MAT257 PSET 5—Question 2

Jonah Chen

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Lemma 1: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable, then $g(t) = f(t, t^k)$ is continuous for $k > 1$.

Proof. For g to be continuous,

$$\forall s \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 : |t - s| < \delta \implies |g(t) - g(s)| = |f(t, t^k) - f(s, s^k)| < \varepsilon$$

As f is continuous on $\mathbb{R}^2$, $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$ hence we know that

$$\forall (x, y) \in \mathbb{R}^2 \, \forall \varepsilon > 0 \exists \delta > 0 : |(x, y) - (a, b)| < \delta \implies |f(x, y) - f(a, b)| < \varepsilon$$

Consider $(a, b) = (s, s^k)$ and $(x, y) = (t, t^k)$. Then,

$$s \in \mathbb{R} \, \forall \varepsilon > 0 \exists \delta > 0 : |(t, t^k) - (s, s^k)| < \delta \implies |f(t, t^k) - f(s, s^k)| = |g(t) - g(s)| < \varepsilon$$

$|(x, y) - (a, b)| = \sqrt{(s - t)^2 + (s^k - t^k)^2} \geq \sqrt{(s - t)^2} = |s - t|$.
$|(x, y) - (a, b)| < \delta \implies |s - t| < \delta$. As this δ exists $\forall \varepsilon > 0 \forall s \in \mathbb{R}$ because f is continuous, g must be continuous. $\square$

(a) Consider any continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, and two other functions $g_1, g_2 : [0, 1] \to \mathbb{R}$ such that

$$g_1(t) = f(t, t^2)$$
$$g_2(t) = f(t, t^3)$$

Note that $g_1(0) = g_2(0) = f(0, 0)$ and $g_1(1) = g_2(1) = f(1, 1)$. As f is continuous on $\mathbb{R}^2$, g_1 and g_2 must be continuous due to Lemma 1. By intermediate value theorem, there exists $t_1 \in (0, 1)$ and $t_2 \in (0, 1)$ such that $g_1(t_1) = g_2(t_2) = \dfrac{f(1, 1) - f(0, 0)}{3}$. Therefore, both points $(t_1, t_1^2), (t_2, t_2^3)$ are mapped to $\dfrac{f(1, 1) - f(0, 0)}{3}$ so f cannot be one-to-one.