MAT257 PSET 15—Question 1

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a) We know that
$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z.$$
And
$$\mathrm{grad}f = \frac{\partial f}{\partial x}\partial_x + \frac{\partial f}{\partial y}\partial_y + \frac{\partial f}{\partial z}\partial_z$$
Hence its component functions of $\mathrm{grad}f$ are $F_1 = \frac{\partial f}{\partial x}, F_2 = \frac{\partial f}{\partial y}, F_3 = \frac{\partial f}{\partial z}$. Then,
$$\omega^1_{\mathrm{grad}f} = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z = df,$$
as desired.

We know that
$$\mathrm{curl}f = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\partial_x + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\partial_y + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\partial_z.$$
Hence, its component functions of $G = \mathrm{curl}f$ are $G_1 = \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, G_2 = \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, G_3 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

$$\begin{aligned}
\mathrm{d}\big(\omega^1_F\big) &= \mathrm{d}(F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z)\\
&= \mathrm{d}x \wedge \left(\frac{\partial F_1}{\partial x}\mathrm{d}x + \frac{\partial F_2}{\partial x}\mathrm{d}y + \frac{\partial F_3}{\partial x}\mathrm{d}z\right) + \mathrm{d}y \wedge \left(\frac{\partial F_1}{\partial y}\mathrm{d}x + \frac{\partial F_2}{\partial y}\mathrm{d}y + \frac{\partial F_3}{\partial y}\mathrm{d}z\right) + \mathrm{d}z \wedge \left(\frac{\partial F_1}{\partial z}\mathrm{d}x + \frac{\partial F_2}{\partial z}\mathrm{d}y + \frac{\partial F_3}{\partial z}\mathrm{d}z\right)\\
&= \frac{\partial F_2}{\partial x}\mathrm{d}x\wedge\mathrm{d}y + \frac{\partial F_3}{\partial x}\mathrm{d}x\wedge\mathrm{d}z + \frac{\partial F_1}{\partial y}\mathrm{d}y\wedge\mathrm{d}x + \frac{\partial F_3}{\partial y}\mathrm{d}y\wedge\mathrm{d}z + \frac{\partial F_1}{\partial z}\mathrm{d}z\wedge\mathrm{d}x + \frac{\partial F_2}{\partial z}\mathrm{d}z\wedge\mathrm{d}y\\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathrm{d}y\wedge\mathrm{d}z + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathrm{d}z\wedge\mathrm{d}x + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y\\
&= G_1\mathrm{d}y\wedge\mathrm{d}z + G_2\mathrm{d}z\wedge\mathrm{d}x + G_3\mathrm{d}x\wedge\mathrm{d}y\\
&= \omega^2_G = \omega^2_{\mathrm{curl}F},
\end{aligned}$$
as desired.

We know that
$$\mathrm{div}F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$
And,
$$\begin{aligned}
\mathrm{d}\big(\omega^2_F\big) &= \mathrm{d}(F_1\mathrm{d}y\wedge\mathrm{d}z + F_2\mathrm{d}z\wedge\mathrm{d}x + F_3\mathrm{d}x\wedge\mathrm{d}y)\\
&= \frac{\partial F_1}{\partial x}\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z + \frac{\partial F_2}{\partial y}\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}x + \frac{\partial F_3}{\partial z}\mathrm{d}z\wedge\mathrm{d}x\wedge\mathrm{d}y\\
&= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\\
&= \mathrm{div}F\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z,
\end{aligned}$$
as desired.

b) $\omega^2_{\mathrm{curl}\,\mathrm{grad}f} = \mathrm{d}\big(\omega^1_{\mathrm{grad}f}\big) = \mathrm{d}(\mathrm{d}f) = \mathrm{d}^2 f = 0 \implies$ the component functions of $\mathrm{curl}\,\mathrm{grad}f$ are zero so $\mathrm{curl}\,\mathrm{grad}f = 0$.

$(\mathrm{div}\,\mathrm{curl}\,F)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z = \mathrm{d}\big(\omega^2_{\mathrm{curl}\,F}\big) = \mathrm{d}\big(\mathrm{d}(\omega^1_F)\big) = d^2(\omega^1_F) = 0 \implies \mathrm{div}\,\mathrm{curl}\,F = 0.$