2.2: Evaluating Determinants by Row Reduction

①

ⓐ $\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = (-2)(4)-(3)(1) = -8-3 = -11$

$\begin{vmatrix} -2 & 1 \\ 3 & 4 \end{vmatrix} = (-2)(4)-(1)(3) = -8-3 = -11$

ⓑ $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{vmatrix} = (2)(2)(6)+(-1)(4)(5)+(3)(1)(-3)-(3)(2)(5)-(2)(4)(-3)-(-1)(1)(6) = 24-20-9-30+24+6 = -5$

$\begin{vmatrix} 2 & 1 & 5 \\ -1 & 2 & -3 \\ 3 & 4 & 6 \end{vmatrix} = (2)(2)(6)+(1)(-3)(3)+(5)(-1)(4)-(5)(2)(3)-(2)(-3)(4)-(1)(-1)(6) = 24-9-20-30+24+6 = -5$

②

ⓐ $(3)(5)(-2) = -30$

ⓑ $(\sqrt{2})(\sqrt{2})(-1)(1) = -2$

ⓒ 0

ⓓ 0

③

ⓐ $(1)(1)(-5)(1) = -5$

ⓑ $-(1)(1)(1)(1) = -1$

ⓒ $(1)(1)(1)(1) = 1$

④ $\begin{vmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix} = -\begin{vmatrix} 3 & 6 & -9 \\ -2 & 1 & 5 \\ 0 & 0 & -2 \end{vmatrix} = -3\begin{vmatrix} 1 & 2 & -3 \\ -2 & 1 & 5 \\ 0 & 0 & -2 \end{vmatrix} = -3\begin{vmatrix} 1 & 2 & -3 \\ 0 & 5 & -1 \\ 0 & 0 & -2 \end{vmatrix}$

$= -3\begin{vmatrix} 1 & 2 & -3 \\ 0 & 5 & -1 \\ 0 & 0 & -2 \end{vmatrix} = (-3)(5)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -1/5 \\ 0 & 0 & -2 \end{vmatrix} = (-3)(5)(-2)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -1/5 \\ 0 & 0 & 1 \end{vmatrix}$

$= (-3)(5)(-2)(1) = 30$

⑤ $$\begin{vmatrix} 0 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 3 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{vmatrix} = (-1)(-5)\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = (-1)(-5)(1) = 5$$

⑥ $$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -1/2 \\ 0 & 13 & 2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 17/2 \end{vmatrix} = (-2)(17/2)\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{vmatrix} = (-2)(17/2)(1) = -17$$

⑦ $$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 5 \end{vmatrix} = (3)(3)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 4/3 \\ 0 & 1 & 5 \end{vmatrix}$$

$$= (3)(3)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 4/3 \\ 0 & 0 & 11/3 \end{vmatrix} = (3)(3)(11/3)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{vmatrix} = (3)(3)(11/3)(1) = 33$$

⑧ $$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 108 & 23 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & -13 \end{vmatrix}$$

$$= (-3)(-13)\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (-3)(-13)(1) = 39$$

⑨ $$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & -1/2 & -1/2 & 1/2 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

$$= (2)\left(\tfrac{-1}{2}\right)\begin{vmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = (2)\left(\tfrac{-1}{2}\right)\begin{vmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \end{vmatrix} = (2)\left(\tfrac{-1}{2}\right)\begin{vmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix}$$

$$= (2)\left(\tfrac{-1}{2}\right)(-1)\begin{vmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{vmatrix} = (2)\left(\tfrac{-1}{2}\right)(-1)\begin{vmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 6 \end{vmatrix}$$

$$= (2)\left(\tfrac{-1}{2}\right)(-1)(6)\begin{vmatrix} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (2)\left(\tfrac{-1}{2}\right)(-1)(6)(1) = 6$$

⑩ $$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1/2 & 1/2 & 1 & 1/2 \\ 2/3 & 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1/2 & 1/2 & 1 & 1/2 \\ 0 & 1 & 1 & 1 \\ 2/3 & 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1/2 & 1/2 & 1 & 1/2 \\ 0 & 1 & 1 & 1 \\ 2/3 & 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 & 0 \end{vmatrix} = -\left(\tfrac{1}{2}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 2/3 & 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 & 0 \end{vmatrix}$$

$$= -\left(\tfrac{1}{2}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 2/3 & 1/3 & 1/3 & 0 \\ -1/3 & 2/3 & 0 & 0 \end{vmatrix} = -\left(\tfrac{1}{2}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1/3 & -1 & -2/3 \\ -1/3 & 2/3 & 0 & 0 \end{vmatrix} = -\left(\tfrac{1}{2}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1/3 & -1 & -2/3 \\ 0 & 1 & 2/3 & 1/3 \end{vmatrix}$$

$$= -\left(\tfrac{1}{2}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2/3 & -1/3 \\ 0 & 1 & 2/3 & 4/3 \end{vmatrix} = -\left(\tfrac{1}{2}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \end{vmatrix} = -\left(\tfrac{1}{2}\right)\left(-\tfrac{2}{3}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & -1/3 & -2/3 \end{vmatrix}$$

$$= -\left(\tfrac{1}{2}\right)\left(-\tfrac{2}{3}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & -1/2 \end{vmatrix} = -\left(\tfrac{1}{2}\right)\left(-\tfrac{2}{3}\right)\left(-\tfrac{1}{2}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (-1)\left(\tfrac{1}{2}\right)\left(-\tfrac{2}{3}\right)\left(-\tfrac{1}{2}\right)(1)$$

$$= -1/6$$

(11) $$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} = -(2)\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & -8 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (-1)(2)(1) = -2$$

(12)

(a) -6

(b) $(3)(-1)(4)(-6) = 72$

(c) -6

(d) $(-3)(-6) = 18$

(13) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$. Consider $A^T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$

Since $\det(A) = \det(A^T)$, we solve for $\det(A^T)$.

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ a & b-a & c \\ a^2 & b^2-a^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = b-a \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

$$= (b-a)(c-a)(c-b) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{vmatrix} = (b-a)(c-a)(c-b)$$

(14)

(a) Proof: The only elementary product from the matrix that can be nonzero is $a_{13}a_{22}a_{31}$. To see that this is so, consider a typical elementary product $a_{1j_1}a_{2j_2}a_{3j_3}$. Since $a_{11} = a_{12} = 0$, we must have $j_1 = 3$ in order to have a nonzero product. If $j_1 = 3$, we must have $j_2 \neq 3$, since no two factors come from the same column. Further, since $a_{21} = 0$, we must have $j_2 = 2$ in order to have a nonzero product. Therefore, $j_3 = 1$. Since $a_{13}a_{22}a_{31}$ is multiplied by -1 in forming the signed elementary product, the determinant is $-a_{13}a_{22}a_{31}$. $\square$

(b) Proof: The only elementary product from the matrix that can be nonzero is $a_{14}a_{23}a_{32}a_{41}$. To see that this is so, consider a typical elementary product $a_{1j_1}a_{2j_2}a_{3j_3}a_{4j_4}$. Since $a_{11} = a_{12} = a_{13} = 0$, we must have $j_1 = 4$ in order to have a nonzero product. If $j_1 = 4$, we must have $j_2 \neq 4$, since no two factors come from the same column. Further, since $a_{21} = a_{22} = 0$, we must have $j_2 = 3$ in order to have a nonzero product. Continuing in this way, we obtain $j_3 = 2$ and $j_4 = 1$. Since $a_{14}a_{23}a_{32}a_{41}$ is multiplied by $+1$ in forming the signed elementary product, the determinant is $a_{13}a_{22}a_{31}$. $\square$

(15)

(a) $\det(B) = ka_{11}a_{22}a_{33} + ka_{12}a_{23}a_{31} + ka_{13}a_{21}a_{32} - ka_{13}a_{22}a_{31} - ka_{11}a_{23}a_{32} - ka_{12}a_{21}a_{33}$

$= k(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33})$

$= k\det(A)$

(b) $\det(B) = a_{21}a_{12}a_{33} + a_{22}a_{13}a_{31} + a_{23}a_{11}a_{32} - a_{23}a_{12}a_{31} - a_{21}a_{13}a_{32} - a_{22}a_{11}a_{33}$

$= -(-a_{21}a_{12}a_{33} - a_{22}a_{13}a_{31} - a_{23}a_{11}a_{32} + a_{23}a_{12}a_{31} + a_{21}a_{13}a_{32} + a_{22}a_{11}a_{33})$

$= -\det(A)$

(16)

(a) -1. This is the identity matrix (with $\det(I) = 1$), with rows 1 and 3 swapped. So $-(1)$, or -1.

(b) 1. This is the identity matrix (with $\det(I) = 1$), with two row swaps (1 and 4) and (2 and 3). So $-(-(1))$, or 1.

(17) $(x)(x+1)(2x-1) = 0$

$x = 0, -1, \frac{1}{2}$

(18)

(a) $x = -3, 1$

(b) No. Because this guy says so.

Also, just hand calculating the determinant, shows that $x = -3$ or $x = 1$.