MAT257 PSET 7—Question 4

Jonah Chen

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Lemma 1: Let A be a closed rectangle, if $f : A \to \mathbb{R}$ then

$$M_A(|f|) - m_A(|f|) \le M_A(f) - m_A(f)$$

Proof. Consider the case $M_A(f) \ge m_A(f) \ge 0$. Then, $|f| = f$ so $M_A(|f|) - m_A(|f|) = M_A(f) - m_A(f)$

Consider the case $0 \ge M_A(f) \ge m_A(f)$.
Then $|f| = -f$ so $M_A(|f|) - m_A(|f|) = -m_A(f) - (-M_A(f)) = M_A(f) - m_A(f)$

Consider the case $M_A(f) > 0 > m_A(f)$.
Then $m_A(|f|) = 0$ and $M_A(|f|) = \max\{M_A(f), -m_A(f)\}$ thus $M_A(|f|) - 0 < M_A(f) - m_A(f)$. □

If f is integrable, then $U(f, P) - L(f, P) < \varepsilon$ for any $\varepsilon > 0$. By lemma 1,

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{S \in P} v(S)[M_S(|f|) - m_S(|f|)] \\ &\le \sum_{S \in P} v(S)[M_S(f) - m_S(f)] = U(f, P) - L(f, P) \end{aligned}$$

Hence, $U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P) < \varepsilon$ so $|f|$ is integrable.

As it was shown (q3) that if f, g are integrable on A, $f \le g \implies \int_A f \le \int_A g$. Consider two cases:

If $\int_A f \ge 0$. As $f \le |f|$, $\left|\int_A f\right| = \int_A f \le \int_A |f|$.

If $\int_A f < 0$. As $-f \le |f|$, $\left|\int_A f\right| = -\int_A f = \int_A -f \le \int_A |f|$.

Hence, $\left|\int_A f\right| \le \int_A |f|$.