MAT257 PSET 8—Question 2

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Given $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}^n$. By way of contradiction, suppose $D_2(D_1 f) \neq D_1(D_2 f)$ for some $x \in A$.

WLOG, assume $D_1(D_2 f(a)) > D_2(D_1 f(a))$. Let

$$g(x) = D_1(D_2 f(x)) - D_2(D_1 f(x))$$

Clearly, $g(a) > 0$. Let the open interval $K = (\, g(a)/2, 2g(a)\,)$. As both partial derivatives are continuous hence g is continuous, $g^{-1}(K) \ni a$ is an open set in A.

For some $a_i < b_i$, let $R := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset g^{-1}(K)$ be a closed rectangle and define $R' := [a_3, b_3] \times \cdots \times [a_n, b_n]$.

Clearly, $x \in R \implies g(x) > 0$. As g is continuous and integrable, and both partial derivatives are continuous and differentiable,

$$\int_R g = \int_{[a_1,b_1]\times[a_2,b_2]\times R'} D_1(D_2 f) - \int_{[a_1,b_1]\times[a_2,b_2]\times R'} D_2(D_1 f) > 0$$

Fubini's theorem states that

$$\int_R D_1(D_2 f) = \int_{R'} \left(\int_{[a_2,b_2]} \left(\int_{[a_1,b_1]} D_1(D_2 f) \right) \right)$$
$$\int_R D_2(D_1 f) = \int_{R'} \left(\int_{[a_1,b_1]} \left(\int_{[a_2,b_2]} D_2(D_1 f) \right) \right)$$

Define $h : [a_3, b_3] \times \cdots \times [a_n, b_n] \to \mathbb{R}$ where $h(x) = f(b_1, b_2, x_3, \ldots, x_n) - f(a_1, a_2, x_3, \ldots, x_n)$. By the 1-dimensional fundamental theorem of calculus,

$$\int_R D_1(D_2 f) = \int_{R'} \left(\int_{[a_2,b_2]} \left(\int_{[a_1,b_1]} D_1(D_2 f) \right) \right) = \int_{R'} h$$
$$\int_R D_2(D_1 f) = \int_{R'} \left(\int_{[a_1,b_1]} \left(\int_{[a_2,b_2]} D_2(D_1 f) \right) \right) = \int_{R'} h$$

Then,

$$\int_R g = \int_{R'} h - \int_{R'} h = 0$$

However, $\displaystyle\int_R g > 0$. This is a contradiction hence $D_2(D_1 f) = D_1(D_2 f)$.