CONTRIBUTION OF QED TO RATIONAL TERMS IN 1-LOOP FEYNMAN DIAGRAMS IN THE STANDARD MODEL

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ABSTRACT

The abstract goes here.

Keywords QFT · 1-loop Feynman Diagrams · Rational Terms · More

Contents

1 Introduction

The introduction goes here.

$$\mathcal{M} = \sum_i d_i \text{Box}_i + \sum_i c_i \text{Triangle}_i + \sum_i b_i \text{Bubble}_i + \sum_i a_i \text{Tadpole}_i + R \tag{1.1}$$

$$R = R_1 + R_2 \tag{1.2}$$

where R_2 is the ϵ-dimensional contribution of dimensional regularization to the amplitude which is just a rational combination of Lorentz tensors and parameters of the theory, i.e. the couplings or masses of the particles in the theory.

We can decompose any m-point 1-loop function $\bar{A}\bar{q})$ in a numerator $\bar{N}(\bar{q})$ and denominators $\bar{D}_i$

$$\bar{A}(\bar{q}) = \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}, \qquad \bar{D}_i = (\bar{q} + \bar{p}_i)^2 - m_i^2, p_0 \neq 0 \tag{1.3}$$

where $\bar{q}$ is the d-dimensional loop momentum and m_i is the mass of the particle corresponding to the propagator with the numerator D_i. The d-dimensional numerator function $\bar{N}(\bar{q})$ can be split in a 4-dimensional and an ϵ-dimensional part

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, q, \epsilon) \tag{1.4}$$

where only $\tilde{N}(\tilde{q}^2, q, \epsilon)$ is interesting to us because it appears in the definitions of rational terms of the form R_2 which are defined as

$$R_2 \equiv \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{\tilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \tag{1.5}$$

which is the ϵ-dimensional contribution to the amplitude in eqn. 1.3 integrated over the d-dimensional loop momentum $\bar{q}$.
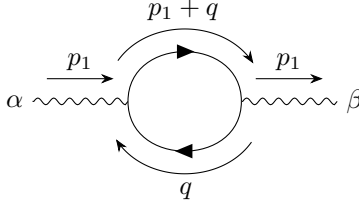
To simplify our calculations later, we can establish a few identities for the manipulation of d-dimensional momenta.

e.g. in reference [1] and [2].

2 R$_2$ in Pure QED

2.1 2-point functions

Photon self-energy

$$= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ ie\gamma^\alpha \frac{i\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1 + q)^2 - m^2} ie\gamma^\beta \frac{i\left(\slashed{q} + m\right)}{q^2 - m^2} \right\}$$

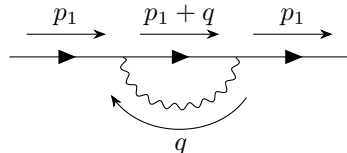$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

From now on bared quantities are d-dimensional, the quantities with a tilde ϵ-dimensional and the normal momenta and gamma matrices 4-dimensional.

$$\bar{N}(\bar{q}) = -e^2 \operatorname{Tr} \left\{ \bar{\gamma}^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m \right) \bar{\gamma}^\beta \left(\bar{\slashed{q}} + m \right) \right\} = -e^2 \operatorname{Tr} \left\{ \gamma^\alpha \left(\slashed{p}_1 + \slashed{q} + m \right) \gamma^\beta \left(\slashed{q} + m \right) + \gamma^\alpha \tilde{\slashed{q}} \gamma^\beta \tilde{\slashed{q}} \right\} \equiv N + \tilde{N}$$

$$\tilde{N} = -e^2 \operatorname{Tr} \left\{ \gamma^\alpha \tilde{\slashed{q}} \gamma^\beta \tilde{\slashed{q}} \right\} \overset{\{\gamma^\mu, \tilde{\gamma}^\nu\} = 0}{=} 4e^2 \tilde{q}^2 g^{\alpha\beta}$$

$$R_2^{\gamma\gamma} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{4e^2}{16\pi^4} \underbrace{\int d^d \bar{q} \frac{\tilde{q}^2}{\bar{D}_1 \bar{D}_0}}_{-i\frac{\pi}{2}(m^2 - p_1^2/3)} = \frac{-ie^2}{8\pi^2} g^{\alpha\beta} \left(2m^2 - \frac{p_1^2}{3} \right) \tag{2.1}$$

Electron self-energy

$$= \int \frac{d^d q}{(2\pi)^d} ie\gamma^\alpha \frac{i\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1 + q)^2 - m^2} ie\gamma^\beta \frac{-ig_{\alpha\beta}}{q^2} = \int \frac{d^d q}{(2\pi)^d} (-e^2) \gamma^\alpha \frac{\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1 + q)^2 - m^2} \gamma_\alpha \frac{1}{q^2}$$
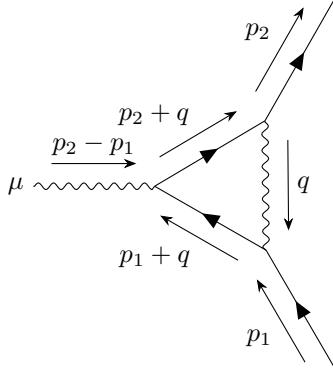
$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = \left(-e^2 \right) \bar{\gamma}^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m \right) \bar{\gamma}_\alpha = -e^2 \left\{ \gamma^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m \right) \gamma_\alpha + \tilde{\gamma}^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m \right) \tilde{\gamma}_\alpha + \gamma^\alpha \tilde{\slashed{q}} \gamma_\alpha + \tilde{\gamma}^\alpha \tilde{\slashed{q}} \tilde{\gamma}_\alpha \right\} \equiv N + \tilde{N}$$

$$\tilde{N} = -e^2 \left\{ \tilde{\gamma}^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m \right) \tilde{\gamma}_\alpha + \gamma^\alpha \tilde{\slashed{q}} \gamma_\alpha + \tilde{\gamma}^\alpha \tilde{\slashed{q}} \tilde{\gamma}_\alpha \right\} = -e^2 \left\{ -\underbrace{\tilde{\gamma}^\alpha \tilde{\gamma}_\alpha}_{=\epsilon} \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} - m \right) - \underbrace{\gamma^\alpha \gamma_\alpha}_{=4} \tilde{\slashed{q}} + \tilde{\gamma}^\alpha \tilde{\slashed{q}} \tilde{\gamma}_\alpha \right\}$$

$$R_2^{\text{ee}} = \frac{1}{(2\pi)^4} \int d^d \tilde{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{-e^2}{(2\pi)^4} \int d^d \tilde{q} \frac{1}{\bar{D}_1 \bar{D}_0} \left(-\epsilon \left(\not{p}_1 + \not{q} - m \right) + \underbrace{\tilde{\not{q}} \left(\ldots \right)}_{=0} \right) =$$

$$= \frac{e^2}{(2\pi)^4} \left\{ \underbrace{\int d^d \tilde{q} \frac{\epsilon \left(\not{p}_1 - m \right)}{\bar{D}_1 \bar{D}_0}}_{= -2\epsilon \frac{i\pi^2}{\epsilon} \left(\not{p}_1 - m \right)} + \underbrace{\int d^d \tilde{q} \frac{\epsilon \not{q}}{\bar{D}_1 \bar{D}_0}}_{= \epsilon \frac{i\pi^2}{\epsilon} \not{p}_1} \right\} = \frac{e^2}{(2\pi)^4} \epsilon \frac{i\pi^2}{\epsilon} \left((-2) \left(\not{p}_1 - m \right) + \not{p}_1 \right) = \frac{-ie^2}{16\pi^2} \left(\not{p}_1 - 2m \right)$$

$$(2.2)$$

2.2 3-point functions

$$= \int \frac{d^d q}{(2\pi)^d} ie\gamma^\beta \frac{i \left(\not{p}_1 + \not{q} + m \right)}{(p_1 + q)^2 - m^2} ie\gamma^\mu \frac{i \left(\not{p}_2 + \not{q} + m \right)}{(p_2 + q)^2 - m^2} ie\gamma^\alpha \frac{-ig_{\alpha\beta}}{q^2}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = e^3 \left\{ \bar{\gamma}^\beta \left(\bar{\not{p}}_1 + \bar{\not{q}} + m \right) \bar{\gamma}^\mu \left(\bar{\not{p}}_2 + \bar{\not{q}} + m \right) \bar{\gamma}_\beta \right\} = e^3 \left\{ \gamma^\beta \left(\not{p}_1 + \not{q} + m \right) \gamma^\mu \left(\not{p}_2 + \not{q} + m \right) \gamma_\beta + \right.$$

$$\left. + \tilde{\gamma}^\beta \left(\not{p}_1 + \not{q} + m \right) \gamma^\mu \left(\not{p}_2 + \not{q} + m \right) \tilde{\gamma}_\beta + \underbrace{\gamma^\beta \tilde{\not{q}} \gamma^\mu \tilde{\not{q}} \gamma_\beta}_{\equiv \text{①}} + \underbrace{\tilde{\gamma}^\beta \tilde{\not{q}} \gamma^\mu \tilde{\not{q}} \tilde{\gamma}_\beta}_{\equiv \text{②}} \right\} \equiv N + \tilde{N}$$

$$\text{①} = \tilde{q}_\rho \tilde{q}_\sigma \gamma^\beta \tilde{\gamma}^\rho \gamma^\mu \tilde{\gamma}^\sigma \gamma_\beta = \tilde{q}_\rho \tilde{q}_\sigma (-1)^3 \tilde{\gamma}^\rho \tilde{\gamma}^\sigma \gamma^\beta \gamma^\mu \gamma_\beta = -2\tilde{\not{q}} \tilde{\not{q}} \gamma^\mu = -2\tilde{q}^2 \gamma^\mu$$

$$\text{②} = \tilde{q}_\rho \tilde{q}_\sigma \tilde{\gamma}^\beta \tilde{\gamma}^\rho \gamma^\mu \tilde{\gamma}^\sigma \tilde{\gamma}_\beta = \tilde{q}_\rho \tilde{q}_\sigma (-1)^2 \gamma^\mu \tilde{\gamma}^\beta \tilde{\gamma}^\rho \tilde{\gamma}^\sigma \tilde{\gamma}_\beta = \tilde{q}^2 \gamma^\mu \tilde{\gamma}^\beta \tilde{\gamma}_\beta = \epsilon \tilde{q}^2 \gamma^\mu$$

$$\tilde{N} = -\epsilon \left(\not{p}_1 + \not{q} - m \right) \gamma^\mu \left(\not{p}_2 + \not{q} - m \right) - (2 - \epsilon) \tilde{q}^2 \gamma^\mu$$

$$R_2^{\gamma\text{ee}} = \frac{1}{(2\pi)^4} \int d^d \tilde{q} \frac{\tilde{N}}{\bar{D}_0 \bar{D}_1 \bar{D}_2} = \frac{1}{(2\pi)^4} \int d^d \tilde{q} \frac{e^3}{\bar{D}_0 \bar{D}_1 \bar{D}_2} \left\{ -\epsilon \left(\not{p}_1 + \not{q} - m \right) \gamma^\mu \left(\not{p}_2 + \not{q} - m \right) - (2 - \epsilon) \tilde{q}^2 \gamma^\mu \right\} =$$

$$= \frac{e^3}{(2\pi)^4} \int d^d \tilde{q} \frac{1}{\bar{D}_0 \bar{D}_1 \bar{D}_2} \left\{ -\epsilon \not{q} \gamma^\mu \not{q} - (2 - \epsilon) \tilde{q}^2 \gamma^\mu \right\} = \frac{e^3}{(2\pi)^4} \left\{ -\epsilon \gamma^\alpha \gamma^\mu \gamma^\beta \left(\frac{-i\pi^2}{2\epsilon} g_{\alpha\beta} \right) - \frac{-i\pi^2}{2} (2 - \epsilon) \gamma^\mu \right\} =$$

$$= \frac{e^3}{(2\pi)^4} \frac{i\pi^2}{2} \left\{ \gamma^\alpha \gamma^\mu \gamma_\alpha - 2\gamma^\mu + O(\epsilon) \right\} = \frac{-ie^3}{8\pi^2} \gamma^\mu \qquad\qquad (2.3)$$

There is one more 3-point function at the 1-loop level which is allowed by the Feynman rules: the 3-point function with only photons as external particles. But it does not contribute to R_2 which we will show now. Because of the symmetry of the 3-point function there are 2 contributing diagrams
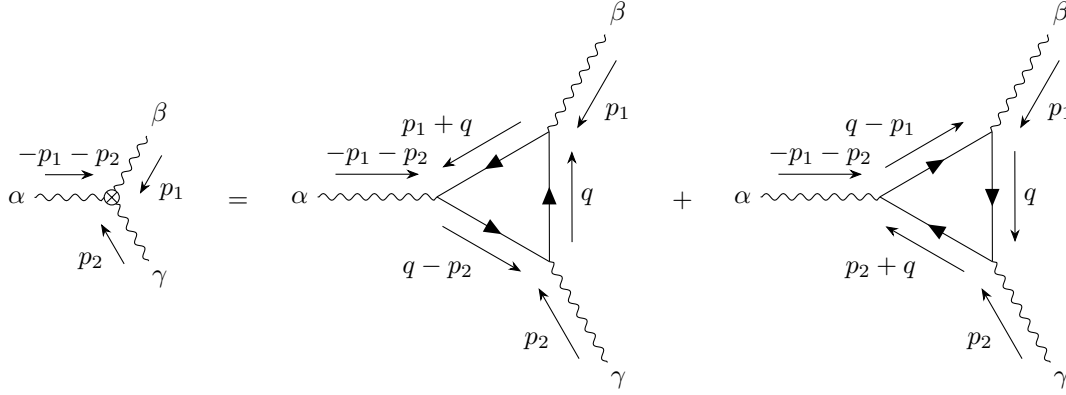
We only calculate the first diagram and then symmetrize the result with $p_1 \leftrightarrow p_2$, $\beta \leftrightarrow \gamma$. Evaluating the first diagram gives

$$= \int \frac{d^d q}{(2\pi)^d} \mathrm{Tr} \left\{ ie\gamma^\beta \frac{i\left(\slashed{q}+m\right)}{q^2-m^2} ie\gamma^\gamma \frac{i\left(\slashed{q}-\slashed{p}_2+m\right)}{(q-p_2)-m^2} ie\gamma^\alpha \frac{i\left(\slashed{q}+\slashed{p}_1+m\right)}{(q+p_1)-m^2} \right\} =$$

$$= \int \frac{d^d q}{(2\pi)^d} e^3 \mathrm{Tr} \left\{ \gamma^\beta \frac{\left(\slashed{q}+m\right)}{q^2-m^2} \gamma^\gamma \frac{\left(\slashed{q}-\slashed{p}_2+m\right)}{(q-p_2)-m^2} \gamma^\alpha \frac{\left(\slashed{q}+\slashed{p}_1+m\right)}{(q+p_1)-m^2} \right\} =$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = e^3 \mathrm{Tr} \left\{ \bar{\gamma}^\beta \left(\bar{\slashed{q}}+m\right) \bar{\gamma}^\gamma \left(\bar{\slashed{q}}-\bar{\slashed{p}}_2+m\right) \bar{\gamma}^\alpha \left(\bar{\slashed{q}}+\bar{\slashed{p}}_1+m\right) \right\} =$$

$$= e^3 \mathrm{Tr} \left\{ \gamma^\beta \left(\slashed{q}+m\right) \gamma^\gamma \left(\slashed{q}-\slashed{p}_2+m\right) \gamma^\alpha \left(\slashed{q}+\slashed{p}_1+m\right) + \gamma^\beta \left(\slashed{q}+\tilde{\slashed{q}}+m\right) \gamma^\gamma \left(\slashed{q}+\tilde{\slashed{q}}-\slashed{p}_2+m\right) \gamma^\alpha \left(\slashed{q}+\tilde{\slashed{q}}+\slashed{p}_1+m\right) \right\} =$$

$$\equiv N + \tilde{N}$$

$$\tilde{N} = e^3 \text{Tr}\left\{ \gamma^\beta \left(\slashed{q} + \tilde{\slashed{q}} + m \right) \gamma^\gamma \left(\slashed{q} + \tilde{\slashed{q}} - \slashed{p}_2 + m \right) \gamma^\alpha \left(\slashed{q} + \tilde{\slashed{q}} + \slashed{p}_1 + m \right) \right\} =$$

$$= e^3 \text{Tr}\left\{ \gamma^\beta \slashed{q} \gamma^\gamma \tilde{\slashed{q}} \gamma^\alpha \slashed{q} + \gamma^\beta \tilde{\slashed{q}} \gamma^\gamma \slashed{q} \gamma^\alpha \tilde{\slashed{q}} + \gamma^\beta \tilde{\slashed{q}} \gamma^\gamma \tilde{\slashed{q}} \gamma^\alpha \slashed{q} + \gamma^\beta \slashed{q} \gamma^\gamma \left(-\slashed{p}_2 \right) \gamma^\alpha \tilde{\slashed{q}} + \gamma^\beta \tilde{\slashed{q}} \gamma^\gamma \tilde{\slashed{q}} \gamma^\alpha \slashed{p}_1 \right\} =$$

$$= -4 e^3 \tilde{q}^2 \left\{ q_\mu \left[\left(g^{\beta\mu} g^{\gamma\alpha} - g^{\beta\gamma} g^{\mu\alpha} + g^{\beta\alpha} g^{\mu\gamma} \right) + \left(g^{\beta\gamma} g^{\alpha\mu} - g^{\beta\mu} g^{\gamma\alpha} + g^{\beta\alpha} g^{\mu\gamma} \right) + \left(g^{\beta\gamma} g^{\mu\alpha} - g^{\beta\alpha} g^{\mu\gamma} + g^{\beta\mu} g^{\alpha\gamma} \right) \right] +$$

$$+ p_{1\mu} \left(g^{\beta\gamma} g^{\alpha\mu} - g^{\beta\alpha} g^{\gamma\mu} + g^{\beta\mu} g^{\alpha\gamma} \right) - p_{2\mu} \left(g^{\beta\gamma} g^{\mu\alpha} - g^{\beta\mu} g^{\alpha\gamma} + g^{\alpha\beta} g^{\gamma\mu} \right) \right\} =$$

$$= -4 e^3 \tilde{q}^2 \left\{ q^\beta g^{\alpha\gamma} + q^\gamma g^{\alpha\beta} + q^\alpha g^{\beta\gamma} + p_1^\alpha g^{\beta\gamma} - p_1^\gamma g^{\alpha\beta} + p_1^\beta g^{\alpha\gamma} - p_2^\alpha g^{\beta\gamma} + p_2^\beta g^{\alpha\gamma} - p_2^\gamma g^{\alpha\beta} \right\}$$

This gives for the R_2 contribution of the first diagram

$$R_2^1 = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_{-2} \bar{D}_0} =$$

$$= \frac{-4 e^3}{(2\pi)^4} \int d^d \bar{q} \frac{1}{\bar{D}_1 \bar{D}_{-2} \bar{D}_0} \left\{ \tilde{q}^2 q^\beta g^{\alpha\gamma} + \tilde{q}^2 q^\gamma g^{\alpha\beta} + \tilde{q}^2 q^\alpha g^{\beta\gamma} + \tilde{q}^2 \left[(p_1 - p_2)^\alpha g^{\beta\gamma} + (p_1 + p_2)^\beta g^{\alpha\gamma} - (p_1 + p_2)^\gamma g^{\alpha\beta} \right] \right\} =$$

$$= \frac{-4 e^3}{(2\pi)^4} \left\{ \frac{i\pi^2}{6} \left[(p_1 - p_2)^\beta g^{\alpha\gamma} + (p_1 - p_2)^\gamma g^{\alpha\beta} + (p_1 - p_2)^\alpha g^{\beta\gamma} \right] - \frac{i\pi^2}{2} \left[(p_1 - p_2)^\alpha g^{\beta\gamma} + (p_1 + p_2)^\beta g^{\alpha\gamma} + \right.$$

$$\left. - (p_1 + p_2)^\gamma g^{\alpha\beta} \right] \right\} =$$

$$= \frac{-4 e^3}{(2\pi)^4} \left\{ g^{\alpha\beta} \left[\frac{i\pi^2}{6} (p_1 - p_2)^\gamma + \frac{i\pi^2}{2} (p_1 + p_2)^\gamma \right] + g^{\beta\gamma} \left[\frac{i\pi^2}{6} (p_1 - p_2)^\alpha - \frac{i\pi^2}{2} (p_1 - p_2)^\alpha \right] + \right.$$

$$+ g^{\alpha\gamma} \left[\frac{i\pi^2}{6} (p_1 - p_2)^\beta - \frac{i\pi^2}{2} (p_1 + p_2)^\beta \right] \right\}$$

$$R_2^2 = R_2^1(p_1 \leftrightarrow p_2, \ \beta \leftrightarrow \gamma) =$$

$$= \frac{-4 e^3}{(2\pi)^4} \left\{ g^{\alpha\gamma} \left[\frac{i\pi^2}{6} (p_2 - p_1)^\beta + \frac{i\pi^2}{2} (p_2 + p_1)^\beta \right] + g^{\beta\gamma} \left[\frac{i\pi^2}{6} (p_2 - p_1)^\alpha - \frac{i\pi^2}{2} (p_2 - p_1)^\alpha \right] + \right.$$

$$+ g^{\alpha\beta} \left[\frac{i\pi^2}{6} (p_2 - p_1)^\gamma - \frac{i\pi^2}{2} (p_2 + p_1)^\gamma \right] \right\} =$$

$$= \frac{-4 e^3}{(2\pi)^4} \left\{ -g^{\alpha\beta} \left[\frac{i\pi^2}{6} (p_1 - p_2)^\gamma + \frac{i\pi^2}{2} (p_1 + p_2)^\gamma \right] - g^{\beta\gamma} \left[\frac{i\pi^2}{6} (p_1 - p_2)^\alpha - \frac{i\pi^2}{2} (p_1 - p_2)^\alpha \right] + \right.$$

$$- g^{\alpha\gamma} \left[\frac{i\pi^2}{6} (p_1 - p_2)^\beta - \frac{i\pi^2}{2} (p_1 + p_2)^\beta \right] \right\} = -R_2^1$$

$$R_2^{3\gamma} = R_2^1 + R_2^2 = R_2^1 - R_2^1 = 0 \tag{2.4}$$

2.3 4-point function

For the 4-point function we have to be more careful. The 1PI contribution at the 1-loop level consists of several diagrams. They are obtained by symmetrizing the external momenta of the diagram as follows

We only calculate one of the diagrams and do the symmetrizing with the result of our calculation, so we only have to evaluate one diagram. The first of the three diagrams gives

$$= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ ie\gamma^\alpha \frac{i\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1 + q)^2 - m^2} ie\gamma^\beta \frac{i\left(\slashed{q} + \slashed{p}_3 + \slashed{p}_1 + m\right)}{(p_3 + p_1 + q)^2 - m^2} \times \right.$$

$$\left. \times ie\gamma^\gamma \frac{i\left(\slashed{q} - \slashed{p}_2 + m\right)}{(q - p_2)^2 - m^2} ie\gamma^\delta \frac{i\left(\slashed{q} + m\right)}{q^2 - m^2} \right\} \equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_{13} \bar{D}_2 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = -e^4 \operatorname{Tr} \left\{ \bar\gamma^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m \right) \bar\gamma^\beta \left(\bar{\slashed{q}} + \bar{\slashed{p}}_1 + \bar{\slashed{p}}_3 + m \right) \bar\gamma^\gamma \left(\bar{\slashed{q}} - \bar{\slashed{p}}_2 + m \right) \bar\gamma^\delta \left(\bar{\slashed{q}} + m \right) \right\} =$$

$$= -e^4 \operatorname{Tr} \left\{ \gamma^\alpha \left(\slashed{p}_1 + \slashed{q} + m \right) \gamma^\beta \left(\slashed{q} + \slashed{p}_1 + \slashed{p}_3 + m \right) \gamma^\gamma \left(\slashed{q} - \slashed{p}_2 + m \right) \gamma^\delta \left(\slashed{q} + m \right) + \right.$$

$$+ \gamma^\alpha \tilde{\slashed{q}} \gamma^\beta \slashed{q} \gamma^\gamma \tilde{\slashed{q}} \gamma^\delta \tilde{\slashed{q}} + \gamma^\alpha \tilde{\slashed{q}} \gamma^\beta \tilde{\slashed{q}} \gamma^\gamma \slashed{q} \gamma^\delta \slashed{q} + \gamma^\alpha \slashed{q} \gamma^\beta \tilde{\slashed{q}} \gamma^\gamma \tilde{\slashed{q}} \gamma^\delta \slashed{q} + \gamma^\alpha \slashed{q} \gamma^\beta \slashed{q} \gamma^\gamma \tilde{\slashed{q}} \gamma^\delta \tilde{\slashed{q}} + \gamma^\alpha \tilde{\slashed{q}} \gamma^\beta \slashed{q} \gamma^\gamma \slashed{q} \gamma^\delta \tilde{\slashed{q}} +$$

$$+ \gamma^\alpha \tilde{\slashed{q}} \gamma^\beta \slashed{q} \gamma^\gamma \tilde{\slashed{q}} \gamma^\delta \slashed{q} + \gamma^\alpha \slashed{q} \gamma^\beta \tilde{\slashed{q}} \gamma^\gamma \slashed{q} \gamma^\delta \tilde{\slashed{q}} \right\} \equiv N + \tilde{N}$$

Where we have used that the trace of an odd number of Dirac matrices is zero. Using (..) and (..) $\tilde{N}$ can be further simplified to

$$\tilde{N} = -e^4 \operatorname{Tr} \left\{ (-1)^{10} \tilde{q}^4 \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta + \tilde{q}^2 \left[(-1)^3 \gamma^\alpha \gamma^\beta \gamma^\gamma \slashed{q} \gamma^\delta \slashed{q} + (-1)^7 \gamma^\alpha \slashed{q} \gamma^\beta \gamma^\gamma \gamma^\delta \slashed{q} + (-1)^{11} \gamma^\alpha \slashed{q} \gamma^\beta \slashed{q} \gamma^\gamma \gamma^\delta + \right. \right.$$

$$\left. \left. + (-1)^7 \gamma^\alpha \gamma^\beta \slashed{q} \gamma^\gamma \slashed{q} \gamma^\delta + (-1)^5 \gamma^\alpha \gamma^\beta \slashed{q} \gamma^\gamma \gamma^\delta \slashed{q} + (-1)^9 \gamma^\alpha \slashed{q} \gamma^\beta \gamma^\gamma \slashed{q} \gamma^\delta \right] \right\} =$$

$$= -e^4 \operatorname{Tr} \left\{ \tilde{q}^4 \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta - \tilde{q}^2 \left(\gamma^\alpha \gamma^\beta \gamma^\gamma \slashed{q} \gamma^\delta \slashed{q} + \gamma^\alpha \slashed{q} \gamma^\beta \gamma^\gamma \gamma^\delta \slashed{q} + \gamma^\alpha \slashed{q} \gamma^\beta \slashed{q} \gamma^\gamma \gamma^\delta + \gamma^\alpha \gamma^\beta \slashed{q} \gamma^\gamma \gamma^\delta \gamma^\delta + \right. \right.$$

$$\left. \left. + \gamma^\alpha \gamma^\beta \slashed{q} \gamma^\gamma \gamma^\delta \slashed{q} + \gamma^\alpha \slashed{q} \gamma^\beta \gamma^\gamma \slashed{q} \gamma^\delta \right) \right\} \equiv N + \tilde{N}$$

Since this expression involves the trace over up to 6 Dirac matrices, the calculation is very cumbersome. We can evaluate this expression with the help of the Mathematica package FeynCalc [3, 4]

```
In[ ]:= FullSimplify[
    TR[a ^ 2 * GA[α].GA[β].GA[γ].GA[δ] - (* a^2 is \tilde{q}^2 from \tilde{q}^4 term,
        other terms are proportional to \tilde{q}^2*q^2 *)
        (GA[α].GS[q].GA[β].GS[q].GA[γ].GA[δ] + GA[α].GA[β].GA[γ].GS[q].GA[δ].GS[q] + GA[α].GA[β].GS[q].GA[γ].GA[δ].GS[q] +
        GA[α].GA[β].GS[q].GA[γ].GS[q].GA[δ] + GA[α].GS[q].GA[β].GA[γ].GA[δ].GS[q] +
        GA[α].GS[q].GA[β].GA[γ].GS[q].GA[δ]))]]
```

$$Out[\circ]:= 4\left(a^2\,\bar{g}^{\alpha\beta}\,\bar{g}^{\gamma\delta} - \left(2\,\bar{q}^2 + a^2\right)\bar{g}^{\alpha\gamma}\,\bar{g}^{\beta\delta} + \bar{g}^{\alpha\delta}\left(\left(2\,\bar{q}^2 + a^2\right)\bar{g}^{\beta\gamma} - 2\,\bar{q}^\beta\,\bar{q}^\gamma\right) - 2\,\bar{q}^\alpha\,\bar{q}^\beta\,\bar{g}^{\gamma\delta} - 2\,\bar{q}^\alpha\,\bar{q}^\delta\,\bar{g}^{\beta\gamma} - 2\,\bar{q}^\gamma\,\bar{q}^\delta\,\bar{g}^{\alpha\beta} + 2\,\bar{q}^2\,\bar{g}^{\alpha\beta}\,\bar{g}^{\gamma\delta}\right)$$

As usual we plug this in the definition of R_2 and evaluate the integrals to get the expression of R_2 for the first of the contributing diagrams.

$$
\begin{aligned}
R_2 &= \frac{1}{(2\pi)^4}\int d^d\bar{q}\,\frac{\tilde{N}}{\bar{D}_0\bar{D}_1\bar{D}_2} = \frac{1}{(2\pi)^4}\int d^d\bar{q}\,\frac{4e^4}{\bar{D}_1\bar{D}_{13}\bar{D}_2\bar{0}}\tilde{q}^2\left\{\left(2q^2 + \tilde{q}^2\right)\left(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\beta}g^{\gamma\delta}\right) + \right. \\
&\quad \left. -2\left(g^{\alpha\beta}q^\gamma q^\delta + g^{\gamma\delta}q^\alpha q^\beta + g^{\alpha\delta}q^\beta q^\gamma + g^{\beta\gamma}q^\alpha q^\delta\right)\right\} = \\
&= \frac{-4e^4}{(2\pi)^4}\left\{\left(2\left(\frac{-i\pi^2}{3}\right) + \left(\frac{-i\pi^2}{6}\right)\right)\left(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\beta}g^{\gamma\delta}\right) - 2\left(\frac{-i\pi^2}{12}\right)\left(g^{\alpha\beta}g^{\gamma\delta} + g^{\gamma\delta}g^{\alpha\beta} + \right.\right. \\
&\quad \left.\left. + g^{\alpha\delta}g^{\beta\gamma} + g^{\beta\gamma}g^{\alpha\delta}\right)\right\} = \frac{ie^4}{4\pi^2}\left\{\frac{5}{6}\left(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\beta}g^{\gamma\delta}\right) - \frac{1}{6}\left(2g^{\alpha\beta}g^{\gamma\delta} + 2g^{\alpha\delta}g^{\beta\gamma} + \right)\right\} = \\
&= \frac{ie^4}{24\pi^2}\left(3g^{\alpha\beta}g^{\gamma\delta} - 5g^{\alpha\gamma}g^{\beta\delta} + 3g^{\beta\gamma}g^{\alpha\delta}\right)
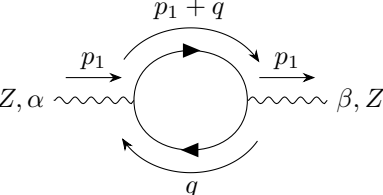\end{aligned}
$$

$$
\begin{aligned}
R_2^{4\gamma} &= 2\left[R_2 + R_2\left(\alpha\leftrightarrow\delta\right) + R_2\left(\alpha\leftrightarrow\beta\right)\right] = \frac{2ie^4}{24\pi^2}\left\{\left(3g^{\alpha\beta}g^{\gamma\delta} - 5g^{\alpha\gamma}g^{\beta\delta} + 3g^{\beta\gamma}g^{\alpha\delta}\right) + \left(3g^{\beta\delta}g^{\alpha\gamma} - 5g^{\gamma\delta}g^{\alpha\beta} + 3g^{\beta\gamma}g^{\alpha\delta}\right) + \right. \\
&\quad \left. + \left(3g^{\alpha\beta}g^{\gamma\delta} - 5g^{\beta\gamma}g^{\alpha\delta} + 3g^{\alpha\gamma}g^{\beta\delta}\right)\right\} = \frac{ie^4}{12\pi^2}\left(g^{\alpha\beta}g^{\gamma\delta} + g^{\alpha\gamma}g^{\beta\delta} + g^{\beta\gamma}g^{\alpha\delta}\right) \quad\quad (2.5)
\end{aligned}
$$

Like the other 3-point function all of the other 4-point functions which are permitted by the Feynman rules vanish. We will not show this here because the calculations for the 4-point functions are quite lengthy.

9

3 QED Contribution to R_2 in the Standard Model

3.1 2-point functions

Z-boson self-energy

$$= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ \frac{ig}{cos\theta_W} \gamma^\alpha (g_V - g_A\gamma_5) \frac{i \left(\not{p}_1 + \not{q} + m \right)}{(p_1 + q)^2 - m^2} \frac{ig}{cos\theta_W} \gamma^\beta \times \right.$$

$$\left. \times (g_V - g_A\gamma_5) \frac{i \left(\not{q} + m \right)}{q^2 - m^2} \right\} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{-g^2}{\cos^2\theta_W} \operatorname{Tr} \left\{ \gamma^\alpha (g_V - g_A\gamma_5) \frac{\left(\not{p}_1 + \not{q} + m \right)}{(p_1 + q)^2 - m^2} \gamma^\beta (g_V - g_A\gamma_5) \frac{\left(\not{q} + m \right)}{q^2 - m^2} \right\}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

Additionally to the four gamma matrices we now also have the fifth gamma matrix. Its extension to d dimensions is not as straightforward as with the four gamma matrices where we just have to change the 4-dimensional metric to the d-dimensional metric in the Clifford algebra $\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}$. This is because chirality is a property of four dimensions.
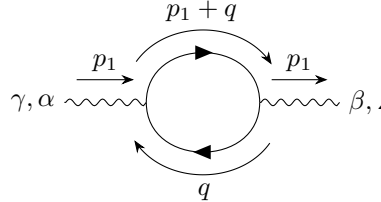
If we still want to impose $\{\gamma_5, \gamma^\mu\} = 0$, then $\operatorname{Tr}\left(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = 0$ for $d \neq 0, 2, 4$ which clashes with $\operatorname{Tr}\left(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = -4i\epsilon^{\alpha\beta\gamma\delta}$ [5]. But the identity is essential in the evaluation of the triangle diagram for the Adler-Bell-Jackiw anomaly. The only definition of γ_5 which is consistent with the chiral anomaly is the definition of 't Hooft and Veltman [6]: $\gamma_5 = i/4! \, \epsilon_{\mu_1\ldots\mu_4}\gamma^{\mu_1}\cdots\gamma^{\mu_4}$. This definition implies $\{\gamma_5, \gamma^\mu\} = 0$ and $[\gamma_5, \tilde{\gamma}^\mu] = 0$.

$$\bar{N}(\bar{q}) = -\frac{g^2}{\cos^2\theta_W} \operatorname{Tr}\left\{ \bar{\gamma}^\alpha (g_V - g_A\gamma_5) \left(\bar{\not{p}}_1 + \bar{\not{q}} + m \right) \bar{\gamma}^\beta (g_V - g_A\gamma_5) \left(\bar{\not{q}} + m \right) \right\} =$$

$$= \frac{-g^2}{\cos^2\theta_W} \operatorname{Tr}\left\{ \gamma^\alpha (g_V - g_A\gamma_5) \left(\not{p}_1 + \not{q} + m \right) \gamma^\beta (g_V - g_A\gamma_5) \left(\not{q} + m \right) + \gamma^\alpha (g_V^2 + g_A^2) \tilde{\not{q}}\gamma^\beta\tilde{\not{q}} \right\} \equiv N + \tilde{N}$$

Where we used $[\gamma_5, \tilde{\gamma}^\mu] = 0$ and the fact that the gamma matrices will be contracted with external momenta.

$$\tilde{N} = \frac{-g^2}{\cos^2\theta_W} (g_V^2 + g_A^2) (-\tilde{q}^2) \operatorname{Tr}\left(\gamma^\alpha\gamma^\beta\right) = \frac{4g^2\tilde{q}^2}{\cos^2\theta_W} (g_V^2 + g_A^2) g^{\alpha\beta}$$

$$R_2^{ZZ} = \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{\tilde{N}}{\bar{D}_1\bar{D}_0} = \frac{4g^2 g^{\alpha\beta}}{(2\pi)^4 \cos^2\theta_W} (g_V^2 + g_A^2) \int d^d\bar{q} \frac{\tilde{q}^2}{\bar{D}_1\bar{D}_0} =$$

$$= \frac{4g^2 g^{\alpha\beta}}{(2\pi)^4 \cos^2\theta_W} (g_V^2 + g_A^2) \left(-\frac{i\pi^2}{2}\right) \left(m^2 - \frac{p_1^2}{3}\right) = \frac{-ig^2}{8\pi^2 \cos^2\theta_W} (g_V^2 + g_A^2) \left(m^2 - \frac{p_1^2}{3}\right) g^{\alpha\beta}$$

$$(3.1)$$

Photon/Z-boson mixed self-energy

$$= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ (-ieQ_f) \gamma^\alpha \frac{i \left(\not{p}_1 + \not{q} + m \right)}{(p_1 + q)^2 - m^2} \frac{ig}{cos\theta_W} \gamma^\beta \times \right.$$

$$\left. \times (g_V - g_A \gamma_5) \frac{i \left(\not{q} + m \right)}{q^2 - m^2} \right\} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{eQ_f g}{\cos \theta_W} \operatorname{Tr} \left\{ \gamma^\alpha \frac{\left(\not{p}_1 + \not{q} + m \right)}{(p_1 + q)^2 - m^2} \gamma^\beta (g_V - g_A \gamma_5) \frac{\left(\not{q} + m \right)}{q^2 - m^2} \right\}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = \frac{eQ_f g}{\cos \theta_W} \operatorname{Tr} \left\{ \bar{\gamma}^\alpha \left(\bar{\not{p}}_1 + \bar{\not{q}} + m \right) \bar{\gamma}^\beta (g_V - g_A \gamma_5) \left(\bar{\not{q}} + m \right) \right\} =$$

$$= \frac{eQ_f g}{\cos \theta_W} \operatorname{Tr} \left\{ \gamma^\alpha \left(\not{p}_1 + \not{q} + m \right) \gamma^\beta (g_V - g_A \gamma_5) \left(\not{q} + m \right) + \gamma^\alpha \tilde{\not{q}} \gamma^\beta g_V \tilde{\not{q}} \right\} \equiv N + \tilde{N}$$

Where we have used $\operatorname{Tr} \left(\gamma^\alpha \gamma^\beta \gamma_5 \right) = 0$.

$$\tilde{N} = \frac{eQ_f g}{cos\theta_W} \operatorname{Tr} \left\{ \gamma^\alpha \tilde{\not{q}} \gamma^\beta g_V \tilde{\not{q}} \right\} = \frac{-4eQ_f g g_V}{\cos \theta_W} \tilde{q}^2 g^{\alpha\beta}$$

$$R_2^{\gamma Z} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{-4eQ_f g g_V}{(2\pi)^4 \cos \theta_W} g^{\alpha\beta} \int d^d \bar{q} \frac{\tilde{q}^2}{\bar{D}_1 \bar{D}_0}$$

$$= \frac{-4eQ_f g g_V}{(2\pi)^4 \cos \theta_W} \left(-\frac{i\pi^2}{2} \right) g^{\alpha\beta} \left(m^2 - \frac{p_1^2}{3} \right) = \frac{ieQ_f g g_V}{8\pi^2 \cos \theta_W} g^{\alpha\beta} \left(m^2 - \frac{p_1^2}{3} \right) \tag{3.2}$$

Gluon self-energy

Because the gluon (just as the photon) couples to a pure vector current, the calculation for the gluon self-energy R_2 is the same as for the photon self-energy R_2 replacing the electric charge generator with the colour charge generator. So, from equation 2.1 with $eQ_f \to g_S T^a$ we get

$$R_2^{gg} = R_2^{\gamma\gamma} (eQ_f \to g_S T^a) = \frac{-ig_S^2}{8\pi^2} \operatorname{Tr} \left(T^a T^b \right) g^{\alpha\beta} \left(2m^2 - \frac{p_1^2}{3} \right) \tag{3.3}$$

3.2 3-point functions

Gluon-quark vertex
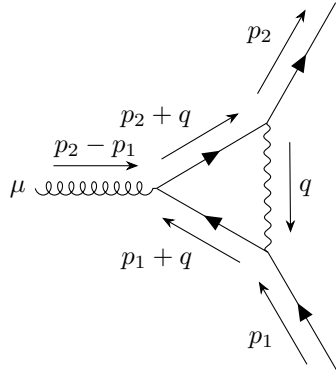
$$\int \frac{d^d q}{(2\pi)^d} \left(-ieQ_q\gamma^\beta\right) \frac{i\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1+q)^2 - m^2} \left(-ig_S\gamma^\mu T^a\right) \frac{i\left(\slashed{p}_2 + \slashed{q} + m\right)}{(p_2+q)^2 - m^2} \left(-ieQ_q\gamma^\alpha\right) \frac{-ig_{\alpha\beta}}{q^2} =$$

$$= \int \frac{d^d q}{(2\pi)^d} - e^2 Q_q^2 g_S \gamma^\beta \frac{\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1+q)^2 - m^2} \gamma^\mu T^a \frac{\left(\slashed{p}_2 + \slashed{q} + m\right)}{(p_2+q)^2 - m^2} \gamma_\beta \frac{1}{q^2} =$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = -e^2 Q_q^2 g_S \left\{\bar{\gamma}^\beta \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m\right) \bar{\gamma}^\mu T^a \left(\bar{\slashed{p}}_2 + \bar{\slashed{q}} + m\right) \bar{\gamma}_\beta\right\} = -e^2 Q_q^2 g_S \left\{\gamma^\beta \left(\slashed{p}_1 + \slashed{q} + m\right) \gamma^\mu T^a \left(\slashed{p}_2 + \slashed{q} + m\right) \gamma_\beta + \right.$$
$$\left. + \gamma^\beta \tilde{\slashed{q}} \gamma^\mu T^a \tilde{\slashed{q}} \gamma_\beta + \tilde{\gamma}^\beta \slashed{q} \gamma^\mu T^a \slashed{q} \tilde{\gamma}_\beta\right\} \equiv N + \tilde{N}$$

$$\tilde{N} = -e^2 Q_q^2 g_S \left\{-\tilde{q}^2 \underbrace{\gamma^\beta \gamma^\mu \gamma_\beta}_{-2\gamma^\mu} T^a - \epsilon q_\alpha q_\beta \gamma^\alpha \gamma^\mu \gamma^\beta T^a\right\} = -e^2 Q_q^2 g_S \left\{2\tilde{q}^2 \gamma^\mu T^a - \epsilon q_\alpha q_\beta \gamma^\alpha \gamma^\mu \gamma^\beta T^a\right\}$$

$$R_2^{gqq} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0} = \frac{-e^2 Q_q^2 g_S}{(2\pi)^4} \int d^d \bar{q} \frac{1}{\bar{D}_1 \bar{D}_2 \bar{D}_0} \left\{2\tilde{q}^2 \gamma^\mu T^a - \epsilon q_\alpha q_\beta \gamma^\alpha \gamma^\mu \gamma^\beta T^a\right\} =$$

$$= \frac{-e^2 Q_q^2 g_S}{(2\pi)^4} \left\{2\left(\frac{-i\pi^2}{2}\right) \gamma^\mu T^a - \epsilon \left(\frac{-i\pi^2}{2\epsilon}\right) \underbrace{g_{\alpha\beta}\gamma^\alpha \gamma^\mu \gamma^\beta}_{-2\gamma^\mu} T^a\right\} = \frac{-e^2 Q_q^2 g_S}{16\pi^4} \left(\frac{-i\pi^2}{2}\right) \left\{2\gamma^\mu T^a + 2\gamma^\mu T^a\right\} =$$

$$= \frac{ie^2 Q_q^2 g_S}{8\pi^2} \gamma^\mu T^a \tag{3.4}$$

Z-fermion vertex
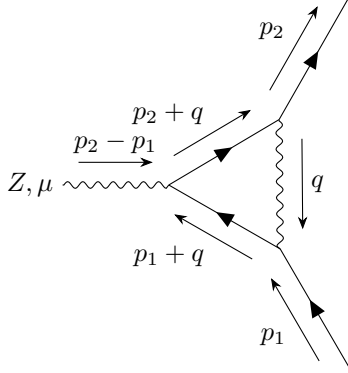
$$= \int \frac{d^d q}{(2\pi)^d} \left(-ieQ_f \gamma^\beta\right) \frac{i\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1+q)^2 - m^2} \frac{ig}{\cos\theta_W} \gamma^\mu \left(g_V - g_A\gamma_5\right) \frac{i\left(\slashed{p}_2 + \slashed{q} + m\right)}{(p_2+q)^2 - m^2} \times$$

$$\times \left(-ieQ_f \gamma^\alpha\right) \frac{-ig_{\alpha\beta}}{q^2} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{e^2 Q_f^2 g}{\cos\theta_W} \gamma^\beta \frac{\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1+q)^2 - m^2} \gamma^\mu \left(g_V - g_A\gamma_5\right) \frac{\left(\slashed{p}_2 + \slashed{q} + m\right)}{(p_2+q)^2 - m^2} \gamma_\beta \frac{1}{q^2} =$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = \frac{e^2 Q_f^2 g}{\cos\theta_W} \left\{ \bar{\gamma}^\beta \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m\right) \bar{\gamma}^\mu \left(g_V - g_A\gamma_5\right) \left(\bar{\slashed{p}}_2 + \bar{\slashed{q}} + m\right) \bar{\gamma}_\beta \right\} = \frac{e^2 Q_f^2 g}{\cos\theta_W} \left\{ \gamma^\beta \left(\slashed{p}_1 + \slashed{q} + m\right) \gamma^\mu \left(g_V - g_A\gamma_5\right) \times \right.$$

$$\times \left(\slashed{p}_2 + \slashed{q} + m\right) \gamma_\beta + \tilde{\gamma}^\beta \left(\slashed{p}_1 + \slashed{q} + m\right) \gamma^\mu \left(g_V - g_A\gamma_5\right) \left(\slashed{p}_2 + \slashed{q} + m\right) \tilde{\gamma}_\beta + \left(\gamma^\beta + \tilde{\gamma}^\beta\right) \tilde{\slashed{q}} \gamma^\mu \left(g_V - g_A\gamma_5\right) \tilde{\slashed{q}} \left(\gamma_\beta + \tilde{\gamma}_\beta\right) \right\} =$$

$$\equiv N + \tilde{N}$$

$$\tilde{N} = \frac{e^2 Q_f^2 g}{\cos\theta_W} \left\{ \left(\slashed{p}_1 + \slashed{q} - m\right) \tilde{\gamma}^\beta \gamma^\mu \tilde{\gamma}_\beta \left(g_V - g_A\gamma_5\right) \left(\slashed{p}_2 + \slashed{q} - m\right) + \gamma^\beta \tilde{\slashed{q}} \gamma^\mu \left(g_V - g_A\gamma_5\right) \tilde{\slashed{q}} \gamma_\beta + \tilde{\gamma}^\beta \tilde{\slashed{q}} \gamma^\mu \left(g_V - g_A\gamma_5\right) \tilde{\slashed{q}} \tilde{\gamma}_\beta \right\} =$$

$$= \frac{e^2 Q_f^2 g}{\cos\theta_W} \left\{ -\epsilon \left(\slashed{p}_1 + \slashed{q} - m\right) \gamma^\mu \left(g_V - g_A\gamma_5\right) \left(\slashed{p}_2 + \slashed{q} - m\right) - \tilde{q}^2 \gamma^\beta \gamma^\mu \gamma_\beta \left(g_V + g_A\gamma_5\right) - \tilde{q}^2 \left(-\epsilon\gamma^\mu\right) \left(g_V - g_A\gamma_5\right) \right\} =$$

$$= \frac{e^2 Q_f^2 g}{\cos\theta_W} \left\{ -\epsilon \left(\slashed{p}_1 + \slashed{q} - m\right) \gamma^\mu \left(g_V - g_A\gamma_5\right) \left(\slashed{p}_2 + \slashed{q} - m\right) + \tilde{q}^2 \left(2\gamma^\mu \left(g_V + g_A\gamma_5\right) + \epsilon\gamma^\mu \left(g_V - g_A\gamma_5\right)\right) \right\} =$$

$$R_2^{Zff} = \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0} = \frac{e^2 Q_f^2 g}{(2\pi)^4 \cos\theta_W} \int d^d\bar{q} \frac{1}{\bar{D}_1 \bar{D}_2 \bar{D}_0} \left\{ -\epsilon\slashed{q}\gamma^\mu \left(g_V - g_A\gamma_5\right) \slashed{q} + \tilde{q}^2 \left(2\gamma^\mu \left(g_V + g_A\gamma_5\right) + \right. \right.$$

$$+ \epsilon\gamma^\mu \left(g_V - g_A\gamma_5\right)\right\} = \frac{e^2 Q_f^2 g}{\cos\theta_W} \left\{ -\epsilon \left(-\frac{i\pi^2}{2\epsilon}\right) g_{\alpha\beta}\gamma^\alpha\gamma^\mu\gamma^\beta \left(g_V + g_A\gamma_5\right) + 2 \left(-\frac{i\pi^2}{2}\right) \gamma^\mu \left(g_V + g_A\gamma_5\right) \right\} =$$

$$= \frac{e^2 Q_f^2 g}{\cos\theta_W} \left(-\frac{i\pi^2}{2}\right) \gamma^\mu \left\{ 2 \left(g_V + g_A\gamma_5\right) + 2 \left(g_V + g_A\gamma_5\right) \right\} = \frac{-ie^2 Q_f^2 g}{8\pi^2 \cos\theta_W} \gamma^\mu \left(g_V + g_A\gamma_5\right) \tag{3.5}$$

where we used that scalar 3-point integrals do not contribute to R_2. The last term in the integral is of order ϵ so it will not contribute in the limit $\epsilon \to 0$

Higgs-fermion Yukawa vertex
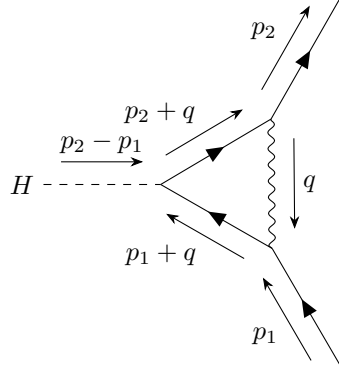
$$= \int \frac{d^d q}{(2\pi)^d} \left(-ieQ_f\gamma^\beta\right) \frac{i\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1+q)^2 - m^2} \left(-\frac{ig}{2}\frac{m}{m_W}\right) \frac{i\left(\slashed{p}_2 + \slashed{q} + m\right)}{(p_2+q)^2 - m^2} \left(-ieQ_f\gamma^\alpha\right) \frac{-ig_{\alpha\beta}}{q^2} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{-e^2 Q_f^2 gm}{2m_W} \gamma^\beta \frac{\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1+q)^2 - m^2} \gamma^\mu \frac{\left(\slashed{p}_2 + \slashed{q} + m\right)}{(p_2+q)^2 - m^2} \gamma_\beta \frac{1}{q^2} =$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = \frac{e^2 Q_f^2 gm}{2m_W} \bar{\gamma}^\beta \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m\right) \left(\bar{\slashed{p}}_2 + \bar{\slashed{q}} + m\right) \bar{\gamma}_\beta =$$

$$= \frac{e^2 Q_f^2 gm}{2m_W} \left\{ \gamma^\beta \left(\slashed{p}_1 + \slashed{q} + m\right) \left(\slashed{p}_2 + \slashed{q} + m\right) \gamma_\beta + \tilde{\gamma}^\beta \left(\slashed{p}_1 + \slashed{q} + m\right) \left(\slashed{p}_2 + \slashed{q} + m\right) \tilde{\gamma}_\beta + \gamma^\beta \tilde{\slashed{q}}\tilde{\slashed{q}} \gamma_\beta \right\} \equiv N + \tilde{N}$$

$$\tilde{N} = -\frac{e^2 Q_f^2 gm}{2m_W} \left\{ \tilde{\gamma}^\beta \left(\slashed{p}_1 + \slashed{q} + m\right) \left(\slashed{p}_2 + \slashed{q} + m\right) \tilde{\gamma}_\beta + \gamma^\beta \tilde{\slashed{q}}\tilde{\slashed{q}} \gamma_\beta \right\} = -\frac{e^2 Q_f^2 gm}{2m_W} \left\{ \tilde{\gamma}^\beta \tilde{\gamma}_\beta \slashed{q}\slashed{q} + \tilde{\slashed{q}}\tilde{\slashed{q}} \gamma^\beta \gamma_\beta \right\}$$

$$R_2 = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_0 \bar{D}_2} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{1}{\bar{D}_1 \bar{D}_0 \bar{D}_2} \left(-\frac{e^2 Q_f^2 gm}{2m_W}\right) \left\{ \tilde{\gamma}^\beta \tilde{\gamma}_\beta \slashed{q}\slashed{q} + \tilde{\slashed{q}}\tilde{\slashed{q}} \gamma^\beta \gamma_\beta \right\} =$$

$$-\frac{e^2 Q_f^2 gm}{2m_W} \left\{ \epsilon \left(-\frac{i\pi^2}{2\epsilon}\right) \gamma^\alpha \gamma^\beta g_{\alpha\beta} \slashed{q}\slashed{q} + \tilde{\slashed{q}}\tilde{\slashed{q}} \gamma^\beta \gamma_\beta + \left(-\frac{i\pi^2}{2}\right) 4 \right\} = \frac{-e^2 Q_f^2 gm}{(2\pi)^4 2m_W} \left(-\frac{i\pi^2}{2}\right) 8 = \frac{ie^2 Q_f^2 gm}{8\pi^2 m_W}$$

4 Pure QED Renormalization

Explain how to express renormalization constants in terms of scalar integrals.

4.1 2-point functions

Photon self-energy

$$= \int \frac{d^4q}{(2\pi)^4} (-1) \, \text{Tr} \left\{ ie\gamma^\alpha \frac{i\left(\not{p}+\not{q}+m\right)}{(p+q)^2-m^2} ie\gamma^\beta \frac{i\left(\not{q}+m\right)}{q^2-m^2} \right\} \equiv \mathcal{A}$$

Let's work on the trace so we can express the numerator of the 2-point function in terms of scalar integrals.

$$\text{Tr}\left\{\gamma^\alpha\left(\not{p}+\not{q}+m\right)\gamma^\beta\left(\not{q}+m\right)\right\} = \text{Tr}\left\{m^2\gamma^\alpha\gamma^\beta + \gamma^\alpha\left(\not{p}+\not{q}\right)\gamma^\beta\not{q}\right\} =$$
$$= 4\left\{m^2 g^{\alpha\beta} + (p+q)_\mu q_\nu \left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right)\right\} =$$
$$= 4\left(m^2 g^{\alpha\beta} + (p+q)^\alpha q^\beta - g^{\alpha\beta}(p+q)\cdot q + g^\alpha(p+q)^\beta\right)$$

$$\mathcal{A} = -4e^2 \int \frac{d^4q}{(2\pi)^4} \frac{m^2 g^{\alpha\beta} + p^\alpha q^\beta + q^\alpha p^\beta + 2q^\alpha q^\beta - g^{\alpha\beta} p\cdot q - g^{\alpha\beta} q^2}{\left((p+q)^2-m^2\right)(q^2-m^2)} =$$
$$= -\frac{4ie^2}{16\pi^2}\left\{m^2 B_0 g^{\alpha\beta} + 2p^{\alpha\beta}B_1 + 2\left(B_{11}p^\alpha p^\beta + B_{00}g^{\alpha\beta}\right) - g^{\alpha\beta}B_1 p^2 - g^{\alpha\beta}\left(4B_{00} + B_{11}p^2\right)\right\} =$$
$$= -\frac{ie^2}{4\pi^2}\left\{g^{\alpha\beta}\left(m^2 B_0 - B_1 p^2 + B_{11}p^2 - 2B_{00}\right) + 2p^\alpha p^\beta \left(B_1 + B_{11}\right)\right\}$$

The arguments of the scalar integrals are suppressed to keep the notation compact. They are the same for all B-functions: $B_i = B_i(p^2, m^2, m^2)$.

The expression can be further simplified using identities between the scalar integrals.

Electron self-energy

$$= \int \frac{d^4q}{(2\pi)^4} ie\gamma^\alpha \frac{i\left(\not{p}+\not{q}+m\right)}{(p+q)^2-m^2} ie\gamma^\beta \frac{-ig_{\alpha\beta}}{q^2} = \int \frac{d^4q}{(2\pi)^4} \left(-e^2\right)\gamma^\alpha \frac{\left(\not{p}+\not{q}+m\right)}{(p+q)^2-m^2}\gamma_\alpha \frac{1}{q^2} \equiv \mathcal{A}$$

With a bit of gamma-matrix algebra the numerator can be written as

$$\gamma^\beta\left(\not{p}+\not{q}+m\right)\gamma_\beta = (p+q)_\alpha \gamma^\beta\gamma^\alpha\gamma_\beta + m\gamma^\beta\gamma_\beta = 4m - 2\left(\not{p}+\not{q}\right)$$
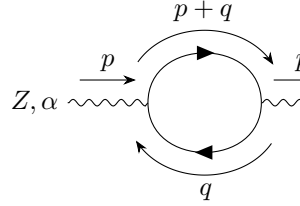
$$\mathcal{A} = -e^2 \int \frac{d^4q}{(2\pi)^4} \frac{4m - 2\left(\not{p} + \not{q}\right)}{\left((p+q)^2 - m^2\right)q^2} =$$

$$= -\frac{ie^2}{16\pi^2}\left[4mB_0 - 2\not{p}\left(B_0 + B_1\right)\right] = \frac{-ie^2}{8\pi^2}\left(2mB_0 - \not{p}\left(B_0 + B_1\right)\right)$$

Where the arguments of the B-functions are suppressed again. They are $B_i = B_i(p^2, 0, m^2)$

5 QED Contribution to the Standard Model Renormalization

5.1 2-point functions

Photon/Z-boson mixed self-energy

$$Z, \alpha \sim\!\!\!\!\sim \quad \overset{p}{\underset{q}{\bigcirc}} \quad \sim\!\!\!\!\sim \beta, \gamma \;=\; \int \frac{d^4q}{(2\pi)^4}\,(-1)\,\mathrm{Tr}\left\{(-ieQ_f)\,\gamma^\alpha \frac{i\left(\slashed{p}+\slashed{q}+m\right)}{(p+q)^2-m^2}\,i\frac{g}{\cos\theta_W}\gamma^\beta\left(g_V-g_A\gamma_5\right)\frac{i\left(\slashed{q}+m\right)}{q^2-m^2}\right\}\equiv\mathcal{A}$$
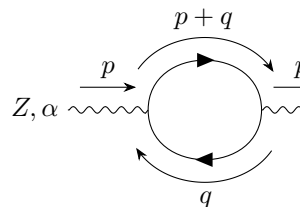
Let's work on the trace so we can express the numerator of the 2-point function in terms of scalar integrals.

$$\mathrm{Tr}\left\{\gamma^\alpha\left(\slashed{p}+\slashed{q}+m\right)\gamma^\beta\left(g_V-g_A\gamma_5\right)\left(\slashed{q}+m\right)\right\}=g_V\,\mathrm{Tr}\left\{\gamma^\alpha\left(\slashed{p}+\slashed{q}\right)\gamma^\beta\slashed{q}+m^2\gamma^\alpha\gamma^\beta\right\}-g_A\,\mathrm{Tr}\left\{\gamma^\alpha\left(\slashed{p}+\slashed{q}\right)\gamma^\beta\gamma_5\slashed{q}\right\}=$$

$$=4g_V\left\{(p+q)_\mu\,q_\nu\left(g^{\alpha\mu}g^{\beta\nu}-g^{\alpha\beta}g^{\mu\nu}+g^{\alpha\nu}g^{\beta\mu}\right)+m^2g^{\alpha\beta}\right\}-4ig_A\,(p+q)_\mu\,q_\nu\epsilon^{\alpha\mu\beta\nu}=$$

$$=4\left\{g_V\left[(p+q)^\alpha\,q^\beta-g^{\alpha\beta}\,(p+q)\cdot q+q^\alpha\,(p+q)^\beta+m^2g^{\alpha\beta}\right]-ig_A\epsilon^{\alpha\mu\beta\nu}p_\mu q_\nu\right\}$$

Where we used that a symmetric tensor contracted with an antisymmetric tensor vanishes.

$$\mathcal{A}=\frac{4Q_f eg}{\cos\theta_W}\int\frac{d^4q}{(2\pi)^4}\frac{g_V\left((p+q)^\alpha\,q^\beta-g^{\alpha\beta}\,(p+q)\cdot q+q^\alpha\,(p+q)^\beta+m^2g^{\alpha\beta}\right)-ig_A\epsilon^{\alpha\mu\beta\nu}p_\mu q_\nu}{\left((p+q)^2-m^2\right)\left(q^2-m^2\right)}=$$

$$=\frac{4Q_f eg}{\cos\theta_W}\frac{i\pi^2}{(2\pi)^4}\left\{-ig_A\epsilon^{\alpha\mu\beta\nu}p_\mu B_1 p_\nu+g_V\left[B_1 p^\alpha p^\beta+B_{00}g^{\alpha\beta}+B_{11}p^\alpha p^\beta-g^{\alpha\beta}\left(B_1 p^2+4B_{00}+B_{11}p^2\right)+\right.\right.$$

$$\left.\left.+B_1 p^\alpha p^\beta+B_{00}g^{\alpha\beta}+B_{11}p^\alpha p^\beta+B_0 m^2 g^{\alpha\beta}\right]\right\}=$$

$$=\frac{iQ_f egg_V}{4\pi^2\cos\theta_W}\left\{2p^\alpha p^\beta\left(B_1+B_{11}\right)+g^{\alpha\beta}\left(m^2 B_0-2B_{00}-p^2\left(B_1+B_{11}\right)\right)\right\}$$

Z-Boson self-energy
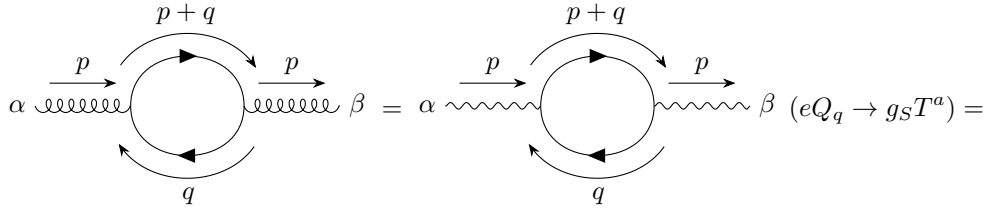
$$Z, \alpha \sim\!\!\!\!\sim \quad \overset{p}{\underset{q}{\bigcirc}} \quad \sim\!\!\!\!\sim \beta, Z \;=\; \int \frac{d^4q}{(2\pi)^4}\,(-1)\,\mathrm{Tr}\left\{\frac{ig}{\cos\theta_W}\gamma^\alpha\left(g_V-g_A\gamma_5\right)\frac{i\left(\slashed{p}+\slashed{q}+m\right)}{(p+q)^2-m^2}\frac{ig}{\cos\theta_W}\gamma^\beta\times\right.$$

$$\left.\times\left(g_V-g_A\gamma_5\right)\frac{i\left(\slashed{q}+m\right)}{q^2-m^2}\right\}\equiv\mathcal{A}$$

Let's work on the trace so we can express the numerator of the 2-point function in terms of scalar integrals.

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$$\text{Tr}\left\{\gamma^\alpha \left(g_V - g_A\gamma_5\right)\left(\slashed{p}+\slashed{q}+m\right)\gamma^\beta \left(g_V - g_A\gamma_5\right)\left(\slashed{q}+m\right)\right\} = \text{Tr}\left\{\gamma^\alpha \left(g_V - g_A\gamma_5\right)^2 \left(\slashed{p}+\slashed{q}-m\right)\gamma^\beta \left(\slashed{q}+m\right)\right\} =$$

$$= \left(g_V^2 + g_A^2\right)\text{Tr}\left\{\gamma^\alpha \left(\slashed{p}+\slashed{q}-m\right)\gamma^\beta \left(\slashed{q}+m\right)\right\} = \left(g_V^2 + g_A^2\right)\text{Tr}\left\{\gamma^\alpha \left(\slashed{p}+\slashed{q}\right)\gamma^\beta \slashed{q} - m^2\gamma^\alpha\gamma^\beta\right\} =$$

$$= \left(g_V^2 + g_A^2\right)\left\{(p+q)_\mu\, q_\nu 4\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right) - 4m^2 g^{\alpha\beta}\right\} =$$

$$= 4\left(g_V^2 + g_A^2\right)\left\{(p+q)^\alpha\, q^\beta - g^{\alpha\beta}\, p+q)\cdot q + q^\alpha\, (p+q)^\beta - m^2 g^{\alpha\beta}\right\}$$

$$\mathcal{A} = \int \frac{d^4q}{(2\pi)^4}\frac{4g^2\left(g_V^2 + g_A^2\right)}{\cos^2\theta_W}\frac{(p+q)^\alpha\, q^\beta - g^{\alpha\beta}\, p+q)\cdot q + q^\alpha\, (p+q)^\beta - m^2 g^{\alpha\beta}}{\left((p+q)^2 - m^2\right)(q^2 - m^2)} =$$

$$= \frac{4g^2\left(g_V^2 + g_A^2\right)}{\cos^2\theta_W}\int \frac{d^4q}{(2\pi)^4}\frac{p^\alpha q^\beta + q^\alpha p^\beta + 2q^\alpha q^\beta - g^{\alpha\beta}\left(m^2 + q\cdot p + q^2\right)}{\left((p+q)^2 - m^2\right)(q^2 - m^2)} =$$

$$= \frac{4g^2\left(g_V^2 + g_A^2\right)}{\cos^2\theta_W}\frac{i\pi^2}{(2\pi)^4}\left\{2p^\alpha p^\beta B_1 + 2\left(B_{00}g^{\alpha\beta} + B_{11}p^\alpha p^\beta\right) - g^{\alpha\beta}\left(m^2 B_0 + B_1 p^2 + 4B_{00} + B_{11}p^2\right)\right\} =$$

$$= \frac{ig^2\left(g_V^2 + g_A^2\right)}{4\pi^2\cos^2\theta_W}\left\{p^\alpha p^\beta 2\left(B_1 + B_{11}\right) - g^{\alpha\beta}\left(m^2 B_0 + 2B_{00} + p^2\left(B_1 + B_{11}\right)\right)\right\}$$

Gluon self-energy

$$= -\frac{ig_S^2 \text{Tr}\left(T^a T^b\right)}{4\pi^2}\left\{2p^\alpha p^\beta \left(B_1 + B_{11}\right) + g^{\alpha\beta}\left(m^2 B_0 - 2B_{00} - p^2\left(B_1 + B_{11}\right)\right)\right\}$$

Appendices

A Important Integrals

In the calculation of R_2 the same integrals appear over and over. The most important ones are [1]

2-point integrals

$$\int d^d\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} = -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{3}\right] + O(\epsilon) \tag{A.1}$$

$$\text{P.P.}\left(\int d^d\bar{q}\,\frac{1}{\bar{D}_i\bar{D}_j}\right) = -2\frac{i\pi^2}{\epsilon} \tag{A.2}$$

$$\text{P.P.}\left(\int d^d\bar{q}\,\frac{q_\mu}{\bar{D}_i\bar{D}_j}\right) = \frac{i\pi^2}{\epsilon}\,(p_i + p_j)_\mu \tag{A.3}$$

3-point integrals

$$\int d^d\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} = -\frac{i\pi^2}{2} + O(\epsilon) \tag{A.4}$$

$$\int d^d\bar{q}\,\frac{\tilde{q}^2 q_\mu}{\bar{D}_i\bar{D}_j\bar{D}_k} = \frac{i\pi^2}{6}\,(p_i + p_j + p_k)_\mu + O(\epsilon) \tag{A.5}$$

$$\text{P.P.}\left(\int d^d\bar{q}\,\frac{q_\mu q_\nu}{\bar{D}_i\bar{D}_j\bar{D}_k}\right) = -\frac{i\pi^2}{2\epsilon}g_{\mu\nu} \tag{A.6}$$

4-point integrals

$$\int d^d\bar{q}\,\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{i\pi^2}{6} + O(\epsilon) \tag{A.7}$$

$$\int d^d\bar{q}\,\frac{\tilde{q}^2 q_\mu q_\nu}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{i\pi^2}{12}g_{\mu\nu} + O(\epsilon) \tag{A.8}$$

$$\int d^d\bar{q}\,\frac{\tilde{q}^2 q^2}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{i\pi^2}{3} + O(\epsilon) \tag{A.9}$$

B Traceology

The following dentities for Dirac matrices are very useful when calculating Feynman diagrams in QED

1. $\text{Tr}\left(\gamma^\alpha\gamma^\beta\right) = dg^{\alpha\beta}$

2. $\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$

3. $\gamma^\alpha\gamma_\alpha = d$

4. $\gamma^\alpha\gamma^\beta\gamma_\alpha = (2 - d)\gamma^\beta$

... ...

n. $\not{a}\not{b} = a \cdot b$

Dirac matrices obey Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_d$ with $g^{\mu\nu}$ the Minkowski metric in d dimensions

$$g^{\mu\nu} = \begin{cases} 1 & \text{for } \mu = \nu = 0 \\ -1 & \text{for } \mu = \nu = 1, 2, \ldots, d-1 \\ 0 & \text{for } \mu \neq \nu \end{cases}$$

Proofs for identities

1. $\mathrm{Tr}\left(\gamma^\alpha \gamma^\beta\right) = dg^{\alpha\beta}$

Proof.

$$\mathrm{Tr}\left(\gamma^\alpha \gamma^\beta\right) = \mathrm{Tr}\left(2g^{\alpha\beta} - \gamma^\beta \gamma^\alpha\right) = 2g^{\alpha\beta}\mathrm{Tr}\left(\mathbb{1}_d\right) - \mathrm{Tr}\left(\gamma^\beta \gamma^\alpha\right) = 2dg^{\alpha\beta} - \mathrm{Tr}\left(\gamma^\alpha \gamma^\beta\right)$$
$$\Rightarrow \mathrm{Tr}\left(\gamma^\alpha \gamma^\beta\right) = dg^{\alpha\beta}$$

$\square$

2. $\mathrm{Tr}\left(\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$

Proof.

$$\mathrm{Tr}\left(\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta\right) = \mathrm{Tr}\left(\left(2g^{\alpha\beta} - \gamma^\beta \gamma^\alpha\right)\gamma^\gamma \gamma^\delta\right) = 2g^{\alpha\beta}\mathrm{Tr}\left(\gamma^\gamma \gamma^\delta\right) - \mathrm{Tr}\left(\gamma^\beta\left(2g^{\alpha\gamma} - \gamma^\gamma \gamma^\alpha\right)\gamma^\delta\right) =$$
$$= 2dg^{\alpha\beta}g^{\gamma\delta} - 2g^{\alpha\gamma}\mathrm{Tr}\left(\gamma^\beta \gamma^\delta\right) + \mathrm{Tr}\left(\gamma^\beta \gamma^\gamma\left(2g^{\alpha\delta} - \gamma^\delta \gamma^\alpha\right)\right) =$$
$$= 2d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) + 2g^{\alpha\delta}\mathrm{Tr}\left(\gamma^\beta \gamma^\gamma\right) - \mathrm{Tr}\left(\gamma^\beta \gamma^\gamma \gamma^\delta \gamma^\alpha\right) =$$
$$= 2d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right) - \mathrm{Tr}\left(\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta\right)$$
$$\Rightarrow \mathrm{Tr}\left(\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$$

$\square$

3. $\gamma^\alpha \gamma_\alpha = d$

Proof.

$$\gamma^\alpha \gamma_\alpha = \frac{1}{2}\left(\gamma^\alpha \gamma_\alpha + \gamma_\alpha \gamma^\alpha\right) = \frac{1}{2}\left\{\gamma^\alpha, \gamma_\alpha\right\} = \frac{1}{2}2g^\alpha_\alpha = d$$

$\square$

4. $\gamma^\alpha \gamma^\beta \gamma_\alpha = (2-d)\gamma^\beta$

Proof.

$$\gamma^\alpha \gamma^\beta \gamma_\alpha = \left(2g^{\alpha\beta} - \gamma^\beta \gamma^\alpha\right)\gamma_\alpha = (2-d)\gamma^\beta$$

$\square$

n. $\slashed{a}\slashed{b} = a \cdot b$

Proof.

$$\slashed{a}\slashed{b} = a_\alpha b_\beta \gamma^\alpha \gamma^\beta = a_\alpha b_\beta\left(2g^{\alpha\beta} - \gamma^\beta \gamma^\alpha\right) = 2a \cdot b - \slashed{a}\slashed{b}$$
$$\Rightarrow \slashed{a}\slashed{b} = a \cdot b$$

$\square$

C   Relation Between Left- & Right-handed Currents and Axial & Vector Currents

A classical Lagrangian permits symmetries which can be implemented by Lie groups G. An element $g \in G$ of a Lie group can be parametrized as $g = \exp\left(i\alpha^a T^a\right)$ where α^a are real parameters and T^a the generators of the Lie group. Noether's theorem predicts a clasically conserved current for each generator of a continuous symmetry. For a field ϕ with trafo $\delta\phi = \phi' - \phi = g\phi - \phi \approx \left(1 + i\alpha^a T^a\right)\phi - \phi = i\alpha^a T^a \phi$ the conserved current can be shown to be

$$j^{\mu a} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\frac{\partial\delta\phi}{\partial\alpha_a}$$

The left- and right-handed part of the currents $j^{\mu a}_{L/R}$ are

$$j^{\mu a}_L = \bar{\psi}_L \gamma^\mu T^a \psi_L = \left(P_L \psi\right)^\dagger \gamma^0 \gamma^\mu T^a P_L \psi \overset{P_L^\dagger = P_L}{=} \psi^\dagger P_L \gamma^0 \gamma^\mu T^a P_L \psi = \bar{\psi}\gamma^\mu T^a P_L^2 \psi \overset{P_L^2 = P_L}{=} \bar{\psi}\gamma^\mu T^a P_L \psi$$

$$j^{\mu a}_R = \bar{\psi}_R \gamma^\mu T^a \psi_R = \bar{\psi}\gamma^\mu T^a P_R \psi$$

where $P_{L/R} = \frac{1}{2}\left(1 \mp \gamma_5\right)$ is the left-/right-handed projector.
From the left- and right-handed currents we can define axial-vector and vector currents

$$j^{\mu a} = j^{\mu a}_R + j^{\mu a}_L = \bar{\psi}\gamma^\mu T^a \left(P_R + P_L\right) = \bar{\psi}\gamma^\mu T^a \psi$$

$$j^{\mu a}_5 = j^{\mu a}_R - j^{\mu a}_L = \bar{\psi}\gamma^\mu T^a \left(P_R - P_L\right)\psi = \bar{\psi}\gamma^\mu T^a \gamma_5 \psi$$

Now we can couple the currents to vector fields to obtain interactions. E.g., the vector coupling in QED is given by the Lagrangian

$$\mathcal{L}^{QED}_{coupl} = eA_\mu j^\mu = eA_\mu \bar{\psi}\gamma^\mu Q_e \psi = -eA_\mu \bar{\psi}\gamma^\mu \psi$$

In general, we can couple any linear combination of currents to a vector field as long as the combination is Lorentz and gauge invariant. E.g., the neutral current in the electroweak theory is a superposition of a vector and an axialvector current

$$\mathcal{L}^{NC}_{coupl} = gZ_\mu \left(g_v j^\mu - g_A j^\mu_5\right)$$

We can use the above relations to express this coupling in terms of right- and left-handed currents

$$\begin{aligned}
\mathcal{L}^{NC}_{coupl} &= gZ_\mu \left(g_V \bar{\psi}\gamma^\mu \psi - g_A \bar{\psi}\gamma^\mu \gamma_5 \psi\right) = \\
&= gZ_\mu \left(g_V \bar{\psi}\gamma^\mu \psi + \frac{g_A}{2}\bar{\psi}\gamma^\mu \psi - \frac{g_A}{2}\bar{\psi}\gamma^\mu \psi - g_A \bar{\psi}\gamma^\mu \gamma_5 \psi + \frac{g_V}{2}\bar{\psi}\gamma^\mu \gamma_5 \psi - \frac{g_V}{2}\bar{\psi}\gamma^\mu \gamma_5 \psi\right) = \\
&= gZ_\mu \left(\left(g_V + g_A\right)\bar{\psi}\gamma^\mu \frac{1}{2}\left(1 - \gamma_5\right)\psi + \left(g_V - g_A\right)\bar{\psi}\gamma^\mu \frac{1}{2}\left(1 + \gamma_5\right)\psi\right) = \\
&= gZ_\mu \left(\left(g_V + g_A\right)\bar{\psi}\gamma^\mu P_L \psi + \left(g_V - g_A\right)\bar{\psi}\gamma^\mu P_R \psi\right) \equiv gZ_\mu \left(g_L j^\mu_L + g_R j^\mu_R\right)
\end{aligned}$$

This gives the following relation between the (axial-)vector and the left-/right-handed couplings

$$g_L = g_V + g_A$$

$$g_R = g_V - g_A$$

References

[1] P Draggiotis, M.V Garzelli, C.G Papadopoulos, and R Pittau. Feynman rules for the rational part of the qcd 1-loop amplitudes. *Journal of High Energy Physics*, 2009(04):072–072, Apr 2009.

[2] Giovanni Ossola, Costas G Papadopoulos, and Roberto Pittau. On the rational terms of the one-loop amplitudes. *Journal of High Energy Physics*, 2008(05):004–004, May 2008.

[3] Rolf Mertig, M. Böhm, and A. Denner. Feyn calc - computer-algebraic calculation of feynman amplitudes. *Computer Physics Communications*, 64:345–359, 06 1991.

[4] Vladyslav Shtabovenko, Rolf Mertig, and Frederik Orellana. New Developments in FeynCalc 9.0. *Comput. Phys. Commun.*, 207:432–444, 2016.

[5] Christian Schubert. On the γ_5–problem of dimensional renormalization. 1993.

[6] Gerard 't Hooft and M. J. G. Veltman. Regularization and Renormalization of Gauge Fields. *Nucl. Phys.*, B44:189–213, 1972.