2.2: Equivalences Involving Quantifiers

Exercise 1:

(a) $\forall x \exists y (Mx \rightarrow (Fxy \wedge Hy))$

$\neg \forall x \exists y (Mx \rightarrow (Fxy \wedge Hy))$

$\exists x \neg \exists y (Mx \rightarrow (Fxy \wedge Hy))$

$\exists x \forall y \neg (\neg Mx \vee (Fxy \wedge Hy))$

$\exists x \forall y (\neg\neg Mx \wedge \neg (Fxy \wedge Hy))$

$\exists x \forall y (Mx \wedge (\neg Fxy \vee \neg Hy))$

There is a math major whose friends don't all need help with their homework.

(b) $\forall x \exists y \forall z (Rxy \wedge Dyz)$

$\neg \forall x \exists y \forall z (Rxy \wedge Dyz)$

$\exists x \neg \exists y \forall z (Rxy \wedge Dyz)$

$\exists x \forall y \neg \forall z (Rxy \wedge Dyz)$

$\exists x \forall y \exists z \neg (Rxy \wedge Dyz)$

$\exists x \forall y \exists z (\neg Rxy \vee \neg Dyz)$

$\exists x \forall y \exists z (Rxy \rightarrow \neg Dyz)$

There is someone such that all of his roomates do not dislike someone.

(c) $\neg (\forall x)((A \cup B) \subseteq C \setminus D)$

$\neg (\forall x)((x \in A \cup B) \rightarrow (x \in C \setminus D))$

$(\exists x) \neg ((\neg x \in A \cup B) \vee (x \in C \setminus D))$

$(\exists x)((x \in A \cup B) \wedge (x \notin C \setminus D))$

There is an x s.t. $x \in A \cup B$ and $x \notin C \setminus D$.

(d) $\neg (\exists x)(\forall y)[y > x \rightarrow (\exists z)(z^2 + 5z = y)]$

$(\forall x)(\exists y) \neg [\neg (y > x) \vee (\exists z)(z^2 + 5z = y)]$

$(\forall x)(\exists y)[(y > x) \wedge \neg (\exists z)(z^2 + 5z = y)]$

$(\forall x)(\exists y)[(y > x) \wedge (\forall z) \neg (z^2 + 5z = y)]$

$(\forall x)(\exists y)[(y > x) \wedge (\forall z)(z^2 + 5z \neq y)]$

For all x, there is a greater y such that for all z, $z^2 + 5z \neq y$.

Exercise 2:

(a) $\neg \exists x (Fx \wedge \neg \exists y (Rxy))$

$\forall x \neg (Fx \wedge \forall y \neg (Rxy))$

$\forall x (\neg Fx \vee \neg \forall y \neg (Rxy))$

$\forall x (\neg Fx \vee \exists y (Rxy))$

$\forall x (Fx \rightarrow \exists y (Rxy))$

Everyone in the freshmen class has a roommate.

ⓑ $\neg(\forall x \exists y (Lxy) \wedge \neg \exists x \forall y (Lxy))$

$\neg \forall x \exists y (Lxy) \vee \exists x \forall y (Lxy)$

$\exists x \forall y (\neg Lxy) \vee \exists x \forall y (Lxy)$

Someone doesn't like anyone or someone likes everyone.

ⓒ $\neg(\forall x ((x \in A) \rightarrow \exists y ((y \in B) \wedge (x \in C \leftrightarrow y \in C))))$

$\exists x \neg (\neg (x \in A) \vee \exists y ((y \in B) \wedge (x \in C \leftrightarrow y \in C)))$

$\exists x ((x \in A) \wedge \forall y \neg ((y \in B) \wedge (x \in C \leftrightarrow y \in C)))$

$\exists x ((x \in A) \wedge \forall y ((y \notin B) \vee (x \notin C \leftrightarrow y \in C)))$

$\exists x ((x \in A) \wedge \forall y ((y \in B) \rightarrow (x \notin C \leftrightarrow y \in C)))$

ⓓ $\neg(\forall y > 0 \exists x (ax^2 + bx + c = y))$

$\neg(\forall y ((y > 0) \rightarrow \exists x (ax^2 + bx + c = y)))$

$\exists y \neg (\neg (y > 0) \vee \exists x (ax^2 + bx + c = y))$

$\exists y ((y > 0) \wedge \forall x \neg (ax^2 + bx + c = y))$

$\exists y ((y > 0) \wedge \forall x (ax^2 + bx + c \neq y))$

Exercise 3:

ⓐ True

ⓑ False

ⓒ True

ⓓ True

Exercise 4:

$\neg \forall x P(x)$

$\neg \forall x \neg\neg P(x)$

$\neg\neg \exists x \neg P(x)$

$\exists x \neg P(x)$

Exercise 5:

$\neg \exists x \in A P(x)$

$\neg \exists x (x \in A \wedge P(x))$

$\forall x \neg (x \in A \wedge P(x))$

$\forall x (x \notin A \vee \neg P(x))$

$\forall x (x \in A \rightarrow \neg P(x))$

Thus, $\forall x \in A \neg P(x)$.

Exercise 6:

$\exists x (P(x) \vee Q(x))$

$\neg\neg \exists x (P(x) \vee Q(x))$

$\neg \forall x \neg (P(x) \vee Q(x))$

$\neg \forall x (\neg P(x) \wedge \neg Q(x))$

$\neg (\forall x \neg P(x) \wedge \forall x \neg Q(x))$

$\neg \forall x \neg P(x) \vee \neg \forall x \neg Q(x)$

$\exists x P(x) \vee \exists x Q(x)$

Exercise 7:

$\exists x (P(x) \rightarrow Q(x))$

$\exists x (\neg P(x) \vee Q(x))$

$\exists x \neg P(x) \vee \exists x Q(x)$

$\neg \forall x P(x) \vee \exists x Q(x)$

$\forall x P(x) \rightarrow \exists x Q(x)$.

Exercise 8:

$\forall x (x \in A \rightarrow P(x)) \wedge \forall x (x \in B \rightarrow P(x))$

$\forall x ((x \in A \rightarrow P(x)) \wedge (x \in B \rightarrow P(x)))$

$\forall x ((x \notin A \vee P(x)) \wedge (x \notin B \vee P(x)))$

$\forall x ((x \notin A \wedge x \notin B) \vee P(x))$

$\forall x (\neg (x \notin A \wedge x \notin B) \rightarrow P(x))$

$\forall x ((x \in A \vee x \in B) \rightarrow P(x))$

$\forall x (A \cup B) P(x)$

Exercise 9:

Let $P(x) = x$ is a person and $Q(x) = x$ is a quilt.

$\forall x (P(x) \vee Q(x)) \Rightarrow$ For all x, x is a person or x is a quilt.

$\forall x P(x) \vee \forall x Q(x) \Rightarrow$ all x are people or all x are quilts.

The propositions are not equivalent.

Exercise 10:

ⓐ $\exists x \in A\, P(x) \vee \exists x \in B\, P(x)$

$\exists x (x \in A \wedge P(x)) \vee \exists x (x \in B \wedge P(x))$

$\exists x ((x \in A \wedge P(x)) \vee (x \in B \wedge P(x)))$

$\exists x ((x \in A \vee x \in B) \wedge P(x))$

$\exists x (x \in A \cup B \wedge P(x))$

$\exists x \in (A \cup B)\, P(x)$

ⓑ No. $\exists x \in A\, P(x) \wedge \exists x \in B\, P(x)$: There is a person in A and a person in B.

$\exists x \in (A \cap B) P(x)$: There is a person who is in both A and B.

Exercise 11:

$A \subseteq B$

$\forall x (x \in A \rightarrow x \in B)$

$\forall x (x \notin A \vee x \in B)$

$\forall x \neg (x \in A \wedge x \notin B)$

$\neg \exists x (x \in A \wedge x \notin B)$

$A \setminus B = \emptyset$

$\neg \exists x (x \in A \wedge x \notin B)$

$\forall x \neg (x \in A \wedge x \notin B)$

$\forall x (x \notin A \vee x \in B)$

Exercise 12:

$C \subseteq A \cup B$

$\forall x (x \in C \rightarrow x \in A \cup B)$

$\forall x (x \notin C \vee x \in A \vee x \in B)$

$\forall x ((x \notin C \vee x \in A) \vee x \in B)$

$\forall x (\neg (x \in C \wedge x \notin A) \vee x \in B)$

$\forall x ((x \in C \wedge x \notin A) \rightarrow x \in B)$

$C \setminus A \subseteq B$

Exercise 13:

ⓐ $A \subseteq B$

$\forall x\,(x \in A \rightarrow x \in B)$

$\forall x\,((x \in A \vee x \in B) \leftrightarrow x \in B)$ (by Exercise 5.11(b))

$\forall x\,(x \in A \cup B \leftrightarrow x \in B)$

$A \cup B = B$

ⓑ $A \subseteq B$

$\forall x(x \in A \rightarrow x \in B)$

$\forall x((x \in A \wedge x \in B) \leftrightarrow x \in A)$ (by Exercise 5.11(a))

$\forall x(x \in A \cap B \leftrightarrow x \in A)$

$A \cap B = A$

Exercise 14:

$A \cap B = \emptyset$

$\neg \exists x(x \in A \wedge x \in B)$

$\forall x \neg(x \in A \wedge x \in B)$

$\forall x(x \notin A \vee x \notin B)$

$\forall x(x \in A \rightarrow x \notin B)$

$\forall x((x \in A \wedge x \notin B) \leftrightarrow x \in A)$ (by Exercise 5.11 (a))

$\forall x(x \in A \setminus B \leftrightarrow x \in A)$

$A \setminus B = A$

Exercise 15:

ⓐ x is the teacher of only one student.

ⓑ Someone is the teacher of only one person.

ⓒ Only one person is the teacher of someone.

ⓓ Only one person is the teacher of someone.

ⓔ Only one person is the teacher of only one person. } equivalent.

ⓕ Only one teacher has only one student