PART II WRITING DOWN LAGRANGIANS

Chapter 5 Continuous Systems

5.1 Lagrangians and Hamiltonians

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \cancel{= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)$$

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

$$\Rightarrow \frac{d}{dt}(p_i \dot{q}_i - L) = 0$$

$$\Rightarrow \frac{d}{dt}H = 0,$$

where $H = p_i \dot{q}_i - L$

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look at Variations of H

$$\delta H = p_i \delta \dot{q}_i + \delta p_i \dot{q}_i - \frac{\partial L}{\partial q_i}\delta q_i - \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i$$

$$= \cancel{p_i \delta \dot{q}_i} + \delta p_i \dot{q}_i - \frac{\partial L}{\partial q_i}\delta q_i - p_i \delta \dot{q}_i$$

$$= \delta p_i \dot{q}_i - \frac{\partial L}{\partial q_i}\delta q_i$$

H is function of q_i and p_i

$$\rightarrow \delta H = \frac{\partial H}{\partial q_i}\delta q_i + \frac{\partial H}{\partial p_i}\delta p_i$$

$$\cancel{\Rightarrow \frac{\partial H}{\partial q_i} = \dot{q}_i , \frac{\partial H}{\partial p_i} =}$$

$$\Rightarrow \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} , \frac{\partial H}{\partial p_i} = \dot{q}_i$$

$$= -\dot{p}_i$$

Poisson Bracket:

$$\{A, B\}_{PB} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

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Note For $F = F(q_i, p_i)$

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i$$

$$= \frac{\partial F}{\partial t} + \{F, H\}_{PB}$$

$$= \{F, H\}_{PB}$$ ~~if F doest~~

if the field is not a function of time.

$\Rightarrow$ if F cons in time $\Rightarrow \{F, H\}_{PB} = 0$.
~ to QM.

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Ex 5.1

$$F=F(\vec{q},\vec{p}) \Rightarrow (\dot{\vec{q}},\dot{\vec{p}}) = (\partial H/\partial \vec{p}, -\partial H/\partial \vec{q})$$

is tangent to $F(\vec{q},\vec{p}) =$ constant.

$$\nabla F \cdot (\dot{\vec{q}},\dot{\vec{p}}) = \left(\frac{\partial F}{\partial \vec{q}}, \frac{\partial F}{\partial \vec{p}}\right)\cdot(\dot{\vec{q}},\dot{\vec{p}})$$

$$= \frac{\partial F}{\partial q_i}\frac{\partial H}{p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$= \{F,H\}_{PB} = 0.$$

$(\dot{\vec{q}},\dot{\vec{p}})$ also tangent to $H(\vec{q},\vec{p})$-const.

$$\nabla H\cdot(\dot{\vec{q}},\dot{\vec{p}}) = \left(\frac{\partial H}{\partial \vec{q}}, \frac{\partial H}{\partial \vec{p}}\right)\cdot\left(\frac{\partial H}{\partial \vec{p}}, \frac{-\partial H}{\partial \vec{q}}\right)$$

$$= \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$= \{H,H\}_{PB}$$

$$= 0.$$

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In QM:

$$\frac{d\langle \hat{F}\rangle}{dt} = \frac{1}{i\hbar}\langle[\hat{F},\hat{H}]\rangle .$$

$$\Rightarrow \quad \{F,H\}_{PB} \xmapsto{QM} \frac{1}{i\hbar}\langle[\hat{F},\hat{H}]\rangle .$$

and $\{A,B\}_{PB} \xmapsto{QM} \frac{1}{i\hbar}\langle[\hat{A},\hat{B}]\rangle .$

Ex 5.2

$$\{q_j, p_k\}_{PB} = \frac{\partial q_j}{\partial q_i}\frac{\partial p_k}{\partial p_i} - \frac{\partial q_j}{\partial p_i}\frac{\partial p_k}{\partial q_i}$$

$$= \delta_{ij}\delta_{ki} - 0$$

$$= \delta_{jk}$$

$\sim$ to $[\hat{q}_j, \hat{p}_k] = i\hbar\delta_{jk}$.