SCALAR ONE-LOOP INTEGRALS

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The completely general one-loop scalar one-, two-, three- and four-point functions are studied. Also an integral occurring in connection with soft bremsstrahlung is considered. Formulas in terms of Spence functions are given. An expansion for Spence functions with complex argument is presented. The whole forms a basis for the calculation of one-loop radiative corrections in the general case, including unstable particles and particles with spin.

1. Introduction

With the advent of gauge theories, field theory, and in particular perturbation theory have gained renewed respectability. In weak, electromagnetic and strong interactions, even in gravitational interactions, diagrams are written down and calculated, and not without success. However, the complications of gauge theories are formidable and it is very hard to go beyond the simplest cases. Past experience is mainly in the domain of quantum electrodynamics and very beautiful work has been done in that domain. However most calculations of radiative corrections concern essentially static quantities which allows the use of methods (such as numerical integration over Feynman parameters) that are for one reason or another not applicable in the general case. Further there exist calculations of e.m. corrections to μ-e scattering [1], $e^+e^- \to \mu^+\mu^-$, $e^+e^- \to e^+e^-$ and similar processes [2, 3]. The diagrams occurring there are special in the sense that some of the internal masses are zero, while other internal masses are equal to certain external masses.

To cope with the complications of weak interactions the general case must be considered. It is the aim of this article to derive a complete set **of** formulas needed for the evaluation of one-loop diagrams with arbitrary internal and external masses. Evidently, this is a quite technical subject, but that cannot be avoided. The lit-

erature on this subject, in particular the four-point function, is rather limited. Wu [4], studying the analytic properties of this function, quotes an explicit formula, but we have not compared this with our results.

In successive sections we will deal with the expressions for the scalar one-loop diagrams with one, two, three and four external lines. Diagrams with more external lines can be reduced to linear combinations of these cases. For completeness an integral occurring in connection with bremsstrahlung is also given. In various appendices a number **of** technical details is given. In particular appendix **A** quotes an expansion useful for the computation **of** Spence functions with complex arguments. Appendix **E** gives some equations for special cases.

Particular emphasis is given to the required analytic structure of the amplitudes. Since there are in general many (logarithmic) branch points there are cuts that must be defined in a precise way. In regions of momentum space where no cuts ever occur (such as Euclidean space) the expressions are simple to derive. In principle the rest can be obtained by analytic continuation but in practice that is hard. Therefore we use different expressions for different regions of momentum space, some of which are considerably lengthier than others.

2. Preliminaries

In this section we will describe a few tricks and establish some conventions.

First there is an identity that holds for products of propagators. Consider the expression

$$\frac{1}{((q+p)^2+m_1^2)((q+p+k)^2+m_2^2)}. \tag{2.1}$$

Multiplication of nominator and denominator with

$$(1-\alpha)((q+p)^2+m_1^2)+\alpha((q+p+k)^2+m_2^2)$$

leads to the identity

$$\frac{1}{((q+p)^2+m_1^2)((q+p+k)^2+m_2^2)}=\sum_{i=1}^{2}\frac{\lambda_i}{((q+l)^2+M^2)((q+l+s_i)^2+M_j^2)}, \tag{2.2}$$

with

$$l=p+\alpha k, \qquad M^2=\alpha(1-\alpha)k^2+\alpha m_2^2+(1-\alpha)m_1^2,$$

$$\lambda_1=\alpha, \qquad s_1=-\alpha k, \qquad M_1=m_1,$$

$$\lambda_2=1-\alpha, \qquad s_2=(1-\alpha)k, \qquad M_2=m_2. \tag{2.3}$$

This identity can be used in several ways. If k is spacelike* then M^2 can be made zero with real α. If one of the masses is zero then things may be arranged such that one Feynman parameter integral can be done easily. (For example, in eq. (5.2), if $f = m_3^2 = 0$ then the substitution $y = xy'$ makes the x-integration trivial.) If k or p are timelike then there are two real solutions α_1, α_2 to the equation

$$l^2 = (p + \alpha k)^2 = 0 \; .$$

The discriminant is $(pk)^2 - p^2 k^2$. By going to the restframe of either p or k it is seen that this is positive.

Thus any diagram can be written as a sum of two diagrams with either some internal or external mass zero.

The above identity can also be used very effectively to test the correctness of a computer program, because any one-loop diagram can be written as a combination of other one-loop diagrams involving a free parameter α. It is important that α is real because the usual procedure of Feynman parametrization, Wick rotation and subsequent four-momentum integration works unchanged only for real α. In terms of the expressions obtained after momentum integration the propagator identity amounts to a transformation in the Feynman parameters, given below.

Let us now turn to questions concerning real and imaginary parts. Usually the masses occurring are given a small imaginary part, i.e., $m^2 \to m^2 - i\varepsilon$. The propagator identity preserves this prescription, since we then also have $M^2 \to M^2 - i\varepsilon$. After evaluating the various integrals we will wind up with logarithms and di-logarithms, and it is important to establish clearly some conventions concerning the singularities of these functions.

The logarithms occurring in this paper have a cut along the negative real axis. With this convention the rule for the logarithm of a product is:

$$\ln(ab) = \ln a + \ln b + \eta(a, b) \; ,$$

$$\eta(a, b) = 2\pi i \{\theta(-\operatorname{Im} a)\theta(-\operatorname{Im} b)\theta(\operatorname{Im} ab) - \theta(\operatorname{Im} a)\theta(\operatorname{Im} b)\theta(-\operatorname{Im} ab)\} \; . \quad (2.4)$$

Important consequences are:

$$\ln(ab) = \ln a + \ln b \; , \qquad \text{if Im } a \text{ and lm } b \text{ have different sign} \; ,$$

$$\ln \frac{a}{b} = \ln a - \ln b \; , \qquad \text{if Im } a \text{ and lm } b \text{ have the same sign.}$$

If A and B are real then:

$$\ln(AB - i\varepsilon) = \ln(A - i\varepsilon') + \ln(B - i\varepsilon/A) \; ,$$

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* In our metric spacelike k implies positive k^2. Further $k_4 = ik_0$, with k_0 real for a physical four-momentum.

where ε' is infinitesimal and has the same sign as ε. It is through this latter identity that $i\varepsilon$ occurs in the final formulas occasionally without a direct association with some mass.

The di-logarithm or Spence function is defined by

$$\mathrm{Sp}\,(x) = -\int_0^1 \mathrm{d}t\,\frac{\ln\,(1-xt)}{t}, \tag{2.5}$$

where x may be complex. Some useful identities and a series development are given in appendix A.

In working with Feynman parameters certain transformations **of** variables are useful. Consider the integral

$$I = \int_{-\infty}^{\infty} \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\,\frac{\delta(\sum x - 1)\theta(x_1)\theta(x_2)\theta(x_3)\theta(x_4)}{[\sum_i m_i^2 x_i + \sum_{i<j} p_{ij}^2 x_i x_j]^2}. \tag{2.6}$$

This is actually the Feynman parameter integral for the one-loop 4-point function to be evaluated later. The projective transformation

$$x_i = \frac{A_i u_i}{\sum A_j u_j}, \tag{2.7}$$

(no summation over i) with real A_i leads to*

$$I = \int \mathrm{d}_4 u\,\frac{A_1 A_2 A_3 A_4 N}{|N|}\,\frac{\delta(\sum u - 1)\prod \theta[(A_i/N)u_i]}{[u_i M_i^2 + \sum_{i<j} q_{ij}^2 u_i u_j]^2}, \tag{2.8}$$

with $N = \sum A_j u_j$ and

$$M_i^2 = m_i^2 A_i^2, \qquad q_{ij}^2 = (p_{ij}^2 + m_i^2 + m_j^2)A_i A_j - m_i^2 A_i^2 - m_j^2 A_j^2. \tag{2.9}$$

No summation over i and f is intended in the last two equations. To derive this we first transformed $x_i = u_i/\sum u_j$, $x_4 = A_4(1 - \sum u_i/A_i)/\sum u_j$ with $i = 1, 2, 3, j = 1, 2, 3,$ 4, and subsequently $u_i = A_i u_i$. A minus sign due to the reversal of the direction of integration of u_4 with respect to x_4 is included. If all A, are positive, one may replace $\theta(A_i u_i/N)$ by $\theta(u_i)$ while N may be taken to be positive. The expression assumes then the same form as the original expression. The transformed four-point function takes a particularly interesting form if we choose $A_i = 1/m_i$. Then all the transformed masses are equal to 1, with

$$q_{ij}^2 = \frac{p_{ij}^2 + (m_i - m_j)^2}{m_i m_j}.$$

Note that these A are positive, so no extra logarithms appear (see below).

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* Without loss of generality A_4 may be taken to be positive.

If one or more of the A_i are negative things are more complicated. First note that the general transformation may be obtained by successive transformations of the same type with all A_i set to one except one of the A_i. Thus consider now this integral with $A_1 = a < 0$, $A_2 = A_3 = A_4 = 1$. Omitting the denominator we have:

$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 = \int d_4 u \frac{A_4 N}{|N|} \delta(\sum u - 1) \theta\left(\frac{A_1 u_1}{N}\right) \theta\left(\frac{u_2}{N}\right) \dots .$$

It is essential to keep track of the direction of integration under this transformation. For instance:

$$0 < x_1 < \frac{a}{a-1} \qquad \text{corresponds to} \qquad 0 > u_1 > -\infty \qquad \text{and} \qquad N > 0,$$

$$\frac{a}{a-1} < x_1 < 1 \qquad \text{corresponds to} \qquad \infty > u_1 > 1 \qquad \text{and} \qquad N < 0.$$

Also extra - sign in view of u_4 integration .

The result is:

$$\left[\int_0^{-\infty} du_1 + \int_\infty^1 du_1 \right] \int_0^{1-u_1} du_2 \int_0^{1-u_1-u_2} du_3 \, a$$

$$= \int_0^1 du_1 \int_0^{1-u_1} du_2 \int_0^{1-u_1-u_2} du_3 \, a - \int_{-\infty}^\infty du_1 \int_0^{1-u_1} du_2 \int_0^{1-u_1-u_2} du_3 \, a .$$

The first term may be rewritten in the same form as the original integral. The second term is new, and generally somewhat easier to evaluate. It gives rise to logarithms but no Spence functions.

Another important transformation of variables is the following. Consider the integral

$$\int_0^1 dx \int_0^x dy \frac{1}{[ax^2 + by^2 + cxy + \dots]} .$$

This is just another version of the Feynman parameters. Now make the substitution $y = y + \alpha x$. The result is

$$\int_0^1 dx \int_{-\alpha x}^{(1-\alpha)x} dy \frac{1}{[by^2 + (c+2\alpha)xy + \dots]} ,$$

where α is chosen such that $b\alpha^2 + c\alpha + a = 0.$ Further

$$
\int_0^1 dx \int_{-\alpha x}^{(1-\alpha)x} dy = \int_0^1 dx \int_0^{(1-\alpha)x} dy - \int_0^1 dx \int_0^{-\alpha x} dy
$$

$$
= \int_0^{(1-\alpha)} dy \int_{y/(1-\alpha)}^1 dx - \int_0^{-\alpha} dy \int_{-y/\alpha}^1 dx .
$$

Trivial transformations lead to the same integration as before, but now with x and y interchanged. However the term x^2 has disappeared from the integrand, and the x-integration becomes easy. Actually this transformation is the Feynman parameter equivalent of the propagator identity given above. Here it is evident that this transformation requires either α real, or (more precisely) that there are no singularities in the complex y plane in the triangle $(0, -\alpha x, (1-\alpha)x),$ which is evidently true if α is real.

The last equations in this section are meant as an illustration of the problems involved with the integration of complex functions. Consider

$$
I(a, b) = \int_0^1 \frac{dx}{ax + b} ,
$$

for arbitrary complex a and b. The primitive of the integrand is $a^{-1}\ln(ax + b),$ and if for $0 < x < 1$ the argument can be negative real one must actually split the integral. It is easier to first divide out a:

$$
I(a, b) = \frac{1}{a} \int_0^1 \frac{dx}{x + b/a} = \frac{1}{a} \ln\left(x + \frac{b}{a}\right)\Big|_0^1 .
$$

No matter what the sign is of Im *(b/a)* the argument of the log never crosses the cut for real *x*. Thus the answer is

$$
I(a, b) = \frac{1}{a}\left[\ln\left(1 + \frac{b}{a}\right) - \ln\frac{b}{a}\right] .
$$

Since $1 + b/a$ and b/a have the same sign for the imaginary part we may combine the two logs to get

$$
I(a, b) = \frac{1}{a}\ln\frac{a + b}{b} ,
$$

which is in fact the standard result. The difficulty with the careless "direct" derivation is that it would give ln *(a+b)*-ln   *b* which is not always the correct answer.

3. The one-point function

The one-point function is given by

$$A(m^2) = \int d_n q \frac{1}{q^2 + m^2 - i\varepsilon} = \frac{i\pi^{n/2}}{(m^2)^{1-n/2}} \Gamma(1 - \tfrac{1}{2}n).$$

As evidenced by this equation we used the dimensional regularization method. Around $n = 4$ we have:

$$A(m^2) = -\Delta m^2 - i\pi^2 m^2 + i\pi^2 m^2 \ln(m^2 - i\varepsilon),$$

with

$$\Delta = -\frac{2\pi^2 i}{n-4} + i\delta, \qquad \delta = \pi^2(\gamma - \ln \pi), \qquad \gamma$$

The quantity δ contains factors arising from the development **of** $\pi^{n/2}$ and $\Gamma(1 - \tfrac{1}{2}n)$ around $n = 4$. The above equation may be considered also as the defining equation for Δ.

4. The two-point function

The one-loop scalar two-point function is given by

$$B(k, m_1, m_2) = \int d_n q \frac{1}{(q^2 + m_1^2 - i\varepsilon)((q+k)^2 + m_2^2 - i\varepsilon)}$$

$$= \Delta - i\pi^2 \int_0^1 dx \ln(-k^2 x^2 + x(k^2 + m_2^2 - m_1^2) + m_1^2 - i\varepsilon).$$

Let now x_1 and x_2 be the roots of the quadratic expression occurring as argument of the log. We have:

$$B(k, m_1, m_2) = \Delta - i\pi^2 [x \ln(-k^2 - i\varepsilon) + (x - x_1) \ln(x - x_1)$$

$$+ (x - x_2) \ln(x - x_2) - 2x]_0^1$$

$$= \Delta - i\pi^2 \left[\ln(-k^2 - i\varepsilon) + \sum_i \left\{ \ln(1 - x_i) - x_i \ln \frac{x_i - 1}{x_i} - 1 \right\} \right],$$

with the same quantity Δ as defined in sect. **3.** We used the fact that x_1 and x_2 have opposite signs for their imaginary parts. **If** $k^2 = 0$ there is only one root, and $\ln(-k^2 - i\varepsilon)$ should be replaced by $\ln(m_2^2 - m_1^2 - i\varepsilon)$. If in addition $m_1^2 = m_2^2$ there are no roots, and the expression in square brackets is simply $\ln(m_1^2 - i\varepsilon)$. If also $m_1^2 = 0$ we have an infrared-divergent expression, to be treated in the appropriate way.

5. The three-point function

The one-loop scalar three-point function is given by

$$C(p_1, p_2, m_1, m_2, m_3) = \int d_n q \frac{1}{(q^2 + m_1^2)((q+p_1)^2 + m_2^2)((q+p_1+p_2)^2 + m_3^2)},$$

(5.1)

where we omitted the *is*. This is a function of 6 variables, i.e., p_1^2, p_2^2, $(p_1+p_2)^2$, m_1^2, m_2^2 and m_3^2. Introducing Feynman parameters we get:

$$C = i\pi^2 \int_0^1 dx \int_0^x dy \, [ax^2 + by^2 + cxy + dx + ey + f]^{-1},$$

(5.2)

with

$$a = -p_2^2, \qquad b = -p_1^2, \qquad c = -2(p_1 p_2) = -(p_1+p_2)^2 + p_1^2 + p_2^2,$$

$$d = m_2^2 - m_3^2 + p_2^2, \qquad e = m_1^2 - m_2^2 + p_1^2 + 2(p_1 p_2), \qquad f = m_3^2 - i\varepsilon.$$

Performing the shift $y = y' + ax$ as described before we get

$$\frac{C}{i\pi^2} = \int_0^1 dy \frac{1-\alpha}{(c+2\alpha b)(1-\alpha)y + d + e\alpha}$$

$$\times \ln \frac{b(1-\alpha)^2 y^2 + (1-\alpha)(c+2\alpha b + e)y + d + e\alpha + f}{\{b(1-\alpha)^2 + (c+2\alpha b)(1-\alpha)\}y^2 + \{e(1-\alpha) + d + e\alpha\}y + f}$$

$$+ \int_0^1 dy \frac{\alpha}{-(c+2\alpha b)\alpha y + d + e\alpha} \ln \frac{b\alpha^2 y^2 - \alpha(c+2\alpha b + e)y + d + e\alpha + f}{\{a^2 b - (c+2\alpha b)\alpha\}y^2 + (e\alpha + d - e\alpha)y + f},$$

(5.3)

with α one of the roots of the equation

$$b\alpha^2 + c\alpha + a = 0.$$

For the moment we suppose that *a, b* and *c* are such that α is real. Then *f* is the only complex quantity, having a negative imaginary part. Next we make the substitutions $y = y'/(1-\alpha)$ and $y = -y'/\alpha$ respectively, and note that the upper terms in the logarithms become identical as well as the factors in front. Subsequently we split up the logarithms with the result

$$\frac{C}{i\pi^2} = \int_{-\alpha}^{1-\alpha} dy \frac{1}{N} \{\ln[by^2 + ey + f + N] - \ln[by_0^2 + ey_0 + f]\}$$

$$- \int\limits_{0}^{1-\alpha} \mathrm{d}y \frac{1}{N} \left\{ \ln \left[by^2 + ey + f + \frac{y}{1-\alpha} N \right] - \ln \left[by_0^2 + ey_0 + f \right] \right\}$$

$$+ \int\limits_{0}^{-\alpha} \mathrm{d}y \frac{1}{N} \left\{ \ln \left[by + ey + f - \frac{y}{\alpha} N \right] - \ln \left[by_0^2 + ey_0 + f \right] \right\}, \tag{5.5}$$

$$N = (c + 2\alpha b)y + d + e\alpha, \qquad y_0 = -(d + e\alpha)/(c + 2\alpha b).$$

We have introduced an extra term in each of the integrals; this extra term is the same for all integrals so that the total contribution of this term is zero. Since y_0 is that value of y for which N is zero we have achieved that the residue of the pole due to $1/N$ is zero, which will enable us to study the behaviour for complex a. However, still taking α to be real, we first make the substitutions $y = y' - \alpha$, $y = (1-\alpha)y'$ and $y = -\alpha y'$ in the three integrals respectively. Working out the arguments of the logarithms we get:

$$\frac{C}{i\pi^2} = \int\limits_{0}^{1} \mathrm{d}y \frac{1}{(c+2\alpha b)y + d + e\alpha + 2a + c\alpha} [\ln \{by^2 + (c+e)y + a + d + f\}$$
$$- \ln \{by_1 + (c+e)y_1 + a + d + f\}]$$

$$- \int\limits_{0}^{1} \mathrm{d}y \frac{1-\alpha}{(c+2\alpha b)(1-\alpha)y + d + e\alpha} [\ln \{(a+b+c)y^2 + (e+d)y + f\}$$
$$- \ln \{(a+b+c)y_2^2 + (e+d)y_2 + f\}]$$

$$- \int\limits_{0}^{1} \mathrm{d}y \frac{\alpha}{-(c+2\alpha b)\alpha y + d + e\alpha} [\ln \{ay^2 + dy + f\} - \ln \{ay_3^2 + dy_3 + f\}]. \tag{5.6}$$

In here y_1, y_2 and y_3 are the values of y for which the denominators in front become zero, i.e., $y_1 = y_0 + \alpha$, $y_2 = y_0/(1-\alpha)$ and $y_3 = -y_0/\alpha$. Of course the terms with y_1, y_2 and y_3 are still equal to $\ln (by_0^2 + ey_0 + f)$, but the above way of writing is slightly more convenient for the evaluation in terms of Spence functions. This derivation is given in appendix B.

The arguments of the logarithms are very similar to that encountered for the two-point function and correspond to the three possible cuts of the triangle diagram. Substituting the expressions in terms of momenta and masses for a, b, c etc., we have:

$$by^2 + (c+e)y + a + d + f = -p_1^2 y^2 + (p_1^2 + m_1^2 - m_2^2)y + m_2^2 - i\varepsilon,$$

$$(a+b+c)y^2 + (e+d)y + f = -(p_1+p_2)^2 y^2 + ((p_1+p_2)^2 + m_1^2 - m_3^2)y + m_3^2 - i\varepsilon,$$

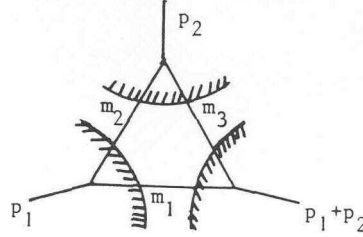$$ay^2 + dy + f = -p_2^2 y^2 + (p_2^2 + m_2^2 - m_3^2)y + m_3^2 - i\varepsilon. \tag{5.7}$$

Fig. 1.

We must now investigate the domain of validity of the above expression. The derivation is valid for real α. Now α is given by

$$\alpha = \frac{(p_1 p_2) \pm \sqrt{(p_1 p_2)^2 - p_1^2 p_2^2}}{-p_1^2},$$

and this is real if p_1 and/or p_2 and/or $p_1 + p_2$ are timelike. Equation (5.2) shows that C is continuous if p_1^2 and p_2^2 become Euclidean, because due to the $i\varepsilon$ the denominator can never become zero, all other terms being real.

Suppose now that p_1^2 is spacelike, i.e., p_1^2 positive, and p_2^2 timelike. Then α is real and our formula is correct. Next we move p_2 through the light cone (where α is still real) into the spacelike region. According to eq. (5.2) we should not meet any singularity in performing this analytic continuation, i.e., no pole should cross an integration contour and also no cuts should be crossed in the logarithms. Consider eq. (5.5). If α becomes complex the integration contours move into the complex y-plane, and in the process the pole due to the factor $1/N$ may move through the integration contour. However, since the residue of this pole is evidently zero this is of no consequence. Next, if α is made complex also y_0 becomes complex, and it may happen that the argument of the logarithms containing y_0 cross the negative real axis. Let us now insert y_0 from eq. (5.5) into $by_0^2 + ey_0 + f$. First we write

$$\alpha = \alpha_1 + \alpha_2, \qquad \alpha_1 = -\frac{c}{2b}, \qquad \alpha_2 = \frac{iw}{2b}, \qquad w = \pm\sqrt{4ab - c^2},$$

$$y_0 = -\frac{d + e\alpha_1 + ie(w/2b)}{iw} = -\frac{d + e\alpha_1}{iw} - \frac{e}{2b}.$$

Then

$$by_0^2 + ey_0 + f = by_0\left\{y_0 + \frac{e}{b}\right\} + f$$

$$= b\left(\frac{d + e\alpha_1}{iw}\right)^2 - \frac{e^2}{4b} + f.$$

Apart from $f = f - i\varepsilon$ this is real, independently of the magnitude of α. Thus $by_0^2 + ey_0 + f$ never crosses the negative real axis. Moreover, since b is negative the real part is positive.

Finally there are the other logarithms. As is clear from eq. (5.6) the arguments are independent of α, and there is never any crossing of the negative real axis because the imaginary part is never zero. We conclude that eq. (5.6) is also correct for spacelike p_1^2 and p_2^2 (and $(p_1 + p_2)^2$).

The situation is more complicated if the internal particles are unstable, i.e., if the m^2 have a negative imaginary part. As long as α is real there are again no problems, but if α is complex then there are difficulties.

If the masses become complex then again the arguments of the logarithms containing y_0 may cross the negative real axis. That in itself is only of consequence if at the same time the location of the pole y_0 is within the triangle $0, -\alpha, 1-\alpha$ in the complex y-plane, because otherwise the jumps across this cut in the three integrals still compensate each other. We thus must show that, if y_0 is within the triangle region, the quantity $by_0^2 + ey_0 + f$ never crosses the negative real axis. To see this we parametrize $y = \lambda(\mu - \alpha)$, with λ and μ between 0 and 1, and consider next

$$by^2 + ey + f + \lambda N.$$

This coincides with $by_0^2 + cy_0 + f$ for $y = y_0$. In terms of λ and μ we obtain

$$b\lambda^2\mu^2 + a\lambda^2 + c\lambda^2\mu + d\lambda + e\lambda\mu + f.$$

•Substituting a, b etc., we find

$$p_1^2\lambda^2\mu(1-\mu) + p_2^2\lambda(1-\lambda)(1-\mu) + (p_1+p_2)^2\lambda\mu(1-\lambda) + m_1^2\lambda\mu + m_2^2\lambda(1-\mu)$$
$$+ m_3^2(1-\lambda).$$

For spacelike p_1^2, p_2^2 and $(p_1 + p_2)^2$ the real part of this expression is always positive. Thus, if y_0 is within the triangle, then $by_0^2 + ey_0 + f$ never crosses the negative real axis, even for complex masses as long as the real part of the masses is positive.

If we consider eq. (5.2) as the defining equation for the three-point function C then there is still a domain where the result eq. (5.6) may be incorrect. This is the region where a and b are positive, with $c^2 - 4ab < 0$. This corresponds to timelike momenta, but such that $(p_1p_2)^2 - p_1^2p_2^2 < 0$. If p_1 and p_2 are real momenta (in the sense that the spatial components are real, and the time component purely imaginary) then this cannot happen. Thus the domain indicated corresponds to the case where either p_1 or p_2 or both are complex even when embedded in the Minkowski space. But then also the derivation of eq. (5.2) from eq. (5.1) is no more valid since at some point the integration momentum q is shifted by a complex momentum P, where P is a linear combination of p_1 and p_2. Thus the forbidden domain of our result eq. (5.6) is within the domain where the derivation of eq. (5.2) is invalid. In other words, it is conceivable that eq. (5.6) is a better representation of the three-point function eq. (5.1) than eq. (5.2). However, it must be noted that

for such values of the momenta the three-point function (5.1) is not unambiguously defined since interchange of external lines would correspond to a shift of the integration variable q in complex space. The $-i\varepsilon$ in the propagators then no longer specify unambiguously which way the poles of the propagators should be passed by the integration contour. The transition to (5.2) is then also not necessarily correct. We could perhaps just as well use (5.6) as a definition of the three-point function in such a region.

We close this section with the observation that anomalous thresholds correspond to the situation where simultaneously the denominator as well as the arguments of the logarithms can be zero. In that case there is a logarithmic singularity in the imaginary part and a jump in the real part of the right-hand side of eq. (5.6). There is no pole, or crossing of a cut associated with this; the logarithmic singularity is maximally equal to ln ε, and the jump arises because the argument of some logarithm moves very rapidly around the cut.

In conclusion, (5.6) is correct for all relevant values of masses and momenta. The integrals are of a standard type and are converted into Spence functions in appendix B.

6. The four-point function

The expression derived for the three-point function is about as nice as one could wish. It is valid for spacelike, lightlike and timelike external masses, and also if the internal masses have a constant imaginary part of a certain sign, which is precisely the sign needed for the case that the internal particles are unstable. Thus, insofar as the momentum dependence of the decay width is neglected, also unstable internal particles can be admitted. Finally the formula is quite compact, there are only 12 Spence functions which appears to be the minimum possible.

The four-point function is not only much more complicated, but moreover there seems to be no simple formula covering the whole domain described above, including complex masses. If the masses are real one can in most cases construct equations giving the four-point function in terms of 24 Spence functions. We will indicate how this can be achieved. However if the masses are complex an untransparent collection of logarithms must be added.

First we will derive an equation valid if for some permutation of the momenta a certain condition holds while all masses are real. In particular this will apply if one of the external momenta or sum of momenta is spacelike. The final result contains in that case 24 Spence functions, in fact it is a sum of two three-point functions. If the condition is not met, a formula in terms of 48 (with some more work 36) Spence functions can be derived using the propagator identity. If also masses are complex then these derivations become untransparent, and we will for that case derive an expression containing 108 Spence functions, but with some limitations on the values of the external momenta.

The one-loop four-point function is defined by

$$D(p_1, p_2, p_3, p_4, m_1, m_2, m_3, m_4)$$

$$= \int \mathrm{d}_n q \frac{1}{(q^2 + m_1^2)((q+p_1)^2 + m_2^2)((q+p_1+p_2)^2 + m_3^2)((q+p_1+p_2+p_3)^2 + m_4^2)} .$$

(6.1)

Using Feynman parameters this may be rewritten in the form quoted in sect. 2:

$$D = i\pi^2 \int \mathrm{d}_4 u \frac{\delta(\sum u - 1)\theta(u_1)\theta(u_2)\theta(u_3)\theta(u_4)}{[\sum m_i^2 u_i + \sum_{i<j} p_{ij}^2 u_i u_j]^2} .$$

(6.2)

Here p_{ij}^2 is the square of the difference of the four-momenta flowing through propagators i and j. Thus for instance $p_{12}^2 = p_1^2$, $p_{13}^2 = (p_1+p_2)^2$, etc. Introducing variables z, x, y this may be cast in the form

$$\frac{D}{i\pi^2} = \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y \int_0^y \mathrm{d}z \, [ax^2 + by^2 + gz^2 + cxy + hxz + jyz + dx + ey + kz + f]^{-2} ,$$

(6.3)

with

$$a = -p_{34}^2 = -p_3^2, \qquad b = -p_{23}^2 = -p_2^2, \qquad g = -p_{12}^2 = -p_1^2 ,$$

$$c = -p_{24}^2 + p_{23}^2 + p_{34}^2 = -2(p_2 p_3), \qquad h = -p_{14}^2 - p_{23}^2 + p_{13}^2 + p_{24}^2 = -2(p_1 p_3) ,$$

$$j = -p_{13}^2 + p_{12}^2 + p_{23}^2 = -2(p_1 p_2) ,$$

$$d = m_3^2 - m_4^2 + p_{34}^2 = m_3^2 - m_4^2 + p_3^2 ,$$

$$e = m_2^2 - m_3^2 + p_{24}^2 - p_{34}^2 = m_2^2 - m_3^2 + 2(p_2 p_3) + p_2^2 ,$$

$$k = m_1^2 - m_2^2 + p_{14}^2 - p_{24}^2 = m_1^2 - m_2^2 + 2(p_1, p_2 + p_3) + p_1^2 ,$$

$$f = m_4^2 - i\varepsilon .$$

(6.4)

An intermediate equation will be useful for later use. From (6.2), with $x = u_4$, $y = u_3$ and $z = u_1$, one has

$$\frac{D}{i\pi^2} = \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int_0^{1-x-y} \mathrm{d}z \, [(a+b+c)x^2 + by^2 + gz^2 + (2b+c)xy - (h+j)xz$$

$$- jyz - \{e+d+2(a+b+c)\}x - (2b+c+e)y + (k+h+j)z + a$$

$$+ b + c + d + e + f]^{-2} .$$

(6.5)

From this equation, eq. (6.3) follows after the substitutions $x = 1 - x'$, $y = x' - y'$, $z = z'$. Conversely, eq. (6.5) obtains from eq. (6.3) by substituting $x = 1 - x'$, $y = l - x' - y', z = z'$.

Now assume all masses to be real. Performing a projective transformation onto eq. (6.2) we obtain again an equation of the form (6.3), (6.4), but with q_{ij}^2 and M_i^2 instead of the p and m, with

$$q_{ij}^2 = (p_{ij}^2 + m_i^2 + m_j^2)A_iA_j - m_i^2A_i^2 - m_i^2A_j^2 \, ,$$

$$M_i^2 = m_i^2 A_i^2 \, . \tag{6.6}$$

We now choose A_1, A_2, A_3 and A_4 in such a way that the coefficients of z^2, xz and yz disappear. Thus we require:

$$q_{12}^2 = 0 \, , \qquad q_{13}^2 = q_{23}^2 \, , \qquad q_{14}^2 = q_{24}^2 \, . \tag{6.7}$$

Solving this for the coefficients A we find:

$$\frac{A_1}{A_2} = \frac{l_{12} \pm \sqrt{l_{12} - 4m_1^2 m_2^2}}{2m_1^2} \, ,$$

$$A_3 = \frac{m_2^2 A_2^2 - m_1^2 A_1^2}{l_{23}A_2 - l_{13}A_1} \, , \qquad A_4 = \frac{m_2^2 A_2^2 - m_1^2 A_1^2}{l_{24}A_2 - l_{14}A_1} \, , \tag{6.8}$$

with the notation

$$l_{ij} = p_{ij}^2 + m_i^2 + m_j^2 \, . \tag{6.9}$$

For the moment we will assume $A_1, \ldots, A_4$ to be real and positive. The extension to negative A's is possible and will be indicated later. The extension to complex A is very intricate, and we have not done it. In principle it should be possible, and result only in the addition of certain logarithms.

After this transformation the calculation becomes trivial. The z-integration gives an expression of the form

$$\frac{D}{i\pi^2} = A_1A_2A_3A_4 \int\limits_0^1 dx \int\limits_0^x dy \frac{1}{k}[\{ax^2 + by^2 + cxy + dx + ey + f\}^{-1}$$
$$- \{ax^2 + by^2 + cxy + dx + (e+k)y + f\}^{-1}], \tag{6.10}$$

where a-f are as in eq. (6.4) but with q and M instead of p and m. This is precisely the sum of two three-point functions, and we can straight away use the equations of sect. 5. Subsequently the coefficients a, b etc., may be expressed in terms of the p, and with the help of the identities

$$-l_{24}A_2A_4 = -l_{14}A_1A_4 + m_1^2A_1^2 - m_2^2A_2^2 \, ,$$

$$-l_{23}A_2A_3 = -l_{13}A_1A_3 + m_1^2A_1^2 - m_2^2A_2^2 \, , \tag{6.11}$$

the following result obtains:

$$\frac{D}{i\pi^2} = \frac{A_1 A_2 A_3 A_4}{k}$$

$$\times \left[-\int_0^1 dy \frac{1-\alpha}{(c+2\alpha b)(1-\alpha)y + d + e\alpha} \{\ln L_{24}(y) - \ln L_{24}(y_1)\} - R^1_{24} \right.$$

$$-\int_0^1 dy \frac{\alpha}{-(c+2\alpha\beta)\alpha y + d + e\alpha} \{\ln L_{34}(y) - \ln L_{34}(y_2)\} + R^2_{34}$$

$$+\int_0^1 dy \frac{1}{(c+2\alpha b)y + d + e\alpha + c\alpha + 2a} \{\ln L_{23}(y) - \ln L_{23}(y_3)\} + R^3_{23}$$

$$+\int_0^1 dy \frac{1-\alpha}{(c+2\alpha b)(1-\alpha)y + d + (e+k)\alpha} \{\ln L_{14}(y) - \ln L_{14}(y_4)\} + R^4_{14}$$

$$+\int_0^1 dy \frac{\alpha}{-(c+2\alpha b)\alpha y + d + (e+k)\alpha} \{\ln L_{34}(y) - \ln L_{34}(y_5)\} - R^5_{34}$$

$$\left. -\int_0^1 dy \frac{1}{(c+2\alpha b)y + d + (e+k)\alpha + c\alpha + 2a} \{\ln L_{13}(y) - \ln L_{13}(y_6)\} - R^6_{13} \right.$$

$$\left. + \theta(-A_1 A_2)S \right], \quad (6.12)$$

$$L_{ij}(y) = (-l_{ij} A_i A_j + m_i^2 A_i^2 + m_j^2 A_j^2)y^2 + (l_{ij} A_i A_j - 2m_j^2 A_j^2)y + m_j^2 A_j^2 - i\varepsilon . \tag{6.13}$$

The y_i are the values of y for which the denominator in front becomes zero. The R^k_{ij} and 5 are zero when $A_1, \dots , A_4$ are positive. As shown in appendix C they are given by:

$$R^k_{ij} = \frac{-\theta(-A_i A_k)}{c+2\alpha b} \int_{-\infty}^{\infty} \frac{1}{y - y_k} \{\ln L_{ij}(y) - \ln L_{ij}(y_k)\}$$

$$= -\frac{\theta(-A_i A_k)}{c+2\alpha b} [\pi^2 + i\pi\theta(\operatorname{Im} y_k)\{2\ln(y_k - y_-) - \ln(y_k - y_-)(y_k - y_+)\}$$

$$- i\pi\theta(-\operatorname{Im} y_k)\{2\ln(y_k - y_+) - \ln(y_k - y_-)(y_k - y_+)\}] . \tag{6.14}$$

In here y_-, y_+ are the roots of the quadratic equation $L_{ij}(y) = 0$, with y_+ having a positive imaginary part (in case the roots are real this imaginary part arises from the $i\varepsilon$ in f). The function 5 is non-zero if A_1 and A_2 have opposite sign; it is given by:

$$S = \varepsilon\left(\frac{A_1}{A_2} - \frac{l_{12}}{2m_1^2}\right)\{\theta(A_3A_4)S_{34}^2 - \theta(-A_3A_4)(S_{24}^3 + S_{23}^4)\}, \tag{6.15}$$

with

$$S_{34}^2 = \frac{-i\pi}{q_{34}^2(x_1 - x_2)}\left\{\ln\left(\frac{x_1 - 1}{x_1}\right) - \ln\left(\frac{x_2 - 1}{x_2}\right)\right\},$$

where the x_i are the solutions of

$$-q_{34}^2 x^2 + (q_{34}^2 + q_{24}^2 - q_{23}^2)x - q_{24}^2 - i\varepsilon = 0,$$

and similarly for the other 5. The function $\varepsilon(\lambda)$ is as usual, i.e., $\varepsilon(\lambda) = +1$ if $\lambda > 0$ and -1 if $\lambda < 0$.

What is the domain of validity of eq. (6.12)? The situation is precisely as in the case of the three-point function. Also the extra terms R (and 5) are evidently free of discontinuities if α becomes complex, again because $L_{ij}(y_k)$ never crosses the real axis. Thus eq. (6.12) is valid provided $A_1, \ldots, A_4$ are real. If the masses are real then this is the case provided the root occurring in A_1 is real. The allowed domain is therefore given by

$$(p_{12}^2 + m_1^2 + m_2^2)^2 > 4m_1^2 m_2^2.$$

Possibly after a permutation we thus have a formula in case that at least one of the p_{ij}^2 is such that

$$(p_{ij}^2 + m_i^2 + m_j^2)^2 \geq 4m_i^2 m_j^2. \tag{6.16}$$

This is true in particular if this momentum is spacelike. In scattering processes often the momentum transfer is spacelike, or else the c.m. energy is large, i.e., above the threshold for producing the intermediate state $[s = -p_{13}^2 = -(p_1 + p_2)^2 > (m_1 + m_3)^2]$. In such cases the equation can be exploited.

There is one gap in the above reasoning, namely we do not know if the quantities a, b and c, given by the q^2 that are complicated functions of the p^2, are such that $c^2 - 4ab > 0$ if one of the corresponding q^2 is timelike. That is, we want to know if there exist real momenta q_a, q_b (in the Minkowski sense) such that $a = -q_a^2$, $b = -q_b^2$ and $c = -2(q_a q_b)$. The answer is yes, it is possible to write down certain linear combinations of the original momenta p with real coefficients that satisfy these relations. See appendix D.

Thus eq. (6.12) has the forbidden domain defined by the non-validity of eq. (6.16) for any choice of i and j. In principle analytic continuation from the allowed into the forbidden domain should be possible, giving rise to some extra logarithms.

We have not done this as it seems quite complicated. We will not dwell here on this subject, nor involve ourselves into alternative derivations, since to our knowledge the complications by far exceed the advantages.

If we allow to deviate from the path of "least number of Spence functions" then things are very easy. Assume that all momenta are inside the forbidden region. Then they are all timelike, and we can use the propagator identity to write the four-point function as a linear combination of two four-point functions each having a lightlike external momentum:

$$
D(p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2, (p_2+p_3)^2, m_1^2, m_2^2, m_3^2, m_4^2]
$$

$$
= \sum_{j=1}^{2} \lambda D[l^2, s^2, t^2, p_4^2, (l+s)^2, (s+t)^2, m_1^2, M^2, m^2, m_4^2], \tag{6.17}
$$

with

$$
l = p_1 + \alpha p_2, \qquad M^2 = \alpha(1-\alpha)p_2^2 + \alpha m_3^2 + (1-\alpha)m_2^2,
$$

while for $f = 1$

$$
\lambda = \alpha, \qquad s = -\alpha p_2, \qquad t = p_2 + p_3, \qquad m = m_2,
$$

and for $f = 2$

$$
\lambda = 1 - \alpha, \qquad s = (1-\alpha)p_2, \qquad t = p_3, \qquad m = m_3.
$$

This equation is valid for any real α. If p_1 and/or p_2 are timelike then α can be chosen such that $l^2 = 0.$ Then eq. (6.12) is valid for each of the terms in the right-hand side of eq. (6.17) with $A_1 = A_2$ in both cases.

Finally we consider the case that also the masses are complex, but always with negative imaginary part. The we can no more use projective transformations, but must rely on the other methods mentioned.

We start from eq. (6.3). First make the substitutions $y = xy'$, $z = xz'$. The result is

$$
\int_0^1 dx \int_0^1 dy \int_0^y dz\, x^2 [x^2(by^2 + gz^2 + jyz) + (ex + cx^2)y + (kx + hx^2)z + ax^2 + dx
$$
$$
+ f]^{-2}
$$

The y-z integral is the same as eq. (5.2) divided by $-i\pi^2$ and differentiated with respect to f. We can then use the result eq. (5.6), by differentiating with respect to f, and providing a minus sign. At this point, to keep things reasonably transparent, we drop the ambition to write down a formula valid for all situations, and we limit ourselves to the case of real α, i.e., at least one timelike momentum. Then the subtraction logarithms in eq. (5.6) can be dropped. Substituting the values of the

various coefficients as given by comparing with eq. (5.2) we obtain:

$$\frac{D}{i\pi^2} = \sum_{j=1}^{3} \int_0^1 dx \int_0^x dy \frac{K}{Gx+Hy+J} \frac{1}{Ax^2+By^2+Cxy+Dx+Ey+F-i\varepsilon}. \quad (6.18)$$

We also performed the substitution $y = y'/x$. The values of the various coefficients are given in table 1.

Table 1

j	1	2	3
A	$a+b+c$	a	a
B	g	$b+g+j$	b
C	$j+h$	$c+h$	c
D	$d+e$	d	d
E	k	$e+k$	e
F	f	f	f
G	$c+2b+h\alpha+j\alpha$	$c+h\alpha$	$c+h\alpha$
H	$j+2\alpha g$	$(j+2\alpha g)(1-\alpha)$	$-(j+2\alpha g)\alpha$
J	$e+k\alpha$	$e+k\alpha$	$e+k\alpha$
K	-1	$1-\alpha$	α

α solution of $g\alpha^2+j\alpha+b=0$, or

$$\alpha = \frac{-j\pm\sqrt{j^2-4bg}}{2g}, \qquad j^2-4bg = 4(p_1 p_2)^2 - 4p_1^2 p_2^2.$$

Eq. (6.18) can be worked out by using the integration shift technique. Thus we shift $y = y+\beta x$, β such that $B\beta^2+C\beta+A=0$. The result is:

$$\frac{D}{i\pi^2} = \sum_{j=1}^{3} \int_0^1 dx \int_{-\beta x}^{(1-\beta)x} \frac{K}{(G+\beta H)x+Hy+J-i\varepsilon}$$

$$\times [By^2+(C+2\beta B)xy+(D+\beta E)x+Ey+F-i\varepsilon]^{-1}. \quad (6.19)$$

We have provided a term $-i\varepsilon$ in the first denominator, which is of no consequence if α is real. For real β we have:

$$\int_0^1 dx \int_{-\beta x}^{(1-\beta)x} dy = \int_0^1 dx \int_0^{(1-\beta)x} dy - \int_0^1 dx \int_0^{-\beta x} dx$$

$$= \int_0^{1-\beta} dy \int_{y/(1-\beta)}^1 dx - \int_0^{-\beta} dy \int_{-y/\beta}^1 dx.$$

The x integral is now simple:

$$\int \mathrm{d}x \frac{1}{Sx+T} \frac{1}{Ux+V} = \int \mathrm{d}x \frac{1}{SV-TU} \left\{ \frac{S}{Sx+T} - \frac{U}{Ux+V} \right\}$$

$$= \frac{1}{SV-TU} \ln \frac{Sx+T}{Ux+V}, \qquad (6.20)$$

$$S = G + \beta H, \qquad T = Hy + J - i\varepsilon,$$

$$U = (C+2\beta B)y + D + \beta E, \qquad V = By^2 + Ey + F - i\varepsilon,$$

$$\beta \text{ solution of } B\beta^2 + C\beta + A = 0.$$

For real β there is never any problem with the logarithms, since all factors have an imaginary part of the same sign. At this point we still assume real masses; otherwise f may be complex with a positive imaginary part. In that case one simply multiplies numerator and denominator of eq. (6.18) with -1. Neither G nor H contain masses. Thus for real α there is no limitation here with respect to complex masses. Recombining the terms coming from the upper limit 1 for x we get

$$\frac{D}{i\pi^2} = \sum_{j=1}^{3} \left[\int_{-\beta}^{1-\beta} \mathrm{d}y \frac{K}{SV-TU} \ln \frac{T+S}{V+U} - \int_{0}^{1-\beta} \mathrm{d}y \frac{K}{SV-TU} \ln \frac{T+Sy/(1-\beta)}{V+Uy/(1-\beta)} \right.$$

$$\left. + \int_{0}^{-\beta} \mathrm{d}y \frac{K}{SV-TU} \ln \frac{T-Sy/\beta}{V-Uy/\beta} \right]. \qquad (6.21)$$

It is instructive to realize that $SV - TU$ can be obtained by taking the second denominator in eq. (6.19), inserting that x for which the first denominator is zero, i.e., $x = -(Hy+J)/(G+\beta H)$, and multiplying with $G + \beta H$. Further, if one makes in eq. (6.21) the substitutions $y = y' - \beta$, $y = (1-\beta)y'$ and $y = -\beta y'$ then all integration limits become 0 and 1, while the β dependence vanishes from the arguments of the logarithms. All this is very similar to what happens in the case of the three-point function. The various factors containing V are then also free of α, and become of the very familiar type as shown in eq. (5.7). The factors containing T are not directly related to any cut of the four-point function, and furthermore contain α.

Eq. (6.21), with the cascade of notations given in eq. (6.20), table 1 and eq. (6.4) is valid for real α and β. We must now try to extend the domain of validity. To this purpose we first note that the denominator $SV - TU$ is a quadratic function in y, and can be decomposed into two terms, $(y-y_+)^{-1}$ and $(y-y_-)^{-1}$. For both $y = y_+$ or $y = y_-$ we have $SV = TU$. Next we subtract terms in such a way that the residues of the poles become zero. At the pole we have

$$\frac{T+S}{V+U} = \frac{T}{V} \left(\frac{1+S/T}{1+U/V} \right) = \frac{T}{V}. \qquad (6.22)$$

Thus from the terms with y_+ we subtract $\ln (T/V)$ in the point y_+, and similarly for the terms containing y_-. At the pole the subtraction terms are equal for all terms in eq. (6.21), so together they give no contribution.

Next we move β into the complex plane. Just like in the case of the three-point function we need to show only that the subtraction terms do not cross any cut. To see that this indeed never happens consider

$$\ln \frac{T+\lambda S}{V+\lambda U}, \qquad \text{with } y = \lambda(\mu-\beta), \qquad 0 \leqslant \lambda, \mu \leqslant 1 . \tag{6.23}$$

This defines y to be in the triangle $0, -\beta, (1-\beta)$. Moreover, at the pole this is equal to $\ln (T/V)$. Working this out we find:

$$T+\lambda S = H\lambda(\mu-\beta)+J+\lambda(G+\beta H) = H\lambda\mu+\lambda G+J,$$
$$V+\lambda U = B\lambda^2\mu^2+A\lambda^2+C\lambda^2\mu+E\lambda\mu+D\lambda+F. \tag{6.24}$$

None of these expressions contain β. The imaginary part of $T+\lambda S$ arises solely from masses in f, and this is made negative as indicated before. The expression $V+\lambda U$ can be worked out for the cases $f = 1$, 2 and 3 with the result:

$$j=1: \quad V+\lambda U = p_1^2\lambda^2\mu(1-\mu)+(p_2+p_3)^2\lambda(1-\lambda)(1-\mu)$$
$$+(p_1+p_2+p_3)^2\lambda\mu(1-\lambda)+m_1^2\lambda\mu+m_2^2\lambda(1-\mu)+m_4^2(1-\lambda)-i\varepsilon,$$
$$j=2: \quad V+\lambda U = (p_1+p_2)^2\lambda^2\mu(1-\mu)+p_3^2\lambda(1-\lambda)(1-\mu)$$
$$+(p_1+p_2+p_3)^2\lambda\mu(1-\lambda)+m_1^2\lambda\mu+m_3^2\lambda(1-\mu)+m_4^2(1-\lambda)-i\varepsilon,$$
$$j=3: \quad V+\lambda U = p_2^2\lambda^2\mu(1-\mu)+p_3^2\lambda(1-\lambda)(1-\mu)$$
$$+(p_2+p_3)^2\lambda\mu(1-\lambda)+m_2^2\lambda\mu+m_3^2\lambda(1-\mu)+m_4^2(1-\lambda)-i\varepsilon.$$

Note that for each of these cases the corresponding β becomes complex only if the momenta occurring in these expressions are all spacelike. In any case, the imaginary part is always negative, also if the masses obtain a negative imaginary part. We therefore can write

$$\ln \frac{T+\lambda S}{V+\lambda U} = \ln (T+\lambda S)-\ln (V+\lambda U), \qquad \text{for } y = \lambda(\mu-\beta), \qquad 0 \leqslant \lambda, \mu \leqslant 1 . \tag{6.25}$$

Furthermore none of the arguments of these logarithms crosses the negative real axis for y in the region indicated.

We conclude that if y_+ or y_- are in the triangle $0, -\beta, 1-\beta$ then the subtraction logarithms never cross a cut, even for complex masses provided their imaginary part is negative.

For completeness we write down the result in terms of $A,\ B$ etc.

$$\frac{D}{i\pi^2} = \sum_{j=1}^{3} \sum_{\eta=\pm} \int_0^1 \frac{dy}{X}$$

$$\times \left[\frac{\eta}{y-\beta-y_\eta} \left\{ \ln \frac{Hy+G+J-i\varepsilon}{By^2+(E+C)y+A+D+F-i\varepsilon} - \ln(``y=y_\eta+\beta\text{''}) \right\} \right.$$

$$+ \frac{-\eta(1-\beta)}{y(1-\beta)-y_\eta} \left\{ \ln \frac{(H+G)y+J-i\varepsilon}{(A+B+C)y^2+(E+D)y+F-i\varepsilon} - \ln(``y=y_\eta/(1-\beta)\text{''}) \right\}$$

$$\left. + \frac{-\eta\beta}{-\beta y-y_\eta} \left\{ \ln \frac{Gy+J-i\varepsilon}{Ay^2+Dy+F-i\varepsilon} - \ln(``y=-y_\eta/\beta\text{''}) \right\} \right]. \tag{6.26}$$

In here the y_η are the solutions of the quadratic equation $SV-TU=0$, while X is given by

$$X = \gamma(y_+ - y_-), \qquad \gamma \text{ coëfficiënt of } y^2 \text{ in } SV-TU \text{ divided by } K.$$

Further

$$B\beta^2 + C\beta + A = 0,$$

$A,\ B,\ C$ etc., given in table 1,

$a,\ b,\ c$ etc., given in eq. (6.4),

$$g\alpha^2 + j\alpha + b = 0.$$

If J complex with positive imaginary part then the whole expression must be given a minus sign, and further one must replace $G,\ H,\ J$ by $-G,\ -H$ and $-f$.

The factor X is the same for all terms, for all j, apart from possibly a minus sign (depending on how one chooses the roots y_η). It is given by the square root of the following expression:

$$X^2 = l_{12}^2 l_{34}^2 + l_{13}^2 l_{24}^2 + l_{14}^2 l_{23}^2$$

$$-2(l_{12}l_{13}l_{24}l_{34} + l_{12}l_{14}l_{23}l_{34} + l_{13}l_{14}l_{23}l_{24})$$

$$+4(m_1^2 l_{23}l_{24}l_{34} + m_2^2 l_{13}l_{14}l_{34} + m_3^2 l_{12}l_{14}l_{24} + m_4^2 l_{12}l_{13}l_{23})$$

$$-4(m_1^2 m_2^2 l_{34}^2 + m_1^2 m_3^2 l_{24}^2 + m_1^2 m_4^2 l_{23}^2 + m_2^2 m_3^2 l_{14}^2 + m_2^2 m_4^2 l_{13}^2 + m_3^2 m_4^2 l_{12}^2)$$

$$+16 m_1^2 m_2^2 m_3^2 m_4^2,$$

with

$$l_{ij} = p_{ij}^2 + m_i^2 + m_j^2. \tag{6.27}$$

As a consequence the formula for the four-point function (and also that for the three-point function) reduces to a sum (or difference) of Spence functions with a common factor in front. In addition there may be certain logarithms.

Eq. (6.26) is valid with only one restriction, namely *a* must be real. If this restriction is to be lifted one must also consider the subtraction logarithms in going from (6.17) to (6.18). This gives rise to terms containing logarithms of *H, G* and *f*. The imaginary parts of the terms containing *H, G* and *f* is now a function of *y*, and nothing can be said on the sign. The split up of the logarithms as done in the foregoing is then no longer possible. Here we will not consider this special case any further.

7. Bremsstrahlung

In computing soft bremsstrahlung one encounters an integral that is essentially a phase-space integral for photons with an energy less than some specified value. This integral has been worked out before [1,2], but both the derivation and the result simplify considerably by the use of techniques inspired by the foregoing. For this reason we present it here, although rather sketchy.

The basic integral is:

$$\mathcal{L}_{ij} = \int' \frac{d_3 k}{k_0} \frac{1}{(p_i k)(p_j k)}, \qquad k_0 = \sqrt{\boldsymbol{k}^2 + \lambda^2}, \qquad |\boldsymbol{k}| < \omega. \tag{7.1}$$

The accent at the integral denotes the region $|\boldsymbol{k}| < \omega$. In here λ is the photon mass, and p_i and p_j refer to four-momenta of the particles that emit the photon. Thus they are timelike. The case that p_i is a multiple of p_j is special (and easy) and is not considered.

The trick is to introduce again a parameter α much like we did before. We write

$$\mathcal{L}_{ij} = \alpha \int' \frac{d_3 k}{k_0} \frac{1}{(kp)(kq)}, \qquad p = \alpha p_i, \qquad q = p_j,$$

and next choose α such that $(p-q)^2 = 0$, i.e., $p-q$ lightlike. There are two solutions with different sign for $p_0 - q_0$, and we choose that solution that gives the same sign to $p_0 - q_0$ as that of q_0.

The next step is to introduce a Feynman parameter *x,* to combine the two factors in the denominator, and to do the integral over *k.* Neglecting terms that go to zero as the photon mass λ goes to zero gives the result:

$$\mathcal{L}_{ij} = -2\pi\alpha (R_1 + R_2),$$

$$R_1 = \int_0^1 dx \frac{1}{q^2 + 2xvl} \ln\left(\frac{2\omega}{\lambda}\right)^2 = \frac{1}{2vl} \ln\frac{q^2 + 2vl}{q^2} \ln\left(\frac{2\omega}{\lambda}\right)^2,$$

$$R_2 = \int_0^1 dx \frac{1}{u^2} \frac{u_0}{|\boldsymbol{u}|} \ln\frac{u_0 - |\boldsymbol{u}|}{u_0 + |\boldsymbol{u}|},$$

with

$$u = q + x(p - q), \qquad l = p_0 - q_0 = \pm |\boldsymbol{p} - \boldsymbol{q}|,$$

$$v = \frac{(q, p - q)}{l} = \frac{p^2 - q^2}{2l}.$$

We now go over to the integration variable $u_0 = q_0 + x(p_0 - q_0)$; in terms of this variable we have

$$u^2 = 2vu_0 - 2vq_0 + q^2,$$

$$\boldsymbol{u}^2 = u^2 + u_0^2 = u_0^2 + 2vu_0 - 2vq_0 + q^2.$$

Subsequently we take the new variable $m = u_0 - |\boldsymbol{u}|$. Then, after some simple algebra:

$$R_2 = - \int_{q_0 - |\boldsymbol{q}|}^{p_0 - |\boldsymbol{p}|} \frac{\mathrm{d}m}{2lv} \left[\frac{1}{m + v} + \frac{1}{m - (2vq_0 - q^2)/v} - \frac{1}{m} \right]$$

$$\times \ln \left(-\frac{m + v}{v} \frac{m}{m - (2vq_0 - q^2)/v} \right).$$

Apart from a term $2/(m + v)$ the factor in square brackets is just the derivative of the logarithm with respect to m. We add and subtract a term $2/(m + v)$, and the first part can be integrated giving the square of a logarithm. The second part (integral of $2/(m + v)$ times the logarithm) is a sum of Spence functions. Using

$$\ln (m - a) = \ln \left(1 - \frac{m + v}{a + v} \right) + \ln (-v - a),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}m} \mathrm{Sp} (A(m + v)) = -\frac{1}{m + v} \ln (1 - A(m + v)),$$

where $Sp(x)$ is a Spence function, see appendix A, we find

$$R_2 = \left[\frac{1}{4lv} \ln^2 \frac{m(m + v)}{2vq_0 - q^2 - mv} - \frac{1}{2lv} \ln^2 \left\{ -\frac{v(m + v)}{v^2 + 2vq_0 - q^2} \right\} \right.$$

$$\left. -\frac{1}{lv} \left\{ -\mathrm{Sp} \left(\frac{m + v}{v} \right) + \mathrm{Sp} \left(\frac{v(m + v)}{v^2 + 2vq_0 - q^2} \right) \right\} \right]_{m = q_0 - |\boldsymbol{q}|}^{m = p_0 - |\boldsymbol{p}|}.$$

After some simple algebra, using the equation relating $\mathrm{Sp}(x)$ to $\mathrm{Sp}(1/x)$ we obtain the final result for $\mathcal{L}_{ij}$ defined by eq. (7.1):

$$
\begin{aligned}
\mathcal{L}_{ij} = -\frac{2\pi\alpha}{vl}\Bigg[&\tfrac{1}{2}\ln\frac{p^2}{q^2}\ln\left(\frac{2\omega}{\lambda}\right)^2 \\
&+ \left\{\tfrac{1}{4}\ln^2\frac{u_0-|\boldsymbol{u}|}{u_0+|\boldsymbol{u}|} + \mathrm{Sp}\left(\frac{v+u_0+|\boldsymbol{u}|}{v}\right) + \mathrm{Sp}\left(\frac{v+u_0-|\boldsymbol{u}|}{v}\right)\right\}_{u=q}^{u=p}\Bigg],
\end{aligned}
\tag{7.3}
$$

$$
p = \alpha p_i, \qquad q = p_j, \qquad l = p_0 - q_0, \qquad v = \frac{p^2-q^2}{2l},
$$

$$
\alpha^2 p_i^2 - 2\alpha(p_i p_j) + p_j^2 = 0,
$$

α such that $(\alpha p_i - p_j)_0$ same sign as p_{j0}.

The approximation of small photon mass λ has been used in deriving the intermediate result eq. (7.2).

8. Conclusions

In this article the basic equations for the evaluation of all one-loop diagrams have been given. Cases where there are momenta in the numerator can be reduced to linear combinations of the integrals given in the foregoing, and also five-point and higher scalar one-loop integrals can be expressed in the ones given here. For the five-point function this goes as follows. There are five denominator factors, involving the four momenta $p_1, \ldots, p_4$. The scalar products (qp_i) as well as q^2 can be expressed in terms of the propagators, masses and external momenta squared, for instance:

$$
2(qp_1) = (q+p_1)^2 + m_2^2 - (q^2+m_1) + m_1^2 - m_2^2 - p_1^2.
$$

Consider now the five-component tensor $p_{4\mu}\varepsilon_{\alpha\beta\gamma\delta}$. Antisymmetrizing in all five indices gives evidently zero because one cannot have an antisymmetrical five tensor in four dimensions. Subsequently we multiply with $p_{1\alpha}p_{2\beta}p_{3\gamma}q_\delta$, and finally with $p_{4\mu}\varepsilon_{\lambda\delta\kappa\nu}p_{1\lambda}p_{2\delta}p_{3\kappa}q_\nu$. Using the well known decomposition of the product of two epsilon tensors in terms of δ-tensors one obtains an identity among the various propagators. Inserting this identity in the numerator of the five-point integral one obtains an identity for the five-point function.

Our equations cover the whole range of possible external momenta, allowing also negative imaginary parts for the masses. There is one small limitation: the formula for the four-point function requires at least one timelike momentum (more precisely: α must be real). Even that can be cured, at the expense of some further complications, but we have not involved that.

In practical applications it is of importance to know something about the order of magnitude of various parts contributing to the final answer. Generally speaking the following can be said. If one or more of the external quantities p_{ij}^2 is large with respect to all masses then certain roots of quadratic equations become very small, very nearly equal to one, or very nearly equal to each other. Thus it becomes difficult to compute precisely the arguments of the Spence functions. This can be overcome by computing these arguments in sufficient precision. But after that there are then no more troubles: there are no particular cancellations among the various Spence functions.

If on the other hand internal masses are very large then the values of the Spence functions are much larger than their sum, the final answer. To deal with that situation would require computation of these Spence functions in high precision, which seems not an easy thing to do. Moreover, even if the basic functions given in this paper are computed correctly, then the expressions for the form factors, i.e., the integrals involving momenta in the numerator, still suffer strong cancellations.

The solution to the large-mass problem is to start from the beginning and develop approximations such as for instance given in ref. [5]. This gives rise to a great number of formulae, not to be quoted here.

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Appendix A

Spence functions

The properties of Spence functions or dilogarithms are given in the literature [6], and we will present only a few equations.

The defining equation is

$$\mathrm{Sp}\,(x) = -\int_0^1 \mathrm{d}t\,\frac{\ln\,(1-xt)}{t}, \tag{A.1}$$

where the cut of the logarithm is along the negative real axis, implying for the Spence function a cut along the positive real axis starting at the point $x = 1$.

Two basic equations, valid also for complex argument, are:

$$\mathrm{Sp}\,(x) = -\mathrm{Sp}\,(1-x) + \tfrac{1}{6}\pi^2 - \ln\,(x)\ln\,(1-x), \tag{A.2}$$

$$\mathrm{Sp}\,(x) = -\mathrm{Sp}\,\left(\frac{1}{x}\right) - \tfrac{1}{6}\pi^2 - \tfrac{1}{2}\ln^2\,(-x). \tag{A.3}$$

We now want a series expansion for this function. With the help of the above equations the argument may be put within a circle with radius 1 in the complex plane, and moreover with the real part less than 0.5. Next we note that the cut of the Spence function is in the same region as the cut of $\ln(1-x)$, so obviously it is of advantage to use $\ln(1-x)$ as expansion parameter. We get:

$$\text{Sp}(x) = -\int_0^x \frac{\ln(1-t)}{t}\,dt = \int_0^{-\ln(1-x)} \frac{u}{e^u-1}\,du,$$

where we used the transformation $t = 1 - e^{-u}$. Expanding the argument of the integral we obtain

$$\text{Sp}(x) = \int_0^z \sum_{n=0}^{\infty} B_n \frac{u^n}{n!} = \sum_{n=0}^{\infty} B_n \frac{z^{n+1}}{(n+1)!}.$$

In here $z = -\ln(1-x)$ and the coefficients B are the Bernouilly numbers,

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30}, \quad B_6 = \tfrac{1}{42}, \quad B_8 = -\tfrac{1}{30},$$

$$B_{10} = \tfrac{5}{66}, \quad B_{12} = -\tfrac{691}{2730}, \quad B_{14} = \tfrac{7}{6}, \quad B_{16} = -\tfrac{3617}{510}, \quad B_{18} = \tfrac{43867}{798}.$$

Computing the series up to and including $n = 14$ gives about 13 decimals accuracy in the worst case.

Appendix B

Reduction to Spence functions

We start from the expression

$$S_3 = \int_0^1 dy \frac{1}{y - y_0}\{\ln(ay^2 + by + c) - \ln(ay_0^2 + by_0 + c)\}, \tag{B.1}$$

where a is real, while b, c and y_0 may be complex, with the restriction that the imaginary part of the argument of the first logarithm has always the same sign for $0 \le y \le 1$.

Let ε and δ be infinitesimal quantities having the *opposite* sign to the imaginary parts of the arguments of the two logarithms. That is, we assume signs as given by

$-i\varepsilon$ and $-i\delta$. Then

$$S_3 = \int_0^1 dy \frac{1}{y - y_0} [\ln (y - y_1)(y - y_2) - \ln (y_0 - y_1)(y_0 - y_2)]$$

$$- \eta \left(a - i\varepsilon, \frac{1}{a - i\delta} \right) \ln \left(\frac{y_0 - 1}{y_0} \right),$$

with η defined in eq. (2.4). Next we split up the logarithms, and note that the sign of the imaginary part of $(y - y_1)(y - y_2)$ is equal to that of $y_1 y_2 = c/a$:

$$S_3 = \int_0^1 dy \frac{1}{y - y_0} \{\ln (y - y_1) - \ln (y_0 - y_1) + \ln (y - y_2) - \ln (y_0 - y_2)\}$$

$$- \left[\eta(-y_1, -y_2) - \eta(y_0 - y_1, y_0 - y_2) - \eta \left(a - i\varepsilon, \frac{1}{a - i\delta} \right) \right] \ln \frac{y_0 - 1}{y_0}. \qquad \text{(B.2)}$$

Consider now

$$R = \int_0^1 dy \frac{1}{y - y_0} \{\ln (y - y_1) - \ln (y_0 - y_1)\}$$

$$= \int_{-y_1}^{1 - y_1} dy \frac{1}{y - y_0 + y_1} \{\ln y - \ln (y_0 - y_1)\}.$$

The residue of the pole is zero. The cut of the logarithm is along the negative real axis, thus outside the triangle $0, -y_1, 1 - y_1$. We may therefore write

$$\int_{-y_1}^{1 - y_1} dy = \int_0^{1 - y_1} dy - \int_0^{-y_1} dy.$$

Making the substitutions $y = (1 - y_1)y'$ and $y = y_1 y'$ we obtain:

$$R = \int_0^1 dy \left\{ \frac{d}{dy} \ln \left(1 + y \frac{1 - y_1}{y_1 - y_0} \right) \right\} \{\ln y(1 - y_1) - \ln (y_0 - y_1)\}$$

$$- \int_0^1 dy \left\{ \frac{d}{dy} \ln \left(1 - y \frac{y_1}{y_1 - y_0} \right) \right\} \{\ln (-y y_1) - \ln (y_0 - y_1)\}.$$

Since y is real and positive there is never any crossing of the cut for any of the logarithms. After partial integration we obtain

$$R = \mathrm{Sp}\left(\frac{y_1-1}{y_1-y_0}\right) - \mathrm{Sp}\left(\frac{y_1}{y_1-y_0}\right) + \ln\frac{1-y_0}{y_1-y_0}\{\ln(1-y_1)-\ln(y_0-y_1)\}$$
$$-\ln\frac{-y_0}{y_1-y_0}\{\ln(-y_1)-\ln(y_0-y_1)\}.$$

After some simplification, using (A.2):

$$R = \mathrm{Sp}\left(\frac{y_0}{y_0-y_1}\right) - \mathrm{Sp}\left(\frac{y_0-1}{y_0-y_1}\right) + \eta\left(-y_1, \frac{1}{y_0-y_1}\right)\ln\frac{y_0}{y_0-y_1}$$
$$-\eta\left(1-y_1, \frac{1}{y_0-y_1}\right)\ln\frac{y_0-1}{y_0-y_1}. \tag{B.3}$$

This completes the calculation of S_3, eq. (B.1).

The evaluation of the expressions encountered in the equation for the four-point function, eq. (6.26), offers no new difficulties. In working out the subtraction logarithms some care is necessary with respect to numerical evaluation of the η-functions. For instance in eq. (6.26) we encounter

$$\ln\frac{Gy+J}{Ay^2+Dy+F},$$

to be evaluated for $y = y_0$. Writing this as the difference of two logarithms we also have an η-function depending on the signs of the imaginary parts of Gy_0+J, $Ay_0^2+Dy_0+F$, and their quotient. In order to avoid nasty cancellations it is best to evaluate this sign of the imaginary part of the quotient separately. This can be done by noting that in the point $y = y_0$ the argument of this logarithm is equal to $T/V = S/U$, which is easier to evaluate, and moreover the same for all three terms in eq. (6.26).

Appendix C

The extra logarithms

We start from eq. (6.3), and will derive the extra logarithm associated with the projective transformation for the case $A_1, A_2, A_3 > 0$ and $A_4 < 0$. After the projective transformation we have the extra term (note that x is associated with u_4, eq. (6.2); we omitted the factor $A_1A_2A_3A_4$):

$$R_4 = \int_{-\Lambda}^{\Lambda} dx \int_0^x dy \int_0^y dz \frac{1}{[ax^2+by^2+cxy+dx+ey+kz+f]^2}, \tag{C.1}$$

to be evaluated in the limit $\Lambda \to \infty$. Doing the z-integration gives

$$R = \frac{1}{k} \int\limits_{-\Lambda}^{\Lambda} dx \int\limits_{0}^{x} dy \left\{ \frac{1}{ax^2 + by^2 + cxy + dx + ey + f} - (e \to e + k) \right\}. \tag{C.2}$$

Each of these terms separately is divergent, so some care is necessary. Now we perform the usual shift $y = y' + \alpha x$, and subsequently exchange the x and y integration:

$$\int\limits_{-\Lambda}^{\Lambda} dx \int\limits_{0}^{\mu x} dy = \int\limits_{0}^{\mu \Lambda} dy \int\limits_{y/\mu}^{\Lambda} dx + \int\limits_{-\mu \Lambda}^{0} dy \int\limits_{-y/\mu}^{-\Lambda} dx,$$

where μ is either $1 - \alpha$ or $-\alpha$. Proceeding carefully one arrives at:

$$R_4 = \frac{1}{k} \int\limits_{-\infty}^{\infty} dy \frac{-\alpha}{-(c + 2b\alpha)\alpha y + d + e\alpha} \{ \ln [ay^2 + dy + f] - \ln (y = y_0) \}$$

$$- \frac{1}{k} \int\limits_{-\infty}^{\infty} dy \frac{1 - \alpha}{(c + 2b\alpha)(1 - \alpha)y + d + e\alpha} \{ \ln [(a + b + c)y^2 + (d + e)y + f]$$

$$- \ln (y = y_0) \}$$

$$- (e \to e + k). \tag{C.3}$$

This corresponds precisely to replacing the limits, 0, 1 by $-\infty, \infty$ in those integrals in eq. (6.12) where the argument of the logarithm contains A_4. This no accident; we do not want to be involved here in a lengthy discussion (see also below), but just note that, for instance, the first term of eq. (6.12) simplifies, at least as far as the logarithm is concerned, if one applies the projective transformation

$$y = \frac{a_3 y'}{(a_3 - a_4)y' + a_4}, \qquad a_3 = \frac{1}{A_3}, \qquad a_4 = \frac{1}{A_4}. \tag{C.4}$$

It is to be noted that in eq. (C.3) the various terms are separately divergent; this can be cured easily by taking for the integrands the symmetrized expression with respect to the reflection $y \to -y$.

Before evaluating the integrals in eq. (C.3) we make a short comment on the case where two of the coefficients A_i are negative. The intermediate result, eq. (6.10), when worked out can be recognized as the projective transform, with coefficients A_2, A_3, A_4 and A_1, A_3, A_4 respectively, of an expression that can be obtained from the starting point, eq. (6.1), as follows. Apply the propagator identity described in sect. 2, and choose x such that $M = 0$. Then parametrize such that this M occurs as m_4^2, i.e., as f in eq. (6.3). If $f = 0$ then the substitutions $y = xy'$,

$z = xz'$ lead to an expression of which the x-integration can be done easily. Compare also appendix D. We thus understand eq. (6.10) as a projective transform of something not having extra terms. Thus the extra terms may be understood at this level, arising from a projective transform with three coefficients. Since a projective transformation is invariant for changing the $A_i \rightarrow -A_i$ we need never consider more than one negative coefficient.

Now the evaluation of the integrals eq. (C.3). This amounts to evaluating the integral

$$R = \tfrac{1}{2} \int\limits_{-\infty}^{\infty} dx \left\{ \frac{1}{x-a} \ln (Ax^2 + Bx + C) - \frac{1}{x+a} \ln (Ax^2 - Bx + C) \right\}, \tag{C.5}$$

with real A, B, possibly complex a, and $C = C - i\varepsilon$, but otherwise real. The roots x_1, x_2 of $Ax^2 + Bx + C$ have imaginary parts such that $\operatorname{Im} x_2 = -\operatorname{Im} x_1$, and we take x_1 to be the root with positive imaginary part. The logarithm may be decomposed and we get:

$$R = \tfrac{1}{2} \int\limits_{-\infty}^{\infty} dx \left[\ln (A - i\varepsilon) \left\{ \frac{1}{x-a} - \frac{1}{x+a} \right\} \right.$$

$$\left. + \frac{1}{x-a} \{ \ln (x - x_1) + \ln (x - x_2) \} - \frac{1}{x+a} \{ \ln (x + x_2) + \ln (x + x_1) \} \right]. \tag{C.6}$$

Take the case $\operatorname{Im} a > 0$, remember $\operatorname{Im} x_1 > 0$. The location of the singularities in the complex x-plane is shown in fig. 2. The integration is from $-\infty$ to ∞ along the real axis. The contour may be closed with a half circle in the upper half plane, and subsequently tightened such that we get the situation shown in fig. 3. In this way we find:

$$R = i\pi \left[\ln (A - i\varepsilon) + \ln (a - x_1) + \ln (a - x_2) - \int\limits_{-\Lambda + x_1}^{x_1} \frac{dx}{x - a} + \int\limits_{-\Lambda - x_2}^{-x_2} \frac{dx}{x + a} \right],$$
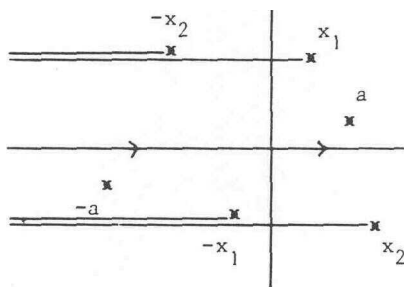$$\tag{C.7}$$

Fig. 2.

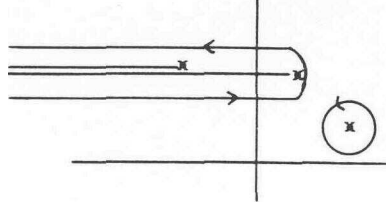Fig. 3.

where the limit $\Lambda \to \infty$ must be taken. One finds:

$$R = i\pi[\ln (A - i\varepsilon) + 2 \ln (a - x_2) - i\pi], \qquad \text{Im } a > 0. \tag{C.8}$$

Similarly, if $\text{Im } a < 0$:

$$R = -i\pi[\ln (A - i\varepsilon) + 2 \ln (a - x_1) + i\pi], \qquad \text{Im } a < 0. \tag{C.9}$$

We still must subtract from this the term needed to make the residue of the pole in $x = a$ or $x = -a$ zero, i.e.,

$$R_s = -\tfrac{1}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}x \left\{ \frac{1}{x - a} \ln (Aa^2 + Ba + C) - \frac{1}{x + a} \ln (Aa^2 + Ba + C) \right\}. \tag{C.10}$$

Together with eq. (C.8) or eq. (C.9) one obtains

$$R - R_s = i\pi\{\ln (a - x_2) - \ln (a - x_1) - i\pi\}, \qquad \text{Im } x_1 > 0. \tag{C.11}$$

To obtain this result one needs to decompose $\ln (Aa^2 + Ba + C - i\varepsilon)$. Now, as shown in the text, the argument is always real, with $C = C - i\varepsilon$. Thus $\ln (A - i\varepsilon)$ may be extracted. But the decomposition in $\ln (a - x_1) + \ln (a - x_2)$ may not be true, and eq. (6.14) is what one obtains if the decomposition is not made. The reader recognizes that eq. (C.11) may contain a jump if $\text{Im } a = \pm\text{Im } x_1$, while eq. (6.14) has no such discontinuities. The derivation of the S-term is slightly more intricate. This term arises if A_1 and A_2 have opposite spin, and very roughly speaking it can be understood as follows. If A_1 is negative, all other A being positive, then there is an extra term. It can be seen as the integral from $-\infty$ to $+\infty$ of the L_{14} and L_{13} term in eq. (6.12), but then only with respect to the contributions coming from infinity. Each of these integrals diverges, and S contains the finite piece that derives from the difference when carefully treated.

The way to obtain these terms is through a regularization procedure. We start from eq. (6.1), and consider the case that $p_{12}^2 = p_1^2$ is timelike, and such that $-p_{12}^2 > (m_1 + m_2)^2$. Now apply the propagator identity to the first two propagators of eq. (6.1) and choose α such that $M^2 = 0$ (see eq. (2.3), with $p = 0$, $k = p_1$). Then we have two terms, each a four-point function with zero mass. Permute things such that this zero mass is the last propagator; we then obtain a formula as eq. (6.3)

with $f = M^2 - i\varepsilon = -i\varepsilon$. Subsequently make the substitutions $y = xy'$ and $z = xz'$, a factor x^2 in the nominator can be cancelled against a similar factor in the denominator. Note that x is always positive so that nothing happens to the $i\varepsilon$ term. Now the x integral is trivial, and doing it one obtains the result

$$\frac{D}{i\pi^2} = (1-\alpha) \int\limits_0^1 du_2\, du_3\, du_4 \frac{1}{(a_3 - a_2)u_3 + (a_4 - a_2)u_4 + a_2 - i\varepsilon}$$

$$\times \frac{\delta(\sum u - 1)}{[\sum p_{ij}^2 u_i u_j + m_i^2 u_i - i\varepsilon]}$$

$$+ \alpha \int\limits_0^1 du_1\, du_3\, du_4 \frac{1}{(a_3 - a_1)u_3 + (a_4 - a_1)u_4 + a_1 - i\varepsilon}$$

$$\times \frac{\delta(\sum u - 1)}{[\sum p_{ij}^2 u_i u_j + m_i^2 u_i - i\varepsilon]} \,, \tag{C.12}$$

with $a_i = 1/A_i$ and $i,j = 2, 3, 4$ in the first term and $1, 3, 4$ in the second term. The coefficients A_i are precisely those of appendix D, eq. (D.5). We changed from the variables y, z to the variables u analogously to the relation between eq. (6.3) and eq. (6.2).

To each of the terms of eq. (6.12) one may apply a projective transformation involving A_2, A_3, A_4 and A_1, A_3 and A_4 respectively, and the result is then precisely eq. (6.10), where one uses $A_1 = (1 - \alpha)k$ and $A_2 = -\alpha k$ with k given in eq. (6.4). This derivation of eq. (6.12) has the advantage that the $i\varepsilon$ prescription is completely clear up to eq. (C.12).

If now A_1 and only A_1 is negative, then we get after the projective transformation the usual extra term in the second term of eq. (C.12), but none in the first. The resulting integral is not convergent, and we must regularize first. To do this we rewrite the second term of eq. (C.12):

$$\int du \frac{\delta(\sum u - 1)}{N} \left\{ \frac{1}{P - i\varepsilon} - \frac{1}{P - i\varepsilon + \Lambda^2 N^2} + \frac{1}{P - i\varepsilon + \Lambda^2 N^2} \right\}, \tag{C.13}$$

with

$$N = (a_3 - a_1)u_3 + (a_4 - a_1)u_4 - i\varepsilon, \qquad P = p_{ij}^2 u_i u_j + m_i^2 u_i,$$

and evaluated in the limit of large Λ. Now take the first two terms together; after the projective transformation the extra term is convergent, and leads to the R-terms in eq. (6.12) as described in the beginning of this appendix, and nothing else survives in the limit of large Λ.

Consider now the last term of eq. (C.13). We perform the projective transformation with coefficients that differ slightly from A_1, A_3 and A_4, in such a way that the extra term takes the form

$$S = \frac{A_1 A_2 A_3 A_4}{k} \int\limits_{-\infty}^{\infty} \mathrm{d}x \int\limits_{0}^{x} \mathrm{d}y \frac{1}{\delta x + 1 - i\varepsilon'}$$

$$\times \frac{1}{ax^2 + by^2 + cxy + dx + ey + f - i\varepsilon + \Lambda^2(\delta x - 1 - i\varepsilon)^2} \, . \tag{C.14}$$

Next we substitute $y = xy'$ and $x = x'/\delta$. For small δ, but with $\delta^2 \Lambda^2$ still very large we find:

$$S = \frac{A_1 A_2 A_3 A_4}{k} \int\limits_{-\infty}^{\infty} \mathrm{d}x \int\limits_{0}^{1} \mathrm{d}y \frac{1}{x + 1 - i\varepsilon'} \frac{1}{x(by^2 + cy + a) - i\varepsilon + \Lambda^2 \delta^2 (x + 1)^2} \, . \tag{C.15}$$

For large Λ the main contribution to this integral comes for the region around $x = -1$. We may therefore put $x = -1$ in the coefficient of the quadratic form in y. For the resulting expression we close the x-integration contour with a half circle in the upper half plane; taking the various poles into account we obtain in the limit of large Λ:

$$S = \frac{A_1 A_2 A_3 A_4}{k} \int\limits_{0}^{1} \mathrm{d}y \frac{-i\pi\varepsilon(\varepsilon')}{by^2 + cy + a - i\varepsilon} \, , \tag{C.16}$$

which can be integrated trivially.

We are left with the question of the sign of ε'. It is left to the reader to verify that the sign is such as to give the result quoted in eq. (6.12), and we concentrate here on the origin of the ε-function occurring in eq. (6.15). To understand this we note the remarkable fact that the replacement of A_i by $-A_i$ is not an invariance of our equations. In view of eq. (2.7) this is indeed a very curious fact. In eq. (C.12) this is however evident; the replacement a_i by $-a_i$ reverses the $i\varepsilon$ prescription, and thus a change of sign of 5, eq. (C.16). The origin of this feature is to be found in the transformation of eq. (6.3) to a similar expression with however g (and h and j) zero. The expression is then linear in z; the transformation moves a pole to infinity. If this pole is in the integration region then we get an extra term. The question is now which of the two poles (that have opposite imaginary parts) is moved to infinity; this depends on the choice of the square root in eq. (6.8). The ε-function in eq. (6.15) tests for this sign.

Appendix D

Projective and conformal transformation

The projective transformations explained in sect. 2 have a direct parallel in terms of a transformation of the momenta of the corresponding diagram. Consider eq. (6.1). We define the new variable r by

$$q_\mu = \frac{r_\mu}{r^2}, \qquad r_\mu = \frac{q_\mu}{q^2}. \tag{D.1}$$

Here we will not worry about singularities of this transformation, which is free of trouble in the Euclidean case. Carrying through this transformation we get a Jacobian $1/r^8$, while for instance for the second propagator we obtain

$$\left(\frac{r}{r^2} + p_1\right)^2 + m_2^2 = \frac{1}{r^2} + 2\frac{(rp_1)}{r^2} + p_1^2 + m_2^2.$$

Subsequently we distribute the four factors r^2 of the Jacobian over the various propagators. The p_1 propagator becomes:

$$(p_1^2 + m_2^2)r^2 + 2(rp_1) + 1 = (p_1^2 + m_2^2)\left\{\left(r + \frac{p_1}{p_1^2 + m_2^2}\right)^2 + \frac{m_2^2}{(p_1^2 + m_2^2)^2}\right\}. \tag{D.3}$$

This is again of the form of a propagator. We so obtain for eq. (6.1):

$$D = A_1 A_2 A_3 A_4 \int d_4 r \frac{1}{(r^2 + M_1^2)((r+q_1)^2 + M_2^2)((r+q_2)^2 + M_3^2)((r+q_3)^2 + M_4^2)},$$

with

$$A_1 = \frac{1}{m_1^2}, \qquad A_2 = \frac{1}{p_1^2 + m_2^2}, \qquad A_3 = \frac{1}{(p_1 + p_2)^2 + m_3^2},$$

$$A_4 = \frac{1}{(p_1 + p_2 + p_3)^2 + m_4^2},$$

$$q_1 = A_2 p_1, \qquad q_2 = A_3(p_1 + p_2), \qquad q_3 = A_4(p_1 + p_2 + p_3),$$

$$M_i^2 = A_i^2 m_i^2.$$

Consider now for instance q_{13}^2:

$$q_{13}^2 = q_2^2 = A_3^2(p_1 + p_2)^2.$$

This is equal to the expression

$$[(p_1 + p_2)^2 + m_1^2 + m_3^2]A_1 A_3 - m_3^2 A_3^2 - m_1^2 A_1^2.$$

Similarly for the other momenta. In other words, the above transformation corresponds to a projective transformation with the A_i indicated.

If m_1^2 is zero the result simplifies considerably, in fact the first denominator becomes simply 1. The resulting expression is simply the integral for the three-point function. Now with the help of the propagator identity one can write a four-point function as the sum of two four-point functions with, however, one of the masses being zero, and as we see from the above each of those can be transformed into a three-point function. This is actually precisely equivalent to the work done in the beginning of sect. 6. In this way one discovers that the momenta q_{ij}^2 of eq. (6.6), with the coefficients A_i as given in eq. (6.8), are indeed the squares of sums of certain momenta q_2, q_3, just like the p_{ij}^2 are derived from the p_i (see eq. (6.4)), with:

$$q_2 = (1-\alpha)(A_3 - A_2)p_1 + A_3 p_2,$$

$$q_3 = (1-\alpha)(A_4 - A_3)p_1 + (A_4 - A_3)p_2 + A_4 p_3, \tag{D.4}$$

$$A_1 = \frac{1}{\alpha^2 p_1^2 + m_1^2}, \qquad A_2 = \frac{1}{(1-\alpha)^2 p_1^2 + m_2^2}, \qquad \frac{A_1}{A_2} = \frac{1-\alpha}{\alpha},$$

$$A_3 = \frac{1}{(1-\alpha)l_{13} + \alpha l_{23}}, \qquad A_4 = \frac{1}{(1-\alpha)l_{14} + \alpha l_{24}}, \qquad l_{ij} = p_{ij}^2 + m_i^2 + m_j^2,$$

$$\alpha \text{ solution of } -\alpha^2 p_1^2 + \alpha(p_1^2 + m_2^2 - m_1^2) + m_1^2 = 0. \tag{D.5}$$

With these q one indeed finds that $q_{23}^2 = q_2^2$, $q_{34}^2 = q_3^2$ and $q_{24}^2 = (q_2 + q_3)^2$ are correctly given by eq. (6.6). The calculation is somewhat cumbersome, but simplifies if one use the identities:

$$(1-\alpha)^2 p_1^2 = \frac{1}{A_2} - m_2^2,$$

$$(1-\alpha)l_{13} - (1-\alpha)l_{23} = \frac{1}{A_3} - l_{23},$$

$$(1-\alpha)l_{12} = \frac{1}{A_2} - 2\alpha m_2^2, \qquad \text{etc.}$$

This demonstrates that for real coefficients A_i the transformed momenta q_{ij}^2 satisfy the properties expected for squares of real (Minkowskian) momenta.

Appendix E

Special cases

In certain cases such as for instance encountered in QED the equations for the one-loop integrals simplify greatly. In this appendix we quote some of these formulas, which are of course already well-known [1, 2].

(a) 3-point function with one very small mass and two external momenta on-mass-shell. Thus, referring to eq. (5.1) we have $p_1^2 = -m_1^2$, $p_2^2 = -m_3^2$ and $m_2^2 = \delta$, with δ small with respect to all other quantities.

The result is:

$$C(-m_1^2, -m_3^2, s, m_1^2, \delta, m_3^2) = -\tfrac{1}{2}i\pi^2(F_1 \ln \delta - F_2),\tag{E.1}$$

$$F_1 = \frac{-1}{s(y_1 - y_2)}\left\{\ln \frac{y_1 - 1}{y_1} - \ln \frac{y_2 - 1}{y_2}\right\},\tag{E.2}$$

$$\begin{aligned}F_2 = F_1 \ln(-s - i\varepsilon) &- \frac{1}{s(y_1 - y_2)}\{\tfrac{1}{2}\ln^2(1 - y_1) - \tfrac{1}{2}\ln^2(-y_1)\\ &- \tfrac{1}{2}\ln^2(1 - y_2) + \tfrac{1}{2}\ln^2(-y_2) - \ln(1 - y_2)\ln(1 - y_1)\\ &+ \ln(-y_2)\ln(-y_1)\\ &+ 2\ln\frac{y_1 - 1}{y_1}\ln(y_1 - y_2) - 2\,\mathrm{Sp}\left(\frac{y_1 - 1}{y_1 - y_2}\right) + 2\,\mathrm{Sp}\left(\frac{y_1}{y_1 - y_2}\right)\},\end{aligned}\tag{E.3}$$

with y_1 and y_2 the roots of the equation

$$-sy^2 + y(s + m_3^2 - m_1^2) + m_1^2 = 0.$$

Terms of order $\sqrt{\delta}$ have been neglected.

This formula obtains easily through the following steps. The starting point is:

$$\frac{C}{i\pi^2} = \int_0^1 dx \int_0^x dy \frac{1}{-sy^2 + 2xy(p_1, p_1 + p_2) - p_1^2 x^2 + \delta(1 - x) - i\varepsilon}.$$

Next substitute $y = xy'$, and use

$$\int_0^1 dx \frac{x}{Ax^2 + \delta(1 - x)} = -\frac{1}{2A}\ln\frac{\delta}{A} + O(\sqrt{-\delta/A}).$$

After this the result follows easily.

(b) Four-point function with two small (and equal) masses, and four external momenta on-mass-shell.

$$\begin{aligned}D(-m_1^2, -m_2^2, -m_2^2, -m_1^2, s, t, m_1^2, \delta, m_2^2, \delta) &= \frac{2i\pi^2}{t}\ln\left|\frac{t}{\delta}\right|\\ &\times \frac{1}{\Lambda}\ln\left|\frac{(m_1 + m_2)^2 + s + \Lambda}{(m_1 + m_2)^2 + s - \Lambda}\right|\\ &- \frac{2\pi^3}{t}\ln\left|\frac{t}{\delta}\right| \cdot \frac{1}{\Lambda}\theta[-s + (m_1 + m_2)^2],\end{aligned}$$

$$\Lambda = \sqrt{s^2 + 2s(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2}.$$

Again terms of order $\sqrt{\delta}$ have been omitted. This equation is valid for negative s, with $-s > (m_1 + m_2)^2$, and positive t, but of course $t \gg \delta$.

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