

# MAT354 Complex Analysis

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## 1 RATIONAL FUNCTIONS

### 1.1 CLASSIFICATION OF RATIONAL FUNCTIONS OF ORDER 2

(up to fractional linear transformations of the source and target):

1. One double pole  $\beta$
2. Two distinct poles  $a, b$

In case 1: Make a fractional linear transformation to move  $\beta$  to  $\infty$

$$z = \beta + \frac{1}{\zeta} \quad (1)$$

We set a rational function with double pole at  $\infty$ , i.e. a polynomial of degree 2

$$w = az^2 + bz + c \quad (2)$$

$$= a \left( z + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} + c \quad (3)$$

$$(4)$$

Making a change of coordinates in the source and the target

$$w_1 = w + \frac{b^2}{4a} - c \quad (5)$$

$$z_1 = z + \frac{b}{2a} \quad (6)$$

$$(7)$$

so we have  $w_1 = z_1^2$

In case 2: Make a fractional linear transformation to move  $a, b$  to  $0, \infty$ .

$$w = \frac{z - b}{z - a} \quad (8)$$

Rational function of order 2 with poles at  $0, \infty$  can be written  $w = Az + B + \frac{C}{z}$ . Make the coefficients of  $z$  and  $1/z$  equal by  $z_1 = \sqrt{\frac{A}{C}}z$  and  $w_1 = \frac{1}{A}(w - B)$  then  $w = z + \frac{1}{z}$ .

## 1.2 RATIONAL FUNCTIONS OF ORDER 1

Fractional linear transformation

$$w = S(z) = \frac{az + b}{cz + d}, ad - bc \neq 0 \quad (9)$$

Note that  $S(\infty) = a/c$  and  $S(-d/c) = \infty$ .

We want to show that all fractional linear transformations can be written as a composition of translation, inversion, homothety

For  $c = 0$ ,  $w = az + b$  which is a translation, homothety.

For  $c \neq 0$ ,

$$\frac{az + b}{cz + d} = \frac{\frac{a}{c}(z + d/c) + b + \frac{bc-ad}{c^2}}{z + d/c} = \frac{a}{c} + \frac{bc-ad}{c^2} \frac{1}{z + d/c} \quad (10)$$

This is a composition of

1. translation:  $z_1 = z + d/c$
2. inversion:  $z_2 = 1/z_1$
3. homothety:  $z_3 = \frac{bc-ad}{c^2} \cdot z_2$
4. translation:  $z_4 = z_3 + a/c$

**Theorem:** Given any 3 distinct points  $z_2, z_3, z_4$ ,  $\exists!$  fractional linear transformation  $S : z_2, z_3, z_4 \mapsto 1, 0, \infty$

*Proof.*

$$S(z) = \begin{cases} \frac{z-z_3}{z-z_4} \bigg/ \frac{z_2-z_3}{z_2-z_4} & \text{otherwise} \\ \frac{z-z_3}{z-z_4} & \text{if } z_2 = \infty \\ \frac{z_2-z_4}{z-z_4} & \text{if } z_3 = \infty \\ \frac{z-z_3}{z_2-z_3} & \text{if } z_4 = \infty \end{cases} \quad (11)$$

Suppose also  $T : z_2, z_3, z_4 \mapsto 1, 0, \infty$ . Consider  $ST^{-1} : 1, 0, \infty \mapsto 1, 0, \infty$ .  $ST^{-1}$  is also a fractional linear transformation  $\frac{az+b}{cz+d}$

Given any pair of circles/lines □

**Definition:** [Cross ratio]

$$(z_1 : z_2 : z_3 : z_4) = S(z_1) \quad (12)$$

is the cross ratio of  $z_1, z_2, z_3, z_4$ .

**Theorem:**

1. If  $z_1, z_2, z_3, z_4$  are distinct points, and  $T$  is a fractional linear transformation, then

$$(z_1 : z_2 : z_3 : z_4) = (Tz_1 : Tz_2 : Tz_3 : Tz_4) \quad (13)$$

2.  $(z_1 : z_2 : z_3 : z_4)$  is real if and only if  $z_1, z_2, z_3, z_4$  lie on a circle or a line.

*Proof.* 1. Let  $Sz = (z : z_2 : z_3 : z_4)$ . Then,  $ST^{-1} : Tz_2, Tz_3, Tz_4 \mapsto 1, 0, \infty$ . Then,  $(Tz_1 : Tz_2 : Tz_3 : Tz_4)$  is by definition equal to  $Tz_1$  under the fractional linear transformation that takes  $Tz_2, Tz_3, Tz_4$  to  $1, 0, \infty$ , which is precisely  $ST^{-1}$ . So,  $(Tz_1 : Tz_2 : Tz_3 : Tz_4) = ST^{-1}(Tz_1) = Sz_1 = (z_1 : z_2 : z_3 : z_4)$ .

2. First, we show the image of the real axis under fractional linear transformation  $T^{-1}$  is either a circle or line.

$w = T^{-1}(z)$  for  $z \in \mathbb{R}$ , we want to see that  $w$  satisfies the equation of a circle or line.

We are interested in all  $w$  such that  $z = Tw = \frac{aw+b}{cw+d}$  is real. If  $z \in \mathbb{R}$ , then  $Tw = \overline{Tw}$  and

$$\frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}} \quad (14)$$

$$(aw+b)(\bar{c}\bar{w}+\bar{d}) = (cw+d)(\bar{a}\bar{w}+\bar{b}) \quad (15)$$

$$\underbrace{(a\bar{c} - \bar{a}c)}_{\text{imaginary}}|w|^2 + \underbrace{(a\bar{d} - \bar{b}c)w + (b\bar{c} - \bar{a}d)}_{\text{imaginary}} + \underbrace{b\bar{d} - \bar{b}d}_{\text{imaginary}} = 0 \quad (16)$$

If  $a\bar{c} - \bar{a}c \neq 0$ , then this is an equation of a circle. If  $a\bar{c} - \bar{a}c = 0$ , then this is an equation of a line.

Next,  $Sz = (z : z_2 : z_3 : z_4)$  is real on the image of the real axis under  $S^{-1}$  and nowhere else.  $S^{-1} : 1, 0, \infty \mapsto z_2, z_3, z_4$

□

Fractional linear transformations  $T$  takes the set of all circles and lines in the complex plane to itself.

Given any pair of circles/lines, there is a fractional linear transformation taking one to the other.

**Example 1 ()**

Fractional linear transformation that takes the upper half plane  $H^+$  to the unit disk  $D$  and the real axis to the unit circle.

We will take  $i$  to  $0$ , so the numerator should be  $z - i$ .  $w = \frac{z-i}{z+i} : i \mapsto 0, 0 \mapsto -1, \infty \mapsto 1, 1 \mapsto -i$

## 2 HOLOMORPHIC FUNCTIONS

- $f(z)$  complex valued functions in an open set  $\Omega \subset \mathbb{C}$  or  $\Omega \subset \mathbb{C} \cup \{\infty\}$
- $f$  is holomorphic if  $\lim_{h \rightarrow 0} \frac{f(z+h)-f(z)}{h}$  exists. i.e. for some  $c \in \mathbb{C}$ ,  $f(z+h) - f(z) = ch + \varphi(h)h$  where  $\varphi(h) \in o(h)$ .
- This is similar to the definition of the derivative from an open set in the plane to an open set in the plane. (writing  $z = x + iy$ ,  $f(z) = u + iv$ ,  $c = a + ib$ ,  $h = \xi + i\eta$  and  $f : (x, y) \mapsto (u, v)$ )
- The derivative at  $z$  takes

$$h \mapsto ch \quad (17)$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad (18)$$

The matrix  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$

- For a function to be holomorphic, it requires an additional constraint than being simply differentiable.  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ . This is the Cauchy-Riemann equations. Or,  $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$ .
- The derivative at  $z$  is a linear transformation  $h \mapsto ch$ .
- The jacobian determinant is  $a^2 + b^2 = |f'(z)|^2$ .
- Consider  $f(x, y)$  differentiable, but complex valued. The differential  $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ . For example,  $z = x + iy$  or  $\bar{z} = x - iy$ . Then,  $dz = dx + idy$  and  $d\bar{z} = dx - idy$ .
- Then we have  $dx = \frac{1}{2}(dz + d\bar{z})$  and  $dy = \frac{1}{2i}(dz - d\bar{z})$ . Then,

$$df = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} \right) dz + \frac{1}{2} \left( \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \right) d\bar{z} \quad (19)$$

- So, we **define**

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} \right) \quad (20)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \right) \quad (21)$$

- Thus, we can write the 1-form  $df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$ .
- $\frac{\partial}{\partial z}$  and  $\frac{\partial}{\partial \bar{z}}$  are defined as the dual basis for  $dz, d\bar{z}$ .
- We can rewrite the Cauchy-Riemann equations as  $\frac{\partial f}{\partial \bar{z}} = 0$ . This means for holomorphic functions, it's **only** a function of  $z$ , **not**  $\bar{z}$ .

**Definition:** [Harmonic Function]  $f(x, y)$  is a **harmonic function** if  $f \in C^2$  and  $\Delta f = 0$ , or  $\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$ . (laplace equation)

- We will see that holomorphic functions are harmonic. (but we need to first show we can differentiate holomorphic functions twice) So, the real and imaginary parts of holomorphic functions are also harmonic.
- Remark:  $\frac{\partial f}{\partial \bar{z}} = 0 \iff \frac{\partial \bar{f}}{\partial z} = 0$ . Why? Consider  $f = u + iv, \bar{f} = u - iv$ .

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \quad (22)$$

$$\frac{\partial \bar{f}}{\partial z} = \frac{1}{2} \left( \frac{\partial \bar{f}}{\partial x} - i \frac{\partial \bar{f}}{\partial y} \right) = \frac{\partial \bar{f}}{\partial z} \quad (23)$$

$$(24)$$

**Lemma 1:** If  $f(z)$  is holomorphic in a connected open set and  $f'(z) = 0$  (identically), then  $f$  is constant.

*Proof.*

$$df = \underbrace{\frac{\partial f}{\partial z}}_0 dz + \underbrace{\frac{\partial f}{\partial \bar{z}}}_{0 \text{ holomorphic}} d\bar{z} = 0 \quad (25)$$

□