MAT354 Complex Analysis

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1 Rational Functions

1.1 Classification of rational functions of order 2

(up to fractional linear transformations of the source and target):

1. One double pole β
2. Two distinct poles a, b

In case 1: Make a fractional linear transformation to move β to ∞

$$z = \beta + \frac{1}{\zeta} \tag{1}$$

We set a rational function with double pole at ∞, i.e. a polynomial of degree 2

$$\begin{aligned} w &= az^2 + bz + c && (2)\\ &= a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c && (3)\\ & && (4) \end{aligned}$$

Making a change of coordinates in the source and the target

$$\begin{aligned} w_1 &= w + \frac{b^2}{4a} - c && (5)\\ z_1 &= z + \frac{b}{2a} && (6)\\ & && (7) \end{aligned}$$

so we have $w_1 = z_1^2$

In case 2: Make a fractional linear transformation to move a, b to $0, \infty$.

$$w = \frac{z - b}{z - a} \tag{8}$$

Rational function of order 2 with poles at $0, \infty$ can be written $w = Az + B + \frac{C}{z}$. Make the coefficients of z and $1/z$ equal by $z_1 = \sqrt{\frac{A}{C}}z$ and $w_1 = \frac{1}{A}(w - B)$ then $w = z + \frac{1}{z}$.

1.2 Rational functions of order 1

Fractional linear transformation

$$w = S(z) = \frac{az+b}{cz+d}, ad - bc \neq 0 \tag{9}$$

Note that $S(\infty) = a/c$ and $S(-d/c) = \infty$.

We want to show that all fractional linear transformations can be written as a composition of translation, inversion, homothety

For $c = 0$, $w = az + b$ which is a translation, homothety.

For $c \neq 0$,

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}(z+d/c) + b + \frac{bc-ad}{c^2}}{z+d/c} = \frac{a}{c} + \frac{bc-ad}{c^2}\frac{1}{z+d/c} \tag{10}$$

This is a composition of

1. translation: $z_1 = z + d/c$
2. inversion: $z_2 = 1/z_1$
3. homethety: $z_3 = \frac{bc-ad}{c^2} \cdot z_2$
4. translation: $z_4 = z_3 + a/c$

Theorem–: Given any 3 distinct points z_2, z_3, z_4, $\exists!$ fractional linear transformation $S : z_2, z_3, z_4 \mapsto 1, 0, \infty$

Proof.

$$S(z) = \begin{cases} \frac{z-z_3}{z-z_4} \Big/ \frac{z_2-z_3}{z_2-z_4} & \text{otherwise} \\ \frac{z-z_3}{z-z_4} & \text{if } z_2 = \infty \\ \frac{z_2-z_4}{z-z_4} & \text{if } z_3 = \infty \\ \frac{z-z_3}{z_2-z_3} & \text{if } z_4 = \infty \end{cases} \tag{11}$$

Suppose also $T : z_2, z_3, z_4 \mapsto 1, 0, \infty$. Consider $ST^{-1} : 1, 0, \infty \mapsto 1, 0, \infty$. ST^{-1} is also a fractional linear transformation $\frac{az_b}{cz+d}$
Given any pair of circles/lines □

Definition–Cross ratio:

$$(z_1 : z_2 : z_3 : z_4) = S(z_1) \tag{12}$$

is the cross ratio of z_1, z_2, z_3, z_4.

Theorem–:

1. If z_1, z_2, z_3, z_4 are distinct points, and T is a fractional linear transformation, then

$$(z_1 : z_2 : z_3 : z_4) = (Tz_1 : Tz_2 : Tz_3 : Tz_4) \tag{13}$$

2. $(z_1 : z_2 : z_3 : z_4)$ is real if and only if z_1, z_2, z_3, z_4 lie on a circle or a line.

Proof. 1. Let $Sz = (z : z_2 : z_3 : z_4)$. Then, $ST^{-1} : Tz_2, Tz_3, Tz_4 \mapsto 1, 0, \infty$. Then, $(Tz_1 : Tz_2 : Tz_3 : Tz_4)$ is by definition equal to Tz_1 under the fractional linear transformation that takes Tz_2, Tz_3, Tz_4 to $1, 0, \infty$, which is precisely ST^{-1}. So, $(Tz_1 : Tz_2 : Tz_3 : Tz_4) = ST^{-1}(Tz_1) = Sz_1 = (z_1 : z_2 : z_3 : z_4)$.

2. First, we show the image of the real axis under fractional linear transformation T^{-1} is either a circle of line.

   $w = T^{-1}(z)$ for $z \in \mathbb{R}$, we want to see that w satisfies the equation of a circle or line.

   We are interested in all w such that $z = Tw = \frac{aw+b}{cw+d}$ is real. If $z \in \mathbb{R}$, then $Tw = \overline{Tw}$ and

$$\frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}} \tag{14}$$

$$(aw+b)(\bar{c}\bar{w}+\bar{d}) = (cw+d)(\bar{a}\bar{w}+\bar{b}) \tag{15}$$

$$\underbrace{(a\bar{c}-\bar{a}c)}_{\text{imaginary}}|w|^2 + \underbrace{(a\bar{d}-\bar{b}c)w + (b\bar{c}-\bar{a}d)}_{\text{imaginary}} + \underbrace{b\bar{d}-\bar{b}d}_{\text{imaginary}} = 0 \tag{16}$$

   If $a\bar{c}-\bar{a}c \neq 0$, then this is an equation of a circle. If $a\bar{c}-\bar{a}c = 0$, then this is an equation of a line.

   Next, $Sz = (z : z_2 : z_3 : z_4)$ is real on the image of the real axis under S^{-1} and nowhere else. $S^{-1} : 1, 0, \infty \mapsto z_2, z_3, z_4$

□

Fractional linear transformations T takes the set of all circles and lines in the complex plane to itself.

Given any pair of circles/lines, there is a fractional linear transformation taking one to the other.

Example 1 ()

Fractional linear transformation that takes the upper half plane H^+ to the unit disk D and the real axis to the unit circle.
We will take i to 0, so the numerator should be $z - i$. $w = \frac{z-i}{z+i} : i \mapsto 0, 0 \mapsto -1, \infty \mapsto 1, 1 \mapsto -i$

2 Holomorphic Functions

- $f(z)$ complex valued functions in an open set $\Omega \subset \mathbb{C}$ or $\Omega \subset \mathbb{C} \cup \{\infty\}$
- f is holomorphic if $\lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$ exists. i.e. for some $c \in \mathbb{C}, f(z+h) - f(z) = ch + \varphi(h)h$ where $\varphi(h) \in o(h)$.
- This is similar to the definition of the derivative from an open set in the plane to an open set in the plane. (writing $z = x + iy, f(z) = u + iv, c = a + ib, h = \xi + i\eta$ and $f : (x, y) \mapsto (u, v)$)
- The derivative at z takes

$$h \mapsto ch \tag{17}$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{18}$$

  The matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$

- For a function to be holomorphic, it requires an additional constraint than being simply differentiable. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. This is the Cauchy-Riemann equations. Or, $\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$.
- The derivative at z is a linear transformation $h \mapsto ch$.
- The jacobian determinant is $a^2 + b^2 = |f'(z)|^2$.
- Consider $f(x, y)$ differentiable, but complex valued. The differential $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$. For example, $z = x + iy$ or $\bar{z} = x - iy$. Then, $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ and $\mathrm{d}\bar{z} = \mathrm{d}x - i\mathrm{d}y$.
- Then we have $\mathrm{d}x = \frac{1}{2}(\mathrm{d}z + \mathrm{d}\bar{z})$ and $\mathrm{d}y = \frac{1}{2i}(\mathrm{d}z - \mathrm{d}\bar{z})$. Then,

$$\mathrm{d}f = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\mathrm{d}z + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\mathrm{d}\bar{z} \tag{19}$$

- So, we **define**

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \tag{20}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{21}$$

- Thus, we can write the 1-form $\mathrm{d}f = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \bar{z}}\mathrm{d}\bar{z}$.
- $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are defined as the dual basis for $\mathrm{d}z, \mathrm{d}\bar{z}$.
- We can rewrite the Cauchy-Riemann equations as $\frac{\partial f}{\partial \bar{z}} = 0$. This means for holomorphic functions, it's **only** a function of z, **not** $\bar{z}$.

Definition–Harmonic Function: $f(x, y)$ is a **harmonic function** if $f \in C^2$ and $\Delta f = 0$, or $\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$. (laplace equation)

- We will see that holomorphic functions are harmonic. (but we need to first show we can differentiate holomorphic functions twice) So, the real and imaginary parts of holomorphic functions are also harmonic.
- Remark: $\frac{\partial f}{\partial \bar{z}} = 0 \iff \frac{\partial \bar{f}}{\partial z} = 0$. Why? Consider $f = u + iv, \bar{f} = u - iv$.

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{22}$$

$$\overline{\frac{\partial f}{\partial \bar{z}}} = \frac{1}{2}\left(\frac{\partial \bar{f}}{\partial x} - i\frac{\partial \bar{f}}{\partial y}\right) = \frac{\partial \bar{f}}{\partial z} \tag{23}$$

$$\tag{24}$$

Lemma 1: If $f(z)$ is holomorphic in a connected open set Ω and $f'(z) = 0$ in Ω, then f is constant.

Proof.

$$\mathrm{d}f = \underbrace{\frac{\partial f}{\partial z}}_{0} \mathrm{d}z + \underbrace{\frac{\partial f}{\partial \bar{z}}}_{0 \text{ holomorphic}} \mathrm{d}\bar{z} = 0 \tag{25}$$

∎

Proposition: Given $f(z)$ is holomorphic in a connected open set Ω, then

1. If $|f(z)|$ is constant, then $f(z)$ is constant.
2. If $\mathrm{Re}(f(z))$ is constant, then $f(z)$ is real.

Proof.

1. $|f(z)|^2 = f(z)\overline{f(z)}$ is constant, so

$$0 = \frac{\partial |f|^2}{\partial z} = \frac{\partial f}{\partial z}\bar{f} + f\cancel{\frac{\partial \bar{f}}{\partial z}} = \frac{\partial f}{\partial z}\bar{z} \tag{26}$$

so either $\bar{f} = 0$ so $f = 0$ thus f is constant or $\frac{\partial f}{\partial z} = 0$ so f is constant.

2. $\mathrm{Re}(f) = f + \bar{f}$ is constant, so

$$0 = \frac{\partial (f + \bar{f})}{\partial z} = \frac{\partial f}{\partial z} + \cancel{\frac{\partial \bar{f}}{\partial z}} = \frac{\partial f}{\partial z} \tag{27}$$

so $\frac{\partial f}{\partial z} = 0$ and f is constant.

∎

2.1 Mapping Properties

Suppose f is holomorphic at some point z_0. The **tangent mapping** of f at z_0 is

$$w = f(z_0) + f'(z_0)(z - z_0), \tag{28}$$

if $f'(z) \neq 0$, then the tangent mapping preserves angles and their orientation.

Definition–Conformal Mapping: A mapping f is **conformal** if f is holomorphic and $f'(z_0) \neq 0$. i.e. if f preserves angles and orientation.

Lemma 2: A $\mathbb{R}$-linear transformation $\mathbb{C} \to \mathbb{C}$ which preserves angles is of the form either $w = cz$ or $w = c\bar{z}$.

Consider $w = f(z)$ in a connected open set Ω. If f is treated as a function from $\mathbb{R}^2 \to \mathbb{R}^2$ has $\det f' \neq 0$ in Ω.

If f preserves angles at every point in Ω, then $\frac{\partial f}{\partial z} = 0$ or $\frac{\partial f}{\partial \bar{z}} = 0$. They cannot be both zero at the same point, as otherwise $\det f' = 0$ at that point. As $f \in C^1$, the partial derivatives are continuous. This means $\{z \in \Omega | \frac{\partial f}{\partial z} = 0\}, \{z \in \Omega | \frac{\partial f}{\partial \bar{z}} = 0\}$ are disjoint sets, and their union is Ω. Since Ω is connected, one of them must be empty.

So, either $\frac{\partial f}{\partial \bar{z}} = 0$ throughout $\Omega \implies f$ is holomorphic, or $\frac{\partial f}{\partial z} = 0$ throughout $\Omega \implies f$ is anti-holomorphic.

Theorem–: f preserves angles at every point in $\Omega \iff f$ is either holomorphic or anti-holomorphic in Ω.

Theorem–Inverse Function: Suppose f is holomorphic in a neighborhood of z_0 and $f'(z_0) \neq 0$. Then there are neighborhoods U of z_0 and V of $w_0 = f(z_0)$ such that f maps U **onto** V, with an inverse $z = g(w)$ which is holomorphic in V. And,

$$g'(w) = \frac{1}{f'(z)}. \tag{29}$$

Proof (to be completed later). We will use the fact that partial derivatives of holomorphic functions are continuous, which we will prove later.
If $f'(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and g' is the inverse, then $g'(w) = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, so g satisfies the cauchy riemann equations and g is holomorphic. □

3 Power Series

- A complex power series $f(w) = \sum_{n=0}^{\infty} a_n w^n$. Note that w is not a complex number, it's just a symbol. Complex power series means $a_n \in \mathbb{C}$.
- Suppose we have another power series $g(z) = \sum_{p=0}^{\infty} b_p z^p$. We want to compose
$$(f \circ g)(z) = a_0 + a_1(b_0 + b_1 z + \cdots) + a_2(b_0 + b_1 z + \cdots)^2 + \cdots \tag{30}$$
- First we need to ask if this even make sense? The answer is yes if $b_0 = 0$. However, in calculus every formal power series is the taylor series of some C^∞ functions, which can be composed. So why do we have this restriction?
- Consider taylor series of $f(g(z))$ at $z = z_0$. Let $w_0 = g(z_0)$ and the taylor series at w_0 is
$$f(w) = \sum_{n=0}^{\infty} a_n (w - w_0)^n. \tag{31}$$
Then we replace w with the taylor series for g at z_0, with $b_0 = w_0$ so these does not have constant term.

Definition–Formal Derivative: We define $f(0) = a_0$ and the **formal derivative** of $f(w)$ as
$$f'(w) = \sum_{n=1}^{\infty} n a_n w^{n-1}. \tag{32}$$

Theorem–Formal inverse function: Given formal power series $f(w) = \sum_{n=0}^{\infty} a_n w^n$. There is a power series $g(z) = \sum_{p=0}^{\infty} b_p z^p$ such that $b_0 = 0$ and $f \circ g = \mathrm{id}$ where $\mathrm{id}(z) = z$ **iff** $f(0) = 0, f'(0) \neq 0$. In that case g is uniquely determined by f and $g \circ f = \mathrm{id}$ also.

Proof by method of undetermined coefficients. We are trying to solve
$$a_0 + a_1(b_1 z + b_2 z^2 + \cdots) + a_2(b_1 z + b_2 z^2 + \cdots)^2 + \cdots = z. \tag{33}$$
We know right away that $a_0 = 0$ and $a_1 b_1 = 1$. so we know that $a_0 = 0$ and $a_1 \neq=$ are necessary conditions. Conversely, they are sufficient to solve for **unique** coefficients of g.
The coefficient of z^n on the LHS is the same as the coefficient of z^n in
$$\cancel{a_0} + a_1 g(z) + \cdots + a_n g(z)^n = a_1 b_n + P(a_2, \ldots, a_n, b_1, \ldots, b_{n-1}). \tag{34}$$
And $b_1 = 1/a_1$, thus b_n can be calculated recursively.
Since $g(0) = 0$ and $g'(0) \neq 0$, there is a unique formal power series $f_1(w)$ s.t. $g \circ f_1 = \mathrm{id}$.
$$f_1 = \mathrm{id} \circ f_1 = (f \circ g) \circ f_1 = f \circ (g \circ f_1) = f \tag{35}$$
□

Proposition: If $f = \sum_{n=0}^{\infty} a_n w^n$ and $g = \sum_{p=0}^{\infty} b_p w^p$ are convergent power series, then $f \circ g$ is also convergent. In fact, take $r > 0$ s.t. $\sum_{p=1}^{\infty} |b_p| r^p < R(f)$ the radius convergence of f. Then,

(1) $R(f \circ g) \geq r$

(2) If $|z| < r$ then $|g(z) < R(f)$.

(3) $f(g(z)) = (f \circ g)(z)$ (by rearrangement of absolute convergent series) where RHS is formal power series composition and LHS is substituting the value of $g(z)$ into f.

Proof of (1).

$$\sum_{n=0}^{\infty} |a_n| \left(\sum_{p=1}^{\infty} |b_p| r^p \right)^n =: \sum_{k=0}^{\infty} \gamma_k r^k < \infty \tag{36}$$

Say $(f \circ g)(z) = \sum c_k z^k$. By triangle inequality, $|c_k| \leq \gamma_k$. As $\sum \gamma_k r^k < \infty$, then $\sum c_k \gamma^k$ is convergent. □

Theorem–Reciprocal: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $a_0 \neq 0$ then there is an unique power series $g(z)$ s.t. $f(z) = g(z) = 1$. If f has a positive radius of convergence, then so does z.

Proof. As $a_0 \neq 0$, then WLOG $a_0 = 1$. Write $f(z) = 1 - h(z)$ then

$$f(z)^{-1} = (1 - h(z))^{-1} = 1 + \sum_{n=1}^{\infty} w^n \quad \text{where } w = h(z). \tag{37}$$

□

Theorem–Inverse function for convergent power series: In the previous statement, if $f(w)$ has a positive radius of convergence, then so does $g(z)$.

Proof. By direct estimate OR follows from inverse function theorem for holomorphic functions once we know holomorphic function has infinite taylor series that converges. □

3.1 Logarithmic Function

- The principal branch of $\log z$ is defined on the largest simply connected set that does not contain zero, which we will choose $\mathbb{C} \setminus (-\infty, 0]$. In this domain, there is a unique value of $\arg z \in (-\pi, \pi)$, we will call it $\mathrm{Arg}(z)$.
- We can show that this is continuous by showing it is continuous on $S' \setminus \{-1\}$. We can show this by its the fact its inverse $z = e^{i\theta}$ is continuous on $[-(\pi - \epsilon), \pi + \epsilon]$ hence the

it's the inverse of an bijection on compact hausdorff space.

- The principal branch of $\log z$ is defined as $\log|z| + i\text{Arg}\, z$, which is continuous on its entire domain $\mathbb{C} \setminus (-\infty, 0]$. Note that this is equal to the real logarithm if $z \in \mathbb{R}$.

Proposition: The power series $f(z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}$ converges if $|z| < 1$ and the sum is equal to the principal branch of $\log(1+z)$.

Proof. The power series $f(z)$ and $g(w) = \sum_{n=1}^{\infty}\frac{w^n}{n!} = e^w - 1$ are inverses. The proof is by MAT157 since the coefficients here are all real with $g(f(z)) = z$ when $|z| < 1$.
We also know that $e^{f(z)} = 1 + z$ and it's the principal branch because $f(0) = \log 1 = 0$ □

Definition–Power:

$$z^\alpha = e^{\alpha \log z} \tag{38}$$

where $\alpha \in \mathbb{C}, z \neq 0$. Note that for fixed α, z^α is a many-valued function of z. This has a branch in any **domain** (connected open subset of $\mathbb{C}$) where $\log$ has a branch. **Any** branch of $\log z$ in Ω defines a branch of z^α.

  - e.g. The **binomial series** $(1+z)^\alpha = e^{\alpha \log(1+z)}$ and its power series expansion in $|z| < 1$ is $\sum \binom{\alpha}{n} z^n$.

Mapping Properties of Holomorphic Functions

- $w = z^\alpha$ for real, positive α maps angles θ to an angle $\alpha\theta$.
- In general z^α is not 1-1 if $\alpha \neq 1$, and is multi-valued if α is fractional.
- Often, we will use a branched covering (mapping $X \to \mathbb{C}$) so we can have a single valued branch. Consider the multi-valued function $w = z^{1/2}$. Consider

$$X = \{(z, w) \in \mathbb{C}^2 | z = w^2\} \tag{39}$$

  X is a manifold (it is a graph of a continuous function) with local coordinate w.

- This multi-valued function $w = z^{1/2}$ lifts to a single valued $(w, z) \mapsto w$ by the covering surface X. X is an example of a **Riemann surface**.

Consider a mapping that takes the upper half plane $H^+ = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$, and consider the mapping that takes $H^+ \to D = \{z \in \mathbb{C} | z| < 1\}$, We can use a fractional linear transformation

$$w = \frac{z-i}{z+i} \tag{40}$$

this takes $i \mapsto 0$. We also know that it maps $\mathbb{R}$ to S^1 as we can pick three points $0, 1, \infty \in \mathbb{R}$ and we know $0 \mapsto -1$. We know this preserves orientations so $1 \mapsto -i$ and $\infty \mapsto 1$.

Now, we want to find a conformal mapping of a circular wedge onto D or H^+.

If circular wedge is formed by two circles intersecting in a and b, first use a fractional linear transformation $\zeta = \frac{z-a}{z-b}$ to map $a \mapsto 0$ and $b \mapsto \infty$. This takes the two circles into rays. Then, we can rotate the region by multiplying a complex number $e^{i\theta}$ and then change the angle by taking $w = e^{i\theta}\zeta^\alpha$ for some power α.

In the case they are degenerate, and only intersect at a, we take $\zeta = \frac{1}{z-a}$ which leads to two parallel lines. Then we can rotate and stretch it so that they become the real line and the line $\mathrm{Im}(z) = \pi$, then $\exp$ will map it to the upper half plane.

Exercise: Find a conformal mapping that takes the complement of the line segment to the interior (or exterior) of the unit disk. We will apply

$$z_1 = \frac{z-1}{z+1} \tag{41}$$

will map the interval $[-1, 1]$ to $(-\infty, 0]$. Then we can apply

$$z_2 = z_1^{1/2} \tag{42}$$

This maps the set to the right half plane. Finally,

$$w = \frac{z_2 - 1}{z_2 + 1} = z - \sqrt{z^2 - 1} \tag{43}$$

will map the right half plane into the interior of the unit disk (flipping the fraction maps to exterior). Check which branch of square root we need to use? Finally, show that $z = \frac{1}{2}(w + 1/w)$

3.2 Mapping Properties of exp and log

- We know that $w = e^z$ is periodic with period $2\pi i$,

$$\begin{aligned} e^z &= e^x e^{iy} && (44) \\ &= e^x(\cos y + i \sin y) && (45) \\ & && (46) \end{aligned}$$

- The exponential maps a vertical line to circle about 0, a horizontal line to a ray through 0, and any other line to a logarithmic spiral.
- The exponential is not injective. To make it single-valued, we need to restrict its domain. The image of e^z on $a < \mathrm{Im}\, z < b$ is a wedge in the complex plane $a < \arg w < b$.
- The logarithm is clearly multi-valued. Can we construct a riemann surface for $w = \log z$?

$$\begin{tikzcd} X \arrow[r, "\text{single value}"] \arrow[d, "\substack{\log z \\ \text{covering}}"'] & \mathbb{C} \\ \mathbb{C} \arrow[ur] & \end{tikzcd}$$

Let $X = \{(z, w) \in \mathbb{C}^2 | z = e^w\}$ then again the single-valued function $(w, z) \to w$ is singled valued.

Now we can try to map the open strip $-i\pi/2 < \mathrm{Im}(z) < i\pi/2$ to the unit disk. First we use $\zeta = e^z$ to map to the right half plane, then $w = \frac{\zeta-1}{\zeta+1}$.

4 Analytic functions

Definition–Analytic Function: A function f is **analytic** in an open set Ω if it has a convergent power series representation at every point $z_0 \in \Omega$.
i.e. $\forall\, z_0 \in \Omega$ there is a power series $\sum a_n(z-z_0)^n$ such that $f(z) = \sum a_n(z-z_0)^n$ when $|z-z_0| < R$ for some $R > 0$.

- If $f(z)$ has convergent power series representation at z_0, then there is a convergent power series $g(z)$ at z_0 such that $g'(z) = f(z)$ in some disk $|z-z_0| < R$, where R is the radius of convergence of f. We know

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \tag{47}$$

$$g(z) = c + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1} \tag{48}$$

$$\tag{49}$$

  The primitive is uniquely determined up to a constant.

- **Question:** Does a convergent power series define an analytic function?

Proposition: If $f(z) = \sum a_n z^n$ is a convergent power series with radius of convergence R, then $f(z)$ is analytic in $|z| < R$.

Proof. Note what we need to show. For any z_0 with $|z_0| < R$, then $f(z)$ has convergent power series representation at z_0 with radius of convergence $R - |z_0|$.

$$f(z) = \sum a_n z^n \tag{50}$$

$$= \sum a_n(z_0 + (z-z_0))^n \tag{51}$$

$$= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k}(z-z_0)^k \tag{52}$$

Note that if we take

$$\sum_{n=0}^{\infty} |a_n| \left(|z_0| + |z-z_0|\right)^n = \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k}|z-z_0|^k \tag{53}$$

We know this series is absolutely convergent. So we can change the order of summation to conclude

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right) (z-z_0)^k \tag{54}$$

□

- We notice that the inner sum

$$\frac{1}{k!} f^{(k)}(z_0) = \sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \tag{55}$$

is the kth derivative of f at z_0, so f is holomorphic.

4.1 Analytic Continuation

Theorem–: Given $f(z)$ is analytic in a domain Ω and $z_0 \in \Omega$, then the following are equivalent

1. $f^{(n)}(z_0) = 0$ for all $n \geq 0$.
2. f is identically 0 in a neighborhood of z_0.
3. f is identically 0 in Ω.

Proof. (3) $\implies$ (1) is trivial, (1) $\implies$ (2) can be shown from the convergent power series representation of f at z_0 (as the coefficient are the derivatives).
To show (2) $\implies$ (3), we define

$$\Omega' = \{z \in \Omega | f = 0 \text{ in a neighborhood of } z \text{ in } \Omega\}. \tag{56}$$

Clearly $\Omega' \neq \emptyset$ because $z_0 \in \Omega$.
Ω' is open by definition.
Ω' is also closed. Take $z \in \overline{\Omega'}$. Then, $f^{(n)}(z) = 0$ for all $n \geq 0$ by continuity. Then $f = 0$ in a neighborhood of z, by (1) $\implies$ (2). So $z \in \Omega'$. thus Ω' is closed.
Hence, $\Omega = \Omega'$. □

Corollary:

1. If f, g are analytic in domain Ω and $f = g$ in a neighborhood of some point then $f = g$ in Ω.
2. The ring $\mathcal{A}(\Omega)$ if analytic functions in a domain Ω is an **integral domain**.

Proof. The proof of (1) is trivial using $h = f - g$. For (2), suppose $f, g \in \mathcal{A}(\Omega)$ and $fg = 0$. Suppose $f \neq 0$ then there is z_0 s.t. f is non-vanishing in a neighborhood in a neighborhood U of z_0. So $g = 0$ in U hence $g = 0$ in Ω. □

- Integral domains are good, but it is better to work with fields. Hence, we will now analyze the zeros and poles.

- Consider f is analytic in a neighborhood of z_0. Then $f(z) = \sum a_n(z - z_0)^n$ is a convergent power series with radius of convergence R. Suppose $f(z_0) = 0$ but $f \neq 0$.
- Let k be the smallest integer s.t. $f^{(k)}(z_0) \neq 0$ (i.e. $a_k \neq 0$) Then, we define g s.t. $f(z) = (z - z_0)^k g(z)$. Then,

$$g(z) = \sum_{n=k}^{\infty} a_n(z - z_0)^{n-k} \tag{57}$$

- k is the **order** or **multiplicity** of the zero at z_0, characterized by $f^{(k)}(z_0) \neq 0$, but $f^{(j)}(z_0) = 0$ for $j < k$.
- This shows the zero is **isolated** meaning $f(z) \neq 0$ in $0 < |z - z_0| < \epsilon$ for any $\epsilon > 0$.
- If we make a local change of variable near z,

$$\zeta = (z - z_0)g(z)^{1/k}. \tag{58}$$

This is a change of coordinates because its derivative is nonzero. Then, $f(z(\zeta)) = \zeta^k$.

- We now consider the quotients of analytic functions $f(z)/g(z)$ where g is not identically zero. $f(z)/g(z)$ is well-defined and analytic in a neighborhood of z_0 if and only if $g(z)$ is analytic in a neighborhood of z_0 where $g(z_0) \neq 0$.
- What if $g(z_0) = 0$? We can try to factor out terms of $z - z_0$ so that $f_1(z_0) \neq 0$ and $g_1(z_0) \neq 0$ and

$$f(z) = (z - z_0)^k f_1(z) \tag{59}$$
$$g(z) = (z - z_0)^l g_1(z) \tag{60}$$

Then

$$\frac{f(z)}{g(z)} = (z - z_0)^{k-l}\frac{f_1(z)}{g_1(z)} \tag{61}$$

We know that $f_1(z)/g_1(z)$ is analytic and nowhere vanishing in a neighborhood of z_0. There are two cases

  - $k \geq l$ then f/g extends to be analytic at z_0.
  - $k < l$ then z_0 is a **pole** of f/g of **order** $l - k$. Then,

$$\left|\frac{f(z)}{g(z)}\right| \to \infty \text{ as } z \to z_0, \tag{62}$$

  so f/g still make sense as a function with values in the Riemann sphere.

> **Definition–Meromorphic Function**: In an open set Ω, a **meromorphic function** is well-defined and analytic in $\Omega \setminus D$ where D is a discrete set, and expressible at in a neighborhood of any point of Ω as the quotient f/g with g is not identically zero.

- Meromorphic functions in domain Ω form a **field**.
- **Exercise:** If $f(z)$ is meromorphic in Ω, then $f'(z)$ is also meromorphic in Ω with the same poles as f. If z_0 is a pole of order k of $f(z)$ then z_0 is a pole of order $k+1$ of $f'(z)$.