MAT257 PSET 13—Question 4

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For convinence, we will sum over repeated multi-indices in the same term.

a) As the basis for V^* is already given, we can easily make a basis for $\Lambda^k(V)$ as $\{I \in \underline{n}_a^k : \varphi_I\}$, where $\underline{n}_a^k$ is the set of ascending multi-index of length k.

   We define $I' \in \underline{n}_a^{n-k}, I' := \underline{n} \setminus I$, where $\underline{n} = \{1, \dots, n\}$.

   For $I = \{i_1, \dots, i_k\}, J = \{j_1, \dots, j_k\} \in \underline{n}_a^k$, define $I + J =: \{i_1, \dots, i_k, j_1, \dots, j_k\}$.

   For $I \in \underline{n}^k$, and let $\tau_I \in S_n$ be the permutation where $\tau_I \underline{n} = I$, define $(-1)^I = (-1)^{\tau_I} = (-1)^{\tau_I^{-1}}$.

   For any $\lambda, \eta \in \Lambda^k(V)$ written in the basis $\lambda = a_I\omega_I, \eta = b_J\omega_J$ and $a, b \in \mathbb{R}$, we define the hodge star operator to be linear, so

$$\star(a\lambda + b\eta) = a \star \lambda + b \star \eta.$$

   Thus, defining this operator on the basis vectors is sufficient.

$$\star\omega_I = (-1)^{I+I'}\omega_{I'}.$$

   Note that for another basis vector, $I \neq J$ i.e. $I \cap J' \neq \emptyset$

$$\implies \omega_I \wedge \star\omega_J = (-1)^{J+J'}\omega_I \wedge \omega_{J'} = 0$$

   Also,

$$\omega_I \wedge \star\omega_I = (-1)^{I+I'}\omega_I \wedge \omega_{I'} = (-1)^{I+I'}\omega_{I+I'} = \omega_n$$

   So, we know that $\omega_I \wedge \star\omega_J = \delta_{IJ}\omega_n$. Also, for $a_I, b_J \in \mathbb{R}$ then due to the bilinearity property of the inner product, $\langle a_I\omega_I, b_J\omega_J\rangle = a_Ib_J\delta_{IJ}$

   Now, we show that the hodge star operator satisfies

$$\begin{aligned}
\lambda \wedge (\star\eta) &= a_I\omega_I \wedge b_J\omega_J \\
&= a_Ib_J(\omega_I \wedge \omega_J) \\
&= a_Ib_J\delta_{IJ}\omega_n \\
&= \langle\lambda, \eta\rangle\omega_n.
\end{aligned}$$

   As the operator is defined as linear, and both $\Lambda^k(V)$ and $\Lambda^{n-k}(V)$ have the same dimension because $\binom{n}{k} = \binom{n}{n-k}$. Using the basis vectors $\{\omega_I\}_{I\in\underline{n}_a^k}$ for $\Lambda^k(V)$ and $\{\star\omega_I\}_{I\in\underline{n}_a^k}$ for $\Lambda^{n-k}(V)$. Then the matrix representing the hodge star operator is the identity matrix, which means it is invertiable. Hence, the hodge star operator is invertible.

b) Using the previous definition, for $n = 3, k = 1$, $\star\omega_1 = \omega_{23}, \star\omega_2 = -\omega_{13}, \star\omega_3 = \omega_{12}$.

   For $n = 4, k = 2$, $\star\omega_{12} = \omega_{34}, \star\omega_{13} = -\omega_{24}, \star\omega_{14} = \omega_{23}, \star\omega_{23} = \omega_{14}, \star\omega_{24} = -\omega_{13}, \star\omega_{34} = \omega_{12}$.

c) Given some alternating tensor $\lambda \in \Lambda^k(V)$ and $\lambda = a_I\omega_I$

$$\begin{aligned}
\star(\star\lambda) &= a_I \star (\star\omega_I) \\
&= (-1)^{I+I'}a_I \star (\omega_{I'}) \\
&= (-1)^{I'+I}(-1)^{I+I'}a_I\omega_I \\
&= (-1)^{I'+I}(-1)^{I+I'}\lambda.
\end{aligned}$$

   Define the permutation $\tau_j \in S_n$ such that $\tau_j\{a_1, \dots, a_n\} = \{a_1, \dots, a_{j-1}, a_{j+1}, a_j, \dots, a_n\}$. It is obvious that $(-1)^{\tau_j} = -1$.

We can calculate the sign of $I' + I$ in terms of the sign of $I + I'$. As I has length k and I' has length $n - k$, for element number $k + 1, \ldots, k + n$, making k swaps between consecutive elements to position $1, \ldots, k$ will convert $I + I'$ to $I' + I$. Performing the inverse operation require the same number of swaps.

This is done by the permutation $(\tau_k \tau_{k-1} \ldots \tau_1)(\tau_{k+1} \tau_k \ldots \tau_2) \ldots (\tau_{n-1} \ldots \tau_{n-k})$. The number of elements to swap is $n - k$, and k swaps per element hence $k(n - k)$ swaps are required to swap $I + I'$ to $I' + I$. Thus,

$$\star(\star\lambda) = (-1)^{I'+I}(-1)^{I+I'}\lambda = ((-1)^{I'+I})^2(-1)^{k(n-k)}(\lambda) = (-1)^{k(n-k)}(\lambda).$$

Hence, $\star \circ \star = (-1)^{k(n-k)}\mathbb{1}$.