MAT257 PSET 3—Question 4

Jonah Chen

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(a) Consider the functions $f_1 : \mathbb{R}^3 \to \mathbb{R}^2, f_2 : \mathbb{R}^2 \to \mathbb{R}, f_3 : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} f_1(x, y, z) &= (\log x, y) \\ f_2(a, b) &= ab \\ f_3(c) &= \exp(c) \end{aligned}$$

The derivative of f_1 is $f_1' = \begin{pmatrix} 1/x & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ by Theorem 2-3(3)

The derivative of f_2 is $f_2' = \begin{pmatrix} b & a \end{pmatrix}$ by Theorem 2-3(5)

The derivative of f_3 is $f_3' = \exp(c)$.

Note that $f_3(f_2(f_1(x, y, z))) = x^y = f(x, y, z)$. By chain rule,

$$\begin{aligned} f'(x, y, z) &= f_3'(f_2(f_1(x, y, z))) \cdot f_2'(f_1(x, y, z)) \cdot f_1'(x, y, z) \\ &= \begin{pmatrix} \exp(y \log x) \end{pmatrix} \begin{pmatrix} y & \log x \end{pmatrix} \begin{pmatrix} 1/x & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} x^{y-1}y & x^y \log x & 0 \end{pmatrix} \end{aligned}$$

(b) Now, $g(x, y, z) = (x^y, z) = (f(x, y, z), z)$, where $f(x, y, z) = x^y$ whose derivative is found in part (a). By Theorem 2-3(3),

$$g'(x, y, z) = \begin{pmatrix} f'(x, y, z) \\ (z)' \end{pmatrix} = \begin{pmatrix} x^{y-1}y & x^y \log x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Consider the function $k : \mathbb{R}^3 \to \mathbb{R}^3$ where $k = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ Then, $k' = k$ by Theorem 2-3(2) as k is a linear map.

Note that $h_1(x, y, z) = (x + y, z, 0)$. Therefore, $h(x, y, z) = f(k(x, y, z)) = (x + y)^z$. By chain rule,

$$\begin{aligned} h'(x, y, z) &= f'(k(x, y, z)) \cdot k'(x, y, z) \\ &= f'(x + y, z, 0) \cdot k(x, y, z) \\ &= \begin{pmatrix} (x + y)^{z-1}z & (x + y)^z \log(x + y) & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (x + y)^{z-1}z & (x + y)^{z-1}z & (x + y)^z \log(x + y) \end{pmatrix} \end{aligned}$$