Mo Tu We Th Fr Sa Su

Memo No.
Date / /

1.3 Functionals

Functional turns function into a number.

Ex 0.1
$$F[f] = \int_0^1 f(x)\,dx$$

$$G[f] = \int_{-a}^{a} 5[f(x)]^2\,dx$$

$$F_x[f] = \int_{-\infty}^{\infty} f(y)\,\delta(y-x)\,dy = f(x).$$

Functional Differentiation

$$\frac{\delta F}{\delta f(x)} = \lim_{\varepsilon \to 0} \frac{F[f(x') + \varepsilon\delta(x-x')] - F[f(x')]}{\varepsilon}$$

Memo No.
Date / /

Mo Tu We Th Fr Sa Su

Ex $I[f] = \int_{-1}^{1} f(x)\,dx$

$$\frac{\delta I[f]}{\delta f(x_0)} = \lim_{\epsilon \to 0} \left[\frac{1}{\epsilon} \left(\int_{-1}^{1} [f(x) + \epsilon\delta(x-x_0)]\,dx - \int_{-1}^{1} f(x)\,dx \right) \right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{-1}^{1} \epsilon\,\delta(x-x_0)\,dx$$

$$= \int_{-1}^{1} \delta(x-x_0)\,dx$$

$$= \begin{cases} 1 & -1 \le x_0 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Memo No.
Date / /

$$\cancel{H[f] = \int_a^b g[f(x)]\,dx}$$

$$J[f] = \int [f(y)]^p \phi(y)\,dy$$

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big[\int [f(y) + \epsilon\delta(y-x)]^p \phi(y)\,dy - \int [f(y)]^p \phi(y)\,dy \Big]$$

$$= p[f(x)]^{p-1}\phi(x)$$

$$H[f] = \int_a^b g[f(x)]\,dx, \quad g' = dg/dx$$

$$\frac{\delta H[f]}{\delta f(x_0)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big[\int g[f(x) + \epsilon\delta(x-x_0)]\,dx - \int g[f(x)]\,dx \Big]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\Big[\int g[f(x)] + \epsilon\delta(x-x_0) g'(f(x))\,dx - \int g[f(x)]\,dx \Big]$$

$$= g'[f(x_0)].$$

Memo No.
Date / /

$$\bar{V}[x] = \frac{1}{\tau}\int_0^{\tau} V[x(t)]\,dt$$

$$\frac{\delta \bar{V}[x]}{\delta(x(t))} = \frac{1}{\tau} V'(x(t))$$

$$J[f] = \int g(f')\,dy, \quad f' = df/dy.$$

$$\frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\int dy\, g\left(\frac{\partial}{\partial y}[f(y) + \epsilon\delta(y-x)]\right) - \int dy\, g\,\frac{\partial f}{\partial y}\right]$$

note $g\left(\frac{\partial}{\partial y}[f(y) + \epsilon\delta(y-x)]\right) = g(f' + \epsilon\delta'(y-x))$
$\approx g(f') + \epsilon\delta'(y-x)\frac{dg(f')}{df'}$

$$\Rightarrow \frac{\delta J[f]}{\delta f(x)} = \int dy\, \delta'(y-x)\frac{dg(f')}{df'}$$

$$= \left[\cancel{\delta(y-x)}\frac{dg(f')}{df'}\right] - \int \delta(y-x)\frac{d}{dy}\left(\frac{dg}{df'}\right)$$

0 if x inside limits.

Memo No.

Date / /

Then $\frac{\delta J[f]}{\delta f(x)} = -\frac{d}{dx}\left(\frac{dg(f')}{df'}\right)$

Thus, $\frac{\delta F[\phi]}{\delta \phi(x)} = -2\frac{\partial^2\phi}{\partial x^2}$, if

$$F[\phi] = \int \left(\frac{\partial\phi}{\partial y}\right)^2 dy.$$

$$-\frac{d}{dx}\left(\frac{d\left[\left(\frac{\partial\phi}{\partial x}\right)^2\right]}{d\frac{\partial\phi}{\partial x}}\right)$$

$$-2\frac{d}{dx}\frac{\partial\phi}{\partial x} = -2\frac{\partial^2\phi}{\partial x^2}.$$

$$\overline{T} = \frac{1}{t}\int_0^T \frac{1}{2}m[\dot{x}(t)]^2$$

$$\frac{\delta T[x]}{\partial(x(t))} = -\frac{m}{t}\ddot{x}(t)$$

Also, $I = \int (\nabla\phi)^2 d^3x \Rightarrow \frac{\delta I}{\delta\phi} = -2\nabla^2\phi.$