MAT257 PSET 15—Question 5

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a) The definition of the exterior derivative is that $\mathrm{d}\omega =: \sum_{i=1}^n \mathrm{d}x_i \wedge \frac{\partial \omega}{\partial x_i}$.

For the given ω,

$$\mathrm{d}\omega = \sum_{j=1}^n \sum_{i=1}^n (-1)^{i-1} \partial_j \left(\frac{x_i}{|x|^p}\right) \mathrm{d}x_j \wedge \mathrm{d}x_1 \wedge \dots \hat{\mathrm{d}x_i} \wedge \dots \mathrm{d}x_n$$

We notice that unless $i = j$, the term is zero as there would be two copies of $\mathrm{d}x_i$ in the alternating wedge product. So we have

$$\mathrm{d}\omega = \sum_{i=1}^n (-1)^{i-1} \partial_i \left(\frac{x_i}{|x|^p}\right) \mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \dots \hat{\mathrm{d}x_i} \wedge \dots \mathrm{d}x_n$$

It would take $i - 1$ swaps to move the first term to the correct place for the sequence to be ascending, which introduces a factor of $(-1)^{i-1}$.

$$\mathrm{d}\omega = \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n \sum_{i=1}^n \partial_i \left(\frac{x_i}{|x|^p}\right)$$

The partial derivative can be computed with quotient rule to be $\frac{1 - px_i^2/|x|^2}{|x|^p}$, thus,

$$\begin{aligned}
\mathrm{d}\omega &= \frac{1}{|x|^p} \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n \sum_{i=1}^n \left(1 - \frac{px_i^2}{|x|^2}\right) \\
&= \frac{1}{|x|^p} \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n \left(n - \frac{p(x_1^2 + \dots + x_n^2)}{|x|^2}\right) \\
&= \frac{n-p}{|x|^p} \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n
\end{aligned}$$

b) For $p = n, \mathrm{d}\omega = 0$.