# Contribution of QED to Rational Terms in 1-loop Feynman Diagrams in the Standard Model

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**Abstract**

The abstract goes here.

***Keywords*** QFT · 1-loop Feynman Diagrams · Rational Terms · More

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The introduction goes here.

e.g. in reference [1] and [2].

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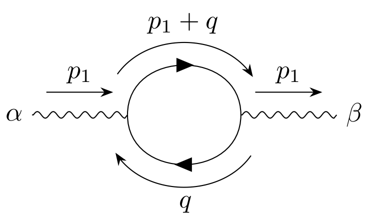



$$= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ ie\gamma^\alpha \frac{i\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} ie\gamma^\beta \frac{i\left(\not{q} + m\right)}{q^2 - m^2} \right\}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

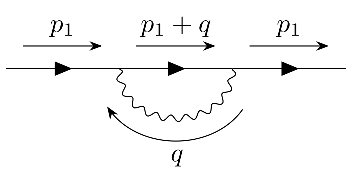
From now on bared quantities are $d$-dimensional, the quantities with a tilde $\epsilon$-dimensional and the normal momenta and gamma matrices 4-dimensional.

$$\bar{N}(\bar{q}) = -e^2 \operatorname{Tr}\left\{\bar{\gamma}^\alpha \left(\not{\bar{p}}_1 + \not{\bar{q}} + m\right) \bar{\gamma}^\beta \left(\not{\bar{q}} + m\right)\right\} = -e^2 \operatorname{Tr}\left\{\gamma^\alpha \left(\not{p}_1 + \not{q} + m\right) \gamma^\beta \left(\not{q} + m\right) + \gamma^\alpha \tilde{\not{q}} \gamma^\beta \tilde{\not{q}}\right\} \equiv N + \tilde{N}$$

$$\tilde{N} = -e^2 \operatorname{Tr}\left\{\gamma^\alpha \tilde{\not{q}} \gamma^\beta \tilde{\not{q}}\right\} \overset{\{\gamma^\mu, \tilde{\gamma}^\nu\}=0}{=} 4e^2 \tilde{q}^2 g^{\alpha\beta}$$

$$R_2^{\gamma\gamma} = \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{4e^2}{16\pi^4} \underbrace{\int d^d\bar{q} \frac{\tilde{q}^2}{\bar{D}_1 \bar{D}_0}}_{-i\frac{\pi}{2}\left(m^2 - p_1^2/3\right)} = \frac{-ie^2}{8\pi^2} g^{\alpha\beta} \left(2m^2 - \frac{p_1^2}{3}\right) \tag{2.1}$$

### Electron self-energy




$$= \int \frac{d^d q}{(2\pi)^d} ie\gamma^\alpha \frac{i\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} ie\gamma^\beta \frac{-ig_{\alpha\beta}}{q^2} = \int \frac{d^d q}{(2\pi)^d} \left(-e^2\right) \gamma^\alpha \frac{\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} \gamma_\alpha \frac{1}{q^2}$$

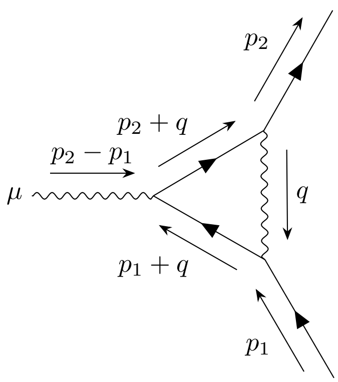
$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = \left(-e^2\right) \bar{\gamma}^\alpha \left(\not{\bar{p}}_1 + \not{\bar{q}} + m\right) \bar{\gamma}_\alpha = -e^2 \left\{\gamma^\alpha \left(\not{\bar{p}}_1 + \not{\bar{q}} + m\right) \gamma_\alpha + \tilde{\gamma}^\alpha \left(\not{\bar{p}}_1 + \not{\bar{q}} + m\right) \tilde{\gamma}_\alpha + \gamma^\alpha \tilde{\not{q}} \gamma_\alpha + \tilde{\gamma}^\alpha \tilde{\not{q}} \tilde{\gamma}_\alpha\right\} \equiv N + \tilde{N}$$

$$\tilde{N} = -e^2 \left\{\tilde{\gamma}^\alpha \left(\not{\bar{p}}_1 + \not{\bar{q}} + m\right) \tilde{\gamma}_\alpha + \gamma^\alpha \tilde{\not{q}} \gamma_\alpha + \tilde{\gamma}^\alpha \tilde{\not{q}} \tilde{\gamma}_\alpha\right\} = -e^2 \left\{ -\underbrace{\tilde{\gamma}^\alpha \tilde{\gamma}_\alpha}_{=\epsilon} \left(\not{\bar{p}}_1 + \not{\bar{q}} - m\right) - \underbrace{\gamma^\alpha \gamma_\alpha}_{=4} \tilde{\not{q}} + \tilde{\gamma}^\alpha \tilde{\not{q}} \tilde{\gamma}_\alpha \right\}$$

$$
\begin{aligned}
R_2^{\mathrm{ee}} &= \frac{1}{(2\pi)^4}\int d^d\bar{q}\,\frac{\tilde{N}}{\bar{D}_1\bar{D}_0} = \frac{-e^2}{(2\pi)^4}\int d^d\bar{q}\,\frac{1}{\bar{D}_1\bar{D}_0}\left(-\epsilon\left(\not{p}_1+\not{q}-m\right)+\underbrace{\tilde{\not{q}}\,(\ldots)}_{=0}\right) = \\
&= \frac{e^2}{(2\pi)^4}\left\{\underbrace{\int d^d\bar{q}\,\frac{\epsilon\left(\not{p}_1-m\right)}{\bar{D}_1\bar{D}_0}}_{=-2\epsilon\frac{i\pi^2}{\epsilon}\left(\not{p}_1-m\right)}+\underbrace{\int d^d\bar{q}\,\frac{\epsilon\not{q}}{\bar{D}_1\bar{D}_0}}_{=\epsilon\frac{i\pi^2}{\epsilon}\not{p}_1}\right\} = \frac{e^2}{(2\pi)^4}\epsilon\frac{i\pi^2}{\epsilon}\left((-2)\left(\not{p}_1-m\right)+\not{p}_1\right) = \frac{-ie^2}{16\pi^2}\left(\not{p}_1-2m\right)
\end{aligned}
\tag{2.2}
$$

### 2.2 3-point function




$$
= \int \frac{d^dq}{(2\pi)^d}\, ie\gamma^\beta \frac{i\left(\not{p}_1+\not{q}+m\right)}{(p_1+q)^2-m^2}\, ie\gamma^\mu \frac{i\left(\not{p}_2+\not{q}+m\right)}{(p_2+q)^2-m^2}\, ie\gamma^\alpha \frac{-ig_{\alpha\beta}}{q^2}
$$

$$
\equiv \int \frac{d^dq}{(2\pi)^d}\frac{\bar{N}}{\bar{D}_1\bar{D}_2\bar{D}_0}
$$

$$
\begin{aligned}
\bar{N}(\bar{q}) = e^3\left\{\bar{\gamma}^\beta\left(\bar{\not{p}}_1+\bar{\not{q}}+m\right)\bar{\gamma}^\mu\left(\bar{\not{p}}_2+\bar{\not{q}}+m\right)\bar{\gamma}_\beta\right\} = e^3\Big\{\gamma^\beta\left(\not{p}_1+\not{q}+m\right)\gamma^\mu\left(\not{p}_2+\not{q}+m\right)\gamma_\beta + \\
+\tilde{\gamma}^\beta\left(\not{p}_1+\not{q}+m\right)\gamma^\mu\left(\not{p}_2+\not{q}+m\right)\tilde{\gamma}_\beta+\underbrace{\gamma^\beta\tilde{\not{q}}\gamma^\mu\tilde{\not{q}}\gamma_\beta}_{\equiv ①}+\underbrace{\tilde{\gamma}^\beta\tilde{\not{q}}\gamma^\mu\tilde{\not{q}}\tilde{\gamma}_\beta}_{\equiv ②}\Big\} \equiv N+\tilde{N}
\end{aligned}
$$

$$
\begin{aligned}
① &= \tilde{q}_\rho\tilde{q}_\sigma\gamma^\beta\tilde{\gamma}^\rho\gamma^\mu\tilde{\gamma}^\sigma\gamma_\beta = \tilde{q}_\rho\tilde{q}_\sigma(-1)^3\tilde{\gamma}^\rho\tilde{\gamma}^\sigma\gamma^\beta\gamma^\mu\gamma_\beta = -2\tilde{\not{q}}\tilde{\not{q}}\gamma^\mu = -2\tilde{q}^2\gamma^\mu \\
② &= \tilde{q}_\rho\tilde{q}_\sigma\tilde{\gamma}^\beta\tilde{\gamma}^\rho\gamma^\mu\tilde{\gamma}^\sigma\tilde{\gamma}_\beta = \tilde{q}_\rho\tilde{q}_\sigma(-1)^2\gamma^\mu\tilde{\gamma}^\beta\tilde{\gamma}^\rho\tilde{\gamma}^\sigma\tilde{\gamma}_\beta = \tilde{q}^2\gamma^\mu\tilde{\gamma}^\beta\tilde{\gamma}_\beta = \epsilon\tilde{q}^2\gamma^\mu
\end{aligned}
$$

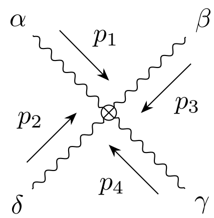
$$
\tilde{N} = -\epsilon\left(\not{p}_1+\not{q}-m\right)\gamma^\mu\left(\not{p}_2+\not{q}-m\right)-(2-\epsilon)\,\tilde{q}^2\gamma^\mu
$$

$$
\begin{aligned}
R_2^{\gamma\mathrm{ee}} &= \frac{1}{(2\pi)^4}\int d^d\bar{q}\,\frac{\tilde{N}}{\bar{D}_0\bar{D}_1\bar{D}_2} = \frac{1}{(2\pi)^4}\int d^d\bar{q}\,\frac{e^3}{\bar{D}_0\bar{D}_1\bar{D}_2}\left\{-\epsilon\left(\not{p}_1+\not{q}-m\right)\gamma^\mu\left(\not{p}_2+\not{q}-m\right)-(2-\epsilon)\,\tilde{q}^2\gamma^\mu\right\} = \\
&= \frac{e^3}{(2\pi)^4}\int d^d\bar{q}\,\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}\left\{-\epsilon\not{q}\gamma^\mu\not{q}-(2-\epsilon)\,\tilde{q}^2\gamma^\mu\right\} = \frac{e^3}{(2\pi)^4}\left\{-\epsilon\gamma^\alpha\gamma^\mu\gamma^\beta\left(\frac{-i\pi^2}{2\epsilon}g_{\alpha\beta}\right)-\frac{-i\pi^2}{2}(2-\epsilon)\,\gamma^\mu\right\} = \\
&= \frac{e^3}{(2\pi)^4}\frac{i\pi^2}{2}\left\{\gamma^\alpha\gamma^\mu\gamma_\alpha-2\gamma^\mu+O(\epsilon)\right\} = \frac{-ie^3}{8\pi^2}\gamma^\mu
\end{aligned}
\tag{2.3}
$$

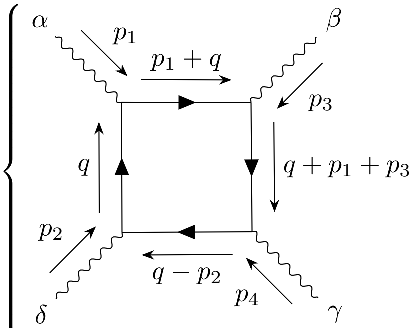
All other possible 3-pt. functions do not contribute to $R_2$ in QED. For example, (do triangle diagram)

### 2.3 4-point function

For the 4-point function we have to be more careful. The 1PI contribution at the 1-loop level consists of several diagrams. They are obtained by symmetrizing the external momenta of the diagram as follows

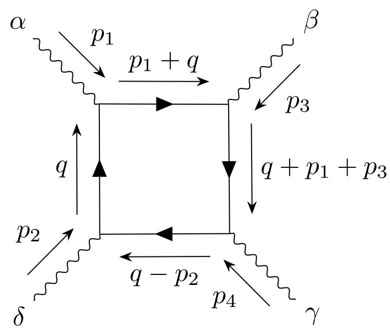



$$= 2 \times \Bigg\{$$




$$+ \ (\alpha \leftrightarrow \beta;\ p_1 \leftrightarrow p_3) \ + \ (\alpha \leftrightarrow \delta;\ p_1 \leftrightarrow p_2) \Bigg\}$$

We only calculate one of the diagrams and do the symmetrizing with the result of our calculation, so we only have to evaluate one diagram. The first of the three diagrams gives




$$= \int \frac{d^d q}{(2\pi)^d} (-1) \, \mathrm{Tr} \left\{ ie\gamma^\alpha \frac{i\left(\not{p}_1 + \not{q} + m\right)}{\left(p_1+q\right)^2 - m^2} ie\gamma^\beta \frac{i\left(\not{q} + \not{p}_3 + \not{p}_1 + m\right)}{\left(p_3+p_1+q\right)^2 - m^2} \times \right.$$

$$\left. \times ie\gamma^\gamma \frac{i\left(\not{q} - \not{p}_2 + m\right)}{\left(q-p_2\right)^2 - m^2} ie\gamma^\delta \frac{i\left(\not{q} + m\right)}{q^2 - m^2} \right\} \equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_{13} \bar{D}_2 \bar{D}_0}$$

$$\begin{aligned}
\bar{N}\left(\bar{q}\right) = & -e^4 \mathrm{Tr} \left\{ \bar{\gamma}^\alpha \left( \bar{\not{p}}_1 + \bar{\not{q}} + m \right) \bar{\gamma}^\beta \left( \bar{\not{q}} + \bar{\not{p}}_1 + \bar{\not{p}}_3 + m \right) \bar{\gamma}^\gamma \left( \bar{\not{q}} - \bar{\not{p}}_2 + m \right) \bar{\gamma}^\delta \left( \bar{\not{q}} + m \right) \right\} = \\
& = -e^4 \mathrm{Tr} \left\{ \gamma^\alpha \left( \not{p}_1 + \not{q} + m \right) \gamma^\beta \left( \not{q} + \not{p}_1 + \not{p}_3 + m \right) \gamma^\gamma \left( \not{q} - \not{p}_2 + m \right) \gamma^\delta \left( \not{q} + m \right) + \right. \\
& + \gamma^\alpha \tilde{\not{q}} \gamma^\beta \tilde{\not{q}} \gamma^\gamma \tilde{\not{q}} \gamma^\delta \tilde{\not{q}} + \gamma^\alpha \tilde{\not{q}} \gamma^\beta \tilde{\not{q}} \gamma^\gamma \not{q} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \tilde{\not{q}} \gamma^\gamma \tilde{\not{q}} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \not{q} \gamma^\gamma \tilde{\not{q}} \gamma^\delta \tilde{\not{q}} + \gamma^\alpha \tilde{\not{q}} \gamma^\beta \not{q} \gamma^\gamma \not{q} \gamma^\delta \tilde{\not{q}} + \\
& \left. + \gamma^\alpha \tilde{\not{q}} \gamma^\beta \not{q} \gamma^\gamma \tilde{\not{q}} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \tilde{\not{q}} \gamma^\gamma \not{q} \gamma^\delta \tilde{\not{q}} \right\} \equiv N + \tilde{N}
\end{aligned}$$

Where we have used that the trace of an odd number of Dirac matrices is zero. Using (..) and (..) $\tilde{N}$ can be further simplified to

$$\begin{aligned}
\tilde{N} = & -e^4 \mathrm{Tr} \left\{ (-1)^{10} \tilde{q}^4 \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta + \tilde{q}^2 \left[ (-1)^3 \gamma^\alpha \gamma^\beta \gamma^\gamma \not{q} \gamma^\delta \not{q} + (-1)^7 \gamma^\alpha \not{q} \gamma^\beta \gamma^\gamma \gamma^\delta \not{q} + (-1)^{11} \gamma^\alpha \not{q} \gamma^\beta \not{q} \gamma^\gamma \gamma^\delta + \right. \right. \\
& \left. \left. + (-1)^7 \gamma^\alpha \gamma^\beta \not{q} \gamma^\gamma \not{q} \gamma^\delta + (-1)^5 \gamma^\alpha \gamma^\beta \not{q} \gamma^\gamma \gamma^\delta \not{q} + (-1)^9 \gamma^\alpha \not{q} \gamma^\beta \gamma^\gamma \not{q} \gamma^\delta \right] \right\} = \\
& = -e^4 \mathrm{Tr} \left\{ \tilde{q}^4 \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta - \tilde{q}^2 \left( \gamma^\alpha \gamma^\beta \gamma^\gamma \not{q} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \gamma^\gamma \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \not{q} \gamma^\gamma \gamma^\delta + \gamma^\alpha \gamma^\beta \not{q} \gamma^\gamma \not{q} \gamma^\delta + \right. \right. \\
& \left. \left. + \gamma^\alpha \gamma^\beta \not{q} \gamma^\gamma \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \gamma^\gamma \not{q} \gamma^\delta \right) \right\} \equiv N + \tilde{N}
\end{aligned}$$

Since this expression involves the trace over up to 6 Dirac matrices, the calculation is very cumbersome. Wcane can evaluate this expression with the help of the Mathematica package FeynCalc [3, 4]

```
In[ ]:= FullSimplify[
  TR[a^2*GA[α].GA[β].GA[γ].GA[δ] - (* a^2 is \tilde{q}^2 from \tilde{q}^4 term,
    other terms are proportional to \tilde{q}^2*q^2 *)
    (GA[α].GS[q].GA[β].GS[q].GA[γ].GA[δ] + GA[α].GA[β].GA[γ].GS[q].GA[δ].GS[q] + GA[α].GA[β].GS[q].GA[γ].GA[δ].GS[q] +
      GA[α].GA[β].GS[q].GA[γ].GS[q].GA[δ] + GA[α].GS[q].GA[β].GA[γ].GA[δ].GS[q] +
      GA[α].GS[q].GA[β].GA[γ].GS[q].GA[δ])]]
```

$$\text{Out[ ]}= 4\left(\bar{a}^2\,\bar{g}^{\alpha\beta}\,\bar{g}^{\gamma\delta} - (2\,\bar{q}^2+\bar{a}^2)\,\bar{g}^{\alpha\gamma}\,\bar{g}^{\beta\delta} + \bar{g}^{\alpha\delta}\left((2\,\bar{q}^2+\bar{a}^2)\,\bar{g}^{\beta\gamma} - 2\,\bar{q}^\beta\,\bar{q}^\gamma\right) - 2\,\bar{q}^\alpha\,\bar{q}^\beta\,\bar{g}^{\gamma\delta} - 2\,\bar{q}^\alpha\,\bar{q}^\delta\,\bar{g}^{\beta\gamma} - 2\,\bar{q}^\gamma\,\bar{q}^\delta\,\bar{g}^{\alpha\beta} + 2\,\bar{q}^2\,\bar{g}^{\alpha\beta}\,\bar{g}^{\gamma\delta}\right)$$

As usual we plug this in the definition of $R_2$ and evaluate the integrals to get the expression of $R_2$ for the first of the contributing diagrams.

$$\begin{aligned}
R_2 =& \frac{1}{(2\pi)^4}\int d^d\bar{q}\frac{\tilde{N}}{\bar{D}_0\bar{D}_1\bar{D}_2} = \frac{1}{(2\pi)^4}\int d^d\bar{q}\frac{4e^4}{\bar{D}_1\bar{D}_{13}\bar{D}_2 0}\tilde{q}^2\left\{\left(2q^2+\tilde{q}^2\right)\left(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\beta}g^{\gamma\delta}\right)+\right.\\
&\left.-2\left(g^{\alpha\beta}q^\gamma q^\delta+g^{\gamma\delta}q^\alpha q^\beta+g^{\alpha\delta}q^\beta q^\gamma+g^{\beta\gamma}q^\alpha q^\delta\right)\right\}=\\
&=\frac{-4e^4}{(2\pi)^4}\left\{\left(2\left(\frac{-i\pi^2}{3}\right)+\left(\frac{-i\pi^2}{6}\right)\right)\left(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\beta}g^{\gamma\delta}\right)-2\left(\frac{-i\pi^2}{12}\right)\left(g^{\alpha\beta}g^{\gamma\delta}+g^{\gamma\delta}g^{\alpha\beta}+\right.\right.\\
&\left.\left.+g^{\alpha\delta}g^{\beta\gamma}+g^{\beta\gamma}g^{\alpha\delta}\right)\right\}=\frac{ie^4}{4\pi^2}\left\{\frac{5}{6}\left(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\beta}g^{\gamma\delta}\right)-\frac{1}{6}\left(2g^{\alpha\beta}g^{\gamma\delta}+2g^{\alpha\delta}g^{\beta\gamma}+\right)\right\}=\\
&=\frac{ie^4}{24\pi^2}\left(3g^{\alpha\beta}g^{\gamma\delta}-5g^{\alpha\gamma}g^{\beta\delta}+3g^{\beta\gamma}g^{\alpha\delta}\right)
\end{aligned}$$

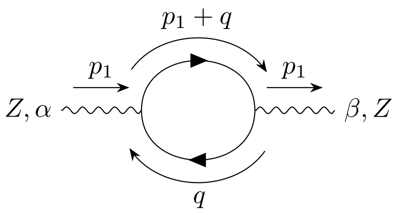
$$\begin{aligned}
R_2^{4\gamma} =& 2\left(R_2+R_2\left(\alpha\leftrightarrow\delta\right)+R_2\left(\alpha\leftrightarrow\beta\right)\right)=\frac{2ie^4}{24\pi^2}\left\{\left(3g^{\alpha\beta}g^{\gamma\delta}-5g^{\alpha\gamma}g^{\beta\delta}+3g^{\beta\gamma}g^{\alpha\delta}\right)+\left(3g^{\beta\delta}g^{\alpha\gamma}-5g^{\gamma\delta}g^{\alpha\beta}+3g^{\beta\gamma}g^{\alpha\delta}\right)+\right.\\
&\left.+\left(3g^{\alpha\beta}g^{\gamma\delta}-5g^{\beta\gamma}g^{\alpha\delta}+3g^{\alpha\gamma}g^{\beta\delta}\right)\right\}=\frac{ie^4}{12\pi^2}\left(g^{\alpha\beta}g^{\gamma\delta}+g^{\alpha\gamma}g^{\beta\delta}+g^{\beta\gamma}g^{\alpha\delta}\right) \qquad (2.4)
\end{aligned}$$

All of the other 4-point functions vanish as well.

# 3 QED Contribution to $R_2$ in the Standard Model

## 3.1 2-point functions

**Z-boson self-energy**




$$
= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ \frac{ig}{cos\theta_W} \gamma^\alpha (g_V - g_A \gamma_5) \frac{i\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} \frac{ig}{cos\theta_W} \gamma^\beta \times \right.
$$

$$
\left. \times (g_V - g_A\gamma_5) \frac{i(\not{q}+m)}{q^2 - m^2} \right\} =
$$

$$
= \int \frac{d^d q}{(2\pi)^d} \frac{-g^2}{\cos^2\theta_W} \operatorname{Tr} \left\{ \gamma^\alpha (g_V - g_A\gamma_5) \frac{\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} \gamma^\beta (g_V - g_A\gamma_5) \frac{(\not{q}+m)}{q^2-m^2} \right\}
$$

$$
\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}
$$

Additionally to the four gamma matrices we now also have the fifth gamma matrix. Its extension to d dimensions is not as straightforward as with the four gamma matrices where we just have to change the 4-dimensional metric to the d-dimensional metric in the Clifford algebra $\left\{\gamma^\alpha, \gamma^\beta\right\} = 2g^{\alpha\beta}$. This is because chirality is a property of four dimensions.
If we still want to impose $\{\gamma_5, \gamma^\mu\} = 0$, then $\operatorname{Tr}\left(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = 0$ for $d \neq 0, 2, 4$ which clashes with $\operatorname{Tr}\left(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = -4i\epsilon^{\alpha\beta\gamma\delta}$ [5]. But the identity is essential in the evaluation of the triangle diagram for the Adler-Bell-Jackiw anomaly. The only definition of $\gamma_5$ which is consistent with the chiral anomaly is the definition of 't Hooft and Veltman [6]: $\gamma_5 = i/4!\, \epsilon_{\mu_1 \ldots \mu_4} \gamma^{\mu_1} \cdots \gamma^{\mu_4}$. This definition implies $\{\gamma_5, \gamma^\mu\} = 0$ and $[\gamma_5, \tilde{\gamma}^\mu] = 0$.

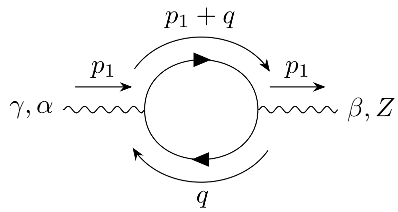
$$
\begin{aligned}
\bar{N}(\bar{q}) &= -\frac{g^2}{\cos^2\theta_W} \operatorname{Tr}\left\{ \bar{\gamma}^\alpha (g_V - g_A\gamma_5) \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right) \bar{\gamma}^\beta (g_V - g_A\gamma_5) (\bar{\not{q}} + m) \right\} = \\
&= \frac{-g^2}{\cos^2\theta_W} \operatorname{Tr}\left\{ \gamma^\alpha (g_V - g_A\gamma_5) \left(\not{p}_1 + \not{q} + m\right) \gamma^\beta (g_V - g_A\gamma_5) (\not{q} + m) + \gamma^\alpha \left(g_V^2 + g_A^2\right) \tilde{\not{q}} \gamma^\beta \tilde{\not{q}} \right\} \equiv N + \tilde{N}
\end{aligned}
$$

Where we used $[\gamma_5, \tilde{\gamma}^\mu] = 0$ and the fact that the gamma matrices will be contracted with external momenta.

$$
\tilde{N} = \frac{-g^2}{\cos^2\theta_W} \left(g_V^2 + g_A^2\right) \left(-\tilde{q}^2\right) \operatorname{Tr}\left(\gamma^\alpha\gamma^\beta\right) = \frac{4g^2\tilde{q}^2}{\cos^2\theta_W} \left(g_V^2 + g_A^2\right) g^{\alpha\beta}
$$

$$
\begin{aligned}
R_2^{ZZ} &= \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{\tilde{N}}{\bar{D}_1\bar{D}_0} = \frac{4g^2 g^{\alpha\beta}}{(2\pi)^4 \cos^2\theta_W} \left(g_V^2 + g_A^2\right) \int d^d\bar{q} \frac{\tilde{q}^2}{\bar{D}_1\bar{D}_0} = \\
&= \frac{4g^2 g^{\alpha\beta}}{(2\pi)^4\cos^2\theta_W} \left(g_V^2 + g_A^2\right) \left(-\frac{i\pi^2}{2}\right) \left(m^2 - \frac{p_1^2}{3}\right) = \frac{-ig^2}{8\pi^2\cos^2\theta_W} \left(g_V^2 + g_A^2\right) \left(m^2 - \frac{p_1^2}{3}\right) g^{\alpha\beta}
\end{aligned}
\tag{3.1}
$$

**Photon/Z-boson mixed self-energy**




$$= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ (-ieQ_f) \gamma^\alpha \frac{i \left( \not{p}_1 + \not{q} + m \right)}{(p_1 + q)^2 - m^2} \frac{ig}{cos\theta_W} \gamma^\beta \times \right.$$

$$\left. \times \, (g_V - g_A \gamma_5) \frac{i \left( \not{q} + m \right)}{q^2 - m^2} \right\} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{eQ_f g}{\cos\theta_W} \operatorname{Tr} \left\{ \gamma^\alpha \frac{\left( \not{p}_1 + \not{q} + m \right)}{(p_1 + q)^2 - m^2} \gamma^\beta (g_V - g_A \gamma_5) \frac{\left( \not{q} + m \right)}{q^2 - m^2} \right\}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

$$\begin{aligned} \bar{N}(\bar{q}) =& \frac{eQ_f g}{\cos\theta_W} \operatorname{Tr} \left\{ \bar{\gamma}^\alpha \left( \bar{\not{p}}_1 + \bar{\not{q}} + m \right) \bar{\gamma}^\beta (g_V - g_A \gamma_5) \left( \bar{\not{q}} + m \right) \right\} = \\ =& \frac{eQ_f g}{\cos\theta_W} \operatorname{Tr} \left\{ \gamma^\alpha \left( \not{p}_1 + \not{q} + m \right) \gamma^\beta (g_V - g_A \gamma_5) \left( \not{q} + m \right) + \gamma^\alpha \tilde{\not{q}} \gamma^\beta g_V \tilde{\not{q}} \right\} \equiv N + \tilde{N} \end{aligned}$$

Where we have used $\operatorname{Tr}\left(\gamma^\alpha \gamma^\beta \gamma_5\right) = 0$.

$$\tilde{N} = \frac{eQ_f g}{cos\theta_W} \operatorname{Tr} \left\{ \gamma^\alpha \tilde{\not{q}} \gamma^\beta g_V \tilde{\not{q}} \right\} = \frac{-4eQ_f g g_V}{\cos\theta_W} \tilde{q}^2 g^{\alpha\beta}$$

$$\begin{aligned} R_2^{\gamma Z} &= \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{-4eQ_f g g_V}{(2\pi)^4 \cos\theta_W} g^{\alpha\beta} \int d^d \bar{q} \frac{\tilde{q}^2}{\bar{D}_1 \bar{D}_0} \\ &= \frac{-4eQ_f g g_V}{(2\pi)^4 \cos\theta_W} \left( -\frac{i\pi^2}{2} \right) g^{\alpha\beta} \left( m^2 - \frac{p_1^2}{3} \right) = \frac{ieQ_f g g_V}{8\pi^2 \cos\theta_W} g^{\alpha\beta} \left( m^2 - \frac{p_1^2}{3} \right) \end{aligned} \tag{3.2}$$

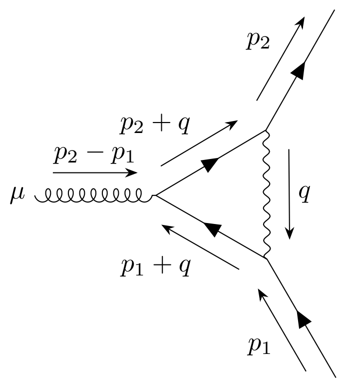
**Gluon self-energy**

Because the gluon (just as the photon) couples to a pure vector current, the calculation for the gluon self-energy $R_2$ is the same as for the photon self-energy $R_2$ replacing the electric charge generator with the colour charge generator. So, from equation 2.1 with $eQ_f \to g_S T^a$ we get

$$R_2^{gg} = R_2^{\gamma\gamma} \left( eQ_f \to g_S T^a \right) = \frac{-ig_S^2}{8\pi^2} \operatorname{Tr}\left( T^a T^b \right) g^{\alpha\beta} \left( 2m^2 - \frac{p_1^2}{3} \right) \tag{3.3}$$

### 3.2 3-point functions

**Gluon-quark vertex**




$$= \int \frac{d^dq}{(2\pi)^d} \left(-ieQ_q\gamma^\beta\right) \frac{i\left(\not p_1+\not q+m\right)}{\left(p_1+q\right)^2-m^2} \left(-ig_S\gamma^\mu T^a\right) \frac{i\left(\not p_2+\not q+m\right)}{\left(p_2+q\right)^2-m^2}\left(-ieQ_q\gamma^\alpha\right)\frac{-ig_{\alpha\beta}}{q^2} =$$

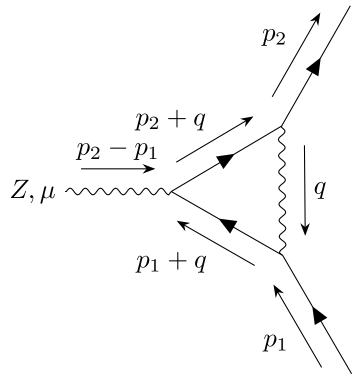
$$= \int \frac{d^dq}{(2\pi)^d} - e^2Q_q^2 g_S\gamma^\beta \frac{\left(\not p_1+\not q+m\right)}{\left(p_1+q\right)^2-m^2}\gamma^\mu T^a \frac{\left(\not p_2+\not q+m\right)}{\left(p_2+q\right)^2-m^2}\gamma_\beta\frac{1}{q^2} =$$
$$\equiv \int \frac{d^dq}{(2\pi)^d}\frac{\bar N}{\bar D_1\bar D_2\bar D_0}$$

$$\bar N\left(\bar q\right) = -e^2Q_q^2g_S\left\{\bar\gamma^\beta\left(\bar{\not p}_1+\bar{\not q}+m\right)\bar\gamma^\mu T^a\left(\bar{\not p}_2+\bar{\not q}+m\right)\bar\gamma_\beta\right\} = -e^2Q_q^2g_S\left\{\gamma^\beta\left(\not p_1+\not q+m\right)\gamma^\mu T^a\left(\not p_2+\not q+m\right)\gamma_\beta + \right.$$
$$\left. +\gamma^\beta\tilde{\not q}\gamma^\mu T^a\tilde{\not q}\gamma_\beta+\tilde\gamma^\beta\not q\gamma^\mu T^a\not q\tilde\gamma_\beta\right\}\equiv N+\tilde N$$

$$\tilde N = -e^2Q_q^2g_S\left\{-\tilde q^2\underbrace{\gamma^\beta\gamma^\mu\gamma_\beta}_{-2\gamma^\mu}T^a-\epsilon q_\alpha q_\beta\gamma^\alpha\gamma^\mu\gamma^\beta T^a\right\} = -e^2Q_q^2g_S\left\{2\tilde q^2\gamma^\mu T^a-\epsilon q_\alpha q_\beta\gamma^\alpha\gamma^\mu\gamma^\beta T^a\right\}$$

$$\begin{aligned}
R_2^{gqq} =& \frac{1}{(2\pi)^4}\int d^d\bar q\frac{\tilde N}{\bar D_1\bar D_2\bar D_0} = \frac{-e^2Q_q^2g_S}{(2\pi)^4}\int d^d\bar q\frac{1}{\bar D_1\bar D_2\bar D_0}\left\{2\tilde q^2\gamma^\mu T^a-\epsilon q_\alpha q_\beta\gamma^\alpha\gamma^\mu\gamma^\beta T^a\right\} = \\
=& \frac{-e^2Q_q^2g_S}{(2\pi)^4}\left\{2\left(\frac{-i\pi^2}{2}\right)\gamma^\mu T^a-\epsilon\left(\frac{-i\pi^2}{2\epsilon}\right)\underbrace{g_{\alpha\beta}\gamma^\alpha\gamma^\mu\gamma^\beta}_{-2\gamma^\mu}T^a\right\} = \frac{-e^2Q_q^2g_S}{16\pi^4}\left(\frac{-i\pi^2}{2}\right)\left\{2\gamma^\mu T^a+2\gamma^\mu T^a\right\} = \\
=& \frac{ie^2Q_q^2g_S}{8\pi^2}\gamma^\mu T^a
\end{aligned} \tag{3.4}$$

**Z-fermion vertex**




$$= \int \frac{d^dq}{(2\pi)^d}\left(-ieQ_f\gamma^\beta\right)\frac{i\left(\not{p}_1+\not{q}+m\right)}{\left(p_1+q\right)^2-m^2}\frac{ig}{\cos\theta_W}\gamma^\mu\left(g_V-g_A\gamma_5\right)\frac{i\left(\not{p}_2+\not{q}+m\right)}{\left(p_2+q\right)^2-m^2}\times$$

$$\times\left(-ieQ_f\gamma^\alpha\right)\frac{-ig_{\alpha\beta}}{q^2}=$$

$$= \int \frac{d^dq}{(2\pi)^d}\frac{e^2Q_f^2g}{\cos\theta_W}\gamma^\beta\frac{\left(\not{p}_1+\not{q}+m\right)}{\left(p_1+q\right)^2-m^2}\gamma^\mu\left(g_V-g_A\gamma_5\right)\frac{\left(\not{p}_2+\not{q}+m\right)}{\left(p_2+q\right)^2-m^2}\gamma_\beta\frac{1}{q^2}=$$

$$\equiv \int \frac{d^dq}{(2\pi)^d}\frac{\bar{N}}{\bar{D}_1\bar{D}_2\bar{D}_0}$$

$$\bar{N}\left(\bar{q}\right)=\frac{e^2Q_f^2g}{\cos\theta_W}\left\{\bar{\gamma}^\beta\left(\bar{\not{p}}_1+\bar{\not{q}}+m\right)\bar{\gamma}^\mu\left(g_V-g_A\gamma_5\right)\left(\bar{\not{p}}_2+\bar{\not{q}}+m\right)\bar{\gamma}_\beta\right\}=\frac{e^2Q_f^2g}{\cos\theta_W}\left\{\gamma^\beta\left(\not{p}_1+\not{q}+m\right)\gamma^\mu\left(g_V-g_A\gamma_5\right)\times\right.$$

$$\left.\times\left(\not{p}_2+\not{q}+m\right)\gamma_\beta+\tilde{\gamma}^\beta\left(\not{p}_1+\not{q}+m\right)\gamma^\mu\left(g_V-g_A\gamma_5\right)\left(\not{p}_2+\not{q}+m\right)\tilde{\gamma}_\beta+\left(\gamma^\beta+\tilde{\gamma}^\beta\right)\tilde{\not{q}}\gamma^\mu\left(g_V-g_A\gamma_5\right)\tilde{\not{q}}\left(\gamma_\beta+\tilde{\gamma}_\beta\right)\right\}=$$

$$\equiv N+\tilde{N}$$

$$\tilde{N}=\frac{e^2Q_f^2g}{\cos\theta_W}\left\{\left(\not{p}_1+\not{q}-m\right)\tilde{\gamma}^\beta\gamma^\mu\tilde{\gamma}_\beta\left(g_V-g_A\gamma_5\right)\left(\not{p}_2+\not{q}-m\right)+\gamma^\beta\tilde{\not{q}}\gamma^\mu\left(g_V-g_A\gamma_5\right)\tilde{\not{q}}\gamma_\beta+\tilde{\gamma}^\beta\tilde{\not{q}}\gamma^\mu\left(g_V-g_A\gamma_5\right)\tilde{\not{q}}\tilde{\gamma}_\beta\right\}=$$

$$=\frac{e^2Q_f^2g}{\cos\theta_W}\left\{-\epsilon\left(\not{p}_1+\not{q}-m\right)\gamma^\mu\left(g_V-g_A\gamma_5\right)\left(\not{p}_2+\not{q}-m\right)-\tilde{q}^2\gamma^\beta\gamma^\mu\gamma_\beta\left(g_V+g_A\gamma_5\right)-\tilde{q}^2\left(-\epsilon\gamma^\mu\right)\left(g_V-g_A\gamma_5\right)\right\}=$$

$$=\frac{e^2Q_f^2g}{\cos\theta_W}\left\{-\epsilon\left(\not{p}_1+\not{q}-m\right)\gamma^\mu\left(g_V-g_A\gamma_5\right)\left(\not{p}_2+\not{q}-m\right)+\tilde{q}^2\left(2\gamma^\mu\left(g_V+g_A\gamma_5\right)+\epsilon\gamma^\mu\left(g_V-g_A\gamma_5\right)\right)\right\}=$$

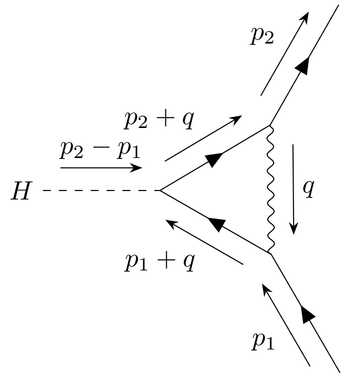
$$R_2^{Zff}=\frac{1}{(2\pi)^4}\int d^d\bar{q}\frac{\tilde{N}}{\bar{D}_1\bar{D}_2\bar{D}_0}=\frac{e^2Q_f^2g}{(2\pi)^4\cos\theta_W}\int d^d\bar{q}\frac{1}{\bar{D}_1\bar{D}_2\bar{D}_0}\left\{-\epsilon\not{q}\gamma^\mu\left(g_V-g_A\gamma_5\right)\not{q}+\tilde{q}^2\left(2\gamma^\mu\left(g_V+g_A\gamma_5\right)+\right.\right.$$

$$\left.\left.+\epsilon\gamma^\mu\left(g_V-g_A\gamma_5\right)\right)\right\}=\frac{e^2Q_f^2g}{\cos\theta_W}\left\{-\epsilon\left(-\frac{i\pi^2}{2\epsilon}\right)g_{\alpha\beta}\gamma^\alpha\gamma^\mu\gamma^\beta\left(g_V+g_A\gamma_5\right)+2\left(-\frac{i\pi^2}{2}\right)\gamma^\mu\left(g_V+g_A\gamma_5\right)\right\}=$$

$$=\frac{e^2Q_f^2g}{\cos\theta_W}\left(-\frac{i\pi^2}{2}\right)\gamma^\mu\left\{2\left(g_V+g_A\gamma_5\right)+2\left(g_V+g_A\gamma_5\right)\right\}=\frac{-ie^2Q_f^2g}{8\pi^2\cos\theta_W}\gamma^\mu\left(g_V+g_A\gamma_5\right)\tag{3.5}$$

where we used that scalar 3-point integrals do not contribute to $R_2$. The last term in the integral is of order $\epsilon$ so it will not contribute in the limit $\epsilon\to 0$

**Higgs-fermion Yukawa vertex**




$$= \int \frac{d^d q}{(2\pi)^d} \left(-ieQ_f\gamma^\beta\right) \frac{i\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} \left(-\frac{ig}{2}\frac{m}{m_W}\right) \frac{i\left(\not{p}_2 + \not{q} + m\right)}{(p_2+q)^2 - m^2} \left(-ieQ_f\gamma^\alpha\right) \frac{-ig_{\alpha\beta}}{q^2} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{-e^2 Q_f^2 gm}{2m_W} \gamma^\beta \frac{\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} \gamma^\mu \frac{\left(\not{p}_2 + \not{q} + m\right)}{(p_2+q)^2 - m^2} \gamma_\beta \frac{1}{q^2} =$$
$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = \frac{e^2 Q_f^2 gm}{2m_W} \bar{\gamma}^\beta \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right) \left(\bar{\not{p}}_2 + \bar{\not{q}} + m\right) \bar{\gamma}_\beta =$$
$$= \frac{e^2 Q_f^2 gm}{2m_W} \left\{ \gamma^\beta \left(\not{p}_1 + \not{q} + m\right) \left(\not{p}_2 + \not{q} + m\right) \gamma_\beta + \tilde{\gamma}^\beta \left(\not{p}_1 + \not{q} + m\right) \left(\not{p}_2 + \not{q} + m\right) \tilde{\gamma}_\beta + \gamma^\beta \tilde{\not{q}}\tilde{\not{q}} \gamma_\beta \right\} \equiv N + \tilde{N}$$

$$\tilde{N} = -\frac{e^2 Q_f^2 gm}{2m_W} \left\{ \tilde{\gamma}^\beta \left(\not{p}_1 + \not{q} + m\right) \left(\not{p}_2 + \not{q} + m\right) \tilde{\gamma}_\beta + \gamma^\beta \tilde{\not{q}}\tilde{\not{q}} \gamma_\beta \right\} = -\frac{e^2 Q_f^2 gm}{2m_W} \left\{ \tilde{\gamma}^\beta \tilde{\gamma}_\beta \not{q}\not{q} + \tilde{\not{q}}\tilde{\not{q}} \gamma^\beta \gamma_\beta \right\}$$

$$R_2 = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_0 \bar{D}_2} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{1}{\bar{D}_1 \bar{D}_0 \bar{D}_2} \left(-\frac{e^2 Q_f^2 gm}{2m_W}\right) \left\{ \tilde{\gamma}^\beta \tilde{\gamma}_\beta \not{q}\not{q} + \tilde{\not{q}}\tilde{\not{q}} \gamma^\beta \gamma_\beta \right\} =$$
$$-\frac{e^2 Q_f^2 gm}{2m_W} \left\{ \epsilon \left(-\frac{i\pi^2}{2\epsilon}\right) \gamma^\alpha \gamma^\beta g_{\alpha\beta} \not{q}\not{q} + \tilde{\not{q}}\tilde{\not{q}} \gamma^\beta \gamma_\beta + \left(-\frac{i\pi^2}{2}\right) 4 \right\} = \frac{-e^2 Q_f^2 gm}{(2\pi)^4\, 2m_W} \left(-\frac{i\pi^2}{2}\right) 8 = \frac{ie^2 Q_f^2 gm}{8\pi^2 m_W}$$

# 4 Pure QED Renormalization

## 4.1 2-point functions

# 5 QED Contribution to the Standard Model Renormalization

## 5.1 2-point functions

# Appendices

## A Important Integrals

In the calculation of $R_2$ the same integrals appear over and over. The most important ones are [1]

#### 2-point integrals

$$\int d^d\bar{q}\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} = -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{3}\right] + O(\epsilon) \tag{A.1}$$

$$\text{P.P.}\left(\int d^d\bar{q}\frac{1}{\bar{D}_i\bar{D}_j}\right) = -2\frac{i\pi^2}{\epsilon} \tag{A.2}$$

$$\text{P.P.}\left(\int d^d\bar{q}\frac{q_\mu}{\bar{D}_i\bar{D}_j}\right) = \frac{i\pi^2}{\epsilon}(p_i + p_j)_\mu \tag{A.3}$$

#### 3-point integrals

$$\int d^d\bar{q}\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} = -\frac{i\pi^2}{2} + O(\epsilon) \tag{A.4}$$

$$\int d^d\bar{q}\frac{1}{\bar{D}_i\bar{D}_j} = \frac{i\pi^2}{6}(p_i + p_j + p_k)_\mu + O(\epsilon) \tag{A.5}$$

$$\text{P.P.}\left(\int d^d\bar{q}\frac{q_\mu q_\nu}{\bar{D}_i\bar{D}_j\bar{D}_k}\right) = -\frac{i\pi^2}{2\epsilon}g_{\mu\nu} \tag{A.6}$$

#### 4-point integrals

$$\int d^d\bar{q}\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{i\pi^2}{6} + O(\epsilon) \tag{A.7}$$

$$\int d^d\bar{q}\frac{\tilde{q}^2 q_\mu q_\nu}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{i\pi^2}{12}g_{\mu\nu} + O(\epsilon) \tag{A.8}$$

$$\int d^d\bar{q}\frac{\tilde{q}^2 q^2}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{i\pi^2}{3} + O(\epsilon) \tag{A.9}$$

## B Traceology

The following dentities for Dirac matrices are very useful when calculating Feynman diagrams in QED

1. $\text{Tr}\left(\gamma^\alpha\gamma^\beta\right) = dg^{\alpha\beta}$
2. $\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$
3. $\gamma^\alpha\gamma_\alpha = d$
4. $\gamma^\alpha\gamma^\beta\gamma_\alpha = (2-d)\gamma^\beta$

... ...

n. $\not{a}\not{b} = a\cdot b$

Dirac matrices obey Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_d$ with $g^{\mu\nu}$ the Minkowski metric in $d$ dimensions

$$g^{\mu\nu} = \begin{cases} 1 & \text{for } \mu = \nu = 0 \\ -1 & \text{for } \mu = \nu = 1, 2, \ldots, d-1 \\ 0 & \text{for } \mu \neq \nu \end{cases}$$

**Proofs for identities**

1. $\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) = dg^{\alpha\beta}$

*Proof.*

$$\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) = \mathrm{Tr}\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right) = 2g^{\alpha\beta}\mathrm{Tr}\left(\mathbb{1}_d\right) - \mathrm{Tr}\left(\gamma^\beta\gamma^\alpha\right) = 2dg^{\alpha\beta} - \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right)$$
$$\Rightarrow \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) = dg^{\alpha\beta}$$

□

2. $\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$

*Proof.*

$$\begin{aligned}\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) &= \mathrm{Tr}\left(\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right)\gamma^\gamma\gamma^\delta\right) = 2g^{\alpha\beta}\mathrm{Tr}\left(\gamma^\gamma\gamma^\delta\right) - \mathrm{Tr}\left(\gamma^\beta\left(2g^{\alpha\gamma} - \gamma^\gamma\gamma^\alpha\right)\gamma^\delta\right) = \\ &= 2dg^{\alpha\beta}g^{\gamma\delta} - 2g^{\alpha\gamma}\mathrm{Tr}\left(\gamma^\beta\gamma^\delta\right) + \mathrm{Tr}\left(\gamma^\beta\gamma^\gamma\left(2g^{\alpha\delta} - \gamma^\delta\gamma^\alpha\right)\right) = \\ &= 2d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) + 2g^{\alpha\delta}\mathrm{Tr}\left(\gamma^\beta\gamma^\gamma\right) - \mathrm{Tr}\left(\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\alpha\right) = \\ &= 2d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right) - \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right)\end{aligned}$$
$$\Rightarrow \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$$

□

3. $\gamma^\alpha\gamma_\alpha = d$

*Proof.*

$$\gamma^\alpha\gamma_\alpha = \frac{1}{2}\left(\gamma^\alpha\gamma_\alpha + \gamma_\alpha\gamma^\alpha\right) = \frac{1}{2}\left\{\gamma^\alpha, \gamma_\alpha\right\} = \frac{1}{2}2g^\alpha_{\ \alpha} = d$$

□

4. $\gamma^\alpha\gamma^\beta\gamma_\alpha = (2-d)\,\gamma^\beta$

*Proof.*

$$\gamma^\alpha\gamma^\beta\gamma_\alpha = \left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right)\gamma_\alpha = (2-d)\,\gamma^\beta$$

□

n. $\not{a}\not{b} = a \cdot b$

*Proof.*

$$\not{a}\not{b} = a_\alpha b_\beta\gamma^\alpha\gamma^\beta = a_\alpha b_\beta\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right) = 2a\cdot b - \not{a}\not{b}$$
$$\Rightarrow \not{a}\not{b} = a \cdot b$$

□

## C Relation Between Left- & Right-handed Currents and Axial & Vector Currents

A classical Lagrangian permits symmetries which can be implemented by Lie groups $G$. An element $g \in G$ of a Lie group can be parametrized as $g = \exp(i\alpha^a T^a)$ where $\alpha^a$ are real parameters and $T^a$ the generators of the Lie group. Noether's theorem predicts a clasically conserved current for each generator of a continuous symmetry. For a field $\phi$ with trafo $\delta\phi = \phi' - \phi = g\phi - \phi \approx (1 + i\alpha^a T^a)\phi - \phi = i\alpha^a T^a \phi$ the conserved current can be shown to be

$$j^{\mu a} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \delta\phi}{\partial \alpha_a}$$

The left- and right-handed part of the currents $j^{\mu a}_{L/R}$ are

$$j^{\mu a}_L = \bar{\psi}_L \gamma^\mu T^a \psi_L = (P_L \psi)^\dagger \gamma^0 \gamma^\mu T^a P_L \psi \overset{P_L^\dagger = P_L}{=} \psi^\dagger P_L \gamma^0 \gamma^\mu T^a P_L \psi = \bar{\psi}\gamma^\mu T^a P_L^2 \psi \overset{P_L^2 = P_L}{=} \bar{\psi}\gamma^\mu T^a P_L \psi$$

$$j^{\mu a}_R = \bar{\psi}_R \gamma^\mu T^a \psi_R = \bar{\psi}\gamma^\mu T^a P_R \psi$$

where $P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$ is the left-/right-handed projector.
From the left- and right-handed currents we can define axial-vector and vector currents

$$j^{\mu a} = j^{\mu a}_R + j^{\mu a}_L = \bar{\psi}\gamma^\mu T^a (P_R + P_L) = \bar{\psi}\gamma^\mu T^a \psi$$

$$j^{\mu a}_5 = j^{\mu a}_R - j^{\mu a}_L = \bar{\psi}\gamma^\mu T^a (P_R - P_L)\psi = \bar{\psi}\gamma^\mu T^a \gamma_5 \psi$$

Now we can couple the currents to vector fields to obtain interactions. E.g., the vector coupling in QED is given by the Lagrangian

$$\mathcal{L}^{QED}_{coupl} = eA_\mu j^\mu = eA_\mu \bar{\psi}\gamma^\mu Q_e \psi = -eA_\mu \bar{\psi}\gamma^\mu \psi$$

In general, we can couple any linear combination of currents to a vector field as long as the combination is Lorentz and gauge invariant. E.g., the neutral current in the electroweak theory is a superposition of a vector and an axialvector current

$$\mathcal{L}^{NC}_{coupl} = gZ_\mu (g_v j^\mu - g_A j^\mu_5)$$

We can use the above relations to express this coupling in terms of right- and left-handed currents

$$\begin{aligned}
\mathcal{L}^{NC}_{coupl} &= gZ_\mu \left(g_V \bar{\psi}\gamma^\mu\psi - g_A \bar{\psi}\gamma^\mu\gamma_5\psi\right) = \\
&= gZ_\mu \left(g_V \bar{\psi}\gamma^\mu\psi + \frac{g_A}{2}\bar{\psi}\gamma^\mu\psi - \frac{g_A}{2}\bar{\psi}\gamma^\mu\psi - g_A\bar{\psi}\gamma^\mu\gamma_5\psi + \frac{g_V}{2}\bar{\psi}\gamma^\mu\gamma_5\psi - \frac{g_V}{2}\bar{\psi}\gamma^\mu\gamma_5\psi\right) = \\
&= gZ_\mu \left((g_V + g_A)\bar{\psi}\gamma^\mu \frac{1}{2}(1-\gamma_5)\psi + (g_V - g_A)\bar{\psi}\gamma^\mu \frac{1}{2}(1+\gamma_5)\psi\right) = \\
&= gZ_\mu \left((g_V + g_A)\bar{\psi}\gamma^\mu P_L \psi + (g_V - g_A)\bar{\psi}\gamma^\mu P_R\psi\right) \equiv gZ_\mu \left(g_L j^\mu_L + g_R j^\mu_R\right)
\end{aligned}$$

This gives the following relation between the (axial-)vector and the left-/right-handed couplings

$$\begin{aligned}
g_L &= g_V + g_A \\
g_R &= g_V - g_A
\end{aligned}$$

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