Chapter 3: Proofs

3.1: Proof Strategies

Exercise 1:

(a) Hypotheses: n is an integer, $n > 1$, n is not prime.
Conclusion: $2^{n}-1$ is not prime
Let $n = 6$. $2^6 - 1 = 63$ (not prime). This only tells us that the theorem is correct in this instance.

(b) $2^{15} - 1 = 32{,}767$. This is not prime and so the conclusion is correct in this instance.

(c) The theorem tells us nothing because n is prime.

Exercise 2:

(a) Hypotheses: $b^2 > 4ac$
Conclusion: $ax^2 + bx + c$ has exactly 2 real solutions.

(b) Because x is not a free variable.

(c) $(-5)^2 > 4(2)(3)$
$2x^2 - 5x + 3$
$= 2x^2 - 2x - 3x + 3$
$= 2x(x-1) + {}^{-}3(x-1)$
$(2x-3)(x-1)$ $\Rightarrow$ The theorem is true in this instance.

(d) $(4)^2 \not> 4(2)(3)$
Thus, the theorem tells us nothing!

Exercise 3:

Hypothesis: $n \in \mathbb{N}$, $n > 2$, n is not prime.
Conclusion: $2n + 13$ is not prime.
Counterexample: Let $n = 8$. Thus, $8 \in \mathbb{N}$, $8 > 2$, and 8 is not prime. But $2(8) + 13 = 29$, and 29 is prime. The theorem is incorrect.

Exercise 4:

Proof: Note that $b - a > 0$. Multiply both sides by $(b+a)$ to get: $(b+a)(b-a) > 0(b+a) = b^2 - a^2 > 0$.

$\square$

Exercise 5:

Proof: Since $a < b < 0$, $a - b < 0$. Multiply both sides by $(a+b)$ (note that $(a+b)$ is negative). Thus $(a+b)(a-b) > 0(a+b)$. That is, $a^2 - b^2 > 0$. Thus, $a^2 > b^2$.

$\square$

Exercise 6:

Proof: Let a and b be real numbers. Suppose that $0 < a < b$. Note that $\frac{1}{ab}$ is positive. Multiply both sides by $\frac{1}{ab}$ to get: $\frac{a}{ab} < \frac{b}{ab}$. Thus, $\frac{1}{b} < \frac{1}{a}$.

$\square$

Exercise 7:

Proof: Let a be any real number. Suppose that $a^3 > a$. Thus, $a^3 - a > 0$. Multiply both sides by (a^2+1): $(a^3-a)(a^2+1) > 0$. That is, $a^5 - a > 0$. Add a to both sides for: $a^5 > a$.

$\square$

Exercise 8:

Proof: Suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$. We will prove the contrapositive. Suppose that $x \notin B$. Note that $x \in A$ and $x \notin B$, that is, $x \in A \setminus B$. Because $A \setminus B \subseteq C \cap D$, $x \in C \cap D$. That is, $x \in C$ and $x \in D$. Thus, if $x \notin B$ then $x \in D$, or if $x \notin D$ then $x \in B$. $\square$

Exercise 9:

Proof: Suppose that $A \cap B \subseteq C \setminus D$. Suppose that $x \in A$. Now suppose that $x \in B$ (for contrapositive). So, $x \in A$ and $x \in B$. Thus, $x \in A \cap B$. Since $A \cap B \subseteq C \setminus D$, $x \in C \setminus D$. That is, $x \in C$ and $x \notin D$. Thus, if $x \in A$, then if $x \in B$, then $x \notin D$. Or, if $x \in A$, then if $x \in D$, then $x \notin B$. $\square$

Exercise 10:

Proof: Suppose that a and b are real numbers. Suppose that $a < b$. Add b to both sides: $a + b < b + b$. That is, $a + b < 2b$. Divide both sides by 2: $\frac{(a+b)}{2} < \frac{2b}{2}$, or $\frac{(a+b)}{2} < b$. $\square$

Exercise 11:

Proof: Suppose that x is a real number and $x \neq 0$. Suppose now that $x = 8$ (for contrapositive). Note that $(\sqrt[3]{8} + 5)/(8^2 + 6) = \frac{7}{70}$, or $\frac{1}{10}$. But $\frac{1}{10} \neq \frac{1}{8}$. That is $(\sqrt[3]{x} + 5)/(x^2 + 6) \neq \frac{1}{x}$. Thus, if $(\sqrt[3]{x} + 5)/(x^2 + 6) = \frac{1}{x}$, then $x \neq 8$. $\square$

Exercise 12:

Proof: Suppose that a, b, c, and d are real numbers, $0 < a < b$, and $d > 0$. Now suppose that $c \leq d$ (for contrapositive). Multiply both sides by a (which is positive): $ac \leq ad$. Multiply both sides of $a < b$ by d (which is positive): $ad < bd$. Thus, $ac \leq ad < bd$. That is, $ac < bd$. Thus, if $ac \geq bd$, then $c > d$. $\square$

Exercise 13:

Proof: Suppose that x and y are real numbers, and $3x+2y \leq 5$. Suppose that $x>1$. Subtract $2y$ from both sides: $3x \leq 5-2y$. Note that since $x>1$, $3x>3$. Thus $3<3x \leq 5-2y$. Subtract 5 from each: $-2<3x-5 \leq -2y$. Divide each side by -2: $1 > -\frac{3x-5}{2} \geq y$. Thus $y<1$.

□

Exercise 14: Proof: Suppose that x and y are real numbers. Suppose that $x^2+y=-3$ and $2x-y=2$. Add the equations: $x^2+2x+1=0$. That is, $(x+1)^2=0$. Thus, $x=-1$.

□

Exercise 15:

Proof: Suppose $x>3$ and $y<2$. Since $0<3<x$, by Theorem 3.1.2, $x^2>9$. Now, multiply both sides of $y<2$ by -2: $-2y>-4$. Now, add the inequalities: $x^2-2y>5$.

□

Exercise 16:

(a) It proved if $x=7$ then $(2x-5)/(x-4)$, NOT if $(2x-5)/(x-4)$ then $x=7$.

(b) Proof: Suppose x is a real number and $x \neq 4$. Suppose $(2x-5)/(x-4)=3$. Thus, $2x-5=3x-12$. Add 12 to both sides: $2x+7=3x$. Subtract $2x$ from both sides: $x=7$.

□

Exercise 17:

(a) x could be -3, in which case $x^2=9$, and you divide by zero.

(b) Let $x=-3$. Thus, $x^2y=(-3)^2y=9y$. Since $x^2y=9y$, we get $9y=9y$, and y could be any real number. Thus, the theorem is incorrect.