Section 3.6: Existence and Uniqueness Proofs

Exercise 1:

Proof: Let x be an arbitrary real number. Let $y = \frac{x}{x^2+1}$. Then:

$$x - y = x - \frac{x}{x^2+1} = \frac{x^3+x}{x^2+1} - \frac{x}{x^2+1} = \frac{x^3}{x^2+1} = x^2\left(\frac{x}{x^2+1}\right) = x^2 y$$

To see that this solution is unique, suppose that $x^2 z = x - z$. Then, $x^2 z + z = x$, and $z(x^2+1) = x$, so $z = \frac{x}{x^2+1} = y$. □

Exercise 2:

Proof: Let $x = 4$, and be an arbitrary real number. Then:

$$xy + x - 4 = 4y + 4 - 4 = 4y$$

To see that this solution is unique, suppose that $zy + z - 4 = 4y$. Then, $zy + z = 4y + 4$, and $z(y+1) = 4(y+1)$, and so $z = 4 = x$. □

Exercise 3:

Proof: Let x be an arbitrary real number, and suppose $x \neq 0$ and $x \neq 1$. Let $y = \frac{x}{1-\frac{1}{x}}$. Then:

$$y - x = \frac{x}{1-\frac{1}{x}} - x = \frac{x}{1-\frac{1}{x}} - \frac{x-1}{1-\frac{1}{x}} = \frac{x-x+1}{1-\frac{1}{x}} = \frac{x}{x-1} = \frac{\frac{x}{1-\frac{1}{x}}}{x} = \frac{y}{x}.$$

Note that because $x \neq 0$ and $x \neq 1$, no denominator above is ever 0.

To see that this solution is unique, suppose that $\frac{z}{x} = z - x$. Then, $x = z - \frac{z}{x}$, and $x = z(1 - \frac{1}{x})$, and so $z = \frac{x}{1-\frac{1}{x}} = y$. □

Exercise 4:

Proof: Let x be an arbitrary real number, and suppose $x \neq 0$. Let $y = \frac{1}{x}$ and let z be an arbitrary real number. Then:

$$zy = z\left(\frac{1}{x}\right) = \frac{z}{x}$$

To see that this solution is unique, suppose that y' is a number with the property: $\forall z \in \mathbb{R}\left(zy' = \frac{z}{x}\right)$. That is, $\forall z \in \mathbb{R}\left(y' = \frac{1}{x}\right)$, and so $\forall z \in \mathbb{R}\left(y' = \frac{1}{x} = y\right)$ □

Exercise 5:

ⓐ Proof: Consider arbitrary x such that $x \in U!\mathcal{F}$. That is, $\exists! A\,(A \in \mathcal{F} \wedge x \in A)$, say A_0. Thus, $A_0 \in \mathcal{F}$ and $x \in A_0$. Accordingly, $\exists A\,(A \in \mathcal{F} \wedge x \in A)$, specifically A_0, and so $x \in \cup \mathcal{F}$. Therefore $U!\mathcal{F} \subseteq \cup \mathcal{F}$. □

ⓑ Proof: Let $\mathcal{F}$ be any family of sets.

($\rightarrow$) Suppose $\bigcup!\mathcal{F} = \bigcup\mathcal{F}$. Consider arbitrary sets A and B such that $A \in \mathcal{F}$ and $B \in \mathcal{F}$, and suppose $A \neq B$. Consider any x such that $x \in A \cap B$, and so $x \in A$ and $x \in B$. Note that $x \in \bigcup\mathcal{F}$ but $x \notin \bigcup!\mathcal{F}$, so $\bigcup!\mathcal{F} \neq \bigcup\mathcal{F}$. But $\bigcup!\mathcal{F} = \bigcup\mathcal{F}$. Contradiction. Therefore, $x \notin A \cap B$. Since x is arbitrary, $A \cap B = \varnothing$, and so $A \neq B \rightarrow A \cap B = \varnothing$. Therefore, $\forall A \in \mathcal{F} \forall B \in \mathcal{F} (A \neq B \rightarrow A \cap B = \varnothing)$, and thus $\mathcal{F}$ is pairwise disjoint.

($\leftarrow$) Suppose $\mathcal{F}$ is pairwise disjoint. That is: $\forall A \in \mathcal{F} \forall B \in \mathcal{F} (A \neq B \rightarrow A \cap B = \varnothing)$.

($\rightarrow$) This is proven in ⓐ.

($\leftarrow$) Consider arbitrary x such that $x \in \bigcup\mathcal{F}$.

Existence. Since $x \in \bigcup\mathcal{F}$, $\forall A (x \in A \wedge A \in \mathcal{F})$.

Uniqueness. Consider arbitrary sets B and C, and suppose $B \in \mathcal{F}$ and $x \in B$ and $C \in \mathcal{F}$ and $x \in C$. Note that since $x \in B$ and $x \in C$, $A \cap B \neq \varnothing$. Since $\forall A \in \mathcal{F} \forall B \in \mathcal{F} (A \neq B \rightarrow A \cap B = \varnothing)$, and since $B \in \mathcal{F}$ and $C \in \mathcal{F}$, $B \neq C \rightarrow B \cap C = \varnothing$. We know that $B \cap C \neq O$, and so $B = C$. Therefore, $\forall B \forall C (((x \in B \wedge B \in \mathcal{F}) \wedge (x \in C \wedge C \in \mathcal{F})) \rightarrow B = C)$.

Therefore, $\exists! A (A \in \mathcal{F} \wedge x \in A)$, and accordingly, $x \in \bigcup!\mathcal{F}$, $\bigcup\mathcal{F} \subseteq \bigcup!\mathcal{F}$.

Therefore: $\bigcup!\mathcal{F} = \bigcup\mathcal{F}$.

□

Exercise 6:

ⓐ Proof: Let U be any set.

Existence. Let $A = \varnothing$. Note that since $\varnothing \subseteq U$, $\varnothing \in \mathcal{P}(U)$. Further, consider arbitrary B such that $B \in \mathcal{P}(U)$. Note that since $A = \varnothing$, $A \cup B = \varnothing \cup B = B$.

Uniqueness. Consider set $A_0 \in \mathcal{P}(U)$. Suppose that $\forall B \in \mathcal{P}(U) (A_0 \cup B = B)$. Thus, $A_0 \cup \varnothing = \varnothing$, and so $A_0 = \varnothing = A$

□

ⓑ Proof: Let U be any set.

Existence. Let $A = U$. Note that since $U \subseteq U$, $A \in \mathcal{P}(U)$. Further, consider arbitrary B such that $B \in \mathcal{P}(U)$. Note that since $A = U$, $A \cup B = U \cup B$. Since $B \in \mathcal{P}(U)$, $B \subseteq U$, and so $A \cup B =$

$U \cup B = A$.

Uniqueness. Consider set $A_0 \in \mathcal{P}(U)$. Suppose that $\forall B \in \mathcal{P}(U)(A_0 \cup B = A_0)$.
Thus, $A_0 \cup U = A_0$. This is only possible if $A_0 = U$. □

Exercise 7:

ⓐ Proof: Let U be any set.
Existence. Let $A = U$. Note that since $U \subseteq U$, $A \in \mathcal{P}(U)$. Further, consider arbitrary B such that $B \in \mathcal{P}(U)$. Note that since $A = U$, $A \cap B = U \cap B$. Since $B \subseteq U$, $A \cap B = U \cap B = B$.
Uniqueness. Consider set $A_0 \in \mathcal{P}(U)$. Suppose that $\forall B \in \mathcal{P}(U)(A_0 \cap B = B)$.
Thus, $A_0 \cap U = U$, and so $A_0 = U = A$. □

ⓑ Proof: Let U be any set.
Existence. Let $A = \varnothing$. Note that since $\varnothing \subseteq U$, $A \in \mathcal{P}(U)$. Further, consider arbitrary B such that $B \in \mathcal{P}(U)$. Note that, since $A = \varnothing$, $A \cap B = \varnothing \cap B = \varnothing$. Thus, $A \cap B = A$.
Uniqueness. Consider set $A_0 \in \mathcal{P}(U)$. Suppose that $\forall B \in \mathcal{P}(U)(A_0 \cap B = A_0)$.
Thus, $A_0 \cap \varnothing = A_0$. That is, $A_0 = \varnothing = A$. □

Exercise 8:

ⓐ Proof: Let U be any set and consider any set $A \in \mathcal{P}(U)$.
Existence. Let $B = U \setminus A$ and consider arbitrary $C \in \mathcal{P}(U)$. Note that since $U \subseteq U$ and $A \subseteq U$, $B \subseteq U$, and so $B \in \mathcal{P}(U)$.
The following equivalences hold: $x \in C \setminus A$ iff $x \in C \wedge x \notin A$
(This line follows since $x \in C$ and $C \subseteq U$) → iff $x \in C \wedge x \in U \wedge x \notin A$.
iff $x \in C \wedge x \in U \setminus A$
iff $x \in C \wedge x \in B$
iff $x \in C \cap B$.
Uniqueness. Consider set $B_0 \in \mathcal{P}(U)$. Suppose $\forall C \in \mathcal{P}(U)(C \setminus A = C \cap B_0)$.
Then since $U \in \mathcal{P}(U)$, $U \setminus A = U \cap B_0$. Since $B_0 \subseteq U$, $U \cap B_0 = B_0$.
Thus, $B_0 = U \setminus A = B$. □

ⓑ Proof: Let U be any set and consider any set $A \in \mathcal{P}(U)$.
Existence. Let $B = U \setminus A$ and consider arbitrary $C \in \mathcal{P}(U)$. Note that

since $U \subseteq U$ and $A \subseteq U$, $B \subseteq U$, and so $B \in \mathcal{P}(U)$. The following equivalences hold: $x \in C \cap A$ iff $x \in C \wedge x \in A$.

iff $x \in C \wedge (x \notin U \vee x \in A)$.

iff $x \in C \wedge x \notin U \setminus A$

iff $x \in C \setminus (U \setminus A)$

iff $x \in C \setminus B_0$.

Uniqueness. Consider set $B_0 \in \mathcal{P}(U)$. Suppose $\forall C \in \mathcal{P}(U)(C \cap A = C \setminus B_0)$

Then since $U \in \mathcal{P}(U)$, $U \cap A = U \setminus B_0$. Since $A \subseteq U$, $A = U \setminus B_0$. Thus, if $x \in B_0$ then $x \notin A$, and so B_0 contains all elements of U that are not in A. Accordingly $B_0 = U \setminus A = B$. □

Exercise 9:

(a) Proof:

Existence. Let $X = \varnothing$ and consider any set A. Note that $A \triangle X = A \triangle \varnothing = A$. Since there is no x such that $x \in \varnothing$, $A \triangle \varnothing = A$. Therefore, $A \triangle X = A$.

Uniqueness. Consider set X_0 and suppose that $\forall A (A \triangle X_0 = A)$.

Note that $x \in A \triangle X_0$ iff $x \in A \cup X_0$ and $x \notin A \cap X_0$.

iff $x \in A \cup X_0$ and $x \notin A$ or $x \notin X_0$.

When $A = \varnothing$, $A \triangle X_0 = A$, or $\varnothing \triangle X_0 = \varnothing$. This is only possible if $X_0 = \varnothing = X$. □

(b) Proof: Consider any set A.

Existence. Let $B = A$. Note then that $A \triangle B = A \triangle A = \varnothing = X$. Thus, $A \triangle B = X$.

Uniqueness. Consider set B_0 and suppose that $A \triangle B_0 = X$. That is $A \triangle B_0 = \varnothing$, or $\forall x (x \notin A \triangle B_0)$, and so $\forall x (x \notin A \cup B_0 \vee x \in A \cap B_0)$. Plainly, for any given x, either x is in neither A nor B_0, or x is in both A and B_0. It must be the case, then, that $B_0 = A = B$. □

(c) Proof: Consider any sets A and B.

Existence. Let $C = A \triangle B$. Note then that $A \triangle C = A \triangle (A \triangle B) = (A \triangle A) \triangle B = \varnothing \triangle B = B$. Thus, $A \triangle C = B$.

Uniqueness. Consider set C_0 and suppose that $A \triangle C_0 = B$. Since A can

be any set, let $A = B \triangle C_0$. Thus, $A \triangle B = (B \triangle C_0) \triangle B =$
$(C_0 \triangle B) \triangle B = C_0 \triangle (B \triangle B) = C_0 \triangle \varnothing = C_0$. That is,
$C_0 = A \triangle B = C$.

$\square$

(d) <u>Proof</u>: Consider any set A.

Existence. Let $B = A$, and let C be any set such that $C \subseteq A$. Note then that $B \triangle C = A \triangle C = (A \setminus C) \cup (C \setminus A)$. Since $C \subseteq A$, $C \setminus A = \varnothing$. Thus, $(A \setminus C) \cup (C \setminus A) = (A \setminus C) \cup \varnothing = A \setminus C$. Therefore, $B \triangle C = A \setminus C$.

Uniqueness. Consider set $B_0 \subseteq A$ and let C be any set such that $C \subseteq A$. Suppose that $B_0 \triangle C = A \setminus C$. Since C is any set such that $C \subseteq A$, let $C = \varnothing$. Thus, $B_0 \triangle \varnothing = A \setminus \varnothing$, or $B_0 = A \setminus \varnothing$. Note that $A \setminus \varnothing = A$, and so $B_0 = A = B$.

$\square$

<u>Exercise 10</u>:

<u>Proof</u>: Suppose A is a set and suppose that A does not have only one element. Two cases:

Case 1. A has more than one element, say $x_0, \cdots, x_n$. Let $\mathcal{F} = \{\{x_0\}, \{\cdots, x_n\}\}$. Here, $\cup \mathcal{F} = A$, but $A \notin \mathcal{F}$. Accordingly, it is not the case that for all $\mathcal{F}$, if $\cup \mathcal{F} = A$ then $A \in \mathcal{F}$.

Case 2. A has no elements (i.e., $A = \varnothing$). Let $\mathcal{F} = \varnothing$. Here, $\cup \mathcal{F} = A$, but $A \notin \mathcal{F}$. Accordingly, it is not the case that for all $\mathcal{F}$, if $\cup \mathcal{F} = A$ then $A \in \mathcal{F}$.

Since these cases are exhaustive, it is not the case that for all $\mathcal{F}$, if $\cup \mathcal{F} = A$ then $A \in \mathcal{F}$. Therefore, by contrapositive, if it is true that if $\cup \mathcal{F} = A$ then $A \in \mathcal{F}$, it must be the case that A has exactly one element.

$\square$

<u>Exercise 11</u>:

<u>Proof</u>: Suppose that $\mathcal{F}$ is a family of sets and for every $\mathcal{G} \subseteq \mathcal{F}$, $\cup \mathcal{G} \in \mathcal{F}$. Note that since $\mathcal{F} \subseteq \mathcal{F}$, $\cup \mathcal{F} \in \mathcal{F}$.

Existence. Let $A = \cup \mathcal{F}$. As noted above, $A \in \mathcal{F}$. Consider arbitrary

B such that $B \in \mathcal{F}$. Note that because $B \in \mathcal{F}$, $B \subseteq \bigcup \mathcal{F}$, and so $B \subseteq A$. Since B is arbitrary, $\forall B \in \mathcal{F}(B \subseteq A)$.

Uniqueness. Consider set A_1 and suppose $A_1 \in \mathcal{F}$ and $\forall B \in \mathcal{F}(B \subseteq A_1)$. Similarly, consider set A_2 and suppose $A_2 \in \mathcal{F}$ and $\forall B \in \mathcal{F}(B \in A_2)$. Since $A_2 \in \mathcal{F}$ and $\forall B \in \mathcal{F}(B \subseteq A_1)$, it follows that $A_2 \subseteq A_1$. Using similar steps, we find that $A_1 \subseteq A_2$. Thus, $A_1 = A_2$, and uniqueness is satisfied.

□

Exercise 12:

ⓐ $\exists x \exists y ((P(x) \wedge P(y) \wedge x \neq y) \wedge \forall z(P(z) \rightarrow (x = z \vee y = z)))$

ⓑ First, prove existence by showing that the property is shared by two different values. Then, prove uniqueness by showing that for all values, if the value has the property, then it must be one of the two values named in the existence proof.

ⓒ Proof:

Existence. Let $x = 0$. Note that $x^3 = 0^3 = 0 = 0^2 = x^2$. Thus, $x^3 = x^2$. Now, let $x = 1$. Note that $x^3 = 1^3 = 1 = 1^2 = x^2$. Thus $x^3 = x^2$. Note also that $1 \neq 0$.

Uniqueness. Consider x_0 such that $x_0^3 = x_0^2$. Thus, $x_0^3 - x_0^2 = 0$, and so $x_0^2(x_0 - 1) = 0$. Accordingly, either $x_0 = 0$ or $x_0 = 1$. Note that these are the two values for x in the existence proof.

Therefore, there are exactly two solutions to the equation $x^3 = x^2$.

□

Exercise 13:

ⓐ Proof:

Existence. Let $c = \frac{9}{4}$.

Existence. Let $x = -\frac{3}{2}$. Note then that $x^2 + 3x + c = 0$ is equivalent to $(\frac{-3}{2})^2 + 3(\frac{-3}{2}) + \frac{9}{4} = 0$, or $0 = 0$. Thus, $x = \frac{-3}{2}$ is a solution to the equation.

Uniqueness. Consider x_0 such that $x_0^2 + 3x_0 + c = 0$, or

$x_0^2 + 3x_0 + \frac{9}{4} = 0$. Applying the quadratic equation yields only one solution—that is, $-\frac{3}{2}$. Thus, $x_0 = -\frac{3}{2} = x$. Thus, x is unique.

Accordingly, there exists a real number c such that the equation $x^2 + 3x + c$ has exactly one solution.

Uniqueness. Consider c_0 such that $x^2 + 3x + c_0 = 0$ has exactly one solution for x. We note that:

$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(c_0)}}{2(1)}$$

Since x has only one solution:

$$\frac{-3 + \sqrt{9 - 4c_0}}{2} = \frac{-3 - \sqrt{9 - 4c_0}}{2}$$

That is: $\sqrt{9 - 4c_0} = -\sqrt{9 - 4c_0}$, which can only be the case if $\sqrt{9 - 4c_0} = 0$. Solving for c_0, we find that $c_0 = \frac{9}{4} = c$. Accordingly, c is unique.

$\square$

6 Proof: Consider any real number x, and consider the equation $x^2 + 3x + c = 0$. The equation is equivalent to $c = -(x^2 + 3x)$. Note that x is a real number and so there is a real number c that solves the equation, specifically: $-(x^2 + 3x)$. Similarly, c must be unique because $-(x^2 + 3x)$ only yields one real number, that is, c. Thus, for every real number x there is a unique real number c such that $x^2 + 3x + c = 0$.