MAT257 PSET 14—Question 4

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Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $\{(e_1)_p, \dots, (e_n)_p\}$ be an orthonormal basis for $T_p\mathbb{R}^n$. As $v_p \in T_p\mathbb{R}^n$, it can be written as a superposition of its basis $v_p = \sum v_i(e_i)_p$.

Since the directional derivative is linear in the direction vector,

$$D_{v_p}f = \sum_{i=1}^n v_i D_{(e_i)_p} f = \sum_{i=1}^n v_i D_i f(p).$$

Using the definition for the gradient,

$$(\mathrm{grad} f)(p) = \sum_{i=1}^n D_i f(p)(e_i)_p$$

and the bilinearity of the inner product,

$$\begin{aligned} \langle (\mathrm{grad} f)(p), v_p \rangle &= \left\langle \sum_{i=1}^n D_i f(p)(e_i)_p, \sum_{j=1}^n v_j(e_j)_p \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n D_i f(p) v_j \langle (e_i)_p, (e_j)_p \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n D_i f(p) v_j \delta_{ij} \\ &= \sum_{i=1}^n D_i f(p) v_i = D_{v_p} f \end{aligned}$$

The directional derivative is defined as $D_{v_p}f = \lim_{t\to 0} \dfrac{f(p+tv) - f(p)}{t}$. This is the rate of increase of the value of f in the direction of v_p. By the cauchy-schwarz inequality,

$$|D_{v_p}f|^2 = |\langle (\mathrm{grad} f)(p), v_p \rangle|^2 \le \langle (\mathrm{grad} f)(p), (\mathrm{grad} f)(p) \rangle \langle v_p, v_p \rangle$$

with equality if v_p and $(\mathrm{grad} f)(p)$ are linearly dependent (i.e. same direction). Hence, the $(\mathrm{grad} f)(p)$ is the direction f is changing fastest at p.