# MAT257 PSET 3—Question 1

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If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then by definition there exists a linear map $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ where

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - \lambda h|}{|h|} = 0$$

Also, as $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, $|\lambda h| \le M|h|$ for some finite $M$. This means that

$$\begin{aligned}
\forall \varepsilon_1 > 0 \exists \delta_1 > 0 : |h| < \delta_1 \implies |f(a+h) - f(a) - \lambda h| &< \varepsilon_1 |h| \\
\implies |f(a+h) - f(a) - \lambda h| &\le |f(a+h) - f(a)| + |\lambda h| \\
&\le |f(a+h) - f(a)| + M|h| \\
&< \varepsilon_1 |h| \\
\implies |f(a+h) - f(a)| < (\varepsilon_1 - M)|h|
\end{aligned}$$

To show the continuity of $f$ at $a$, it is required that $\lim_{x\to a} f(x) = f(a)$. Define $h := x - a$, so, $\lim_{h\to 0} f(a+h) = f(a)$. This means we need to show

$$\forall \varepsilon_2 > 0 \, \exists \, \delta_2 > 0 : |h| < \delta_2 \implies |f(a+h) - f(a)| < \varepsilon_2$$

As $f$ is differentiable, we can choose any $\varepsilon_1 > 0$ and there must exist $\delta_1 > 0$. So, choose $\varepsilon_1 = M + \epsilon_2$