MAT257 PSET 15—Question 2

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Lemma 1: Let open sets $U, V \subset \mathbb{R}^n$ be diffeomorphic with the diffeomorphism $g : U \to V$. Then, any k-form on U can be expressed as the pullback using g of some k-form on V, and any k-form on V can be expressed as the pullback using g^{-1} of some k-form on U.

Specifically, $\forall \nu \in \Omega^k(V)$, then $\nu = (g^{-1})^* \mu$ where $\mu = g^* \nu \in \Omega^k(U)$,
and $\forall \mu \in \Omega^k(U)$, then $\mu = g^* \nu$ where $\nu = (g^{-1})^* \mu \in \Omega^k(V)$

Proof. Let $u_1, \ldots, u_k \in T_p U$ and $v_1 = g_* u_1, \ldots, v_k = g_* u_k \in T_q V$ where $p \in U$ and $q = g(p)$.
Consider some $\nu \in \Omega^k(V)$, then $\nu = (g^{-1})^* \mu$ where $\mu = g^* \nu \in \Omega^k(U)$.

$$\nu(v_1, \ldots, v_k) = (g^{-1})^* \mu(v_1, \ldots, v_k) = \mu(u_1, \ldots, u_k) = g^* \nu(u_1, \ldots, u_k) = \nu(v_1, \ldots, v_k).$$

Similarly, consider some $\mu \in \Omega^k(U)$, then $\mu = g^* \nu$ where $\nu = (g^{-1})^* \mu \in \Omega^k(V)$.

$$\mu(u_1, \ldots, u_k) = g^* \nu(u_1, \ldots, u_k) = \nu(v_1, \ldots, v_k) = (g^{-1})^* \mu(v_1, \ldots, v_k) = \mu(u_1, \ldots, u_k).$$

□

If there are no closed forms on V, then the statement is trivial.

By lemma 1, for every closed $k+1$ form on ν on V, there is a corresponding $k+1$ form $\mu = g^* \nu$ on U such that $\nu = (g^{-1})^* \mu$.

$$\mathrm{d}\mu = \mathrm{d}(g^* \nu) = g^*(\mathrm{d}\nu) = g^*(0) = 0.$$

Hence, μ is a closed $k+1$ form on U. Since every closed form in U is exact, then there must exist $\eta \in \Omega^k(U) : \mathrm{d}\eta = \mu$. Note that $\lambda = (g^{-1})^* \eta$ is a k form on V.

$$\begin{aligned} \mathrm{d}\eta &= \mu \\ (g^{-1})^*(\mathrm{d}\eta) &= (g^{-1})^* \mu \\ \mathrm{d}\big((g^{-1})^* \eta\big) &= (g^{-1})^* \mu \\ \mathrm{d}\lambda &= \nu \end{aligned}$$

As a λ can be found for any closed form ν on V, then every closed form on V is also exact.