MAT257 PSET 7—Question 6

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If A is a countable union of open intervals, A is an open set. Thus, A^C is closed.

First, we show that $A^C \cap [0,1] \subset \mathrm{bd}A$ and thus $A \cup (A^C \cap [0,1]) \supset [0,1]$.

Consider any point $a \in [0,1] : a \notin A$. Because of the density of $\mathbb{Q}$, there exists a rational number b in any open set containing a. As $[0,1] \cap \mathbb{Q} \subset A$, then $b \in A$. Since any open set about a has a point in A, which is b; and a point in A^C, which is a. Hence, $a \in \mathrm{bd}A$.

As the union of two closed sets is a closed set, $A^C \cap [0,1]$ is closed. It is also clearly bounded hence $A^C \cap [0,1]$ is compact.

By way of contradiction, suppose $\mathrm{bd}A$ is measure zero. Then $A^C \cap [0,1]$ which is contained in $\mathrm{bd}A$ must also be measure zero, and because $A^C \cap [0,1]$ is also compact, $A^C \cap [0,1]$ must be content zero.

So, there exists a finite collection of sets $D = \{(u_i, v_i)\}$ for $i = 1, \ldots, n$ that covers $A^C \cap [0,1]$ s.t. $\displaystyle\sum_{i=1}^{n}(v_i - u_i) < \varepsilon$ for any $\varepsilon > 0$.

Choose $\varepsilon = \dfrac{1}{2}\left(1 - \displaystyle\sum_{i=1}^{\infty}(b_i - a_i)\right)$. Then, $\displaystyle\sum_{i=1}^{\infty}(b_i - a_i) + \sum_{i=1}^{n}(v_i - u_i) < \sum_{i=1}^{\infty}(b_i - a_i) + \varepsilon < 1$. However, a set with volume 1 cannot be contained in a set with volume less than 1. Thus, $[0,1] \not\subseteq A \cup (A^C \cap [0,1])$ which is a contradiction. Hence, $A^C \cap [0,1]$ is not content 0 and hence $\mathrm{bd}A$ is not measure zero.