MAT257 PSET 10—Question 1

Jonah Chen

Let $U = \{(r, \phi, \theta) : 0 < r < a, 0 < \phi < \pi, 0 < \theta < 2\pi\}$ be open set and g be the coordinate transformation

$$g(r, \phi, \theta) = (r \cos\phi \cos\theta, r \cos\phi \sin\theta, r \sin\phi).$$

Then, $g(U) = \{(x, y, z) : x^2 + y^2 + z^2 < a^2, z > 0\} = V$ and if $f(x, y, z) = z$ then $f(g(r, \phi, \theta)) = r \sin\phi$. g' is also continuous on all of $\mathbb{R}^3$. Using mathematica, it is possible to compute $\det(g') = r^2 \cos\phi$. Thus, there are no $x \in U$ where $\det(g') = 0$. By the change of variables theorem,

$$\int_{g(U)} z = \int_U (f \circ g)|\det(g')| = \int_U r^3 \sin\phi |\cos\phi|$$

Then, by fubini's theorem, we can integrate over the compact set $\overline{U}$

$$\int_{\overline{U}} r^3 \sin\phi |\cos\phi| = \int_0^a r^3 \mathrm{d}r \left(\int_0^\pi \sin\phi |\cos\phi| \mathrm{d}\phi \left(\int_0^{2\pi} \mathrm{d}\theta \right) \right) = \frac{\pi a^4}{2}.$$

As U has content zero boundary, so

$$\int_U (f \circ g)|\det(g')| = \int_{\overline{U}} (f \circ g)|\det(g')| = \frac{\pi a^4}{2}.$$