Reduction of One-loop Tensor Form-Factors to Scalar Integrals: A General Scheme

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Abstract

A general method for reducing tensor form factors, that appear in one-loop calculations in dimensional regularization, to scalar integrals is presented. The method is an extension of the reduction scheme introduced by Passarino and Veltman [1] and is applicable in all regions of parameter space including those where kinematic Gram determinant vanishes. New relations between the the form factors that valid for vanishing Gram determinant play a key rôle in the extended scheme.

1 Introduction

The evaluation of one-loop tensor integrals, that was pioneered by Brown and Feynman [2] and systematized in the work of Passarino and Veltman [1], is by now a mature science. The latter authors defined a set of generic tensor form factors in terms of which any one-loop tensor integral could be written. Thus, for example, the 3-point tensor integral with two Lorentz indices,

$$C_{\mu\nu}(p_1, p_2; m_1^2, m_2^2, m_3^2) = \int \frac{d^n q}{i\pi^2} \frac{q_\mu q_\nu}{[q^2 + m_1^2][(q + p_1)^2 + m_2^2][(q + p_1 + p_2)^2 + m_3^2]}, \quad (1)$$

is written in terms of generic tensor form factors $C_{ij}(p_1^2, p_2^2, p_5^2; m_1^2, m_2^2, m_3^2)$ defined by

$$C_{\mu\nu}(p_1, p_2; m_1^2, m_2^2, m_3^2) = p_{1\mu}p_{1\mu}C_{21} + p_{2\mu}p_{2\nu}C_{22} + \{p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu}\}C_{23} + \delta_{\mu\nu}C_{24}$$
(2)

where $p_5 = p_1 + p_2$ and we have omitted the arguments of the C_{ij} 's in eq.(2). This choice for defining the C_{ij} 's has the obvious advantage of its great simplicity. Passarino and Veltman showed how the C_{ij} 's and their analogous 4-point form factors, D_{ij} could be reduced to scalar integrals and 2-point form factors.

In ref.[3] it was shown how to reduce the C_{ij} , D_{ij} to unique form consisting entirely of scalar integrals. The advantages of doing this are manifold. The resulting expressions are often simpler in form, containing just a few scalar integrals with coefficients that are rational functions of the masses and momenta. All non-trivial analytic structure, such as cuts in the complex plane, are isolated in the scalar integrals which can be evaluated separately taking care of these technicalities [4, 5]. Having a general and well-defined procedure for reducing tensor form factors to scalar integrals lends itself to automation and means that quite lengthy calculations can be undertaken. In such calculations it is essential that stringent checks be performed as to the correctness of the final results. These require the comparison of two algebraic expressions whose equality can only be determined if both have been fully reduced to a unique form containing only scalar integrals with no remaining tensor form factors [6, 7].

Passarino and Veltman applied their reduction scheme numerically to the to the process $e^+e^- \to \mu^+\mu^-$ and in doing so highlighted a significant disadvantage of their approach. It is sometimes the case that strong numerical cancellations occur during reduction. These authors, for example, found it necessary to calculate to 40 significant figures in order to obtain sufficient accuracy. Performing the reduction algebraically as in ref.[3] has the advantage of eliminating many of the strong numerical cancellations and hence leads to improved numerical stability. It can also lead to rather compact and convenient expressions for physical observables.

Still the reduction scheme of Passarino and Veltman has other vexing problems. The reduction formulas for 3- and 4-point functions involve the inversion of kinematic matrices and therefore it breaks down when the determinants of these matrices, Gram determinants, vanish. In ref.[3] it was shown how to extend the Passarino-Veltman reduction scheme to treat situations of vanishing Gram determinants. This was based on the observation [8] that in such cases, the *n*-point form factors reduce to (n-1)-point form factors. This extended reduction scheme was implemented in the REDUCE program LERG-I [3]. Even here, the procedure broke down for certain special cases of external momenta and internal masses. In ref.[9] extensions were made to enlarge the region of parameter space for which the reduction could be performed but some problematic holes nevertheless remain. The extended scheme was implemented in both REDUCE [9] and Mathematica [10]. In this paper we present an alternative reduction scheme for the case of vanishing Gram determinant derived directly from the original Passarino-Veltman approach. New techniques involving differentiation with respect to mass arguments are introduced that make this and earlier schemes viable everywhere in parameter space, except perhaps where true infrared divergences or mass singularities appear.

Although expressions that have been algebraically reduced to scalar integrals exhibit improved numerical stability over those in which the tensor form factors are calculated numerically in a hierarchical fashion, they can still become problematic as kinematic determinants become small. This problem was addressed by van Oldenborgh and Vermaseren [11] who employed a different tensor basis, constructed from so-called inverse vectors, to define the form factors. This approach has the effect of simplifying the contraction of Lorentz indices and yields expressions that can exhibit improved numerical stability for small Gram determinants. Since the form factors that they obtain are linear combinations of those defined by Passarino and Veltman,

their method can then be viewed as a way of arranging expressions to reduce numerical instabilities but it does not show, as they state, that the approach of ref.[3] "is most suited for numerical application".

Ezawa et al. [12] have performed the reduction of the tensor form factors using an orthonormal tensor basis.

Pittau [13] has proposed a reduction method, based on γ -algebra, that avoids the appearance of Gram determinants and is applicable for loop integrals with at least two massless external legs.

More recently Campbell et al. [14] have pointed out that the Gram determinants are artifacts of the procedure used to reduce tensor form factors to scalar integrals. They show how to write Feynman parameter representations of tensor integral form factors in terms of scalar integrals in higher dimensions and derivatives of ordinary scalar integrals. The various Feynman parameter integrals are separately finite and highly stable as the condition for vanishing Gram determinant is approached. In practice, however, evaluating the Feynman parameter integrals yields terms between which there is eventually a strong additive cancellation and numerical instabilities do set in although their appearance is much delayed.

Other tensor integral reduction schemes have been put forward. In quantum gravity, the computation of tensor integrals by reduction to scalar integrals was pioneered by Capper and Liebbrandt [15] both for the graviton self-energy and the fictitious particle loop. Davydychev has [16] obtained reduction formulas from relations between integrals in different numbers of dimensions. Recursion relations obtained recently by Tarasov [17] have completed the picture to make this a practical approach but here again problems arise when the Gram determinant vanishes. Bern et al.[18] derive tensor integrals by differentiation of Feynman parameter representations of scalar integrals and introduced many of the basic techniques employed in ref.[14].

All reduction schemes have certain strengths and weaknesses and may be more conveniently applied in certain circumstances than others. In the scheme we present here we take the point of view that one important advantage of Passarino-Veltman reduction is the simplicity in the way the formfactors are defined (c.f. eq.(2)). This simplicity may come at the price of numerical instability near phase space boundaries but even there appropriate Taylor series expansion of the scalar form factors may eliminate it. In such cases other methods which employ more numerically stable form factors may be more easily applied. The methods given here can also lead to lengthy algebraic expressions when there are many different masses and momenta in the problem. These expressions can be costly in terms of computer storage and processing time although such issues are of ever-declining importance.

In this work we consider the form factors, C_{ij} and D_{ij} , as well-defined analytic functions of their arguments everywhere in parameter space. Feynman parameter representations exist for these functions which, can be expected to be analytic, at least in the physical region, except for isolated singularities. The reduction scheme is one way of obtaining an expression for, or value of, these functions in a given region of parameter space. Note that this means, for example, that C_{21} and C_{22} with arguments $C_{ij}(p^2, p^2, 0; m_1^2, m_2^2, m_2^3)$ remain well-defined entities even though, with $p_1 = p_2$, eq.(2) does not uniquely specify them. It also means that relations such as (2) are valid everywhere in the parameter space. This approach lends itself well to implementations with computer algebra.

Our aim here is to obtain valid analytic expressions for the tensor form factors in terms of scalar integrals for all regions of the parameter space where those form factors do not experience genuine singularities. We do not address the problem of numerical stability, since it has been intensely studied elsewhere, but concentrate rather on on the region where the Gram determinant exactly vanishes. A new ingredient is the use of additional relations between tensor integrals and scalar integrals that are valid only for vanishing Gram determinant. These relations actually follow directly and straightforwardly from the original reduction formulas given by Passarino and Veltman.

At first sight it might seem somewhat strange that there is such difficulty in evaluating the tensor form factors for vanishing Gram determinant. After all one could, in principle, vary one of the arguments away from region of vanishing Gram determinant and then use l'Hôpital's rule to take the limit. One finds, however, that this procedure regenerates a vanishing Gram determinant in the denominator of the resulting expressions. The new relations between the form factors allow l'Hôpital's rule to be applied should it be necessary in extremely singular situations.

2 Notation

In dimensional regularization the dimensionality of space-time, n, is continued away from n=4. Quartic, quadratic and logarithmic divergences correspond to poles at even integer $n \geq 0$, 2 and 4 respectively. One introduces

the logarithmically divergent constant,

$$\Delta = \pi^{\frac{n}{2} - 2} \Gamma\left(2 - \frac{n}{2}\right) \tag{3}$$

$$\rightarrow \frac{-2}{n-4} - \gamma - \ln \pi \tag{4}$$

as $n \to 4$ where γ is Euler's constant.

Note that it is essential to keep careful track of background terms that may be generated when Δ is multiplied by polynomials in n. For example

$$(n-a)\Delta \rightarrow -2 + (4-a)\Delta$$

as $n \to 4$.

A detailed exposition of dimensional regularization is to be found in refs [1, 19] along with results for certain standard integrals. For our purposes we need in addition,

$$\int d^{n}p \, \frac{p_{\kappa}p_{\lambda}p_{\mu}p_{\nu}}{(p^{2}+2k\cdot p+m^{2})^{\alpha}} = \frac{i\pi^{\frac{n}{2}}}{(m^{2}-k^{2})^{\alpha-\frac{n}{2}}} \frac{1}{\Gamma(\alpha)} \left\{ \Gamma\left(\alpha-\frac{n}{2}\right) k_{\kappa}k_{\lambda}k_{\mu}k_{\nu} + \frac{1}{2}\Gamma\left(\alpha-1-\frac{n}{2}\right)(m^{2}-k^{2})\{kk\delta\}_{\kappa\lambda\mu\nu} \right\} + \frac{1}{4}\Gamma\left(\alpha-2-\frac{n}{2}\right)(m^{2}-k^{2})^{2}\{\delta\delta\}_{\kappa\lambda\mu\nu} \right\}$$
(5)

In defining tensor integrals we shall adopt the notation of ref.[1] in which braces denote symmetrization with respect to Lorentz indices. Thus for example,

$$\{p\delta\}_{\mu\nu\alpha} = p_{\mu}\delta_{\nu\alpha} + p_{\nu}\delta_{\mu\alpha} + p_{\alpha}\delta_{\mu\nu} \tag{6}$$

$$\{\delta\delta\}_{\mu\nu\alpha\beta} = \delta_{\mu\nu}\delta_{\alpha\beta} + \delta_{\mu\alpha}\delta_{\nu\beta} + \delta_{\mu\beta}\delta_{\nu\alpha} \tag{7}$$

We will use the notation $B_0(i, j)$ and $C_0(i, j, k)$ in context to denote the two- and three-point scalar integrals constructed from the *i*-th, *j*-th and *k*-th denominators of some tensor integral.

3 One- and two-point form factors

Following Passarino and Veltman [1] we use a metric in which the squares of time-like momenta are negative.

The one-point scalar integral, $A(m^2)$, is quadratically divergent and defined as,

$$A(m^2) = \int \frac{d^n q}{i\pi^2} \frac{1}{q^2 + m^2}$$
 (8)

$$= \frac{\pi^{\frac{n}{2}-2}}{(m^2)^{1-\frac{n}{2}}} \Gamma\left(1 - \frac{n}{2}\right) \tag{9}$$

The two-point tensor form factors, $B_{ij}(p^2; m_1^2, m_2^2)$, are defined in Appendix A. These form factors are not independent of one another and may all be reduced to the form,

$$B_{ij}(p^2; m_1^2, m_2^2) \rightarrow \alpha_1 A(m_2^2) + \beta_{12} B_0(p^2; m_1^2, m_2^2) + \beta_1 B_0(0; m_1^2, m_1^2) + \beta_2 B_0(0; m_2^2, m_2^2) + \beta_0 \quad (10)$$

in which α_1 and the β 's are rational functions of the momenta and masses, p^2 , m_1^2 , m_2^2 ,

$$B_0(p^2; m_1^2, m_2^2) = \int \frac{d^n q}{i\pi^2} \frac{1}{[q^2 + m_1^2][(q+p)^2 + m_2^2]}$$
(11)

and

$$B_0(0; m^2, m^2) = \Delta - \ln m^2 \tag{12}$$

The coefficient α_1 is non-zero only for quadratically divergent B_{ij} . The form factor B_{43} is quartically divergent and requires the introduction of an additional term but we do not make use of it explicitly in the the present work. In addition, the partial derivatives of any form factor B_{ij} with respect to any of its three arguments can be written in the form (10).

By performing the usual Feynman parameterization of the integral (11) one obtains,

$$B_{i1}(p^2; m_1^2, m_2^2) = (-)^i \pi^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \ (13)^{n-2} dz = (-)^i \pi^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \ (13)^{n-2} dz = (-)^i \pi^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \ (13)^{n-2} dz = (-)^i \pi^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \ (13)^{n-2} dz = (-)^i \pi^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \ (13)^{n-2} dz = (-)^i \pi^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \int_0^1 dz \ z^i [az^2 + bz + c]^{\frac{n}{2} - 2} \Gamma(2 - \frac{n}{2}) \Gamma(2 - \frac{n$$

$$= (-)^{i} \int_{0}^{1} dz \ z^{i} [\Delta - \ln(az^{2} + bz + c)]$$
 (14)

where $B_{01} \equiv B_0$, $B_{11} \equiv B_1$ and $a = -p^2$, $b = p^2 - m_1^2 + m_2^2$, $c = m_1^2 - i\epsilon$. Eq.(14) follows from eq.(13) by dropping terms of $\mathcal{O}(n-4)$ and higher. Such terms will be discarded without further comment in what follows. In the region of n = 4 these integrals become

$$B_{i1}(p^2; m_1^2, m_2^2) = \frac{(-)^i}{i+1} \left\{ \Delta - \ln(a+b+c) + \int_0^1 dz \frac{z^{i+1}(2az+b)}{az^2 + bz + c} \right\}$$
(15)

The integrand can be decomposed as.

$$\frac{z^{i+1}(2az+b)}{az^2+bz+c} = \sum_{j=0}^{i} \alpha_j z^j + \frac{(\beta_1 z + \beta_0)(2az+b)}{az^2+bz+c}$$
(16)

Carrying out the integration for all but the term proportional to β_1 and comparing with (12) and (15) for i = 0 yields,

$$B_{i1}(p^{2}; m_{1}^{2}, m_{2}^{2}) = \frac{(-)^{i}}{i+1} \left\{ \beta_{1} B_{0}(p^{2}; m_{1}^{2}, m_{2}^{2}) + \beta_{0} B_{0}(0; m_{1}^{2}, m_{1}^{2}) + (1 - \beta_{0} - \beta_{1}) B_{0}(0, m_{2}^{2}, m_{2}^{2}) + \sum_{i=0}^{i} \frac{1}{j+1} \alpha_{j} \right\}$$
(17)

Full expressions for the form factors B_{ij} can be found in ref.[3] and will not be repeated here.

The partial derivative of the $B_{i1}(p^2; m_1^2, m_2^2)$ with respect to p^2 form factors arise in practice in wavefunction renormalization factors and those with respect to the mass arguments are needed in intermediate steps in the reduction to scalar integrals at kinematic boundaries. Because of the reduction (10), it follows that one only needs expressions for the partial derivatives of B_0 and it turns out that the partial derivatives themselves take the form (10). From eq.(15) one has for the region of n = 4,

$$\frac{\partial B_0}{\partial p^2}(p^2; m_1^2, m_2^2) = -\int_0^1 dz \frac{z(1-z)}{az^2 + bz + c}$$
 (18)

$$\frac{\partial B_0}{\partial m_1^2}(p^2; m_1^2, m_2^2) = -\int_0^1 dz \frac{(1-z)}{az^2 + bz + c}$$
 (19)

The partial derivative with respect to m_2^2 easily follows since B_0 is symmetric in its mass arguments.

Writing,

$$\frac{z(z-1)}{az^2 + bz + c} = \alpha_0 + \beta_1 \frac{z(2az+b)}{az^2 + bz + c} + \beta_0 \frac{(2az+b)}{az^2 + bz + c}$$
(20)

in (18) for which,

$$\alpha_0 = -\frac{b}{a} \frac{2a+b}{b^2-4ac}, \qquad \beta_1 = \frac{1}{a} \frac{b^2-2ac+ab}{b^2-4ac}, \qquad \beta_0 = \frac{c}{a} \frac{2a+b}{b^2-4ac}$$

and

$$\frac{1-z}{az^2+bz+c} = \alpha_0 + \beta_1 \frac{z(2az+b)}{az^2+bz+c} + \beta_0 \frac{(2az+b)}{az^2+bz+c}$$
 (21)

in (19) for which,

$$\alpha_0 = \frac{2(2a+b)}{b^2 - 4ac}, \qquad \beta_1 = -\frac{b+2c}{b^2 - 4ac}, \qquad \beta_0 = -\frac{2a+b}{b^2 - 4ac}$$

the integrals expressions for the partial derivatives are easily reduced to the form (10). Full expressions for α_0 , β_1 , β_0 and $-(\beta_1 + \beta_0)$ in terms of p^2 , m_1^2 , m_2^2 are to be found in Appendix A.

4 Three-point form factors

The 3-point scalar form factor $C_0(p_1^2, p_2^2, p_5^2; m_1^2, m_2^2, m_3^2)$ is a function of three external momenta squared and three internal masses. Following ref.[1] we define,

$$C_0(p_1^2, p_2^2, p_5^2; m_1^2, m_2^2, m_3^2) = \int \frac{d^n q}{i\pi^2} \frac{1}{[q^2 + m_1^2][(q + p_1)^2 + m_2^2][(q + p_1 + p_2)^2 + m_3^2]}$$
(22)

with $p_5 = p_1 + p_2$.

This may be written as the Feynman parameter integral,

$$C_{0}(p_{1}^{2}, p_{2}^{2}, p_{5}^{2}; m_{1}^{2}, m_{2}^{2}, m_{3}^{2}) = 2 \int_{0}^{1} dz_{1} \int_{0}^{1} dz_{2} \int_{0}^{1} dz_{3} \delta(1 - z_{1} - z_{2} - z_{3}) \times \int \frac{d^{n}q}{i\pi^{2}} \frac{1}{[q^{2} + p_{1}^{2}z_{1}z_{2} + p_{5}^{2}z_{1}z_{3} + p_{5}^{2}z_{1}z_{3} + m_{1}^{2}z_{1} + m_{2}^{2}z_{2} + m_{2}^{2}z_{3}]}$$
(23)

that clearly displays the symmetry properties of the arguments and from which it may be shown that,

$$C_{0}(p_{1}^{2}, p_{2}^{2}, p_{5}^{2}; m_{1}^{2}, m_{2}^{2}, m_{3}^{2}), C_{0}(p_{5}^{2}, p_{2}^{2}, p_{1}^{2}; m_{1}^{2}, m_{3}^{2}, m_{2}^{2}), C_{0}(p_{1}^{2}, p_{5}^{2}, p_{2}^{2}; m_{2}^{2}, m_{1}^{2}, m_{3}^{2}), C_{0}(p_{2}^{2}, p_{5}^{2}, p_{1}^{2}; m_{2}^{2}, m_{3}^{2}, m_{1}^{2}), C_{0}(p_{5}^{2}, p_{1}^{2}, p_{5}^{2}; m_{3}^{2}, m_{1}^{2}, m_{2}^{2}), C_{0}(p_{2}^{2}, p_{1}^{2}, p_{5}^{2}; m_{3}^{2}, m_{2}^{2}, m_{1}^{2}), C_{0}(p_{2}^{2}, p_{1}^{2}, p_{5}^{2}; m_{3}^{2}, m_{2}^{2}, m_{1}^{2}), C_{0}(p_{2}^{2}, p_{1}^{2}, p_{5}^{2}; m_{3}^{2}, m_{2}^{2}, m_{1}^{2}), C_{0}(p_{2}^{2}, p_{1}^{2}, p_{5}^{2}; m_{3}^{2}, m_{1}^{2}, m_{1}^{2}), C_{0}(p_{2}^{2}, p_{1}^{2}, p_{2}^{2}; m_{3}^{2}, m_{1}^{2}, m_{1}^{2}), C_{0}(p_{2}^{2}, p_{1}^{2}, p_{2}^{2}; m_{3}^{2}, m_{1}^{2}, m_{1}^{2}, m_{1}^{2}), C_{0}(p_{2}^{2}, p_{1}^{2}, p_{2}^{2}; m_{3}^{2}, m_{1}^{2}, m_{1}^{2}, m_{1}^{2}), C_{0}(p_{2}^{2}, p_{1}^{2}, p_{2}^{2}; m_{3}^{2}, m_{1}^{2}, m_{1}^{2}, m_{1}^{2}, m_{1}^{2}), C_{0}(p_{2}^{2}, p_{1}^{2}, p_{2}^{2}; m_{1}^{2}, m_{1}^{$$

are all equal.

In a similar way as for the two-point form factors, $B_{ij}(p^2, m_1^2, m_2^2)$, one can define a sequence of three-point tensor form factors, $C_{ij}(p_1^2, p_2^2, p_5^2; m_1^2, m_2^2, m_3^2)$ as shown in ref.[1]. Passarino and Veltman [1] have shown how to reduce to expressions involving B_{ij} and lower C_{ij} form factors. For certain of the tensor three-point form factors, this method yields two apparently distinct expressions. In ref.[1], these expressions were evaluated numerically and their agreement was used as a powerful check of the approach. The method that they employ can be made use of to algebraically reduce tensor three-point form factors to the two- and three-point scalar integrals.

$$C_{ij}(p_1^2, p_2^2, p_5^2; m_1^2, m_2^2, m_3^2) \rightarrow \alpha_1 C_0(p_1^2, p_2^2, p_5^2; m_1^2, m_2^2, m_3^2) + \beta_{12} B_0(1, 2) + \beta_{13} B_0(1, 3) + \beta_{23} B_0(2, 3) + \beta_1 B_0(0; m_1^2, m_1^2) + \beta_2 B_0(0; m_2^2, m_2^2) + \beta_3 B_0(0; m_3^2, m_3^2) + \beta_0$$
(25)

where α_1 and the β 's are again rational functions of the masses and momenta. The method of ref.[1] involves inverting the matrix,

$$X = \begin{pmatrix} p_1^2 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2^2 \end{pmatrix} \tag{26}$$

and consequently breaks down with the vanishing of the determinant at kinematic boundaries. We will now specify the properties of the external momenta, p_1 , p_2 and p_5 for which the determinant vanishes, that is when

$$p_1^2 p_2^2 - (p_1 \cdot p_2)^2 = -\frac{1}{4} (p_1^4 + p_2^4 + p_5^4 - 2p_1^2 p_2^2 - 2p_1^2 p_5^2 - 2p_2^2 p_5^2) = 0$$
 (27)

If $p_1 = (\vec{p_1}, iE_1)$, $p_2 = (\vec{p_2}, iE_2)$ and θ is the angle between their 3-vector components then,

$$\cos \theta = \left(\frac{E_1}{|\vec{p_1}|}\right) \left(\frac{E_2}{|\vec{p_2}|}\right) \pm \sqrt{\left[1 - \left(\frac{E_1}{|\vec{p_1}|}\right)^2\right] \left[1 - \left(\frac{E_2}{|\vec{p_2}|}\right)^2\right]} \tag{28}$$

From the requirement that $\cos \theta$ be real and $|\cos \theta| \leq 1$ it may be deduced that,

- If one of the momenta p_1 , p_2 , p_5 is time-like then the other two are as well. Moreover $p_1 = \alpha p_2$ for some real constant α .
- If two of the momenta are light-like then the third is as well. Moreover $p_1 = \alpha p_2$ for some real constant α .
- If one of the momenta is space—like then the other two are either both space—like or one is space—like and the other is light—like. In this case p_1 and p_2 can be strictly linearly independent.

5 The Derivative of C_0

In the reduction scheme presented in this paper, it is sometimes necessary to obtain the expression for the derivative of C_0 with respect to one of its mass arguments. For completeness the expressions for the derivative of C_0 with respect to any of its six arguments are derived in this section.

The scalar integral C_0 has a Feynman parameter representation given by

$$C_0(p_1^2, p_2^2, p_5^2, m_1^2, m_2^2, m_3^2) = \int_0^1 dx \int_0^x dy \, \frac{1}{ax^2 + by^2 + cxy + dx + ey + f - i\epsilon}$$
(29)

where,

$$\begin{array}{ll} a=-p_2^2, & b=-p_1^2, & c=p_1^2+p_2^2-p_5^2, \\ d=m_2^2-m_3^2+p_2^2, & e=m_1^2-m_2^2+p_5^2-p_2^2, & f=m_3^2. \end{array}$$

The derivative of C_0 with respect to any of it's six arguments is,

$$C_0' = \int_0^1 dx \int_0^x dy \, \frac{\alpha_{x^2} x^2 + \alpha_{y^2} y^2 + \alpha_{xy} xy + \alpha_x x + \alpha_y y + \alpha_1}{(ax^2 + by^2 + cxy + dx + ey + f - i\epsilon)^2}$$
(30)

where the coefficients, α , are given by

Derivative w.r.t.	α_{x^2}	α_{y^2}	α_{xy}	α_x	α_y	α_1
p_{1}^{2}	0	1	-1	0	0	0
p_2^2	1	0	-1	-1	1	0
p_5^2	0	0	1	0	-1	0
m_1^2	0	0	0	0	-1	0
m_2^2	0	0	0	-1	1	0
$m_3^{ar{2}}$	0	0	0	1	0	-1

It is convenient to define as basis set integrals, I_i , for the decomposition of C'_0 as follows

$$I_{1} \equiv \int_{0}^{1} dx \int_{0}^{x} dy \, \frac{y(2by + cx + e)}{D^{2}} \qquad I_{2} \equiv \int_{0}^{1} dx \int_{0}^{x} dy \, \frac{y(2ax + cy + d)}{D^{2}}$$

$$I_{3} \equiv \int_{0}^{1} dx \int_{0}^{x} dy \, \frac{x(2by + cx + e)}{D^{2}} \qquad I_{4} \equiv \int_{0}^{1} dx \int_{0}^{x} dy \, \frac{2ax + cy + d}{D^{2}}$$

$$I_{5} \equiv \int_{0}^{1} dx \int_{0}^{x} dy \, \frac{2by + cx + e}{D^{2}} \qquad I_{6} \equiv \int_{0}^{1} dx \int_{0}^{x} dy \, \frac{dx + ey + 2f}{D^{2}}$$

where $D = ax^2 + by^2 + cxy + dx + ey + f - i\epsilon$. It is easy to show that the I_i 's may be written in terms of C_0 and the derivatives of B_0 's.

$$I_{1} = C_{0} + \frac{\partial B_{0}(1,3)}{\partial m_{1}^{2}}$$

$$I_{2} = \frac{\partial B_{0}(1,2)}{\partial m_{1}^{2}} - \frac{\partial B_{0}(1,3)}{\partial m_{1}^{2}}$$

$$I_{3} = \frac{\partial B_{0}(1,3)}{\partial m_{1}^{2}} - \frac{\partial B_{0}(2,3)}{\partial m_{2}^{2}}$$

$$I_{4} = \frac{\partial B_{0}(1,2)}{\partial m_{1}^{2}} + \frac{\partial B_{0}(1,2)}{\partial m_{2}^{2}} - \frac{\partial B_{0}(1,3)}{\partial m_{1}^{2}} - \frac{\partial B_{0}(1,3)}{\partial m_{3}^{2}}$$

$$I_{5} = \frac{\partial B_{0}(1,3)}{\partial m_{1}^{2}} + \frac{\partial B_{0}(1,3)}{\partial m_{3}^{2}} - \frac{\partial B_{0}(2,3)}{\partial m_{2}^{2}} - \frac{\partial B_{0}(2,3)}{\partial m_{3}^{2}}$$

$$I_{6} = -\frac{\partial B_{0}(1,2)}{\partial m_{1}^{2}} - \frac{\partial B_{0}(1,2)}{\partial m_{2}^{2}}$$

The derivative of B_0 may be expressed in terms of B_0 's as shown in eq.(A.7). Note however that this equation will yield an ill-defined $B_0(0; 0, 0)$ if m_1^2 is set equal to zero. Such terms will generally cancel without any special treatment

in expressions for physical quantities. It may, in cases where infrared divergences or mass singularities are present, be convenient to carry a non-zero m_1 through intermediate steps and set it to zero only in the final results. With the above relations in hand the reduction of C'_0 is straightforward.

$$C_0' = \beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3 + \beta_4 I_4 + \beta_5 I_5 + \beta_6 I_6 \tag{31}$$

The equation for the β_i 's is obtained by equating the coefficients of x^2 , y^2 , etc., in the above.

$$\begin{pmatrix}
0 & 0 & c & 0 & 0 & 0 \\
2b & c & 0 & 0 & 0 & 0 \\
c & 2a & 2b & 0 & 0 & 0 \\
0 & 0 & e & 2a & c & d \\
e & d & 0 & c & 2b & e \\
0 & 0 & 0 & d & e & 2f
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4 \\
\beta_5 \\
\beta_6
\end{pmatrix} =
\begin{pmatrix}
\alpha_{x^2} \\
\alpha_{y^2} \\
\alpha_{xy} \\
\alpha_x \\
\alpha_y \\
\alpha_1
\end{pmatrix}$$
(32)

This matrix equation may be solved by first determining β_1 , β_2 , and β_3 from the relation

$$\begin{pmatrix} 0 & 0 & c \\ 2b & c & 0 \\ c & 2a & 2b \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha_{x^2} \\ \alpha_{y^2} \\ \alpha_{xy} \end{pmatrix}$$
(33)

and then obtaining β_4 , β_5 , and β_6 using,

$$\begin{pmatrix} 2a & c & d \\ c & 2b & e \\ d & e & 2f \end{pmatrix} \begin{pmatrix} \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} = \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & e \\ e & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$
(34)

It should be noted that the derivative of C_0 with respect to any of its mass arguments reduces to B_0 's. In this case, $\alpha_{x^2} = \alpha_{y^2} = \alpha_{xy} = 0$, and consequently $\beta_1 = \beta_2 = \beta_3 = 0$. Since I_4 , I_5 , and I_6 can be expressed in terms of just B_0 's, it follows that $\partial C_0(p_1^2, p_2^2, p_5^2, m_1^2, m_2^2, m_3^2)/\partial m_i^2$ can also be expressed in terms of B_0 's.

The expressions for the β_i 's, for the particular case of the derivative with respect to p_1^2 , are given below.

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \frac{1}{4ab - c^2} \begin{pmatrix} 2a + c \\ -2b - c \\ 0 \end{pmatrix}$$
 (35)

$$\begin{pmatrix} \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} = \frac{2bd + cd - 2ae - ce}{2(4ab - c^2)(-bd^2 + cde - ae^2 + 4abf - c^2f)} \begin{pmatrix} de - 2cf \\ 4af - d^2 \\ cd - 2ae \end{pmatrix}$$
(36)

The expressions for the derivatives with respect to the other two momentum arguments may be obtained from the symmetry properties of C_0 (eq.(24)).

The expressions for the β_i 's for $\partial C_0/\partial m_1^2$ are, $\beta_1=\beta_2=\beta_3=0$ and,

$$\begin{pmatrix} \beta_4 \\ \beta_5 \\ \beta_6 \end{pmatrix} = \frac{-1}{-bd^2 + cde - ae^2 + 4abf - c^2 f} \begin{pmatrix} de - 2cf \\ 4af - d^2 \\ cd - 2ae \end{pmatrix}$$
(37)

Again the derivatives with respect to the other masses may be obtained using the symmetry properties of C_0 (eq.(24)).

In case the matrix that needs to be inverted to obtain $\partial C_0/\partial m_i^2$ (eq.(34)), has zero determinant, the simple solution is to obtain the expression for the derivative with one of the one of the masses shifted away from its value, and since the resulting expression will contain only B_0 's, l'Hôpital's rule or Taylor series expansion may be applied to obtain the limit. Recall that the derivative of C_0 with the most general set of arguments, with respect to any of its mass arguments is expressible in terms of B_0 's. The expressions for $\partial B_0/\partial m_i^2$ are given in Appendix A.

6 Reduction of C_{ij} Form Factors

The Passarino-Veltman scheme for the reduction of all C_{ij} form factors to the basic scalar integrals C_0 and B_0 breaks down when the kinematic or Gram determinant, vanishes. The Gram determinant will be denoted, \mathcal{D} , and when it vanishes all the C_{ij} form factors, including the scalar integral C_0 , can be reduced to linear combinations of B_0 's as long as they are at most logarithmically divergent. Expressions for the quadratically divergent C_{ij} 's contain the quadratically divergent scalar integral, A.

The reduction for C_0 is obtained as follows. The Passarino-Veltman formula for reducing the scalar integrals C_{11} and C_{12} to C_0 's and B_0 's in the notation of ref.[1] is,

$$\begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} = X^{-1} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \frac{1}{\mathcal{D}} \begin{pmatrix} p_2^2 & -p_1 \cdot p_2 \\ -p_1 \cdot p_2 & p_1^2 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$
(38)

where,

$$R_{1} = \frac{1}{2} \{ f_{1}C_{0} + B_{0}(1,3) - B_{0}(2,3) \}$$

$$R_{2} = \frac{1}{2} \{ f_{2}C_{0} + B_{0}(1,2) - B_{0}(1,3) \}$$

$$f_{1} = m_{1}^{2} - m_{2}^{2} - p_{1}^{2}$$

$$f_{2} = m_{2}^{2} - m_{3}^{2} + p_{1}^{2} - p_{5}^{2}$$

and \mathcal{D} is the determinant of the matrix X,

$$\mathcal{D} = p_1^2 p_2^2 - (p_1 \cdot p_2)^2. \tag{39}$$

When $\mathcal{D} = 0$, in order for C_{11} and C_{12} to exist, the following equations must be satisfied,

$$p_2^2 R_1 - p_1 \cdot p_2 R_2 = 0 (40)$$

$$-p_1 \cdot p_2 R_1 + p_1^2 R_2 = 0 (41)$$

Substituting for R_1 and R_2 one obtains,

$$C_0 = \frac{(p_1 \cdot p_2)B_0(1,2) - (p_1 \cdot p_2 + p_2^2)B_0(1,3) + p_2^2B_0(2,3)}{p_2^2f_1 - (p_1 \cdot p_2)f_2}$$
(42)

$$C_0 = \frac{p_1^2 B_0(1,2) - (p_1^2 + p_1 \cdot p_2) B_0(1,3) + (p_1 \cdot p_2) B_0(2,3)}{(p_1 \cdot p_2) f_1 - p_1^2 f_2}$$
(43)

respectively. Equations (42) and (43) give explicit expressions for C_0 when the kinematic determinant vanishes. Although both of these expressions are equivalent, it is useful to keep both expressions at hand because one encounters situations where the denominator of one of the expressions vanishes but that of the other does not. For example, when $p_2^2 = 0$ and $p_1 \cdot p_2 = 0$, the denominator of the expression in (42) vanishes but that of (43) need not.

The eq.s (42) and (43) are new formulas valid when $\mathcal{D} = 0$ and represent an alternative to the method described in ref.s[3, 9].

At this point we introduce some notation to make the equations more concise. The cofactor matrix of the matrix X, will be called \bar{X} . Thus,

$$\bar{X} = \begin{pmatrix} \bar{x}_{11} & \bar{x}_{12} \\ \bar{x}_{21} & \bar{x}_{22} \end{pmatrix} = \begin{pmatrix} p_2^2 & -p_1 \cdot p_2 \\ -p_1 \cdot p_2 & p_1^2 \end{pmatrix}$$

We also define,

$$Y_1 = \bar{x}_{11}f_1 + \bar{x}_{12}f_2 \tag{44}$$

$$Y_2 = \bar{x}_{21}f_1 + \bar{x}_{22}f_2 \tag{45}$$

In this notation the two equations for C_0 , (42) and (43), can be written concisely as,

$$C_0 = \frac{1}{Y_i} \{ -\bar{x}_{i2} B_0(1,2) + (\bar{x}_{i2} - \bar{x}_{i1}) B_0(1,3) + \bar{x}_{i1} B_0(2,3) \}$$
 (46)

where i = 1, 2. These eq.s(46) run into difficulties when both of the Y_i vanish. In what follows it will be shown how to bypass such problems by differentiation with respect to mass arguments. It can be demonstrated that the vanishing of both Y_i 's is a necessary condition for the breakdown of the method described in ref.[9] and many of the techniques we introduce here can be used to extend that method as well.

To obtain the reductions for C_{11} and C_{12} when $\mathcal{D} = 0$, one could attempt to obtain the limit of the rational functions in eq.(38) using l'Hôpital's rule but we find that when the partial derivative of C_0 with respect to one of its momentum arguments, is expressed in terms of C_0 and the B_0 's, the Gram determinant, eq.(39), appears in the denominator and the application of l'Hôpital's rule in this manner does not yield a limit. Instead, the reductions for C_{11} and C_{12} when $\mathcal{D} = 0$ can be obtained by looking at the general formulas for the C_{2j} form factors,

$$\begin{pmatrix} C_{21} \\ C_{23} \end{pmatrix} = X^{-1} \begin{pmatrix} R_3 \\ R_5 \end{pmatrix} = \frac{\bar{X}}{\mathcal{D}} \begin{pmatrix} \frac{1}{2} \{ f_1 C_{11} + B_1(1,3) + B_0(2,3) \} - C_{24} \\ \frac{1}{2} \{ f_2 C_{11} + B_1(1,2) - B_1(1,3) \} \end{pmatrix}$$
$$\begin{pmatrix} C_{23} \\ C_{22} \end{pmatrix} = X^{-1} \begin{pmatrix} R_4 \\ R_6 \end{pmatrix} = \frac{\bar{X}}{\mathcal{D}} \begin{pmatrix} \frac{1}{2} \{ f_1 C_{12} + B_1(1,3) - B_1(2,3) \} \\ \frac{1}{2} \{ f_2 C_{12} - B_1(1,3) \} - C_{24} \end{pmatrix}$$

Hence when $\mathcal{D}=0$, we must have,

$$\bar{x}_{i1}R_3 + \bar{x}_{i2}R_5 = 0 (47)$$

$$\bar{x}_{i1}R_4 + \bar{x}_{i2}R_6 = 0 \tag{48}$$

Substituting for R_3 , R_4 , R_5 , and R_6 one obtains,

$$Y_iC_{11} = -\bar{x}_{i2}B_1(1,2) + (\bar{x}_{i2} - \bar{x}_{i1})B_1(1,3) - \bar{x}_{i1}B_0(2,3) + 2\bar{x}_{i1}C_{24}$$
 (49)

$$Y_i C_{12} = (\bar{x}_{i2} - \bar{x}_{i1}) B_1(1,3) + \bar{x}_{i1} B_1(2,3) + 2\bar{x}_{i2} C_{24}$$

$$(50)$$

Unlike the case for C_0 where the right hand side did not include any C_{ij} form factors, here the logarithmically divergent C_{24} is encountered on the right hand side. The appearance of C_{24} causes no difficulties because it is given in closed form by,

$$C_{24} = \frac{1}{4} - \frac{1}{4} \{ f_1 C_{11} + f_2 C_{12} + 2m_1^2 C_0 - B_0(2,3) \}$$
 (51)

Note that this is not affected by a vanishing denominator when $\mathcal{D} = 0$. Substituting this expression for C_{24} into (49) and (50) one obtains,

$$\begin{pmatrix} \bar{x}_{i1}f_1 + 2Y_i & \bar{x}_{i1}f_2 \\ \bar{x}_{i2}f_1 & 2Y_i + \bar{x}_{i2}f_2 \end{pmatrix} \begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} = \begin{pmatrix} \overline{R}_{11} \\ \overline{R}_{12} \end{pmatrix}$$
(52)

where,

$$\overline{R}_{11} = -2m_1^2 \bar{x}_{i1} C_0 - 2\bar{x}_{i2} B_1(1,2) + 2(\bar{x}_{i2} - \bar{x}_{i1}) B_1(1,3)
- \bar{x}_{i1} B_0(2,3) + \bar{x}_{i1}
\overline{R}_{12} = -2m_1^2 \bar{x}_{i2} C_0 + 2(\bar{x}_{i2} - \bar{x}_{i1}) B_1(1,3) + 2\bar{x}_{i1} B_1(2,3)
+ \bar{x}_{i2} B_0(2,3) + \bar{x}_{i2}$$

Once the substitution for C_0 is made using eq.(46) and B_1 is written in terms of B_0 [3], the expressions for C_{11} and C_{12} will be given purely in terms of B_0 's.

The formulas for the reduction for C_{2j} , C_{3j} , and C_{4j} form factors, when $\mathcal{D}=0$, are obtained in a similar manner from the general formulas for the reduction of the C_{3j} , C_{4j} , and C_{5j} form factors respectively. The difference between the C_{2j} and higher form factors, and the ones treated above, C_0 and C_{1j} , is that the former have form factors that multiply tensors that have one or more $\delta_{\mu\nu}$. All the form factors that multiply tensors containing one or more factors of $\delta_{\mu\nu}$, have formulas that do not suffer from the problem of a vanishing denominator. These form factors are C_{24} , C_{35} , C_{36} , C_{46} , C_{47} , C_{48} , and C_{49} . The formula for the reduction of C_{24} is given above in eq.(51). The formulas for C_{35} , C_{36} , C_{46} , etc. are given in Appendix B. The rest of the form factors, those that multiply tensors without any $\delta_{\mu\nu}$'s in them, are given by formulas similar to (52) but involving larger matrices. The form factors C_{21} , C_{22} , and C_{23} are given by a set of three simultaneous linear equations, C_{31} to C_{34} by a set of four equations, and C_{41} to C_{45} by a set of five equations. The relevant formulas appear in Appendix C.

The reduction formulas for these form factors have denominators that are proportional to powers of Y_i . As discussed earlier, when only one of the Y_i 's is equal to zero no problem arises but when both Y_1 and Y_2 are equal to zero, the limit of these rational functions has to be obtained using l'Hôpital's rule. We discuss two separate cases here:

- At least one of the \bar{x}_{ij} is not zero: In this case the limit is easily obtained using l'Hôpital's rule or Taylor series expansion where the differentiation is performed with respect to one of the mass arguments.
- All of the \$\bar{x}_{ij}\$ are zero: This condition arises if and only if all three of the external momenta squared are equal to zero. In this case the \$Y_i\$ cannot be shifted away from zero by shifting one of the mass arguments. The reduction formulas for this case are obtained as follows. The derivation for \$C_0\$ is given below. The rest of the form factors are obtained in the same manner. To obtain the reduction for \$C_0\$, we go back to eq.(38). When \$p_1^2 = p_2^2 = p_5^2 = 0\$, note that every element of "\$X^{-1}\$" is of the \$0/0\$ form. Applying l'Hôpital's rule on every element of \$X^{-1}\$, using \$p_1^2\$, \$p_2^2\$, and \$p_5^2\$ as independent variables and differentiating with respect to \$p_5^2\$, eq.(38) becomes,

$$\begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} = \frac{-1}{p_1 \cdot p_2} \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$
 (53)

Since $p_1 \cdot p_2 = 0$ and C_{11} and C_{12} are assumed to exist we must have that $R_1 = 0$ and $R_2 = 0$. Substituting for R_1 and R_2 one obtains,

$$C_0 = \frac{1}{f_1} \{ B_0(2,3) - B_0(1,3) \} = \frac{1}{f_2} \{ B_0(1,3) - B_0(1,2) \}$$
 (54)

Substituting $p_1^2 = p_2^2 = p_5^2 = 0$ in (54) one obtains,

$$C_0(0,0,0,m_1^2,m_2^2,m_3^2) = \frac{B_0(0,m_2^2,m_3^2) - B_0(0,m_1^2,m_3^2)}{m_1^2 - m_2^2}$$
 (55)

$$= \frac{B_0(0, m_1^2, m_3^2) - B_0(0, m_1^2, m_2^2)}{m_2^2 - m_3^2}$$
 (56)

If in the above formulas the denominator vanishes, the limit is obtained by differentiating with respect to one of the masses.

6.1 Example

The 1-loop QED contribution to the anomalous magnetic moment (g-2) of the electron was first calculated by Schwinger [20]. Because of the many singularities involved this is one of the most pathological cases that can be encountered. The result can be derived in terms of 3-point form factors to be

$$\frac{g-2}{2} = -\left(\frac{me}{2\pi}\right)^2 \left(C_{11} + C_{12}\right) \tag{57}$$

in which the arguments of the form factors are $C_{ij}(-m^2, 0, -m^2; \lambda^2, m^2, m^2)$ where m is the electron mass and e is its charge. Although absent in the final result, infrared divergences may appear in intermediate steps and are regulated by giving the photon a mass, λ . The given set of momenta leads to a vanishing Gram determinant and for the given masses causes the extended method, described in ref.[9], to break down. In the method described here the denominators Y_i in eq.s (44) and (45) both vanish. To alleviate this m_3 can be shifted away from m and the reduction performed in the usual way. l'Hôpital's or Taylor series expansion is then used to take the limit $m_3 \to m$. This same approach works when applied to the method of ref.[9] and involves differentiation with respect to the mass arguments of B_0 by means of the formulas given in Appendix A. The photon mass, λ can then be safely set to zero. Note that the form factors C_{12} , C_{22} and C_{23} all blow up as $m_3 \to m$, even with a finite photon mass, but that none of these occur in the expression for the physical quantity (g-2). Following this procedure one obtains

$$C_{11} + C_{12} = \frac{1}{2m^2} \left[1 + B_0(0; m^2, m^2) - B_0(-m^2; 0, m^2) \right]$$
 (58)

using the result

$$B_0(-m^2; 0, m^2) = B_0(0; m^2, m^2) + 2$$
 (59)

this gives

$$\frac{g-2}{2} = \frac{\alpha}{2\pi}$$

in agreement with the well-known result.

7 Reduction of D_{ij} Form Factors

The derivation of the reduction formulas for the D_{ij} form factors when the kinematic determinant, which we again call \mathcal{D} , vanishes, is obtained in an identical manner as for the C_{ij} form factor reductions. The kinematic determinant, \mathcal{D} , is given by,

$$\mathcal{D} = p_1^2 p_2^2 p_3^2 - p_1^2 (p_2 \cdot p_3)^2 - p_2^2 (p_1 \cdot p_3)^2 - p_3^2 (p_1 \cdot p_2)^2 + 2(p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_3)$$

To obtain the formula for D_0 when $\mathcal{D} = 0$, we start with the general formula for the reduction of the D_{1j} form factors.

$$\begin{pmatrix} D_{11} \\ D_{12} \\ D_{13} \end{pmatrix} = \frac{1}{\mathcal{D}} \begin{pmatrix} \bar{x}_{11} & \bar{x}_{12} & \bar{x}_{13} \\ \bar{x}_{21} & \bar{x}_{22} & \bar{x}_{23} \\ \bar{x}_{31} & \bar{x}_{32} & \bar{x}_{33} \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}$$
(60)

where,

$$R_1 = \{f_1 D_0 + C_0(1,3,4) - C_0(2,3,4)\}/2 \tag{61}$$

$$R_2 = \{f_2 D_0 + C_0(1, 2, 4) - C_0(1, 3, 4)\}/2 \tag{62}$$

$$R_3 = \{f_3 D_0 + C_0(1, 2, 3) - C_0(1, 2, 4)\}/2$$
(63)

$$f_1 = m_1^2 - m_2^2 - p_1^2$$
, $f_2 = m_2^2 - m_3^2 + p_1^2 - p_5^2$, $f_3 = m_3^2 - m_4^2 - p_4^2 + p_5^2$.

The elements of the co-factor matrix, \bar{X} , are,

$$\begin{pmatrix} \bar{x}_{11} & \bar{x}_{12} & \bar{x}_{13} \\ \bar{x}_{21} & \bar{x}_{22} & \bar{x}_{23} \\ \bar{x}_{31} & \bar{x}_{32} & \bar{x}_{33} \end{pmatrix} = \begin{pmatrix} p_2^2 p_3^2 - p_{23}^2 & p_{13} p_{23} - p_{12} p_3^2 & p_{12} p_{23} - p_2^2 p_{13} \\ p_{13} p_{23} - p_{12} p_3^2 & p_1^2 p_3^2 - p_{13}^2 & p_{12} p_{13} - p_1^2 p_{23} \\ p_{12} p_{23} - p_2^2 p_{13} & p_{12} p_{13} - p_1^2 p_{23} & p_1^2 p_2^2 - p_{12}^2 \end{pmatrix}$$

$$(64)$$

where p_{ij} is used to represent $p_i \cdot p_j$. Using the same argument as in section 6, when $\mathcal{D} = 0$, in order for D_{11} , D_{12} , and D_{13} to exist, the following relation must hold.

$$D_0 = \frac{1}{Y_i} \{ -\bar{x}_{i3} C_0(1, 2, 3) + (\bar{x}_{i3} - \bar{x}_{i2}) C_0(1, 2, 4) + (\bar{x}_{i2} - \bar{x}_{i1}) C_0(1, 3, 4) + \bar{x}_{i1} C_0(2, 3, 4) \}$$
 (65)

where,

$$Y_i = \bar{x}_{i1}f_1 + \bar{x}_{i2}f_2 + \bar{x}_{i3}f_3$$

and i can take on the values 1, 2 or 3.

The reduction formulas for the D_{1j} and higher form factors when $\mathcal{D}=0$ are obtained as in the case of the C_{ij} form factors. Here again the D_{ij} form factors that multiply tensors containing one or more $\delta_{\mu\nu}$'s, are given by formulas which do not suffer from a vanishing denominator for any set of momenta or masses. The rest of the form factors are obtained by solving simultaneous linear equations. These form factors have denominators which are proportional to powers of Y_i .

As in the case of the C_{ij} form factors, when all of the Y_i are zero, the limit of the rational functions are obtained using l'Hôpital's rule or Taylor series expansion. Again there are two cases to consider:

- At least one of the \bar{x}_{ij} are non-zero: The limit is obtained using l'Hôpital's rule where the differentiation is performed with respect to one of the mass arguments.
- All the \bar{x}_{ij} are zero: This case is treated as for the C_{ij} form factors, but there is one difference to note. Recall that in the case of the C_{ij} form factors, all the \bar{x}_{ij} vanished if and only if all the external momenta squared were zero, but here all the \bar{x}_{ij} could vanish even when none of the external momenta squared are zero. An example of this is when all the external momenta are proportional to one another. Again we use the same technique described in the earlier section. Apply l'Hôpital's rule to each element of " X^{-1} ". Using p_1^2 , p_2^2 , p_3^2 , p_{12} , p_{13} , and p_{23} as independent variables and differentiating with respect to p_1^2 one obtains,

$$\frac{\bar{X}}{\mathcal{D}} \to \frac{1}{p_2^2 p_3^2 - p_{23}^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & p_3^2 & -p_{23} \\ 0 & -p_{23} & p_2^2 \end{pmatrix}$$
(66)

If the relation (66) is again of the 0/0 form, ie. if $p_2^2 = p_3^2 = p_{23} = 0$, then the differentiation is performed again, but this time with respect to, say p_2^2 , to obtain,

$$\frac{\bar{X}}{\mathcal{D}} \to \frac{1}{p_3^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
 (67)

Proceeding with the assumption that (66) is not of the 0/0 form, eq. (60) becomes,

$$\begin{pmatrix} D_{11} \\ D_{12} \\ D_{13} \end{pmatrix} = \frac{1}{p_2^2 p_3^2 - p_{23}^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & p_3^2 & -p_{23} \\ 0 & -p_{23} & p_2^2 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix}$$
(68)

Since all the \bar{x}_{ij} are zero, $p_2^2 p_3^2 - p_{23}^2 = 0$ and therefore we must have,

$$p_3^2 R_2 - p_{23} R_3 = 0$$
 and $p_{23} R_2 - p_2^2 R_3 = 0$

Substituting for R_2 and R_3 one obtains the expression for D_0 . If (66) is of the 0/0 form, then using (67) we get $R_3 = 0$, and hence the expression for D_0 . The expression obtained for D_0 will be in terms of C_0 , but these C_0 's necessarily reduce to B_0 's. That is, the vanishing of all the \bar{x}_{ij} is a sufficient condition for the reduction of D_0 , and hence all D form factors, to B_0 's.

8 Alternate Derivation of the Reduction Formulas

The formulas obtained above for the reduction of the C_{ij} and D_{ij} form factors when their respective kinematic determinants vanish, may be obtained in another manner described below.

Consider the derivation of the reduction formulas for the C_{ij} form factors. The reduction for C_0 is obtained as before but for the C_1 form factors, the derivation proceeds in an identical manner as in section 6 only up to eq.(50). The C_{24} form factor can be handled differently. Instead of using eq.(51), one can use the relation,

$$C_{24} = \frac{1}{Y_i} \{ -\bar{x}_{i2} B_{22}(1,2) + (\bar{x}_{i2} - \bar{x}_{i1}) B_{22}(1,3) + \bar{x}_{i1} B_{22}(2,3) \}$$
 (69)

Eq.(69) is only valid when $\mathcal{D} = 0$, and it is derived from the general formulas for the reduction of C_{35} and C_{36} form factors, given by [1],

$$\begin{pmatrix} C_{35} \\ C_{36} \end{pmatrix} = \frac{\bar{X}}{\mathcal{D}} \begin{pmatrix} \frac{1}{2} \{ f_1 C_{24} + B_{22}(1,3) - B_{22}(2,3) \} \\ \frac{1}{2} \{ f_2 C_{24} + B_{22}(1,2) - B_{22}(1,3) \} \end{pmatrix}$$
(70)

When (69) is substituted for C_{24} in eq.s (49) and (50), the formulas for C_{11} and C_{12} are obtained in terms of B_{ij} form factors which can be eventually reduced to B_0 's.

The reduction of the D_{ij} form factors proceeds in the same manner. The advantage of this method is that one obtains formulas that are not coupled. That is, one does not have to solve simultaneous equations to obtain formulas for the individual form factors, as was necessary in the previous method. The disadvantage here becomes evident when this method is used for obtaining the formulas for the higher form factors. Note that to obtain the formulas for the C_{1i} form factors we had to use the general formulas for the reduction of C_{3j} form factors. In the formula for the reduction of the C_{2j} form factors, C_{35} and C_{36} show up. From the general formulas for the C_{4i} form factors one obtains a formula for C_{35} and C_{36} in terms of B_{ij} form factors and the quadratically divergent C_{49} . Then from the general formulas for C_{5i} form factors one can obtain a formula for C_{49} in terms of B_{ij} form factors alone. Specifically the B_{ij} form factors that appear in the formula for C_{49} is the quartically divergent form factor, B_{43} . That is, the quadratically divergent C_{49} is obtained as the difference of quartically divergent B_{43} . Substituting this formula for C_{49} into the formulas for the C_{35} and C_{36} form factors, gives C_{35} and C_{36} purely in terms of B_{ij} form factors which can then be substituted into the formulas for the C_{2i} form factors to give C_{21} , C_{22} and C_{23} in terms of B_{ij} form factors alone. Similarly, to obtain the formulas for the C_{3i} form factors one has to drive general formulas for the C_{7i} form factors, which contain the highly divergent B_{64} . The situation is even worse for the C_{4j} form factors and similar problems are faced in the case of the D_{ij} form factors.

Therefore the previous derivation is simpler since to obtain the formulas for C_{ij} form factors, we only have to look at the general formulas of the $C_{i+1,j}$ form factors, and likewise for the D_{ij} form factors. Also, one does not have to deal with the highly divergent form factors which show up in this alternate derivation. But no matter how the reduction formulas are derived, once the formulas are reduced to B_0 's and C_0 's, they take on unique forms.

9 Comparison with Earlier Work

The method employed in ref.s[3, 9, 10] to treat the reduction of scalar integrals when the Gram determinant vanished yielded spurious logarithmic

divergences that were observed to cancel in physical results. Thus, for example, the logarithmic divergences that appeared in the expressions for C_{11} and C_{12} , canceled in practical situations when the $C_{\mu\ldots}$ were constructed and the indices were contracted with the appropriate Lorentz tensors. In this section we will show how the cancellation occurs, list all the situations under which the cancellation necessarily occurs and also give the situation under which this cancellation might not occur.

The formula given for the C_{1j} form factors in Ref [3] is,

$$C_{11} = \alpha_{12}B_1(1,2) + \alpha_{13}B_1(1,3) - \alpha_{23}B_0(2,3)$$
 (71)

$$C_{12} = \alpha_{13}B_1(1,3) + \alpha_{23}B_1(2,3) \tag{72}$$

where the α_{ij} are,

$$\alpha_{12} = -\bar{x}_{22}/Y_2, \quad \alpha_{13} = (\bar{x}_{22} - \bar{x}_{21})/Y_2, \quad \alpha_{23} = \bar{x}_{21}/Y_2$$

Comparing eq.s (71) and (72) with the complete expressions given in eq.s (49) and (50), it is evident that the terms involving the logarithmically divergent C_{24} are missing from the expressions in eq.s (71) and (72).

The reason why the terms involving C_{24} cancel in practice is shown below. If one constructs $C_{\mu} = p_{1\mu}C_{11} + p_{2\mu}C_{12}$ using C_{11} and C_{12} from eq.s (49) and (50), then one obtains,

$$C_{\mu} = \frac{1}{Y_i} \Big(-p_{1\mu} \bar{x}_{i2} B_1(1,2) + (p_{1\mu} + p_{2\mu}) (\bar{x}_{i2} - \bar{x}_{i1}) B_1(1,3)$$

$$+ p_{2\mu} \bar{x}_{i1} B_1(2,3) - p_{1\mu} \bar{x}_{i1} B_0(2,3) + 2(p_{1\mu} \bar{x}_{i1} + p_{2\mu} \bar{x}_{i2}) C_{24} \Big)$$
 (73)

Note that the term involving C_{24} is proportional to the light-like vector

$$P_{\mu} = p_{1\mu}\bar{x}_{i1} + p_{2\mu}\bar{x}_{i2} \tag{74}$$

We note that for all higher rank tensor integrals, $C_{\mu\nu}$, $C_{\mu\nu\lambda}$, etc., the terms that are missing from the expressions in ref.[3] are always proportional to the vector P_{α} , where α is one of the indices of the tensor integral.

Remembering that at the outset we have assumed $\mathcal{D} = 0$, we note the following properties of the vector P_{μ}

- $P_{\mu} = 0$ if $p_1 = \alpha p_2$ for some real α .
- $\bullet \ p_{1\mu} \cdot P_{\mu} = 0.$

$$\bullet \ p_{2\mu} \cdot P_{\mu} = 0.$$

It was stated in section 4 that when $\mathcal{D} = 0$, it is necessarily true that $p_1 = \alpha p_2$, unless at least two of the three external momenta, were spacelike. Therefore, as long as the external momenta are not space-like, P_{μ} is identically zero. If at least two of the external momenta are space-like, P_{μ} is not identically zero, but if it is dotted with any of the external momenta or any linear combination of them, it vanishes. Under either of these conditions, which is almost always the case, the term involving C_{24} vanishes. The only condition under which this term can survive is if at least two of the external momenta are space-like and it is contracted with a vector which is linearly independent of the external momenta of the three point function. Such conditions may occur for the process $e^+e^- \to \nu_e\bar{\nu}_e\gamma$ produced by tchannel exchange of of a W-boson. There the one-loop corrections to the $W^+W^-\gamma$ vertex can yield a form-factor having two external space-like momenta (i.e. those of the W's) and one external light-like momentum (i.e. that of the external photon). In such a case the method used in ref.s[3, 9, 10] might yield incorrect results but would be indicated by the non-cancellation of the extra spurious diverges. In the program LERG-I this would also be indicated by the appearance of arbitrary constants in physical results.

In the case of the D_{ij} form factors, the situation is similar. The "missing terms" are always proportional to the vector $p_{1\mu}\bar{x}_{i1} + p_{2\mu}\bar{x}_{i2} + p_{3\mu}\bar{x}_{i3}$. The properties of this vector are identical to the one above. It is identically zero if any two of $p_{1\mu}$, $p_{2\mu}$, and $p_{3\mu}$ are linearly dependent, and it vanishes if it is dotted with any of its external momenta, and hence any linear combination of them. Here again, the only condition under which the "missing terms" can survive, is if all three external momenta, $p_{1\mu}$, $p_{2\mu}$, and $p_{3\mu}$, are linearly independent and if $D_{\mu\ldots}$ is not contracted with any vector which is linearly dependent with the external momenta of the four-point tensor integral.

10 Summary and Conclusion

We have extended the method for the reduction of Lorentz tensor form factors to scalar integrals, introduced by Passarino and Veltman, to regions of parameter space including those of vanishing Gram determinants. No attempt has been made to handle numerical instabilities that may arise and the region of vanishing Gram determinant is approached since this problem

as been amply discussed in the literature. If required the methods described here can be used in this critical region by employing a Taylor series expansion.

The technique presented here may also be applied in the reduction of fivepoint and higher functions [21] when their Gram determinants vanish. This is necessarily the case for six-point and higher functions in four space-time dimensions.

In the foregoing it is implicitly assumed that the calculations are performed in a covariant gauge. In axial gauge, for example, new denominators appear in the Feynman integrals that are not immediately amenable to the methods described here.

Once again it should be emphasized that many of the formulas that have been derived in this paper are valid only for the divergent and finite parts of the integrals and do not necessarily apply to the terms of $\mathcal{O}(n-4)$ and higher. Thus the results cannot be used for the insertion of subdiagrams to form 2-loop or higher diagrams.

The reduction of tensor form factors to scalar integrals can also be implemented at the 2-loop level [22] but new complications arise. Whereas at 1-loop there is just a single topology for each of the 2-, 3- and 4-point scalar integrals, this is no longer true at 2-loops. For the 2-point function at 2-loops there are 20 possible topologies for the scalar integrals. Some of these can be calculated in closed form [23] but it is known that general case is not expressible in terms of polylogarithms and so are probably best calculated numerically.

Appendix A

The 2-point form factors, B_{ij} , are defined following Passarino and Veltman [1] via the relations

$$B_0; B_{\mu}; B_{\mu\nu}; B_{\mu\nu\alpha}; B_{\mu\nu\alpha\beta}(p; m_1, m_2) = \int \frac{d^n q}{i\pi^2} \frac{1; q_{\mu}; q_{\mu}q_{\nu}; q_{\mu}q_{\nu}q_{\alpha}; q_{\mu}q_{\nu}q_{\alpha}q_{\beta}}{[q^2 + m_1^2][(q+p)^2 + m_2^2]}$$

By invoking Lorentz covariance one may define the form factor,

$$B_{\mu} = p_{\mu}B_1$$

Here and in what follows the arguments of the B_{ij} 's are taken to be $B_{ij}(p^2; m_1^2, m_2^2)$.

Similarly,

$$B_{\mu\nu} = p_{\mu}p_{\nu}B_{21} + \delta_{\mu\nu}B_{22}$$

$$B_{\mu\nu\alpha} = p_{\mu}p_{\nu}p_{\alpha}B_{31} + \{p\delta\}_{\mu\nu\alpha}B_{32}$$

$$B_{\mu\nu\alpha\beta} = p_{\mu}p_{\nu}p_{\alpha}p_{\beta}B_{41} + \{pp\delta\}_{\mu\nu\alpha\beta}B_{42} + \{\delta\delta\}_{\mu\nu\alpha\beta}B_{43}$$

The braces are used to denote the symmetrized product in the Lorentz indices. The form factors B_{i1} (we take $B_0 \equiv B_{01}$, $B_1 \equiv B_{11}$) are all logarithmically divergent as signaled by their having poles at even integer $n \geq 4$.

A and B_{i2} are quadratically divergent with poles at even integer $n \geq 2$. B_{43} is quartically divergent with poles at even integer $n \geq 0$.

Expressions for the derivative of B_0 with respect to its momentum argument can be found in ref.[3] in the form (10) and various special cases and recursion relations are to be found in ref.[9]. Here we give the expressions for the derivatives with respect to both the momentum and mass arguments. Neither derivative exists at the physical threshold, $p^2 = -(m_1 + m_2)^2$.

Away from threshold

$$\frac{\partial B_0}{\partial p^2}(p^2; m_1^2, m_2^2) = -\frac{p^2 m_1^2 + p^2 m_2^2 + m_1^4 - 2m_1^2 m_2^2 + m_2^4}{p^2 D} B_0(p^2; m_1^2, m_2^2)
+ m_1^2 \frac{(p^2 - m_2^2 + m_1^2)}{p^2 D} B_0(0; m_1^2, m_1^2)
+ m_2^2 \frac{(p^2 - m_1^2 + m_2^2)}{p^2 D} B_0(0; m_2^2, m_2^2)
- \frac{(p^2 - m_1^2 + m_2^2)(p^2 - m_2^2 + m_1^2)}{p^2 D} \tag{A.1}$$

provided the denominator $p^2D \neq 0$. Here

$$D = (p^4 + m_1^4 + m_2^4 + 2p^2m_1^2 + 2p^2m_2^2 - 2m_1^2m_2^2)$$

$$= [p^2 + (m_1 + m_2)^2][p^2 + (m_1 - m_2)^2].$$
(A.2)

For $p^2 = 0$ and $m_1^2 \neq m_2^2$

$$\frac{\partial B_0}{\partial p^2}(p^2; m_1^2, m_2^2) \Big|_{p^2=0} = -\frac{m_1^2 + m_2^2}{2(m_1^2 - m_2^2)^2} - \frac{m_1^2 m_2^2}{(m_1^2 - m_2^2)^3} [B_0(0; m_1^2, m_1^2) - B_0(0; m_2^2, m_2^2)] \quad (A.4)$$

which reduces to

$$\left. \frac{\partial B_0}{\partial p^2} (p^2; m_1^2, m_2^2) \right|_{p^2 = 0} = -\frac{1}{6m^2} \tag{A.5}$$

when $m^2 = m_1^2 = m_2^2$.

For the special case, $p^2 = -(m_1 - m_2)^2$, for which D vanishes

$$\frac{\partial B_0}{\partial p^2} (p^2; m_1^2, m_2^2) \Big|_{p^2 = -(m_1 - m_2)^2} \\
= -\frac{2}{p^2} \left\{ 1 - \frac{(m_1^2 - m_2^2)}{4p^2} \left[B_0(0; m_1^2, m_1^2) - B_0(0; m_2^2, m_2^2) \right] \right\} \Big|_{p^2 = -(m_1 - m_2)^2} (A.6)$$

Provided $D \neq 0$ the derivative with respect to m_1^2 is

$$\frac{\partial B_0}{\partial m_1^2}(p^2; m_1^2, m_2^2) = \frac{p^2 + m_1^2 - m_2^2}{D} B_0(p^2; m_1^2, m_2^2)
- \frac{p^2 + m_2^2 + m_1^2}{D} B_0(0; m_1^2, m_1^2) + \frac{2m_2^2}{D} B_0(0; m_2^2, m_2^2)
- 2\frac{(p^2 + m_1^2 - m_2^2)}{D}.$$
(A.7)

When $p^2 = -(m_1 - m_2)^2$

$$\frac{\partial B_0}{\partial m_1^2} (p^2; m_1^2, m_2^2) \Big|_{p^2 = -(m_1 - m_2)^2} \\
= \frac{1}{m_1(m_1 - m_2)} - \frac{1}{2p^2} \left[B_0(0; m_1^2, m_1^2) - B_0(0; m_2^2, m_2^2) \right] \Big|_{p^2 = -(m_1 - m_2)^2} .$$
(A.8)

For $p^2 = 0$ and $m_1^2 \neq m_2^2$ eq.(A.7) becomes

$$\frac{\partial B_0}{\partial m_1^2} (p^2; m_1^2, m_2^2) \Big|_{p^2 = -(m_1 - m_2)^2}
= -\frac{1}{(m_1^2 - m_2^2)} - \frac{m_2^2}{(m_2^2 - m_1^2)^2} \left[B_0(0; m_1^2, m_1^2) - B_0(0; m_2^2, m_2^2) \right]$$
(A.9)

which reduces to

$$\left. \frac{\partial B_0}{\partial p^2} (p^2; m_1^2, m_2^2) \right|_{p^2 = 0} = -\frac{1}{2m^2} \tag{A.10}$$

for $m_1^2 = m_2^2 = m^2$.

The partial derivative with respect to m_2^2 can be obtained from the fact that B_0 is symmetric in its mass arguments.

Appendix B

The following are the reduction formulas for the C_{ij} form factors that multiply tensors which have one $\delta_{\mu\nu}$ in them. These form factors are logarithmically divergent.

$$C_{24} = -\frac{1}{2} \left[-\frac{1}{2} + \frac{1}{2} \{ f_1 C_{11} + f_2 C_{12} + 2m_1^2 C_0 - B_0(2,3) \} \right]$$

$$C_{35} = -\frac{1}{3} \left[\frac{1}{3} + \frac{1}{2} \{ f_1 C_{21} + f_2 C_{23} + 2m_1^2 C_{11} + B_0(2,3) \} \right]$$

$$C_{36} = -\frac{1}{3} \left[\frac{1}{6} + \frac{1}{2} \{ f_1 C_{23} + f_2 C_{22} + 2m_1^2 C_{12} - B_1(2,3) \} \right]$$

$$C_{46} = -\frac{1}{4} \left[-\frac{1}{4} + \frac{1}{2} \{ f_1 C_{31} + f_2 C_{33} + 2m_1^2 C_{21} - B_0(2,3) \} \right]$$

$$C_{47} = -\frac{1}{4} \left[-\frac{1}{12} + \frac{1}{2} \{ f_1 C_{34} + f_2 C_{32} + 2m_1^2 C_{22} - B_{21}(2,3) \} \right]$$

$$C_{48} = -\frac{1}{4} \left[-\frac{1}{8} + \frac{1}{2} \{ f_1 C_{33} + f_2 C_{34} + 2m_1^2 C_{23} + B_1(2,3) \} \right]$$

All the logarithmically divergent C_{ij} form factors given above can be written using a single formula, with a change in notation, as follows,

$$C_{mn1} = \frac{-1}{m+n} \left[[C_{mn1}] + \frac{1}{2} \{ f_1 C_{\{m+1\}n0} + f_2 C_{m\{n+1\}0} + 2m_1^2 C_{mn0} + (-1)^{m+1} B_{n0}(2,3) \} \right]$$

where,

$$[C_{mn1}] = \lim_{n \to 4} (n-4)C_{mn1} = \frac{(-1)^{m+n+1}}{(m+n+2)(n+1)}$$

This notation will become evident with an example.

$$C_{\mu\nu} = p_{1\mu}p_{1\nu}C_{200} + p_{2\mu}p_{2\nu}C_{020} + \{p_1p_2\}_{\mu\nu}C_{110} + \delta_{\mu\nu}C_{001}$$

The first index of C counts the number of p_1 's in the tensor that it multiplies, the second index, the number of p_2 's and the third index, the number of δ 's. In the case of the B_{ij} form factors, the first index counts the number of p's, and the second index, the number of δ 's.

Finally the only quadratically divergent form factor C_{49} is given by,

$$C_{49} = -\frac{1}{4} \left[\frac{1}{2} f_1 C_{35} + \frac{1}{2} f_2 C_{36} + m_1^2 C_{24} - B_{22}(2,3) + \frac{1}{4} p_2^2 B_{21}(2,3) + \frac{1}{4} (p_2^2 - m_2^2 + m_3^2) B_1(2,3) - \frac{1}{4} m_2^2 B_0(2,3) + \frac{1}{48} (p_1^2 - p_2^2 + p_5^2 + 4m_1^2 - 2m_2^2 - 2m_3^2) \right]$$

The D_{ij} form factors that multiply tensors that have one $\delta_{\mu\nu}$ in them are

given by,

$$D_{27} = -1 \left[\frac{1}{2} \{ f_1 D_{11} + f_2 D_{12} + f_3 D_{13} + 2m_1^2 D_0 - C_0(2, 3, 4) \} \right]$$

$$D_{311} = -\frac{1}{2} \left[\frac{1}{2} \{ f_1 D_{21} + f_2 D_{24} + f_3 D_{25} + 2m_1^2 D_{11} + C_0(2, 3, 4) \} \right]$$

$$D_{312} = -\frac{1}{2} \left[\frac{1}{2} \{ f_1 D_{24} + f_2 D_{22} + f_3 D_{26} + 2m_1^2 D_{12} - C_{11}(2, 3, 4) \} \right]$$

$$D_{313} = -\frac{1}{2} \left[\frac{1}{2} \{ f_1 D_{25} + f_2 D_{26} + f_3 D_{23} + 2m_1^2 D_{13} - C_{12}(2, 3, 4) \} \right]$$

$$D_{416} = -\frac{1}{3} \left[\frac{1}{2} \{ f_1 D_{31} + f_2 D_{34} + f_3 D_{35} + 2m_1^2 D_{21} - C_0(2, 3, 4) \} \right]$$

$$D_{417} = -\frac{1}{3} \left[\frac{1}{2} \{ f_1 D_{36} + f_2 D_{32} + f_3 D_{38} + 2m_1^2 D_{22} - C_{21}(2, 3, 4) \} \right]$$

$$D_{418} = -\frac{1}{3} \left[\frac{1}{2} \{ f_1 D_{37} + f_2 D_{39} + f_3 D_{33} + 2m_1^2 D_{23} - C_{22}(2, 3, 4) \} \right]$$

$$D_{419} = -\frac{1}{3} \left[\frac{1}{2} \{ f_1 D_{34} + f_2 D_{36} + f_3 D_{310} + 2m_1^2 D_{24} + C_{11}(2, 3, 4) \} \right]$$

$$D_{420} = -\frac{1}{3} \left[\frac{1}{2} \{ f_1 D_{35} + f_2 D_{310} + f_3 D_{37} + 2m_1^2 D_{25} + C_{12}(2, 3, 4) \} \right]$$

$$D_{421} = -\frac{1}{3} \left[\frac{1}{2} \{ f_1 D_{310} + f_2 D_{38} + f_3 D_{39} + 2m_1^2 D_{26} - C_{23}(2, 3, 4) \} \right]$$

This set of D_{ij} form factors given above can be written using a single formula, with a change in notation, as,

$$D_{mnu1} = \frac{-1}{m+n+u+1} \left[\frac{1}{2} \{ f_1 D_{\{m+1\}nu0} + f_2 D_{m\{n+1\}u0} + f_3 D_{mn\{u+1\}0} + 2m_1^2 D_{mnu0} + (-1)^{m+1} C_{nu0}(2,3,4) \} \right]$$

The notation is as described in the case of the C form factors. The first, second, and third index of the D_{ij} form factor count the number of p_1 's, p_2 's, and p_3 's respectively, in the tensor that it multiplies, and the last index counts the number of δ 's.

Finally, the only logarithmically divergent D_{ij} form factor, D_{422} , is given

by,

$$D_{422} = -\frac{1}{3} \left[-\frac{1}{4} + \frac{1}{2} \{ p_1^2 D_{416} + p_2^2 D_{417} + p_3^2 D_{418} + 2(p_1 \cdot p_2) D_{419} + 2(p_1 \cdot p_3) D_{420} + 2(p_2 \cdot p_3) D_{421} + m_1^2 D_{27} - C_{24}(2, 3, 4) \} \right]$$

Appendix C

The following are the reduction formulas for the C_{ij} form factors when $\mathcal{D} = 0$. The formulas for C_0 and the C_{1j} form factors are given in section 6. The formulas for the C_{ij} form factors that multiply tensors that have one or more $\delta_{\mu\nu}$ are given in Appendix B. The rest of the C_{ij} form factors are given here.

C_{2j} Form Factors

$$\begin{pmatrix} 2\bar{x}_{i1}f_1 + 3Y_i & 2\bar{x}_{i1}f_2 & 0\\ \bar{x}_{i2}f_1 & 4Y_i & \bar{x}_{i1}f_2\\ 0 & 2\bar{x}_{i2}f_1 & 3Y_i + 2\bar{x}_{i2}f_2 \end{pmatrix} \begin{pmatrix} C_{21}\\ C_{23}\\ C_{22} \end{pmatrix} = \begin{pmatrix} \overline{R}_{21}\\ \overline{R}_{23}\\ \overline{R}_{22} \end{pmatrix}$$

$$\overline{R}_{21} = -4m_1^2 \bar{x}_{i1} C_{11} - 3\bar{x}_{i2} B_{21}(1,2) + 3(\bar{x}_{i2} - \bar{x}_{i1}) B_{21}(1,3)
+ \bar{x}_{i1} B_0(2,3) - \frac{4}{3} \bar{x}_{i1}
\overline{R}_{23} = -2m_1^2 (\bar{x}_{i2} C_{11} + \bar{x}_{i1} C_{12}) + 3(\bar{x}_{i2} - \bar{x}_{i1}) B_{21}(1,3) - 2\bar{x}_{i1} B_{11}(2,3)
- \bar{x}_{i2} B_0(2,3) - \bar{x}_{i1}/3 - 2\bar{x}_{i2}/3
\overline{R}_{22} = -4m_1^2 \bar{x}_{i2} C_{12} + 3(\bar{x}_{i2} - \bar{x}_{i1}) B_{21}(1,3) + 3\bar{x}_{i1} B_{21}(2,3)
+ 2\bar{x}_{i2} B_{11}(2,3) - \frac{2}{3} \bar{x}_{i2}$$

C_{3j} Form Factors

$$\begin{pmatrix} 3\bar{x}_{i1}f_1 + 4Y_i & 3\bar{x}_{i1}f_2 & 0 & 0\\ \bar{x}_{i2}f_1 & \bar{x}_{i1}f_1 + 5Y_i & 2\bar{x}_{i1}f_2 & 0\\ 0 & 2\bar{x}_{i2}f_1 & 5Y_i + \bar{x}_{i2}f_2 & \bar{x}_{i1}f_2\\ 0 & 0 & 3\bar{x}_{i2}f_1 & 4Y_i + 3\bar{x}_{i2}f_2 \end{pmatrix} \times \mathbf{C}_3 = \overline{\mathbf{R}}_3$$

where,

$$\{\mathbf{C}_{3}\}^{T} = (C_{31} \ C_{33} \ C_{34} \ C_{32}) \quad \{\overline{\mathbf{R}}_{3}\}^{T} = (\overline{R}_{31} \ \overline{R}_{33} \ \overline{R}_{34} \ \overline{R}_{32})$$

$$\overline{R}_{31} = -6m_{1}^{2}\bar{x}_{i1}C_{21} - 4\bar{x}_{i2}B_{31}(1,2) + 4(\bar{x}_{i2} - \bar{x}_{i1})B_{31}(1,3)$$

$$-2\bar{x}_{i1}B_{0}(2,3) + \frac{3}{2}\bar{x}_{i1}$$

$$\overline{R}_{33} = -2m_{1}^{2}(\bar{x}_{i2}C_{21} + 2\bar{x}_{i1}C_{23}) + 4(\bar{x}_{i2} - \bar{x}_{i1})B_{31}(1,3) + 2\bar{x}_{i1}B_{1}(2,3)$$

$$+\bar{x}_{i2}B_{0}(2,3) + \frac{1}{2}\bar{x}_{i1} + \frac{1}{2}\bar{x}_{i2}$$

$$\overline{R}_{34} = -2m_{1}^{2}(2\bar{x}_{i2}C_{23} + \bar{x}_{i1}C_{22}) + 4(\bar{x}_{i2} - \bar{x}_{i1})B_{31}(1,3) - 3\bar{x}_{i1}B_{21}(2,3)$$

$$-2\bar{x}_{i2}B_{1}(2,3) + \frac{1}{6}\bar{x}_{i1} + \frac{1}{2}\bar{x}_{i2}$$

$$\overline{R}_{32} = -6m_{1}^{2}C_{22} + 4(\bar{x}_{i2} - \bar{x}_{i1})B_{31}(1,3) + 4\bar{x}_{i1}B_{31}(2,3)$$

$$+3\bar{x}_{i2}B_{21}(2,3) + \frac{1}{2}\bar{x}_{i2}$$

C_{4j} Form Factors

$$\begin{pmatrix} 4\bar{x}_{i1}f_1 + 5Y_i & 4\bar{x}_{i1}f_2 & 0 & 0 & 0\\ \bar{x}_{i2}f_1 & 2\bar{x}_{i1}f_1 + 6Y_i & 3\bar{x}_{i1}f_2 & 0 & 0\\ 0 & 2\bar{x}_{i2}f_1 & 7Y_i & 2\bar{x}_{i1}f_2 & 0\\ 0 & 0 & 3\bar{x}_{i2}f_1 & 6Y_i + 2\bar{x}_{i2}f_2 & \bar{x}_{i1}f_2\\ 0 & 0 & 0 & 4\bar{x}_{i2}f_1 & 5Y_i + 4\bar{x}_{i2}f_2 \end{pmatrix} \times \mathbf{C}_4 = \mathbf{\overline{R}}_4$$

where,

$$\{\mathbf{C}_{4}\}^{T} = (C_{41} \ C_{43} \ C_{45} \ C_{44} \ C_{42})$$

$$\{\overline{\mathbf{R}}_{4}\}^{T} = (\overline{R}_{41} \ \overline{R}_{43} \ \overline{R}_{45} \ \overline{R}_{44} \ \overline{R}_{42})$$

$$\overline{R}_{41} = -8m_1^2 \bar{x}_{i1} C_{31} - 5\bar{x}_{i2} B_{41}(1,2) + 5(\bar{x}_{i2} - \bar{x}_{i1}) B_{41}(1,3)
+ \bar{x}_{i1} B_0(2,3) - \frac{8}{5} \bar{x}_{i1}
\overline{R}_{43} = -2m_1^2 (\bar{x}_{i2} C_{31} + 3\bar{x}_{i1} C_{33}) + 5(\bar{x}_{i2} - \bar{x}_{i1}) B_{41}(1,3) - 2\bar{x}_{i1} B_1(2,3)
- \bar{x}_{i2} B_0(2,3) - \frac{3}{5} \bar{x}_{i1} - \frac{2}{5} \bar{x}_{i2}
\overline{R}_{45} = -4m_1^2 (\bar{x}_{i2} C_{33} + \bar{x}_{i1} C_{34}) + 5(\bar{x}_{i2} - \bar{x}_{i1}) B_{41}(1,3) + 3\bar{x}_{i1} B_{21}(2,3)
+ 2\bar{x}_{i2} B_1(2,3) - \frac{2}{15} (2\bar{x}_{i1} + 3\bar{x}_{i2})
\overline{R}_{44} = -2m_1^2 (3\bar{x}_{i2} C_{34} + \bar{x}_{i1} C_{32}) + 5(\bar{x}_{i2} - \bar{x}_{i1}) B_{41}(1,3) - 4\bar{x}_{i1} B_{31}(2,3)
- 3\bar{x}_{i2} B_{21}(2,3) - \frac{1}{10} \bar{x}_{i1} - \frac{2}{5} \bar{x}_{i2}
\overline{R}_{42} = -8m_1^2 \bar{x}_{i2} C_{32} + 5(\bar{x}_{i2} - \bar{x}_{i1}) B_{41}(1,3) + 5\bar{x}_{i1} B_{41}(2,3)
+ 4\bar{x}_{i2} B_{31}(2,3) - \frac{2}{5} \bar{x}_{i2}$$

Appendix D

The following are the formulas for the D_{ij} form factor reductions when $\mathcal{D} = 0$. The formula for the reduction of D_0 is given in section 7. The formulas for the reduction of all D_{ij} form factors which multiply tensors that have one or more $\delta_{\mu\nu}$ in them are given in Appendix B. The formulas for the rest of the D_{ij} form factor reductions are given below.

D_{1j} Form Factors

$$\begin{pmatrix} Y_i + \bar{x}_{i1} f_1 & \bar{x}_{i1} f_2 & \bar{x}_{i1} f_3 \\ \bar{x}_{i2} f_1 & Y_i + \bar{x}_{i2} f_2 & \bar{x}_{i2} f_3 \\ \bar{x}_{i3} f_1 & \bar{x}_{i3} f_2 & Y_i + \bar{x}_{i3} f_3 \end{pmatrix} \begin{pmatrix} D_{11} \\ D_{12} \\ D_{13} \end{pmatrix} = \begin{pmatrix} \overline{R}_{11} \\ \overline{R}_{12} \\ \overline{R}_{13} \end{pmatrix}$$

$$\overline{R}_{11} = -2m_1^2 \bar{x}_{i1} D_0 - \bar{x}_{i3} C_{11}(1,2,3) + (\bar{x}_{i3} - \bar{x}_{i2}) C_{11}(1,2,4)
+ (\bar{x}_{i2} - \bar{x}_{i1}) C_{11}(1,3,4)
\overline{R}_{12} = -2m_1^2 \bar{x}_{i2} D_0 - \bar{x}_{i3} C_{12}(1,2,3) + (\bar{x}_{i3} - \bar{x}_{i2}) C_{12}(1,2,4)
+ (\bar{x}_{i2} - \bar{x}_{i1}) C_{11}(1,3,4) + \bar{x}_{i1} C_{11}(2,3,4) + \bar{x}_{i2} C_0(2,3,4)
\overline{R}_{13} = -2m_1^2 \bar{x}_{i3} D_0 + (\bar{x}_{i3} - \bar{x}_{i2}) C_{12}(1,2,4) + (\bar{x}_{i2} - \bar{x}_{i1}) C_{12}(1,3,4)
+ \bar{x}_{i1} C_{12}(2,3,4) + \bar{x}_{i3} C_0(2,3,4)$$

The D_{2j} and higher D_{ij} form factor formulas involve large matrices which do not fit neatly in the page and so the equations are not written in the matrix form.

D_{2j} Form Factors

$$(Y_i + \bar{x}_{i1}f_1)D_{21} + \bar{x}_{i1}f_2D_{24} + \bar{x}_{i1}f_3D_{25} = -2m_1^2\bar{x}_{i1}D_{11} - \bar{x}_{i3}C_{21}(1,2,3) + (\bar{x}_{i3} - \bar{x}_{i2})C_{21}(1,2,4) + (\bar{x}_{i2} - \bar{x}_{i1})C_{21}(1,3,4)$$

$$\bar{x}_{i2}f_{1}D_{21} + (3Y_{i} - \bar{x}_{i3}f_{3})D_{24} + \bar{x}_{i2}f_{3}D_{25} + \bar{x}_{i1}f_{2}D_{22} + \bar{x}_{i1}f_{3}D_{26} = -2m_{1}^{2}(\bar{x}_{i2}D_{11} + \bar{x}_{i1}D_{12}) + 2(\bar{x}_{i2} - \bar{x}_{i1})C_{21}(1,3,4) - 2\bar{x}_{i3}C_{23}(1,2,3) + 2(\bar{x}_{i3} - \bar{x}_{i2})C_{23}(1,2,4) - \bar{x}_{i1}C_{11}(2,3,4) - \bar{x}_{i2}C_{0}(2,3,4)$$

$$\bar{x}_{i3}f_{1}D_{21} + \bar{x}_{i3}f_{2}D_{24} + (3Y_{i} - \bar{x}_{i2}f_{2})D_{25} + \bar{x}_{i1}f_{2}D_{26} + \bar{x}_{i1}f_{3}D_{23} = -2m_{1}^{2}(\bar{x}_{i3}D_{11} + \bar{x}_{i1}D_{13}) + 2(\bar{x}_{i3} - \bar{x}_{i2})C_{23}(1, 2, 4) + 2(\bar{x}_{i2} - \bar{x}_{i1})C_{23}(1, 3, 4) - \bar{x}_{i1}C_{12}(2, 3, 4) - \bar{x}_{i3}C_{0}(2, 3, 4)$$

$$\bar{x}_{i2}f_{1}D_{24} + (Y_{i} + \bar{x}_{i2}f_{2})D_{22} + \bar{x}_{i2}f_{3}D_{26} =$$

$$-2m_{1}^{2}\bar{x}_{i2}D_{12} + (\bar{x}_{i2} - \bar{x}_{i1})C_{21}(1,3,4) + \bar{x}_{i1}C_{21}(2,3,4)$$

$$-\bar{x}_{i3}C_{22}(1,2,3) + (\bar{x}_{i3} - \bar{x}_{i2})C_{22}(1,2,4) + \bar{x}_{i2}C_{11}(2,3,4)$$

$$\bar{x}_{i3}f_{1}D_{24} + \bar{x}_{i2}f_{1}D_{25} + \bar{x}_{i3}f_{2}D_{22} + (3Y_{i} - \bar{x}_{i1}f_{1})D_{26} + \bar{x}_{i2}f_{3}D_{23} = -2m_{1}^{2}(\bar{x}_{i3}D_{12} + \bar{x}_{i2}D_{13}) + 2(\bar{x}_{i3} - \bar{x}_{i2})C_{22}(1, 2, 4) + 2(\bar{x}_{i2} - \bar{x}_{i1}) C_{23}(1, 3, 4) + 2\bar{x}_{i1}C_{23}(2, 3, 4) + \bar{x}_{i3}C_{11}(2, 3, 4) + \bar{x}_{i2}C_{12}(2, 3, 4)$$

$$\bar{x}_{i3}f_1D_{25} + \bar{x}_{i3}f_2D_{26} + (Y_i + \bar{x}_{i3}f_3)D_{23} = -2m_1^2\bar{x}_{i3}D_{13} + (\bar{x}_{i3} - \bar{x}_{i2})C_{22}(1, 2, 4) + (\bar{x}_{i2} - \bar{x}_{i1})C_{22}(1, 3, 4) + \bar{x}_{i1}C_{22}(2, 3, 4) + \bar{x}_{i3}C_{12}(2, 3, 4)$$

D_{3i} Form Factors

$$(Y_i + \bar{x}_{i1}f_1)D_{31} + \bar{x}_{i1}f_2D_{34} + \bar{x}_{i1}f_3D_{35} = -2m_1^2\bar{x}_{i1}D_{21} - \bar{x}_{i3}C_{31}(1,2,3) + (\bar{x}_{i3} - \bar{x}_{i2})C_{31}(1,2,4) + (\bar{x}_{i2} - \bar{x}_{i1})C_{31}(1,3,4)$$

$$\bar{x}_{i2}f_{1}D_{31} + (3Y_{i} + 2\bar{x}_{i1}f_{1} + \bar{x}_{i2}f_{2})D_{34}
+ \bar{x}_{i2}f_{3}D_{35} + 2\bar{x}_{i1}f_{2}D_{36} + 2\bar{x}_{i1}f_{3}D_{310} =
- 2m_{1}^{2}(\bar{x}_{i2}D_{21} + 2\bar{x}_{i1}D_{24}) - 3\bar{x}_{i3}C_{33}(1, 2, 3)
+ 3(\bar{x}_{i3} - \bar{x}_{i2})C_{33}(1, 2, 4) + 3(\bar{x}_{i2} - \bar{x}_{i1})C_{31}(1, 3, 4)
+ \bar{x}_{i1}C_{11}(2, 3, 4) + \bar{x}_{i2}C_{0}(2, 3, 4)$$

$$\bar{x}_{i3}f_{1}D_{31} + \bar{x}_{i3}f_{2}D_{34}$$

$$+ (3Y_{i} + 2\bar{x}_{i1}f_{1} + \bar{x}_{i3}f_{3})D_{35} + 2\bar{x}_{i1}f_{2}D_{310} + 2\bar{x}_{i1}f_{3}D_{37} =$$

$$- 2m_{1}^{2}(\bar{x}_{i3}D_{21} + 2\bar{x}_{i1}D_{25}) + 3(\bar{x}_{i3} - \bar{x}_{i2})C_{33}(1, 2, 4)$$

$$+ 3(\bar{x}_{i2} - \bar{x}_{i1})C_{33}(1, 3, 4) + \bar{x}_{i1}C_{12}(2, 3, 4) + \bar{x}_{i3}C_{0}(2, 3, 4)$$

$$2\bar{x}_{i2}f_{1}D_{34} + (4Y_{i} + \bar{x}_{i2}f_{2} - \bar{x}_{i3}f_{3})D_{36}$$

$$+ 2\bar{x}_{i2}f_{3}D_{310} + \bar{x}_{i1}f_{2}D_{32} + \bar{x}_{i1}f_{3}D_{38} =$$

$$- 2m_{1}^{2}(2\bar{x}_{i2}D_{24} + \bar{x}_{i1}D_{22}) - 3\bar{x}_{i3}C_{34}(1, 2, 3)$$

$$+ 3(\bar{x}_{i3} - \bar{x}_{i2})C_{34}(1, 2, 4) + 3(\bar{x}_{i2} - \bar{x}_{i1})C_{31}(1, 3, 4)$$

$$- 2\bar{x}_{i1}C_{21}(2, 3, 4) - 2\bar{x}_{i2}C_{11}(2, 3, 4)$$

$$\bar{x}_{i3}f_{1}D_{34} + \bar{x}_{i2}f_{1}D_{35} + \bar{x}_{i3}f_{2}D_{36}$$

$$+ 4Y_{i}D_{310} + \bar{x}_{i2}f_{3}D_{37} + \bar{x}_{i1}f_{2}D_{38} + \bar{x}_{i1}f_{3}D_{39} =$$

$$- 2m_{1}^{2}(\bar{x}_{i3}D_{24} + \bar{x}_{i2}D_{25} + \bar{x}_{i1}D_{26}) + 3(\bar{x}_{i2} - \bar{x}_{i1})C_{33}(1, 3, 4)$$

$$+ 3(\bar{x}_{i3} - \bar{x}_{i2})C_{34}(1, 2, 4) - 2\bar{x}_{i1}C_{23}(2, 3, 4)$$

$$- \bar{x}_{i3}C_{11}(2, 3, 4) - \bar{x}_{i2}C_{12}(2, 3, 4)$$

$$2\bar{x}_{i3}f_{1}D_{35} + 2\bar{x}_{i3}f_{2}D_{310} + (4Y_{i} - \bar{x}_{i2}f_{2} + \bar{x}_{i3}f_{3})D_{37} + \bar{x}_{i1}f_{2}D_{39} + \bar{x}_{i1}f_{3}D_{33} = -2m_{1}^{2}(2\bar{x}_{i3}D_{25} + \bar{x}_{i1}D_{23}) + 3(\bar{x}_{i3} - \bar{x}_{i2})C_{34}(1, 2, 4) + 3(\bar{x}_{i2} - \bar{x}_{i1})C_{34}(1, 3, 4) - 2\bar{x}_{i1}C_{22}(2, 3, 4) - 2\bar{x}_{i3}C_{12}(2, 3, 4)$$

$$\bar{x}_{i2}f_1D_{36} + (Y_i + \bar{x}_{i2}f_2)D_{32} + \bar{x}_{i2}f_3D_{38} = -2m_1^2\bar{x}_{i2}D_{22} + (\bar{x}_{i2} - \bar{x}_{i1})C_{31}(1,3,4) + \bar{x}_{i1}C_{31}(2,3,4) -\bar{x}_{i3}C_{32}(1,2,3) + (\bar{x}_{i3} - \bar{x}_{i2})C_{32}(1,2,4) + \bar{x}_{i2}C_{21}(2,3,4)$$

$$\bar{x}_{i3}f_{1}D_{36} + 2\bar{x}_{i2}f_{1}D_{310} + \bar{x}_{i3}f_{2}D_{32} + (3Y_{i} + 2\bar{x}_{i2}f_{2} + \bar{x}_{i3}f_{3})D_{38} + 2\bar{x}_{i2}f_{3}D_{39} = -2m_{1}^{2}(\bar{x}_{i3}D_{22} + 2\bar{x}_{i2}D_{26}) + 3(\bar{x}_{i3} - \bar{x}_{i2})C_{32}(1, 2, 4) + 3(\bar{x}_{i2} - \bar{x}_{i1})C_{33}(1, 3, 4) + 3\bar{x}_{i1}C_{33}(2, 3, 4) + \bar{x}_{i3}C_{21}(2, 3, 4) + 2\bar{x}_{i2}C_{23}(2, 3, 4)$$

$$2\bar{x}_{i3}f_{1}D_{310} + \bar{x}_{i2}f_{1}D_{37} + 2\bar{x}_{i3}f_{2}D_{38} + (3Y_{i} + \bar{x}_{i2}f_{2} + 2\bar{x}_{i3}f_{3})D_{39} + \bar{x}_{i2}f_{3}D_{33} = -2m_{1}^{2}(2\bar{x}_{i3}D_{26} + \bar{x}_{i2}D_{23}) + 3(\bar{x}_{i3} - \bar{x}_{i2})C_{32}(1, 2, 4) + 3(\bar{x}_{i2} - \bar{x}_{i1})C_{34}(1, 3, 4) + 3\bar{x}_{i1}C_{34}(2, 3, 4) + \bar{x}_{i2}C_{22}(2, 3, 4) + 2\bar{x}_{i3}C_{23}(2, 3, 4)$$

$$\bar{x}_{i3}f_{1}D_{37} + \bar{x}_{i3}f_{2}D_{39} + (Y_{i} + \bar{x}_{i3}f_{3})D_{33} = -2m_{1}^{2}\bar{x}_{i3}D_{23} + (\bar{x}_{i3} - \bar{x}_{i2})C_{32}(1, 2, 4) + (\bar{x}_{i2} - \bar{x}_{i1})C_{32}(1, 3, 4) + \bar{x}_{i1}C_{32}(2, 3, 4) + \bar{x}_{i3}C_{22}(2, 3, 4)$$

D_{4j} Form Factors

$$(Y_i + \bar{x}_{i1}f_1)D_{41} + \bar{x}_{i1}f_2D_{44} + \bar{x}_{i1}f_3D_{45} = -2m_1^2\bar{x}_{i1}D_{31} - \bar{x}_{i3}C_{41}(1,2,3) + (\bar{x}_{i3} - \bar{x}_{i2})C_{41}(1,2,4) + (\bar{x}_{i2} - \bar{x}_{i1})C_{41}(1,3,4)$$

$$\bar{x}_{i2}f_{1}D_{41} + (5Y_{i} + 2\bar{x}_{i1}f_{1} - \bar{x}_{i3}f_{3})D_{44} + \bar{x}_{i2}f_{3}D_{45} + 3\bar{x}_{i1}f_{2}D_{410} + 3\bar{x}_{i1}f_{3}D_{413} = -2m_{1}^{2}(\bar{x}_{i2}D_{31} + 3\bar{x}_{i1}D_{34}) - 4\bar{x}_{i3}C_{43}(1,2,3) + 4(\bar{x}_{i3} - \bar{x}_{i2})C_{43}(1,2,4) + 4(\bar{x}_{i2} - \bar{x}_{i1})C_{41}(1,3,4) - \bar{x}_{i1}C_{11}(2,3,4) - \bar{x}_{i2}C_{0}(2,3,4)$$

$$\bar{x}_{i3}f_{1}D_{41} + \bar{x}_{i3}f_{2}D_{44}$$

$$+ (5Y_{i} + 2\bar{x}_{i1}f_{1} - \bar{x}_{i2}f_{2})D_{45} + 3\bar{x}_{i1}f_{2}D_{413} + 3\bar{x}_{i1}f_{3}D_{411} =$$

$$- 2m_{1}^{2}(\bar{x}_{i3}D_{31} + 3\bar{x}_{i1}D_{35}) + 4(\bar{x}_{i3} - \bar{x}_{i2})C_{43}(1, 2, 4)$$

$$+ 4(\bar{x}_{i2} - \bar{x}_{i1})C_{43}(1, 3, 4) - \bar{x}_{i1}C_{12}(2, 3, 4) - \bar{x}_{i3}C_{0}(2, 3, 4)$$

$$\bar{x}_{i2}f_1D_{44} + (3Y_i - \bar{x}_{i3}f_3)D_{410} + \bar{x}_{i2}f_3D_{413} + \bar{x}_{i1}f_2D_{46} + \bar{x}_{i1}f_3D_{414} = -2m_1^2(\bar{x}_{i2}D_{34} + \bar{x}_{i1}D_{36}) - 2\bar{x}_{i3}C_{45}(1,2,3) + 2(\bar{x}_{i3} - \bar{x}_{i2})C_{45}(1,2,4) + 2(\bar{x}_{i2} - \bar{x}_{i1})C_{41}(1,3,4) + \bar{x}_{i1}C_{21}(2,3,4) + \bar{x}_{i2}C_{11}(2,3,4)$$

$$\bar{x}_{i3}f_{1}D_{44} + \bar{x}_{i2}f_{1}D_{45} + \bar{x}_{i3}f_{2}D_{410}$$

$$+ (5Y_{i} + \bar{x}_{i1}f_{1})D_{413} + \bar{x}_{i2}f_{3}D_{411} + 2\bar{x}_{i1}f_{2}D_{414} + 2\bar{x}_{i1}f_{3}D_{415} =$$

$$- 2m_{1}^{2}(\bar{x}_{i3}D_{34} + \bar{x}_{i2}D_{35} + 2\bar{x}_{i1}D_{310})$$

$$+ 4(\bar{x}_{i3} - \bar{x}_{i2})C_{45}(1, 2, 4) + 4(\bar{x}_{i2} - \bar{x}_{i1})C_{43}(1, 3, 4)$$

$$+ 2\bar{x}_{i1}C_{23}(2, 3, 4) + \bar{x}_{i3}C_{11}(2, 3, 4) + \bar{x}_{i2}C_{12}(2, 3, 4)$$

$$\bar{x}_{i3}f_{1}D_{45} + \bar{x}_{i3}f_{2}D_{413} + (3Y_{i} - \bar{x}_{i2}f_{2})D_{411} + \bar{x}_{i1}f_{2}D_{415} + \bar{x}_{i1}f_{3}D_{48} = -2m_{1}^{2}(\bar{x}_{i3}D_{35} + \bar{x}_{i1}D_{37}) + 2(\bar{x}_{i3} - \bar{x}_{i2})C_{45}(1, 2, 4) +2(\bar{x}_{i2} - \bar{x}_{i1})C_{45}(1, 3, 4) + \bar{x}_{i1}C_{22}(2, 3, 4) + \bar{x}_{i3}C_{12}(2, 3, 4)$$

$$3\bar{x}_{i2}f_{1}D_{410} + (5Y_{i} + 2\bar{x}_{i2}f_{2} - \bar{x}_{i3}f_{3})D_{46}$$

$$+ 3\bar{x}_{i2}f_{3}D_{414} + \bar{x}_{i1}f_{2}D_{42} + \bar{x}_{i1}f_{3}D_{47} =$$

$$- 2m_{1}^{2}(3\bar{x}_{i2}D_{36} + \bar{x}_{i1}D_{32}) - 4\bar{x}_{i3}C_{44}(1, 2, 3)$$

$$+ 4(\bar{x}_{i3} - \bar{x}_{i2})C_{44}(1, 2, 4) + 4(\bar{x}_{i2} - \bar{x}_{i1})C_{41}(1, 3, 4)$$

$$- 3\bar{x}_{i1}C_{31}(2, 3, 4) - 3\bar{x}_{i2}C_{21}(2, 3, 4)$$

$$\bar{x}_{i3}f_{1}D_{410} + 2\bar{x}_{i2}f_{1}D_{413} + \bar{x}_{i3}f_{2}D_{46}$$

$$+ (5Y_{i} + \bar{x}_{i2}f_{2})D_{414} + 2\bar{x}_{i2}f_{3}D_{415} + \bar{x}_{i1}f_{2}D_{47} + \bar{x}_{i1}f_{3}D_{412} =$$

$$- 2m_{1}^{2}(\bar{x}_{i3}D_{36} + 2\bar{x}_{i2}D_{310} + \bar{x}_{i1}D_{38})$$

$$+ 4(\bar{x}_{i3} - \bar{x}_{i2})C_{44}(1, 2, 4) + 4(\bar{x}_{i2} - \bar{x}_{i1})C_{43}(1, 3, 4)$$

$$- 3\bar{x}_{i1}C_{33}(2, 3, 4) - \bar{x}_{i3}C_{21}(2, 3, 4) - 2\bar{x}_{i2}C_{23}(2, 3, 4)$$

$$2\bar{x}_{i3}f_{1}D_{413} + \bar{x}_{i2}f_{1}D_{411} + 2\bar{x}_{i3}f_{2}D_{414} + (5Y_{i} + \bar{x}_{i3}f_{3})D_{415} + \bar{x}_{i2}f_{3}D_{48} + \bar{x}_{i1}f_{2}D_{412} + \bar{x}_{i1}f_{3}D_{49} = -2m_{1}^{2}(2\bar{x}_{i3}D_{310} + \bar{x}_{i2}D_{37} + \bar{x}_{i1}D_{39}) + 4(\bar{x}_{i3} - \bar{x}_{i2})C_{44}(1, 2, 4) + 4(\bar{x}_{i2} - \bar{x}_{i1})C_{45}(1, 3, 4) -3\bar{x}_{i1}C_{34}(2, 3, 4) - \bar{x}_{i2}C_{22}(2, 3, 4) - 2\bar{x}_{i3}C_{23}(2, 3, 4)$$

$$3\bar{x}_{i3}f_{1}D_{411} + 3\bar{x}_{i3}f_{2}D_{415}$$

$$+ (5Y_{i} - \bar{x}_{i2}f_{2} + 2\bar{x}_{i3}f_{3})D_{48} + \bar{x}_{i1}f_{2}D_{49} + \bar{x}_{i1}f_{3}D_{43} =$$

$$- 2m_{1}^{2}(3\bar{x}_{i3}D_{37} + \bar{x}_{i1}D_{33}) + 4(\bar{x}_{i3} - \bar{x}_{i2})C_{44}(1, 2, 4)$$

$$+ 4(\bar{x}_{i2} - \bar{x}_{i1})C_{44}(1, 3, 4) - 3\bar{x}_{i1}C_{32}(2, 3, 4) - 3\bar{x}_{i3}C_{22}(2, 3, 4)$$

$$\bar{x}_{i2}f_{1}D_{46} + (Y_{i} + \bar{x}_{i2}f_{2})D_{42} + \bar{x}_{i2}f_{3}D_{47} = -2m_{1}^{2}\bar{x}_{i2}D_{32} - \bar{x}_{i3}C_{42}(1,2,3) + (\bar{x}_{i3} - \bar{x}_{i2})C_{42}(1,2,4) + (\bar{x}_{i2} - \bar{x}_{i1})C_{41}(1,3,4) + \bar{x}_{i1}C_{41}(2,3,4) + \bar{x}_{i2}C_{31}(2,3,4)$$

$$\bar{x}_{i3}f_{1}D_{46} + 3\bar{x}_{i2}f_{1}D_{414} + \bar{x}_{i3}f_{2}D_{42}$$

$$+ (5Y_{i} - \bar{x}_{i1}f_{1} + 2\bar{x}_{i2}f_{2})D_{47} + 3\bar{x}_{i2}f_{3}D_{412} =$$

$$- 2m_{1}^{2}(\bar{x}_{i3}D_{32} + 3\bar{x}_{i2}D_{38}) + 4(\bar{x}_{i3} - \bar{x}_{i2})C_{42}(1, 2, 4)$$

$$+ 4(\bar{x}_{i2} - \bar{x}_{i1})C_{43}(1, 3, 4) + 4\bar{x}_{i1}C_{43}(2, 3, 4)$$

$$+ \bar{x}_{i3}C_{31}(2, 3, 4) + 3\bar{x}_{i2}C_{33}(2, 3, 4)$$

$$\bar{x}_{i3}f_{1}D_{414} + \bar{x}_{i2}f_{1}D_{415} + \bar{x}_{i3}f_{2}D_{47}$$

$$+ (3Y_{i} - \bar{x}_{i1}f_{1})D_{412} + \bar{x}_{i2}f_{3}D_{49} =$$

$$- 2m_{1}^{2}(\bar{x}_{i3}D_{38} + \bar{x}_{i2}D_{39}) + 2(\bar{x}_{i3} - \bar{x}_{i2})C_{42}(1, 2, 4)$$

$$+ 2(\bar{x}_{i2} - \bar{x}_{i1})C_{45}(1, 3, 4) + 2\bar{x}_{i1}C_{45}(2, 3, 4)$$

$$+ \bar{x}_{i3}C_{33}(2, 3, 4) + \bar{x}_{i2}C_{34}(2, 3, 4)$$

$$3\bar{x}_{i3}f_{1}D_{415} + \bar{x}_{i2}f_{1}D_{48} + 3\bar{x}_{i3}f_{2}D_{412}$$

$$+ (5Y_{i} - \bar{x}_{i1}f_{1} + 2\bar{x}_{i3}f_{3})D_{49} + \bar{x}_{i2}f_{3}D_{43} =$$

$$- 2m_{1}^{2}(3\bar{x}_{i3}D_{39} + \bar{x}_{i2}D_{33}) + 4(\bar{x}_{i3} - \bar{x}_{i2})C_{42}(1, 2, 4)$$

$$+ 4(\bar{x}_{i2} - \bar{x}_{i1})C_{44}(1, 3, 4) + 4\bar{x}_{i1}C_{44}(2, 3, 4)$$

$$+ \bar{x}_{i2}C_{32}(2, 3, 4) + 3\bar{x}_{i3}C_{34}(2, 3, 4)$$

$$\bar{x}_{i3}f_{1}D_{48} + \bar{x}_{i3}f_{2}D_{49} + (Y_{i} + \bar{x}_{i3}f_{3})D_{43} = -2m_{1}^{2}\bar{x}_{i3}D_{33} + (\bar{x}_{i3} - \bar{x}_{i2})C_{42}(1, 2, 4) + (\bar{x}_{i2} - \bar{x}_{i1})C_{42}(1, 3, 4) + \bar{x}_{i1}C_{42}(2, 3, 4) + \bar{x}_{i3}C_{32}(2, 3, 4)$$

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