MAT257 PSET 7—Question 1

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(a) For a subrectangle S and $y \in S$, $m_S(f) \leq f(y)$ and $m_S(y) \leq g(y)$ so $m_S(f) + m_S(g) \leq f(y) + g(y)$. Since this is true for any $y \in S$, $m_S(f) + m_S(g) \leq m_S(f+g)$.

Similarly, $M_S(f) \geq f(y)$ and $M_S(g) \geq g(y)$ so $M_S(f) + M_S(g) \geq M_S(f+g)$.

For any partition P of A,

$$L(f+g,P) = \sum_{S\in P} m_S(f+g)v(S) \geq \sum_{S\in P} m_S(f)v(S) + \sum_{S\in P} m_S(g)v(S) = L(f,P) + L(g,P)$$
$$U(f+g,P) = \sum_{S\in P} M_S(f+g)v(S) \leq \sum_{S\in P} M_S(f)v(S) + \sum_{S\in P} M_S(g)v(S) = U(f,P) + U(g,P)$$

(b) To show $f+g$ is integrable on A, and given $\varepsilon > 0$, we need to find a partition P such that $U(f+g,P) - L(f+g,P) < \varepsilon$.

As f and g are integrable on A, there exists partitions of A, P_f and P_g, such that $U(f,P_f) - L(f,P_f) < \varepsilon/2$ and $U(g,P_g) - L(g,P_g) < \varepsilon/2$.

Let P be the refinement of P_f and P_g. Then,

$$U(f,P) - L(f,P) \leq U(f,P_f) - L(f,P_f) < \varepsilon/2$$
$$U(g,P) - L(g,P) \leq U(g,P_g) - L(g,P_g) < \varepsilon/2$$

For $f+g$, $L(f,P) + L(g,P) \leq L(f+g,P) \leq U(f+g,P) \leq U(f,P) + U(g,P)$. Hence, $U(f+g,P) - L(f+g,P) < \varepsilon$.

For any P_f and P_g of A, let P be the refinement of P_f and P_g. Then,

$$L(f,P) \geq L(f,P_f)$$
$$L(g,P) \geq L(g,P_g)$$
$$U(f,P) \leq U(f,P_f)$$
$$U(g,P) \leq U(g,P_g)$$

So,

$$\sup_{P_f} L(f,P_f) + \sup_{P_g} L(g,P_g) = \sup_P \{L(f,P) + L(g,P)\}$$
$$\inf_{P_f} U(f,P_f) + \inf_{P_g} U(g,P_g) = \inf_P \{U(f,P) + U(g,P)\}$$

Then,

$$\int_A f + \int_A g = \sup_P \{L(f,P) + L(g,P)\} \leq \sup_P L(f+g,P) \leq \inf_P U(f+g,P) \leq \inf_P \{U(f,P) + U(g,P)\} = \int_A f + \int_A g$$

Thus, $\sup_P L(f+g,P) = \inf_P U(f+g,P) = \int_A f+g = \int_A f + \int_A g$.

(c) Case $c = 0 : cf = 0 \implies \int_A 0 = 0 \int_A f = 0$ as f is integrable on A.

Case $c > 0$: For any partition P of A,

$$L(cf, P) = \sum_{S \in P} m_S(cf)v(S) = c \sum_{S \in P} m_S(cf)v(S) = cL(f, P)$$
$$U(cf, P) = \sum_{S \in P} M_S(cf)v(S) = c \sum_{S \in P} M_S(cf)v(S) = cU(f, P)$$

Then, $\sup L(cf, P) = c \sup L(f, P)$ and $\inf U(cf, P) = c \inf U(f, P)$. As f is integrable on A,

$$\sup L(f, P) = \inf U(f, P) = \int_A f \implies \sup L(cf, P) = \inf U(cf, P) = \int_A cf = c \int_A f$$

Case $c < 0$. Then, for any partition P of A,

$$L(cf, P) = \sum_{S \in P} m_S(cf)v(S) = c \sum_{S \in P} M_S(f)v(S) = cU(f, P)$$
$$U(cf, P) = \sum_{S \in P} M_S(cf)v(S) = c \sum_{S \in P} m_S(cf)v(S) = cL(f, P)$$

Then, $\sup L(cf, P) = c \inf U(f, P)$ and $\inf U(cf, P) = c \sup L(f, P)$. As f is integrable on A,

$$\sup L(f, P) = \inf U(f, P) = \int_A f \implies \sup L(cf, P) = \inf U(cf, P) = \int_A cf = c \int_A f$$