MAT257 PSET 7—Question 5

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(a) By way of contradiction, suppose there is an unbounded set $A \subset \mathbb{R}^n$ that is of content zero. Then, there are finitely many closed rectangles $D_i = \displaystyle\prod_{j=1}^{M} [a_i^j, b_i^j]$ such that both $\displaystyle\bigcup_{i=1}^{M} D_i \supset A$ and $\displaystyle\sum_{i=1}^{M} v(D_i) < 1$.

Choose $r^2 = \displaystyle\sum_{i=1}^{N} \max_{j=1,\ldots,M} \{(a_i^j)^2, (b_i^j)^2\}$. Then, $B_r(0) \supset \displaystyle\bigcup_{i=1}^{N} D_i$.

However, since A is unbounded $\exists\, a \in A$ where $|a| > r$ for any r. Then, $a \notin \displaystyle\bigcup_{i=1}^{M} D_i$ which is a contradiction. So, A cannot be of content zero.

(b) Consider the natural numbers $\mathbb{N}$. Clearly $\mathbb{N}$ is not of content zero because it is not bounded.

Given $\varepsilon > 0$, consider the sequence of closed rectangles $D_i = [n - \frac{\varepsilon}{2^{i+2}}, n + \frac{\varepsilon}{2^{i+2}}]$.

The natural number n is contained in D_n and $\displaystyle\sum_{i=1}^{\infty} v(D_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2} < \varepsilon$. Hence, $\mathbb{N}$ is of measure zero but not of content zero.