MAT257 PSET 18—Question 1

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a) Consider the 1-manifold-with-boundary $M = \mathbb{R}_+$ with the usual orientation and an atlas with one coordinate chart (the identity map from $\mathbb{R}_+ \to \mathbb{R}_+$), and the 0-form $\omega = x$, then $\mathrm{d}\omega = \mathrm{d}x$. We know $\partial M = -(0)$, so

$$\int_{\partial M} \omega = \int_{-(0)} x = 0 \tag{1}$$

However, the other side

$$\int_M \mathrm{d}\omega = \int_M \mathrm{d}x = \int_{\mathbb{R}+} 1 \tag{2}$$

This integral does not exist, hence it is not equal to 1 and Stokes' theorem does not hold for non-compact manifolds. (which is well known as it's a single variable integral, but this can also be shown by using a partition of unity and showing the series does not converge)

Proposition: Let ω be a k-form supported on a compact set $A \subset M$. Then,

$$\int_M \omega = \int_A \omega \tag{3}$$

Proof. Consider □

Then, if ω has compact support, $\mathrm{d}\omega$ has the same support. Then,

$$\int_M \mathrm{d}\omega = \int_{\mathrm{supp}(\omega)} \mathrm{d}\omega = \int_{\partial \mathrm{supp}(\omega)} \omega \tag{4}$$

b) Given an exact k-form ω on a k-manifold-without-boundary M, by definition, $\omega = \mathrm{d}\eta$ for some $k-1$-form η. If M is compact and oriented, Stokes' theorem holds.

$$\int_M \omega = \int_M \mathrm{d}\eta = \int_{\partial M} \eta = \int_{\emptyset} \eta = 0 \tag{5}$$

If M is not compact however, consider $M = \mathbb{R}$, and a closed form $\omega = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}\mathrm{d}x = \mathrm{d}\eta$ where $\eta = \dfrac{1}{2}\,\mathrm{erf}(x/\sqrt{2})$.

However, the integral

$$\int_M \omega = 1 \tag{6}$$