Exercise 11:

(a) The lines never intersect.

(b) The lines intersect at one point.

(c) The lines coincide.

Exercise 12:

If there is one solution, all three lines intersect at the same point. Eliminating one of the three lines will not affect this solution.

If there are infinitely many solutions, all three lines coincide. Getting rid of one line would still result in two coinciding lines.

Exercise 13:

$$ax + by = 0$$
$$cx + dy = 0$$
$$ex + fy = 0$$

Either the lines coincide or they intersect at one point: $(0, 0)$.

1.2: Gaussian Elimination

Exercise 1: a, b, c, d, h, i, j

Exercise 2: a, b, d, e.

Exercise 3:

(a) Both

(b) Neither

(c) Both

(d) Row-echelon form

(e) Neither

(f) Both

Exercise 4:

(a) $x_1 = -3$
$x_2 = 0$
$x_3 = 7$

(b) $x_1 = 8 + 7t$ $x_4 = t$
$x_2 = 2 - 3t$
$x_3 = -5 - t$

(c) $x_1 = -2 + 6s - 3t$
$x_3 = 7 - 4t$
$x_4 = 8 - 5t$
$x_2 = s$
$x_5 = t$

(d) This system is inconsistent.

Exercise 5:

(a) $x_1 - 3x_2 - 4x_3 = 7$
$x_2 + 2x_3 = 2$
$x_3 = 5$

$x_1 - 3x_2 - 4x_3 = 7$
$x_2 = -8$ Substitute 3rd equation into 2nd.
$x_3 = 5$

$x_1 - 3(-8) - 4(5) = 7$ Substitute 2nd equation into 1st.
$x_1 + 24 - 20 = 7$
$x_1 = 3$

$x_1 = 3,\ x_2 = -8,\ x_3 = 5$

ⓑ Let $x_4 = t$. Then $x_3 = 2 - t$. Therefore:

$$x_2 = 4x_3 - 9x_4 = 4(2-t) - 9t = 8 - 13t = 3 \Rightarrow -5 + 13t$$

$$x_1 = 8x_3 - 5x_4 = 8(2-t) - 5t = 16 - 13t = 6 \Rightarrow -10 + 13t$$

ⓒ Let $x_5 = t$. Then $x_4 = 9 - 3t$. Also, let $x_2 = s$. Therefore:

$$x_3 = 5 - x_4 - 6x_5 = 5 - 9 + 3t - 6t = -4 - 3t$$

$$x_1 = -3 - 7x_2 + 2x_3 + 8x_5 = -3 - 7s - 8 - 6t + 8t = 2t - 7s - 11$$

ⓓ The system is inconsistent.

Exercise 6:

ⓐ

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \xrightarrow[\text{row 2}]{\text{1×row 1 added to}} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix} \xrightarrow[\text{row 3}]{\text{-3×row 1 added to}} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \xrightarrow{\text{-1×row 2}} \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \xrightarrow[\text{row 1}]{\text{-1×row 2 added to}}$$

$$\begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \xrightarrow[\text{row 3}]{\text{10×row 2 added to}} \begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \xrightarrow{-\frac{1}{52}\text{×row 3}} \underset{\text{REF}}{\begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}} \xrightarrow[\text{row 2}]{\text{5×row 3 added to}} \begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[\text{row 1}]{\text{-7×row 3 added to}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{ RREF}$$

$x_1 = 3$
$x_2 = 1$
$x_3 = 2$

ⓑ

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \xrightarrow{\frac{1}{2}\text{×row 1}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \xrightarrow[\text{row 2}]{\text{2×row 1 added to}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 8 & 1 & 4 & -1 \end{bmatrix} \xrightarrow[\text{row 3}]{\text{-8×row 1 added to}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{bmatrix} \xrightarrow{\text{row 2+row 3}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{7}\text{×row 2}} \underset{\text{REF}}{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}} \xrightarrow[\text{row 1}]{\text{-1×row 2 added to}} \underset{\text{RREF}}{\begin{bmatrix} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

Let $x_3 = t$. Then:

$x_1 = -\frac{1}{7} - \frac{3}{7}t$

$x_2 = \frac{1}{7} - \frac{4}{7}t$

ⓒ $\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \xrightarrow[\text{row 2}]{-2\times\text{row 1 added to}} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \xrightarrow[\text{row 3}]{1\times\text{row 1 added to}} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \xrightarrow[\text{row 4}]{-3\times\text{row 1 added to}} \begin{bmatrix} 1 & -1 & 2 & -1 & 1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$

$\xrightarrow[\text{row 2 and row 3}]{\text{switch}} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \xrightarrow[\text{row 3}]{-3\times\text{row 2 added to}} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \xrightarrow[\text{row 4}]{-3\times\text{row 2 added to}} \underset{\text{REF}}{\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \xrightarrow[\text{row 1}]{\text{row 2 added to}} \underset{\text{RREF}}{\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$

Let $w = s$ and $z = t$. Therefore:

$x = -1 + s$

$y = 2t$

ⓓ $\begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix} \xrightarrow[\text{and row 2}]{\text{switch row 1}} \begin{bmatrix} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix} \xrightarrow{\frac{1}{3}\times\text{row 1}} \begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{bmatrix} \xrightarrow[\text{row 3}]{-6\times\text{row 1 added to}} \begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{bmatrix}$

$\xrightarrow{-\frac{1}{2}\times\text{row 2}} \begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{bmatrix} \xrightarrow[\text{row 3}]{6\times\text{row 2 added to}} \underset{\text{REF}}{\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{bmatrix}}$ The system is inconsistent.

Exercise 7:

ⓐ $\underset{\text{REF}}{\begin{bmatrix} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}}$ $x_1 = 17 - 7x_3$, $x_2 = -9 + 5x_3$, $x_3 = 2$

$x_1 = 17 - 7(2) = 3$, $x_2 = -9 + 5(2) = 1$

ⓑ $\underset{\text{REF}}{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}}$ $x_1 = -x_2 - x_3$, $x_2 = \frac{1}{7} - \frac{4}{7}x_3$

$x_1 = -\frac{1}{7} + \frac{4}{7}x_3 - x_3$

$x_1 = -\frac{1}{7} - \frac{3}{7}x_3$

Let $x_3 = t$. Therefore: $x_1 = -\frac{1}{7} - \frac{3}{7}t$, $x_2 = \frac{1}{7} - \frac{4}{7}t$

ⓒ $\underset{\text{REF}}{\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$ $x = -1 + y - 2z + w$, $y = 2z$

$x = -1 + 2z - 2z + w$

$x = -1 + w$

Let $w = s$ and $z = t$. Therefore: $x = -1 + s$, $y = 2t$

ⓓ $\underset{\text{REF}}{\begin{bmatrix} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{bmatrix}}$ $a = -\frac{2}{3} - 2b + c$, $b = -\frac{1}{2} + \frac{3}{2}c$, $0 = 6$

The system is inconsistent.

Exercise 8:

ⓐ $\begin{bmatrix} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2} \times R1} \begin{bmatrix} 1 & -\frac{3}{2} & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow[+R2]{-2 \times R1} \begin{bmatrix} 1 & -\frac{3}{2} & -1 \\ 0 & 4 & 3 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow[+R3]{-3 \times R1} \begin{bmatrix} 1 & -\frac{3}{2} & -1 \\ 0 & 4 & 3 \\ 0 & \frac{13}{2} & 4 \end{bmatrix} \xrightarrow{\frac{1}{4} \times R2}$

$\begin{bmatrix} 1 & -\frac{3}{2} & -1 \\ 0 & 1 & \frac{3}{4} \\ 0 & \frac{13}{2} & 4 \end{bmatrix} \xrightarrow[+R3]{-\frac{13}{2} \times R2} \begin{bmatrix} 1 & -\frac{3}{2} & -1 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & -\frac{7}{8} \end{bmatrix} \xrightarrow{-\frac{8}{7} \times R3} \begin{bmatrix} 1 & -\frac{3}{2} & -1 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{bmatrix}$ REF

The system is inconsistent.

ⓑ $\begin{bmatrix} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ -6 & -4 & 2 & 30 \end{bmatrix} \xrightarrow[+R3]{2 \times R1} \begin{bmatrix} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3} \times R1} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R2]{-5 \times R1} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & -\frac{1}{3} & \frac{11}{3} & 25 \\ 3 & 1 & 3 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R3]{-3 \times R1}$

$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & -\frac{1}{3} & \frac{11}{3} & 25 \\ 0 & -1 & 4 & 26 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & -1 & 4 & 26 \\ 0 & -\frac{1}{3} & \frac{11}{3} & 25 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-1 \times R2} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & 1 & -4 & -26 \\ 0 & -\frac{1}{3} & \frac{11}{3} & 25 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R3]{\frac{1}{3} \times R2} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & \frac{7}{3} & \frac{49}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{3}{7} \times R3}$

$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R1]{-\frac{2}{3} \times R2} \begin{bmatrix} 1 & 0 & \frac{7}{3} & \frac{37}{3} \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R1]{-\frac{7}{3} \times R3} \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R2]{4 \times R3} \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ RREF

REF

$x_1 = -4, \; x_2 = 2, \; x_3 = 7$

ⓒ $\begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & -6 \end{bmatrix} \xrightarrow{\frac{1}{4} \times R1} \begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \\ -2 & 4 & -6 \end{bmatrix} \xrightarrow[+R2]{-3 \times R1} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ -2 & 4 & -6 \end{bmatrix} \xrightarrow[+R3]{2 \times R1} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ REF/RREF

$x_1 = 3 + 2x_2$

Let $x_2 = t$. Therefore: $x_1 = 3 + 2t$

ⓓ $\begin{bmatrix} 0 & 10 & -4 & 1 & 1 \\ 1 & 4 & -1 & 1 & 2 \\ 3 & 2 & 1 & 2 & 5 \\ -2 & -8 & 2 & -2 & -4 \\ 1 & -6 & 3 & 0 & 1 \end{bmatrix} \xrightarrow[+R4]{2 \times R2} \begin{bmatrix} 0 & 10 & -4 & 1 & 1 \\ 1 & 4 & -1 & 1 & 2 \\ 3 & 2 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 3 & 0 & 1 \end{bmatrix} \xrightarrow{R4 \leftrightarrow R5} \begin{bmatrix} 0 & 10 & -4 & 1 & 1 \\ 1 & 4 & -1 & 1 & 2 \\ 3 & 2 & 1 & 2 & 5 \\ 1 & -6 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2}$

$\begin{bmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 3 & 2 & 1 & 2 & 5 \\ 1 & -6 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R3]{-3 \times R1} \begin{bmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & -10 & 4 & -1 & -1 \\ 1 & -6 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R3]{1 \times R2} \begin{bmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R3 \leftrightarrow R4}$

$\begin{bmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 1 & -6 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R3]{-1 \times R1} \begin{bmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & -10 & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2 + R3} \begin{bmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 10 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{10} \times R2} \begin{bmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 1 & -\frac{2}{5} & \frac{1}{10} & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ REF

$$\xrightarrow[+R_1]{-4\times R_2}\begin{bmatrix} 1 & 0 & 3/5 & 3/5 & 8/5 \\ 0 & 1 & -2/5 & 1/10 & 1/10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
RREF

$x = \frac{8}{5} - \frac{3}{5}z - \frac{3}{5}w$

$y = \frac{1}{10} + \frac{2}{5}z - \frac{1}{10}w$

Let $z = t$ and $w = s$. Therefore: $x = \frac{8}{5} - \frac{3}{5}t - \frac{3}{5}s$, $y = \frac{1}{10} + \frac{2}{5}t - \frac{1}{10}s$

Exercise 9:

ⓐ $\begin{bmatrix} 1 & -3/2 & -1 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}$
REF

$x_1 = -1 + \frac{3}{2}x_2$, $x_2 = \frac{3}{4}$, $0 = 1$

The system is inconsistent.

ⓑ $\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & -5 \\ 0 & 1 & -4 & -26 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
REF

$x_1 = -5 - \frac{2}{3}x_2 + \frac{1}{3}x_3$, $x_2 = -26 + 4x_3$, $x_3 = 7$

$x_1 = -5 - \frac{2}{3}(2) + \frac{1}{3}(7)$, $x_2 = -26 + 4(7)$

$x_1 = -4$, $x_2 = 2$

ⓒ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
REF

$x_1 = 3 + 2x_2$

Let $x_2 = t$. Therefore: $x_1 = 3 + 2t$

ⓓ $\begin{bmatrix} 1 & 4 & -1 & 1 & 2 \\ 0 & 1 & -\frac{2}{5} & \frac{1}{10} & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
REF

$x = 2 - 4y + z - w$, $y = \frac{1}{10} + \frac{2}{5}z - \frac{1}{10}w$

$x = 2 - 4\left(\frac{1}{10} + \frac{2}{5}z - \frac{1}{10}w\right) + z - w$

$x = 2 - \frac{4}{10} - \frac{8}{5}z + \frac{2}{5}w + z - w$

$x = \frac{8}{5} - \frac{3}{5}z - \frac{3}{5}w$

Let $z = t$ and $w = s$. Therefore: $x = \frac{8}{5} - \frac{3}{5}t - \frac{3}{5}s$, $y = \frac{1}{10} + \frac{2}{5}t - \frac{1}{10}s$

Exercise 10:

ⓐ $$\begin{bmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{bmatrix} \xrightarrow{\frac{1}{5}\times R_1} \begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} & 0 \\ -2 & 1 & 3 & 1 \end{bmatrix} \xrightarrow[+R_2]{2\times R_1} \begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} & 0 \\ 0 & \frac{1}{5} & \frac{27}{5} & 0 \end{bmatrix} \xrightarrow{5\times R_2} \begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} & 0 \\ 0 & 1 & 27 & 0 \end{bmatrix} \xrightarrow[+R_1]{\frac{2}{5}\times R_2}$$
REF

$\begin{bmatrix} 1 & 0 & 12 & 0 \\ 0 & 1 & 27 & 0 \end{bmatrix}$
RREF

$x_1 = -12x_3$, $x_2 = -27x_3$

Let $x_3 = t$. therefore: $x_1 = -12t$, $x_2 = -27t$

(b)

$$\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{bmatrix} \xrightarrow[+R_2]{-1\times R_1} \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 1 & -12 & -11 & -16 & 5 \end{bmatrix} \xrightarrow[+R_3]{-1\times R_1} \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{bmatrix} \xrightarrow[+R_3]{2\times R_2} \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{5}\times R_2} \underset{\text{REF}}{\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 1 & \frac{6}{5} & \frac{6}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}} \xrightarrow[+R_1]{2\times R_2} \underset{\text{RREF}}{\begin{bmatrix} 1 & 0 & \frac{17}{5} & -\frac{8}{5} & \frac{7}{5} \\ 0 & 1 & \frac{6}{5} & \frac{6}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}}$$

The system is inconsistent.

(c)

$$\begin{bmatrix} 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 2 & 4 & 1 & 7 & 0 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 2 & 4 & 1 & 7 & 0 & 7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 1 & 2 & -1 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}\times R_1} \begin{bmatrix} 1 & 2 & \frac{1}{2} & \frac{7}{2} & 0 & \frac{7}{2} \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 1 & 2 & -1 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & \frac{1}{2} & \frac{7}{2} & 0 & \frac{7}{2} \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow[+R_4]{-1\times R_2} \begin{bmatrix} 1 & 2 & \frac{1}{2} & \frac{7}{2} & 0 & \frac{7}{2} \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -1 & 1 & -3 \end{bmatrix} \xrightarrow{R_3+R_4} \underset{\text{REF}}{\begin{bmatrix} 1 & 2 & \frac{1}{2} & \frac{7}{2} & 0 & \frac{7}{2} \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \xrightarrow[+R_1]{-\frac{1}{2}\times R_2} \begin{bmatrix} 1 & 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R_1]{-2\times R_3} \begin{bmatrix} 1 & 2 & 0 & 0 & 3 & -6 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[+R_2]{-3\times R_3} \underset{\text{RREF}}{\begin{bmatrix} 1 & 2 & 0 & 0 & 3 & -6 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

$u = -6 - 2v - 3y$, $w = -2 - y$, $x = 3 + y$

Let $v = t$ and $y = s$. Therefore: $u = -6 - 2t - 3s$, $w = -2 - s$, $x = 3 + s$

Exercise 11:

(a)

$$\underset{\text{REF}}{\begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} & 0 \\ 0 & 1 & 27 & 0 \end{bmatrix}}$$

$x_1 = \frac{2}{5}x_2 - \frac{6}{5}x_3$, $x_2 = -27x_3$

$x_1 = \frac{2}{5}(-27x_3) - \frac{6}{5}x_3$

$x_1 = -12x_3$

Let $x_3 = t$. Therefore: $x_1 = -12t$, $x_2 = -27t$

(b)

$$\underset{\text{REF}}{\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 1 & \frac{6}{5} & \frac{6}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}}$$

$x_1 = 1 + 2x_2 - x_3 + 4x_4$, $x_2 = \frac{1}{5} - x_2 - \frac{6}{5}x_3 - \frac{6}{5}x_4$, $0 = 6$

The system is inconsistent.

(c)

$$\underset{\text{REF}}{\begin{bmatrix} 1 & 2 & \frac{1}{2} & \frac{7}{2} & 0 & \frac{7}{2} \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

$u = \frac{7}{2} - 2v - \frac{1}{2}w - \frac{7}{2}x$, $w = 7 - 3x + 2y$, $x = 3 + y$

$u = \frac{7}{2} - 2v - \frac{1}{2}(-2-y) - \frac{7}{2}(3+y)$, $w = 7 - 3(3+y) + 2y$

$u = \frac{7}{2} - 2v + 1 + \frac{y}{2} - \frac{21}{2} - \frac{7}{2}y$, $w = 7 - 9 - 3y + 2y$

$u = -6 - 2v - 3y$, $w = -2 - y$

Let $v = t$ and $y = s$. Therefore: $u = -6 - 2t - 3s$, $w = -2 - s$, $x = 3 + s$

Exercise 12:

ⓐ Has nontrivial solutions.

ⓑ Only the trivial solution.

ⓒ Has nontrivial solutions.

ⓓ Has nontrivial solutions.

Exercise 13:

ⓐ

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}\times R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[+R_2]{-1\times R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{2}{3}\times R_2} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow[+R_3]{-1\times R_2} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow[+R_1]{-\frac{1}{2}\times R_2} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}\times R_3} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[+R_1]{-2\times R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$x_1 = 0,\ x_2 = 0,\ x_3 = 0$

ⓑ

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}\times R_1} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -1 & 1 & -1 & 0 \end{bmatrix} \xrightarrow[+R_2]{-5\times R_1} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} & 0 \end{bmatrix} \xrightarrow{-\frac{3}{8}\times R_2} \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix} \xrightarrow[+R_1]{-\frac{1}{3}\times R_2}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{bmatrix}$$

$x_1 = -\frac{1}{4}x_3,\ x_2 = -\frac{1}{4}x_3 - x_4$

Let $x_3 = t$ and $x_4 = s$. Therefore: $x_1 = -\frac{1}{4}t,\ x_2 = -\frac{1}{4}t - s$

ⓒ

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \xrightarrow[+R_3]{-2\times R_1} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \xrightarrow[+R_4]{2\times R_1} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}\times R_2} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \xrightarrow[+R_3]{-3\times R_2} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \xrightarrow[+R_4]{-1\times R_2} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix} \xrightarrow[+R_4]{10\times R_3} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$w = y + 3z,\ x = -y - 2z,\ z = 0$

$w = y \qquad x = -y$

Let $y = t$. Therefore: $w = t,\ x = -t,\ z = 0$.

Exercise 14:

ⓐ

$$\begin{bmatrix} 2 & -1 & -3 & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}\times R_1} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ -1 & 2 & -3 & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{9}{2} & 0 \\ 1 & 1 & 4 & 0 \end{bmatrix} \xrightarrow[+R_3]{-1\times R_1} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 3/2 & -9/2 & 0 \\ 0 & 3/2 & 11/2 & 0 \end{bmatrix} \xrightarrow{\frac{2}{3}\times R_2}$$

$$\begin{bmatrix} 1 & -1/2 & -3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 3/2 & 11/2 & 0 \end{bmatrix} \xrightarrow[+R_3]{-\frac{3}{2}\times R_2} \begin{bmatrix} 1 & -1/2 & -3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix} \xrightarrow{\frac{1}{10}\times R_3} \begin{bmatrix} 1 & -1/2 & -3/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$x = \frac{1}{2}y + \frac{3}{2}z$, $y = 3z$, $z = 0$

$x = 0$, $y = 0$ Only the trivial solution.

ⓑ

$$\begin{bmatrix} 0 & 1 & 3 & -2 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2} \times R_1} \begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix} \xrightarrow[+R_3]{-2 \times R_1} \begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 2 & 6 & -4 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix} \xrightarrow[+R_3]{-2 \times R_2} \begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ -4 & -3 & 5 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R_3]{4 \times R_1} \begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & -1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & \frac{1}{2} & -2 & \frac{3}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R_1]{-\frac{1}{2} \times R_2} \begin{bmatrix} 1 & 0 & -\frac{7}{2} & \frac{5}{2} & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$u = \frac{7}{2}w - \frac{5}{2}x$, $v = 2x - 3w$

Let $w = t$ and $x = s$. Therefore: $u = \frac{7}{2}t - \frac{5}{2}s$, $v = 2s - 3t$.

ⓒ

$$\begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 2 & -4 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[+R_2]{-1 \times R_1} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 2 & -4 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[+R_4]{-2 \times R_1} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 0 & -10 & 1 & -1 & 0 \\ 1 & -2 & -1 & 1 & 0 \end{bmatrix} \xrightarrow[+R_5]{-1 \times R_1} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -2 & -2 & -1 & 0 \\ 0 & -10 & 1 & -1 & 0 \\ 0 & -5 & -1 & 0 & 0 \end{bmatrix} \xrightarrow[+R_3]{2 \times R_2} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & -10 & 1 & -1 & 0 \\ 0 & -5 & -1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[+R_4]{10 \times R_2} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & -5 & -1 & 0 & 0 \end{bmatrix} \xrightarrow[+R_5]{5 \times R_2} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2} \times R_3} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 21 & -11 & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix} \xrightarrow[+R_4]{-21 \times R_3} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{41}{2} & 0 \\ 0 & 0 & 9 & -5 & 0 \end{bmatrix} \xrightarrow[+R_5]{-9 \times R_3} \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 41/2 & 0 \\ 0 & 0 & 0 & 17/2 & 0 \end{bmatrix}$$

$$\xrightarrow[+R_1]{-3 \times R_2} \begin{bmatrix} 1 & 0 & -6 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 41/2 & 0 \\ 0 & 0 & 0 & 17/2 & 0 \end{bmatrix}$$

$x_1 = 6x_3 - 5x_4$, $x_2 = x_4 - 2x_3$, $x_3 = \frac{3}{2}x_4$, $\frac{41}{2}x_4 = 0$, $\frac{17}{2}x_4 = 0$

$x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$

Only the trivial solution.

Exercise 15:

ⓐ

$$\begin{bmatrix} 2 & -1 & 3 & 4 & 9 \\ 1 & 0 & -2 & 7 & 11 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix} \xrightarrow{\frac{1}{2} \times R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 2 & \frac{9}{2} \\ 1 & 0 & -2 & 7 & 11 \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix} \xrightarrow[+R_2]{-1 \times R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 2 & \frac{9}{2} \\ 0 & \frac{1}{2} & -\frac{7}{2} & 5 & \frac{13}{2} \\ 3 & -3 & 1 & 5 & 8 \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix} \xrightarrow[+R_3]{-3 \times R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 2 & \frac{9}{2} \\ 0 & \frac{1}{2} & -\frac{7}{2} & 5 & \frac{13}{2} \\ 0 & -\frac{3}{2} & -\frac{7}{2} & -1 & -\frac{11}{2} \\ 2 & 1 & 4 & 4 & 10 \end{bmatrix} \xrightarrow[+R_4]{-2 \times R_1}$$

$$\begin{bmatrix} 1 & -1/2 & 3/2 & 2 & 9/2 \\ 0 & 1/2 & -7/2 & 5 & 13/2 \\ 0 & -3/2 & -7/2 & -1 & -11/2 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{2 \times R_2} \begin{bmatrix} 1 & -1/2 & 3/2 & 2 & 9/2 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & -3/2 & -7/2 & -1 & -11/2 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[+R_1]{\frac{1}{2} \times R_2} \begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & -3/2 & -7/2 & -1 & -11/2 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[+R_3]{\frac{3}{2} \times R_2} \begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & -14 & 14 & 14 \\ 0 & 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[+R_4]{-2 \times R_2} \begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & -14 & 14 & 14 \\ 0 & 0 & 15 & -20 & -25 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{14} \times R_3} \begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 15 & -20 & -25 \end{bmatrix} \xrightarrow[+R_4]{-15 \times R_3} \begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -5 & -10 \end{bmatrix} \xrightarrow{-\frac{1}{5} \times R_4} \begin{bmatrix} 1 & 0 & -2 & 7 & 11 \\ 0 & 1 & -7 & 10 & 13 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$I_1 = 11 + 2I_3 - 7I_4,\ I_2 = 13 + 7I_3 - 10I_4,\ I_3 = -1 + I_4,\ I_4 = 2$

$I_1 = -1,\quad I_2 = 0,\quad I_3 = 1$

(b)
$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[+R_4]{-2\times R_1} \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{bmatrix} \xrightarrow[+R_4]{-3\times R_2} \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \xrightarrow[+R_4]{-1\times R_3} \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{-3}\times R_3} \begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R_1]{2\times R_2}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R_1]{-2\times R_3} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[+R_2]{-1\times R_3} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$z_1 = -z_2 - z_5,\ z_3 = -z_5,\ z_4 = 0.$

Let $z_2 = s$ and $z_5 = t$. Therefore: $z_1 = -s - t,\ z_3 = -t,\ z_4 = 0.$

Exercise 16:

(a)
$$\begin{bmatrix} 2 & 1 & a \\ 3 & 6 & b \end{bmatrix} \xrightarrow{\frac{1}{2}\times R_1} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2}a \\ 3 & 6 & b \end{bmatrix} \xrightarrow[+R_2]{-3\times R_1} \begin{bmatrix} 1 & 1/2 & \frac{1}{2}a \\ 0 & 9/2 & -\frac{3}{2}a + b \end{bmatrix} \xrightarrow{\frac{2}{9}\times R_2} \begin{bmatrix} 1 & 1/2 & \frac{1}{2}a \\ 0 & 1 & -\frac{1}{3}a + \frac{2}{9}b \end{bmatrix}$$

$x = \frac{1}{2}a - \frac{1}{2},\quad y = -\frac{1}{3}a + \frac{2}{9}b$

$x = \frac{1}{2}(a-1),\quad y = -\frac{1}{3}(a - \frac{2}{3}b)$

(b)
$$\begin{bmatrix} 1 & 1 & 1 & a \\ 2 & 0 & 2 & b \\ 0 & 3 & 3 & c \end{bmatrix} \xrightarrow[+R_2]{-2\times R_1} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -2 & 0 & -2a+b \\ 0 & 3 & 3 & c \end{bmatrix} \xrightarrow{-2\times R_2} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a + \frac{-b}{2} \\ 0 & 3 & 3 & c \end{bmatrix} \xrightarrow[+R_3]{-3\times R_2} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a + \frac{-b}{2} \\ 0 & 0 & 3 & -3a + \frac{3b}{2} + c \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}\times R_3} \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a + \frac{-b}{2} \\ 0 & 0 & 1 & -a + \frac{b}{2} + \frac{c}{3} \end{bmatrix}$$

$x_1 = a - x_2 - x_3,\ x_2 = a + \frac{-b}{2},\ x_3 = -a + \frac{b}{2} + \frac{c}{3}$

$x_1 = a - (a + \frac{-1}{2}b) - (-a + \frac{1}{2}b + \frac{1}{3}c)$

$x_1 = a - a + \frac{1}{2}b + a - \frac{1}{2}b - \frac{1}{3}c$

$x_1 = a - \frac{1}{3}c$

Exercise 17:

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \xrightarrow[+R_2]{-3\times R_1} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \xrightarrow[+R_3]{-4\times R_1} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix} \xrightarrow{-\frac{1}{7}\times R_2}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix} \xrightarrow[+R_3]{7\times R_2} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

$x = 4 - 2y + 3z$, $y = \frac{10}{7} + 2z$, $(a^2-16)z = a-4$

No solutions: $a = -4$

One solution: $a \neq \pm 4$

Infinite solutions: $a = 4$

Exercise 18:

$$\begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[+R_3]{-1\times R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & -29 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -29 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow[+R_3]{-2\times R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -29 \\ 0 & -5 & -1 \end{bmatrix} \xrightarrow{-1\times R_3} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -29 \\ 0 & 5 & 1 \end{bmatrix} \xrightarrow[+R_3]{2\times R_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -29 \\ 0 & 1 & -57 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -57 \\ 0 & -2 & -29 \end{bmatrix} \xrightarrow[+R_1]{-3\times R_2} \begin{bmatrix} 1 & 0 & 173 \\ 0 & 1 & -57 \\ 0 & -2 & -29 \end{bmatrix} \xrightarrow[+R_3]{2\times R_2} \begin{bmatrix} 1 & 0 & 173 \\ 0 & 1 & -57 \\ 0 & 0 & -143 \end{bmatrix} \xrightarrow{-\frac{1}{143}\times R_3} \begin{bmatrix} 1 & 0 & 173 \\ 0 & 1 & -57 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[+R_2]{57\times R_3} \begin{bmatrix} 1 & 0 & 173 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[+R_1]{-173\times R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 19:

One: $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \xrightarrow[+R_2]{-2\times R_1} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

Two: $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}\times R_1} \begin{bmatrix} 1 & \frac{7}{2} \\ 1 & 3 \end{bmatrix} \xrightarrow[+R_2]{-1\times R_1} \begin{bmatrix} 1 & 7/2 \\ 0 & -1/2 \end{bmatrix} \xrightarrow{-2\times R_2} \begin{bmatrix} 1 & 7/2 \\ 0 & 1 \end{bmatrix}$

Exercise 20:

$$\begin{bmatrix} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{bmatrix} \xrightarrow{\frac{1}{2}\times R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{bmatrix} \xrightarrow[+R_2]{-4\times R_1} \begin{bmatrix} 1 & -1/2 & 3/2 & 3/2 \\ 0 & 4 & -8 & -4 \\ 6 & -3 & 1 & 9 \end{bmatrix} \xrightarrow[+R_3]{-6\times R_1} \begin{bmatrix} 1 & -1/2 & 3/2 & 3/2 \\ 0 & 4 & -8 & -4 \\ 0 & 0 & -8 & 0 \end{bmatrix} \xrightarrow{\frac{1}{4}\times R_2} \begin{bmatrix} 1 & -1/2 & 3/2 & 3/2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -8 & 0 \end{bmatrix} \xrightarrow[+R_1]{\frac{1}{2}\times R_2}$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -8 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{8}\times R_3} \begin{bmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[+R_2]{2\times R_3} \begin{bmatrix} 1 & 0 & 1/2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[+R_1]{-\frac{1}{2}\times R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\sin\alpha = 1$, $\cos\beta = -1$, $\tan\gamma = 0$

Thus, $\alpha = \frac{\pi}{2}$, $\beta = \pi$, $\gamma = 0$

Exercise 21:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{bmatrix} \xrightarrow[+R_2]{-2\times R_1} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ -1 & -5 & 5 & 0 \end{bmatrix} \xrightarrow{R_1+R_3} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 8 & 0 \end{bmatrix} \xrightarrow[+R_3]{3\times R_2} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{-1\times R_3} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[+R_1]{-2\times R_2} \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[+R_2]{3\times R_3} \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[+R_1]{-9\times R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\sin\alpha = 0$, $\cos\beta = 0$, $\tan\gamma = 0$

$\alpha = 0, \pi, 2\pi$; $\beta = \frac{\pi}{2}, \frac{3\pi}{2}$; $\gamma = 0, \pi, 2\pi$

There are: $3 \times 2 \times 3$ solutions $=$ 18 solutions.

Exercise 22:

$$\begin{bmatrix} \lambda-3 & 1 & 0 \\ 1 & \lambda-3 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & \lambda-3 & 0 \\ \lambda-3 & 1 & 0 \end{bmatrix} \xrightarrow[+R_2]{-(\lambda-3)\times R_1} \begin{bmatrix} 1 & \lambda-3 & 0 \\ 0 & 1-(\lambda-3)^2 & 0 \end{bmatrix}$$

If $\lambda = 2$: $x = y$

If $\lambda = 4$: $x = -y$

Otherwise, there is only the trivial solution.

Exercise 23:

Let $\lambda = 1$. Therefore:

$$\begin{aligned} x_1 - x_2 \quad &= 0 \\ 2x_1 - 2x_2 + x_3 &= 0 \\ -2x_1 + 2x_2 \quad &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ -2 & 2 & 0 & 0 \end{bmatrix} \xrightarrow[+R_2]{-2\times R_1} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 2 & 0 & 0 \end{bmatrix} \xrightarrow[+R_3]{2\times R_1} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_2$, $x_3 = 0$. Let $x_2 = t$. Therefore: $x_1 = t$, $x_3 = 0$.

Let $\lambda = 2$. Therefore:

$$\begin{aligned} -x_2 \quad &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \\ -2x_1 + 2x_2 - x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -2 & 2 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} -2 & 2 & -1 & 0 \\ 2 & -3 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}\times R_1} \begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 2 & -3 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \xrightarrow[+R_2]{-2\times R_1} \begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \xrightarrow{-1\times R_2} \begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3} \begin{bmatrix} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -\frac{1}{2}x_3$, $x_2 = 0$. Let $x_3 = t$. Therefore:

$x_1 = -\frac{1}{2}t$, $x_2 = 0$.

Exercise 24:

Let $u = \frac{1}{x}$, $v = \frac{1}{y}$, $w = \frac{1}{z}$. Thus:
$$u + 2v - 4w = 1$$
$$2u + 3v + 8w = 0$$
$$-u + 9v + 10w = 5$$

$$\begin{bmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & 8 & 0 \\ -1 & 9 & 10 & 5 \end{bmatrix} \xrightarrow[+R_2]{-2 \times R_1} \begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & -1 & 16 & -2 \\ -1 & 9 & 10 & 5 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & -1 & 16 & -2 \\ 0 & 11 & 6 & 6 \end{bmatrix} \xrightarrow{-1 \times R_2} \begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 11 & 6 & 6 \end{bmatrix} \xrightarrow[+R_3]{-11 \times R_2} \begin{bmatrix} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 182 & -16 \end{bmatrix} \xrightarrow[+R_1]{-2 \times R_2}$$

$$\begin{bmatrix} 1 & 0 & 28 & -3 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 182 & -16 \end{bmatrix} \xrightarrow{\frac{1}{182} \times R_3} \begin{bmatrix} 1 & 0 & 28 & -3 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 1 & \frac{-8}{91} \end{bmatrix} \xrightarrow[+R_2]{16 * R_3} \begin{bmatrix} 1 & 0 & 28 & -3 \\ 0 & 1 & 0 & 54/91 \\ 0 & 0 & 1 & \frac{-8}{91} \end{bmatrix} \xrightarrow[+R_1]{-28 \times R_3} \begin{bmatrix} 1 & 0 & 0 & -7/13 \\ 0 & 1 & 0 & 54/91 \\ 0 & 0 & 1 & -8/91 \end{bmatrix}$$

$u = \frac{-7}{13}$, $v = \frac{54}{91}$, $w = \frac{-8}{91}$

Thus: $x = -\frac{13}{7}$, $y = \frac{91}{54}$, $z = -\frac{91}{8}$

Exercise 25:

$d = 10$, $7 = a + b + c + d$, $-11 = 27a + 9b + 3c + d$, $-14 = 64a + 16b + 4c + d$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 7 \\ 27 & 9 & 3 & 1 & -11 \\ 64 & 16 & 4 & 1 & -14 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix} \xrightarrow[\text{RREF}]{\text{To}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 10 \end{bmatrix}$$

Thus: $a = 1$, $b = -6$, $c = 2$, $d = 10$.

Exercise 26:

$ax^2 + ay^2 + bx + cy + d = x^2 + y^2 + \frac{b}{a}x + \frac{c}{a}y + \frac{d}{a} = 0$

Let $\frac{b}{a} = u$, $\frac{c}{a} = v$, $\frac{d}{a} = w$. Thus: $x^2 + y^2 + ux + vy + w = 0$

$16 + 25 - 4u + 5v + w = 0$, $4 + 49 - 2u + 7v + w = 0$, $16 + 9 + 4u - 3v + w = 0$

$-4u + 5v + w = -41$, $-2u + 7v + w = -53$, $4u - 3v + w = -25$

$$\begin{bmatrix} -4 & 5 & 1 & -41 \\ -2 & 7 & 1 & -53 \\ 4 & -3 & 1 & -25 \end{bmatrix} \xrightarrow[\text{RREF}]{\text{to}} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -29 \end{bmatrix}$$

$u = -2$, $v = -4$, $w = -29$

$\frac{b}{a} = -2$, $\frac{c}{a} = -4$, $\frac{d}{a} = -29$

Since the curve is a circle, $a = 1$. Thus: $b = 2$, $c = -4$, $d = -29$.

Exercise 27:

(a) Suppose that $ad-bc \neq 0$. Suppose that $a \neq 0$. Thus:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\frac{1}{a}\times R_1} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow[+R_2]{-c\times R_1} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d-\frac{cb}{a} \end{bmatrix} \xrightarrow{\frac{1}{(d-\frac{cb}{a})}\times R_2} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} \xrightarrow[+R_1]{-\frac{b}{a}\times R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Suppose that $a=0$. Therefore, $b \neq 0$ and $c \neq 0$. Thus:

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \xrightarrow{\frac{1}{c}\times R_1} \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & b \end{bmatrix} \xrightarrow{\frac{1}{b}\times R_2} \begin{bmatrix} 1 & \frac{d}{c} \\ 0 & 1 \end{bmatrix} \xrightarrow[+R_1]{-\frac{d}{c}\times R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $(a=0) \vee (a \neq 0)$, the RREF is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\square$

(b) Suppose that $ad-bc \neq 0$.

$$\begin{bmatrix} a & b & k \\ c & d & l \end{bmatrix} \xrightarrow{\text{Part (a)}} \begin{bmatrix} 1 & 0 & \bar{k} \\ 0 & 1 & \bar{l} \end{bmatrix}$$

Thus: $x=\bar{k}$ and $y=\bar{l}$, where $\bar{k}$ and $\bar{l}$ are both constants. Thus, there is only one solution.

Exercise 28:

$x+y+z=1$

$x+y+z=2.$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{\text{To RREF}} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the system is inconsistent.

Exercise 29:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & p & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & p & q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Where p and q are any real numbers.

Exercise 30:

(a) 3 lines which intersect at one point, at least 2 are distinct, (b) Three identical lines.

Exercise 31:

(a) False. A matrix always has a single RREF.

(b) True. A matrix may have multiple REFs.

(c) False. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(d) False. It would have no solutions.

Exercise 32:

(a) False. $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$

(b) False $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) False $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) False $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

1.3: Matrices and Matrix Operations

Exercise 1:

(a) $(4\times5)(4\times5)$ undefined

(b) $(4\times5)(5\times2)+(4\times2)$

$(4\times2)+(4\times2)$ defined (4×2)

(c) $(4\times5)(5\times2)+(4\times5)$

$(4\times2)+(4\times5)$ undefined

(d) $(4\times5)(4\times5)+(4\times5)$ undefined.

(e) $(5\times4)((4\times5)+(4\times5))$

$(5\times4)(4\times5)$ defined (5×5)

(f) $(5\times4)((4\times5)(5\times2))$

$(5\times4)(4\times2)$ defined (5×2)

(g) $(5\times4)^T(4\times5)$

$(4\times5)(4\times5)$ undefined.

(h) $((4\times5)^T+(5\times4))(4\times2)$

$((5\times4)+(5\times4))(4\times2)$

$(5\times4)(4\times2)$ defined (5×2)

Exercise 2:

$a-b=8$

$b+c=1$

$3d+c=7$

$2a-4d=6$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 8 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 7 \\ 2 & 0 & 0 & -4 & 6 \end{bmatrix} \xrightarrow[\text{RREF}]{\text{To}} \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$a=5,\ b=-3,\ c=4,\ d=1$