MAT257 PSET 4—Question 4

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October 15, 2021

(a) Note that $a + te_i = \begin{pmatrix} a_1 & \dots & a_i + t & \dots & a_n \end{pmatrix}$. Then,

$$\lim_{t\to 0} \frac{f(a+te_i) - f(a)}{t} = \lim_{t\to 0} \frac{f(a_1, \dots, a_i + t, \dots, a_n) - f(a_1, \dots, a_n)}{t} = D_i f(a)$$

by definition on Spivak pp.25.

(b) Using the definition of the directional derivative

$$D_x f(a) = \lim_{h\to 0} \frac{f(a+xh) - f(a)}{h}$$

Consider making a change of variables $k := h/t$ for $t \neq 0$ (the direction derivative with respect to the 0 vector does not make sense). Note that as $h \to 0$, $k \to 0$, and h can be written as tk. Rewriting the limit results in

$$D_x f(a) = \lim_{k\to 0} \frac{f(a+xtk) - f(a)}{tk} = \frac{1}{t} \lim_{k\to 0} \frac{f(a+xtk) - f(a)}{k} = \frac{1}{t} D_{tx} f(a)$$

Multiplying both sides by t yields $D_{tx} f(a) = t D_x f(a)$.

(c) As f is differentiable at a,

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - Df(a)h|}{|h|} = 0$$

Make the substitution $h = tx$ for nonzero vector x. As $h \to 0, t \to 0$. Then,

$$\lim_{t\to 0} \frac{|f(a+tx) - f(a) - tDf(a)x|}{|t|} = 0$$

Rearranging the definition of the directional derivative, $\lim_{t\to 0} \frac{|f(a+tx) - f(a) - tD_x f(a)|}{|t|} = 0$. Since the derivative is unique, $Df(a)x = D_x f(a)$.

Since $Df(a)$ is a linear operator, $Df(a)(x+y) = Df(a)(x) + Df(a)(y)$. So, $D_{x+y} f(a) = D_x f(a) + D_y f(a)$.