MAT257 PSET 13—Question 3

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For $\omega \in \Lambda^{n-k}(V)$ and $\lambda \in \Lambda^k(V)$. Define the function $\psi_k : \Lambda^{n-k}(V) \to (\Lambda^k(V))^*$ as

$$\psi_k(\omega)(\lambda) = \chi(\omega \wedge \lambda)$$

We claim ψ_k is an isomorphism. Given any $\omega, \xi \in \Lambda^{n-k}(V)$ and $a, b \in \mathbb{R}$,

$$\begin{aligned}
\psi_k(a\omega + b\xi)(\lambda) &= \chi((a\omega + b\xi) \wedge \lambda) \\
&= \chi(a\omega \wedge \lambda + b\xi \wedge \lambda) \\
&= a\chi(\omega \wedge \lambda) + b\chi(\xi \wedge \lambda) \\
&= a\psi_k(\omega) + b\psi_k(\xi)
\end{aligned}$$

Hence, ψ_k is linear. As ψ_k is a linear map between $\binom{n}{k}$ dimensional vector spaces, we need to show that ψ_k is 1-to-1, which is equivalent to showing the null space contains only the zero vector.

By way of contradiction, suppose $\exists\, \omega \neq 0 \in \Lambda^{n-k}(V) : \psi_k(\omega) = 0$. Then, for any $\lambda \in \Lambda^k(V)$, $\psi_k(\omega)(\lambda) = \chi(\omega \wedge \lambda) = 0$. As χ is an isomorphism, $\omega \wedge \lambda = 0$ so for any n vectors $v_1, \ldots, v_n \in V$,

$$\omega \wedge \lambda(v_1, \ldots, v_n) = \frac{1}{n!(n-k)!} \sum \omega(v_1, \ldots, v_k)\lambda(v_{k+1}, \ldots, v_n)$$

(I can't really figure the rest out)