3.2: Proofs Involving Negations and Conditionals

Exercise 1:

(a) Proof: Suppose $P \to Q$ and $Q \to R$. Suppose P. Since $P \to Q$ and P, Q must be true. Since $Q \to R$ and Q, R must be true. Therefore $P \to R$ must be true. □

(b) Proof: Suppose $\neg R \to (P \to \neg Q)$. Suppose that P is true. Further suppose that Q is true. Thus $P \wedge Q$ is true, which can be expressed as $\neg(P \to \neg Q)$. Since $\neg(P \to \neg Q)$ and $\neg R \to (P \to \neg Q)$, $\neg\neg R$ is true, or R is true. Therefore $Q \to R$. Therefore $P \to (Q \to R)$. □

Exercise 2:

(a) Proof: Suppose $P \to Q$ and $R \to \neg Q$. Suppose P. Since P and $P \to Q$, Q must be true, that is, $\neg\neg Q$ must be true. Since $R \to \neg Q$ and $\neg\neg Q$, $\neg R$ must be true. Therefore, $P \to \neg R$. □

(b) Proof: Suppose P. Suppose further Q. Since R and Q, $Q \wedge P$ must be true. $Q \wedge P$ can be expressed as follows: $\neg(\neg Q \vee \neg P)$, and thus, $\neg(Q \to \neg P)$. Thus, $Q \to \neg(Q \to \neg P)$. □

Exercise 3:

Proof: Suppose $A \subseteq C$, and B and C are disjoint. Suppose that $x \in A$. Since $x \in A$ and $A \subseteq C$, $x \in C$. Since $x \in C$, and B and C are disjoint, $x \notin B$. Thus, if $x \in A$ then $x \notin B$. □

Exercise 4:

Proof: Suppose $A \setminus B$ is disjoint from C, and $x \in A$. Suppose $x \in C$. Since $x \in C$ and $A \setminus B$ is disjoint from C, $x \notin A \setminus B$. That is, $\neg(x \in A \wedge x \notin B)$, or $x \notin A$ or $x \in B$. Since $x \notin A$ or $x \in B$, and $x \in A$, $x \in B$. Thus, if $x \in C$ then $x \in B$. □

Exercise 5:

Proof: Suppose $x \in A \setminus B$ and $x \in B \setminus C$. Since $x \in A \setminus B$, $x \in A$ and $x \notin B$. Since $x \in B \setminus C$, $x \in B$ and $x \notin C$. Thus, $x \in B$ and $x \notin B$. Therefore it cannot be the case that both $x \in A \setminus B$ and $x \in B \setminus C$. □

Exercise 6:

Proof: Suppose $A \cap C \subseteq B$ and $a \in C$. Suppose $a \in A \setminus B$. Since $a \in A \setminus B$, $a \in A$ and $a \notin B$. Since $a \in A$ and $a \in C$, $a \in A \cap C$. Since $A \cap C \subseteq B$ and $a \in A \cap C$, $a \in B$. But $a \notin B$. Thus, $a \notin A \setminus B$. □

Exercise 7:

Proof: Suppose $A \subseteq B$, $a \in A$, and $a \notin B \setminus C$. Suppose $a \notin C$. Since $a \notin B \setminus C$, $a \notin B \vee a \in C$. Since $a \in A$ and $A \subseteq B$, $a \in B$. Since $a \notin B \vee a \in C$, and $a \in B$, $a \in C$. But $a \notin C$. Therefore $a \in C$. □

Exercise 8:

Proof: Suppose $y + x = 2y - x$, and x and y are not both zero. Suppose $y = 0$. Thus, $0 + x = 2(0) - x$, or $x = -x$, or $1 = -1$. Therefore, $y \neq 0$. □

Exercise 9:

Proof: Suppose a and b are nonzero real numbers. Suppose that $a < 1/a < b < 1/b$. Since $a < \frac{1}{a}$, either $a < -1$ or $0 < a < 1$. Similarly, $b < -1$ or $0 < b < 1$. Thus, we have four cases:

- Case 1. $a < -1$ and $b < -1$: Because $\frac{1}{a} < \frac{1}{b}$, and a and b are both negative, it follows that $a > b$. However, $a < b$. Thus, this case is impossible.
- Case 2. $a < -1$ and $0 < b < 1$: Since $a < -1$, $a < \frac{1}{a}$, which is also negative. Since $0 < b < 1$, $b < \frac{1}{b}$, but they are both positive. Thus, $a < \frac{1}{a} < b < \frac{1}{b}$.
- Case 3. $0 < a < 1$ and $b < -1$: If $b < -1$, but a is positive then $b < a$. But $b > a$. Thus, this case is impossible.
- Case 4: $0 < a < 1$ and $0 < b < 1$: Since $a < b$, $\frac{1}{b} < \frac{1}{a}$. But $\frac{1}{b} > \frac{1}{a}$. Thus, this case is impossible.

Since Case 2, is the only possible case, $a<-1$. $\square$

Exercise 10:

Proof: Suppose x and y are real numbers. Suppose $x^2y = 2x + y$. Suppose $y \neq 0$. Suppose $x=0$. Since $x=0$ and $x^2y = 2x+y$, $(0)^2y = 2(0)+y$, and so $0 = y$. But $y \neq 0$. Therefore, $x \neq 0$. Therefore, if $y \neq 0$, then $x \neq 0$. Therefore, if $x^2y = 2x+y$, then if $y \neq 0$, then $x \neq 0$. $\square$

Exercise 11:

Proof: Suppose that x and y are real numbers. Suppose $x \neq 0$. Suppose $y = (3x^2+2y)/(x^2+2)$. The equation can be rewritten: $yx^2 + 2y = 3x^2 + 2y$, or $yx^2 = 3x^2$. Since $x \neq 0$, both sides can be divided by x^2 to yield $y=3$. Thus, $x \neq 0$, then if $y = (3x^2+2y)/(x^2+2)$ then $y=3$. $\square$

Exercise 12:

(a) The negation of $x \neq 3$ and $y \neq 8$ is not $x=3$ and $y=8$, it is $x=3$ or $y=8$.
(b) If $x=3$ and $y=7$, then $x+y=10$. Thus, the theorem is incorrect.

Exercise 13:

(a) The premises $x \notin B$ and $B \subseteq C$ do not imply $x \notin C$.
(b) Let $A = \{a\}$, $B = \varnothing$, and $C = \{a, b\}$. $A \subseteq C$, $B \subseteq C$, and $a \in A$, but $a \notin B$. Thus, the theorem is incorrect.

Exercise 14:

P	Q	$P \to Q$	$\neg Q$	$\neg P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

(Modus tollens).

The argument is valid.

Exercise 15:

P	Q	R	$P \rightarrow$	$(Q \rightarrow R)$	$\neg R$	$\rightarrow$	$(P \rightarrow$	$\neg Q)$
T	T	T	T	T	F	T	F	F
T	T	F	F	F	T	F	F	F
T	F	T	T	T	F	T	T	T
T	F	F	T	T	T	T	T	T
F	T	T	T	T	F	T	T	F
F	T	F	T	F	T	T	T	F
F	F	T	T	T	F	T	T	T
F	F	F	T	T	T	T	T	T

The argument is valid.

Exercise 16:

(a)

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$P \rightarrow R$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

The argument is valid.

(b)

P	Q	R	¬R → (P → ¬Q)				P → (Q → R)	
T	T	T	F	T	F	F	T	T
T	T	F	T	F	F	F	F	F
T	F	T	F	T	T	T	T	T
T	F	F	T	T	T	T	T	T
F	T	T	F	T	T	F	T	T
F	T	F	T	T	T	F	T	F
F	F	T	F	T	T	T	T	T
F	F	F	T	T	T	T	T	T
				↑			↑	

The argument is valid.

Exercise 17:

(a)

	P	Q	R	P → Q	R → ¬Q		P → ¬R	
	T	T	T	T	F	F	F	F
→	T	T	F	T	T	F	T	T
	T	F	T	F	T	T	F	F
	T	F	F	F	T	T	T	T
	F	T	T	T	F	F	T	F
→	F	T	F	T	T	F	T	T
→	F	F	T	T	T	T	T	F
→	F	F	F	T	T	T	T	T
				↑	↑		↑	

The argument is valid.

(b)

	P	Q	$Q \to \neg(Q \to \neg P)$			
→	T	T	T	T	F	F
→	T	F	T	F	T	F
	F	T	T	F	T	T
	F	F	T	F	T	T

The argument is valid.

Exercise 18:

Proof: Suppose $x^2+y=13$ and $x \neq 3$. Suppose $y=4$. Substituting this into the equation $x^2+y=13$, we get $x^2+4=13$, so $x=3$ or -3. For $x=-3$, there is no contradiction. Thus, the proof cannot be modified to prove that if $x^2+y=13$ and $x\neq 3$ then $y \neq 4$.