MAT257 PSET 9—Question 2

Jonah Chen

Lemma 1: If A is an arbitrary subset of $\mathbb{R}^n$, then A is contained in a countable union of compact sets.

Proof. The any point in $\mathbb{R}^n$ is contained in $D = \bigcup_{n=1}^{\infty} \{|x| \leq n\}$, hence any arbiturary subset of $\mathbb{R}^n$ is also contained in D. All sets $\{|x| \leq n\}$ are closed and bounded, so they are compact. $\square$

If f is smooth at every $a \in A$, then $\forall a \in A\, \exists$ open set U_a and a function $g_a \in C^\infty : U_a \to \mathbb{R}$ s.t. $x \in U_a \implies g(x) = f(x)$.

From the lemma, A is a countable union of compact sets $D_1, D_2, \ldots$.

For $x \in D_i$ there is an open set U_x where the corresponding C^∞ function g_x agrees with f on $U_x \cap A$.

Let $\{U_x : x \in D_i\}$ be an open cover of D_i. Since D_i is compact, there is a finite subcover $\mathcal{U}_i = \{U_{x_1}, \ldots, U_{x_n}\}$. This is true for every i.

Let $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$. This is a countable collection of open sets because it is a countable union of finite collections. $\mathcal{U}$ also covers A.

Now, consider the open cover $\mathcal{V} = \{U \in \mathcal{U} : U \cap A \neq \emptyset\} \equiv \{V_{x_i}, V_{x_2}, \ldots\}$.

By the partition of unity theorem, there exists a C^∞ partition of unity $\Phi = \{\varphi_i\}$ of $\mathcal{V}$ such that φ_i is subordinate to V_{x_i}.

As $\mathcal{V}$ is an open cover of A, then $V = \bigcup_{E \in \mathcal{V}} E$ is an open set containing A.

Define $F : V \to \mathbb{R}$

$$F(x) = \sum_{i=1}^{\infty} g_{x_i}(x)\, \varphi_i(x)$$

F is smooth as g_{x_i} and φ_i are smooth for all i. Since $g_{x_i} = f$ on the support of φ_i, then $x \in A \implies F(x) = f(x)$.