Contribution of QED to Rational Terms in 1-loop Feynman Diagrams in the Standard Model

Jonathan Kley
Department of Physics
Technical University of Munich
James-Franck-Str. 1, 85748 Garching
jonathan.kley@tum.de

January 21, 2020

ABSTRACT

For years it has been of high interest to automate the calculation of Feynman diagrams in the Standard Model (SM) in order to be able to keep up with the increasing precision of experimental data. A lot of efforts have been made in the early 2000s to make progress in this direction where one of the big milestones was the proof that any amplitude in the SM can be decomposed into a box-, a triangle-, a bubble-, a tadpole-subdiagram and a rational term which can be of two types. In this paper we investigate the ϵ-dimensional contribution of dimensional regularisation to this rational term. If all of the coefficients in the composition into the mentioned subdiagrams alongside the rational terms are known we obtain an effective treelike theory giving us 1-loop results for the whole SM.

This paper is structured as follows: After a short introduction to the subject, we first calculate R_2 in pure QED as a warm-up and then the QED contribution to R_2 in the SM. Then, we renormalize QED in terms of scalar integrals and calculate the contribution of QED to the renormalization of the SM. Once this is done, we are able to implement an algorithm to automate calculations in the SM and can compare our results to the results from other references.

Keywords QFT · 1-loop Feynman Diagrams · Rational Terms · More

Contents

1 Introduction[1]

In the early 2000s a lot of effort has been put into automating loop calculations in order to keep up with the increasing accuracy of high precision measurements. Early attempts were mostly based on Passarino-Veltman tensor reduction while newer developments changed their focus to unitarity arguments. Instead of performing the whole tensor reduction process, now the problem is substituted by finding the coefficients of the scalar integrals appearing in the 1-loop amplitudes. This is possible because the basis of scalar integrals is know in terms of boxes, triangles, bubbles and tadpoles. This leads to a master formula for any one-loop amplitude

$$\mathcal{M} = \sum_i d_i \text{Box}_i + \sum_i c_i \text{Triangle}_i + \sum_i b_i \text{Bubble}_i + \sum_i a_i \text{Tadpole}_i + R \tag{1.1}$$

The first attempts to extract the coefficients in equation 1.1 failed at providing a systematic procedure to completely determine them. Finally, the OPP method was brought forward which makes it possible to find all of the coefficients for a given theory. But with the OPP method a new class of terms arise (denoted by R in equation 1.1) which are not the coefficient of one of the four types of diagrams. They can be split into two categories

$$R = R_1 + R_2 \tag{1.2}$$

where R_1 can be computed alongside the coefficients of the scalar integrals in OPP.
R_2 on the other hand is the ϵ-dimensional contribution of dimensional regularization to the amplitude. Any 1-loop m-point function $\bar{A}(\bar{q})$ can be decomposed in a numerator $\bar{N}(\bar{q})$ and denominators $\bar{D}_i$

$$\bar{A}(\bar{q}) = \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}, \qquad \bar{D}_{\pm i} = (\bar{q} \pm \bar{p}_i)^2 - m_i^2, p_0 \neq 0 \tag{1.3}$$

where $\bar{q}$ is the d-dimensional loop momentum and m_i is the mass of the particle corresponding to the propagator with numerator D_i. The d-dimensional numerator function $\bar{N}(\bar{q})$ can be split in a 4-dimensional and an ϵ-dimensional part

$$\bar{N}(\bar{q}) = N(q) + \widetilde{N}(\widetilde{q}^2, q, \epsilon) \tag{1.4}$$

where $\widetilde{N}(\widetilde{q}^2, q, \epsilon)$ is of interest to us because it makes up the rational terms of the form R_2 which are defined as

$$R_2 \equiv \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}(\widetilde{q}^2, q, \epsilon)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \tag{1.5}$$

R_2 is just a rational combination of Lorentz tensors and parameters of the theory, i.e. the couplings or masses of the particles in the theory. The R_2 contribution can be added to the theory by introducing tree-level like Feynman Rules similarly to counterterms in perturbative renormalization procedures.
To compute R_2 we first have to extract the ϵ-dimensional part of the amplitude by splitting the d-dimensional Lorentz tensors appearing in the amplitude into a 4-dimensional and an ϵ-dimensional part

$$\bar{A}^{\mu_1 \cdots \mu_n} = A^{\mu_1 \cdots \mu_n} + \widetilde{A}^{\mu_1 \cdots \mu_n}. \tag{1.6}$$

To simplify our calculations later, we can establish a few identities for the manipulation of d-dimensional Lorentz tensors. If we contract a d-dimensional tensor with an observable Lorentz tensor (like the momentum of an external particle) only the 4-dimensional part survives, e.g. for a loop momentum $\bar{q}^\mu$ and an external momentum p^μ

$$\bar{q} \cdot p = q \cdot p. \tag{1.7}$$

Thus, if an amplitude transforms with indices $\mu_1, \ldots, \mu_n$ under a Lorentz transformation, the tensors in the amplitude bearing these indices will only appear as 4-dimensional.

---

[1]The whole introduction is based on [1, 2].

Since, we want to perform calculations in QED which contains a fermion, we have to extend the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$ to d dimensions. This is straightforward by promoting $\gamma^\mu \to \bar{\gamma}^\mu$ and extending the Minkowski metric to d dimensions by adding additional -1s on the diagonal for the extra spatial dimensions. We have

$$\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2\bar{g}^{\mu\nu}\mathbb{1}_d \tag{1.8}$$

If we want to preserve the Clifford algebra separately in 4 and ϵ dimensions this implies

$$\{\gamma^\mu, \widetilde{\gamma}^\nu\} = 0 \tag{1.9}$$

As opposed to QED the Standard Model is a chiral theory, i.e. it couples differently to left- and right-handed currents. This means that also axial-vector currents appear in the theory which are formulated with the fifth gamma matrix. The extension of γ_5 to d dimensions is not as straightforward as with the four gamma matrices. This is because chirality is a property of four dimensions.

If we also want to impose $\{\gamma_5, \gamma^\mu\} = 0$ for $d \neq 4$, then $\mathrm{Tr}\left(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = 0$ for $d \neq 0, 2, 4$ which clashes with $\mathrm{Tr}\left(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = -4i\epsilon^{\alpha\beta\gamma\delta}$ in four dimensions [3]. But the identity is essential in the evaluation of the triangle diagram for the Adler-Bell-Jackiw anomaly. The only definition of γ_5 which is consistent with the chiral anomaly is the definition of 't Hooft and Veltman [4]: $\gamma_5 = i/4!\, \epsilon_{\mu_1\ldots\mu_4}\gamma^{\mu_1}\cdots\gamma^{\mu_4}$. This definition implies

$$\{\gamma_5, \gamma^\mu\} = 0 \text{ and } [\gamma_5, \widetilde{\gamma}^\mu] = 0. \tag{1.10}$$
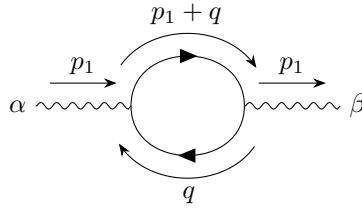
2 R_2 in Pure QED

Before we calculate anything in the Standard Model, let us start with the R_2 of pure QED. We have to consider all n-point functions up to $n = 4$ which are allowed by the Feynman rules of QED and calculate their contribution to equation 1.5.

2.1 2-point functions

The Feynman rules of QED allow two 2-point functions; the self-energy diagrams of the photon and the electron. Let us start with the photon self-energy which has the simplest Lorentz structure and therefore an easy to evaluate numerator function.

Photon self-energy
The photon 2-point function is given by

$$= \int \frac{d^d\bar{q}}{(2\pi)^d} (-1) \, \mathrm{Tr}\left\{ ie\bar{\gamma}^\alpha \frac{i\left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m\right)}{(p_1+q)^2 - m^2} ie\bar{\gamma}^\beta \frac{i\left(\bar{\slashed{q}} + m\right)}{q^2 - m^2} \right\}$$

$$\equiv \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1\bar{D}_0}$$

Where we defined the numerator and denominator functions in the last step. Now we can extract the ϵ-dimensional contribution from the numerator

$$\bar{N}(\bar{q}) = -e^2 \, \mathrm{Tr}\left\{ \bar{\gamma}^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m\right) \bar{\gamma}^\beta \left(\bar{\slashed{q}} + m\right) \right\} = -e^2 \, \mathrm{Tr}\left\{ \gamma^\alpha \left(\slashed{p}_1 + \slashed{q} + m\right) \gamma^\beta \left(\slashed{q} + m\right) + \gamma^\alpha\widetilde{\slashed{q}}\gamma^\beta\widetilde{\slashed{q}} \right\} \equiv N + \widetilde{N}$$

Here, the first term is the 4-dimensional numerator which also appears in normal loop calculations and the second term is the ϵ-dimensional part which we need for the calculation of R_2. We have used that the gamma matrices are contracted with external momenta and therefore only the 4-dimensional part of the gamma matrices survives.
We can now evaluate the trace and get

$$\widetilde{N} = -e^2 \, \mathrm{Tr}\left\{ \gamma^\alpha\widetilde{\slashed{q}}\gamma^\beta\widetilde{\slashed{q}} \right\} = 4e^2\widetilde{q}^2 g^{\alpha\beta}$$
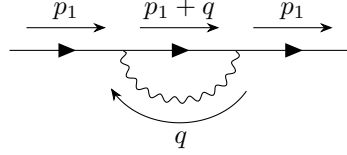
Where we have used that the ϵ-dimensional gamma matrices anticommute with the 4-dimensional gamma matrices and the trace identity for 2 gamma matrices (equation C.1) which can be found alongside a proof in appendix C. Plugging the expression for $\widetilde{N}$ in equation 1.5 gives

$$R_2^{\gamma\gamma} = \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{\widetilde{N}}{\bar{D}_1\bar{D}_0} = \frac{4e^2}{16\pi^4} \underbrace{\int d^d\bar{q} \frac{\widetilde{q}^2}{\bar{D}_1\bar{D}_0}}_{-i\frac{\pi}{2}\left(2m^2 - p_1^2/3\right)} = \frac{-ie^2}{8\pi^2} g^{\alpha\beta}\left(2m^2 - \frac{p_1^2}{3}\right) \tag{2.1}$$

In the last step we have used the 2-point integral B.1.

Electron self-energy

The other 2-point function in QED is the electron 2-point function which is given by

$$= \int \frac{d^d q}{(2\pi)^d} ie\gamma^\alpha \frac{i\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1 + q)^2 - m^2} ie\gamma^\beta \frac{-ig_{\alpha\beta}}{q^2} = \int \frac{d^d q}{(2\pi)^d} \left(-e^2\right) \gamma^\alpha \frac{\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1 + q)^2 - m^2} \gamma_\alpha \frac{1}{q^2}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

Now we extract again the ϵ-dimensional part from the numerator function we defined in the last step. We get

$$\bar{N}(\bar{q}) = -e^2 \bar{\gamma}^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m\right) \bar{\gamma}_\alpha = -e^2 \left\{ \gamma^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m\right) \gamma_\alpha + \widetilde{\gamma}^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m\right) \widetilde{\gamma}_\alpha + \gamma^\alpha \widetilde{\slashed{q}} \gamma_\alpha + \widetilde{\gamma}^\alpha \widetilde{\slashed{q}} \widetilde{\gamma}_\alpha \right\} \equiv N + \widetilde{N}$$

Here, the first term is again the normal 4-dimensional numerator and the rest the ϵ-dimensional part we are interested in. It can be simplified further

$$\widetilde{N} = -e^2 \left\{ \widetilde{\gamma}^\alpha \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m\right) \widetilde{\gamma}_\alpha + \gamma^\alpha \widetilde{\slashed{q}} \gamma_\alpha + \widetilde{\gamma}^\alpha \widetilde{\slashed{q}} \widetilde{\gamma}_\alpha \right\} = -e^2 \left\{ -\underbrace{\widetilde{\gamma}^\alpha \widetilde{\gamma}_\alpha}_{=\epsilon} \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} - m\right) - \underbrace{\gamma^\alpha \gamma_\alpha}_{=4} \widetilde{\slashed{q}} + \widetilde{\gamma}^\alpha \widetilde{\slashed{q}} \widetilde{\gamma}_\alpha \right\}$$

where we have used again that the ϵ-dimensional gamma matrices anticommute with the 4-dimensional gamma matrices and equation C.5 to simplify the expression. Plugging $\widetilde{N}$ into the definition of R_2 we get

$$R_2^{ee} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{-e^2}{(2\pi)^4} \int d^d \bar{q} \frac{1}{\bar{D}_1 \bar{D}_0} \left(-\epsilon \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} - m\right) + \underbrace{\widetilde{\slashed{q}}(\ldots)}_{=0} \right) =$$
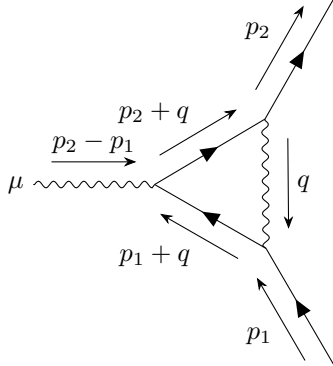
$$= \frac{e^2}{(2\pi)^4} \left\{ \underbrace{\int d^d \bar{q} \frac{\epsilon \left(\bar{\slashed{p}}_1 - m\right)}{\bar{D}_1 \bar{D}_0}}_{=-2\epsilon \frac{i\pi^2}{\epsilon}\left(\slashed{p}_1 - m\right)} + \underbrace{\int d^d \bar{q} \frac{\epsilon \slashed{q}}{\bar{D}_1 \bar{D}_0}}_{=\epsilon \frac{i\pi^2}{\epsilon} \slashed{p}_1} \right\} = \frac{e^2}{(2\pi)^4} \epsilon \frac{i\pi^2}{\epsilon} \left((-2) \left(\slashed{p}_1 - m\right) + \slashed{p}_1 \right) = \frac{-ie^2}{16\pi^2} \left(\slashed{p}_1 - 2m\right)$$

$$(2.2)$$

Where we have used that the integral over an odd function in q integrated over the whole space vanishes at the end of the first line. We also used the 2-point integrals B.2 and B.3.

These are all the 2-point functions that are allowed by the Feynman rules of QED. So we continue with the 3-point functions now.

2.2 3-point functions

There are two possible 3-point functions, the 1PI contribution to the electron-photon vertex at 1-loop in QED and the 3-photon triangle diagram. Let us start with the electron-photon vertex which is given by

$$= \int \frac{d^d q}{(2\pi)^d} ie\gamma^\beta \frac{i\left(\slashed{p}_1 + \slashed{q} + m\right)}{(p_1 + q)^2 - m^2} ie\gamma^\mu \frac{i\left(\slashed{p}_2 + \slashed{q} + m\right)}{(p_2 + q)^2 - m^2} ie\gamma^\alpha \frac{-ig_{\alpha\beta}}{q^2}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}$$

We get for $\bar{N}$

$$\bar{N}(\bar{q}) = e^3 \left\{ \bar{\gamma}^\beta \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m\right) \bar{\gamma}^\mu \left(\bar{\slashed{p}}_2 + \bar{\slashed{q}} + m\right) \bar{\gamma}_\beta \right\} = e^3 \left\{ \gamma^\beta \left(\slashed{p}_1 + \slashed{q} + m\right) \gamma^\mu \left(\slashed{p}_2 + \slashed{q} + m\right) \gamma_\beta + \right.$$

$$\left. + \widetilde{\gamma}^\beta \left(\slashed{p}_1 + \slashed{q} + m\right) \gamma^\mu \left(\slashed{p}_2 + \slashed{q} + m\right) \widetilde{\gamma}_\beta + \underbrace{\gamma^\beta \widetilde{\slashed{q}} \gamma^\mu \widetilde{\slashed{q}} \gamma_\beta}_{\equiv \text{①}} + \underbrace{\widetilde{\gamma}^\beta \widetilde{\slashed{q}} \gamma^\mu \widetilde{\slashed{q}} \widetilde{\gamma}_\beta}_{\equiv \text{②}} \right\} \equiv N + \widetilde{N}$$

Where we have used that the gamma matrices are contracted with external momenta and therefore only the 4-dimensional part survives. The last 3 terms define $\widetilde{N}$. Let us work on ① and ② separately. Using the fact that 4-dimensional and ϵ-dimensional gamma matrices anticommute and equations C.6 and C.8 we get

$$\text{①} = \widetilde{q}_\rho \widetilde{q}_\sigma \gamma^\beta \widetilde{\gamma}^\rho \gamma^\mu \widetilde{\gamma}^\sigma \gamma_\beta = \widetilde{q}_\rho \widetilde{q}_\sigma (-1)^3 \widetilde{\gamma}^\rho \widetilde{\gamma}^\sigma \gamma^\beta \gamma^\mu \gamma_\beta = 2\widetilde{\slashed{q}} \widetilde{\slashed{q}} \gamma^\mu = 2\widetilde{q}^2 \gamma^\mu$$

And for the other term

$$\text{②} = \widetilde{q}_\rho \widetilde{q}_\sigma \widetilde{\gamma}^\beta \widetilde{\gamma}^\rho \gamma^\mu \widetilde{\gamma}^\sigma \widetilde{\gamma}_\beta = \widetilde{q}_\rho \widetilde{q}_\sigma (-1)^2 \gamma^\mu \widetilde{\gamma}^\beta \widetilde{\gamma}^\rho \widetilde{\gamma}^\sigma \widetilde{\gamma}_\beta = \widetilde{q}_\rho \widetilde{q}_\sigma \gamma^\mu \left(\epsilon \widetilde{\gamma}^\rho \widetilde{\gamma}^\sigma + 2 [\widetilde{\gamma}^\rho, \widetilde{\gamma}^\sigma]\right) = \epsilon \widetilde{q}^2 \gamma^\mu$$

where we have used that the 4- and ϵ-dimensional gamma matrices anticommute and equation C.7 for $d = \epsilon$. And in the last step we used equation C.8 which also implies $[\widetilde{\slashed{q}}, \widetilde{\slashed{q}}] = 0$.
Hence, after summing all the terms we have

$$\widetilde{N} = -e^3 \epsilon \left(\slashed{p}_1 + \slashed{q} - m\right) \gamma^\mu \left(\slashed{p}_2 + \slashed{q} - m\right) + (2 + \epsilon) \widetilde{q}^2 \gamma^\mu$$

We can again plug this in the definition of R_2 and get

$$R_2^{\gamma ee} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}}{\bar{D}_0 \bar{D}_1 \bar{D}_2} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{e^3}{\bar{D}_0 \bar{D}_1 \bar{D}_2} \left\{ -\epsilon \left(\slashed{p}_1 + \slashed{q} - m\right) \gamma^\mu \left(\slashed{p}_2 + \slashed{q} - m\right) + (2 + \epsilon) \widetilde{q}^2 \gamma^\mu \right\} =$$

$$= \frac{e^3}{(2\pi)^4} \int d^d \bar{q} \frac{1}{\bar{D}_0 \bar{D}_1 \bar{D}_2} \left\{ -\epsilon \slashed{q} \gamma^\mu \slashed{q} + (2 + \epsilon) \widetilde{q}^2 \gamma^\mu \right\} = \frac{e^3}{(2\pi)^4} \left\{ -\epsilon \gamma^\alpha \gamma^\mu \gamma^\beta \left(\frac{-i\pi^2}{2\epsilon} g_{\alpha\beta}\right) + \frac{-i\pi^2}{2} (2 + \epsilon) \gamma^\mu \right\} =$$

$$= \frac{e^3}{(2\pi)^4} \frac{i\pi^2}{2} \left\{ \gamma^\alpha \gamma^\mu \gamma_\alpha - 2\gamma^\mu + O(\epsilon) \right\} = \frac{-ie^3}{8\pi^2} \gamma^\mu \tag{2.3}$$

Where we used the 3-point integrals B.4 and B.6 and in the last step equation C.6.

As already mentioned, there is one more 3-point function at the 1-loop level which is permitted by the Feynman rules: the 3-point function with only photons as external particles. But it does not contribute to R_2 which we will show now. Because of the symmetry of the 3-point function there are 2 contributing diagrams
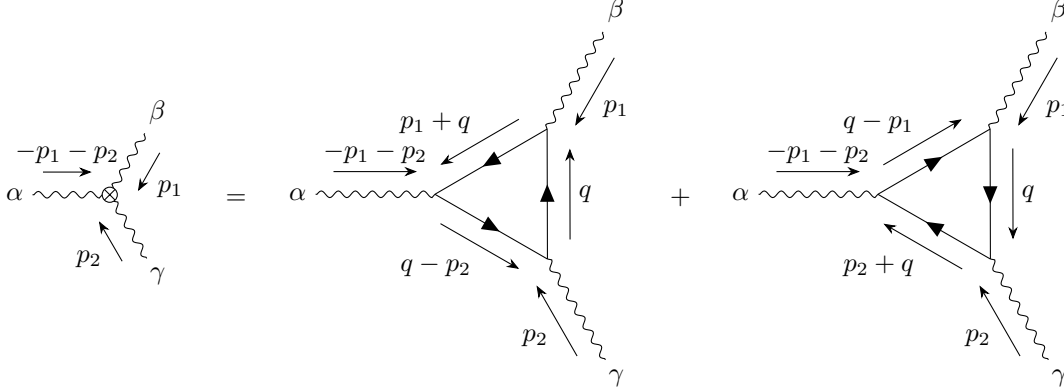
We only calculate the first diagram and then symmetrize the result with $p_1 \leftrightarrow p_2$, $\beta \leftrightarrow \gamma$. Evaluating the first diagram gives

$$= \int \frac{d^d q}{(2\pi)^d} \mathrm{Tr}\left\{ ie\gamma^\beta \frac{i\left(\slashed{q}+m\right)}{q^2-m^2} ie\gamma^\gamma \frac{i\left(\slashed{q}-\slashed{p}_2+m\right)}{(q-p_2)-m^2} ie\gamma^\alpha \frac{i\left(\slashed{q}+\slashed{p}_1+m\right)}{(q+p_1)-m^2} \right\} =$$

$$= \int \frac{d^d q}{(2\pi)^d} e^3 \mathrm{Tr}\left\{ \gamma^\beta \frac{\left(\slashed{q}+m\right)}{q^2-m^2} \gamma^\gamma \frac{\left(\slashed{q}-\slashed{p}_2+m\right)}{(q-p_2)-m^2} \gamma^\alpha \frac{\left(\slashed{q}+\slashed{p}_1+m\right)}{(q+p_1)-m^2} \right\} =$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_{-2} \bar{D}_0}$$

From here we can again extract the d-dimensional numerator

$$\bar{N}(\bar{q}) = e^3 \mathrm{Tr}\left\{ \bar{\gamma}^\beta \left(\bar{\slashed{q}}+m\right) \bar{\gamma}^\gamma \left(\bar{\slashed{q}}-\bar{\slashed{p}}_2+m\right) \bar{\gamma}^\alpha \left(\bar{\slashed{q}}+\bar{\slashed{p}}_1+m\right) \right\} =$$

$$= e^3 \mathrm{Tr}\left\{ \gamma^\beta \left(\slashed{q}+m\right) \gamma^\gamma \left(\slashed{q}-\slashed{p}_2+m\right) \gamma^\alpha \left(\slashed{q}+\slashed{p}_1+m\right) + \gamma^\beta \left(\slashed{q}+\widetilde{\slashed{q}}+m\right) \gamma^\gamma \left(\slashed{q}+\widetilde{\slashed{q}}-\slashed{p}_2+m\right) \gamma^\alpha \left(\slashed{q}+\widetilde{\slashed{q}}+\slashed{p}_1+m\right) \right\} =$$

$$\equiv N + \widetilde{N}$$

We used again that the gamma matrices are contracted with external momenta. The second term is $\widetilde{N}$ which can be further simplified as follows

$$
\begin{aligned}
\widetilde{N} &= e^3 \mathrm{Tr}\left\{\gamma^\beta \left(\slashed{q} + \widetilde{\slashed{q}} + m\right)\gamma^\gamma \left(\slashed{q} + \widetilde{\slashed{q}} - \slashed{p}_2 + m\right)\gamma^\alpha \left(\slashed{q} + \widetilde{\slashed{q}} + \slashed{p}_1 + m\right)\right\} = \\
&= e^3 \mathrm{Tr}\left\{\gamma^\beta \slashed{q}\gamma^\gamma \widetilde{\slashed{q}}\gamma^\alpha \widetilde{\slashed{q}} + \gamma^\beta \widetilde{\slashed{q}}\gamma^\gamma \slashed{q}\gamma^\alpha \widetilde{\slashed{q}} + \gamma^\beta \widetilde{\slashed{q}}\gamma^\gamma \widetilde{\slashed{q}}\gamma^\alpha \slashed{q} + \gamma^\beta \widetilde{\slashed{q}}\gamma^\gamma \left(-\slashed{p}_2\right)\gamma^\alpha \widetilde{\slashed{q}} + \gamma^\beta \widetilde{\slashed{q}}\gamma^\gamma \widetilde{\slashed{q}}\gamma^\alpha \slashed{p}_1\right\} = \\
&= -4e^3 \widetilde{q}^2 \left\{q_\mu \left[\left(g^{\beta\mu}g^{\gamma\alpha} - g^{\beta\gamma}g^{\mu\alpha} + g^{\beta\alpha}g^{\mu\gamma}\right) + \left(g^{\beta\gamma}g^{\alpha\mu} - g^{\beta\mu}g^{\gamma\alpha} + g^{\beta\alpha}g^{\mu\gamma}\right) + \left(g^{\beta\gamma}g^{\mu\alpha} - g^{\beta\alpha}g^{\mu\gamma} + g^{\beta\mu}g^{\alpha\gamma}\right)\right] + \right. \\
&\quad \left. + p_{1\mu}\left(g^{\beta\gamma}g^{\alpha\mu} - g^{\beta\alpha}g^{\gamma\mu} + g^{\beta\mu}g^{\alpha\gamma}\right) - p_{2\mu}\left(g^{\beta\gamma}g^{\gamma\mu} - g^{\beta\mu}g^{\alpha\gamma} + g^{\alpha\beta}g^{\gamma\mu}\right)\right\} = \\
&= -4e^3 \widetilde{q}^2 \left\{q^\beta g^{\alpha\gamma} + q^\gamma g^{\alpha\beta} + q^\alpha g^{\beta\gamma} + p_1^\alpha g^{\beta\gamma} - p_1^\gamma g^{\alpha\beta} + p_1^\beta g^{\alpha\gamma} - p_2^\alpha g^{\beta\gamma} + p_2^\beta g^{\alpha\gamma} - p_2^\gamma g^{\alpha\beta}\right\}
\end{aligned}
$$

From the second to the third line we have used that 4- and ϵ-dimensional gamma matrices anticommute as well as equations C.2 and C.8.

This gives for the R_2 contribution of the first diagram

$$
\begin{aligned}
R_2^1 &= \frac{1}{(2\pi)^4}\int d^d\bar{q}\,\frac{\widetilde{N}}{\bar{D}_1\bar{D}_{-2}\bar{D}_0} = \\
&= \frac{-4e^3}{(2\pi)^4}\int d^d\bar{q}\,\frac{1}{\bar{D}_1\bar{D}_{-2}\bar{D}_0}\left\{\widetilde{q}^2 q^\beta g^{\alpha\gamma} + \widetilde{q}^2 q^\gamma g^{\alpha\beta} + \widetilde{q}^2 q^\alpha g^{\beta\gamma} + \widetilde{q}^2\left[(p_1 - p_2)^\alpha g^{\beta\gamma} + (p_1 + p_2)^\beta g^{\alpha\gamma} - (p_1 + p_2)^\gamma g^{\alpha\beta}\right]\right\} = \\
&= \frac{-4e^3}{(2\pi)^4}\left\{\frac{i\pi^2}{6}\left[(p_1 - p_2)^\beta g^{\alpha\gamma} + (p_1 - p_2)^\gamma g^{\alpha\beta} + (p_1 - p_2)^\alpha g^{\beta\gamma}\right] - \frac{i\pi^2}{2}\left[(p_1 - p_2)^\alpha g^{\beta\gamma} + (p_1 + p_2)^\beta g^{\alpha\gamma} + \right.\right. \\
&\quad \left.\left. - (p_1 + p_2)^\gamma g^{\alpha\beta}\right]\right\} = \\
&= \frac{-4e^3}{(2\pi)^4}\left\{g^{\alpha\beta}\left[\frac{i\pi^2}{6}(p_1 - p_2)^\gamma + \frac{i\pi^2}{2}(p_1 + p_2)^\gamma\right] + g^{\beta\gamma}\left[\frac{i\pi^2}{6}(p_1 - p_2)^\alpha - \frac{i\pi^2}{2}(p_1 - p_2)^\alpha\right] + \right. \\
&\quad \left. + g^{\alpha\gamma}\left[\frac{i\pi^2}{6}(p_1 - p_2)^\beta - \frac{i\pi^2}{2}(p_1 + p_2)^\beta\right]\right\}
\end{aligned}
$$

Here we have used the 3-point integrals B.4 and B.5.

To obtain the contribution of the second diagram we can simply exchange $p_1 \leftrightarrow p_2$, $\beta \leftrightarrow \gamma$ in the result of the first diagram.

$$
\begin{aligned}
R_2^2 &= R_2^1(p_1 \leftrightarrow p_2,\ \beta \leftrightarrow \gamma) = \\
&= \frac{-4e^3}{(2\pi)^4}\left\{g^{\alpha\gamma}\left[\frac{i\pi^2}{6}(p_2 - p_1)^\beta + \frac{i\pi^2}{2}(p_2 + p_1)^\beta\right] + g^{\beta\gamma}\left[\frac{i\pi^2}{6}(p_2 - p_1)^\alpha - \frac{i\pi^2}{2}(p_2 - p_1)^\alpha\right] + \right. \\
&\quad \left. + g^{\alpha\beta}\left[\frac{i\pi^2}{6}(p_2 - p_1)^\gamma - \frac{i\pi^2}{2}(p_2 + p_1)^\gamma\right]\right\} = \\
&= \frac{-4e^3}{(2\pi)^4}\left\{-g^{\alpha\beta}\left[\frac{i\pi^2}{6}(p_1 - p_2)^\gamma + \frac{i\pi^2}{2}(p_1 + p_2)^\gamma\right] - g^{\beta\gamma}\left[\frac{i\pi^2}{6}(p_1 - p_2)^\alpha - \frac{i\pi^2}{2}(p_1 - p_2)^\alpha\right] + \right. \\
&\quad \left. - g^{\alpha\gamma}\left[\frac{i\pi^2}{6}(p_1 - p_2)^\beta - \frac{i\pi^2}{2}(p_1 + p_2)^\beta\right]\right\} = -R_2^1
\end{aligned}
$$

Now we add up both terms to get the full contribution of the photon triangle diagram. We get

$$
R_2^{3\gamma} = R_2^1 + R_2^2 = R_2^1 - R_2^1 = 0 \tag{2.4}
$$

9

2.3 4-point function

For the 4-point function we have to be more careful. The 1PI contribution at the 1-loop level consists of several diagrams. They are obtained by symmetrizing the external momenta of the diagram as follows

We only calculate one of the diagrams and do the symmetrizing with the result of our calculation, so we only have to evaluate one diagram. The first of the three diagrams gives

$$= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ ie\gamma^\alpha \frac{i\left(\not{p}_1 + \not{q} + m\right)}{(p_1 + q)^2 - m^2} ie\gamma^\beta \frac{i\left(\not{q} + \not{p}_3 + \not{p}_1 + m\right)}{(p_3 + p_1 + q)^2 - m^2} \times \right.$$

$$\left. \times ie\gamma^\gamma \frac{i\left(\not{q} - \not{p}_2 + m\right)}{(q - p_2)^2 - m^2} ie\gamma^\delta \frac{i\left(\not{q} + m\right)}{q^2 - m^2} \right\} \equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_{13} \bar{D}_{-2} \bar{D}_0}$$

where $D_{13} = (p_3 + p_1 + q)^2 - m^2$. From this we get for the d-dimensional numerator the following

$$\bar{N}(\bar{q}) = -e^4 \operatorname{Tr} \left\{ \bar{\gamma}^\alpha \left(\bar{\not{p}}_1 + \bar{\not{q}} + m \right) \bar{\gamma}^\beta \left(\bar{\not{q}} + \bar{\not{p}}_1 + \bar{\not{p}}_3 + m \right) \bar{\gamma}^\gamma \left(\bar{\not{q}} - \bar{\not{p}}_2 + m \right) \bar{\gamma}^\delta \left(\bar{\not{q}} + m \right) \right\} =$$

$$= -e^4 \operatorname{Tr} \left\{ \gamma^\alpha \left(\not{p}_1 + \not{q} + m \right) \gamma^\beta \left(\not{q} + \not{p}_1 + \not{p}_3 + m \right) \gamma^\gamma \left(\not{q} - \not{p}_2 + m \right) \gamma^\delta \left(\not{q} + m \right) + \right.$$

$$+ \gamma^\alpha \widetilde{\not{q}} \gamma^\beta \widetilde{\not{q}} \gamma^\gamma \widetilde{\not{q}} \gamma^\delta \widetilde{\not{q}} + \gamma^\alpha \widetilde{\not{q}} \gamma^\beta \widetilde{\not{q}} \gamma^\gamma \not{q} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \widetilde{\not{q}} \gamma^\gamma \widetilde{\not{q}} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \not{q} \gamma^\gamma \widetilde{\not{q}} \gamma^\delta \widetilde{\not{q}} + \gamma^\alpha \widetilde{\not{q}} \gamma^\beta \not{q} \gamma^\gamma \not{q} \gamma^\delta \widetilde{\not{q}} +$$

$$+ \gamma^\alpha \widetilde{\not{q}} \gamma^\beta \not{q} \gamma^\gamma \widetilde{\not{q}} \gamma^\delta \not{q} + \gamma^\alpha \not{q} \gamma^\beta \widetilde{\not{q}} \gamma^\gamma \not{q} \gamma^\delta \widetilde{\not{q}} \right\} \equiv N + \widetilde{N}$$

Where all of the terms besides the first one define $\widetilde{N}$. Furthermore, we have used that the gamma matrices are contracted with external momenta and that the trace of an odd number of Dirac matrices is zero. Using the fact that 4- and ϵ-dimensional gamma matrices anticommute as well as equation C.8, $\widetilde{N}$ can be further

simplified to

$$\widetilde{N} = -e^4 \mathrm{Tr}\left\{(-1)^{10}\,\widetilde{q}^4 \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta + \widetilde{q}^2\left[(-1)^3\,\gamma^\alpha \gamma^\beta \gamma^\gamma \slashed{q}\gamma^\delta \slashed{q} + (-1)^7\,\gamma^\alpha \slashed{q}\gamma^\beta \gamma^\gamma \gamma^\delta \slashed{q} + (-1)^{11}\,\gamma^\alpha \slashed{q}\gamma^\beta \slashed{q}\gamma^\gamma \gamma^\delta + \right.\right.$$

$$\left.\left. + (-1)^7\,\gamma^\alpha \gamma^\beta \slashed{q}\gamma^\gamma \slashed{q}\gamma^\delta + (-1)^5\,\gamma^\alpha \gamma^\beta \slashed{q}\gamma^\gamma \gamma^\delta \slashed{q} + (-1)^9\,\gamma^\alpha \slashed{q}\gamma^\beta \gamma^\gamma \slashed{q}\gamma^\delta \right] \right\} =$$

$$= -e^4 \mathrm{Tr}\left\{\widetilde{q}^4 \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta - \widetilde{q}^2\left(\gamma^\alpha \gamma^\beta \gamma^\gamma \slashed{q}\gamma^\delta \slashed{q} + \gamma^\alpha \slashed{q}\gamma^\beta \gamma^\gamma \gamma^\delta \slashed{q} + \gamma^\alpha \slashed{q}\gamma^\beta \slashed{q}\gamma^\gamma \gamma^\delta + \gamma^\alpha \gamma^\beta \slashed{q}\gamma^\gamma \slashed{q}\gamma^\delta + \right.\right.$$

$$\left.\left. + \gamma^\alpha \gamma^\beta \slashed{q}\gamma^\gamma \gamma^\delta \slashed{q} + \gamma^\alpha \slashed{q}\gamma^\beta \gamma^\gamma \slashed{q}\gamma^\delta\right)\right\}$$

Since this expression involves the trace over up to 6 Dirac matrices, the calculation is very cumbersome. We can evaluate this expression with the help of the Mathematica package FeynCalc [5, 6]

```
In[ ]:= FullSimplify[
    TR[a ^ 2 * GA[α].GA[β].GA[γ].GA[δ] - (* a^2 is \tilde{q}^2 from \tilde{q}^4 term,
      other terms are proportional to \tilde{q}^2*q^2 *)
     (GA[α].GS[q].GA[β].GS[q].GA[γ].GA[δ] + GA[α].GA[β].GA[γ].GS[q].GA[δ].GS[q] + GA[α].GA[β].GS[q].GA[γ].GA[δ].GS[q] +
       GA[α].GA[β].GS[q].GA[γ].GS[q].GA[δ] + GA[α].GS[q].GA[β].GA[γ].GA[δ].GS[q] +
       GA[α].GS[q].GA[β].GA[γ].GS[q].GA[δ])]]
```

$$\text{Out[ ]= } 4\left(a^2\,\bar{g}^{\alpha\beta}\,\bar{g}^{\gamma\delta} - \left(2\,\bar{q}^2 + a^2\right)\bar{g}^{\alpha\gamma}\,\bar{g}^{\beta\delta} + \bar{g}^{\gamma\delta}\left(\left(2\,\bar{q}^2 + a^2\right)\bar{g}^{\beta\gamma} - 2\,\bar{q}^\beta\,\bar{q}^\gamma\right) - 2\,\bar{q}^\alpha\,\bar{q}^\beta\,\bar{g}^{\gamma\delta} - 2\,\bar{q}^\gamma\,\bar{q}^\delta\,\bar{g}^{\beta\gamma} - 2\,\bar{q}^\gamma\,\bar{q}^\delta\,\bar{g}^{\alpha\beta} + 2\,\bar{q}^2\,\bar{g}^{\alpha\beta}\,\bar{g}^{\gamma\delta}\right)$$

As usual we plug this in the definition of R_2 and evaluate the integrals to get the expression of R_2 for the first of the contributing diagrams.

$$R_2 = \frac{1}{(2\pi)^4}\int d^d\bar{q}\,\frac{\widetilde{N}}{\bar{D}_1 \bar{D}_{13} \bar{D}_{-2} \bar{D}_0} = \frac{1}{(2\pi)^4}\int d^d\bar{q}\,\frac{4e^4}{\bar{D}_1 \bar{D}_{13} \bar{D}_{2} \bar{0}}\widetilde{q}^2\left\{\left(2q^2 + \widetilde{q}^2\right)\left(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\beta}g^{\gamma\delta}\right) + \right.$$

$$\left. -2\left(g^{\alpha\beta}q^\gamma q^\delta + g^{\gamma\delta}q^\alpha q^\beta + g^{\alpha\delta}q^\beta q^\gamma + g^{\beta\gamma}q^\alpha q^\delta\right)\right\} =$$

$$= \frac{-4e^4}{(2\pi)^4}\left\{\left(2\left(\frac{-i\pi^2}{3}\right) + \left(\frac{-i\pi^2}{6}\right)\right)\left(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\beta}g^{\gamma\delta}\right) - 2\left(\frac{-i\pi^2}{12}\right)\left(g^{\alpha\beta}g^{\gamma\delta} + g^{\gamma\delta}g^{\alpha\beta} + \right.\right.$$

$$\left.\left. + g^{\alpha\delta}g^{\beta\gamma} + g^{\beta\gamma}g^{\alpha\delta}\right)\right\} = \frac{ie^4}{4\pi^2}\left\{\frac{5}{6}\left(g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\beta}g^{\gamma\delta}\right) - \frac{1}{6}\left(2g^{\alpha\beta}g^{\gamma\delta} + 2g^{\alpha\delta}g^{\beta\gamma} + \right)\right\} =$$

$$= \frac{ie^4}{24\pi^2}\left(3g^{\alpha\beta}g^{\gamma\delta} - 5g^{\alpha\gamma}g^{\beta\delta} + 3g^{\beta\gamma}g^{\alpha\delta}\right)$$

Where we have used the 4-point integrals B.7, B.8 and B.9. This is independent of momenta, so we only have to symmetrize the indices to get the full 4-photon R_2.

$$R_2^{4\gamma} = 2\left[R_2 + R_2\left(\alpha \leftrightarrow \delta\right) + R_2\left(\alpha \leftrightarrow \beta\right)\right] = \frac{2ie^4}{24\pi^2}\left\{\left(3g^{\alpha\beta}g^{\gamma\delta} - 5g^{\alpha\gamma}g^{\beta\delta} + 3g^{\beta\gamma}g^{\alpha\delta}\right) + \left(3g^{\beta\delta}g^{\alpha\gamma} - 5g^{\gamma\delta}g^{\alpha\beta} + 3g^{\beta\gamma}g^{\alpha\delta}\right) + \right.$$

$$\left. + \left(3g^{\alpha\beta}g^{\gamma\delta} - 5g^{\beta\gamma}g^{\alpha\delta} + 3g^{\alpha\gamma}g^{\beta\delta}\right)\right\} = \frac{ie^4}{12\pi^2}\left(g^{\alpha\beta}g^{\gamma\delta} + g^{\alpha\gamma}g^{\beta\delta} + g^{\beta\gamma}g^{\alpha\delta}\right) \tag{2.5}$$

Like for the 3-point functions all of the other 4-point functions which are permitted by the Feynman rules vanish. We will not show this here because the calculations for the 4-point functions are quite lengthy. We have derived the complete set of R_2 in pure QED. Now we can go to the more complex Standard Model to see how QED contributes to the rational terms in the full Standard Model.
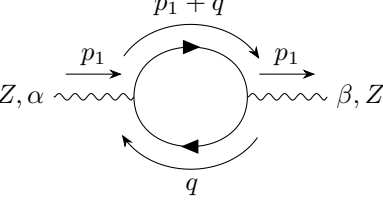
3   QED Contribution to R$_2$ in the Standard Model

As already mentioned above, the Standard Model is a chiral theory which will make the calculations more complicated. What is also new in the Standard Model is that the uncharged (in the sense of the whole gauge group) gauge boson can now fluctuate to a different uncharged gauge boson which leads to a lot more diagrams for the 2-point functions. Let us start again with the simplest correlation functions, the 2-point functions.

3.1   2-point functions

Z-boson self-energy
One of the 2-point functions in the Standard Model which gets a contribution from QED at 1-loop level is the Z-boson 2-point function. The fermionic loop correction is given by

$$= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ \frac{ig}{cos\theta_W} \gamma^\alpha (g_V - g_A \gamma_5) \frac{i \left(\slashed{p}_1 + \slashed{q} + m_f \right)}{(p_1 + q)^2 - m_f^2} \frac{ig}{cos\theta_W} \gamma^\beta \times \right.$$

$$\left. \times (g_V - g_A \gamma_5) \frac{i \left(\slashed{q} + m_f \right)}{q^2 - m_f^2} \right\} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{-g^2}{\cos^2 \theta_W} \operatorname{Tr} \left\{ \gamma^\alpha (g_V - g_A \gamma_5) \frac{\left(\slashed{p}_1 + \slashed{q} + m_f \right)}{(p_1 + q)^2 - m_f^2} \gamma^\beta (g_V - g_A \gamma_5) \frac{\left(\slashed{q} + m_f \right)}{q^2 - m_f^2} \right\}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

From the d-dimensional numerator function we can as usual extract the ϵ-dimensional part we are interested in.

$$\bar{N}(\bar{q}) = -\frac{g^2}{\cos^2 \theta_W} \operatorname{Tr} \left\{ \bar{\gamma}^\alpha (g_V - g_A \gamma_5) \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m_f \right) \bar{\gamma}^\beta (g_V - g_A \gamma_5) \left(\bar{\slashed{q}} + m_f \right) \right\} =$$

$$= \frac{-g^2}{\cos^2 \theta_W} \operatorname{Tr} \left\{ \gamma^\alpha (g_V - g_A \gamma_5) \left(\slashed{p}_1 + \slashed{q} + m_f \right) \gamma^\beta (g_V - g_A \gamma_5) \left(\slashed{q} + m_f \right) + \gamma^\alpha \left(g_V^2 + g_A^2 \right) \widetilde{\slashed{q}} \gamma^\beta \widetilde{\slashed{q}} \right\} \equiv N + \widetilde{N}$$

Where we used $[\gamma_5, \widetilde{\gamma}^\mu] = 0$ and the fact that the gamma matrices will be contracted with external momenta. We can further simplify $\widetilde{N}$ by evaluating the trace over the gamma matrices
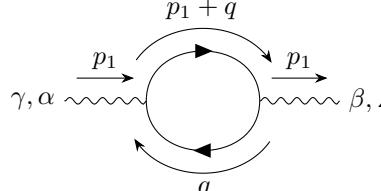
$$\widetilde{N} = \frac{-g^2}{\cos^2 \theta_W} \left(g_V^2 + g_A^2 \right) \left(-\widetilde{q}^2 \right) \operatorname{Tr} \left(\gamma^\alpha \gamma^\beta \right) = \frac{4 g^2 \widetilde{q}^2}{\cos^2 \theta_W} \left(g_V^2 + g_A^2 \right) g^{\alpha\beta}$$

Plugging this into the definition of R_2 gives

$$R_2^{ZZ} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{4 g^2 g^{\alpha\beta}}{(2\pi)^4 \cos^2 \theta_W} \left(g_V^2 + g_A^2 \right) \int d^d \bar{q} \frac{\widetilde{q}^2}{\bar{D}_1 \bar{D}_0} =$$

$$= \frac{4 g^2 g^{\alpha\beta}}{(2\pi)^4 \cos^2 \theta_W} \left(g_V^2 + g_A^2 \right) \left(-\frac{i\pi^2}{2} \right) \left(2 m_f^2 - \frac{p_1^2}{3} \right) = \frac{-i g^2}{8\pi^2 \cos^2 \theta_W} \left(g_V^2 + g_A^2 \right) \left(2 m_f^2 - \frac{p_1^2}{3} \right) g^{\alpha\beta}$$

$$(3.1)$$

Photon/Z-boson mixed self-energy

Since the Z-boson is a singlet under $SU(2)_L \times U(1)_Y$ it can also oscillate into a photon via a fermion loop. The amplitude of this process is given by

$$= \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr} \left\{ (-ieQ_f) \gamma^\alpha \frac{i \left(\not{p}_1 + \not{q} + m_f \right)}{(p_1 + q)^2 - m_f^2} \frac{ig}{cos\theta_W} \gamma^\beta \times \right.$$

$$\left. \times \, (g_V - g_A\gamma_5) \frac{i \left(\not{q} + m_f \right)}{q^2 - m_f^2} \right\} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{eQ_f g}{\cos\theta_W} \operatorname{Tr} \left\{ \gamma^\alpha \frac{\left(\not{p}_1 + \not{q} + m_f \right)}{(p_1 + q)^2 - m_f^2} \gamma^\beta (g_V - g_A\gamma_5) \frac{\left(\not{q} + m_f \right)}{q^2 - m_f^2} \right\}$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_0}$$

We simplify the d-dimensional denominator function

$$\bar{N}(\bar{q}) = \frac{eQ_f g}{\cos\theta_W} \operatorname{Tr} \left\{ \bar{\gamma}^\alpha \left(\bar{\not{p}}_1 + \bar{\not{q}} + m_f \right) \bar{\gamma}^\beta (g_V - g_A\gamma_5) \left(\bar{\not{q}} + m_f \right) \right\} =$$

$$= \frac{eQ_f g}{\cos\theta_W} \operatorname{Tr} \left\{ \gamma^\alpha \left(\not{p}_1 + \not{q} + m_f \right) \gamma^\beta (g_V - g_A\gamma_5) \left(\not{q} + m_f \right) + \gamma^\alpha \widetilde{\not{q}} \gamma^\beta g_V \widetilde{\not{q}} \right\} \equiv N + \widetilde{N}$$

Where we have used $\operatorname{Tr} \left(\gamma^\alpha \gamma^\beta \gamma_5 \right) = 0$ and the fact that the gamma matrices are contracted with external momenta. The last term defines $\widetilde{N}$ and can be further simplified by evaluating the trace

$$\widetilde{N} = \frac{eQ_f g}{cos\theta_W} \operatorname{Tr} \left\{ \gamma^\alpha \widetilde{\not{q}} \gamma^\beta g_V \widetilde{\not{q}} \right\} = \frac{-4eQ_f g g_V}{\cos\theta_W} \widetilde{q}^2 g^{\alpha\beta}$$

With this we can calculate the R_2 for the mixed photon/Z-boson self-energy.

$$R_2^{\gamma Z} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}}{\bar{D}_1 \bar{D}_0} = \frac{-4eQ_f g g_V}{(2\pi)^4 \cos\theta_W} g^{\alpha\beta} \int d^d \bar{q} \frac{\widetilde{q}^2}{\bar{D}_1 \bar{D}_0} =$$

$$= \frac{-4eQ_f g g_V}{(2\pi)^4 \cos\theta_W} \left(-\frac{i\pi^2}{2} \right) g^{\alpha\beta} \left(2m_f^2 - \frac{p_1^2}{3} \right) = \frac{ieQ_f g g_V}{8\pi^2 \cos\theta_W} g^{\alpha\beta} \left(2m_f^2 - \frac{p_1^2}{3} \right) \tag{3.2}$$
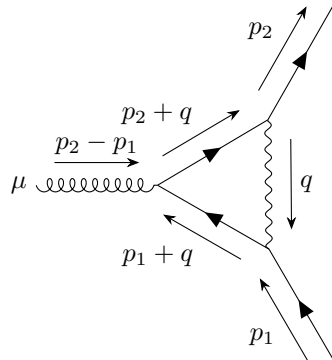
Gluon self-energy

Because the gluon (just as the photon) couples to a pure vector current, the calculation for the gluon self-energy R_2 is the same as for the photon self-energy R_2 replacing the electric charge generator with the colour charge generator and noticing that gluons only couple to quarks. So, from equation 2.1 with $eQ_f \rightarrow g_S T^a$ and $m_e \equiv m \rightarrow m_q$ we get

$$R_2^{gg} = R_2^{\gamma\gamma} \left(eQ_f \rightarrow g_S T^a, \, m \rightarrow m_q \right) = \frac{-ig_S^2}{8\pi^2} \operatorname{Tr} \left(T^a T^b \right) g^{\alpha\beta} \left(2m_q^2 - \frac{p_1^2}{3} \right) \tag{3.3}$$

3.2 3-point functions

Gluon-quark vertex

For the 3-point functions let us continue where we left of: the QCD part of the Standard Model. The quark-gluon vertex gets a photonic correction at 1-loop which is given by

$$= \int \frac{d^d q}{(2\pi)^d} \left(-ieQ_q\gamma^\beta\right) \frac{i\left(\slashed{p}_1 + \slashed{q} + m_q\right)}{(p_1+q)^2 - m_q^2} \left(-ig_S\gamma^\mu T^a\right) \frac{i\left(\slashed{p}_2 + \slashed{q} + m_q\right)}{(p_2+q)^2 - m_q^2} \left(-ieQ_q\gamma^\alpha\right) \frac{-ig_{\alpha\beta}}{q^2} =$$

$$= \int \frac{d^d q}{(2\pi)^d} - e^2 Q_q^2 g_S \gamma^\beta \frac{\left(\slashed{p}_1 + \slashed{q} + m_q\right)}{(p_1+q)^2 - m_q^2} \gamma^\mu T^a \frac{\left(\slashed{p}_2 + \slashed{q} + m_q\right)}{(p_2+q)^2 - m_q^2} \gamma_\beta \frac{1}{q^2} =$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0} =$$

Extracting the d-dimensional numerator gives

$$\bar{N}(\bar{q}) = -e^2 Q_q^2 g_S \left\{ \bar{\gamma}^\beta \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m_q\right) \bar{\gamma}^\mu T^a \left(\bar{\slashed{p}}_2 + \bar{\slashed{q}} + m_q\right) \bar{\gamma}_\beta \right\} = -e^2 Q_q^2 g_S \left\{ \gamma^\beta \left(\slashed{p}_1 + \slashed{q} + m_q\right) \gamma^\mu T^a \left(\slashed{p}_2 + \slashed{q} + m_q\right) \gamma_\beta + \right.$$

$$\left. + \gamma^\beta \widetilde{\slashed{q}} \gamma^\mu T^a \widetilde{\slashed{q}} \gamma_\beta + \widetilde{\gamma}^\beta \slashed{q} \gamma^\mu T^a \slashed{q} \widetilde{\gamma}_\beta \right\} \equiv N + \widetilde{N}$$

The last two terms define $\widetilde{N}$. We have used the fact that the gamma matrices are contracted with external momenta and equation C.4.
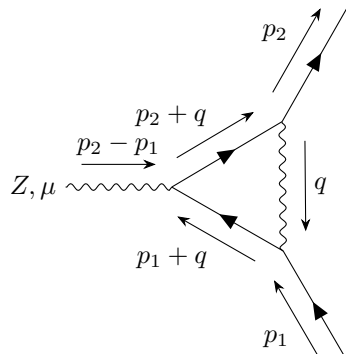
$$\widetilde{N} = -e^2 Q_q^2 g_S \left\{ (-1) \widetilde{q}^2 \gamma^\beta \gamma^\mu \gamma_\beta T^a + (-1)^3 \widetilde{\gamma}^\beta \widetilde{\gamma}_\beta q_\alpha q_\beta \gamma^\alpha \gamma^\mu \gamma^\beta T^a \right\} = -e^2 Q_q^2 g_S \left\{ 2\widetilde{q}^2 \gamma^\mu T^a - \epsilon q_\alpha q_\beta \gamma^\alpha \gamma^\mu \gamma^\beta T^a \right\}$$

Where we have used the fact that 4- and d-dimensional gamma matrices anticommute and equations C.5 with $d = \epsilon$ and C.6 with $d = 4$. Now we can again compute R_2

$$R_2^{gqq} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\widetilde{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0} = \frac{-e^2 Q_q^2 g_S}{(2\pi)^4} \int d^d \bar{q} \frac{1}{\bar{D}_1 \bar{D}_2 \bar{D}_0} \left\{ 2\widetilde{q}^2 \gamma^\mu T^a - \epsilon q_\alpha q_\beta \gamma^\alpha \gamma^\mu \gamma^\beta T^a \right\} =$$

$$= \frac{-e^2 Q_q^2 g_S}{(2\pi)^4} \left\{ 2\left(\frac{-i\pi^2}{2}\right) \gamma^\mu T^a - \epsilon \left(\frac{-i\pi^2}{2\epsilon}\right) \underbrace{g_{\alpha\beta}\gamma^\alpha \gamma^\mu \gamma^\beta}_{-2\gamma^\mu} T^a \right\} = \frac{-e^2 Q_q^2 g_S}{16\pi^4} \left(\frac{-i\pi^2}{2}\right) \left\{ 2\gamma^\mu T^a + 2\gamma^\mu T^a \right\} =$$

$$= \frac{ie^2 Q_q^2 g_S}{8\pi^2} \gamma^\mu T^a \tag{3.4}$$

Where we have used integrals B.4 and B.6 as well as equation C.6 with $d = 4$.

Z-fermion vertex

$$
\begin{aligned}
&= \int \frac{d^d q}{(2\pi)^d} \left(-ieQ_f\gamma^\beta\right) \frac{i\left(\slashed{p}_1 + \slashed{q} + m_f\right)}{(p_1+q)^2 - m_f^2} \frac{ig}{\cos\theta_W}\gamma^\mu\left(g_V - g_A\gamma_5\right) \frac{i\left(\slashed{p}_2 + \slashed{q} + m_f\right)}{(p_2+q)^2 - m_f^2} \times
\end{aligned}
$$

$$
\times \left(-ieQ_f\gamma^\alpha\right) \frac{-ig_{\alpha\beta}}{q^2} =
$$

$$
= \int \frac{d^d q}{(2\pi)^d} \frac{e^2 Q_f^2 g}{\cos\theta_W}\gamma^\beta \frac{\left(\slashed{p}_1 + \slashed{q} + m_f\right)}{(p_1+q)^2 - m_f^2}\gamma^\mu\left(g_V - g_A\gamma_5\right) \frac{\left(\slashed{p}_2 + \slashed{q} + m_f\right)}{(p_2+q)^2 - m_f^2}\gamma_\beta \frac{1}{q^2} =
$$

$$
\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1 \bar{D}_2 \bar{D}_0}
$$

We start by splitting the 4- and ϵ-dimensional contributions in the numerator function

$$
\bar{N}(\bar{q}) = \frac{e^2 Q_f^2 g}{\cos\theta_W}\left\{\bar{\gamma}^\beta\left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m_f\right)\bar{\gamma}^\mu\left(g_V - g_A\gamma_5\right)\left(\bar{\slashed{p}}_2 + \bar{\slashed{q}} + m_f\right)\bar{\gamma}_\beta\right\} = \frac{e^2 Q_f^2 g}{\cos\theta_W}\left\{\gamma^\beta\left(\slashed{p}_1 + \slashed{q} + m_f\right)\gamma^\mu\left(g_V - g_A\gamma_5\right)\times\right.
$$

$$
\times\left(\slashed{p}_2 + \slashed{q} + m_f\right)\gamma_\beta + \widetilde{\gamma}^\beta\left(\slashed{p}_1 + \slashed{q} + m_f\right)\gamma^\mu\left(g_V - g_A\gamma_5\right)\left(\slashed{p}_2 + \slashed{q} + m_f\right)\widetilde{\gamma}_\beta + \left(\gamma^\beta + \widetilde{\gamma}^\beta\right)\widetilde{\slashed{q}}\gamma^\mu\left(g_V - g_A\gamma_5\right)\widetilde{\slashed{q}}\left(\gamma_\beta + \widetilde{\gamma}_\beta\right)\left.\right\} =
$$

$$
\equiv N + \widetilde{N}
$$

Then we simplify $\widetilde{N}$

$$
\widetilde{N} = \frac{e^2 Q_f^2 g}{\cos\theta_W}\left\{\left(\slashed{p}_1 + \slashed{q} - m_f\right)\widetilde{\gamma}^\beta\gamma^\mu\widetilde{\gamma}_\beta\left(g_V - g_A\gamma_5\right)\left(\slashed{p}_2 + \slashed{q} - m_f\right) + \gamma^\beta\widetilde{\slashed{q}}\gamma^\mu\left(g_V - g_A\gamma_5\right)\widetilde{\slashed{q}}\gamma_\beta + \widetilde{\gamma}^\beta\widetilde{\slashed{q}}\gamma^\mu\left(g_V - g_A\gamma_5\right)\widetilde{\slashed{q}}\widetilde{\gamma}_\beta\right\} =
$$

$$
= \frac{e^2 Q_f^2 g}{\cos\theta_W}\left\{-\epsilon\left(\slashed{p}_1 + \slashed{q} - m_f\right)\gamma^\mu\left(g_V - g_A\gamma_5\right)\left(\slashed{p}_2 + \slashed{q} - m_f\right) - \widetilde{q}^2\gamma^\beta\gamma^\mu\gamma_\beta\left(g_V + g_A\gamma_5\right) - \widetilde{q}^2\left(-\epsilon\gamma^\mu\right)\left(g_V - g_A\gamma_5\right)\right\} =
$$

$$
= \frac{e^2 Q_f^2 g}{\cos\theta_W}\left\{-\epsilon\left(\slashed{p}_1 + \slashed{q} - m_f\right)\gamma^\mu\left(g_V - g_A\gamma_5\right)\left(\slashed{p}_2 + \slashed{q} - m_f\right) + \widetilde{q}^2\left(2\gamma^\mu\left(g_V + g_A\gamma_5\right) + \epsilon\gamma^\mu\left(g_V - g_A\gamma_5\right)\right)\right\} =
$$

$$
= \frac{e^2 Q_f^2 g}{\cos\theta_W}\left\{-\epsilon\slashed{q}\gamma^\mu\left(g_V - g_A\gamma_5\right)\slashed{q} + \widetilde{q}^2\left(2\gamma^\mu\left(g_V + g_A\gamma_5\right)\right)\right\}
$$

where we used the fact that 4- and ϵ-dimensional gamma matrices anticommute as well as equations C.5 and C.8 from line one to two. In the last step we dropped the terms which are not proportional to the loop momentum because they don't contribute to R_2 after integration (c.f. equations B.4 - B.6). Now we can plug
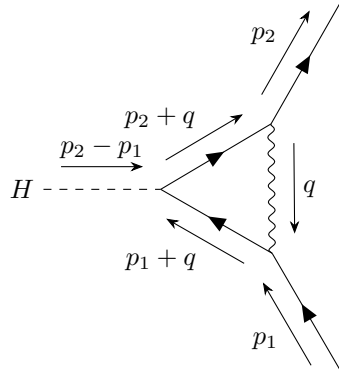
15

$\widetilde{N}$ in the definition of R_2 and evaluate the integrals

$$R_2^{Zff} = \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{\widetilde{N}}{\bar{D}_1\bar{D}_2\bar{D}_0} = \frac{e^2Q_f^2 g}{(2\pi)^4 \cos\theta_W} \int d^d\bar{q} \frac{1}{\bar{D}_1\bar{D}_2\bar{D}_0} \left\{ -\epsilon\bar{\slashed{q}}\gamma^\mu (g_V - g_A\gamma_5)\slashed{q} + \widetilde{q}^2 (2\gamma^\mu (g_V + g_A\gamma_5)) \right\} =$$

$$= \frac{e^2Q_f^2 g}{(2\pi)^4 \cos\theta_W} \left\{ -\epsilon\left(-\frac{i\pi^2}{2\epsilon}\right) g_{\alpha\beta}\gamma^\alpha\gamma^\mu\gamma^\beta (g_V + g_A\gamma_5) + 2\left(-\frac{i\pi^2}{2}\right)\gamma^\mu (g_V + g_A\gamma_5) \right\} =$$

$$= \frac{e^2Q_f^2 g}{(2\pi)^4 \cos\theta_W} \left(-\frac{i\pi^2}{2}\right)\gamma^\mu \left\{ 2(g_V + g_A\gamma_5) + 2(g_V + g_A\gamma_5) \right\} = \frac{-ie^2Q_f^2 g}{8\pi^2 \cos\theta_W}\gamma^\mu (g_V + g_A\gamma_5) \qquad (3.5)$$

Here, we used equation C.6 as well as the integrals B.4 and B.6.

Higgs-fermion Yukawa vertex

Finally, we also have the Higgs scalar in the Standard Model which couples via the Yukawa couplings to the fermions giving them their mass.

$$= \int \frac{d^d q}{(2\pi)^d} (-ieQ_f\gamma^\beta) \frac{i(\slashed{p}_1 + \slashed{q} + m_f)}{(p_1 + q)^2 - m_f^2} \left(-\frac{ig}{2}\frac{m_f}{m_W}\right) \frac{i(\slashed{p}_2 + \slashed{q} + m_f)}{(p_2 + q)^2 - m_f^2} (-ieQ_f\gamma^\alpha) \frac{-ig_{\alpha\beta}}{q^2} =$$

$$= \int \frac{d^d q}{(2\pi)^d} \frac{-e^2Q_f^2 g m_f}{2m_W}\gamma^\beta \frac{(\slashed{p}_1 + \slashed{q} + m_f)}{(p_1 + q)^2 - m_f^2}\gamma^\mu \frac{(\slashed{p}_2 + \slashed{q} + m_f)}{(p_2 + q)^2 - m_f^2}\gamma_\beta \frac{1}{q^2} =$$

$$\equiv \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}}{\bar{D}_1\bar{D}_2\bar{D}_0}$$

As always we start by extracting the ϵ-dimensional part from the numerator function

$$\bar{N}(\bar{q}) = \frac{e^2Q_f^2 g m_f}{2m_W}\bar{\gamma}^\beta \left(\bar{\slashed{p}}_1 + \bar{\slashed{q}} + m_f\right)\left(\bar{\slashed{p}}_2 + \bar{\slashed{q}} + m_f\right)\bar{\gamma}_\beta =$$

$$= \frac{e^2Q_f^2 g m_f}{2m_W} \left\{ \gamma^\beta \left(\slashed{p}_1 + \slashed{q} + m_f\right)\left(\slashed{p}_2 + \slashed{q} + m_f\right)\gamma_\beta + \widetilde{\gamma}^\beta \left(\slashed{p}_1 + \slashed{q} + m_f\right)\left(\slashed{p}_2 + \slashed{q} + m_f\right)\widetilde{\gamma}_\beta + \gamma^\beta\widetilde{\slashed{q}}\widetilde{\slashed{q}}\gamma_\beta \right\} \equiv N + \widetilde{N}$$

Then, we simplify the expression

$$\widetilde{N} = -\frac{e^2Q_f^2 g m_f}{2m_W} \left\{ \widetilde{\gamma}^\beta \left(\slashed{p}_1 + \slashed{q} + m_f\right)\left(\slashed{p}_2 + \slashed{q} + m_f\right)\widetilde{\gamma}_\beta + \gamma^\beta\widetilde{\slashed{q}}\widetilde{\slashed{q}}\gamma_\beta \right\} = -\frac{e^2Q_f^2 g m_f}{2m_W} \left\{ \widetilde{\gamma}^\beta\widetilde{\gamma}_\beta\slashed{q}\slashed{q} + \widetilde{q}^2\gamma^\beta\gamma_\beta \right\} =$$

$$= -\frac{e^2Q_f^2 g m_f}{2m_W} \left\{ \epsilon\slashed{q}\slashed{q} + 4\widetilde{q}^2 \right\}$$

where we used the fact that 4- and ϵ-dimensional gamma matrices anticommute and equations C.5 and C.8. Then, plugging $\widetilde{N}$ in the definition of R_2 gives

$$
\begin{aligned}
R_2^{Hff} &= \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{\widetilde{N}}{\bar{D}_1 \bar{D}_0 \bar{D}_2} = \frac{1}{(2\pi)^4} \int d^d\bar{q} \frac{1}{\bar{D}_1 \bar{D}_0 \bar{D}_2} \left(-\frac{e^2 Q_f^2 g m_f}{2m_W} \right) \left\{ \epsilon \slashed{q}\slashed{q} + 4\widetilde{q}^2 \right\} = \\
&= -\frac{e^2 Q_f^2 g m_f}{2m_W} \left\{ \epsilon \left(-\frac{i\pi^2}{2\epsilon} \right) \gamma^\alpha \gamma^\beta g_{\alpha\beta} + 4 \left(-\frac{i\pi^2}{2} \right) \right\} = \frac{-e^2 Q_f^2 g m_f}{(2\pi)^4 \, 2m_W} \left(-\frac{i\pi^2}{2} \right)(4+4) = \\
&= \frac{i e^2 Q_f^2 g m_f}{8\pi^2 m_W}
\end{aligned}
\tag{3.6}
$$

where we used the equation C.6 and the integrals B.4 and B.6.

4 Perturbative Renormalization in Terms of Scalar Integrals

Explain how to express renormalization constants in terms of scalar integrals.

We start from the QED Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^0_{\mu\nu}F_0^{\mu\nu} + \bar{\psi}_0\left(i\slashed{\partial} - m_0\right)\psi_0 - e_0\bar{\psi}_0\slashed{A}_0\psi_0 \tag{4.1}$$

where $F_0^{\mu\nu} = \partial^\mu A_0^\nu - \partial^\nu A_0^\mu$. Now, we reinterpret the fields and parameters in the Lagrangian as "bare" fields and parameters which are given by the actual "renormalized" quantities times a renormalization constant

$$\psi_0 = \sqrt{Z_2}\psi$$
$$A_0^\mu = \sqrt{Z_3}A^\mu$$
$$m_0 = Z_m m$$
$$e_0 = Z_e e\mu^{-\frac{\epsilon}{2}} \tag{4.2}$$

The renormalization constants Z_i absorb the divergences which appear in loop calculations. We can split them as $Z_i = 1 + \delta_i$ to extract the renormalized Lagrangian which is divergence free and the so called counter-term Lagrangian which absorbs the divergences

$$\mathcal{L} = -\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} + iZ_2\bar{\psi}\slashed{\partial}\psi - Z_m Z_2 m\bar{\psi}\psi - eZ_1\bar{\psi}\slashed{A}\psi =$$
$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\slashed{\partial} - m\right)\psi - e\bar{\psi}\slashed{A}\psi - \frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + i\delta_2\bar{\psi}\slashed{\partial}\psi - \left(\delta_m + \delta_2\right)m\bar{\psi}\psi - e\delta_1\bar{\psi}\slashed{A}\psi \equiv \mathcal{L}_{ren} + \mathcal{L}_{ct} \tag{4.3}$$

where $Z_1 = Z_e Z_2\sqrt{Z_3}\mu^{-\frac{\epsilon}{2}}$.

The counter term Lagrangians gives the following new Feynman rules

$$\alpha \xrightarrow{\;\;p\;\;}\!\!\!\otimes\!\!\!\sim\!\!\beta \;\; = i\left(p^\alpha p^\beta - g^{\alpha\beta}p^2\right)\delta_3$$

$$\xrightarrow{\;\;p\;\;}\!\!\!\otimes\!\!\!\longrightarrow \;\; = i\left(\slashed{p}\delta_2 - \left(\delta_m + \delta_2\right)m\right)$$

$$\alpha \sim\!\!\!\otimes \;\; = -ie\gamma^\mu\delta_1 \tag{4.4}$$

We can use these new Feynman rules to calculate the Z_i in order to be able to make predictions with perturbative calculations. These renormalization conditions can be obtained by calculating the dressed propagators and requiring that the propagators have a pole at the physical mass.

Let's start with the electron propagator. The dressed propagator is given by a sum of so called 1-particle irreducible insertions (i.e. insertions of subdiagrams which do not fall apart when one of the internal lines is cut) as follows

where the empty circles on the right represent renormalized 1-PI interactions and the appropriate counter terms. This gives

$$iS_0(\not{p}) = iS(\not{p}) + iS(\not{p})i\Sigma'(\not{p})iS(\not{p}) + iS(\not{p})i\Sigma'(\not{p})iS(\not{p})i\Sigma'(\not{p})iS(\not{p}) + \dots \tag{4.5}$$

where $i\Sigma'(\not{p}) = i\Sigma(\not{p}) + i\left(\delta_2\not{p} - (\delta_2 + \delta_m)m\right)$, $iS_0 = \frac{i}{\not{p} - m_0}$ and $iS = \frac{i}{\not{p} - m}$. Now we can sum the geometric series in $i\Sigma'(\not{p})iS(\not{p})$ which yields

$$\frac{i}{\not{p} - m_0} = \frac{i}{\not{p} - m + \left(\Sigma(\not{p}) + \delta_2\not{p} - (\delta_2 + \delta_m)m\right)} \tag{4.6}$$

By requiring the dressed propagator to have a pole at the physical mass $\not{p} = m_{\text{phys}} = m$ we obtain

$$m - m + \Sigma(m) + \delta_2 m - (\delta_2 + \delta_m)m = 0 \tag{4.7}$$

$$\Rightarrow \delta_m = \frac{1}{m}\Sigma(m) \tag{4.8}$$

We also want the propagator to have a residue of unity at the pole. This gives the renormalization condition for the electron field

$$\text{Res}_{\not{p}=m}\left(S(\not{p})\right) = \text{Res}_{\not{p}=m}\left(\frac{1}{\not{p} - m + \left(\Sigma(\not{p}) + \delta_2\not{p} - (\delta_2 + \delta_m)m\right)}\right) =$$

$$= \lim_{\not{p}\to m} \frac{\not{p} - m}{\not{p} - m + \left(\Sigma(\not{p}) + \delta_2\not{p} - (\delta_2 + \delta_m)m\right)} \overset{\text{L'H}}{=} \lim_{\not{p}\to m} \frac{1}{1 + \frac{d\Sigma}{d\not{p}} + \delta_2} \overset{!}{=} 1$$

$$\Rightarrow \delta_2 = -\frac{d\Sigma(\not{p})}{d\not{p}}\bigg|_{\not{p}=m} \tag{4.9}$$

Z_1 and Z_2 are related by symmetry, so we do not have to evaluate the electron-photon 3-point function. It was first shown by Ward in 1950 that $Z_1 = Z_2$ [7].

The only remaining renormalization constant from equations 4.2 is therefore Z_3. It can be obtained from the dressed photon propagator in the same way we obtained the electron field renormalization from the electron propagator. The dressed photon operator is given by

where the empty circles are again insertions of 1-Pi diagrams and the appropriate counter term. So, we have

$$iS_0^{\alpha\beta}(p^2) = iS^{\alpha\beta}(p^2) + \left[iS(p^2)i\Pi'(p^2)iS(p^2)\right]^{\alpha\beta} + \left[iS(p^2)i\Pi'(p^2)iS(p^2)i\Pi'(p^2)iS(p^2)\right]^{\alpha\beta} + \dots \tag{4.10}$$

with $iS_0^{\alpha\beta} = \frac{-i}{p^2}\left(g^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}\right) = iS^{\alpha\beta}$ and $i\Pi'^{\alpha\beta} = i\Pi^{\alpha\beta} + i\delta_3\left(p^\alpha p^\beta - g^{\alpha\beta}p^2\right)$. Due to gauge invariance and the respective Ward identity we must have $\Pi^{\alpha\beta} = \left(p^\alpha p^\beta - p^2 g^{\alpha\beta}\right)\Pi(p^2)$, since the Ward identity demands $p_\alpha\Pi^{\alpha\beta} = 0 = \left(p^2 p^\beta - p^2 p^\beta\right)\Pi(p^2)$ ✓. So in total we have $i\Pi^{\alpha\beta} = \left(p^\alpha p^\beta - p^2 g^{\alpha\beta}\right)\left(\Pi(p^2) + \delta_3\right) \equiv \left(p^\alpha p^\beta - p^2 g^{\alpha\beta}\right)\Pi'(p^2)$.

Now we can sum the geometric series in $i\Pi'(p^2)iS(p^2)$ which yields

$$\frac{-i}{p^2}\left(g^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}\right) = \left(g^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}\right)\frac{-i}{p^2\left(1 + \Pi(p^2) + \delta_3\right)} \tag{4.11}$$

By requiring the propagator to have a residue of unity at the physical photon mass $p^2 = 0$ we get

$$\text{Res}_{p^2=0}\left(S(p^2)\right) = \text{Res}_{p^2=0}\left(\frac{1}{p^2\left(1 + \Pi(p^2) + \delta_3\right)}\right) =$$

$$= \lim_{p^2 \to 0}\frac{p^2}{p^2\left(1 + \Pi(p^2) + \delta_3\right)} = \frac{1}{1 + \Pi(0) + \delta_3} \overset{!}{=} 1$$

$$\Rightarrow \delta_3 = -\Pi(0) \tag{4.12}$$

The renormalization procedure for the whole Standard Model is obviously a lot more involved, since there are a lot more fields and parameters in the theory. But it still follows the same lines as for the simpler QED case. The whole derivation for the renormalization conditions of the electroweak part of the Standard Model can be found in [8]. We will use the results from there and calculate the needed self-energies in section 4.2.

We now have to calculate all of the necessary 2-point functions to evaluate the renormalization constants. Since our goal is to automate 1-loop calculations in QED and their contributions to the Standard Model it is convenient to express the results in terms of scalar integrals (see Appendix B) which can be easily implemented and evaluated numerically. In the following all momenta will be d-dimensional until we take the limit $d \to 4$, so we will not indicate the dimensionality of the momenta by bars. Let us start by defining the following 2-point functions

$$B = \int \frac{d^d q}{i\pi^2}\frac{1}{(q^2 - m_0^2)\left((p+q)^2 - m_1^2\right)} \equiv B_0$$

$$B^\mu = \int \frac{d^d q}{i\pi^2}\frac{q^\mu}{(q^2 - m_0^2)\left((p+q)^2 - m_1^2\right)} \equiv B_1 p^\mu$$

$$B^{\mu\nu} = \int \frac{d^d q}{i\pi^2}\frac{q^\mu q^\nu}{(q^2 - m_0^2)\left((p+q)^2 - m_1^2\right)} \equiv B_{00}g^{\mu\nu} + B_{11}p^\mu p^\nu \tag{4.13}$$

These are all of the integrals we will need in the following. The same procedure can obviously be done for any n-point function. On the right-hand side of the equations we used the Lorentz invariance of the tensor integrals and decomposed them into the appropriate tensors times a so-called form factor. From equation 4.13 we can get the following expressions

$$p_\mu B^\mu = \int \frac{d^d q}{i\pi^2}\frac{p \cdot q}{(q^2 - m_0^2)\left((p+q)^2 - m_1^2\right)} = B_1 p^2$$

$$g_{\mu\nu}B^{\mu\nu} = \int \frac{d^d q}{i\pi^2}\frac{q^2}{(q^2 - m_0^2)\left((p+q)^2 - m_1^2\right)} = dB_{00} + B_{11}p^2$$

$$p_\mu p_\nu B^{\mu\nu} = \int \frac{d^d q}{i\pi^2}\frac{(p \cdot q)^2}{(q^2 - m_0^2)\left((p+q)^2 - m_1^2\right)} = p^2 B_{00} + B_{11}p^4$$

(!!!!!!!!!!!!!!!!!!maybe use identity for $B_{00}d + B_{11}p^2$ in terms of A and B functions!!!!!!!!!!!!!!!) which are useful for our calculations. We also have to be careful about taking the limit $d \to 4$ since $d = O(\epsilon)$ and $B_i = O(\epsilon^{-1})$. E.g., for B_0 we have in dimensional regularisation with the well-known expression for B_0 and $d = 4 - \epsilon$
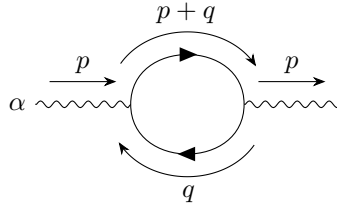
$$d \cdot B_0 = (4 - \epsilon)\left(\frac{2}{\epsilon} - \int_0^1 dx \log\left(\frac{\Delta(x, m_i^2, p^2)}{4\pi e^{-\gamma_E}\mu^2}\right)\right) \to 4B_0 - \frac{2\epsilon}{\epsilon} = 4B_0 - 2 \tag{4.14}$$

The same can be done for the other form factors. This yields [9]

$$d \cdot B_0 \to 4B_0 - 2$$

$$d \cdot B_1 \to 4B_1 + 1$$

$$d \cdot B_{00} \to 4B_{00} - \frac{1}{6}\left(p_1^2 - 3\left(m_0^2 + m_1^2\right)\right)$$

$$d \cdot B_{11} \to 4B_{11} - \frac{2}{3}$$

$$(4.15)$$

4.1 Renormalization of Pure QED

Photon self-energy

$$\alpha \sim\!\!\!\sim\!\!\!\sim \bigcirc \sim\!\!\!\sim\!\!\!\sim \beta \; = \int \frac{d^d q}{(2\pi)^d} (-1) \operatorname{Tr}\left\{ ie\gamma^\alpha \frac{i\left(\not p + \not q + m\right)}{(p+q)^2 - m^2} ie\gamma^\beta \frac{i\left(\not q + m\right)}{q^2 - m^2} \right\} \equiv i\Pi^{\alpha\beta}$$

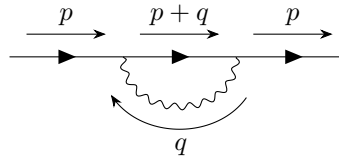Let's work on the trace so we can express the numerator of the 2-point function in terms of scalar integrals.

$$\operatorname{Tr}\left\{ \gamma^\alpha \left(\not p + \not q + m\right) \gamma^\beta \left(\not q + m\right) \right\} = \operatorname{Tr}\left\{ m^2 \gamma^\alpha \gamma^\beta + \gamma^\alpha \left(\not p + \not q\right) \gamma^\beta \not q \right\} =$$

$$= d\left\{ m^2 g^{\alpha\beta} + (p+q)_\mu q_\nu \left(g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu} + g^{\alpha\nu} g^{\beta\mu} \right) \right\} =$$

$$= d\left(m^2 g^{\alpha\beta} + (p+q)^\alpha q^\beta - g^{\alpha\beta} (p+q) \cdot q + g^\alpha (p+q)^\beta \right)$$

$$i\Pi^{\alpha\beta} = -e^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d\left\{ m^2 g^{\alpha\beta} + p^\alpha q^\beta + q^\alpha p^\beta + 2q^\alpha q^\beta - g^{\alpha\beta} p \cdot q - g^{\alpha\beta} q^2 \right\}}{\left((p+q)^2 - m^2\right)(q^2 - m^2)} =$$

$$= -\frac{ie^2}{16\pi^2} d\left\{ m^2 B_0 g^{\alpha\beta} + 2p^\alpha p^\beta B_1 + 2\left(B_{11} p^\alpha p^\beta + B_{00} g^{\alpha\beta} \right) - g^{\alpha\beta} B_1 p^2 - g^{\alpha\beta} \left(d \cdot B_{00} + B_{11} p^2 \right) \right\} =$$

$$= -\frac{ie^2}{4\pi^2} \left\{ g^{\alpha\beta} \left(m^2 B_0 - B_1 p^2 + B_{11} p^2 - 2B_{00} \right) + 2p^\alpha p^\beta \left(B_1 + B_{11} \right) \right\}$$

The arguments of the scalar integrals are suppressed to keep the notation compact. They are the same for all B-functions: $B_i = B_i(p^2, m^2, m^2)$.

The expression can be further simplified using identities between the scalar integrals.

Electron self-energy

$$= \int \frac{d^d q}{(2\pi)^d} ie\gamma^\alpha \frac{i\left(\not p + \not q + m\right)}{(p+q)^2 - m^2} ie\gamma^\beta \frac{-ig_{\alpha\beta}}{q^2} = \int \frac{d^4 q}{(2\pi)^4} \left(-e^2\right) \gamma^\alpha \frac{\left(\not p + \not q + m\right)}{(p+q)^2 - m^2} \gamma_\alpha \frac{1}{q^2} \equiv i\Sigma(\not p)$$

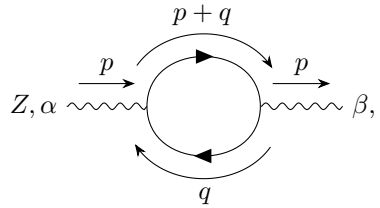With a bit of gamma-matrix algebra the numerator can be written as

$$\gamma^\beta \left(\not{p} + \not{q} + m\right) \gamma_\beta = (p+q)_\alpha \gamma^\beta \gamma^\alpha \gamma_\beta + m \gamma^\beta \gamma_\beta = d \cdot m + (2-d)\left(\not{p} + \not{q}\right)$$

$$i\Sigma(\not{p}) = -e^2 \int \frac{d^d q}{(2\pi)^d} \frac{d \cdot m + (2-d)\left(\not{p} + \not{q}\right)}{\left((p+q)^2 - m^2\right) q^2} = -\frac{ie^2}{16\pi^2} \left\{ d \cdot m B_0 + (2-d) \not{p} (B_0 + B_1) \right\} \rightarrow$$

$$\rightarrow \frac{-ie^2}{16\pi^2} \left\{ m(4B_0 - 2) + 2\not{p}(B_0 + B_1) - (4B_0 - 2 + 4B_1 + 1) \right\} = \frac{-ie^2}{16\pi^2} \left\{ m(4B_0 - 2) - \not{p}(2(B_0 + B_1) - 1) \right\}$$

Where the arguments of the B-functions are suppressed again. They are $B_i = B_i(p^2, 0, m^2)$

4.2 QED Contribution to the Renormalization of the Standard Model

Photon/Z-boson mixed self-energy

$$= \int \frac{d^4 q}{(2\pi)^4} (-1) \operatorname{Tr} \left\{ (-ieQ_f) \gamma^\alpha \frac{i\left(\not{p} + \not{q} + m\right)}{(p+q)^2 - m^2} i \frac{g}{\cos\theta_W} \gamma^\beta (g_V - g_A \gamma_5) \frac{i\left(\not{q} + m\right)}{q^2 - m^2} \right\} \equiv i\Pi_{\gamma Z}^{\alpha\beta}$$
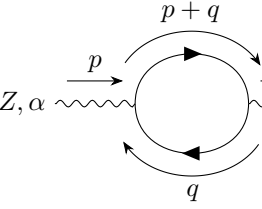
Let's work on the trace so we can express the numerator of the 2-point function in terms of scalar integrals.

$$\operatorname{Tr} \left\{ \gamma^\alpha \left(\not{p} + \not{q} + m\right) \gamma^\beta (g_V - g_A \gamma_5)\left(\not{q} + m\right) \right\} = g_V \operatorname{Tr} \left\{ \gamma^\alpha \left(\not{p} + \not{q}\right) \gamma^\beta \not{q} + m^2 \gamma^\alpha \gamma^\beta \right\} - g_A \operatorname{Tr} \left\{ \gamma^\alpha \left(\not{p} + \not{q}\right) \gamma^\beta \gamma_5 \not{q} \right\} =$$

$$= 4g_V \left\{ (p+q)_\mu q_\nu \left(g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\beta} g^{\mu\nu} + g^{\alpha\nu} g^{\beta\mu} \right) + m^2 g^{\alpha\beta} \right\} - 4ig_A (p+q)_\mu q_\nu \epsilon^{\alpha\mu\beta\nu} =$$

$$= 4 \left\{ g_V \left[(p+q)^\alpha q^\beta - g^{\alpha\beta} (p+q)\cdot q + q^\alpha (p+q)^\beta + m^2 g^{\alpha\beta} \right] - ig_A \epsilon^{\alpha\mu\beta\nu} p_\mu q_\nu \right\}$$

Where we used that a symmetric tensor contracted with an antisymmetric tensor vanishes.

$$i\Pi_{\gamma Z}^{\alpha\beta} = \frac{4Q_f eg}{\cos\theta_W} \int \frac{d^4 q}{(2\pi)^4} \frac{g_V \left((p+q)^\alpha q^\beta - g^{\alpha\beta}(p+q)\cdot q + q^\alpha (p+q)^\beta + m^2 g^{\alpha\beta} \right) - ig_A \epsilon^{\alpha\mu\beta\nu} p_\mu q_\nu}{\left((p+q)^2 - m^2\right)(q^2 - m^2)} =$$

$$= \frac{4Q_f eg}{\cos\theta_W} \frac{i\pi^2}{(2\pi)^4} \left\{ -ig_A \epsilon^{\alpha\mu\beta\nu} p_\mu B_1 p_\nu + g_V \left[B_1 p^\alpha p^\beta + B_{00} g^{\alpha\beta} + B_{11} p^\alpha p^\beta - g^{\alpha\beta}\left(B_1 p^2 + 4B_{00} + B_{11} p^2\right) + \right. \right.$$

$$\left. \left. + B_1 p^\alpha p^\beta + B_{00} g^{\alpha\beta} + B_{11} p^\alpha p^\beta + B_0 m^2 g^{\alpha\beta} \right] \right\} =$$

$$= \frac{iQ_f egg_V}{4\pi^2 \cos\theta_W} \left\{ 2p^\alpha p^\beta (B_1 + B_{11}) + g^{\alpha\beta} \left(m^2 B_0 - 2B_{00} - p^2 (B_1 + B_{11}) \right) \right\}$$
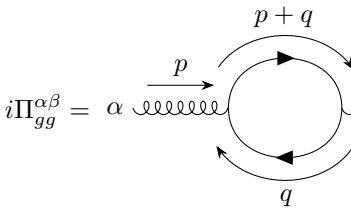
Z-Boson self-energy

$$Z, \alpha \;\; \beta, Z \;= \int \frac{d^4q}{(2\pi)^4}(-1)\,\mathrm{Tr}\left\{ \frac{ig}{\cos\theta_W}\gamma^\alpha\left(g_V - g_A\gamma_5\right)\frac{i\left(\not p + \not q + m\right)}{(p+q)^2 - m^2}\frac{ig}{\cos\theta_W}\gamma^\beta \times \right.$$

$$\left. \times \left(g_V - g_A\gamma_5\right)\frac{i\left(\not q + m\right)}{q^2 - m^2} \right\} \equiv i\Pi_{ZZ}^{\alpha\beta}$$

Let's work on the trace so we can express the numerator of the 2-point function in terms of scalar integrals.

$$\mathrm{Tr}\left\{\gamma^\alpha\left(g_V - g_A\gamma_5\right)\left(\not p + \not q + m\right)\gamma^\beta\left(g_V - g_A\gamma_5\right)\left(\not q + m\right)\right\} = \mathrm{Tr}\left\{\gamma^\alpha\left(g_V - g_A\gamma_5\right)^2\left(\not p + \not q - m\right)\gamma^\beta\left(\not q + m\right)\right\} =$$

$$= \left(g_V^2 + g_A^2\right)\mathrm{Tr}\left\{\gamma^\alpha\left(\not p + \not q - m\right)\gamma^\beta\left(\not q + m\right)\right\} = \left(g_V^2 + g_A^2\right)\mathrm{Tr}\left\{\gamma^\alpha\left(\not p + \not q\right)\gamma^\beta\not q - m^2\gamma^\alpha\gamma^\beta\right\} =$$

$$= \left(g_V^2 + g_A^2\right)\left\{(p+q)_\mu q_\nu 4\left(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\beta\mu}\right) - 4m^2 g^{\alpha\beta}\right\} =$$

$$= 4\left(g_V^2 + g_A^2\right)\left\{(p+q)^\alpha q^\beta - g^{\alpha\beta}(p+q)\cdot q + q^\alpha(p+q)^\beta - m^2 g^{\alpha\beta}\right\}$$

$$i\Pi_{ZZ}^{\alpha\beta} = \int \frac{d^4q}{(2\pi)^4}\frac{4g^2\left(g_V^2 + g_A^2\right)}{\cos^2\theta_W}\frac{(p+q)^\alpha q^\beta - g^{\alpha\beta}(p+q)\cdot q + q^\alpha(p+q)^\beta - m^2 g^{\alpha\beta}}{\left((p+q)^2 - m^2\right)(q^2 - m^2)} =$$

$$= \frac{4g^2\left(g_V^2 + g_A^2\right)}{\cos^2\theta_W}\int \frac{d^4q}{(2\pi)^4}\frac{p^\alpha q^\beta + q^\alpha p^\beta + 2q^\alpha q^\beta - g^{\alpha\beta}\left(m^2 + q\cdot p + q^2\right)}{\left((p+q)^2 - m^2\right)(q^2 - m^2)} =$$

$$= \frac{4g^2\left(g_V^2 + g_A^2\right)}{\cos^2\theta_W}\frac{i\pi^2}{(2\pi)^4}\left\{2p^\alpha p^\beta B_1 + 2\left(B_{00}g^{\alpha\beta} + B_{11}p^\alpha p^\beta\right) - g^{\alpha\beta}\left(m^2 B_0 + B_1 p^2 + 4B_{00} + B_{11}p^2\right)\right\} =$$

$$= \frac{ig^2\left(g_V^2 + g_A^2\right)}{4\pi^2\cos^2\theta_W}\left\{p^\alpha p^\beta 2\left(B_1 + B_{11}\right) - g^{\alpha\beta}\left(m^2 B_0 + 2B_{00} + p^2\left(B_1 + B_{11}\right)\right)\right\}$$

Gluon self-energy

$$i\Pi_{gg}^{\alpha\beta} = \;\alpha \qquad \beta \;=\; \alpha \qquad \beta \;\; \left(eQ_q \to g_S T^a\right) =$$

$$= -\frac{ig_S^2\,\mathrm{Tr}\left(T^a T^b\right)}{4\pi^2}\left\{2p^\alpha p^\beta\left(B_1 + B_{11}\right) + g^{\alpha\beta}\left(m^2 B_0 - 2B_{00} - p^2\left(B_1 + B_{11}\right)\right)\right\}$$

Appendices

A Feynman Rules

In this appendix all of the Feynman rules which were used for the calculations are listed. The Feynman rules are given for the whole Standard Model, but the pure QED Feynman rules can be obtained by taking $Q_f \to Q_e = -1$ and $m_f \to m_e \equiv m$.

Propagator

$$\xrightarrow{\quad p \quad} = \frac{i\left(\not{p} + m_f\right)}{p^2 - m_f^2 + i\epsilon} \qquad\qquad \alpha \sim\!\!\!\xrightarrow{\quad p \quad}\!\!\!\sim \beta = \frac{-ig^{\alpha\beta}}{p^2 + i\epsilon} \qquad \text{in 't Hooft-Feynman gauge}$$

Interactions

$$= \frac{-ig}{2}\frac{m_f}{m_W}$$

$$\gamma,\alpha = -ieQ_f\gamma^\alpha$$

$$\text{Z,}\alpha = \frac{ig}{\cos\theta_W}\gamma^\alpha\left(g_V - g_A\gamma_5\right)$$

$$\text{a,}\alpha = ig_S T^a_{ij}\gamma^\alpha$$

B Important Integrals

In the calculation of R_2 we have to evaluate 2-,3- and 4-point functions. They can be reduced to a set of integrals which are knwon in a general form. The integrals we need are [1]

2-point integrals

$$\int d^d\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} = -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{3}\right] + O(\epsilon) \tag{B.1}$$

$$\text{P.P.}\left(\int d^d\bar{q}\,\frac{1}{\bar{D}_i\bar{D}_j}\right) = -2\frac{i\pi^2}{\epsilon} \tag{B.2}$$

$$\text{P.P.}\left(\int d^d\bar{q}\,\frac{q_\mu}{\bar{D}_i\bar{D}_j}\right) = \frac{i\pi^2}{\epsilon}\left(p_i + p_j\right)_\mu \tag{B.3}$$

3-point integrals

$$\int d^d \bar{q} \frac{\tilde{q}^2}{\bar{D}_i \bar{D}_j \bar{D}_k} = -\frac{i\pi^2}{2} + O(\epsilon) \tag{B.4}$$

$$\int d^d \bar{q} \frac{\tilde{q}^2 q_\mu}{\bar{D}_i \bar{D}_j \bar{D}_k} = \frac{i\pi^2}{6} (p_i + p_j + p_k)_\mu + O(\epsilon) \tag{B.5}$$

$$\text{P.P.} \left(\int d^d \bar{q} \frac{q_\mu q_\nu}{\bar{D}_i \bar{D}_j \bar{D}_k} \right) = -\frac{i\pi^2}{2\epsilon} g_{\mu\nu} \tag{B.6}$$

4-point integrals

$$\int d^d \bar{q} \frac{\tilde{q}^4}{\bar{D}_i \bar{D}_j \bar{D}_k \bar{D}_l} = -\frac{i\pi^2}{6} + O(\epsilon) \tag{B.7}$$

$$\int d^d \bar{q} \frac{\tilde{q}^2 q_\mu q_\nu}{\bar{D}_i \bar{D}_j \bar{D}_k \bar{D}_l} = -\frac{i\pi^2}{12} g_{\mu\nu} + O(\epsilon) \tag{B.8}$$

$$\int d^d \bar{q} \frac{\tilde{q}^2 q^2}{\bar{D}_i \bar{D}_j \bar{D}_k \bar{D}_l} = -\frac{i\pi^2}{3} + O(\epsilon) \tag{B.9}$$

C   Identities for Gamma Matrices

In a theory with fermions the Dirac matrices appear as the generators of the spinor representation of the Poincaré algebra. The following identities for Dirac matrices are very useful when evaluating Feynman diagrams

1. $\text{Tr} \left(\gamma^\alpha \gamma^\beta \right) = d g^{\alpha\beta}$ $\hfill$ (C.1)
2. $\text{Tr} \left(\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta \right) = d \left(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma} \right)$ $\hfill$ (C.2)
3. $\text{Tr} \left(\gamma^{\alpha_1} \gamma^{\alpha_2} \cdots \gamma^{\alpha_n} \right) = 0 \quad$ for n odd $\hfill$ (C.3)
4. $\text{Tr} \left(\gamma^\alpha \gamma^\beta \gamma_5 \right) = 0$ $\hfill$ (C.4)
5. $\gamma^\alpha \gamma_\alpha = d$ $\hfill$ (C.5)
6. $\gamma^\alpha \gamma^\beta \gamma_\alpha = (2 - d) \gamma^\beta$ $\hfill$ (C.6)
7. $\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma_\alpha = d \gamma^\beta \gamma^\gamma + 2 \left[\gamma^\gamma, \gamma^\beta \right]$ $\hfill$ (C.7)
8. $\not{a}\not{b} = a \cdot b$ $\hfill$ (C.8)

The Dirac matrices obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbb{1}_d$ with $g^{\mu\nu}$ the Minkowski metric in d dimensions

$$g^{\mu\nu} = \begin{cases} 1 & \text{for } \mu = \nu = 0 \\ -1 & \text{for } \mu = \nu = 1, 2, \ldots, d - 1 \\ 0 & \text{for } \mu \neq \nu \end{cases}$$

In a few of the identities also the fifth gamma matrix appears which is defined as $\gamma_5 = i/4! \, \epsilon_{\mu_1 \ldots \mu_d} \gamma^{\mu_1} \cdots \gamma^{\mu_d}$. It is hermitian, traceless and self-inverse: $\gamma_5^2 = \mathbb{1}_d$.

Proofs for identities

1. $\text{Tr}\left(\gamma^\alpha\gamma^\beta\right) = dg^{\alpha\beta}$

Proof.

$$\text{Tr}\left(\gamma^\alpha\gamma^\beta\right) = \text{Tr}\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right) = 2g^{\alpha\beta}\text{Tr}\left(\mathbb{1}_d\right) - \text{Tr}\left(\gamma^\beta\gamma^\alpha\right) = 2dg^{\alpha\beta} - \text{Tr}\left(\gamma^\alpha\gamma^\beta\right)$$

$$\Rightarrow \text{Tr}\left(\gamma^\alpha\gamma^\beta\right) = dg^{\alpha\beta}$$

□

2. $\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$

Proof.

$$\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = \text{Tr}\left(\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right)\gamma^\gamma\gamma^\delta\right) = 2g^{\alpha\beta}\text{Tr}\left(\gamma^\gamma\gamma^\delta\right) - \text{Tr}\left(\gamma^\beta\left(2g^{\alpha\gamma} - \gamma^\gamma\gamma^\alpha\right)\gamma^\delta\right) =$$

$$= 2dg^{\alpha\beta}g^{\gamma\delta} - 2g^{\alpha\gamma}\text{Tr}\left(\gamma^\beta\gamma^\delta\right) + \text{Tr}\left(\gamma^\beta\gamma^\gamma\left(2g^{\alpha\delta} - \gamma^\delta\gamma^\alpha\right)\right) =$$

$$= 2d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) + 2g^{\alpha\delta}\text{Tr}\left(\gamma^\beta\gamma^\gamma\right) - \text{Tr}\left(\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\alpha\right) =$$

$$= 2d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right) - \text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right)$$

$$\Rightarrow \text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = d\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)$$

□

3. $\text{Tr}\left(\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}\right) = 0$ for n odd

Proof.

$$\text{Tr}\left(\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}\right) = \text{Tr}\left(\gamma_5^2\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}\right) = (-1)^n\,\text{Tr}\left(\gamma_5\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}\gamma_5\right) \overset{\text{n odd}}{=} -\text{Tr}\left(\gamma_5^2\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}\right) =$$

$$= -\text{Tr}\left(\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}\right)$$

$$\Rightarrow \text{Tr}\left(\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}\right) = 0 \quad \text{for n odd}$$

□

4. $\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma_5\right) = 0$

Proof.

$$\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma_5\right) \overset{5.}{=} \frac{1}{d}\text{Tr}\left(\gamma_\mu\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_5\right) \overset{\{\gamma^\mu,\gamma_5\}=0}{=} \frac{(-1)^3}{d}\text{Tr}\left(\gamma_\mu\gamma_5\gamma^\mu\gamma^\alpha\gamma^\beta\right) =$$

$$= -\frac{1}{d}\text{Tr}\left(\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu\gamma_5\right) \overset{7.}{=} -\frac{1}{d}\text{Tr}\left(\left(d\gamma^\beta\gamma^\gamma + 2\left[\gamma^\gamma,\gamma^\beta\right]\right)\gamma_5\right) = -\left(1 + \frac{2}{d}\right)\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma_5\right) + \frac{2}{d}\text{Tr}\left(\gamma^\beta\gamma^\alpha\gamma_5\right) =$$

$$= -\left(1 + \frac{2}{d}\right)\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma_5\right) + \frac{2}{d}\text{Tr}\left(\left(2g^{\beta\alpha} - \gamma^\alpha\gamma^\beta\right)\gamma_5\right) = -\left(1 + \frac{4}{d}\right)\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma_5\right) + \frac{4}{d}g^{\beta\alpha}\underbrace{\text{Tr}\left(\gamma_5\right)}_{=0} =$$

$$= -\left(1 + \frac{4}{d}\right)\text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma_5\right)$$

$$\Rightarrow \text{Tr}\left(\gamma^\alpha\gamma^\beta\gamma_5\right) = 0$$

□

5. $\gamma^\alpha \gamma_\alpha = d$

Proof.

$$\gamma^\alpha \gamma_\alpha = \frac{1}{2} \left(\gamma^\alpha \gamma_\alpha + \gamma_\alpha \gamma^\alpha \right) = \frac{1}{2} \{ \gamma^\alpha, \gamma_\alpha \} = \frac{1}{2} 2 g^\alpha_\alpha = d$$

$\square$

6. $\gamma^\alpha \gamma^\beta \gamma_\alpha = (2 - d) \gamma^\beta$

Proof.

$$\gamma^\alpha \gamma^\beta \gamma_\alpha = \left(2 g^{\alpha\beta} - \gamma^\beta \gamma^\alpha \right) \gamma_\alpha = (2 - d) \gamma^\beta$$

$\square$

7. $\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma_\alpha = d \gamma^\beta \gamma^\gamma + 2 \left[\gamma^\gamma, \gamma^\beta \right]$

Proof.

$$\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma_\alpha = \left(2 g^{\alpha\beta} - \gamma^\beta \gamma^\alpha \right) \gamma^\gamma \gamma_\alpha \overset{4.}{=} 2 \left(\gamma^\gamma \gamma^\beta - \gamma^\beta \gamma^\gamma \right) + d \gamma^\beta \gamma^\gamma = d \gamma^\beta \gamma^\gamma + 2 \left[\gamma^\gamma, \gamma^\beta \right]$$

$\square$

8. $\slashed{a} \slashed{b} = a \cdot b$

Proof.

$$\slashed{a} \slashed{b} = a_\alpha b_\beta \gamma^\alpha \gamma^\beta = a_\alpha b_\beta \left(2 g^{\alpha\beta} - \gamma^\beta \gamma^\alpha \right) = 2 a \cdot b - \slashed{b} \slashed{a}$$

$$\Rightarrow \slashed{a} \slashed{b} = a \cdot b$$

$\square$

D   Relation Between Left- & Right-handed Currents and Axial & Vector Currents

A classical Lagrangian permits symmetries which can be implemented by Lie groups G. An element $g \in G$ of a Lie group can be parametrized as $g = \exp \left(i \alpha^a T^a \right)$ where α^a are real parameters and T^a the generators of the Lie group. Noether's theorem predicts a clasically conserved current for each generator of a continuous symmetry. For a field ϕ with trafo $\delta\phi = \phi' - \phi = g\phi - \phi \approx \left(1 + i \alpha^a T^a \right) \phi - \phi = i \alpha^a T^a \phi$ the conserved current can be shown to be

$$j^{\mu a} = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi \right)} \frac{\partial \delta \phi}{\partial \alpha_a}$$

The left- and right-handed part $j^{\mu a}_{L/R}$ of a fermionic current are

$$j^{\mu a}_L = \bar{\psi}_L \gamma^\mu T^a \psi_L = \left(P_L \psi \right)^\dagger \gamma^0 \gamma^\mu T^a P_L \psi \overset{P_L^\dagger = P_L}{=} \psi^\dagger P_L \gamma^0 \gamma^\mu T^a P_L \psi = \bar{\psi} \gamma^\mu T^a P_L^2 \psi \overset{P_L^2 = P_L}{=} \bar{\psi} \gamma^\mu T^a P_L \psi$$

$$j^{\mu a}_R = \bar{\psi}_R \gamma^\mu T^a \psi_R = \bar{\psi} \gamma^\mu T^a P_R \psi$$

where $P_{L/R} = \frac{1}{2} \left(1 \mp \gamma_5 \right)$ is the left-/right-handed projector.

From the left- and right-handed currents we can define axial-vector and vector currents

$$j^{\mu a} = j^{\mu a}_R + j^{\mu a}_L = \bar{\psi} \gamma^\mu T^a \left(P_R + P_L \right) = \bar{\psi} \gamma^\mu T^a \psi$$

$$j^{\mu a}_5 = j^{\mu a}_R - j^{\mu a}_L = \bar{\psi} \gamma^\mu T^a \left(P_R - P_L \right) \psi = \bar{\psi} \gamma^\mu T^a \gamma_5 \psi$$

Now we can couple the currents to vector fields to obtain interactions. E.g., the vector coupling in QED is given by the Lagrangian

$$\mathcal{L}_{coupl}^{QED} = eA_\mu j^\mu = eA_\mu \bar{\psi}\gamma^\mu Q_e \psi = -eA_\mu \bar{\psi}\gamma^\mu \psi$$

In general, we can couple any linear combination of currents to a vector field as long as the combination is Lorentz and gauge invariant. E.g., the neutral current in the electroweak theory is a superposition of a vector and an axialvector current

$$\mathcal{L}_{coupl}^{NC} = gZ_\mu \left(g_V j^\mu - g_A j_5^\mu \right)$$

We can use the above relations to express this coupling in terms of right- and left-handed currents

$$\mathcal{L}_{coupl}^{NC} = gZ_\mu \left(g_V \bar{\psi}\gamma^\mu \psi - g_A \bar{\psi}\gamma^\mu \gamma_5 \psi \right) =$$
$$= gZ_\mu \left(g_V \bar{\psi}\gamma^\mu \psi + \frac{g_A}{2}\bar{\psi}\gamma^\mu \psi - \frac{g_A}{2}\bar{\psi}\gamma^\mu \psi - g_A \bar{\psi}\gamma^\mu \gamma_5 \psi + \frac{g_V}{2}\bar{\psi}\gamma^\mu \gamma_5 \psi - \frac{g_V}{2}\bar{\psi}\gamma^\mu \gamma_5 \psi \right) =$$
$$= gZ_\mu \left((g_V + g_A)\bar{\psi}\gamma^\mu \frac{1}{2}(1-\gamma_5)\psi + (g_V - g_A)\bar{\psi}\gamma^\mu \frac{1}{2}(1+\gamma_5)\psi \right) =$$
$$= gZ_\mu \left((g_V + g_A)\bar{\psi}\gamma^\mu P_L \psi + (g_V - g_A)\bar{\psi}\gamma^\mu P_R \psi \right) \equiv gZ_\mu \left(g_L j_L^\mu + g_R j_R^\mu \right)$$

This gives the following relation between the (axial-)vector and the left-/right-handed couplings

$$g_L = g_V + g_A$$
$$g_R = g_V - g_A$$

References

[1] P Draggiotis, M.V Garzelli, C.G Papadopoulos, and R Pittau. Feynman rules for the rational part of the qcd 1-loop amplitudes. *Journal of High Energy Physics*, 2009(04):072–072, Apr 2009.

[2] Giovanni Ossola, Costas G Papadopoulos, and Roberto Pittau. On the rational terms of the one-loop amplitudes. *Journal of High Energy Physics*, 2008(05):004–004, May 2008.

[3] Christian Schubert. On the γ_5–problem of dimensional renormalization. 1993.

[4] Gerard 't Hooft and M. J. G. Veltman. Regularization and Renormalization of Gauge Fields. *Nucl. Phys.*, B44:189–213, 1972.

[5] Rolf Mertig, M. Böhm, and A. Denner. Feyn calc - computer-algebraic calculation of feynman amplitudes. *Computer Physics Communications*, 64:345–359, 06 1991.

[6] Vladyslav Shtabovenko, Rolf Mertig, and Frederik Orellana. New Developments in FeynCalc 9.0. *Comput. Phys. Commun.*, 207:432–444, 2016.

[7] J. C. Ward. An identity in quantum electrodynamics. *Phys. Rev.*, 78:182–182, Apr 1950.

[8] Ansgar Denner. Techniques for calculation of electroweak radiative corrections at the one loop level and results for W physics at LEP-200. *Fortsch. Phys.*, 41:307–420, 1993.

[9] A. Denner and S. Dittmaier. Reduction schemes for one-loop tensor integrals. *Nuclear Physics B*, 734(1-2):62–115, Jan 2006.