MAT257 PSET 13—Question 1

Jonah Chen

For convinence, we will sum over repeated indices in the same term from 1 to 3. Let ε_{ijk} be the levi-civita tensor.

Given the standard basis for $\mathbb{R}^3, \{e_1, e_2, e_3\}$ and its dual basis $\{\varphi_1, \varphi_2, \varphi_3\}$, we can identify a basis for $\Lambda^1(\mathbb{R}^3)$: $\{\varphi_1, \varphi_2, \varphi_3\}$ and for $\Lambda^2(\mathbb{R}^3)$: $\{k = 1, 2, 3 : \Phi_k := \frac{1}{2}\varepsilon_{ijk}(\varphi_i \wedge \varphi_j)\} = \{\varphi_2 \wedge \varphi_3, \varphi_3 \wedge \varphi_1, \varphi_1 \wedge \varphi_2\}$.

Using these bases, the wedge product $\wedge : \Lambda^1(\mathbb{R}^3) \times \Lambda^1(\mathbb{R}^3) \to \Lambda^2(\mathbb{R}^3)$ is the cross product $P : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, P(x, y) = x \times y$.

Let $\omega, \eta \in \Lambda^1(\mathbb{R}^3)$,

$$\omega \wedge \eta = (a_i\varphi_i) \wedge (b_j\varphi_j) = a_ib_j(\varphi_i \wedge \varphi_j) = \varepsilon_{ijk}a_ib_j\Phi_k$$

Similarly, for $x = a_ie_i, y = b_je_j \in \mathbb{R}^3$, their cross product is

$$x \times y = (a_ie_i) \times (b_je_j) = \varepsilon_{ijk}a_ib_je_k$$

As $\{\varphi_i\}$ are the basis for $\Lambda^1(\mathbb{R}^3)$ and $\{\Phi_i\}$ are the basis for $\Lambda^2(\mathbb{R}^3)$, these forms are equivalent.