3.3 : Proofs Involving Quantifiers

Exercise 1:

Proof: Suppose $\exists x(P(x) \to Q(x))$. Thus, $P(x_0) \to Q(x_0)$ for some x_0. Suppose $\forall x P(x)$. Therefore, $P(x_0)$. Through modus ponens, $Q(x_0)$. Therefore $\exists x Q(x)$, namely x_0. $\square$

Exercise 2:

Proof: Suppose that A and $B \setminus C$ are disjoint. Suppose further that $x \in A \cap B$ for arbitrary x. Since A and $B \setminus C$ are disjoint, $x \notin A$, or $x \notin B \setminus C$. That is, $x \notin A$, or $x \notin B$ or $x \in C$. But $x \in A$ and $x \in B$. It follows that $x \in C$. Since x is arbitrary, $A \cap B \subseteq C$. $\square$

Exercise 3:

Proof: Suppose $A \subseteq B \setminus C$. Let x be arbitrary, and let $x \in A$. Now suppose (for contradiction) that A and C are not disjoint. Thus there is some element, x_0, such that $x_0 \in A$ and $x_0 \in C$. Since $A \subseteq B \setminus C$, $x_0 \in B \setminus C$, that is, $x_0 \in B$ and $x_0 \notin C$. This contradicts the fact that $x_0 \in C$. Therefore, A and C must be disjoint.

$\square$

Exercise 4:

Proof: Suppose $A \subseteq \mathcal{P}(A)$. Let x be arbitrary, and let $x \in \mathcal{P}(A)$. That is, $x \subseteq A$. Since $x \subseteq A \subseteq \mathcal{P}(A)$, $x \subseteq \mathcal{P}(A)$. Thus, $x \in \mathcal{P}(\mathcal{P}(A))$. Since x is arbitrary, $\mathcal{P}(A) \subseteq \mathcal{P}(\mathcal{P}(A))$.

$\square$

Exercise 5:

(a) $\{\varnothing\}$

(b) $\{\varnothing, \{\varnothing\}\}$

Exercise 6:

(a) Proof: Let x be any real number except 1. Let $y = \frac{-2x-1}{1-x}$. So $y(1-x) = -2x-1$, or $y - yx = -2x - 1$. That is, $y = -2x + yx - 1$, and so $y + 1 = x(-2+y)$. Finally, we can see that $\frac{y+1}{y-2} = x$. Therefore, there is a real number y such that $\frac{y+1}{y-2} = x$, namely $\frac{-2x-1}{1-x}$.

$\square$

(b) Proof: Suppose that there is a real number y such that $\frac{y+1}{y-2} = x$, and suppose (for contradiction), that $x = 1$. So: $\frac{y+1}{y-2} = 1$, or $y+1 = y-2$. Thus, $y = y-3$. This means that $y \neq y$. But we know that $y = y$. Thus, $x \neq 1$.

$\square$

Exercise 7:

Proof: Let x be any real number and suppose that $x > 2$. Let $y = \frac{x+\sqrt{x^2-4}}{2}$ (this is fine since $x > 2$). Consider $y + \frac{1}{y}$, or $\frac{x+\sqrt{x^2-4}}{2} + \frac{2}{x\sqrt{x^2-4}}$. Finding a common denominator yields: $\frac{2x^2+2x\sqrt{x^2-4}-4}{2(x+\sqrt{x^2-4})} + \frac{4}{2(x+\sqrt{x^2-4})}$, or $\frac{2x^2+2x\sqrt{x^2-4}}{2(x+\sqrt{x^2-4})}$. Simplification yields x. Therefore, there is a real number y such that $y + \frac{1}{y} = x$. $\square$

Exercise 8:

Proof: Suppose that $\mathcal{F}$ is a family of sets and let A be any set such that $A \in \mathcal{F}$. Consider any x such that $x \in A$. Because $x \in A$ and $A \in \mathcal{F}$, there is some set in $\mathcal{F}$ that itself contains x (namely A). Thus, $x \in \bigcup\mathcal{F}$. Because x is arbitrary, $A \subseteq \bigcup\mathcal{F}$. $\square$

Exercise 9:

Proof: Suppose that $\mathcal{F}$ is a family of sets and let A be any set such that $A \in \mathcal{F}$. Consider any x such that $x \in \bigcap\mathcal{F}$. Because $x \in \bigcap\mathcal{F}$ and $A \in \mathcal{F}$, $x \in A$. Thus, because x is arbitrary, $\bigcap\mathcal{F} \subseteq A$. $\square$

Exercise 10:

Proof: Suppose that $\mathcal{F}$ is a nonempty family of sets, B is a set, and $\forall A \in \mathcal{F}(B \subseteq A)$. Consider any x such that $x \in B$. Note that $B \subseteq A$ for every ~~set~~ A in $\mathcal{F}$. Thus, $x \in A$ for all A in $\mathcal{F}$. Since A is any set in $\mathcal{F}$, it follows that $x \in \bigcap\mathcal{F}$. Therefore $B \subseteq \bigcap\mathcal{F}$. $\square$

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Exercise 11:

Proof: Suppose that $\mathcal{F}$ is a family of sets and $\emptyset \in \mathcal{F}$. Consider any x such that $x \in \bigcap\mathcal{F}$. Thus, x is in every set in $\mathcal{F}$. But $\emptyset \in \mathcal{F}$ and $\emptyset$ has no elements. Therefore $\bigcap\mathcal{F} = \emptyset$. $\square$

Exercise 12:

Proof: Suppose that $\mathcal{F}$ and $\mathcal{G}$ are families of sets and $\mathcal{F} \subseteq \mathcal{G}$. Consider any x such that $x \in \bigcup \mathcal{F}$. Thus, x is in some set A in $\mathcal{F}$. Since $A \in \mathcal{F}$, and $\mathcal{F} \subseteq \mathcal{G}$, $A \in \mathcal{G}$. Thus, ~~there~~ is some set (A) in $\mathcal{G}$ such that x is in that set. Therefore $x \in \bigcup \mathcal{G}$, and so $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$. □

Problem 13:

Proof: Suppose that $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets and $\mathcal{F} \subseteq \mathcal{G}$. Consider any x such that $x \in \bigcap \mathcal{G}$. Thus, for all $A \in \mathcal{G}$, $x \in A$. Since all sets in $\mathcal{F}$ are in $\mathcal{G}$, it follows that for all $A \in \mathcal{F}$, $x \in A$. Therefore $x \in \bigcap \mathcal{F}$, and so $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$. □

Exercise 14:

Proof: Suppose that $\{A_i \mid i \in I\}$ is an indexed family of sets and $x \in \bigcup_{i \in I} \mathcal{P}(A_i)$. Let x exist in $\mathcal{P}(A_t)$ where $t \in I$. Thus, $x \in \mathcal{P}(A_t)$, and so $x \subseteq A_t$. Thus, $x \subseteq \bigcup_{i \in I} A_i$. And so, $x \in \mathcal{P}(\bigcup_{i \in I} A_i)$. Therefore, $\bigcup_{i \in I} \mathcal{P}(A_i) \subseteq \mathcal{P}(\bigcup_{i \in I} A_i)$ □

Exercise 15:

Proof: Suppose that $\{A_i \mid i \in I\}$ is an indexed family of sets and $I \neq \emptyset$. Note that $x \in \bigcap_{i \in I} \mathcal{P}(A_i)$ ~~then~~ only if $x \in \mathcal{P}(A_i)$ for all $i \in I$; ~~that~~ is $x \subseteq A_i$ for all $i \in I$. Consider any y such that $y \in \bigcap_{i \in I} A_i$. That is, $y \in A_i$ for all $i \in I$. Since $y \in A_i$ for all $i \in I$, $y \subseteq A_i$ for all $i \in I$. Thus, $y \in \bigcap_{i \in I} \mathcal{P}(A_i)$. Therefore $\bigcap_{i \in I} A_i \in \bigcap_{i \in I} \mathcal{P}(A_i)$. □

Exercise 16:

Proof: Suppose $\mathcal{F} \subseteq \mathcal{P}(B)$. Consider any x such that $x \in \bigcup \mathcal{F}$. Thus, for some $A \in \mathcal{F}$, $x \in A$. Note that since $A \in \mathcal{F}$, $A \in \mathcal{P}(B)$. That is $A \subseteq B$. Since $x \in A$, and $A \subseteq B$, $x \in B$. Thus, if $x \in \bigcup \mathcal{F}$ then $x \in B$, and so $\bigcup \mathcal{F} \subseteq B$. □

Exercise 17:

Proof: Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets, and every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$. Consider any x such that $x \in \bigcup\mathcal{F}$. That is, there is some $A \in \mathcal{F}$ such that $x \in A$. Note that A is a subset of every element of $\mathcal{G}$. Consider any set B such that $B \in \mathcal{G}$. Note that $A \subseteq B$. Since $x \in A$, $x \in B$ as well. Since B is an arbitrary set in $\mathcal{G}$, it follows that $x \in \bigcap\mathcal{G}$, and so $\bigcup\mathcal{F} \subseteq \bigcap\mathcal{G}$.

□

Exercise 18:

ⓐ Proof: Suppose that $a \mid b$ and $a \mid c$. Since $a \mid b$, we can choose some arbitrary integer m such that $ma = b$. Similarly, since $a \mid b$, we can choose some arbitrary integer n such that $na = c$. Note that $b + c = ma + na$; or $b + c = a(m+n)$. Since m and n are both integers, $m+n$ is an integer. Thus, $a \mid b+c$.

□

ⓑ Proof: Suppose that $ac \mid bc$ and $c \neq 0$. Since $ac \mid bc$, we can choose some arbitrary integer m such that $mac = bc$. That is, $ma = b$. Since m is an integer, $a \mid b$.

□

Exercise 19:

ⓐ Proof: Let x and y be any real numbers. Let $z = \frac{y-x}{2}$ and consider $x+z$. That is $x + \frac{y-x}{2} = \frac{2x}{2} + \frac{y-x}{2} = \frac{y+x}{2} = \frac{2y}{2} + \frac{-y+x}{2} = y - \frac{y-x}{2}$ $= y - z$. Thus, for all real numbers, there is a real number z such that $x + z = y - z$.

□

ⓑ The above proof would not be correct if "real number" were changed to "integer" because we assume z to be $\frac{y-x}{2}$. If y and x were both 1, then z would be $1/2$, which is not an integer.

Exercise 20:

It should be that there is at least one real number x such that $x^2 < 0$.

Exercise 21:

(a). The proof only proves that $\exists x \in B (x \neq 0)$.

(b) Counterexample: Let $A = \{1, 2, 3\}$ and $B = \{0, 1, 2, 3\}$. Here, $\forall x \in A (x \neq 0)$ and $A \subseteq B$, but it is not the case that $\forall x \in B (x \neq 0)$.

Exercise 22:

x needs to be introduced prior to y, but they are introduced at the same time.

Exercise 23:

(a) The assumption should be suppose $\mathcal{F}$ ~~or~~ $\mathcal{G}$ is not disjoint.

(b) Counterexample: Let $\cup\mathcal{F} = \{1\}$ and $\cup\mathcal{G} = \{0\}$, and let $\mathcal{F} = \{\{1, 0\}, \{1, 2\}\}$ and $\mathcal{G} = \{\{1, 0\}, \{2, 0\}\}$. Note that $\cup\mathcal{F}$ and $\cup\mathcal{G}$ are disjoint, but $\mathcal{F}$ and $\mathcal{G}$ are not disjoint (they have $\{1, 0\}$ in common).

Exercise 24:

(a) x and y are not arbitrary since $x = r = y$.

(b) Counterexample: Let $x = 1$ and $y = 0$. Here, $x^2 + xy - 2y^2 = 1 + 0 - 0 = 1 \neq 0$. Thus, the theorem is incorrect.

Exercise 25:

Proof: Consider any arbitrary real number x. Let $y = 2x$ and consider arbitrary z. Note that $y = 2x = \frac{2xz}{z} = \frac{x^2 + 2xz + z^2 - x^2 - z^2}{z} = \frac{(x+z)^2 - (x^2 + z^2)}{z}$. That is, $yz = (x+z)^2 - (x^2 + z^2)$. Thus, for all x, there is a y such that for all z, $yz = (x+z)^2 - (x^2 + z^2)$. $\square$

Exercise 26:

(a) For a goal of $\forall x P(x)$, you must assume x is arbitrary and prove $P(x)$.
For a given of $\exists x P(x)$, you should let $x = x_0$ such that $P(x_0)$ is true.
In both cases, you may not make further assumptions about, or restrict x/x_0.
For a goal of $\exists x P(x)$, you must find some a such that $P(a)$ is true.
For a given of $\forall x P(x)$, you may assume $P(a)$ is true.
In both cases, x must be assumed to be a specific value.

(b) When we have a goal $\forall x P(x)$, a proof by contradiction would require us to assume $\neg\forall x P(x)$, or $\exists x \neg P(x)$. Here, x must be unspecific to proceed. When we have a goal $\exists x P(x)$, a proof by contradiction would require us to assume $\neg\exists x P(x)$, or $\forall x \neg P(x)$. Here to proceed, we need to let x be specific.