3.4: Proofs Involving Conjunctions and Biconditionals

Exercise 1:

Proof:

($\rightarrow$) Suppose $\forall x(P(x) \wedge Q(x))$. Let y be arbitrary. Since $\forall x(P(x) \wedge Q(x))$, $P(y) \wedge Q(y)$. Thus, $P(y)$. Because y is arbitrary, $\forall x P(x)$. By similar reasoning, $\forall x Q(x)$. Therefore, $\forall x P(x) \wedge \forall x Q(x)$.

($\leftarrow$) Suppose $\forall x P(x) \wedge \forall x Q(x)$. Let y be arbitrary. Since $\forall x P(x)$, $P(y)$, and similarly, since $\forall x Q(x)$, $Q(y)$. Because both $P(y)$ and $Q(y)$, $P(y) \wedge Q(y)$. Because y is arbitrary, $\forall x(P(y) \wedge Q(y))$.

Therefore, $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$. $\square$

Exercise 2:

Proof: Suppose that $A \subseteq B$ and $A \subseteq C$. Thus, $\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in A \rightarrow x \in C)$. Consider any y such that $y \in A$. Since $\forall x(x \in A \rightarrow x \in B)$, $y \in A \rightarrow y \in B$. Since $y \in A$, it follows that $y \in B$. By similar reasoning, $y \in C$. Because $y \in B$ and $y \in C$, $y \in B \cap C$. Thus, $y \in A \rightarrow y \in B \cap C$. Since y is arbitrary, $\forall x(x \in A \rightarrow x \in B \cap C)$. Therefore, $A \subseteq B \cap C$ $\square$

Exercise 3:

Proof: Suppose $A \subseteq B$. That is, $\forall x (x \in A \rightarrow x \in B)$. Consider an arbitrary set, C. Now, consider any y such that $y \in C \setminus B$. Thus, $y \in C \wedge y \notin B$, and it follows that $y \notin B$. Recall that $\forall x (x \in A \rightarrow x \in B)$, and so $y \in A \rightarrow y \in B$. Because $y \notin B$, it follows that $y \notin A$. Since $y \in C$ and $y \notin A$, $y \in C \setminus A$. Thus, $y \in C \setminus B \rightarrow y \in C \setminus A$. Since y is arbitrary, $\forall x (x \in C \setminus B \rightarrow x \in C \setminus A)$. Therefore $C \setminus B \subseteq C \setminus A$. $\square$

Exercise 4:

Proof: Suppose $A \subseteq B$ and $A \nsubseteq C$. That is, $\forall x (x \in A \rightarrow x \in B)$ and $\neg \forall x (x \in A \rightarrow x \in C)$. Through equivalences: $\exists x \neg (x \in A \rightarrow x \in C)$; $\exists x \neg (x \notin A \vee x \in C)$; and $\exists x (x \in A \wedge x \notin C)$. Let x_0 be any x such that $x_0 \in A \wedge x_0 \notin C$. Since $\forall x (x \in A \rightarrow x \in B)$, $x_0 \in A \rightarrow x_0 \in B$. Since $x_0 \in A$, it follows that $x_0 \in B$. Thus, $x_0 \in B$ and $x_0 \notin C$, or $x_0 \in B \wedge x_0 \notin C$. Accordingly, $\exists x (x \in B \wedge x \notin C)$. By equivalences: $\exists x \neg (x \notin B \vee x \in C)$; $\exists x \neg (x \in B \rightarrow x \in C)$; and $\neg \forall x (x \in B \rightarrow x \in C)$. Therefore, $B \nsubseteq C$. $\square$

Exercise 5:

Proof: Suppose $A \subseteq B \setminus C$ and $A \neq \emptyset$. That is, $\forall x (x \in A \rightarrow x \in B \setminus C)$; or $\forall x (x \in A \rightarrow x \in B \wedge x \notin C)$. Also, since $A \neq \emptyset$, $\exists x (x \in A)$. Let x_0 be any x such that $x_0 \in A$. Since, $\forall x (x \in A \rightarrow x \in B \wedge x \notin C)$, $x_0 \in A \rightarrow x_0 \in B \wedge x_0 \notin C$. Note that $x_0 \in A$, and so $x_0 \in B \wedge x_0 \notin C$. Thus, $\exists x (x \in B \wedge x \notin C)$. See Exercise 4 for equivalences resulting in $B \nsubseteq C$. $\square$

Exercise 6:

Proof: Let x be arbitrary. Then:

$$\begin{aligned} x \in A \setminus (B \cap C) &\text{ iff } x \in A \wedge x \notin B \cap C \\ &\text{ iff } x \in A \wedge (x \notin B \vee x \notin C) \\ &\text{ iff } (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C) \\ &\text{ iff } x \in (A \setminus B) \cup (A \setminus C) \end{aligned}$$

Thus, $\forall x (x \in A \setminus (B \cap C) \leftrightarrow x \in (A \setminus B) \cup (A \setminus C))$, so $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ $\square$

Exercise 7:

Proof: Consider any sets A and B. Let C be an arbitrary set.

($\rightarrow$) Suppose $C \in \mathcal{P}(A \cap B)$. Thus, $C \subseteq A \cap B$. That is, $\forall x(x \in C \rightarrow x \in A \cap$ Let y be any y such that $y \in C$. Since $\forall x(x \in C \rightarrow x \in A \cap B)$, $y \in C \rightarrow y \in A \cap B$. Since $y \in C$, $y \in A \cap B$. That is $y \in A$ and $y \in B$. It follows, then, that $y \in C \rightarrow y \in A$ and $y \in C \rightarrow y \in B$. Since y is arbitrary: $\forall x(x \in C \rightarrow x \in A)$ and $\forall x(x \in C \rightarrow x \in B)$. That is, $C \subseteq A$ and $C \subseteq B$. Accordingly, $C \in \mathcal{P}(A)$ and $C \in \mathcal{P}(B)$; or $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Since C is arbitrary: $\forall x(x \in \mathcal{P}(A \cap B) \rightarrow x \in \mathcal{P}(A) \cap \mathcal{P}(B))$. Therefore: $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

($\leftarrow$) Suppose $C \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Thus, $C \in \mathcal{P}(A)$ and $C \in \mathcal{P}(B)$; or $C \subseteq A$ and $C \subseteq B$; or $\forall x(x \in C \rightarrow x \in A)$ and $\forall x(x \in C \rightarrow x \in B)$. Let y be any y such that $y \in C$. Since $\forall x(x \in C \rightarrow x \in A)$, $y \in C \rightarrow y \in A$. Similarly, $y \in C \rightarrow y \in B$. Since $y \in C$, $y \in A$ and $y \in B$. Thus, $y \in A \cap B$. Accordingly, $y \in C \rightarrow y \in A \cap B$. Since y is arbitrary, $\forall x(x \in C \rightarrow x \in A \cap B)$, or $C \subseteq A \cap B$. Thus, $C \in \mathcal{P}(A \cap B)$. Since C is arbitrary: $\forall x(x \in \mathcal{P}(A) \cap \mathcal{P}(B) \rightarrow x \in \mathcal{P}(A \cap B))$. Therefore: $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Therefore, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

$\square$

Exercise 8:

Proof: Consider any sets A and B.

($\rightarrow$) Suppose $A \subseteq B$. Consider any x such that $x \in \mathcal{P}(A)$. So, $x \subseteq A$. Since $A \subseteq B$ and $x \subseteq A$, it follows that $x \subseteq B$. Thus, $x \in \mathcal{P}(B)$. Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

($\leftarrow$) Suppose $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Consider any x such that $x \in A$. Let y be some set in $\mathcal{P}(A)$ such that $x \in y$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, $y \in \mathcal{P}(B)$. Thus, $y \subseteq B$. Since $x \in y$, $x \in B$. Therefore, $A \subseteq B$.

Therefore $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

$\square$

Exercise 9:

Proof: Suppose that x and y are odd integers. Thus, there is some integer k such that $x = 2k+1$ and some integer j such that $y = 2j+1$. So, $xy = (2k+1)(2j+1) = 4kj + 2k + 2j + 1 = 2(2kj + k + j) + 1$. Let $i = 2kj + k + j$. Because k and j are both integers, i is also an integer. Thus, $xy = 2i + 1$, where i is an integer. Accordingly, xy is an odd integer. $\square$

Exercise 10:

Proof: Suppose that x and y are odd integers. Thus, there is some integer k such that $x = 2k+1$ and some integer j such that $y = 2j+1$. So, $x - y = (2k+1) - (2j+1) = 2k + 1 - 2j - 1 = 2k - 2j = 2(k-j)$. Let $i = k - j$. Because k and j are both integers, i is also an integer. Thus, $x - y = 2i$, where i is an integer. Accordingly, $x - y$ is an even integer. $\square$

Exercise 11:

Proof: Let n be an arbitrary integer.

($\rightarrow$) Suppose that n is odd. That is, there is some integer k such that $n = 2k+1$. Note that $n^3 = (2k+1)(2k+1)(2k+1) = (4k^2+4k+1)(2k+1) = 8k^3 + 4k^2 + 8k^2 + 4k + 2k + 1 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$. Let $i = 4k^3 + 6k^2 + 3k$. Note that because k is an integer, i is also an integer. Thus, $n^3 = 2i + 1$ where i is an integer. Accordingly, n^3 is odd.

($\leftarrow$) Suppose that n is even. That is, there is some integer k such that $n = 2k$. Note that $n^3 = (2k)(2k)(2k) = 8k^3 = 2(4k^3)$. Let $i = 4k^3$. Because k is an integer, i is also an integer. Thus, $n^3 = 2i$ where i is an integer. Accordingly, n^3 is even.

Therefore, for every integer n, n^3 is even iff n is even. $\square$

Exercise 12:

ⓐ The proof assumes that the integer K in both $m=2K$ and $n=2K+1$ is the same integer — this is only true in a small set of cases.

ⓑ Counterexample: Let $n=4$ and $m=2$. $n^2-m^2=16-4=12$, but $n+m=4+2=6$. Thus, the "theorem" is not correct.

Exercise 13:

Proof: Let x be an arbitrary real number.

($\rightarrow$) Suppose that $\exists y \in \mathbb{R}(x+y=xy)$. Thus, there is some $y_0 \in \mathbb{R}$ such that $x+y_0=xy_0$. Now suppose, for contradiction, that $x=1$. Thus, $1+y_0=(1)y_0$; or $1+y_0=y_0$. Thus, $1=0$ — a contradiction. Therefore $x \neq 1$.

($\leftarrow$) Suppose that $x \neq 1$. Let $y_0=\frac{x}{x-1}$ (which is fine since $x \neq 1$).

So: $x+y_0=x+\frac{x}{x-1}=\frac{x(x-1)}{x-1}+\frac{x}{x-1}=\frac{x^2-x+x}{x-1}=\frac{x^2}{x-1}$

$=x\left(\frac{x}{x-1}\right)=xy_0$. Thus, there is some y (y_0) such that $x+y=xy$.

Accordingly, $\exists y \in \mathbb{R}(x+y=xy)$.

Therefore: $\forall x \in \mathbb{R}[\exists y \in \mathbb{R}(x+y=xy) \leftrightarrow x \neq 1]$. □

Exercise 14:

Proof: Let $z=1$ and consider arbitrary x such that $x>0$.

($\rightarrow$) Suppose that $\exists y \in \mathbb{R}(y-x=\frac{y}{x})$. Thus, there is some y_0 such that $y_0-x=\frac{y_0}{x}$. Suppose (for contradiction) that $x=z=1$. Thus, $y_0-1=\frac{y_0}{1}$, or $y_0-y_0-1=0$, or $-1=0$. Therefore, $x \neq z$.

($\leftarrow$) Suppose that $x \neq z$. Let $y_0=\frac{x^2}{x-1}$. Note that:

$y_0-x=\frac{x^2}{x-1}-x=\frac{x^2-(x^2-x)}{x-1}=\frac{x}{x-1}=\frac{x^2}{x(x-1)}=\frac{x^2}{x-1}\left(\frac{1}{x}\right)=\frac{\frac{x^2}{x-1}}{x}=\frac{y_0}{x}$.

Thus, $\exists y \in \mathbb{R}(y-x=\frac{y}{x})$; namely, y_0.

Therefore: $\exists z \in \mathbb{R}\, \forall x \in \mathbb{R}^+[\exists y \in \mathbb{R}(y-x=\frac{y}{x}) \leftrightarrow x \neq z]$

□

Exercise 15:

Proof: Suppose B is a set and $\mathcal{F}$ is a family of sets. Consider arbitrary x such that $x \in \bigcup\{A \setminus B \mid A \in \mathcal{F}\}$. Thus, there is a set A such that $A \in \mathcal{F}$ and $x \in A$ and $x \notin B$. It follows that $A \subseteq B$. Accordingly, $A \notin \mathscr{P}(B)$. Thus, $A \in \mathcal{F}$ and $A \notin \mathscr{P}(B)$. Since, x is an element of A, it follows that $x \in \bigcup(\mathcal{F} \setminus \mathscr{P}(B))$. Because x is arbitrary, $\bigcup\{A \setminus B \mid A \in \mathcal{F}\} \subseteq \bigcup(\mathcal{F} \setminus \mathscr{P}(B))$. $\square$

Exercise 16:

Proof: Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets and every element of $\mathcal{F}$ is disjoint from some element of $\mathcal{G}$. Suppose further that $\bigcup\mathcal{F}$ and $\bigcap\mathcal{G}$ are not disjoint. Thus, there is some x such that $x \in \bigcup\mathcal{F}$ and $x \in \bigcap\mathcal{G}$. Accordingly, there is some set $A \in \mathcal{F}$ such that $x \in A$. Also, x is in all sets in $\mathcal{G}$. Accordingly, A is not disjoint from any set in $\mathcal{G}$, as they both have x as an element. Thus, there is an element of $\mathcal{F}$ that is not disjoint from an element of $\mathcal{G}$. Contradiction. Thus, $\bigcup\mathcal{F}$ and $\bigcap\mathcal{G}$ must be disjoint. $\square$

Exercise 17:

Proof: Let A be any set.

($\rightarrow$) Consider any x such that $x \in A$. There is some set B such that $B \subseteq A$ and $x \in B$. Since $B \subseteq A$, $B \in \mathscr{P}(A)$. Accordingly, since $x \in B$, $x \in \bigcup\mathscr{P}(A)$.

($\leftarrow$) Consider any x such that $x \in \bigcup\mathscr{P}(A)$. Thus, there is some set B such that $B \in \mathscr{P}(A)$ and $x \in B$. Because $B \in \mathscr{P}(A)$, $B \subseteq A$. Thus, since $x \in B$, $x \in A$.

Therefore, $A = \bigcup\mathscr{P}(A)$ $\square$

Exercise 18:

Suppose that $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

(a) Proof: Consider any x such that $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$. Note that x is in a set, say B, such that $B \in \mathcal{F}$ and $B \in \mathcal{G}$. It follows, then, that $x \in \bigcup\mathcal{F}$ and $x \in \bigcup\mathcal{G}$. Accordingly, $x \in (\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G})$. Therefore, $\bigcup(\mathcal{F} \cap \mathcal{G}) \subseteq (\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G})$. $\square$

(b) The author of the proof concludes that $\exists A \in \mathcal{F}(x \in A)$ and $\exists A \in \mathcal{G}(x \in A)$ from $x \in \bigcup\mathcal{F}$ and $x \in \bigcup\mathcal{G}$. The sets containing x in $\mathcal{F}$ and containing x in $\mathcal{G}$ may not be the same. Thus, it is faulty for the author to assume that they are the same.

(c) Counterexample: Let $\mathcal{F} = \{A, B\}$ and $\mathcal{G} = \{A, C\}$. Further, let $A = \{y\}$; $B = \{x\}$, and; $C = \{x\}$. Note that $\mathcal{F} \cap \mathcal{G} = \{A\}$ and so $\bigcup(\mathcal{F} \cap \mathcal{G}) = \{y\}$. But $\bigcup\mathcal{F} = \{y, x\}$; $\bigcup\mathcal{G} = \{y, x\}$, and so; $(\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G}) = \{y, x\}$. Note that $\bigcup(\mathcal{F} \cap \mathcal{G}) \neq (\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G})$.

$\square$

Exercise 19:

Proof: Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

($\rightarrow$) Suppose $(\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$. Consider arbitrary sets A and B such that $A \in \mathcal{F}$ and $B \in \mathcal{G}$. Consider further any x such that $x \in A \cap B$. That is, $x \in A$ and $x \in B$. Further, because $A \in \mathcal{F}$ and $B \in \mathcal{G}$, $x \in \bigcup\mathcal{F}$ and $x \in \bigcup\mathcal{G}$. Accordingly, $x \in (\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G})$. Note that $(\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$ and so $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$. Thus, $A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$. Since A and B are arbitrary, $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G}))$.

($\leftarrow$) Suppose $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G}))$. Consider any x such that $x \in (\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G})$. Note that $x \in (\bigcup\mathcal{F})$ and $x \in (\bigcup\mathcal{G})$. Thus, there is some $A_0 \in \mathcal{F}$ such that $x \in A_0$ and some $B_0 \in \mathcal{G}$ such that $x \in B_0$. Because $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G}))$, $A_0 \cap B_0 \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$. Note that $x \in A_0 \cap B_0$ and so $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$. Because x is arbitrary, $(\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$.

Therefore, $(\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$ iff $\forall A \in \mathcal{F} \forall B \in \mathcal{G}(A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G}))$.

$\square$

Exercise 20:

Proof: Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

($\rightarrow$) Suppose $\bigcup\mathcal{F}$ and $\bigcup\mathcal{G}$ are disjoint. Consider arbitrary sets $A \in \mathcal{F}$

and $B \in \mathcal{G}$. Since $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint, none of the sets in $\mathcal{F}$ have any elements in common with any set in $\mathcal{G}$. Thus, A has no element in common with B, and so A and B are disjoint.

($\leftarrow$) Suppose that for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are disjoint. Now, suppose (for contradiction) that $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are not disjoint. Thus, there is some $x \in \cup\mathcal{F}$ such that $x \in \cup\mathcal{G}$. That is, there is a set in $\mathcal{F}$, say A, that shares an element, x, with some set in $\mathcal{G}$, say B. Thus, there is an $A \in \mathcal{F}$ and a $B \in \mathcal{G}$ such that A and B are not disjoint. But, for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are disjoint: Contradiction. Thus, $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint.

Therefore, $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint iff for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are disjoint.

$\square$

Exercise 21: Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

(a) Proof: Consider any x such that $x \in (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$. Thus, $x \in \cup\mathcal{F}$ and $x \notin \cup\mathcal{G}$. That is, there is some set $A \in \mathcal{F}$ such that $x \in A$, and for all $B \in \mathcal{G}$ it is not the case that $x \in B$. Accordingly, $x \in A$ and $x \notin B$ for at least one $A \in \mathcal{F}$ and for all $B \in \mathcal{G}$. Thus, x is in one of the sets of $\mathcal{F} \setminus \mathcal{G}$. Therefore, $x \in \cup(\mathcal{F} \setminus \mathcal{G})$. Because x was arbitrary: $(\cup\mathcal{F}) \setminus (\cup\mathcal{G}) \subseteq \cup(\mathcal{F} \setminus \mathcal{G})$. $\square$

(b) The conclusion that $x \notin \cup\mathcal{G}$ from premises $x \in A$ and $A \notin \mathcal{G}$ is not valid. Just because $A \notin \mathcal{G}$ does not mean there could not be another set $B \in \mathcal{G}$ such that $x \in B$. If this were the case, $x \in \cup\mathcal{G}$ even though $A \notin \mathcal{G}$.

(c) Proof:

($\rightarrow$) Suppose $\cup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$. Consider arbitrary A such that $A \in (\mathcal{F} \setminus \mathcal{G})$ and arbitrary B such that $B \in \mathcal{G}$. Note that $A \in \mathcal{F}$ and $A \notin \mathcal{G}$. Now suppose, for contradiction, that $A \cap B \neq \emptyset$. That is, $\exists x \in A \cap B$, say x_0. Note that $x_0 \in A$ and $x_0 \in B$. And since $A \in \mathcal{F}$, $A \notin \mathcal{G}$, and $x_0 \in A$, $\exists A \in \mathcal{F} \setminus \mathcal{G}\,(x \in A)$. That is, $x_0 \in \cup(\mathcal{F} \setminus \mathcal{G})$. Since $\cup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$, $x_0 \in (\cup\mathcal{F}) \setminus (\cup\mathcal{G})$; or $x_0 \in \cup\mathcal{F}$ and $x_0 \notin \cup\mathcal{G}$. But since $x_0 \in B$, and

$B \in \mathcal{G}$, $x_0 \in \bigcup\mathcal{G}$. Contradiction. Thus, $A \cap B = \emptyset$. Since A and B were arbitrary, $\forall A \in (\mathcal{F} \setminus \mathcal{G}) \forall B \in \mathcal{G} (A \cap B = \emptyset)$.

($\leftarrow$) Suppose $\forall A \in (\mathcal{F} \setminus \mathcal{G}) \forall B \in \mathcal{G} (A \cap B \neq \emptyset)$. Consider arbitrary x such that $x \in \bigcup(\mathcal{F} \setminus \mathcal{G})$. So, $\exists S \in \mathcal{F} \setminus \mathcal{G} (x \in S)$, say S_0. That is: $S_0 \in \mathcal{F}$, $S_0 \notin \mathcal{G}$, and $x \in S_0$. Consider arbitrary set T such that $T \in \mathcal{G}$. Since $S_0 \in \mathcal{F} \setminus \mathcal{G}$ and $T \in \mathcal{G}$, it follows that $S_0 \cap T \neq \emptyset$. Since $x \in S_0$, $x \notin T$. Since T is an arbitrary set in $\mathcal{G}$, x is in no set in $\mathcal{G}$, and so $x \notin \bigcup\mathcal{G}$. But since $x \in S_0$ and $S_0 \in \mathcal{F}$, $x \in \bigcup\mathcal{F}$. Thus, $x \in (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$. Since x is arbitrary, $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$.

Therefore: $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$ iff $\forall A \in (\mathcal{F} \setminus \mathcal{G}) \forall B \in \mathcal{G} (A \cap B = \emptyset)$.

□

(d) Counterexample: Consider the following sets: $\mathcal{F} = \{A\}$; $\mathcal{G} = \{B\}$; $A = \{x\}$; and $B = \{x\}$. Note the following: $A \in \mathcal{F}$, $A \notin \mathcal{G}$, and $x \in A$; so $A \in \mathcal{F} \setminus \mathcal{G}$ and $x \in A$; so $\exists S \in \mathcal{F} \setminus \mathcal{G} (x \in S)$; and so $x \in \bigcup(\mathcal{F} \setminus \mathcal{G})$. Also note that: $A \in \mathcal{F}$ and $x \in A$; so $\exists S \in \mathcal{F} (x \in S)$; so $x \in \bigcup\mathcal{F}$; and $B \in \mathcal{G}$ and $x \in B$; so $\exists S \in \mathcal{G} (x \in S)$; so $x \in \bigcup\mathcal{G}$; and since $x \in (\bigcup\mathcal{F}) \cap (\bigcup\mathcal{G})$, $x \notin (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$. Therefore: $\bigcup(\mathcal{F} \setminus \mathcal{G}) \neq (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$.

□

Exercise 22:

Proof: Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Further suppose there is no $A \in \mathcal{F}$ such that for all $B \in \mathcal{G}$, $A \not\subseteq B$. That is, $\neg \exists A \in \mathcal{F} (\forall B \in \mathcal{G} (A \not\subseteq B))$. By equivalence: $\forall A \in \mathcal{F} \neg (\forall B \in \mathcal{G} (A \not\subseteq B))$; $\forall A \in \mathcal{F} (\exists B \in \mathcal{G} (A \subseteq B))$. Consider arbitrary x such that $x \in \bigcup\mathcal{F}$. Thus, $\exists S \in \mathcal{F} (x \in S)$, say A_0. Thus, $A_0 \in \mathcal{F}$ and $x \in A_0$. Since $A_0 \in \mathcal{F}$, $\exists B \in \mathcal{G} (A_0 \subseteq B)$, say B_0. Thus, $B_0 \in \mathcal{G}$ and $A_0 \subseteq B_0$. Since $x \in A_0$ and $A_0 \subseteq B_0$, $x \in B_0$. Thus, $\exists T \in \mathcal{G} (x \in T)$, and so $x \in \bigcup\mathcal{G}$. Since x was arbitrary, $\bigcup\mathcal{F} \subseteq \bigcup\mathcal{G}$. By contraposition: if $\bigcup\mathcal{F} \not\subseteq \bigcup\mathcal{G}$, then there is some $A \in \mathcal{F}$ such that for all $B \in \mathcal{G}$, $A \not\subseteq B$.

□

Exercise 23:

(a) Rules used: rule for proving $\forall x P(x)$; rule for proving $P \wedge Q$; rule for proving $\exists x P(x)$; rule for using a given of $\exists x P(x)$; rule for proving $P \leftrightarrow Q$.

(b) Proof: Suppose B is a set, $\{A_i \mid i \in I\}$ is an indexed family of sets, and $I \neq \emptyset$. Consider any x such that $x \in B \setminus (\bigcup_{i\in I} A_i)$. The following equivalences apply:

$$x \in B \wedge \neg \exists i \in I (x \in A_i).$$
$$x \in B \wedge \forall i \in I \neg (x \in A_i)$$
$$x \in B \wedge \forall i \in I (x \notin A_i)$$
$$\forall i \in I (x \in B \wedge x \notin A_i)$$
$$\forall i \in I (x \in B \setminus A_i)$$
$$x \in \bigcap_{i \in I} (B \setminus A_i)$$

Since x is arbitrary, $B \setminus (\bigcup_{i\in I} A_i) = \bigcap_{i\in I} (B \setminus A_i)$. □

(c) Proof: Suppose B is a set, $\{A_i \mid i \in I\}$ is an indexed family of sets, and $I \neq \emptyset$. Consider any x such that $x \in B \setminus (\bigcap_{i\in I} A_i)$. The following equivalences apply:

$$x \in B \wedge \neg \forall i \in I (x \in A_i)$$
$$x \in B \wedge \exists i \in I \neg (x \in A_i)$$
$$x \in B \wedge \exists i \in I (x \notin A_i)$$
$$\exists i \in I (x \in B \wedge x \notin A_i)$$
$$\exists i \in I (x \in B \setminus A_i)$$
$$x \in \bigcup_{i \in I} (B \setminus A_i)$$

Since x is arbitrary, $B \setminus (\bigcap_{i\in I} A_i) = \bigcup_{i\in I} (B \setminus A_i)$. □

Exercise 24:

(a) Proof: Suppose $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ are indexed families of sets and $I \neq \emptyset$. Consider any x such that $x \in \bigcup_{i\in I} (A_i \setminus B_i)$. So, $\exists i \in I (x \in A_i \wedge x \notin B_i)$, say i_0. Thus, $x \in A_{i_0}$ and $x \notin B_{i_0}$. Accordingly, $\exists i \in I (x \in A_i)$. Also, $\neg \forall i \in I (x \in B_i)$, or $\exists i \in I (x \notin B_i)$. Thus, $x \in (\bigcup_{i\in I} A_i)$ and $x \notin (\bigcap_{i\in I} B_i)$. Therefore: $x \in (\bigcup_{i\in I} A_i) \setminus (\bigcap_{i \in I} B_i)$. Because x is arbitrary, $\bigcup_{i\in I} (A_i \setminus B_i) \subseteq (\bigcup_{i\in I} A_i) \setminus (\bigcap_{i\in I} B_i)$. □

ⓑ <u>Counterexample</u>: Consider the following sets: $A_1 = \{x\}$; $A_2 = \{y\}$; $B_1 = \{x\}$; and $B_2 = \{z\}$. Note the following: $x \in A_1$ and $x \in B_1$; and $x \notin A_2$ and $x \notin B_2$; so $\neg \exists i \in I (x \in A_i \wedge x \notin B_i)$; so $\neg \exists i \in I (x \in A_i \setminus B_i)$; and so $x \notin \bigcup_{i \in I} (A_i \setminus B_i)$. Also note that: $x \in A_1$ and $x \notin B_2$; so $\exists i \in I (x \in A_i)$ and $\neg \forall i \in I (x \in B_i)$; so $x \in (\bigcup_{i \in I} A_i)$ and $x \notin (\bigcap_{i \in I} B_i)$; and so $x \in (\bigcup_{i \in I} A_i) \setminus (\bigcap_{i \in I} B_i)$. Therefore: $\bigcup_{i \in I} (A_i \setminus B_i) \neq (\bigcup_{i \in I} A_i) \setminus (\bigcap_{i \in I} B_i)$. □

<u>Exercise 25</u>:

ⓐ <u>Proof</u>: Suppose $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ are indexed families of sets. Consider any x such that $x \in \bigcup_{i \in I} (A_i \cap B_i)$. That is, $\exists i \in I (x \in A_i \cap B_i)$. Thus, there is an i, say i_0, such that $x \in A_{i_0} \cap B_{i_0}$; or $x \in A_{i_0}$ and $x \in B_{i_0}$. Since $x \in A_{i_0}$, $\exists i \in I (x \in A_i)$, and since $x \in B_{i_0}$, $\exists i \in I (x \in B_i)$. That is, $x \in (\bigcup_{i \in I} A_i) \cap (\bigcup_{i \in I} B_i)$. Since x is arbitrary, $\bigcup_{i \in A} (A_i \cap B_i) \subseteq (\bigcup_{i \in I} A_i) \cap (\bigcup_{i \in I} B_i)$. □

ⓑ <u>Counterexample</u>: Consider the following sets: $A_1 = \{x\}$; $A_2 = \{y\}$; $B_1 = \{z\}$; $B_2 = \{x\}$. Note the following: $x \in A_1$ and $x \notin B_1$; and $x \notin B_1$ and $x \in B_2$; and so $\neg \exists i \in I (x \in A_i \wedge x \in B_i)$; and so $\neg \exists i \in I (x \in A_i \cap B_i)$; and so $x \notin \bigcup_{i \in I} (A_i \cap B_i)$. Also note that: $x \in A_1$ and $x \in B_2$; so $\exists i \in I (x \in A_i)$ and $\exists i \in I (x \in B_i)$; and so $x \in \bigcup_{i \in I} A_i$ and $x \in \bigcup_{i \in I} B_i$; and so $x \in (\bigcup_{i \in I} A_i) \cap (\bigcup_{i \in I} B_i)$. Therefore $\bigcup_{i \in I} (A_i \cap B_i) \neq (\bigcup_{i \in I} A_i) \cap (\bigcup_{i \in I} B_i)$. □

<u>Exercise 26</u>:

<u>Proof</u>: Consider any integers a and b, and let $c = ab$. Note that c is an integer since a and b are integers. Since $ba = c$, $\exists m \in \mathbb{Z} (ma = c)$, or $a \mid c$. Similarly, since $ab = c$, $\exists n \in \mathbb{Z} (nb = c)$, or $b \mid c$. Thus, $a \mid c \wedge b \mid c$, and accordingly, $\exists c \in \mathbb{Z} (a \mid c \wedge b \mid c)$. Since a and b are arbitrary integers, we can conclude that for all integers a and b, there is an integer c such that $a \mid c$ and $b \mid c$. □

Exercise 27:

ⓐ Proof: Consider arbitrary integer n.

($\rightarrow$) Suppose that $15|n$. Thus, there is some integer a such that $15a = n$. Note that: $3(5a) = n$ and $5(3a) = n$. Also note that $5a$ and $3a$ are integers because 5, 3, and a are all integers. Accordingly, $3|n$ and $5|n$.

($\leftarrow$) Suppose that $3|n$ and $5|n$. Thus there is some integer a such that $3a = n$ and some integer b such that $5b = n$. Then, $3a = 5b$, or $\frac{3a}{5} = b$. Note that since b is an integer, $5|a$, since $3 \nmid 5$. Thus, there is an integer c such that $5c = a$. Since $3a = n$, it follows that $3(5c) = n$, or $15c = n$. Since c is an integer, $15|n$.

Thus, for every integer n, $15n$ iff $3|n$ and $5|n$. □

ⓑ Counterexample: Let $n = 30$. Note that since $5 \times 6 = n$, $6|n$; and since $3 \times 10 = n$, $10|n$. However, there is no integer m such that $60m = n$ (as m would need to be $\frac{1}{2}$). Thus, $60 \nmid n$. Therefore, it is not the case that for every integer n, $60|n$ iff $6|n$ and $3|n$.

□

Case 2. Suppose $x \in B$. Thus, $x \in B \setminus C$ and so $x \in A \cup (B \setminus C)$

Since these cases are exhaustive, $x \in A \cup (B \setminus C)$. Since x is an arbitrary element of $(A \cup B) \setminus C$, $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$. $\square$

Exercise 3:

Proof: Suppose A and B are sets.

($\rightarrow$) Consider any x such that $x \in A \setminus (A \setminus B)$. Thus, $x \in A$ and $x \notin A \setminus B$. And so, $x \notin A$ or $x \in B$. But $x \in A$, so $x \in B$. Thus, $x \in A \cap B$. Since x is an arbitrary element of $A \setminus (A \setminus B)$, $A \setminus (A \setminus B) \subseteq A \cap B$.

($\leftarrow$) Consider any x such that $x \in A \cap B$. Thus, $x \in A$ and $x \in B$. Since $x \in B$, $x \notin A$ or $x \in B$. That is $x \notin (A \setminus B)$. Thus, $x \in A \setminus (A \setminus B)$. Since x is an arbitrary element of $A \cap B$, $A \cap B \subseteq A \setminus (A \setminus B)$.

Therefore: $A \setminus (A \setminus B) = A \cap B$ $\square$

Exercise 4:

Proof: Suppose A, B, and C are sets.

($\rightarrow$) Consider any x such that $x \in A \setminus (B \setminus C)$. Thus, $x \in A$ and $x \notin B \setminus C$; so $x \notin B$ or $x \in C$. Two cases:

Case 1. $x \notin B$. So, $x \in A \setminus B$, and so $x \in (A \setminus B) \cup (A \cap C)$.

Case 2. $x \in C$. So, $x \in A \cap C$, and so $x \in (A \setminus B) \cup (A \cap C)$.

Since these cases are exhaustive, $x \in (A \setminus B) \cup (A \cap C)$. Since x is an arbitrary element of $A \setminus (B \setminus C)$, $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)$.

($\leftarrow$) Consider any x such that $x \in (A \setminus B) \cup (A \cap C)$. Two cases:

Case 1. $x \in A \setminus B$. Thus, $x \in A$ and $x \notin B$. It follows that $x \notin B$ or $x \in C$. That is, $x \notin B \setminus C$. So, $x \in A \setminus (B \setminus C)$.

Case 2. $x \in A \cap C$. Thus, $x \in A$ and $x \in C$. It follows that $x \notin B$ or $x \in C$. That is, $x \notin B \setminus C$. So, $x \in A \setminus (B \setminus C)$.

Since these cases are exhaustive, $x \in A \setminus (B \setminus C)$. Since x is an arbitrary element of $(A \setminus B) \cup (A \cap C)$, $(A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C)$.

Therefore: $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$. $\square$