$[\implies]$. C is compact, therefore every open cover $\{U\}_{\alpha\in I}$ has a finite subcover $\{U\}_{\alpha\in I'}$. Let $T = \bigcup_{\alpha\in I'} U_\alpha$. By definition of a subcover, $C \subset T$. If $\{U\}_{\alpha\in I}$ is closed under union of pairs, then $T \in \{U\}_{\alpha\in I}$.

$[\impliedby]$ Consider the contrapositive case. If a set C is not compact, then there is at least one open cover that is closed under unions without a set T such that $C \subset T$.

Now, consider $C = \mathbb{R}^+$, the positive real numbers, and the cover $U = \{n \in \mathbb{Z}^+ : (0, n)\}$. U is closed under unions because given any $n, m \in \mathbb{Z}^+, (0, n) \cup (0, m) = (0, \max\{n, m\})$ and $\max\{n, m\} \in \mathbb{Z}^+$.

Assume there is a set $T \in U$ s.t. $C \subset T$. As $T \in U, T = (0, M)$ for some $M \in \mathbb{Z}^+$. However, consider the number $b = M + \frac{1}{3} \in \mathbb{R}^+$. Clearly, $b \notin T$. Thus, there cannot be a set T in this cover where $C \subset T$.