# Reducing full one-loop amplitudes to scalar integrals at the integrand level

Giovanni Ossola\*, Costas G. Papadopoulos† and Roberto Pittau<sup>‡§</sup>

Institute of Nuclear Physics, NCSR "DEMOKRITOS", 15310 Athens, Greece.

ABSTRACT: We show how to extract the coefficients of the 4-, 3-, 2- and 1-point one-loop scalar integrals from the full one-loop amplitude of arbitrary scattering processes. In a similar fashion, also the rational terms can be derived. Basically no information on the analytical structure of the amplitude is required, making our method appealing for an efficient numerical implementation.

<sup>\*</sup>e-mail: ossola@inp.demokritos.gr

<sup>&</sup>lt;sup>†</sup>e-mail: costas.papadopoulos@cern.ch

<sup>&</sup>lt;sup>‡</sup>Permanent address: Dipartimento di Fisica Teorica, Univ. di Torino and INFN, sez. di Torino, Italy.

<sup>§</sup>e-mail: roberto.pittau@to.infn.it

#### Contents

1.	Introduction		1
2.	The spurious terms		4
	2.1	The 4-point like spurious term	5
	2.2	The 3-point like spurious terms	7
	2.3	The 2-point like spurious terms	8
	2.4	The 1-point like spurious terms	11
	2.5	The 0-point like spurious term	11
3.	Ext	racting the coefficients of the scalar loop functions	12
	3.1	The coefficient of the 4-point functions	12
	3.2	The coefficient of the 3-point functions	14
	3.3	The coefficient of the 2-point functions	15
	3.4	The coefficient of the 1-point functions	16
4.	Reconstructing the rational part of the amplitude		17
<b>5</b> .	Applications and tests		18
6.	Conclusions		21
Α.	A. The system for the coefficients of the 3-point functions		
в.	3. The system for the coefficients of the 2-point functions		

#### 1. Introduction

With the ongoing evolution of the experimental programs of the LHC and the International Linear Collider, high precision predictions for multi-particle processes are urgently needed. In the last years we have seen a remarkable progress in the theoretical description of multi-particle processes at tree-order, thanks to very efficient recursive algorithms [1]. Nevertheless the current need of precision goes beyond tree order and therefore a similar description at the one loop level is more than desirable.

The computation of the one-loop matrix elements seems to be notoriously difficult. The development of the main ingredients to accomplish this task, started almost 30 years ago with the pioneering works of 't Hooft and Veltman [2] and Passarino and Veltman [3].

Still nowadays very few *complete* calculations with more than 3 particles [4] in the final state exist <sup>1</sup>.

The problem arises because of two reasons: the complexity of Feynman graph representation at the one-loop level and the way the reduction of n-point one-loop integrals in terms of scalar 4-, 3-, 2- and 1-point functions is performed. For the former, recursive equations seem to be a very promising tool. For the latter, it would be desirable to have a reduction of the full (sub-)amplitude instead of the individual tensor one-loop integrals. In fact, a method that will be based on the minimum possible analytical information about the one-loop amplitude will be more adequate, in principle, for an efficient numerical implementation<sup>2</sup>.

During the last years we have seen some very interesting developments. In the front of the reduction of tensor integrals a new method at the integrand level has been worked out by Pittau, and del Aguila [7]. This method will be the starting point of our work. On the other hand the idea of cut constructibility [8] in computing one-loop amplitudes (and not just integrals), has been proven very efficient in getting many very well known results. In the last two years a new development in computing one-loop amplitudes has been initiated by the work of Britto, Cachazo and Feng, [9] and combined with the unitarity method, resulted to a series of remarkable results concerning QCD one-loop multi-particle amplitudes [10]. Moreover the introduction of quadruple cuts allowed a simplified algebraic approach, at least for the coefficient of the box function.

In this paper we propose a reduction of arbitrary one-loop (sub-)amplitude at the integrand level by exploiting the set of kinematical equations for the integration momentum [11], that extend the already used quadruple, triple and double cuts. In contrast to the usual method of cut-contractibility, it is also possible to reconstruct the full rational terms of the amplitude. The method requires a minimal information about the form of the one-loop (sub-)amplitude and therefore it is well suited for a numerical implementation. It also gives rise to very interesting simplifications of well known results.

In [7] it has been shown, by explicitly reconstructing denominators, that the *integrand* of any m-point one-loop amplitude can be rewritten as <sup>3</sup>

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}, \quad \bar{D}_i = (\bar{q} + p_i)^2 - m_i^2, \quad p_0 \neq 0,$$
(1.1)

where we use a bar to denote objects living in  $n = 4 + \epsilon$  dimensions and where the numerator N(q) can be cast in the form

$$N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i$$

$$+ \sum_{i_0 < i_1 < i_2}^{m-1} \left[ c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i$$

<sup>&</sup>lt;sup>1</sup>For a review of reduction methods, see [5].

<sup>&</sup>lt;sup>2</sup>For other attempts towards a direct numerical implementation see also [6].

<sup>&</sup>lt;sup>3</sup>We assume  $p_0 \neq 0$ , for that choice allows a completely symmetric treatment of all denominators  $\bar{D}_i$ .

$$+ \sum_{i_{0} < i_{1}}^{m-1} \left[ b(i_{0}i_{1}) + \tilde{b}(q; i_{0}i_{1}) \right] \prod_{i \neq i_{0}, i_{1}}^{m-1} \bar{D}_{i}$$

$$+ \sum_{i_{0}}^{m-1} \left[ a(i_{0}) + \tilde{a}(q; i_{0}) \right] \prod_{i \neq i_{0}}^{m-1} \bar{D}_{i}$$

$$+ \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_{i}. \tag{1.2}$$

The quantities  $d(i_0i_1i_2i_3)$  are the coefficients of the 4-point loop functions with the four denominators  $\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}\bar{D}_{i_3}$ . Analogously, the  $c(i_0i_1i_2)$ ,  $b(i_0i_1)$  and  $a(i_0)$  are the coefficients of all possible 3-point, 2-point and 1-point loop functions, respectively.

The "spurious" terms d,  $\tilde{c}$ , b,  $\tilde{a}$  and  $\tilde{P}$  still depend on q. They are defined by the requirement that they should vanish upon integration over  $d^n\bar{q}$ , as we shall see later.

Notice that no coefficient of scalar functions with more that four denominators appear. This is due to the fact that scalar functions with m > 4 can always be expressed as a linear combination of 4-point functions and, possibly, extra  $\tilde{d}$  terms [12].

All q's in the numerator N(q) of Eq. (1.2) are 4-dimensional. If n-dimensional q's are needed, they can be split into 4-dimensional and  $\epsilon$ -dimensional parts as explained in [7]. Eq. (1.2) is then applicable to the purely 4-dimensional terms, that are usually the most difficult to compute.

Since the scalar 1-, 2-, 3-, 4-point functions are known, the only knowledge of the existence of the decomposition of Eq. (1.2) allows one to reduce the problem of calculating  $A(\bar{q})$  to the algebraical problem of extracting all possible coefficients in Eq. (1.2) by computing N(q) a sufficient number of times, at different values of q, and then inverting the system.

In carrying out this program two problems arise. First the explicit knowledge of the spurious terms is needed, secondly the size of the system should be kept manageable. To illustrate this second point, for m=6 and without counting any spurious term, there are 56 independent scalar loop functions and it is clearly advisable to avoid the inversion of a  $56 \times 56$  matrix. Our solution to this is basically singling out particular choices of q such that, systematically, 4, 3, 2 or 1 among all possible denominators  $\bar{D}_i$  vanishes. Then, as we shall see in Section 3, the system of equations becomes "triangular": first one solves for all possible 4-point functions, then for the 3-point functions and so on.

Notice that the described procedure can be performed at the amplitude level. One does not need to repeat the work for all Feynman diagrams, provided their sum is known. This circumstance is particularly appealing when our method is used together with some recursion relation to build up N(q). We postpone this problem to a future publication and, in this paper, we suppose to know N(q).

A last comment is in order. In reconstructing the denominators, there is a mismatch between the 4-dimensional q in N(q) and the n-dimensional denominators  $\bar{D}_i$ . To compensate for this it suffices to replace  $m_i^2 \to m_i^2 - \tilde{q}^2$  in all the coefficient of Eq. (1.2), where  $\tilde{q}^2$  is the (n-4)-dimensional part of  $\bar{q}^2$  [7]. The coefficients of the various powers of  $\tilde{q}^2$ , obtained through this mass shift, are the coefficients of the extra integrals introduced in [7], which give rise to nothing but the rational part of the amplitude [13].

In this paper we show how to extract, with the help of Eq. (1.2), the coefficients of the loop functions, including the rational terms, from any amplitude. In Section 2 we list and compute the needed spurious terms, in Section 3 we show how to get, in a systematic fashion, the coefficients of all 4-, 3-, 2- and 1-point scalar integrals, while Section 4 deals with the problem of adding the missing rational terms. Finally, in Section 5, we present some practical applications and tests we made on our method.

## 2. The spurious terms

Before any attempt of extracting  $d(i_0i_1i_2i_3)$ ,  $c(i_0i_1i_2)$ ,  $b(i_0i_1)$  and  $a(i_0)$ , one should explicitly know the q dependence of the spurious terms  $\tilde{d}(q;i_0i_1i_2i_3)$ ,  $\tilde{c}(q;i_0i_1i_2)$ ,  $\tilde{b}(q;i_0i_1)$ ,  $\tilde{a}(q;i_0)$  and  $\tilde{P}(q)$ . This can be achieved by decomposing any 4-dimensional q appearing in the numerator of Eq. (1.1) in terms of a convenient basis of massless 4-momenta [7], the coefficients of which either reconstruct denominators or vanish upon integration over  $d^n\bar{q}$ . The first terms gives rise to the d, c, b and a coefficients in Eq. (1.2). The second ones to all the additional spurious terms. The latter category is further classified, in Eq. (1.2), according to the number of the remaining denominators. In the following, we shall call the  $\tilde{d}$ ,  $\tilde{c}$ ,  $\tilde{b}$ ,  $\tilde{a}$ ,  $\tilde{P}$  terms 4,3,2,1,0-point like spurious terms, respectively. As we shall see, the actual form of them generally depends on the maximum possible rank of the loop tensors in the amplitude.

Since the decomposition makes use of the momenta appearing in the denominators we set, for simplicity,  $i_0 = 0$ ,  $i_1 = 1$ ,  $i_2 = 2$  and  $i_3 = 3$  and derive explicit formulae for  $\tilde{d}(q;0123)$ ,  $\tilde{c}(q;012)$ ,  $\tilde{b}(q;01)$ ,  $\tilde{a}(q;0)$  and  $\tilde{P}(q)$ . By relabeling the indices, all the other spurious terms are easily derived. Before carrying out this program, we recall some basic results of Ref. [7], also with the aim to set up our notation.

The explicit decomposition reads

$$q^{\mu} = -p_0^{\mu} + \frac{\beta}{\gamma} D^{\mu} - \frac{1}{2\gamma} Q^{\mu},$$

$$D^{\mu} = \frac{1}{\beta} [2[(q+p_0) \cdot \ell_1] \ell_2^{\mu} + 2[(q+p_0) \cdot \ell_2] \ell_1^{\mu}],$$

$$Q^{\mu} = [(q+p_0) \cdot \ell_3] \ell_4^{\mu} + [(q+p_0) \cdot \ell_4] \ell_3^{\mu},$$
(2.1)

where  $\ell_1$  and  $\ell_2$  are massless 4-vector satisfying the relations

$$k_1 = \ell_1 + \alpha_1 \ell_2 \,, \quad k_2 = \ell_2 + \alpha_2 \ell_1 \,,$$
 (2.2)

with

$$k_i = p_i - p_0. (2.3)$$

Furthermore, in spinorial notation,

$$\ell_3^{\mu} = <\ell_1 | \gamma^{\mu} | \ell_2 |, \quad \ell_4^{\mu} = <\ell_2 | \gamma^{\mu} | \ell_1 | \quad \text{with} \quad (\ell_3 \cdot \ell_4) = -4(\ell_1 \cdot \ell_2).$$
 (2.4)

The solution to Eq. (2.2) reads

$$\ell_{1} = \beta(k_{1} - \alpha_{1}k_{2}), \quad \ell_{2} = \beta(k_{2} - \alpha_{2}k_{1}),$$

$$\beta = 1/(1 - \alpha_{1}\alpha_{2}), \quad \alpha_{i} = \frac{k_{i}^{2}}{\gamma},$$

$$\gamma \equiv 2(\ell_{1} \cdot \ell_{2}) = (k_{1} \cdot k_{2}) \pm \sqrt{\Delta}, \quad \Delta = (k_{1} \cdot k_{2})^{2} - k_{1}^{2}k_{2}^{2}.$$
(2.5)

A last comment is in order. We make the assumption, always realized in practical calculations, to compute the amplitude  $A(\bar{q})$  in a gauge where the maximum rank of the appearing loop tensors is never higher than the number of denominators, as it happens, for example, in the renormalizable gauge. This choice limits the number of the spurious terms. For instance, there is no  $\tilde{P}$  in such a case. However, the fact that all gauges are equivalent, leads us to the following conjecture

One can always limit her/himself to the spurious terms appearing in the renoramalizable gauge, because, in physical amplitudes, the contributions coming from the tensors of higher rank should add up to zero.

This conjecture, being rather strong, has to be checked in practical calculations. We are now ready to derive the q dependence of the spurious terms.

#### 2.1 The 4-point like spurious term

By iteratively using Eq. (2.1), only one possible integrand that vanishes upon integration is left, namely

$$\tilde{d}(q;0123) = \tilde{d}(0123) T(q),$$
(2.6)

where  $\tilde{d}(0123)$  is a constant (namely does not depend on q) and

$$T(q) \equiv Tr[(\not q + \not p_0) \not \ell_1 \not \ell_2 \not k_3 \gamma_5].$$
 (2.7)

To prove this statement, let us call  $N^{(3)}(q)$  the numerator of a term containing the four denominators  $\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3$ .  $N^{(3)}(q)$  is necessarily a polynomial in q, whose maximum degree we will denote by  $j_{max}$ . Notice that  $j_{max}$  is also the maximum rank of the 4-point tensors that appear when performing a standard reduction procedure. Being interested in terms in which the four original denominators are not canceled out by the numerator function, we can systematically neglect all denominators that are reconstructed from  $N^{(3)}(q)^4$ . In particular, by expressing back  $\ell_1$  and  $\ell_2$  in terms of  $k_1$  and  $k_2$ , as shown in Eq. (2.5), one obtains [7]

$$D^{\mu} = \frac{2}{\beta} (p_0 \cdot \ell_1) \ell_2^{\mu} + \frac{2}{\beta} (p_0 \cdot \ell_2) \ell_1^{\mu} + (\bar{D}_1 - \bar{D}_0 + h_1) r_2^{\mu} + (\bar{D}_2 - \bar{D}_0 + h_2) r_1^{\mu},$$

$$h_i = (m_i^2 - p_i^2) - (m_0^2 - p_0^2), \quad r_1^{\mu} = \ell_1^{\mu} - \alpha_1 \ell_2^{\mu}, \quad r_2^{\mu} = \ell_2^{\mu} - \alpha_2 \ell_1^{\mu}. \tag{2.8}$$

<sup>&</sup>lt;sup>4</sup>We will also ignore terms proportional to powers of  $\tilde{q}^2$ , for they give rise to rational parts in the amplitude that can be treated separately (see Section 4).

Therefore

$$D^{\mu} = F^{\mu} + \sum_{i=0}^{2} \mathcal{O}(\bar{D}_i), \qquad (2.9)$$

where

$$F^{\mu} \equiv \frac{2}{\beta} (p_0 \cdot \ell_1) \ell_2^{\mu} + \frac{2}{\beta} (p_0 \cdot \ell_2) \ell_1^{\mu} + h_1 r_2^{\mu} + h_2 r_1^{\mu}, \qquad (2.10)$$

so that

$$q^{\mu} = -p_0^{\mu} + \frac{\beta}{\gamma} F^{\mu} - \frac{1}{2\gamma} Q^{\mu} + \sum_{i=0}^{2} \mathcal{O}(\bar{D}_i).$$
 (2.11)

We used the notation  $\mathcal{O}(\bar{D}_i)$  to indicate terms in which one of the denominators  $\bar{D}_i$  has been explicitly reconstructed, and can therefore be neglected, as far as the construction of spurious terms is concerned. By replacing each q appearing in  $N^{(3)}(q)$  by the r.h.s. of Eq. (2.11), the only possible numerators of degree  $j_{max}$ , which preserve all four denominators, turn out to be

$$[(q+p_0)\cdot\ell_3]^{j_{max}}, \quad [(q+p_0)\cdot\ell_4]^{j_{max}}, \quad [(q+p_0)\cdot\ell_3]^{j_3}[(q+p_0)\cdot\ell_4]^{j_4}, \quad (2.12)$$

with  $j_3 + j_4 = j_{max}$ . The rank of such terms can be reduced with the help of the two following identities <sup>5</sup>:

$$[(q+p_0)\cdot \ell_3][(q+p_0)\cdot \ell_4] = \beta(q+p_0)^{\alpha}D_{\alpha} - \gamma(q+p_0)^2,$$

$$[(q+p_0)\cdot \ell_{3(4)}][(q+p_0)\cdot \ell_{3(4)}] = \frac{1}{(k_3\cdot \ell_{4(3)})} \left\{ [\gamma(q+p_0)^2 - \beta(q+p_0)^{\alpha}D_{\alpha}](k_3\cdot \ell_{3(4)}) - 2\left[\gamma[(q+p_0)\cdot k_3] - \beta k_3^{\alpha}D_{\alpha}\right][(q+p_0)\cdot \ell_{3(4)}] \right\}. \quad (2.13)$$

In fact, the insertion of Eqs. (2.9) and (2.11) in Eq. (2.13), together with the identities

$$(q+p_0)^2 = \bar{D}_0 + m_0^2 - \tilde{q}^2, \quad 2(q \cdot k_3) = \bar{D}_3 - \bar{D}_0 + h_3,$$
 (2.14)

with  $h_3$  given in Eq. (2.8), gives

$$[(q+p_0)\cdot \ell_3][(q+p_0)\cdot \ell_4] = \frac{2\beta^2 F^2 - \beta F \cdot Q}{2\gamma} - \gamma m_0^2 + \sum_{i=0}^2 \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2)$$

$$[(q+p_0)\cdot \ell_{3(4)}][(q+p_0)\cdot \ell_{3(4)}] = \frac{1}{(k_3\cdot \ell_{4(3)})} \left\{ \left[ \gamma m_0^2 - \frac{2\beta^2 F^2 - \beta F \cdot Q}{2\gamma} \right] (k_3\cdot \ell_{3(4)}) - 2\left[ \gamma \left( p_0 \cdot k_3 + \frac{h_3}{2} \right) - \beta k_3 \cdot F \right] [(q+p_0)\cdot \ell_{3(4)}] \right\}$$

$$+ \sum_{i=0}^3 \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2). \tag{2.15}$$

<sup>&</sup>lt;sup>5</sup>A demonstration can be found in [7]. Notice that the factor 2 in front of the last term of the second equation is missing in that paper.

The two equations (2.15), when introduced in the numerators of Eq. (2.12), reduce their rank from  $j_{max}$  to  $j_{max}-1$ , up to contributions containing less denominators or proportional to  $\tilde{q}^2$ . By applying the described procedure  $j_{max}-1$  times  $N^{(3)}(q)$  takes the form

$$N^{(3)}(q) = \eta_0 + \eta_3 \left[ (q + p_0) \cdot \ell_3 \right] + \eta_4 \left[ (q + p_0) \cdot \ell_4 \right] + \sum_{i=0}^{3} \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2), \qquad (2.16)$$

where the coefficients  $\eta_0$ ,  $\eta_3$  and  $\eta_4$  do not depend on q. The denominators still hidden in  $[(q+p_0)\cdot\ell_{3,(4)}]$  can be further extracted by using the identity [7]

$$[(q+p_0)\cdot \ell_{3,4}] = \frac{1}{(k_3\cdot \ell_{4,3})} \left\{ \beta \, k_3^{\alpha} D_{\alpha} - \gamma [(q+p_0)\cdot k_3] \pm \frac{T(q)}{2} \right\}$$

$$= \frac{1}{(k_3\cdot \ell_{4,3})} \left\{ \beta \, k_3 \cdot F - \gamma \left[ p_0 \cdot k_3 + \frac{h_3}{2} \right] \pm \frac{T(q)}{2} \right\}$$

$$+ \sum_{i=0}^{3} \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2) , \qquad (2.17)$$

with T(q) given in Eq. (2.7). Therefore the final expression for  $N^{(3)}(q)$  reads

$$N^{(3)}(q) = d(0123) + \tilde{d}(0123) T(q) + \sum_{i=0}^{3} \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2).$$
 (2.18)

The statement that  $\tilde{d}(q;0123)$  must have the form given in Eq. (2.6) is then equivalent to the

Theorem:

$$\int d^n \bar{q} \frac{T(q)}{\bar{D}_0 \bar{D}_1 \bar{D}_2 \bar{D}_3} = 0.$$
 (2.19)

The proof trivially follows by making the shift  $q \to q - p_0$  in the integration variable and by noticing that  $T(q) \propto \epsilon(q, \ell_1, \ell_2, k_3)$ . In fact, the resulting rank one 4-point function can only be proportional to  $k_1$ ,  $k_2$  and  $k_3$  and each term vanishes when contracted with the  $\epsilon$  tensor.

#### 2.2 The 3-point like spurious terms

To derive  $\tilde{c}(q;012)$  we make use of the Theorems:

$$\int d^{n} \bar{q} \frac{[(q+p_{0}) \cdot \ell_{3}]^{j}}{\bar{D}_{0} \bar{D}_{1} \bar{D}_{2}} = 0,$$

$$\int d^{n} \bar{q} \frac{[(q+p_{0}) \cdot \ell_{4}]^{j}}{\bar{D}_{0} \bar{D}_{1} \bar{D}_{2}} = 0, \quad \forall j = 1, 2, 3, \cdots$$
(2.20)

Once again, they are proven by shifting q and performing a tensor decomposition.

Such integrals come from the only possible terms in the decomposition of 3-point tensor *integrands* that do not reconstruct denominators [7]. The proof closely follows the

reasoning of the 4-point case. Calling  $N^{(2)}(q)$  the numerator of a term containing the three denominators  $\bar{D}_0\bar{D}_1\bar{D}_2$ , and being  $j_{max}$  its maximum rank, one applies Eqs. (2.8)-(2.11) to cast  $N^{(2)}(q)$  in a form where all rank  $j_{max}$  terms with three denominators are proportional to the three numerators given in Eq. (2.12). The first two terms are not further reducible, while the iterative use of the first of Eqs. (2.15) reduces the third one to a sum of contributions proportional to  $[(q+p_0)\cdot \ell_3]^j$  and  $[(q+p_0)\cdot \ell_4]^j$  separately, with  $j < j_{max}$ . Then

$$N^{(2)}(q) = c(012) + \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j}(012)[(q+p_0) \cdot \ell_3]^j + \tilde{c}_{2j}(012)[(q+p_0) \cdot \ell_4]^j \right\}$$

$$+ \sum_{i=0}^{2} \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2), \qquad (2.21)$$

where c(012),  $\tilde{c}_{1j}(012)$  and  $\tilde{c}_{2j}(012)$  are constants. Therefore  $\tilde{c}(q;012)$  must have the form

$$\tilde{c}(q;012) = \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j}(012)[(q+p_0) \cdot \ell_3]^j + \tilde{c}_{2j}(012)[(q+p_0) \cdot \ell_4]^j \right\}. \tag{2.22}$$

In the renormalizable gauge,  $j_{max} = 3$ . In order to illustrate this fact, a simple argument can be used. Let us consider the reduction of a m-point function of rank m. In the decomposition of Eq. (1.2), the 3-point like spurious terms involve (m-3) reconstructed denominators  $\bar{D}_i$ . Each one of them can be obtained from a power of q in the numerator N(q) by means for example of Eq. (2.11). This leaves at most m - (m-3) = 3 powers of q available for the construction of the  $\tilde{c}(q)$  term. The same reasoning can be applied to determine  $j_{max} = 2$  and  $j_{max} = 1$  respectively for 2-point and 1-point like spurious terms.

#### 2.3 The 2-point like spurious terms

To derive  $\tilde{b}(q;01)$  it is convenient to rewrite q in a basis expressed in terms of an auxiliary arbitrary 4-vector v not parallel to  $k_1$  <sup>6</sup>:

$$q^{\mu} = -p_0^{\mu} + y_1 k_1^{\mu} + y_n n^{\mu} + y_7 \ell_7^{\mu} + y_8 \ell_8^{\mu}.$$
 (2.23)

In a way similar to Eqs. (2.2) and (2.4),  $k_1$  and v are decomposed in terms of two massless 4-vectors  $\ell_{5,6}$ ,

$$k_1 = \ell_5 + \alpha_5 \ell_6, \qquad v = \ell_6 + \alpha_6 \ell_5,$$
 (2.24)

and  $\ell_{7,8}$  defined as follows

$$\ell_7^{\mu} = \langle \ell_5 | \gamma^{\mu} | \ell_6 \rangle, \quad \ell_8^{\mu} = \langle \ell_6 | \gamma^{\mu} | \ell_5 \rangle.$$
 (2.25)

Then n is taken to be

$$n = \ell_5 - \alpha_5 \ell_6 \,, \tag{2.26}$$

<sup>&</sup>lt;sup>6</sup> We normalize v such that  $(\ell_7 \cdot \ell_8) = -4(\ell_5 \cdot \ell_6) = -k_1^2$ . Then  $\alpha_5 = 2$ ,  $\ell_5 = (k_1 + n)/2$  and  $\ell_6 = (k_1 - n)/4$ .

so that it satisfies the two following properties

$$n \cdot k_1 = 0$$
, and  $n^2 = -k_1^2$ . (2.27)

Finally, one computes

$$q^{\mu} = -p_0^{\mu} + \frac{[(q+p_0) \cdot k_1]}{k_1^2} k_1^{\mu} - \frac{[(q+p_0) \cdot n]}{k_1^2} n^{\mu} + \frac{[(q+p_0) \cdot \ell_8]}{(\ell_7 \cdot \ell_8)} \ell_7^{\mu} + \frac{[(q+p_0) \cdot \ell_7]}{(\ell_7 \cdot \ell_8)} \ell_8^{\mu}.$$
(2.28)

By using this basis, the spurious terms can be determined with the help of the following Theorems:

$$\int d^{n}\bar{q} \, \frac{[(q+p_{0}) \cdot \ell_{7}]^{j}[(q+p_{0}) \cdot n]^{i}}{\bar{D}_{0}\bar{D}_{1}} = 0,$$

$$\int d^{n}\bar{q} \, \frac{[(q+p_{0}) \cdot \ell_{8}]^{j}[(q+p_{0}) \cdot n]^{i}}{\bar{D}_{0}\bar{D}_{1}} = 0,$$

$$\int d^{n}\bar{q} \, \frac{[(q+p_{0}) \cdot n]^{2j-1}}{\bar{D}_{0}\bar{D}_{1}} = 0,$$

$$\int d^{n}\bar{q} \, \frac{[(q+p_{0}) \cdot n]^{2j} - r_{j}\{[(q+p_{0}) \cdot k_{1}]^{2} - (q+p_{0})^{2}k_{1}^{2}\}^{j}}{\bar{D}_{0}\bar{D}_{1}} = 0, \quad r_{1} = \frac{1}{3}, \quad r_{2} = \frac{1}{5}, \cdots,$$

$$\forall j = 1, 2, 3, \cdots \quad \text{and} \quad i = 0, 1, 2 \cdots$$
(2.29)

Again shifting q and decomposing allows an easy proof. As for the fourth line of Eq. (2.29), both terms in the numerator, after tensor decomposition, are proportional to  $k_1^{2j}$ . The factors  $r_j$  are chosen such that their sum is zero.

Now we are ready to prove that only the terms given in Eq. (2.29) contribute to  $\tilde{b}(q;01)$ . First one rewrites

$$q^{\mu} = G^{\mu} - \frac{1}{k_1^2} \{ [(q+p_0) \cdot n] n^{\mu} + [(q+p_0) \cdot \ell_7] \ell_8^{\mu} + [(q+p_0) \cdot \ell_8] \ell_7^{\mu} \} + \sum_{i=0}^1 \mathcal{O}(\bar{D}_i) ,$$

$$G^{\mu} \equiv -p_0^{\mu} + \frac{k_1^{\mu}}{k_1^2} \left[ (p_0 \cdot k_1) + \frac{h_1}{2} \right] ,$$
(2.30)

with  $h_1$  given in Eq. (2.8). Then, calling  $N^{(1)}(q)$  the numerator of a term containing the two denominators  $\bar{D}_0\bar{D}_1$ , and being  $j_{max}$  its maximum rank, one replaces each q appearing in  $N^{(1)}(q)$  by the r.h.s. of Eq. (2.30). Therefore, the only possible generated numerators of degree  $j_{max}$  are

$$[(q+p_0)\cdot \ell_7]^j, \qquad [(q+p_0)\cdot \ell_8]^j, [(q+p_0)\cdot \ell_7]^{j_7}[(q+p_0)\cdot n]^{j_n}, \qquad [(q+p_0)\cdot \ell_8]^{j_8}[(q+p_0)\cdot n]^{j_n}, [(q+p_0)\cdot \ell_7]^{i_7}[(q+p_0)\cdot \ell_8]^{i_8},$$
(2.31)

with  $j = j_{max}$ ,  $j_{7,8} + j_n = j_{max}$  and  $i_7 + i_8 = j_{max}$ . Due to the first two lines of Eq. (2.29), the first four terms of Eq. (2.31) directly give rise to 2-point like spurious terms. Conversely,

the last one can be further reduced by means of the following identity

$$[(q+p_0)\cdot \ell_7][(q+p_0)\cdot \ell_8] = 4[(q+p_0)\cdot \ell_5][(q+p_0)\cdot \ell_6] - 2(\ell_5\cdot \ell_6)(q+p_0)^2$$

$$= \frac{1}{2}\left\{\left((p_0\cdot k_1) + \frac{h_1}{2}\right)^2 - k_1^2 m_0^2 - [(q+p_0)\cdot n]^2\right\}$$

$$+ \sum_{i=0}^{1} \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2), \qquad (2.32)$$

where we have used the results of footnote 6 and reconstructed denominators as in Eq. (2.14). Then, the last term of Eq. (2.31) can be put in the form

$$[(q+p_0)\cdot \ell_7]^{i_7}[(q+p_0)\cdot \ell_8]^{i_8} = \sum_{i=0}^1 \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2)$$

$$+ \begin{cases} [(q+p_0)\cdot \ell_8]^{i_8-i_7} \sum_{i=0}^{i_7} \delta_i [(q+p_0)\cdot n]^{2i}, & \text{if } i_7 \leq i_8, \\ [(q+p_0)\cdot \ell_7]^{i_7-i_8} \sum_{i=0}^{i_8} \delta_i [(q+p_0)\cdot n]^{2i}, & \text{if } i_8 < i_7, \end{cases}$$

$$(2.33)$$

where the  $\delta_i$  are constants. All pieces generated by the previous equation are given by the first five terms of Eq. (2.31), but now with  $j \leq j_{max}$  and  $j_{7,8} + j_n \leq j_{max}$ . Therefore, apart from  $[(q+p_0)\cdot n]^j$ , they are again taken into account by the first two lines of Eq. (2.29); in other words, they contribute to  $\tilde{b}(q;01)$ . If j is odd, due to the third line of Eq. (2.29),  $[(q+p_0)\cdot n]^j$  also gives rise to spurious 2-point terms. On the contrary, in order to get the contribution to  $\tilde{b}(q;01)$  in the case when j is even, one has first to subtract powers of

$$[(q+p_0)\cdot k_1]^2 - (q+p_0)^2 k_1^2, \qquad (2.34)$$

as performed in the last line of Eq. (2.29). Incidentally, when adding this piece back, a contribution to b(01) is generated, since

$$[(q+p_0)\cdot k_1]^2 - (q+p_0)^2 k_1^2 = \left[ \left( (p_0\cdot k_1) + \frac{h_1}{2} \right)^2 - k_1^2 m_0^2 \right] + \sum_{i=0}^1 \mathcal{O}(\bar{D}_i) + \mathcal{O}(\tilde{q}^2).$$
(2.35)

In summary, no other 2-point like spurious terms are possible besides those listed in Eq. (2.29).

In the renormalizable gauge there are at most two powers of q in the numerator. By applying the above reasonings with  $j_{max} = 2$  and  $j_{max} = 1$  gives rise to eight possible spurious  $\tilde{b}$  terms:

$$\tilde{b}(q;01) = \tilde{b}_{11}(01)[(q+p_0)\cdot\ell_7] + \tilde{b}_{21}(01)[(q+p_0)\cdot\ell_8] + \tilde{b}_{12}(01)[(q+p_0)\cdot\ell_7]^2 + \tilde{b}_{22}(01)[(q+p_0)\cdot\ell_8]^2 + \tilde{b}_{0}(01)[(q+p_0)\cdot n] + \tilde{b}_{00}(01)K(q;01)$$

+ 
$$\tilde{b}_{01}(01)[(q+p_0)\cdot \ell_7][(q+p_0)\cdot n]$$
  
+  $\tilde{b}_{02}(01)[(q+p_0)\cdot \ell_8][(q+p_0)\cdot n]$ , with

$$K(q;01) = \left\{ [(q+p_0) \cdot n]^2 - \frac{[(q+p_0) \cdot k_1]^2 - (q+p_0)^2 k_1^2}{3} \right\}.$$
 (2.36)

#### 2.4 The 1-point like spurious terms

First we decompose

$$q^{\mu} = -p_0^{\mu} + y k^{\mu} + y_n n^{\mu} + y_7 \ell_7^{\mu} + y_8 \ell_8^{\mu}, \qquad (2.37)$$

where k is an arbitrary 4-vector and n,  $\ell_7$  and  $\ell_8$  are built up from k and v as in the 2-point case. Then we make use of the following

## Theorems:

$$\int d^{n} \bar{q} \frac{\prod_{i=1}^{2n-1} (q+p_{0}) \cdot v_{i}}{\bar{D}_{0}} = 0,$$

$$\int d^{n} \bar{q} \frac{\prod_{i=1}^{2n} (q+p_{0}) \cdot v_{i} - r_{n} (q+p_{0})^{2n} g_{\mu_{1} \mu_{2} \cdots \mu_{2n}} v_{1}^{\mu_{1}} v_{2}^{\mu_{2}} \cdots v_{2n}^{\mu_{2n}}}{\bar{D}_{0}} = 0,$$

$$r_{n} = (g_{\mu_{1} \mu_{2} \cdots \mu_{2n}} g^{\mu_{1} \mu_{2}} \cdots g^{\mu_{2n-1} \mu_{2n}})^{-1}, \quad \forall n = 1, 2, 3, \cdots, \tag{2.38}$$

for any 4-vector  $v_i$  and where  $g_{\mu_1\mu_2...\mu_{2n}}$  is the symmetrized product of n metric tensors. The proof is just a direct consequence of the fact that

$$\int d^n \bar{q} \, \frac{q^{\mu_1} q^{\mu_2} \cdots q^{\mu_{2n-1}}}{(\bar{q}^2 - m_0^2)} = 0 \quad \text{and} \quad \int d^n \bar{q} \, \frac{q^{\mu_1} q^{\mu_2} \cdots q^{\mu_{2n}}}{(\bar{q}^2 - m_0^2)} \propto g^{\mu_1 \mu_2 \cdots \mu_{2n}} \,. \tag{2.39}$$

In the renormalizable gauge at most rank one 1-point functions appear. Therefore

$$\tilde{a}(q;0) = \tilde{a}_1(0)[(q+p_0) \cdot k] + \tilde{a}_2(0)[(q+p_0) \cdot n] + \tilde{a}_3(0)[(q+p_0) \cdot \ell_7] + \tilde{a}_4(0)[(q+p_0) \cdot \ell_8].$$
(2.40)

#### 2.5 The 0-point like spurious term

As already pointed out,  $\tilde{P}(q) = 0$  in the renormalizable gauge. We can prove this statement simply by counting the powers of q in Eq. (1.2). The last term on the r.h.s. contains 2m powers of q in the m reconstructed denominators  $\bar{D}_i$ . Since N(q) on the l.h.s. is at most of rank m and the other terms on the r.h.s. contain at most 2m - 1 powers of q, in order to satisfy Eq. (1.2) we should have  $\tilde{P}(q) = 0$ .

In more general gauges,  $\tilde{P}(q)$  gives a contribution, polynomial in q, to the integrand of the amplitude. Terms like that vanish, upon integration, in dimensional regularization. This is why we classified  $\tilde{P}(q)$  among the spurious term. Off course  $\tilde{P}(q)$  is never needed, in any gauge, if all the other coefficients in Eq. (1.2) can be determined without making use of its actual form. This is the case, as we shall see in the next Section. However, if necessary, after all the other coefficients are known,  $\tilde{P}(q)$  is easily computed as the difference between N(q) and the first four lines of the r.h.s. of Eq. (1.2), divided by all the denominators.

## 3. Extracting the coefficients of the scalar loop functions

Being interested here in the coefficients of the scalar loop functions, we can set everywhere

$$\tilde{q}^2 = 0, (3.1)$$

so that, in particular

$$\bar{D}_i \to D_i \equiv (q + p_i)^2 - m_i^2$$
 (3.2)

The "error" induced by the above replacement is at the level of the rational part of the amplitude, as we shall see in Section 4, where we will also use the fact that the  $\tilde{q}^2$  terms are always connected to the masses in the denominators to reconstruct the information we are missing with the replacement of Eq. (3.2).

Our approach to the problem is suggested by the structure of Eq. (1.2) itself. Choosing particular values of q such that 4, 3, 2 or 1 denominators vanish allows one to reduce the problem to the solution of simpler sub-systems of equations. To illustrate how this works we concentrate again on the particular choice  $i_0 = 0$ ,  $i_1 = 1$ ,  $i_2 = 2$  and  $i_3 = 3$  and derive explicit formulae for the coefficients d(0123), c(012), b(01) and a(0) as well as for the coefficients of the corresponding spurious terms <sup>7</sup>. As we shall see, the latter information is also needed when iterating the algorithm: in order to solve for the c(012) coefficient, one has to know the coefficients of all terms with 4 denominators, including the spurious ones. In order to solve for b(01), one needs, in addition, all the terms with 3 denominators and so on.

## 3.1 The coefficient of the 4-point functions

We look for a q such that

$$D_0 = D_1 = D_2 = D_3 = 0. (3.3)$$

By writing the 4-vector q as

$$q^{\mu} = -p_0^{\mu} + \sum_{i=1}^{4} x_i \,\ell_i^{\mu} \,, \tag{3.4}$$

with  $\ell_i$  given in Eqs. (2.2)-(2.5), one obtains a system of equations for the  $x_i$ :

$$0 = \gamma(x_1x_2 - 4x_3x_4) - d_0$$

$$0 = d_0 - d_1 + \gamma(x_1\alpha_1 + x_2)$$

$$0 = d_0 - d_2 + \gamma(x_2\alpha_2 + x_1)$$

$$0 = d_0 - d_3 + 2\left[x_1(k_3 \cdot \ell_1) + x_2(k_3 \cdot \ell_2) + x_3(k_3 \cdot \ell_3) + x_4(k_3 \cdot \ell_4)\right],$$
(3.5)

where  $k_i$  is given in Eq. (2.3) and

$$d_i \equiv m_i^2 - k_i^2 \,. \tag{3.6}$$

<sup>&</sup>lt;sup>7</sup>For any other choice of indices the procedure is the same.

There are two possible solutions

$$(q_0^{\pm})^{\mu} = -p_0^{\mu} + x_1^0 \ell_1^{\mu} + x_2^0 \ell_2^{\mu} + x_3^{\pm} \ell_3^{\mu} + x_4^{\pm} \ell_4^{\mu}, \tag{3.7}$$

with

$$x_{1}^{0} = \frac{\beta}{\gamma} [d_{2} - \alpha_{2}d_{1} - d_{0}(1 - \alpha_{2})],$$

$$x_{2}^{0} = \frac{\beta}{\gamma} [d_{1} - \alpha_{1}d_{2} - d_{0}(1 - \alpha_{1})],$$

$$A x_{3}^{\pm 2} + B x_{3}^{\pm} - C = 0,$$

$$x_{4}^{\pm} = \frac{C}{x_{3} \pm},$$
(3.8)

and

$$A = -\frac{(k_3 \cdot \ell_3)}{(k_3 \cdot \ell_4)}, \quad B = \frac{d_3 - d_0 - 2x_1^0(k_3 \cdot \ell_1) - 2x_2^0(k_3 \cdot \ell_2)}{2(k_3 \cdot \ell_4)},$$

$$C = \frac{1}{4} \left( x_1^0 x_2^0 - \frac{d_0}{\gamma} \right). \tag{3.9}$$

Notice that we *need* two solutions to be able to determine both d(0123) and  $\tilde{d}(0123)$ , and that the existence of more than one solution is a consequence of the quadratic nature of the system in Eq. (3.5). By putting both  $q_0^{\pm}$  in Eq. (1.2), and recalling the form of  $\tilde{d}(q;0123)$  given in Eq. (2.6), one finds

$$N(q_0^{\pm}) = [d(0123) + \tilde{d}(0123) T(q_0^{\pm})] \prod_{i \neq 0,1,2,3} D_i(q_0^{\pm}).$$
 (3.10)

Then, by defining

$$R(q_0^{\pm}) \equiv \frac{N(q_0^{\pm})}{\prod_{i \neq 0, 1, 2, 3} D_i(q_0^{\pm})},$$
(3.11)

it is possible to extract d and d

$$d(0123) = \frac{R(q_0^-)T(q_0^+) - R(q_0^+)T(q_0^-)}{T(q_0^+) - T(q_0^-)},$$

$$\tilde{d}(0123) = \frac{R(q_0^+) - R(q_0^-)}{T(q_0^+) - T(q_0^-)}.$$
(3.12)

Notice that, in terms of  $x_{3,4}^{\pm}$ , one rewrites

$$T(q_0^{\pm}) = 2\gamma \left[ x_3^{\pm}(k_3 \cdot \ell_3) - x_4^{\pm}(k_3 \cdot \ell_4) \right] ,$$
  

$$T(q_0^{+}) = -T(q_0^{-}) .$$
(3.13)

Then

$$d(0123) = \frac{1}{2} [R(q_0^+) + R(q_0^-)],$$

$$\tilde{d}(0123) = \frac{1}{2} \frac{R(q_0^+) - R(q_0^-)}{T(q^+)}.$$
(3.14)

The two above equations do not depend on the rank of the tensors in the amplitude. When N(q) = 1, they allow a trivial decomposition of any m-point scalar loop function with m > 4 to boxes, as we shall see in Section 5.

## 3.2 The coefficient of the 3-point functions

At this stage all d and  $\tilde{d}$  coefficients are known. When q is such that

$$D_0 = D_1 = D_2 = 0$$
 and  $D_i \neq 0 \ \forall i \neq 0, 1, 2$  (3.15)

Eq. (1.2) reads

$$N(q) - \sum_{2 < i_3} [d(012i_3) + \tilde{d}(q; 012i_3)] \prod_{i \neq 0, 1, 2, i_3} D_i(q)$$

$$\equiv R'(q) \prod_{i \neq 0, 1, 2} D_i(q) = [c(012) + \tilde{c}(q; 012)] \prod_{i \neq 0, 1, 2} D_i(q), \qquad (3.16)$$

and one can extract c(012), together with all the six  $\tilde{c}_{ij}(012)$  coefficients of Eq. (2.22), by computing R'(q) at seven different q's that fulfill Eq. (3.15). For a q written as in Eq. (3.4) that happens when

$$x_1 = x_1^0 x_2 = x_2^0 x_3 x_4 = C, (3.17)$$

where  $x_{1,2}^0$  and C are given in Eqs. (3.8) and (3.9). There is now an infinite number of solutions, which we parametrize by imposing the extra condition

$$(q+p_0) \cdot \ell_3 = \pm \sqrt{C}e^{i\pi/k}(\ell_3 \cdot \ell_4), \quad k=1,2,3,\cdots.$$
 (3.18)

Then

$$x_4 = \pm \sqrt{C}e^{i\pi/k} \equiv x_{4k}^{\pm}, \quad x_3 = \frac{C}{x_{4k}^{\pm}} = \pm \sqrt{C}e^{-i\pi/k} \equiv x_{3k}^{\pm},$$
 (3.19)

and the complete solution reads

$$(q_k^{\pm})^{\mu} = -p_0^{\mu} + x_1^0 \ell_1^{\mu} + x_2^0 \ell_2^{\mu} + x_{3k}^{\pm} \ell_3^{\mu}, + x_{4k}^{\pm} \ell_4^{\mu}. \tag{3.20}$$

Finally, with the help of the relation

$$[(q_k^{\pm} + p_0) \cdot \ell_4][(q_k^{\pm} + p_0) \cdot \ell_3] = C(\ell_3 \cdot \ell_4)^2, \qquad (3.21)$$

one writes

$$R'(q_k^{\pm}) = \sum_{j=-3}^{3} c_j [\pm e^{i\pi/k}]^j, \qquad (3.22)$$

with

$$\tilde{c}_{1j}(012) = \frac{c_j}{C^{j/2} (\ell_3 \cdot \ell_4)^j}, \quad \tilde{c}_{2j}(012) = \frac{c_{-j}}{C^{j/2} (\ell_3 \cdot \ell_4)^j} \text{ and } c(012) = c_0. (3.23)$$

In Appendix A, we explicitly determine all  $c_j$ 's of Eq. (3.22) by choosing 7 different solutions of the form given in Eq. (3.20).

#### 3.3 The coefficient of the 2-point functions

At this stage all d,  $\tilde{d}$ , c and  $\tilde{c}$  coefficients are known. When q is such that

$$D_0 = D_1 = 0 \quad \text{and} \quad D_i \neq 0 \quad \forall i \neq 0, 1$$
 (3.24)

Eq. (1.2) reads

$$N(q) - \sum_{1 < i_2 < i_3} [d(01i_2i_3) + \tilde{d}(q;01i_2i_3)] \prod_{i \neq 0,1,i_2,i_3} D_i$$

$$- \sum_{1 < i_2} [c(01i_2) + \tilde{c}(q;01i_2)] \prod_{i \neq 0,1,i_2} D_i$$

$$\equiv R''(q) \prod_{i \neq 0,1} D_i(q) = [b(01) + \tilde{b}(q;01)] \prod_{i \neq 0,1} D_i(q), \qquad (3.25)$$

and one can extract b(01) together with all the eight  $\tilde{b}(01)$  coefficients of Eq. (2.36), by computing R''(q) at nine different q's that fulfill Eq. (3.24). That happens when, for a q written as in Eq. (2.23),

$$y_1 = \frac{d_1 - d_0}{2k_1^2} \equiv y_1^0,$$

$$y_n^2 = (y_1^0)^2 - \frac{d_0 + 2k_1^2 y_7 y_8}{k_1^2},$$
(3.26)

where we used our normalization condition  $(\ell_7 \cdot \ell_8) = -k_1^2$ . We impose now, as an extra requirement, that K(q;01) defined in Eq. (2.36) vanishes. This implies

$$y_1 = y_1^0,$$

$$y_n = \pm \sqrt{\frac{1}{3} \left( (y_1^0)^2 - \frac{d_0}{k_1^2} \right)} \equiv \pm \sqrt{F},$$

$$y_7 = \frac{F}{y_8}.$$
(3.27)

We fix the remaining freedom by imposing

$$(q+p_0) \cdot \ell_7 = \pm \sqrt{F} e^{i\pi/k} (\ell_7 \cdot \ell_8),$$
 (3.28)

that implies

$$y_8 = \pm \sqrt{F} e^{i\pi/k} \equiv y_{8k}^{\pm}$$

$$y_7 = \frac{F}{y_{9k}^{\pm}} = \pm \sqrt{F} e^{-i\pi/k} \equiv y_{7k}^{\pm}.$$
(3.29)

Then, this class of solutions can be parametrized as follows

$$(q_{lk}^{\pm})^{\mu} = -p_0^{\mu} + y_1^0 k_1^{\mu} + l\sqrt{F} n^{\mu} + y_{7k}^{\pm} \ell_7^{\mu} + y_{8k}^{\pm} \ell_8^{\mu}, \quad \text{with } l = \pm 1.$$
 (3.30)

Since

$$(q+p_0) \cdot \ell_8 = \frac{F(\ell_7 \cdot \ell_8)^2}{(q+p_0) \cdot \ell_7}, \tag{3.31}$$

we can rewrite (see Eq. (2.36))

$$R''(q_{lk}^{\pm}) = l \sum_{j=-1}^{1} \beta_j \left[ \pm e^{i\pi/k} \right]^j + \sum_{j=-2}^{2} b_j \left[ \pm e^{i\pi/k} \right]^j, \tag{3.32}$$

where

$$\tilde{b}_{1j}(01) = b_0 \qquad \tilde{b}_{1j}(01) = \frac{b_j}{F^{j/2} (\ell_7 \cdot \ell_8)^j} \qquad \tilde{b}_{2j}(01) = \frac{b_{-j}}{F^{j/2} (\ell_7 \cdot \ell_8)^j} 
\tilde{b}_{01}(01) = \frac{\beta_1}{F(\ell_7 \cdot \ell_8)^2} \qquad \tilde{b}_{02}(01) = \frac{\beta_{-1}}{F(\ell_7 \cdot \ell_8)^2} \qquad \tilde{b}_{0}(01) = \frac{\beta_0}{F^{1/2} (\ell_7 \cdot \ell_8)}.$$
(3.33)

To determine the last coefficient  $\tilde{b}_{00}(01)$  we need to introduce a different solution, for which K(q;01) does not vanish. We call  $q_0$  such a solution and choose it satisfying the condition  $y_7 = y_8 = 0$ . Then, by insertion in Eqs. (3.26) and (2.36), one finds

$$q_0^{\mu} = -p_0^{\mu} + y_1^0 k_1^{\mu} + \sqrt{3F} n^{\mu}, \qquad (3.34)$$

and

$$R''(q_0) = b(01) + \tilde{b}_0(01) [(q_0 + p_0) \cdot n] + \tilde{b}_{00}(01) K(q_0; 01)$$
  
=  $b(01) - \tilde{b}_0(01) \sqrt{3F}k_1^2 + 2\tilde{b}_{00}(01) k_1^4 F.$  (3.35)

The complete solution for all the coefficients is given in Appendix B.

### 3.4 The coefficient of the 1-point functions

In massless theories all 1-point functions, namely all tadpoles, vanish, also implying that, in such cases, one does not need to know all the  $\tilde{b}$  coefficients: only the coefficients of the scalar 2-point functions are needed, each of them can be determined in terms of just four solutions of Eq. (3.24) (see Appendix B). However, in general, also the coefficients of the tadpoles are required. Therefore we show how to extract them.

At this stage we assume to know all the  $d,\ \tilde{d},\ c,\ \tilde{c},\ b$  and  $\tilde{b}$  coefficients and, when q is such that

$$D_0 = 0 \quad \text{and} \quad D_i \neq 0 \quad \forall i \neq 0 \,, \tag{3.36}$$

Eq. (1.2) reads

$$N(q) - \sum_{0 < i_1 < i_2 < i_3} [d(0i_1i_2i_3) + \tilde{d}(q;0i_1i_2i_3)] \prod_{i \neq 0, i_1, i_2, i_3} D_i$$
$$- \sum_{0 < i_1 < i_2} [c(0i_1i_2) + \tilde{c}(q;0i_1i_2)] \prod_{i \neq 0, i_1, i_2} D_i$$

$$-\sum_{0 < i_1} [b(0i_1) + \tilde{b}(q;0i_1)] \prod_{i \neq 0, i_1} D_i$$

$$\equiv R'''(q) \prod_{i \neq 0} D_i(q) = [a(0) + \tilde{a}(q;0)] \prod_{i \neq 0} D_i(q). \tag{3.37}$$

Then, we parametrize q as in Eq. (2.37) and choose two solutions of Eq. (3.36) such that  $y_n = y_7 = y_8 = 0$ :

$$(q_0^{\pm})^{\mu} = -p_0^{\mu} \pm \sqrt{\frac{d_0}{k^2}} \, k^{\mu} \,. \tag{3.38}$$

Therefore, by recalling Eq. (2.40), we can write

$$R'''(q_0^+) = a(0) + \tilde{a}_1(0)\sqrt{\frac{d_0}{k^2}}k^2$$

$$R'''(q_0^-) = a(0) - \tilde{a}_1(0)\sqrt{\frac{d_0}{k^2}}k^2,$$
(3.39)

so that

$$a(0) = \frac{R'''(q_0^+) + R'''(q_0^-)}{2}.$$
(3.40)

Notice that  $\tilde{a}(q;0)$  is never needed, because it would be necessary only to extract  $\tilde{P}(q)$ , that, as already observed, is irrelevant.

# 4. Reconstructing the rational part of the amplitude

Until now we have assumed  $\tilde{q}^2 = 0$ . As already discussed, this is enough to reconstruct the coefficients of the 4-3-2-1-point loop functions, but rational parts are missing. In the renormalizable gauge the only possible contributions to those rational terms come from the following extra scalar integrals introduced in [7] <sup>8</sup>

$$\int d^n \bar{q} \frac{\tilde{q}^4}{\bar{D}_i \bar{D}_j \bar{D}_k \bar{D}_l} = -\frac{i\pi^2}{6} + \mathcal{O}(\epsilon) ,$$

$$\int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_i \bar{D}_j \bar{D}_k} = -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon) ,$$

$$\int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_i \bar{D}_j} = -\frac{i\pi^2}{2} \left[ m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{3} \right] + \mathcal{O}(\epsilon) .$$
(4.1)

We checked that they reproduce the rational terms listed in [13]. Therefore, in our language, the coefficients of the integrals in Eq. (4.1) are just the coefficients of the maximum powers of  $\tilde{q}^2$  contained in the d(ijkl), c(ijk) and b(ij) once  $\tilde{q}^2$  is reintroduced through the mass shift

$$m_i^2 \to m_i^2 - \tilde{q}^2. \tag{4.2}$$

<sup>&</sup>lt;sup>8</sup>The powers of  $\tilde{q}^2$  are dictated by the maximum rank of the loop tensors in  $A(\bar{q})$ .

With the above replacement the coefficients get a dependence on  $\tilde{q}^2$  and one can expand:

$$d(ijkl; \tilde{q}^{2}) = d(ijkl) + \tilde{q}^{2}d^{(2)}(ijkl) + \tilde{q}^{4}d^{(4)}(ijkl),$$

$$c(ijk; \tilde{q}^{2}) = c(ijk) + \tilde{q}^{2}c^{(2)}(ijk),$$

$$b(ij; \tilde{q}^{2}) = b(ij) + \tilde{q}^{2}b^{(2)}(ij).$$
(4.3)

 $d^{(4)}(ijkl)$ ,  $c^{(2)}(ijk)$  and  $b^{(2)}(ij)$  are then the coefficients of the first, second and third integral of Eq. (4.1), respectively. They can be either computed numerically

$$d^{(4)}(ijkl) = \lim_{\tilde{q}^2 \to \infty} \frac{d(ijkl; \tilde{q}^2)}{\tilde{q}^4},$$

$$c^{(2)}(ijk) = \lim_{\tilde{q}^2 \to \infty} \frac{c(ijk; \tilde{q}^2)}{\tilde{q}^2},$$

$$b^{(2)}(ij) = \lim_{\tilde{q}^2 \to \infty} \frac{b(ij; \tilde{q}^2)}{\tilde{q}^2},$$

$$(4.4)$$

or as solutions of systems obtained by evaluating Eq. (4.3) at different  $\tilde{q}^2$ . For instance:

$$d^{(4)}(ijkl) = \frac{d(ijkl;1) + d(ijkl;-1) - 2d(ijkl)}{2},$$

$$c^{(2)}(ijk) = c(ijk;1) - c(ijk),$$

$$b^{(2)}(ij) = b(ij;1) - b(ij).$$
(4.5)

A small example of the described approach is given in the next Section.

### 5. Applications and tests

We tested the whole method on the reduction of a rank four 4-point tensor integral

$$\int d^n \bar{q} \frac{q^\mu q^\nu q^\rho q^\sigma}{\bar{D}_0 \bar{D}_1 \bar{D}_2 \bar{D}_3} \tag{5.1}$$

to scalar functions. We have been able to correctly extract the coefficients of the scalar integrals. In addition, when reducing the tensor in Eq. (5.1) with the techniques of [7], we have been able to also test the coefficients of the spurious terms.

As for the rational terms, we explicitly show here, as an illustrative example, the extraction of the coefficient of

$$\int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_0 \bar{D}_1 \bar{D}_2} \tag{5.2}$$

from

$$\int d^n \bar{q} \frac{q^\mu q^\nu}{\bar{D}_0 \bar{D}_1 \bar{D}_2} \,, \tag{5.3}$$

where, for simplicity, we have put  $p_0 = 0$ . Then, in this case

$$N(q) = q^{\mu}q^{\nu} = R'(q). \tag{5.4}$$

We first observe that, in the limit  $\tilde{q}^2 \to \infty$ , one can use the following asymptotic form of the solutions given in Eq. (3.20)

$$(q_1^{\pm})^{\mu} = \mp \sqrt{C_{\infty}} \left( \ell_3^{\mu} + \ell_4^{\mu} \right), \quad (q_2^{\pm})^{\mu} = \mp i \sqrt{C_{\infty}} \left( \ell_3^{\mu} - \ell_4^{\mu} \right)$$
with  $C_{\infty} \equiv \lim_{\tilde{q}^2 \to \infty} C = \frac{\tilde{q}^2}{4\gamma}$ . (5.5)

Therefore, one obtains, for the coefficient  $c_0$  in Eq. (A.4)

$$\lim_{\tilde{q}^2 \to \infty} \frac{c_0}{\tilde{q}^2} = \frac{\ell_3^{\mu} \ell_4^{\nu} + \ell_4^{\mu} \ell_3^{\nu}}{4\gamma}, \tag{5.6}$$

Then

$$c^{(2)}(012) = \frac{\ell_3^{\mu} \ell_4^{\nu} + \ell_4^{\mu} \ell_3^{\nu}}{4\gamma}, \tag{5.7}$$

in agreement with the result obtained in [7].

Another rather straightforward application of the method is the reduction of the scalar n-point functions, with  $n \geq 5$ , in terms of box functions. It will allow the reader to follow the reasoning of the reduction in a simple case. It should be mentioned that the content of this derivation, amazingly enough, goes back to the year 1965, in the work of Melrose [14] and Källén and Toll [11].

In the conventional approach one can easily prove [14,15] the following relation

$$\begin{vmatrix} I^{N} - I^{N-1}(0) - I^{N-1}(1) \dots - I^{N-1}(N-1) \\ 1 & Y_{00} & Y_{01} \dots & Y_{0N-1} \\ 1 & Y_{10} & Y_{11} \dots & Y_{1N-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & Y_{N-10} & Y_{N-11} \dots & Y_{N-1N-1} \end{vmatrix} = 0,$$
 (5.8)

where  $I^N$  is the N-point scalar function

$$I^{N} = \int d^{n} \bar{q} \frac{1}{\bar{D}_{0} \cdots \bar{D}_{N-1}}, \qquad (5.9)$$

 $I^{N-1}(i)$  is the N-1-point function with the i-th propagator missing and

$$Y_{ij} = m_i^2 + m_j^2 - (p_i - p_j)^2, \quad i = 0, \dots, N \quad j = 0, \dots, N.$$
 (5.10)

By repeated use of Eq. (5.8) we may express the N-point function in terms of 4-point functions with coefficients expressible in terms of the determinants of Y matrices. For instance

$$N = 5: I^5 = -\sum_{i=0}^4 \frac{\det_i(Y^{(5)})}{\det(Y^{(5)})} I^4(i), (5.11)$$

where  $Y^{(5)}$  is the  $5 \times 5$  Y matrix, and  $det_i$  represents the determinant of matrix Y where all elements of the *i*-th column have been replaced by 1.

Similarly we obtain

$$N = 6: I^6 = -\sum_{i=0}^{5} \frac{\det_i(Y^{(6)})}{\det(Y^{(6)})} I^5(i). (5.12)$$

Let us now see how these formula get simplified using the method described so far. For the 5-point function we get

$$I^5 = \sum_{i=0}^4 d_i I^4(i) \tag{5.13}$$

with

$$d_i = \frac{1}{2} \left( \frac{1}{D_i(q_{(i)}^+)} + \frac{1}{D_i(q_{(i)}^-)} \right) , \qquad (5.14)$$

whereas the 6-point function reads

$$I^{6} = \sum_{i < j} \sum_{i,j=0}^{5} d_{ij} I^{4}(ij), \qquad (5.15)$$

where  $I^4(ij)$  is obtained from  $I^6$  by dropping the propagators  $\bar{D}_i$  and  $\bar{D}_j$  and where

$$d_{ij} = \frac{1}{2} \left( \frac{1}{D_i(q_{(ij)}^+)D_j(q_{(ij)}^+)} + \frac{1}{D_i(q_{(ij)}^-)D_j(q_{(ij)}^-)} \right). \tag{5.16}$$

In the above equations  $q_{(i)}^{\pm}$  are the two solutions given in Eq. (3.7) when all the propagators, except  $D_i$ , are zero. Analogously  $q_{(ij)}^{\pm}$  are the solutions when  $D_i$  and  $D_j$  are the only non vanishing propagators.

It is quite straightforward to prove that

$$\frac{1}{2} \left( \frac{1}{D_i(q_{(i)}^+)} + \frac{1}{D_i(q_{(i)}^-)} \right) = -\frac{\det_i(Y^{(5)})}{\det(Y^{(5)})}. \tag{5.17}$$

This is because the Gram determinant of  $q, p_1, p_2, p_3, p_4, p_5$  should be zero, which results to

$$\begin{vmatrix} 2D_{0} + Y_{00} & D_{1} - D_{0} + Y_{10} - Y_{00} & D_{2} - D_{0} + Y_{20} - Y_{00} & \dots & D_{5} - D_{0} + Y_{20} - Y_{00} \\ D_{1} - D_{0} + Y_{10} - Y_{00} & Y_{11} - Y_{10} - Y_{01} + Y_{00} & Y_{12} - Y_{10} - Y_{02} + Y_{00} & \dots & Y_{15} - Y_{10} - Y_{05} + Y_{00} \\ D_{2} - D_{0} + Y_{20} - Y_{00} & Y_{21} - Y_{20} - Y_{01} + Y_{00} & Y_{22} - Y_{20} - Y_{02} + Y_{00} & \dots & Y_{25} - Y_{20} - Y_{05} + Y_{00} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{5} - D_{0} + Y_{20} - Y_{00} & Y_{51} - Y_{50} - Y_{01} + Y_{00} & Y_{52} - Y_{50} - Y_{02} + Y_{00} & \dots & Y_{55} - Y_{50} - Y_{05} + Y_{00} \end{vmatrix} = 0$$

$$(5.18)$$

when on takes into account that

$$2q^2 = 2D_0 + 2m_0^2,$$

$$2q \cdot p_i = D_i - D_0 + Y_{0i} - Y_{00}$$

and

$$2p_i \cdot p_j = Y_{ij} - Y_{i0} - Y_{0j} + Y_{00}$$
.

Taking Eq. (5.18) at the point  $q = q_{(i)}$  we end up with a second order equation for  $D_i$  given by

$$aD_i^2 + bD_i + c = 0,$$

with  $b = -2det_i(Y^{(5)})$  and  $c = det(Y^{(5)})$ . An analytical proof for arbitrary N, can be found in [14].

## 6. Conclusions

We have shown how computing the *integrand* of any one-loop amplitude at special values of the integration momentum allows the one-shot reconstruction of all the coefficients of the scalar loop functions and of the rational terms. Then, by simply multiplying those coefficients by the known scalar integrals, the computation of the amplitude becomes trivial. Our method should be particularly useful in the case when recursive techniques are used to numerically compute the integrand. We plan to investigate this subject in the near future.

# Acknowledgments

We thank Pierpaolo Mastrolia and Andre van Hameren for useful discussions. RP and GO acknowledge the financial support of the ToK Program "ALGOTOOLS" (MTKD-CD-2004-014319). The research of RP was also supported in part by MIUR under contract 2004021808\_009.

### Appendices

## A. The system for the coefficients of the 3-point functions

We choose the following seven solutions (see Eq. (3.20))

$$q_1^{\pm}, \quad q_2^{\pm}, \quad q_3^{\pm} \quad \text{and} \quad q_6^{+},$$
 (A.1)

and define the combinations

$$T^{\pm}(q_k) \equiv \frac{R'(q_k^+) \pm R'(q_k^-)}{2}$$
. (A.2)

Then, from Eq. (3.22), one obtains, for the even powers of j, the system

$$\begin{cases}
T^{+}(q_{1}) = +c_{-2} +c_{0} +c_{2} \\
T^{+}(q_{2}) = -c_{-2} +c_{0} -c_{2} \\
T^{+}(q_{3}) = +c_{-2}e^{-2i\pi/3} +c_{0} +c_{2}e^{2i\pi/3}
\end{cases} ,$$
(A.3)

whose solution is

$$c_0 = \frac{T^+(q_1) + T^+(q_2)}{2},$$

$$c_{\pm 2} = \left[\frac{T^+(q_1) - T^+(q_2)}{2} - e^{\pm 2i\pi/3}(T^+(q_3) - c_0)\right] \frac{1}{1 - e^{\mp 2i\pi/3}}.$$
(A.4)

For the odd powers of j one gets instead

$$\begin{cases}
T^{-}(q_{3}) = -c_{-3} + c_{-1}e^{-i\pi/3} + c_{1}e^{i\pi/3} - c_{3} \\
T^{-}(q_{2}) = +ic_{-3} -ic_{-1} +ic_{1} -ic_{3} \\
T^{-}(q_{1}) = -c_{-3} -c_{-1} -c_{1} -c_{3} \\
T^{0}(q_{6}) = -ic_{-3} +c_{-1}e^{-i\pi/6} +c_{1}e^{i\pi/6} +ic_{3}
\end{cases}$$
(A.5)

where

$$T^{0}(q_{k}) \equiv R'(q_{k}^{+}) - \sum_{j=-1}^{1} c_{2j} [e^{i\pi/k}]^{2j}$$
(A.6)

is known because the coefficients  $c_{0,\pm 2}$  have already been determined. The solution reads

$$c_{\pm 1} = \frac{[T^{-}(q_3) - T^{-}(q_1)](1 + e^{\pm i\pi/3}) \mp i[T^{0}(q_6) + T^{-}(q_2)](1 + e^{\mp i\pi/3})}{3},$$

$$c_{\pm 3} = -c_{\mp 1} - \frac{T^{-}(q_1) \mp iT^{-}(q_2)}{2}.$$
(A.7)

#### B. The system for the coefficients of the 2-point functions

We choose the following nine solutions (see Eq. (3.30))

$$q_{+11}^+, q_{+12}^+, q_{+13}^+, q_{1-1}^-, q_{1-2}^-, q_0,$$
 (B.1)

and build up the combinations

$$S^{\pm}(q_k^{\pm}) \equiv \frac{R''(q_{+1k}^{\pm}) \pm R''(q_{-1k}^{\pm})}{2}.$$
 (B.2)

Therefore the  $\beta$  coefficients satisfy the system

$$\begin{cases}
S^{-}(q_{1}^{+}) = -\beta_{-1} + \beta_{0} - \beta_{1} \\
S^{-}(q_{2}^{+}) = -i\beta_{-1} + \beta_{0} + i\beta_{1} \\
S^{-}(q_{3}^{+}) = \beta_{-1}e^{-i\pi/3} + \beta_{0} \beta_{1}e^{i\pi/3}
\end{cases} ,$$
(B.3)

whose solution reads

$$\beta_{\pm 1} = \frac{(1 + e^{\mp i\pi/3})[S^{-}(q_{2}^{+}) - S^{-}(q_{1}^{+})] - (1 \mp i)[S^{-}(q_{3}^{+}) - S^{-}(q_{1}^{+})]}{\pm i(3 - \sqrt{3})},$$

$$\beta_{0} = S^{-}(q_{1}^{+}) + (\beta_{-1} + \beta_{1}). \tag{B.4}$$

Next, by defining

$$T^{\pm}(q_k) \equiv \frac{S^+(q_k^+) \pm S^+(q_k^-)}{2},$$
 (B.5)

one finds

$$\begin{cases}
T^{-}(q_{1}) = -b_{-1} & -b_{1} \\
T^{-}(q_{2}) = -ib_{-1} & +ib_{1}
\end{cases}$$
(B.6)

whose solution is

$$b_{\pm 1} = -\frac{1}{2} \left[ T^{-}(q_1) \pm i T^{-}(q_2) \right] . \tag{B.7}$$

Now that  $\beta_0$ ,  $\beta_{-1}$ ,  $\beta_1$ ,  $b_{-1}$  and  $b_1$  are known we define

$$T^{0}(q_{3}) \equiv R''(q_{+13}^{+}) - \sum_{j=-1}^{1} \beta_{j} \left[ e^{i\pi/3} \right]^{j} - b_{-1} e^{-i\pi/3} - b_{1} e^{i\pi/3},$$
 (B.8)

then

$$\begin{cases}
T^{+}(q_{1}) = +b_{-2} +b_{0} +b_{2} \\
T^{+}(q_{2}) = -b_{-2} +b_{0} -b_{2} \\
T^{0}(q_{3}) = +b_{-2}e^{-2i\pi/3} +b_{0} +b_{2}e^{2i\pi/3}
\end{cases}$$
(B.9)

This system is analogous to the system in Eq. (A.3) with the replacements  $c_i \to b_i$  and  $T^+(q_3) \to T^0(q_3)$ . Therefore its solution can be directly read from Eq. (A.4):

$$b_0 = \frac{T^+(q_1) + T^+(q_2)}{2},$$

$$b_{\pm 2} = \left[ \frac{T^+(q_1) - T^+(q_2)}{2} - e^{\pm 2i\pi/3} (T^0(q_3) - b_0) \right] \frac{1}{1 - e^{\mp 2i\pi/3}}.$$
(B.10)

Finally, from Eq. (3.35) one obtains the last coefficient

$$\tilde{b}_{00}(01) = \frac{R''(q_0) - b_0 - \sqrt{3}\beta_0}{2k_1^4 F}.$$
(B.11)

## References

- [1] F. A. Berends and W. T. Giele, Nucl. Phys. B **306**, 759 (1988);
  - F. A. Berends, W. T. Giele and H. Kuijf, Phys. Lett. B 232, 266 (1989);
  - F. A. Berends, H. Kuijf, B. Tausk and W. T. Giele, Nucl. Phys. B 357, 32 (1991);
  - F. Caravaglios and M. Moretti, Phys. Lett. B 358, 332 (1995) [arXiv:hep-ph/9507237];
  - P. Draggiotis, R. H. Kleiss and C. G. Papadopoulos, Phys. Lett. B 439, 157 (1998) [arXiv:hep-ph/9807207];
  - A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082];
  - P. D. Draggiotis, R. H. Kleiss and C. G. Papadopoulos, Eur. Phys. J. C 24, 447 (2002) [arXiv:hep-ph/0202201];
  - M. L. Mangano, M. Moretti, F. Piccinini, R. Pittau and A. D. Polosa, JHEP **0307**, 001 (2003) [arXiv:hep-ph/0206293];
  - C. G. Papadopoulos and M. Worek, arXiv:hep-ph/0512150;
  - C. Duhr, S. Hoche and F. Maltoni, arXiv:hep-ph/0607057.

- [2] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B 153 (1979) 365.
- [3] G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.
- [4] A. Denner, S. Dittmaier, M. Roth and L. H. Wieders, Phys. Lett. B 612 (2005) 223[arXiv:hep-ph/0502063];
  - A. Denner, S. Dittmaier, M. Roth and L. H. Wieders, Nucl. Phys. B **724** (2005) 247 [arXiv:hep-ph/0505042];
  - G. Belanger, F. Boudjema, J. Fujimoto, T. Ishikawa, T. Kaneko, K. Kato and Y. Shimizu, Phys. Rept. **430** (2006) 117 [arXiv:hep-ph/0308080];
  - G. Montagna, F. Piccinini, O. Nicrosini, G. Passarino and R. Pittau, Nucl. Phys. B **401** (1993) 3.
- $[5]\,$  A. Denner and S. Dittmaier, Nucl. Phys. B  $\mathbf{734}$  (2006) 62 [arXiv:hep-ph/0509141].
- [6] A. Ferroglia, M. Passera, G. Passarino and S. Uccirati, Nucl. Phys. B 650 (2003) 162 [arXiv:hep-ph/0209219];
  - W. T. Giele and E. W. N. Glover, arXiv:hep-ph/0402152;
  - T. Binoth, J. P. Guillet and G. Heinrich, Nucl. Phys. B **572** (2000) 361 [arXiv:hep-ph/9911342];
  - G. Duplancic and B. Nizic, arXiv:hep-ph/0303184;
  - T. Binoth, G. Heinrich and N. Kauer, Nucl. Phys. B 654 (2003) 277 [arXiv:hep-ph/0210023];
  - D. E. Soper, Phys. Rev. D 62 (2000) 014009 [arXiv:hep-ph/9910292],
  - Phys. Rev. D **64** (2001) 034018 [arXiv:hep-ph/0103262].
  - Z. Nagy and D. E. Soper, JHEP 0309 (2003) 055 [arXiv:hep-ph/0308127];
  - A. van Hameren, J. Vollinga and S. Weinzierl, Eur. Phys. J. C **41** (2005) 361 [arXiv:hep-ph/0502165].
- [7] F. del Aguila and R. Pittau, JHEP 0407 (2004) 017 [arXiv:hep-ph/0404120];
   R. Pittau, arXiv:hep-ph/0406105.
- [8] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B 435 (1995) 59 [arXiv:hep-ph/9409265].
- [9] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **725** (2005) 275 [arXiv:hep-th/0412103].
- [10] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, arXiv:hep-ph/0607014 and arXiv:hep-ph/0604195;
  - Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D 73 (2006) 065013 [arXiv:hep-ph/0507005].
- [11] G. Källén and J. Toll, J. Math. Phys. 6, 299 (1965).
- [12] W. L. van Neerven and J. A. M. Vermaseren, Phys. Lett. B 137 (1984) 241.
- Z. G. Xiao, G. Yang and C. J. Zhu, arXiv:hep-ph/0607015;
   X. Su, Z. G. Xiao, G. Yang and C. J. Zhu, arXiv:hep-ph/0607016.
- [14] D. B. Melrose, Nuovo Cim. **40** (1965) 181.
- [15] A. Denner, Fortsch. Phys. **41** (1993) 307.