2.1 : The Determinant Function

①

ⓐ 5 ⓑ 9 ⓒ 6 ⓓ 10 ⓔ 0 ⓕ 2

②

ⓐ odd ⓑ odd ⓒ even ⓓ even ⓔ even ⓕ e

③ $(3 \times 4) - (5 \times (-2))$

$= 12 + 10 = 22$

④ $(4 \times 2) - (1 \times 8)$

$= 8 - 8 = 0$

⑤ $(-5 \times -2) - (6 \times (-7))$

$= 10 + 42 = 52$

⑥ $(\sqrt{2} \times \sqrt{3}) - (4 \times \sqrt{6})$

$= \sqrt{6} - 4\sqrt{6}$

$= -3\sqrt{6}$

⑦ $(a-3)(a-2) - (5 \times (-3))$

$= a^2 - 5a + 6 + 15$

$= a^2 - 5a + 21$

⑧ $((-2) \times 1 \times 4) + (7 \times (-2) \times 3) + (6 \times 5 \times 8) - (6 \times 1 \times 3) - ((-2) \times (-2) \times 8) - (7 \times 5 \times 4)$

$= -8 - 42 + 240 - 18 - 32 - 140$

$= 0$

⑨ $((-2) \times 5 \times 2) + (1 \times (-7) \times 1) + (4 \times 3 \times 6) - (4 \times 5 \times 1) - ((-2) \times (-7) \times 6) - (1 \times 3 \times 2)$

$= -20 - 7 + 72 - 20 - 84 - 6$

$= -65$

⑩ $((-1)\times 0\times 2)+(1\times(-5)\times 1)+(2\times 3\times 7)-(2\times 0\times 1)-((-1)\times(-5)\times 7)-(1\times 3\times 2)$

$= 0-5+42-0-35-6$

$= -4$

⑪ $(3\times(-1)\times(-4))+(0\times 5\times 1)+(0\times 2\times 9)-(0\times(-1)\times 1)-(3\times 5\times 9)-(0\times 2\times(-4))$

$= 12+0+0-0-135-0$

$= -123$

⑫ $(c\times 1\times 2)+((-4)\times c^2\times 4)+(3\times 2\times(c-1))-(3\times 1\times 4)-(c\times c^2\times(c-1))-((-4)\times 2\times 2)$

$= 2c-16c^2+6c-6-12-c^4+c^3+16$

$= -c^4+c^3-16c^2+8c-2$

⑬

ⓐ $(\lambda-2)(\lambda+4)-((-5)\times 1)=0$

$\lambda^2+2\lambda-8+5=0$

$\lambda^2+2\lambda-3=0$

$(\lambda-1)(\lambda+3)=0$

$\lambda=1,-3$

ⓑ $((\lambda-4)\times\lambda\times(\lambda-1))+0+0-0-((\lambda-4)\times 2\times 3)-0=0$

$\lambda^3-\lambda^2-4\lambda^2+4\lambda-6\lambda+24=0$

$\lambda^3-5\lambda^2-2\lambda+24=0$

$\lambda=-2,3,4$

⑭ (1, 2, 3, 4): even — (2, 1, 3, 4): odd

(1, 2, 4, 3): odd — (2, 1, 4, 3): even

(1, 3, 2, 4): odd — (2, 3, 1, 4): even

(1, 3, 4, 2): even — (2, 3, 4, 1): odd

(1, 4, 2, 3): even — (2, 4, 1, 3): odd

(1, 4, 3, 2): odd — (2, 4, 3, 1): even

$(3,1,2,4)$: even $\quad$ $(4,1,2,3)$: odd

$(3,1,4,2)$: odd $\quad$ $(4,1,3,2)$: even

$(3,2,1,4)$: odd $\quad$ $(4,2,1,3)$: even

$(3,2,4,1)$: even $\quad$ $(4,2,3,1)$: odd

$(3,4,1,2)$: even $\quad$ $(4,3,1,2)$: odd

$(3,4,2,1)$: odd $\quad$ $(4,3,2,1)$: even

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$$a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44} + a_{11}a_{23}a_{34}a_{42}$$
$$+ a_{11}a_{24}a_{32}a_{43} - a_{11}a_{24}a_{33}a_{42} - a_{12}a_{21}a_{33}a_{44} + a_{12}a_{21}a_{34}a_{43}$$
$$+ a_{12}a_{23}a_{31}a_{44} - a_{12}a_{23}a_{34}a_{41} - a_{12}a_{24}a_{31}a_{43} + a_{12}a_{24}a_{33}a_{41}$$
$$+ a_{13}a_{21}a_{32}a_{44} - a_{13}a_{21}a_{34}a_{42} - a_{13}a_{22}a_{31}a_{44} + a_{13}a_{22}a_{34}a_{41}$$
$$+ a_{13}a_{24}a_{31}a_{42} - a_{13}a_{24}a_{32}a_{41} - a_{14}a_{21}a_{32}a_{43} + a_{14}a_{21}a_{33}a_{42}$$
$$+ a_{14}a_{22}a_{31}a_{43} - a_{14}a_{22}a_{33}a_{41} - a_{14}a_{23}a_{31}a_{42} + a_{14}a_{23}a_{32}a_{41}$$

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$$\det\begin{bmatrix} 4 & -9 & 9 & 2 \\ -2 & 5 & 6 & 4 \\ 1 & 2 & -5 & -3 \\ 1 & -2 & 0 & -2 \end{bmatrix} = 275$$

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(a) Note that there is only one permutation such that the elementary product $\neq 0$. That is: $a_{15}a_{24}a_{33}a_{42}a_{51}$. This permutation is even, and so the signed elementary product is positive. The determinant is therefore:

$$(-3)\times(-4)\times(-1)\times 2\times 5 = -120$$

(b) As above, there is only one permutation such that the elementary product $\neq 0$. That is $a_{11}a_{25}a_{33}a_{44}a_{52}$. This permutation is odd, and so the signed elementary product is negative.

The determinant is therefore:

$$-(5 \times (-4) \times 3 \times 1 \times (-2)) = -120$$

(18) First, we calculate: $\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = x(1-x) - 3(-1) = x - x^2 + 3$
$= -x^2 + x + 3$

Next: $\begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix} = (1)(x)(x-5) + 0 + (-3)(2)(3) - (-3)(x)(1) - (1)(-6)(3) - 0$
$= x^2 - 5x - 18 + 3x + 18$
$= x^2 - 2x$

Thus: $-x^2 + x + 3 = x^2 - 2x$
$2x^2 - 3x - 3 = 0$
$x = \frac{3}{4} \pm \frac{\sqrt{33}}{4}$

(19) $\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ \sin\theta - \cos\theta & \sin\theta + \cos\theta & 1 \end{vmatrix}$

$= \sin\theta \sin\theta + 0 + 0 - 0 - 0 - (-\cos\theta)(\cos\theta)$
$= \sin^2\theta + \cos^2\theta = 1$

(20)

Proof: Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$.

($\rightarrow$) Suppose A and B commute. That is, $AB = BA$, and so:

$\begin{bmatrix} ad & ae + bf \\ 0 & cf \end{bmatrix} = \begin{bmatrix} da & db + ec \\ 0 & fc \end{bmatrix}$. Note then that $ae + bf = db + ec$.

So, $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = b(d-f) - e(a-c) = bd - bf - ea + ec$
$= -bf - ea + bd + ec$
$= -bf - ea + ea + bf = 0.$

($\leftarrow$) Suppose $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$. That is: $b(d-f) - e(a-c) = 0$
$bd - bf - ea + ec = 0$

Note that: $bd + ec = ea + bf$

So, $AB = \begin{bmatrix} ad & ae + bf \\ 0 & cf \end{bmatrix}$ and $BA = \begin{bmatrix} da & db + ec \\ 0 & fc \end{bmatrix}$

Since $db+ec=ae+bf$, $AB=BA$. Therefore, A and B commute.

$\square$

(21) Because it is the sum of the products of integers.

(22) For any n such that $n>1$, the determinant would be 0. There are an equal number of odd and even permutations, and all the elementary products are 1. Thus, there are an equal number of 1's and -1's, and so the determinant is 0.

(23)

(a) Because each signed elementary product will be 0, since each row must be represented in each elementary product (including the 0 row).

(b) Because each signed elementary product will be 0, since each column must be represented in each elementary product (including the 0 column).

(24) The determinant is the product of the diagonal entries.

(25) For both, the determinant is the product of the diagonal entries.