ⓑ $(A+B)(A-B) = A^2 - AB + BA - B^2$

Exercise 33:

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}, \begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$

Exercise 34:

ⓐ If A is not singular, then A^T is not singular.

Thus, if A is invertible then A^T is invertible.

ⓑ Suppose A is invertible. So, $AA^{-1} = A^{-1}A = I$

We must show that $A^T(A^{-1})^T = (A^{-1})^T A^T = I$

Note that: $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$

$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$

Thus, the statement is true.

Exercise 35:

ⓐ Sometimes False. $(AB)^2 = (AB)(AB) = ABAB \neq A^2B^2$

ⓑ $(A-B)^2 = (A-B)(A-B) = A^2 - AB - BA + B^2$

$(B-A)^2 = (B-A)(B-A) = B^2 - BA - AB + A^2$ } Always true

ⓒ $(AB^{-1})(BA^{-1}) = A(B^{-1}B)A^{-1} = AI_nA^{-1} = AA^{-1} = I_n \Rightarrow$ Always true

ⓓ Sometimes false. E.g., when $A = B$.

$\boxed{\text{1.5: Elementary Matrices and a Method For Finding } A^{-1}}$

Exercise 1:

ⓐ Yes ⓑ No ⓒ Yes ⓓ Yes ⓔ No ⓕ Yes ⓖ No.

Exercise 2:

ⓐ $3R_1$ added to R_2 ⓑ $\frac{1}{3} \times R_3$ ⓒ $R_1 \leftrightarrow R_4$ ⓓ $\frac{1}{7}R_3$ added to R_1

Exercise 3:

ⓐ $$\underset{E_1}{\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}}\underset{A}{\begin{bmatrix}3&4&1\\2&-7&-1\\8&1&5\end{bmatrix}}=\underset{B}{\begin{bmatrix}8&1&5\\2&-7&-1\\3&4&1\end{bmatrix}}$$

ⓑ $$\underset{E_2}{\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}}\underset{B}{\begin{bmatrix}8&1&5\\2&-7&-1\\3&4&1\end{bmatrix}}=\underset{A}{\begin{bmatrix}3&4&1\\2&-7&-1\\8&1&5\end{bmatrix}}$$

ⓒ $$\underset{E_3}{\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}}\underset{A}{\begin{bmatrix}3&4&1\\2&-7&-1\\8&1&5\end{bmatrix}}=\underset{C}{\begin{bmatrix}3&4&1\\2&-7&-1\\2&-7&3\end{bmatrix}}$$

ⓓ $$\underset{E_4}{\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}}\underset{C}{\begin{bmatrix}3&4&1\\2&-7&-1\\2&-7&3\end{bmatrix}}=\underset{A}{\begin{bmatrix}3&4&1\\2&-7&-1\\8&1&5\end{bmatrix}}$$

Exercise 4:

No. This requires more than one elementary row operation.

Exercise 5:

ⓐ $$\left[\begin{array}{cc|cc}1&4&1&0\\2&7&0&1\end{array}\right]\xrightarrow[+R_2]{-2\times R_1}\left[\begin{array}{cc|cc}1&4&1&0\\0&-1&-2&1\end{array}\right]\xrightarrow{R_2\times -1}\left[\begin{array}{cc|cc}1&4&1&0\\0&1&2&-1\end{array}\right]\xrightarrow[+R_1]{-4\times R_2}\left[\begin{array}{cc|cc}1&0&-7&4\\0&1&2&-1\end{array}\right]$$

ⓑ $$\left[\begin{array}{cc|cc}-3&6&1&0\\4&5&0&1\end{array}\right]\xrightarrow{-\frac{1}{3}\times R_1}\left[\begin{array}{cc|cc}1&-2&-\frac{1}{3}&0\\4&5&0&1\end{array}\right]\xrightarrow[+R_2]{-4\times R_1}\left[\begin{array}{cc|cc}1&-2&-\frac{1}{3}&0\\0&13&\frac{4}{3}&1\end{array}\right]\xrightarrow{\frac{1}{13}\times R_2}\left[\begin{array}{cc|cc}1&-2&-\frac{1}{3}&0\\0&1&\frac{4}{39}&\frac{1}{13}\end{array}\right]$$

$$\xrightarrow[+R_1]{2\times R_2}\left[\begin{array}{cc|cc}1&0&-\frac{5}{39}&\frac{2}{13}\\0&1&\frac{4}{39}&\frac{1}{13}\end{array}\right]$$

ⓒ $$\left[\begin{array}{cc|cc}6&-4&1&0\\-3&2&0&1\end{array}\right]\xrightarrow[+R_1]{2\times R_2}\left[\begin{array}{cc|cc}0&0&1&2\\-3&2&0&1\end{array}\right]$$ Not invertible.

Exercise 6:

ⓐ $$\left[\begin{array}{ccc|ccc}3&4&-1&1&0&0\\1&0&3&0&1&0\\2&5&-4&0&0&1\end{array}\right]\xrightarrow{\frac{1}{3}\times R_1}\left[\begin{array}{ccc|ccc}1&\frac{4}{3}&-\frac{1}{3}&\frac{1}{3}&0&0\\1&0&3&0&1&0\\2&5&-4&0&0&1\end{array}\right]\xrightarrow[+R_2]{-1\times R_1}\left[\begin{array}{ccc|ccc}1&\frac{4}{3}&-\frac{1}{3}&\frac{1}{3}&0&0\\0&-\frac{4}{3}&\frac{10}{3}&-\frac{1}{3}&1&0\\2&5&-4&0&0&1\end{array}\right]\xrightarrow[+R_3]{-2\times R_1}$$

$$\left[\begin{array}{ccc|ccc}1&\frac{4}{3}&-\frac{1}{3}&\frac{1}{3}&0&0\\0&-\frac{4}{3}&\frac{10}{3}&-\frac{1}{3}&1&0\\0&\frac{7}{3}&-\frac{10}{3}&-\frac{2}{3}&0&1\end{array}\right]\xrightarrow{R_2\times -\frac{3}{4}}\left[\begin{array}{ccc|ccc}1&\frac{4}{3}&-\frac{1}{3}&\frac{1}{3}&0&0\\0&1&-\frac{5}{2}&\frac{1}{4}&-\frac{3}{4}&0\\0&\frac{7}{3}&-\frac{10}{3}&-\frac{2}{3}&0&1\end{array}\right]\xrightarrow[+R_3]{-\frac{7}{3}\times R_2}\left[\begin{array}{ccc|ccc}1&\frac{4}{3}&-\frac{1}{3}&\frac{1}{3}&0&0\\0&1&-\frac{5}{2}&\frac{1}{4}&-\frac{3}{4}&0\\0&0&\frac{5}{2}&-\frac{5}{4}&\frac{7}{4}&1\end{array}\right]\xrightarrow[+R_1]{-\frac{4}{3}\times R_2}$$

$$\left[\begin{array}{ccc|ccc}1&0&3&0&1&0\\0&1&-\frac{5}{2}&1/4&-3/4&0\\0&0&\frac{5}{2}&-5/4&7/4&1\end{array}\right]\xrightarrow{R_3\times\frac{2}{5}}\left[\begin{array}{ccc|ccc}1&0&3&0&1&0\\0&1&-5/2&1/4&-3/4&0\\0&0&1&-1/2&7/10&2/5\end{array}\right]\xrightarrow[+R_2]{\frac{5}{2}\times R_3}\left[\begin{array}{ccc|ccc}1&0&3&0&1&0\\0&1&0&-1&1&1\\0&0&1&-1/2&7/10&2/5\end{array}\right]\xrightarrow[+R_1]{-3\times R_3}\left[\begin{array}{ccc|ccc}1&0&0&3/2&-11/10&-6/5\\0&1&0&-1&1&1\\0&0&1&-1/2&7/10&2/5\end{array}\right]$$

(b) $$\left[\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1\times R_1} \left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_2]{-2\times R_1} \left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_3]{4\times R_1} \left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$ The matrix is not invertible.

(c) $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_3]{-1\times R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[+R_3]{-1\times R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}\times R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \xrightarrow[+R_1]{-1\times R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \xrightarrow[+R_2]{-1\times R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

(d) $$\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}\times R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_2]{-2\times R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_3]{-2\times R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[+R_1]{-3\times R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 7/2 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[+R_3]{-1\times R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 7/2 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow[+R_1]{-3\times R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

(e) $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_3]{-1\times R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}\times R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right] \xrightarrow[+R_1]{-1\times R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array}\right]$$

Exercise 7:

(a) $$\left[\begin{array}{ccc|ccc} 1/5 & 1/5 & -2/5 & 1 & 0 & 0 \\ 1/5 & 1/5 & 1/10 & 0 & 1 & 0 \\ 1/5 & -4/5 & 1/10 & 0 & 0 & 1 \end{array}\right] \xrightarrow{5\times R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 1/5 & 1/5 & 1/10 & 0 & 1 & 0 \\ 1/5 & -4/5 & 1/10 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_2]{-\frac{1}{5}\times R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & 1/2 & -1 & 1 & 0 \\ 1/5 & -4/5 & 1/10 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_3]{-\frac{1}{5}\times R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & 1/2 & -1 & 1 & 0 \\ 0 & -1 & 1/2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & -1 & 1/2 & -1 & 0 & 1 \\ 0 & 0 & 1/2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{-1\times R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -1/2 & 1 & 0 & -1 \\ 0 & 0 & 1/2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{2\times R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 5 & 0 & 0 \\ 0 & 1 & -1/2 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right] \xrightarrow[+R_1]{-1\times R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3/2 & 4 & 0 & 1 \\ 0 & 1 & -1/2 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[+R_1]{\frac{3}{2}\times R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & -1/2 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right] \xrightarrow[+R_2]{\frac{1}{2}\times R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 2 & 0 \end{array}\right]$$

(b) $$\left[\begin{array}{ccc|ccc} \sqrt{2} & 3\sqrt{2} & 0 & 1 & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{\sqrt{2}}\times R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \sqrt{2}/2 & 0 & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_2]{4\sqrt{2}\times R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \sqrt{2}/2 & 0 & 0 \\ 0 & 13\sqrt{2} & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13\sqrt{2}}\times R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & \sqrt{2}/2 & 0 & 0 \\ 0 & 1 & 0 & 4\sqrt{2}/26 & \sqrt{2}/26 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_1]{-3\times R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \sqrt{2}/26 & -3\sqrt{2}/26 & 0 \\ 0 & 1 & 0 & 4\sqrt{2}/26 & \sqrt{2}/26 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

(c)
$$\left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 1&3&0&0&0&1&0&0\\ 1&3&5&0&0&0&1&0\\ 1&3&5&7&0&0&0&1 \end{array}\right] \xrightarrow[+R_2]{-1\times R_1} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&3&0&0&-1&1&0&0\\ 1&3&5&0&0&0&1&0\\ 1&3&5&7&0&0&0&1 \end{array}\right] \xrightarrow[+R_3]{-1\times R_1} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&3&0&0&-1&1&0&0\\ 0&3&5&0&-1&0&1&0\\ 1&3&5&7&0&0&0&1 \end{array}\right]$$

$$\xrightarrow[+R_4]{-1\times R_1} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&3&0&0&-1&1&0&0\\ 0&3&5&0&-1&0&1&0\\ 0&3&5&7&-1&0&0&1 \end{array}\right] \xrightarrow[+R_3]{-1\times R_2} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&3&0&0&-1&1&0&0\\ 0&0&5&0&0&-1&1&0\\ 0&3&5&7&-1&0&0&1 \end{array}\right] \xrightarrow[+R_4]{-1\times R_2} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&3&0&0&-1&1&0&0\\ 0&0&5&0&0&-1&1&0\\ 0&0&5&7&0&-1&0&1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}\times R_2} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&1&0&0&-1/3&1/3&0&0\\ 0&0&5&0&0&-1&1&0\\ 0&0&5&7&0&-1&0&1 \end{array}\right] \xrightarrow[+R_4]{-1\times R_3} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&1&0&0&-1/3&1/3&0&0\\ 0&0&5&0&0&-1&1&0\\ 0&0&0&7&0&0&-1&1 \end{array}\right] \xrightarrow{\frac{1}{5}\times R_3} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&1&0&0&-\frac{1}{3}&\frac{1}{3}&0&0\\ 0&0&1&0&0&-\frac{1}{5}&\frac{1}{5}&0\\ 0&0&0&7&0&0&-1&1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}\times R_4} \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&1&0&0&-\frac{1}{3}&\frac{1}{3}&0&0\\ 0&0&1&0&0&-\frac{1}{5}&\frac{1}{5}&0\\ 0&0&0&1&0&0&-\frac{1}{7}&\frac{1}{7} \end{array}\right]$$

(d)
$$\left[\begin{array}{cccc|cccc} -8&17&2&1/3&1&0&0&0\\ 4&0&2/5&-9&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ -1&13&4&2&0&0&0&1 \end{array}\right]$$

The matrix is not invertible because it cannot be reduced to I_4.

(e)
$$\left[\begin{array}{cccc|cccc} 0&0&2&0&1&0&0&0\\ 1&0&0&1&0&1&0&0\\ 0&-1&3&0&0&0&1&0\\ 2&1&5&-3&0&0&0&1 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{cccc|cccc} 1&0&0&1&0&1&0&0\\ 0&0&2&0&1&0&0&0\\ 0&-1&3&0&0&0&1&0\\ 2&1&5&-3&0&0&0&1 \end{array}\right] \xrightarrow[+R_4]{-2\times R_1} \left[\begin{array}{cccc|cccc} 1&0&0&1&0&1&0&0\\ 0&0&2&0&1&0&0&0\\ 0&-1&3&0&0&0&1&0\\ 0&1&5&-5&0&-2&0&1 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1&0&0&1&0&1&0&0\\ 0&1&5&-5&0&-2&0&1\\ 0&-1&3&0&0&0&1&0\\ 0&0&2&0&1&0&0&0 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{cccc|cccc} 1&0&0&1&0&1&0&0\\ 0&1&5&-5&0&-2&0&1\\ 0&0&8&-5&0&-2&1&1\\ 0&0&2&0&1&0&0&0 \end{array}\right] \xrightarrow{R_3\leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1&0&0&1&0&1&0&0\\ 0&1&5&-5&0&-2&0&1\\ 0&0&2&0&1&0&0&0\\ 0&0&8&-5&0&-2&1&1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}\times R_3} \left[\begin{array}{cccc|cccc} 1&0&0&1&0&1&0&0\\ 0&1&5&-5&0&-2&0&1\\ 0&0&1&0&1/2&0&0&0\\ 0&0&8&-5&0&-2&1&1 \end{array}\right] \xrightarrow[+R_4]{-8\times R_3} \left[\begin{array}{cccc|cccc} 1&0&0&1&0&1&0&0\\ 0&1&5&-5&0&-2&0&1\\ 0&0&1&0&1/2&0&0&0\\ 0&0&0&-5&-4&-2&1&1 \end{array}\right] \xrightarrow{-\frac{1}{5}\times R_4} \left[\begin{array}{cccc|cccc} 1&0&0&1&0&1&0&0\\ 0&1&5&-5&0&-2&0&1\\ 0&0&1&0&1/2&0&0&0\\ 0&0&0&1&4/5&2/5&-1/5&-1/5 \end{array}\right]$$

$$\xrightarrow[+R_2]{-5\times R_3} \left[\begin{array}{cccc|cccc} 1&0&0&1&0&1&0&0\\ 0&1&0&-5&-5/2&-2&0&1\\ 0&0&1&0&1/2&0&0&0\\ 0&0&0&1&4/5&2/5&-1/5&-1/5 \end{array}\right] \xrightarrow[+R_1]{-1\times R_4} \left[\begin{array}{cccc|cccc} 1&0&0&0&-4/5&3/5&1/5&1/5\\ 0&1&0&-5&-5/2&-2&0&1\\ 0&0&1&0&1/2&0&0&0\\ 0&0&0&1&4/5&2/5&-1/5&-1/5 \end{array}\right] \xrightarrow[+R_2]{5\times R_4} \left[\begin{array}{cccc|cccc} 1&0&0&0&-4/5&3/5&1/5&1/5\\ 0&1&0&0&3/2&0&-1&0\\ 0&0&1&0&1/2&0&0&0\\ 0&0&0&1&4/5&2/5&-1/5&-1/5 \end{array}\right]$$

Exercise 8:

(a)
$$\begin{bmatrix} 1/k_1&0&0&0\\ 0&1/k_2&0&0\\ 0&0&1/k_3&0\\ 0&0&0&1/k_4 \end{bmatrix}$$

(b) $\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & K_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & K_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & K_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ K_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} K_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & K_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & K_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & K_1 & 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|cccc} K_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & K_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & K_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & K_1 & 1 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow[\text{row by } K_n]{\text{divide each}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{K_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{K_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{K_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{K_1} & 0 & 0 & 0 \end{array}\right]$

(c) $\left[\begin{array}{cccc|cccc} K & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & K & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & K & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & K & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{row by } K]{\text{divide each}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/K & 0 & 0 & 0 \\ 1/K & 1 & 0 & 0 & 0 & 1/K & 0 & 0 \\ 0 & 1/K & 1 & 0 & 0 & 0 & 1/K & 0 \\ 0 & 0 & 1/K & 1 & 0 & 0 & 0 & 1/K \end{array}\right] \xrightarrow[+R_2]{-1/K \times R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/K & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/K^2 & 1/K & 0 & 0 \\ 0 & 1/K & 1 & 0 & 0 & 0 & 1/K & 0 \\ 0 & 0 & 1/K & 1 & 0 & 0 & 0 & 1/K \end{array}\right]$

$\xrightarrow[+R_3]{-1/K \times R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/K & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/K^2 & 1/K & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/K^3 & -1/K^2 & 1/K & 0 \\ 0 & 0 & 1/K & 1 & 0 & 0 & 0 & 1/K \end{array}\right] \xrightarrow[+R_4]{-\frac{1}{K} \times R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/K & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/K^2 & 1/K & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/K^3 & -1/K^2 & 1/K & 0 \\ 0 & 0 & 0 & 1 & -1/K^4 & 1/K^3 & -1/K^2 & 1/K \end{array}\right]$

Exercise 9:

(a) $\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -5 & 2 & 0 & 1 \end{array}\right] \xrightarrow[+R_2]{5 \times R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & 5 & 1 \end{array}\right] \xrightarrow{\frac{1}{2} \times R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 5/2 & 1/2 \end{array}\right]$ $\quad A^{-1} = E_2 E_1 = \begin{bmatrix} 1 & 0 \\ 5/2 & 1/2 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad E_1 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

(b) $A^{-1} = \begin{bmatrix} 1 & 0 \\ 5/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Exercise 10:

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 4 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 7 \\ 4 & 5 & -3 \\ 2 & -1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 4 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 4/3 & 5/3 & -1 \\ 1 & -4 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 4 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 9 & -6 \\ 4 & 5 & -3 \\ 1 & -4 & 7 \end{bmatrix}$

Exercise 11:

$\begin{bmatrix} 0 & 1 & 7 & 8 \\ 1 & 3 & 3 & 8 \\ -2 & -5 & 1 & -8 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 3 & 8 \\ 0 & 1 & 7 & 8 \\ -2 & -5 & 1 & -8 \end{bmatrix} \xrightarrow[+R_3]{2 \times R_1} \begin{bmatrix} 1 & 3 & 3 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 1 & 7 & 8 \end{bmatrix} \xrightarrow[+R_3]{-R_2} \begin{bmatrix} 1 & 3 & 3 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\overset{A}{\begin{bmatrix} 0 & 1 & 7 & 8 \\ 1 & 3 & 3 & 8 \\ -2 & -5 & 1 & -8 \end{bmatrix}} = \overset{E}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{F}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \overset{G}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \overset{R}{\begin{bmatrix} 1 & 3 & 3 & 8 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$

Exercise 12:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$ be an elementary matrix. Note that A cannot be the result of interchanging two rows. Thus, two cases:

- A is the result of multiplying a row by a factor n. Note that if row 3 is multiplied by a factor, a and b would be zero. If rows 1 and 2 were multiplied by a factor (i.e., 1), then a and b would be zero. Thus, at least one entry in row 3 is zero.
- A is the result of adding a multiple of one row to another row. If rows 1 or 2 is the manipulated row, then a and b are zero. If row 3 is the manipulated row, then a is zero if $n(R_2)$ is added to row 3, and b is zero if $n(R_1)$ is added to Row 3. Thus, at least one entry in row 3 is zero.

$\square$

Exercise 13:

$$\left[\begin{array}{ccccc|ccccc} 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & d & 0 & e & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccccc|ccccc} b & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & e & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{b} \times R_1} \left[\begin{array}{ccccc|ccccc} 1 & 0 & \frac{c}{b} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & e & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{a} \times R_2} \left[\begin{array}{ccccc|ccccc} 1 & 0 & \frac{c}{b} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & d & 0 & e & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[+R_3]{-d \times R_2} \left[\begin{array}{ccccc|ccccc} 1 & 0 & \frac{c}{b} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & -\frac{d}{a} & 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{e} \times R_3} \left[\begin{array}{ccccc|ccccc} 1 & 0 & \frac{c}{b} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{d}{ae} & 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[+R_5]{-h \times R_3} \left[\begin{array}{ccccc|ccccc} 1 & 0 & \frac{c}{b} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{d}{ae} & 0 & 1 & 0 & 0 \\ 0 & 0 & f & 0 & g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{hd}{ae} & 0 & -h & 0 & 1 \end{array}\right]$$ the matrix is uninvertible.

Exercise 14:

Let A be any $m \times n$ matrix. Reducing A to RREF can be accomplished by a series of the three elementary row operations, each of which can be represented by an elementary matrix. Let C be the product of these matrices (i.e., $C = E_k \ldots E_3 E_2 E_1$). Thus, CA is in RREF. Moreover, C is invertible because it is a product of elementary matrices (Theorem 1.5.3).

$\square$

Exercise 15:

Proof: Suppose A is an invertible matrix and B is row equivalent to A. Thus, there is some finite sequence of elementary row operations by which A can be obtained from B (and vice versa). Thus, $B = E_k \dots E_3 E_2 E_1 A$. Note that, since A is invertible, and each elementary matrix is invertible, B is also invertible because it is the product of invertible matrices.

□

Exercise 16:

(a) Proof: Let A and B be $m \times n$ matrices.

($\Rightarrow$) Suppose that A and B are row equivalent. Thus, B can be reduced to A by a series of elementary row operations. Since B can be reduced to A with elementary row operations and every matrix has exactly one RREF, B can be reduced to A and then to A's RREF through a finite number of elementary row operations. Thus, B and A have the same RREF.

($\Leftarrow$) Suppose that A and B have the same RREF. Thus, A can be obtained from B through a finite series of elementary row operations (i.e., $A \Rightarrow \text{RREF} \Rightarrow B$) Thus, A and B are row equivalent.

□

(b)
$$\underset{A}{\begin{bmatrix}1&2&3\\1&4&1\\2&1&9\end{bmatrix}} \xrightarrow[+R_2]{-1\times R_1} \begin{bmatrix}1&2&3\\0&2&-2\\2&1&9\end{bmatrix} \xrightarrow[+R_3]{-2\times R_1} \begin{bmatrix}1&2&3\\0&2&-2\\0&-3&3\end{bmatrix} \xrightarrow{\frac{1}{2}\times R_2} \begin{bmatrix}1&2&3\\0&1&-1\\0&-3&3\end{bmatrix} \xrightarrow[+R_3]{3\times R_2} \begin{bmatrix}1&2&3\\0&1&-1\\0&0&0\end{bmatrix}$$

$$\xrightarrow[+R_1]{-2\times R_2} \begin{bmatrix}1&0&5\\0&1&-1\\0&0&0\end{bmatrix} \xrightarrow[+R_3]{R_2} \begin{bmatrix}1&0&5\\0&1&-1\\0&1&-1\end{bmatrix} \xrightarrow[+R_3]{R_1} \begin{bmatrix}1&0&5\\0&1&-1\\1&1&4\end{bmatrix} \xrightarrow{2\times R_2} \underset{B}{\begin{bmatrix}1&0&5\\0&2&-2\\1&1&4\end{bmatrix}}$$

Thus, A and B are row equivalent.

$$\underset{B}{\begin{bmatrix}1&0&5\\0&2&-2\\1&1&4\end{bmatrix}} \xrightarrow{\frac{1}{2}\times R_2} \begin{bmatrix}1&0&5\\0&1&-1\\1&1&4\end{bmatrix} \xrightarrow[+R_3]{-1\times R_1} \begin{bmatrix}1&0&5\\0&1&-1\\0&1&-1\end{bmatrix} \xrightarrow[+R_3]{-1\times R_2} \begin{bmatrix}1&0&5\\0&1&-1\\0&0&0\end{bmatrix} \xrightarrow[+R_1]{2\times R_2} \begin{bmatrix}1&2&3\\0&1&-1\\0&0&0\end{bmatrix}$$

$$\xrightarrow[+R_3]{-3\times R_2} \begin{bmatrix}1&2&3\\0&1&-1\\0&-3&3\end{bmatrix} \xrightarrow{2\times R_2} \begin{bmatrix}1&2&3\\0&2&-2\\0&-3&3\end{bmatrix} \xrightarrow[+R_3]{2\times R_1} \begin{bmatrix}1&2&3\\0&2&-2\\2&1&9\end{bmatrix} \xrightarrow[+R_2]{R_1} \underset{A}{\begin{bmatrix}1&2&3\\1&4&1\\2&1&9\end{bmatrix}}$$

□

Exercise 17:

Proof: Suppose that the Elementary Matrix E results from performing a certain row operation on I_m and A is an $m \times n$ matrix. We must show that the product EA is the matrix that results when this same row operation is performed on A. Let us represent A as a row of columns $A = [c_1, c_2, \ldots, c_n]$. Let us also represent E as a column of rows $r_1, \ldots, r_m$. To multiply E by A, we need to multiply each r_a by each c_b.

Three cases:

- Case 1: Suppose that E is obtained from I_m by adding the j^{th} row multiplied by x to the i^{th} row. Then each row r_a except r_i has 1 on the (a,a)-place and zeroes everywhere else. The row r_i has 1 on the (i,i)-place and x on the (i,j)-place and zeroes everywhere else. If a is not equal to i then the element $EA(a,b)$, the product of r_a and c_b, will be equal to the a^{th} entry of c_b, that is, $A(a,b)$. Thus, $EA(a,b) = A(a,b)$ wherever $a \neq i$.
Thus, all rows of EA except for the i^{th} row coincide with the corresponding rows of A. Let us examine the i^{th} row. The entries of this row are products of r_i and c_b $(b = 1, \ldots, n)$. Since r_i has 1 on the i^{th} place and x on the j^{th} place and zero everywhere else, the product of r_i and c_b is the sum of the i^{th} entry, and the j^{th} entry of c_b multiplied by x. Thus, $EA(i,b) = A(i,b) + xA(j,b)$ for every $b = 1, \ldots, n$. Therefore the i^{th} row of EA is obtained by adding x times the j^{th} row of A to the i^{th} row of A.
- Case 2: Suppose that E is obtained from I_m by multiplying the i^{th} row by x (where $x \neq 0$). Then each row r_a except r_i has 1 on the (a,a)-place and zeroes everywhere else. The row r_i has x on the (i,i)-place and zeroes everywhere else. If $a \neq i$, then the element $EA(a,b)$, the product of r_a and c_b, will be equal to the a^{th} entry of c_b, that is $A(a,b)$. Thus, $EA(a,b) = A(a,b)$ whenever $a \neq i$. Thus, all rows of EA except for the i^{th} row coincide with the corresponding rows of A. Let us examine the i^{th} row. The entries of this row are products of r_i and c_b $(b = 1, \ldots, n)$. Since r_i has x on the i^{th} place and zeroes everywhere else, the product of r_i and c_b is the i^{th} entry of c_b multiplied by x. Thus, $EA(i,b) = xA(i,b)$ for every $b = 1, \ldots, n$. Therefore the i^{th} row of EA is obtained by multiplying the i^{th} row of A by x (where $x \neq 0$).

• Case 3: Suppose that E is obtained from I_m by swapping the i^{th} row and the j^{th} row. Then each row r_a except r_i and r_j has 1 on the (a,a)-place and zeroes everywhere else. The row r_i has 1 on the (i,j)-place and zeroes everywhere else. The row r_j has 1 on the (j,i)-place and zeroes everywhere else. If $a \neq i, j$, then the element $EA(a,b)$, the product of r_a and c_b, will be equal to the a^{th} entry of c_b, that is $A(a,b)$. Thus, $EA(a,b) = A(a,b)$ whenever $a \neq i, j$. Thus, all rows of EA except for the i^{th} and j^{th} rows coincide with the corresponding rows of A. Let us examine the i^{th} row. The entries of this row are products of r_i and c_b $(b = 1, \dots, n)$. Since r_i has 1 on the j^{th} place and zeroes everywhere else, the product of r_i and c_b is the j^{th} entry of c_b. Thus, $EA(i,b) = A(j,b)$ for every $b = 1, \dots, n$. Therefore the i^{th} row of EA is the j^{th} row of A. Let us examine the j^{th} row. The entries of this row are products of r_j and c_b $(b = 1, \dots, n)$. Since r_j has 1 on the i^{th} place and zeroes everywhere else, the product of r_j and c_b is the i^{th} entry of c_b. Thus, $EA(j,b) = A(i,b)$ for every $b = 1, \dots, n$. Therefore the j^{th} row of EA is the i^{th} row of A.

$\square$

Exercise 18:

Let the sequence of row operations be $1, \dots, n$, where the operations are applied in order from 1 to n. Let "opposite operation" be defined as follows:

- If a row operation swaps rows i and j, the opposite operation swaps rows i and j.
- If a row operation multiplies row i by x, the opposite operation multiplies row i by $\frac{1}{x}$.
- If a row operation adds row j multiplied by x to row i, the opposite operation adds row j multiplied by $-x$ to row i.

To find A, start with I, and starting with n and ending with 1, apply the corresponding opposite operation in sequence to I.

Exercise 19:

(a) False. Per Theorem 1.5.3, noninvertible square matrices cannot be expressed as a product of elementary matrices.

(b) False. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Note that A and B are both elementary matrices.

$AB = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Leftarrow$ Not an elementary matrix.

(c) True. Note that the operation in question is an elementary row operation. Thus, the result of this operation on A can be expressed by EA, where E is the result of performing the operation on I_n (Theorem 1.5.1). Note that E is invertible (thm 1.5.2). Since the product of invertible matrices is invertible, the resulting matrix is invertible.

(d) True. $AB = 0$

$A^{-1}(AB) = A^{-1}0$

$(A^{-1}A)B = A^{-1}0$

$IB = A^{-1}0$

$B = 0$

Exercise 20:

(a) True. A in its RREF has at least one row of zeros (see b). Thus, there is at least one arbitrary variable in x. Thus, $Ax = 0$ has infinitely many solutions.

(b) True. Per Theorem 1.5.3, if A is singular, then the RREF of A is not I. Per Theorem 1.4.3, the RREF of a square matrix either is I or has a row of zeros. Since the RREF is not I, then it must have a row of zeros.

ⓒ True. Since A^{-1} is expressible as the product of elementary matrices, it is invertible (Theorem 1.5.3). Thus, A is also invertible (as it is the inverse of A^{-1}). Thus, $Ax = 0$ only has the trivial solution (Theorem 1.5.3).

ⓓ False. B and A have the same RREF (as B can be reduced back to A with the inverse operation), and A has a unique RREF. Per (b), A's RREF has at least one row of zeros, and thus so does the RREF of B. Since the RREF is not I_n, B must be singular (per Theorem 1.5.3).

Exercise 21:

No. Let $a = 1$, $b = 0$, $c = 1$, $d = 0$. There is no A such that $A\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$. Thus, the following must be true:

1. $w + x = 0$
2. $0 = 0$
3. $y + z = 1$
4. $0(y) + 0(z) = 1$

Based on 4, A cannot exist.

$\boxed{\text{1.6 : Further Results on Systems of Equations and Invertibility}}$

Exercise 1:

$$A = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6-5}\begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}$$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \end{bmatrix} = \begin{bmatrix} 12-9 \\ -10+9 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Exercise 2:

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -5 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-20+6}\begin{bmatrix} -5 & 3 \\ -2 & 4 \end{bmatrix} = \frac{1}{-14}\begin{bmatrix} -5 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 5/14 & -3/14 \\ 2/14 & -4/14 \end{bmatrix} = \begin{bmatrix} 5/14 & -3/14 \\ 1/7 & -2/7 \end{bmatrix}$$

By Theorem 1.6.2: $x = A^{-1}b = \begin{bmatrix} 5/14 & -3/14 \\ 1/7 & -2/7 \end{bmatrix}\begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -15/14 + -27/14 \\ -3/7 + -18/7 \end{bmatrix} = \begin{bmatrix} -42/14 \\ -21/7 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$