MAT347 Abstract Algebra

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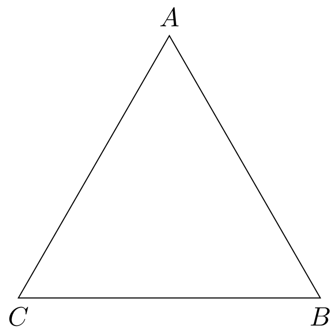
1 Groups

Groups are generally associated with symmetries. Consider the equilateral triangle:



We know that there are six symmetries of the triangle:

- Identity transformation (do nothing) denoted as id or e
- Two rotations ($A \to B \to C \to A$ and $A \to C \to B \to A$)
- Three reflections $A \leftrightarrow B$, $A \leftrightarrow C$, $B \leftrightarrow C$

Note that these symmetries preserve the structure of the triangle, hence the composition of two symmetries must also be a symmetry. Let

- ρ be the rotation $A \to B \to C \to A$
- σ be the reflections $B \leftrightarrow C$

Note that $\rho\sigma$ is the $A \leftrightarrow C$ reflection and $\sigma\rho$ is the $A \leftrightarrow B$ reflection. Hence they may not be commutative.

We also know that all symmetries can be reversed. α has an inverse α^{-1} such that $\alpha\alpha^{-1} = \alpha^{-1}\alpha = e$. These inspires the following definition:

Definition: A **group** is a set G with a composition

$$G \times G \to G \quad (1)$$
$$(g, h) \mapsto g \cdot h \quad (2)$$

Satisfying:

- Associativity: $(g \cdot h) \cdot k = g \cdot (h \cdot k)$

- Identity: $\exists\, e \in G$ such that $g \cdot e = e \cdot g = g$ for all $g \in G$
- Inverse: $\forall\, g \in G$, $\exists\, g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$

Examples:

- $\mathbb{Z}$ with $+$ is a group. It is associative, $e = 0$ and $g^{-1} = -g$.
- $\mathbb{Z}/n\mathbb{Z}$ with addition modulo n.
- $SL(n, F)$ all $n \times n$ matrices with determinant 1 over a field F.