Note the following:

1. As C is compact, C is bounded; thus $\exists r$ s.t. the closed ball of radius r centered at the origin contains C.
2. As U is open, U^C is closed. As $C \subset U$, $C \cap U^C = \emptyset$. Given what was shown in $2b$, there exists $d > 0$ such that $|y - x| \geq d$ for all $x \in U^C$ and $y \in C$.

Consider the set

$$S = \bigcup_{x \in U^C} \{y : |y - x| < \frac{d}{2}\}$$

Note that

- S is clearly open as it is a union of open balls, thus S^C is closed.
- Every point in U^C is contained in S since $\frac{d}{2} > 0$. Hence, $S^C \subset U$.
- No point in C is contained in S because [2.]. Thus, $C \subset S^C$.

Now, let $B = \{x : |x| \leq r + 1\}$ and consider the set $D = S^C \cap B$. Note that D is bounded because it is contained in the ball of radius $r + 1$ centered at the origin, and D is closed as it is the intersection of two closed sets S^C and B. Thus, D is compact.

Also, $D \subset U$ as $S^C \subset U$.

Finally, to show $\mathrm{int} D \supset C$, we must show $\forall c \in C$, there is an open ball Y about c s.t. $y \subset D$. This can be done by showing all points in the open ball Y of radius $q = \min\{\frac{d}{2}, 1\}$ centered at c is contained in B and not contained in S.

- As c is contained in the closed ball of radius r, a open ball of radius $q \leq 1$ about any point in C is contained in a closed ball of radius $r + 1$.
- For any $p \in Y$ and any $x \in U^C$, we must have $|p - c| < \frac{d}{2}$ and $|x - c| \geq d$ as $c \in C$ by [2.]. By triangle inequality,

  $|x - c| \leq |x - p| + |p - c| \implies |x - c| - |p - c| \leq |x - p| \implies |x - p| > \frac{d}{2}$.

  If $p \in S$, $|x - p| < \frac{d}{2}$ for some $x \in U^C$. However, $|x - p| > \frac{d}{2}$, thus $p \notin S$.