Exercise 29:

Let $A=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$ and $B=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$. Note that $AB=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$.

Thus, it is possible to have $AB=I$ without B being the inverse of A.

Exercise 30:

Let A and B be square matrices of the same size. If A or B is not invertible, then AB is not invertible.

1.7: Diagonal, Triangular, and Symmetric Matrices

Exercise 1:

(a) Invertible $\begin{bmatrix}\frac{1}{2}&0\\0&-\frac{1}{5}\end{bmatrix}$ (b) Singular (c) Invertible $\begin{bmatrix}-1&0&0\\0&\frac{1}{2}&0\\0&0&3\end{bmatrix}$

Exercise 2:

(a) $\begin{bmatrix}6&3\\4&-1\\4&10\end{bmatrix}$ (b) $\begin{bmatrix}8&-2&6\\-1&-2&0\\-20&4&-8\end{bmatrix}\begin{bmatrix}-3&0&0\\0&5&0\\0&0&2\end{bmatrix}=\begin{bmatrix}-24&-10&12\\3&-10&0\\60&20&-16\end{bmatrix}$

Exercise 3:

(a) $A^2=\begin{bmatrix}1&0\\0&4\end{bmatrix}$ $A^{-2}=\begin{bmatrix}1&0\\0&\frac{1}{4}\end{bmatrix}$ $A^{-k}=\begin{bmatrix}1&0\\0&\frac{1}{(-2)^k}\end{bmatrix}$

(b) $A^2=\begin{bmatrix}\frac{1}{4}&0&0\\0&1/9&0\\0&0&1/16\end{bmatrix}$ $A^{-2}=\begin{bmatrix}4&0&0\\0&9&0\\0&0&16\end{bmatrix}$ $A^{-k}=\begin{bmatrix}2^k&0&0\\0&3^k&0\\0&0&4^k\end{bmatrix}$

Exercise 4:

Matrices (b) and (c) are symmetric.

Exercise 5:

(a) Invertible

(b) Singular

Exercise 6:

$$\begin{aligned} a-2b+2c&=3\\ 2a+b+c&=0\\ a+\quad\;\; c&=-2 \end{aligned} \Longrightarrow \begin{bmatrix} 1 & -2 & 2 & 3\\ 2 & 1 & 1 & 0\\ 1 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 11\\ 0 & 1 & 0 & -9\\ 0 & 0 & 1 & -13 \end{bmatrix}$$

Thus: $a=11$, $b=-9$, $c=-13$.

Exercise 7:

$$\begin{aligned} a+b-1=0 &\Rightarrow a+b=1\\ 2a-3b-7=0 &\Rightarrow 2a-3b=7 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 1 & 1\\ 2 & -3 & 7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2\\ 0 & 1 & -1 \end{bmatrix}$$

Thus: $a=2$, $b=-1$

Exercise 8:

ⓐ The matrices do not commute.

ⓑ The matrices commute.

Exercise 9:

$$\begin{bmatrix} 2 & 1\\ 1 & -5 \end{bmatrix}\begin{bmatrix} a & b\\ b & d \end{bmatrix} = \begin{bmatrix} 2a+b & 2b+d\\ a-5b & b-5d \end{bmatrix}$$

Note that $d=a-7b$. Thus: $\begin{bmatrix} 2a+b & 2b+a-7b\\ a-5b & b-5d \end{bmatrix}$

That is: $\begin{bmatrix} 2a+b & a-5b\\ a-5b & b-5d \end{bmatrix}$. Because the product of A and B is symmetric, A and B commute.

Exercise 10:

ⓐ $A=\begin{bmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1 \end{bmatrix}$

ⓑ $A=\begin{bmatrix} \pm\frac{1}{3} & 0 & 0\\ 0 & \pm\frac{1}{2} & 0\\ 0 & 0 & \pm 1 \end{bmatrix}$

Exercise 11:

ⓐ $B=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad D=\begin{bmatrix} 3 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 7 \end{bmatrix}$

ⓑ There are other possible factorizations. For instance: $B=\begin{bmatrix} -a_{11} & a_{12} & a_{13}\\ -a_{21} & a_{22} & a_{23}\\ -a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad D=\begin{bmatrix} -3 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 7 \end{bmatrix}$

Exercise 12:

$$AB = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} 2 & -8 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 12 & 17 \\ 0 & 2 & 10 \\ 0 & 0 & -12 \end{bmatrix}$$

The product is upper-triangular (as stated by Theorem 1.7.1(b)).

Exercise 13:

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -1 & 2 & 1/4 \\ 0 & 1 & 3/4 \\ 0 & 0 & -1/4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -8 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1/2 & 2 & -2/3 \\ 0 & 1/2 & -1/6 \\ 0 & 0 & 1/3 \end{bmatrix}$$

Both A^{-1} and B^{-1} are upper-triangular (as stated by Theorem 1.7.1(d)).

Exercise 14:

(a) $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ A is invertible. $A^T = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

Thus, $AA^T = A^TA = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix}$ This matrix is invertible.

(b) $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 1 & -7 \\ 3 & -7 & 4 \end{bmatrix}$ A is invertible. $A^T = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 1 & -7 \\ 3 & -7 & 4 \end{bmatrix}$

Thus, $AA^T = A^TA = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 1 & -7 \\ 3 & -7 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ -2 & 1 & -7 \\ 3 & -7 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -25 & 29 \\ -25 & 54 & -41 \\ 29 & -41 & 74 \end{bmatrix}$ This matrix is invertible.

Exercise 15:

(a) Proof: Let A be a symmetric matrix. Note that the factors A and A commute. Since the product of two symmetric matrices is symmetric iff the matrices commute, and A is a symmetric matrix, the product of AA is symmetric. Since $AA = A^2$, the product of A^2 is symmetric. □

(b) Proof: Let A be a symmetric matrix. Since A^2 is symmetric (see above), and kA is symmetric for any scalar k, $2A^2$ is symmetric (Theorem 1.7.2(c)). Similarly, $-3A$ is symmetric (Theorem 1.7.2(c)). Since $2A^2$ and $-3A$ are symmetric matrices of the same size, $2A^2 - 3A$ is symmetric (Theorem 1.7.2(b)). Similarly, $2A^2 - 3A + I$ is symmetric, since I is symmetric (Theorem 1.7.2(b)). □

Exercise 16:

ⓐ Proof: Let A be a symmetric matrix. Suppose that k is any nonnegative integer. Consider $(A^k)^T$. Note that $(A^k)^T = (AA\ldots A)^T$ with k factors of A. Since the transpose of a product of any number of matrices is equal to the product of their transposes in the reverse order (Theorem 1.4.9 (d)), $(A^k)^T = \underbrace{A^TA^T\cdots A^T}_{k\ \text{factors}}$. But since A is symmetric, $A = A^T$. Thus, $(A^k)^T = \overbrace{AA\cdots A}^{k\ \text{factors}} = A^k$. Therefore, A^k is symmetric for any nonnegative integer k.

$\square$

ⓑ Proof: Let A be a symmetric matrix, and let $p(x)$ be a polynomial. From Part (a), since A is symmetric, so is A^k for any nonnegative integer k. Note that $I = A^0$ is also symmetric. Next, if A is symmetric, and c is any real scalar, then cA is also symmetric (Theorem 1.7.2(c)). Also, if A and B are symmetric, then so is $A+B$. These facts allow us to conclude if $p(x)$ is any real polynomial, and A is symmetric, then $p(A)$ is a symmetric matrix.

$\square$

Exercise 17:

Proof: Let A be an upper triangular matrix and let $p(x)$ be a polynomial. Per Theorem 1.7.1 (b), since A is upper triangular, so is AA, or A^2. By induction, A^k is upper triangular for $k = 1, 2, 3, \ldots$ Note that $I = A^0$ is upper triangular. Next, if A is upper triangular, and c is any real scalar, then cA is also upper triangular. Also, if A and B are upper triangular, then so is $A+B$. These facts allow us to conclude if $p(x)$ is any real polynomial, and A is upper triangular, then $P(A)$ is an upper triangular matrix.

$\square$

Exercise 18:

Proof: Suppose $A^TA = A$. Note that $A^T = (A^TA)^T = A^TA = A$ (Theorem 1.4.9(d)). Thus, A is symmetric. Also, since $A^TA = A$ and $A^T = A$, we have $AA = A$, and therefore $A^2 = A$.

$\square$

Exercise 19:

Let $A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$. We have: $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}^2 - 3\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$

That is: $\begin{bmatrix} x^2 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & z^2 \end{bmatrix} - \begin{bmatrix} 3x & 0 & 0 \\ 0 & 3y & 0 \\ 0 & 0 & 3z \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 0$

Which gives us: $x^2 - 3x - 4 = 0$ or $(x-4)(x+1) = 0$, or $x = 4, -1$

$y^2 - 3y - 4 = 0$ or $(y-4)(y+1) = 0$, or $y = 4, -1$

$z^2 - 3z - 4 = 0$ or $(z-4)(z+1) = 0$, or $z = 4, -1$

Therefore there are 8 possibilities for x, y, z respectively: $(4,4,4)$, $(-1,4,4)$, $(4,-1,4)$, $(4,-1,-1)$, $(4,4,-1)$, $(-1,-1,4)$, $(-1,4,-1)$, $(-1,-1,-1)$.

Exercise 20:

(a) Yes (b) No (unless $n=1$) (c) Yes (d) No (unless $n=1$).

Exercise 21:

If i and j can be substituted for each other without changing the result of the equation, then $A = [a_{ij}]$ is symmetric.

Exercise 22:

(a) Proof: Suppose A is an invertible skew-symmetric matrix. Consider $(A^{-1})^T$. Note that $(A^{-1})^T = (A^T)^{-1}$ (Theorem 1.4.10). Since $A^T = -A$, $(A^{-1})^T = (A^T)^{-1} = -A^{-1}$. Thus, A^{-1} is skew-symmetric. □

(b) Proof: Suppose A and B are skew-symmetric matrices, and k is any scalar.

- A^T: Consider $(A^T)^T$. Since $A^T = -A$, $(A^T)^T = -A^T$, and thus A^T is skew-symmetric.
- $A+B$: Consider $(A+B)^T$. Note that $(A+B)^T = A^T + B^T$ (Theorem 1.4.9(b)). Since $A^T = -A$ and $B^T = -B$, $A^T + B^T = -A + -B$, or $-(A+B)$ (Theorem

1.4.1(h)). Thus, $A+B$ is skew-symmetric.

- $A-B$: Consider $(A-B)^T$. Note that $(A-B)^T = A^T - B^T$ (Theorem 1.4.9(b)). Since $A^T = -A$ and $B^T = -B$, $A^T - B^T = -A - -B$, or $-(A-B)$ (Theorem 1.4.1(i)). Thus, $A-B$ is skew-symmetric.
- kA: Consider $(kA)^T$. Note that $(kA)^T = kA^T$ (Theorem 1.4.9(c)). Since $A^T = -A$, $kA^T = k(-A)$, or $-kA$. Thus, kA is skew-symmetric.

□

(c) Proof: Let A be any square matrix. Note the identity $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$. Let $P = \frac{1}{2}(A+A^T)$ and let $Q = \frac{1}{2}(A-A^T)$.

- Consider P^T. $P^T = [\frac{1}{2}(A+A^T)]^T = (\frac{1}{2}A + \frac{1}{2}(A^T))^T$ {∵ Theorem 1.4.1(h)} $= (\frac{1}{2}A)^T + (\frac{1}{2}A^T)^T$ {∵ Theorem 1.4.9(b)} $= \frac{1}{2}A^T + \frac{1}{2}(A^T)^T$ {∵ Theorem 1.4.9(c)} $= \frac{1}{2}A^T + \frac{1}{2}A$ {∵ Theorem 1.4.9(a)} $= \frac{1}{2}(A+A^T)$ {∵ Theorem 1.4.1(h)} $= P$. Therefore, P is symmetric.
- Consider Q^T. $Q^T = [\frac{1}{2}(A-A^T)]^T = [\frac{1}{2}A - \frac{1}{2}(A^T)]^T$ {∵ Theorem 1.4.1(i)} $= (\frac{1}{2}A - \frac{1}{2}A^T)^T$ {∵ Theorem 1.4.9(c)} $= (\frac{1}{2}A)^T - (\frac{1}{2}A^T)^T$ {∵ Theorem 1.4.9(b)} $= \frac{1}{2}A^T - \frac{1}{2}A$ {∵ Theorem 1.4.9(a) and (c)} $= -\frac{1}{2}(A-A^T)$ {∵ Theorem 1.4.1(i)} $= -Q$

Therefore Q is skew-symmetric.

∴ A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

□

Exercise 23:

Proof: Let A and B be commuting skew-symmetric matrices. Consider $(AB)^T$. $(AB)^T = B^T A^T$ {∵ Theorem 1.4.9(d)}. $B^T A^T = (-B)(-A)$ {∵ A and B are skew-symmetric} $= BA = AB$ {∵ A and B commute}. Thus, $(AB)^T = AB$ and so, the product is symmetric, not skew-symmetric.

□

Exercise 24:

(a) Step 1: $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ ∴ $\begin{bmatrix} 1 & 0 & 0 & 1 \\ -2 & 3 & 0 & -2 \\ 2 & 4 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ ∴ $y = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Step 2: $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ ∴ $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 4 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 7/4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$ ∴ $x = \begin{bmatrix} 7/4 \\ 1 \\ -1/2 \end{bmatrix}$

ⓑ Step 1: $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & -2 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 2 \end{bmatrix}$ $\therefore \begin{bmatrix} 2 & 0 & 0 & 4 \\ 4 & 1 & 0 & -5 \\ -3 & -2 & 3 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -6 \end{bmatrix}$ $\therefore y = \begin{bmatrix} 2 \\ -13 \\ -6 \end{bmatrix}$

Step 2: $\begin{bmatrix} 3 & -5 & 2 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -13 \\ -6 \end{bmatrix}$ $\therefore \begin{bmatrix} 3 & -5 & 2 & 2 \\ 0 & 4 & 1 & -13 \\ 0 & 0 & 2 & -6 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & -5/2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ $\therefore x = \begin{bmatrix} -3/2 \\ -5/2 \\ -3 \end{bmatrix}$

Exercise 25:

Let $A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ $\therefore A^2 = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x^2 & xy + yz \\ 0 & z^2 \end{bmatrix} = \begin{bmatrix} x^2 & y(x+z) \\ 0 & z^2 \end{bmatrix}$

$A^3 = \begin{bmatrix} x^2 & y(x+z) \\ 0 & z^2 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x^3 & x^2y + yz(x+z) \\ 0 & z^3 \end{bmatrix} = \begin{bmatrix} 1 & 30 \\ 0 & -8 \end{bmatrix}$

$\therefore x^3 = 1$

$x^2y + yz(x+z) = y(x^2 + z(x+z)) = 30$

$z^3 = -8$

Thus, $x = 1$ and $z = -2$. $\therefore y(1 + -2(1-2)) = 30$

$\therefore 3y = 30$

$\therefore y = 10$ $\therefore A = \begin{bmatrix} 1 & 10 \\ 0 & -2 \end{bmatrix}$

Exercise 26:

Maximum number of entries: $n + (n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(n+1)}{2}$

Exercise 27:

Theorem: Let A and B be any diagonal matrices of the same size. For each i, $AB_{ii} = A_{ii} \times B_{ii}$. All other entries in AB are 0.

Exercise 28:

Let A be a square matrix, and D be a diagonal matrix such that $AD = I$. A must be a diagonal matrix with entries that are reciprocals of the corresponding entries in D.

Exercise 29:

(a) $v = 0$

$v + w = 0$

$v + w + x = 0$

$v + w + x + y = 0$

$v + w + x + y + z = 0$; $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

(b) Each successive row can be solved through substitution, starting at row 1 (which has only one unknown).

(c) We can call this procedure: "front-substitution"

Exercise 30:

(a) True. Contrapositive of Theorem 1.7.4 is: if AA^T is not invertible, then A is not invertible. That is, A is singular.

(b) False. Consider: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) True. If $Ax = 0$ has only the trivial solution, then A is invertible {∵ Theorem 1.6.4 (b)}. Thus, A^T is invertible {∵ Theorem 1.4.10}. Therefore $A^Tx = 0$ has only the trivial solution {∵ Theorem 1.6.4 (b) and (a)}.

(d) False. Consider $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$. Note that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, A^2 is symmetric but A is not symmetric.