MAT257 PSET 3—Question 6

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As $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc, f(a,b,c,d) = ad - bc$. Define a function $g : \mathbb{R}^4 \to \mathbb{R}$ where $g(a,b,c,d) := (d, -c, -b, a)$, thus, $g(a,b,c,d)(h) = dh_1 - ch_2 - bh_3 + ah_4$. If g is the derivative of f, it must be true that

$$\lim_{h\to 0} \frac{|f(a+h_1, b+h_2, c+h_3, d+h_4) - f(a,b,c,d) - g(a,b,c,d)h|}{|h|} = 0$$

Substituting the definition for f and the guess for f',

$$\begin{aligned}
&\lim_{h\to 0} \frac{|(a+h_1)(d+h_4) - (b+h_2)(c+h_3) - ad - bc - dh_1 + ch_2 + bh_3 - ah_4|}{|h|} \\
&= \lim_{h\to 0} \frac{|ad + ah_4 + h_1 d + h_1 h_4 - bc - bh_3 - h_2 c - h_2 h_3 - ad + bc - dh_1 + ch_2 + bh_3 - ah_4|}{|h|} \\
&= \lim_{h\to 0} \frac{|h_1 h_4 - h_2 h_3|}{|h|}
\end{aligned}$$

Invoking the triangle inequality, this limit must be less than or equal to $\lim_{h\to 0} \frac{|h_1 h_4|}{|h|} + \lim_{h\to 0} \frac{|h_2 h_3|}{|h|}$ Note that $|h| > |h_1|$ and $|h| > |h_2|$. Then,

$$\begin{aligned}
\lim_{h\to 0} \frac{|h_1 h_4|}{|h|} &\le \lim_{h\to 0} \frac{|h_1 h_4|}{|h_1|} = \lim_{h\to 0} |h_4| = 0 \\
\lim_{h\to 0} \frac{|h_2 h_3|}{|h|} &\le \lim_{h\to 0} \frac{|h_2 h_3|}{|h_2|} = \lim h \to 0 |h_3| = 0
\end{aligned}$$

as when $h \to 0$, all its components $h_i \to 0$ for $i = 1, 2, 3, 4$. Therefore, the initial limit

$$\begin{aligned}
\lim_{h\to 0} \frac{|f(a+h_1, b+h_2, c+h_3, d+h_4) - f(a,b,c,d) - g(a,b,c,d)h|}{|h|} &\le 0 + 0 \\
\lim_{h\to 0} \frac{|f(a+h_1, b+h_2, c+h_3, d+h_4) - f(a,b,c,d) - g(a,b,c,d)h|}{|h|} &= 0
\end{aligned}$$

Hence, f is differentiable and $f' = g$ where $g(a,b,c,d) = (d, -c, -b, a)$.