MAT257 PSET 5—Question 3

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October 22, 2021

(a) Suppose f is not 1-1. Then, there exists two distinct numbers $a, b \in \mathbb{R}$ such that $f(a) = f(b)$. WLOG suppose $a < b$. As f is differentiable on $\mathbb{R}$, f is continuous on $[a, b]$ and f is differentiable on (a, b). Therefore, by special case of mean value theorem (Rolle's), $\exists c \in (a, b)$ where $f'(c) = 0$, which contridicts $f'(a) \neq 0 \forall a \in \mathbb{R}$.

(b) Clearly, f is not 1-1 as $f(0, 0) = f(0, 2\pi) = (0, 0)$.

We can find $f'(x, y)$ by using the partial derivatives of f.

$$f'(x, y) = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix} = e^x \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix}$$

This is just a nonzero quantity e^x multipled by a rotation matrix by angle y. Thus, the inverse should be

$$(f')^{-1}(x, y) = e^{-y} \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix}$$

We can verify that this is true for all $(x, y) \in \mathbb{R}^2$ by verifying that $(f') \circ (f')^{-1} = I$ for all x and y. Thus,

$$\begin{aligned} (f')(f')^{-1}(x, y) &= e^x \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix} e^{-y} \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix} \\ &= \begin{pmatrix} \cos^2 y + \sin^2 y & -\sin y \cos y + \sin y \cos y \\ -\sin y \cos y + \sin x \cos y & \cos^2 y + \sin^2 y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \end{aligned}$$