## 3.5: Proofs Involving Disjunctions

Exercise 1:

Proof: Suppose $A$, $B$, and $C$ are sets. Consider any $x$ such that $x \in A \cap (B \cup C)$. Thus, $x \in A$ and $x \in B \cup C$. Two cases:

Case 1. Suppose $x \in B$. Thus, $x \in A \cap B$ and so $x \in (A \cap B) \cup C$.

Case 2. Suppose $x \in C$. Thus, $x \in (A \cap B) \cup C$.

Since these cases are exhaustive, $x \in (A \cap B) \cup C$. Since $x$ is an arbitrary element of $A \cap (B \cup C)$, $A \cap (B \cup C) \subseteq (A \cap B) \cup C$. $\square$

Exercise 2:

Proof: Suppose $A$, $B$, and $C$ are sets. Consider any $x$ such that $x \in (A \cup B) \setminus C$. That is, $x \in A \cup B$ and $x \notin C$. Two cases:

Case 1. Suppose $x \in A$. Thus, $x \in A \cup (B \setminus C)$.

Case 2. Suppose $x \in B$. Thus, $x \in B \setminus C$ and so $x \in A \cup (B \setminus C)$

Since these cases are exhaustive, $x \in A \cup (B \setminus C)$. Since $x$ is an arbitrary element of $(A \cup B) \setminus C$, $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$. □

Exercise 3:

Proof: Suppose $A$ and $B$ are sets.

($\rightarrow$) Consider any $x$ such that $x \in A \setminus (A \setminus B)$. Thus, $x \in A$ and $x \notin A \setminus B$. And so, $x \notin A$ or $x \in B$. But $x \in A$, so $x \in B$. Thus, $x \in A \cap B$. Since $x$ is an arbitrary element of $A \setminus (A \setminus B)$, $A \setminus (A \setminus B) \subseteq A \cap B$.

($\leftarrow$) Consider any $x$ such that $x \in A \cap B$. Thus, $x \in A$ and $x \in B$. Since $x \in B$, $x \notin A$ or $x \in B$. That is $x \notin (A \setminus B)$. Thus, $x \in A \setminus (A \setminus B)$. Since $x$ is an arbitrary element of $A \cap B$, $A \cap B \subseteq A \setminus (A \setminus B)$.

Therefore: $A \setminus (A \setminus B) = A \cap B$ □

Exercise 4:

Proof: Suppose $A$, $B$, and $C$ are sets.

($\rightarrow$) Consider any $x$ such that $x \in A \setminus (B \setminus C)$. Thus, $x \in A$ and $x \notin B \setminus C$; so $x \notin B$ or $x \in C$. Two cases:

Case 1. $x \notin B$. So, $x \in A \setminus B$, and so $x \in (A \setminus B) \cup (A \cap C)$.

Case 2. $x \in C$. So, $x \in A \cap C$, and so $x \in (A \setminus B) \cup (A \cap C)$.

Since these cases are exhaustive, $x \in (A \setminus B) \cup (A \cap C)$. Since $x$ is an arbitrary element of $A \setminus (B \setminus C)$, $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)$.

($\leftarrow$) Consider any $x$ such that $x \in (A \setminus B) \cup (A \cap C)$. Two cases:

Case 1. $x \in A \setminus B$. Thus, $x \in A$ and $x \notin B$. It follows that $x \notin B$ or $x \in C$. That is, $x \notin B \setminus C$. So, $x \in A \setminus (B \setminus C)$.

Case 2. $x \in A \cap C$. Thus, $x \in A$ and $x \in C$. It follows that $x \notin B$ or $x \in C$. That is, $x \notin B \setminus C$. So, $x \in A \setminus (B \setminus C)$.

Since these cases are exhaustive, $x \in A \setminus (B \setminus C)$. Since $x$ is an arbitrary element of $(A \setminus B) \cup (A \cap C)$, $(A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C)$.

Therefore: $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$. □

Exercise 5:

Proof: Suppose $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Consider arbitrary $x$ such that $x \in A$. Thus, $x \in A \cup C$ and so $x \in B \cup C$. Two cases:

Case 1. $x \in B$.

Case 2. $x \in C$. So, $x \in A \cap C$, and so $x \in B \cap C$. Thus, $x \in B$.

Since these cases are exhaustive, $x \in B$. Since $x$ is an arbitrary element of $A$, $A \subseteq B$. □

Exercise 6:

Proof: Suppose $A \triangle B \subseteq A$. Consider arbitrary $x$ such that $x \notin A$. Thus, $x \notin A \triangle B$. That is, $x \notin A \cup B$ or $x \in A \cap B$. Since $x \notin A$, $x \notin A \cap B$. Thus, $x \notin A \cup B$, and so $x \notin A$ and $x \notin B$. Of note: $x \notin B$. We have proven that if $x \notin A$ then $x \notin B$. Since $x$ is an arbitrary non-element of $A$, it follows that $B \subseteq A$. □

Exercise 7:

Proof: Suppose $A$, $B$, and $C$ are sets.

($\rightarrow$) Suppose $A \cup C \subseteq B \cup C$. Consider arbitrary $x$ such that $x \in A \setminus C$. So, $x \in A$ and $x \notin C$, and so $x \in A \cup C$. Thus, $x \in B \cup C$, and since $x \notin C$, $x \in B$. Since $x \in B$ and $x \notin C$, $x \in B \setminus C$. Since $x$ is an arbitrary element of $A \setminus C$, $A \setminus C \subseteq B \setminus C$.

($\leftarrow$) Suppose $A \setminus C \subseteq B \setminus C$, and consider arbitrary $x$ such that $x \in A \cup C$. Two cases:

Case 1. $x \in C$. Clearly, $x \in B \cup C$.

Case 2. $x \notin C$. Since $x \in A \cup C$, $x \in A$, and so $x \in A \setminus C$. It follows, then, that $x \in B \setminus C$. Thus, $x \in B$, and so $x \in B \cup C$.

Since these cases are exhaustive, $x \in B \cup C$. Since $x$ is an arbitrary element of $A \cup C$, $A \cup C \subseteq B \cup C$.

Therefore: $A \cup C \subseteq B \cup C$ iff $A \setminus C \subseteq B \setminus C$. □

Exercise 8:

Proof: Suppose $A$ and $B$ are sets. Consider any $x$ such that $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$.
Two cases:

Case 1. $x \in \mathcal{P}(A)$. Thus, $x \subseteq A$, and so $x \subseteq A \cup B$. That is, $x \in \mathcal{P}(A \cup B)$.

Case 2. $x \in \mathcal{P}(B)$. Thus, $x \subseteq B$, and so $x \subseteq A \cup B$. That is, $x \in \mathcal{P}(A \cup B)$.

Since these cases are exhaustive, $x \in \mathcal{P}(A \cup B)$. Since $x$ is an arbitrary element of $\mathcal{P}(A) \cup \mathcal{P}(B)$, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. □

Exercise 9:

Proof: Suppose $A$ and $B$ are sets and suppose $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.
Note that since $A \cup B \subseteq A \cup B$, $A \cup B \in \mathcal{P}(A \cup B)$. Thus, $A \cup B \in \mathcal{P}(A) \cup \mathcal{P}($
Two cases:

Case 1. $A \cup B \in \mathcal{P}(A)$. So, $A \cup B \subseteq A$. Thus, $B \subseteq A$.

Case 2. $A \cup B \in \mathcal{P}(B)$. So, $A \cup B \subseteq B$. Thus, $A \subseteq B$.

Because these cases are exhaustive, either $A \subseteq B$ or $B \subseteq A$. □

Exercise 10:

Proof: Suppose $x$ and $y$ are real numbers and $x \neq 0$. Consider: $y + \frac{1}{x} = 1 + \frac{y}{x}$.
The following equivalences follow: $\frac{xy+1}{x} = \frac{x+y}{x}$

$$xy + 1 = x + y$$
$$xy - x - y + 1 = 0$$
$$x(y-1) - (y-1) = 0$$
$$(x-1)(y-1) = 0$$

Accordingly, either $x = 1$ or $y = 1$, else $(x-1)(y-1) \neq 0$. □

Exercise 11:

Proof: Let $x$ be any real number. Suppose that $|x-3| > 3$.

Two cases:

Case 1. $x - 3 \geq 0$. Thus, $|x-3| = x - 3$. Since $|x-3| > 3$, $x - 3 > 3$. This is equivalent to $x > 6$, and so $x^2 > 6x$.

Case 2. $x - 3 < 0$. Thus, $|x-3| = 3 - x$. Since $|x-3| > 3$, $3 - x > 3$. Note that it must be true that $x < 0$. Since $x < 0$, $x^2 > 0$. Also, since $x < 0$, $6x < 0$. Accordingly, $x^2 > 6x$.

Since these cases are exhaustive, $x^2 > 6x$.

$\square$

Exercise 12:

Proof: Let $x$ be any real number.

($\rightarrow$) Suppose $|2x-6| > x$. Two cases:

Case 1. $2x - 6 \geq 0$. Thus, $|2x-6| = 2x - 6$. Since $|2x-6| > x$, $2x - 6 > x$. That is, $x > 6$, or $x - 4 > 2$. Since, $x - 4 \geq 0$ (because $x > 6$), $|x-4| > 2$.

Case 2. $2x - 6 < 0$. Thus, $|2x-6| = 6 - 2x$. Since $|2x-6| > x$, $6 - 2x > x$. That is $3x < 6$, or $x < 2$. Note that $x - 4 < -2$, or $4 - x > 2$. Since $x - 4 < 0$ (because $x < 2$), $4 - x = |x-4|$. Therefore $|x-4| > 2$.

Since these cases are exhaustive, $|x-4| > 2$.

($\leftarrow$) Suppose $|x-4| > 2$. Two cases:

Case 1. $x - 4 \geq 0$. Thus, $|x-4| = x - 4$. Since $|x-4| > 2$, $x - 4 > 2$, and so $x > 6$. Then, $2x > 12$ and $2x - 6 > 6$. Since $x > 6$, it must be true that $2x - 6 > x$. Since $2x - 6 \geq 0$ (because $x > 6$), $|2x-6| > x$.

Case 2. $x - 4 < 0$. Thus, $|x-4| = 4 - x$. Since $|x-4| > 2$, $4 - x > 2$, and so $x < 2$. Then $2x < 4$ and $2x - 6 < -2$.

That is: $-2x+6>2$. Since $x<2$, $-2x+6>x$, or $6-2x>x$.
Since $2x-6<0$ (because $x<2$), $6-2x=|2x-6|$.
Therefore $|2x-6|>x$.

Since these cases are exhaustive, $|2x-6|>x$.

Therefore: $|2x-6|>x$ iff $|x-4|>2$.

□

Exercise 13:

(a) Proof: Consider any real numbers a and b.

($\rightarrow$) Suppose $|a|\leq b$. Two cases:

Case 1. $a\geq 0$. Thus $|a|=a$, and so $a\leq b$. Since $a\geq 0$ and $a\leq b$, $-b\leq a\leq b$.

Case 2. $a<0$. Thus $|a|=-a$, and so $-a\leq b$. That is, $-b\leq a$. Since $a<0$ and $-b\leq a$, $-b\leq a\leq b$.

Since these cases are exhaustive, $-b\leq a\leq b$.

($\leftarrow$) Suppose $-b\leq a\leq b$. Two cases:

Case 1. $a\geq 0$. Here, $a=|a|$. Since $a\leq b$, $|a|\leq b$.

Case 2. $a<0$. Here, $-a=|a|$. Note that because $-b\leq a$, $-a\leq b$. Thus $|a|\leq b$.

Since these cases are exhaustive, $|a|\leq b$. □

(b) Proof: Consider any real number $x$. Let $x=a$ and $|x|=b$. Note that $|a|\leq b$. Thus, per Part (a), $-b\leq a\leq b$. Therefore, $-|x|\leq x\leq |x|$.

□

(c) Proof: Consider any real numbers $x$ and $y$, and consider the following cases:

Case 1. $x\geq 0$ and $y\geq 0$. Note that $x+y\geq 0$. Accordingly, $x=|x|$, $y=|y|$, and $x+y=|x+y|$. Also note that $x+y\leq x+y$. Thus, $|x+y|\leq |x|+|y|$.

Case 2. $x < 0$ and $y < 0$. Note that $x + y < 0$. Accordingly, $|x| = -x$, $|y| = -y$, and $|x + y| = -(x + y)$. Note that $-x - y \leq -x - y$. That is: $-(x + y) \leq -x + -y$. Using the equalities above: $|x + y| \leq |x| + |y|$.

Case 3. $x \geq 0$ and $y < 0$. Thus, $|x| = x$ and $|y| = -y$. Two cases:

Case 1. $|x| \geq |y|$. Here, $x + y \geq 0$, and so $|x+y| = x+y$. Since $y < 0$, $x + y \leq x - y$. Therefore, $|x+y| \leq |x| + |y|$

Case 2. $|x| < |y|$. Here $x + y < 0$, and so $|x+y| = -x - y$. Since $x \geq 0$, $-x - y \leq x - y$. Thus, $|x+y| \leq |x| + |y|$.

Since these cases are exhaustive, $|x+y| \leq |x| + |y|$.

Case 4. $x < 0$ and $y \geq 0$. Thus, $|x| = -x$ and $|y| = y$. Two cases:

Case 1. $|x| > |y|$. Here, $x + y < 0$, and so $|x+y| = -(x+y)$. Since $y \geq 0$, $-x - y \leq -x + y$. Thus, $|x+y| \leq |x| + |y|$.

Case 2. $|x| \leq |y|$. Here, $x + y \geq 0$, and so $|x+y| = x+y$. Since $x < 0$, $x + y \leq -x + y$. Therefore, $|x+y| < |x| + |y|$.

Since these cases are exhaustive, $|x+y| \leq |x| + |y|$.

Since these cases are exhaustive, $|x+y| \leq |x| + |y|$

□

(d) Proof: Consider any real numbers $x$ and $y$. Note that $|x| = |(x+y) + (-y)|$. Per the Triangle Inequality (proven in Part (c)): $|x| = |(x+y) + (-y)| \leq |x+y| + |y|$, or $|x| - |y| \leq |x+y|$.

□

Exercise 14:

Proof: Consider any integer $x$. Two cases.

Case 1. $x$ is even. Thus, there is some integer $k$ such that $x = 2k$. Note that $x^2 + x = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$. Since $k$ is an integer, $2k^2 + k$ is also an integer. Thus, there is some integer $i$ such that $2i = x^2 + x$. Accordingly, $x^2 + x$ is even.

Case 2. $x$ is odd. Thus, there is some integer $k$ such that $x=2k+1$.
Note that $x^2+x=(2k+1)^2+(2k+1)=4k^2+6k+2=2(2k^2+3k+1)$. Since $k$ is an integer, $2k^2+3k+1$ is also an integer. Thus, there is some integer $i$ such that $2i=x^2+x$. Accordingly, $x^2+x$ is even.

Since these cases are exhaustive, $x^2+x$ is even.

$\square$

Exercise 15:

Proof: Consider any integer $x$. Two cases.

Case 1. $x$ is even. Then $x=2k$ for some integer $k$, so $x^4=16k^4$. Clearly, the remainder when $x^4$ is divided by 8 is 0.

Case 2. $x$ is odd. Then $x=2k+1$ for some integer $k$, so
$x^4=16k^4+32k^3+24k^2+8k+1=8k(2k^3+4k^2+3k+1)+1$
Clearly, the remainder when $x^4$ is divided by 8 is 1.

Since these cases are exhaustive, the remainder when $x^4$ is divided by 8 is either 0 or 1.

$\square$

Exercise 16:

(a) Proof: Consider arbitrary $x$.

($\rightarrow$) Suppose $x\in\bigcup(\mathcal{F}\cup\mathcal{G})$. Thus, there is some set $A\in\mathcal{F}\cup\mathcal{G}$ such that $x\in A$. Note that since $A\in\mathcal{F}\cup\mathcal{G}$, we have two cases:

Case 1. $A\in\mathcal{F}$. Since $x\in A$, $x\in\bigcup\mathcal{F}$. It follows that $x\in(\bigcup\mathcal{F})\cup(\bigcup\mathcal{G})$.

Case 2. $A\in\mathcal{G}$. Since $x\in A$, $x\in\bigcup\mathcal{G}$. It follows that $x\in(\bigcup\mathcal{F})\cup(\bigcup\mathcal{G})$.

Since these cases are exhaustive, $x\in(\bigcup\mathcal{F})\cup(\bigcup\mathcal{G})$.

($\leftarrow$) Suppose $x\in(\bigcup\mathcal{F})\cup(\bigcup\mathcal{G})$. Two cases.

Case 1. $x\in\bigcup\mathcal{F}$. Thus, there is some set $A\in\mathcal{F}$ such that $x\in A$. Note that $A\in\mathcal{F}\cup\mathcal{G}$, and thus $x\in\bigcup(\mathcal{F}\cup\mathcal{G})$.

Case 2. $x\in\bigcup\mathcal{G}$. Thus, there is some set $A\in\mathcal{G}$ such that $x\in A$. Note that $A\in\mathcal{F}\cup\mathcal{G}$, and thus $x\in\bigcup(\mathcal{F}\cup\mathcal{G})$.

Since these cases are exhaustive, $x\in\bigcup(\mathcal{F}\cup\mathcal{G})$.

Therefore $\bigcup(\mathcal{F} \cup \mathcal{G}) = (\bigcup\mathcal{F}) \cup (\bigcup\mathcal{G})$.

(b) Proof: Consider arbitrary $x$.

($\rightarrow$) Suppose $x \in \bigcap(\mathcal{F} \cup \mathcal{G})$. Thus, $x$ is in every set in $\mathcal{F} \cup \mathcal{G}$. That is $x \in \bigcap\mathcal{F}$ and $x \in \bigcap\mathcal{G}$, or $x \in (\bigcap\mathcal{F}) \cap (\bigcap\mathcal{G})$.

($\leftarrow$) Suppose $x \in (\bigcap\mathcal{F}) \cap (\bigcap\mathcal{G})$. Thus, $x \in \bigcap\mathcal{F}$ and $x \in \bigcap\mathcal{G}$, or $\forall A \in \mathcal{F}(x \in A)$ and $\forall B \in \mathcal{G}(x \in B)$. Accordingly, $x$ is in every set in $\mathcal{F} \cup \mathcal{G}$. Therefore, $x \in \bigcap(\mathcal{F} \cup \mathcal{G})$.

Therefore $\bigcap(\mathcal{F} \cup \mathcal{G}) = (\bigcap\mathcal{F}) \cap (\bigcap\mathcal{G})$.

$\square$

Exercise 17:

(a) Proof: Suppose $\mathcal{F}$ is a nonempty family of sets and $B$ is a set.

($\rightarrow$) Consider any $x$ such that $x \in B \cup (\bigcup\mathcal{F})$. Two cases.

Case 1. $x \in B$. Note that $B \in \{B\}$, and so $B \in \mathcal{F} \cup \{B\}$. Also note that $\exists A \in (\mathcal{F} \cup \{B\})(x \in A)$, and thus $x \in \bigcup(\mathcal{F} \cup \{B\})$.

Case 2. $x \in \bigcup\mathcal{F}$. That is, $\exists A \in \mathcal{F}(x \in A)$. Note that $A \in \mathcal{F} \cup \{B\}$. Since $\exists C \in (\mathcal{F} \cup \{B\})(x \in C)$, $x \in \bigcup(\mathcal{F} \cup \{B\})$.

Since these cases are exhaustive, $x \in \bigcup(\mathcal{F} \cup \{B\})$.

($\leftarrow$) Consider any $x$ such that $x \in \bigcup(\mathcal{F} \cup \{B\})$. Thus, $\exists A \in (\mathcal{F} \cup \{B\})(x \in A)$. Two cases

Case 1. $A \in \mathcal{F}$. Thus, $x \in \bigcup\mathcal{F}$, and so $x \in B \cup (\bigcup\mathcal{F})$.

Case 2. $A \in \{B\}$. Thus, $A = B$ and so $x \in B$. Accordingly, $x \in B \cup (\bigcup\mathcal{F})$.

Since these cases are exhaustive, $x \in B \cup (\bigcup\mathcal{F})$.

Therefore: $B \cup (\bigcup\mathcal{F}) = \bigcup(\mathcal{F} \cup \{B\})$.

$\square$

(b) <u>Proof</u>: Suppose $\mathcal{F}$ is a nonempty family of sets and $B$ is a set.

($\to$) Consider any $x$ such that $x \in B \cup (\bigcap \mathcal{F})$. Two cases.

Case 1. $x \in B$. Thus, for all $A$, $x \in B \cup A$, and certainly $\forall A \in \mathcal{F}(x \in B \cup A)$. Accordingly, $x \in \bigcap_{A \in \mathcal{F}} (B \cup A)$.

Case 2. $x \in \bigcap \mathcal{F}$. That is, $\forall A \in \mathcal{F}(x \in A)$. It follows that $\forall A \in \mathcal{F}(x \in B \cup A)$. Therefore $x \in \bigcap_{A \in \mathcal{F}} (B \cup A)$.

Since these cases are exhaustive, $x \in \bigcap_{A \in \mathcal{F}} (B \cup A)$.

($\leftarrow$) Consider any $x$ such that $x \in \bigcap_{A \in \mathcal{F}} (B \cup A)$. Thus, $\forall A \in \mathcal{F}(x \in B \cup A)$. Two cases.

Case 1. $x \in B$. Thus, $x \in B \cup (\bigcap \mathcal{F})$.

Case 2. $x \notin B$. Consider arbitrary $A \in \mathcal{F}$. As noted above, $x \in B \cup A$. But $x \notin B$, so $x \in A$. Since $A$ is arbitrary, $\forall A \in \mathcal{F}(x \in A)$. That is, $x \in \bigcap \mathcal{F}$. It follows that $x \in B \cup (\bigcap \mathcal{F})$.

Since these cases are exhaustive, $x \in B \cup (\bigcap \mathcal{F})$.

Therefore $B \cup (\bigcap \mathcal{F}) = \bigcap_{A \in \mathcal{F}} (B \cup A)$.

$\square$

(c) <u>Proof</u>: Suppose $\mathcal{F}$ is a nonempty family of sets and $B$ is a set.

($\to$) Consider any $x$ such that $x \in B \cap (\bigcup \mathcal{F})$. That is, $x \in B$ and $x \in \bigcup \mathcal{F}$. It follows that $\exists A \in \mathcal{F}(x \in A)$, say $A_0$. Thus, $x \in B \cap A_0$, and so $\exists A \in \mathcal{F}(x \in B \cap A)$. Accordingly, $x \in \bigcup_{A \in \mathcal{F}} (B \cap A)$.

($\leftarrow$) Consider any $x$ such that $x \in \bigcup_{A \in \mathcal{F}} (B \cap A)$. That is, $\exists A \in \mathcal{F}(x \in B \cap A)$, say $A_0$. Thus, $x \in B \cap A_0$, or $x \in B$ and $x \in A_0$. Note that since $x \in A_0$ and $A_0 \in \mathcal{F}$, $x \in \bigcup \mathcal{F}$. Thus, $x \in B \cap (\bigcup \mathcal{F})$.

Therefore $B \cap (\bigcup \mathcal{F}) = \bigcup_{A \in \mathcal{F}} (B \cap A)$.

$\square$

<u>Proof</u>: Suppose $\mathcal{F}$ is a nonempty family of sets and $B$ is a set.

($\to$) Consider any $x$ such that $x \in B \cap (\bigcap \mathcal{F})$. That is $x \in B$ and $x \in \bigcap \mathcal{F}$. It follows that $\forall A \in \mathcal{F}(x \in A)$. Consider arbitrary $A \in \mathcal{F}$. Note that $x \in A$ and so $x \in B \cap A$. Since $A$ is an arbitrary member of $\mathcal{F}$, $\forall A \in \mathcal{F}(x \in B \cap A)$. Therefore, $x \in \bigcap_{A \in \mathcal{F}} (B \cap A)$.

($\leftarrow$) Consider any $x$ such that $x \in \bigcap_{A \in \mathcal{F}} (B \cap A)$. That is $\forall A \in \mathcal{F}(x \in B \cap A)$.

Consider arbitrary $A \in \mathcal{F}$. Note that $x \in B \cap A$, or $x \in B$ and $x \in A$. Note that since $A$ is an arbitrary member of $\mathcal{F}$ and $x \in A$, $x \in \bigcap \mathcal{F}$. Therefore, $x \in B \cap (\bigcap \mathcal{F})$.

Therefore $B \cap (\bigcap \mathcal{F}) = \bigcap_{A \in \mathcal{F}} (B \cap A)$.

$\square$

## Exercise 18:

Proof: Suppose $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are nonempty families of sets and for every $A \in \mathcal{F}$ and every $B \in \mathcal{G}$, $A \cup B \in \mathcal{H}$. Consider any $x$ such that $x \in \bigcap \mathcal{H}$. Note then, that $\forall C \in \mathcal{H} (x \in C)$. Two cases.

Case 1. $x \in \bigcap \mathcal{F}$. Clearly, $x \in (\bigcap \mathcal{F}) \cup (\bigcap \mathcal{G})$.

Case 2. $x \notin \bigcap \mathcal{F}$. Thus, there is some set $A$ in $\mathcal{F}$ such that $x \notin A$, say $A_0$. It is the case that for every $A \in \mathcal{F}$ and every $B \in \mathcal{G}$, $A \cup B \in \mathcal{H}$. Note also that $x$ is an element of all sets in $\mathcal{H}$. Thus, since $A_0 \cup B \in \mathcal{H}$ $x \in A_0 \cup B$. But $x \notin A_0$, so $x \in B$ for every $B \in \mathcal{G}$. That is, $x \in \bigcap \mathcal{G}$. Therefore, $x \in (\bigcap \mathcal{F}) \cup (\bigcap \mathcal{G})$.

Since these cases are exhaustive, $x \in (\bigcap \mathcal{F}) \cup (\bigcap \mathcal{G})$, and thus $\bigcap \mathcal{H} \subseteq (\bigcap \mathcal{F}) \cup (\bigcap \mathcal{G})$.

$\square$

## Exercise 19:

Proof: Suppose $A$ and $B$ are sets. Consider arbitrary $x$.

($\rightarrow$) Suppose $x \in A \triangle B$. So $x \in A$ or $x \in B$, but not both.

($\rightarrow$) Suppose $x \in A$. It follows that $x \notin B$.

($\leftarrow$) Suppose $x \notin B$. It follows that $x \in A$.

Therefore $x \in A \leftrightarrow x \notin B$.

($\leftarrow$) Suppose $x \in A \leftrightarrow x \notin B$. Two cases.

Case 1. $x \in A$. Thus, $x \notin B$. Accordingly, $x \in (A \cup B) \setminus (A \cap B)$.

Case 2. $x \notin A$. Thus, $x \in B$. Accordingly, $x \in (A \cup B) \setminus (A \cap B)$.

Since these cases are exhaustive, $x \in A \triangle B$.

Therefore $x \in A \triangle B \leftrightarrow (x \in A \leftrightarrow x \notin B)$. Since $x$ is arbitrary, $\forall x (x \in A \triangle B \leftrightarrow (x \in A \leftrightarrow x \notin B))$.

$\square$

## Exercise 20:

Proof: Suppose $A$, $B$, and $C$ are sets.

($\to$) Suppose $A \triangle B$ and $C$ are disjoint.

($\to$) Consider any $x$ such that $x \in A \cap C$. Thus, $x \in A$ and $x \in C$. Because $x \in C$, $x \notin A \triangle B$. That is, $x \notin A \cup B$ or $x \in A \cap B$. We know that $x \in A$, so $x \in A \cup B$. It follows that $x \in A \cap B$, and so $x \in B$, and in turn, $x \in B \cap C$.

($\leftarrow$) Consider any $x$ such that $x \in B \cap C$. This is essentially the same as the opposite direction.

Therefore $A \cap C = B \cap C$.

($\leftarrow$) Suppose $A \cap C = B \cap C$. Consider any $x$ such that $x \in (A \triangle B) \cap C$. Note that $x \in A \triangle B$ and $x \in C$, and so $x \in A \cup B$ and $x \notin A \cap B$. That is: $x \notin A$ or $x \notin B$. Two cases.

Case 1. $x \notin A$. Thus, $x \in B$. Note that $x \notin A \cap C$ but $x \in B \cap C$. Accordingly, $A \cap C \neq B \cap C$.

Case 2. $x \notin B$. Thus, $x \in A$. Note that $x \in A \cap C$ but $x \notin B \cap C$. Accordingly, $A \cap C \neq B \cap C$.

Because these cases are exhaustive, $A \cap C \neq B \cap C$. But $A \cap C = B \cap C$. Contradiction. Thus, $(A \triangle B) \cap C = \emptyset$ (i.e., the sets are disjoint.

Therefore, $A \triangle B$ and $C$ are disjoint iff $A \cap C = B \cap C$.

$\square$

## Exercise 21:

Proof: Suppose $A$, $B$, and $C$ are sets.

($\to$) Suppose $A \triangle B \subseteq C$.

($\to$) Consider any $x$ such that $x \in A \cup C$. Two cases.

Case 1. $x \in C$. Clearly $x \in B \cup C$.

Case 2. $x \in A$. Two cases.

Case 1. $x \in B$. Clearly, $x \in B \cup C$.

Case 2. $x \notin B$. Since $x \in A$ and $x \notin B$, $x \in A \triangle B$, and thus $x \in C$. Then, $x \in B \cup C$.

Since these cases are exhaustive, $x \in B \cup C$.

Since these cases are exhaustive, $x \in B \cup C$.

($\leftarrow$) Consider any $x$ such that $x \in B \cup C$. Follow the same steps as above to prove $x \in A \cup C$.

Therefore, $A \cup C = B \cup C$.

($\Leftarrow$) Suppose $A \cup C = B \cup C$. Consider any $x$ such that $x \in A \triangle B$.

Thus, $x \in A \cup B$ and $x \notin A \cap B$. That is, $x \notin A$ or $x \notin B$. Two cases.

Case 1. $x \notin A$. Thus, $x \in B$, and so $x \in B \cup C$. Since $A \cup C = B \cup C$, $x \in A \cup C$.

Since $x \notin A$, $x \in C$.

Case 2. $x \notin B$. Thus, $x \in A$, and so $x \in A \cup C$. Since $A \cup C = B \cup C$, $x \in B \cup C$.

Since $x \notin B$, $x \in C$.

Since these cases are exhaustive, $x \in C$.

Therefore: $A \triangle B \subseteq C$ iff $A \cup C = B \cup C$.

$\square$

## Exercise 22:

Proof: Suppose $A$, $B$, and $C$ are sets.

($\rightarrow$) Suppose $C \subseteq A \triangle B$. Consider any $x$ such that $x \in C$. Thus, $x \in A \triangle B$.

That is $x \in A \cup B$ and $x \notin A \cap B$. Thus, $C \subseteq A \cup B$.

Now, consider any $x$ such that $x \in A \cap B \cap C$. Thus, $x \in A$, $x \in B$, and $x \in C$.

Since $x \in C$, $x \in A \triangle B$, and so $x \in A \cup B$ and $x \notin A \cap B$. That is, $x \notin A$ or $x \notin B$. Two cases.

Case 1. $x \notin A$. So $x \notin A \cap B \cap C$.

Case 2. $x \notin B$. So $x \notin A \cap B \cap C$.

Since these cases are exhaustive, $x \notin A \cap B \cap C$. But $x \in A \cap B \cap C$.

Contradiction. Thus, $A \cap B \cap C = \varnothing$.

Therefore: $C \subseteq A \cup B$ and $A \cap B \cap C = \varnothing$.

($\leftarrow$) Suppose $C \subseteq A \cup B$ and $A \cap B \cap C = \varnothing$. Consider any $x$ such that $x \in C$.

Thus, $x \in A \cup B$. Since $A \cap B \cap C = \varnothing$ and $x \in C$, either $x \notin A$ or $x \notin B$.

Two cases.

Case 1. $x \notin A$. Thus, $x \notin A \cap B$.

Case 2. $x \notin B$. Thus, $x \notin A \cap B$.

Since these cases are exhaustive, $x \notin A \cap B$.

Accordingly, $x \in A \cup B$ and $x \notin A \cap B$, and so $x \in A \triangle B$.

Therefore: $C \subseteq A \triangle B$ iff $C \subseteq A \cup B$ and $A \cap B \cap C = \varnothing$.

$\square$

## Exercise 23:

ⓐ <u>Proof</u>: Suppose $A, B,$ and $C$ are sets. Consider any $x$ such that $x \in A \setminus C$. That is $x \in A$ and $x \notin C$. Two cases.

Case 1. $x \notin B$. Thus, $x \in A \setminus B$, and so $x \in (A \setminus B) \cup (B \setminus C)$.

Case 2. $x \in B$. Thus, $x \in B \setminus C$, and so $x \in (A \setminus B) \cup (B \setminus C)$.

Since these cases are exhaustive, $x \in (A \setminus B) \cup (B \setminus C)$.

Therefore: $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$.

□

ⓑ <u>Proof</u>: Suppose $A, B,$ and $C$ are sets. Consider any $x$ such that $x \in A \triangle C$. That is $x \in A \cup C$ and $x \notin A \cap C$, or $x \notin A$ or $x \notin C$. Two cases.

Case 1. $x \in B$. Two cases.

Case 1.1. $x \notin A$. Note that $x \in A \triangle B$, and so $x \in (A \triangle B) \cup (B \triangle C)$.

Case 1.2. $x \notin C$. Note that $x \in B \triangle C$, and so $x \in (A \triangle B) \cup (B \triangle C)$.

Since these cases are exhaustive, $x \in (A \triangle B) \cup (B \triangle C)$.

Case 2. $x \notin B$. Two cases.

Case 2.1. $x \notin A$. Thus, $x \in C$. Accordingly, $x \in B \triangle C$, and so $x \in (A \triangle B) \cup (B \triangle C)$.

Case 2.2. $x \notin C$. Thus, $x \in A$. Accordingly, $x \in A \triangle B$, and so $x \in (A \triangle B) \cup (B \triangle C)$.

Since these cases are exhaustive, $x \in (A \triangle B) \cup (B \triangle C)$.

Since these cases are exhaustive, $x \in (A \triangle B) \cup (B \triangle C)$.

Therefore: $A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$.

□

## Exercise 24:

ⓐ <u>Proof</u>: Suppose $A, B,$ and $C$ are sets. Consider any $x$ such that $x \in (A \cup B) \triangle C$. That is, $x \in (A \cup B) \cup C$ and $x \notin (A \cup B) \cap C$, so $x \notin A \cup B$ or $x \notin C$. Two cases.

Case 1. $x \notin A \cup B$. That is, $x \notin A$ and $x \notin B$. Thus, $x \in C$. Accordingly, $x \in A \triangle C$, and so $x \in (A \triangle C) \cup (B \triangle C)$.

Case 2. $x \notin C$. So $x \in A \cup B$. Two cases.

Case 2.1. $x \in A$. Thus, $x \in A \triangle C$, and so $x \in (A \triangle C) \cup (B \triangle C)$.

Case 2.2. $x \in B$. Thus, $x \in B \triangle C$, and so $x \in (A \triangle C) \cup (B \triangle C)$.

Since these cases are exhaustive, $x \in (A \triangle C) \cup (B \triangle C)$.

Since these cases are exhaustive $x \in (A \triangle C) \cup (B \triangle C)$.

Therefore $(A \cup B) \triangle C \subseteq (A \triangle C) \cup (B \triangle C)$.

□

ⓑ <u>Counterexample</u>: Consider the sets $A = \{x\}$, $B = \{y\}$, and $C = \{x\}$. Note that $x \in (A \triangle C) \cup (B \triangle C)$, but $x \notin (A \cup B) \triangle C$. Thus, $(A \triangle C) \cup (B \triangle C) \neq (A \cup B) \triangle C$.

$\square$

<u>Exercise 25</u>:

ⓐ <u>Proof</u>: Suppose $A$, $B$, and $C$ are sets. Consider any $x$ such that $x \in (A \triangle C) \cap (B \triangle C)$. That is, $x \in A \triangle C$ and $x \in B \triangle C$, or $x \in A \cup C$ and $x \notin A \cap C$ and $x \in B \cup C$ and $x \notin B \cap C$. Accordingly, $x \notin A$ or $x \notin C$ <u>and</u> $x \notin B$ or $x \notin C$. Two cases.

Case 1. $x \in C$. Thus, $x \notin A$ and $x \notin B$, and so $x \notin A \cap B$. Also, $x \in ((A \cap B) \cup C) \setminus ((A \cap B) \cap C)$. Therefore, $x \in (A \cap B) \triangle C$.

Case 2. $x \notin C$. Thus, $x \in A$ and $x \in B$, and so $x \in A \cap B$. Also, $x \in ((A \cap B) \cup C) \setminus ((A \cap B) \cap C)$. Therefore, $x \in (A \cap B) \triangle C$.

Since these cases are exhaustive, $x \in (A \cap B) \triangle C$.
Therefore, $(A \triangle C) \cap (B \triangle C) \subseteq (A \cap B) \triangle C$.

$\square$

ⓑ <u>Counterexample</u>: Consider the sets $A = \{x\}$, $B = \{x\}$, and $C = \{y\}$. Note that $x \in (A \cap B) \triangle C$, but $x \notin (A \triangle B) \cap (B \triangle C)$. Thus, $(A \cap B) \triangle C \not\subseteq (A \triangle B) \cap (B \triangle C)$.

$\square$

<u>Exercise 26</u>:

<u>Proof</u>: Suppose $A$, $B$, and $C$ are sets. Consider any $x$ such that $x \in (A \triangle C) \setminus (B \triangle C)$. That is, $x \in A \triangle C$ and $x \notin B \triangle C$. Thus, $x \in A \cup C$ and $x \notin A \cap C$, so $x \notin A$ or $x \notin C$. Also, $x \notin B$ and $x \notin C$ <u>or</u> $x \in B$ and $x \in C$. Two cases.

Case 1. $x \in A$. Thus, $x \notin C$, and so, $x \notin B$. Since $x \in A \setminus B$ and $x \notin C$, $x \in (A \setminus B) \triangle C$.

Case 2. $x \in C$. Thus, $x \notin A$, and so $x \in B$. Note that $x \notin A \setminus B$ and $x \in C$. Thus, $x \in (A \setminus B) \triangle C$.

Since these cases are exhaustive, $x \in (A \setminus B) \triangle C$.
Therefore: $(A \triangle C) \setminus (B \triangle C) \subseteq (A \setminus B) \triangle C$.

$\square$

Proof: Suppose $A$, $B$, and $C$ are sets. Consider any $x$ such that $x \in (A \setminus B) \triangle C$. That is $x \in (A \setminus B) \cup C$ and $x \notin (A \setminus B) \cap C$. That is, $x \notin A \setminus B$ or $x \notin C$. Two cases.

Case 1. $x \in A \setminus B$. That is $x \in A$ and $x \notin B$. Also, $x \notin C$. Note that $x \in A \triangle C$ but $x \notin B \triangle C$. Accordingly, $x \in (A \triangle C) \setminus (B \triangle C)$.

Case 2. $x \in C$. Then, $x \notin A \setminus B$, so $x \notin A$ or $x \in B$. Note that $x \in A \triangle C$ but $x \notin B \triangle C$. Accordingly, $x \in (A \triangle C) \setminus (B \triangle C)$.

Since these cases are exhaustive, $x \in (A \triangle C) \setminus (B \triangle C)$.

Therefore: $(A \setminus B) \triangle C \subseteq (A \triangle C) \setminus (B \triangle C)$.

□

We've proven that $(A \setminus B) \triangle C = (A \triangle C) \setminus (B \triangle C)$.

Exercise 27:

The proof is not correct. It only proves that either $x < 6$ or $0 < x$, not that $0 < x < 6$. The proof can be fixed by proving $0 < x < 6$ in each case. In Case 1, $x \geq 3$ and so $x < 0$. In Case 2, $x < 3$ and so $x < 6$. Thus, the theorem is correct.

Exercise 28:

The proof is correct. The proof aims to prove a conditional statement so it first supposes the antecedent. It then uses existential instantiation. It also takes a disjunction, where one disjunct is known to be false to conclude the other disjunct.

Exercise 29:

The proof is correct. The proof uses existential and universal generalization, and uses exhaustive cases that come to the same conclusion to make that conclusion.

## Exercise 30:

Proof: Suppose that $\forall x P(x) \to \exists x Q(x)$. That is $\neg \forall x P(x)$ or $\exists x Q(x)$. Two cases.

Case 1. $\neg \forall x P(x)$. That is $\exists x \neg P(x)$, say $x_0$. Note then that $\neg P(x_0) \vee Q(x_0)$, or $P(x_0) \to Q(x_0)$. Accordingly, $\exists x (P(x) \to Q(x))$.

Case 2. $\exists x Q(x)$. Let that $x$ be $x_0$, and so $Q(x_0)$. Note then that $\neg P(x_0) \vee Q(x_0)$, or $P(x_0) \to Q(x_0)$. Accordingly, $\exists x (P(x) \to Q(x))$.

Since these cases are exhaustive, $\exists x (P(x) \to Q(x))$.

$\square$

## Exercise 31:

The proof is incorrect. In each case, it is incorrectly concluded that $\forall x \in A (x \in B)$ for Case 1 and $\forall x \in A (x \in C)$ for Case 2. The theorem is incorrect and thus the proof cannot be fixed. Consider $A = \{x, y\}$, $B = \{x\}$, and $C = \{y\}$. Note that $A \subseteq B \cup C$, but $A \not\subseteq B$ and $A \not\subseteq C$.

## Exercise 32:

Proof: Suppose $A$, $B$, and $C$ are sets and $A \subseteq B \cup C$. Consider arbitrary $x$ such that $x \in A$. Thus, $x \in B \cup C$. Two cases.

Case 1. $x \in B$. Then clearly, $A \subseteq B$.

Case 2. $x \in C$. Since $x \in A$ and $x \in C$, $x \in A \cap C$, and thus $A \cap C \neq \emptyset$.

Since these cases are exhaustive, either $A \subseteq B$ or $A \cap C \neq \emptyset$.

$\square$

## Exercise 33:

Proof: Assume the universe of discourse is not $\emptyset$. Two cases.

Case 1. $\forall x P(x)$. Note that this is equivalent to saying $\forall y P(y)$. Thus, $\neg P(x) \vee \forall y P(y)$, or $P(x) \to \forall y P(y)$ for all $x$. Thus, it is the case that $\exists x (P(x) \to \forall y P(y))$.

Case 2. $\neg \forall x P(x)$. That is, $\exists x \neg P(x)$, say $x_0$. Thus, $\neg P(x_0)$, and so $\neg P(x_0) \vee \forall y P(y)$, or $P(x_0) \to \forall y P(y)$. Therefore, $\exists x (P(x) \to \forall y P(y))$.

Since these cases are exhaustive, $\exists x (P(x) \to \forall y P(y))$.

$\square$