MAT354 Complex Analysis

Jonah Chen

1 Rational Functions

1.1 Classification of rational functions of order 2

(up to fractional linear transformations of the source and target):

1. One double pole β
2. Two distinct poles a, b

In case 1: Make a fractional linear transformation to move β to ∞

$$z = \beta + \frac{1}{\zeta} \tag{1}$$

We set a rational function with double pole at ∞, i.e. a polynomial of degree 2

$$\begin{aligned} w &= az^2 + bz + c \qquad (2) \\ &= a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \qquad (3) \\ & \qquad (4) \end{aligned}$$

Making a change of coordinates in the source and the target

$$\begin{aligned} w_1 &= w + \frac{b^2}{4a} - c \qquad (5) \\ z_1 &= z + \frac{b}{2a} \qquad (6) \\ & \qquad (7) \end{aligned}$$

so we have $w_1 = z_1^2$

In case 2: Make a fractional linear transformation to move a, b to $0, \infty$.

$$w = \frac{z-b}{z-a} \tag{8}$$

Rational function of order 2 with poles at $0, \infty$ can be written $w = Az + B + \frac{C}{z}$. Make the coefficients of z and $1/z$ equal by $z_1 = \sqrt{\frac{A}{C}}z$ and $w_1 = \frac{1}{A}(w - B)$ then $w = z + \frac{1}{z}$.

1.2 Rational functions of order 1

Fractional linear transformation

$$w = S(z) = \frac{az+b}{cz+d}, ad - bc \neq 0 \tag{9}$$

Note that $S(\infty) = a/c$ and $S(-d/c) = \infty$.

We want to show that all fractional linear transformations can be written as a composition of translation, inversion, homothety

For $c = 0$, $w = az + b$ which is a translation, homothety.

For $c \neq 0$,

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}(z+d/c) + b + \frac{bc-ad}{c^2}}{z+d/c} = \frac{a}{c} + \frac{bc-ad}{c^2}\frac{1}{z+d/c} \tag{10}$$

This is a composition of

1. translation: $z_1 = z + d/c$
2. inversion: $z_2 = 1/z_1$
3. homethety: $z_3 = \frac{bc-ad}{c^2} \cdot z_2$
4. translation: $z_4 = z_3 + a/c$

Theorem–: Given any 3 distinct points z_2, z_3, z_4, $\exists!$ fractional linear transformation $S : z_2, z_3, z_4 \mapsto 1, 0, \infty$

Proof.

$$S(z) = \begin{cases} \frac{z-z_3}{z-z_4} \Big/ \frac{z_2-z_3}{z_2-z_4} & \text{otherwise} \\ \frac{z-z_3}{z-z_4} & \text{if } z_2 = \infty \\ \frac{z_2-z_4}{z-z_4} & \text{if } z_3 = \infty \\ \frac{z-z_3}{z_2-z_3} & \text{if } z_4 = \infty \end{cases} \tag{11}$$

Suppose also $T : z_2, z_3, z_4 \mapsto 1, 0, \infty$. Consider $ST^{-1} : 1, 0, \infty \mapsto 1, 0, \infty$. ST^{-1} is also a fractional linear transformation $\frac{az_b}{cz+d}$
Given any pair of circles/lines □

Definition–Cross ratio:

$$(z_1 : z_2 : z_3 : z_4) = S(z_1) \tag{12}$$

is the cross ratio of z_1, z_2, z_3, z_4.

Theorem–:

1. If z_1, z_2, z_3, z_4 are distinct points, and T is a fractional linear transformation, then
$$(z_1 : z_2 : z_3 : z_4) = (Tz_1 : Tz_2 : Tz_3 : Tz_4) \tag{13}$$
2. $(z_1 : z_2 : z_3 : z_4)$ is real if and only if z_1, z_2, z_3, z_4 lie on a circle or a line.

Proof.

1. Let $Sz = (z : z_2 : z_3 : z_4)$. Then, $ST^{-1} : Tz_2, Tz_3, Tz_4 \mapsto 1, 0, \infty$. Then, $(Tz_1 : Tz_2 : Tz_3 : Tz_4)$ is by definition equal to Tz_1 under the fractional linear transformation that takes Tz_2, Tz_3, Tz_4 to $1, 0, \infty$, which is precisely ST^{-1}. So, $(Tz_1 : Tz_2 : Tz_3 : Tz_4) = ST^{-1}(Tz_1) = Sz_1 = (z_1 : z_2 : z_3 : z_4)$.
2. First, we show the image of the real axis under fractional linear transformation T^{-1} is either a circle of line.

   $w = T^{-1}(z)$ for $z \in \mathbb{R}$, we want to see that w satisfies the equation of a circle or line.

   We are interested in all w such that $z = Tw = \frac{aw+b}{cw+d}$ is real. If $z \in \mathbb{R}$, then $Tw = \overline{Tw}$ and
$$\frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}} \tag{14}$$
$$(aw+b)(\bar{c}\bar{w}+\bar{d}) = (cw+d)(\bar{a}\bar{w}+\bar{b}) \tag{15}$$
$$(\underbrace{a\bar{c}-\bar{a}c}_{\text{imaginary}})|w|^2 + \underbrace{(a\bar{d}-\bar{b}c)w + (b\bar{c}-\bar{a}d)}_{\text{imaginary}} + \underbrace{b\bar{d}-\bar{b}d}_{\text{imaginary}} = 0 \tag{16}$$

   If $a\bar{c}-\bar{a}c \neq 0$, then this is an equation of a circle. If $a\bar{c}-\bar{a}c = 0$, then this is an equation of a line.

   Next, $Sz = (z : z_2 : z_3 : z_4)$ is real on the image of the real axis under S^{-1} and nowhere else. $S^{-1} : 1, 0, \infty \mapsto z_2, z_3, z_4$

□

Fractional linear transformations T takes the set of all circles and lines in the complex plane to itself.

Given any pair of circles/lines, there is a fractional linear transformation taking one to the other.

Example 1 ()

Fractional linear transformation that takes the upper half plane H^+ to the unit disk D and the real axis to the unit circle.
We will take i to 0, so the numerator should be $z - i$. $w = \frac{z-i}{z+i} : i \mapsto 0, 0 \mapsto -1, \infty \mapsto 1, 1 \mapsto -i$

2 Holomorphic Functions

- $f(z)$ complex valued functions in an open set $\Omega \subset \mathbb{C}$ or $\Omega \subset \mathbb{C} \cup \{\infty\}$
- f is holomorphic if $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists. i.e. for some $c \in \mathbb{C}, f(z+h) - f(z) = ch + \varphi(h)h$ where $\varphi(h) \in o(h)$.
- This is similar to the definition of the derivative from an open set in the plane to an open set in the plane. (writing $z = x+iy, f(z) = u+iv, c = a+ib, h = \xi + i\eta$ and $f : (x,y) \mapsto (u,v)$)
- The derivative at z takes

$$h \mapsto ch \tag{17}$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{18}$$

The matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$

- For a function to be holomorphic, it requires an additional constraint than being simply differentiable. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. This is the Cauchy-Riemann equations. Or, $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$.
- The derivative at z is a linear transformation $h \mapsto ch$.
- The jacobian determinant is $a^2 + b^2 = |f'(z)|^2$.
- Consider $f(x,y)$ differentiable, but complex valued. The differential $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$. For example, $z = x+iy$ or $\bar{z} = x - iy$. Then, $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - i\mathrm{d}y$.
- Then we have $\mathrm{d}x = \frac{1}{2}(\mathrm{d}z + \mathrm{d}\bar{z})$ and $\mathrm{d}y = \frac{1}{2i}(\mathrm{d}z - \mathrm{d}\bar{z})$. Then,

$$\mathrm{d}f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\mathrm{d}z + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\mathrm{d}\bar{z} \tag{19}$$

- So, we **define**

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \tag{20}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{21}$$

- Thus, we can write the 1-form $\mathrm{d}f = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \bar{z}}\mathrm{d}\bar{z}$.
- $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are defined as the dual basis for $\mathrm{d}z, \mathrm{d}\bar{z}$.
- We can rewrite the Cauchy-Riemann equations as $\frac{\partial f}{\partial \bar{z}} = 0$. This means for holomorphic functions, it's **only** a function of z, **not** $\bar{z}$.

Definition–Harmonic Function: $f(x, y)$ is a **harmonic function** if $f \in C^2$ and $\Delta f = 0$, or $\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$. (laplace equation)

- We will see that holomorphic functions are harmonic. (but we need to first show we can differentiate holomorphic functions twice) So, the real and imaginary parts of holomorphic functions are also harmonic.
- Remark: $\frac{\partial f}{\partial \bar{z}} = 0 \iff \frac{\partial \bar{f}}{\partial z} = 0$. Why? Consider $f = u + iv, \bar{f} = u - iv$.

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{22}$$
$$\overline{\frac{\partial f}{\partial \bar{z}}} = \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x} - i\frac{\partial \bar{f}}{\partial y}\right) = \frac{\partial \bar{f}}{\partial z} \tag{23}$$
$$\tag{24}$$

Lemma 1: If $f(z)$ is holomorphic in a connected open set Ω and $f'(z) = 0$ in Ω, then f is constant.

Proof.

$$\mathrm{d}f = \underbrace{\frac{\partial f}{\partial z}}_{0}\mathrm{d}z + \underbrace{\frac{\partial f}{\partial \bar{z}}}_{0 \text{ holomorphic}}\mathrm{d}\bar{z} = 0 \tag{25}$$

□

Proposition: Given $f(z)$ is holomorphic in a connected open set Ω, then

1. If $|f(z)|$ is constant, then $f(z)$ is constant.
2. If $\mathrm{Re}(f(z))$ is constant, then $f(z)$ is real.

Proof. 1. $|f(z)|^2 = f(z)\overline{f(z)}$ is constant, so

$$0 = \frac{\partial |f|^2}{\partial z} = \frac{\partial f}{\partial z}\bar{f} + f\cancel{\frac{\partial \bar{f}}{\partial z}} = \frac{\partial f}{\partial z}\bar{z} \tag{26}$$

so either $\bar{f} = 0$ so $f = 0$ thus f is constant or $\frac{\partial f}{\partial z} = 0$ so f is constant.

2. $\mathrm{Re}(f) = f + \bar{f}$ is constant, so

$$0 = \frac{\partial (f + \bar{f})}{\partial z} = \frac{\partial f}{\partial z} + \cancel{\frac{\partial \bar{f}}{\partial z}} = \frac{\partial f}{\partial z} \tag{27}$$

so $\frac{\partial f}{\partial z} = 0$ and f is constant.

□

2.1 Mapping Properties

Suppose f is holomorphic at some point z_0. The **tangent mapping** of f at z_0 is

$$w = f(z_0) + f'(z_0)(z - z_0), \tag{28}$$

if $f'(z) \neq 0$, then the tangent mapping preserves angles and their orientation.

Definition–Conformal Mapping: A mapping f is **conformal** if f is holomorphic and $f'(z_0) \neq 0$. i.e. if f preserves angles and orientation.

Lemma 2: A $\mathbb{R}$-linear transformation $\mathbb{C} \to \mathbb{C}$ which preserves angles is of the form either $w = cz$ or $w = c\bar{z}$.

Consider $w = f(z)$ in a connected open set Ω. If f is treated as a function from $\mathbb{R}^2 \to \mathbb{R}^2$ has $\det f' \neq 0$ in Ω.

If f preserves angles at every point in Ω, then $\frac{\partial f}{\partial z} = 0$ or $\frac{\partial f}{\partial \bar{z}} = 0$. They cannot be both zero at the same point, as otherwise $\det f' = 0$ at that point. As $f \in C^1$, the partial derivatives are continuous. This means $\{z \in \Omega | \frac{\partial f}{\partial z} = 0\}, \{z \in \Omega | \frac{\partial f}{\partial \bar{z}} = 0\}$ are disjoint sets, and their union is Ω. Since Ω is connected, one of them must be empty.

So, either $\frac{\partial f}{\partial \bar{z}} = 0$ throughout $\Omega \implies f$ is holomorphic, or $\frac{\partial f}{\partial z} = 0$ throughout $\Omega \implies f$ is anti-holomorphic.

Theorem–: f preserves angles at every point in $\Omega \iff f$ is either holomorphic or anti-holomorphic in Ω.

Theorem–Inverse Function: Suppose f is holomorphic in a neighborhood of z_0 and $f'(z_0) \neq 0$. Then there are neighborhoods U of z_0 and V of $w_0 = f(z_0)$ such that f maps U **onto** V, with an inverse $z = g(w)$ which is holomorphic in V. And,

$$g'(w) = \frac{1}{f'(z)}. \tag{29}$$

Proof (to be completed later). We will use the fact that partial derivatives of holomorphic functions are continuous, which we will prove later.
If $f'(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and g' is the inverse, then $g'(w) = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, so g satisfies the cauchy riemann equations and g is holomorphic. $\square$