MAT257 PSET 10—Question 2

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Lemma 1: If $A \subset B \subset \mathbb{R}^n$, and a function $f : B \to \mathbb{R}$ where $f \geq 0$ is integrable on A and B then $\int_B f = b, \int_A f = a$. Then, $b \geq a$.

Proof. Let $(\mathcal{U}, \Phi)$ be an admissible open cover and partition of unity for A. Then, as A is integrable, then $\sum_{\varphi \in \Phi} \int \varphi f$ converges absolutely to a.

As $A \subset B$, there is some open cover of B, $\mathcal{V} = \mathcal{U} \cup \mathcal{W}$ where $\mathcal{W}$ is an open cover of $B \setminus A$. Let Ψ be a partition of unity for $B \setminus A$ subordinate to $\mathcal{W}$. Similarly, $\chi := \Phi \cup \Psi$ is a partiton of unity for B subordinate to $\mathcal{V}$.

Note that as $f \geq 0$, $\sum_{\psi \in \Psi} \int \psi f \geq 0$. Then, as f is also integrable on B, $\sum_{\beta \in \chi} \int \beta f$ converges absolutely to b. This sum can be rewritten as $b = \sum_{\phi \in \Phi} \int \phi f + \sum_{\psi \in \Psi} \int_B \psi f \geq \sum_{\phi \in \Phi} \int \phi f = a$. $\square$

Let the coordinate transformation $g(r, \theta) = (r \cos\theta, r \sin\theta)$. Let $V_1 = (0, 1) \times (0, 2\pi)$ and $V_2 = (1, \infty) \times (0, 2\pi)$. Then, $g(V_1) = U_1$ and $g(V_2) = U_2$. Also, $\det(g') = r$, which is nonzero for any $x \in U_1$ or $x \in U_2$. Note that $|\det(g')| = \det(g')$ as $r > 0$ for any $x \in U_1$ or $x \in U_2$. Also, $f \circ g = \frac{1}{r^2}$. Then, suppose these integrals exist and have a value of

$$I_1 \equiv \int_{U_1} f = \int_{V_1} (f \circ g) |\det g'| = \int_{V_1} \frac{1}{r}$$

$$I_2 \equiv \int_{U_2} f = \int_{V_2} (f \circ g) |\det g'| = \int_{V_2} \frac{1}{r}$$

Let $W_n := (2^{-n}, 1) \subset V_1$. As all W_n are jordan measureable and f is bounded on all W_n. Then, we can use Fubini's theorem to integrate on the closure of W_n to obtain the same result:

$$I_1 \equiv \int_{V_1} f \geq \int_{W_n} f = \int_{\overline{W_n}} f = \int_0^{2\pi} \mathrm{d}\theta \int_{2^{-n}}^1 \mathrm{d}r \frac{1}{r} = 2\pi n \log 2 \text{ for any } n.$$

However, for $n = 1 + \left\lceil \frac{I_1}{2\pi \log 2} \right\rceil$, this inequality is false. Hence, I_1 cannot exist.

Now, let $W_n := (1, 2^n) \subset V_2$. As all W_n are jordan measurable and f is bounded on all W_n. Then, we can use Fubini's theorem to integrate on the closure of W_n to obtain the same result:

$$I_2 \equiv \int_{V_2} f \geq \int_{W_n} f = \int_{\overline{W_n}} f = \int_0^{2\pi} \mathrm{d}\theta \int_1^{2^n} \mathrm{d}r \frac{1}{r} = 2\pi n \log 2 \text{ for any } n.$$

However, for $n = 1 + \left\lceil \frac{I_2}{2\pi \log 2} \right\rceil$, this inequality is false. Hence, I_2 cannot exist.