2.3: Properties of the Determinant Function

①

ⓐ $kA = 2\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}-2 & 4\\ 6 & 8\end{bmatrix}$

$\det(kA) = (-2)(8) - (4)(6) = -16 - 24 = -40$

$\det(A) = (-1)(4) - (2)(3) = -4 - 6 = -10$

$K^n \det(A) = (2)^2(-10) = (4)(-10) = -40$

Thus: $\det(kA) = k^n \det(A) = -40$

ⓑ $kA = -2\begin{bmatrix}2 & -1 & 3\\ 3 & 2 & 1\\ 1 & 4 & 5\end{bmatrix} = \begin{bmatrix}-4 & 2 & -6\\ -6 & -4 & -2\\ -2 & -8 & -10\end{bmatrix}$

$\det(kA) = (-4)(-4)(-10) + (2)(-2)(-2) + (-6)(-6)(-8) - (-6)(-4)(-2) - (-4)(-2)(-8) - (2)(-6)(-10)$

$= -160 + 8 - 288 + 48 + 64 - 120 = -448$

$\det(A) = (2)(2)(5) + (-1)(1)(1) + (3)(3)(4) - (3)(2)(1) - (2)(1)(4) - (-1)(3)(5)$

$= 20 - 1 + 36 - 6 - 8 + 15 = 56$

$k^n \det(A) = (-2)^3(56) = (-8)(56) = -488$

Thus: $\det(kA) = k^n \det(A) = -488$

② $AB = \begin{bmatrix}2 & 1 & 0\\ 3 & 4 & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & -1 & 3\\ 7 & 1 & 2\\ 5 & 0 & 1\end{bmatrix} = \begin{bmatrix}9 & -1 & 8\\ 31 & 1 & 17\\ 10 & 0 & 2\end{bmatrix}$

$\det(AB) = -170$ $\quad \det(A) = 10$ $\quad \det(B) = -17$

$\det(A)\det(B) = -170$

Thus: $\det(AB) = \det(A)\det(B) = -170$

③ Columns 2 and 4 are proportional, so per Theorem 2.2.5, $\det(A) = 0$.

④

ⓐ $\begin{vmatrix}1 & 0 & -1\\ 9 & -1 & 4\\ 8 & 9 & -1\end{vmatrix} = -124 \neq 0$. The matrix is invertible.

ⓑ $\begin{vmatrix}4 & 2 & 8\\ -2 & 1 & -4\\ 3 & 1 & 6\end{vmatrix} = 0$. The matrix is singular.

ⓒ The determinant is 0 because there is a column of 0. Thus, the matrix is singular.

ⓓ The determinant is 0 because there is a column of 0. Thus, the matrix is singular.

5

(a) $\det(3A) = 3^3 \det(A) = (27)(-7) = -189$

(b) $\det(A^{-1}) = \frac{1}{\det(A)} = -\frac{1}{7}$

(c) $\det(2A^{-1}) = 2^3 \det(A^{-1}) = 8\left(\frac{1}{\det(A)}\right) = 8\left(-\frac{1}{7}\right) = -\frac{8}{7}$

(d) $\det((2A)^{-1}) = \frac{1}{\det(2A)} = \frac{1}{8\det(A)} = \frac{1}{(8)(-7)} = -\frac{1}{56}$

(e) $\det(A) = -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = -\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = -7.$

Thus, $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = 7$

6 When $x=0$, the first and third rows are proportional, and so the determinant is 0. When $x=2$, the first and second rows are proportional, and so the determinant is 0.

7 Per Theorem 2.3.1, $\det\begin{bmatrix} b+c & c+a & b+a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} c & a & b \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix} + \det\begin{bmatrix} b & c & a \\ a & b & c \\ 1 & 1 & 1 \end{bmatrix}$

$= (cb+ac+ba-a^2-b^2-c^2) + (b^2+c^2+a^2-ba-cb-ac)$

$= 0$

8 $\begin{vmatrix} a_1 & b_1 & a_1+b_1+c_1 \\ a_2 & b_2 & a_2+b_2+c_2 \\ a_3 & b_3 & a_3+b_3+c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= 0 + 0 + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

9 $\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_1-b_1 & c_1 \\ a_2 & a_2-b_2 & c_2 \\ a_3 & a_3-b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_1-b_1 & c_1 \\ b_2 & a_2-b_2 & c_2 \\ b_3 & a_3-b_3 & c_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & -b_1 & c_1 \\ a_2 & -b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & -b_1 & c_1 \\ b_2 & -b_2 & c_2 \\ b_3 & -b_3 & c_3 \end{vmatrix}$

$= 0 - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0$

$= -2\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(10)

$$\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} b_1t & b_2t & b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1t & a_2t & a_3t \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} b_1t & b_2t & b_3t \\ a_1t & a_2t & a_3t \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} b_1t & b_2t & b_3t \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= 0 + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - t^2 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + 0$$

$$= (1-t^2) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(11)

$$\begin{vmatrix} a_1 & b_1+ta_1 & c_1+rb_1+sa_1 \\ a_2 & b_2+ta_2 & c_2+rb_2+sa_2 \\ a_3 & b_3+ta_3 & c_3+rb_3+sa_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1+rb_1+sa_1 \\ a_2 & b_2 & c_2+rb_2+sa_2 \\ a_3 & b_3 & c_3+rb_3+sa_3 \end{vmatrix} + \begin{vmatrix} a_1 & ta_1 & c_1+rb_1+sa_1 \\ a_2 & ta_2 & c_2+rb_2+sa_2 \\ a_3 & ta_3 & c_3+rb_3+sa_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & rb_1 \\ a_2 & b_2 & rb_2 \\ a_3 & b_3 & rb_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & sa_1 \\ a_2 & b_2 & sa_2 \\ a_3 & b_3 & sa_3 \end{vmatrix} + \begin{vmatrix} a_1 & ta_1 & c_1 \\ a_2 & ta_2 & c_2 \\ a_3 & ta_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & ta_1 & rb_1 \\ a_2 & ta_2 & rb_2 \\ a_3 & ta_3 & rb_3 \end{vmatrix} + \begin{vmatrix} a_1 & ta_1 & sa_1 \\ a_2 & ta_2 & sa_2 \\ a_3 & ta_3 & sa_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0 + 0 + 0 + 0 + 0$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(12)

(a) $(k-3)(k-2)-(-2)(-2) = k^2-5k+2 = 0$

$$k = \frac{5 \pm \sqrt{25-8}}{2} = \frac{5 \pm \sqrt{17}}{2}$$

(b) $2+12k+36-4k-18-12 = 8k+8 = 8(k+1) = 0$

$k = -1$

(13) $\begin{bmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{bmatrix}$ is singular iff $\begin{vmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{vmatrix} = 0$

$$\begin{vmatrix} \sin^2\alpha & \sin^2\beta & \sin^2\gamma \\ \cos^2\alpha & \cos^2\beta & \cos^2\gamma \\ 1 & 1 & 1 \end{vmatrix} = \sin^2\alpha\cos^2\beta + \sin^2\beta\cos^2\gamma + \sin^2\gamma\cos^2\alpha - \sin^2\gamma\cos^2\beta - \sin^2\alpha\cos^2\gamma - \sin^2\beta\cos^2\alpha$$

$$= \sin^2\alpha(1-\sin^2\beta) + \sin^2\beta(1-\sin^2\gamma) + \sin^2\gamma(1-\sin^2\alpha) - \sin^2\gamma(1-\sin^2\beta) - \sin^2\alpha(1-\sin^2\gamma) - \sin^2\beta(1-\sin^2\alpha)$$

$$= \sin^2\alpha - \sin^2\alpha\sin^2\beta + \sin^2\beta - \sin^2\beta\sin^2\gamma + \sin^2\gamma - \sin^2\gamma\sin^2\alpha - \sin^2\gamma + \sin^2\gamma\sin^2\beta - \sin^2\alpha + \sin^2\alpha\sin^2\gamma - \sin^2\beta + \sin^2\beta\sin^2\alpha$$

$$= 0$$

(14)

(a) $\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\lambda\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$\lambda\begin{bmatrix}x_1\\x_2\end{bmatrix}-\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}-\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}\lambda-1&-2\\-2&\lambda-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

(b) $\begin{bmatrix}2&3\\4&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\lambda\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$\lambda\begin{bmatrix}x_1\\x_2\end{bmatrix}-\begin{bmatrix}2&3\\4&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}-\begin{bmatrix}2&3\\4&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}\lambda-2&-3\\-4&\lambda-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

(c) $\begin{bmatrix}3&1\\-5&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\lambda\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$\lambda\begin{bmatrix}x_1\\x_2\end{bmatrix}-\begin{bmatrix}3&1\\-5&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}-\begin{bmatrix}3&1\\-5&-3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}\lambda-3&-1\\5&\lambda+3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

(15)

(a)

(i) $\det(\lambda I-A)=\begin{vmatrix}\lambda-1&-2\\-2&\lambda-1\end{vmatrix}=0$

$(\lambda-1)(\lambda-1)-4=0$

$\lambda^2-2\lambda-3=0$

(ii) $(\lambda-3)(\lambda+1)=0$

$\lambda=3$ and $\lambda=-1$

(iii) If $\lambda=3$:

$\begin{bmatrix}2&-2\\-2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

So: $x_1-x^2=0$

$x_1=t, x_2=t$

$\mathbf{x}=\begin{bmatrix}t\\t\end{bmatrix}$

If $\lambda=-1$:

$\begin{bmatrix}-2&-2\\-2&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

So: $x_1+x_2=0$

$x_1=-t, x_2=t$

$\mathbf{x}=\begin{bmatrix}-t\\t\end{bmatrix}$

(b)

(i) $\det(\lambda I-A)=\begin{vmatrix}\lambda-2&-3\\-4&\lambda-3\end{vmatrix}=0$

$(\lambda-2)(\lambda-3)-12=0$

$\lambda^2-5\lambda-6=0$

(ii) $(\lambda-6)(\lambda+1)=0$

$\lambda=6$ and $\lambda=-1$

(iii) If $\lambda=6$:

$\begin{bmatrix}4&-3\\-4&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

So: $x_1-\frac{3}{4}x_2=0$

$x_1=\frac{3}{4}t,\ x_2=t$

$\mathbf{x}=\begin{bmatrix}3/4t\\t\end{bmatrix}$

If $\lambda=-1$:

$\begin{bmatrix}-3&-3\\-4&-4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

So: $x_1+x_2=0$

$x_1=-t, x_2=t$

$\mathbf{x}=\begin{bmatrix}-t\\t\end{bmatrix}$

(c)

(i) $\det(\lambda I-A)=\begin{vmatrix}\lambda-3&-1\\5&\lambda+3\end{vmatrix}=0$

$(\lambda-3)(\lambda+3)+5=0$

$\lambda^2-4=0$

(ii) $(\lambda+2)(\lambda-2)=0$

$\lambda=-2$ and $\lambda=2$

(iii) If $\lambda=-2$:

$\begin{bmatrix}-5&-1\\5&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

So: $x_1+\frac{1}{5}x_2=0$

$x_1=\frac{-1}{5}t,\ x_2=t$

$\mathbf{x}=\begin{bmatrix}-1/5t\\t\end{bmatrix}$

If $\lambda=2$:

$\begin{bmatrix}-1&-1\\5&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

So: $x_1+x_2=0$

$x_1=-t, x_2=t$

$\mathbf{x}=\begin{bmatrix}-t\\t\end{bmatrix}$

(16)

__Proof__: Let A and B be $n\times n$ matrices, and suppose A is invertible. Per Theorem 2.3.3, $\det(A)\neq 0$. Note that:

$$\begin{aligned}\det(B)&=\frac{1}{\det(A)}\det(B)\det(A) && \text{This is defined because } \det(A)\neq 0.\\ &=\det(A^{-1})\det(B)\det(A) && \text{Theorem 2.3.5}\\ &=\det(A^{-1}BA) && \text{Theorem 2.3.4}\end{aligned}$$

□

(17)

(a) $\begin{vmatrix}a_1+b_1&c_1+d_1\\a_2+b_2&c_2+d_2\end{vmatrix}=\begin{vmatrix}a_1&c_1\\a_2+b_2&c_2+d_2\end{vmatrix}+\begin{vmatrix}b_1&d_1\\a_2+b_2&c_2+d_2\end{vmatrix}$

$$= \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 \\ b_2 & d_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}$$

(b) $$\begin{vmatrix} a_1+b_1 & c_1+d_1 & e_1+f_1 \\ a_2+b_2 & c_2+d_2 & e_2+f_2 \\ a_3+b_3 & c_3+d_3 & e_3+f_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & e_1 \\ a_2+b_2 & c_2+d_2 & e_2+f_2 \\ a_3+b_3 & c_3+d_3 & e_3+f_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & f_1 \\ a_2+b_2 & c_2+d_2 & e_2+f_2 \\ a_3+b_3 & c_3+d_3 & e_3+f_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & c_1 & e_1 \\ a_2 & c_2 & e_2 \\ a_3+b_3 & c_3+d_3 & e_3+f_3 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & e_1 \\ b_2 & d_2 & f_2 \\ a_3+b_3 & c_3+d_3 & e_3+f_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & f_1 \\ a_2 & c_2 & e_2 \\ a_3+b_3 & c_3+d_3 & e_3+f_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & f_1 \\ b_2 & d_2 & f_2 \\ a_3+b_3 & c_3+d_3 & e_3+f_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & c_1 & e_1 \\ a_2 & c_2 & e_2 \\ a_3 & c_3 & e_3 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & e_1 \\ a_2 & c_2 & e_2 \\ b_3 & d_3 & f_3 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & e_1 \\ b_2 & d_2 & f_2 \\ a_3 & c_3 & e_3 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & e_1 \\ b_2 & d_2 & f_2 \\ b_3 & d_3 & f_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & f_1 \\ a_2 & c_2 & e_2 \\ a_3 & c_3 & e_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & f_1 \\ a_2 & c_2 & e_2 \\ b_3 & d_3 & f_3 \end{vmatrix}$$

$$+ \begin{vmatrix} b_1 & d_1 & f_1 \\ b_2 & d_2 & f_2 \\ a_3 & c_3 & e_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & f_1 \\ b_2 & d_2 & f_2 \\ b_3 & d_3 & f_3 \end{vmatrix}$$

18

Proof: Consider any square matrix A.

($\rightarrow$) Suppose A is invertible. Note that $\det(A) \neq 0$ and $\det(A^T) = \det(A)$.
Then $\det(A^T A) \neq 0$ and so $A^T A$ is invertible.

($\leftarrow$) Suppose $A^T A$ is invertible. Note that $\det(A^T A) \neq 0$. It follows that $\det(A) \neq 0$ and $\det(A^T) \neq 0$. Thus, A is invertible.

$\square$

19 Case 2. If E results from interchanging two rows of I_n, then by Theorem 1.5.1 EB results from B by interchanging two rows; so from Theorem 2.2.3b we have:

$$\det(EB) = -\det(B)$$

But from Theorem 2.2.4(b) we have $\det(E) = -1$, so

$$\det(EB) = \det(E)\det(B)$$

Case 3. If E results from adding a multiple of one row of I_n to another, then by Theorem 1.5.1 EB results from adding a multiple of one row of B to another; so from Theorem 2.2.3c we have:

$$\det(EB) = \det(B)$$

But from Theorem 2.2.4(c) we have $\det(E)=1$, so

$$\det(EB)=\det(E)\det(B).$$

$\square$

(20) $\det(AB)$ and $\det(BA)$ must be equal. Note:

$$\begin{aligned}\det(AB) &= \det(A)\det(B) && \text{Theorem 2.3.4}\\ &= \det(B)\det(A)\\ &= \det(BA)\end{aligned}$$

(21) If one or both factors of AB are singular then either $\det(A)=0$ or $\det(B)=0$. Note that $\det(A)\det(B)=0=\det(AB)$. Thus, AB is singular.

(22)

(a) Sometimes false:

$$\det(2A)=2^n\det(A)$$

Thus, when $n\neq 1$ the statement is false.

(b) Always true:

Let $A=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$. So, $\det(A)=ad-cb$ and $\det(A)^2=(ad-cb)^2$

$$=(ad-cb)(ad-cb)=a^2d^2-abcd-abcd+c^2b^2$$
$$=a^2d^2-2abcd+c^2b^2$$

And $A^2=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\begin{bmatrix} a & b\\ c & d\end{bmatrix}=\begin{bmatrix} a^2+bc & ab+bd\\ ca+dc & bc+d^2\end{bmatrix}$

And so $\det(A^2)=(a^2+bc)(bc+d^2)-(ca+dc)(ab+bd)$

$$=a^2bc+a^2d^2+b^2c^2+bcd^2-a^2bc-abcd-abcd-bcd^2$$
$$=a^2d^2+c^2b^2-2abcd.$$

Therefore: $\det(A^2)=\det(A)^2$

(c) Sometimes false:

Let $A=\begin{bmatrix} 1 & 2\\ 3 & 4\end{bmatrix}$. Note that $I+A=\begin{bmatrix} 2 & 2\\ 3 & 5\end{bmatrix}$. So: $\det(I+A)=4$

Note that $\det(A)=-2$, and $1+\det(A)=-1$

So: $\det(I+A)\neq 1+\det(A)$

(d) Always true:
Suppose $\det(A) = 0$. By Theorem 2.3.6, it is not the case that homogeneous system $A\mathbf{x} = 0$ has only the trivial solution. Since homogeneous systems must either have only the trivial solution or infinitely many solutions, the latter is true. Thus, $A\mathbf{x} = 0$ has infinitely many solutions.

(23)

(a) Always true:
This follows from Theorem 2.3.6.

(b) Always true:
This follows from Theorem 2.3.6.

(c) Sometimes false:
Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So, $\det(A) = 1 - 0 = 1$. Now, reverse the columns of A to yield $A' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. So, $\det(A') = 0 - 1 = -1$. Here, the determinant of A is changed if the columns are written in reverse order.

(d) Always true:
Let A be any square matrix. Note: $\det(AA^T) = \det(A)\det(A^T)$. But $\det(A) = \det(A^T)$. Thus, $\det(AA^T) = \det(A)\det(A)$, or $\det(A)^2$. Since, $\det(A)^2$ is a square, $\det(AA^T) \neq -1$. Therefore, there is no square matrix A such that $\det(AA^T) = -1$.