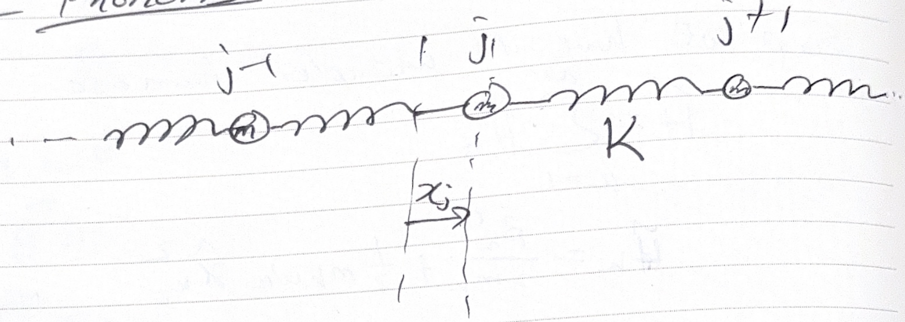
2.4 Phonons



$R_j = ja$

$$\hat{H} = \sum_j \frac{\hat{p}_j^2}{2m} + \frac{1}{2}K(\hat{x}_{j+1} - \hat{x}_j)^2$$

Ex $x_j =$

p_j

$\tilde{\hat{x}}_k$

$\hat{p}_k$

Need

$\Longrightarrow$

Note

$[x$

$[\hat{x}$

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Ex $$x_j = \frac{1}{\sqrt{N}} \sum_k \tilde{x}_k e^{ikja}$$

$$p_j = \frac{1}{\sqrt{N}} \sum_k \tilde{p}_k e^{ikja}$$

$$\tilde{x}_k = \frac{1}{\sqrt{N}} \sum_j x_j e^{-ikja}$$

$$\tilde{p}_k = \frac{1}{\sqrt{N}} \sum_j p_j e^{-ikja}.$$

Need $e^{ikja} = e^{ik(j+N)a}$

$$\Longrightarrow k = \frac{2\pi m}{Na}, \; m \in \mathbb{Z} \text{ w/ } m \in [-N/2, N/2].$$

Note
$$\sum_j e^{ikja} = N \delta_{k,0}$$

$$[x_j, p_{j'}] = i\hbar \delta_{jj'}$$

$$\cancel{[\tilde{x}_k, \tilde{p}_{k'}] = \frac{1}{N} \sum_j \sum_{j'}}$$

$\sum x_j, p_j$

$j q$

$\tilde{p}_k e^{ikjq})\left(\frac{1}{\sqrt{N}}\sum_{k'} \tilde{p}_{k'} e^{ik'jq}\right)$

$\sum_k \sum_{k'} \hat{p}_k \hat{p}_{k'} e^{i(k+k')jq}$

$= \sum_k \sum_{k'} \hat{p}_k \hat{p}_{k'} \delta_{k',-k}$

$N \sum_k \hat{p}_k \hat{p}_{-k}$

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$\sum_j (x_j - \mu$

$= \sum_j x_j +$

$\sum_j x_j^2 =$

$=$

$=$

$\sum_j \sum_k$

$= \sum_j \sum_k$

$= \sum_j \sum_k$

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$$[\tilde{x}_k, \tilde{p}_{k'}] = \frac{1}{N}\sum_j\sum_{j'} e^{-ikja} e^{ik'j'a} [x_j, p_{j'}]$$

$$= \frac{i\hbar}{N}\sum_j e^{-i(k+k')ja}$$

$$= i\hbar\,\delta_{k,k'}$$

$$\sum_j p_j^2 = \sum_j \left(\frac{1}{\sqrt{N}}\sum_k \tilde{p}_k e^{ikja}\right)\left(\frac{1}{\sqrt{N}}\sum_{k'} \tilde{p}_{k'} e^{ik'ja}\right)$$

$$= \frac{1}{N}\sum_j\sum_k\sum_{k'} \hat{p}_k \hat{p}_{k'} e^{i(k+k')ja}$$

$$= \cancel{N}\sum_k\sum_{k'} \hat{p}_k \hat{p}_{k'}\,\delta_{k,-k}$$

$$= \cancel{N}\sum_k \hat{p}_k \hat{p}_{-k}$$

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$$\sum_j (x_j+1 -$$

$$= \sum_j x_{j+1}^2$$

$$\sum_j x_j^2 = \sum$$

$$=$$

$$=$$

$$\sum_j \sum_k$$

$$= \sum_j \sum_k$$

$$= \sum_j \sum\sum_{k'k}$$

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$$\sum_j (x_{j+1}-x_j)^2 = \frac{1}{N}\sum_j\sum_k\sum_{k'} \tilde{x}_k\hat{x}_{k'} e^{i(k+k')jq} (e^{ikq}-1)(e^{ik'q}-1)$$

$$= \sum_j x_{j+1}^2 + x_j^2 - 2x_{j+1}x_j$$

$$\sum_j x_j^2 = \sum_j\sum_k\sum_{k'} \hat{x}_k\hat{x}_{k'}\, e^{i(k+k')jq}$$

$$= \sum_k\sum_{k'} \hat{x}_k\hat{x}_{k'}\,\delta_{k',-k}$$

$$= \sum_k \hat{x}_k\hat{x}_{-k}$$

$$\sum_j\sum_k\sum_{k'} \hat{x}_k\hat{x}_{k'} e^{i(k+k')jq} + \hat{x}_k\hat{x}_{k'} e^{i(k+k')(j+1)q}$$

$$- 2\hat{x}_k\hat{x}_{k'}\, e^{ikjq}\, e^{i(k')(j+1)q}$$

$$= \sum_j\sum_k\sum_{k'} \hat{x}_k\hat{x}_{k'} e^{i(k+k')jq} + \hat{x}_k\hat{x}_{k'} e^{i(k+k')jq}\, e^{i(k+k')q}$$

$$- 2\hat{x}_k\hat{x}_{k'}\, e^{i(k+k')j}\, e^{i(kjq+k'q)}$$

$$= \sum_j\sum_k\sum_{k'} \hat{x}_k\hat{x}_{k'} e^{i(k+k')jq}\left[1+e^{i(k+k')q} - 2e^{i(kjq+k'q)}\right]$$

$$\cancel{\sum_j \sum_k \sum_{k'}} \quad \cancel{\sum_j \sum_k \tilde{x}_k - \tilde{x}_k}$$

$$\frac{1}{\sqrt{N}} \sum_j \sum_k \hat{x}_k e^{ik(j+1)a} - x_k e^{i(kja)}$$

$$= \frac{1}{\sqrt{N}} \sum_j \sum_k \tilde{x}_k \left(e^{ik(j+1)a} - e^{ikja}\right)$$

$$\sum_j (x_{j+1} - x_j)^2 = \frac{1}{N} \sum_j \sum_k \sum_{k'} \tilde{x}_k \tilde{x}_{k'} \left(e^{ik(j+1)a} - e^{ikja}\right)\left(e^{ik'(j+1)a} - e^{ik'ja}\right)$$

$$\cancel{= \sum_j \sum_k \sum_{k'} \tilde{x}_k \tilde{x}_{k'}}$$

$$\cancel{= \sum_j \sum_k \sum_{k'} \tilde{x}_k \tilde{x}_{k'} \left[e^{i(k+k')ja} \left(e^{ika} e^{ika} e^{-ikja} e^{-ik'ja} \dots\right)\right.}$$

$$\left.(\cdot - - -)\right]$$

$$\cancel{= \sum_j \sum_k \sum_{k'} \hat{x}_k \hat{x}_{k'} \left[e^{i(k+k')ja} \left(e^{ika} e^{-ik'ja} - e^{-ikja}\right)\left(e^{ik'a} e^{-ikja} - e^{-ikja}\right)\right]}$$

$= \frac{1}{N} \sum_j \sum_k$

Finally.

$= \frac{1}{N} \sum_k \hat{x}$

$= \frac{1}{N} \sum_k \tilde{x}$

$= \frac{1}{N} \sum_k \tilde{x}$

$= \frac{1}{N} \sum_k \tilde{x}$

$\Rightarrow \hat{H} =$

where

$\hat{H} =$

Call the
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$$= \frac{1}{N}\sum_{j} \sum_{k}\sum_{k'} \hat{x}_k \hat{x}_{k'} \left[e^{i(k+k')ja} \left[e^{ika} - 1 \right] \left[e^{ik'a} - 1 \right] \right]$$

Finally.

$$= \sum_{k} \hat{x}_k \hat{x}_{-k} (e^{ika} - 1)(e^{-ika} - 1)$$

$$= \sum_{k} \hat{x}_k \hat{x}_{-k} (1 + 1 - e^{ika} - e^{-ika})$$

$$= \sum_{k} \hat{x}_k \hat{x}_{-k} (2 - 2\cos ka)$$

$$= \sum_{k} \hat{x}_k \hat{x}_{-k} \, 4\sin^2 \frac{ka}{2}$$

$$\Rightarrow \hat{H} = \sum_{k} \left[\frac{1}{2m} \hat{p}_k \hat{p}_{-k} + \frac{1}{2} m \omega_k^2 \hat{x}_k \hat{x}_{-k} \right],$$

where $\omega_k^2 = (4K/m)\sin^2(ka/2)$

$$\hat{H} = \sum_{k=1}^{N} \hbar \omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right).$$

Call the modes labelled w/ k phonons. Each has integer multiples of $\hbar\omega_k$ as energy.