taylor series for g at z_0, with $b_0 = w_0$ so these does not have constant term.

Definition–Formal Derivative: We define $f(0) = a_0$ and the **formal derivative** of $f(w)$ as
$$f'(w) = \sum_{n=1}^{\infty} n a_n w^{n-1}. \tag{32}$$

Theorem–Formal inverse function: Given formal power series $f(w) = \sum_{n=0}^{\infty} a_n w^n$. There is a power series $g(z) = \sum_{p=0}^{\infty} b_p z^p$ such that $b_0 = 0$ and $f \circ g = \mathrm{id}$ where $\mathrm{id}(z) = z$ **iff** $f(0) = 0, f'(0) \neq 0$. In that case g is uniquely determined by f and $g \circ f = \mathrm{id}$ also.

Proof by method of undetermined coefficients. We are trying to solve
$$a_0 + a_1(b_1 z + b_2 z^2 + \cdots) + a_2(b_1 z + b_2 z^2 + \cdots)^2 + \cdots = z. \tag{33}$$
We know right away that $a_0 = 0$ and $a_1 b_1 = 1$. so we know that $a_0 = 0$ and $a_1 \neq=$ are necessary conditions. Conversely, they are sufficient to solve for **unique** coefficients of g.
The coefficient of z^n on the LHS is the same as the coefficient of z^n in
$$\cancel{a_0} + a_1 g(z) + \cdots + a_n g(z)^n = a_1 b_n + P(a_2, \ldots, a_n, b_1, \ldots, b_{n-1}). \tag{34}$$
And $b_1 = 1/a_1$, thus b_n can be calculated recursively.
Since $g(0) = 0$ and $g'(0) \neq 0$, there is a unique formal power series $f_1(w)$ s.t. $g \circ f_1 = \mathrm{id}$.
$$f_1 = \mathrm{id} \circ f_1 = (f \circ g) \circ f_1 = f \circ (g \circ f_1) = f \tag{35}$$
□

Proposition: If $f = \sum_{n=0}^{\infty} a_n w^n$ and $g = \sum_{p=0}^{\infty} b_p w^p$ are convergent power series, then $f \circ g$ is also convergent. In fact, take $r > 0$ s.t. $\sum_{p=1}^{\infty} |b_p| r^p < R(f)$ the radius convergence of f. Then,

(1) $R(f \circ g) \geq r$

(2) If $|z| < r$ then $|g(z) < R(f)$.

(3) $f(g(z)) = (f \circ g)(z)$ (by rearrangement of absolute convergent series) where RHS is formal power series composition and LHS is substituting the value of $g(z)$ into f.

Proof of (1).

$$\sum_{n=0}^{\infty} |a_n| \left(\sum_{p=1}^{\infty} |b_p| r^p \right)^n =: \sum_{k=0}^{\infty} \gamma_k r^k < \infty \tag{36}$$

Say $(f \circ g)(z) = \sum c_k z^k$. By triangle inequality, $|c_k| \leq \gamma_k$. As $\sum \gamma_k r^k < \infty$, then $\sum c_k \gamma^k$ is convergent. □

Theorem–Reciprocal: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $a_0 \neq 0$ then there is an unique power series $g(z)$ s.t. $f(z) = g(z) = 1$. If f has a positive radius of convergence, then so does z.

Proof. As $a_0 \neq 0$, then WLOG $a_0 = 1$. Write $f(z) = 1 - h(z)$ then

$$f(z)^{-1} = (1 - h(z))^{-1} = 1 + \sum_{n=1}^{\infty} w^n \quad \text{where } w = h(z). \tag{37}$$

□

Theorem–Inverse function for convergent power series: In the previous statement, if $f(w)$ has a positive radius of convergence, then so does $g(z)$.

Proof. By direct estimate OR follows from inverse function theorem for holomorphic functions once we know holomorphic function has infinite taylor series that converges. □

3.1 Logarithmic Function

- The principal branch of $\log z$ is defined on the largest simply connected set that does not contain zero, which we will choose $\mathbb{C} \setminus (-\infty, 0]$. In this domain, there is a unique value of $\arg z \in (-\pi, \pi)$, we will call it $\operatorname{Arg}(z)$.
- We can show that this is continuous by showing it is continuous on $S' \setminus \{-1\}$. We can show this by its the fact its inverse $z = e^{i\theta}$ is continuous on $[-(\pi - \epsilon), \pi + \epsilon]$ hence the

it's the inverse of an bijection on compact hausdorff space.

- The principal branch of $\log z$ is defined as $\log|z| + i\text{Arg}\, z$, which is continuous on its entire domain $\mathbb{C} \setminus (-\infty, 0]$. Note that this is equal to the real logarithm if $z \in \mathbb{R}$.

Proposition: The power series $f(z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}$ converges if $|z| < 1$ and the sum is equal to the principal branch of $\log(1+z)$.

Proof. The power series $f(z)$ and $g(w) = \sum_{n=1}^{\infty}\frac{w^n}{n!} = e^w - 1$ are inverses. The proof is by MAT157 since the coefficients here are all real with $g(f(z)) = z$ when $|z| < 1$.
We also know that $e^{f(z)} = 1 + z$ and it's the principal branch because $f(0) = \log 1 = 0$ ☐

Definition–Power:

$$z^\alpha = e^{\alpha \log z} \tag{38}$$

where $\alpha \in \mathbb{C}, z \neq 0$. Note that for fixed α, z^α is a many-valued function of z. This has a branch in any **domain** (connected open subset of $\mathbb{C}$) where $\log$ has a branch. **Any** branch of $\log z$ in Ω defines a branch of z^α.

  - e.g. The **binomial series** $(1+z)^\alpha = e^{\alpha \log(1+z)}$ and its power series expansion in $|z| < 1$ is $\sum \binom{\alpha}{n} z^n$.

Mapping Properties of Holomorphic Functions

- $w = z^\alpha$ for real, positive α maps angles θ to an angle $\alpha\theta$.
- In general z^α is not 1-1 if $\alpha \neq 1$, and is multi-valued if α is fractional.
- Often, we will use a branched covering (mapping $X \to \mathbb{C}$) so we can have a single valued branch. Consider the multi-valued function $w = z^{1/2}$. Consider

$$X = \{(z, w) \in \mathbb{C}^2 | z = w^2\} \tag{39}$$

  X is a manifold (it is a graph of a continuous function) with local coordinate w.

- This multi-valued function $w = z^{1/2}$ lifts to a single valued $(w, z) \mapsto w$ by the covering surface X. X is an example of a **Riemann surface**.

Consider a mapping that takes the upper half plane $H^+ = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$, and consider the mapping that takes $H^+ \to D = \{z \in \mathbb{C} | z| < 1\}$, We can use a fractional linear transformation

$$w = \frac{z - i}{z + i} \tag{40}$$

this takes $i \mapsto 0$. We also know that it maps $\mathbb{R}$ to S^1 as we can pick three points $0, 1, \infty \in \mathbb{R}$ and we know $0 \mapsto -1$. We know this preserves orientations so $1 \mapsto -i$ and $\infty \mapsto 1$.

Now, we want to find a conformal mapping of a circular wedge onto D or H^+.

If circular wedge is formed by two circles intersecting in a and b, first use a fractional linear transformation $\zeta = \frac{z-a}{z-b}$ to map $a \mapsto 0$ and $b \mapsto \infty$. This takes the two circles into rays. Then, we can rotate the region by multiplying a complex number $e^{i\theta}$ and then change the angle by taking $w = e^{i\theta}\zeta^\alpha$ for some power α.

In the case they are degenerate, and only intersect at a, we take $\zeta = \frac{1}{z-a}$ which leads to two parallel lines. Then we can rotate and stretch it so that they become the real line and the line $\mathrm{Im}(z) = \pi$, then $\exp$ will map it to the upper half plane.

Exercise: Find a conformal mapping that takes the complement of the line segment to the interior (or exterior) of the unit disk. We will apply

$$z_1 = \frac{z-1}{z+1} \tag{41}$$

will map the interval $[-1, 1]$ to $(-\infty, 0]$. Then we can apply

$$z_2 = z_1^{1/2} \tag{42}$$

This maps the set to the right half plane. Finally,

$$w = \frac{z_2 - 1}{z_2 + 1} = z - \sqrt{z^2 - 1} \tag{43}$$

will map the right half plane into the interior of the unit disk (flipping the fraction maps to exterior). Check which branch of square root we need to use? Finally, show that $z = \frac{1}{2}(w + 1/w)$

3.2 Mapping Properties of exp and log

- We know that $w = e^z$ is periodic with period $2\pi i$,

$$\begin{aligned} e^z &= e^x e^{iy} && (44) \\ &= e^x(\cos y + i \sin y) && (45) \\ & && (46) \end{aligned}$$

- The exponential maps a vertical line to circle about 0, a horizontal line to a ray through 0, and any other line to a logarithmic spiral.
- The exponential is not injective. To make it single-valued, we need to restrict its domain. The image of e^z on $a < \mathrm{Im}\, z < b$ is a wedge in the complex plane $a < \arg w < b$.
- The logarithm is clearly multi-valued. Can we construct a riemann surface for $w = \log z$?

$$\begin{tikzcd} X \arrow[r, "\text{single value}"] \arrow[d, "\substack{\log z\\ \text{covering}}"'] & \mathbb{C} \\ \mathbb{C} \arrow[ur] & \end{tikzcd}$$

Let $X = \{(z, w) \in \mathbb{C}^2 | z = e^w\}$ then again the single-valued function $(w, z) \to w$ is singled valued.

Now we can try to map the open strip $-i\pi/2 < \operatorname{Im}(z) < i\pi/2$ to the unit disk. First we use $\zeta = e^z$ to map to the right half plane, then $w = \frac{\zeta-1}{\zeta+1}$.

4 Analytic functions

Definition–Analytic Function: A function f is **analytic** in an open set Ω if it has a convergent power series representation at every point $z_0 \in \Omega$.
i.e. $\forall\, z_0 \in \Omega$ there is a power series $\sum a_n(z-z_0)^n$ such that $f(z) = \sum a_n(z-z_0)^n$ when $|z-z_0| < R$ for some $R > 0$.

- If $f(z)$ has convergent power series representation at z_0, then there is a convergent power series $g(z)$ at z_0 such that $g'(z) = f(z)$ in some disk $|z-z_0| < R$, where R is the radius of convergence of f. We know

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{47}$$

$$g(z) = c + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1} \tag{48}$$

$$\tag{49}$$

  The primitive is uniquely determined up to a constant.

- **Question:** Does a convergent power series define an analytic function?

  Proposition: If $f(z) = \sum a_n z^n$ is a convergent power series with radius of convergence R, then $f(z)$ is analytic in $|z| < R$.

  Proof. Note what we need to show. For any z_0 with $|z_0| < R$, then $f(z)$ has convergent power series representation at z_0 with radius of convergence $R - |z_0|$.

$$f(z) = \sum a_n z^n \tag{50}$$

$$= \sum a_n (z_0 + (z-z_0))^n \tag{51}$$

$$= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k}(z-z_0)^k \tag{52}$$

  Note that if we take

$$\sum_{n=0}^{\infty} |a_n| \left(|z_0| + |z-z_0|\right)^n = \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k}|z-z_0|^k \tag{53}$$

  We know this series is absolutely convergent. So we can change the order of summation to conclude

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right) (z-z_0)^k \tag{54}$$

□

- We notice that the inner sum

$$\frac{1}{k!}f^{(k)}(z_0) = \sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \tag{55}$$

is the kth derivative of f at z_0, so f is holomorphic.

4.1 Analytic Continuation

Theorem–: Given $f(z)$ is analytic in a domain Ω and $z_0 \in \Omega$, then the following are equivalent

1. $f^{(n)}(z_0) = 0$ for all $n \geq 0$.
2. f is identically 0 in a neighborhood of z_0.
3. f is identically 0 in Ω.

Proof. (3) $\implies$ (1) is trivial, (1) $\implies$ (2) can be shown from the convergent power series representation of f at z_0 (as the coefficient are the derivatives).
To show (2) $\implies$ (3), we define

$$\Omega' = \{z \in \Omega | f = 0 \text{ in a neighborhood of } z \text{ in } \Omega\}. \tag{56}$$

Clearly $\Omega' \neq \emptyset$ because $z_0 \in \Omega$.
Ω' is open by definition.
Ω' is also closed. Take $z \in \overline{\Omega'}$. Then, $f^{(n)}(z) = 0$ for all $n \geq 0$ by continuity. Then $f = 0$ in a neighborhood of z, by (1) $\implies$ (2). So $z \in \Omega'$. thus Ω' is closed.
Hence, $\Omega = \Omega'$. □

Corollary:

1. If f, g are analytic in domain Ω and $f = g$ in a neighborhood of some point then $f = g$ in Ω.
2. The ring $\mathcal{A}(\Omega)$ if analytic functions in a domain Ω is an **integral domain**.

Proof. The proof of (1) is trivial using $h = f - g$. For (2), suppose $f, g \in \mathcal{A}(\Omega)$ and $fg = 0$. Suppose $f \neq 0$ then there is z_0 s.t. f is non-vanishing in a neighborhood in a neighborhood U of z_0. So $g = 0$ in U hence $g = 0$ in Ω. □

- Integral domains are good, but it is better to work with fields. Hence, we will now analyze the zeros and poles.

- Consider f is analytic in a neighborhood of z_0. Then $f(z) = \sum a_n(z-z_0)^n$ is a convergent power series with radius of convergence R. Suppose $f(z_0) = 0$ but $f \neq 0$.
- Let k be the smallest integer s.t. $f^{(k)}(z_0) \neq 0$ (i.e. $a_k \neq 0$) Then, we define g s.t. $f(z) = (z-z_0)^k g(z)$. Then,

$$g(z) = \sum_{n=k}^{\infty} a_n(z-z_0)^{n-k} \tag{57}$$

- k is the **order** or **multiplicity** of the zero at z_0, characterized by $f^{(k)}(z_0) \neq 0$, but $f^{(j)}(z_0) = 0$ for $j < k$.
- This shows the zero is **isolated** meaning $f(z) \neq 0$ in $0 < |z-z_0| < \epsilon$ for any $\epsilon > 0$.
- If we make a local change of variable near z,

$$\zeta = (z-z_0)g(z)^{1/k}. \tag{58}$$

This is a change of coordinates because its derivative is nonzero. Then, $f(z(\zeta)) = \zeta^k$.
- We now consider the quotients of analytic functions $f(z)/g(z)$ where g is not identically zero. $f(z)/g(z)$ is well-defined and analytic in a neighborhood of z_0 if and only if $g(z)$ is analytic in a neighborhood of z_0 where $g(z_0) \neq 0$.
- What if $g(z_0) = 0$? We can try to factor out terms of $z - z_0$ so that $f_1(z_0) \neq 0$ and $g_1(z_0) \neq 0$ and

$$f(z) = (z-z_0)^k f_1(z) \tag{59}$$

$$g(z) = (z-z_0)^l g_1(z) \tag{60}$$

Then

$$\frac{f(z)}{g(z)} = (z-z_0)^{k-l}\frac{f_1(z)}{g_1(z)} \tag{61}$$

We know that $f_1(z)/g_1(z)$ is analytic and nowhere vanishing in a neighborhood of z_0. There are two cases
  - $k \geq l$ then f/g extends to be analytic at z_0.
  - $k < l$ then z_0 is a **pole** of f/g of **order** $l-k$. Then,

$$\left|\frac{f(z)}{g(z)}\right| \to \infty \text{ as } z \to z_0, \tag{62}$$

  so f/g still make sense as a function with values in the Riemann sphere.

> **Definition–Meromorphic Function**: In an open set Ω, a **meromorphic function** is well-defined and analytic in $\Omega \setminus D$ where D is a discrete set, and expressible at in a neighborhood of any point of Ω as the quotient f/g with g is not identically zero.

- Meromorphic functions in domain Ω form a **field**.
- **Exercise:** If $f(z)$ is meromorphic in Ω, then $f'(z)$ is also meromorphic in Ω with the same poles as f. If z_0 is a pole of order k of $f(z)$ then z_0 is a pole of order $k+1$ of $f'(z)$.

5 Cauchy's Integral Formula

- Review of integration over curves: Let $\Omega \subset \mathbb{R}^2$ be an open set. A curve $\gamma : [a, b] \to \Omega$ where $\gamma(t) = (x(t), y(t))$.

- Let a differential 1 form $\omega = P\mathrm{d}x + Q\mathrm{d}y$ where P, Q are continuous function on Ω. Let $F(t) = P(x(t), y(t)) + Q(x(t), y(t))$ Then,

$$\int_\gamma \omega = \int_a^b F(t)\mathrm{d}t. \tag{63}$$

because

$$\int_\gamma \omega = \int_a^b \gamma^* \omega \tag{64}$$
$$\gamma^* P = P \circ \gamma \tag{65}$$
$$\gamma^*(\mathrm{d}x) = \mathrm{d}(\gamma^* x) - \mathrm{d}((x \circ \gamma)(t)) = \mathrm{d}(x(t)) \tag{66}$$

- We also know that this integral is independent of the choice of parametrization of γ. Suppose $\delta(u) = \gamma(t(u))$ where $t : [c, d] \to [a, b]$ where $t(c) = a, t(d) = b, t'(u) > 0$. Then, $\delta^* \omega = F(t(u))t'(u)\mathrm{d}u$, then

$$\int_\delta \omega = \int_\gamma \omega \tag{67}$$

by integration by substitution (MAT157).

- If the change of parametrization is orientation reversing (where $t(c) = b, t(d) = a, t'(u) < 0$) then

$$\int_\delta \omega = -\int_\gamma \omega, \tag{68}$$

as the integral from c to d becomes an integral from $t(c) = b$ to $t(d) = a$, which is the negative of the integral from a to b.

- If $\gamma_i = \gamma|_{[t_{i-1}, t_i]}$, where $a = t_0 < t_1 < \cdots < t_n = b$, then

$$\int_\gamma \omega = \sum_{i=1}^n \int_{\gamma_i} \omega. \tag{69}$$

so $\int_\gamma \omega$ makes sense if γ is piecewise C^1.

- When we have a closed curve $\gamma(a) = \gamma(b)$, the integral is independent of the choice of initial and final points.''

Lemma 3: Any two points of a connected open subset $\Omega \subset \mathbb{R}^2$ can be joined by a piecewise C^1 curve.

Proof. Fix $a \in \Omega$. Let $E = \{b \in \Omega | a$ and b can be joined by piecewise C^1 curve$\}$. If $b \in E$ then a neighborhood of b is in E. Hence, E is open.
Let $b \in \overline{E}$. Then any neighborhood of b will intersect E. Then, $\exists c \in E$ in this neighborhood. We can join c and b by a straight line, and a and c by a piecewise C^1 curve as $b \in E$. Hence, $b \in E$ and E is closed.
Clearly, $E \neq \emptyset$ because $a \in E$. Thus, $E = \Omega$. □

- The **primitive of** ω is a C^1 function F on Ω s.t. $\omega = \mathrm{d}F$. Then,

$$\int_\gamma \mathrm{d}F = \int_{\partial\gamma} F = F(\gamma(b)) - F(\gamma(a)). \tag{70}$$

- If Ω is connected and $dF = 0$, then F is constant.
- As the primitive is easy to integrate, we want to ask given ω, can we find a primitive?

Proposition: ω has a primitive **iff** $\int_\gamma \omega = 0$ for every piecewise C^1 closed curve γ.

Proof $\Rightarrow$. Suppose ω has a primitive F. Then for any γ

$$\int_\gamma \omega = \int_\gamma \mathrm{d}F = \int_{\partial\gamma} F = F(\gamma(b)) - F(\gamma(a)) = 0. \tag{71}$$

because γ is closed. □

Proof $\Leftarrow$. Fix a base point $x_0 \in \Omega$. Take any piecewise C^1 curve γ from x_0 to x. Then, define

$$F(x) = \int_\gamma \omega. \tag{72}$$

This is well defined as if we choose another curve δ from x_0 to x, then the curve γ then δ is a closed curve, and by the hypothesis it is 0 so $F(x)$ is independent of the choice of γ.
Now, we need to show F is C^1. If h is small enough, then we pick γ from x_0 to x, then a straight line segment x to $x + h$.

$$F(x+h) - F(x) = \int_x^{x+h} P(t,y)\mathrm{d}t \tag{73}$$

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} P(t,y)\mathrm{d}t = P(x,y). \tag{74}$$

by the fundamental theorem of calculus. This can be repeated for the other partial derivatives, and hence F is C^1. □

Definition–: ω is **closed** if ω locally has a primitive. This is equivalent to

1. $\int_\gamma \omega = 0$ whenever γ is the boundary of a sufficiently small rectangle R in Ω.
2. If γ is the boundary of any rectangle in Ω, because any rectangle can be split up into many small rectangles.

So every closed differential form ω **in a disk** has a primitive.

Note that closed differential form in a domain Ω need not have a primitive. Note the following counterexample. Let $\Omega = \mathbb{C} \setminus \{0\}$ and $\omega = \mathrm{d}z/z$. Clearly, ω is closed because locally at each point of ω there is a branch of $\log$ in a neighborhood of that point, which is a primitive.

However, ω does not have a global primitive. Let $\gamma(t) = e^{it}$ with $t \in [0, 2\pi]$. Then, $z = e^{it}$ so $\mathrm{d}z = ie^{it}\mathrm{d}t$ and

$$\int_\gamma \frac{\mathrm{d}z}{z} = \int_0^{2\pi i} \mathrm{d}t = 2\pi i. \tag{75}$$

Note that the example need not be complex. The imaginary part was

$$\mathrm{d}t = \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2}, \tag{76}$$

where $t = \arctan(y/x)$, which was proven to not have a global primitive in MAT257.

Theorem–: Aside: assume P and Q are continuous with the following continuous partial derivatives $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ in some neighborhood of a closed rectangle A and let $\gamma = \partial A$. Then, Green's theorem says

$$\int_\gamma \underbrace{P\mathrm{d}x + Q\mathrm{d}y}_{\omega} = \int_A \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y}_{\mathrm{d}\omega}. \tag{77}$$

Proof. We can apply Stoke's theorem, but that is too easy. Let $A = [a_1, a_2] \times [b_1, b_2]$ then

$$\int_A \frac{\partial Q}{\partial x} \mathrm{d}x \wedge \mathrm{d}y = \int_{b_1}^{b_2} \left(\int_{a_1}^{a_2} \frac{\partial Q}{\partial x} \mathrm{d}x \right) \mathrm{d}y \tag{78}$$

$$= \int_{b_1}^{b_2} (Q(a_2, y) - Q(a_1, y)) \tag{79}$$

$$= \int_\gamma Q\mathrm{d}y. \tag{80}$$

We can repeat the same argument for $\frac{\partial P}{\partial y}$. □

Green's theorem tells us that if $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ exists and are continuous then $\int_\gamma = 0$ for a sufficiently small rectangle A iff $\int_A = \left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y} \mathrm{d}x \wedge \mathrm{d}y \right)$ for sufficiently small rectangle A iff $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

This means that our definition of closed is equivalent to the MAT257 definition ($d\omega = 0$) when ω is C^1.

Theorem–Cauchy: If $f(z)$ is a holomorphic function in an open subset $\Omega \subseteq \mathbb{C}$ then the differential form $f(z)dz$ is closed.

Proof (with additional assumption). Let $z = x + iy$, and additionally assume $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous. Then,

$$f(z)dz = f(z)dx + if(z)dy. \tag{81}$$

By Green's theorem, it is sufficient to show that $\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}$. This is the Cauchy-Riemann equation which holds because f is holomorphic.
Warning: we do not want to use this assumption, because we will use Cauchy's theorem to prove that the partial derivatives are continuous. If we accept this proof, we accept **circular reasoning**. □

Proof (complete). It suffices to show $\mu(R) = \int_\gamma f(z)dz = 0$ when γ is the boundary of any rectangle $R \subset \Omega$. Let's divide R into 4 equal parts R_i; each with an oriented boundary γ_i. Then,

$$\int_\gamma f(z)dz = \sum_{i=1}^{4} \int_{\gamma_i} f(z)dz. \tag{82}$$

So, for at least one i,

$$\left|\int_{\gamma_i} f(z)dz\right| \geq \frac{1}{4}\left|\int_\gamma f(z)dz\right|, \tag{83}$$

for this i, we define $\gamma^{(1)} =: \gamma_i, R^{(1)} = R_i$. Hence, $|\mu(R^{(1)})| = \frac{1}{4}|\mu(R)|$. If we continue to subdivide, we have $R \supset R^{(1)} \supset R^{(2)} \supset \cdots$ and $|\mu(R^{(n+1)})| \geq \frac{1}{4}|\mu(R^{(n)})|$. Then,

$$|\mu(R^{(n)})| = \int_{\gamma^{(k)}} f(z)dz \geq \frac{1}{4^k}\left|\int_\gamma f(z)dz\right| = \frac{1}{4^k}|\mu(R)|. \tag{84}$$

Then, we know $\exists! z_0 \in \bigcup_k R^{(k)}$. Since f is holomorphic at z_0 then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varphi(z)|z - z_0| \tag{85}$$

where $\lim_{z\to z_0} \varphi(z) = 0$ i.e. $\forall \varepsilon > 0\, \exists \delta > 0$ s.t $|z - z_0| < \delta \implies |\varphi(z)| < \varepsilon$.

$$\int_{\gamma^{(k)}} f(z)dz = \int_{\gamma^{(k)}} f(z_0)dz + \int_{\gamma^{(k)}} f'(z_0)(z - z_0)dz + \int_{\gamma^{(k)}} \varphi(z)|z - z_0|dz. \tag{86}$$

The first two integrals are zero because their integrands have primitives. Given $\varepsilon > 0$, if $|z - z_0| < \delta$ then

$$\left|\int_{\gamma^{(k)}} \varphi(z)|z - z_0|dz\right| \leq \varepsilon \int_{\gamma^{(k)}} |z - z_0|dz \tag{87}$$

$$\leq \varepsilon \cdot \text{diameter}(R^{(k)}) \cdot \text{perimeter}(R^{(k)}) \tag{88}$$

$$= \varepsilon \cdot \frac{1}{2^k}\text{diameter}(R) \cdot \frac{1}{2^k}\text{perimeter}(R). \tag{89}$$

$$= \varepsilon \cdot \frac{1}{4^k}\text{diameter}(R) \cdot \text{perimeter}(R). \tag{90}$$

We know that

$$|\mu(R)| \leq 4^k \left| \int_{\gamma^{(k)}} f(z)\mathrm{d}z \right| \leq \varepsilon \cdot \text{diameter}(R) \cdot \text{perimeter}(R). \tag{91}$$

for any $\varepsilon > 0$, so $\mu(R) = 0$. □

Corollary: If $f(z)$ is a holomorphic function in an open $\Omega \subset \mathbb{C}$ locally has a primitive, **which is also holomorphic**.

Proof. Consider the local primitive $F(z)$. Then, by definition $f(z)\mathrm{d}z = \mathrm{d}F$. We know

$$\mathrm{d}F = \frac{\partial F}{\partial z}\mathrm{d}z + \frac{\partial F}{\partial \bar{z}}\mathrm{d}\bar{z}. \tag{92}$$

We know $\frac{\partial F}{\partial \bar{z}} = 0$ because $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ are linearly independent, so F is holomorphic. □

Corollary: In Cauchy's theorem, it is enough to assume that f is continuous in Ω and holomorphic outside a line parallel to the real axis.

Proof. Consider the following three cases:

1. If the rectangle does not intersect the line, then f is holomorphic in the rectangle.
2. If the boundary of the rectangle intersects the line, integrate over a slightly smaller rectangle that does not intersect the line. By continuity, the integral is the same as the difference between the rectangles approach zero.
3. If the line intersects the interior of the rectangle, then split the region into two and integrate over the boundary of each of the smaller rectangle.

This corollary extends to a finite union of lines. □

Theorem–: Closed differential form ω in a **simply-connected** open subset Ω of $\mathbb{R}^2$ has a global primitive.

Next time we will show that a closed differential form $\omega = P\mathrm{d}x + Q\mathrm{d}y$ in an open $\Omega \subset \mathbb{R}^2$ always has a **primitive along a curve** $\gamma(t), t \in [a, b]$. i.e. a continuous function $f(t)$ s.t. for any $t_0 \in [a, b]$ then

1. $\exists$ a primitive F of ω in a neighborhood of $\gamma(t)$ s.t. $f(t) = F(\gamma(t))$ for t sufficiently close to t_0.

5.1 Homotopy

We want to know what is a primitive of a closed differential form? We know this is not always the case, but

Proposition: $\Omega \subset \mathbb{R}^2$ open and ω closed differential form in Ω. Let $\gamma : [a,b] \to \Omega$ be a continuous curve. Then, there is a continuous function $f(t)$ on $[a,b]$ s.t. for every $t_0 \in [a,b]$ there is a primitive F of ω in a neighborhood of $\gamma(t_0)$ s.t. $f(t) = F(\gamma(t))$ for t in a neighborhood of t_0. f is uniquely determined up to a constant.

Proof (Uniqueness). Suppose f_1, f_2 are primitives of ω along γ. Then in a neighborhood of t_0, then $f_1(t) - f_2(t) = F_1(\gamma(t)) - F_2(\gamma(t))$. F_1 and F_2 are local primitives of ω in the same neighborhood, so they differ by a constant. Since $f_1 - f_2$ is locally constant and $f_1 - f_2$ is continuous, $f_1 - f_2$ is constant. □

Proof (Existence). There is a primitive $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$ s.t. every $\gamma([t_{i-1}, t_i])$ lies in an open disk U_i in which ω has a primitive F_i. The partition is finite because the curve is compact. We know that $U_i \cap U_{i+1}$ is connected, that $F_{i+1} - F_i$ is constant in $U_i \cap U_{i+1}$. Hence, we can choose the constants step-by-step to make $F_i = F_{i+1}$ in $U_i \cap U_{i+1}$. Then define $f(t) = F_i(\gamma(t))$. □

If γ is piecewise C^1 and f is a primitive of ω along γ, then $\int_\gamma \omega = f(b) - f(a)$. Consider a partition $\gamma_i =: \gamma|_{[t_{i-1}, t_i]}$. Then,

$$\int_\gamma \omega = \sum_i \int_{\gamma_i} \omega = \sum_i \big(F_i(\gamma(t_i)) - F_i(\gamma(t_{i-1}))\big) = f(b) - f(a). \tag{93}$$

So we can define $\int_\gamma \omega$ where γ is a continuous curve as $f(b) - f(a)$.

Example: Let γ be a closed curve about the origin. What is $\int_\gamma \mathrm{d}z/z = f(b) - f(a) =$ the difference between two branches of $\log$ at $\gamma(a) = \gamma(b)$, which is $2\pi i n$ for $n \in \mathbb{Z}$.

Let the real differential form $\int_\gamma \frac{x\mathrm{d}y - y\mathrm{d}x}{x^2+y^2} = 2\pi n$, the difference between two branches of $\arctan$. This is called the **variation of $\arg z$ along γ**. We use this definition even if γ is not closed.

Definition–Homotopy: $\gamma_0, \gamma_1 : [0,1] \to \Omega$ continuous curves with the same endpoints are **homotopic with fixed endpoints in Ω** if there is a continuous function $\gamma : [0,1]^2 \to \Omega$ such that.

$$\gamma(0,t) = \gamma_0(t), \tag{94}$$
$$\gamma(1,t) = \gamma_1(t), \tag{95}$$
$$\gamma(s,0) = \gamma_0(0) = \gamma_1(0), \tag{96}$$
$$\gamma(s,1) = \gamma_0(1) = \gamma_1(1). \tag{97}$$

$\gamma_0, \gamma_1 : [0,1] \to \Omega$ that are continuous closed curves are **homotopic as closed curves** in Ω if there is a continuous function $\gamma : [0,1]^2 \to \Omega$ such that.

$$\gamma(0,t) = \gamma_0(t), \tag{98}$$
$$\gamma(1,t) = \gamma_1(t), \tag{99}$$
$$\gamma(s,0) = \gamma(s,1). \tag{100}$$

If γ_1 is constant, then γ_0 is **homotopic to a point**.

Definition–Simply Connected: Ω is **simply connected** if every closed curve in Ω is **null homotopic** (homotopic to a point). There are the following equivalent definitions

- Any two curves with the same endpoints are homotopic with fixed endpoints.

Theorem–: If ω is a closed differential form in Ω and $\gamma_0, \gamma_1 : [0,1] \to \Omega$ are continuous curves which are homotopic (either with fixed endpoints or as closed curves), then $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$.

Proof (Hint). What we started today by showing that a differential form always has a primitive along a curve. A curve is a mapping from $[0,1] \to \Omega$. There is nothing in the argument that required us to work on an interval. We can also work on a rectangle. What's true is that it always exists a primitive that exists from $[0,1]^2 \to \Omega$ If we extend a proof to a square, this theorem is a very simple conclusion. $\square$

Lemma 4: Suppose ω is a closed form in Ω and $\gamma : [a,b] \times [c,d] \to \Omega$ is a continuous function. Then, there is a continuous function $f : [a,b] \times [c,d] \to \mathbb{C}$ s.t. for every point $(s_0, t_0) \in [a,b] \times [c,d]$ there is a local primitive F of ω defined in a neighborhood of $\gamma(s_0, t_0)$ s.t. $f = F(\gamma(s,t))$ in a neighborhood of (s_0, t_0). Moreover, f is unique up to the addition of a constant.

Proof. Choose partitions $\{s_i\}$ of $[a,b]$ and $\{t_j\}$ of $[c,d]$ s.t. γ maps every $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ into an open disk U_{ij} where ω has a primitive F_{ij}.
For a fixed j, there is a primitive f_j along $\gamma|_{[a,b]\times[t_{j-1},t_j]}$ like before. ($F_{ij}, F_{i+1,j}$ differ by a constant in $U_{ij} \cap U_{i+1,j}$ So, we can adjust these constants one at a time to make them equal in the intersection.)
Then we can define f_j as $F_{ij} \circ \gamma$ in $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$. For each j, on $[a,b] \times \{t_j\}$. On that line, both f_j and f_{j+1} are primitives along a curve, so they differ by a constant. So, we can again adjust these constants so $f_j = f_{j+1}$ on $[a,b] \times \{t_j\}$. Then we can define f as f_j in $[a,b] \times [t_{j-1}, t_j]$. $\square$

Proof (fixed endpoints). We have homotopy $\gamma : [0,1] \times [0,1] \to \Omega$ with fixed endpoints. Let f be a primitive of ω along γ. We know f is constant on $t = 0$ and $t = 1$ because γ is constant. So, $f(0,0) = f(1,0)$ and $f(0,1) = f(1,1)$. We know f is a primitive of ω, so

$$\int_{\gamma_0} \omega = f(0,1) - f(0,0) \tag{101}$$
$$\int_{\gamma_1} \omega = f(1,1) - f(1,0), \tag{102}$$

thus they are equal. □

Corollary: In a simply connected open set, any closed form has a primitive.

Definition–Star-Shaped: Ω is **star-shaped** if $\exists a \in \Omega$ s.t. for every $x \in \Omega$, the line segment $\{(1-t)a + tx | t \in [0,1]\} \subseteq \Omega$.

We will present some examples:

1. A star shaped open set Ω is simply connected. For any curve, $\gamma : [0,1] \to \Omega$, choose $\gamma(s,t) = a + s(\gamma(t) - a)$ which is a homotopy between γ and the point a.

2. Closed form $\omega = \mathrm{d}z/z$ has a primitive in any simply connected open set Ω not containing 0. i.e. $\log z$ has a branch in any simply connected open set not containing 0. Hence, we can define

$$\log z = w_0 + \int_{z_0}^{z} \frac{\mathrm{d}z'}{z'} \qquad \text{where } e^{w_0} = z_0. \tag{103}$$

3. $\mathbb{C} \setminus \{0\}$ is not simplify connected. S^1 is not homotopic to a point in $\mathbb{C} \setminus \{0\}$ because $\int_{S^1} \mathrm{d}z/z = 2\pi i \neq 0$.

Definition–Winding Number: Let $\gamma : [0,1] \to \mathbb{C}$ be a closed curve and a be a point not on a. The **winding number** of γ w.r.t a is

$$w(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - a}. \tag{104}$$

Observe the following properties

1. $w(\gamma, a) \in \mathbb{Z}$.

2. Fix a, then $w(\gamma, a)$ is invariant under homotopy of γ not passing through a (by the previous theorem).

3. In particular, if γ lies in a simply connected open set not containing a, then $w(\gamma, a) = 0$.

4. Fix γ, then $w(\gamma, a)$ is constant on connected components of $\mathbb{C} \setminus \gamma([0,1])$. It suffices to show that $w(\gamma, a)$ is locally constant. A small shift of a is equivalent to the same shift of γ in the opposite direction, which is a homotopy not passing through a.

5. If γ is a circle described in **positive sense** (w.r.t to the center $c, w(\gamma, c) = 1$.) Then, $w(\gamma, a) = 0$ when a is outside γ and $w(\gamma, a) = 1$ when a is inside γ.

Theorem–Cauchy's Integral Formula: For an open set $\Omega \subseteq \mathbb{C}$, a point $a \in \mathbb{C}$ a holomorphic function in Ω, $f(z)$ and a closed curve $\gamma : [0,1] \to \Omega$ not containing a that is homotopic to a point in Ω. Then,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} \mathrm{d}z = w(\gamma, a) \cdot f(a). \tag{105}$$

Proof. Let

$$g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & \text{if } z \neq a \\ f'(a) & \text{if } z = a \end{cases}. \tag{106}$$

We know $g(z)$ is continuous, and holomorphic when $z \neq a$. By Cauchy's theorem, $g(z)\mathrm{d}z$ is closed. Thus,

$$0 = \int_\gamma g(z)\mathrm{d}z = \int_\gamma \frac{f(z)-f(a)}{z-a} \mathrm{d}z \tag{107}$$

i.e.

$$\int_\gamma \frac{f(z)}{z-a} \mathrm{d}z = f(a) \int_\gamma \frac{\mathrm{d}z}{z-a} = f(a) \cdot 2\pi i w(\gamma, a). \tag{108}$$

□

Corollary: If $f(z)$ is holomorphic in a neighborhood of a closed disk D and $\gamma = \partial D$ then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} = \begin{cases} f(a) & \text{if } a \in D \\ 0 & \text{otherwise} \end{cases} \tag{109}$$

Corollary: A holomorphic function $f(z)$ in an open set in D is infinitely differentiable in D. Fix some smaller disk containing z with boundary γ. By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z}. \tag{110}$$

Then we can differentiate under the integral sign so

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}}{\mathrm{d}z} \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z)^2}. \tag{111}$$

In fact,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)\mathrm{d}\zeta}{(\zeta - z)^{n+1}}. \tag{112}$$

Theorem–Morera: If $f(z)\mathrm{d}z$ is a closed form, then $f(z)$ is holomorphic.

Proof. If $f(z)\mathrm{d}z$ is a closed form, then $f(z)$ locally has a primitive $g(z)$ which is holomorphic. Then, $f(z) = g'(z)$ is also holomorphic. □

Corollary: A continuous function which is holomorphic except on a finitely many line is holomorphic everywhere.

We can wrap up the previous theorem as follows. Consider $f(z)$ is a continuous function in Ω. Then, the following are equivalent

1. $f(z)$ is holomorphic in Ω.
2. $f(z)\mathrm{d}z$ is closed.
3. $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\mathrm{d}\xi$ when z is in the interior of a closed disk D in Ω, with oriented boundary γ.

$(1) \implies (2)$ is cauchy's theorem, $(2) \implies (3)$ is cauchy's integral formula, $(3) \implies (2)$ is the corollary, and $(2) \implies (1)$ is Morera's theorem.

Applications of Cauchy's Integral Formula: Let $f(z)$ be holomorphic in the disk $|z| < R$. Let r be any number $0 < r < R$, then if $z < r$ then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta \qquad f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta. \tag{113}$$

This allows us to write the power series expansion of f at 0 $f(z) = \sum a_n z^n$.

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}}\mathrm{d}\zeta. \tag{114}$$

Theorem–: $f(z)$ has a convergent power series expansion in $|z| < R$.

Proof. Note that

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\left(1 - \frac{z}{\zeta}\right)^{-1} = \frac{1}{\zeta}\sum_{n=0}^{\infty}\frac{z^n}{\zeta^n}, \tag{115}$$

this is a geometric series which converges for $|z| < |\zeta| = r$. Then, we can substitute this expression

$$f(z) = \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \frac{z^n f(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta. \tag{116}$$

For fixed z, and $|z| < r$, this is uniformly convergent by comparison with geometric series. So, the integral and the sum commute.

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta. \tag{117}$$

□

Corollary: Every holomorphic function is analytic.

We can write the power series expansion as a fourier series by using $z = re^{i\theta}$, then

$$f(z) = \sum_{m=0}^{\infty} a_m r^m e^{im\theta}. \tag{118}$$

We can multiply both sides by $e^{in\theta}$ and integrate from 0 to 2π, then

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{in\theta}\mathrm{d}\theta. \tag{119}$$

We can also see this by integration by substitution $\zeta = re^{i\theta}, \mathrm{d}\zeta = ire^{i\theta}\mathrm{d}\theta$. This integral formula gives an upper bound for the taylor coefficients. Let $M(r) = \sup_\theta |f(re^{i\theta})|$. Then. $|a_n r^n| \leq |M(r)|$ so $|a_n| \leq |M(r)|/r^n$. These are called **Cauchy's Inequalities**.

Theorem–Liouville: A bounded holomorphic function in $\mathbb{C}$ is constant.

Proof. $M(r) \leq M$ for some M since f is bounded. So, $|a_n| \leq M/r^n$ for all $r > 0$, so if $n \geq 1$, then $a_n = 0$ and $f(z) = a_0$ is constant. □

Corollary: **Fundamental theorem of algebra.** Any non-constant polynomial on $\mathbb{C}$ has a root.

Proof. By contradiction suppose a polynomial $P(z)$ has no root. Then, $1/P(z)$ is a holomorphic function on $\mathbb{C}$ and its bounded. Therefore, it is constant so $P(z)$ is constant. □

Theorem–Schwartz's reflection principle: Let $\Omega \subseteq \mathbb{C}$ be a domain that is symmetric with respect to the real axis. Let $\Omega^+ = \{z \in \Omega | \operatorname{Im}(z) \geq 0\}, \Omega^- = \{z \in \Omega | \operatorname{Im}(z) \leq 0\}$. Let $f(z)$ be continuous on Ω^+, real on $\Omega \cap \mathbb{R}$ and holomorphic in $\Omega^+ \setminus \mathbb{R}$, then $f(z)$ extends to a holomorphic function in Ω (and it is unique by the principle of analytic continuation).

Proof. Define $g(z) = \begin{cases} f(z) & z \in \Omega^+ \\ \overline{f(\bar{z})} & z \in \Omega^- \end{cases}$. Then we know $g(z)$ is holomorphic in $\Omega \setminus \mathbb{R}$. g is continuous on Ω also therefore g is holomorphic in Ω. □

A more general form of this result can apply to any line (with the image on another line). Similarly to a circle.

Consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$, for $r > 0$ small enough, we know $a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})e^{in\theta}\mathrm{d}\theta$. So, $f(0) = a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\mathrm{d}\theta$, the mean value of $f(z)$ on $|z| = r$. This is known as the **mean value property**. If $f(z)$ is a holomorphic function in an open $\Omega \subset \mathbb{C}$, then it has the mean value property. i.e. for any closed disk in Ω, with center z_0, then $f(z_0) =$ the mean value on the boundary. Note that the real and imaginary parts of a function with the mean value property also has the mean value property.

Theorem–Maximum modulus principle: If f is a continuous complex-valued function with the mean value property in an open $\Omega \subset \mathbb{C}$, and $|f|$ has a local maximum at a point $a \in \Omega$, then it is constant in a neighborhood of a. (note that if f is holomorphic and Ω is connected, then that implies f is constant)

Proof. If $f(a) = 0$ then the fact is trivial. So, we can assume $f(a) \neq 0$, then we can assume $f(a)$ is real and positive (as we can multiply f by a constant).
For $r > 0$, small enough (so the local maximum is the maximum on the neighborhood), consider $M(r) = \sup_{\theta\in[0,2\pi)} |f(a + re^{i\theta})| \leq f(a) \leq M(r)$ as f has a local maximum at a but also satisfies the mean value property. So $f(a) = M(r)$
Let

$$g(z) = \mathrm{Re}(f(a) - f(z)) = f(a) - \mathrm{Re}(f(z)) = \sup_{\theta\in[0,2\pi)} |f(a + re^{i\theta})| - \mathrm{Re}(f(z)). \tag{120}$$

This tells us that $g(z) \geq 0$ on $|z - a| - r$ because the modulus is greater or equal to the real part. Also note that $g(z) = 0$ iff $f(z) = f(a)$. Therefore, the mean value of $g(z)$ on $|z - a| = r$ is 0. Since $g(z)$ is continuous and $g(z) \geq 0$ on $|z - a| = r$, then $g = 0$ on $|z - a| = r$. Therefore, $f(z) = f(a)$ on $|z - a| = r$. This holds for all small enough r, so f is constant on a neighborhood of a. □

Corollary: Suppose Ω is a bounded domain in $\mathbb{C}$. Let $f(z)$ be a continuous complex-valued function on $\overline{\Omega}$ with the mean value property in Ω. Let $M = \sup_{z\in\overline{\Omega}\setminus\Omega} |f(z)|$. (on the frontier) Then, $f(z) \leq M$ for $z \in \overline{\Omega}$. And, $|f(z_0)| = M$ for $z_0 \in \Omega$, then f is constant.

Proof. Let $M' = \max_{z\in\overline{\Omega}} |f(z)|$. M' exists because $\overline{\Omega}$ is compact, and M' is attained in a point $a \in \overline{\Omega}$. If $a \in$ frontier, we are done. If $a \in \Omega$, by the maximum modulus principle $\{z \in \Omega : f(z) = f(a)\}$ is open. It is also closed, so it is all of Ω. □

Theorem–Schwartz's Lemma: $f(z)$ is a holomorphic function in $|z| < 1$ with values $|f(z)| < 1$ and $f(0) = 0$. Then,

1. $|f(z)| \leq |z|$
2. If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $f(z) = \lambda z$ with $|\lambda| = 1$.

Proof. We know $f(z)$ has a convergent power series expansion and because $f(0) = 0$ we know $a_0 = 0$, and $f(z)/z$ is still holomorphic. As $|f(z)| < 1$, we know

$$\left|\frac{f(z)}{z}\right| < \frac{1}{r} \qquad \text{on } |z| = r \tag{121}$$

Therefore on $|z| \leq r$ by corollary, we know $|f(z)/z| < 1/r$ for all r, we know $|z| \leq r < 1$. Taking the limit as $r \to 1$ we have $|f(z)| \leq |z|$.
If $|f(z_0) = |z_0|$ at some $z_0 \neq 0$ then $f(z)/z$ attains its maximum modulus at an interior point, so it must be a constant λ. □

Definition–Laurent Expansion: Given a holomorphic function $f(z)$ in an **annulus**, $0 \leq R_2 < |z| < R_1 \leq \infty$ it has a convergent **Laurent expansion** in the annulus

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \underbrace{\sum_{n=1}^{\infty} \frac{a_{-n}}{z^n}}_{\text{converge if} |z| > R_2} + \underbrace{\sum_{n=0}^{\infty} a_n z^n}_{\text{converge if } z < R_1} . \tag{122}$$

If we let $\zeta = 1/z$, and we have a holomorphic function $f(\zeta)$ for $|\zeta| < 1/R_2$.

Proof (Existance). Consider $R_2 < r_2 < r_1 < R_1$. Let γ_1 be the circle $|z| = r_1$ and γ_2 be the circle $|z| = r_2$. By Cauchy's integral formula we can write

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta. \tag{123}$$

We know the first integral

$$\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \sum_{n=0}^{\infty} a_n z^n \quad \text{where } a_n = \int_{\gamma_1} \frac{f(\zeta)}{\zeta} \mathrm{d}\zeta. \tag{124}$$

Consider the second integral, but in this case $|z| > |\zeta|$, and use the same technique used to get the power series for the first integral. We can write

$$\frac{1}{\zeta - z} = -\frac{1}{z}\frac{1}{1 - \zeta/z} = -\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}} = -\sum_{n<0} \frac{z^n}{\zeta^{n+1}}. \tag{125}$$

This is uniformly and absolutely convergent on $|\zeta| = r_2$. Thus, we get

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n \qquad \text{where } a_n = \begin{cases} \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta & \text{if } n \geq 0 \\ \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta & \text{if } n < 0 \end{cases} \tag{126}$$

□

A holomorphic functions $f(z)$ in a **punctured disk** e.g. $0 < |z| < r$ has an **isolated singularity** and $z = 0$ if it can't be extended to a holomorphic function in $|z| < r$. In this situation, we know there is still a Laurent expansion of $f(z)$ in $0 < |z| < r$. This allows us to distinguish between the two things:

Definition–:

- A **pole** when there are finite number of $n < 0$ where $a_n \neq 0$ in the Laurent expansion.
- An **essential singularity** when there are infinitely many $n < 0$ where $a_n \neq 0$ in the Laurent expansion.

In the case of a pole, we know $z^n f(z)$ extends to a holomorphic function $g(z)$ at 0 for some n, then $f(z) = g(z)/z^n$ thus f is meromorphic.

Proposition: A holomorphic function on a punctured disk can be extended to a holomorphic function in the disk iff f is bounded in a neighborhood of 0.

Proof. Consider $f(re^{i\theta})$. Using the Laurent expansion, we know

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r e^{in\theta}. \tag{127}$$

with

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(re^{i\theta}) \mathrm{d}\theta. \tag{128}$$

Let $M(r) = \sup_{|z|=r} |f(z)|$. then $|a_n| r^n \leq M(r)$, i.e. $|a_n| \leq M(r)/r^n, \forall n \in \mathbb{Z}$.
If f is bounded in a punctured neighborhood $\iff$ $M(r) \leq M$. If $n < 0$, then $|a_n| \leq M r^{-n}$ so as $r \to 0, r^{-n} \to 0$ so $|a_n| = 0$ i.e. f is holomorphic. $\square$

If 0 is a pole, then $\lim_{z\to 0} f(z) = \infty$. A meromorphic function is a holomorphic function with values in the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Theorem–Weierstrass: If z_0 (WLOG $z_0 = 0$) is an essential singularity of f, then for any $\varepsilon > 0$, then $f(0 < |z| < \varepsilon)$ is dense in $\mathbb{C}$.

Proof. Otherwise, there is some $a \in \mathbb{C}, \delta > 0$ s.t. $|f(z) - a| > \delta$ if $0 < |z| < \varepsilon$.
Let $g(z) = \frac{1}{f(z)-a}$. We know that g is holomorphic in $0 < |z| < \varepsilon$ and it is bounded by $1/\delta$. Therefore, by proposition it is holomorphic in $|z| < \varepsilon$.
We know $f(z) = a + \frac{1}{g(z)}$, a quotient of holomorphic functions, so 0 is a pole and this is a contradiction. $\square$

In fact, **Picard's big theorem** states that $f(0 < |z| < \varepsilon) = \mathbb{C}$ or $\mathbb{C} \setminus \{b\}$ for one value $b \in \mathbb{C}$.

At infinity: Suppose f is holomorphic in $|z| > r$.

- f is holomorphic at ∞ if $f(1/z)$ is holomorphic in $z < 1/R$.
- f has a pole at ∞ if $f(1/z)$ has a pole at 0.
- f has an essential singularity at ∞ if $f(1/z)$ has an essential singularity at 0.

Consider the laurent expansion in $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. If f has a pole at ∞, there are only finitely many nonzero a_n when $n > 0$.

Exercise: Let $f(z)$ be a holomorphic function in $R_2 < |z| < |R_1$ and let γ be a closed curve in the annulus. Compute $\int_\gamma f(z)\mathrm{d}z$.

We know $f(z)$ has a laurent expansion,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \frac{a_{-1}}{z} + \underbrace{\sum_{n\neq -1} a_n z^n}_{\text{has primitive}} \tag{129}$$

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma \frac{a_{-1}}{z}\mathrm{d}z = 2\pi i \cdot a_{-1} \cdot w(\gamma, 0). \tag{130}$$

In particular, if f is holomorphic in a punctured disk $0 < |z| < \varepsilon$ and γ be a circle around 0 in the punctured disk, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\mathrm{d}z = a_{-1}. \tag{131}$$

Definition–Residue: The **residue** of the differential form ω at a is defined as

$$\frac{1}{2\pi i}\int_\gamma \omega \tag{132}$$

where γ is a positively oriented circle around a.

- We shown that the residue $f(z)\mathrm{d}z$ at 0 is a_{-1}.
- **Positively oriented w.r.t ∞ means in the negative sense**. At ∞, let $z = 1/z'$ so $f(z)\mathrm{d}z = -\frac{1}{z^{2\prime}} f(1/z')\mathrm{d}z'$ where γ is a small circle

$$-\frac{1}{2\pi i}\int_\gamma \frac{1}{z'^2} f(1/z')\mathrm{d}z' = -a_{-1}. \tag{133}$$

Theorem–Residue: Let f be a meromorphic function in Ω and let a be a point in Ω. Then

Theorem–Argument Principle: $f(z)$ meromorphic function in a neighborhood of a point a, how do we compute the residue of f'/f at a?
We don't know if $f(z)$ has a pole at a or vanishes at a, but because it is meromorphic we can always write $f(z) = (z-a)^k g(z)$ (if $k < 0, f$ has a pole, if $k > 0, f$ has a zero)
We can take the logarithmic derivative of f then

$$\frac{f'}{f} = \frac{g'}{g} + \frac{k}{z-a}. \tag{134}$$

As $g(a) \neq 0$, so g'/g is holomorphic at a. Thus, the residue of f'/f is k.
The **argument principle** states $f(z)$ is a non-constant meromorphic function in an open set Ω. Let $K \subset \Omega$ be a compact set with oriented boundary Γ. Given a, assume there are no zeros of $f(z) - a$ and no poles of $f(z)$ on Γ. Then,

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z) - a} \mathrm{d}z = Z - P, \tag{135}$$

where Z is the number of zeros of $f(z) - a$ and P is the number of poles of $f(z)$ in P, counted with multiplicity.

Proof. By residue theorem and example. □

Theorem–: $f(z)$ non-constant holomorphic function in a neighborhood of $z = z_0$ where z_0 is a root of order k of $f(z) - a$, where $a \in \mathbb{C}$.
Then for every sufficiently small neighborhood U of z_0, and every b sufficiently close to a, $f(z) - b$ has k simple roots in U.

Proof. Take U small enough so $f(z) - a$ has no other zeros at z_0 in $\overline{U}$ and $f'(z) \neq 0$ in $\overline{U} \setminus \{z_0\}$. Let γ be the positively oriented boundary of U. Then consider

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - b} \mathrm{d}z \tag{136}$$

Write $\zeta = f(z)$, and

$$\frac{1}{2\pi i} \int_{f\circ\gamma} \frac{\mathrm{d}\zeta}{\zeta - b} = w(f \circ \gamma, b) \tag{137}$$

which is constant as a function of b in a connected component of $f \circ \gamma$, which is an open set. By the argument principle,

$$w(f \circ \gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} \mathrm{d}z = k, \tag{138}$$

by the argument principle, $f(z) - b$ has k roots inside γ and they are all simple because $f'(z) \neq 0$. Note z_0 cannot be a root of $f(z) - b$ because $b \neq a$. □

Theorem–Rouche's: $f(z), g(z)$ are holomorphic functions in an open set Ω, let $K \subset \Omega$ be a compact set with an oriented boundary Γ. If $|f(z) - g(z)| < |f(z)|$ on Γ, then $f(z), g(z)$ have the same number of zeros in K, counted with multiplicity.

Proof. If $|f(z) - g(z)| < |f(z)$, then

$$\left|1 - \frac{g(z)}{f(z)}\right| < 1 \quad \text{on } \Gamma. \tag{139}$$

We are interested in the meromorphic function

$$F(z) = \frac{g(z)}{f(z)} \tag{140}$$

whose values on Γ lie in the open disk $|z - 1| < 1$. Then consider the integral and apply the argument principle

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{F'(z)}{F(z)} \mathrm{d}z = Z - P, \tag{141}$$

where Z is the number of zeros of $F(z)$ (which is also the number of zeros of g) and P is the number of poles of $F(z)$ (which is also the number of zeros in f) in K. Note that $\Gamma = \bigcup \Gamma_i$, and the integral is the sum of the winding numbers of the images of $F \circ \Gamma$ at 0, but all Γ_is lie in the disk $|z - 1| < 1$, so their winding numbers about 0 is 0. Thus, $Z - P = 0$ so g and f has the same number of zeros in K. $\square$

Evaluation of definite integrals by residue calculus. (this is analogous to integration in first year calculus, with the difference that we will not be finding the primitives) We will instead compute these integrals as sums of residues of suitable holomorphic functions.