MAT257 PSET 5—Question 1

Jonah Chen

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Lemma 1: If $f : A \to \mathbb{R}^n$ is continuously differentiable 1-1 function with a invertiable derivative $\forall x \in A$, then $x \in A \implies \exists X \ni x$ such that $f(X)$ is open. Moreover, for any open subset $Y \subset X$, $f(Y)$ is open.

Proof. Let $g|_X : f(X) \to X$ such that $g(x) = f^{-1}(x)$. As f satisfies the hypotheses of the inverse function theorem about x, a continuous function $g|_X$ must exist for some $X \ni x$. By definition of continuity, the preimage of any open set is open. So, for any open set $Y \subset X$, $(g|_X)^{-1}(Y) = f(Y)$ is an open set. Since $X \subset X$ and X is open, $f(X)$ is also open. $\square$

Consider an open set $B \subset A$. As f and any $x \in B$ satisfies the hypotheses of the Lemma 1, $\exists X \ni x$ such that $f(X \cap B)$ is open since $X \cap B$ is open because the intersection of two open sets X and B is an open set, and $X \cap B \subset X$.

As this is satisfied for all points $x \in B$, consider the set

$$D = \bigcup_{x \in B} f(X \cap B)$$

As D is a union of open sets, D is an open set. Note that $f(A \cup B) = f(A) \cup f(B)$ thus $D = f(B)$. So, $f(B)$ is an open set.

As f satisfies the hypotheses of the inverse function theorem for all $x \in A$, then $\exists X \ni x$ such that $f^{-1} : f(X) \to X$ is differentiable with the derivative being

$$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$$

As all $x \in A$ allow f to satisfy the hypotheses of the inverse value theorem, this formula for the derivative of the inverse of f is valid for all $y \in f(A)$ since f is one-to-one. That means f^{-1} is differentiable for all $y \in f(A)$.