Chapter 1 Supplementary Exercises

Exercise 1:

$$\begin{bmatrix} 3/5 & -4/5 & x \\ 4/5 & 3/5 & y \end{bmatrix} \xrightarrow{\frac{5}{3}\times R_1} \begin{bmatrix} 1 & -4/3 & \frac{5}{3}x \\ 4/5 & 3/5 & y \end{bmatrix} \xrightarrow[+R_2]{-\frac{4}{5}\times R_1} \begin{bmatrix} 1 & -4/3 & \frac{5}{3}x \\ 0 & \frac{5}{3} & y-\frac{4}{3}x \end{bmatrix} \xrightarrow{\frac{3}{5}\times R_2} \begin{bmatrix} 1 & -4/3 & \frac{5}{3}x \\ 0 & 1 & \frac{3}{5}y-\frac{4}{5}x \end{bmatrix} \xrightarrow[+R_1]{\frac{4}{3}\times R_2}$$

$$\begin{bmatrix} 1 & 0 & \frac{4}{5}y+\frac{3}{5}x \\ 0 & 1 & \frac{3}{5}y-\frac{4}{5}x \end{bmatrix} \qquad \begin{aligned} x' &= \frac{4}{5}y + \frac{3}{5}x \\ y' &= \frac{3}{5}y - \frac{4}{5}x \end{aligned}$$

Exercise 2:

$$\begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \end{bmatrix} \xrightarrow{\frac{1}{\cos\theta}\times R_1} \begin{bmatrix} 1 & -\tan\theta & \frac{x}{\cos\theta} \\ \sin\theta & \cos\theta & y \end{bmatrix} \xrightarrow[+R_2]{-\sin\theta\times R_1} \begin{bmatrix} 1 & -\tan\theta & \frac{x}{\cos\theta} \\ 0 & \frac{1}{\cos\theta} & -x\tan\theta + y \end{bmatrix} \xrightarrow{\cos\theta\times R_2}$$

$$\begin{bmatrix} 1 & -\tan\theta & \frac{x}{\cos\theta} \\ 0 & 1 & -x\sin\theta + y\cos\theta \end{bmatrix} \xrightarrow[+R_1]{\tan\theta\times R_2} \begin{bmatrix} 1 & 0 & x\cos\theta + y\sin\theta \\ 0 & 1 & -x\sin\theta + y\cos\theta \end{bmatrix} \qquad \begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned}$$

Exercise 3:

This system is denoted:
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= 0 \end{aligned}$$

Thus, we get:
$$\begin{aligned} a_{11} - a_{12} + a_{13} + 2a_{14} &= 0 \\ 2a_{21} \qquad + 3a_{23} - a_{24} &= 0 \end{aligned}$$

So:
$$\begin{bmatrix} 1 & -1 & 1 & 2 & 0 \\ 2 & 0 & 3 & -1 & 0 \end{bmatrix} \xrightarrow[+R_2]{-2\times R_1} \begin{bmatrix} 1 & -1 & 1 & 2 & 0 \\ 0 & 2 & 1 & -5 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}\times R_2} \begin{bmatrix} 1 & -1 & 1 & 2 & 0 \\ 0 & 1 & 1/2 & -5/2 & 0 \end{bmatrix} \xrightarrow[+R_1]{1\times R_2}$$

$$\begin{bmatrix} 1 & 0 & 3/2 & -1/2 & 0 \\ 0 & 1 & 1/2 & -5/2 & 0 \end{bmatrix}$$

This implies: $a_{11} = -\frac{3}{2}a_{13} + \frac{1}{2}a_{14}$ and $a_{12} = -\frac{1}{2}a_{13} + \frac{5}{2}a_{14}$

And also: $a_{21} = -\frac{3}{2}a_{23} + \frac{1}{2}a_{24}$ and $a_{22} = -\frac{1}{2}a_{23} + \frac{5}{2}a_{24}$

So, we let $a_{13} = a_{14} = -1$, $a_{23} = 0$, $a_{24} = 2$. Thus, $a_{11} = 1$, $a_{12} = -2$, $a_{21} = 1$, $a_{22} = 5$. The system becomes:

$$\begin{aligned} x_1 - 2x_2 - x_3 - x_4 &= 0 \\ x_1 + 5x_2 \qquad + 2x_4 &= 0. \end{aligned}$$

Exercise 4:

$$x + y + z = 13$$
$$x + 5y + 10z = 83$$

$$\begin{bmatrix} 1 & 1 & 1 & 13 \\ 1 & 5 & 10 & 83 \end{bmatrix} \xrightarrow[+R_2]{-1\times R_1} \begin{bmatrix} 1 & 1 & 1 & 13 \\ 0 & 4 & 9 & 70 \end{bmatrix} \xrightarrow{\frac{1}{4}\times R_2} \begin{bmatrix} 1 & 1 & 1 & 13 \\ 0 & 1 & 9/4 & 70/4 \end{bmatrix} \xrightarrow[+R_1]{-1\times R_2} \begin{bmatrix} 1 & 0 & -5/4 & -9/2 \\ 0 & 1 & 9/4 & 70/4 \end{bmatrix}$$

So: $x = \frac{5}{4}z - \frac{9}{2} = 1.25z - 4.5$

$y = \frac{70}{4} - \frac{9}{4}z = 17.5 - 2.25z$

Note that x, y, and z must be integers.

Let $z = 6$. Thus, $x = 3$ and $y = 4$.

Thus, we have 3 pennies, 4 nickels, and 6 dimes.

Exercise 5:

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 1 & 5 & 10 & 44 \end{bmatrix} \xrightarrow[+R_2]{-1\times R_1} \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 4 & 9 & 35 \end{bmatrix} \xrightarrow{\frac{1}{4}\times R_2} \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 9/4 & 35/4 \end{bmatrix} \xrightarrow[+R_1]{-1\times R_2} \begin{bmatrix} 1 & 0 & -5/4 & 1/4 \\ 0 & 1 & 9/4 & 35/4 \end{bmatrix}$$

So: $x = \frac{5}{4}z + \frac{1}{4}$ and $y = \frac{35}{4} - \frac{9}{4}z$

That is: $x = 1.25z + 0.25$ and $y = 8.75 - 2.25z$

Let $z = 3$. Thus, $x = 4$ and $y = 2$.

Exercise 6:

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & (a^2-4) & a-2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & (-2a-3)(a+2) \end{bmatrix}.$$

Infinite solutions: $a = -2, -\frac{3}{2}$

One solution: Never

No solutions: $a \neq -2, -\frac{3}{2}$

Exercise 7:

$$\begin{bmatrix} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & \frac{2-b}{a} \\ 0 & 1 & 0 & \frac{b-2}{a} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus: $x_1 = \frac{2-b}{a}$

$x_2 = \frac{b-2}{a}$

$x_3 = 1$

ⓐ Unique solution: $a \neq 0$, $b \neq 2$

ⓑ One-parameter solution: $a \neq 0$, $b = 2$

ⓒ Two-parameter Solution: $a = 0$, $b = 2$

ⓓ No solution: $a = 0$, $b \neq 2$.

Exercise 8:

$$\begin{bmatrix} y & -2 & 3y & 8 \\ 2y & -3 & 2y & 7 \\ -y & 1 & 2y & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & \frac{5}{y} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & \frac{3}{y} \end{bmatrix}$$

Thus: $x = \frac{5}{y} = \frac{5}{9}$

$\sqrt{y} = 3 \Rightarrow y = 9$

$z = \frac{3}{y} = \frac{3}{9} = \frac{1}{3}$

Exercise 9:

K must be $2 \times 2 \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$

Note that: $AK = \begin{bmatrix} a+4c & b+4d \\ -2+3c & -2b+3d \\ a-2c & b-2d \end{bmatrix}$

and so: $AKB = \begin{bmatrix} 2a+8c & b+4d & -b-4d \\ -4a+6 & -2b+3d & 2b-3d \\ 2a-4c & b-2d & -b+2d \end{bmatrix} = \begin{bmatrix} 8 & 6 & -6 \\ 6 & -1 & 1 \\ -4 & 0 & 0 \end{bmatrix}$

Thus, we have:

$2a + 8c = 8$ $\qquad$ $b + 4d = 6$

$-4a + 6c = 6$ $\qquad$ $b - 2d = 0$

Solve: $a = 0, c = 1$ $\qquad$ $b = 2, d = 1$

Thus, $K = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$

Exercise 10:

$$\begin{aligned} a - b \quad &= 3 \\ b + 2c &= 1 \\ a \quad - 2c &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus: $a = 2$, $b = -1$, $c = 1$.

Exercise 11:

ⓐ Let $A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}$. Thus: $A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$

So: $X = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ 6 & 0 & 1 \end{bmatrix}$

ⓑ Note that X is 2×2: Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

So: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a+3b & -a & 2a+b \\ c+3d & -c & 2c+d \end{bmatrix} = \begin{bmatrix} -5 & -1 & 0 \\ 6 & -3 & 7 \end{bmatrix}$

Thus: $a = 1$, $b = -2$, $c = 3$, $d = 1$.

So: $X = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$

ⓒ Note that X is 2×2: Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

So: $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3a+c & 3b+d \\ -a+2c & -b+2d \end{bmatrix} - \begin{bmatrix} a+2b & 4a \\ c+2d & 4c \end{bmatrix} =$

$= \begin{bmatrix} 2a-2b+c & -4a+3b+d \\ -a+c-2d & -b-4c+2d \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 5 & 4 \end{bmatrix}$

$$\begin{bmatrix} 2 & -2 & 1 & 0 & 2 \\ -4 & 3 & 0 & 1 & -2 \\ -1 & 0 & 1 & -2 & 5 \\ 0 & -1 & -4 & 2 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 & -113/37 \\ 0 & 1 & 0 & 0 & -160/37 \\ 0 & 0 & 1 & 0 & -20/37 \\ 0 & 0 & 0 & 1 & -46/37 \end{bmatrix}$$

Thus: $X = \begin{bmatrix} \frac{-113}{37} & \frac{-160}{37} \\ \frac{-20}{37} & \frac{-46}{37} \end{bmatrix}$

Exercise 12:

ⓐ $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -4 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 1 \\ -3 & 5 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$Y = A \quad X \qquad\qquad Z = B \quad Y$

$Z = BY = BAX = CX$

$$C = BA = \begin{bmatrix} 4 & -1 & 1 \\ -3 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 3 & 1 & -4 \\ -2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -7 & 11 \\ 14 & 10 & -26 \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -1 & -7 & 11 \\ 14 & 10 & -26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$Z = C \quad X$

ⓑ $z_1 = -x_1 - 7x_2 + 11x_3$

$z_2 = 14x_1 + 10x_2 - 26x_3$

ⓒ $z_1 = 4(x_1 - x_2 + x_3) - (3x_1 + x_2 - 4x_3) + (-2x_1 - 2x_2 + 3x_3)$

$= 4x_1 - 4x_2 + 4x_3 - 3x_1 - x_2 + 4x_3 - 2x_1 - 2x_2 + 3x_3$

$= -x_1 - 7x_2 + 11x_3$

$z_2 = -3(x_1 - x_2 + x_3) + 5(3x_1 + x_2 - 4x_3) - (-2x_1 - 2x_2 + 3x_3)$

$= -3x_1 + 3x_2 - 3x_3 + 15x_1 + 5x_2 - 20x_3 + 2x_1 + 2x_2 - 3x_3$

$= 14x_1 + 10x_2 - 26x_3$

Exercise 13:

If A is $m \times n$ and B is $n \times p$: $m \times n \times p$ multiplication operations and $m \times (n-1) \times p$ addition operations are needed to calculate the matrix product AB.

Exercise 14:

ⓐ Suppose A is a square matrix and $A^4 = 0$.

($\Rightarrow$) Consider $(I - A)(I + A + A^2 + A^3)$.

$$= I + A + A^2 + A^3 - A - A^2 - A^3 - A^4$$

$$= I - A^4 = I - 0 = I$$

($\Leftarrow$) Now consider $(I + A + A^2 + A^3)(I - A)$

$$= I - A + A - A^2 + A^2 - A^3 + A^3 - A^4$$

$$= I - A^4 = I - 0 = I$$

Thus, $I + A + A^2 + A^3 = (I - A)^{-1}$

ⓑ Suppose A is a square matrix and $A^{n+1} = 0$.

($\Rightarrow$) Consider $(I - A)(I + A + A^2 + \cdots + A^n)$

$$= I + A + A^2 + \cdots + A^n - A - A^2 - A^3 - \cdots - A^{n+1}$$

$$= I - A^{n+1} = I - 0 = I$$

($\Leftarrow$) Now consider $(I + A + A^2 + \cdots + A^n)(I - A) = I - A + A - A^2 + A^2 - A^3 + \cdots + A^n - A^{n+1}$

$$= I - A^{n+1} = I - 0 = I$$

Thus, $I + A + A^2 + \cdots + A^n = (I - A)^{-1}$

Exercise 15:

For $p(x)$ to pass through $(1,2)$, $(-1,6)$, and $(2,3)$, the following equations must be consistent:

$$2 = a + b + c$$

$$6 = a - b + c$$

$$3 = 4a + 2b + c$$

Augmented Matrix: $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \\ 4 & 2 & 1 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Thus: $a = 1$, $b = -2$, $c = 3$.

Exercise 16:

For $p(x)$ to pass through $(-1, 0)$ and have a horizontal tangent at $(2, -9)$, the following equations must be consistent:

$$0 = a - b + c$$
$$-9 = 4a + 2b + c$$
$$0 = 4a + b$$

Augmented Matrix: $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 4 & 2 & 1 & -9 \\ 4 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{bmatrix}$

Thus: $a = 1$, $b = -4$, $c = -5$

Exercise 17:

Proof: Let J_n be the $n \times n$ matrix each of whose entries is 1. Suppose that $n > 1$. Consider $(I - J_n)(I - \frac{1}{n-1}J_n)$.

$$= I^2 - \frac{1}{n-1} - J_n + \frac{1}{n-1}J^2_n$$
$$= I - \frac{n}{n-1}J_n + \frac{n}{n-1}J_n$$
$$= I.$$

Thus, by Theorem 1.6.3, $(I - \frac{1}{n-1}J_n) = (I - J_n)^{-1}$.

$\square$

Exercise 18:

Proof: Consider square matrix A such that $A^3 + 4A^2 - 2A + 7I = 0$

Note that $0 = 0^T$

$$= (A^3 + 4A^2 - 2A + 7I)^T$$
$$= (A^3)^T + (4A^2)^T - (2A)^T + (7I)^T$$
$$= (A^T)^3 + 4(A^T)^2 - 2(A^T) + 7I$$

Thus, A^T satisfies $(A^T)^3 + 4(A^T)^2 - 2(A^T) + 7I = 0$

$\square$

Exercise 19:

Proof: Suppose B is invertible.

($\Rightarrow$) Now suppose $AB^{-1} = B^{-1}A$.

Thus, the following equivalences hold:

$$AB^{-1}B = B^{-1}AB$$
$$AI = B^{-1}AB$$
$$A = B^{-1}AB$$
$$BA = BB^{-1}AB$$
$$BA = IAB$$
$$BA = AB$$

($\Leftarrow$) Now, suppose $AB = BA$.

Thus, the following equivalences hold:

$$ABB^{-1} = BAB^{-1}$$
$$AI = BAB^{-1}$$
$$A = BAB^{-1}$$
$$B^{-1}A = B^{-1}BAB^{-1}$$
$$B^{-1}A = IAB^{-1}$$
$$B^{-1}A = AB^{-1}$$

Therefore $AB^{-1} = B^{-1}A$ iff $AB = BA$. $\square$

Exercise 20:

Proof: Suppose A is invertible. Two cases: $A+B$ is invertible or $A+B$ is singular.

- Case 1: Suppose $A+B$ is invertible. Note that $(A+B)A^{-1} = AA^{-1} + BA^{-1} = I + BA^{-1}$. Thus, since $I + BA^{-1}$ can be expressed as the product of invertible matrices (i.e., $A+B$ and A^{-1}), $I + BA^{-1}$ is invertible. (Theorem 1.4.6).
- Case 2: Suppose $A+B$ is singular. Suppose, for contradiction, that $I+BA^{-1}$ is invertible. Note that $(I+BA^{-1})A = A + BA^{-1}A = A+B$. Thus, since $A+B$ can be expressed as the product of invertible matrices (i.e., $I+BA^{-1}$ and A), $A+B$ is invertible. But $A+B$ is singular. Thus, $I+BA^{-1}$ is singular.

Therefore, $A+B$ and $I+BA^{-1}$ are both invertible or both not invertible. $\square$

Exercise 21:

Proof: let A and B both be $n \times n$ matrices:

(a) $tr(A+B) = (A_{11}+B_{11}) + (A_{22}+B_{22}) + \cdots + (A_{nn}+B_{nn})$
$= (A_{11}+A_{22}+\cdots+A_{nn}) + (B_{11}+B_{22}+\cdots+B_{nn})$.
$= tr(A) + tr(B)$.

(b) $tr(kA) = kA_{11} + kA_{22} + \cdots + kA_{nn}$
$= k(A_{11}+A_{22}+\cdots+A_{nn})$.
$= k\,tr(A)$

(c) $tr(A^T) = A^T_{11} + A^T_{22} + \cdots + A^T_{nn}$
$= A_{11} + A_{22} + \cdots + A_{nn}$
$= tr(A)$.

(d) $tr(AB) = (A_{11}B_{11} + A_{12}B_{21} + \cdots + A_{1n}B_{n1})$
$+ (A_{21}A_{12} + A_{22}B_{22} + \cdots + A_{2n}B_{n2}) + \cdots$
$+ (A_{n1}B_{1n} + A_{n2}B_{2n} + \cdots + A_{nn}B_{nn})$
$= (B_{11}A_{11} + B_{12}A_{21} + \cdots + B_{1n}A_{n1})$
$+ (B_{21}A_{12} + B_{22}A_{22} + \cdots + B_{2n}A_{n2}) + \cdots$
$+ (B_{n1}A_{1n} + B_{n2}A_{2n} + \cdots + B_{nn}A_{nn}) = tr(BA)$

□

Exercise 22:

Proof: Consider $tr(AB-BA)$. Based on the identities proven in Exercise 21:
$tr(AB-BA) = tr(AB) + tr(-BA) = tr(AB) - tr(BA) = tr(AB) - tr(AB) = 0$.
Note that $tr(I)$ is always nonzero. Therefore there are no square matrices such that $AB - BA = I$.

□

Exercise 23:

Proof: Suppose A is an $m \times n$ matrix, and B is an $n \times 1$ matrix each of whose entries is $1/n$. Thus: $AB = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix} = \begin{bmatrix} A_{11}/n + A_{12}/n + \cdots + A_{1n}/n \\ A_{21}/n + A_{22}/n + \cdots + A_{2n}/n \\ \vdots \\ A_{m1}/n + A_{m2}/n + \cdots + A_{mn}/n \end{bmatrix} =$

$$\begin{bmatrix} (A_{11}+A_{12}+\cdots+A_{1n})/n \\ (A_{21}+A_{22}+\cdots+A_{2n})/n \\ \vdots \\ (A_{m1}+A_{m2}+\cdots+A_{mn})/n \end{bmatrix}$$

Therefore, $AB = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}$ where F_i is the average of the entries in the i^{th} row of A.

$\square$

Exercise 24:

(a) Proof: Let the entries of A be arbitrary differentiable functions of x, and let A be size $m \times n$. Note that:

$$\frac{d}{dx}(kA) = \begin{bmatrix} (ka_{11}(x))' & (ka_{12}(x))' & \cdots & (ka_{1n}(x))' \\ (ka_{21}(x))' & (ka_{22}(x))' & \cdots & (ka_{2n}(x))' \\ \vdots & \vdots & & \vdots \\ (ka_{m1}(x))' & (ka_{m2}(x))' & \cdots & (ka_{mn}(x))' \end{bmatrix} = \begin{bmatrix} k(a_{11}(x))' & k(a_{12}(x))' & \cdots & k(a_{1n}(x))' \\ k(a_{21}(x))' & k(a_{22}(x))' & \cdots & k(a_{2n}(x))' \\ \vdots & \vdots & & \vdots \\ k(a_{m1}(x))' & k(a_{m2}(x))' & \cdots & k(a_{mn}(x))' \end{bmatrix}$$

$$= k \begin{bmatrix} a_{11}'(x) & a_{12}'(x) & \cdots & a_{1n}'(x) \\ a_{21}'(x) & a_{22}'(x) & \cdots & a_{2n}'(x) \\ \vdots & \vdots & & \vdots \\ a_{m1}'(x) & a_{m2}'(x) & \cdots & a_{mn}'(x) \end{bmatrix} = k\frac{dA}{dx}$$

$\square$

(b) Proof: Let the entries of A and B be arbitrary differentiable functions of x, and let A and B both be size $m \times n$. Note that:

$$\frac{d}{dx}(A+B) = \begin{bmatrix} (a_{11}(x)+b_{11}(x))' & (a_{12}(x)+b_{12}(x))' & \cdots & (a_{1n}(x)+b_{1n}(x))' \\ (a_{21}(x)+b_{21}(x))' & (a_{22}(x)+b_{22}(x))' & \cdots & (a_{2n}(x)+b_{2n}(x))' \\ \vdots & \vdots & & \vdots \\ (a_{m1}(x)+b_{m1}(x))' & (a_{m2}(x)+b_{m2}(x))' & \cdots & (a_{mn}(x)+b_{mn}(x))' \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}'(x)+b_{11}'(x) & a_{12}'(x)+b_{12}'(x) & \cdots & a_{1n}'(x)+b_{1n}'(x) \\ a_{21}'(x)+b_{21}'(x) & a_{22}'(x)+b_{22}'(x) & \cdots & a_{2n}'(x)+b_{2n}'(x) \\ \vdots & \vdots & & \vdots \\ a_{m1}'(x)+b_{m1}'(x) & a_{m2}'(x)+b_{m2}'(x) & \cdots & a_{mn}'(x)+b_{mn}'(x) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}'(x) & a_{12}'(x) & \cdots & a_{1n}'(x) \\ a_{21}'(x) & a_{22}'(x) & \cdots & a_{2n}'(x) \\ \vdots & \vdots & & \vdots \\ a_{m1}'(x) & a_{m2}'(x) & \cdots & a_{mn}'(x) \end{bmatrix} + \begin{bmatrix} b_{11}'(x) & b_{12}'(x) & \cdots & b_{1n}'(x) \\ b_{21}'(x) & b_{22}'(x) & \cdots & b_{2n}'(x) \\ \vdots & \vdots & & \vdots \\ b_{m1}'(x) & b_{m2}'(x) & \cdots & b_{mn}'(x) \end{bmatrix} = \frac{dA}{dx} + \frac{dB}{dx}$$

$\square$

ⓒ Proof: Let the entries of A and B be arbitrary differentiable functions of x, let A be $m \times n$ and B be $n \times p$. Note that:

$$\frac{d}{dx}AB = \begin{bmatrix} (a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1n}b_{n1})' & (a_{11}b_{12}+a_{12}b_{22}+\cdots+a_{1n}b_{n2})' & \cdots & (a_{11}b_{1p}+a_{12}b_{2p}+\cdots+a_{1n}b_{np})' \\ (a_{21}b_{11}+a_{22}b_{21}+\cdots+a_{2n}b_{n1})' & (a_{21}b_{12}+a_{22}b_{22}+\cdots+a_{2n}b_{n2})' & \cdots & (a_{21}b_{1p}+a_{22}b_{2p}+\cdots+a_{2n}b_{np})' \\ \vdots & \vdots & & \vdots \\ (a_{m1}b_{11}+a_{m2}b_{21}+\cdots+a_{mn}b_{n1})' & (a_{m1}b_{12}+a_{m2}b_{22}+\cdots+a_{mn}b_{n2})' & \cdots & (a_{m1}b_{1p}+a_{m2}b_{2p}+\cdots+a_{mn}b_{np})' \end{bmatrix}$$

$$= \begin{bmatrix} (a_{11}b_{11})'+(a_{12}b_{21})'+\cdots+(a_{1n}b_{n1})' & (a_{11}b_{12})'+(a_{12}b_{22})'+\cdots+(a_{1n}b_{n2})' & \cdots & (a_{11}b_{1p})'+(a_{12}b_{2p})'+\cdots+(a_{1n}b_{np})' \\ (a_{21}b_{11})'+(a_{22}b_{21})'+\cdots+(a_{2n}b_{n1})' & (a_{21}b_{12})'+(a_{22}b_{22})'+\cdots+(a_{2n}b_{n2})' & \cdots & (a_{21}b_{1p})'+(a_{22}b_{2p})'+\cdots+(a_{2n}b_{np})' \\ \vdots & \vdots & & \vdots \\ (a_{m1}b_{11})'+(a_{m2}b_{21})'+\cdots+(a_{mn}b_{n1})' & (a_{m1}b_{12})'+(a_{m2}b_{22})'+\cdots+(a_{mn}b_{n2})' & \cdots & (a_{m1}b_{1p})'+(a_{m2}b_{2p})'+\cdots+(a_{mn}b_{np})' \end{bmatrix}$$

Also:

$$\frac{dA}{dx}B = \begin{bmatrix} a_{11}'b_{11}+a_{12}'b_{21}+\cdots+a_{1n}'b_{n1} & a_{11}'b_{12}+a_{12}'b_{22}+\cdots+a_{1n}'b_{n2} & \cdots & a_{11}'b_{1p}+a_{12}'b_{2p}+\cdots+a_{1n}'b_{np} \\ a_{21}'b_{11}+a_{22}'b_{21}+\cdots+a_{2n}'b_{n1} & a_{21}'b_{12}+a_{22}'b_{22}+\cdots+a_{2n}'b_{n2} & \cdots & a_{21}'b_{1p}+a_{22}'b_{2p}+\cdots+a_{2n}'b_{np} \\ \vdots & \vdots & & \vdots \\ a_{m1}'b_{11}+a_{m2}'b_{21}+\cdots+a_{mn}'b_{n1} & a_{m1}'b_{12}+a_{m2}'b_{22}+\cdots+a_{mn}'b_{n2} & \cdots & a_{m1}'b_{1p}+a_{m2}'b_{2p}+\cdots+a_{mn}'b_{np} \end{bmatrix}$$

and:

$$A\frac{dB}{dx} = \begin{bmatrix} a_{11}b_{11}'+a_{12}b_{21}'+\cdots+a_{1n}b_{n1}' & a_{11}b_{12}'+a_{12}b_{22}'+\cdots+a_{1n}b_{n2}' & \cdots & a_{11}b_{1p}'+a_{12}b_{2p}'+\cdots+a_{1n}b_{np}' \\ a_{21}b_{11}'+a_{22}b_{21}'+\cdots+a_{2n}b_{n1}' & a_{21}b_{12}'+a_{22}b_{22}'+\cdots+a_{2n}b_{n2}' & \cdots & a_{21}b_{1p}'+a_{22}b_{2p}'+\cdots+a_{2n}b_{np}' \\ \vdots & \vdots & & \vdots \\ a_{m1}b_{11}'+a_{m2}b_{21}'+\cdots+a_{mn}b_{n1}' & a_{m1}b_{12}'+a_{m2}b_{22}'+\cdots+a_{mn}b_{n2}' & \cdots & a_{m1}b_{1p}'+a_{m2}b_{2p}'+\cdots+a_{mn}b_{np}' \end{bmatrix}$$

Now, note that in applying the product rule $\left(\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)\right)$ to the entries in $\frac{d}{dx}AB$ (the simplified matrix), we are left with $\frac{dA}{dx}B + A\frac{dB}{dx}$. Therefore, $\frac{d}{dx}AB = \frac{dA}{dx}B + A\frac{dB}{dx}$. $\square$

Exercise 25:

Proof: We begin by noting that $AA^{-1} = I$. Thus, the following equivalences hold:

$$\frac{d}{dx}(AA^{-1}) = \frac{d}{dx}(I)$$
$$\frac{dA}{dx}A^{-1} + A\frac{dA^{-1}}{dx} = 0$$
$$A\frac{dA^{-1}}{dx} = -\frac{dA}{dx}A^{-1}$$
$$A^{-1}A\frac{dA^{-1}}{dx} = -A^{-1}\frac{dA}{dx}A^{-1}$$
$$I\frac{dA^{-1}}{dx} = -A^{-1}\frac{dA}{dx}A^{-1}$$
$$\frac{dA^{-1}}{dx} = -A^{-1}\frac{dA}{dx}A^{-1}$$

$\square$

Exercise 26:

$$\frac{x^2+x-2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1} \implies x^2+x-2 = A(x^2+1)+(Bx+C)(3x-1)$$

$$\implies x^2+x-2 = Ax^2+A+3Bx^2-Bx+3Cx-C \implies x^2+x-2 = Ax^2+3Bx^2-Bx+3Cx+A-C$$

Thus:
$$\begin{aligned} x^2 &= Ax^2+3Bx^2 \\ x &= -Bx+3Cx \\ -2 &= A-C \end{aligned} \implies \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -1 & 3 & 1 \\ 1 & 0 & -1 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -7/5 \\ 0 & 1 & 0 & 4/5 \\ 0 & 0 & 1 & 3/5 \end{bmatrix}$$

Therefore: $A = -7/5$, $B = 4/5$, $C = 3/5$

Exercise 27:

(a) Proof: Let $P^T = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} & \frac{1}{4} & \frac{5}{12} & \frac{5}{12} \end{bmatrix}$. Note that $P = \begin{bmatrix} 3/4 \\ 1/6 \\ 1/4 \\ 5/12 \\ 5/12 \end{bmatrix}$.

Consider $P^TP = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} & \frac{1}{4} & \frac{5}{12} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/6 \\ 1/4 \\ 5/12 \\ 5/12 \end{bmatrix} = \begin{bmatrix} \frac{9}{16} + \frac{1}{36} + \frac{1}{16} + \frac{25}{144} + \frac{25}{144} \end{bmatrix} = 1.$

Now, consider $H = I - 2PP^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 3/4 \\ 1/6 \\ 1/4 \\ 5/12 \\ 5/12 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{6} & \frac{1}{4} & \frac{5}{12} & \frac{5}{12} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 9/16 & 1/8 & 3/16 & 5/16 & 5/16 \\ 1/8 & 1/36 & 1/24 & 5/72 & 5/72 \\ 3/16 & 1/24 & 1/16 & 5/48 & 5/48 \\ 5/16 & 5/72 & 5/48 & 25/144 & 25/144 \\ 5/16 & 5/72 & 5/48 & 25/144 & 25/144 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 9/8 & 1/4 & 3/8 & 5/8 & 5/8 \\ 1/4 & 1/18 & 1/12 & 5/36 & 5/36 \\ 3/8 & 1/12 & 1/8 & 5/24 & 5/24 \\ 5/8 & 5/36 & 5/24 & 25/72 & 25/72 \\ 5/8 & 5/36 & 5/24 & 25/72 & 25/72 \end{bmatrix} = \begin{bmatrix} -1/8 & -1/4 & -3/8 & -5/8 & -5/8 \\ -1/4 & 17/18 & -1/12 & -5/36 & -5/36 \\ -3/8 & -1/12 & 7/8 & -5/24 & -5/24 \\ -5/8 & -5/36 & -5/24 & 47/72 & -25/72 \\ -5/8 & -5/36 & -5/24 & -25/72 & 47/72 \end{bmatrix}$$

$\square$

ⓑ <u>Proof:</u> Let H be any Householder matrix. Note that since $H = I - 2PP^T$:

$$H^T = (I - 2PP^T)^T$$
$$H^T = I^T - (2PP^T)^T \quad \text{(Theorem 1.4.9 (b))}$$
$$H^T = I - 2(PP^T)^T \quad \text{(Theorem 1.4.9 (c))}$$
$$H^T = I - 2(P^T)^T P^T \quad \text{(Theorem 1.4.9 (d))}$$
$$H^T = I - 2PP^T \quad \text{(Theorem 1.4.9 (a))}$$
$$H^T = H$$

And note that:

$$H^T H = H^2 \quad \text{(above proof)}$$
$$H^T H = (I - 2PP^T)^2$$
$$H^T H = I^2 - 2PP^T - 2PP^T + (2PP^T)^2$$
$$H^T H = I - 4PP^T + 4PP^T PP^T$$
$$H^T H = I - 4PP^T + 4PP^T \quad (\text{since } P^T P = 1)$$
$$H^T H = I$$

□

ⓒ $H = H^T$

$$\begin{bmatrix} -1/8 & -1/4 & -3/8 & -5/8 & -5/8 \\ -1/4 & 17/18 & -1/12 & -5/36 & -5/36 \\ -3/8 & -1/12 & 7/8 & -5/24 & -5/24 \\ -5/8 & -5/36 & -5/24 & 47/72 & -25/72 \\ -5/8 & -5/36 & -5/24 & -25/72 & 47/72 \end{bmatrix} = \begin{bmatrix} -1/8 & -1/4 & -3/8 & -5/8 & -5/8 \\ -1/4 & 17/18 & -1/12 & -5/36 & -5/36 \\ -3/8 & -1/12 & 7/8 & -5/24 & -5/24 \\ -5/8 & -5/36 & -5/24 & 47/72 & -25/72 \\ -5/8 & -5/36 & -5/24 & -25/72 & 47/72 \end{bmatrix}$$

$H^T H = I$

$$\begin{bmatrix} -1/8 & -1/4 & -3/8 & -5/8 & -5/8 \\ -1/4 & 17/18 & -1/12 & -5/36 & -5/36 \\ -3/8 & -1/12 & 7/8 & -5/24 & -5/24 \\ -5/8 & -5/36 & -5/24 & 47/72 & -25/72 \\ -5/8 & -5/36 & -5/24 & -25/72 & 47/72 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Exercise 28:

ⓐ Proof: Assuming the following inverses exist:

$$(C^{-1}+D^{-1})^{-1} = C(C+D)^{-1}D$$
$$(C^{-1}+D^{-1})^{-1}(C^{-1}+D^{-1}) = C(C+D)^{-1}D(C^{-1}+D^{-1})$$
$$I = C(C+D)^{-1}(DC^{-1}+DD^{-1})$$
$$I = C(C+D)^{-1}(DC^{-1}+I)$$
$$IC = C(C+D)^{-1}(DC^{-1}+I)C$$
$$C = C(C+D)^{-1}(DC^{-1}C+C)$$
$$C = C(C+D)^{-1}(DI+C)$$
$$C = C(C+D)^{-1}(D+C)$$
$$C = C(C+D)^{-1}(C+D)$$
$$C = CI$$
$$C = C$$

$\square$

ⓑ Proof: Assuming the following inverses exist:

$$(I+CD)^{-1}C = C(I+DC)^{-1}$$
$$(I+CD)(I+CD)^{-1}C = (I+CD)C(I+DC)^{-1}$$
$$IC = (IC+CDC)(I+DC)^{-1}$$
$$C(I+DC) = (C+CDC)(I+DC)^{-1}(I+DC)$$
$$C+CDC = (C+CDC)I$$
$$C+CDC = C+CDC$$

$\square$

ⓒ Proof: Assuming the following inverses exist:

$$(C+DD^T)^{-1}D = C^{-1}D(I+D^TC^{-1}D)^{-1}$$
$$(C+DD^T)(C+DD^T)^{-1}D = (C+DD^T)C^{-1}D(I+D^TC^{-1}D)^{-1}$$
$$D(I+D^TC^{-1}D) = (C+DD^T)C^{-1}D(I+D^TC^{-1}D)^{-1}(I+D^TC^{-1}D)$$
$$D+DD^TC^{-1}D = (C+DD^T)C^{-1}D$$
$$D+DD^TC^{-1}D = CC^{-1}D+DD^TC^{-1}D$$
$$D+DD^TC^{-1}D = D+DD^TC^{-1}D$$

$\square$

Exercise 29:

ⓐ Proof: Suppose that $a \neq b$.

Consider: $a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n = \dfrac{a^{n+1} - b^{n+1}}{a-b}$

$(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n)(a-b) = a^{n+1} - b^{n+1}$

$(a^{n+1} + a^n b + a^{n-1}b^2 + \cdots + a^2 b^{n-1} + ab^n) + (-a^n b - a^{n-1}b^2 - a^{n-2}b^3 - \cdots - ab^n - b^{n+1}) = a^{n+1} - b^{n+1}$

$a^{n+1} - b^{n+1} = a^{n+1} - b^{n+1}$.

Thus the equivalence holds. $\square$

ⓑ Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 1 & 0 & c \end{bmatrix}$.

Consider A^n. Note that $A^n = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ d & 0 & c \end{bmatrix}$

where $d = a^n + a^{n-1}c + a^{n-2}c^2 + \cdots + a^2c^{n-2} + ac^{n-1} + c^n$.

That is: $d = \dfrac{a^{n+1} - c^{n+1}}{a-c}$ (by Part ⓐ)

Therefore: $A^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ \frac{a^{n+1}-c^{n+1}}{a-c} & 0 & c^n \end{bmatrix}$.