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MASSLESS SPINORS IN MORE THAN FOUR DIMENSIONS

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We consider fermions in theories of higher dimensional gravity where the four-dimensional gauge group is embedded in the invariance group of d dimensional ($d > 4$) Lorentz and general co-ordinate transformations. It is a necessary condition for obtaining massless chiral fermions from dimensional reduction that the d dimensional spinor does not admit a mass term consistent with Lorentz and general co-ordinate transformations. This is the case for a Weyl spinor for $d = 6$, $8 \bmod 8$, a Majorana spinor for $d = 9 \bmod 8$ or a Majorana-Weyl spinor for $d = 2 \bmod 8$.

It is an old idea [1] that a unified picture of gravitational and gauge interactions could arise from higher dimensional general relativity. An acceptable unified gauge group for the weak, electromagnetic and strong interactions can be obtained in a rather straightforward manner [2]. For example, a gauge group SO(10) [3] arises naturally if the ground state for nine of the supplementary dimensions is a nine-dimensional hypersphere. It has been argued [4] that pure higher dimensional gravity including invariants of second (or higher) order in the curvature tensor could indeed account for such a ground state.

One of the most important problems to make such theories realistic is to explain the existence of massless or nearly massless spinors. Neglecting the spontaneous breakdown of the weak interaction gauge group $SU(2)_L \times U(1)_Y$, we observe several generations of massless fermions. These fermions belong to chiral representations which means that left-handed and right-handed fermions transform differently with respect to the gauge group $SU(2)_L \times U(1)_Y$. They all carry a non-zero hypercharge. This guarantees the fermions remaining massless as long as $SU(2)_L \times U(1)_Y$ is unbroken. No $SU(2)_L \times U(1)_Y$ invariant Dirac mass term can be constructed for chiral representations and the non-zero hypercharge forbids Majorana mass terms. (On the contrary, vector-like representations have the same transformation properties for left and right-handed fermions and mass terms are not forbidden by the gauge symmetry.) How can we obtain chiral representations from dimensional reduction of higher dimensional gravity? We will not answer this question in this paper. We only show how the dimension of the space is critical for having a chance of getting chiral fermions. We give necessary conditions without which all fermions come out to be vector-like. In this paper we restrict ourselves to pure higher dimensional gravity without supplementary gauge or scalar interactions. These are the simplest theories and for us the main motivation for considering higher dimensional theories is that pure higher dimensional gravity is enough to explain four-dimensional gauge and scalar interactions, without introducing such interactions in higher dimensions.

To start with, we have to choose a spinor representation of the group of d-dimensional local Lorentz and co-ordinate transformations (gen_d transformations). Dimensional reduction consists first in reducing this spinor representation with respect to the symmetry group of the ground state*. (For example, a 13-dimensional world with ground state $S^9 \times \mathcal{M}^4$ has the ground state symmetry $SO(10) \times$ (four-dimensional inhomogeneous Lorentz transformations).) Then one integrates over the $d-4$ supplementary co-ordinates in order to obtain an effective four-dimensional theory. For a compact $d-4$ dimensional "internal" space the reduction of the d-dimensional spinor gives infinitely many representations of the ground state symmetry. Most (if not all) of these four-dimensional fermions are massive, but one may hope that in some cases a finite number of massless chiral fermions could be left.

However, no chiral fermions in four dimensions can be obtained from dimensional reduction if a mass term of the d-dimensional spinor is not forbidden by gen_d invariance. This follows from the observation that a non-vanishing d-dimensional mass term would contribute a non-vanishing mass to *all* four-dimensional fermions. Therefore, if a d-dimensional mass term is allowed by gen_d invariance, four-dimensional mass terms are allowed for all fermions consistent with any subgroup of gen_d. Especially, such mass terms cannot be forbidden by gauge invariance if the gauge group is a subgroup of gen_d**. As a consequence, all four-dimensional fermions must, in this case, belong to vector-like representations. Note that this argument does not assume that there is actually a non-vanishing bare mass in d dimensions, but only that such a mass term is not forbidden by gen_d invariance. The representation content of the theory does not depend of course on the numerical value of the d-dimensional mass term. A theory with a vanishing d-dimensional mass but allowing the construction of a mass term consistent with gen_d invariance does not exclude the existence of massless fermions in four dimensions. But, in this case, these fermions belong to vector-like representations. This applies in particular to cases where a d-dimensional mass term is forbidden by other symmetries than gen_d invariance (for example, supersymmetry). In conclusion, in higher-dimensional mod-

* Note that in this approach the gauge group is a subgroup of gen_d transformations, but in general not a subgroup of the d-dimensional Lorentz transformations or the d-dimensional co-ordinate transformations alone. The ground state symmetry must leave the vielbein of the ground state invariant and this links Lorentz and co-ordinate transformations.

** Our argument does not apply to cases where the gauge group is connected with other symmetries than gen_d invariance, as for example supplementary gauge symmetries in higher dimensions or hidden local symmetries in the case of supersymmetry. For the first example it can be generalized easily if a mass term is possible consistent with gen_d and d-dimensional gauge invariance.

els where the gauge group is a subgroup of gen_d, any attempt of constructing a realistic theory with massless *chiral* fermions in four dimensions should start with a gen_d representation not permitting a mass term in d dimensions.

Fortunately, the irreducible massive gen_d representations often become reducible in the massless case. The irreducible massless gen_d representations then obey supplementary constraints and no mass term can be constructed consistent with gen_d invariance. Two examples of such constraints are Weyl and Majorana conditions which characterize the transformation properties with respect to the d-dimensional local Lorentz group $\mathfrak{L}_d$. (There may be other constraints for massless spinors involving both $\mathfrak{L}_d$ and general co-ordinate transformations.) In the remainder of this note we describe Weyl and Majorana spinors in d dimensions and classify the dimensions where no mass term can be constructed for irreducible $\mathfrak{L}_d$ representations.

The Dirac spinor in d dimensions has $2^{[d/2]}$ complex components. For d even, the Dirac spinor is reducible with respect to $\mathfrak{L}_d$ and decomposes into two $2^{d/2-1}$ component Weyl spinors. Majorana spinors are eigenstates of an appropriately defined charge conjugation operator. For one time-like and $d-1$ space-like dimensions, they exist for $d = 2,3,4,8,9 \bmod 8$. They can be represented by a spinor with $2^{[d/2]}$ real components. Majorana-Weyl spinors obey both the Weyl and Majorana conditions and exist for $d = 2 \bmod 8$. For one time-like dimension, mass terms for irreducible $\mathfrak{L}_d$ spinors are forbidden for $d = 2,6,8,9 \bmod 8$.

Many of these properties are known from general group theoretical arguments [5] about the irreducible representations of $\mathfrak{L}_d$. We reproduce them here using explicitly the properties of the Dirac matrices in d dimensions. The Dirac matrices obey the Clifford algebra

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn} \tag{1}$$

and can be represented by $2^{[d/2]} \times 2^{[d/2]}$ unitary matrices. Here η^{mn} is diagonal with t eigenvalues $+1$ and s eigenvalues -1, $s + t = d$. The time-like Dirac matrices are then Hermitian $(\gamma^{0\dagger} = \gamma^0)$ and the space-like Dirac matrices antiHermitian $(\gamma^{i\dagger} = -\gamma^i)$. An infinitesimal $\mathfrak{L}_d$ transformation of a $2^{[d/2]}$ component Dirac spinor is given by

$$\begin{aligned} \delta\psi &= -\tfrac{1}{2}\varepsilon_{mn}\Sigma^{mn}\psi, \\ \Sigma^{mn} &= -\tfrac{1}{4}[\gamma^m, \gamma^n]. \end{aligned} \tag{2}$$

In even dimensions, there exists a γ^5-like matrix

$$\bar{\gamma} = \eta\gamma^0\gamma^1 \dots \gamma^{d-1} \tag{3}$$

with

$$\{\bar{\gamma}, \gamma^m\} = 0, \qquad [\bar{\gamma}, \Sigma^{mn}] = 0. \tag{4}$$

The requirement $\bar{\gamma}^2 = 1$ implies

$$\eta^2 = (-1)^{(s-t)/2} \tag{5}$$

and one has $\bar{\gamma}^\dagger = \bar{\gamma}$. Then $\frac{1}{2}(1 \pm \bar{\gamma})$ are projection operators. The Dirac spinor is reducible and decomposes into two inequivalent Weyl spinors

$$\psi_+ = \tfrac{1}{2}(1+\bar{\gamma})\psi, \qquad \psi_- = \tfrac{1}{2}(1-\bar{\gamma})\psi. \tag{6}$$

Furthermore, one can always find (see below) a matrix B satisfying

$$\Sigma^{mn*} = B\Sigma^{mn}B^{-1}. \tag{7}$$

This defines the charge conjugate spinor

$$\psi^c = B^{-1}\psi^* = \mathcal{C}\psi \tag{8}$$

which has the same Lorentz transformation properties as ψ:

$$\delta\psi^c = -\tfrac{1}{2}\varepsilon_{mn}\Sigma^{mn}\psi^c. \tag{9}$$

Thus $\mathcal{C}$ commutes with Σ^{mn} (remember that $\mathcal{C}$ involves the operation of complex conjugation):

$$[\mathcal{C}, \Sigma^{mn}] = 0. \tag{10}$$

If we can find B fulfilling $\mathcal{C}^2 = 1$, $\frac{1}{2}(1 \pm \mathcal{C})$ are again projection operators and the Dirac spinor can be reduced into eigenstates of $\mathcal{C}$ (Majorana spinors):

$$\psi_A = \tfrac{1}{2}(1+\mathcal{C})\psi, \qquad \psi_B = \tfrac{1}{2}(1-\mathcal{C})\psi. \tag{11}$$

Majorana spinors can always be represented by real $2^{[d/2]}$ component spinors with real generators Σ^{mn}. This can be seen most easily by representing a Dirac spinor by a $2^{[d/2+1]}$ real component spinor with corresponding real $2^{[(d/2+1)]} \times 2^{[(d/2)+1]}$ matrices Σ^{mn}. (For example, the complex spinor $\psi_1 + i\psi_2$ can be represented by (ψ_1, ψ_2) with twice the number of components. The operation of complex conjugation of this spinor corresponds to multiplication with the matrix diag$(1, -1)$.) In this representation $\mathcal{C}$ is a real matrix and the projected spinors ψ_A and ψ_B are real $2^{[d/2]}$ component spinors*. However, the condition $\mathcal{C}^2 = 1$ is equivalent to $BB^* = 1$ and this cannot be fulfilled in arbitrary dimensions.

* Note that not every real $2^{[d/2]}$ component spinor is a Majorana spinor. It is trivial that every complex $2^{(d/2)-1}$ component Weyl spinor can be represented by a $2^{d/2}$ component real spinor with a corresponding real representation of the generators Σ^{mn}. But this does not imply that an appropriate charge conjugation operation $\mathcal{C}$ with $\mathcal{C}^2 = 1$ exists for the Dirac representation, where Σ^{mn} can be represented by the commutator of γ^m matrices.

For even dimensions, one may have Majorana-Weyl spinors. This further requires $\mathcal{C}$ to commute with $\bar{\gamma}$. One has

$$\bar{\gamma}=(-2)^{d/2}\eta\Sigma^{01}\Sigma^{23}\dots\Sigma^{d-2,d-1} \tag{12}$$

and eqs. (5) and (7) yield

$$\mathcal{C}\bar{\gamma}=(-1)^{(s-t)/2}\bar{\gamma}\mathcal{C}. \tag{13}$$

For $\frac{1}{2}(s-t)$ odd, $\mathcal{C}$ flips the helicity and Weyl spinors belong to complex representations. (This means that a representation is not equivalent to its complex conjugate representation. Well known examples are left-handed spinors for $d=4$, $t=1$ where the charge conjugate spinor is right-handed or the 16 spinor of SO(10) ($d=10$, $t=0$) where the complex conjugate spinor transforms as $\overline{16}$.) For $\frac{1}{2}(s-t)$ even, Weyl spinors belong to representations which are equivalent to their complex conjugates. For $B^*B=1$ the representation is real (Majorana-Weyl spinor) whereas otherwise it is called pseudoreal.

To find out the properties of B we first consider the case of an even number of dimensions. Since the matrices $-(\gamma^m)$, $-(\gamma^m)^{\mathrm{T}}$ and $(\gamma^m)^\dagger$ obey the same Clifford algebra as the γ^m and there is only one irreducible representation of the Clifford algebra by complex $2^{d/2}\times 2^{d/2}$ matrices up to equivalence transformations, there exist matrices B_1, C_1 and D_1 with

$$\begin{aligned}\gamma^{m*}&=-B_1\gamma^m B_1^{-1},\\ \gamma^{m\mathrm{T}}&=-C_1\gamma^m C_1^{-1},\\ \gamma^{m\dagger}&=D_1\gamma^m D_1^{-1}.\end{aligned} \tag{14}$$

An appropriate normalization of these matrices (they are only defined in eq. (14) up to a constant) and their involutive properties give [6]

$$\begin{aligned}&B_1^\dagger B_1=1, \qquad B_1^* B_1=\varepsilon_1=\pm 1,\\ &D_1^\dagger D_1=1, \qquad D_1^\dagger=D_1,\\ &C_1=B_1D_1, \qquad C_1^\dagger C_1=1,\\ &C_1=\delta_1 C_1^{\mathrm{T}}=\pm C_1^{\mathrm{T}}.\end{aligned} \tag{15}$$

The constant δ_1 has been calculated [6] to be

$$\delta_1=\begin{cases}+1 & \text{for } d=6,8 \bmod 8\\ -1 & \text{for } d=2,4 \bmod 8.\end{cases} \tag{16}$$

To determine the relation between ε_1 and δ_1, we can use the properties of D_1: For $t=0$, $D_1 \equiv \bar{\gamma}$ fulfills (14) and one has $B_1\bar{\gamma} = \delta_1\varepsilon_1\bar{\gamma}^* B_1 = \delta_1\varepsilon_1(-1)^{d/2}B_1\bar{\gamma}$. For $t=1$, we choose $D_1 \equiv \gamma^0$ and use the relation $B_1\gamma^0 = \delta_1\varepsilon_1(\gamma^0)^* B_1 = \delta_1\varepsilon_1 B_1\gamma^0$. We thus obtain

$$\varepsilon_1 = \begin{cases} (-1)^{d/2}\delta_1 & \text{for } t=0, \\ -\delta_1 & \text{for } t=1. \end{cases} \tag{17}$$

The matrix B_1 fulfills the definition of B in eq. (7). This choice of B is however not unique. Any matrix B_1A obeys eq. (7), provided A is invertible and commutes with Σ^{mn}. The most general choice for B is

$$B = B_1(a + b\bar{\gamma}). \tag{18}$$

Indeed, we could have started with matrices B_2, C_2 determined by

$$\gamma^{m*} = B_2\gamma^m B_2^{-1}, \qquad \gamma^{mT} = C_2\gamma^m C_2^{-1}, \tag{19}$$

with

$$B_2 = B_1\bar{\gamma}, \qquad C_2 = C_1\bar{\gamma},$$

$$B_2^* B_2 = \varepsilon_2 = \pm 1, \qquad C_2^T = \delta_2 C_2 = \pm C_2, \tag{20}$$

and

$$\delta_2 = (-1)^{d/2}\delta_1, \qquad \varepsilon_2 = (-1)^{(s-t)/2}\varepsilon_1. \tag{21}$$

The values of δ_1, δ_2, ε_1 and ε_2 as a function of d are resumed in table 1.

For $d = 4 + 2t \bmod 8$ one cannot have $B^*B = 1$ and no Majorana spinor exists. For $d = 8 + 2t \bmod 8$, $B^*B = 1$ can be fulfilled (for example for $B = B_1$). Since $\mathcal{C}$ commutes with $\bar{\gamma}$, one has Majorana-Weyl spinors. For $d = 2 + 2t \bmod 8$ we choose

TABLE 1
Values of δ_1, δ_2, ε_1 and ε_2

			$t=0$		$t=1$	
	δ_1	δ_2	ε_1	ε_2	ε_1	ε_2
$d = 2 \bmod 8$	-1	$+1$	$+1$	-1	$+1$	$+1$
$d = 3 \bmod 8$	-1	–	–	-1	$+1$	–
$d = 4 \bmod 8$	-1	-1	-1	-1	$+1$	-1
$d = 5 \bmod 8$	–	-1	-1	–	–	-1
$d = 6 \bmod 8$	$+1$	-1	-1	$+1$	-1	-1
$d = 7 \bmod 8$	$+1$	–	–	$+1$	-1	–
$d = 8 \bmod 8$	$+1$	$+1$	$+1$	$+1$	-1	$+1$
$d = 9 \bmod 8$	–	$+1$	$+1$	–	–	$+1$

$B=B_1$ with $B^*B=1$. In these dimensions $\mathcal{C}$ commutes with $i\gamma^m$ and the γ^m can be represented as pure imaginary matrices. For $d=6+2t \bmod 8$, the condition $B^*B=1$ is fulfilled for $B=B_2$. Now, $\mathcal{C}$ commutes with γ^m which can be represented as real matrices. For $d=2+2t \bmod 4$, $\mathcal{C}$ anticommutes with $\bar{\gamma}$. In a basis where $\bar{\gamma}$ is diagonal,

$$\bar{\gamma}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

B^{-1} has the form

$$\begin{pmatrix}0 & \tilde{E}\\ E & 0\end{pmatrix}$$

where $\mathcal{C}^2=1$ requires $\tilde{E}=(E^{-1})^*$. In this basis, a Majorana spinor $\mathcal{C}\psi\equiv\psi$ has the form

$$\begin{pmatrix}\chi\\ E\chi^*\end{pmatrix}.$$

It is completely described by the complex $2^{d/2-1}$ component spinor χ which shows the equivalence of Weyl and Majorana spinors in these dimensions.

Finally, we investigate the existence of Majorana spinors in an odd number of dimensions $d=D+1$. The Clifford algebra in D dimensions is easily enlarged to $D+1$ dimensions (the extra dimension is assumed to be space-like) by adding to the D γ matrices, the matrix $\gamma^{D+1}\equiv i\bar{\gamma}$. We can still find a matrix B obeying (7), but not both matrices B_1 and B_2 obey eqs. (14) and (19) in $D+1$ dimensions. Indeed, eq. (13) implies

$$-(\gamma^{D+1})^*=(-1)^{(s-t-1)/2}B_1\gamma^{D+1}B_1^{-1}=(-1)^{(s-t-1)/2}B_2\gamma^{D+1}B_2^{-1}. \quad (22)$$

For $d=1+2t \bmod 4$, one has $B\equiv B_1$ whereas for $d=3+2t \bmod 4$, B is given by B_2. The values of $B^*B=\varepsilon$ are also given in table 1. For $t=1$, Majorana spinors therefore exist for $d=3,9 \bmod 8$, whereas for $d=5,7 \bmod 8$, the only irreducible representations of the Lorentz group are Dirac spinors. In conclusion, for $t=1$, Majorana spinors exist for $d=2,3,4,8,9 \bmod 8$.

Next we turn to the construction of Lorentz invariants bilinear in the spinor field. These include kinetic terms, mass terms and possible couplings to other fields. One constructs a spinor $\tilde{\psi}$ which is the transpose of ψ up to a matrix C:

$$\tilde{\psi}=\psi^{\mathrm{T}}C. \quad (23)$$

If the infinitesimal Lorentz transformation of $\tilde{\psi}$ is given by

$$\delta\tilde{\psi}=\tfrac{1}{2}\varepsilon_{mn}\tilde{\psi}\Sigma^{mn} \quad (24)$$

the bilinears $\tilde{\psi}\psi$, $\tilde{\psi}\gamma^m\psi$ and $\tilde{\psi}\Gamma^k\psi$ transform as a Lorentz scalar, a vector and a totally antisymmetric tensor of rank k respectively (Γ^k denotes the totally antisymmetrized product of k γ matrices). Eq. (24) requires

$$(\Sigma^{mn})^{\mathrm{T}} = -C\Sigma^{mn}C^{-1}. \tag{25}$$

For even dimensions, this condition is met by the above-introduced matrices C_1 or C_2 (eqs. (14),(19)). For odd dimensions, we must choose $C \equiv C_1$ for $d = 3 \bmod 4$ and $C \equiv C_2$ for $d = 5 \bmod 4$, where C_1 and C_2 are the corresponding matrices in $d-1$ (even) dimensions. If the spinor is not a Majorana spinor, its charge conjugate spinor ψ^c can be used to construct further invariants. Since

$$\bar{\psi} = (\psi^c)^{\mathrm{T}}C = \psi^\dagger B^* C = \varepsilon\psi^\dagger D \tag{26}$$

transforms like $\tilde{\psi}$ $\left[D = B^{-1}C \text{ with } (\Sigma^{mn})^\dagger = -D\Sigma^{mn}D^{-1}\right]$, $\bar{\psi}$ can be used instead of $\tilde{\psi}$ to construct scalars, vectors, etc.*. For Dirac spinors, all totally antisymmetric tensors can be constructed as spinor bilinears. For Weyl and Majorana spinors, however, the supplementary constraints forbid some of the antisymmetric tensors to be contained in the product of two spinors.

First consider the Weyl spinors: using the relation

$$\bar{\gamma}^{\mathrm{T}} = (-1)^{d/2}C\bar{\gamma}C^{-1}, \tag{27}$$

one finds for the Weyl spinor $\psi_+ = \frac{1}{2}(1+\bar{\gamma})\psi$:

$$\tilde{\psi}_+ = \tilde{\psi}\tfrac{1}{2}\left(1 + (-1)^{d/2}\bar{\gamma}\right), \qquad \bar{\psi}_+ = \bar{\psi}\tfrac{1}{2}\left(1 + (-1)^{s}\bar{\gamma}\right). \tag{28}$$

For $d = 2 \bmod 4$, the bilinear $\tilde{\psi}_+\Gamma^k\psi_+$ therefore vanishes identically if Γ^k is a product of an even number of γ^m matrices. In these dimensions, the direct product of two identical spinors only contains antisymmetric tensors of an odd rank. In particular, no singlet can be constructed out of two Weyl spinors of a given representation**. For $t = 1$, the Weyl spinors are self-conjugate representations ($\mathcal{C}$ commutes with $\bar{\gamma}$). This implies that $\bar{\psi}_+\psi_+$ also vanishes and no mass term is possible for Weyl spinors. For $d = 4 \bmod 4$, $\tilde{\psi}_+\Gamma^k\psi_+$ vanishes for k odd. However, for $t = 1$, $\mathcal{C}$ anticommutes with $\bar{\gamma}$ and the antisymmetric tensors of odd rank can be constructed out of $\bar{\psi}_+\Gamma^k\psi_+$.

For invariants constructed with $\tilde{\psi}$ and ψ there is a further constraint coming from the Pauli principle. Due to the antisymmetry in the exchange of two identical spinors, $\psi^{\mathrm{T}}C\Gamma^k\psi$ vanishes identically if the matrix $(C\Gamma^k)$ is symmetric. This applies especially to Majorana spinors where all bilinears are constructed from $\tilde{\psi}$ and ψ. The

* For $d = 4$, $t = 1$, $B = B_1$ one has $\varepsilon = 1$, $D = \gamma^0$, and our definition of $\bar{\psi}$ coincides with the usual one.

** Any mass term necessarily involves two inequivalent Weyl spinors.

following relations are easily found:

$$(C_1\Gamma^k)^{\mathrm{T}} = \varepsilon_1(-1)^{k(k+1)/2}C_1\Gamma^k,$$
$$(C_2\Gamma^k)^{\mathrm{T}} = \varepsilon_2(-1)^{k(k-1)/2}C_2\Gamma^k, \tag{29}$$

and the symmetry of some $C_i\Gamma^k$ is displayed in table 2. For $t = 1$, a mass term for a Weyl (or equivalently a Majorana) spinor is allowed for $d = 4 \bmod 8$, but forbidden for $d = 8 \bmod 8$. No mass term is possible for a Majorana spinor for $d = 9 \bmod 8$. We list in table 3 the possible mass terms, vector couplings and couplings to second and third rank antisymmetric tensors in the case of only one irreducible spinor ($t = 1$). Note that a coupling to a third rank antisymmetric tensor is always possible, which is required for a consistent coupling to gravity.

Finally we construct the kinetic terms for the different irreducible spinors and derive the corresponding field equations. The gen_d covariant derivative is given by

$$D_m\psi = e_m{}^\mu\left(\partial_\mu - \tfrac{1}{2}\omega_{\mu np}\Sigma^{np}\right)\psi, \tag{30}$$

TABLE 2
Symmetry Properties of $C\Gamma^k$

	C_1	C_2	$C_1\gamma^m$	$C_2\gamma^m$	$C_1\Sigma^{mn}$	$C_2\Sigma^{mn}$	$C_1\gamma^{mnp}$	$C_2\gamma^{mnp}$
$d = 2 \bmod 8$	a	s	s	s	s	a	a	a
$d = 3 \bmod 8$	a	–	s	–	s	–	a	–
$d = 4 \bmod 8$	a	a	s	a	s	s	a	s
$d = 5 \bmod 8$	–	a	–	a	–	s	–	s
$d = 6 \bmod 8$	s	a	a	a	a	s	s	s
$d = 7 \bmod 8$	s	–	a	–	a	–	s	–
$d = 8 \bmod 8$	s	s	a	s	a	a	s	a
$d = 9 \bmod 8$	–	s	–	s	–	a	–	a

TABLE 3
Allowed mass terms and couplings for one irreducible spinor ($t = 1$)

Dimension	Type of spinor	Mass term	Vector coupling	Σ^{mn} coupling	γ^{mnp} coupling
$d = 2 \bmod 8$	Majorana-Weyl	–	–	–	×
$d = 3 \bmod 8$	Majorana	×	–	–	×
$d = 4 \bmod 8$	Weyl/Majorana	×	×	–	×
$d = 5 \bmod 8$	Dirac	×	×	×	×
$d = 6 \bmod 8$	Weyl	–	×	–	×
$d = 7 \bmod 8$	Dirac	×	×	×	×
$d = 8 \bmod 8$	Weyl/Majorana	–	×	×	×
$d = 9 \bmod 8$	Majorana	–	–	×	×

where $e_m{}^\mu$ is the inverse vielbein and $\omega_{\mu np}$ the spin connection. Considering the irreducible parts of the spin connection $\omega^m{}_{mn}$ and $\tilde{\omega}_{mnp} = \omega_{[mnp]}$ (the symbol $[mnp]$ means total antisymmetrization in the indices m, n and p) and defining $\gamma^{mnp} = \gamma^{[m}\gamma^n\gamma^{p]}$ and $\gamma^\mu = \gamma^m e_m{}^\mu$ one has

$$\gamma^m D_m \psi = \left(\gamma^\mu \partial_\mu + \tfrac{1}{2}\omega^m{}_{mn}\gamma^n + \tfrac{1}{4}\tilde{\omega}_{mnp}\gamma^{mnp}\right)\psi. \tag{31}$$

Then $\tilde{\psi}\gamma^m D_m \psi$, $\bar{\psi}\gamma^m D_m \psi$ transform as gen_d scalars.

First we consider the case where ψ^c is different from ψ. This applies to Dirac spinors ($d = 5, 7 \bmod 8$) and Weyl spinors in $d = 4, 6, 8 \bmod 8$. (In the following we always assume $t = 1$). For $d = 5, 6, 7 \bmod 8$, $C\gamma^m$ is antisymmetric and the construction of a kinetic term based on $e\tilde{\psi}\gamma^\mu\partial_\mu\psi$ is not possible, since this term only consists of a total derivative plus a non-derivative term. (e is the determinant of the vielbein.) For $d = 4, 8 \bmod 8$, $e\tilde{\psi}_+\gamma^\mu\partial_\mu\psi_+$ vanishes identically due to the Weyl constraint. Therefore, the kinetic term must be constructed from $\bar{\psi}$. Noting that

$$i\bar{\psi}\gamma^m\psi + \text{h.c.} = 0,$$

$$i\bar{\psi}\gamma^{mnp}\psi + \text{h.c.} = 2i\bar{\psi}\gamma^{mnp}\psi, \tag{32}$$

one has the Hermitian kinetic term

$$\mathcal{L}_{\text{kin}} = \tfrac{1}{2}ie\bar{\psi}\gamma^\mu\partial_\mu\psi + \text{h.c.} + \tfrac{1}{4}ie\bar{\psi}\gamma^{mnp}\tilde{\omega}_{mnp}\psi. \tag{33}$$

(For Weyl spinors, replace ψ by ψ_+ or ψ_-.) From

$$\tfrac{1}{2}ie\bar{\psi}\gamma^\mu\partial_\mu\psi + \text{h.c.} = ie\bar{\psi}\gamma^\mu\partial_\mu\psi + \tfrac{1}{2}ie\bar{\psi}\gamma^m\omega^n{}_{nm}\psi - \tfrac{1}{2}i\partial_\mu\left(e\bar{\psi}\gamma^\mu\psi\right) \tag{34}$$

follows

$$\mathcal{L}_{\text{kin}} = ie\bar{\psi}\gamma^m D_m \psi + \text{total derivative}, \tag{35}$$

and the field equations of a spinor coupled to gravity are*

$$iD\gamma^m D_m \psi_{(+)} + \text{mass term} = 0. \tag{36}$$

The mass term vanishes for Weyl spinors in $d = 6, 8 \bmod 8$. For $d = 5, 7 \bmod 8$, it has the form $-mD\psi$ and for a Weyl spinor ψ_+ in $d = 4 \bmod 8$ it is given by $-mC^{-1}\psi_+^*$.

In the case of Majorana spinors, the kinetic term must be constructed with $\tilde{\psi}$. This is possible since for $d = 2, 3, 4, 8, 9 \bmod 8$, $C\gamma^m$ is symmetric. (For $d = 4 \bmod 8$, one

* Note that the field equation is not always of the form of the standard Dirac equation. For Weyl spinors neither γ^m nor D can be represented by $2^{d/2-1} \times 2^{d/2-1}$ matrices, but this is possible for the product $D\gamma^m$ since it commutes with $\bar{\gamma}$.

has to choose $C \equiv C_1$ and for $d = 8 \bmod 8$ $C \equiv C_2$.) The term $i\tilde{\psi}\gamma^\mu \partial_\mu \psi$ is Hermitian and the kinetic term is

$$\mathcal{L}_{\text{kin}} = \tfrac{1}{2} ie\tilde{\psi}\gamma^\mu \partial_\mu \psi + \tfrac{1}{8} ie\tilde{\psi}\gamma^{mnp}\tilde{\omega}_{mnp}\psi. \tag{37}$$

Note that a term $\tilde{\psi}\gamma^n \omega^m{}_{mn}\psi$ vanishes due to the symmetry of $C\gamma^m$. Nevertheless, due to the identity $\partial_\mu \gamma^\mu = \left(-\Gamma_{\mu\nu}{}^\nu + \omega^m{}_{m\mu}\right)\gamma^\mu$ $\left(\Gamma_{\mu\nu}{}^\rho \text{ is the Christoffel symbol}\right)$, a term involving $\omega^m{}_{mn}$ appears in the field equation which is given by $iC\gamma^m D_m \psi +$ mass term $= 0$. The mass term vanishes for a Majorana spinor in $d = 2, 8, 9 \bmod 8$ and is given by $-mC\psi$ for $d = 3, 4 \bmod 8$.

In conclusion, we have shown that a necessary condition for obtaining massless chiral fermions from dimensional reduction of higher dimensional gravity is to start with a spinor representation not admitting a mass term consistent with Lorentz and general coordinate transformations in d dimensions. Good candidates for such spinors are Weyl spinors for $d = 6, 8 \bmod 8$, a Majorana spinor for $d = 9 \bmod 8$ or a Majorana-Weyl spinor for $d = 2 \bmod 8$. It is interesting to note that the massless irreducible spinors in $d = 16, 17, 18$ all have 256 real components which corresponds to the number of real components of four-fermion families within SO(10) unification. However, the dimensional reduction for Weyl and/or Majorana spinors has to be carried out explicitly in order to see if chiral massless four-dimensional fermions* exist in these cases or not.

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Note Added

After completion of this work, the author became aware of a preprint by G. Chapline and R. Slansky [7] also discussing Majorana conditions in higher dimensions.

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* The only case of massless chiral fermions from dimensional reduction of pure higher dimensional gravity we know of so far [3], is for Weyl (or Majorana-Weyl) spinors for $d = 4n + 2$ with a flat compact $d - 4$ dimensional internal manifold which admits an $SO(4n - 2)$ gauge symmetry. However, this manifold is not a very satisfactory ground state.

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