CONTRIBUTION OF QED TO RATIONAL TERMS IN 1-LOOP FEYNMAN DIAGRAMS IN THE STANDARD MODEL

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ABSTRACT

The abstract goes here.

***K*eywords** QFT · 1-loop Feynman Diagrams · Rational Terms · More

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1 Introduction[1]

In the early 2000s a lot of effort has been put into automating loop calculations in order to keep up with the increasing accuracy of high precision measurements. Early attempts were mostly based on Passarino-Veltman tensor reduction while newer developments changed their focus to unitarity arguments. Instead of performing the whole tensor reduction process, now the problem is substituted by finding the coefficients of the scalar integrals appearing in the 1-loop amplitudes. This is possible because the basis of scalar integrals is know in terms of boxes, triangles, bubbles and tadpoles. This leads to a master formula for any one-loop amplitude

$$\mathcal{M} = \sum_i d_i \mathrm{Box}_i + \sum_i c_i \mathrm{Triangle}_i + \sum_i b_i \mathrm{Bubble}_i + \sum_i a_i \mathrm{Tadpole}_i + R \tag{1.1}$$

The first attempts to extract the coefficients in equation 1.1 failed at providing a systematic procedure to completely determine them. Finally, the OPP method was brought forward which makes it possible to find all of the coefficients for a given theory. But with the OPP method a new class of terms arise (denoted by R in equation 1.1) which are not the coefficient of one of the four types of diagrams. They can be split into two categories

$$R = R_1 + R_2 \tag{1.2}$$

where R_1 can be computed alongside the coefficients of the scalar integrals in OPP.
R_2 on the other hand is the ϵ-dimensional contribution of dimensional regularization to the amplitude. Any m-point 1-loop function $\bar{A}\bar{q})$ can be decomposed in a numerator $\bar{N}(\bar{q})$ and denominators $\bar{D}_i$

$$\bar{A}(\bar{q}) = \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}, \qquad \bar{D}_i = (\bar{q} + \bar{p}_i)^2 - m_i^2, p_0 \neq 0 \tag{1.3}$$

where $\bar{q}$ is the d-dimensional loop momentum and m_i is the mass of the particle corresponding to the propagator with the numerator D_i. The d-dimensional numerator function $\bar{N}(\bar{q})$ can be split in a 4-dimensional and an ϵ-dimensional part

$$\bar{N}(\bar{q}) = N(q) + \widetilde{N}(\tilde{q}^2, q, \epsilon) \tag{1.4}$$

where $\widetilde{N}(\tilde{q}^2, q, \epsilon)$ is of interest to us because it makes up the rational terms of the form R_2 which are defined as

$$R_2 \equiv \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \tag{1.5}$$

R_2 is just a rational combination of Lorentz tensors and parameters of the theory, i.e. the couplings or masses of the particles in the theory. The R_2 contribution can be added to the theory by introducing tree-level like Feynman Rules similarly to counterterms in perturbative renormalization procedures.
To compute R_2 we first have to extract the ϵ-dimensional part of the amplitude by splitting the d-dimensional Lorentz tensors appearing in the amplitude into a 4-dimensional and an ϵ-dimensional part

$$\bar{A}^{\mu_1 \cdots \mu_n} = A^{\mu_1 \cdots \mu_n} + \widetilde{A}^{\mu_1 \cdots \mu_n}. \tag{1.6}$$

To simplify our calculations later, we can establish a few identities for the manipulation of d-dimensional Lorentz tensors. If we contract a d-dimensional tensor with an observable Lorentz tensor (like the momentum of an external particle) only the 4-dimensional part survives, e.g. for a loop momentum $\bar{q}^\mu$ and an external momentum p^μ

$$\bar{q} \cdot p = q \cdot p. \tag{1.7}$$

[1] The whole introduction is based on [1, 2].

Thus, if an amplitude transforms with indices $\mu_1, \ldots, \mu_n$ under a Lorentz transformation, the tensors in the amplitude bearing these indices will only appear as 4-dimensional.
Since, we want to perform calculations in QED which contains a fermion, we have to extend the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$ to d dimensions. This is straightforward by promoting $\gamma^\mu \rightarrow \bar{\gamma}^\mu$ and extending the Minkowski metric to d dimensions by adding additional -1s on the diagonal for the extra spatial dimensions. We have

$$\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2\bar{g}^{\mu\nu}\mathbb{1}_d \tag{1.8}$$

If we want to preserve the Clifford algebra separately in 4 and ϵ dimensions this implies

$$\{\gamma^\mu, \widetilde{\gamma}^\nu\} = 0 \tag{1.9}$$

As opposed to QED the Standard Model is a chiral theory, i.e. it couples differently to left- and right-handed currents. This means that also axial-vector currents appear in the theory which are formulated with the fifth gamma matrix. The extension of γ_5 to d dimensions is not as straightforward as with the four gamma matrices. This is because chirality is a property of four dimensions.
If we also want to impose $\{\gamma_5, \gamma^\mu\} = 0$ for $d \neq 4$, then $\mathrm{Tr}\left(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = 0$ for $d \neq 0, 2, 4$ which clashes with $\mathrm{Tr}\left(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = -4i\epsilon^{\alpha\beta\gamma\delta}$ in four dimensions [3]. But the identity is essential in the evaluation of the triangle diagram for the Adler-Bell-Jackiw anomaly. The only definition of γ_5 which is consistent with the chiral anomaly is the definition of 't Hooft and Veltman [4]: $\gamma_5 = i/4!\ \epsilon_{\mu_1 \ldots \mu_4}\gamma^{\mu_1} \cdots \gamma^{\mu_4}$. This definition implies

$$\{\gamma_5, \gamma^\mu\} = 0 \text{ and } [\gamma_5, \widetilde{\gamma}^\mu] = 0. \tag{1.10}$$

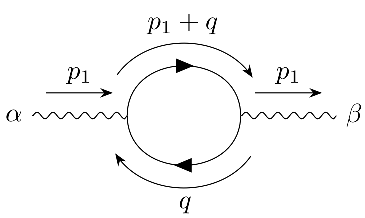
2 R$_2$ in Pure QED

Before we calculate anything in the Standard Model, let us start with the R_2 of pure QED. We have to consider all n-point functions up to $n = 4$ which are allowed by the Feynman rules of QED and calculate their contribution to equation 1.5.

2.1 2-point functions

The Feynman rules of QED allow two 2-point functions; the self-energy diagrams of the photon and the electron. Let us start with the photon self-energy which has the simplest Lorentz structure and therefore an easy to evaluate numerator function.

Photon self-energy
The photon 2-point function is given by



$$= \int \frac{d^d\bar{q}}{(2\pi)^d} (-1) \, \mathrm{Tr} \left\{ ie\bar{\gamma}^\alpha \frac{i\left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right)}{(p_1+q)^2 - m^2} ie\bar{\gamma}^\beta \frac{i\left(\bar{\not{q}} + m\right)}{q^2 - m^2} \right\}$$

$$\equiv \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1\bar{D}_0}$$

Where we defined the numerator and denominator functions in the last step. Now we can extract the ϵ-dimensional contribution from the numerator

$$\bar{N}(\bar{q}) = -e^2 \, \mathrm{Tr}\left\{\bar{\gamma}^\alpha \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right) \bar{\gamma}^\beta \left(\bar{\not{q}} + m\right)\right\} = -e^2 \, \mathrm{Tr}\left\{\gamma^\alpha \left(\not{p}_1 + \not{q} + m\right) \gamma^\beta \left(\not{q} + m\right) + \gamma^\alpha \tilde{\not{q}} \gamma^\beta \tilde{\not{q}}\right\} \equiv N + \widetilde{N}$$

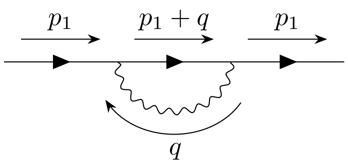
Here, the first term is the 4-dimensional numerator which also appears in normal loop calculations and the second term is the ϵ-dimensional part which we need for the calculation of R_2.
We can now evaluate the trace and get

$$\widetilde{N} = -e^2 \, \mathrm{Tr}\left\{\gamma^\alpha \tilde{\not{q}} \gamma^\beta \tilde{\not{q}}\right\} = 4e^2 \tilde{q}^2 g^{\alpha\beta}$$

Where we have used that the ϵ-dimensional gamma matrices anti-commute with the 4-dimensional gamma matrices and the trace identity for 2 gamma matrices which can be found alongside a proof in appendix B.
Plugging the expression for $\widetilde{N}$ in equation 1.5 gives

$$R_2^{\gamma\gamma} = \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{\widetilde{N}}{\bar{D}_1\bar{D}_0} = \frac{4e^2}{16\pi^4} \underbrace{\int d^d\bar{q} \frac{\tilde{q}^2}{\bar{D}_1\bar{D}_0}}_{-i\frac{\pi}{2}\left(m^2 - p_1^2/3\right)} = \frac{-ie^2}{8\pi^2} g^{\alpha\beta} \left(2m^2 - \frac{p_1^2}{3}\right) \tag{2.1}$$

Electron self-energy
The other 2-point function in QED is the electron 2-point function which is given by



$$= \int \frac{d^dq}{(2\pi)^d} ie\gamma^\alpha \frac{i\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} ie\gamma^\beta \frac{-ig_{\alpha\beta}}{q^2} = \int \frac{d^dq}{(2\pi)^d} \left(-e^2\right) \gamma^\alpha \frac{\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} \gamma_\alpha \frac{1}{q^2}$$

$$\equiv \int \frac{d^dq}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

Now we extract again the ϵ-dimensional part from the numerator function we defined in the last step. We get

$$\bar{N}(\bar{q}) = \left(-e^2\right) \bar{\gamma}^\alpha \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right) \bar{\gamma}_\alpha = -e^2 \left\{ \gamma^\alpha \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right) \gamma_\alpha + \tilde{\gamma}^\alpha \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right) \tilde{\gamma}_\alpha + \gamma^\alpha \tilde{\not{q}} \gamma_\alpha + \tilde{\gamma}^\alpha \tilde{\not{q}} \tilde{\gamma}_\alpha \right\} \equiv N + \tilde{N}$$

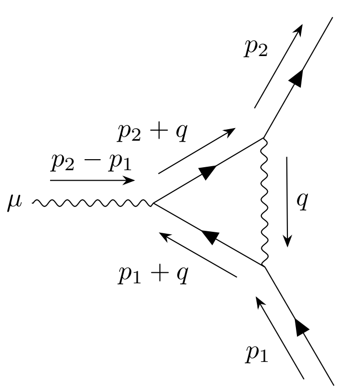
Here, the first term is again the normal 4-dimensional numerator and the rest the ϵ-dimensional part we are interested in.

$$\tilde{N} = -e^2 \left\{ \tilde{\gamma}^\alpha \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right) \tilde{\gamma}_\alpha + \gamma^\alpha \tilde{\not{q}} \gamma_\alpha + \tilde{\gamma}^\alpha \tilde{\not{q}} \tilde{\gamma}_\alpha \right\} = -e^2 \left\{ -\underbrace{\tilde{\gamma}^\alpha \tilde{\gamma}_\alpha}_{=\epsilon} \left(\bar{\not{p}}_1 + \bar{\not{q}} - m\right) - \underbrace{\gamma^\alpha \gamma_\alpha}_{=4} \tilde{\not{q}} + \tilde{\gamma}^\alpha \tilde{\not{q}} \tilde{\gamma}_\alpha \right\}$$

$$R_2^{\text{ee}} = \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{\tilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{-e^2}{(2\pi)^4} \int d^d\bar{q} \frac{1}{\bar{D}_1 \bar{D}_0} \left(-\epsilon \left(\not{p}_1 + \not{q} - m\right) + \underbrace{\tilde{\not{q}}\,(\ldots)}_{=0} \right) =$$

$$= \frac{e^2}{(2\pi)^4} \left\{ \underbrace{\int d^d\bar{q} \frac{\epsilon\left(\not{p}_1 - m\right)}{\bar{D}_1 \bar{D}_0}}_{=-2\epsilon\frac{i\pi^2}{\epsilon}\left(\not{p}_1 - m\right)} + \underbrace{\int d^d\bar{q} \frac{\epsilon \not{q}}{\bar{D}_1 \bar{D}_0}}_{=\epsilon\frac{i\pi^2}{\epsilon}\not{p}_1} \right\} = \frac{e^2}{(2\pi)^4} \epsilon \frac{i\pi^2}{\epsilon} \left((-2)\left(\not{p}_1 - m\right) + \not{p}_1 \right) = \frac{-ie^2}{16\pi^2} \left(\not{p}_1 - 2m\right) \tag{2.2}$$

2.2 3-point functions



$$= \int \frac{d^dq}{(2\pi)^d} ie\gamma^\beta \frac{i\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} ie\gamma^\mu \frac{i\left(\not{p}_2 + \not{q} + m\right)}{(p_2+q)^2 - m^2} ie\gamma^\alpha \frac{-ig_{\alpha\beta}}{q^2}$$

$$\equiv \int \frac{d^dq}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = e^3 \left\{ \bar{\gamma}^\beta \left(\bar{\not{p}}_1 + \bar{\not{q}} + m \right) \bar{\gamma}^\mu \left(\bar{\not{p}}_2 + \bar{\not{q}} + m \right) \bar{\gamma}_\beta \right\} = e^3 \left\{ \gamma^\beta \left(\not{p}_1 + \not{q} + m \right) \gamma^\mu \left(\not{p}_2 + \not{q} + m \right) \gamma_\beta + \right.$$

$$\left. + \widetilde{\gamma}^\beta \left(\not{p}_1 + \not{q} + m \right) \gamma^\mu \left(\not{p}_2 + \not{q} + m \right) \widetilde{\gamma}_\beta + \underbrace{\gamma^\beta \widetilde{\not{q}} \gamma^\mu \widetilde{\not{q}} \gamma_\beta}_{\equiv ①} + \underbrace{\widetilde{\gamma}^\beta \widetilde{\not{q}} \gamma^\mu \widetilde{\not{q}} \widetilde{\gamma}_\beta}_{\equiv ②} \right\} \equiv N + \widetilde{N}$$

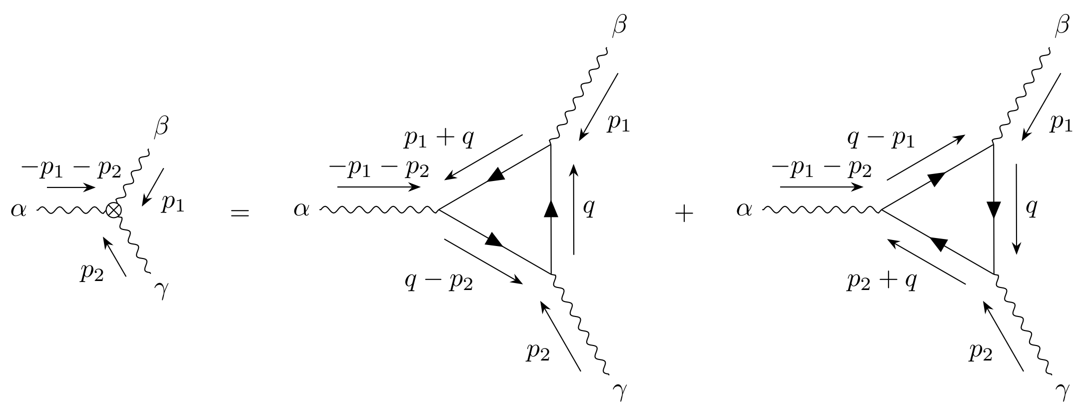
$$① = \widetilde{q}_\rho \widetilde{q}_\sigma \gamma^\beta \widetilde{\gamma}^\rho \gamma^\mu \widetilde{\gamma}^\sigma \gamma_\beta = \widetilde{q}_\rho \widetilde{q}_\sigma \left(-1\right)^3 \widetilde{\gamma}^\rho \widetilde{\gamma}^\sigma \gamma^\beta \gamma^\mu \gamma_\beta = -2 \widetilde{\not{q}} \widetilde{\not{q}} \gamma^\mu = -2 \widetilde{q}^2 \gamma^\mu$$

$$② = \widetilde{q}_\rho \widetilde{q}_\sigma \widetilde{\gamma}^\beta \widetilde{\gamma}^\rho \gamma^\mu \widetilde{\gamma}^\sigma \widetilde{\gamma}_\beta = \widetilde{q}_\rho \widetilde{q}_\sigma \left(-1\right)^2 \gamma^\mu \widetilde{\gamma}^\beta \widetilde{\gamma}^\rho \widetilde{\gamma}^\sigma \widetilde{\gamma}_\beta = \widetilde{q}^2 \gamma^\mu \widetilde{\gamma}^\beta \widetilde{\gamma}_\beta = \epsilon \widetilde{q}^2 \gamma^\mu$$

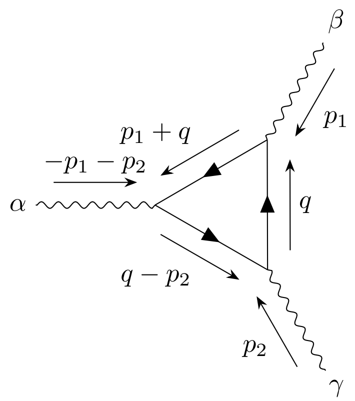
$$\widetilde{N} = -\epsilon \left(\not{p}_1 + \not{q} - m \right) \gamma^\mu \left(\not{p}_2 + \not{q} - m \right) - \left(2 - \epsilon\right) \widetilde{q}^2 \gamma^\mu$$

$$\begin{aligned} R_2^{\gamma ee} =& \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}}{\bar{D}_0 \bar{D}_1 \bar{D}_2} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{e^3}{\bar{D}_0 \bar{D}_1 \bar{D}_2} \left\{ -\epsilon \left(\not{p}_1 + \not{q} - m \right) \gamma^\mu \left(\not{p}_2 + \not{q} - m \right) - \left(2 - \epsilon\right) \widetilde{q}^2 \gamma^\mu \right\} = \\ =& \frac{e^3}{(2\pi)^4} \int d^d \bar{q} \frac{1}{\bar{D}_0 \bar{D}_1 \bar{D}_2} \left\{ -\epsilon \not{q} \gamma^\mu \not{q} - \left(2 - \epsilon\right) \widetilde{q}^2 \gamma^\mu \right\} = \frac{e^3}{(2\pi)^4} \left\{ -\epsilon \gamma^\alpha \gamma^\mu \gamma^\beta \left(\frac{-i\pi^2}{2\epsilon} g_{\alpha\beta} \right) - \frac{-i\pi^2}{2} \left(2 - \epsilon\right) \gamma^\mu \right\} = \\ =& \frac{e^3}{(2\pi)^4} \frac{i\pi^2}{2} \left\{ \gamma^\alpha \gamma^\mu \gamma_\alpha - 2\gamma^\mu + O(\epsilon) \right\} = \frac{-ie^3}{8\pi^2} \gamma^\mu \end{aligned} \tag{2.3}$$

There is one more 3-point function at the 1-loop level which is allowed by the Feynman rules: the 3-point function with only photons as external particles. But it does not contribute to R_2 which we will show now. Because of the symmetry of the 3-point function there are 2 contributing diagrams



We only calculate the first diagram and then symmetrize the result with $p_1 \leftrightarrow p_2$, $\beta \leftrightarrow \gamma$. Evaluating the first diagram gives



$$= \int \frac{d^d q}{(2\pi)^d} \mathrm{Tr} \left\{ ie\gamma^\beta \frac{i\left(\slashed{q} + m\right)}{q^2 - m^2} ie\gamma^\gamma \frac{i\left(\slashed{q} - \slashed{p}_2 + m\right)}{(q - p_2) - m^2} ie\gamma^\alpha \frac{i\left(\slashed{q} + \slashed{p}_1 + m\right)}{(q + p_1) - m^2} \right\} =$$

$$= \int \frac{d^d q}{(2\pi)^d} e^3 \mathrm{Tr} \left\{ \gamma^\beta \frac{\left(\slashed{q} + m\right)}{q^2 - m^2} \gamma^\gamma \frac{\left(\slashed{q} - \slashed{p}_2 + m\right)}{(q - p_2) - m^2} \gamma^\alpha \frac{\left(\slashed{q} + \slashed{p}_1 + m\right)}{(q + p_1) - m^2} \right\} =$$
$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}$$

$$\begin{aligned}
\bar{N}(\bar{q}) &= e^3 \mathrm{Tr} \left\{ \bar{\gamma}^\beta \left(\bar{\slashed{q}} + m\right) \bar{\gamma}^\gamma \left(\bar{\slashed{q}} - \bar{\slashed{p}}_2 + m\right) \bar{\gamma}^\alpha \left(\bar{\slashed{q}} + \bar{\slashed{p}}_1 + m\right) \right\} = \\
&= e^3 \mathrm{Tr} \left\{ \gamma^\beta \left(\slashed{q} + m\right) \gamma^\gamma \left(\slashed{q} - \slashed{p}_2 + m\right) \gamma^\alpha \left(\slashed{q} + \slashed{p}_1 + m\right) + \gamma^\beta \left(\slashed{q} + \widetilde{\slashed{q}} + m\right) \gamma^\gamma \left(\slashed{q} + \widetilde{\slashed{q}} - \slashed{p}_2 + m\right) \gamma^\alpha \left(\slashed{q} + \widetilde{\slashed{q}} + \slashed{p}_1 + m\right) \right\} = \\
&\equiv N + \widetilde{N}
\end{aligned}$$

$$\begin{aligned}
\widetilde{N} &= e^3 \mathrm{Tr} \left\{ \gamma^\beta \left(\slashed{q} + \widetilde{\slashed{q}} + m\right) \gamma^\gamma \left(\slashed{q} + \widetilde{\slashed{q}} - \slashed{p}_2 + m\right) \gamma^\alpha \left(\slashed{q} + \widetilde{\slashed{q}} + \slashed{p}_1 + m\right) \right\} = \\
&= e^3 \mathrm{Tr} \left\{ \gamma^\beta \slashed{q} \gamma^\gamma \widetilde{\slashed{q}} \gamma^\alpha \widetilde{\slashed{q}} + \gamma^\beta \widetilde{\slashed{q}} \gamma^\gamma \slashed{q} \gamma^\alpha \widetilde{\slashed{q}} + \gamma^\beta \widetilde{\slashed{q}} \gamma^\gamma \widetilde{\slashed{q}} \gamma^\alpha \slashed{q} + \gamma^\beta \widetilde{\slashed{q}} \gamma^\gamma \left(-\slashed{p}_2\right) \gamma^\alpha \widetilde{\slashed{q}} + \gamma^\beta \widetilde{\slashed{q}} \gamma^\gamma \widetilde{\slashed{q}} \gamma^\alpha \slashed{p}_1 \right\} = \\
&= -4e^3 \widetilde{q}^2 \left\{ q_\mu \left[\left(g^{\beta\mu} g^{\gamma\alpha} - g^{\beta\gamma} g^{\mu\alpha} + g^{\beta\alpha} g^{\mu\gamma}\right) + \left(g^{\beta\gamma} g^{\alpha\mu} - g^{\beta\mu} g^{\gamma\alpha} + g^{\beta\alpha} g^{\mu\gamma}\right) + \left(g^{\beta\gamma} g^{\mu\alpha} - g^{\beta\alpha} g^{\mu\gamma} + g^{\beta\mu} g^{\alpha\gamma}\right) \right] + \right. \\
&\left. + p_{1\mu} \left(g^{\beta\gamma} g^{\alpha\mu} - g^{\beta\alpha} g^{\gamma\mu} + g^{\beta\mu} g^{\alpha\gamma}\right) - p_{2\mu} \left(g^{\beta\gamma} g^{\mu\alpha} - g^{\beta\mu} g^{\alpha\gamma} + g^{\alpha\beta} g^{\gamma\mu}\right) \right\} = \\
&= -4e^3 \widetilde{q}^2 \left\{ q^\beta g^{\alpha\gamma} + q^\gamma g^{\alpha\beta} + q^\alpha g^{\beta\gamma} + p_1^\alpha g^{\beta\gamma} - p_1^\gamma g^{\alpha\beta} + p_1^\beta g^{\alpha\gamma} - p_2^\alpha g^{\beta\gamma} + p_2^\beta g^{\alpha\gamma} - p_2^\gamma g^{\alpha\beta} \right\}
\end{aligned}$$

This gives for the R_2 contribution of the first diagram

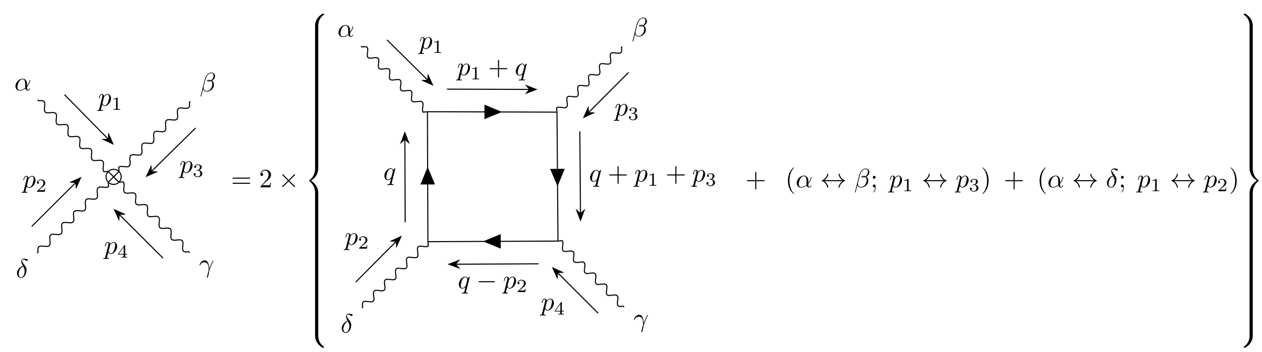
$$
\begin{aligned}
R_2^1 &= \frac{1}{(2\pi)^4}\int d^d\bar{q}\,\frac{\widetilde{N}}{\bar{D}_1\bar{D}_{-2}\bar{D}_0} = \\
&= \frac{-4e^3}{(2\pi)^4}\int d^d\bar{q}\,\frac{1}{\bar{D}_1\bar{D}_{-2}\bar{D}_0}\left\{\tilde{q}^2 q^\beta g^{\alpha\gamma} + \tilde{q}^2 q^\gamma g^{\alpha\beta} + \tilde{q}^2 q^\alpha g^{\beta\gamma} + \tilde{q}^2\left[(p_1-p_2)^\alpha g^{\beta\gamma} + (p_1+p_2)^\beta g^{\alpha\gamma} - (p_1+p_2)^\gamma g^{\alpha\beta}\right]\right\} = \\
&= \frac{-4e^3}{(2\pi)^4}\left\{\frac{i\pi^2}{6}\left[(p_1-p_2)^\beta g^{\alpha\gamma} + (p_1-p_2)^\gamma g^{\alpha\beta} + (p_1-p_2)^\alpha g^{\beta\gamma}\right] - \frac{i\pi^2}{2}\left[(p_1-p_2)^\alpha g^{\beta\gamma} + (p_1+p_2)^\beta g^{\alpha\gamma} + \right.\right. \\
&\left.\left. - (p_1+p_2)^\gamma g^{\alpha\beta}\right]\right\} = \\
&= \frac{-4e^3}{(2\pi)^4}\left\{g^{\alpha\beta}\left[\frac{i\pi^2}{6}(p_1-p_2)^\gamma + \frac{i\pi^2}{2}(p_1+p_2)^\gamma\right] + g^{\beta\gamma}\left[\frac{i\pi^2}{6}(p_1-p_2)^\alpha - \frac{i\pi^2}{2}(p_1-p_2)^\alpha\right] + \right. \\
&\left. + g^{\alpha\gamma}\left[\frac{i\pi^2}{6}(p_1-p_2)^\beta - \frac{i\pi^2}{2}(p_1+p_2)^\beta\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
R_2^2 &= R_2^1(p_1\leftrightarrow p_2,\ \beta\leftrightarrow\gamma) = \\
&= \frac{-4e^3}{(2\pi)^4}\left\{g^{\alpha\gamma}\left[\frac{i\pi^2}{6}(p_2-p_1)^\beta + \frac{i\pi^2}{2}(p_2+p_1)^\beta\right] + g^{\beta\gamma}\left[\frac{i\pi^2}{6}(p_2-p_1)^\alpha - \frac{i\pi^2}{2}(p_2-p_1)^\alpha\right] + \right. \\
&\left. + g^{\alpha\beta}\left[\frac{i\pi^2}{6}(p_2-p_1)^\gamma - \frac{i\pi^2}{2}(p_2+p_1)^\gamma\right]\right\} = \\
&= \frac{-4e^3}{(2\pi)^4}\left\{-g^{\alpha\beta}\left[\frac{i\pi^2}{6}(p_1-p_2)^\gamma + \frac{i\pi^2}{2}(p_1+p_2)^\gamma\right] - g^{\beta\gamma}\left[\frac{i\pi^2}{6}(p_1-p_2)^\alpha - \frac{i\pi^2}{2}(p_1-p_2)^\alpha\right] + \right. \\
&\left. - g^{\alpha\gamma}\left[\frac{i\pi^2}{6}(p_1-p_2)^\beta - \frac{i\pi^2}{2}(p_1+p_2)^\beta\right]\right\} = -R_2^1
\end{aligned}
$$

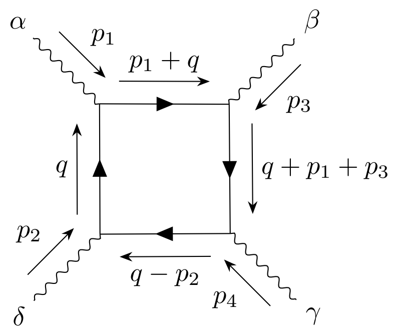
$$
R_2^{3\gamma} = R_2^1 + R_2^2 = R_2^1 - R_2^1 = 0 \tag{2.4}
$$

2.3 4-point function

For the 4-point function we have to be more careful. The 1PI contribution at the 1-loop level consists of several diagrams. They are obtained by symmetrizing the external momenta of the diagram as follows



We only calculate one of the diagrams and do the symmetrizing with the result of our calculation, so we only have to evaluate one diagram. The first of the three diagrams gives



$$= \int \frac{d^dq}{(2\pi)^d} (-1) \,\mathrm{Tr}\left\{ ie\gamma^\alpha \frac{i\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} ie\gamma^\beta \frac{i\left(\not{q} + \not{p}_3 + \not{p}_1 + m\right)}{(p_3+p_1+q)^2 - m^2} \times \right.$$

$$\left. \times ie\gamma^\gamma \frac{i\left(\not{q} - \not{p}_2 + m\right)}{(q-p_2)^2 - m^2} ie\gamma^\delta \frac{i\left(\not{q} + m\right)}{q^2 - m^2} \right\} \equiv \int \frac{d^dq}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_{13} \bar{D}_2 \bar{D}_0}$$

$$\begin{aligned}
\bar{N}(\bar{q}) &= -e^4 \mathrm{Tr}\left\{ \bar{\gamma}^\alpha \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right) \bar{\gamma}^\beta \left(\bar{\not{q}} + \bar{\not{p}}_1 + \bar{\not{p}}_3 + m\right) \bar{\gamma}^\gamma \left(\bar{\not{q}} - \bar{\not{p}}_2 + m\right) \bar{\gamma}^\delta \left(\bar{\not{q}} + m\right) \right\} = \\
&= -e^4 \mathrm{Tr}\left\{ \gamma^\alpha \left(\not{p}_1 + \not{q} + m\right) \gamma^\beta \left(\not{q} + \not{p}_1 + \not{p}_3 + m\right) \gamma^\gamma \left(\not{q} - \not{p}_2 + m\right) \gamma^\delta \left(\not{q} + m\right) + \right. \\
&+ \gamma^\alpha \tilde{\not{q}} \gamma^\beta \tilde{\not{q}} \gamma^\gamma \tilde{\not{q}} \gamma^\delta \tilde{\not{q}} + \gamma^\alpha \tilde{\not{q}} \gamma^\beta \tilde{\not{q}} \gamma^\gamma \not{q} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \tilde{\not{q}} \gamma^\gamma \tilde{\not{q}} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \not{q} \gamma^\gamma \tilde{\not{q}} \gamma^\delta \tilde{\not{q}} + \gamma^\alpha \tilde{\not{q}} \gamma^\beta \not{q} \gamma^\gamma \not{q} \gamma^\delta \tilde{\not{q}} + \\
&\left. + \gamma^\alpha \tilde{\not{q}} \gamma^\beta \not{q} \gamma^\gamma \tilde{\not{q}} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \tilde{\not{q}} \gamma^\gamma \not{q} \gamma^\delta \tilde{\not{q}} \right\} \equiv N + \widetilde{N}
\end{aligned}$$

Where we have used that the trace of an odd number of Dirac matrices is zero. Using (..) and (..) $\widetilde{N}$ can be further simplified to

$$\begin{aligned}
\widetilde{N} =& -e^4 \mathrm{Tr}\left\{ (-1)^{10} \tilde{q}^4 \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta + \tilde{q}^2 \left[(-1)^3 \gamma^\alpha \gamma^\beta \gamma^\gamma \not{q} \gamma^\delta \not{q} + (-1)^7 \gamma^\alpha \not{q} \gamma^\beta \gamma^\gamma \gamma^\delta \not{q} + (-1)^{11} \gamma^\alpha \not{q} \gamma^\beta \not{q} \gamma^\gamma \gamma^\delta + \right.\right. \\
&\left.\left. + (-1)^7 \gamma^\alpha \gamma^\beta \not{q} \gamma^\gamma \not{q} \gamma^\delta + (-1)^5 \gamma^\alpha \gamma^\beta \not{q} \gamma^\gamma \gamma^\delta \not{q} + (-1)^9 \gamma^\alpha \not{q} \gamma^\beta \gamma^\gamma \not{q} \gamma^\delta \right] \right\} = \\
=& -e^4 \mathrm{Tr}\left\{ \tilde{q}^4 \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta - \tilde{q}^2 \left(\gamma^\alpha \gamma^\beta \gamma^\gamma \not{q} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \gamma^\gamma \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \not{q} \gamma^\gamma \gamma^\delta + \gamma^\alpha \gamma^\beta \not{q} \gamma^\gamma \not{q} \gamma^\delta + \right.\right. \\
&\left.\left. + \gamma^\alpha \gamma^\beta \not{q} \gamma^\gamma \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \gamma^\gamma \not{q} \gamma^\delta \right) \right\} \equiv N + \widetilde{N}
\end{aligned}$$

Since this expression involves the trace over up to 6 Dirac matrices, the calculation is very cumbersome. We can evaluate this expression with the help of the Mathematica package FeynCalc [5, 6]

```
In[ ]:= FullSimplify[
  TR[a ^ 2 * GA[α].GA[β].GA[γ].GA[δ] - (* a^2 is \tilde{q}^2 from \tilde{q}^4 term,
    other terms are proportional to \tilde{q}^2*q^2 *)
    (GA[α].GS[q].GA[β].GS[q].GA[γ].GA[δ] + GA[α].GA[β].GA[γ].GS[q].GA[δ].GS[q] + GA[α].GA[β].GS[q].GA[γ].GA[δ].GS[q] +
      GA[α].GA[β].GS[q].GA[γ].GS[q].GA[δ] + GA[α].GS[q].GA[β].GA[γ].GA[δ].GS[q] +
      GA[α].GS[q].GA[β].GA[γ].GS[q].GA[δ])]]
```

Out[ ]= $4\left(a^2\, \bar{g}^{\alpha\beta}\, \bar{g}^{\gamma\delta} - \left(2\,\bar{q}^2 + a^2\right) \bar{g}^{\alpha\gamma}\, \bar{g}^{\beta\delta} + \bar{g}^{\alpha\delta}\left(\left(2\,\bar{q}^2 + a^2\right) \bar{g}^{\beta\gamma} - 2\,\bar{q}^\beta\, \bar{q}^\gamma\right) - 2\,\bar{q}^\alpha\, \bar{q}^\beta\, \bar{g}^{\gamma\delta} - 2\,\bar{q}^\alpha\, \bar{q}^\delta\, \bar{g}^{\beta\gamma} - 2\,\bar{q}^\gamma\, \bar{q}^\delta\, \bar{g}^{\alpha\beta} + 2\,\bar{q}^2\, \bar{g}^{\alpha\beta}\, \bar{g}^{\gamma\delta}\right)$

As usual we plug this in the definition of R_2 and evaluate the integrals to get the expression of R_2 for the first of the contributing diagrams.

$$
\begin{aligned}
R_2 =& \frac{1}{(2\pi)^4}\int d^d\bar{q}\frac{\widetilde{N}}{\bar{D}_0\bar{D}_1\bar{D}_2} = \frac{1}{(2\pi)^4}\int d^d\bar{q}\frac{4e^4}{\bar{D}_1\bar{D}_{13}\bar{D}_2\bar{0}}\widetilde{q}^2\left\{\left(2q^2+\widetilde{q}^2\right)\left(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\beta}g^{\gamma\delta}\right)+\right.\\
&\left.-2\left(g^{\alpha\beta}q^\gamma q^\delta+g^{\gamma\delta}q^\alpha q^\beta+g^{\alpha\delta}q^\beta q^\gamma+g^{\beta\gamma}q^\alpha q^\delta\right)\right\}=\\
&=\frac{-4e^4}{(2\pi)^4}\left\{\left(2\left(\frac{-i\pi^2}{3}\right)+\left(\frac{-i\pi^2}{6}\right)\right)\left(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\beta}g^{\gamma\delta}\right)-2\left(\frac{-i\pi^2}{12}\right)\left(g^{\alpha\beta}g^{\gamma\delta}+g^{\gamma\delta}g^{\alpha\beta}+\right.\right.\\
&\left.\left.+g^{\alpha\delta}g^{\beta\gamma}+g^{\beta\gamma}g^{\alpha\delta}\right)\right\}=\frac{ie^4}{4\pi^2}\left\{\frac{5}{6}\left(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\beta}g^{\gamma\delta}\right)-\frac{1}{6}\left(2g^{\alpha\beta}g^{\gamma\delta}+2g^{\alpha\delta}g^{\beta\gamma}+\right)\right\}=\\
&=\frac{ie^4}{24\pi^2}\left(3g^{\alpha\beta}g^{\gamma\delta}-5g^{\alpha\gamma}g^{\beta\delta}+3g^{\beta\gamma}g^{\alpha\delta}\right)
\end{aligned}
$$

This is independent of momenta, so we only have to symmetrize the indices to get the complete 4-photon R_2.

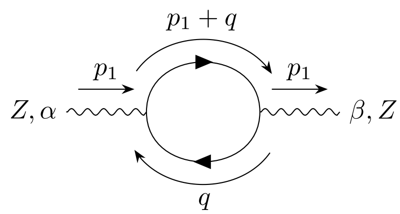
$$
\begin{aligned}
R_2^{4\gamma} =& 2\left[R_2+R_2\left(\alpha\leftrightarrow\delta\right)+R_2\left(\alpha\leftrightarrow\beta\right)\right]=\frac{2ie^4}{24\pi^2}\left\{\left(3g^{\alpha\beta}g^{\gamma\delta}-5g^{\alpha\gamma}g^{\beta\delta}+3g^{\beta\gamma}g^{\alpha\delta}\right)+\left(3g^{\beta\delta}g^{\alpha\gamma}-5g^{\gamma\delta}g^{\alpha\beta}+3g^{\beta\gamma}g^{\alpha\delta}\right)+\right.\\
&\left.+\left(3g^{\alpha\beta}g^{\gamma\delta}-5g^{\beta\gamma}g^{\alpha\delta}+3g^{\alpha\gamma}g^{\beta\delta}\right)\right\}=\frac{ie^4}{12\pi^2}\left(g^{\alpha\beta}g^{\gamma\delta}+g^{\alpha\gamma}g^{\beta\delta}+g^{\beta\gamma}g^{\alpha\delta}\right) \qquad (2.5)
\end{aligned}
$$

Like for the 3-point functions all of the other 4-point functions which are permitted by the Feynman rules vanish. We will not show this here because the calculations for the 4-point functions are quite lengthy.

3 QED Contribution to R_2 in the Standard Model

3.1 2-point functions

Z-boson self-energy



$$= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ \frac{ig}{cos\theta_W} \gamma^\alpha (g_V - g_A \gamma_5) \frac{i\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} \frac{ig}{cos\theta_W} \gamma^\beta \times \right.$$

$$\left. \times\, (g_V - g_A\gamma_5) \frac{i\left(\not{q} + m\right)}{q^2 - m^2} \right\} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{-g^2}{\cos^2\theta_W} \operatorname{Tr} \left\{ \gamma^\alpha (g_V - g_A\gamma_5) \frac{\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} \gamma^\beta (g_V - g_A\gamma_5) \frac{\left(\not{q} + m\right)}{q^2 - m^2} \right\}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

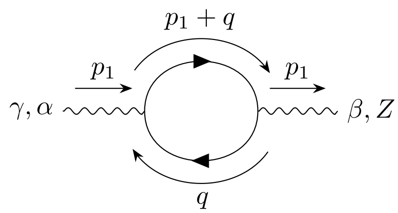
$$\bar{N}(\bar{q}) = -\frac{g^2}{\cos^2\theta_W} \operatorname{Tr}\left\{ \bar{\gamma}^\alpha (g_V - g_A\gamma_5) \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right) \bar{\gamma}^\beta (g_V - g_A\gamma_5) \left(\bar{\not{q}} + m\right) \right\} =$$
$$= \frac{-g^2}{\cos^2\theta_W} \operatorname{Tr}\left\{ \gamma^\alpha (g_V - g_A\gamma_5) \left(\not{p}_1 + \not{q} + m\right) \gamma^\beta (g_V - g_A\gamma_5) \left(\not{q} + m\right) + \gamma^\alpha \left(g_V^2 + g_A^2\right) \tilde{\not{q}} \gamma^\beta \tilde{\not{q}} \right\} \equiv N + \widetilde{N}$$

Where we used $[\gamma_5, \widetilde{\gamma}^\mu] = 0$ and the fact that the gamma matrices will be contracted with external momenta.

$$\widetilde{N} = \frac{-g^2}{\cos^2\theta_W} \left(g_V^2 + g_A^2\right) \left(-\widetilde{q}^2\right) \operatorname{Tr}\left(\gamma^\alpha \gamma^\beta\right) = \frac{4g^2 \widetilde{q}^2}{\cos^2\theta_W} \left(g_V^2 + g_A^2\right) g^{\alpha\beta}$$

$$R_2^{ZZ} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{4g^2 g^{\alpha\beta}}{(2\pi)^4 \cos^2\theta_W} \left(g_V^2 + g_A^2\right) \int d^d \bar{q} \frac{\widetilde{q}^2}{\bar{D}_1 \bar{D}_0} =$$
$$= \frac{4g^2 g^{\alpha\beta}}{(2\pi)^4 \cos^2\theta_W} \left(g_V^2 + g_A^2\right) \left(-\frac{i\pi^2}{2}\right) \left(m^2 - \frac{p_1^2}{3}\right) = \frac{-ig^2}{8\pi^2 \cos^2\theta_W} \left(g_V^2 + g_A^2\right) \left(m^2 - \frac{p_1^2}{3}\right) g^{\alpha\beta} \tag{3.1}$$

Photon/Z-boson mixed self-energy



$$= \int \frac{d^d q}{(2\pi)^d} (-1) \, \mathrm{Tr} \left\{ (-ieQ_f) \, \gamma^\alpha \frac{i \left(\not{p}_1 + \not{q} + m \right)}{(p_1 + q)^2 - m^2} \frac{ig}{cos\theta_W} \gamma^\beta \times \right.$$

$$\left. \times \, (g_V - g_A \gamma_5) \frac{i \left(\not{q} + m \right)}{q^2 - m^2} \right\} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{eQ_f g}{\cos\theta_W} \mathrm{Tr} \left\{ \gamma^\alpha \frac{\left(\not{p}_1 + \not{q} + m \right)}{(p_1 + q)^2 - m^2} \gamma^\beta \left(g_V - g_A \gamma_5 \right) \frac{\left(\not{q} + m \right)}{q^2 - m^2} \right\}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

$$\begin{aligned} \bar{N}(\bar{q}) =& \frac{eQ_f g}{\cos\theta_W} \, \mathrm{Tr} \left\{ \bar{\gamma}^\alpha \left(\bar{\not{p}}_1 + \bar{\not{q}} + m \right) \bar{\gamma}^\beta \left(g_V - g_A \gamma_5 \right) \left(\bar{\not{q}} + m \right) \right\} = \\ =& \frac{eQ_f g}{\cos\theta_W} \mathrm{Tr} \left\{ \gamma^\alpha \left(\not{p}_1 + \not{q} + m \right) \gamma^\beta \left(g_V - g_A \gamma_5 \right) \left(\not{q} + m \right) + \gamma^\alpha \widetilde{\not{q}} \gamma^\beta g_V \widetilde{\not{q}} \right\} \equiv N + \widetilde{N} \end{aligned}$$

Where we have used $\mathrm{Tr} \left(\gamma^\alpha \gamma^\beta \gamma_5 \right) = 0$.

$$\widetilde{N} = \frac{eQ_f g}{cos\theta_W} \mathrm{Tr} \left\{ \gamma^\alpha \widetilde{\not{q}} \gamma^\beta g_V \widetilde{\not{q}} \right\} = \frac{-4eQ_f g g_V}{\cos\theta_W} \widetilde{q}^2 g^{\alpha\beta}$$

$$\begin{aligned} R_2^{\gamma Z} &= \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{-4eQ_f g g_V}{(2\pi)^4 \cos\theta_W} g^{\alpha\beta} \int d^d \bar{q} \frac{\widetilde{q}^2}{\bar{D}_1 \bar{D}_0} \\ &= \frac{-4eQ_f g g_V}{(2\pi)^4 \cos\theta_W} \left(-\frac{i\pi^2}{2} \right) g^{\alpha\beta} \left(m^2 - \frac{p_1^2}{3} \right) = \frac{ieQ_f g g_V}{8\pi^2 \cos\theta_W} g^{\alpha\beta} \left(m^2 - \frac{p_1^2}{3} \right) \end{aligned} \tag{3.2}$$

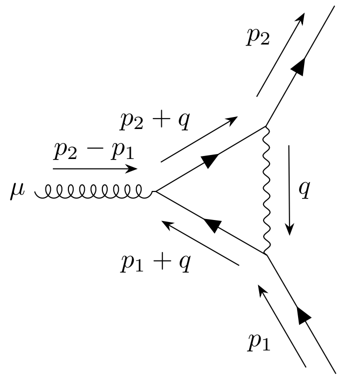
Gluon self-energy

Because the gluon (just as the photon) couples to a pure vector current, the calculation for the gluon self-energy R_2 is the same as for the photon self-energy R_2 replacing the electric charge generator with the colour charge generator. So, from equation 2.1 with $eQ_f \to g_S T^a$ we get

$$R_2^{gg} = R_2^{\gamma\gamma} \left(eQ_f \to g_S T^a \right) = \frac{-ig_S^2}{8\pi^2} \mathrm{Tr} \left(T^a T^b \right) g^{\alpha\beta} \left(2m^2 - \frac{p_1^2}{3} \right) \tag{3.3}$$

3.2 3-point functions

Gluon-quark vertex



$$= \int \frac{d^d q}{(2\pi)^d} \left(-ieQ_q\gamma^\beta\right) \frac{i\left(\not{p}_1 + \not{q} + m\right)}{\left(p_1+q\right)^2 - m^2} \left(-ig_S\gamma^\mu T^a\right) \frac{i\left(\not{p}_2 + \not{q} + m\right)}{\left(p_2+q\right)^2 - m^2} \left(-ieQ_q\gamma^\alpha\right) \frac{-ig_{\alpha\beta}}{q^2} =$$

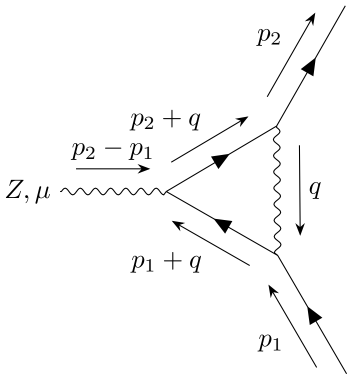
$$= \int \frac{d^d q}{(2\pi)^d} - e^2 Q_q^2 g_S \gamma^\beta \frac{\left(\not{p}_1 + \not{q} + m\right)}{\left(p_1+q\right)^2 - m^2} \gamma^\mu T^a \frac{\left(\not{p}_2 + \not{q} + m\right)}{\left(p_2+q\right)^2 - m^2} \gamma_\beta \frac{1}{q^2} =$$
$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}$$

$$\bar{N}(\bar{q}) = -e^2 Q_q^2 g_S \left\{ \bar{\gamma}^\beta \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right) \bar{\gamma}^\mu T^a \left(\bar{\not{p}}_2 + \bar{\not{q}} + m\right) \bar{\gamma}_\beta \right\} = -e^2 Q_q^2 g_S \left\{ \gamma^\beta \left(\not{p}_1 + \not{q} + m\right) \gamma^\mu T^a \left(\not{p}_2 + \not{q} + m\right) \gamma_\beta + \right.$$
$$\left. + \gamma^\beta \widetilde{\not{q}} \gamma^\mu T^a \widetilde{\not{q}} \gamma_\beta + \widetilde{\gamma}^\beta \not{q} \gamma^\mu T^a \not{q} \widetilde{\gamma}_\beta \right\} \equiv N + \widetilde{N}$$

$$\widetilde{N} = -e^2 Q_q^2 g_S \left\{ -\widetilde{q}^2 \underbrace{\gamma^\beta \gamma^\mu \gamma_\beta}_{-2\gamma^\mu} T^a - \epsilon q_\alpha q_\beta \gamma^\alpha \gamma^\mu \gamma^\beta T^a \right\} = -e^2 Q_q^2 g_S \left\{ 2\widetilde{q}^2 \gamma^\mu T^a - \epsilon q_\alpha q_\beta \gamma^\alpha \gamma^\mu \gamma^\beta T^a \right\}$$

$$\begin{aligned} R_2^{gqq} =& \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0} = \frac{-e^2 Q_q^2 g_S}{(2\pi)^4} \int d^d \bar{q} \frac{1}{\bar{D}_1 \bar{D}_2 \bar{D}_0} \left\{ 2\widetilde{q}^2 \gamma^\mu T^a - \epsilon q_\alpha q_\beta \gamma^\alpha \gamma^\mu \gamma^\beta T^a \right\} = \\ =& \frac{-e^2 Q_q^2 g_S}{(2\pi)^4} \left\{ 2\left(\frac{-i\pi^2}{2}\right) \gamma^\mu T^a - \epsilon \left(\frac{-i\pi^2}{2\epsilon}\right) \underbrace{g_{\alpha\beta} \gamma^\alpha \gamma^\mu \gamma^\beta}_{-2\gamma^\mu} T^a \right\} = \frac{-e^2 Q_q^2 g_S}{16\pi^4} \left(\frac{-i\pi^2}{2}\right) \left\{ 2\gamma^\mu T^a + 2\gamma^\mu T^a \right\} = \\ =& \frac{ie^2 Q_q^2 g_S}{8\pi^2} \gamma^\mu T^a \end{aligned} \tag{3.4}$$

Z-fermion vertex



$$= \int \frac{d^d q}{(2\pi)^d} \left(-ieQ_f\gamma^\beta\right) \frac{i\left(\not p_1 + \not q + m\right)}{(p_1+q)^2 - m^2} \frac{ig}{\cos\theta_W}\gamma^\mu \left(g_V - g_A\gamma_5\right) \frac{i\left(\not p_2+\not q+m\right)}{(p_2+q)^2-m^2}\times$$

$$\times\left(-ieQ_f\gamma^\alpha\right)\frac{-ig_{\alpha\beta}}{q^2} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{e^2Q_f^2 g}{\cos\theta_W}\gamma^\beta \frac{\left(\not p_1 + \not q + m\right)}{(p_1+q)^2 - m^2}\gamma^\mu \left(g_V - g_A\gamma_5\right) \frac{\left(\not p_2+\not q+m\right)}{(p_2+q)^2-m^2}\gamma_\beta\frac{1}{q^2} =$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar N}{\bar D_1 \bar D_2 \bar D_0}$$

$$\bar N(\bar q) = \frac{e^2Q_f^2 g}{\cos\theta_W}\left\{\bar\gamma^\beta\left(\bar{\not p}_1+\bar{\not q}+m\right)\bar\gamma^\mu\left(g_V-g_A\gamma_5\right)\left(\bar{\not p}_2+\bar{\not q}+m\right)\bar\gamma_\beta\right\} = \frac{e^2Q_f^2 g}{\cos\theta_W}\left\{\gamma^\beta\left(\not p_1+\not q+m\right)\gamma^\mu\left(g_V-g_A\gamma_5\right)\times\right.$$

$$\left.\times\left(\not p_2+\not q+m\right)\gamma_\beta + \widetilde\gamma^\beta\left(\not p_1+\not q+m\right)\gamma^\mu\left(g_V-g_A\gamma_5\right)\left(\not p_2+\not q+m\right)\widetilde\gamma_\beta + \left(\gamma^\beta+\widetilde\gamma^\beta\right)\widetilde{\not q}\gamma^\mu\left(g_V-g_A\gamma_5\right)\widetilde{\not q}\left(\gamma_\beta+\widetilde\gamma_\beta\right)\right\} =$$

$$\equiv N+\widetilde N$$

$$\widetilde N = \frac{e^2Q_f^2 g}{\cos\theta_W}\left\{\left(\not p_1+\not q-m\right)\widetilde\gamma^\beta\gamma^\mu\widetilde\gamma_\beta\left(g_V-g_A\gamma_5\right)\left(\not p_2+\not q-m\right) + \gamma^\beta\widetilde{\not q}\gamma^\mu\left(g_V-g_A\gamma_5\right)\widetilde{\not q}\gamma_\beta + \widetilde\gamma^\beta\widetilde{\not q}\gamma^\mu\left(g_V-g_A\gamma_5\right)\widetilde{\not q}\widetilde\gamma_\beta\right\} =$$

$$= \frac{e^2Q_f^2 g}{\cos\theta_W}\left\{-\epsilon\left(\not p_1+\not q-m\right)\gamma^\mu\left(g_V-g_A\gamma_5\right)\left(\not p_2+\not q-m\right) - \widetilde q^2\gamma^\beta\gamma^\mu\gamma_\beta\left(g_V+g_A\gamma_5\right) - \widetilde q^2\left(-\epsilon\gamma^\mu\right)\left(g_V-g_A\gamma_5\right)\right\} =$$

$$= \frac{e^2Q_f^2 g}{\cos\theta_W}\left\{-\epsilon\left(\not p_1+\not q-m\right)\gamma^\mu\left(g_V-g_A\gamma_5\right)\left(\not p_2+\not q-m\right) + \widetilde q^2\left(2\gamma^\mu\left(g_V+g_A\gamma_5\right)+\epsilon\gamma^\mu\left(g_V-g_A\gamma_5\right)\right)\right\} =$$

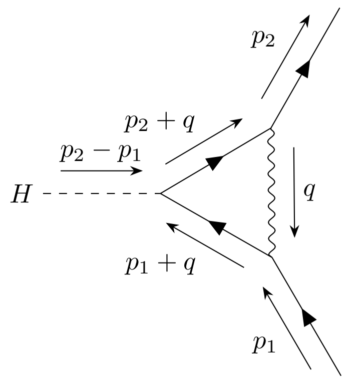
$$R_2^{Zff} = \frac{1}{(2\pi)^4}\int d^d\bar q\frac{\widetilde N}{\bar D_1\bar D_2\bar D_0} = \frac{e^2Q_f^2 g}{(2\pi)^4\cos\theta_W}\int d^d\bar q\frac{1}{\bar D_1\bar D_2\bar D_0}\left\{-\epsilon\not q\gamma^\mu\left(g_V-g_A\gamma_5\right)\not q + \widetilde q^2\left(2\gamma^\mu\left(g_V+g_A\gamma_5\right)+\right.\right.$$

$$\left.\left.+\epsilon\gamma^\mu\left(g_V-g_A\gamma_5\right)\right)\right\} = \frac{e^2Q_f^2 g}{\cos\theta_W}\left\{-\epsilon\left(-\frac{i\pi^2}{2\epsilon}\right)g_{\alpha\beta}\gamma^\alpha\gamma^\mu\gamma^\beta\left(g_V+g_A\gamma_5\right) + 2\left(-\frac{i\pi^2}{2}\right)\gamma^\mu\left(g_V+g_A\gamma_5\right)\right\} =$$

$$= \frac{e^2Q_f^2 g}{\cos\theta_W}\left(-\frac{i\pi^2}{2}\right)\gamma^\mu\left\{2\left(g_V+g_A\gamma_5\right)+2\left(g_V+g_A\gamma_5\right)\right\} = \frac{-ie^2Q_f^2 g}{8\pi^2\cos\theta_W}\gamma^\mu\left(g_V+g_A\gamma_5\right) \tag{3.5}$$

where we used that scalar 3-point integrals do not contribute to R_2. The last term in the integral is of order ϵ so it will not contribute in the limit $\epsilon \to 0$

Higgs-fermion Yukawa vertex



$$= \int \frac{d^d q}{(2\pi)^d} \left(-ieQ_f\gamma^\beta\right) \frac{i\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} \left(-\frac{ig}{2}\frac{m}{m_W}\right) \frac{i\left(\not{p}_2 + \not{q} + m\right)}{(p_2+q)^2 - m^2} \left(-ieQ_f\gamma^\alpha\right) \frac{-ig_{\alpha\beta}}{q^2} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{-e^2Q_f^2 gm}{2m_W} \gamma^\beta \frac{\left(\not{p}_1 + \not{q} + m\right)}{(p_1+q)^2 - m^2} \gamma^\mu \frac{\left(\not{p}_2 + \not{q} + m\right)}{(p_2+q)^2 - m^2} \gamma_\beta \frac{1}{q^2} =$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1\bar{D}_2\bar{D}_0}$$

$$\bar{N}(\bar{q}) = \frac{e^2Q_f^2 gm}{2m_W} \bar{\gamma}^\beta \left(\bar{\not{p}}_1 + \bar{\not{q}} + m\right)\left(\bar{\not{p}}_2 + \bar{\not{q}} + m\right)\bar{\gamma}_\beta =$$

$$= \frac{e^2Q_f^2 gm}{2m_W} \left\{\gamma^\beta \left(\not{p}_1 + \not{q} + m\right)\left(\not{p}_2 + \not{q} + m\right)\gamma_\beta + \widetilde{\gamma}^\beta \left(\not{p}_1 + \not{q} + m\right)\left(\not{p}_2 + \not{q} + m\right)\widetilde{\gamma}_\beta + \gamma^\beta \widetilde{\not{q}}\widetilde{\not{q}}\gamma_\beta\right\} \equiv N + \widetilde{N}$$

$$\widetilde{N} = -\frac{e^2Q_f^2 gm}{2m_W} \left\{\widetilde{\gamma}^\beta \left(\not{p}_1 + \not{q} + m\right)\left(\not{p}_2 + \not{q} + m\right)\widetilde{\gamma}_\beta + \gamma^\beta \widetilde{\not{q}}\widetilde{\not{q}}\gamma_\beta\right\} = -\frac{e^2Q_f^2 gm}{2m_W}\left\{\widetilde{\gamma}^\beta\widetilde{\gamma}_\beta \not{q}\not{q} + \widetilde{\not{q}}\widetilde{\not{q}}\gamma^\beta\gamma_\beta\right\}$$

$$R_2 = \frac{1}{(2\pi)^4}\int d^d\bar{q} \frac{\widetilde{N}}{\bar{D}_1\bar{D}_0\bar{D}_2} = \frac{1}{(2\pi)^4}\int d^d\bar{q}\frac{1}{\bar{D}_1\bar{D}_0\bar{D}_2}\left(-\frac{e^2Q_f^2 gm}{2m_W}\right)\left\{\widetilde{\gamma}^\beta\widetilde{\gamma}_\beta \not{q}\not{q} + \widetilde{\not{q}}\widetilde{\not{q}}\gamma^\beta\gamma_\beta\right\} =$$

$$-\frac{e^2Q_f^2 gm}{2m_W}\left\{\epsilon\left(-\frac{i\pi^2}{2\epsilon}\right)\gamma^\alpha\gamma^\beta g_{\alpha\beta}\not{q}\not{q} + \widetilde{\not{q}}\widetilde{\not{q}}\gamma^\beta\gamma_\beta + \left(-\frac{i\pi^2}{2}\right)4\right\} = \frac{-e^2Q_f^2 gm}{(2\pi)^4\, 2m_W}\left(-\frac{i\pi^2}{2}\right)8 = \frac{ie^2Q_f^2 gm}{8\pi^2 m_W}$$

4 Perturbative Renormalization in Terms of Scalar Integrals

Explain how to express renormalization constants in terms of scalar integrals.

We start from the QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{0}_{\mu\nu}F_0^{\mu\nu} + \bar{\psi}_0\left(i\partial\!\!\!/ - m_0\right)\psi_0 - e_0\bar{\psi}_0 A\!\!\!/_0\psi_0 \tag{4.1}$$

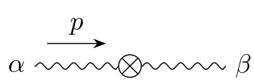
where $F_0^{\mu\nu} = \partial^\mu A_0^\nu - \partial^\nu A_0^\mu$. Now, we reinterpret the fields and parameters in the Lagrangian as "bare" fields and parameters which are given by the actual "renormalized" quantities times a renormalization constant

$$\begin{aligned} \psi_0 &= \sqrt{Z_2}\psi \\ A_0^\mu &= \sqrt{Z_3}A^\mu \\ m_0 &= Z_m m \\ e_0 &= Z_e e \mu^{-\frac{\epsilon}{2}} \end{aligned} \tag{4.2}$$

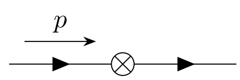
The renormalization constants Z_i absorb the divergences which appear in loop calculations. We can split them as $Z_i = 1 + \delta_i$ to extract the renormalized Lagrangian which is divergence free and the so called counter-term Lagrangian which absorbs the divergences

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} + iZ_2\bar{\psi}\partial\!\!\!/\psi - Z_m Z_2 m\bar{\psi}\psi - eZ_1\bar{\psi}A\!\!\!/\psi = \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\bar{\psi}A\!\!\!/\psi - \frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + i\delta_2\bar{\psi}\partial\!\!\!/\psi - \left(\delta_m + \delta_2\right)m\bar{\psi}\psi - e\delta_1\bar{\psi}A\!\!\!/\psi \equiv \mathcal{L}_{ren} + \mathcal{L}_{ct} \end{aligned} \tag{4.3}$$

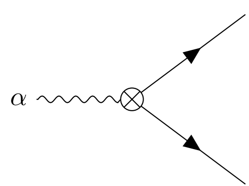
where $Z_1 = Z_e Z_2\sqrt{Z_3}\mu^{-\frac{\epsilon}{2}}$.
The counter term Lagrangians gives the following new Feynman rules



$$= i\left(p^\alpha p^\beta - g^{\alpha\beta}p^2\right)\delta_3$$

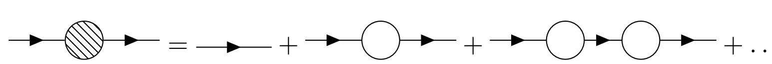


$$= i\left(p\!\!\!/\delta_2 - \delta_m\right)$$



$$= -ie\gamma^\mu\delta_1 \tag{4.4}$$

We can use these new Feynman rules to calculate the Z_i in order to be able to make predictions with perturbative calculations. These renormalization conditions can be obtained by calculating the dressed propagators and requiring that the propagators have a pole at the physical mass.
Let's start with the electron propagator. The dressed propagator is given by a sum of so called 1-particle irreducible insertions (i.e. insertions of subdiagrams which do not fall apart when one of the internal lines is cut) as follows



where the empty circles on the right represent renormalized 1-PI interactions and the appropriate counter terms. This gives

$$iS_0(\not{p}) = iS(\not{p}) + iS(\not{p})i\Sigma'(\not{p})iS(\not{p}) + iS(\not{p})i\Sigma'(\not{p})iS(\not{p})i\Sigma'(\not{p})iS(\not{p}) + \dots \tag{4.5}$$

where $i\Sigma'(\not{p}) = i\Sigma(\not{p}) + i\left(\delta_2\not{p} - (\delta_2 + \delta_m)\, m\right)$, $iS_0 = \frac{i}{\not{p}-m_0}$ and $iS = \frac{i}{\not{p}-m}$. Now we can sum the geometric series in $i\Sigma'(\not{p})$ which yields

$$\frac{i}{\not{p} - m_0} = \frac{i}{\not{p} - m + \left(\Sigma(\not{p}) + \delta_2\not{p} - (\delta_2 + \delta_m)\, m\right)} \tag{4.6}$$

By requiring the dressed propagator to have a pole at the physical mass $\not{p} = m_{\text{phys}} = m$ we obtain

$$m - m + \Sigma(m) + \delta_2 m - (\delta_2 + \delta_m)\, m = 0 \tag{4.7}$$

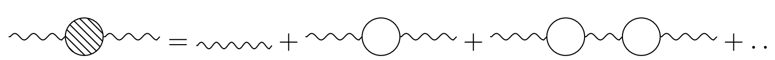
$$\Rightarrow \delta_m = \frac{1}{m}\Sigma(m) \tag{4.8}$$

We also want the propagator to have a residue of unity at the pole. This gives the the renormalization condition for the electron field

$$\begin{aligned}\text{Res}_{\not{p}=m}\left(S(\not{p})\right) &= \text{Res}_{\not{p}=m}\left(\frac{1}{\not{p} - m + \left(\Sigma(\not{p}) + \delta_2\not{p} - (\delta_2 + \delta_m)\, m\right)}\right) = \\ &= \lim_{\not{p}\to m} \frac{\not{p} - m}{\not{p} - m + \left(\Sigma(\not{p}) + \delta_2\not{p} - (\delta_2 + \delta_m)\, m\right)} \overset{\text{L'H}}{=} \lim_{\not{p}\to m} \frac{1}{1 + \frac{d\Sigma}{d\not{p}} + \delta_2} \overset{!}{=} 1\end{aligned}$$

$$\Rightarrow \delta_2 = -\left.\frac{d\Sigma(\not{p})}{d\not{p}}\right|_{\not{p}=m} \tag{4.9}$$

Z_1 and Z_2 are related by symmetry, so we do not have to evaluate the electron-photon 3-point function. It was first shown by Ward in 1950 that $Z_1 = Z_2$ [8]. The only remaining renormalization constant from equations 4.2 is Z_3. It can be obtained from the dressed photon propagator in the same way we obtained the electron field renormalization from the electron propagator. The dressed photon operator is given by



where the empty circles are again insertions of 1-Pi diagrams and the appropriate counter term. So, we have

$$iS_0^{\alpha\beta}(p^2) = iS^{\alpha\beta}(p^2) + \left[iS(p^2)i\Pi'(p^2)iS(p^2)\right]^{\alpha\beta} + \left[iS(p^2)i\Pi'(\not{p})iS(p^2)i\Pi'(p^2)iS(p^2)\right]^{\alpha\beta} + \dots \tag{4.10}$$

with $iS_0^{\alpha\beta} = \frac{-i}{p^2}\left(g^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}\right) = iS^{\alpha\beta}$ and $i\Pi'^{\alpha\beta} = i\Pi^{\alpha\beta} + i\delta_3\left(p^\alpha p^\beta - g^{\alpha\beta}p^2\right)$. Due to gauge invariance and the respective Ward identity we must have $\Pi^{\alpha\beta} = \left(p^\alpha p^\beta - p^2 g^{\alpha\beta}\right)\Pi(p^2)$, since the Ward identity demands $p_\alpha \Pi^{\alpha\beta} = 0 = \left(p^2 p^\beta - p^2 p^\beta\right)\Pi(p^2) \checkmark$.
Now we can sum the geometric series in $i\Pi'(p^2)$ which yields

$$\frac{-i}{p^2}\left(g^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}\right) = \left(g^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}\right)\frac{-i}{p^2\left(1 + \Pi(p^2) + \delta_3\right)} \tag{4.11}$$

By requiring the propagator to have a pole at the physical photon mass $p^2 = 0$ we get

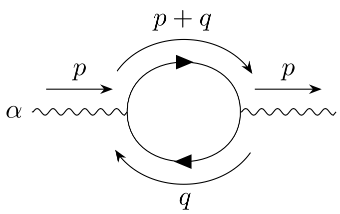
$$\delta_3 = -\Pi(0) \tag{4.12}$$

The renormalization procedure for the whole Standard Model is obviously a lot more involved, since there are a lot more fields and parameters in the theory. But it still follows the same lines as for the simpler QED case. The whole derivation for the renormalization conditions of the electroweak part of the Standard Model can be found in [7]. We will use the results from there and calculate the needed self-energies.

4.1 Pure QED Renormalization

We now have to calculate the self-energy of the photon and the electron to evaluate the renormalization constants. Since our goal is to automate 1-loop calculations in QED and their contributions to the Standard Model it is convenient to express the results in terms of scalar integrals (see Appendix A) which can be easily implemented.

Photon self-energy



$$= \int \frac{d^4q}{(2\pi)^4} (-1) \, \mathrm{Tr} \left\{ ie\gamma^\alpha \frac{i\left(\not p + \not q + m\right)}{(p+q)^2 - m^2} ie\gamma^\beta \frac{i\left(\not q + m\right)}{q^2 - m^2} \right\} \equiv i\Pi^{\alpha\beta}$$

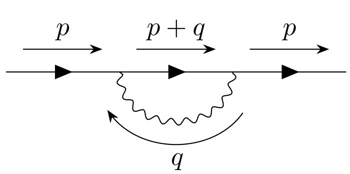
Let's work on the trace so we can express the numerator of the 2-point function in terms of scalar integrals.

$$\begin{aligned}
&\mathrm{Tr}\left\{\gamma^\alpha \left(\not p + \not q + m\right)\gamma^\beta \left(\not q + m\right)\right\} = \mathrm{Tr}\left\{m^2\gamma^\alpha\gamma^\beta + \gamma^\alpha \left(\not p + \not q\right)\gamma^\beta \not q\right\} = \\
&= 4\left\{m^2 g^{\alpha\beta} + (p+q)_\mu q_\nu \left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right)\right\} = \\
&= 4\left(m^2 g^{\alpha\beta} + (p+q)^\alpha q^\beta - g^{\alpha\beta}(p+q)\cdot q + g^\alpha (p+q)^\beta\right)
\end{aligned}$$

$$\begin{aligned}
i\Pi^{\alpha\beta} &= -4e^2 \int \frac{d^4q}{(2\pi)^4} \frac{m^2 g^{\alpha\beta} + p^\alpha q^\beta + q^\alpha p^\beta + 2q^\alpha q^\beta - g^{\alpha\beta} p\cdot q - g^{\alpha\beta} q^2}{\left((p+q)^2 - m^2\right)(q^2 - m^2)} = \\
&= -\frac{4ie^2}{16\pi^2}\left\{m^2 B_0 g^{\alpha\beta} + 2p^{\alpha\beta} B_1 + 2\left(B_{11}p^\alpha p^\beta + B_{00}g^{\alpha\beta}\right) - g^{\alpha\beta}B_1 p^2 - g^{\alpha\beta}\left(4B_{00} + B_{11}p^2\right)\right\} = \\
&= -\frac{ie^2}{4\pi^2}\left\{g^{\alpha\beta}\left(m^2 B_0 - B_1 p^2 + B_{11}p^2 - 2B_{00}\right) + 2p^\alpha p^\beta \left(B_1 + B_{11}\right)\right\}
\end{aligned}$$

The arguments of the scalar integrals are suppressed to keep the notation compact. They are the same for all B-functions: $B_i = B_i(p^2, m^2, m^2)$.
The expression can be further simplified using identities between the scalar integrals.

Electron self-energy



$$= \int \frac{d^4q}{(2\pi)^4} ie\gamma^\alpha \frac{i\left(\not p + \not q + m\right)}{(p+q)^2 - m^2} ie\gamma^\beta \frac{-ig_{\alpha\beta}}{q^2} = \int \frac{d^4q}{(2\pi)^4}\left(-e^2\right)\gamma^\alpha \frac{\left(\not p + \not q + m\right)}{(p+q)^2 - m^2}\gamma_\alpha \frac{1}{q^2} \equiv i\Sigma(\not p)$$

With a bit of gamma-matrix algebra the numerator can be written as

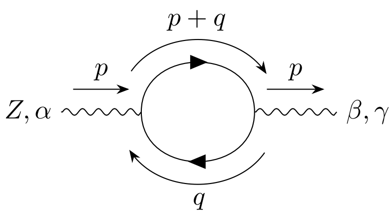
$$\gamma^\beta \left(\not p + \not q + m\right)\gamma_\beta = (p+q)_\alpha \gamma^\beta\gamma^\alpha\gamma_\beta + m\gamma^\beta\gamma_\beta = 4m - 2\left(\not p + \not q\right)$$

$$
\begin{aligned}
i\Sigma(\not{p}) &= -e^2 \int \frac{d^4q}{(2\pi)^4} \frac{4m - 2\left(\not{p} + \not{q}\right)}{\left((p+q)^2 - m^2\right) q^2} = \\
&= -\frac{ie^2}{16\pi^2}\left[4mB_0 - 2\not{p}\left(B_0 + B_1\right)\right] = \frac{-ie^2}{8\pi^2}\left(2mB_0 - \not{p}\left(B_0 + B_1\right)\right)
\end{aligned}
$$

Where the arguments of the B-functions are suppressed again. They are $B_i = B_i(p^2, 0, m^2)$

4.2 QED Contribution to the Standard Model Renormalization

Photon/Z-boson mixed self-energy



$$
= \int \frac{d^4q}{(2\pi)^4} (-1)\,\mathrm{Tr}\left\{ \left(-ieQ_f\right)\gamma^\alpha \frac{i\left(\not{p} + \not{q} + m\right)}{(p+q)^2 - m^2} i\frac{g}{\cos\theta_W}\gamma^\beta\left(g_V - g_A\gamma_5\right)\frac{i\left(\not{q} + m\right)}{q^2 - m^2}\right\} \equiv i\Pi^{\alpha\beta}_{\gamma Z}
$$

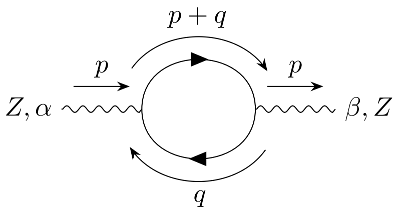
Let's work on the trace so we can express the numerator of the 2-point function in terms of scalar integrals.

$$
\begin{aligned}
&\mathrm{Tr}\left\{\gamma^\alpha\left(\not{p} + \not{q} + m\right)\gamma^\beta\left(g_V - g_A\gamma_5\right)\left(\not{q} + m\right)\right\} = g_V\,\mathrm{Tr}\left\{\gamma^\alpha\left(\not{p} + \not{q}\right)\gamma^\beta\not{q} + m^2\gamma^\alpha\gamma^\beta\right\} - g_A\,\mathrm{Tr}\left\{\gamma^\alpha\left(\not{p} + \not{q}\right)\gamma^\beta\gamma_5\not{q}\right\} = \\
&= 4g_V\left\{(p+q)_\mu q_\nu\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right) + m^2 g^{\alpha\beta}\right\} - 4ig_A\,(p+q)_\mu q_\nu \epsilon^{\alpha\mu\beta\nu} = \\
&= 4\left\{g_V\left[(p+q)^\alpha q^\beta - g^{\alpha\beta}(p+q)\cdot q + q^\alpha (p+q)^\beta + m^2 g^{\alpha\beta}\right] - ig_A\epsilon^{\alpha\mu\beta\nu}p_\mu q_\nu\right\}
\end{aligned}
$$

Where we used that a symmetric tensor contracted with an antisymmetric tensor vanishes.

$$
\begin{aligned}
i\Pi^{\alpha\beta}_{\gamma Z} &= \frac{4Q_f eg}{\cos\theta_W}\int\frac{d^4q}{(2\pi)^4}\frac{g_V\left((p+q)^\alpha q^\beta - g^{\alpha\beta}(p+q)\cdot q + q^\alpha(p+q)^\beta + m^2 g^{\alpha\beta}\right) - ig_A\epsilon^{\alpha\mu\beta\nu}p_\mu q_\nu}{\left((p+q)^2 - m^2\right)\left(q^2 - m^2\right)} = \\
&= \frac{4Q_f eg}{\cos\theta_W}\frac{i\pi^2}{(2\pi)^4}\left\{-ig_A\epsilon^{\alpha\mu\beta\nu}p_\mu B_1 p_\nu + g_V\left[B_1 p^\alpha p^\beta + B_{00}g^{\alpha\beta} + B_{11}p^\alpha p^\beta - g^{\alpha\beta}\left(B_1 p^2 + 4B_{00} + B_{11}p^2\right) + \right.\right. \\
&\left.\left. + B_1 p^\alpha p^\beta + B_{00}g^{\alpha\beta} + B_{11}p^\alpha p^\beta + B_0 m^2 g^{\alpha\beta}\right]\right\} = \\
&= \frac{iQ_f egg_V}{4\pi^2\cos\theta_W}\left\{2p^\alpha p^\beta\left(B_1 + B_{11}\right) + g^{\alpha\beta}\left(m^2 B_0 - 2B_{00} - p^2\left(B_1 + B_{11}\right)\right)\right\}
\end{aligned}
$$

Z-Boson self-energy



$$= \int \frac{d^4q}{(2\pi)^4} (-1) \,\mathrm{Tr} \left\{ \frac{ig}{\cos\theta_W} \gamma^\alpha \left(g_V - g_A\gamma_5\right) \frac{i\left(\not{p} + \not{q} + m\right)}{(p+q)^2 - m^2} \frac{ig}{\cos\theta_W} \gamma^\beta \times \right.$$

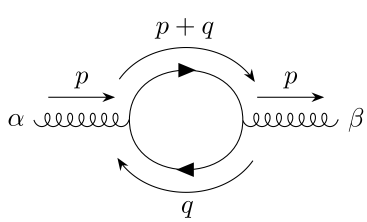
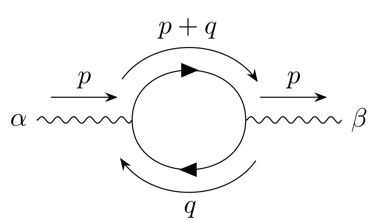
$$\left. \times \left(g_V - g_A\gamma_5\right) \frac{i\left(\not{q} + m\right)}{q^2 - m^2} \right\} \equiv i\Pi^{\alpha\beta}_{ZZ}$$

Let's work on the trace so we can express the numerator of the 2-point function in terms of scalar integrals.

$$\mathrm{Tr}\left\{\gamma^\alpha \left(g_V - g_A\gamma_5\right)\left(\not{p} + \not{q} + m\right)\gamma^\beta\left(g_V - g_A\gamma_5\right)\left(\not{q} + m\right)\right\} = \mathrm{Tr}\left\{\gamma^\alpha\left(g_V - g_A\gamma_5\right)^2\left(\not{p} + \not{q} - m\right)\gamma^\beta\left(\not{q} + m\right)\right\} =$$
$$= \left(g_V^2 + g_A^2\right)\mathrm{Tr}\left\{\gamma^\alpha\left(\not{p} + \not{q} - m\right)\gamma^\beta\left(\not{q} + m\right)\right\} = \left(g_V^2 + g_A^2\right)\mathrm{Tr}\left\{\gamma^\alpha\left(\not{p} + \not{q}\right)\gamma^\beta\not{q} - m^2\gamma^\alpha\gamma^\beta\right\} =$$
$$= \left(g_V^2 + g_A^2\right)\left\{(p+q)_\mu q_\nu 4\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right) - 4m^2 g^{\alpha\beta}\right\} =$$
$$= 4\left(g_V^2 + g_A^2\right)\left\{(p+q)^\alpha q^\beta - g^{\alpha\beta}(p+q)\cdot q + q^\alpha (p+q)^\beta - m^2 g^{\alpha\beta}\right\}$$

$$i\Pi^{\alpha\beta}_{ZZ} = \int \frac{d^4q}{(2\pi)^4} \frac{4g^2\left(g_V^2 + g_A^2\right)}{\cos^2\theta_W} \frac{(p+q)^\alpha q^\beta - g^{\alpha\beta}(p+q)\cdot q + q^\alpha(p+q)^\beta - m^2 g^{\alpha\beta}}{\left((p+q)^2 - m^2\right)\left(q^2 - m^2\right)} =$$
$$= \frac{4g^2\left(g_V^2 + g_A^2\right)}{\cos^2\theta_W} \int \frac{d^4q}{(2\pi)^4} \frac{p^\alpha q^\beta + q^\alpha p^\beta + 2q^\alpha q^\beta - g^{\alpha\beta}\left(m^2 + q\cdot p + q^2\right)}{\left((p+q)^2 - m^2\right)\left(q^2 - m^2\right)} =$$
$$= \frac{4g^2\left(g_V^2 + g_A^2\right)}{\cos^2\theta_W} \frac{i\pi^2}{(2\pi)^4} \left\{2p^\alpha p^\beta B_1 + 2\left(B_{00}g^{\alpha\beta} + B_{11}p^\alpha p^\beta\right) - g^{\alpha\beta}\left(m^2 B_0 + B_1 p^2 + 4B_{00} + B_{11}p^2\right)\right\} =$$
$$= \frac{ig^2\left(g_V^2 + g_A^2\right)}{4\pi^2\cos^2\theta_W} \left\{p^\alpha p^\beta 2\left(B_1 + B_{11}\right) - g^{\alpha\beta}\left(m^2 B_0 + 2B_{00} + p^2\left(B_1 + B_{11}\right)\right)\right\}$$

Gluon self-energy

$i\Pi^{\alpha\beta}_{gg} =$  $=$  $\left(eQ_q \to g_S T^a\right) =$

$$= -\frac{ig_S^2 \mathrm{Tr}\left(T^a T^b\right)}{4\pi^2} \left\{2p^\alpha p^\beta\left(B_1 + B_{11}\right) + g^{\alpha\beta}\left(m^2 B_0 - 2B_{00} - p^2\left(B_1 + B_{11}\right)\right)\right\}$$

Appendices

A Important Integrals

In the calculation of R_2 we have to evaluate 2-,3- and 4-point functions. They can be reduced to a set of integrals which are knwon in a general form. The integrals we need are [1]

2-point integrals

$$\int d^d\bar{q}\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} = -\frac{i\pi^2}{2}\left[m_i^2+m_j^2-\frac{\left(p_i-p_j\right)^2}{3}\right]+O(\epsilon) \tag{A.1}$$

$$\text{P.P.}\left(\int d^d\bar{q}\frac{1}{\bar{D}_i\bar{D}_j}\right) = -2\frac{i\pi^2}{\epsilon} \tag{A.2}$$

$$\text{P.P.}\left(\int d^d\bar{q}\frac{q_\mu}{\bar{D}_i\bar{D}_j}\right) = \frac{i\pi^2}{\epsilon}\left(p_i+p_j\right)_\mu \tag{A.3}$$

3-point integrals

$$\int d^d\bar{q}\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} = -\frac{i\pi^2}{2}+O(\epsilon) \tag{A.4}$$

$$\int d^d\bar{q}\frac{\tilde{q}^2 q_\mu}{\bar{D}_i\bar{D}_j\bar{D}_k} = \frac{i\pi^2}{6}\left(p_i+p_j+p_k\right)_\mu+O(\epsilon) \tag{A.5}$$

$$\text{P.P.}\left(\int d^d\bar{q}\frac{q_\mu q_\nu}{\bar{D}_i\bar{D}_j\bar{D}_k}\right) = -\frac{i\pi^2}{2\epsilon}g_{\mu\nu} \tag{A.6}$$

4-point integrals

$$\int d^d\bar{q}\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{i\pi^2}{6}+O(\epsilon) \tag{A.7}$$

$$\int d^d\bar{q}\frac{\tilde{q}^2 q_\mu q_\nu}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{i\pi^2}{12}g_{\mu\nu}+O(\epsilon) \tag{A.8}$$

$$\int d^d\bar{q}\frac{\tilde{q}^2 q^2}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} = -\frac{i\pi^2}{3}+O(\epsilon) \tag{A.9}$$

B Traceology

In a theory with fermions the Dirac matrices appear as the generators of the spinor representation of the Poincaré algebra. The following identities for Dirac matrices are very useful when evaluating Feynman diagrams

1. $\text{Tr}\left(\gamma^\alpha\gamma^\beta\right) = dg^{\alpha\beta}$
2. $\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\delta}g^{\beta\gamma}\right)$
3. $\gamma^\alpha\gamma_\alpha = d$
4. $\gamma^\alpha\gamma^\beta\gamma_\alpha = (2-d)\,\gamma^\beta$

... ...

n. $\not{a}\not{b} = a \cdot b$

The Dirac matrices obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_d$ with $g^{\mu\nu}$ the Minkowski metric in d dimensions

$$g^{\mu\nu} = \begin{cases} 1 & \text{for } \mu = \nu = 0 \\ -1 & \text{for } \mu = \nu = 1, 2, \dots, d-1 \\ 0 & \text{for } \mu \neq \nu \end{cases}$$

Proofs for identities

1. $\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) = dg^{\alpha\beta}$

Proof.

$$\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) = \mathrm{Tr}\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right) = 2g^{\alpha\beta}\mathrm{Tr}\left(\mathbb{1}_d\right) - \mathrm{Tr}\left(\gamma^\beta\gamma^\alpha\right) = 2dg^{\alpha\beta} - \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right)$$
$$\Rightarrow \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) = dg^{\alpha\beta}$$

□

2. $\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$

Proof.

$$\begin{aligned} \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) &= \mathrm{Tr}\left(\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right)\gamma^\gamma\gamma^\delta\right) = 2g^{\alpha\beta}\mathrm{Tr}\left(\gamma^\gamma\gamma^\delta\right) - \mathrm{Tr}\left(\gamma^\beta\left(2g^{\alpha\gamma} - \gamma^\gamma\gamma^\alpha\right)\gamma^\delta\right) = \\ &= 2dg^{\alpha\beta}g^{\gamma\delta} - 2g^{\alpha\gamma}\mathrm{Tr}\left(\gamma^\beta\gamma^\delta\right) + \mathrm{Tr}\left(\gamma^\beta\gamma^\gamma\left(2g^{\alpha\delta} - \gamma^\delta\gamma^\alpha\right)\right) = \\ &= 2d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) + 2g^{\alpha\delta}\mathrm{Tr}\left(\gamma^\beta\gamma^\gamma\right) - \mathrm{Tr}\left(\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\alpha\right) = \\ &= 2d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right) - \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) \end{aligned}$$
$$\Rightarrow \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$$

□

3. $\gamma^\alpha\gamma_\alpha = d$

Proof.

$$\gamma^\alpha\gamma_\alpha = \frac{1}{2}\left(\gamma^\alpha\gamma_\alpha + \gamma_\alpha\gamma^\alpha\right) = \frac{1}{2}\{\gamma^\alpha, \gamma_\alpha\} = \frac{1}{2}2g^\alpha_{\ \alpha} = d$$

□

4. $\gamma^\alpha\gamma^\beta\gamma_\alpha = (2-d)\gamma^\beta$

Proof.

$$\gamma^\alpha\gamma^\beta\gamma_\alpha = \left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right)\gamma_\alpha = (2-d)\gamma^\beta$$

□

n. $\not{a}\not{b} = a \cdot b$

Proof.

$$\not{a}\not{b} = a_\alpha b_\beta\gamma^\alpha\gamma^\beta = a_\alpha b_\beta\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right) = 2a \cdot b - \not{a}\not{b}$$

$$\Rightarrow \not{a}\not{b} = a \cdot b$$

□

C Relation Between Left- & Right-handed Currents and Axial & Vector Currents

A classical Lagrangian permits symmetries which can be implemented by Lie groups G. An element $g \in G$ of a Lie group can be parametrized as $g = \exp(i\alpha^a T^a)$ where α^a are real parameters and T^a the generators of the Lie group. Noether's theorem predicts a clasically conserved current for each generator of a continuous symmetry. For a field ϕ with trafo $\delta\phi = \phi' - \phi = g\phi - \phi \approx (1 + i\alpha^a T^a)\phi - \phi = i\alpha^a T^a \phi$ the conserved current can be shown to be

$$j^{\mu a} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\partial \delta\phi}{\partial \alpha_a}$$

The left- and right-handed part $j^{\mu a}_{L/R}$ of a fermionic current are

$$j^{\mu a}_L = \bar{\psi}_L \gamma^\mu T^a \psi_L = (P_L\psi)^\dagger \gamma^0 \gamma^\mu T^a P_L \psi \overset{P_L^\dagger = P_L}{=} \psi^\dagger P_L \gamma^0 \gamma^\mu T^a P_L \psi = \bar{\psi}\gamma^\mu T^a P_L^2 \psi \overset{P_L^2 = P_L}{=} \bar{\psi}\gamma^\mu T^a P_L \psi$$

$$j^{\mu a}_R = \bar{\psi}_R \gamma^\mu T^a \psi_R = \bar{\psi}\gamma^\mu T^a P_R \psi$$

where $P_{L/R} = \frac{1}{2}(1 \mp \gamma_5)$ is the left-/right-handed projector.
From the left- and right-handed currents we can define axial-vector and vector currents

$$j^{\mu a} = j^{\mu a}_R + j^{\mu a}_L = \bar{\psi}\gamma^\mu T^a (P_R + P_L) = \bar{\psi}\gamma^\mu T^a \psi$$

$$j^{\mu a}_5 = j^{\mu a}_R - j^{\mu a}_L = \bar{\psi}\gamma^\mu T^a (P_R - P_L)\psi = \bar{\psi}\gamma^\mu T^a \gamma_5 \psi$$

Now we can couple the currents to vector fields to obtain interactions. E.g., the vector coupling in QED is given by the Lagrangian

$$\mathcal{L}^{QED}_{coupl} = eA_\mu j^\mu = eA_\mu \bar{\psi}\gamma^\mu Q_e \psi = -eA_\mu \bar{\psi}\gamma^\mu \psi$$

In general, we can couple any linear combination of currents to a vector field as long as the combination is Lorentz and gauge invariant. E.g., the neutral current in the electroweak theory is a superposition of a vector and an axialvector current

$$\mathcal{L}^{NC}_{coupl} = gZ_\mu (g_V j^\mu - g_A j^\mu_5)$$

We can use the above relations to express this coupling in terms of right- and left-handed currents

$$\begin{aligned}
\mathcal{L}^{NC}_{coupl} &= gZ_\mu \left(g_V \bar{\psi}\gamma^\mu \psi - g_A \bar{\psi}\gamma^\mu \gamma_5 \psi\right) = \\
&= gZ_\mu \left(g_V \bar{\psi}\gamma^\mu \psi + \frac{g_A}{2}\bar{\psi}\gamma^\mu \psi - \frac{g_A}{2}\bar{\psi}\gamma^\mu \psi - g_A \bar{\psi}\gamma^\mu \gamma_5 \psi + \frac{g_V}{2}\bar{\psi}\gamma^\mu \gamma_5 \psi - \frac{g_V}{2}\bar{\psi}\gamma^\mu \gamma_5 \psi\right) = \\
&= gZ_\mu \left((g_V + g_A)\bar{\psi}\gamma^\mu \frac{1}{2}(1 - \gamma_5)\psi + (g_V - g_A)\bar{\psi}\gamma^\mu \frac{1}{2}(1 + \gamma_5)\psi\right) = \\
&= gZ_\mu \left((g_V + g_A)\bar{\psi}\gamma^\mu P_L \psi + (g_V - g_A)\bar{\psi}\gamma^\mu P_R \psi\right) \equiv gZ_\mu (g_L j^\mu_L + g_R j^\mu_R)
\end{aligned}$$

This gives the following relation between the (axial-)vector and the left-/right-handed couplings

$$\begin{aligned}
g_L &= g_V + g_A \\
g_R &= g_V - g_A
\end{aligned}$$

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