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1.3 Functionals

Functional turns function into a number.

$$\text{Ex. } \mathcal{F}[f] = \int_0^1 f(x) dx$$

$$\mathcal{G}[f] = \int_{-\infty}^{\infty} S[f(x)]^2 dx$$

$$\mathcal{F}_{\alpha}[f] = \int_{-\infty}^{\infty} f(y) \delta(y - x) dy = f(x).$$

Functional Differentiation

$$\frac{\delta F}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{F[f(x) + \epsilon \delta(x-x')] - F[f(x')]}{\epsilon}$$



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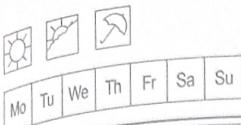
$$\text{Ex } E[f] = \int_{-1}^1 f(x) dx$$

$$\frac{\delta E[f]}{\delta f(x_0)} = \lim_{\epsilon \rightarrow 0} \left[\bar{E} \left(\int_{-1}^1 [f(x) + \delta(x-x_0)] dx \right) - \int_{-1}^1 f(x) dx \right]$$

$$= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{-1}^1 \delta(x-x_0) dx$$

$$= \int_{-1}^1 \delta(x-x_0) dx$$

$$= \begin{cases} 1 & -1 \leq x_0 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$



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$$H[f] = \int_a^b g[f(x)] dx$$

$$J[f] = \int [f(y)]^p \phi(y) dy$$

$$\begin{aligned} \frac{\delta J[f]}{\delta f(x)} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int [f(y) + \epsilon \delta(y-x)]^p \phi(y) dy \right. \\ &\quad \left. - \int [f(y)]^p \phi(y) dy \right] \\ &= p [f(x)]^{p-1} \phi(x) \end{aligned}$$

$$H[f] = \int_a^b g[f(x)] dx, \quad g' = dg/dx$$

$$\begin{aligned} \frac{\delta H[f]}{\delta f(x_0)} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int g[f(x) + \epsilon \delta(x-x_0)] dx \right. \\ &\quad \left. - \int g[f(x)] dx \right] \end{aligned}$$

$$\begin{aligned} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int [g(f(x)) + f'(x-x_0)g'(f(x))] dx \right. \\ &\quad \left. - \int g[f(x)] dx \right] \\ &= g'[f(x_0)]. \end{aligned}$$



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$$J[x] = \frac{1}{T} \int_0^T \sqrt{x(t)} dt$$

$$\frac{\delta \bar{J}[x]}{\delta x(t)} = \frac{1}{T} \sqrt{x(t)}$$

$$J[f] = \int g(\epsilon) dy, f' = df/dy.$$

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\int dy g \left(\frac{\partial}{\partial y} [f(y) + \epsilon \delta(y-x)] \right) - \int dy g \frac{\partial f}{\partial y} \right]$$

Note $g \left(\frac{\partial}{\partial y} [f(y) + \epsilon \delta(y-x)] \right) = g(f' + \epsilon \delta'(y-x))$
 $\approx g(f') + \epsilon \delta'(y-x) \frac{dg}{df}$

$$\Rightarrow \frac{\delta J[f]}{\delta f(x)} = \int dy \delta'(y-x) \frac{dg(f')}{df} \\ = \left[\delta_{\epsilon}(y-x) \frac{dg(f')}{df} \right] - \int \delta(y-x) \frac{dg}{dy} dy$$

or if x inside limits.



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$$\text{Then } \frac{\delta J[f]}{\delta f(x)} = -\frac{d}{dx} \left(\frac{dy(f)}{df} \right)$$

$$\text{Thus, } \frac{\delta F[\phi]}{\delta \phi(x)} = -2 \frac{\partial^2 \phi}{\partial x^2}, \text{ if}$$

$$F[\phi] = \int \left(\frac{\partial \phi}{\partial y} \right)^2 dy.$$

$$- \frac{d}{dx} \left(\frac{d \left(\frac{\partial \phi}{\partial y} \right)}{d \frac{\partial \phi}{\partial x}} \right)$$

$$-2 \frac{d}{dx} \frac{\partial \phi}{\partial x} = -2 \frac{\partial^2 \phi}{\partial x^2}.$$

$$\bar{T} = \frac{1}{T} \int_0^T \frac{1}{2} m [\ddot{x}(t)]^2 dt$$

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$$\frac{\delta \bar{T}[x]}{\delta x(t)} = -\frac{m}{T} \ddot{x}(t)$$

$$\text{Also, } I = \int (\nabla \phi)^2 d^3x \Rightarrow \frac{\delta I}{\delta \phi} = -2 \nabla^2 \phi.$$