MAT257 PSET 6—Question 2

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(a) Consider functions $F:\mathbb{R}^3\to\mathbb{R}^2, E:U\to\mathbb{R}^2$ s.t. $F=(G,H), E=(g,h)$ and $U\subset\mathbb{R}$ is an open set containing -1.

Then, $F(2,-1,1)=0$ and $E(-1)=(2,1)$.

F is C^1 since G and H are C^1. So, for the implicit function theorem to guarantee the existance of E (hence guarantee the existance of g and h in U), it is also required that $\dfrac{\partial F}{\partial(x,u)}$ to be invertible at $(2,-1,1)$.

$$\det\left(\frac{\partial F}{\partial(x,u)}(2,-1,1)\right)=\det\begin{pmatrix}\pi_1 f' & 2u\\ u & 3u^2+x\end{pmatrix}=\det\begin{pmatrix}\pi_1 f'(2,-1) & 2\cdot 1\\ 1 & 3\cdot 1^2+2\end{pmatrix}=5\pi_1 f'(2,-1)-2\neq 0$$

Thus, $\pi_1 f'(2,-1)\neq\dfrac{2}{5}$ must be true to ensure the existance of g and h on an open set U about $y=-1$.

(b) As the implicit function theorem applies,

$$E'=-\left(\frac{\partial F}{\partial(x,u)}\right)^{-1}\frac{\partial F}{\partial y}=-\begin{pmatrix}\pi_1 f' & 2u\\ u & 3u^2+x\end{pmatrix}^{-1}\begin{pmatrix}\pi_2 f'\\ 9y^2\end{pmatrix}$$
$$E'(g(-1),-1,h(-1))=E'(2,-1,1)=-\begin{pmatrix}1 & 2\\ 1 & 5\end{pmatrix}^{-1}\begin{pmatrix}-3\\ 9\end{pmatrix}=\begin{pmatrix}11\\ -4\end{pmatrix}$$

Recalling the definition of E, $g'(-1)=11$ amd $h'(-1)=-4$.