If $A \subset \mathbb{R}^n$ is not closed, then $A^C$ is not open. Thus, $\exists y \in A^C$ where each open ball containing $y$ intersects $A$. Define $f : A \to \mathbb{R}$ where $f(x) = \dfrac{1}{|x-y|}$.

First, we show $f$ is unbounded. By way of contradiction, assume there is an upper bound $M$ s.t. $f(x) \leq M \, \forall x \in A$.

Consider the open ball of radius $\dfrac{1}{M}$ about $y$: $B = \{x \in \mathbb{R}^n : |x-y| < \dfrac{1}{M}\}$. Let $S = B \cap A$. $S$ is not empty because its an open ball containing $y$; thus, it must intersect $A$. Take any point $s \in S$. $f(s) = \dfrac{1}{|s-y|} > M$. This contradicts that $f(x)$ is bounded above by $M$. Thus, $f$ must be unbounded.

Next, we show $f$ is continuous. It suffices to show that the preimage of every open interval on $\mathbb{R}$ is open in $A$, as every open set on $\mathbb{R}$ is the union of open intervals.

Consider the interval $D = (a, b) \subset \mathbb{R}$. Then,

$$f^{-1}(D) = \{x \in A : a < \frac{1}{|x-y|} < b\} = \underbrace{\{x \in A : \frac{1}{|x-y|} > a\}}_{D_1} \cap \underbrace{\{x \in A : \frac{1}{|x-y|} < b\}}_{D_2}$$

Note that if $a \leq 0 \implies D_1 = \emptyset$ and $b \leq 0 \implies D_2 = A$, which are both open in $A$.

If $a > 0$, $D_1$ is the intersection between $A$ and open ball with radius $\dfrac{1}{a}$ about $y$, which is open in $A$ by definition.

If $b > 0$, $D_2$ is the intersection between $A$, and the complement of the closed ball of radius $\dfrac{1}{b}$ (i.e. $\{x \in \mathbb{R}^n : |x-y| \leq \dfrac{1}{b}\}$), which is an open set because its the complement of a closed set. Thus, $D_2$ is open in $A$.

Since for any $D \subset \mathbb{R}$, $D_1$ and $D_2$ are open in $A$, the intersection of two open sets is an open set. Thus, $D$ open in $\mathbb{R} \implies f^{-1}(D)$ open in $A$, thus $f$ is continuous.