ⓒ True. Since A^{-1} is expressible as the product of elementary matrices, it is invertible (Theorem 1.5.3). Thus, A is also invertible (as it is the inverse of A^{-1}). Thus, $Ax = 0$ only has the trivial solution (Theorem 1.5.3).

ⓓ False. B and A have the same RREF (as B can be reduced back to A with the inverse operation), and A has a unique RREF. Per (b), A's RREF has at least one row of zeros, and thus so does the RREF of B. Since the RREF is not I_n, B must be singular (per Theorem 1.5.3).

Exercise 21:

No. Let $a=1$, $b=0$, $c=1$, $d=0$. There is no A such that $A\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$

Let $A = \begin{bmatrix}w & x\\ y & z\end{bmatrix}$. Thus, the following must be true:

1. $w + x = 0$
2. $0 = 0$
3. $y + z = 1$
4. $0(y) + 0(z) = 1$

Based on 4, A cannot exist.

1.6: Further Results on Systems of Equations and Invertibility

Exercise 1:

$A = \begin{bmatrix}1 & 1\\ 5 & 6\end{bmatrix}$ $x = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$ $b = \begin{bmatrix}2\\ 9\end{bmatrix}$

$A^{-1} = \frac{1}{6-5}\begin{bmatrix}6 & -1\\ -5 & 1\end{bmatrix} = \begin{bmatrix}6 & -1\\ -5 & 1\end{bmatrix}$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix}6 & -1\\ -5 & 1\end{bmatrix}\begin{bmatrix}2\\ 9\end{bmatrix} = \begin{bmatrix}12-9\\ -10+9\end{bmatrix} = \begin{bmatrix}3\\ -1\end{bmatrix}$

Exercise 2:

$A = \begin{bmatrix}4 & -3\\ 2 & -5\end{bmatrix}$ $x = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$ $b = \begin{bmatrix}-3\\ 9\end{bmatrix}$

$A^{-1} = \frac{1}{-20+6}\begin{bmatrix}-5 & 3\\ -2 & 4\end{bmatrix} = \frac{1}{-14}\begin{bmatrix}-5 & 3\\ -2 & 4\end{bmatrix} = \begin{bmatrix}5/14 & -3/14\\ 2/14 & -4/14\end{bmatrix} = \begin{bmatrix}5/14 & -3/14\\ 1/7 & -2/7\end{bmatrix}$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix}5/14 & -3/14\\ 1/7 & -2/7\end{bmatrix}\begin{bmatrix}-3\\ 9\end{bmatrix} = \begin{bmatrix}-15/14 + -27/14\\ -3/7 + -18/7\end{bmatrix} = \begin{bmatrix}-42/14\\ -21/7\end{bmatrix} = \begin{bmatrix}-3\\ -3\end{bmatrix}$

Exercise 3:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix}$$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4+3 \\ 1+3 \\ 8-3-12 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -7 \end{bmatrix}$

Exercise 4:

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix}$$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2-1 \\ -6+5-10 \\ 6-5+15 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$

Exercise 5:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -4 \\ -4 & 1 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/5 & 0 & -1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & -1/5 & 0 \end{bmatrix}$$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix} 1/5 & 0 & -1/5 \\ 3/5 & 1/5 & 1/5 \\ 1/5 & -1/5 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3+2 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$

Exercise 6:

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 1 & 4 & 4 \\ 1 & 3 & 7 & 9 \\ -1 & -2 & -4 & -6 \end{bmatrix} \quad x = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 7 \\ 4 \\ 6 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 6 & -3 & 4 & 1 \\ 1 & 0 & 1 & 1 \\ -3 & 1 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 6 & -3 & 4 & 1 \\ 1 & 0 & 1 & 1 \\ -3 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 7 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ -21+16+6 \\ 4+6 \\ 7-8-6 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 10 \\ -7 \end{bmatrix}$

Exercise 7:

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6-5}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 - 5b_2 \\ -b_1 + 3b_2 \end{bmatrix}$

Exercise 8:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -15/2 & 1/2 & 5/2 \\ 1/2 & 1/2 & -1/2 \\ 5/2 & -1/2 & -1/2 \end{bmatrix}$$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix} -15/2 & 1/2 & 5/2 \\ 1/2 & 1/2 & -1/2 \\ 5/2 & -1/2 & -1/2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2}b_1 + \frac{1}{2}b_2 + \frac{5}{2}b_3 \\ \frac{1}{2}b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3 \\ \frac{5}{2}b_1 - \frac{1}{2}b_2 - \frac{1}{2}b_3 \end{bmatrix}$

Exercise 9:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -1/3 & 1/3 & 1 \\ 1/3 & -1/3 & 0 \\ 2/3 & 1/3 & -1 \end{bmatrix}$$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix} -1/3 & 1/3 & 1 \\ 1/3 & -1/3 & 0 \\ 2/3 & 1/3 & -1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}b_1 + \frac{1}{3}b_2 + b_3 \\ \frac{1}{3}b_1 - \frac{1}{3}b_2 \\ \frac{2}{3}b_1 + \frac{1}{3}b_2 - b_3 \end{bmatrix}$

(a) $x = \begin{bmatrix} -\frac{1}{3}(-1) + \frac{1}{3}(3) + (4) \\ \frac{1}{3}(-1) - \frac{1}{3}(3) \\ \frac{2}{3}(-1) + \frac{1}{3}(3) - (4) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + 1 + 4 \\ -\frac{1}{3} - 1 \\ -\frac{2}{3} + 1 - 4 \end{bmatrix} = \begin{bmatrix} 16/3 \\ -4/3 \\ -11/3 \end{bmatrix}$

(b) $x = \begin{bmatrix} -\frac{1}{3}(5) + \frac{1}{3}(0) + (0) \\ \frac{1}{3}(5) - \frac{1}{3}(0) \\ \frac{2}{3}(5) + \frac{1}{3}(0) - (0) \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} \\ \frac{5}{3} \\ \frac{10}{3} \end{bmatrix}$

(c) $x = \begin{bmatrix} -\frac{1}{3}(-1) + \frac{1}{3}(-1) + (3) \\ \frac{1}{3}(-1) - \frac{1}{3}(-1) \\ \frac{2}{3}(-1) + \frac{1}{3}(-1) - (3) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} - \frac{1}{3} + 3 \\ -\frac{1}{3} + \frac{1}{3} \\ -\frac{2}{3} - \frac{1}{3} - 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$

Exercise 10:

$$\left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & -1 & 5 & -1 \\ 1 & -1 & 1 & 3 & 0 & -1 \\ 1 & 1 & 0 & 4 & 0 & 3 \end{array}\right] \xrightarrow[+R_2]{-1\times R_1} \left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & -1 & 5 & -1 \\ 0 & -3 & 0 & 4 & -5 & 0 \\ 1 & 1 & 0 & 4 & 0 & 3 \end{array}\right] \xrightarrow[+R_3]{-1\times R_1} \left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & -1 & 5 & -1 \\ 0 & -3 & 0 & 4 & -5 & 0 \\ 0 & -1 & -1 & 5 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}\times R_2} \left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & -1 & 5 & -1 \\ 0 & 1 & 0 & -4/3 & 5/3 & 0 \\ 0 & -1 & -1 & 5 & -5 & 4 \end{array}\right] \xrightarrow[+R_3]{1\times R_2} \left[\begin{array}{ccc|c|c|c} 1 & 2 & 1 & -1 & 5 & -1 \\ 0 & 1 & 0 & -4/3 & 5/3 & 0 \\ 0 & 0 & -1 & 11/3 & -10/3 & 4 \end{array}\right] \xrightarrow[+R_1]{-2\times R_2} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 1 & 5/3 & 5/3 & -1 \\ 0 & 1 & 0 & -4/3 & 5/3 & 0 \\ 0 & 0 & -1 & 11/3 & -10/3 & 4 \end{array}\right]$$

$$\xrightarrow{-1\times R_3} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 1 & 5/3 & 5/3 & -1 \\ 0 & 1 & 0 & -4/3 & 5/3 & 0 \\ 0 & 0 & 1 & -11/3 & 10/3 & -4 \end{array}\right] \xrightarrow[+R_1]{-1\times R_3} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 16/3 & -5/3 & 3 \\ 0 & 1 & 0 & -4/3 & 5/3 & 0 \\ 0 & 0 & 1 & -11/3 & 10/3 & -4 \end{array}\right]$$

(a) (b) (c)

Exercise 11:

$$\left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 3 & 2 & 4 & 5 \end{array}\right] \xrightarrow[+R_2]{-3\times R_1} \left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 17 & 1 & 11 \end{array}\right] \xrightarrow{\frac{1}{17}\times R_2} \left[\begin{array}{cc|c|c} 1 & -5 & 1 & -2 \\ 0 & 1 & 1/17 & 11/17 \end{array}\right] \xrightarrow[+R_1]{5\times R_2} \left[\begin{array}{cc|c|c} 1 & 0 & 22/17 & 21/17 \\ 0 & 1 & 1/17 & 11/17 \end{array}\right]$$

(a) (b)

Exercise 12:

$$\left[\begin{array}{ccc|c|c} -1 & 4 & 1 & 0 & -3 \\ 1 & 9 & -2 & 1 & 4 \\ 6 & 4 & -8 & 0 & -5 \end{array}\right] \xrightarrow{-1\times R_1} \left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 1 & 9 & -2 & 1 & 4 \\ 6 & 4 & -8 & 0 & -5 \end{array}\right] \xrightarrow[+R_2]{-1\times R_1} \left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 13 & -1 & 1 & 1 \\ 6 & 4 & -8 & 0 & -5 \end{array}\right] \xrightarrow[+R_3]{-6\times R_1}$$

$$\left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 13 & -1 & 1 & 1 \\ 0 & 28 & -2 & 0 & -23 \end{array}\right] \xrightarrow{\frac{1}{13}\times R_2} \left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 1 & -1/13 & 1/13 & 1/13 \\ 0 & 28 & -2 & 0 & -23 \end{array}\right] \xrightarrow[+R_3]{-28\times R_2} \left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 1 & -1/13 & 1/13 & 1/13 \\ 0 & 0 & 2/13 & -28/13 & \frac{-327}{13} \end{array}\right] \xrightarrow{\frac{13}{2}\times R_3} \left[\begin{array}{ccc|c|c} 1 & -4 & -1 & 0 & 3 \\ 0 & 1 & -1/13 & 1/13 & 1/13 \\ 0 & 0 & 1 & -14 & \frac{-327}{2} \end{array}\right]$$

$$\xrightarrow[+R_1]{4\times R_2} \left[\begin{array}{ccc|c|c} 1 & 0 & -17/13 & 4/13 & 43/13 \\ 0 & 1 & -1/13 & 1/13 & 1/13 \\ 0 & 0 & 1 & -14 & \frac{-327}{2} \end{array}\right] \xrightarrow[+R_2]{\frac{1}{13}\times R_3} \left[\begin{array}{ccc|c|c} 1 & 0 & -17/13 & 4/13 & 43/13 \\ 0 & 1 & 0 & -1 & -25/2 \\ 0 & 0 & 1 & -14 & -\frac{327}{2} \end{array}\right] \xrightarrow[+R_1]{\frac{17}{13}\times R_3} \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -18 & -\frac{421}{2} \\ 0 & 1 & 0 & -1 & -\frac{25}{2} \\ 0 & 0 & 1 & -14 & -\frac{327}{2} \end{array}\right]$$

(a) (b)

Exercise 13:

$$\left[\begin{array}{cc|c|c|c|c} 4 & -7 & 0 & -4 & -1 & -5 \\ 1 & 2 & 1 & 6 & 3 & 1 \end{array}\right] \xrightarrow{\frac{1}{4}\times R_1} \left[\begin{array}{cc|c|c|c|c} 1 & -7/4 & 0 & -1 & -1/4 & -5/4 \\ 1 & 2 & 1 & 6 & 3 & 1 \end{array}\right] \xrightarrow[+R_2]{-1\times R_1} \left[\begin{array}{cc|c|c|c|c} 1 & -7/4 & 0 & -1 & -1/4 & -5/4 \\ 0 & 15/4 & 1 & 7 & 13/4 & 9/4 \end{array}\right]$$

$$\xrightarrow{\frac{4}{15}\times R_2} \left[\begin{array}{cc|c|c|c|c} 1 & -7/4 & 0 & -1 & -1/4 & -5/4 \\ 0 & 1 & 4/15 & \frac{28}{15} & 13/15 & 3/5 \end{array}\right] \xrightarrow[+R_1]{\frac{7}{4}\times R_2} \left[\begin{array}{cc|c|c|c|c} 1 & 0 & 7/15 & 34/15 & 19/15 & -1/5 \\ 0 & 1 & 4/15 & 28/15 & 13/15 & 3/5 \end{array}\right]$$

(a) (b) (c) (d)

Exercise 14:

$$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ -1 & -2 & 0 & 0 & 1 & -1 \\ 2 & 5 & 4 & -1 & 1 & 0 \end{array}\right] \xrightarrow[+R_2]{1\times R_1} \left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 2 & 5 & 4 & -1 & 1 & 0 \end{array}\right] \xrightarrow[+R_3]{-2\times R_1} \left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & -1 & -6 & -3 & 1 & 2 \end{array}\right] \xrightarrow[+R_3]{1\times R_2}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 3 & 5 & 1 & 0 & -1 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & 0 & -1 & -2 & 2 & 0 \end{array}\right] \xrightarrow[+R_1]{-3\times R_2} \left[\begin{array}{ccc|c|c|c} 1 & 0 & -10 & -2 & -3 & 5 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & 0 & -1 & -2 & 2 & 0 \end{array}\right] \xrightarrow{-1\times R_3} \left[\begin{array}{ccc|c|c|c} 1 & 0 & -10 & -2 & -3 & 5 \\ 0 & 1 & 5 & 1 & 1 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right] \xrightarrow[+R_2]{-5\times R_3}$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & -10 & -2 & -3 & 5 \\ 0 & 1 & 0 & -9 & 11 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right] \xrightarrow[+R_1]{10\times R_3} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 18 & -23 & 5 \\ 0 & 1 & 0 & -9 & 11 & -2 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

$$\begin{array}{cccccc} & & & \text{(a)} & \text{(b)} & \text{(c)} \end{array}$$

Exercise 15:

$$\left[\begin{array}{ccc|c|c} 1 & -2 & 1 & -2 & 1 \\ 2 & -5 & 1 & 1 & -1 \\ 3 & -7 & 2 & -1 & 0 \end{array}\right] \xrightarrow[+R_2]{-2\times R_1} \left[\begin{array}{ccc|c|c} 1 & -2 & 1 & -2 & 1 \\ 0 & -1 & -1 & 5 & -3 \\ 3 & -7 & 2 & -1 & 0 \end{array}\right] \xrightarrow[+R_3]{-3\times R_1} \left[\begin{array}{ccc|c|c} 1 & -2 & 1 & -2 & 1 \\ 0 & -1 & -1 & 5 & -3 \\ 0 & -1 & -1 & 5 & -3 \end{array}\right] \xrightarrow[+R_3]{-1\times R_2} \left[\begin{array}{ccc|c|c} 1 & -2 & 1 & -2 & 1 \\ 0 & -1 & -1 & 5 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-1\times R_2}$$

$$\left[\begin{array}{ccc|c|c} 1 & -2 & 1 & -2 & 1 \\ 0 & 1 & 1 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[+R_1]{2\times R_2} \left[\begin{array}{ccc|c|c} 1 & 0 & 3 & -12 & 7 \\ 0 & 1 & 1 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) $x_1 = -12 - 3s$, $x_2 = -5 - s$, $x_3 = s$

(b) $x_1 = 7 - 3s$, $x_2 = 3 - s$; $x_3 = s$

Exercise 16:

$$\left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{array}\right] \xrightarrow[+R_2]{-\frac{1}{2}\times R_1} \left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 0 & 0 & -\frac{1}{2}b_1 + b_2 \end{array}\right] \xrightarrow{\frac{1}{6}\times R_1} \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{1}{6}b_1 \\ 0 & 0 & -\frac{1}{2}b_1 + b_2 \end{array}\right]$$

Must satisfy condition: $-\frac{1}{2}b_1 + b_2 = 0$.

That is, $b_2 = \frac{1}{2}b_1$.

$Ax = b$ is consistent iff b is a matrix of the form: $b = \begin{bmatrix} b_1 \\ \frac{1}{2}b_1 \end{bmatrix}$

Exercise 17:

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right] \xrightarrow[+R_2]{-4\times R_1} \left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right] \xrightarrow[+R_3]{3\times R_1} \left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & -3 & 12 & 3b_1 + b_3 \end{array}\right] \xrightarrow[+R_3]{R_2} \left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1 + b_2 \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}\times R_2} \left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 1 & -4 & \frac{-4b_1 + b_2}{3} \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right] \xrightarrow[+R_1]{2\times R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & b_1 + \frac{-8b_1 + b_2}{3} \\ 0 & 1 & -4 & \frac{-4b_1 + b_2}{3} \\ 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$$

Must satisfy condition: $-b_1 + b_2 + b_3 = 0$

That is: $b_3 = b_1 - b_2$

$Ax+b$ is consistent iff b is a matrix of the form: $b = \begin{bmatrix} b_1 \\ b_2 \\ b_1 - b_2 \end{bmatrix}$

Exercise 18:

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -4 & 5 & 2 & b_2 \\ -4 & 7 & 4 & b_3 \end{array}\right] \xrightarrow[+R_2]{4\times R_1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & 4b_1+b_2 \\ -4 & 7 & 4 & b_3 \end{array}\right] \xrightarrow[+R_3]{4\times R_1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -3 & -2 & 4b_1+b_2 \\ 0 & -1 & 0 & 4b_1+b_3 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -1 & 0 & 4b_1+b_3 \\ 0 & -3 & -2 & 4b_1+b_2 \end{array}\right]$$

$$\xrightarrow{-1\times R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -4b_1-b_3 \\ 0 & -3 & -2 & 4b_1+b_2 \end{array}\right] \xrightarrow[+R_3]{3\times R_2} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & 1 & 0 & -4b_1-b_3 \\ 0 & 0 & -2 & -8b_1+b_2-3b_3 \end{array}\right] \xrightarrow[+R_1]{2\times R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -7b_1-2b_3 \\ 0 & 1 & 0 & -4b_1-b_3 \\ 0 & 0 & -2 & -8b_1+b_2-3b_3 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}\times R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -7b_1-2b_3 \\ 0 & 1 & 0 & -4b_1-b_3 \\ 0 & 0 & 1 & \frac{8b_1-b_2+3b_3}{2} \end{array}\right] \xrightarrow[+R_1]{R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7b_1-2b_3+\frac{8b_1-b_2+3b_3}{2} \\ 0 & 1 & 0 & -4b_1-b_3 \\ 0 & 0 & 1 & \frac{8b_1-b_2+3b_3}{2} \end{array}\right].$$

There are no restrictions here.

Exercise 19:

$$\left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right] \xrightarrow[+R_2]{2\times R_1} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right] \xrightarrow[+R_3]{3\times R_1} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & -1 & 11 & 5 & 3b_1+b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right]$$

$$\xrightarrow[+R_4]{-4\times R_1} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1+b_2 \\ 0 & -1 & 11 & 5 & 3b_1+b_3 \\ 0 & 1 & -11 & -5 & -4b_1+b_4 \end{array}\right] \xrightarrow{-1\times R_2} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & -1 & 11 & 5 & 3b_1+b_3 \\ 0 & 1 & -11 & -5 & -4b_1+b_4 \end{array}\right] \xrightarrow[+R_3]{R_2} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 1 & -11 & -5 & -4b_1+b_4 \end{array}\right]$$

$$\xrightarrow[+R_4]{-1\times R_2} \left[\begin{array}{cccc|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right] \xrightarrow[+R_1]{R_2} \left[\begin{array}{cccc|c} 1 & 0 & -8 & -3 & -b_1-b_2 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right]$$

Must satisfy conditions: $b_1-b_2+b_3=0$ and $-2b_1+b_2+b_4=0$.

That is: $b_3=b_2-b_1$ and $b_4=2b_1-b_2$

$Ax+b$ is consistent iff b is a matrix of the form: $b=\begin{bmatrix} b_1 \\ b_2 \\ b_2-b_1 \\ 2b_1-b_2 \end{bmatrix}$

Exercise 20:

(a) Since A is 3×3 and x is 3×1, Ax is 3×1. Thus, x can be subtracted from both sides: $Ax-x=0$ (Theorem 1.4.2(b)). Note that since $x=Ix$; $Ax-Ix=0$. That is, $(A-I)x=0$ (Theorem 1.4.1(g)).

Since $Ax = x$ is equivalent to $(A - I)x = 0$, we can use the latter to solve for x.

That is, $\left(\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

Thus, $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & -2 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$. Or, $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 1 & -2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$.

Thus, $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

ⓑ $Ax = 4x$

Note that $Ax - 4x = 0$. Thus, $Ax - 4Ix = 0$. That is $(A - 4I)x = 0$
(Theorem 1.4.1(g)).

So, $\left(\begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$. Thus, $\begin{bmatrix} -2 & 1 & 2 \\ 2 & -2 & -2 \\ 3 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$.

Or, $\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 2 & -2 & -2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Thus, $x = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix}$ or $x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

<u>Exercise 21:</u>

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$. Note that A is invertible and $A^{-1} = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix}$

Thus: $A^{-1}(AX) = A^{-1}\begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$.

And: $(A^{-1}A)X = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix}\begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$

Therefore: $X = \begin{bmatrix} 11 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & -35 \end{bmatrix}$

Exercise 22:

ⓐ Only the trivial solution. Thus, the matrix is invertible.

ⓑ Has nontrivial solutions. Thus, the matrix is singular.

Exercise 23:

Proof: Let $Ax=0$ be a homogeneous system of n linear equations in n unknowns that has only the trivial solution. Per Theorem 1.6.4, A is invertible. Note that per Theorem 1.4.6, a product of invertible matrices is invertible. Let k be any positive integer. Note that $A^k = AA\cdots A$ (where there are k As multiplied in sequence). Per Theorem 1.4.6, this product (A^k) is invertible. Thus, Per Theorem 1.6.4, the system $A^k x = 0$ has only the trivial solution.

□

Exercise 24:

Proof: Let $Ax=0$ be a homogeneous system of n linear equations in n unknowns, and let Q be an invertible $n\times n$ matrix.

(⇒) Suppose that $Ax=0$ has only the trivial solution. Per Theorem 1.6.4, A is invertible. Since, A is invertible, QA is also invertible as the product of invertible matrices (Theorem 1.4.6). Thus, $(QA)x=0$ has only the trivial solution (Theorem 1.6.4).

(⇐) Suppose that $(QA)x=0$ has only the trivial solution. Thus, QA is invertible (Theorem 1.6.4). Thus $Q^{-1}(QA)$ is invertible (as the product of invertible matrices), that is, $(Q^{-1}Q)A = IA = A$ is invertible. Thus, $Ax=0$ has only the trivial solution (Theorem 1.6.4).

Therefore $Ax=0$ has only the trivial solution iff. $(QA)x=0$ has only the trivial solution.

□

Exercise 25:

Proof: Let $Ax=b$ be any consistent system of linear equations and let x_1 be a fixed solution. Thus, $Ax=b$ and $Ax_1=b$. Now let A_0 be a solution to $Ax=0$. So: $Ax - Ax_1 = A(x-x_1) = b-b=0$. Thus, $x_0 = x - x_1$, and thus, $x = x_1 + x_0$. Using this equation: $A(x_1+x_0) = Ax_1 + Ax_0 = b+0 = b$. Thus, every matrix in the form $x = x_1 + x_0$ is a solution to $Ax = b$.

$\square$

Exercise 26:

Proof: Assume that $AB = I$. First we prove that B is invertible by showing that $Bx = 0$ only has the trivial solution. Let x_0 be any solution of this system. If we multiply both sides on the left by A, we obtain $ABx_0 = A0$ or $Ix_0 = 0$ or $x_0 = 0$. Thus, the system of equations only has the trivial solution, and B is invertible. Multiply $AB = I$ by B on the left to obtain: $BAB = BI = B$. Now, multiply $BAB = B$ on the right by B^{-1} to obtain: $BABB^{-1} = BAI = BA = BB^{-1} = I$. Since $BA = I$, then $B = A^{-1}$ (Theorem 1.6.3(a)).

$\square$

Exercise 27:

ⓐ Let A be an $n \times n$ matrix and let b be an $n \times 1$ matrix. Note that $Ax = b$ has exactly one solution for every $n \times 1$ matrix b iff A is invertible (Theorem 1.6.4). The equation $x = Ax + b$ can be written as follows: $x - Ax = b$, or $Ix - Ax = b$, or $(I - A)x = b$ (per Theorem 1.4.1(f)). Thus, if $(I - A)$ is invertible, then $x = Ax + b$ has a unique solution for x.

ⓑ Suppose that $(I - A)$ is invertible. The solution can be expressed as follows: $(I - A)^{-1}(I - A)x = (I - A)^{-1}b$, that is: $x = (I - A)^{-1}b$.

Exercise 28:

Suppose that A is an invertible $n \times n$ matrix. The system of equations $Ax = x$ can be written: $Ax - x = 0$, or $Ax - Ix = 0$, or $(A - I)x = 0$ (Theorem 1.4.1(f)). Note that $(A - I)x = 0$ has only the trivial solution iff $(A - I)$ is invertible (Theorem 1.6.4). Thus, x need not have a unique solution.

Exercise 29:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Note that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Thus, it is possible to have $AB = I$ without B being the inverse of A.

Exercise 30:

Let A and B be square matrices of the same size. If A or B is not invertible, then AB is not invertible.

1.7: Diagonal, Triangular, and Symmetric Matrices

Exercise 1:

(a) Invertible $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}$ (b) Singular (c) Invertible $\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Exercise 2:

(a) $\begin{bmatrix} 6 & 3 \\ 4 & -1 \\ 4 & 10 \end{bmatrix}$ (b) $\begin{bmatrix} 8 & -2 & 6 \\ -1 & -2 & 0 \\ -20 & 4 & -8 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -24 & -10 & 12 \\ 3 & -10 & 0 \\ 60 & 20 & -16 \end{bmatrix}$

Exercise 3:

(a) $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ $A^{-2} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ $A^{-k} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{(-2)^k} \end{bmatrix}$

(b) $A^2 = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/16 \end{bmatrix}$ $A^{-2} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$ $A^{-k} = \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}$

Exercise 4:

Matrices (b) and (c) are symmetric.

Exercise 5:

(a) Invertible

(b) Singular