# Contents

# 1 What is an Electric Circuit

**Definition**: An electric circuit is an interconnection of circuit elements (incl. Conductors, semi-conductors, non-conductors).

# 2 Electrical Variables

To help us analyze electric circuits, we define several electrical variables:

## 2.1 Current

**Definition**: Electric currents: The "movement" or rate of change of electrical charge.

$$i \equiv \frac{dq}{dt} \tag{1}$$

1. S.I. unit: Ampere = Columb per second.
2. The current has a direction, and its direction is defined as the direction of positive charge. (Can be shown with $\leftarrow$ or $\rightarrow$)

- When performing circuit analysis, the direction of positive current is either given or we are required to guess a direction of positive current and verify it later.
- A positive current means the direction you guessed is correct. A negative current means the actual direction is the direction opposiing the current.

## 2.2 Voltage

**Definition**: Voltage: The energy required for 1C of charge between point A and B in a circuit is called the voltage between points A and B.

$$V \equiv \frac{dw}{dq} \tag{2}$$

1. S.I. unit: Volt = Joule per columb.
2. The voltage also have polarity (+ or $-$). Positive polarity means energy is consumed when the charge moves from A to B.

- If the polarity is not given, guess a polarity
- a positive voltage indicates the guess is correct and a negative voltage indicates the guess is incorrect.

## 2.3 Power

**Definition**: Power: The rate of delivering or absorbing energy.

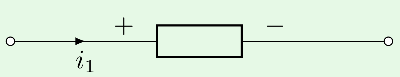
$$p \equiv \frac{dw}{dt} \tag{3}$$

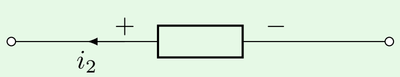
1. S.I. unit: Watt = Joule per second.
2. Using chain rule,

$$p = \frac{dw}{dt} = \frac{dw}{dq}\frac{dq}{dt} = iv \tag{4}$$

- **Note:** When a problem asks for a current or voltage, the direction/polarity **must** be indicated otherwise full credit will not be given.

**Definition**: For a pair of v and i, PSC holds if the current direction **enters** the positive side of voltage polarity. If PSC holds, $p = +vi$; else $p = -vi$



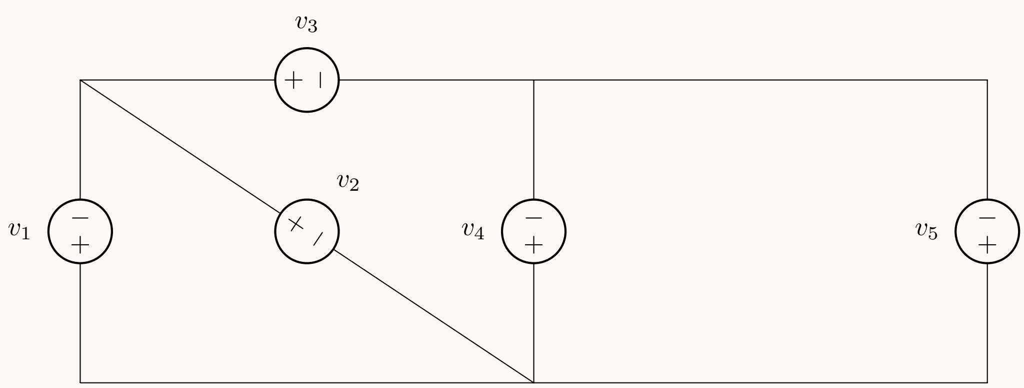
PSC holds on the left and does not hold on the right.

- The consequences of this convention are
  1. $p > 0 \implies$ power is absorbed
  2. $p < 0 \implies$ power is delivered (generated by the circuit element)

## 2.4 Examples

**Example 1 (1)**

Identify the devices that generate power in the circuit below and find the unknowns in the table.

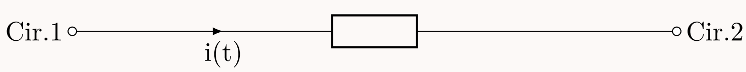



**Theorem: Conservation of Power** states that the algebraic sum of the power of all elements in a circuit is zero. (algebraic means the signs are preserved)

$$\sum p = 0 \tag{5}$$

**Proof**: Will not be presented at this point in the course.

**Example 2 (2)**




Given the above circuit and

$$v(t) = 50(1 - e^{-5000t})\mathrm{V} \tag{6}$$
$$i(t) = 10e^{-5000t}\mathrm{A} \tag{7}$$

Find the total energy transfered to this device after $t = 0$.

**Solution 1 (2):** PSC holds. Thus,

$$p(t) = v(t)i(t) \tag{8}$$
$$p(t) = 50(1 - e^{-5000t})10(e^{-5000t})\mathrm{W} \tag{9}$$

From the definition of power,

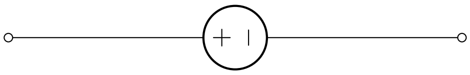
$$p(t) = \frac{dw}{dt} \therefore dw = p(t)dt \tag{10}$$
$$w = \int_0^\infty p(t)\mathrm{d}t = \int_0^\infty 50(1 - e^{-5000t})10(e^{-5000t})\mathrm{d}t = \frac{1}{20}\mathrm{J} = 50\mathrm{mJ} \tag{11}$$

# 3 Circuit Elements

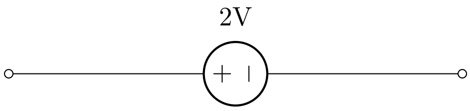
## 3.1 Independent sources

1. *Independent voltage sources:* A circuit element that has a specific voltage independent of the current that flows through it.
   - For example, the voltage can be a fixed voltage like $v_s = 2\text{V}$ or a variable voltage source like $v_s = 2\sin(50t + 2)$.
   - On diagrams, generic voltage source is shown on the left and fixed (DC) voltage sources is shown on the right. For DC sources, the uppercase letter $V$ is commonly used.


   - For DC source, the longer lines denotes positive voltage polarity and shorter lines denote negative voltage polarity.
   - An sinsudal voltage is drawn as

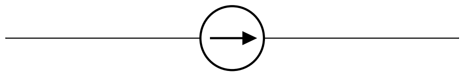



   - Questions
     (a) What is the direction of $i$ for the following voltage source?




     Answer: You cannot determine.
     (b) Does a voltage source always generate power?
     Answer: No. Take the voltage source above. Take a current $i_1 = 2\text{A} \rightarrow$. Then, PSC holds and $p = vi = (2\text{V})(2\text{A}) = 4\text{W}$. Since $p > 0$, the voltage source absorbs power.

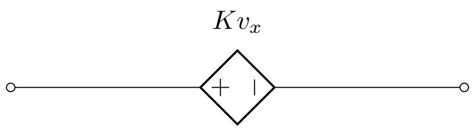
2. *Independent current source:* It gives a specific current independent of the voltage across it. Can be constant like $i_s = 5\text{A}$ or time dependent like $i_s = 10\cos(15t)$
   - The independent current source is drawn as



   - Questions
     (a) What is the polarity of voltage across a current source?
     (b) Does a current source always generate power?
   - Do **not** let the word *source* decieve you. A source is not associated with generating power.

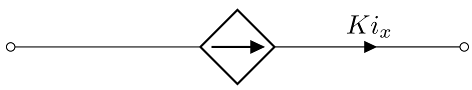
## 3.2 Dependent Sources

1. *Voltage-dependent voltage source:*




   The voltage of this source depends on voltages somewhere else in the circuit. i.e. $v_s = Kv_s$

2. *Current-dependent voltage source:*




   The voltage of this source depends on the current somewhere else in a circuit i.e. $v_s = ki_x$. Note k has the dimensions Current/Voltage.

3. *Voltage-dependent current source:*
4. *Current-dependent current source:*

### 3.3 Resistor

A resistor is a circuit element that keeps the ratio between the voltage and current constant.

$$R \equiv \frac{v}{i} \tag{12}$$

- S.I. unit: Volt per Amp or Ohm ($\Omega$).
- The resistor is drawn like this. If PSC holds, $v = Ri$ else $v = -Ri$. This is also known as Ohm's Law.

- The inverse of the resistence $R$ is called the conductance $G$

$$G \equiv \frac{1}{R} = \pm \frac{i}{v} \tag{13}$$

- The S.I. unit for conductance is 1/Ohm, also refered to as "mho" or "siemens" (Si).
- We could not find a generic relation for the power of a source, since $P = vi$ (assuming PSC holds). Since for a independent voltage source, we don't know the current and vise versa. For a resistor however, assume PSC holds,

$$P = vi = (Ri)i = Ri^2 \tag{14}$$

$$= v\left(\frac{v}{R}\right) = \frac{v^2}{R} \tag{15}$$

  If PSC doesn't hold,

$$P = -vi = -(-Ri)i = Ri^2 \tag{16}$$

$$= -v\left(-\frac{v}{R}\right) = \frac{v^2}{R} \tag{17}$$

  Thus, the power of a resistor is independent of whether or not PSC holds. For physical resistors, $R > 0$ hence $P > 0$ which means power is always absorbed.

### 3.4 Short Circuit

A short circuit is the limiting behavior for a resistor as $R \to 0$. Since $v = Ri$, the voltage is zero independent of the current. This is denoted with a solid line:

- Other names include zero ohm path or ideal conductor.
- **In analysis, all parts of a circuit that are connected using ideal conductor can be considered the same point in the circuit.**

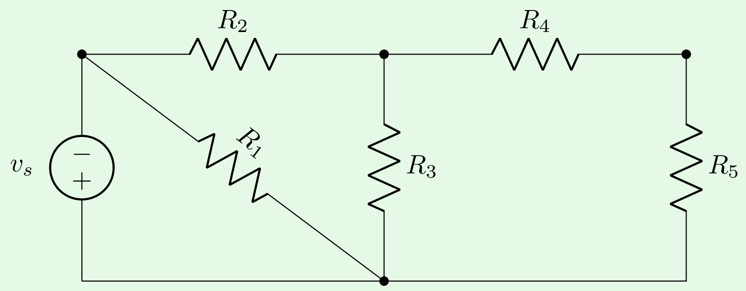
### 3.5 Open Circuit

The limiting behavior for a resistor as $R \to \infty$. Since $v = Ri$, the current is zero independent of the voltage.

For example, the air between the ground and transmission line can be treated like a open circuit, since air is not a conductor.

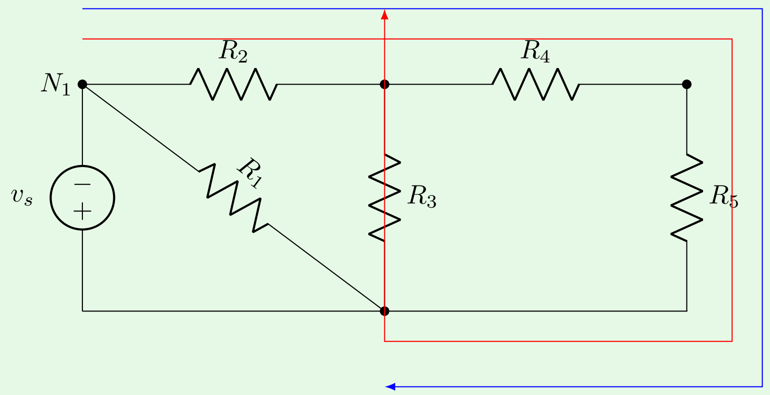
# 4 Circuit Analysis Definitions

**Definition**: A **Node** is a junction of two or more circuit elements.



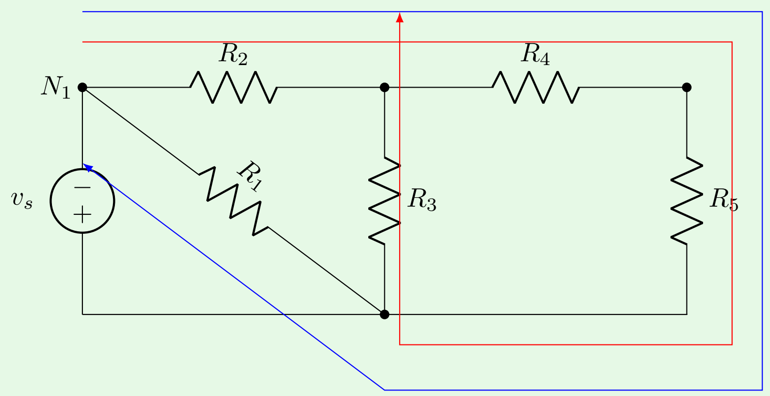

The labelled points are nodes: top-left connects $v_s$, $R_1$ and $R_2$; top-center connects $R_2$, $R_3$, and $R_4$; top-right connects $R_4$ and $R_5$; as well as the region at the bottom (that are connected by short circuit as the junction of $v_s$, $R_1$, $R_3$, $R_5$).

**Definition**: Start moving from one node towards the other nodes. As long as no node is passed (**unless** it is a loop when start and end nodes are the same) more than once, the set of nodes passed is called a **Path**.




The blue one is a path, the red one is not.

**Definition**: If the beginning and end of a path is one node, that path is a **loop**.



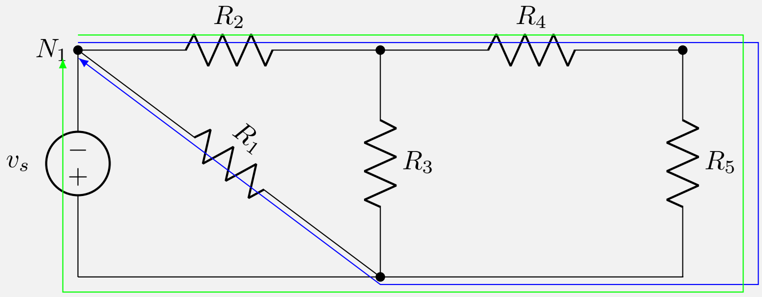

The blue one is a loop, the red one is not a loop.

## 4.1 Examples

**Example 3 (1)**

Identify the loops in the circuit above circuit

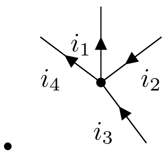
**Solution 2 (1):** Arrows are too hard to draw




# 5 Circuit Analysis Laws

## 5.1 Kirchoff's Current Law (KCL)

- KCL states that the algebraic sum of the currents entering a node is zero. This is based on the conservation of charge which will not be covered in this course.




- Define a sign convention for the algebraic sum. Either positive sign for current for entering the node or leaving the node is sufficient.
- For instance, define the sign convention as positive current entering the node. Then, KCL requires

$$i_1 - i_2 + i_3 - i_4 = 0 \tag{18}$$

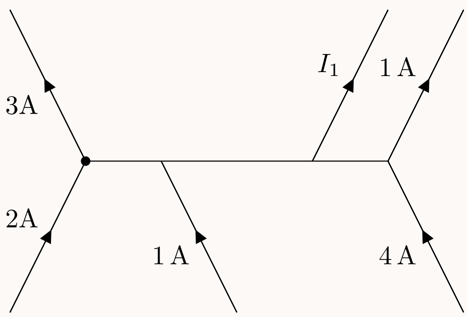
- An equivalent statement of KCL is: "The sum of the currents entering a node is equal to the sum of the currents leaving the node." This statement would require

$$i_2 + i_3 = i_1 + i_4 \tag{19}$$

which is equivalent to (18).

**Example 4 (2)**

Find $I_1$ in the following circuit

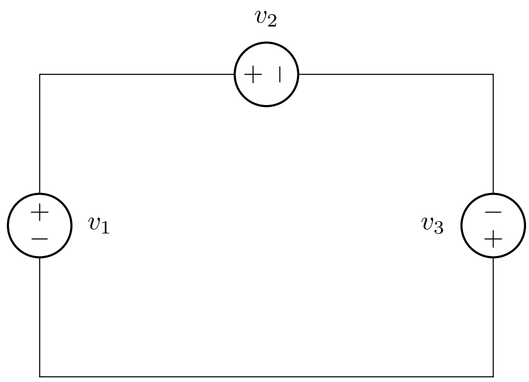



**Solution 3 (2):** All the currents are entering/leaving one node. KCL requires the currents entering and leaving the node to be equal. Therefore,

$$2 + 1 + 4 = 3 + I_1 + 1 \implies I_1 = 3\,\mathrm{A} \tag{20}$$

## 5.2 Kirchoff's Voltage Law (KVL)

- KVL states that the algebraic sum of the voltages around any loop is zero.
    1. If the loop direction enters a voltage source from positive voltage polarity, voltage used for KVL is positive.
    2. Else, the voltage used for KVL is negative.




- For the same loop, the starting point and direction of performing the analysis is irrelevent. i.e. starting from the top left in the clockwise direction, KVL requires

$$+v_2 - v_3 - v_1 = 0 \tag{21}$$

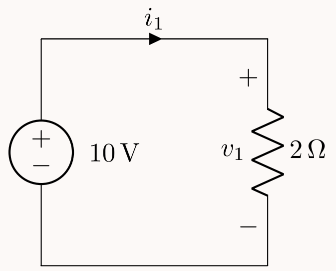
Starting from the top right in the anti-clockwise direction,

$$-v_2 + v_1 + v_3 = 0 \tag{22}$$

- When writing KVL for circuits with resistors, it can be combined with Ohm's law in the following fashion. Note that this is independent of voltage polarity defined for the resistor.
    1. When the current direction entering a resistor is the same as the loop direction, the voltage of the resistor for KVL is $+iR$
    2. When the current direction entering a resistor is opposing the loop direction, the voltage of the resistor for KVL is $-iR$

**Example 5 (1)**

Use KVL and Ohm's law to find the current of the resistor.




**Solution 4 (1):** Applying KVL in the clockwise direction,

$$v_1 - 10 = 0 \implies v_1 = 10\,\text{V} \tag{23}$$

Using Ohm's law. Since current enters the + polarity, PSC holds. Thus,

$$v_1 = 2i_1 \implies 10 = 2i_1 \implies i_1 = 5\,\text{A} \tag{24}$$

Using the "shortcut", again applying KVL in a clockwise direction. Since the current direction is the same as the loop direction,

$$+2i_1 - 10 = 0 \implies i_1 = 5\,\text{A} \tag{25}$$

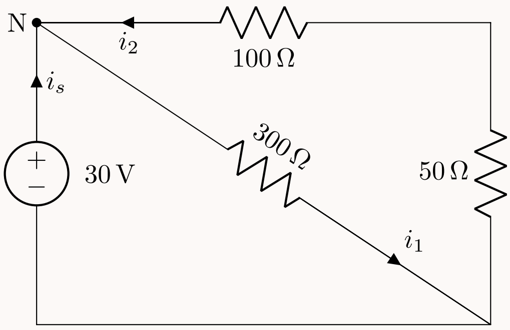
# 6 Circuit Analysis Tips

- If a circuit **does not** include a voltage-dependent current source or a current-dependent voltage source, you may use any of the following units systems to perform your analysis.
  1. V, A, Ω, W
  2. V, mA, kΩ, mW
  3. kV, mA, Ω, W
  4. Other consistent units can be used but are not as common.
- It is simpler to use a consistent system of units to analyze the circuits. Thus, units consistent with the voltage-dependent current source or a current-dependent voltage source should be used.

## 6.1 Examples

**Example 6 (2)**

Find the power of the voltage source.




**Solution 5 (2):** Since PSC does not hold for the voltage source,

$$P = -30i_s \tag{26}$$

KVL for the bottom left loop (clockwise)

$$-30 + 300i_1 = 0 \implies i_1 = 0.1\,\mathrm{A} \tag{27}$$

KVL for the outer loop (clockwise)

$$-30 - 100i_2 - 50i_2 = 0 \implies i_2 = -0.2\,\mathrm{A} \tag{28}$$

KCL at node N

$$i_s + i_2 = i_1 \implies i_s = i_1 - i_2 = 0.1 - (-0.2) = 0.3\,\mathrm{A} \tag{29}$$

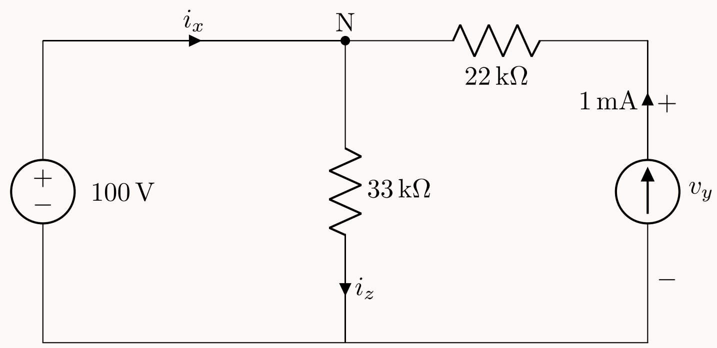
Find the power

$$P = -30i_s = -30(0.3) = -9\,\mathrm{W} \tag{30}$$

Since $P < 0$, power is generated.

**Example 7 (3)**

Find the power of each source.




**Solution 6 (3):** KVL for the left loop (clockwise)

$$-100 + 33i_z = 0 \implies i_z = 3\,\mathrm{mA} \tag{31}$$

KCL for N, $i_x$ and $1\,\mathrm{mA}$ are entering N and $i_z = 3\,\mathrm{mA}$ is leaving N.

$$i_x + 1 = i_z \implies i_x = 2\,\mathrm{mA} \tag{32}$$

For the voltage source, PSC does not hold. Thus,

$$P = -vi = -100 \times 2 = -200\,\mathrm{mW} \tag{33}$$

KVL for outer loop (clockwise)

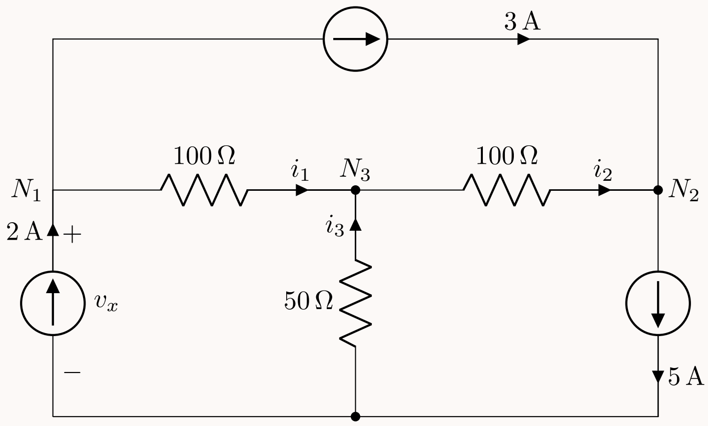
$$-100 - (1 \times 22) + v_y = 0 \implies v_y = 122\,\mathrm{V} \tag{34}$$

For the current source, PSC does not hold. Thus,

$$P = -vi = -122 \times 1 = -122\,\mathrm{mW} \tag{35}$$

**Example 8 (4)**

Find the power of each source.




**Solution 7 (4):** KCL at $N_1$

$$2 = 3 + i_1 \implies i_1 = -1\,\mathrm{A} \tag{36}$$

KCL at $N_2$

$$3 + i_2 = 5 \implies i_2 = 2\,\mathrm{A} \tag{37}$$

KCL at $N_2$

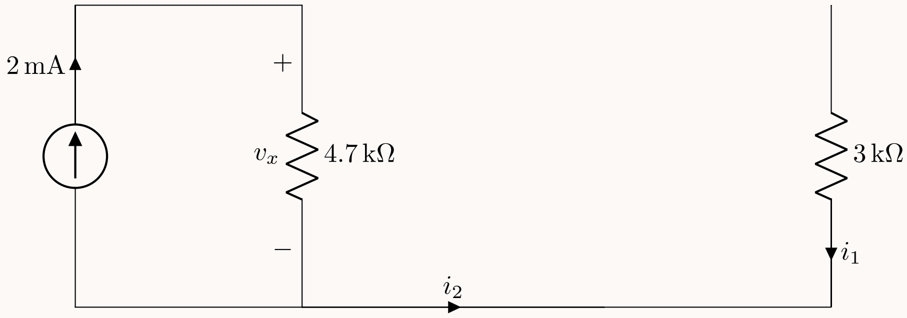
$$i_1 + i_3 = i_2 \implies i_3 = 3\,\mathrm{A} \tag{38}$$

KVL for the bottom left loop (clockwise)

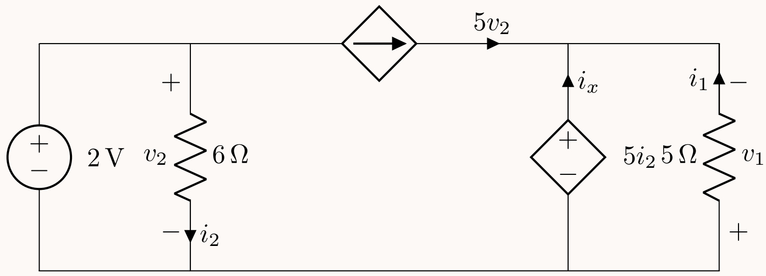
$$-v_x + 100i_1 - 50i_3 = 0 \implies v_x = 100 \times -1 - 50 \times 3 = -250\,\mathrm{V} \tag{39}$$

The rest is left as an exercise.

**Example 9 (5)**




**Example 10 (6)**




**Solution 8 (6):** Use KVL on the left most loop (clockwise).

$$-2 + 6i_2 = 0 \implies i_2 = 0.33\,\text{A} \tag{40}$$

Using Ohm's law on the resistor, PSC holds thus

$$v_2 = 6i_2 = 2\,\text{V} \tag{41}$$

The current provided by the dependent current source would be 10 A and the voltage provided by the dependent voltage source would be 1.67 V.
Using KVL on the right most loop (anti-clockwise),

$$v_1 + 5i_2 = 0 \implies v_1 = -5i_2 = -1.67\,\text{V} \tag{42}$$

Using Ohm's law on this resistor to find $i_1$. As PSC holds,

$$i_1 = v_1/5 = 0.33\,\text{A} \tag{43}$$

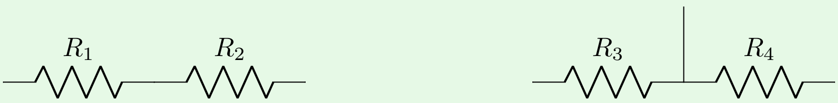
Finally using KCL at the junction,

$$i_x + i_1 + 5v_2 = 0 \implies i_x = -10.33\,\text{A} \tag{44}$$

# 7 Equivalent circuits for parallel and series resistors

## 7.1 Series Connected Resistors

**Definition**: Series Connection: Two circuit elements are connect in series if and only if they are connected back to back, and at their point of connection, there is no other current path.



$R_1$ and $R_2$ are connected in series. $R_3$ and $R_4$ are also connected in series since the path between the two resistence is a open circuit, thus the current is zero so it is not a current path.

**Theorem**: For resistors in series, the equivalent resistence is

$$R_{eq} = \sum_k R_k \tag{45}$$

**Proof**: Take a circuit with 2 resistors in series.
Using KVL (clockwise),

$$-v_{tot} + v_1 + v_2 = 0 \tag{46}$$

Using Ohm's law,

$$v_1 = R_1 i_{tot} \tag{47}$$

$$v_2 = R_2 i_{tot} \tag{48}$$

Combining the results,

$$v_{tot} = v_1 + v_2 = (R_1 + R_2) i_{tot} \tag{49}$$

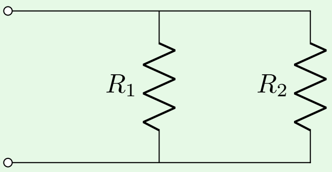
For an equvilent resistor from Ohm's law,

$$v_{tot} = R_{eq} i_{tot} \implies R_{eq} = R_1 + R_2 \tag{50}$$

From induction, the equivalent resistence of any series connected resistors is equal to the sum of their resistences.

## 7.2 Parellel Connected Resistors

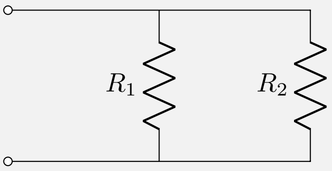
**Definition**: Parallel connection: Two circuit elements are connected in parallel if they share two common nodes.




**Theorem**: For parellel resistors, the equivalent resistence is

$$R_{eq} = \left[\sum_k \frac{1}{R_k}\right]^{-1} \tag{51}$$

**Proof**:




Write KCL at N

$$i_{tot} = i_1 + i_2 \tag{52}$$

Using KVL

$$-v_{tot} + v_1 = 0 \implies v_{tot} = v_1 = v_2 \tag{53}$$

Using Ohm's law, PSC holds

$$i_{tot} = \frac{v_1}{R_1} + \frac{v_2}{R_2} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) v_{tot} \tag{54}$$

Using ohm's law for the equivalent resistor,

$$i_{tot} = \frac{v_{tot}}{R_{eq}} \implies \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \implies R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \tag{55}$$

Consider two resistors in parellel with $R_2 = 0$. Thus,

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} = \frac{0}{R_1} = 0 \tag{56}$$

You can consider that the current is "sane". When given an option to flow with no resistence, it will take that path. Thus, if $R_2 = 0$, no current will flow through $R_1$ and the equivalent resistence is 0

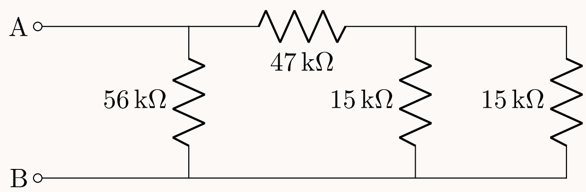
**Corollary:** Recall the definition of conductance is $G = 1/R$. Thus,

$$G_{eq} = \sum_k G_k \tag{57}$$

## 7.3 Examples

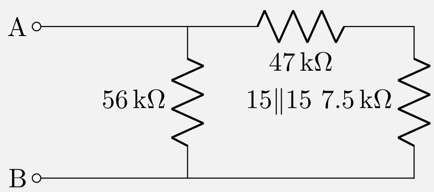
**Example 11 (1)**

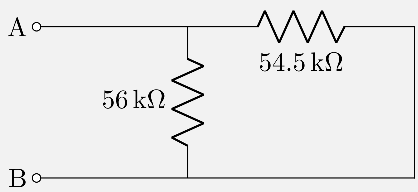
Find the equivalent resistence between A and B




**Solution 9 (1):** Simplify the circuit in several steps (starting from the right)

1. The two right most resistors are correct in parellel.




2.

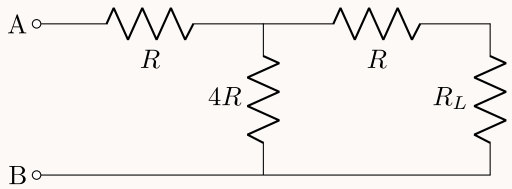
3.

$$R_{eq} = 56 \| 54.5 = \frac{56 \times 54.5}{56 + 54.5} = 27.62\,\mathrm{k\Omega} \tag{58}$$

Note that in the original circuit, the 15 kΩ resistors are **not** connected in series to the 47 kΩ resistor as there exist (one) current path between them.

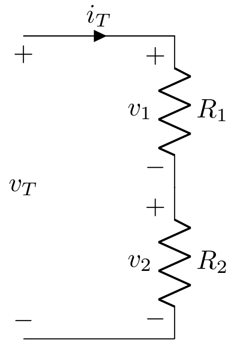
**Example 12 (2)**

Select $R$ such that $R_{AB} = R_L$




# 8 Voltage Division

Suppose there are two resistor connected in series. With voltage through both as $v_T$ and $i_T$.




Then the voltage division principle states

$$v_1 = \frac{R_1}{R_1 + R_2} v_T \tag{59}$$

In general for $N$ series connected resistors,

$$v_i = \frac{R_i}{\sum_k R_k} v_T \tag{60}$$

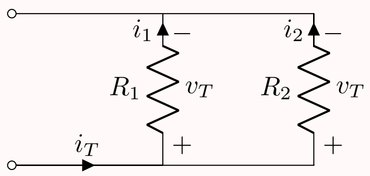
Note the polarity of the voltage $v_i$ **must** be the same as $v_T$, otherwise the $-$ sign must be introduced.

As a consequence, the resistor with the higher resistence will have the higher voltage.

**Proof**: The equivalent resistence of the two resistors is $R_T = R_1 + R_2$. Using Ohm's law, $v_T = i_T(R_1 + R_2)$.

# 9 Current Division

Derivation: Suppose there are two resistors connected in parellel.




Use Ohm's Law

$$v_T = R_1 i_1 \tag{61}$$

$$v_T = \frac{R_1 R_2}{R_1 + R_2} \tag{62}$$

Since the LHS is the same, the RHS must be the same as well. Thus,

$$i_1 = \frac{R_2}{R_1 + R_2} i_T \tag{63}$$

Equation (63) is known as the current division principle. This can also be expressed in terms of conductances

$$i_1 = \frac{G_1}{G_1 + G_2} i_T \tag{64}$$

For the general case of $N$ parellel resistors, it is possible to use the equation with the conductances similar to the voltage division principle.

$$i_i = \frac{G_i}{\sum_k G_k} i_T \tag{65}$$

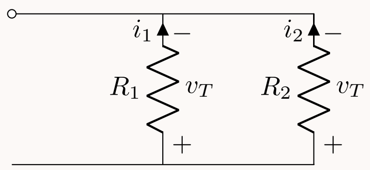
It is possible to write the other resistors as an equivalent resistor with resistence $R'_{eq}$, then the current at the resistor of interest $R$ is

$$i = \frac{R'_{eq}}{R + R'_{eq}} i_T \tag{66}$$

The directions of $i_T$ and $i_1$ **must** be similar. Otherwise, the $-$ sign must be introduced.

**Example 13 ()**

Find $v_0$ and $i_0$




# 10 Nodal Analysis

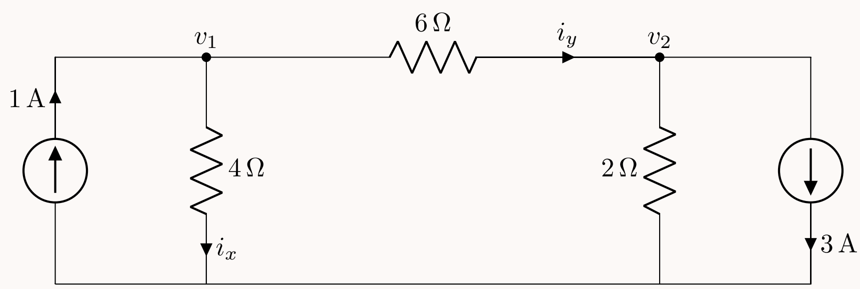
- An algorithmic approcah to solve circuits.
- Objective: Find node voltages.
- Methodology: KCL for all the nod in terms of the **node voltage**: The voltage with respect to a reference node or ground node.
- The common convention is to assume a negative sign for current entering node and positive sign for current leaving node. It is also customary to write the current leaving the node at a resistor.

**Definition**:

1. $V_{AB}$ is the voltage with the $+$ polarity at point A and $-$ polarity at point B.
2. $V_A$ is the voltage with the $+$ polarity at point A and $-$ at the reference node or ground.
3. The voltage of the ground node is equal to zero.

**Example 14 (1)**

Perform nodal analysis on the following circuit.



**Solution 10 (1):** Using ohm's law, $i_x = v_1/4$, $i_y = (v_1 - v_2)/6$. (labelled for analysis of node 1)
Execute KCL for node 1

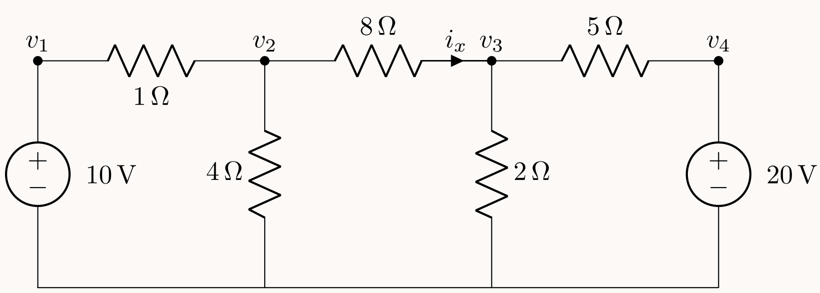
$$-1 + \frac{v_1}{4} + \frac{v_1 - v_2}{6} = 0 \tag{67}$$

Execute KCL for node 2. It is recommended to start fresh for each node.

$$+3 + \frac{v_2}{2} + \frac{v_2 - v_1}{6} = 0 \tag{68}$$

Now there are two equations for two unknowns. This can be solved using computer to obtain $v_1 = 2/3\,\mathrm{V}, v_2 = -13/2\,\mathrm{V}$. Note that $v_1$ represents the voltage measured by using a voltmeter with the positive terminal of node 1 and negative terminal at the ground.
The power of nodal analysis is that now it is possible to easily find the voltage between any two points in the circuit. For example, the voltage of the $3\,\mathrm{A}$ is just $v_2 - 0 = -13/2\,\mathrm{A}$

**Example 15 (2)**




**Solution 11 ():** In this circuit, there are voltage sources! Hence, we do not know the current of the at certain nodes thus writing KCL writing there would not be productive. However, the voltage at nodes 1 and 4 are actually known!

$$v_1 = +10\,\mathrm{V} \tag{69}$$
$$v_2 = +20\,\mathrm{V} \tag{70}$$

**Note** to pay attention to the direction the voltage sources are oriented.
KCL for node 2

$$\frac{v_2 - v_1}{1} + \frac{v_2 - 0}{4} + \frac{v_2 - v_3}{8} = 0 \tag{71}$$

KCL at node 3

$$\frac{v_3 - v_4}{5} + \frac{v_3 + 0}{2} + \frac{v_3 - v_2}{8} = 0 \tag{72}$$

It may seem like there are too many unknowns, but $v_1$ and $v_2$ are already known. Hence, only $v_2 = 7.82\,\mathrm{V}, v_3 = 6.03\,\mathrm{V}$ needs to be found.
To find $i_x$, use the ohm's law

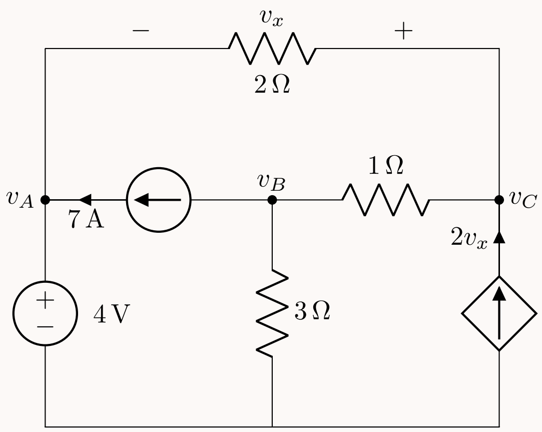
$$i_x = \frac{v_2 - v_3}{8} = 0.223\,\mathrm{A} \tag{73}$$

## 10.1 Circuits with Dependent Sources

- Before writing the KCL for dependent source, first write the parameters it depends on in terms of the *node voltages*.

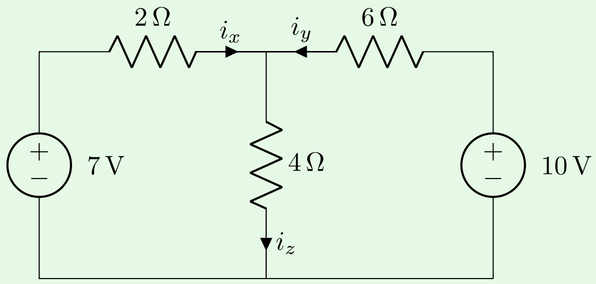
**Example 16 ()**

Perform nodal analysis on the circuit.




## 11 Mesh Analysis

**Definition**: A **Mesh** is a loop that does not include any other loops inside it.




1. A **mesh current** is a hypothetical circulating current about eash mesh. Note this is **not** a real current, just a variable for circuit analysis.
2. A **branch current** is a physical current that flows through a branch.

- Branch currents can be expressed in terms of mesh currents. Note that if the mesh current is in the same direction as the branch current, it can be added directly otherwise a negative sign must be included.
- The $2\,\Omega$ resistor only belongs to the mesh with mesh current $i_1$. Thus, $i_x = i_1$
- The $6\,\Omega$ resistor only belongs to the mesh with mesh current $i_2$. However, the $i_y$ is in the opposite direction as the mesh current; thus, $i_y = -i_2$
- The $4\,\Omega$ resistor belongs to both meshes. $i_z$ is in the same direction as the mesh current $i_1$ but opposite direction of the mesh current $i_2$; thus, $i_z = i_1 - i_2$.

- Mesh analysis is a different circuit analysis technique that is generally more efficient for circuits with lots of series connections.
- The objective for mesh analysis is to find all the mesh currents.
- Methodology for mesh analysis:
  1. Identify all the meshes in the circuit.
  2. Associate a mesh current with each mesh. It is conventional to choose a clockwise direction for all the mesh current.
  3. If there are dependent sources, identify the parameters they depend on based on the mesh currents.
  4. Identify current sources and the meshes they effect.
     (a) If at the periphery (part of only one mesh), the current of that mesh is determined.

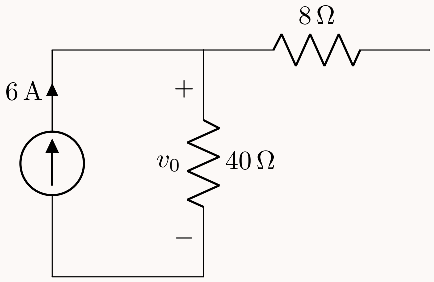
(b) If not at the periphery, the current sources should provide 1 equation. The other equation will come from KVL of a larger loop (*supermesh*) including to meshes the current source effects.

5. Write KVL for all the meshes in terms of the mesh current. It is conventional to write KVL in the clockwise direction.

- For mesh 1:
$$-7 + 2i_1 + 4(i_1 - i_2) = 0 \tag{74}$$
For mesh 2:
$$4(i_2 - i_1) + 6i_2 + 10 = 0 \tag{75}$$
Solving the equations, $i_1 = 5/22\,\mathrm{A}, i_2 = 29/22\,\mathrm{A}$. The branch currents can be found.

- For meshes with current sources, KVL can no longer be written for those meshes. However, knowing that the current at the current source is constant, they provide a constraint for the mesh currents; hence, the circuit is still determined.

**Example 17 ()**

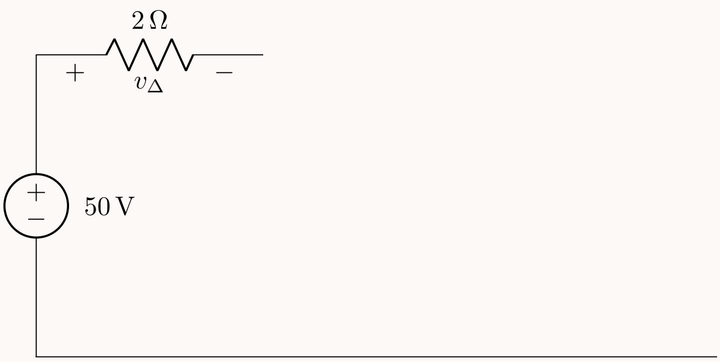



## 11.1 Circuits with Dependent Sources

- For analysis with dependent sources, the parameters the dependent sources depend on must be written first in terms of mesh current.

**Example 18 ()**

Use mesh analysis to find which sources are generating power.




**Solution 12 ():** First, write parameter the dependent sources depend on in terms of mesh currents.

$$v_\Delta = +2i_1 \tag{76}$$
$$i_\Delta = +i_2 \tag{77}$$

A dependent current source is on the periphery of the circuit. Thus, it is unhelpful to write KVL for that mesh as the current of that mesh is already known, as

$$i_3 = -1.7(2i_1) = 3.4i_1 \tag{78}$$

Now, KVL can be written for mesh 1 and mesh 2 respectively

$$0 = -50 + 2i_1 + 4(i_1 - i_2) + 9(i_2) \tag{79}$$
$$0 = -9(i_2) + 4(i_2 - i_1) + 5i_2 + 20(i_2 - i_3) \tag{80}$$

Solving the equation to obtain $i[\,] = \{-5, 16, 17\}$A. Then, $v_\Delta = -10\,\mathrm{V}, i_\Delta = 16\,\mathrm{A}$ For the $50\,\mathrm{V}$ independent source,

$$P = -50i_1 = -50(-5) = 250\,\mathrm{W} \tag{81}$$

For the $9i_\Delta$ dependent source,

$$P = -(i_2 - i_1)(9i_\Delta) = -3024\,\mathrm{W} \tag{82}$$

For the $1.7v_\Delta$ current source, firstly find the KVL for mesh 3 to find the voltage across the current source.

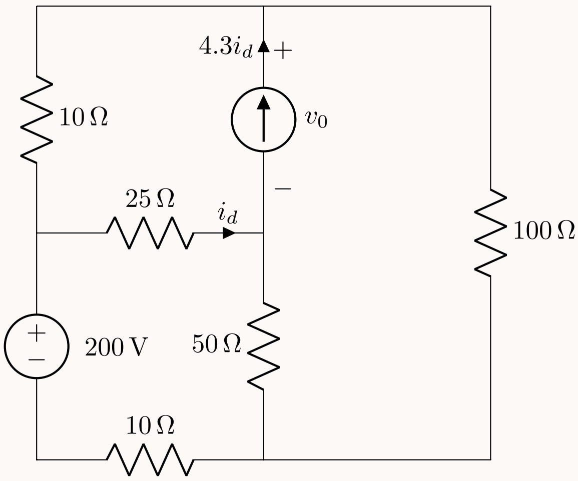
$$+20(i_3 - i_2) + v_y = 0 \implies v_y = -20\,\mathrm{V} \tag{83}$$

$$P = -v_y(1.7v_\Delta) = -340\,\mathrm{W} \tag{84}$$

Thus, only the dependent voltage source and dependent current sources are generating power.

**Example 19 ()**

Use mesh analysis to find $v_0$.




**Solution 13 ():** Firstly find $i_d = i_1 - i_2$ as it is a parameter for a dependent source. Then write KVL for mesh 1.

$$0 = 10i_1 - 200 + 25i_1 + 50i_1 \tag{85}$$

This current source is not at the periphery of the circuit as it is part of more than one mesh. For the current source,

$$4.3i_d = 4.3(i_1 - i_2) = i_3 - i_2 \tag{86}$$

Write KVL for the "supermesh".

$$0 = +50(i_3 - i_1) + 25(i_2 - i_1) + 10i_2 + 100i_3 \tag{87}$$

$i[\,] = \{4.6, 5.7, 0.97\}$A. Mesh analysis is complete. Finding $v_0$ is trivial by writing KVL for mesh 2.

$$0 = v_0 + 25(i_2 - i_1) + 10i_2 = 0 \implies v_0 = -84.5\,\mathrm{V} \tag{88}$$

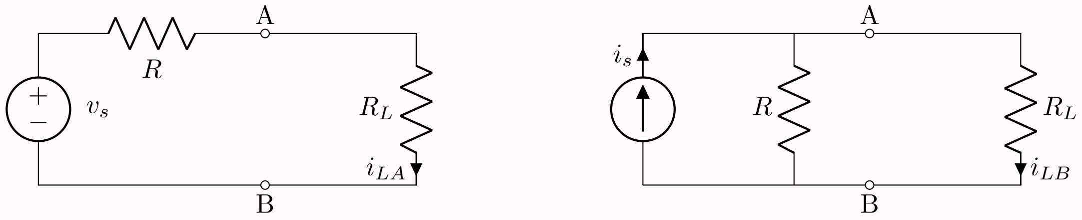
## 12 Source Transformation

- A voltage source in series with a resistor is equivalent to a current source in parellel with the same resistor. One can be transformed to the other using source transformation with magnitude

$$v_s = Ri_s \tag{89}$$

- The direction of $i_s$ and $v_s$ do **not** conform to PSC.

Derivation: Consider the following two circuits.




Using Ohm's law on for the circuit on the left and current division for the circuit on the right,

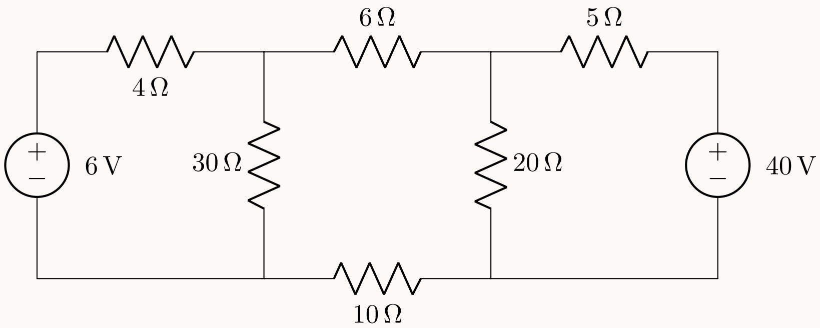
$$i_{LA} = \frac{v_s}{R + R_L} \tag{90}$$

$$i_{LB} = i_s \frac{R}{R + R_L} \tag{91}$$

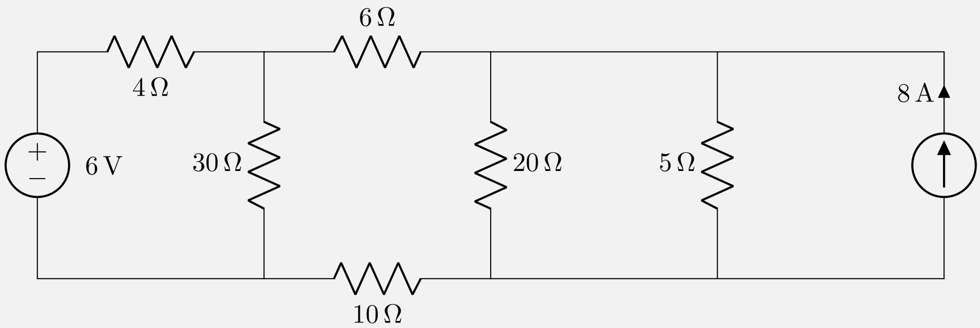
For the two sources to be equivalent, $i_{LA} = i_{LB}$. Thus, $v_s = Ri_s$. Note that $v_s$ and $i_s$ does not conform to PSC, as noted above. As $R_L$ cancels out in the expression, the two circuits are equivalent regardless of what is connected to terminals A and B.

**Example 20 (1)**

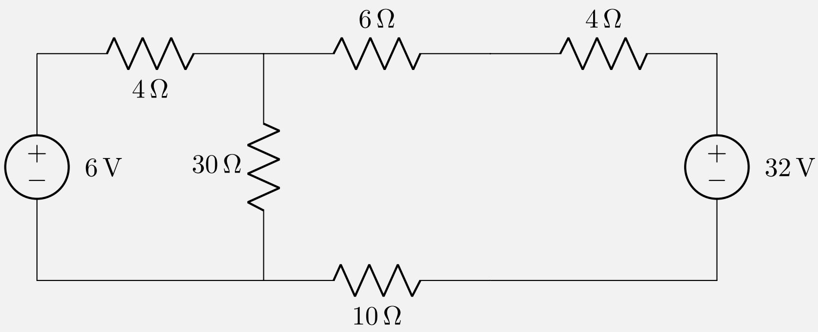
Use the Source Transformation to find the power of the 6 V voltage source.



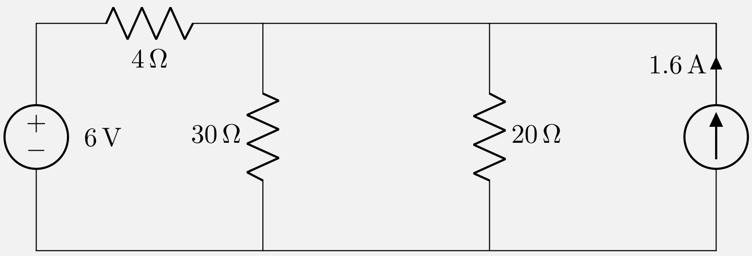

**Solution 14 (1):** First, simplify the circuit that is connected to the 6 V voltage source. Firstly, simplify the series connection of 40 V voltage source and 5 Ω resistor with a parellel connection of a current source and resistor.



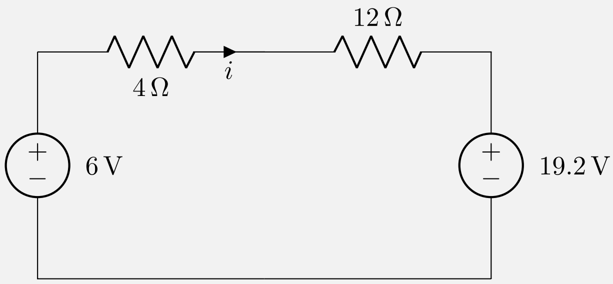

Now, the 5 Ω resistor is connected in parellel to 20 Ω resistor. The equivalent resistence is 4 Ω. Then, the current source connected parellel to the 4 Ω resistor can be transformed by a voltage source connected series to the 4 Ω resistor.




Now the 6 Ω resistor, 4 Ω resistor, 32 V voltage source, 10 Ω resistor are connected in series. Because the order of the circuit elements connected in series are not relevent to the terminals, the three resistors can be combined to a equivalent resistor of 20 Ω. Now the source transformation can be applied




The 30 Ω and 20 Ω resistors are connected in parellel. Using a source transformation the circuit can be simplified to a simple circuit.



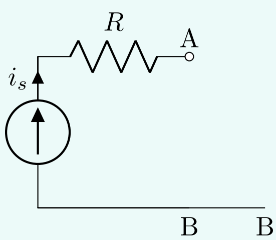

Now writing KVL clockwise for the circuit,

$$-6 + 4i + 12i + 19.2 = 0 \implies i = -0.825\,\mathrm{A} \tag{92}$$

Since the current direction does not conform to PSC for the 6 V source,

$$P = -vi = 4.95\,\mathrm{W} \tag{93}$$

**Corollary**: For resistors connected in parellel with a voltage source or in series with a current source can be eliminated from the circuit. This is done by opening the resistor for parellel connection and shorting the resistor for series connection.




## 13 Thevenin and Norton Equivilant Circuit

Just as an interconnection of resistors can be replaced by a equivalent resistence, an interconnection of resistors and sources can be replaced by an equivalent circuit

## 13.1 Thevenin equivalent circuit

**Theorem:** For any linear electrical network containing only voltage sources, current sources and resistances can be replaced at terminals A-B by an equivalent combination of a voltage source $V_{th}$ in a series connection with a resistance $R_{th}$.

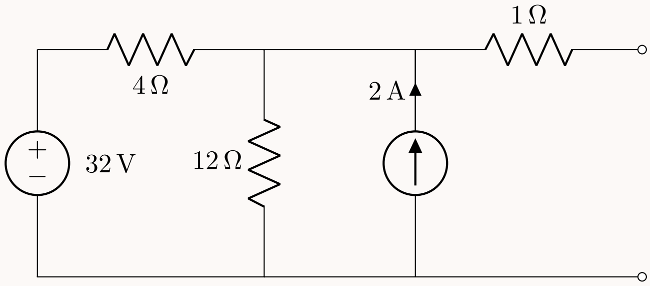
- The Thevenin voltage $V_{th}$ is the open circuit voltage between A and B in the original circuit.
- There are three different methods of finding the Thevenin resistence $R_{th}$.

### 13.1.1 Method 1: For circuits without dependent sources

1. Deactivate all the independent sources in the circuit.
   - Deactivate a voltage source by shorting it.
   - Deactivate a current source by opening it.
2. Find the equivalent resistance, which would be the Thevenin resistance.

**Example 21 ()**

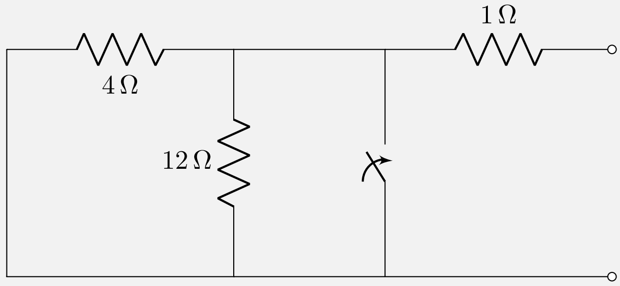
Find the thevenin equivalent of the circuit connected to $R_L$




**Solution 15 ():** Firstly find the open circuit voltage using nodal analysis.

$$\frac{v_1 - 32}{4} \tag{94}$$

To find $R_{th}$, deactivate all the sources




Now the $4\,\Omega$ resistor and $12\,\Omega$ resistor are connected in parellel. Thus, the equivalent resistence in the circuit is $4\,\Omega$, which is the Thevenin resistance.

## 13.2 Norton Equivalent Circuit

- Norton equivalent circuit is the dual of the Thevenin equivalent circuit with a resistor of resistance $R_N$ and current source with current $I_N$ connected in parellel.

- To find the equivalent resistance, $R_N = R_{th}$
- There are two methods to find the Norton current $I_N$
  1. $I_N$ is equal to the short circuit current between the two terminals. The direction of the Norton current must be consistent with the direction of $i_{sc}$.
  2. Perform source transformation on Thevenin equivalent circuit; thus $I_N = V_{th}/R_{th}$ with the direction of the current source not conforming to PSC of the voltage source (in Thevenin equivilant circuit).
- Note that you **cannot** use the same circuit analysis for short circuit current and open circuit voltage.

**Example 22 ()**

Find the Norton equivilant circuit for previous example

**Solution 16 ():** KVL @1

$$\frac{v_1 - 32}{4} + \frac{v_1}{12} - 2 + \frac{v_1}{1} = 0 \implies v_1 = 7.5\,\mathrm{V} \tag{95}$$

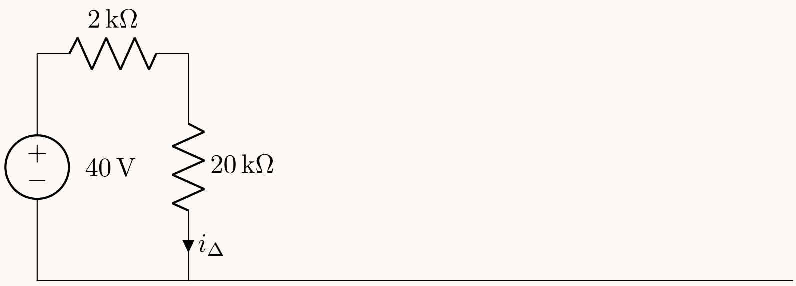
The short circuit current is the same as the current in the $1\,\Omega$ resistor. Thus, $i_{sc} = v_1/1 = 7.5\,\mathrm{A}$.

### 13.2.1 Method 2: For circuits that includes at least one independent source

$$R_{th} = \frac{v_{oc}}{i_{sc}} \tag{96}$$

Where $v_{oc}$ and $i_{sc}$ **must** conform to PSC.

**Example 23 ()**




## 13.3 Method 3: Most general method

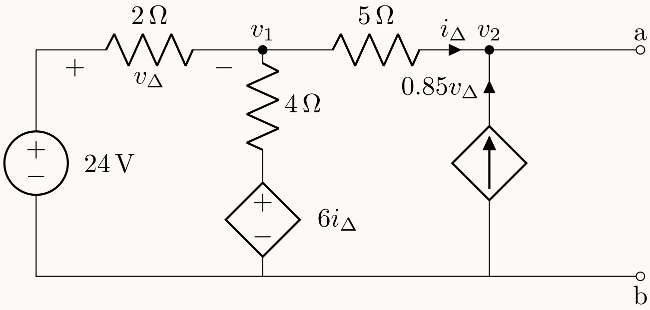
- If there are no independent source in the circuit, the circuit is not energized; thus, $v_{oc} = 0, i_{sc} = 0$. This calls for a new method.
  1. Deactivate all independent sources.
  2. Connect a test current source between the terminals you want to find the Thevenin resistance, terminals A and B. For simplicity, common practice is using the test current source of 1 A.
  3. Find the voltage across the test current source using any circuit analysis technique.

$$R_{th} = \frac{V_T}{I_T} \tag{97}$$

  Note that PSC must **not** hold for $V_T$ and $I_T$.
- Note that the test source is only added to the circuit for evaluating $R_{th}$ and must be disconnected after the calculation.
- Note that some texts add a test voltage source instead of a test current source. This method also works but you may have to do the long division.

**Example 24 ()**

Find the Thevenin equivilent of the circuit connected to $R_L$




**Solution 17 ():** Using nodal analysis, first write the dependent sources as a function of the node voltages.

$$v_\Delta = 24 - v_1 \tag{98}$$

$$i_\Delta = \frac{v_1 - v_2}{5} \tag{99}$$

$$\tag{100}$$

KCL at node 1:

$$\frac{v_1 - 24}{2} + \frac{v_1 - v_3}{4} + \frac{v_1 - v_2}{5} = 0 \tag{101}$$

KCL at node 2:

$$\frac{v_2 - v_1}{5} - 0.85(24 - v_1) + 0 = 0 \tag{102}$$

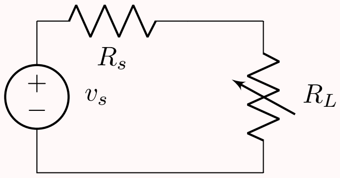
Node 3:

$$v_3 = 6i_\Delta = 6\left(\frac{v_1 - v_2}{5}\right) \tag{103}$$

$v_1 = 5.538\,\mathrm{V}, v_2 = 84\,\mathrm{V}, v_3 = -94.154\,\mathrm{V} \therefore v_{th} = v_2 = 84\,\mathrm{V}$. To find the thevenin resistence, deactivate the independent source

## 13.4 Maximum Power Transfer

**Derivation**:




Consider this circuit. Choose $R_L$ such that maximum power is transfered. Note the power relation for the resistor.

$$P_L = R_L i_L^2 \tag{104}$$

$$v_s = (R_s + R_L) i_L \tag{105}$$

$$P_L = R_L \frac{v_s^2}{(R_s + R_L)^2} \tag{106}$$

To maximize, find the partial derivative with respect to $R_L$

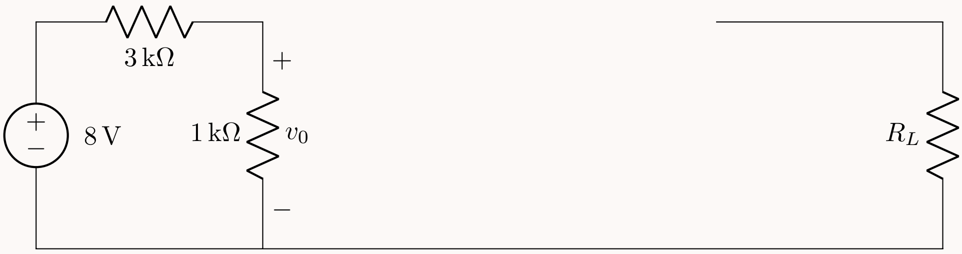
$$\frac{\partial P_L}{\partial R_L} = v_s^2 \frac{R_S - R_L}{(R_S + R_L)^3} = 0 \tag{107}$$

To maximize power transfer, $R_S = R_L$. Thus, the maximum power transfer is

$$P_L = \frac{v_s^2}{4R_L} \tag{108}$$

**Example 25 ()**

Find $R_L$ that results in maximum power transfer.



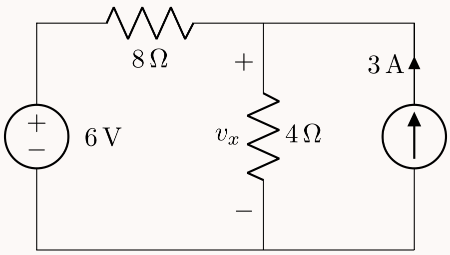

## 14 Superposition Principle

**Definition**: A linear circuit that consists of independent sources, linear dependent sources, and linear elements.

**Theorem**: Superposition Principle: The response of a linear circuit to multiple independent sources is equal to the algebraic sum of the responses caused by each independent source acting alone.

**Example 26 ()**

Consider the following circuit and find $v_x$



**Solution 18 ():** First deactivate the current source. Then, using voltage division principle

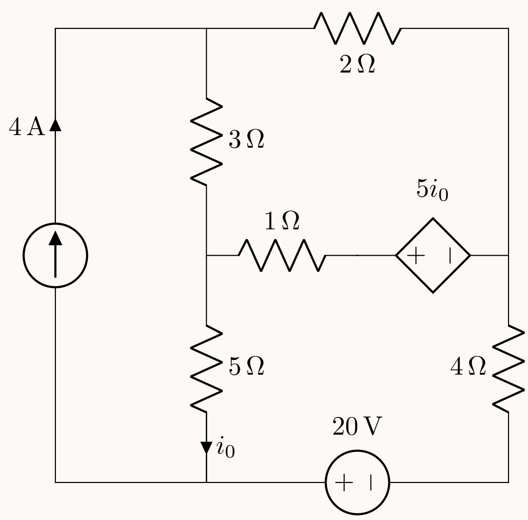
$$v_{x1} = 6\frac{4}{4+8} = 2\,\mathrm{V} \tag{109}$$

Then, deactivate the voltage source. Then, using current division principle

$$i_{x1} = 3\frac{8}{4+8} = 2\,\mathrm{A} \tag{110}$$

From ohm's law, $v_{x2} = iR = 4 \times 2 = 8\,\mathrm{V}$. Then, the total voltage would be $8 + 2 = 10\,\mathrm{V}$. This result can be confirmed using nodal analysis.

**Example 27 ()**




- **Note: Never** deactivate the dependent sources when using superposition principle.

**Solution 19 ():** Phase I: Deactivate the voltage source and perform mesh analysis.

# 15 Operational Amplifier: Op-Amp

# 16 Capacitors

**Definition**: A capacitor is a circuit element that consists of two conducting plates that are seperated by an insulator. It stores energy in the elecric field.

- When connected to a voltage source, positive and negative charges ($\pm q$) are deposited on the conductive plates.

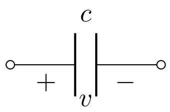
$$q = cv \tag{111}$$

- Note $q$ and $v$ can be time dependent, but $C$ must be constant. This is known as the capacitance.
- The S.I. unit for capacitance is columb per volt, called a "Farad" with the symbol F. Practically, 1 F is a very large capacitance and not often seen in normal circuits. In practice, the capacitances are usually expressed in µF,mF,nF, or pF.

- Differentiating both sides of (111),

$$i = c\dot{v} \tag{112}$$

- In a circuit, a capacitor is drawn with this symbol




- **Note** that (111) and (112) holds when PSC is held. Otherwise, an minus sign must be introduced to these equations.
- If the capacitance and the current is known, the voltage betweeen two different times can be found with

$$v(t_2) - v(t_1) = \frac{1}{C}\int_{t_1}^{t_2} i(t)\mathrm{d}t \tag{113}$$

- This equation is often written with $t_2 = t, t_1 = 0$

$$v(t) = v(0) + \frac{1}{C}\int_{0}^{t} i(\tau)\mathrm{d}\tau \tag{114}$$

- From (112), we note the following:
  1. If the voltage is constant (DC), then it doesn't change with respect to time hence $\dot{v} = 0$. Then, the current becomes zero and the capacitor behaves like an open circuit.
  2. The voltage must be a differentiable function of time. Hence at undifferentiable points, the current is undefined.

## 16.1 Energy of a capacitor

**Derivation:** Since the current is known in terms of the voltage,

$$P = vi = Cv\dot{v} \tag{115}$$

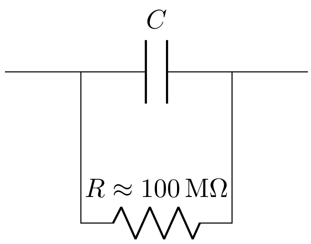
It is also known that $P = \dot{w}$. Integrating both sides from $t_1$ to $t_2$,

$$w(t_2) - w(t_1) = C\int_{t_1}^{t_2} v\mathrm{d}v = \frac{1}{2}C(v(t_2)^2 - v(t_1)^2) \tag{116}$$

Again, using $t_2 = t, t_1 = 0, w(t_1) = 0$, then

$$w(t) = \frac{1}{2}cv(t)^2 \tag{117}$$

- An ideal capacitor does not dissipate energy. It only stores energy then delivers the energy.
- A real capacitor has some leakage resistance. This is modelled by connecting the leakage resistance in parellel with an ideal capacitor.




**Example 28 ()**

Determine the voltage across a $2\,\mu\mathrm{F}$ capacitor if the current through it is $i(t) = 6\exp(-3000t)\mathrm{mA}$, and $v(0) = 0$

**Solution 20 ():**

$$v(t) = v(0) + \frac{1}{2} \tag{118}$$

### 16.2 Equivalent Capacitances

Derivation: For parellel connected capacitors,
For series connected capacitors, use KVL to find:

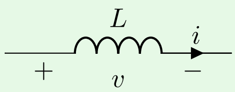
$$v = v_1 + v_2 + \cdots + v_n \tag{119}$$

Then,

$$v(t) = \sum v_i(0) + (\frac{1}{C_i}) \int_0^t i(\tau) \mathrm{d}t \tag{120}$$

## 17 Inductors

**Definition**: An inductor consists of a coil of conducting wire. The core of this coil could be air, iron.... Inductors store energy in its magnetic field with the given properties: (If $v$ and $i$ conform to PSC)




$$v = L\dot{i} \tag{121}$$

- $L$ is called the inductance. The unit for inductance is Henry (H).
- From (121),
  1. For DC circuits, the inductor voltage is zero, and an inductor behaves like a short-circuit.
  2. The current of an inductor must be differentiable with respect to time.
- If the inductance and the voltage (as a function of time) is known, then the difference in current at two different times is

$$i(t_2) - i(t_1) = \frac{1}{L} \int_{t_1}^{t_2} v(t) \mathrm{d}t \tag{122}$$

### 17.1 Energy of an inductor

Derivation:

$$P = vi = Li\dot{i} \tag{123}$$
$$P = \dot{w} \tag{124}$$
$$\tag{125}$$

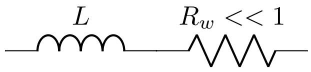
Integrating both sides yield

$$w(t_2) - w(t_1) = \frac{1}{2} L(i(t_2)^2 + i(t_1)^2) \tag{126}$$

If the energy is initially uncharged, $w(0) = 0$, the energy stored in an inductor at time $t$ is

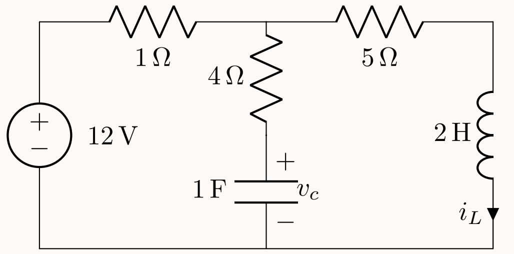
$$w(t) = \frac{1}{2} Li(t)^2 \tag{127}$$

- An ideeal inductor does not dissipate energy like an ideal capacitor. It only stores and delivers energy.
- A real inductor includes a series resistor:




**Example 29 ()**

Find the energy stored in the capacitor and inductor at steady state DC.




**Solution 21 ():** At steady state, open the capacitor and short the inductor. Write KVL on the outer loop.

$$6i_L = 12 \implies i_L = 2\,\mathrm{A} \tag{128}$$

The energy stored in the inductor is

$$w_L = \frac{1}{2}Li_L^2 = \frac{1}{2} \times 2 \times 2^2 = 4\,\mathrm{J} \tag{129}$$

Write KVL for the right loop,

$$v_c = 5i_L = 10\,\mathrm{V} \tag{130}$$

The energy stored in the capacitor is

$$w_C = \frac{1}{2}Cv^2 = \frac{1}{2} \times 1 \times 10^2 = 50\,\mathrm{J} \tag{131}$$

### 17.2 Equivalent Inductances

**Derivation:** In series,

$$L_{eq} = L_1 + L_2 + \cdots + L_n \tag{132}$$

In parellel,

$$L_{eq} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n} \tag{133}$$

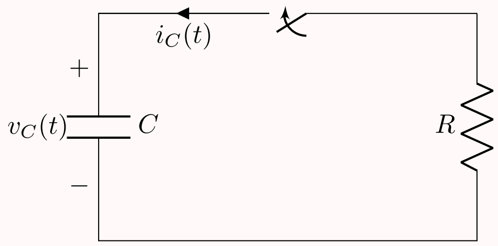
## 18 First order circuits

**Definition**: A first order circuit is a circuit where the voltage and current relations is can be described by a system of first order differential equations.

- The passive elements of these circuits are either R & C or R & L.

### 18.1 Source-free RC Circuits

**Derivation:** The capacitor has been connected to an arbiturary external circuit before time $t = 0$ such that it gains a voltage $v_C(0^-) = V_0$. At time $t = 0$, the capacitor is disconnected to the external circuit and the switch is closed.




We can write KVL for this loop (counter clockwise)

$$v_C(t) + i_C(t)R = 0 \tag{134}$$

For a capacitor, $i_C = C\dot{v}_C$. Substituting into the previous relation,

$$v_C + RC\dot{v}_C = 0 \tag{135}$$

$R$ and $C$ are time-independent, thus, this is a simple seperable differential equation and the general solution is

$$v_C(t) = A\exp\left(-\frac{t}{RC}\right) \tag{136}$$

for arbiturary constant A. Noting that $v_C(0^-) = V_0$ and the voltage across a capacitor cannot change abruptly, the capacitor voltage is

$$v_C(t) = V_0\exp\left(-\frac{t}{RC}\right) \tag{137}$$

From (134), note that the initial current in the capacitor is $i_C(0) = -\frac{V_0}{R} \equiv I_0$. Thus, the current flowing through the capacitor is

$$i_C(t) = I_0\exp\left(-\frac{t}{RC}\right) \tag{138}$$

**Definition**: We define the *time constant*

$$\tau \equiv RC \tag{139}$$

as it has the dimensions of time.

- This indicate how long it takes for the tangent line (of the voltage) to cross the time axis.
- This is also the amount of time for the voltage or current to drop to $1/e$ of its original value.
- A larger time constant means the voltage and current decays slower.
- There is some initial energy in the circuit that gradually dissipate through the resistor

$$w(0) = \frac{1}{2}CV_0^2 \tag{140}$$

- If a capacitor that is connected to an interconnection of resistors (in series), the capacitor voltage can be found by substituting the Thevenin resistance for $R$ in (137).