As A is the union of open intervals, A is an open set. Thus, $\mathrm{bd}A \cap A = \emptyset$.

Next, show $\mathrm{bd}A \subset [0, 1]$.

- If $x > 1$, consider the open rectangle $R = (1, x + 1) \ni x$. Clearly, $A \cap R = A \cap (1, x + 1) = \emptyset$. Hence, $x \notin \mathrm{bd}A$.
- If $x < 0$, consider the open rectangle $R = (x - 1, 0) \ni x$. Clearly, $A \cap R = A \cap (x - 1, 0) = \emptyset$. Hence, $x \notin \mathrm{bd}A$.

Thus, $x \in \mathrm{bd}A \implies 0 \leq x \leq 1 \implies \mathrm{bd}A \subset [0, 1]$.

Consider $x \in \mathrm{bd}A$. $x \notin A$ because $\mathrm{bd}A \cap A = \emptyset$ and $x \in [0, 1]$ because $\mathrm{bd}A \subset [0, 1]$. Hence, $[0, 1] \setminus A \supset \mathrm{bd}A$.

Next, show $[0, 1] \setminus A \subset \mathrm{bd}A$. For any $x \in [0, 1] : x \notin A$, consider every open rectangle $R = (a, b) \ni x$ where $a < b$.

Due to the density of $\mathbb{Q}$ over $\mathbb{R}$, there exists a rational in every open interval.

- If $0 \leq a < b \leq 1$, $\exists y \in \mathbb{Q} : y \in (a, b) \subset (0, 1)$.
- If $a < 0 < b < 1$, $(a, b) = (a, 0) \cup \{0\} \cup (0, b)$. Hence, $\exists y \in \mathbb{Q} : y \in (0, b) \subset (0, 1)$.
- If $0 < a < 1 < b$, $(a, b) = (a, 1) \cup \{1\} \cup (1, b)$. Hence, $\exists y \in \mathbb{Q} : y \in (a, 1) \subset (0, 1)$.
- If $a < 0 < 1 < b$, $(a, b) = (a, 0) \cup \{0\} \cup (0, 1) \cup \{1\} \cup (1, b)$. Hence, $\exists y \in \mathbb{Q} : y \in (0, 1)$.

Since all rationals in $(0, 1)$ are contained in A, $y \in A$. Hence, $x \in R \cap A^C$ and $y \in R \cap A$ for every open rectangle $R \ni x$.

Hence, $x \in [0, 1] \setminus A \implies x \in \mathrm{bd}A$ and $[0, 1] \setminus A \subset \mathrm{bd}A$.

As it was shown that $[0, 1] \setminus A \subset \mathrm{bd}A$ and $[0, 1] \setminus A \supset \mathrm{bd}A$, then $[0, 1] \setminus A = \mathrm{bd}A$ as desired.