Geometric Approach to Elementary Physics

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1 Introduction

I will write the introduction later.

2 Mathematical Preliminaries

We will be mostly working with vectors in three-dimensional space here. First, several concepts must be introduced that serves as the basis for this approach to elementary physics. Note that unless otherwise stated, these concepts are applicable to any finite dimensional vector space over the real numbers. We will use $\mathbb{R}^n$ to denote a $n-$dimensional vector space.

2.1 The Real Inner Product

Definition: In $\mathbb{R}^n$, we will define a function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denoted $\langle\, .\, ,\, .\,\rangle$ called the *inner product* (or dot product) with the following properties:

1. Linearity of the second argument:
$$\langle \mathbf{x}, a\mathbf{y} + b\mathbf{z}\rangle = a\langle \mathbf{x}, \mathbf{y}\rangle + b\langle \mathbf{x}, \mathbf{z}\rangle \tag{2.1}$$
For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $a, b \in \mathbb{R}$.
2. Positive Definiteness:
$$\langle \mathbf{x}, \mathbf{x}\rangle \geq 0 \tag{2.2}$$
For any $\mathbf{x} \in \mathbb{R}^n$, with equality holding if and only if $\mathbf{x} = \mathbf{0}$, the zero vector.

3. Commutativity[a]:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle \tag{2.3}$$

For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

[a] This is only true for the inner product in a real vector space.

Proposition: For any two non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ implies $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.

Proof. Consider the contrapositive case. If $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent, $\mathbf{y} = a\mathbf{x}$ for some $a \neq 0 \in \mathbb{R}$. Then, $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, a\mathbf{x} \rangle = a\langle \mathbf{x}, \mathbf{x} \rangle = a(0) = 0$. This means $\mathbf{x} = 0$. However, $\mathbf{x}$ is non-zero; therefore, $\mathbf{x}$ and $\mathbf{y}$ are not linearly dependent. □

With this definition of the inner product, we can carefully select a basis for $\mathbb{R}^n$ called an orthonormal basis which will be used for the majority of this discussion. As $\mathbb{R}^n$ is a $n-$dimensional vector space, the basis consists of n vectors.

Definition: For $\mathbb{R}^n$, we define an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2 \dots \mathbf{e}_n\}$ such that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} \tag{2.4}$$

Note that now, every vector in $\mathbb{R}^n$ can be written in this basis so that $\mathbf{v} = v^i \mathbf{e}_i$, where v^i are the *coordinates* or $\mathbf{v}$. The norm or the length of a vector can also be defined.

Definition: The norm of a vector $\mathbf{x} \in \mathbb{R}^n$ is

$$|\mathbf{x}| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \tag{2.5}$$

Using an orthonormal basis, this definition is consistent with the normal pythagorean definition of distance.

$$|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x^i x^j \delta_{ij}} = \sqrt{\sum_i (x^i)^2} \tag{2.6}$$

2.2 Multivectors and Exterior Product

Definition: The *exterior product* is a binary operation $\Lambda(\mathbb{R}^n) \times \Lambda(\mathbb{R}^n) \to \Lambda(\mathbb{R}^n)$ with the following properties

1. Multilinearity:

$$(a\vec{x} + b\vec{y}) \wedge \vec{z} = a(\vec{x} \wedge \vec{z}) + b(\vec{y} \wedge \vec{z}) \tag{2.7}$$
$$\vec{x} \wedge (a\vec{y} + b\vec{z}) = a(\vec{x} \wedge \vec{y}) + b(\vec{x} \wedge \vec{z}) \tag{2.8}$$

For any $\vec{x}, \vec{y}, \vec{z} \in \Lambda(\mathbb{R}^n)$ and $a, b \in \mathbb{R}$

2. Associativity:

$$(\vec{x} \wedge \vec{y}) \wedge \vec{z} = \vec{x} \wedge (\vec{y} \wedge \vec{z}) \tag{2.9}$$

For any $\vec{x}, \vec{y}, \vec{z} \in \Lambda(\mathbb{R}^n)$

3. Antisymmetry:

$$\vec{x} \wedge \vec{y} = -\vec{y} \wedge \vec{x} \tag{2.10}$$

For any $\vec{x}, \vec{y} \in \Lambda(\mathbb{R}^n)$

Definition: Firstly, we will make an intermediate definition. Define $G_k(\mathbb{R}^n)$ for $k = 1, 2, \dots, n$ recursively with

$$G_0(\mathbb{R}^n) := \mathbb{R} \tag{2.11}$$
$$G_1(\mathbb{R}^n) := \mathbb{R}^n \tag{2.12}$$
$$G_{i+1}(\mathbb{R}^n) := \{(\vec{x} \wedge \vec{y})\ \forall \vec{x} \in G_i(\mathbb{R}^n), \vec{y} \in G_1(\mathbb{R}^n)\} \qquad i > 1 \tag{2.13}$$

An *exterior algebra* on the vector space $\mathbb{R}^n$ denoted $\Lambda(\mathbb{R}^n)$ is

$$\Lambda(\mathbb{R}^n) := \bigoplus_{k=0}^{n} G_k(\mathbb{R}^n) \tag{2.14}$$

A *multivector* $\vec{x}$ is an element of the exterior algebra. Note that $G_k(\mathbb{R}^n)$ is known as the set of all multivectors of grade k over the vector space $\mathbb{R}^n$.

Proposition: $G_k(\mathbb{R}^n)$ is a $\binom{n}{k}-$dimensional vector space over $\mathbb{R}$. For $k > 0$, $G_k(\mathbb{R}^n)$ admits a *standard basis*

$$E_k = \{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\} \tag{2.15}$$

Proof. For $k = 0$, it is well known that $\mathbb{R}$ is a one-dimensional vector space. For $k = 1$, by definition, $\mathbb{R}^n$ is a $n-$dimensional vector space.

For $k > 1$, we will first show that $G_k(\mathbb{R}^n)$ is a vector space. From the recursive definition of $G_k(\mathbb{R}^n)$, any element $\vec{x} \in G_k(\mathbb{R}^n)$ can be expressed as $\vec{x} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge ... \wedge \mathbf{x}_k$ due to associativity of the exterior product.
We can show that it satisfies all the vector space axioms.
To show E_k is a basis, we will first show that its elements are linearly independent. Suppose a superposition of the vectors

$$\lambda^{i_1 i_2 \ldots i_k}(\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k}) = 0 \tag{2.16}$$

Using the multilinearity of the exterior product,

$$\lambda^{i_1 i_2 \ldots i_k}\mathbf{e}_{i_1} \wedge \lambda^{i_1 i_2 \ldots i_k}\mathbf{e}_{i_2} \wedge \cdots \wedge \lambda^{i_1 i_2 \ldots i_k}\mathbf{e}_{i_k} = 0 \tag{2.17}$$

Next we can show the $G_k(\mathbb{R}^n) = \mathrm{span}(E_k)$. Since the elements of $E_k \subset G_k(\mathbb{R}^n)$ and $G_k(\mathbb{R}^n)$ is a vector space, $\mathrm{span}(E_k) \subseteq G_k(\mathbb{R}^n)$. Now given any $\vec{x} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge ... \wedge \mathbf{x}_k \in G_k(\mathbb{R}^n)$. The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ can be written as their coordinates with respect to the standard basis of $\mathbb{R}^n$.

$$\vec{x} = x_1^i \mathbf{e}_i \wedge x_2^i \mathbf{e}_i \wedge \cdots \wedge x_k^i \mathbf{e}_i \tag{2.18}$$

Using multilinearity of the exterior product,

$$\vec{x} = \lambda^{i_1 i_2 \ldots i_k}(\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_k}) \tag{2.19}$$

where

$$\lambda^{i_1 i_2 \ldots i_k} = \tag{2.20}$$

□

Corollary: $\Lambda(\mathbb{R}^n)$ is a 2^n-dimensional vector space over $\mathbb{R}$.

Proof. By the definition of $\Lambda(\mathbb{R}^n)$ in (2.14), the direct sum of vector spaces is a vector space. The dimension of the direct sum of vector spaces is the sum of the dimensions of the vector spaces. It is well known that the sum of the $n-$th row of the pascal triangle is 2^n. □

2.3 Dot Product and Geometric Product

Definition: The *geometric product* is the algebra operation in $\Lambda(\mathbb{R}^n)$ that must satisfy the following axioms

1. Closure:

$$\vec{x}\vec{y} \in \Lambda(\mathbb{R}^n) \tag{2.21}$$

2. Existence of identity:

$$1\vec{x} = \vec{x}1 = \vec{x} \tag{2.22}$$

3. Associativity:

$$\vec{x}(\vec{y}\vec{z}) = (\vec{x}\vec{y})\vec{z} \tag{2.23}$$

4. Distributivity:

$$\vec{x}(\vec{y}+\vec{z}) = \vec{x}\vec{y}+\vec{x}\vec{z} \tag{2.24}$$
$$(\vec{x}+\vec{y})\vec{z} = \vec{x}\vec{z}+\vec{y}\vec{z} \tag{2.25}$$

5.

3 Geometric Algebra in Three Dimensional Space

We live in three dimensions and the vast majority of physics takes place in three dimensional space. As we are using this as the mathematical basis to do physics, we should familiarize ourselves with the mathematics in three dimensions in particular as it would be used extensively.

3.1 Basis and Shorthand Notation

We first notice that $\Lambda(\mathbb{R}^3) = G_0(\mathbb{R}^3) \oplus G_1(\mathbb{R}^3) \oplus G_2(\mathbb{R}^3) \oplus G_3(\mathbb{R}^3)$ is an eight-dimensional vector space. We will assign some terminology to each of these subspaces.

- Elements of $G_0(\mathbb{R}^3)$ are called *scalars* and their basis consists of the real number one $E_0 = \{1\}$. Quantities like time and charge are scalars.
- Elements of $G_1(\mathbb{R}^3)$ are called *vectors* and their basis is the familiar standard basis for $\mathbb{R}^3$, which we will redefine as $\mathbf{e}_1 \equiv \hat{x}, \mathbf{e}_2 \equiv \hat{y}, \mathbf{e}_3 \equiv \hat{z}$. Then, the standard basis would be $E_1 = \{\hat{x}, \hat{y}, \hat{z}\}$. Quantities like force, momentum, and position are vectors.
- Elements of $G_2(\mathbb{R}^3)$ are called *psudovectors*. The standard basis is called $E_2 = \{\hat{y}\hat{z}, \hat{z}\hat{x}, \hat{x}\hat{y}\} \equiv \{I\hat{x}, I\hat{y}, I\hat{z}\}$. These are called psudovectors because they the the basis also consists of three elements. Quantities like angular momentum, torque, and magnetic field are psudovectors.
- Elements of $G_3(\mathbb{R}^3)$ are called *psudoscalars*. The standard basis only consists of one element $I \equiv \hat{x}\hat{y}\hat{z}$. Quantities like electric and magnetic flux are psudoscalars.