MAT257 PSET 6—Question 4

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For any vector $x \in \mathbb{R}^k \times \mathbb{R}^n$ define $x_X := \begin{pmatrix} x_1 & \dots & x_k \end{pmatrix} \in \mathbb{R}^k$ and $x_Y = \begin{pmatrix} x_{k+1} & \dots & x_{k+n} \end{pmatrix} \in \mathbb{R}^n$.

Then, $f(a) = f(a_X, a_Y) = 0$. WLOG, assume $\dfrac{\partial f}{\partial a_Y}$ has rank n hence invertible (the choice of at least one set of n coordinates for a_Y must make this true as $f'(a)$ is rank n).

Define $R := \det\left(\dfrac{\partial f}{\partial a_Y}\right)$ which is nonzero as $\dfrac{\partial f}{\partial a_Y}$ is full rank. Define $g(x_X, x_Y) := (x_X, f(x_X, x_Y))$.

Then, $Dg(a) = \begin{pmatrix} I_{k\times k} & 0 \\ \dfrac{\partial f}{\partial a_X} & \dfrac{\partial f}{\partial a_Y} \end{pmatrix}$ and hence $\det(Dg(a)) = R \neq 0$.

As g is also C^1 on $\mathbb{R}^k \times \mathbb{R}^n$ because f is C^1 on $\mathbb{R}^k \times \mathbb{R}^n$, and $\mathbb{R}^k \times \mathbb{R}^n$ contains a, g satisfies the hypotheses of the inverse function theorem.

Hence, $\exists$ an open set $V \ni a$ and an open set $U \ni g(a) = (a_X, 0)$ such that $g : V \to U$ has a continuous inverse $g^{-1} : U \to V$.

As U is an open set, $\exists$ an open ball $B \subset U$ of radius $r > 0$ centered at $(a_X, 0)$.

For any $c \in \mathbb{R}^n$ where $|c| < r$, $|(a_X, c) - (a_X, 0)| < r$ so $(a_X, c) \in B \subset U \implies g^{-1}(a_X, c)$ exists.

Thus, $g(g^{-1}(a_X, c)) = (a_X, c) \implies f(g^{-1}(a_X, c)) = c$ by the definition of g so the solution to the equation $f(x) = c$ is

$$x = g^{-1}(a_X, c)$$

for any c sufficiently close to 0.