2.3: More Operations On Sets

Exercise 1:

ⓐ $\forall x(x \in \mathcal{F} \to x \in \mathscr{P}(A))$.

$\forall x(x \in \mathcal{F} \to \forall y(y \in x \to y \in A))$

ⓑ $\forall x(x \in A \to x \in \{2n+1 \mid n \in \mathbb{N}\})$.

$\forall x(x \in A \to \exists n \in \mathbb{N}(x = 2n+1))$

ⓒ $\forall x(x \in \{n^2+n+1 \mid n \in \mathbb{N}\} \to x \in \{2n+1 \mid n \in \mathbb{N}\})$

$\forall n \in \mathbb{N} \exists m \in \mathbb{N}(n^2+n+1 = 2m+1)$

ⓓ $\neg\forall x(x \in \mathscr{P}(\bigcup_{i \in I} A_i) \to x \in \bigcup_{i \in I} \mathscr{P}(A_i))$.

$\exists x \neg(\neg(x \in \mathscr{P}(\bigcup_{i \in I} A_i)) \vee (x \in \bigcup_{i \in I} \mathscr{P}(A_i)))$.

$\exists x(x \in \mathscr{P}(\bigcup_{i \in I} A_i) \wedge \neg(x \in \bigcup_{i \in I} \mathscr{P}(A_i)))$.

$\exists x(\forall y(y \in x \to y \in \bigcup_{i \in I} A_i) \wedge \neg(\exists i \in I(x \in \mathscr{P}(A_i))))$.

$\exists x(\forall y(y \in x \to \exists i \in I(y \in A_i)) \wedge \forall i \in I \exists y(y \notin x \wedge y \in A_i))$

Exercise 2:

ⓐ $x \in \bigcup \mathcal{F} \wedge x \notin \bigcup \mathcal{G}$

$\exists A \in \mathcal{F}(x \in A) \wedge \neg\exists A \in \mathcal{G}(x \in A)$

$\exists A \in \mathcal{F}(x \in A) \wedge \forall A \in \mathcal{G}(x \notin A)$

ⓑ $\forall y(y \in \{x \in B \mid x \notin C\} \to y \in A)$

$\forall y((y \in B \wedge y \notin C) \to y \in A)$

ⓒ $\forall i \in I(x \in (A_i \cup B_i))$

$\forall i \in I(x \in A_i \vee x \in B_i)$

ⓓ $\forall i \in I(x \in A_i) \vee \forall i \in I(x \in B_i)$

Exercise 3:

$\mathscr{P}(\{\emptyset\}) = \{x \mid x \subseteq \{\emptyset\}\}$

$= \{\{\emptyset\}, \emptyset\}$

Exercise 4:

$\bigcap \mathcal{F} = \{red, blue\}$

$\bigcup \mathcal{F} = \{red, green, blue, orange, purple\}$

Exercise 5:

$\bigcap \mathcal{F} = \varnothing$

$\bigcup \mathcal{F} = \{3, 7, 12, 5, 16, 23\}$

Exercise 6:

ⓐ $A_2 = \{2, 3, 1, 4\}$

$A_3 = \{3, 4, 2, 6\}$

$A_4 = \{4, 5, 3, 8\}$

$A_5 = \{5, 6, 4, 10\}$

ⓑ $\bigcap_{i \in I} A_i = \{4\}$

$\bigcup_{i \in I} A_i = \{2, 3, 1, 4, 6, 5, 8, 10\}$

Exercise 7:

$\bigcup_{y \in Y} A_y = \{$Bach, Goethe, Hume, Mozart, Washington$\}$

$\bigcap_{y \in Y} A_y = \{$Goethe, Hume, Washington$\}$

Exercise 8:

ⓐ $A_2 = \{2, 4\}$; $A_3 = \{3, 6\}$; $B_2 = \{2, 3\}$; $B_3 = \{3, 4\}$

ⓑ $\bigcap_{i \in I} (A_i \cup B_i) = \{3, 4\}$

$(\bigcap_{i \in I} A_i) \cup (\bigcap_{i \in I} B_i) = \{3\}$

ⓒ The statements are not equivalent.

Exercise 9:

ⓐ $x \in \bigcup_{i \in I} (A_i \setminus B_i)$

$\exists i \in I \, (x \in A_i \setminus B_i)$

$\exists i \in I \, (x \in A_i \wedge x \notin B_i)$

$x \in \left(\bigcup_{i \in I} A_i\right) \setminus \left(\bigcup_{i \in I} B_i\right)$

$x \in \left(\bigcup_{i \in I} A_i\right) \wedge x \notin \left(\bigcup_{i \in I} B_i\right)$

$\exists i \in I (x \in A_i) \wedge \forall i \in I (x \notin B_i)$

$x \in \left(\bigcup_{i \in I} A_i\right) \setminus \left(\bigcap_{i \in I} B_i\right)$

$x \in \left(\bigcup_{i \in I} A_i\right) \wedge x \notin \left(\bigcap_{i \in I} B_i\right)$

$\exists i \in I (x \in A_i) \wedge \exists i \in I (x \notin B_i)$

ⓑ $\bigcup_{i \in I} (A_i \setminus B_i) = \{4, 6\}$

$\left(\bigcup_{i \in I} A_i\right) \setminus \left(\bigcup_{i \in I} B_i\right) = \{2, 3, 4, 6\} \setminus \{2, 3, 4\} = \{6\}$

$\left(\bigcup_{i \in I} A_i\right) \setminus \left(\bigcap_{i \in I} B_i\right) = \{2, 3, 4, 6\} \setminus \{3\} = \{2, 4, 6\}$

The statements are not equivalent.

Exercise 10:

$x \in \bigcup_{i \in I} (A_i \cap B_i)$

$\exists i \in I \, (x \in A_i \cap B_i)$

$\exists i \in I \, (x \in A_i \wedge x \in B_i)$

$x \in \left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{i \in I} B_i\right)$

$\exists i \in I (x \in A_i) \wedge \exists i \in I (x \in B_i)$

$I = \{1, 2\}$, $A_i = \{i\}$, $B_i = \{i - 1\}$

$A_1 = \{1\}$ $B_1 = \{0\}$

$A_2 = \{2\}$ $B_2 = \{1\}$

$\bigcup_{i \in I} (A_i \cap B_i) = \emptyset$

$\left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{i \in I} B_i\right) = \{1, 2\} \cap \{0, 1\} = \{1\}$

The statements are not equivalent.

Exercise 11:

$x \in \mathcal{P}(A \cap B)$

$\forall y (y \in x \to y \in A \cap B)$

$\forall y (y \in x \to (y \in A \wedge y \in B))$

$\forall y \in x (y \in A \wedge y \in B)$

$x \in \mathcal{P}(A) \cap \mathcal{P}(B)$

$x \in \mathcal{P}(A) \wedge x \in \mathcal{P}(B)$

$\forall y (y \in x \to y \in A) \wedge \forall y (y \in x \to y \in B)$

$\forall y \in x (y \in A) \wedge \forall y \in x (y \in B)$

$\forall y \in x (y \in A \wedge y \in B)$.

Exercise 12:

$x \in \mathcal{P}(A \cup B)$

$\forall y (y \in x \to y \in A \cup B)$

$\forall y (y \in x \to (y \in A \vee y \in B))$

$\forall y \in x (y \in A \vee y \in B)$

$x \in \mathcal{P}(A) \cup \mathcal{P}(B)$

$x \in \mathcal{P}(A) \vee x \in \mathcal{P}(B)$

$\forall y (y \in x \to y \in A) \vee \forall y (y \in x \to y \in B)$

$\forall y \in x (y \in A) \vee \forall y \in x (y \in B)$

Let $A = \{1, 2\}$, $B = \{2, 3\}$

$A \cup B = \{1, 2, 3\}$

$\mathcal{P}(A \cup B) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}, \varnothing\}$

$\mathcal{P}(A) = \{\{1\}, \{2\}, \{1,2\}, \varnothing\}$

$\mathcal{P}(B) = \{\{2\}, \{3\}, \{2,3\}, \varnothing\}$

$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \varnothing\}$

The sets are not equivalent.

Exercise 13:

ⓐ $x \in \bigcup_{i \in I} (A_i \cup B_i)$

$\exists i \in I (x \in A_i \cup B_i)$

$\exists i \in I (x \in A_i \vee x \in B_i)$

$\exists i \in I (x \in A_i) \vee \exists i \in I (x \in B_i)$

$x \in \bigcup_{i \in I} A_i \vee x \in \bigcup_{i \in I} B_i$

$x \in \left(\bigcup_{i \in I} A_i\right) \cup \left(\bigcup_{i \in I} B_i\right)$

(b) $x \in (\bigcap \mathcal{F}) \cap (\bigcap \mathcal{G})$

$x \in (\bigcap \mathcal{F}) \wedge x \in (\bigcap \mathcal{G})$

$\forall A \in \mathcal{F}(x \in A) \wedge \forall A \in \mathcal{G}(x \in A)$

$\forall A (A \in \mathcal{F} \rightarrow x \in A) \wedge \forall A (A \in \mathcal{G} \rightarrow x \in A)$

$\forall A (A \notin \mathcal{F} \vee x \in A) \wedge \forall A (A \notin \mathcal{G} \vee x \in A)$

$\forall A ((A \notin \mathcal{F} \wedge A \notin \mathcal{G}) \vee x \in A)$

$\forall A (\neg (A \in \mathcal{F} \vee A \in \mathcal{G}) \vee x \in A)$

$\forall A ((A \in \mathcal{F} \vee A \in \mathcal{G}) \rightarrow x \in A)$

$\forall A \in (\mathcal{F} \cup \mathcal{G})(x \in A)$

$x \in \bigcap (\mathcal{F} \cup \mathcal{G})$

(c) $x \in \bigcap_{i \in I} (A_i \setminus B_i)$

$\forall i \in I (x \in A_i \setminus B_i)$

$\forall i \in I (x \in A_i \wedge x \notin B_i)$

$\forall i \in I (x \in A_i) \wedge \forall i \in I (x \notin B_i)$

$\forall i \in I (x \in A_i) \wedge \neg \exists i \in I (x \in B_i)$

$x \in \bigcap_{i \in I} A_i \wedge x \notin \bigcup_{i \in I} B_i$

$x \in (\bigcap_{i \in I} A_i) \setminus (\bigcup_{i \in I} B_i)$

Exercise 14:

(a) $B_3 = A_{1,3} \cup A_{2,3} = \{1,3,4\} \cup \{2,3,5\} = \{1,2,3,4,5\}$

$B_4 = A_{1,4} \cup A_{2,4} = \{1,4,5\} \cup \{2,4,6\} = \{1,2,4,5,6\}$

(b) $\bigcap_{j \in J} B_j = \bigcap_{j = J} (\bigcup_{i \in I} A_{i,j}) = \{1,2,4,5\}$

(c) $\bigcup_{i \in I} (A_{i,3} \cap A_{i,4}) = (A_{1,3} \cap A_{1,4}) \cup (A_{2,3} \cap A_{2,4})$

$= (\{1,3,4\} \cap \{1,4,5\}) \cup (\{2,3,5\} \cap \{2,4,6\})$

$= \{1,4\} \cup \{2\} = \{1,2,4\}$

They are not equal.

ⓓ $x \in \bigcap_{j\in J}\left(\bigcup_{i\in I} A_{i,j}\right)$

$\forall j \in J\left(x \in \left(\bigcup_{i\in I} A_{i,j}\right)\right)$

$\forall j \in J \exists i \in I (x \in A_{i,j})$

They are not equivalent.

$x \in \bigcup_{i\in I}\left(\bigcap_{j\in J} A_{i,j}\right)$

$\exists i \in I\left(x \in \left(\bigcap_{j\in J} A_{i,j}\right)\right)$

$\exists i \in I \forall j \in J (x \in A_{i,j})$

Exercise 15:

ⓐ Proof: Suppose that $\mathcal{F} = \varnothing$. Note that $\bigcup\mathcal{F} = \{x \mid \exists A(A \in \mathcal{F} \wedge x \in A)\}$. Here, the only subset of $\mathcal{F}$ is $\varnothing$, and $\varnothing$ has no elements. Thus, there is no A such that $A \in \mathcal{F}$, and so $\bigcup\mathcal{F} = \varnothing$. The statement $x \in \bigcup\mathcal{F}$ must be false because $\bigcup\mathcal{F}$ has no elements. Also, note that $\bigcup\varnothing = \varnothing$. $\square$

ⓑ Suppose that $\mathcal{F} = \varnothing$. Note that $\bigcap\mathcal{F} = \{x \mid \forall A(A \in \mathcal{F} \rightarrow x \in A)\}$. $\varnothing$ has no elements. Thus, $A \in \mathcal{F} \rightarrow x \in A$ is vacuously true (because there is no A such that $A \in \mathcal{F}$), and so, no matter what x is, $x \in \bigcap\mathcal{F}$. Note that $\bigcap\varnothing = \mathcal{U}$ $\square$

Exercise 16:

ⓐ Applying the fact to the set R: $(R \in R \leftrightarrow R \notin R)$.

ⓑ A set of all sets cannot be a universe of discourse.