MAT257 PSET 3—Question 2

Jonah Chen

October 8, 2021

(a) First consider when $t \geq 0$. $h(t) = f(tx) = |tx|g(tx/|tx|) = txg(x/|x|) = tf(x)$.

Next, consider when $t < 0$. $h(t) = f(tx) = |tx|g(tx/|tx|) = -t|x|g(-x/|x|) = t|x|g(x/|x|) = tf(x)$, as $g(y) = g(-y)$.

Thus, $\forall t, h(t) = tf(x)$. As $f(x)$ is independent of t, h is differentiable and $h'(t) = f(x)$.

(b) Firstly, note that $f(0) = 0$ by definition. We will show the countrapositive case. If g is nonzero, then f is not differentiable at $(0, 0)$.

Given g is nonzero, $\exists a \in S^1, b \neq 0 : g(a) = b$. Wwe also know that $g(0,1) = g(1,0) = g(0,-1) = g(-1,0) = 0$ as $g(-x) = -g(x)$. Thus, a cannot be any of these four points.

Consider the line passing through the origin and a. Then, f restricted to that line is $f(ta) = |t|g(ta/|ta|) = tg(a) = tb$. Also consider the line passing through the origin and the point $0, 1$ that is given by $f(t(0,1)) = |t|g((0,1)/|(0,1)|) = 0$. As f is a constant function when restricted to the line passing through the origin and $(0, 1)$, $f'(0, 0)$ must be 0 if it exists. Assuming the derivative exists,

$$\lim_{h \to 0} \frac{|f(h) - 0h|}{|h|} = 0 \implies \lim_{t \to 0} \frac{|f(ta)|}{|ta|} = 0$$

However, we know that $f(ta) = tb$ and $|a| = 1$. Therefore,

$$\lim_{t \to 0} \frac{|f(ta)|}{|ta|} = \lim_{t \to 0} \frac{|tb|}{|t|} = |b|$$

As b is defined as being nonzero, this cannot be true. Thus, the derivative must not exist at $(0, 0)$ unless $g(a)$ is identically 0. In that case, $f(x) = |x| \cdot 0 = 0$, which is a constant function. Therefore, $f'(0, 0) = 0$ when $g = 0$.