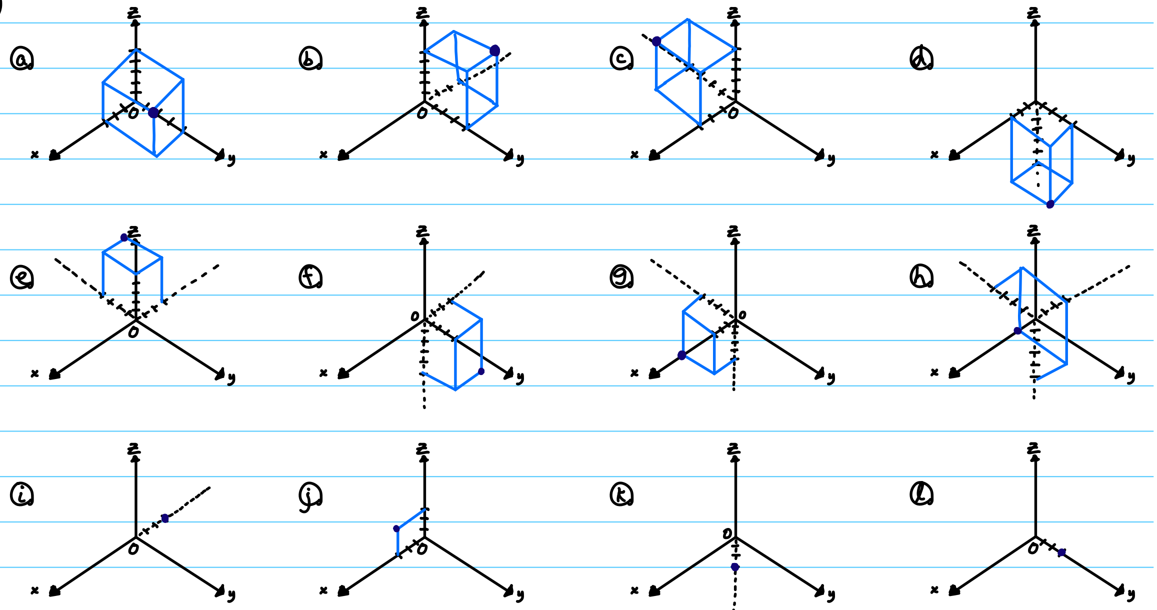
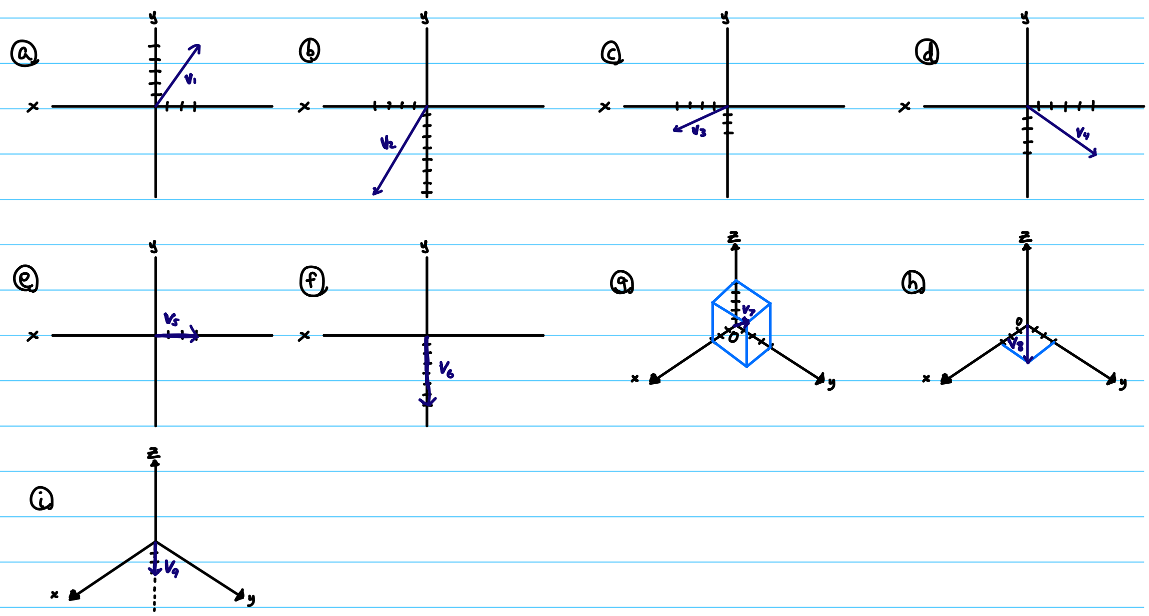
3.1: Vectors in 2-Space and 3-Space

①



②



③

ⓐ $P_1(4,8)$, $P_2(3,7)$

$x' = x - k$, $y' = y - \ell$

$x' = 3 - 4$, $y' = 7 - 8$

$\overrightarrow{P_1P_2} = (-1,-1)$

ⓑ $P_1(3,-5)$, $P_2(-4,-7)$

$x' = x - k$, $y' = y - \ell$

$x' = (-4) - 3$, $y' = (-7) - (-5)$

$\overrightarrow{P_1P_2} = (-7,-2)$

ⓒ $P_1(-5,0)$, $P_2(-3,1)$

$x' = x - k$, $y' = y - \ell$

$x' = (-3) - (-5)$, $y' = 1 - 0$

$\overrightarrow{P_1P_2} = (2, 1)$

ⓓ $P_1(0,0)$, $P_2(a,b)$

$x' = x - k$, $y' = y - \ell$

$x' = a - 0$, $y' = b - 0$

$\overrightarrow{P_1P_2} = (a, b)$

(e) $P_1(3,-7,2),\ P_2(-2,5,-4)$

$x' = x-k,\ y' = y-\ell,\ z' = z-m$

$x' = (-2)-3,\ y' = 5-(-7),\ z' = (-4)-2$

$\overrightarrow{P_1P_2} = (-5,12,-6)$

(f) $P_1(-1,0,2),\ P_2(0,-1,0)$

$x' = x-k,\ y' = y-\ell,\ z' = z-m$

$x' = 0-(-1),\ y' = (-1)-0,\ z' = 0-2$

$\overrightarrow{P_1P_2} = (1,-1,-2)$

(g) $P_1(a,b,c),\ P_2(0,0,0)$

$x' = x-k,\ y' = y-\ell,\ z' = z-m$

$x' = 0-a,\ y' = 0-b,\ z' = 0-c$

$\overrightarrow{P_1P_2} = (-a,-b,-c)$

(h) $P_1(0,0,0),\ P_2(a,b,c)$

$x' = x-k,\ y' = y-\ell,\ z' = z-m$

$x' = a-0,\ y' = b-0,\ z' = c-0$

$\overrightarrow{P_1P_2} = (a,b,c)$

4

(a) $Q = (-1,3,-5)+(6,7,-3) = (5,10,-8)$

(b) $Q = (-1,3,-5)-(6,7,-3) = (-7,-4,-2)$

5

(a) $P = (3,0,-5)-(4,-2,-1) = (-1,2,-4)$

(b) $P = (3,0,-5)+(4,-2,-1) = (7,-2,-6)$

6

(a) $\mathbf{v}-\mathbf{w} = (4,0,-8)-(6,-1,-4) = (-2,1,-4)$

(b) $6\mathbf{u}+2\mathbf{v} = 6(-3,1,2)+2(4,0,-8) = (-10,6,-4)$

(c) $-\mathbf{v}+\mathbf{u} = -(4,0,-8)+(-3,1,2) = (-7,1,10)$

(d) $5(\mathbf{v}-4\mathbf{u}) = 5((4,0,-8)-4(-3,1,2)) = (80,-20,-80)$

(e) $-3(\mathbf{v}-8\mathbf{w}) = -3((4,0,-8)-8(6,-1,-4))$

$= -3(-44,8,24) = (132,-24,-72)$

(f) $(2\mathbf{u}-7\mathbf{w})-(8\mathbf{v}+\mathbf{u}) = (2(-3,1,2)-7(6,-1,-4))-(8(4,0,-8)+(-3,1,2))$

$= ((-6,2,4)-(42,-7,-28))-((32,0,-64)+(-3,1,2))$

$= (-48,9,32)-(29,1,-62) = (-77,8,94)$

7 $2\mathbf{u}-\mathbf{v}+\mathbf{x} = 7\mathbf{x}+\mathbf{w}$

$2(-3,1,2)-(4,0,-8)+x = 7x+(6,-1,-4)$

$(-6,2,4)-(4,0,-8)+(x_1,x_2,x_3) = (7x_1,7x_2,7x_3)+(6,-1,-4)$

$(-10+x_1,\ 2+x_2,\ 12+x_3) = (7x_1+6,\ 7x_2-1,\ 7x_3-4)$

That is: $-10+x_1 = 7x_1+6 \implies 6x_1 = -16 \implies x_1 = \frac{-8}{3}$

$2+x_2 = 7x_2-1 \implies 6x_2 = 3 \implies x_2 = \frac{1}{2}$

$12+x_3 = 7x_3-4 \implies 6x_3 = 16 \implies x_3 = \frac{8}{3}$

Therefore: $x = \left(\frac{-8}{3}, \frac{1}{2}, \frac{8}{3}\right)$

8 $c_1\mathbf{u}+c_2\mathbf{v}+c_3\mathbf{w} = (2,0,4)$

$c_1(-3,1,2)+c_2(4,0,-8)+c_3(6,-1,-4) = (2,0,4)$

$(-3c_1,c_1,2c_1)+(4c_2,0,-8c_2)+(6c_3,-c_3,-4c_3) = (2,0,4)$

$(-3c_1+4c_2+6c_3,\ c_1-c_3,\ 2c_1-8c_2-4c_3) = (2,0,4)$

That is: $-3c_1 + 4c_2 + 6c_3 = 2$
$c_1 - c_3 = 0$
$2c_1 - 8c_2 - 4c_3 = 4$

$$\Rightarrow \begin{bmatrix} -3 & 4 & 6 & 2 \\ 1 & 0 & -1 & 0 \\ 2 & -8 & -4 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Therefore $c_1 = 2, c_2 = -1, c_3 = 2$

9 $c_1(-2,9,6) + c_2(-3,2,1) + c_3(1,7,5) = (0,5,4)$
$(-2c_1, 9c_1, 6c_1) + (-3c_2, 2c_2, c_2) + (c_3, 7c_3, 5c_3) = (0,5,4)$
$(-2c_1 - 3c_2 + c_3, 9c_1 + 2c_2 + 7c_3, 6c_1 + c_2 + 5c_3) = (0,5,4)$

That is: $-2c_1 - 3c_2 + c_3 = 0$
$9c_1 + 2c_2 + 7c_3 = 5$
$6c_1 + c_2 + 5c_3 = 4$

$$\Rightarrow \begin{bmatrix} -2 & -3 & 1 & 0 \\ 9 & 2 & 7 & 5 \\ 6 & 1 & 5 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Line 3 implies: $0c_1 + 0c_2 + 0c_3 = 1$
or: $0 = 1$

Thus, there do not exist scalars c_1, c_2, c_3 such that $c_1(-2,9,6) + c_2(3,2,1) + c_3(1,7,5) = (0,5,4)$.

10 $c_1(1,2,0) + c_2(2,1,1) + c_3(0,3,1) = (0,0,0)$
$(c_1, 2c_1, 0) + (2c_2, c_2, c_2) + (0, 3c_3, c_3) = (0,0,0)$
$(c_1 + 2c_2, 2c_1 + c_2 + 3c_3, c_2 + c_3) = (0,0,0)$

That is: $c_1 + 2c_2 = 0$
$2c_1 + c_2 + 3c_3 = 0$
$c_2 + c_3 = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Only the trivial solution works.
So $c_1 = c_2 = c_3 = 0$

11

a $\left(\frac{(2+7)}{2}, \frac{(3-4)}{2}, \frac{(-2+1)}{2}\right) = \left(\frac{9}{2}, \frac{-1}{2}, \frac{-1}{2}\right)$

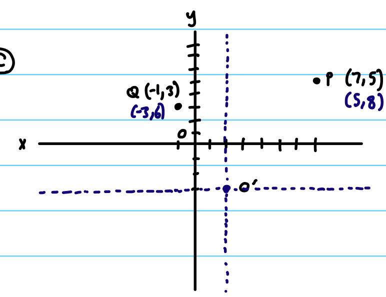
b $Q - P = (7,-4,1) - (2,3,-2) = (5,-7,3)$
$\frac{3}{4}(Q-P) = \left(\frac{15}{4}, \frac{-21}{4}, \frac{9}{4}\right)$
$P + \frac{3}{4}(Q-P) = (2,3,-2) + \left(\frac{15}{4}, \frac{-21}{4}, \frac{9}{4}\right) = \left(\frac{23}{4}, \frac{-9}{4}, \frac{1}{4}\right)$

12 $0' = (2,-3) = (k, l)$

a $xy: (7,5)$
$x' = 7 - 2 = 5$
$y' = 5 - (-3) = 8$
$x'y': (5,8)$

b $x'y' = (-3, 6)$
$x = -3 + 2 = -1$
$y = 6 + (-3) = 3$
$xy: (-1,3)$

c



⑬ <u>Proof</u>: Suppose that an xyz-coordinate system is translated to obtain an $x'y'z'$-coordinate system. Let $\mathbf{v}$ be a vector whose components are $\mathbf{v} = (v_1, v_2, v_3)$ in the xyz-system. Note that the translation from xyz to $x'y'z'$ does not affect the relative positioning of the initial and terminal points of $\mathbf{v}$. To exhibit this, let A be the initial point of $\mathbf{v}$ and let B be the terminal point of $\mathbf{v}$ in the xyz-system. Note that $\mathbf{v} = (B_x - 0,\ B_y - 0,\ B_z - 0)$.

So: $A' = (0-k,\ 0-\ell,\ 0-m) = (-k, -\ell, -m)$

$B' = (B_x - k,\ B_y - \ell,\ B_z - m)$

Then: $\mathbf{v}' = (B_x - k - (-k),\ B_y - \ell - (-\ell),\ B_z - m - (-m))$

$= (B_x, B_y, B_z) = \mathbf{v}$.

□

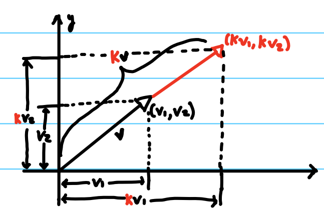
⑭ $\mathbf{u} = ((1)\cos 30,\ (1)\sin 30) = (\cos 30, \sin 30) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$\mathbf{v} = ((1)\cos 240,\ (1)\sin 240) = (\cos 240,\ \sin 240) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$\mathbf{u}+\mathbf{v} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \left(\frac{\sqrt{3}-1}{2}, \frac{1-\sqrt{3}}{2}\right)$

$\mathbf{u}-\mathbf{v} = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) - \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \left(\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}+1}{2}\right)$

⑮ <u>Proof</u>: Suppose that $\mathbf{v} = (v_1, v_2)$ and $k > 0$. Refer to the figure below:



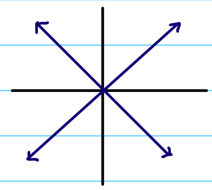
Observe the similar triangles, and so $\frac{v_1}{v_2} = \frac{kv_1}{kv_2} = k\left(\frac{v_1}{v_2}\right)$. The factor of the sides of the $k\mathbf{v}$ triangle is k, and thus multiplying $\mathbf{v}$ by k scales its components by k, proving that $k\mathbf{v} = (kv_1, kv_2)$.

□

⑯ Beginning at Figure 3.1.13, we would first position $\overrightarrow{OP_2}$ and $\overrightarrow{OP_1}$ so their initial points coincide. $(\overrightarrow{OP_2} - \overrightarrow{OP_1})$ is the vector from the terminal point of $\overrightarrow{OP_1}$ to the terminal point of $\overrightarrow{OP_2}$. $\frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1})$ compresses that vector (its components) by $1/2$. Finally, take the sum of that vector and $\overrightarrow{OP_1}$ by using the parallelogram method. This yields $\mathbf{u} = \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1})$.

□

⑰



18 Continual appication of tail-to-tip vector addition