MAT257 PSET 10—Question 5

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Define $h : \mathbb{R}^3_{u,\theta,\varphi} \to \mathbb{R}^3_{r,\theta,z}$ so that $h(u,\theta,\varphi) := (u\cos\varphi + b, \theta, u\sin\varphi)$ and $B = (0,a)\times(0,2\pi)\times(0,2\pi)$. Clearly, g and h are C^1 so $g \circ h$ is also C^1. The determinant of $(g\circ h)'$ is

$$\det((g\circ h)') = \det((g'\circ h)\cdot h') = \det(g'\circ h)\det(h') = (b + u\cos\varphi)(u).$$

Within B, u is always positive and $b > u$ so the determinent is always positive.

Note that

$$h(B) = \{(r,\theta,z) : 0 < (r-b)^2 + z^2 < a^2 \text{ and } 0 < \theta < 2\pi\} = \operatorname{int} A.$$

As $T = g(A), \operatorname{int} T = g(\operatorname{int} A) = (g\circ h)(B)$ because g is continuous. As 1 is bounded and $\operatorname{bd} T$ is content zero, the integral over T is the same as the integral over the interior of T.

By the change of variables theorem,

$$\int_T 1 = \int_{\operatorname{int} T} 1 = \int_{(g\circ h)(B)} 1 = \int_B |\det((g\circ h)')| = \int_B |\det((g'\circ h)\cdot h')| = \int_B (b + u\cos\varphi)(u)$$

we can integrate over the closure of B to get the same answer. As the closure of B is closed, we can use Fubini's theorem to get

$$\int_T 1 = \int_{\overline{B}} u(b + u\cos\varphi) = \int_0^{2\pi} \mathrm{d}\theta \int_0^a \mathrm{d}u \int_0^{2\pi} \mathrm{d}\varphi (b + u\cos\varphi)(u) = 2\pi^2 a^2 b$$