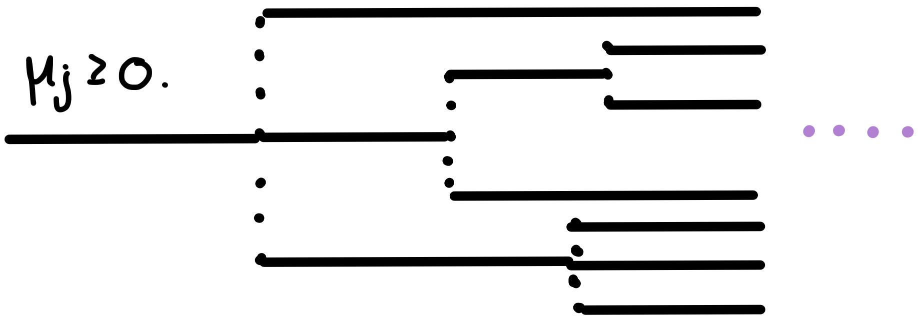
Motivation II :

The Volkonskii and the Lamperti transformations

Continuous-time Bienaymé-Galton-Watson Process: Z

CTMC on $\mathbb{N}=\{0,1,2,\dots\}$ w/ jump rates: $k \mapsto k+j-1 \quad \lambda \cdot k \; \mu_j$

μ: offspring distribution : $\sum_{j=0}^{\infty} \mu_j = 1$, $\mu_j \geq 0$.



Jump sizes: Same as for a continuous time random walk X

Jump intensity: proportional to the state

$$\boxed{Z_t^k = k + X_{\int_0^t Z_s^k \, ds}}$$

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Continuous-time Bienaymé-Galton-Watson Process: Z

CTMC on $\mathbb{N} = \{0,1,2,\dots\}$ w/ jump rates: $k \mapsto k+\xi-1$ $\quad \lambda \cdot k \quad \mu_j$

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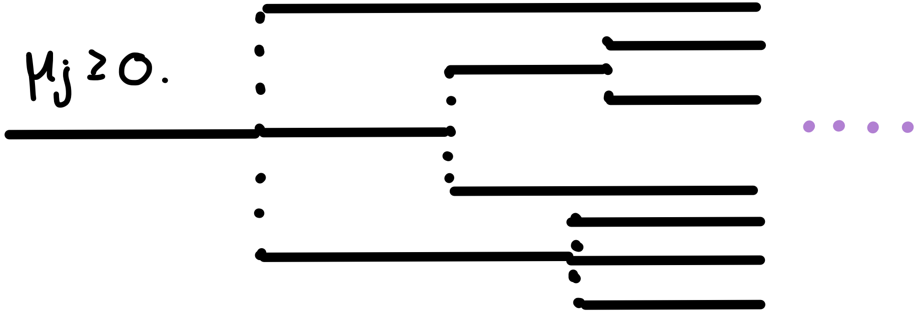


With the possible exception of Itô's method, the preceding result is, from a purely probabilistic point of view, the most satisfactory resolution of a martingale problem which we will present. It is not so much that we are surprised that one can handle the situation dealt with in Corollary 6.5.5 (indeed, considering Feller's magnificent success in understanding one-dimensional, time-homogeneous diffusions, it would have been very disappointing if we had not been able to), but that

(Volkonskii)

we are able to do so without having to invoke anything from the theory of partial differential equations. Unfortunately, we will not get off so lightly when we insist on being more ambitious.

Volkonskii: $Z_t = X_{\int_0^t a(Z_s)\,ds}$ $\qquad$ Grators: $A_Z f(x) = a(x) A_X f(x)$

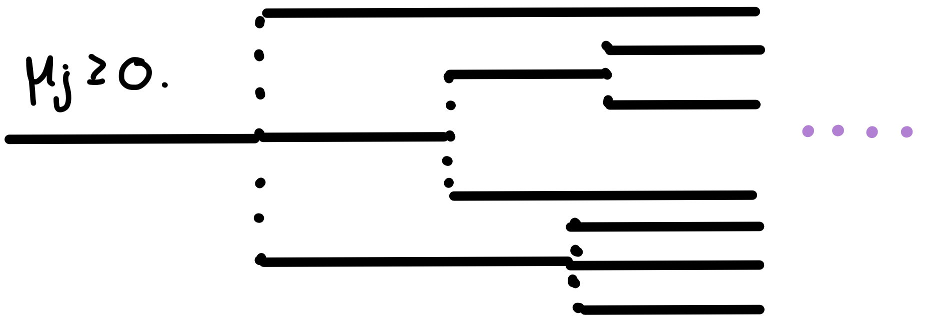
Motivation II :

The Volkonskii and the Lamperti transformations

Continuous-time Bienaymé-Galton-Watson Process w/ immigration Z

CTMC on $\mathbb{N}=\{0,1,2,\dots\}$ w/ jump rates: $k \mapsto k+j-1$ $\quad \lambda\cdot(k\,\mu_j+\nu_{j-1})$

μ: offspring distribution : $\sum_{j=0}^{\infty}\mu_j=1$, $\mu_j\geq 0$.

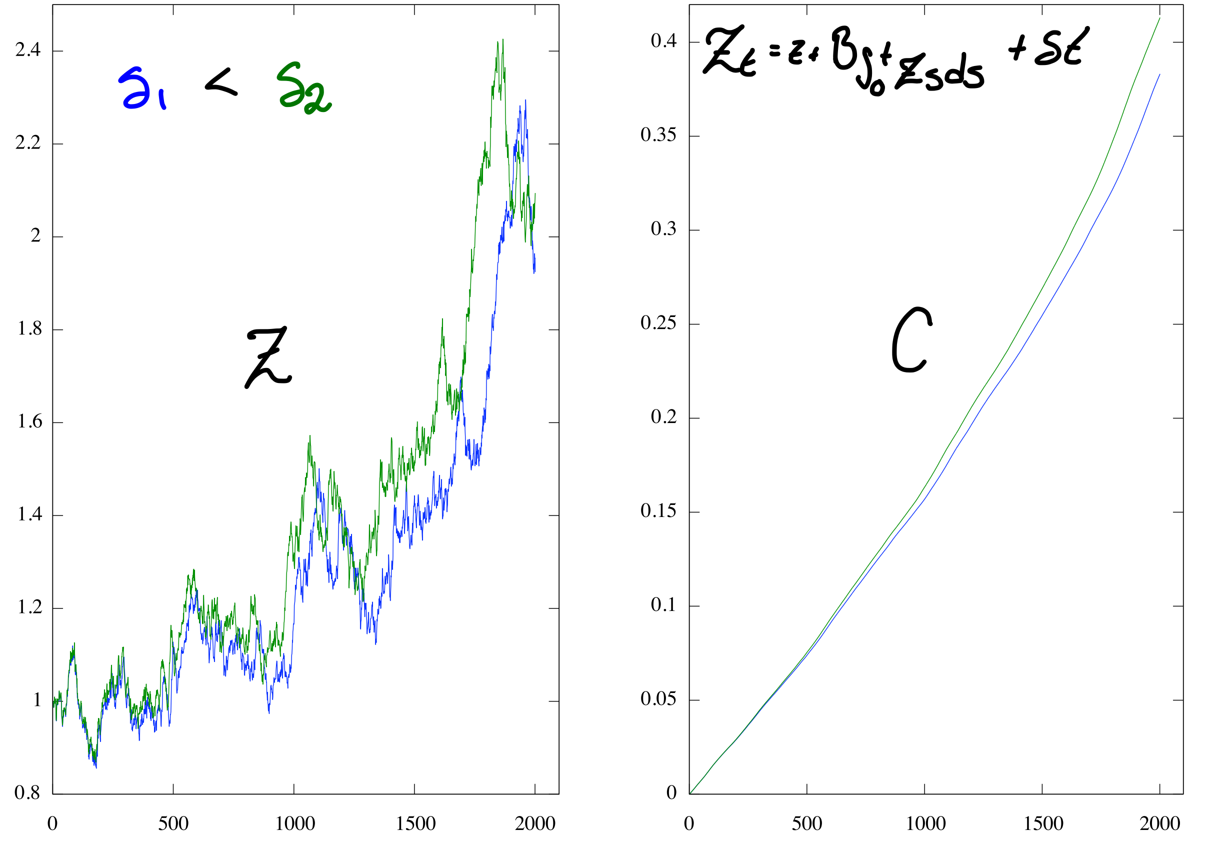


ν: immigration distribution

X : $RW(\lambda,\tilde{\mu})$ $\qquad$ Y : $RW(\lambda,\nu)$

$$\boxed{Z_t^k = k + X_{\int_0^t Z_s\,ds} + Y_t}$$

Grators: $A_Z f(z) = z\,A_X f(z) + A_Y f(z)$



δ1 < δ2
Z
Zt = z + B ∫0t Zs ds + δt
C
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
500
1000
1500
2000

A Lamperti type transformation for continuous state branching processes with immigration

Caballero, Pérez-Garmendia, UB, AoP, 2013

$$Z_t = z + X_{\int_0^t Z_s\,ds} + Y_t \quad \left\{ \begin{array}{lll} X & \text{Lévy process} \quad \Delta X_t \geq 0 & E(e^{-\lambda X_t}) = e^{t\Psi(\lambda)} \\ \perp\!\!\!\perp & & \\ Y & \text{" } \qquad \text{" } \quad Y\uparrow & E(e^{-\lambda Y_t}) = e^{t\Phi(\lambda)} \end{array} \right.$$

Deterministic theorem: Let f, g be càdlàg, $g\uparrow\uparrow$, $\Delta f \geq 0$.

Then: $\exists!\, h$ s.t.: $\boxed{h(t) = f\left(\int_0^t h(s)\,ds\right) + g(t)}$.

Existence + uniqueness $\Leftarrow$ Monotonicity: either g or $\tilde g$ $\uparrow\uparrow$

Let (h,c) and $(\tilde h, \tilde c)$ solve: $h = f\circ c + g \quad c(t) = \int_0^t h(s)\,ds \ldots$

If $f \leq \tilde f$ and $g \leq \tilde g$ then $c \leq \tilde c$. Lema: $c\uparrow$ y $h \geq 0$ $\quad (\bar h, \bar c) : \in C_T(\alpha f, \alpha g(\alpha\cdot))$

$\alpha > 1 \quad \bar c(t) = \tilde c(\alpha t) \quad \overset{\to \bar h}{\bar c'(t)} = \alpha \tilde c'(\alpha t) = \alpha\left[\tilde f \circ \tilde c(\alpha t) + \tilde g(\alpha t)\right]$

$\tau = \inf\{t \geq 0 : c(t) > \bar c(t)\} \quad c(0) = \bar c(0) = 0 \Rightarrow c \leq \bar c$ en $[0,\tau]$, $c(\tau) = \bar c(\tau)$

$\bar c'(\tau) = \alpha\left[\underbrace{\tilde f \circ \bar c(\tau) + \tilde g(\alpha\tau)}\right] > \alpha\left[f\circ c(\tau) + g(\tau)\right] > c'(\tau) \Rightarrow \bar c > c$ en vec. derecha de τ.

$h(\tau) = c'(\tau) \geq 0$. $\quad (\Rightarrow c \leq \bar c \;\; \forall \alpha > 1 \Rightarrow c \leq \tilde c$!

Future work

5) Infinite dimensional formulations:

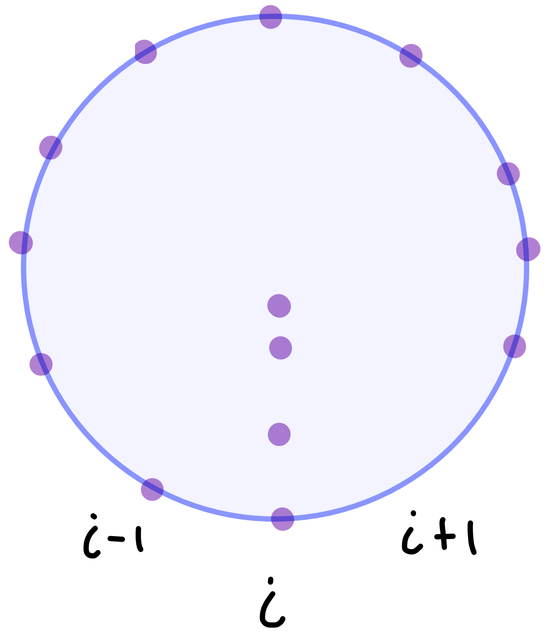
From: $Z_t^i = z_i + B^i \circ C^i(t) + \left(C_t^{i+1} + C_t^{i-1} - 2C_t^i\right)$

$$C_t^i = \int_0^t Z_s^i \, ds$$

$\alpha > 1$

$\tau = \inf\{t \ge 0 : C^i(t) > \tilde{C}^i(\alpha t)$ p.a. $i\}$.

$\tau < \infty \Rightarrow$!



Comparación.

$h_i(t) = f_i \circ C_i + \Delta C_i + g_i(t)$

$C_i(t) = \int_0^t h_i(s) \, ds$

$\Delta C_i = C_{i+1} + C_{i-1} - 2C_i$.

(h, C) $(\tilde{h}, \tilde{C})$. $g_i \uparrow\uparrow \forall i$

$f_i \le \tilde{f}_i$, $g_i \le \tilde{g}_i$

$\Rightarrow C_i \le \tilde{C}_i \; \forall i \; (\forall t)$.