Chapter 2 Supplementary Exercises

① $A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$ $A_1 = \begin{bmatrix} x & -4/5 \\ y & 3/5 \end{bmatrix}$ $A_2 = \begin{bmatrix} 3/5 & x \\ 4/5 & y \end{bmatrix}$

$\det(A) = (\frac{3}{5})(\frac{3}{5}) + (\frac{4}{5})(\frac{4}{5}) = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$

$\det(A_1) = (\frac{3}{5})(x) + (\frac{4}{5})(y) = \frac{3}{5}x + \frac{4}{5}y$

$\det(A_2) = (\frac{3}{5})(y) - (\frac{4}{5})(x) = -\frac{4}{5}x + \frac{3}{5}y$

$x' = \frac{\det(A_1)}{\det(A)} = \frac{3}{5}x + \frac{4}{5}y$

$y' = \frac{\det(A_2)}{\det(A)} = -\frac{4}{5}x + \frac{3}{5}y$

② $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $A_1 = \begin{bmatrix} x & -\sin\theta \\ y & \cos\theta \end{bmatrix}$ $A_2 = \begin{bmatrix} \cos\theta & x \\ \sin\theta & y \end{bmatrix}$

$\det(A) = \sin^2\theta + \cos^2\theta = 1$

$\det(A_1) = x\cos\theta + y\sin\theta$

$\det(A_2) = -x\sin\theta + y\cos\theta$

$x' = \frac{\det(A_1)}{\det(A)} = x\cos\theta + y\sin\theta$

$y' = \frac{\det(A_2)}{\det(A)} = -x\sin\theta + y\cos\theta$

③ Proof: Note that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution iff $\det(A) = 0$ (Theorem 2.3.6). Here,

$A = \begin{bmatrix} 1 & 1 & \alpha \\ 1 & 1 & \beta \\ \alpha & \beta & 1 \end{bmatrix}$. Accordingly, $\det(A) = \begin{vmatrix} 1 & \beta \\ \beta & 1 \end{vmatrix} - \begin{vmatrix} 1 & \beta \\ \alpha & 1 \end{vmatrix} + \alpha \begin{vmatrix} 1 & 1 \\ \alpha & \beta \end{vmatrix} = 1 - \beta^2 - (1 - \alpha\beta) + \alpha(\beta - \alpha) =$
$1 - \beta^2 - 1 + \alpha\beta + \alpha\beta - \alpha^2 = -\alpha^2 + 2\alpha\beta - \beta^2 = -(\alpha - \beta)^2$

($\rightarrow$) Suppose that the system has a nontrivial solution. That is, $\det(A) = 0$, and so:

$$-(\alpha-\beta)^2 = 0$$
$$(\alpha-\beta)^2 = 0$$
$$\alpha - \beta = 0$$
$$\alpha = \beta.$$

($\leftarrow$) Suppose that $\alpha = \beta$. Then:

$$-(\alpha-\beta)^2 = -(\alpha-\alpha)^2 = -(0)^2 = 0$$

Since $\det(A) = 0$, the system has a nontrivial solution.

Therefore: the system has a nontrivial solution iff $\alpha = \beta$. □

④ 2. Logical placing of 1's then 0's reveals that 3 isn't possible.

⑤ ⓐ Note that: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = R$

Thus: $a = R\sin\alpha$; $b = R\sin\beta$; $c = R\sin\gamma$

The Following equivalances hold:

(i) $b\cos\gamma + c\cos\beta = R\sin\beta\cos\gamma + R\sin\gamma\cos\beta$
$$= R(\sin\beta\cos\gamma + \sin\gamma\cos\beta)$$
$$= R\sin(\beta+\gamma)$$
$$= R\sin(180-\alpha)$$
$$= R\sin\alpha$$
$$= a$$

(ii) $c\cos\alpha + a\cos\gamma = R\sin\gamma\cos\alpha + R\sin\alpha\cos\gamma$
$$= R(\sin\gamma\cos\alpha + \sin\alpha\cos\gamma)$$
$$= R\sin(\gamma+\alpha)$$
$$= R\sin(180-\beta)$$
$$= R\sin\beta$$
$$= b$$

(iii) $a\cos\beta + b\cos\alpha = R\sin\alpha\cos\beta + R\sin\beta\cos\alpha$
$$= R(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$$
$$= R\sin(\alpha+\beta)$$
$$= R\sin(180-\gamma)$$
$$= R\sin\gamma$$
$$= c$$

Then: $\begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ Let $A = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$ and $A_{\cos\alpha} = \begin{bmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{bmatrix}$

$$\det(A) = c(ab) + b(ca) = abc + abc = 2abc$$

$$\det(A_{\cos\alpha}) = b(ba) - a(a^2-c^2) = ab^2 - a(a^2-c^2) = a(b^2 - a^2 + c^2)$$

Thus: $\cos\alpha = \dfrac{a(b^2+c^2-a^2)}{2abc} = \dfrac{b^2+c^2-a^2}{2bc}$

(b) $A_{\cos\beta} = \begin{bmatrix} 0 & a & b \\ c & b & a \\ b & c & 0 \end{bmatrix}$ $A_{\cos\gamma} = \begin{bmatrix} 0 & c & a \\ c & 0 & b \\ b & a & c \end{bmatrix}$

$$\det(A_{\cos\beta}) = a(ab) + b(c^2-b^2) = a^2b + b(c^2-b^2) = b(a^2+c^2-b^2)$$

$$\det(A_{\cos\gamma}) = -c(c^2-b^2) + a(ca) = c(b^2-c^2) + a^2c = c(b^2-c^2+a^2)$$

Thus: $\cos\beta = \dfrac{b(a^2+c^2-b^2)}{2abc} = \dfrac{a^2+c^2-b^2}{2ac}$; $\cos\gamma = \dfrac{c(a^2+b^2-c^2)}{2abc} = \dfrac{a^2+b^2-c^2}{2ab}$

⑥ $x(1-\lambda)-2y=0$

$x-y(1+\lambda)=0$

$$\begin{array}{ccc} A & x & b \\ \begin{bmatrix} 1-\lambda & -2 \\ 1 & -1-\lambda \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix} = & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

$\det(A) = (1-\lambda)(-1-\lambda)+2$

$= \lambda^2 - 1 + 2 = \lambda^2 + 1$

$\det(A_x) = \begin{vmatrix} 0 & -2 \\ 0 & -1-\lambda \end{vmatrix} = 0$

$\det(A_y) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 0 \end{vmatrix} = 0$

$x = \frac{\det(A_x)}{\det(A)} = \frac{0}{1} = 0$

$y = \frac{\det(A_y)}{\det(A)} = \frac{0}{1} = 0$

⑦ Proof: Suppose A is invertible. Then by Theorem 2.3.6, $\det(A) \neq 0$. Since $\det(A) \neq 0$, $\frac{1}{\det(A)} \neq 0$ and therefore $\det(A^{-1}) \neq 0$ (Theorem 2.3.5). Note that $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$, and so $\det(A)A^{-1} = \operatorname{adj}(A)$. Then, $\det(\det(A)A^{-1}) = \det(\operatorname{adj}(A))$. Since $\det(A)$ is a nonzero scalar, $\det(\det(A)A^{-1}) \neq 0$ and thus $\det(\operatorname{adj}(A)) \neq 0$ (see Theorem 1.4.8). Therefore $\operatorname{adj}(A)$ is invertible. The following equivalences then hold:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$
$$A^{-1}[\operatorname{adj}(A)]^{-1} = \frac{1}{\det(A)}$$
$$[\operatorname{adj}(A)]^{-1} = A\frac{1}{\det(A)} = \frac{1}{\det(A)} A$$

Also:

$$A = \frac{1}{\det(A^{-1})} \operatorname{adj}(A^{-1})$$
$$A = \det(A) \operatorname{adj}(A^{-1})$$
$$\frac{1}{\det(A)} A = \operatorname{adj}(A^{-1})$$

□

⑧ Proof: Suppose A is an $n \times n$ matrix. Note that $A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$. Then the following equations are implied:

$$A^{-1}\det(A) = \operatorname{adj}(A)$$
$$I \det(A) = A \operatorname{adj}(A)$$
$$\det(I \det(A)) = \det(A \operatorname{adj}(A))$$
$$\det(A)^n \det(I) = \det(A)\det(\operatorname{adj}(A))$$
$$\det(A)^{n-1} = \det(\operatorname{adj}(A))$$

□

(9) Proof: Suppose that $f_1(x), f_2(x), g_1(x), g_2(x)$ are differentiable functions and $W = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}$.
Then $W = f_1(x)g_2(x) - f_2(x)g_1(x)$, and so: $\frac{dW}{dx} = f_1(x)g_2'(x) + f_1'(x)g_2(x) - f_2(x)g_1'(x) - f_2'(x)g_1(x)$.
Rearranged: $f_1'(x)g_2(x) - f_2'(x)g_1(x) + f_1(x)g_2'(x) - f_2(x)g_1'(x)$. Therefore $\frac{dW}{dx} = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}$ □

(10)

(a) Area ABC = area $ADEC$ + area $CEFB$ − area $ADFB$

$$= \tfrac{1}{2}(x_3 - x_1)(y_1 + y_3) + \tfrac{1}{2}(x_2 - x_3)(y_2 + y_3) - \tfrac{1}{2}(x_2 - x_1)(y_1 + y_2)$$

$$= \tfrac{1}{2}(x_3y_1 + x_3y_3 - x_1y_1 - x_1y_3 + x_2y_2 + x_2y_3 - x_3y_2 - x_3y_3 - x_2y_1 - x_2y_2 + x_1y_1 + x_1y_2)$$

$$= \tfrac{1}{2}(x_1y_2 + x_3y_1 + x_2y_3 - x_3y_2 - x_1y_3 - x_2y_1)$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(b) Area $ABC = \frac{1}{2}\begin{vmatrix} 3 & 3 & 1 \\ 4 & 0 & 1 \\ -2 & -1 & 1 \end{vmatrix} = \frac{1}{2}(-19) = -9.5$.

Accordingly, the area of ABC is 9.5.

(11) Proof: Suppose the entries in each row of an $n \times n$ matrix A add up to zero.
Note that $A\mathbf{x} = 0$ has two solutions: (1) the $n \times 1$ matrix, each of whose entries is zero, and; (2) the $n \times 1$ matrix, each of whose entries is one. Thus, $A\mathbf{x} = 0$ does not only have the trivial solution. Then, per Theorem 2.3.6, $\det(A) = 0$. □

(12) Note that each time rows are swapped, the determinant is multiplied by -1. We can use this fact to find $\det(B)$: $\det(B) = (-1)^{\frac{n(n-1)}{2}} \det(A)$

(13)

(a) The i^{th} and j^{th} columns of A^{-1} will be interchanged.
(b) The i^{th} column of A^{-1} is multiplied by a nonzero scalar $\frac{1}{c}$.
(c) $-c$ times the j^{th} column is added to the i^{th} column

(14) Proof: Let A be an $n \times n$ matrix, and suppose that B_1 is obtained by adding the same number t to each entry in the i^{th} row of A and B_2 is obtained by subtracting t from each entry in the i^{th} row of A. Note that A, B_1, and B_2 all differ only in a single row: row i. Also, 2 times the i^{th} row of A is obtained by adding corresponding entries in the i^{th} rows of B_1 and B_2. Apply Theorem 2.3.1 and pull the row i common factor 2 out of the det function:

$$2\det(A) = \det(A) + \det(B), \text{ or } \det(A) = \tfrac{1}{2}[\det(B_1) + \det(B_2)]$$ □

(15)

(a) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & \lambda - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda - a_{33} \end{bmatrix}$$

$\det(\lambda I - A) = (\lambda - a_{11})((\lambda - a_{22})(\lambda - a_{33}) - a_{23}a_{32}) + a_{12}(-a_{21}(\lambda - a_{33}) - a_{23}a_{31}) - a_{13}(a_{21}a_{32} + a_{31}(\lambda - a_{22}))$

$= (\lambda - a_{11})(\lambda^2 - a_{33}\lambda - a_{22}\lambda + a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{21}a_{33} - a_{21}\lambda - a_{23}a_{31}) - a_{13}(a_{21}a_{32} + a_{31}\lambda - a_{31}a_{22})$

$= \lambda^3 - a_{33}\lambda^2 - a_{22}\lambda^2 + a_{22}a_{33}\lambda - a_{23}a_{32}\lambda - a_{11}\lambda^2 + a_{11}a_{33}\lambda + a_{11}a_{22}\lambda - a_{11}a_{22}a_{33} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} - a_{12}a_{21}\lambda - a_{12}a_{23}a_{31}$
$- a_{13}a_{21}a_{32} - a_{13}a_{31}\lambda + a_{13}a_{31}a_{22}$

$= \lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32})\lambda - (a_{11}a_{22}a_{33}$
$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31})$

(b) $\lambda^3 - tr(A)\lambda^2 + (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32})\lambda - \det(A)$

(16) Let $A = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$. Note that $\det(A) = \det(A)^T$, and so, we can find $\det(A)^T$ instead of $\det(A)$. Note that:

$$\det(A)^T = \begin{vmatrix} \sin\alpha & \sin\beta & \sin\gamma \\ \cos\alpha & \cos\beta & \cos\gamma \\ \sin(\alpha+\delta) & \sin(\beta+\delta) & \sin(\gamma+\delta) \end{vmatrix} = \sin\delta\cos\delta \begin{vmatrix} \sin\alpha\cos\delta & \sin\beta\cos\delta & \sin\gamma\cos\delta \\ \cos\alpha\sin\delta & \cos\beta\sin\delta & \cos\gamma\sin\delta \\ \sin(\alpha+\delta) & \sin(\beta+\delta) & \sin(\gamma+\delta) \end{vmatrix}$$

$$= \sin\delta\cos\delta \begin{vmatrix} \sin\alpha\cos\delta & \sin\beta\cos\delta & \sin\gamma\cos\delta \\ \sin(\alpha+\delta) & \sin(\beta+\delta) & \sin(\gamma+\delta) \\ \sin(\alpha+\delta) & \sin(\beta+\delta) & \sin(\gamma+\delta) \end{vmatrix} = \sin\delta\cos\delta(0) = 0$$

This step results from the fact that for any x and y, $\sin(x+y) = \sin x\cos y + \cos x \sin y$. We added Row 1 to Row 2.

(17)

Let $A = \begin{bmatrix} 2 & 1 & 3 & 7 & 5 \\ 3 & 8 & 7 & 9 & 8 \\ 3 & 4 & 1 & 6 & 2 \\ 4 & 0 & 2 & 2 & 3 \\ 7 & 9 & 1 & 5 & 4 \end{bmatrix}$. Note that you can add a multiple of any column to another column without changing the determinant. Thus, we add the following to column 5:

$10 \times$ column 4; $100 \times$ column 3; $1{,}000 \times$ column 2; $10{,}000 \times$ column 1.

This results in the following matrix, say, A':

$$A' = \begin{bmatrix} 2 & 1 & 3 & 7 & 21{,}375 \\ 3 & 8 & 7 & 9 & 38{,}798 \\ 3 & 4 & 1 & 6 & 34{,}162 \\ 4 & 0 & 2 & 2 & 40{,}223 \\ 7 & 9 & 1 & 5 & 79{,}154 \end{bmatrix}$$

Finally, let:

$$A'' = \begin{bmatrix} 2 & 1 & 3 & 7 & 1{,}125 \\ 3 & 8 & 7 & 9 & 2{,}042 \\ 3 & 4 & 1 & 6 & 1{,}798 \\ 4 & 0 & 2 & 2 & 2{,}117 \\ 7 & 9 & 1 & 5 & 4{,}116 \end{bmatrix}$$

Note that $\det(A) = \det(A') = 19 \det(A'')$, and so $\det(A)$ is divisible by 19.

(18)

(a)

$$\begin{aligned} x_2 + 9x_3 &= \lambda x_1 \\ x_1 + 4x_2 - 7x_3 &= \lambda x_2 \\ x_1 \qquad - 3x_3 &= \lambda x_3 \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 9 \\ 1 & 4 & -7 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 9 \\ 1 & 4 & -7 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 9 \\ 1 & 4 & -7 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda & -1 & -9 \\ -1 & \lambda-4 & 7 \\ -1 & 0 & \lambda+3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & -9 \\ -1 & \lambda-4 & 7 \\ -1 & 0 & \lambda+3 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & -9 \\ -1 & \lambda-4 & 7 \\ -1 & 0 & \lambda+3 \end{vmatrix} = 0$$

$$= -\begin{vmatrix} -\lambda & 1 & 9 \\ 1 & 4-\lambda & -7 \\ 1 & 0 & -\lambda-3 \end{vmatrix} = 0$$

$$= -(-\lambda^3 + \lambda^2 + 22\lambda - 40) = 0$$

$$= (\lambda-2)(\lambda-4)(\lambda+5) = 0$$

Eigenvalues: $\lambda = 2, 4, -5$

Eigenvectors:

$\lambda = 2$: $\begin{bmatrix} 2 & -1 & -9 \\ -1 & -2 & 7 \\ -1 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \mathbf{x} = \begin{bmatrix} 5t \\ t \\ t \end{bmatrix}$

$\lambda = 4$: $\begin{bmatrix} 4 & -1 & -9 \\ -1 & 0 & 7 \\ -1 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \mathbf{x} = \begin{bmatrix} 7t \\ 19t \\ t \end{bmatrix}$

$\lambda = -5$: $\begin{bmatrix} -5 & -1 & -9 \\ -1 & -9 & 7 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \mathbf{x} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix}$

(b)

$$x_2 + x_3 = \lambda x_1$$
$$x_1 - x_3 = \lambda x_2$$
$$x_1 + 5x_2 + 3x_3 = \lambda x_3$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & 1 \\ -1 & -5 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & 1 \\ -1 & -5 & \lambda-3 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & 1 \\ -1 & -5 & \lambda-3 \end{vmatrix} = 0$$

$$= (\lambda - 1)^3 = 0$$

Eigenvalue: $\lambda = 1$

Eigenvector:

$\lambda = 1$: $\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & -5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \mathbf{x} = \begin{bmatrix} \frac{1}{2}t \\ -\frac{1}{2}t \\ t \end{bmatrix}$