MAT257 PSET 7—Question 7

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First, we show that if $f : [a, b] \to \mathbb{R}$ is an increasing function, then for any finite set of distinct points $x_1, \ldots, x_n$

$$\sum_{i=1}^{n} o(f, x_i) < f(b) - f(a)$$

where $o(f, x) = \lim_{r\to 0} M_{B_r(x)}(f) - m_{B_r(x)}(f)$. For an increasing function, $x > y \implies f(x) > f(y)$ so

1. $M_{B_r(x)}(f) = f(x + r), m_{B_r(x)}(f) = f(x - r)$
2. $o(f, x) = \lim_{r\to 0} f(x + r) - f(x - r)$
3. $\forall s, t > 0$, $o(f, x) < f(x + s) - f(x - t)$

WLOG, assume $x_i < x_{i+1}$. Then, letting $a = x_0, b = x_{n+1}$

$$\sum_{i=1}^{n} o(f, x_i) < \sum_{i=1}^{n} f(x_n + [x_{n+1} - x_n]) - f(x_n - [x_n - x_{n-1}]) = \sum_{i=1}^{n} f(x_{i+1}) - f(x_{i-1}) = f(b) - f(a)$$

Define the set

$$D_n = \{x \in [a, b] : o(f, x) > 1/n\} \text{ for } n = 1, 2, 3, \ldots$$

As f is discontinuous at $x \iff o(f, x) > 0$, $\bigcup_{n=1}^{\infty} D_n$ contains the set of discontinuities of f.

Moreover, each D_n is a discrete set because $\frac{|D_n|}{n} \le \sum_{x\in D_n} o(f, x) < f(b) - f(a) \implies |D_n| < n(f(b) - f(a))$. So, each D_n is measure zero. A countable union of measure zero sets is measure zero. Therefore, $\bigcup_{n=1}^{\infty} D_n$ is measure zero hence the set of discontinuities of f is measure zero.