1.4: Inverses; Rules of Matrix Arithmetic

Exercise 1:

ⓐ $A + \left(\begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}\right)$

$= \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 8 & -5 & -2 \\ 1 & 8 & 6 \\ 7 & -2 & 15 \end{bmatrix} = \begin{bmatrix} 10 & -6 & 1 \\ 1 & 12 & 11 \\ 5 & -1 & 19 \end{bmatrix}$

$\left(\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix}\right) + C$

$\begin{bmatrix} 10 & -4 & -2 \\ 0 & 5 & 7 \\ 2 & -6 & 10 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 10 & -6 & 1 \\ 1 & 12 & 11 \\ 5 & -1 & 19 \end{bmatrix}$

ⓑ $\left(\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix}\right) C$

$\begin{bmatrix} 28 & -28 & 6 \\ 20 & -31 & 38 \\ 0 & -21 & 36 \end{bmatrix}\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} = \begin{bmatrix} -10 & -222 & 26 \\ 83 & -67 & 278 \\ 87 & 33 & 240 \end{bmatrix}$

$A\left(\begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix}\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}\right)$

$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} -18 & -62 & -33 \\ 7 & 17 & 22 \\ 11 & -27 & 38 \end{bmatrix} = \begin{bmatrix} -10 & -222 & 26 \\ 83 & -67 & 278 \\ 87 & 33 & 240 \end{bmatrix}$

ⓒ $(4-7)\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} =$

$-3\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 6 & -9 \\ -3 & -21 & -12 \\ -9 & -15 & -27 \end{bmatrix}$

$4\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} + -7\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} =$

$\begin{bmatrix} 0 & -8 & 12 \\ 4 & 28 & 16 \\ 12 & 20 & 36 \end{bmatrix} + \begin{bmatrix} 0 & 14 & -21 \\ -7 & -49 & -28 \\ -21 & -35 & -63 \end{bmatrix} = \begin{bmatrix} 0 & 6 & -9 \\ -3 & -21 & -12 \\ -9 & -15 & -27 \end{bmatrix}$

ⓓ $4\left(\begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix} - \begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}\right)$

$4\begin{bmatrix} 8 & -1 & -8 \\ -1 & -6 & -2 \\ 1 & -12 & -3 \end{bmatrix} = \begin{bmatrix} 32 & -4 & -32 \\ -4 & -24 & -8 \\ 4 & -48 & -12 \end{bmatrix}$

$4\begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix} - 4\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} =$

$\begin{bmatrix} 32 & -12 & -20 \\ 0 & 4 & 8 \\ 16 & -28 & 24 \end{bmatrix} + \begin{bmatrix} 0 & 8 & -12 \\ -4 & -28 & -16 \\ -12 & -20 & -36 \end{bmatrix} = \begin{bmatrix} 32 & -4 & -32 \\ -4 & -24 & -8 \\ 4 & -48 & -12 \end{bmatrix}$

Exercise 2:

ⓐ $4\begin{bmatrix} -18 & -62 & -33 \\ 7 & 17 & 22 \\ 11 & -27 & 38 \end{bmatrix} =$

$\begin{bmatrix} -72 & -248 & -132 \\ 28 & 68 & 88 \\ 44 & -108 & 152 \end{bmatrix}$

$\left(4\begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix}\right) \times C =$

$\begin{bmatrix} 32 & -12 & -20 \\ 0 & 4 & 8 \\ 16 & -28 & 24 \end{bmatrix}\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} =$

$\begin{bmatrix} -72 & -248 & -132 \\ 28 & 68 & 88 \\ 44 & -108 & 152 \end{bmatrix}$

$B \times \left(4\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}\right) =$

$\begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix}\begin{bmatrix} 0 & -8 & 12 \\ 4 & 28 & 16 \\ 12 & 20 & 36 \end{bmatrix} =$

$\begin{bmatrix} -72 & -248 & -132 \\ 28 & 68 & 88 \\ 44 & -108 & 152 \end{bmatrix}$

ⓑ $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 8 & -1 & -8 \\ -1 & -6 & -2 \\ 1 & -12 & -3 \end{bmatrix} =$

$\begin{bmatrix} 20 & -32 & -23 \\ 1 & -84 & -23 \\ -13 & -52 & 2 \end{bmatrix}$

$\begin{bmatrix} 28 & -28 & 6 \\ 20 & -31 & 38 \\ 0 & -21 & 36 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}$

$\begin{bmatrix} 28 & -28 & 6 \\ 20 & -31 & 38 \\ 0 & -21 & 36 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & 4 & 29 \\ 19 & 53 & 61 \\ 13 & 31 & 34 \end{bmatrix} = \begin{bmatrix} 20 & -32 & -23 \\ 1 & -84 & -23 \\ -13 & -52 & 2 \end{bmatrix}$

ⓒ $\begin{bmatrix} 8 & -5 & -2 \\ 1 & 8 & 6 \\ 7 & -2 & 15 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix} =$

$\begin{bmatrix} 20 & -30 & -9 \\ -10 & 37 & 67 \\ -16 & 0 & 71 \end{bmatrix}$

$\begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix} =$

$\begin{bmatrix} 26 & -25 & -11 \\ -4 & 6 & 13 \\ -4 & -26 & 1 \end{bmatrix} + \begin{bmatrix} -6 & -5 & 2 \\ -6 & 31 & 54 \\ -12 & 26 & 70 \end{bmatrix} = \begin{bmatrix} 20 & -30 & -9 \\ -10 & 37 & 67 \\ -16 & 0 & 71 \end{bmatrix}$

ⓓ $4\left(-7\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}\right) =$

$4\begin{bmatrix} 0 & 14 & -21 \\ -7 & -49 & -28 \\ -21 & -35 & -63 \end{bmatrix} = \begin{bmatrix} 0 & 56 & -84 \\ -28 & -196 & -112 \\ -84 & -140 & -252 \end{bmatrix}$

$-28\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 56 & -84 \\ -28 & -196 & -112 \\ -84 & -140 & -252 \end{bmatrix}$

Exercise 3:

ⓐ $\begin{bmatrix} 2 & 0 & -2 \\ -1 & 4 & 1 \\ 3 & 5 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix} = A$

ⓑ $\begin{bmatrix} 10 & -4 & -2 \\ 0 & 5 & 7 \\ 2 & -6 & 10 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -2 \\ -1 & 4 & 1 \\ 3 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 4 \\ -3 & 1 & -7 \\ -5 & 2 & 6 \end{bmatrix}$

$\begin{bmatrix} 10 & 0 & 2 \\ -4 & 5 & -6 \\ -2 & 7 & 10 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ -4 & 5 & -6 \\ -2 & 7 & 10 \end{bmatrix}$

ⓒ $\left(4\begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}\right)^T = 4 \times \begin{bmatrix} 0 & 1 & 3 \\ -2 & 7 & 5 \\ 3 & 4 & 9 \end{bmatrix}$

$\begin{bmatrix} 0 & -8 & 12 \\ 4 & 28 & 16 \\ 12 & 20 & 36 \end{bmatrix}^T = \begin{bmatrix} 0 & 4 & 12 \\ -8 & 28 & 20 \\ 12 & 16 & 36 \end{bmatrix}$

$\begin{bmatrix} 0 & 4 & 12 \\ -8 & 28 & 20 \\ 12 & 16 & 36 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 12 \\ -8 & 28 & 20 \\ 12 & 16 & 36 \end{bmatrix}$

(d) $\begin{bmatrix} 28 & -28 & 6 \\ 20 & -31 & 38 \\ 0 & -21 & 36 \end{bmatrix}^T = \begin{bmatrix} 8 & 0 & 4 \\ -3 & 1 & -7 \\ -5 & 2 & 6 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ -1 & 4 & 1 \\ 3 & 5 & 4 \end{bmatrix}$

$\begin{bmatrix} 28 & 20 & 0 \\ -28 & -31 & -21 \\ 6 & 38 & 36 \end{bmatrix} = \begin{bmatrix} 28 & 20 & 0 \\ -28 & -31 & -21 \\ 6 & 38 & 36 \end{bmatrix}$

Exercise 4:

(a) $A^{-1} = \frac{1}{6-5} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

(b) $B^{-1} = \frac{1}{8+12} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 4 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}$

(c) $C^{-1} = \frac{1}{-6+8} \begin{bmatrix} -1 & -4 \\ 2 & 6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & -4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -1/2 & -2 \\ 1 & 3 \end{bmatrix}$

(d) $\frac{1}{6} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

Exercise 5:

(a) $(A^{-1})^{-1} = \frac{1}{6-5} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

(b) $\left(\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}\right)^{-1} = \left(\begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & 1/10 \end{bmatrix}\right)^T$

$\frac{1}{8+12} \begin{bmatrix} 4 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & -1/5 \\ 3/20 & 1/10 \end{bmatrix}$

$\begin{bmatrix} 1/5 & -1/5 \\ 3/20 & 1/10 \end{bmatrix} = \begin{bmatrix} 1/5 & -1/5 \\ 3/20 & 1/10 \end{bmatrix}$

Exercise 6:

(a) $\left(\begin{bmatrix} 10 & -5 \\ 18 & -7 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1/5 & 3/20 \\ -1/5 & 1/10 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

$\frac{1}{-70+90} \begin{bmatrix} -7 & 5 \\ -18 & 10 \end{bmatrix} = \begin{bmatrix} -7/20 & 1/4 \\ -9/10 & 1/2 \end{bmatrix}$

$\frac{1}{20} \begin{bmatrix} -7 & 5 \\ -18 & 10 \end{bmatrix} = \begin{bmatrix} -7/20 & 1/4 \\ -9/10 & 1/2 \end{bmatrix}$

$\begin{bmatrix} -7/20 & 1/4 \\ -9/10 & 1/2 \end{bmatrix} = \begin{bmatrix} -7/20 & 1/4 \\ -9/10 & 1/2 \end{bmatrix}$

(b) $\left(\begin{bmatrix} 70 & 45 \\ 122 & 79 \end{bmatrix}\right)^{-1} = \begin{bmatrix} -1/2 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1/5 & 3/20 \\ -1/5 & 1/10 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

$\frac{1}{5530-5490} \begin{bmatrix} 79 & -45 \\ -122 & 70 \end{bmatrix} = \begin{bmatrix} -1/2 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -7/20 & 1/4 \\ -9/10 & 1/2 \end{bmatrix}$

$\begin{bmatrix} 79/40 & -9/8 \\ -61/20 & 7/4 \end{bmatrix} = \begin{bmatrix} 79/40 & -9/8 \\ -61/20 & 7/4 \end{bmatrix}$

Exercise 7:

(a) $(A^{-1})^{-1} = \frac{1}{10+3} \begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5/13 & 1/13 \\ -3/13 & 2/13 \end{bmatrix} = A$

(b) $7A = \frac{1}{6-7} \begin{bmatrix} -2 & -7 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$

$A = \frac{1}{7} \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2/7 & 1 \\ 1/7 & 3/7 \end{bmatrix}$

(c) $5A^T = \frac{1}{-6+5}\begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}$

$A^T = \frac{1}{5}\begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} -2/5 & -1/5 \\ 1 & 3/5 \end{bmatrix}$

$A = \begin{bmatrix} -2/5 & 1 \\ -1/5 & 3/5 \end{bmatrix}$

(d) $I + 2A = \frac{1}{-5-8}\begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix} = \frac{1}{-13}\begin{bmatrix} 5 & -2 \\ -4 & -1 \end{bmatrix} = \begin{bmatrix} -5/13 & 2/13 \\ 4/13 & 1/13 \end{bmatrix}$

$2A = \begin{bmatrix} -5/13 & 2/13 \\ 4/13 & 1/13 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -18/13 & 2/13 \\ 4/13 & -12/13 \end{bmatrix}$

$A = \frac{1}{2}\begin{bmatrix} -18/13 & 2/13 \\ 4/13 & -12/13 \end{bmatrix} = \begin{bmatrix} -9/13 & 1/13 \\ 2/13 & -6/13 \end{bmatrix}$

Exercise 8:

$A^3 = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 28 & 1 \end{bmatrix}$

$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 \\ -7/2 & 1 \end{bmatrix}$

$A^2 - 2A + I = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$

Exercise 9:

(a) $p(A) = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

(b) $p(A) = 2\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2\begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 22 & 8 \\ 16 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 19 & 7 \\ 14 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 7 \\ 14 & 6 \end{bmatrix}$

(c) $p(A) = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix} - \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 35 & 13 \\ 26 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 39 & 13 \\ 26 & 13 \end{bmatrix}$

Exercise 10:

(a) $p_1(A) = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$p_1(A) = \begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$

$p_1(A) = \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix}$

$p_2(A) = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$p_2(A) = \begin{bmatrix} 6 & 1 \\ 2 & 4 \end{bmatrix}$

$p_2(A)p_3(A) = \begin{bmatrix} 6 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix}$

$p_3(A) = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$p_3(A) = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix}$

(b) Let A be any square matrix. Note that $p(x) = x^2 - 9$, and so $p_1(x) = (x+3)(x-3)$. Thus, $p_1(x) = p_2(x)\,p_3(x)$. Therefore, $p_1(A) = p_2(A)\,p_3(A)$.

Exercise 11:

$$\frac{1}{\cos^2\theta+\sin^2\theta}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

Exercise 12:

$$\frac{1}{\frac{1}{4}(e^x+e^{-x})^2-\frac{1}{4}(e^x-e^{-x})^2}\begin{bmatrix}\frac{1}{2}(e^x+e^{-x}) & -\frac{1}{2}(e^x-e^{-x})\\ -\frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x})\end{bmatrix}$$

$$=\frac{1}{\frac{1}{4}((e^x+e^{-x})^2-(e^x-e^{-x})^2)}\begin{bmatrix}\frac{1}{2}(e^x+e^{-x}) & -\frac{1}{2}(e^x-e^{-x})\\ -\frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x})\end{bmatrix}$$

$$=\frac{1}{\frac{1}{4}\left(e^{2x}+e^{-2x}+2e^xe^{-x}-e^{2x}-e^{-2x}+2e^xe^{-x}\right)}\begin{bmatrix}\frac{1}{2}(e^x+e^{-x}) & -\frac{1}{2}(e^x-e^{-x})\\ -\frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x+e^{-x})\end{bmatrix}$$

$$=\frac{1}{\frac{1}{4}\left(4e^xe^{-x}\right)}\begin{bmatrix}\frac{1}{2}(e^x+e^{-x}) & -\frac{1}{2}(e^x-e^{-x})\\ -\frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x-e^{-x})\end{bmatrix}$$

$$=\frac{1}{e^xe^{-x}}\begin{bmatrix}\frac{1}{2}(e^x+e^{-x}) & -\frac{1}{2}(e^x-e^{-x})\\ -\frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x-e^{-x})\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(e^x+e^{-x}) & -\frac{1}{2}(e^x-e^{-x})\\ -\frac{1}{2}(e^x-e^{-x}) & \frac{1}{2}(e^x-e^{-x})\end{bmatrix}$$

Exercise 13:

$$A^{-1}=\begin{bmatrix}\frac{1}{a_{11}} & 0 & \cdots & 0\\ 0 & \frac{1}{a_{22}} & \cdots & 0\\ \vdots & & & \vdots\\ 0 & 0 & \cdots & \frac{1}{a_{nn}}\end{bmatrix}$$ Note that $AA^{-1}=A^{-1}A=\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}$

Exercise 14:

$A^2-3A+I=0$

$I=3A-A^2$

$A^{-1}I=3AA^{-1}-A^2A^{-1}$

$A^{-1}=3I-A$

Exercise 15:

ⓐ Let A be any matrix with a row of zeros. For any matrix B, note that AB has a row of zeros. Thus, $AB \neq I$. Thus, A is not invertible.

(b) Let A be any matrix with a column of zeros. For any matrix B, note that BA has a column of zeros. Thus $BA \neq I$. Thus, A is not invertible.

Exercise 16:

No. Consider for, example the matrices: $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$. Their sum is not invertible.

Exercise 17:

Suppose that A is invertible. Thus, $A \neq 0$ because 0 is not invertible, and so A has at least one non-zero entry (i.e., a_{mn}). Futher suppose that $B \neq 0$. For similar reasons, B has one non-zero entry (i.e., b_{ij}). Note that AB has at least one non-zero entry (at row m and column j of AB). Thus, $AB \neq 0$. But $AB = 0$. Therefore, $B = 0$.

Exercise 18:

$$BA^{-1} + AC = 0$$
$$AC = -BA^{-1}$$
$$C = -A^{-1}BA^{-1}$$

Exercise 19:

(a) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$
$$-A^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix}$$
$$-A^{-1}B = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}$$
$$-A^{-1}BA^{-1} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ -1 & 0 & 1/2 & 1/2 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$
$$-A^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$-A^{-1} \cdot B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$-A^{-1}BA^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 20:

ⓐ $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $A^T=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$. Thus $A=A^T$

ⓑ $A=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}$ $-A=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$ $A^T=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}$. Thus, $-A=A^T$

Exercise 21:

ⓐ $A=BB^T=(B^T)^TB^T=(BB^T)^T=A^T$

$A=B+B^T=(B^T)^T+B^T=(B^T+B)^T=(B+B^T)^T=A^T$

ⓑ Let $A=B-B^T$.

Thus, $A^T=(B-B^T)^T=B^T-(B^T)^T=B^T-B=-(B-B^T)=-A$

Exercise 22: $(AB)^T=B^TA^T$

$(A^n)^T=(A^{n-1}A)^T=A^T(A^{n-1})^T=A^T(A^{n-2}A)^T=A^TA^T(A^{n-2})^T=\dots=(A^T)^n$

Exercise 23:

$$\begin{bmatrix}1&0&1\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}x_{11}&x_{12}&x_{13}\\x_{21}&x_{22}&x_{23}\\x_{31}&x_{32}&x_{33}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$x_{11}+x_{31}=1$

$x_{12}+x_{32}=0$

$x_{13}+x_{33}=0$

$x_{11}+x_{21}=0$

$x_{12}+x_{22}=1$

$x_{13}+x_{23}=0$

$x_{21}+x_{31}=0$

$x_{22}+x_{32}=0$

$x_{23}+x_{33}=1$

Solving the above system of equations gives the matrix, $A^{-1}=\begin{bmatrix}1/2&1/2&-1/2\\-1/2&1/2&1/2\\1/2&-1/2&1/2\end{bmatrix}$

Exercise 24:

(a) Proof (b): $A+(B+C)=(A+B)+C$

- Same size: To form $B+C$, the matrices B and C must have the same size, let us say, $m\times n$. A must also be of size $m\times n$ to form $A+(B+C)$. Since A, B, and C all are of size $m\times n$, $(A+B)+C$ is also of size $m\times n$.
- Same entries: Suppose that $A=[a_{ij}]$, $B=[b_{ij}]$, and $C=[c_{ij}]$. We want to show that corresponding entries of $A+(B+C)$ and $(A+B)+C$ are equal. That is, $[A+(B+C)]_{ij}=[(A+B)+C]_{ij}$ for all values of i and j. From the definition of matrix addition we have:

$$\begin{aligned}[A+(B+C)]_{ij} &= a_{ij}+(b_{ij}+c_{ij})\\ &=(a_{ij}+b_{ij})+c_{ij}\\ &=[(A+B)+C]_{ij}\end{aligned}$$

Thus, the matrices are equivalent. $\square$

(b) Proof (i): $a(B-C)=aB-aC$

- Same size: To form $B-C$, the matrices B and C must have the same size, let us say $m\times n$. Since a is a scalar, it does not change the matrix's size, and so $a(B-C)$ is of size $m\times n$. Since B is of size $m\times n$, aB is of size $m\times n$. Since C is of size $m\times n$, aC is of size $m\times n$. Thus, $aB-aC$ is of size $m\times n$.
- Same entries: Suppose that $B=[b_{ij}]$ and $C=[c_{ij}]$. We want to show that corresponding entries of $a(B-C)$ and $aB-aC$ are equal. That is, $[a(B-C)]_{ij}=[aB-aC]_{ij}$ for all values of i and j. From the definition of matrix subtraction we have:

$$\begin{aligned}[a(B-C)]_{ij} &= a(b_{ij}-c_{ij})\\ &= ab_{ij}-ac_{ij}\\ &=[aB-aC]_{ij}\end{aligned}$$

Thus, the matrices are equivalent. $\square$

(c) Proof (m): $a(BC) = (aB)C = B(aC)$.

- Same size: To form BC, B must have the same number of columns as the number of C's rows. Let B be of size $m \times n$ and C be of size $n \times p$. Thus, BC is of size $m \times p$, and $a(BC)$ is of size $m \times p$. Note that aB is of size $m \times n$ and so, $(aB)C$ is of size $m \times p$. Also, aC is of size $n \times p$, so $B(aC)$ is of size $m \times p$.
- Same entries: Suppose that $B = [b_{ij}]$ and $C = [c_{ij}]$. We want to show that corresponding entries of $a(BC)$, $(aB)C$, and $B(aC)$ are equal. That is, $[a(BC)]_{ij} = [(aB)C]_{ij} = [B(aC)]_{ij}$ for all values i and j. From the definition of matrix multiplication we have

$$[a(BC)]_{ij} = a(b_{i1}c_{1j} + b_{i2}c_{2j} + \cdots + b_{in}c_{nj})$$
$$= ab_{i1}c_{1j} + ab_{i2}c_{2j} + \cdots + ab_{in}c_{nj}$$
$$[(aB)C]_{ij} = ab_{i1}(c_{1j}) + ab_{i2}(c_{2j}) + \cdots + ab_{in}(c_{nj})$$
$$= ab_{i1}c_{1j} + ab_{i2}c_{2j} + \cdots + ab_{in}c_{nj}.$$
$$[B(aC)]_{ij} = b_{i1}(ac_{1j}) + b_{i2}(ac_{2j}) + \cdots + b_{in}(ac_{nj})$$
$$= ab_{i1}c_{1j} + ab_{i2}c_{2j} + \cdots + ab_{in}c_{nj}.$$

Thus, the matrices are equivalent.

□

Exercise 25:

Proof (f): $A(B-C) = AB - AC$.

By (d), $A(B+(-C)) = AB + A(-C)$. By (f), $A(-C) = -AC$.
Thus, $A(B-C) = AB - AC$.

□

Exercise 26:

(a) Proof: $A + O = O + A = A$

- Same Size: To form $A+O$, A and O must be of the same size, let's say $m \times n$. Thus, $A+O$ is of size $m \times n$. Similarly, $O + A$ is of size $m \times n$. As stated above, A is of size $m \times n$.
- Same entries: Suppose that $A = [a_{ij}]$. We want to show that corresponding entries of $A+O$, $O+A$, and A are equal. That is, $[A+O]_{ij} = [O+A]_{ij} = [A]_{ij}$ for all values of i and j.

From the definition of matrix addition we have $[A+O]_{ij} = a_{ij} + 0 = a_{ij} = [A]_{ij}$

$$[O+A]_{ij} = 0 + a_{ij} = a_{ij} = [A]_{ij}$$

Thus the matrices are equivalent. $\square$

(b) Proof: $A - A = O$

- Same size: A (and thus, $A-A$) and O are the same size because it is assumed that O is a matrix of the same size as A.
- Same entries: Suppose that $A = [a_{ij}]$. We want to show that corresponding entries of $A-A$ and O are equal. That is $[A-A]_{ij} = [O]_{ij}$ for all values of i and j. From the definition of matrix subtraction we have

$$\begin{aligned}[A-A]_{ij} &= a_{ij} - a_{ij} \\ &= 0 \\ &= [O]_{ij}\end{aligned}$$

Thus the matrices are equivalent. $\square$

(c) Proof: $O - A = -A$

- Same size: To form $O-A$, O and A must be the same size, let's say $m \times n$. Thus, $O-A$ is of size $m \times n$. $-A$ is also of size $m \times n$, because A is of size $m \times n$.
- Same entries: Suppose that $A = [a_{ij}]$. We want to show that corresponding entries of $O-A$ and $-A$ are equal. That is $[O-A]_{ij} = [-A]_{ij}$ for all values of i and j. From the definition of matrix subtraction we have

$$\begin{aligned}[O-A]_{ij} &= 0 - a_{ij} \\ &= -a_{ij} \\ &= [-A]_{ij}\end{aligned}$$

Thus the matrices are equivalent. $\square$

(d) Proof: $AO = O$; $OA = O$

- Same size: To form AO, A must have the same number of columns as the number of rows in O. Let's say A is of size $m \times n$ and O is of size $n \times p$. So, AO is of size $m \times p$. The matrix O on the right is of size $m \times p$ because O is just a notation for a zero matrix of any size. For these same reasons, the second equation matrices are also of the same size.

- Same entries: Suppose that $A = [a_{ij}]$. We want to show that corresponding entries of AO and O are equal. That is $[AO]_{ij} = [O]_{ij}$. From the definition of matrix multiplication we have

$$\begin{aligned}[AO]_{ij} &= a_{i1}O_{1j} + a_{i2}O_{2j} + \cdots + a_{in}O_{nj} \\ &= a_{i1}(0) + a_{i2}(0) + \cdots + a_{in}(0) \\ &= 0 \\ &= [O]_{ij}\end{aligned}$$

Similarly

$$\begin{aligned}[OA]_{ij} &= O_{i1}a_{1j} + O_{i2}a_{2j} + \cdots + O_{in}a_{nj} \\ &= 0(a_{1j}) + 0(a_{2j}) + \cdots + 0(a_{nj}) \\ &= 0 \\ &= [O]_{ij}\end{aligned}$$

Thus, the matrices are equivalent. $\square$

Exercise 27:

(a) Proof: Let A be any square matrix, and r and s be any nonnegative integers.

- Same size: Let A be any size (i.e., $n \times n$). Note that multiplying an $n \times n$ matrix by another $n \times n$ matrix always yields an $n \times n$ matrix. Thus, A to any power yields an $n \times n$ matrix. Thus, $A^r A^s$ yields an $n \times n$ matrix, and A^{r+s} yields an $n \times n$ matrix. Similarly, A^r is $n \times n$, and so $(A^r)^s$ is also $n \times n$. A^{rs} is $n \times n$ because it is A to some power.
- Same entries:
  - $A^r A^s = (\underbrace{AA\cdots A}_{r \text{ factors}})(\underbrace{AA\cdots A}_{s \text{ factors}}) = \underbrace{AA\cdots A}_{r+s \text{ factors}} = A^{r+s}.$
  - $(A^r)^s = \underbrace{A^r A^r \cdots A^r}_{s \text{ factors}} = \underbrace{(\underbrace{AA\cdots A}_{r \text{ factors}})(\underbrace{AA\cdots A}_{r \text{ factors}})\cdots(\underbrace{AA\cdots A}_{r \text{ factors}})}_{s \text{ factors}} = (\underbrace{AA\cdots A}_{r \times s \text{ factors}}) = A^{r \cdot s}.$

$\square$

(b) Proof: Let A be any invertible square matrix, and $r < 0$ and $s < 0$. Let $\alpha = -r$ and $\beta = -s$.

- $A^r A^s = A^{-\alpha} A^{-\beta} = (A^{-1})^{\alpha}(A^{-1})^{\beta} = (A^{-1})^{\alpha+\beta} = A^{-(\alpha+\beta)} = A^{-(-r+-s)} = A^{r+s}$
- $(A^r)^s = (A^{-\alpha})^{-\beta} = ((A^{-\alpha})^{-1})^{\beta} = (((A^{-1})^{\alpha})^{-1})^{\beta} = (((A^{-1})^{-1})^{\alpha})^{\beta} = (A^{\alpha})^{\beta} = A^{\alpha \times \beta} = A^{-r \times -s} = A^{r \times s}$

$\square$

Exercise 28:

Suppose that A is invertible and K is a nonzero scalar.

$n \geq 0$ • $(kA)^n = \underbrace{(kA)(kA)\cdots(kA)}_{n \text{ factors}} = \underbrace{(kk\cdots k)}_{n \text{ factors}}\underbrace{(AA\cdots A)}_{n \text{ factors}} = K^n A^n$

$n < 0$ • Let $\alpha = -n$.

$(kA)^n = (kA)^{-\alpha} = ((kA)^{-1})^{\alpha} = \underbrace{(kA)^{-1}(kA)^{-1}\cdots(kA)^{-1}}_{\alpha \text{ factors}} = \underbrace{(\frac{1}{k}A^{-1})(\frac{1}{k}A^{-1})\cdots(\frac{1}{k}A^{-1})}_{\alpha \text{ factors}} = (\frac{1}{k}A^{-1})^{\alpha}$

$= (k^{-1}A^{-1})^{\alpha} = k^{-\alpha}A^{-\alpha} = k^n A^n$

$\square$

Exercise 29:

ⓐ Suppose that A is invertible and $AB = AC$.

Note that if: $AB = AC$

$$A^{-1}(AB) = A^{-1}(AC)$$
$$(A^{-1}A)B = (A^{-1}A)C$$
$$IB = IC$$
$$B = C$$

ⓑ The matrix A in Exercise 3 is not invertible.

Exercise 30:

Proof: $A(BC) = (AB)C$

• Same Size: To form $A(BC)$, B must have the same number of columns as the number of rows in C. Thus, let B be of size $n \times p$ and C be of size $p \times q$. By similar logic, let A be of size $m \times n$. Thus, $A(BC)$ is of size $m \times q$. $(AB)C$ is also of size $m \times q$.

• Same entries: Let $l_{ij} = a_{i1}[BC]_{1j} + a_{i2}[BC]_{2j} + \cdots + a_{in}[BC]_{nj}$ and let $r_{ij} = [AB]_{i1}c_{1j} + [AB]_{i2}c_{2j} + \cdots + [AB]_{ip}c_{pj}$. We must show that $l_{ij} = r_{ij}$.

Note that: $l_{ij} = a_{i1}(b_{11}c_{1j} + b_{12}c_{2j} + \cdots + b_{1p}c_{pj})$
$+ a_{i2}(b_{21}c_{1j} + b_{22}c_{2j} + \cdots + b_{2p}c_{pj})$
$+ \cdots$
$+ a_{in}(b_{n1}c_{1j} + b_{n2}c_{2j} + \cdots + b_{np}c_{pj})$

$$
\begin{aligned}
&= a_{i1}b_{11}c_{1j} + a_{i1}b_{12}c_{2j} + \cdots + a_{i1}b_{1p}c_{pj} \\
&+ a_{i2}b_{21}c_{1j} + a_{i2}b_{22}c_{2j} + \cdots + a_{i2}b_{2p}c_{pj} \\
&+ \cdots + a_{in}b_{n1}c_{1j} + a_{in}b_{n2}c_{2j} + \cdots + a_{in}b_{np}c_{pj} \\
&= (a_{i1}b_{11})c_{1j} + (a_{i1}b_{12})c_{2j} + \cdots + (a_{i1}b_{1p})c_{pj} \\
&+ (a_{i2}b_{21})c_{1j} + (a_{i2}b_{22})c_{2j} + \cdots + (a_{i2}b_{2p})c_{pj} \\
&+ \cdots + (a_{in}b_{n1})c_{1j} + (a_{in}b_{n2})c_{2j} + \cdots + (a_{in}b_{np})c_{pj}. \\
&= (a_{i1}b_{11})c_{1j} + (a_{i2}b_{21})c_{1j} + \cdots + (a_{in}b_{n1})c_{1j} \\
&+ (a_{i1}b_{12})c_{2j} + (a_{i2}b_{22})c_{2j} + \cdots + (a_{in}b_{n2})c_{2j} \\
&+ \cdots + (a_{i1}b_{1p})c_{pj} + (a_{i2}b_{2p})c_{pj} + \cdots + (a_{in}b_{np})c_{pj}. \\
&= (a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{in}b_{n1})c_{1j} \\
&+ (a_{i1}b_{12} + a_{i2}b_{22} + \cdots + a_{in}b_{n2})c_{2j} \\
&+ \cdots + (a_{i1}b_{1p} + a_{i2}b_{2p} + \cdots + a_{in}b_{np})c_{pj} \\
&= [AB]_{i1}c_{1j} + [AB]_{i2}c_{2j} + \cdots + [AB]_{ip}c_{pj} = r_{ij}.
\end{aligned}
$$

$\square$

Exercise 31:

ⓐ Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

Note: $A + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; $(A+B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Also: $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $2AB = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$; $B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Thus, $A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$(A+B)^2 \neq A^2 + 2AB + B^2$

ⓑ $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$

Exercise 32:

ⓐ Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

Note: $A + B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$; $A - B = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$; $(A+B)(A-B) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$

Also: $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$; $A^2 - B^2 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$

$(A+B)(A-B) \neq A^2 - B^2$

(b) $(A+B)(A-B) = A^2 - AB + BA - B^2$

Exercise 33:

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$, $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$, $\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$, $\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$, $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$

Exercise 34:

(a) If A is not singular, then A^T is not singular.
Thus, if A is invertible then A^T is invertible.

(b) Suppose A is invertible. So, $AA^{-1} = A^{-1}A = I$
We must show that $A^T(A^{-1})^T = (A^{-1})^T A^T = I$
Note that: $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$
$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$

Thus, the statement is true.

Exercise 35:

(a) Sometimes False. $(AB)^2 = (AB)(AB) = ABAB \neq A^2B^2$

(b) $(A-B)^2 = (A-B)(A-B) = A^2 - AB - BA + B^2$
$(B-A)^2 = (B-A)(B-A) = B^2 - BA - AB + A^2$ } Always true

(c) $(AB^{-1})(BA^{-1}) = A(B^{-1}B)A^{-1} = A I_n A^{-1} = AA^{-1} = I_n \Rightarrow$ Always true

(d) Sometimes false. E.g., when $A = B$.

$\boxed{\text{1.5: Elementary Matrices and a Method For Finding } A^{-1}}$

Exercise 1:

(a) Yes (b) No (c) Yes (d) Yes (e) No (f) Yes (g) No.

Exercise 2:

(a) $3R_1$ added to R_2 (b) $\frac{1}{3} \times R_3$ (c) $R_1 \leftrightarrow R_4$ (d) $\frac{1}{7}R_3$ added to R_1