MAT257 PSET 7—Question 2

Jonah Chen

November 19, 2021

Given an integrable function $f : A \to \mathbb{R}$, any partition P of A, and any subrectangle $S \in P$, and some $\varepsilon > 0$.

Let P' be a partition of A where $U(f, P') - L(f, P') < \varepsilon$.

Let Q be the refinement of P and P' and $Q' = \{R \in Q : R \subset S\} \subset Q$ be a partition of S. As P is used to refine Q,

$$\sum_{R \in Q'} v(R) = v(S)$$

Then, $U(f, Q) - L(f, Q) \le U(f, P') - L(f, P') < \dfrac{\varepsilon}{v(S)}$.

$$\varepsilon > U(f, Q) - L(f, Q) = \sum_{R \in Q} v(R)[M_R(f) - m_R(f)] \ge \sum_{R \in Q'} v(R)[M_R(f) - m_R(f)] = U(f|_S, Q') - L(f|_S, Q')$$

Hence, $f|_S$ is integrable on S.

To prove the other direction, for any partition P of A and $f|_S$ is integrable for any subrectangle $S \in P$. Meaning there is some partition P_S of each subrectangle and $\varepsilon > 0$ where $U(f|_S, P_S) - L(f|_S, P_S) < \varepsilon / v(S)$.

Let P' be a partition of A that is the refinement of all $\{P_S : S \in P\}$. Then,

$$U(f|_S, P') - L(f|_S, P') \le U(f|_S, P_S) - L(f|_S, P_S) < \varepsilon \frac{v(S)}{v(R)}$$

. As P' refines all P_S, each rectangle of P' is contained in a rectangle of some partition P_S. So,

$$\begin{aligned}
U(f, P') - L(f, P') &= \sum_{R \in P'} v(R)[M_R(f) - m_R(f)] \\
&= \sum_{S \in P} \sum_{R \in P_S} v(R)[M_R(f) - m_R(f)] \\
&\le \sum_{S \in P} U(f|_S, P_S) - L(f|_S, P_S) \\
&\le \sum_{S \in P} \varepsilon \frac{v(S)}{v(R)} = \varepsilon
\end{aligned}$$

Hence, f is integrable on A.

Given that P' refines all of P_S, then

$$\sum_{S \in P} \int_S f|_S = \sum_{S \in P} \sup_{P_S} L(f|_S, P_S) \le \sup_{P'} L(f, P') \le \inf_{P'} U(f, P') \le \sum_{S \in P} \inf_{P_S} U(f|_S, P_S) = \sum_{S \in P} \int_S f|_S$$

So, $\sup\limits_{P'} L(f, P') = \inf\limits_{P'} U(f, P') = \displaystyle\int_A f = \sum_{S \in P} \int_S f|_S$.