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Fusing Gauge Theory Tree Amplitudes Into Loop Amplitudes

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ABSTRACT

We identify a large class of one-loop amplitudes for massless particles that can be constructed via unitarity from tree amplitudes, without any ambiguities. One-loop amplitudes for massless supersymmetric gauge theories fall into this class; in addition, many non-supersymmetric amplitudes can be rearranged to take advantage of the result. As applications, we construct the one-loop amplitudes for n-gluon scattering in $N = 1$ supersymmetric theories with the helicity configuration of the Parke-Taylor tree amplitudes, and for six-gluon scattering in $N = 4$ super-Yang-Mills theory for all helicity configurations.

1. Introduction

While the computation of amplitudes in perturbative QCD is central to the study of jet and jet-associated physics at current and future hadron colliders, and thereby important to the prospects of discovering new physics there, such computations are not easy. Recent years have nonetheless seen a number of improvements in techniques both at tree level and at the one-loop level, which have allowed the computation of a more extensive set of matrix elements than previously possible. These techniques include the spinor helicity basis [1]; recurrence relations for amplitudes [2,3,4,5]; and string-based techniques for one-loop amplitudes [6,7,8]. Collinear limits are useful in constructing ansätze for amplitudes with an arbitrary number of gluons, both at tree level [9,10] and at loop level [11]. They also provide strong checks on results obtained by other means.

Recently we gave a formula [12,13] for all-multiplicity one-loop amplitudes in an $N = 4$ supersymmetric gauge theory, with the helicity configuration of the Parke-Taylor [9] tree-level amplitudes (which have maximal helicity violation). The computation of this formula utilized a new technique, based on the observation that for an $N = 4$ supersymmetric amplitude with all external gluons, all integrals that enter are determined by their absorptive parts (that is, their cuts), and thus the amplitude can be computed from the interference of appropriate tree amplitudes. The choice of external helicities in turn restricts the set of tree helicity amplitudes that appear, and thereby renders the computation tractable even for an arbitrary number of external legs. These amplitudes constitute part of the computation of the corresponding one-loop amplitudes in QCD: the latter amplitudes have a natural decomposition [8,14] into three pieces: an $N = 4$ supersymmetric amplitude, the contribution of an internal $N = 1$ supersymmetric chiral matter multiplet, and the contribution of an internal scalar.

In this paper, we show that a larger class of amplitudes can be calculated solely from the knowledge of their cuts. In particular, the collection of amplitudes that can be calculated in this manner — amplitudes which we shall term *cut-constructible* — includes all purely massless one-loop amplitudes for which the loop integrals satisfy certain power-counting criteria, given in section 3. As we shall show, all color-ordered amplitudes in massless supersymmetric gauge theories with trivial superpotential are cut-constructible. (We expect that the result to carry over to all amplitudes in massless supersymmetric theories.) This observation is also useful in non-supersymmetric theories, where sums or differences of certain contributions to amplitudes satisfy the criterion, allowing one to trade a harder calculation for an easier one.

As applications of the method, we compute the one-loop maximally helicity-violating

n-gluon amplitudes in $N = 1$ supersymmetric gauge theory, and those one-loop six-gluon amplitudes in $N = 4$ supersymmetric gauge theory that were not previously computed in ref. [12].

For amplitudes that are not cut-constructible, such as scalar-loop contributions to n-gluon amplitudes, one may still use the cuts to determine efficiently all terms with logarithms and dilogarithms. We will present an explicit scalar-loop example with an arbitrary number of external legs. In this case, there are additional polynomial terms which cannot be directly determined from the cuts, although the form of the missing polynomials is restricted by the simple factorization properties of color-ordered amplitudes in the soft and collinear limits. (The sorts of universal functions that enter into these limits also provide a powerful means of summarizing the integrations over singular regions of phase space in the construction of numerical programs [15].) A technique utilizing collinear factorization has been used to construct an ansatz for the all-plus helicity configuration with an arbitrary number of external legs [16,11], which was subsequently proven by Mahlon via recursive techniques [5]. Recursive techniques have also been used to calculate a variety of other non-cut-constructible amplitudes [4,5]. In this way the collinear and recursive techniques complement the cutting method discussed in this paper.

In section 2 we briefly review salient features of color and supersymmetry decompositions of QCD amplitudes. Section 3 contains a "uniqueness" result which shows that a one-loop amplitude whose diagrams all satisfy a certain power-counting criterion can be completely determined from its cuts. A general discussion of the applicability of this result to massless supersymmetric gauge theories is given in section 4. A concrete application follows in section 5: by systematically determining all the cuts, we construct the maximally helicity-violating one-loop amplitudes in $N = 1$ supersymmetric gauge theory, for an arbitrary number of external gluons. In section 6 we present those six-gluon amplitudes in $N = 4$ supersymmetric gauge theory that have not appeared previously. In section 7 we show how one can apply cutting methods to non-supersymmetric theories. Section 8 contains our conclusions.

2. Review

At tree level, amplitudes in an $U(N_c)$ or $SU(N_c)$ gauge theory with n external gluons can be written in terms of color-ordered partial amplitudes A_n^{tree}, multiplied by an associated color trace [17,18]. In order to obtain the full amplitude, one must sum over all

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non-cyclic permutations,

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) \, A_n^{\text{tree}}(k_{\sigma(1)}^{\lambda_{\sigma(1)}}, \ldots, k_{\sigma(n)}^{\lambda_{\sigma(n)}}) \,, \quad (2.1)$$

where k_i, λ_i, and a_i are respectively the momentum, helicity ($\pm$), and color index of the i-th external gluon, g is the coupling constant, and S_n/Z_n is the set of non-cyclic permutations of $\{1, \ldots, n\}$. The $U(N_c)$ $(SU(N_c))$ generators T^a are the set of hermitian (traceless hermitian) $N_c \times N_c$ matrices, normalized so that $\text{Tr}\left(T^a T^b\right) = \delta^{ab}$. The color decomposition (2.1) can be derived in conventional field theory by using

$$f^{abc} = -\frac{i}{\sqrt{2}} \, \text{Tr}\left(\left[T^a, T^b\right] T^c\right), \quad (2.2)$$

where the T^a may by either $SU(N_c)$ matrices or $U(N_c)$ matrices. The structure constants f^{abc} vanish when any index belongs to the $U(1)$, which is generated by the matrix $T^{a_{U(1)}} \equiv 1/\sqrt{N_c}$; therefore the partial amplitudes satisfy the $U(1)$ decoupling identities [18,2]

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \varepsilon_i, a_i\}_{i=1}^{n-1}; k_n, \varepsilon_n, a_{U(1)}) = 0 \,. \quad (2.3)$$

One may perform a similar color decomposition for one-loop amplitudes; in this case, there are up to two traces over color matrices [19], and one must also sum over the different spins J of the internal particles circulating in the loop. When all internal particles transform as color adjoints, the result takes the form

$$\mathcal{A}_n\left(\{k_i, \lambda_i, a_i\}\right) = g^n \sum_{J=0,\frac{1}{2},1} n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \text{Gr}_{n;c}\left(\sigma\right) A_{n;c}^{[J]}(\sigma), \quad (2.4)$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x, and n_J is the number of particles of spin J. The leading color-structure factor

$$\text{Gr}_{n;1}(1) = N_c \, \text{Tr}\left(T^{a_1} \cdots T^{a_n}\right) \quad (2.5)$$

is just N_c times the tree color factor, and the subleading color structures are given by

$$\text{Gr}_{n;c}(1) = \text{Tr}\left(T^{a_1} \cdots T^{a_{c-1}}\right) \text{Tr}\left(T^{a_c} \cdots T^{a_n}\right). \quad (2.6)$$

S_n is the set of all permutations of n objects, and $S_{n;c}$ is the subset leaving $\text{Gr}_{n;c}$ invariant. Once again it is convenient to use $U(N_c)$ matrices; the extra $U(1)$ decouples from all final results [19]. (For internal particles in the fundamental ($N_c + \bar{N}_c$) representation, only the

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single-trace color structure ($c = 1$) would be present, and the corresponding color factor would be smaller by a factor of N_c. In this case the $U(1)$ gauge boson will *not* decouple from the partial amplitude, so one should only sum over $SU(N_c)$ indices when color-summing the cross-section.) In each case the massless spin-J particle is taken to have two helicity states whether it is a gauge boson, a Weyl fermion, or a complex scalar.

In the next-to-leading-order correction to the cross-section, summed over colors, the leading contribution for large N_c comes from $A_{n;1}^{[J]}$; the remaining partial amplitudes $A_{n;c}^{[J]}$ ($c > 1$) produce corrections which are down by a factor of $1/N_c^2$ [19]. For amplitudes whose external legs are purely in the adjoint representation, we showed in section 7 of ref. [12] how to obtain $A_{n;c}^{[J]}$ as a sum over permutations of the leading contribution $A_{n;1}^{[J]}$. Therefore, it is sufficient to calculate the $A_{n;1}^{[J]}$. These color-ordered partial amplitudes involve a limited class of loop integrals, those where the external legs (including attached trees we amputate to leave off-shell legs) are ordered sequentially.

We also draw lessons from string-based techniques for computing one-loop amplitudes in gauge theories [6,7], used previously to calculate all one-loop five-gluon helicity amplitudes [8]. The techniques, though independent of the conventional Feynman-diagram expansion, can be understood using a reorganization of a conventional approach [20,21] utilizing the background-field method [22] and a second-order formalism for internal fermions. Such an approach has also proven useful for the computation of effective actions [23]. The string-based techniques further reveal that gluon amplitudes are most naturally written in a form [8,14] that reflects the simplicity of contributions from different supersymmetry multiplets,

$$
\begin{aligned}
A_{n;1}^{[0]} &= c_\Gamma \left(V_n^s A_n^{\text{tree}} + i F_n^s \right) , \\
A_{n;1}^{[1/2]} &= -c_\Gamma \left((V_n^f + V_n^s) A_n^{\text{tree}} + i(F_n^f + F_n^s) \right) , \\
A_{n;1}^{[1]} &= c_\Gamma \left((V_n^g + 4V_n^f + V_n^s) A_n^{\text{tree}} + i(F_n^g + 4F_n^f + F_n^s) \right) ,
\end{aligned}
\tag{2.7}
$$

where, with $\epsilon = (4 - D)/2$ the dimensional regularization parameter,

$$
r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \qquad c_\Gamma = \frac{r_\Gamma}{(4\pi)^{2-\epsilon}} ,
\tag{2.8}
$$

and we have assumed use of a supersymmetry preserving regulator [24,6]. In equation (2.7) the superscripts g, f and s reflect the supersymmetric decomposition, whereas the split into V and F type pieces indicates the singularities as $\epsilon \to 0$.

All singular pieces (poles in ϵ) are proportional to the tree amplitude, and hence are contained in the V_n factors; the F_n terms, in contrast, are finite as $\epsilon \to 0$ and may involve different spinor-product forms [1] than the tree. While V_n contains only momentum invariants but no spinor products, the ratio of F_n to the tree amplitude will in general

contain not just momentum invariants but spinor products as well. (The latter could be replaced in such a phase-free ratio by combinations of momentum invariants and contracted antisymmetric tensors such as $\varepsilon(1,2,3,4) \equiv 4i\varepsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho k_4^\sigma$, but the latter combinations cannot be eliminated from the F_n.) There is of course some freedom in shifting terms between V_n and F_n.

The organization (2.7) in terms of g, f and s pieces is equivalent to calculating the fermion and gluon loop contributions in terms of the scalar loop contributions plus the contributions from supersymmetric multiplets. In an $N = 4$ super-Yang-Mills theory, summing over the contributions from one gluon, four Weyl fermions and three complex (or six real) scalars, all functions except V_n^g and F_n^g cancel from eq. (2.7) and the amplitudes are

$$A_{n;1}^{N=4} \equiv A_{n;1}^{[1]} + 4A_{n;1}^{[1/2]} + 3A_{n;1}^{[0]} = c_\Gamma \left(V_n^g \, A_n^{\text{tree}} + iF_n^g\right). \tag{2.9}$$

For an $N = 1$ chiral multiplet, containing one scalar and one Weyl fermion, only the functions V_n^f and F_n^f survive,

$$A_{n;1}^{N=1 \text{ chiral}} \equiv A_{n;1}^{[1/2]} + A_{n;1}^{[0]} = -c_\Gamma \left(V_n^f A_n^{\text{tree}} + iF_n^f\right). \tag{2.10}$$

We can thus rewrite equation (2.7) in terms of the $N = 4$ supersymmetric amplitude, the contribution of an $N = 1$ supersymmetric chiral multiplet, and the contribution of a complex scalar,

$$\begin{aligned} A_{n;1}^{[1/2]} &= A_{n;1}^{N=1 \text{ chiral}} - A_{n;1}^{[0]}, \\ A_{n;1}^{[1]} &= A_{n;1}^{N=4} - 4A_{n;1}^{N=1 \text{ chiral}} + A_{n;1}^{[0]}. \end{aligned} \tag{2.11}$$

These equations, which hold for supersymmetry preserving regulators [24,6], may be converted to the 't Hooft-Veltman (HV) [25] or conventional [26] schemes by accounting for the $[\epsilon]$-scalar differences between the schemes. In the 't Hooft-Veltman or conventional scheme we must remove $[\epsilon]$-scalars propagating in the gluon loop. For the amplitudes with external gluon helicities in four dimensions ($\pm$) we modify the above relation to

$$\begin{aligned} A_{n;1}^{[1/2]}\Big|_{\text{HV, conventional}} &= A_{n;1}^{N=1 \text{ chiral}} - A_{n;1}^{[0]}, \\ A_{n;1}^{[1]}\Big|_{\text{HV, conventional}} &= A_{n;1}^{N=4} - 4A_{n;1}^{N=1 \text{ chiral}} + (1-\epsilon)A_{n;1}^{[0]}, \end{aligned} \tag{2.12}$$

where the quantities on the right-hand-side are ones computed in this paper. In the conventional scheme there are additional amplitudes with external $[\epsilon]$-helicities [27,6]. For the case of n external gluons these are given by the corresponding tree amplitudes with $[\epsilon]$-helicities (which are proportional to ϵ) multiplied by the universal singularities [15,28] contained in the V_n.

Given the explicit results of section 5 for $A_{n;1}^{N=1\ \mathrm{chiral}}$ in the maximally helicity-violating (MHV) configurations, and of ref. [12] for the corresponding $N = 4$ amplitudes, only the computation of the scalar contribution $A_{n;1}^{[0]}$ remains in order to obtain the full QCD amplitude for this choice of helicities.

3. A Uniqueness Result

In this section, we shall prove that a certain class of amplitudes can be determined entirely from the knowledge of their cuts; we call such amplitudes *cut-constructible*. The class consists of color-ordered one-loop amplitudes in massless theories for which a diagrammatic representation exists where all the loop-momentum integrals satisfy the following power-counting criterion: the m-point integrals have at most $m-2$ powers of the loop momentum in the numerator of the integrand, and the two-point (bubble) integrals have at most one power of the loop momentum. In other words, in the dimensionally-regulated integral

$$I_m[P(p^\mu)] = (-1)^{m+1} i\,(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\, \frac{P(p^\mu)}{p^2\,(p-K_1)^2\,(p-K_1-K_2)^2 \cdots (p+K_m)^2}\,,$$

$$(3.1)$$

where K_i are (sums of) external momenta k_i for the amplitude, the loop-momentum polynomial $P(p^\mu)$ should have degree at most $m-2$, except for $m = 2$ when it should be at most linear.

We refer to this result as a "uniqueness" result because the issue is whether the cuts uniquely determine an amplitude. An equivalent question is whether one can rule out terms in an amplitude that are pure "polynomials" — actually, rational functions — in the kinematic invariants, which cannot be detected by any cut. By inspecting the integrals appearing in cut-constructible amplitudes, we shall show that they have no such additive polynomial ambiguity.

In general, the power counting associated with a given amplitude depends on the specific gauge choice, regulator, and diagrammatic organization. In order to apply the uniqueness result, we need only find one organization of the diagrams satisfying the power-counting criterion. It is often convenient to use string-based diagrams or background-field-gauge (superspace) diagrams [29], because they can satisfy the power-counting criterion even when the corresponding diagrams in conventional Feynman gauge do not. Furthermore, amplitudes which satisfy the power-counting criterion with a supersymmetry preserving regulator [24,6] will generally not do so in the 't Hooft-Veltman [25] or conventional [26] schemes.

In a previous paper [12] we considered the case of $N = 4$ supersymmetric amplitudes, where the loop integrals satisfy an even more stringent power-counting criterion (m-point integrals have at most $m - 4$ powers of the loop-momentum). The proof of uniqueness presented here is an extension of the proof given in ref. [12]. An outline of the proof follows. We first note that any one-loop m-point tensor integral in $4 - 2\epsilon$ dimensions with $m \geq 4$ can be reduced to a combination of scalar box integrals, tensor triangle integrals, and tensor bubble integrals, using standard integral reduction formulae [30,31,32]. (A tensor integral means any integral with non-constant $P(p^\mu)$, while a scalar integral has $P(p^\mu) = 1$.) When applied to integrals obeying the power-counting criterion, the reduction formulae maintain the discrepancy of two units between the number of legs in the integral and the degree of the loop-momentum polynomial, so that only scalar box integrals and *linear* triangle and bubble integrals appear. (In the $N = 4$ case the triangle and bubble integrals were absent.) The linear bubble integrals arise directly from internal propagator diagrams, in amplitudes with external fermions. We define a set of integral functions $\mathcal{F}_n$ related to these integrals, such that any cut-constructible n-point amplitude A_n can be written as a linear combination of the elements of $\mathcal{F}_n$,

$$A_n = \sum_{i | I_i \in \mathcal{F}_n} c_i I_i \ , \tag{3.2}$$

with coefficients c_i allowed to be arbitrary *rational* functions of the momentum invariants. Finally, direct inspection of the integrals in $\mathcal{F}_n$ shows that any cut-free linear combination of these integrals must vanish, and therefore the cuts contain sufficient information to completely determine A_n.

We do not review in detail the general techniques [30,31,32] for reducing dimensionally-regulated ($m \geq 4$)-point tensor integrals to lower-point and lower-degree integrals, because we need only two particular features. First, scalar ($m > 4$)-point integrals can be reduced to a linear combination of scalar box integrals [31,32], provided that at least four of the external momentum vectors are kept in four dimensions.[†] Second, in the reduction

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[†]   There is a technical complication having to do with the number of independent kinematic variables for n-point processes in four dimensions. Momentum conservation leaves $n(n-3)/2$ kinematic invariants, e.g. the $t_i^{[r]}$ defined below, but for $n \geq 6$ only $3n - 10$ of these are actually independent in four dimensions, due to constraints imposed by the vanishing of various Gram determinants [33]. In explicit loop calculations we wish to restrict all external momenta to four dimensions, so that we can use four-dimensional tree amplitudes to evaluate cuts of the loop amplitudes. However, then the Gram-determinantal constraints would complicate our argument that the cut-constructible amplitudes are uniquely determined, which implicitly assumes that the $t_i^{[r]}$ are all independent. A solution is to restrict exactly four momenta to $D = 4$ — this is the minimum number needed for the ($m > 4$)-point reduction formulae to hold [31] — but leave the rest free in $D = 4 - 2\epsilon$, thus removing the Gram-determinantal constraints and validating the uniqueness argument.

of $(m > 4)$-point tensor integrals, the degree of the loop-momentum polynomial always shrinks along with the number of legs. The reduction procedure involves replacing a loop momentum in the numerator by a linear combination of inverse propagators $(p - p_i)^2$, which are cancelled against factors in the denominator to reduce the number of legs of the integral, plus a constant (in the loop momentum), $p^\mu = \sum_i \alpha_i^\mu (p - p_i)^2 + \alpha_0^\mu$. The reduction thus maintains the criterion of having two fewer powers of loop momentum in the integrand than the number of external legs. Reducing the (quadratic and linear) tensor box integrals in the same way [30], one arrives finally at the aforementioned linear combination of scalar box integrals, linear triangle integrals and linear bubble integrals.

As shown in appendix I, the linear bubble integrals can be reduced to the corresponding scalar bubble integrals, and the linear triangle integrals can be reduced to scalar triangle integrals plus scalar bubbles. The momenta flowing out of the legs of the integrals, K_i are sums of external momenta k_i of the original n-point integral. Assuming that the amplitude is color-ordered (as is the case for all leading-color amplitudes), the momentum invariants encountered as arguments of the integrals all involve sums of color-adjacent momenta,

$$t_i^{[r]} \equiv (k_i + k_{i+1} + \cdots + k_{i+r-1})^2, \tag{3.3}$$

where momentum labels are taken mod n. In particular, the external legs of the box, triangle, and bubble integrals may be off-shell, or massive $(K_i^2 \neq 0)$. Thus the set $\mathcal{F}_n$ of functions that can appear in color-ordered cut-constructible n-point amplitudes $(n > 4)$ is

$$\mathcal{F}_n \equiv \{I_{4:r,r',r'';i}^{4m}, I_{4:r,r';i}^{3m}, I_{4:r;i}^{2m\,h}, I_{4:r;i}^{2m\,e}, I_{4:i}^{1m}, I_{3:r,r';i}^{3m}, I_{3:r;i}^{2m}, I_{3:i}^{1m}, I_{2:r;i}\}. \tag{3.4}$$

The elements of $\mathcal{F}_n$ are given in eqs. (I.16), (I.4), (I.5), (I.8) and (I.2) and are depicted in fig. 1, fig. 2, and fig. 3. The indices labeling the ordered external momenta k_i increase in the clockwise direction in the figure. Note that these functions are not linearly independent in general.
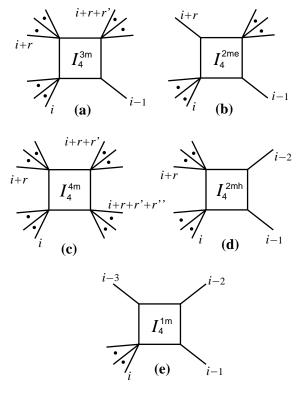
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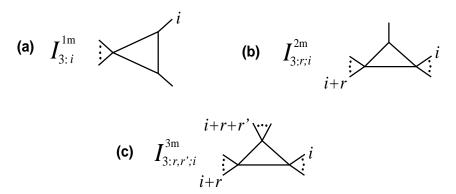Figure 1. The box integrals that may appear.

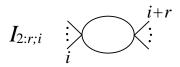Figure 2. The triangle integrals that may appear.

Figure 3. The bubble integrals that may appear.

Any cut-constructible amplitude A_n is a linear combination of functions in $\mathcal{F}_n$, as in equation (3.2), where the coefficients c_i are rational functions of the kinematic invariants, arising in part from the diagrammatic rules and in part from the integral reduction procedure. ('Kinematic invariants' here includes the spinor products [1] used to give compact forms for helicity amplitudes.) The functions in $\mathcal{F}_n$ contain logarithms and dilogarithms whose arguments depend only on the kinematic invariants $t_i^{[r]}$.

We wish to show that any such amplitude A_n is uniquely determined by its cuts. To be more precise, given two linear combinations of integral functions possessing the same cuts,

$$\sum_{i|I_i \in \mathcal{F}_n} c_i I_i \bigg|_{\text{cuts}} = \sum_{i|I_i \in \mathcal{F}_n} c_i' I_i \bigg|_{\text{cuts}} , \tag{3.5}$$

their difference must be a rational function of the momentum invariants,

$$\sum_{i|I_i \in \mathcal{F}_n} (c_i - c_i') \, I_i = \text{ rational}, \tag{3.6}$$

and we wish to show that the rational function appearing on the right-hand side of (3.6) must vanish.

A cursory examination of the set of functions shows why one might expect this to be true. The cuts uniquely determine the coefficients of the logarithms and dilogarithms, and the set of functions is dominated in content by these functions. At $\mathcal{O}(\epsilon^0)$, rational terms only appear in a small number of places — in the bubbles and in the π^2 terms in the boxes — and they are always associated with logarithms and dilogarithms.

We shall prove the result by inspecting the different channels that can possess cuts, working always at $\mathcal{O}(\epsilon^0)$, and demonstrating that the different functional dependence in the different channels either uniquely singles out one of the functions, or else singles out a set of functions lacking cut-free parts. We first illustrate the method using the four-point case. In this case,

$$\mathcal{F}_4 = \{I_4^{0\text{m}}, I_{3:3}^{1\text{m}}, I_{3:2}^{1\text{m}}, I_{2:2;1}, I_{2:2;4}\} , \tag{3.7}$$

is the set of scalar integrals depicted in fig. 4. (Other scalar integral functions are equivalent to these: $I_{3:1}^{1\text{m}} = I_{3:3}^{1\text{m}}$, etc.) As discussed in appendix I, the tensor integrals that appear in an explicit evaluation of a cut-constructible four-point amplitude, such as triangle integrals with one power of loop momentum in the numerator of their integrands, can be written as a linear combination of this set of integrals. Because we are only considering color-ordered partial amplitudes, the legs of all diagrams follow the same ordering, say, $1, 2, 3, 4$. For this simple case, we make the kinematic variable dependence explicit with

the notation

$$I_4^{0m}(s,t) \equiv I_4^{0m} \ , \qquad I_3^{1m}(s) \equiv I_{3:3}^{1m} \ , \qquad I_3^{1m}(t) \equiv I_{3:2}^{1m} \ ,$$

$$I_2(s) \equiv I_{2:2;1} \ , \qquad\qquad I_2(t) \equiv I_{2:2;4} \ , \qquad\qquad (3.8)$$

where $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$ are the usual Mandelstam variables.
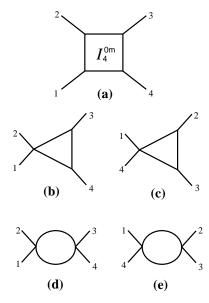

Figure 4. The integral functions that may appear in a four-point calculation.

The five integral functions in $\mathcal{F}_4$ are given explicitly in equations (I.15), (I.4) and (I.2) of appendix I. Each one contains at $\mathcal{O}(\epsilon^0)$ a logarithm or a product of logarithms unique to that function, as shown in Table 1. Therefore the only way to satisfy eq. (3.6) is to choose the coefficient of every integral to vanish, $c_i - c_i' = 0$. In other words, one cannot construct a non-vanishing cut-free massless four-point integral, and hence amplitude, that satisfies the power-counting criterion. This result applies, for example, to one-loop four-gluon amplitudes in a supersymmetric gauge theory.

	Integral	Unique Function
a	$I_4^{0m}(s,t)$	$\ln(-s)\ln(-t)$
b	$I_3^{1m}(s)$	$\ln(-s)^2$
c	$I_3^{1m}(t)$	$\ln(-t)^2$
d	$I_2(s)$	$\ln(-s)$
e	$I_2(t)$	$\ln(-t)$

Table 1: The set of integral functions that may appear in a cut-constructible massless four-point amplitude, together with the independent logarithms.

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The proof for the n-point case is similar, although more involved. We will again show, by inspecting the integral functions in $\mathcal{F}_n$, that the only solutions to (3.6) have zero right-hand side. (Note that, as in the four-point case, some of the integral functions have multiple names, so we need not count these twice; $I_{4:r;i}^{2\mathrm{m}\,e} = I_{4:n-r-2;i+r+1}^{2\mathrm{m}\,e}$, while $I_{2:r;i}$, $I_{3:r,r';i}^{3\mathrm{m}}$ and $I_{4:r,r',r'';i}^{4\mathrm{m}}$ respectively have two-, three- and four-fold naming degeneracies.) We successively inspect the integrals in kinematic regimes where one of the kinematic invariants, say s, is taken to be large, while the others remain fixed. In this regime we isolate terms of the form $\ln(-s)\ln(-s')$. The region of large kinematic invariants is useful because it simplifies the functional form of the integrals, particularly the three-mass triangle and the four-mass box. For example,

$$\mathrm{Li}_2\left(1 - \frac{ss'}{tt'}\right) \to -\ln(-s)\ln(-s') + \cdots \tag{3.9}$$

where Li_2 is the dilogarithm [34]. In these kinematic regimes the terms involving square-root arguments also reduce to simple products of logarithms. In taking certain kinematic invariants to be large, we do not take the limit of infinite s, but keep the full functional form in the coefficients $c_i - c_i'$.

With successive judicious choices of s one can isolate a single coefficient at a time from the sum, for certain of the functions in $\mathcal{F}_n$; that is only a single integral contains the chosen $\ln(-s)\ln(-s')$, after noting the vanishing of prior coefficients. The coefficient of the isolated function must therefore be zero as well, in order to obtain a cut-free result (3.6). We display in table 2 the order in which we examine the integrals for large $t_i^{[r]}$ and the product of logarithms thereby isolated.

Consider the first entry, the three-mass box $I_{4:r,r';i}^{3\mathrm{m}}$. For a color-ordered amplitude with the ordering of legs shown in fig. 1a we examine the regime where the kinematic variable $t_i^{[r]}$ is large. The combination $\ln(-t_i^{[r]})\ln(-t_{i+r+r'}^{[n-r-r'-1]})$ appears only in the single integral function $I_{4:r,r';i}^{3\mathrm{m}}$; in fact, these two kinematic variables appear together only in this function. (For non-color ordered amplitudes more functions can contain this combination, complicating the situation.) One then deduces that the coefficients of the three-mass boxes in eq. (3.6) are zero. The ordering of the table is such that at each stage only a single function contains the pairs s and s' together. In checking the behavior for large kinematics one merely checks that the coefficient of $\ln(-s)\ln(-s')$ is nonzero, thereby requiring the corresponding difference of coefficients $c_i - c_i'$ to be set to zero. We can then proceed through the table, at each turn forcing additional differences of coefficients to vanish.

Upon reaching the end of the table, we are left with three kinds of integrals: one- and two-mass triangle integrals, and bubble integrals. The former two contain no rational

pieces at all at $\mathcal{O}(\epsilon^0)$, but only logarithms squared; whatever linear combination makes the logarithms vanish eliminates these integrals entirely. This leaves us only with bubble integrals; while these do contain rational pieces at $\mathcal{O}(\epsilon^0)$, requiring the coefficient of each of the single powers of logarithms of independent invariants to vanish eliminates the bubble integrals, leaving us with the desired result, that the rational function on the right-hand side of equation (3.6) indeed vanishes.

	Integral	Unique Function
a	$I^{3\mathrm{m}}_{4:r,r';i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r+r'}^{[n-r-r'-1]})$
b	$I^{2\mathrm{m}\,e}_{4:r;i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r+1}^{[n-r-2]})$
c	$I^{4\mathrm{m}}_{4:r,r',r'';i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r+r'}^{[r'']})$
d	$I^{2\mathrm{m}\,h}_{4:r;i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r}^{[n-r-1]})$
e	$I^{1\mathrm{m}}_{4;i}$	$\ln(-t_i^{[r]})\ln(-t_i^{[r+1]})$
f	$I^{3\mathrm{m}}_{3:r,r';i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r}^{[r']})$

Table 2: Following the ordering shown and taking large $t_i^{[r]}$ makes the proof of uniqueness of the cuts straightforward.

For integrals and thus amplitudes that fail to satisfy the power-counting criterion given at the beginning of the section, for example m-point loop diagrams containing terms with m powers of the loop momentum, the result should not be expected to hold, and in fact can be shown explicitly to be violated. Here, after a Passarino-Veltman reduction, one has additional functions beyond those mentioned previously, for example bubble integrals with up to *two* powers of the loop momentum in the numerator of the integrand. Denote the loop momentum by p^μ; the bubble integrals with loop-momentum polynomial 1 and $p^\mu p^\nu$, defined in equation (I.1), are respectively

$$
\begin{aligned}
I_2[1](K) &= \frac{r_\Gamma}{\epsilon(1-2\epsilon)}(-K^2)^{-\epsilon}\,, \\
I_2[p^\mu p^\nu](K) &= \frac{r_\Gamma(-K^2)^{-\epsilon}}{\epsilon(1-2\epsilon)}\left[\frac{K^\mu K^\nu}{3}\left(1+\frac{\epsilon}{6}\right) - \frac{\eta^{\mu\nu}K^2}{12}\left(1+\frac{2\epsilon}{3}\right)\right],
\end{aligned}
\tag{3.10}
$$

where K^μ is the momentum flowing out one side of the bubble. The linear combination

$$
\left(\frac{K^\mu K^\nu}{3} - \frac{\eta^{\mu\nu}K^2}{12}\right)I_2[1](K) - I_2[p^\mu p^\nu](K) = -\frac{r_\Gamma}{18}(K^\mu K^\nu - \eta^{\mu\nu}K^2)
\tag{3.11}
$$

provides a solution to eq. (3.6). The presence of such a combination within the amplitude thus cannot be determined from the cuts. Such combinations of integrals do occur in

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QCD calculations. Simple examples are the four-gluon amplitudes [6] where all external helicities are the same or where one leg is of opposite helicity. In these two cases the cuts all vanish but the amplitudes do not; furthermore, they are not equal, showing that the cuts in this case do not uniquely determine the amplitude. (The integrals in these amplitudes do violate the required power-counting criterion.)

4. Supersymmetric Theories

In this section, we show that the power-counting criterion of section 3 is satisfied by all color-ordered one-loop amplitudes in massless supersymmetric gauge theories with no superpotential. Thus such amplitudes can be computed solely from the knowledge of their cuts. Color-ordering is irrelevant to the power-counting, but it is required for the uniqueness argument of section 3 to be valid. The leading-color contributions to a given amplitude in $SU(N_c)$ (super-) gauge theory are always color-ordered, and in many cases the subleading-color contributions can be expressed as sums over permutations of color-ordered objects [12,13,35]. While we expect that the result extends to the full amplitude, as well as to amplitudes in all massless supersymmetric theories (i.e. including nonvanishing terms in the superpotential), we shall not present an argument for such an extension here.

We shall use ordinary (as opposed to superspace) diagrams, and background-field gauge. We always assume the use of a supersymmetric regulator [24,6]. Use of ordinary diagrams reveals that the power-counting criterion does not always require supersymmetric cancellations, and one can see how to apply it to certain nonsupersymmetric calculations as well. (At the end of the section we sketch how a superspace argument would proceed.) The presence of trees attached to the loop does not change the power-counting of the loop integrand; hence we amputate external trees in favor of taking legs off-shell, that is we restrict our attention to the computation of the one-loop effective action.

If all external legs are gluons, we can write a compact determinantal formula for the color-ordered (and color-stripped) effective action (one should think of this object as the generating functional for the amputated color-ordered diagrams),

$$\Gamma^{\text{SUSY}} = \ln \det {}_{[1]}^{-1/2}[D^2\eta_{\alpha\beta} - g(\Sigma_{\mu\nu})_{\alpha\beta}F^{\mu\nu}] + \ln \det {}_{[0]}[D^2] + \frac{1}{2}\ln \det {}_{[1/2]}^{1/2}[D^2 - \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}]$$
$$+ n_m\left(\frac{1}{2}\ln \det {}_{[1/2]}^{1/2}[D^2 - \frac{g}{2}\sigma_{\mu\nu}F^{\mu\nu}] + \ln \det {}_{[0]}^{-1}[D^2]\right),$$

(4.1)

where $\det {}_{[J]}$ is the one-loop determinant for a particle of spin J in the loop. This formula assumes two-component fields, the fermions being taken to be Majorana to allow a

definition of the determinant; D is the covariant derivative, n_m is the number of $N = 1$ chiral matter multiplets, $\frac{1}{2}\sigma_{\mu\nu}$ ($\Sigma_{\mu\nu}$) are the spin-$\frac{1}{2}$ (spin-1) Lorentz generators, and we have used the fact that the contribution of a Weyl fermion in a non-chiral theory is half that of a Dirac fermion. The fermion determinant is written in a second-order form [20], making the similarity of the fermion and gluon loop contributions clear. Note that the gluon determinant contains Lorentz indices and the fermion determinant spinor indices, and the effective action implicitly includes traces over these indices.

We now extract the m-point contribution by differentiating m times with respect to the background field, thereby expanding the logarithms in equation (4.1). It is convenient to think of performing this expansion in two stages: first, expanding with respect to the field strength $F^{\mu\nu}$, and only then expanding with respect to the background field. Expanding the determinants,

$$\ln \det{}_{[1]}^{-1/2}[D^2 \eta_{\alpha\beta} - g(\Sigma_{\mu\nu})_{\alpha\beta} F^{\mu\nu}] = -2 \ln \det{}_{[0]}[D^2] + \cdots ,$$
$$\frac{1}{2} \ln \det{}_{[1/2]}^{1/2}[D^2 - \frac{g}{2}\sigma_{\mu\nu} F^{\mu\nu}] = +\ln \det{}_{[0]}[D^2] + \cdots ,$$

(4.2)

in the first stage of the expansion it is clear that terms with no Fs cancel within each supermultiplet. Furthermore, terms with a lone F also vanish, because the trace (over Lorentz indices) of the accompanying Lorentz generator (σ or Σ) vanishes. Thus each term must contain at least two Fs. Whereas the terms generated by expanding D^2 ($A \cdot \partial$, etc.) may have up to one power of the loop momentum per background field, F does not contain any powers of the loop momentum, but rather only powers of external momenta. Thus each insertion of F reduces by one the number of powers of loop momenta in the numerator of the loop integrand, from a maximum of m to a maximum of $m - 2$. We conclude that the effective action for external gluons, and hence the amplitudes whose external legs are all gluons, satisfies the desired power-counting criterion in a supersymmetric theory. (For amplitudes containing only external gluons, formula (7.2) of ref. [12] ensures that the entire amplitude — subleading as well as leading-color terms — may be determined in this fashion.)

Next consider one-particle irreducible graphs with multiple pairs of external fermion lines. For each pair of external fermions there is one fermion line running along the loop. Using the standard first-order formalism for the fermions, the interaction vertex for the fermion line with a gauge boson or a scalar does not contain the loop momentum ℓ^μ, but the rationalized Dirac propagator ($\sim \ell\!\!\!/ / \ell^2$) contributes one power of the loop momentum to the numerator. Thus each fermion line running along the loop leads to a reduction of the maximum degree of the loop-momentum polynomial by one. Graphs with two or

more pairs of external fermions automatically satisfy the power-counting criterion, without requiring any supersymmetric cancellation.

Graphs with only one pair of external fermion lines require us to find a reduction of the degree of the loop-momentum polynomial by one more unit. If there is also (at least) one external scalar line, this can be achieved within a single diagram. For example, if the scalar couples directly to the fermion line running around the loop, then there is a factor of the form

$$\ell\!\!\!/(a + b\gamma_5)(\ell\!\!\!/ - k\!\!\!/_s) \;=\; (a - b\gamma_5)\ell^2 \;+\; \mathcal{O}(\ell)\,, \tag{4.3}$$

from the Yukawa coupling and adjacent propagators. We can use the factor of ℓ^2 to cancel one of the propagators in the denominator of the m-point integral, thus converting it to an $(m-1)$-point integral with a loop polynomial of maximum degree $m - 3 = (m-1) - 2$. (One power is lost due to the external fermion pair, and one due to the ℓ^2 cancellation.) The power-counting criterion is then satisfied. If the scalar does not couple directly to the fermion line, then in the leading loop-momentum terms it is still possible to find a factor of ℓ^2 to cancel against the propagator, again establishing the criterion.

If there are no external scalars, but just two external fermions and $m - 2$ gluons, then supersymmetric cancellations are required. For example in the pure super-Yang-Mills case with only gluinos and gluons, the graphs that cancel are shown in fig. 5. Since the maximum degree of the loop momentum polynomial is $m - 1$ for each graph, we only need to show that they cancel to leading order in ℓ. It is straightforward to show this by repeated use of the identity

$$\ell\!\!\!/\gamma_\mu\ell\!\!\!/ \;=\; 2\ell_\mu\ell\!\!\!/ - \gamma_\mu\ell^2. \tag{4.4}$$

The second term in (4.4) can be ignored by the cancelled-propagator argument used above, while the first term has converted a gluon emission off a fermion line into the form of a gluon emission off a scalar line, or off a gluon line. (In background-field gauge, the latter two vertices have the same form to leading order in ℓ.) In this way the two graphs where the fermion line runs around the loop two different ways can be made to look identical at leading order, apart from an overall minus sign, so that the sum cancels. This argument breaks down if there are no external gluons, i.e. just a fermion propagator bubble, because there is no second diagram to cancel against. However such a linear bubble is permitted by the uniqueness result.
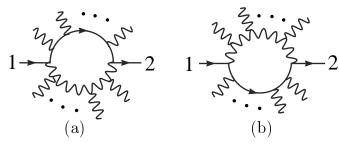
Figure 5. Two graphs with external fermions between which cancellations occur in satisfying the power-counting criterion.

Finally there are the one-particle irreducible graphs with no external fermions, but including (two or more) external scalars. For a fermion line running all the way around the loop, the power-counting criterion is satisfied (with one exception) by virtue of equation (4.3) applied twice, which reduces a degree-m m-point integral to a degree $m - 4$ $(m - 2)$-point integral. The exception is for exactly two *adjacent* external scalars; in this case a supersymmetric cancellation occurs against corresponding graphs with gluons and scalars in the loop. The graphs with gluons and scalars in the loop can also be shown to satisfy the criterion without invoking supersymmetry (with the same exception); however, here a few graphs with four-point vertices must be added to the "parent diagram" with only three-point vertices.

Supersymmetric cancellations of the type described above should be manifest in superspace. For n-gluon amplitudes in an $N = 4$ supersymmetric theory, where only a vector superfield V circulates in the loop, a superspace argument is straightforward to construct [29,12]. With chiral superfields, the chiral constraints lead to large numbers of supercovariant derivatives D_α and $\overline{D}_{\dot\alpha}$ in the supergraphs, which should be removed by D-algebra, because they correspond to high-degree loop-momentum polynomials. In brief, the commutator $[D^2, \overline{D}^2] \propto \partial^2$ can be used to cancel a propagator just as in the ordinary diagram analysis performed above. Combining this fact with D-integration-by-parts, in order to remove D's on the loop in favor of D's on external legs, one can drastically reduce the degree of the loop-momentum polynomials, and it should be possible to verify the power-counting criterion in this way too.

In conclusion we have shown that in a supersymmetric theory the "cut-constructible" conditions may be met. We have avoided the use of elegant superspace arguments in order to illustrate how components of non-supersymmetric amplitudes can satisfy the conditions also.

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5. All-Multiplicity $N = 1$ Supersymmetric MHV Amplitudes

In this section we apply the unitarity result of the previous section to obtain explicit formulas for "maximally helicity-violating" (MHV) gluon amplitudes in an $N = 1$ supersymmetric gauge theory. Recall that, in the convention where all momenta are taken as outgoing, supersymmetric all-gluon amplitudes where all the gluons, or all but one, have identical helicities, vanish on account of a supersymmetry Ward identity [36],

$$A_n^{\text{SUSY}}(1^\pm, 2^+, \ldots, n^+) = 0 . \tag{5.1}$$

The MHV amplitudes — with two opposite-helicity gluons — are thus the simplest non-vanishing amplitudes we can consider. We shall denote these MHV n-gluon amplitudes by

$$A_{ij}(1, 2, \ldots, n) \equiv A_{n;1}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) . \tag{5.2}$$

The corresponding amplitudes with predominantly negative helicities may of course be obtained by complex conjugation.

In previous work [12], we computed the amplitudes with this helicity configuration in an $N = 4$ supersymmetric theory. In this section, we present the result for the $N = 1$ supersymmetric chiral matter multiplet contribution to the n-gluon MHV amplitudes. The contribution to an n-gluon amplitude from any supersymmetric multiplet is a linear combination of the $N = 4$ amplitude and $N = 1$ chiral multiplet contributions. For example, using eq. (2.11), the amplitude in a pure $N = 1$ supersymmetric nonabelian gauge theory (a theory with just the vector supermultiplet, containing a gluon and a gluino) is given by the linear combination

$$A_{n;1}^{N=1 \text{ vector}}(1, 2, \ldots, n) = A_{n;1}^{N=4}(1, 2, \ldots, n) - 3A_{n;1}^{N=1 \text{ chiral}}(1, 2, \ldots, n). \tag{5.3}$$

As discussed in section 2, the contribution of an $N = 1$ supersymmetric chiral multiplet contribution is also one of the three components of a QCD n-gluon amplitude. To obtain the gluon scattering amplitude in QCD we need one further contribution, essentially the contribution of an internal scalar loop; this remaining component contains rational functions of the invariants that cannot be fixed by the cut techniques discussed in this paper.

Consider the cut in the channel $t_{m_1}^{[m_2-m_1+1]} \equiv (k_{m_1} + k_{m_1+1} + \cdots + k_{m_2-1} + k_{m_2})^2$ for the loop amplitude $A_{n;1}^{N=1 \text{ chiral}}(1, 2, \ldots, n)$, depicted in fig. 6 and given by [37]
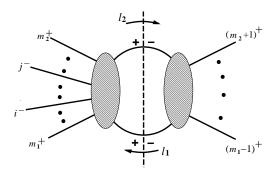
$$i \int d^D \text{LIPS}(-\ell_1, \ell_2) \, A^{\text{tree}}(-\ell_1, m_1, \ldots, m_2, \ell_2) \, A^{\text{tree}}(-\ell_2, m_2 + 1, \ldots, m_1 - 1, \ell_1) , \tag{5.4}$$

where the integration is over D-dimensional Lorentz-invariant phase-space. The intermediate states with momenta ℓ_1 and ℓ_2 are fermions or scalars, the remaining external states are gluons. The overall sign is unimportant, since it is a simple matter to fix the overall sign in final expressions from the known universal ultraviolet and infrared singularities [28]. Instead of evaluating the phase-space integrals we evaluate the off-shell integral

$$\int \frac{d^D \ell_1}{(2\pi)^D} \; A^{\text{tree}}(-\ell_1, m_1, \ldots, m_2, \ell_2) \frac{1}{\ell_2^2} \; A^{\text{tree}}(-\ell_2, m_2+1, \ldots, m_1-1, \ell_1) \frac{1}{\ell_1^2}\bigg|_{\text{cut}}. \quad (5.5)$$

whose cut in this channel is (5.4). This replacement is valid only in this channel. In evaluating this off-shell integral, we may substitute $\ell_1^2 = \ell_2^2 = 0$ in the numerator; terms with ℓ_1^2 or ℓ_2^2 in the numerator do not produce a cut in this channel because the ℓ_1^2 or ℓ_2^2 cancels a cut propagator. We emphasize that the cuts are evaluated not for a lone diagram at a time, but for the whole amplitude.
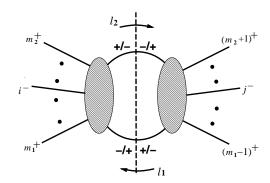

Figure 6. The cuts needed to obtain the MHV amplitudes.

In quoting final results for the amplitudes we do so in the dimensional reduction [24] or 'four-dimensional helicity' [6] schemes (which for practical purposes are equivalent at

one-loop). In these schemes, after using Passarino-Veltman reduction [30], the coefficients of all integral functions do not depend on ϵ since all tensor and spinor manipulations are performed in four dimensions. In reconstructing the full amplitudes from cuts we therefore take the coefficients to be free of ϵ, and automatically obtain results in this scheme. This scheme choice has the additional advantage of preserving supersymmetry, which is necessary for the power-counting criterion to be satisfied. At the end of the calculation, one can convert to more conventional schemes by accounting for the $[\epsilon]$-scalars as discussed near eq. (2.12).

There is one subtlety we must address. We perform the cut integration in $4 - 2\epsilon$ dimensions, yet the tree amplitudes on either side are evaluated in four dimensions. Since the calculation contains divergences, one might worry that this $\mathcal{O}(\epsilon)$ discrepancy in the integrand might lead to errors in the final result. In fact, this is not the case, and in performing the cut calculations through $\mathcal{O}(\epsilon^0)$ there is no need to track which momenta in the numerator are four-dimensional and which are $(4 - 2\epsilon)$-dimensional. (Momenta in the denominator must of course be $(4 - 2\epsilon)$-dimensional to regulate singularities.) Ultraviolet and non-overlapping infrared singularities lead only to single poles in ϵ; in this case an $\mathcal{O}(\epsilon)$ error in the numerator of eq. (5.5) could only change the final result by a 'polynomial' at $\mathcal{O}(\epsilon^0)$. Such singularities thus do not affect the cuts. The case of overlapping soft and collinear singularities is a bit more subtle, since here there is a double pole in ϵ, and correspondingly there are $\ln(s)/\epsilon$-type terms. One might worry that these can 'feed down' and affect the $\mathcal{O}(\epsilon^0)$ cuts. To see that this is not the case, one may employ the same prescription for handling dimensional regularization with spinor helicity advocated by Mahlon [38,4]. This prescription separates the integration in (5.5) as follows,

$$\int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} = -\frac{\epsilon(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \int \frac{d^4\ell}{(2\pi)^4} \int_0^\infty d\ell_\epsilon^2 (\ell_\epsilon^2)^{-1-\epsilon} \ . \tag{5.6}$$

A discrepancy caused by cavalier handling of ℓ_ϵ contributions leads to a discrepancy containing an explicit ℓ_ϵ in the numerator. A lone power of ℓ_ϵ will lead to a vanishing integral, because all external momenta with which ℓ_ϵ can be contracted may be taken to be four-dimensional. Integrals with two or more extra powers of ℓ_ϵ in the numerator have only ultraviolet, not infrared, singularities (and can be interpreted as integrals in $D > 4$ [39]). Thus, the 'error' induced by incorrect ℓ_ϵ terms in the numerator is of the form $\epsilon \times (1/\epsilon) \times$ polynomial, which we can again neglect since we are extracting the cuts alone to order ϵ^0. (Errors in the spinor helicity manipulations due to a mismatch of dimensions do introduce errors in the finite cut-free polynomials; explicit examples of this phenomenon have been provided by Mahlon [4,5].)

To evaluate the integral (5.5) we need explicit formulae for the tree amplitudes, preferably in a compact form. To obtain the $N = 1$ chiral multiplet contributions we need formulae for tree amplitudes with two external fermions or scalars and up to $n-2$ external gluons. For the helicity configuration (5.2) the necessary tree-level results exist. There are two cases to consider, as shown in fig. 6. In the first case, the two negative helicities are on the same side of the cut. Here the cut vanishes since there is no assignment of helicity for the intermediate scalars or fermions where the tree on the right-side does not vanish. Tree amplitudes with all positive-helicity gluons and two external fermions or scalars vanish by helicity conservation if the two (outgoing) fermions or scalars have the same helicity, and vanish by a supersymmetry Ward identity [36] if they have opposite helicity. (For complex scalars the two 'helicities' mean particles and antiparticles respectively.) The second case, where the two external negative helicities are on opposite sides of the cut, has two possible intermediate helicity configurations as shown in fig. 6b. The tree amplitudes on either side of the cut are the MHV amplitudes with two external fermions (Λ) or scalars (ϕ). These can be obtained using the supersymmetry Ward identities [36]

$$
\begin{aligned}
A^{\text{tree}}(\Lambda_1^-, g_2^+, \ldots, g_j^-, \ldots, \Lambda_n^+) &= \frac{\langle j\, n\rangle}{\langle j\, 1\rangle}\, A^{\text{tree}}(g_1^-, g_2^+, \ldots, g_j^-, \ldots, g_n^+), \\
A^{\text{tree}}(\phi_1^-, g_2^+, \ldots, g_j^-, \ldots, \phi_n^+) &= \frac{\langle j\, n\rangle^2}{\langle j\, 1\rangle^2}\, A^{\text{tree}}(g_1^-, g_2^+, \ldots, g_j^-, \ldots, g_n^+),
\end{aligned}
\tag{5.7}
$$

where '$\ldots$' denotes positive-helicity gluons and the Parke-Taylor n-gluon MHV tree amplitudes [9,40] are

$$
A_{ij}^{\text{tree}}(1, 2, \ldots, n) = i\, \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}\,.
\tag{5.8}
$$

The result (5.8) is written in terms of spinor inner-products, $\langle j\, l\rangle = \langle j^-|l^+\rangle = \bar{u}_-(k_j)u_+(k_l)$ and $[j\, l] = \langle j^+|l^-\rangle = \bar{u}_+(k_j)u_-(k_l)$, where $u_\pm(k)$ is a massless Weyl spinor with momentum k and chirality $\pm$ [1,41]. We thus have explicit formulae for all the required tree amplitudes.

The evaluation of the cuts is similar to (although somewhat more complicated than) the evaluation of the cuts for the $N = 4$ MHV amplitudes presented in ref. [12]. In performing the calculation there is no need to track the overall sign, which is fixed by the pole terms in ϵ. The off-shell integral reduces to a sum of scalar box and triangle integrals; we shall not present the details of this reduction.

Let us consider first the special case where the two negative-helicity gluons are adjacent. In this case the calculation leads to a result which does not contain box integral functions and is particularly simple. For convenience we take the negative-helicity legs to

be the first and second ($i = 1$ and $j = 2$), obtaining for the unrenormalized amplitude [13],

$$
A_{1\,2}^{N=1 \text{ chiral}}(1,\ldots,n) = \frac{c_\Gamma(\mu^2)^\epsilon A_{1\,2}^{\text{tree}}(1,\ldots,n)}{2} \left\{ \left(\mathrm{K}_0(t_2^{[2]}) + \mathrm{K}_0(t_n^{[2]}) \right) \right.
$$
$$
\left. - \frac{1}{t_1^{[2]}} \sum_{m=4}^{n-1} \frac{\mathrm{L}_0\left(-t_2^{[m-2]}/(-t_2^{[m-1]}) \right)}{t_2^{[m-1]}} \left(\mathrm{tr}_+[\not{k}_1\not{k}_2\not{k}_m\not{q}_{m,1}] - \mathrm{tr}_+[\not{k}_1\not{k}_2\not{q}_{m,1}\not{k}_m] \right) \right\}
$$

$$(5.9)$$

where K_0 and L_0 are integral functions defined in appendix II,

$$
q_{m,l} = \begin{cases} \displaystyle\sum_{i=m}^{l} k_i, & m \le l, \\[2ex] \displaystyle\sum_{i=m}^{n} k_i + \sum_{i=1}^{l} k_i, & m > l, \end{cases}
$$

$$(5.10)$$

μ is the renormalization scale,

$$
\mathrm{tr}_+[\rho] \equiv \frac{1}{2} \mathrm{tr}[(1+\gamma_5)\rho] \; ;
$$

$$(5.11)$$

and c_Γ was defined in equation (2.8).

The expression (5.9) has the correct cuts in all channels, as does the more general expression (5.12) we present below. Since it is written in terms of the integral functions, K_0 and L_0, the unitarity result of section 3 implies it is the complete answer including all rational, cut-free parts. Note that the L_0's appearing in equation (5.9) do not form an independent set, and therefore the result may be expressed in a number of seemingly inequivalent ways. We have chosen the form in equation (5.9) in order to make the reflection (anti-)symmetry of this color-ordered amplitude for n even (odd) manifest. We give a schematic representation of this amplitude in fig. 7, where the coefficients c_m are the ones appearing in eq. (5.9). (As discussed in appendix II the function L_0 can be regarded as arising from a two mass triangle integral.)

Figure 7. Schematic representation of the $N = 1$ chiral amplitude with legs 1 and 2 negative helicities and the rest positive helicities. For the boundary terms ($m = 3, n$), L_0 reduces to K_0.

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The general case in which the two negative-helicity legs need not be adjacent yields a more complicated expression. Choose the first leg to be a negative-helicity leg ($i = 1$) and label the second negative-helicity leg by j; the color-ordered amplitude is then

$$A_{1j}^{N=1 \text{ chiral}}(1, \ldots, n) = \frac{c_\Gamma (\mu^2)^\epsilon A_{1j}^{\text{tree}}(1, \ldots, n)}{2}$$

$$\times \left\{ \sum_{m_1=2}^{j-1} \sum_{m_2=j+1}^{n} b_{m_1,m_2}^j \, \text{M}_0 \left(t_{m_1+1}^{[m_2-m_1]}, t_{m_1}^{[m_2-m_1]}; t_{m_1+1}^{[m_2-m_1-1]}, t_{m_2+1}^{[n+m_1-m_2-1]} \right) \right.$$

$$+ \sum_{2 \le m < j} \sum_{a \in \hat{\mathcal{C}}_m} c_{m,a}^j \frac{\text{L}_0((-t_{m+1}^{[a-m]})/(-t_m^{[a-m+1]}))}{t_m^{[a-m+1]}}$$

$$+ \sum_{j < m \le n} \sum_{a \in \mathcal{C}_m} c_{m,a}^j \frac{\text{L}_0((-t_{a+1}^{[m-a]})/(-t_{a+1}^{[m-a-1]}))}{t_{a+1}^{[m-a-1]}}$$

$$\left. + \frac{c_{2,n}^j}{t_1^{[2]}} \, \text{K}_0(t_1^{[2]}) + \frac{c_{n,1}^j}{t_n^{[2]}} \, \text{K}_0(t_n^{[2]}) + \frac{c_{j+1,j-1}^j}{t_j^{[2]}} \, \text{K}_0(t_j^{[2]}) + \frac{c_{j-1,j}^j}{t_{j-1}^{[2]}} \, \text{K}_0(t_{j-1}^{[2]}) \right\}$$

(5.12)

where the various integral functions are defined in appendix II and

$$b_{m_1,m_2}^j = -2 \frac{\text{tr}_+[\not{k}_1 \not{k}_j \not{k}_{m_1} \not{k}_{m_2}] \, \text{tr}_+[\not{k}_1 \not{k}_j \not{k}_{m_2} \not{k}_{m_1}]}{[(k_1 + k_j)^2]^2 \, [(k_{m_1} + k_{m_2})^2]^2}$$

$$= 2 \frac{\langle 1\, m_1 \rangle \langle 1\, m_2 \rangle \langle j\, m_1 \rangle \langle j\, m_2 \rangle}{\langle 1\, j \rangle^2 \langle m_1\, m_2 \rangle^2} ,$$

(5.13)

$$c_{m,a}^j = \frac{(\text{tr}_+[\not{k}_1 \not{k}_j \not{k}_m \not{q}_{m,a}] - \text{tr}_+[\not{k}_1 \not{k}_j \not{q}_{m,a} \not{k}_m])}{[(k_1 + k_j)^2]^2} \left(\frac{\text{tr}_+[\not{k}_m \not{k}_{a+1} \not{k}_j \not{k}_1]}{(k_m + k_{a+1})^2} - \frac{\text{tr}_+[\not{k}_m \not{k}_a \not{k}_j \not{k}_1]}{(k_m + k_a)^2} \right) ,$$

$$= \frac{(\text{tr}_+[\not{k}_1 \not{k}_j \not{k}_m \not{q}_{m,a}] - \text{tr}_+[\not{k}_1 \not{k}_j \not{q}_{m,a} \not{k}_m])}{[(k_1 + k_j)^2]^2} \langle m\, 1 \rangle [1\, j] \langle j\, m \rangle \, \mathcal{S}_m(a, a+1) ,$$

(5.14)

$$\mathcal{S}_k(i, j) = \frac{\langle i\, j \rangle}{\langle i\, k \rangle \langle k\, j \rangle} ,$$

(5.15)

$$\mathcal{C}_m = \begin{cases} \{1, 2, \ldots, j-2\}, & m = j+1, \\ \{1, 2, \ldots, j-1\}, & j+1 < m < n, \\ \{2, \ldots, j-1\}, & m = n, \end{cases} \qquad \hat{\mathcal{C}}_m = \begin{cases} \{j, j+1, \ldots, n-1\}, & m = 2, \\ \{j, j+1, \ldots, n\}, & 2 < m < j-1, \\ \{j+1, j+2, \ldots, n\}, & m = j-1. \end{cases}$$

(5.16)

The reader may verify that equation (5.12) reduces to equation (5.9) for $j = 2$. These equations agree for $n = 5$ with previously published results [8]; note that in equations (5.9) and (5.12), we have effectively combined V_n and F_n / A_n^{tree}.

We may observe that

$$\frac{c_{2,n}^j}{t_1^{[2]}} + \frac{c_{n,1}^j}{t_n^{[2]}} = 1\,,$$

$$\frac{c_{j+1,j-1}^j}{t_j^{[2]}} + \frac{c_{j-1,j}^j}{t_{j-1}^{[2]}} = 1\,,$$

(5.17)

and that as a result, equation (5.12) has the correct $1/\epsilon$ singularity since the pole terms, contained in the K_0 functions, are

$$c_\Gamma \frac{A_{1j}^{\mathrm{tree}}(1,\ldots,n)}{2} \left(\frac{c_{2,n}^j}{t_1^{[2]}} + \frac{c_{n,1}^j}{t_n^{[2]}} + \frac{c_{j+1,j-1}^j}{t_j^{[2]}} + \frac{c_{j-1,j}^j}{t_{j-1}^{[2]}} \right) \frac{1}{\epsilon}$$

$$= c_\Gamma \frac{1}{\epsilon} A_{ij}^{\mathrm{tree}}(1,\ldots,n)\,.$$

(5.18)

The amplitude may be described in terms of a set of $(D = 6)$ scalar two-mass box integrals and linear two mass triangle integrals (plus the boundary terms of single mass integrals) shown schematically in fig. 8. In appendix III, we check the collinear limits of these expressions (indeed, equation (5.9) was first constructed from a knowledge of these limits).
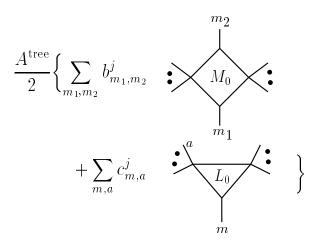
Figure 8. Schematic representation of the general $N = 1$ chiral loop amplitude with two negative helicity external legs.

6. $N = 4$ Supersymmetric Non-MHV Six-Gluon Amplitudes

We calculated the $N = 4$ supersymmetric MHV one-loop n-gluon amplitudes $A_{jk}^{N=4}(1,\ldots,n)$ in a previous paper [12]. They are proportional to the tree amplitude,

$$A_{jk}^{N=4}(1,\ldots,n) \;=\; c_\Gamma \times A_{jk}^{\mathrm{tree}}(1,\ldots,n) \times V_n^g(t_i^{[r]})\,,$$

(6.1)

where the "universal", cyclically symmetric function V_n^g is independent of the locations j, k of the negative helicities, and contains no spinor inner products, but only the momentum invariants $t_i^{[r]}$. (For the $N = 4$ four-point amplitude this simple structure was first observed by Green, Schwarz and Brink, who obtained the amplitude as the low-energy limit of a superstring amplitude [42].) One might wonder whether the *non*-MHV $N = 4$ supersymmetric amplitudes also have this structure. Non-MHV helicity configurations first appear in six-point amplitudes, for which there are three distinct configurations, $(+ + + - - -)$, $(+ + - + - -)$, and $(+ - + - + -)$. In this section we calculate these three $N = 4$ supersymmetric six-gluon amplitudes from their unitarity cuts. We will find that the structure of non-MHV amplitudes is more elaborate than that in eq. (6.1).

From our previous work [12] we know that $N = 4$ amplitudes can be expressed as linear combinations of scalar box integrals only. It will turn out that the coefficient of each box integral in each of the three non-MHV helicity configurations can be obtained (using various symmetries) from the cut in the t_{123} channel $[t_{ijl} = (k_i + k_j + k_l)^2]$ of the first configuration, $A_{6;1}^{N=4}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-)$. We thus begin by evaluating this cut.

6.1 The t_{123} cut of $A_{6;1}^{N=4}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-)$

Using the supersymmetry Ward identities at tree-level, it is easy to see that only the gluon loop contributes to this cut, and in fact the cut is given by a product of two five-point, pure-glue, MHV tree amplitudes. However, unlike the cuts in MHV loop amplitudes evaluated in ref. [12] and in section 5, in this case one of the MHV amplitudes is complex conjugated, since it is for the helicity configuration $(+ + - - -)$ instead of $(- - + + +)$. Thus the t_{123} cut is given by the cut hexagon integral

$$
\begin{aligned}
C_{123} &\equiv i \int d^D\text{LIPS}(-\ell_1, \ell_2)\, A_5^{\text{tree}}((-\ell_1)^-, 1^+, 2^+, 3^+, \ell_2^-)\, A_5^{\text{tree}}((-\ell_2)^+, 4^-, 5^-, 6^-, \ell_1^+) \\
&= -\frac{i}{\langle 1\,2 \rangle\, \langle 2\,3 \rangle\, [4\,5]\, [5\,6]} \int d^D\text{LIPS}(-\ell_1, \ell_2) \frac{\langle \ell_1\, \ell_2 \rangle^3 [\ell_1\, \ell_2]^3}{\langle \ell_1\, 1 \rangle\, \langle 3\, \ell_2 \rangle\, [\ell_2\, 4]\, [6\, \ell_1]} \\
&= -\frac{i(t_{123})^3}{\langle 1\,2 \rangle\, \langle 2\,3 \rangle\, [4\,5]\, [5\,6]} \int d^D\text{LIPS}(-\ell_1, \ell_2) \frac{[3\, \ell_2]\, \langle \ell_2\, 4 \rangle\, [1\, \ell_1]\, \langle \ell_1\, 6 \rangle}{(\ell_1 - k_1)^2 (\ell_2 + k_3)^2 (\ell_2 - k_4)^2 (\ell_1 + k_6)^2},
\end{aligned}
$$

(6.2)

using $\langle \ell_1\, \ell_2 \rangle\, [\ell_1\, \ell_2] = (\ell_1 - \ell_2)^2 = t_{123}$. (We suppress an overall factor of $\mu^{2\epsilon}$.)

There are several ways to reduce this cut hexagon integral to a linear combination of cut scalar box integrals. We choose to Feynman parametrize the integral, letting

$$
\begin{aligned}
\ell_1 &= q + a_2 k_1 + a_3(k_1 + k_2) + a_4(k_1 + k_2 + k_3) - a_5(k_5 + k_6) - a_6 k_6, \\
\ell_2 &= \ell_1 - k_1 - k_2 - k_3,
\end{aligned}
$$

(6.3)

with $\sum_{i=1}^{6} a_i = 1$. The numerator polynomial in eq. (6.2) becomes, after a little spinor simplification,

$$
\begin{aligned}
[3\,\ell_2]&\,\langle\ell_2\,4\rangle\,[1\,\ell_1]\,\langle\ell_1\,6\rangle \\
&\to \langle 3^+|\gamma_\mu|4^+\rangle\langle 1^+|\gamma_\nu|6^+\rangle\, q^\mu q^\nu \\
&\quad + \text{ linear term in } q^\mu \\
&\quad + \Big([3\,5]\,\langle 5\,4\rangle\,a_6 + ([3\,5]\,\langle 5\,4\rangle + [3\,6]\,\langle 6\,4\rangle)a_1 - [3\,2]\,\langle 2\,4\rangle\,a_2\Big) \\
&\quad \times \Big([1\,2]\,\langle 2\,6\rangle\,a_3 + ([1\,2]\,\langle 2\,6\rangle + [1\,3]\,\langle 3\,6\rangle)a_4 - [1\,5]\,\langle 5\,6\rangle\,a_5\Big)\,.
\end{aligned}
\tag{6.4}
$$

Now we can employ formula (VII.8) of ref. [39], for $n = 6$, to express the two-parameter ("reduced") hexagon integrals arising from the third term in (6.4), $\hat{I}_6[a_i a_j]$, in terms of scalar hexagon and pentagon integrals in $6 - 2\epsilon$ dimensions, $\hat{I}_6^{D=6-2\epsilon}$ and $\hat{I}_5^{D=6-2\epsilon\,(\ell)}$, and the desired scalar box integrals, $\hat{I}_4^{(\ell,p)}$,

$$
\begin{aligned}
\hat{I}_6[a_i a_j] &= \frac{\eta_{ij}\hat{\Delta}_6 + 2\epsilon\gamma_i\gamma_j}{2N_6\hat{\Delta}_6}\,\hat{I}_6^{D=6-2\epsilon} \\
&\quad + \frac{2\epsilon}{4N_6^2}\sum_{\ell=1}^{6}\left[\eta_{i\ell}\gamma_j + \eta_{j\ell}\gamma_i - \frac{\eta_{i\ell}\eta_{j\ell}\gamma_\ell}{\eta_{\ell\ell}} - \frac{\gamma_i\gamma_j\gamma_\ell}{\hat{\Delta}_6}\right]\hat{I}_5^{D=6-2\epsilon\,(\ell)} \\
&\quad + \frac{1}{4N_6^2}\sum_{\ell,p=1}^{6}\left[\frac{\eta_{ip}\eta_{j\ell}\eta_{\ell\ell} - \eta_{i\ell}\eta_{j\ell}\eta_{\ell p}}{\eta_{\ell\ell}}\right]\hat{I}_4^{(\ell,p)}\,.
\end{aligned}
\tag{6.5}
$$

The definitions of the reduced integrals $\hat{I}_n$ and the kinematical quantities $\hat{\Delta}_6$, γ_i, η_{ij} and N_6 appearing in eq. (6.5) are given in ref. [39]. The index ℓ on the "daughter" integral $\hat{I}_5^{D=6-2\epsilon\,(\ell)}$ means that the propagator between external legs $(\ell-1)$ (mod 6) and ℓ should be omitted; that propagator and the one between $(p-1)$ (mod 6) and p should be omitted for $\hat{I}_4^{(\ell,p)}$. (In this formula, four-dimensional external kinematics should be taken only at the very end of the calculation, or one encounters unphysical singularities; this point is discussed in ref. [39].)

The first two terms on the right-hand side of eq. (6.5) can be dropped: the first term cancels (to $\mathcal{O}(\epsilon)$) against a $(6-2\epsilon)$-dimensional hexagon integral from integrating the first term in (6.4); the second term is $\mathcal{O}(\epsilon)$. Combining eqs. (6.4) and (6.5), the cut hexagon integral is expressed in terms of scalar box integrals $I_4^{(\ell,p)}$ as,

$$
\begin{aligned}
C_{123} &= \frac{i}{(4\pi)^{2-\epsilon}}\,\mathop{\mathrm{Im}}_{t_{123}>0}\Bigg[c^{(2,3)}\,I_4^{(2,3)} + c^{(5,6)}\,I_4^{(5,6)} \\
&\quad + c^{(2,5)}\,I_4^{(2,5)} + c^{(3,6)}\,I_4^{(3,6)} + c^{(2,6)}\,I_4^{(2,6)} + c^{(3,5)}\,I_4^{(3,5)}\Bigg]\,,
\end{aligned}
\tag{6.6}
$$

where the box coefficients are

$$c^{(\ell,p)} = \frac{(t_{123})^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,[4\,5]\,[5\,6]} \times \sum_{i,j=1}^{6} v^i\,(M^{(\ell,p)})_{ij}\,w^j \tag{6.7}$$

with

$$(M^{(\ell,p)})_{ij} \equiv \frac{\alpha_\ell\alpha_p}{4N_6^2}\left[\frac{\eta_{ip}\eta_{j\ell}\eta_{\ell\ell} - \eta_{i\ell}\eta_{j\ell}\eta_{\ell p}}{\eta_{\ell\ell}} + (\ell \leftrightarrow p)\right] \tag{6.8}$$

and

$$v^i \equiv \big(([3\,5]\,\langle 5\,4\rangle + [3\,6]\,\langle 6\,4\rangle)\alpha_1, -[3\,2]\,\langle 2\,4\rangle\,\alpha_2, 0, 0, 0, [3\,5]\,\langle 5\,4\rangle\,\alpha_6\big),$$
$$w^j \equiv \big(0, 0, [1\,2]\,\langle 2\,6\rangle\,\alpha_3, ([1\,2]\,\langle 2\,6\rangle + [1\,3]\,\langle 3\,6\rangle)\alpha_4, -[1\,5]\,\langle 5\,6\rangle\,\alpha_5, 0\big). \tag{6.9}$$

(The kinematical quantities α_i are also defined in ref. [39].) Of the $6\cdot 5/2 = 15$ box integrals $I_4^{(\ell,p)}$, only the six appearing in eq. (6.6) have cuts in the t_{123} channel.

It is convenient to rewrite eq. (6.6) in terms of the scalar box F functions defined in equation (I.19) of Appendix I,

$$C_{123} = -2\,i\,c_\Gamma \mathop{\mathrm{Im}}_{t_{123} > 0}\left[\frac{c^{(2,3)}}{s_{45}s_{56}}F_{6:1}^{1\mathrm{m}} + \frac{c^{(5,6)}}{s_{12}s_{23}}F_{6:4}^{1\mathrm{m}}\right.$$
$$+ \frac{c^{(2,5)}}{t_{612}t_{123} - s_{12}s_{45}}F_{6:2;1}^{2\mathrm{m}\,e} + \frac{c^{(3,6)}}{t_{123}t_{234} - s_{23}s_{56}}F_{6:2;2}^{2\mathrm{m}\,e} \tag{6.10}$$
$$\left. + \frac{c^{(2,6)}}{s_{34}t_{123}}F_{6:2;5}^{2\mathrm{m}\,h} + \frac{c^{(3,5)}}{s_{61}t_{123}}F_{6:2;2}^{2\mathrm{m}\,h}\right].$$

Upon simplifying the rather messy form (6.7) for the quantities $c^{(\ell,p)}$, we find that the coefficients of the "easy-two-mass" box functions $F_{6:2;1}^{2\mathrm{m}\,e}$, $F_{6:2;2}^{2\mathrm{m}\,e}$ vanish, and the others are all equal:

$$C_{123} = c_\Gamma B_0 \times \mathop{\mathrm{Im}}_{t_{123} > 0}\left[F_{6:1}^{1\mathrm{m}} + F_{6:4}^{1\mathrm{m}} + F_{6:2;2}^{2\mathrm{m}\,h} + F_{6:2;5}^{2\mathrm{m}\,h}\right]. \tag{6.11}$$

where the coefficient B_0 is defined as

$$B_0 \equiv -2\,i\,\frac{c^{(2,3)}}{s_{45}s_{56}}, \tag{6.12}$$

and is given by

$$B_0 = i\,\frac{([1\,2]\,\langle 2\,4\rangle + [1\,3]\,\langle 3\,4\rangle)\,([3\,1]\,\langle 1\,6\rangle + [3\,2]\,\langle 2\,6\rangle)\,(t_{123})^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,[4\,5]\,[5\,6]\,(t_{123}t_{345} - s_{12}s_{45})\,(t_{123}t_{234} - s_{23}s_{56})}. \tag{6.13}$$

In deriving this expression, we used the vanishing of the Gram determinant for any subsystem of five vectors, after integral reductions.

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Recall the form of the $(+ + + - - -)$ tree amplitude from refs. [18,41],

$$A_6^{\text{tree}}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-) = i \left[\frac{\beta^2}{t_{234} s_{23} s_{34} s_{56} s_{61}} + \frac{\gamma^2}{t_{345} s_{34} s_{45} s_{61} s_{12}} + \frac{\beta \gamma t_{123}}{s_{12} s_{23} s_{34} s_{45} s_{56} s_{61}} \right],$$

$$\beta \equiv [2\,3] \langle 5\,6 \rangle \left([1\,2] \langle 2\,4 \rangle + [1\,3] \langle 3\,4 \rangle \right),$$

$$\gamma \equiv [1\,2] \langle 4\,5 \rangle \left([3\,1] \langle 1\,6 \rangle + [3\,2] \langle 2\,6 \rangle \right).$$

$$(6.14)$$

The coefficient B_0 is related to the third term in this expression: their ratio is a real quantity, and can be written without complex spinor products, as a simple rational function in $s_{i,i+1}$ and $t_{i,i+1,i+2}$.

Having simplified the coefficient B_0, we turn to the linear combination of box functions appearing in eq. (6.11). Denoting this combination by $W_6^{(1)}$, and its cyclic permutations by $W_6^{(i)}$, we have explicitly,

$$
\begin{aligned}
W_6^{(i)} &\equiv F_{6:i}^{1\mathrm{m}} + F_{6:i+3}^{1\mathrm{m}} + F_{6:2;i+1}^{2\mathrm{m}\,h} + F_{6:2;i+4}^{2\mathrm{m}\,h} \\
&= -\frac{1}{2\epsilon^2} \sum_{j=1}^{6} \left(\frac{\mu^2}{-s_{j,j+1}} \right)^{\epsilon} - \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i,i+1}} \right) \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+1,i+2}} \right) \\
&\quad - \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+3,i+4}} \right) \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+4,i+5}} \right) + \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+2,i+3}} \right) \ln\left(\frac{-t_{i,i+1,i+2}}{-s_{i+5,i}} \right) \\
&\quad + \frac{1}{2} \ln\left(\frac{-s_{i,i+1}}{-s_{i+3,i+4}} \right) \ln\left(\frac{-s_{i+1,i+2}}{-s_{i+4,i+5}} \right) + \frac{1}{2} \ln\left(\frac{-s_{i-1,i}}{-s_{i,i+1}} \right) \ln\left(\frac{-s_{i+1,i+2}}{-s_{i+2,i+3}} \right) \\
&\quad + \frac{1}{2} \ln\left(\frac{-s_{i+2,i+3}}{-s_{i+3,i+4}} \right) \ln\left(\frac{-s_{i+4,i+5}}{-s_{i+5,i}} \right) + \frac{\pi^2}{3} .
\end{aligned}
$$

$$(6.15)$$

It is amusing that all dilogarithms cancel out of $W_6^{(i)}$. Also note that $W_6^{(i)}$ is invariant under a cyclic permutation by three units; hence there are only three independent objects, $W_6^{(1)}$, $W_6^{(2)}$, and $W_6^{(3)}$.

6.2 Remaining $t_{i,i+1,i+2}$ cuts of $A_{6;1}^{N=4}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-)$, and the full amplitude

Fortunately, no other integrals have to be performed explicitly to get the other two $(+++---)$ $t_{i,i+1,i+2}$ cuts, nor for the $t_{i,i+1,i+2}$ cuts of the other two non-MHV six-gluon amplitudes. They can all be related to the cut C_{123} defined in equation (6.2), and thereby to B_0 given in equation (6.13).

First consider the t_{234} cut of $A_{6;1}^{N=4}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-)$. There are two contributions to this cut, one where the intermediate helicities are the same, so only gluons contribute, and one where the helicities of the intermediate states are opposite, so one

sums over the entire $N = 4$ multiplet. The first contribution is given by

$$
\begin{aligned}
C_{234}^{(a)} &\equiv i \int d^D \text{LIPS}(-\ell_1, \ell_2) \, A_5^{\text{tree}}((-\ell_1)^-, 2^+, 3^+, 4^-, \ell_2^-) \, A_5^{\text{tree}}((-\ell_2)^+, 5^-, 6^-, 1^+, \ell_1^+) \\
&= -\frac{i([2\,3] \langle 5\,6 \rangle)^4}{t_{234} \, [2\,3][3\,4] \langle 5\,6 \rangle \langle 6\,1 \rangle} \int d^D \text{LIPS}(-\ell_1, \ell_2) \frac{1}{[\ell_1\, 2][4\, \ell_2] \langle \ell_2\, 5 \rangle \langle 1\, \ell_1 \rangle} \\
&= \left(\frac{[2\,3] \langle 5\,6 \rangle}{t_{234}} \right)^4 \times [C_{123}^\dagger]|_{j \to j+1} \,,
\end{aligned}
$$

(6.16)

where † means complex conjugating spinor products, $\langle i\, j \rangle \leftrightarrow [j\, i]$ (without complex conjugating factors of i), and where the subscript $j \to j+1$ means applying a cyclic permutation of the six momenta k_i, $\{1, 2, 3, 4, 5, 6\} \to \{2, 3, 4, 5, 6, 1\}$.

The second contribution is given in terms of the contribution of scalar intermediate states, multiplied by a correction factor $\tilde{\rho}^2$ similar to that used in evaluating $N = 4$ MHV cuts [12],

$$
\begin{aligned}
\tilde{\rho}^2 &\equiv \left(\frac{[1\, \ell_1] \langle \ell_1\, 4 \rangle}{[1\, \ell_2] \langle \ell_2\, 4 \rangle} \right)^2 - 4 \left(\frac{[1\, \ell_1] \langle \ell_1\, 4 \rangle}{[1\, \ell_2] \langle \ell_2\, 4 \rangle} \right) + 6 - 4 \left(\frac{[1\, \ell_1] \langle \ell_1\, 4 \rangle}{[1\, \ell_2] \langle \ell_2\, 4 \rangle} \right)^{-1} + \left(\frac{[1\, \ell_1] \langle \ell_1\, 4 \rangle}{[1\, \ell_2] \langle \ell_2\, 4 \rangle} \right)^{-2} \\
&= \frac{\left(\langle 1^+ | (\ell\!\!\!/_1 - \ell\!\!\!/_2) | 4^+ \rangle \right)^4}{\langle \ell_1\, 4 \rangle^2 \langle \ell_2\, 4 \rangle^2 [\ell_1\, 1]^2 [\ell_2\, 1]^2} \\
&= \frac{\left(\langle 1^+ | (k\!\!\!/_2 + k\!\!\!/_3) | 4^+ \rangle \right)^4}{\langle \ell_1\, 4 \rangle^2 \langle \ell_2\, 4 \rangle^2 [\ell_1\, 1]^2 [\ell_2\, 1]^2} \,,
\end{aligned}
$$

(6.17)

$$
\begin{aligned}
C_{234}^{(b)} &\equiv i \int d^D \text{LIPS}(-\ell_1, \ell_2) \, A_5^{\text{tree}}((-\ell_1), 2^+, 3^+, 4^-, \ell_2) \, A_5^{\text{tree}}((-\ell_2), 5^-, 6^-, 1^+, \ell_1) \times \tilde{\rho}^2 \\
&= -\frac{i\left(\langle 1^+ | (k\!\!\!/_2 + k\!\!\!/_3) | 4^+ \rangle \right)^4}{t_{234} \, \langle 2\,3 \rangle \langle 3\,4 \rangle [5\,6][6\,1]} \int d^D \text{LIPS}(-\ell_1, \ell_2) \frac{1}{\langle \ell_1\, 2 \rangle \langle 4\, \ell_2 \rangle [\ell_2\, 5][1\, \ell_1]} \\
&= \left(\frac{\langle 1^+ | (k\!\!\!/_2 + k\!\!\!/_3) | 4^+ \rangle}{t_{234}} \right)^4 \times [C_{123}]|_{j \to j+1} \,,
\end{aligned}
$$

(6.18)

where the legs carrying momenta ℓ_1 and ℓ_2 are intermediate scalars.

Adding up these two contributions, and noticing that the remaining t_{345} cut is related by a reflection symmetry, we can now write down the full amplitude:

$$
A_{6;1}^{N=4}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-) = c_\Gamma \left[B_1 \, W_6^{(1)} + B_2 \, W_6^{(2)} + B_3 \, W_6^{(3)} \right],
$$

(6.19)

where

$$B_1 = B_0 ,$$

$$B_2 = \left(\frac{\langle 1^+|(\not{k}_2 + \not{k}_3)|4^+\rangle}{t_{234}}\right)^4 B_0|_{j\to j+1} + \left(\frac{[2\,3]\,\langle 5\,6\rangle}{t_{234}}\right)^4 B_0^\dagger|_{j\to j+1} ,$$

$$B_3 = \left(\frac{\langle 3^+|(\not{k}_1 + \not{k}_2)|6^+\rangle}{t_{345}}\right)^4 B_0|_{j\to j-1} + \left(\frac{[1\,2]\,\langle 4\,5\rangle}{t_{345}}\right)^4 B_0^\dagger|_{j\to j-1} .$$

(6.20)

There are several consistency checks one may apply to equation (6.19). The pole terms in ϵ represent infrared divergences, which should have a universal form, proportional to the corresponding tree amplitude [15,28],

$$A_{6;1}^{N=4}(1,2,3,4,5,6)|_{\text{singular}} = c_\Gamma \left[-\frac{1}{\epsilon^2} \sum_{j=1}^{6} \left(\frac{\mu^2}{-s_{j,j+1}}\right)^\epsilon \right] A_6^{\text{tree}}(1,2,3,4,5,6) .$$

(6.21)

This behavior for the result (6.19) can be verified using the explicit form (6.15) for $W_6^{(i)}$, and the relation

$$B_1 + B_2 + B_3 = 2\, A_6^{\text{tree}}(1^+, 2^+, 3^+, 4^-, 5^-, 6^-) ,$$

(6.22)

which we have verified numerically; this verifies the overall sign in eq. (6.19). In addition, we have checked numerically that the result in eq. (6.19) is equal to that given by a direct calculation using the string-based methods of ref. [6].

In appendix III, we verify that these amplitudes have the expected behavior in the limit that two neighboring (in the sense of color-ordering) momenta become collinear, thereby isolating a two-particle singular invariant ($s_{i,i+1} \to 0$). One might also wonder how loop amplitudes factorize in multi-particle channels, i.e. as $t_i^{[r]} \to 0$ for $r > 2$. It is easy to see from the supersymmetry Ward identities that supersymmetric MHV amplitudes cannot have multi-particle poles, so the first amplitudes for which this question arises are in fact non-MHV supersymmetric amplitudes (and also the non-supersymmetric amplitudes $A_{n;1}^{[0]}(1^-, 2^+, \ldots, n^+)$ constructed by Mahlon [5], which we will not study here). In the $(+ + + - - -)$ amplitude, one expects and finds poles in the three-particle channels, $t_{234} \to 0$ and $t_{345} \to 0$. However, naive factorization does not hold in these limits, due to the presence in $W_6^{(i)}$ of logarithms of kinematic invariants which get caught "across" the pole. For example, as $t_{234} \to 0$, the invariants s_{12}, s_{45}, t_{123} and t_{345} each have at least one argument on each side of the pole, i.e. at least one argument belonging to $\{k_2, k_3, k_4\}$ and one to $\{k_5, k_6, k_1\}$. This lack of naive factorization is present even at the level of the universal singular terms which come from soft virtual gluon exchange. One might

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suspect that naive factorization holds for the $N = 1$ chiral contribution and for the scalar contribution, since these contributions do not have virtual gluons. We have examined the three-particle poles of the $(+ + + - - -)$ $N = 1$ chiral contribution, calculated from its cuts, and have found that a simple factorization does hold there.

6.3 The remaining $N = 4$ non-MHV amplitudes

The analysis for the remaining two $N = 4$ supersymmetric non-MHV six-gluon amplitudes is identical to that presented for $(+ + + - - -)$, so we merely quote the results, in terms of the function B_0 given in eq. (6.13):

$$A_{6;1}^{N=4}(1^+, 2^+, 3^-, 4^+, 5^-, 6^-) = c_\Gamma \left[D_1 \, W_6^{(1)} + D_2 \, W_6^{(2)} + D_3 \, W_6^{(3)} \right], \qquad (6.23)$$

where

$$D_1 = \left(\frac{\langle 4^+|(\not{k}_1 + \not{k}_2)|3^+\rangle}{t_{123}} \right)^4 B_0 + \left(\frac{[1\,2]\,\langle 5\,6\rangle}{t_{123}} \right)^4 B_0^\dagger ,$$

$$D_2 = \left(\frac{\langle 1^+|(\not{k}_2 + \not{k}_4)|3^+\rangle}{t_{234}} \right)^4 B_0|_{j \to j+1} + \left(\frac{[2\,4]\,\langle 5\,6\rangle}{t_{234}} \right)^4 B_0^\dagger|_{j \to j+1} , \qquad (6.24)$$

$$D_3 = \left(\frac{\langle 4^+|(\not{k}_1 + \not{k}_2)|6^+\rangle}{t_{345}} \right)^4 B_0|_{j \to j-1} + \left(\frac{[1\,2]\,\langle 3\,5\rangle}{t_{345}} \right)^4 B_0^\dagger|_{j \to j-1} ,$$

and

$$A_{6;1}^{N=4}(1^+, 2^-, 3^+, 4^-, 5^+, 6^-) = c_\Gamma \left[G_1 \, W_6^{(1)} + G_2 \, W_6^{(2)} + G_3 \, W_6^{(3)} \right], \qquad (6.25)$$

where

$$G_1 = \left(\frac{\langle 5^+|(\not{k}_4 + \not{k}_6)|2^+\rangle}{t_{123}} \right)^4 B_0 + \left(\frac{[1\,3]\,\langle 4\,6\rangle}{t_{123}} \right)^4 B_0^\dagger ,$$

$$G_2 = \left(\frac{\langle 3^+|(\not{k}_2 + \not{k}_4)|6^+\rangle}{t_{234}} \right)^4 B_0^\dagger|_{j \to j+1} + \left(\frac{[5\,1]\,\langle 2\,4\rangle}{t_{234}} \right)^4 B_0|_{j \to j+1} , \qquad (6.26)$$

$$G_3 = \left(\frac{\langle 1^+|(\not{k}_6 + \not{k}_2)|4^+\rangle}{t_{345}} \right)^4 B_0^\dagger|_{j \to j-1} + \left(\frac{[3\,5]\,\langle 6\,2\rangle}{t_{345}} \right)^4 B_0|_{j \to j-1} .$$

We have checked (numerically) the relations required for the pole terms in ϵ to be correctly given by (6.21):

$$\begin{aligned} D_1 + D_2 + D_3 &= 2\, A_6^{\text{tree}}(1^+, 2^+, 3^-, 4^+, 5^-, 6^-), \\ G_1 + G_2 + G_3 &= 2\, A_6^{\text{tree}}(1^+, 2^-, 3^+, 4^-, 5^+, 6^-). \end{aligned} \qquad (6.27)$$

We have taken the tree amplitudes from refs. [18,41].

7. Non-Supersymmetric Theories.

The cut uniqueness result may also be applied to non-supersymmetric amplitudes, allowing one to trade harder calculations for easier ones. The supersymmetric decomposition of the n-gluon amplitudes in eq. (2.11) provides one such example. Instead of computing QCD amplitudes with gluons and quarks in the loop we compute amplitudes with an $N = 4$ multiplet, $N = 1$ chiral multiplet and scalar in the loop. The supersymmetric amplitudes are cut-constructible, leaving the easier-to-compute scalar loop contribution to be obtained by other means (see below).

One can apply the same technique to amplitudes with external fermions. Consider the color-ordered partial amplitudes with two massless external quarks and $n - 2$ external gluons. An example of a one-particle irreducible diagram contributing to such an amplitude is depicted in fig. 9a. Such an m-point graph has a loop-momentum polynomial of maximum degree $m - 1$, as discussed in section 4. If one adds to this diagram an identical diagram but with the gluon replaced by a scalar (depicted in fig.9b), suitably adjusts the scalar-fermion Yukawa coupling, and works in background-field Feynman gauge [22], then the leading loop-momentum terms cancel. (From the γ-matrix algebra the diagram with a gluon in the loop has a minus sign relative to the case of a scalar in the loop.) Thus the sum is cut-constructible, and so (after calculating the cuts for the sum) one can again replace a more difficult calculation with a gluon in the loop by a simpler calculation with a scalar in the loop.
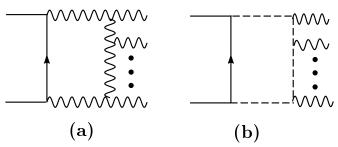

(a)   (b)

Figure 9. Diagrams with identical leading loop momentum behavior; the wavy lines are gluons, the straight lines are fermions and the dashed lines are scalars.

Amplitudes with four or more external fermions can be treated similarly. Indeed, from section 4 we know that the only diagrams requiring cancellation of the leading loop-momentum behavior are those where fermions are "pinched off" onto external trees, such that the one-particle irreducible part of the diagram has either zero or two external fermion lines.

Although the scalar loop contribution to n-gluon amplitudes is not cut-constructible (since the m-point loop integrals contain m powers of loop momentum) we can still fix all terms containing cuts. As an explicit example, by following the same procedure as for the $N = 1$ chiral multiplet, we obtain the scalar loop contribution for two negative helicity adjacent legs ($i = 1$ and $j = 2$),

$$
\begin{aligned}
A^{[0]}_{1\,2}(1,\ldots,n) \;=\;& \frac{1}{3} A^{N=1 \ \text{chiral}}_{1\,2}(1,\ldots,n) \\
& - \frac{c_\Gamma A^{\text{tree}}_{1\,2}(1,\ldots,n)}{3} \frac{1}{(t_1^{[2]})^3} \sum_{m=4}^{n-1} \frac{\text{L}_2\left(-t_2^{[m-2]}/(-t_2^{[m-1]})\right)}{(t_2^{[m-1]})^3} \\
& \times \left((\text{tr}_+[\not{k}_1\not{k}_2\not{k}_m\not{q}_{m,1}])^2 \ \text{tr}_+[\not{k}_1\not{k}_2\not{q}_{m,1}\not{k}_m] - \text{tr}_+[\not{k}_1\not{k}_2\not{k}_m\not{q}_{m,1}] \, (\text{tr}_+[\not{k}_1\not{k}_2\not{q}_{m,1}\not{k}_m])^2 \right) \\
& + \text{polynomials.}
\end{aligned}
$$

(7.1)

Although the cut analysis is quite similar to the $N = 1$ chiral case, here we cannot conclude that there are no missing polynomial (rational function) terms. Indeed, the known five-gluon amplitude with a scalar in the loop [8] contains polynomials not reproduced by this formula. Furthermore, if we ignore missing polynomials, this formula has incorrect factorization properties; in particular, there should be poles in multi-particle channels ($t_i^{[r]} \sim 0$ for $r \geq 3$), which are lacking from the non-polynomial terms in (7.1). The appearance of multi-particle poles significantly complicates the structure of the polynomial terms, but these pieces should still be amenable to recursive [4,5] or collinear bootstrap [11,12] techniques.

8. Conclusions

In a previous paper we demonstrated that $N = 4$ super-Yang-Mills amplitudes are constructible solely from their cuts and we explicitly computed n-point one-loop amplitudes with maximal helicity violation. In this paper we extended the 'cut-constructibility' to include all massless amplitudes which satisfy a power-counting criterion. In particular, $N = 1$ supersymmetric gauge theory amplitudes satisfy this criterion, when using string-based [6,7] or background field diagrams [22,20]. Once the power-counting criterion has been checked, the one-loop amplitudes can be obtained directly from tree amplitudes using the cuts.

It is convenient to calculate cuts in terms of the imaginary parts of one-loop integrals that would have been encountered in a direct calculation. This makes it straightforward to write down an analytic expression with the correct cuts in all channels, avoiding the need

to perform dispersion integrals to reconstruct the full amplitude. Once this expression is written in terms of the integral functions appearing in a direct calculation, it is fixed uniquely so long as the power-counting criterion is satisfied.

Using this method, we calculated all maximally helicity-violating one-loop n-gluon amplitudes in $N = 1$ super-QCD as well as the $N = 4$ six-gluon helicity amplitudes for all helicity configurations that were not presented in ref. [12]. The $N = 4$ and $N = 1$ amplitudes form two of three components of a QCD n-gluon amplitude [8,14], with the third component the scalar loop contribution. The scalar loop contributions contain polynomial pieces which must be obtained by other means, such as string-based [7], collinear [11], or recursive [4,5] techniques. The replacement of more difficult calculations with easier ones can also be carried out for amplitudes with external fermions.

Gauge theory loop calculations are difficult to perform because of the algebraic complexity which in most cases disappears near the end of a calculation, with relatively simple final results. By fusing together tree amplitudes we are utilizing simplifications already performed at tree level. When applicable, the technique presented here is remarkably simple relative to conventional techniques, as there is no explosion in the size of intermediate expressions. We expect this method to have wide applications and to be useful in the continuing effort to calculate the gauge theory loop amplitudes required by experiment.

Acknowledgements

Appendix I. Integrals

In this appendix, we list the integral functions which occur, after performing a Passarino-Veltman reduction, in explicit calculations of amplitudes satisfying the power-counting criterion given in section 3, namely that all m-point integrals ($m > 2$) contain at most $m - 2$ powers of the loop momentum in the numerator of the integrand, and that all two-point integrals contain at most a single power of the loop momentum. (That is, loop integrals with m or $m - 1$ powers of the loop momentum are excluded for $m > 2$, as is the tensor bubble integral.) We consider only purely massless processes; thus while the integral functions appearing may have off-shell external legs (which we shall refer to as massive legs, since $K^2 \neq 0$), internal legs are strictly massless. As discussed in Section 2, a Passarino-Veltman

reduction may be used to re-express any integral satisfying the power-counting criterion (and thus any amplitude all of whose loop integrals satisfy the criterion), in terms of

(a) scalar boxes, triangles and bubbles; and,

(b) triangles with one or two external massive (off-shell) legs and with a single power of loop momentum in the numerator, and bubbles with a single power of the loop momentum in the numerator.

However, as we shall see the integrals in category (b) are all linear combinations (with coefficients taken to be rational functions of the momentum invariants) of integrals in category (a).

I.1 Bubble Integrals

The bubble integral is defined by

$$I_2[P(p^\mu)] = -i\,(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{P(p^\mu)}{p^2(p-K)^2}\,, \tag{I.1}$$

where $P(p^\mu)$ is a polynomial in the loop momentum p^μ and $K = \sum_{l=i}^{i+r-1} k_l$ is the total momentum flowing out of one side, r being the number of external legs clustered on one side of the bubble starting at leg i, as depicted in fig. 3. It is straightforward to evaluate the integrals of this type belonging to the cut-constructible class (scalar and vector),

$$I_{2:r;i} \equiv I_2[1] = \frac{r_\Gamma}{\epsilon(1-2\epsilon)}(-t_i^{[r]})^{-\epsilon}$$
$$= r_\Gamma\left(\frac{1}{\epsilon} - \ln(-t_i^{[r]}) + 2\right) + \mathcal{O}(\epsilon)\,, \tag{I.2}$$
$$I_2[p^\mu] = \frac{K^\mu}{2}\,I_{2:r;i}\,,$$

where r_Γ was defined in equation (2.8). Observe that the two terms not containing logarithms are linked to the logarithm before the expansion in ϵ; furthermore, one cannot cancel the logarithms in any linear combination of the scalar and vector integrals without also cancelling the cut-free parts. For the degenerate case with $r = 1$, the standard prescription in dimensional regularization is to take I_2 to vanish for massless external legs; this is interpreted as a cancellation of ultra-violet and infrared divergences [43].

I.2 Triangle Integrals

The triangle integrals are defined by

$$I_3[P(p^\mu)] = i\,(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{P(p^\mu)}{p^2\,(p-K_1)^2\,(p+K_3)^2}\,, \tag{I.3}$$

where $P(p^\mu)$ is again a polynomial of the loop momentum and the external momentum arguments $K_{1...3}$ in equation (I.3) are sums of external momenta k_i that are the arguments of the n-point amplitude. There are three types of scalar triangle integrals that can appear in an n-point calculation depending on how many legs are massive (off-shell), as depicted in fig. 2.

The one-mass triangle in fig. 2a depends only on the momentum invariant of the massive leg, $t_{i+2}^{[n-2]} = t_i^{[2]}$,

$$I_{3;i}^{1m} = \frac{r_\Gamma}{\epsilon^2}(-t_i^{[2]})^{-1-\epsilon} \ . \tag{I.4}$$

This integral does not contain any finite polynomials in the region $t_i^{[2]} < 0$.

The next integral function shown in fig. 2b is the two-mass triangle integral,

$$I_{3:r;i}^{2m}(-t_i^{[r]}, -t_{i+r}^{[n-r-1]}) = \frac{r_\Gamma}{\epsilon^2} \frac{(-t_i^{[r]})^{-\epsilon} - (-t_{i+r}^{[n-r-1]})^{-\epsilon}}{(-t_i^{[r]}) - (-t_{i+r}^{[n-r-1]})} \ . \tag{I.5}$$

Note that the functions in eqs. (I.4) and (I.5) are linear combinations of the set of functions

$$G(-t_i^{[r]}) = \frac{(-t_i^{[r]})^{-\epsilon}}{\epsilon^2} \ . \tag{I.6}$$

Conversely, the functions $G(-t_i^{[r]})$ can also be written as linear combinations of the one and two-mass triangle scalar integrals. For example,

$$(t_i^{[2]} - t_{i+2}^{[n-3]})\frac{I_{3:2;i}^{2m}}{r_\Gamma} - t_i^{[2]}\frac{I_{3;i}^{1m}}{r_\Gamma} = G(-t_{i+2}^{[n-3]}) = G(-t_{i-1}^{[3]}) \ . \tag{I.7}$$

The final scalar triangle is the three-mass integral function depicted in fig. 2c. The evaluation of this integral is more involved, and can be obtained from refs. [44,39]

$$I_{3:r,r';i}^{3m} = \frac{i}{\sqrt{\Delta_3}} \sum_{j=1}^{3} \left[\text{Li}_2\left(-\left(\frac{1+i\delta_j}{1-i\delta_j}\right)\right) - \text{Li}_2\left(-\left(\frac{1-i\delta_j}{1+i\delta_j}\right)\right) \right] + \mathcal{O}(\epsilon), \tag{I.8}$$

where

$$\delta_1 = \frac{t_i^{[r]} - t_{i+r}^{[r']} - t_{i+r+r'}^{[n-r-r']}}{\sqrt{\Delta_3}} \ ,$$

$$\delta_2 = \frac{-t_i^{[r]} + t_{i+r}^{[r']} - t_{i+r+r'}^{[n-r-r']}}{\sqrt{\Delta_3}} \ , \tag{I.9}$$

$$\delta_3 = \frac{-t_i^{[r]} - t_{i+r}^{[r']} + t_{i+r+r'}^{[n-r-r']}}{\sqrt{\Delta_3}} \ ,$$

and

$$\Delta_3 \equiv -(t_i^{[r]})^2 - (t_{i+r}^{[r']})^2 - (t_{i+r+r'}^{[n-r-r']})^2 + 2t_i^{[r]}t_{i+r}^{[r']} + 2t_{i+r}^{[r']}t_{i+r+r'}^{[n-r-r']} + 2t_{i+r+r'}^{[n-r-r']}t_i^{[r]} \ . \quad \text{(I.10)}$$

There are three different ways of labeling any three-mass triangle $I_{3:r,r';i}^{\mathrm{3m}} = I_{3:r',n-r-r';i+r}^{\mathrm{3m}} = I_{3:n-r-r',r;i+r+r'}^{\mathrm{3m}}$; we only need to keep the distinct cases.

We must also consider the single and double mass triangles that have a single power of the loop momenta in their numerator. The single mass case with the same kinematics as in fig. 2 gives,

$$I_{3;i}^{\mathrm{1m}}[p^\mu] = r_\Gamma K_1^\mu \frac{(-t_i^{[2]})^{-1-\epsilon}}{\epsilon^2(1-2\epsilon)} - r_\Gamma(K_1^\mu + K_2^\mu)\frac{(-t_i^{[2]})^{-1-\epsilon}}{\epsilon(1-2\epsilon)} \ , \quad \text{(I.11)}$$

where $K_1 = k_i$, $K_2 = k_{i+1}$ and $K_3 = k_{i+2} + \cdots + k_{i+n-1}$. The second term is proportional to $I_{2:2;i}$ (defined in eq. (I.2)). Since

$$\frac{1}{\epsilon^2(1-2\epsilon)} = \frac{1}{\epsilon^2} + \frac{2}{\epsilon(1-2\epsilon)} \ , \quad \text{(I.12)}$$

the first term is a linear combination of $I_2(t_i^{[2]}) \equiv I_{2:2;i}$ and $I_3^{\mathrm{1m}}(t_i^{[2]}) \equiv I_{3;i}^{\mathrm{1m}}$. Hence $I_{3;i}^{\mathrm{1m}}[p^\mu]$ can be excluded from the set of integral functions. The two-mass linear triangle is

$$I_{3:r;i}^{\mathrm{2m}}[p^\mu] = r_\Gamma K_1^\mu \frac{(-t_i^{[r]})^{-\epsilon} - (-t_{i+r}^{[n-r-1]})^{-\epsilon}}{\epsilon(1-2\epsilon)(t_i^{[r]} - t_{i+r}^{[n-r-1]})}$$
$$- r_\Gamma(K_1^\mu + K_2^\mu)\left\{ \frac{t_i^{[r]}((-t_i^{[r]})^{-\epsilon} - (-t_{i+r}^{[n-r-1]})^{-\epsilon})}{\epsilon^2(1-2\epsilon)(t_i^{[r]} - t_{i+r}^{[n-r-1]})^2} + \frac{(-t_{i+r}^{[n-r-1]})^{-\epsilon}}{\epsilon(1-2\epsilon)(t_i^{[r]} - t_{i+r}^{[n-r-1]})} \right\} \ , \quad \text{(I.13)}$$

where $K_1 = k_i + \cdots + k_{i+r-1}$, $K_2 = k_{i+r} + \cdots + k_{i+n-2}$, and $K_3 = k_{i+n-1}$. By the same reasoning this is also a linear combination of the scalar integrals.

Finally, we can simplify the three-mass linear triangle using equation (42) of ref. [45] since none of the coefficients are singular in this case, and the three-mass linear triangle can also be re-expressed as a sum of scalar triangle and bubble integrals.

I.3 Box Integrals

The scalar box integrals considered here have vanishing internal masses, but may have from zero to four nonvanishing external masses. Again by external masses we mean off-shell legs with $K^2 \neq 0$. These integrals are defined and given in ref. [39] (the four-mass box was computed by Denner, Nierste, and Scharf [46]) and are shown in fig. 1 and in fig. 4(a).

The scalar box integral is

$$I_4 = -i\,(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2\,(p-K_1)^2\,(p-K_1-K_2)^2\,(p+K_4)^2}\ . \tag{I.14}$$

The external momentum arguments $K_{1\ldots4}$ in equation (I.14) are sums of external momenta k_i that are the arguments of the n-point amplitude.

The no-mass box is depicted in fig. 4(a) and is through $\mathcal{O}(\epsilon^0)$

$$I_4^{\rm 0m}[1] \;=\; r_\Gamma\,\frac{1}{st}\left\{\frac{2}{\epsilon^2}\Big[(-s)^{-\epsilon}+(-t)^{-\epsilon}\Big]-\ln^2\left(\frac{-s}{-t}\right)-\pi^2\right\}, \tag{I.15}$$

where $s=(k_1+k_2)^2$ and $t=(k_2+k_3)^2$ are the usual Mandelstam variables. This function appears only in four-point amplitudes with massless particles. In the proof, presented in Section 3, that the cuts uniquely determine the amplitude, this box appears only as a special case.

With the labeling of legs shown in fig. 1 (that is, expressing the functions in terms of the invariants $t_i^{[r]}$ of the n-point amplitude), the scalar box integrals I_4 expanded through order $\mathcal{O}(\epsilon^0)$ for the different cases reduce to

$$
\begin{aligned}
I_{4:i}^{\rm 1m} \;=\;& \frac{-2r_\Gamma}{t_{i-3}^{[2]}t_{i-2}^{[2]}}\left\{-\frac{1}{\epsilon^2}\Big[(-t_{i-3}^{[2]})^{-\epsilon}+(-t_{i-2}^{[2]})^{-\epsilon}-(-t_i^{[n-3]})^{-\epsilon}\Big]\right. \\
& \left.+\,\mathrm{Li}_2\left(1-\frac{t_i^{[n-3]}}{t_{i-3}^{[2]}}\right)\;+\;\mathrm{Li}_2\left(1-\frac{t_i^{[n-3]}}{t_{i-2}^{[2]}}\right)\;+\;\frac{1}{2}\ln^2\left(\frac{t_{i-3}^{[2]}}{t_{i-2}^{[2]}}\right)\;+\;\frac{\pi^2}{6}\right\},
\end{aligned} \tag{I.16a}
$$

$$
\begin{aligned}
I_{4:r;i}^{\rm 2me} \;=\;& \frac{-2r_\Gamma}{t_{i-1}^{[r+1]}t_i^{[r+1]}-t_i^{[r]}t_{i+r+1}^{[n-r-2]}}\left\{-\frac{1}{\epsilon^2}\Big[(-t_{i-1}^{[r+1]})^{-\epsilon}+(-t_i^{[r+1]})^{-\epsilon}-(-t_i^{[r]})^{-\epsilon}-(-t_{i+r+1}^{[n-r-2]})^{-\epsilon}\Big]\right. \\
& +\,\mathrm{Li}_2\left(1-\frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right)\;+\;\mathrm{Li}_2\left(1-\frac{t_i^{[r]}}{t_i^{[r+1]}}\right)\;+\;\mathrm{Li}_2\left(1-\frac{t_{i+r+1}^{[n-r-2]}}{t_{i-1}^{[r+1]}}\right) \\
& \left.+\,\mathrm{Li}_2\left(1-\frac{t_{i+r+1}^{[n-r-2]}}{t_i^{[r+1]}}\right)\;-\;\mathrm{Li}_2\left(1-\frac{t_i^{[r]}t_{i+r+1}^{[n-r-2]}}{t_{i-1}^{[r+1]}t_i^{[r+1]}}\right)\;+\;\frac{1}{2}\ln^2\left(\frac{t_{i-1}^{[r+1]}}{t_i^{[r+1]}}\right)\right\},
\end{aligned} \tag{I.16b}
$$

$$
\begin{aligned}
I_{4:r;i}^{\rm 2mh} \;=\;& \frac{-2r_\Gamma}{t_{i-2}^{[2]}t_{i-1}^{[r+1]}}\left\{-\frac{1}{\epsilon^2}\Big[(-t_{i-2}^{[2]})^{-\epsilon}+(-t_{i-1}^{[r+1]})^{-\epsilon}-(-t_i^{[r]})^{-\epsilon}-(-t_{i+r}^{[n-r-2]})^{-\epsilon}\Big]\right. \\
& -\,\frac{1}{2\epsilon^2}\frac{(-t_i^{[r]})^{-\epsilon}(-t_{i+r}^{[n-r-2]})^{-\epsilon}}{(-t_{i-2}^{[2]})^{-\epsilon}}\;+\;\frac{1}{2}\ln^2\left(\frac{t_{i-2}^{[2]}}{t_{i-1}^{[r+1]}}\right) \\
& \left.+\,\mathrm{Li}_2\left(1-\frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right)\;+\;\mathrm{Li}_2\left(1-\frac{t_{i+r}^{[n-r-2]}}{t_{i-1}^{[r+1]}}\right)\right\},
\end{aligned} \tag{I.16c}
$$

$$I_{4:r,r',i}^{3\mathrm{m}} = \frac{-2r_\Gamma}{t_{i-1}^{[r+1]}t_i^{[r+r']} - t_i^{[r]}t_{i+r+r'}^{[n-r-r'-1]}}\Bigg\{$$

$$- \frac{1}{\epsilon^2}\Big[(-t_{i-1}^{[r+1]})^{-\epsilon} + (-t_i^{[r+r']})^{-\epsilon} - (-t_i^{[r]})^{-\epsilon} - (-t_{i+r}^{[r']})^{-\epsilon} - (-t_{i+r+r'}^{[n-r-r'-1]})^{-\epsilon}\Big]$$

$$- \frac{1}{2\epsilon^2}\frac{(-t_i^{[r]})^{-\epsilon}(-t_{i+r}^{[r']})^{-\epsilon}}{(-t_i^{[r+r']})^{-\epsilon}} - \frac{1}{2\epsilon^2}\frac{(-t_{i+r}^{[r']})^{-\epsilon}(-t_{i+r+r'}^{[n-r-r'-1]})^{-\epsilon}}{(-t_{i-1}^{[r+1]})^{-\epsilon}} + \frac{1}{2}\ln^2\left(\frac{t_{i-1}^{[r+1]}}{t_i^{[r+r']}}\right)$$

$$+ \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_{i+r+r'}^{[n-r-r'-1]}}{t_i^{[r+r']}}\right) - \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}t_{i+r+r'}^{[n-r-r'-1]}}{t_{i-1}^{[r+1]}t_i^{[r+r']}}\right)\Bigg\},$$

$$\tag{I.16d}$$

$$I_{4:r,r',r'',i}^{4\mathrm{m}} = \frac{-r_\Gamma}{t_i^{[r+r']}\,t_{i+r}^{[r'+r'']}\,\rho}\Bigg\{- \mathrm{Li}_2\left(\tfrac{1}{2}(1 - \lambda_1 + \lambda_2 + \rho)\right) + \mathrm{Li}_2\left(\tfrac{1}{2}(1 - \lambda_1 + \lambda_2 - \rho)\right)$$

$$- \mathrm{Li}_2\left(-\tfrac{1}{2\lambda_1}(1 - \lambda_1 - \lambda_2 - \rho)\right) + \mathrm{Li}_2\left(-\tfrac{1}{2\lambda_1}(1 - \lambda_1 - \lambda_2 + \rho)\right)$$

$$- \frac{1}{2}\ln\left(\frac{\lambda_1}{\lambda_2^2}\right)\ln\left(\frac{1 + \lambda_1 - \lambda_2 + \rho}{1 + \lambda_1 - \lambda_2 - \rho}\right)\Bigg\},$$

$$\tag{I.16e}$$

where

$$\rho \equiv \sqrt{1 - 2\lambda_1 - 2\lambda_2 + \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2}\,, \tag{I.17}$$

and

$$\lambda_1 = \frac{t_i^{[r]}\,t_{i+r+r'}^{[r'']}}{t_i^{[r+r']}\,t_{i+r}^{[r'+r'']}}\,, \qquad\qquad \lambda_2 = \frac{t_{i+r}^{[r']}\,t_{i+r+r'+r''}^{[n-r-r'-r'']}}{t_i^{[r+r']}\,t_{i+r}^{[r'+r'']}}\,. \tag{I.18}$$

There are different ways of labeling $I_{4:r,r',r'',i}^{4\mathrm{m}}$ and $I_{4:r;i}^{2\mathrm{me}}$; once again we need only keep the distinct cases.

For the explicit six-point amplitudes presented in section 6, it is convenient to define scalar box functions F in which prefactors have been removed from the scalar box integrals I_4:

$$I_{4:i}^{1\mathrm{m}} = -2r_\Gamma\frac{F_{n:i}^{1\mathrm{m}}}{t_{i-3}^{[2]}t_{i-2}^{[2]}}\,, \tag{I.19a}$$

$$I_{4:r;i}^{2\mathrm{me}} = -2r_\Gamma\frac{F_{n:r;i}^{2\mathrm{m}\,e}}{t_{i-1}^{[r+1]}t_i^{[r+1]} - t_i^{[r]}t_{i+r+1}^{[n-r-2]}}\,, \tag{I.19b}$$

$$I_{4:r;i}^{2\mathrm{mh}} = -2r_\Gamma\frac{F_{n:r;i}^{2\mathrm{m}\,h}}{t_{i-2}^{[2]}t_{i-1}^{[r+1]}}\,, \tag{I.19c}$$

$$I_{4:r,r',i}^{3\mathrm{m}} = -2r_\Gamma\frac{F_{n:r,r';i}^{3\mathrm{m}}}{t_{i-1}^{[r+1]}t_i^{[r+r']} - t_i^{[r]}t_{i+r+r'}^{[n-r-r'-1]}}\,, \tag{I.19d}$$

$$I_{4:r,r',r'',i}^{4\mathrm{m}} = -2\frac{F_{n:r,r',r'';i}^{4\mathrm{m}}}{t_i^{[r+r']}\,t_{i+r}^{[r'+r'']}\,\rho}\,. \tag{I.19e}$$

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Appendix II. Functions Used in the $N = 1$ Supersymmetric Amplitudes

Here we define the functions appearing in the $N = 1$ supersymmetric MHV amplitudes. In addition to the definitions we demonstrate that these functions, with the appropriate arguments, are linear combinations of the integral functions of Appendix I.

$$
\mathrm{K}_0(s) = \left(-\ln(-s) + 2 + \frac{1}{\epsilon}\right) + \mathcal{O}(\epsilon) = \frac{1}{\epsilon(1 - 2\epsilon)}(-s)^{-\epsilon} \, ,
$$

$$
\mathrm{L}_0(r) = \frac{\ln(r)}{1 - r} \, ,
$$

$$
\mathrm{M}_0(s_1, s_2; m_1^2, m_2^2) = \mathrm{Li}_2\left(1 - \frac{m_1^2 m_2^2}{s_1 s_2}\right) - \mathrm{Li}_2\left(1 - \frac{m_1^2}{s_1}\right) - \mathrm{Li}_2\left(1 - \frac{m_1^2}{s_2}\right) - \mathrm{Li}_2\left(1 - \frac{m_2^2}{s_1}\right)
$$
$$
- \mathrm{Li}_2\left(1 - \frac{m_2^2}{s_2}\right) - \frac{1}{2}\ln^2\left(\frac{s_1}{s_2}\right) \, .
$$

$$(\mathrm{II}.1)$$

The function $\mathrm{K}_0(s)$ is simply proportional to the scalar bubble function

$$
\mathrm{K}_0(t_i^{[r]}) = \frac{I_{2:r;i}}{r_\Gamma} \, .
\tag{II.2}
$$

In the amplitudes it is the combination

$$
\frac{\mathrm{L}_0\left((-t_i^{[r]})/(-t_i^{[r+1]})\right)}{t_i^{[r+1]}}
\tag{II.3}
$$

which appears. This has several representations; it can be expressed as a linear combination of bubble functions,

$$
\frac{\mathrm{L}_0\left((-t_i^{[r]})/(-t_i^{[r+1]})\right)}{t_i^{[r+1]}} = -\frac{\ln(-t_i^{[r]}) - \ln(-t_i^{[r+1]})}{t_i^{[r+1]} - t_i^{[r]}}
$$
$$
= \frac{1}{t_i^{[r+1]} - t_i^{[r]}}\left(I_{2:r;i} - I_{2:r+1;i}\right) \, .
\tag{II.4}
$$

Secondly it is a Feynman parameter integral, for a two mass triangle integral,

$$
\frac{\mathrm{L}_0\left((-t_i^{[r]})/(-t_i^{[r+1]})\right)}{t_i^{[r+1]}} = \frac{1}{r_\Gamma} I_{3:r,i}^{2m}[a_2]
\tag{II.5}
$$

where $t_i^{[r]}$ and $t_{i+r}^{[n-r-1]} = t_i^{[r+1]}$ are the momentum invariants of the massive legs and a_2 is the Feynman parameter for the leg between the two massive external legs. We mention

this representation as it arises naturally when one carries out the calculation of the $N = 1$ chiral multiplet amplitude in a manner analogous to the $N = 4$ calculation in ref. [12].

The M_0 is a linear combination of the scalar 'easy two-mass' box integral (with the masses on diagonally opposite legs) and certain triangle integrals. As discussed in Appendix I the function

$$G(s) \equiv \frac{1}{\epsilon^2}(-s)^{-\epsilon} \tag{II.6}$$

is a linear combination of integral functions. Using this function we have

$$
\begin{aligned}
\mathrm{M}_0(t_{i-1}^{[r+1]}, t_i^{[r+1]}; t_i^{[r]}, t_{i+r+1}^{[n-r-2]}) &= \frac{t_{i-1}^{[r+1]} t_i^{[r+1]} - t_i^{[r]} t_{i+r+1}^{[n-r-2]}}{2r_\Gamma} I_{4:r;i}^{2\mathrm{me}} \\
&+ G(t_i^{[r]}) + G(t_{i+r+1}^{[n-r-2]}) - G(t_{i-1}^{[r+1]}) - G(t_i^{[r+1]}) .
\end{aligned}
\tag{II.7}
$$

It is in fact precisely the combination that yields the $D = 6$ 'easy two-mass' box integral [45] multiplied by its denominator and a constant factor. The $D = 6$ box integrals arise here from the reduction of box integrals with insertions of loop momentum polynomials to scalar box integrals and triangle integrals. It provides a slightly preferable representation to the $D = 4$ box integral because it is manifestly free of soft divergences, as should be the case for contributions of internal $N = 1$ matter supermultiplets. (Only internal gluons can give rise to soft divergences.)

In the case $r = n - 3$ the two mass box reduces to a single mass box, because the momentum invariant giving rise to the mass of the second massive leg is zero, $t_{i+r+1}^{[n-r-2]} = 0$. This corresponds to setting one of the last two arguments of M_0 to zero whence,

$$\mathrm{M}_0(s_1, s_2; m_1^2, 0) = -\mathrm{Li}_2\left(1 - \frac{m_1^2}{s_1}\right) - \mathrm{Li}_2\left(1 - \frac{m_1^2}{s_2}\right) - \frac{1}{2}\ln^2\left(\frac{s_1}{s_2}\right) - \frac{\pi^2}{6} , \tag{II.8}$$

so that

$$
\begin{aligned}
\mathrm{M}_0(t_{i-1}^{[n-2]}, t_i^{[n-2]}; t_i^{[n-3]}, 0) &= \frac{t_i^{[n-2]} t_{i-1}^{[n-2]}}{2r_\Gamma} I_{4:i}^{1\mathrm{m}} \\
&+ G(t_i^{[r]}) - G(t_{i-1}^{[r+1]}) - G(t_i^{[r+1]}) ,
\end{aligned}
\tag{II.9}
$$

and the function corresponds to a single mass box integral, plus G functions with various arguments.

As we have expressed the functions K_0, L_0 and M_0 in terms of the set of integral functions, any cut constructible amplitude which is written in terms of these functions and has the correct cuts, automatically has the correct polynomial terms.

Appendix III. Collinear Limits

In this appendix, we verify that the $N = 1$ MHV n-point amplitudes and (one of) the $N = 4$ non-MHV six-point amplitudes have the expected collinear limits. In general, one expects [11,12] the collinear limits of the leading-color one-loop partial amplitudes to have the following form,

$$
A_{n;1}^{\text{loop}} \xrightarrow{a \| b} \sum_{\lambda = \pm} \Bigg(\text{Split}_{-\lambda}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) \, A_{n-1;1}^{\text{loop}}(\ldots (a+b)^{\lambda} \ldots)
$$
$$
+ \, \text{Split}_{-\lambda}^{\text{loop}}(a^{\lambda_a}, b^{\lambda_b}) \, A_{n-1}^{\text{tree}}(\ldots (a+b)^{\lambda} \ldots) \Bigg), \tag{III.1}
$$

where $\text{Split}^{\text{tree}}$ and $\text{Split}^{\text{loop}}$ are universal tree and loop "splitting amplitudes", given in the appendix of ref. [12], and the two collinear legs are adjacent in the color ordering.

The $N = 1$ chiral multiplet contribution to the loop splitting amplitudes for gluons vanishes, while in an $N = 4$ supersymmetric theory, the loop splitting function is proportional to the tree splitting function,

$$
\text{Split}_{-\lambda}^{\text{loop}}(a^{\lambda_a}, b^{\lambda_b}) \; = \; c_\Gamma \times \text{Split}_{-\lambda}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) \times r_S^{\text{SUSY}}(z, s_{ab}), \tag{III.2}
$$

where $s_{ab} = (k_a + k_b)^2$, and

$$
r_S^{\text{SUSY}}(z, s) \; = \; -\frac{1}{\epsilon^2} \left(\frac{\mu^2}{z(1-z)(-s)} \right)^{\epsilon} + 2 \ln z \, \ln(1-z) - \frac{\pi^2}{6} \tag{III.3}
$$

is independent of the helicities.

III.1 $N = 1$ MHV Collinear Limits

We begin with the $N = 1$ MHV amplitudes appearing in eq. (5.12). In all collinear limits the net loop splitting amplitude for an $N = 1$ chiral multiplet vanishes, so that defining

$$
V_n^{N=1 \, \text{chiral}} \; = \; \frac{A_{1j}^{N=1 \, \text{chiral}}(1, \ldots, n)}{c_\Gamma A_{1j}^{\text{tree}}(1, \ldots, n)}, \tag{III.4}
$$

the expected behavior of V_n in the collinear limits is simply

$$
V_n^{N=1 \, \text{chiral}} \; \rightarrow \; V_{n-1}^{N=1 \, \text{chiral}}. \tag{III.5}
$$

Denote the collinear legs by c and $c + 1$, and the momentum fraction within the fused momentum k_P by z (so that $k_c = z k_P$ and $k_{c+1} = (1-z) k_P$). Then relabel the fused leg $P \to c$, and shift the labels of legs $c + 2, c + 3, \ldots, n$ down by one, in order to recover the

standard labeling $1, 2, \ldots, n - 1$ for $(n - 1)$-point kinematics. For $2 < j < n$, so that the two negative helicities are not adjacent, and using the reflection symmetry of the $N = 1$ MHV amplitudes, we need only consider two cases, $1 < c, c + 1 < j$ and $c = 1$.

In the case $1 < c, c + 1 < j$, we note that

$$b^j_{c,m_2} \to b^j_{c,m_2-1}, \qquad b^j_{c+1,m_2} \to b^j_{c,m_2-1}, \qquad (\text{III.6})$$

and also that

$$\mathrm{M}_0 \left(t^{[m_2-c]}_{c+1}, t^{[m_2-c]}_c; \ t^{[m_2-c-1]}_{c+1}, t^{[n+c-m_2-1]}_{m_2+1} \right) + \mathrm{M}_0 \left(t^{[m_2-c-1]}_{c+2}, t^{[m_2-c-1]}_{c+1}; t^{[m_2-c-2]}_{c+2}, t^{[n+c-m_2]}_{m_2+1} \right)$$
$$\to \mathrm{M}_0 \left(t^{[m_2-c-1]}_{c+1}, t^{[m_2-c-1]}_c; t^{[m_2-c-2]}_{c+1}, t^{[n+c-m_2-1]}_{m_2} \right) , \qquad (\text{III.7})$$

where the ts on the right-hand side refer to the daughter kinematics. Other M_0s reduce trivially to corresponding ones for the daughter kinematics. As a result, the M_0 terms satisfy equation (III.5) on their own. Furthermore,

$$\begin{aligned}
c^j_{m,c} &\to 0 , \\
c^j_{m,c+1} &\to c^j_{m,c} , \\
c^j_{c,a} &\to z c^j_{c,a-1} , \\
c^j_{c+1,a} &\to (1 - z) c^j_{c,a-1} ,
\end{aligned} \qquad (\text{III.8})$$

all other coefficients reducing in a trivial manner to the corresponding ones for the reduced kinematics. For the generic case $2 < c < j - 2$, the pair of L_0s whose coefficients are $c^j_{c,a}$ and $c^j_{c+1,a}$ combine to produce the appropriate L_0 for V_{n-1},

$$z \frac{\mathrm{L}_0 \left(t^{[a-c]}_{c+1} / t^{[a-c+1]}_c \right)}{t^{[a-c+1]}_c} + (1 - z) \frac{\mathrm{L}_0 \left(t^{[a-c-1]}_{c+2} / t^{[a-c]}_{c+1} \right)}{t^{[a-c]}_{c+1}} \to \frac{\mathrm{L}_0 \left(t^{[a-c-1]}_{c+1} / t^{[a-c]}_c \right)}{t^{[a-c]}_c} . \qquad (\text{III.9})$$

The remaining L_0s as well as the K_0s reduce trivially to those appearing in V_{n-1}. In the boundary cases $c = 2$ and $c + 1 = j - 1$, an L_0 combines with a K_0 to produce the desired K_0. In the case $c = 2$, for example, there is no $c^j_{c,n}$ term in the double sum; instead,

$$c^j_{3,n} \frac{\mathrm{L}_0 \left(t^{[n-3]}_4 / t^{[n-2]}_3 \right)}{t^{[n-2]}_3} + \frac{c^j_{2,n}}{t^{[2]}_1} \mathrm{K}_0 \left(t^{[2]}_1 \right) \to \frac{c^j_{2,n-1}}{t^{[2]}_1} \mathrm{K}_0 \left(t^{[2]}_1 \right). \qquad (\text{III.10})$$

In the second case, $c = 1$. Here, b^j_{c+1,m_2} and $c^j_{c+1,a}$ (as well as $c^j_{m,c}$) vanish in the collinear limit, while again $c^j_{m,c+1} \to c^j_{m,c}$, so that all terms except the $c^j_{3,n}$, $c^j_{2,n}$, $c^j_{n,2}$, and

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$c_{n,1}^j$ terms reduce trivially to the corresponding terms in V_{n-1}. For the remaining terms, note that

$$c_{3,n}^j \to c_{2,n-1}^j \,, \qquad \frac{c_{2,n}^j}{t_1^{[2]}} \to 1 \,, \qquad c_{n,2}^j \to c_{n-1,1}^j \,, \qquad c_{n,1}^j \to 0 \,. \qquad \text{(III.11)}$$

Combining these equations with equation (5.17), we can see that the singular $\ln(-t_1^{[2]}) = \ln(-P^2)$ terms of the collinear limit cancel; furthermore, the remaining parts of the L_0 functions in the $c_{3,n}^j$ and $c_{n,2}^j$ terms reproduce the K_0 functions with $c_{2,n-1}^j$ and $c_{n-1,1}^j$ coefficients respectively.

Finally we examine the collinear limit $1 \parallel 2$ in the amplitude A_{12} with adjacent negative-helicity gluons. In this case all four of the terms on the right-hand-side of (III.1) vanish due to the supersymmetry Ward identity (5.1), so the limit should be nonsingular, and it is so, by virtue of the factors of $\langle 1\,2 \rangle$ in the tree amplitude. This completes the explicit verification of the collinear limits of the $N = 1$ MHV amplitudes.

III.2 $N = 4$ Non-MHV Collinear Limits

Verification of the behavior (III.1) for the six-point non-MHV $N = 4$ amplitudes hinges on the collinear behavior of the coefficient B_0 and the functions $W_6^{(i)}$. When two like-helicity gluons become collinear, B_0 has a singularity. For example, in the limit $5 \parallel 6$ (with $k_5 + k_6 = k_P$),

$$\begin{aligned}
B_0 &\xrightarrow{5\parallel6} \text{Split}_+^{\text{tree}}(5^-, 6^-) \times A_5^{\text{tree}}(1^+, 2^+, 3^+, 4^-, P^-) \,, \\
B_0^\dagger &\xrightarrow{5\parallel6} \text{Split}_-^{\text{tree}}(5^+, 6^+) \times A_5^{\text{tree}}(1^-, 2^-, 3^-, 4^+, P^+) \,,
\end{aligned} \qquad \text{(III.12)}$$

where

$$\text{Split}_+^{\text{tree}}(5^-, 6^-) = -\frac{1}{\sqrt{z(1-z)}\,[5\,6]} \,, \qquad \text{Split}_-^{\text{tree}}(5^+, 6^+) = \frac{1}{\sqrt{z(1-z)}\,\langle 5\,6 \rangle} \,. \qquad \text{(III.13)}$$

The singular behavior of B_0 in the limits $1 \parallel 2$, $2 \parallel 3$ and $4 \parallel 5$ is related to eq. (III.12) by symmetries. In the limit that two opposite-helicity gluons become collinear ($3 \parallel 4$ or $6 \parallel 1$), B_0 is nonsingular. The only result needed regarding the $W_6^{(i)}$ is that the sum of two of the three $W_6^{(i)}$'s has a simple collinear limit in two of the six collinear channels:

$$\begin{aligned}
W_6^{(3)} + W_6^{(1)} &\xrightarrow{s\to0} V_5^g + r_S^{\text{SUSY}}(z, s), & s &= s_{45}, s_{12}, \\
W_6^{(1)} + W_6^{(2)} &\xrightarrow{s\to0} V_5^g + r_S^{\text{SUSY}}(z, s), & s &= s_{23}, s_{56}, \\
W_6^{(2)} + W_6^{(3)} &\xrightarrow{s\to0} V_5^g + r_S^{\text{SUSY}}(z, s), & s &= s_{34}, s_{61},
\end{aligned} \qquad \text{(III.14)}$$

where V_5^g is the universal, cyclicly symmetric function appearing in the $N = 4$ supersymmetric five-gluon amplitudes,

$$V_5^g = \sum_{i=1}^{5} -\frac{1}{\epsilon^2}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^\epsilon + \sum_{i=1}^{5} \ln\left(\frac{-s_{i,i+1}}{-s_{i+1,i+2}}\right) \ln\left(\frac{-s_{i+2,i+3}}{-s_{i-2,i-1}}\right) + \frac{5}{6}\pi^2 , \qquad \text{(III.15)}$$

which should be evaluated in the appropriate five-point kinematics.

Up to symmetries, there are only two inequivalent collinear limits for the $(+++---)$ amplitude, say $5^- \parallel 6^-$ and $6^- \parallel 1^+$. Here we will check the latter limit, which is more intricate in that both helicities of the intermediate gluon P enter. In the limit $6 \parallel 1$, B_0 is nonsingular. The singular behavior of B_2 is

$$B_2 \xrightarrow{6\parallel1} \left(\frac{(1-z)^{1/2}\,[P\,5]\,\langle 5\,4\rangle}{s_{5P}}\right)^4 \left[\text{Split}_+^{\text{tree}}(5^-,6^-) \times A_5^{\text{tree}}(1^+,2^+,3^+,4^-,P^-)\right]\big|_{j\to j+1}$$

$$+ \left(\frac{z^{1/2}\,[2\,3]\,\langle 5\,P\rangle}{s_{5P}}\right)^4 \left[\text{Split}_-^{\text{tree}}(5^+,6^+) \times A_5^{\text{tree}}(1^-,2^-,3^-,4^+,P^+)\right]\big|_{j\to j+1}$$

$$= (1-z)^2\,\text{Split}_+^{\text{tree}}(6^-,1^-) \times \frac{\langle 4\,5\rangle^4}{\langle 5\,P\rangle^4}\,A_5^{\text{tree}}(2^+,3^+,4^+,5^-,P^-)$$

$$+ z^2\,\text{Split}_-^{\text{tree}}(6^+,1^+) \times \frac{[2\,3]^4}{[5\,P]^4}\,A_5^{\text{tree}}(2^-,3^-,4^-,5^+,P^+)$$

$$= \text{Split}_-^{\text{tree}}(6^-,1^+) \times A_5^{\text{tree}}(2^+,3^+,4^-,5^-,P^+)$$

$$+ \text{Split}_+^{\text{tree}}(6^-,1^+) \times A_5^{\text{tree}}(2^+,3^+,4^-,5^-,P^-).$$

$$\text{(III.16)}$$

Using symmetries, the singular behavior of B_3 is identical to (III.16). Using eq. (III.16), the third line of (III.14), and the form (6.1) of the (MHV) five-gluon amplitudes, the expected behavior (III.1) for $6 \parallel 1$ is verified.

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