Section 3.7: More Examples of Proofs

Exercise 1:

Proof: Suppose $\mathcal{F}$ is a family of sets.

Existence: Let $A = \cup\mathcal{F}$. To prove property (a), let S be an arbitrary set such that $S \in \mathcal{F}$. Further, consider arbitrary x such that $x \in S$. Note that since $x \in S$ and $S \in \mathcal{F}$, $x \in \cup\mathcal{F} = A$. Because x is arbitrary, $S \subseteq A$; that is $S \in \mathcal{P}(A)$. Because S is arbitrary, $\mathcal{F} \subseteq \mathcal{P}(A)$.

To prove property (b), consider arbitrary set B and suppose that $\mathcal{F} \subseteq \mathcal{P}(B)$. Further, consider arbitrary x such that $x \in A$. That is, $x \in \cup\mathcal{F}$ or $\exists S \in \mathcal{F}(x \in S)$, say S_0. Thus, $S_0 \in \mathcal{F}$ and $x \in S_0$. Note, then, that $S_0 \in \mathcal{P}(B)$, or $S_0 \subseteq B$. Since $x \in S_0$, it follows that $x \in B$. Since x is arbitrary, it follows that $A \subseteq B$. Thus, $\mathcal{F} \subseteq \mathcal{P}(B) \rightarrow A \subseteq B$, and since B is arbitrary, $\forall B(\mathcal{F} \subseteq \mathcal{P}(B) \rightarrow A \subseteq B)$.

Uniqueness: Suppose A_1 and A_2 both have properties (a) and (b). In other words: $\mathcal{F} \subseteq \mathcal{P}(A_1)$ and $\mathcal{F} \subseteq \mathcal{P}(A_2)$, and $\forall B(\mathcal{F} \subseteq \mathcal{P}(B) \rightarrow A_1 \subseteq B)$ and $\forall B(\mathcal{F} \subseteq \mathcal{P}(B) \rightarrow A_2 \subseteq B)$. Thus: $\mathcal{F} \subseteq \mathcal{P}(A_2) \rightarrow A_1 \subseteq A_2$ and $\mathcal{F} \subseteq \mathcal{P}(A_1) \rightarrow A_2 \subseteq A_1$. Since both antecedents are true, $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$. Thus, $A_1 = A_2$. □

Exercise 2:

Proof:

Existence: Let $m=1$. To prove property (a), consider arbitrary positive real number x. Since x is positive, $\frac{x}{x+1}$ is always less than 1. Thus, $\frac{x}{x+1} < 1 = m$. Since x is arbitrary: $\forall x \in \mathbb{R}^+(\frac{x}{x+1} < m)$.

To prove property (b), consider any positive real number y and suppose that $\forall x \in \mathbb{R}^+(\frac{x}{x+1} < m)$, or $\forall x \in \mathbb{R}^+(\frac{x}{x+1} < y)$. Note that $\lim_{x\to\infty}\left(\frac{x}{x+1}\right) = 1$ and so 1 is the smallest positive real number that $\frac{x}{x+1}$ will never reach. Thus, $1 = m \leq 1$, and so $\forall x \in \mathbb{R}^+(\frac{x}{x+1} < m) \rightarrow m \leq y$. Since y is arbitrary: $\forall y \in \mathbb{R}^+(\forall x \in \mathbb{R}^+(\frac{x}{x+1} < m) \rightarrow m \leq y)$.

Uniqueness. Suppose m_1 and m_2 both have properties (a) and (b), and let m_1 and m_2 be any positive real numbers. Accordingly, it follows that: $\forall x \in \mathbb{R}^+(\frac{x}{x+1} < m_1)$ and $\forall x \in \mathbb{R}^+(\frac{x}{x+1} < m_2)$, and

$\forall y \in \mathbb{R}^+ \left(\forall x \in \mathbb{R}^+ \left(\frac{x}{x+1} < m_1\right) \rightarrow m_1 \leq y\right)$ and $\forall y \in \mathbb{R}^+ \left(\forall x \in \mathbb{R}^+ \left(\frac{x}{x+1} < m_2\right) \rightarrow m_2 \leq y\right)$. Thus, $\forall x \in \mathbb{R}^+ \left(\frac{x}{x+1} < m_1\right) \rightarrow m_1 \leq m_2$ and $\forall x \in \mathbb{R}^+ \left(\frac{x}{x+1} < m_2\right) \rightarrow m_2 \leq m_1$. Note that both antecedents are true, and so, $m_1 \leq m_2$ and $m_2 \leq m_1$. Therefore, $m_1 = m_2$.

$\square$

Exercise 3:

Theorem: If A and B are sets, $\mathcal{P}(A \setminus B) \setminus (\mathcal{P}(A) \setminus \mathcal{P}(B)) = \{\varnothing\}$.

Proof: Let A and B be arbitrary sets.

($\rightarrow$) Consider any S such that $S \in \mathcal{P}(A \setminus B) \setminus (\mathcal{P}(A) \setminus \mathcal{P}(B))$, and consider arbitrary x such that $x \in S$. Thus, $S \in \mathcal{P}(A \setminus B)$ and $S \notin \mathcal{P}(A) \setminus \mathcal{P}(B)$. So: $S \subseteq A \setminus B$ and $S \notin \mathcal{P}(A) \vee S \in \mathcal{P}(B)$, which means $S \nsubseteq A$ or $S \subseteq B$. Since $S \subseteq A \setminus B$, $x \in A \setminus B$, i.e., $x \in A$ and $x \notin B$. Two cases:

Case 1. $S \nsubseteq A$. Thus, there is some x_0 such that $x_0 \in S$ and $x_0 \notin A$. But $x_0 \in A$. Contradiction. Therefore, $S = \varnothing$ and so $S \in \{\varnothing\}$.

Case 2. $S \subseteq B$. Note that since $x \notin B$, $x \notin S$. But $x \in S$. Contradiction. Therefore, $S = \varnothing$ and so $S \in \{\varnothing\}$.

Since these cases are exhaustive, $S \in \{\varnothing\}$.

($\leftarrow$) Consider any S such that $S \in \{\varnothing\}$. Note, then, that $S = \varnothing$. Note that $\forall x (x \in S \rightarrow (x \in A \wedge x \notin B))$ is vacuously true, and so $\forall x (x \in S \rightarrow x \in A \setminus B)$; $S \subseteq A \setminus B$, and; $S \in \mathcal{P}(A \setminus B)$. Note also that $\forall x (x \in S \rightarrow x \in B)$ is also vacuously true, and so the following hold: $\exists x (x \in S \wedge x \notin A) \vee \forall x (x \in S \rightarrow x \in B)$; $S \nsubseteq A \vee S \subseteq B$; $S \notin \mathcal{P}(A) \vee S \in \mathcal{P}(B)$, and; $S \notin \mathcal{P}(A) \setminus \mathcal{P}(B)$. Thus, $S \in \mathcal{P}(A \setminus B) \setminus (\mathcal{P}(A) \setminus \mathcal{P}(B))$

Therefore: $\mathcal{P}(A \setminus B) \setminus (\mathcal{P}(A) \setminus \mathcal{P}(B)) = \{\varnothing\}$.

$\square$

Exercise 4:

Proof: Suppose A, B, and C are sets.

($a \rightarrow b$) Suppose $(A \triangle C) \cap (B \triangle C) = \varnothing$. Consider arbitrary x such that $x \in A \cap B$, that is $x \in A$ and $x \in B$. Note that there is no x such that $x \in (A \triangle C)$ and $x \in (B \triangle C)$, that is, there is no x such that $x \in A$ or $x \in C$ (but not both) and $x \in B$ or $x \in C$ (but not both).

Restated: $\forall x (x \in A \cap C \vee x \notin A \cup C \vee x \in B \cap C \vee x \notin B \cup C)$.

Because $x \in A$ and $x \in B$, either $x \in A \cap C$ or $x \in B \cap C$. Either way,

$x \in C$. Therefore, $A \cap B \subseteq C$. Now consider arbitrary y such that $y \in C$. It follows that $y \in A \cap C$ or $y \in B \cap C$. Thus, either $x \in A$ and $x \in C$ <u>or</u> $x \in B$ and $x \in C$. Either way, $x \in A \cup B$. Therefore $C \subseteq A \cup B$. Accordingly: $A \cap B \subseteq C \subseteq A \cup B$.

($b \rightarrow c$) Suppose $A \cap B \subseteq C \subseteq A \cup B$, and consider any x such that $x \in A \triangle C$, that is, $x \in A \leftrightarrow x \notin C$.

($\rightarrow$) Suppose that $x \notin A$. Then $x \in C$ and so $x \in A \cup B$. But $x \notin A$, so $x \in B$.

($\leftarrow$) Suppose that $x \in B$. Further suppose $x \in A$. Thus, $x \in A \cap B$ and so $x \in C$. It follows that $x \notin A \triangle C$. But $x \in A \triangle C$. Contradiction. Thus, $x \notin A$.

Therefore: $x \in A \triangle B$. Since x is arbitrary, $A \triangle C \subseteq A \triangle B$.

($c \rightarrow a$) Suppose $A \triangle C \subseteq A \triangle B$, and consider any x such that $x \in (A \triangle C) \cap (B \triangle C)$. That is, $x \in A \leftrightarrow x \notin C$ <u>and</u> $x \in B \leftrightarrow x \notin C$.

Two cases:

Case 1. $x \in C$. Thus, $x \notin A$ and $x \notin B$, and so $x \notin A \triangle B$. It follows that $x \notin A \triangle C$. Since $x \notin A$, $x \notin C$. Contradiction. Therefore: $(A \triangle C) \cap (B \triangle C) = \varnothing$.

Case 2. $x \notin C$. Thus, $x \in A$ and $x \in B$. Since $x \in A \triangle C$, $x \in A \triangle B$. But $x \notin A \triangle B$. Contradiction. Therefore: $(A \triangle C) \cap (B \triangle C) = \varnothing$.

Since these cases are exhaustive, $(A \triangle C) \cap (B \triangle C) = \varnothing$. □

<u>Exercise 5</u>:

<u>Proof</u>: Suppose $\{A_i \mid i \in I\}$ is a family of sets, and $\mathcal{P}(\bigcup_{i \in I} A_i) \subseteq \bigcup_{i \in I} \mathcal{P}(A_i)$. Note that $\bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} A_i$, and so $\bigcup_{i \in I} A_i \in \mathcal{P}(\bigcup_{i \in I} A_i)$. Then $\bigcup_{i \in I} A_i \in \bigcup_{i \in I} \mathcal{P}(A_i)$. Thus, there is some $i \in I$ such that $\bigcup_{i \in I} A_i \subseteq A_i$. Let j be arbitrary and $j \in I$. Note, then, that $A_j \subseteq \bigcup_{i \in I} A_i$ (Exercise 8, §3.3). Therefore $A_j \subseteq A_i$. □

<u>Exercise 6</u>:

For the following proofs: suppose $\mathcal{F}$ is a nonempty family of sets, let $I = \bigcup \mathcal{F}$ and $J = \bigcap \mathcal{F}$, suppose that $J \neq \varnothing$ and so for all $X \in \mathcal{F}$, $X \neq \varnothing$, and $I \neq \varnothing$, and $\{A_i \mid i \in I\}$ is an indexed family of sets.

(a) Proof: Consider arbitrary x. Note that:

$x \in \bigcup_{i\in I} A_i$ iff $\exists i \in I\,(x \in A_i)$

iff for some $i_0 \in I$, $x \in A_{i_0}$

iff for some $i_0 \in \bigcup \mathcal{F}$, $x \in A_{i_0}$

iff for some i_0, $\exists X \in \mathcal{F}\,(i_0 \in X)$ and $x \in A_{i_0}$

iff for some i_0 and X_0, $X_0 \in \mathcal{F}$, $i_0 \in X_0$, and $x \in A_{i_0}$.

iff for some X_0, $\exists i \in X_0\,(x \in A_i)$ and $X_0 \in \mathcal{F}$

iff $\exists X \in \mathcal{F}\,(\exists i \in X\,(x \in A_i))$

iff $\exists X \in \mathcal{F}\,(x \in \bigcup_{i \in X} A_i)$

iff $x \in \bigcup_{X \in \mathcal{F}} (\bigcup_{i \in X} A_i)$ □

(b) Proof: Consider arbitrary x. Note that:

$x \in \bigcap_{i \in I} A_i$ iff $\forall i \in I\,(x \in A_i)$

iff for all i, if $i \in I$, then $x \in A_i$

iff for all i, if $i \in \bigcup \mathcal{F}$, then $x \in A_i$

iff for all i, if $\exists X \in \mathcal{F}\,(i \in X)$, then $x \in A_i$

iff $\forall i\,(\exists X \in \mathcal{F}\,(i \in X) \rightarrow x \in A_i)$

iff $\forall i\,(\exists X\,(X \in \mathcal{F} \wedge i \in X) \rightarrow x \in A_i)$

iff $\forall i\,(\neg \exists X (X \in \mathcal{F} \wedge i \in X) \vee x \in A_i)$

iff $\forall i\,(\forall X (X \notin \mathcal{F} \vee i \notin X) \vee x \in A_i)$

iff $\forall i\,\forall X\,(X \notin \mathcal{F} \vee i \notin X \vee x \in A_i)$

iff $\forall X\,(X \notin \mathcal{F} \vee \forall i (i \notin X \vee x \in A_i))$

iff $\forall X\,(X \in \mathcal{F} \rightarrow \forall i\,(i \notin X \vee x \in A_i))$

iff $\forall X\,(X \in \mathcal{F} \rightarrow \forall i\,(i \in X \rightarrow x \in A_i))$

iff $\forall X\,(X \in \mathcal{F} \rightarrow \forall i \in X (x \in A_i))$

iff $\forall X \in \mathcal{F}\,(\forall i \in X\,(x \in A_i))$

iff $\forall X \in \mathcal{F}\,(x \in \bigcap_{i \in X} A_i)$

iff $x \in \bigcap_{X \in \mathcal{F}} (\bigcap_{i \in X} A_i)$ □

(c) Proof: Consider arbitrary x such that $x \in \bigcup_{i \in J} A_i$, that is, $\exists i \in J\,(x \in A_i)$, or $\exists i\,(i \in J \wedge x \in A_i)$. Then: $\exists i\,((i \in \bigcap \mathcal{F} \wedge x \in A_i)$, and so: $\exists i\,(\forall X \in \mathcal{F}\,(i \in X) \wedge x \in A_i)$, and $\exists i\,(\forall X\,(X \in \mathcal{F} \rightarrow i \in X) \wedge x \in A_i)$, say i_0. Therefore, $\forall X\,(X \in \mathcal{F} \rightarrow i_0 \in X) \wedge x \in A_{i_0}$, or $\forall X\,(X \in \mathcal{F} \rightarrow (i_0 \in X \wedge x \in A_{i_0}))$. Accordingly, $\forall X\,(X \in \mathcal{F} \rightarrow \exists i \in X\,(x \in A_i))$, and $\forall X \in \mathcal{F}\,(\exists i \in X\,(x \in A_i))$, and $\forall X \in \mathcal{F}\,(x \in \bigcup_{i \in X} A_i)$.

Therefore, $x \in \bigcap_{X \in \mathcal{F}} \left(\bigcup_{i \in X} A_i \right)$.

$\square$

Counterexample: Let $\mathcal{F} = \{\{1,2\},\{2,3\}\}$. Then $I = \{1,2,3\}$ and $J = \{2\}$. Now, let $A_1 = \{a,b,c\}$, $A_2 = \{b,c,d\}$, and $A_3 = \{c,d,a\}$. Finally, let $x = a$. Note that: $x \in A_1$ and $x \in A_3$, but $x \notin A_2$. Accordingly, $\forall X \in \mathcal{F}(\exists i \in X(x \in A_i))$, so $\forall X \in \mathcal{F}(x \in \bigcup_{i \in X} A_i)$, and so $x \in \bigcap_{X \in \mathcal{F}}(\bigcup_{i \in X} A_i)$. Also note that: $\forall i \in \cap\mathcal{F}(x \notin A_i)$, that is, $\neg \exists i \in \cap \mathcal{F}(x \in A_i)$, or $x \notin \bigcup_{i \in J} A_i$. Therefore: $\bigcap_{X \in \mathcal{F}}(\bigcup_{i \in X} A_i) \not\subseteq \bigcup_{i \in J} A_i$.

$\square$

(d) Proof: Consider arbitrary x such that $x \in \bigcup_{X \in \mathcal{F}}(\bigcap_{i \in X} A_i)$, that is, $\exists X \in \mathcal{F}(x \in \bigcap_{i \in X} A_i)$, or $\exists X \in \mathcal{F} \forall i \in X(x \in A_i)$, say X_0. So: $X_0 \in \mathcal{F} \wedge \forall i \in X_0 (x \in A_i)$. Consider arbitrary i and suppose $i \in \cap\mathcal{F}$, that is $\forall X \in \mathcal{F}(i \in X)$, and since $X_0 \in \mathcal{F}$, $i \in X_0$. Since $\forall i \in X_0 (x \in A_i)$, $x \in A_i$. Thus, $i \in \cap\mathcal{F} \rightarrow x \in A_i$, and since i is arbitrary, $\forall i(i \in \cap\mathcal{F} \rightarrow x \in A_i)$. Then, $\forall i \in \cap\mathcal{F}(x \in A_i)$ and thus, $x \in \bigcap_{i \in J} A_i$. Therefore: $\bigcup_{X \in \mathcal{F}}(\bigcap_{i \in X} A_i) \subseteq \bigcap_{i \in J} A_i$.

$\square$

Counterexample: Let $\mathcal{F} = \{\{1,2\},\{2,3\}\}$. Then $I = \{1,2,3\}$ and $J = \{2\}$. Now, let $A_1 = \{a,b,c\}$, $A_2 = \{g\}$ and $A_3 = \{d,e,f\}$. Finally, let $x = g$. Note that $x \in A_2$, but $x \notin A_1$ and $x \notin A_3$. Since $x \in A_2$, $\forall i \in J(x \in A_i)$, so $x \in \bigcap_{i \in J} A_i$. Since $x \notin A_1$ and $x \notin A_3$, $\forall X(X \in \mathcal{F} \rightarrow \exists i(i \in X \wedge x \notin A_i))$. Equivalently: $\neg \exists X(X \in \mathcal{F} \wedge \forall i(i \in X \rightarrow x \in A_i))$, so $\neg \exists X \in \mathcal{F}(x \in \bigcap_{i \in X} A_i)$ and thus, $x \notin \bigcup_{X \in \mathcal{F}}(\bigcap_{i \in X} A_i)$. Therefore $\bigcap_{i \in J} A_i \not\subseteq \bigcup_{X \in \mathcal{F}}(\bigcap_{i \in X} A_i)$.

$\square$

Exercise 7:

Proof: Suppose $\epsilon > 0$. Let $\delta = \frac{\epsilon}{3}$, which is also clearly positive. Let x be an arbitrary real number, and suppose that $0 < |x-2| < \delta$. Then:

$$\left| \frac{3x^2-12}{x-2} - 12 \right| = \left| \frac{3(x+2)(x-2)}{x-2} - 12 \right| = |3x+6-12| = |3x-6|$$
$$= 3|x-2| < 3\delta = 3\left(\frac{\epsilon}{3}\right) = \epsilon.$$

$\square$

Exercise 8:

Proof: Suppose $\lim_{x \to c} f(x) = L$ and $L > 0$. That is:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \left(0 < |x-c| < \delta \rightarrow |f(x) - L| < \epsilon\right)$$

Let $\epsilon = L$. Note that: $\exists \delta > 0 \forall x(0 < |x-c| < \delta \rightarrow |f(x) - L| < L)$, say δ_0. So: $\forall x(0 < |x-c| < \delta_0 \rightarrow |f(x) - L| < L)$. Consider arbitrary real number x

and suppose that $0 < |x-c| < \delta_0$. Thus, $|f(x)-L| < L$, or $-L < f(x)-L < L$. Accordingly, $0 < f(x) < 2L$. Since x is arbitrary, for all x, if $0<|x-c|<\delta_0$ then $f(x) > 0$. Also, there is some number $\delta > 0$ such that for all x, if $0 < |x-c| < \delta$ then $f(x) > 0$.

$\square$

Exercise 9:

Prop: Suppose that $\lim_{x\to c} f(x) = L$. That is:

$$\forall \epsilon > 0 \exists \delta > 0 \forall x \left(0 < |x-c| < \delta \to |f(x) - L| < \epsilon\right)$$

Suppose that ϵ is an arbitrary positive real number: $\exists \delta > 0 \forall x (0<|x-c|<\delta \to |f(x)-L| < \epsilon)$, say δ_0. That is: $\forall x(0<|x-c|<\delta_0 \to |f(x)-L|<\epsilon)$. Consider arbitrary x. Note that $0<|x-c|<\delta_0 \to |f(x)-L|<\epsilon$. Suppose that $0<|x-c|<\delta_0$. Then: $|f(x)-L| < \epsilon$, so $7|f(x)-L| < 7\epsilon$, and $|7f(x)-7L| < 7\epsilon$. Since ϵ is positive, $|7f(x)-7L| < \epsilon'$, where $\epsilon' = 7\epsilon$. Note that since x is arbitrary: $\forall x(0<|x-c|<\delta_0 \to |7f(x)-7L|<\epsilon')$. It follows that $\exists\delta>0\forall x(0<|x-c|<\delta_0 \to |7f(x)-7L|<\epsilon')$. Also note that since ϵ is arbitrary, so is ϵ'. Therefore: $\forall\epsilon>0\exists\delta>0\forall x(0<|x-c|<\delta_0 \to |7f(x)-7L|<\epsilon)$, and so if $\lim_{x\to c} f(x) = L$ then $\lim_{x\to c} 7f(x) = 7L$.

$\square$

Exercise 10:

The proof and theorem are correct. It argues by cases and existential instantiation.