Basic combinatory objects, uniform preorders and partial combinatory algebras

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Dedicated to the Memory of Pieter Hofstra

Remembering Pieter

Overview

Pieter Hofstra's Basic Combinatory objects

Basic Combinatory Objects

In his 2006 paper *"All realizability is relative"*[1], Pieter Hofstra introduced the notion of *basic combinatory object* (building on work with van Oosten on *ordered combinatory algebras*).

Definition

A **basic combinatory object (BCO)** is a set A equipped with a partial order $\leq$ and a set $\mathcal{F}_A$ of 'computable' partial endofunctions with down-closed domain, such that

1. $\exists i \in \mathcal{F} \, \forall a \in A \, . \, i(a) \leq a$
2. $\forall f, g \in \mathcal{F} \, \exists h \in \mathcal{F} \, \forall a \in \mathrm{dom}(g \circ f) \, . \, h(a) \leq g(f(a))$

BCOs form a locally ordered category **BCO** which admits a full embedding

$$\mathrm{fam} : \mathbf{BCO} \hookrightarrow \mathbf{IOrd}$$

into the locally ordered category $\mathbf{IOrd} = [\mathbf{Set}^{\mathrm{op}}, \mathbf{Ord}]$ of **Set-indexed preorders**, given by $\mathrm{fam}(A)(J) = (A^J, \leq)$ where

$$\varphi \leq \psi \quad :\Leftrightarrow \quad \exists f \in \mathcal{F}_A \, \forall j \in J \, . \, f(\varphi(j)) \leq \psi(j)$$

for $\varphi, \psi : J \to A$.

[1] Hofstra, Pieter JW. "All realizability is relative." *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 141. No. 2. Cambridge University Press, 2006.

Basic Combinatory objects II – finite meets

BCOs are closed under products in $[\mathbf{Set}^{\mathrm{op}}, \mathbf{Ord}]$, thus $\mathrm{fam}(A)$ is an indexed meet-semilattice iff

$$A \to A \times A \qquad \text{and} \qquad A \to 1$$

have right adjoints

$$(- \wedge -) : A \to A \times A \qquad \text{and} \qquad \top : A \to 1$$

in **BCO**. We call such BCOs **cartesian**.

Basic Combinatory objects III – existential quantification

- Say that an indexed preorder $P : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ **admits existential quantification**, if the reindexing maps $f^* : P(I) \to P(J)$ have left adjoints $\exists_f : P(J) \to P(I)$ for all $f : J \to$, subject to the **Beck–Chevalley condition.**
- Denote by $\exists$-**IOrd** the subcategory of **IOrd** on indexed preorders admitting $\exists$ and indexed monotone maps preserving $\exists$.
- Pieter Hofstra showed that
  1. the forgetful functor $\exists$-**IOrd** $\to$ **IOrd** is 2-monadic, and
  2. the induced (lax idempotent) 2-monad $D : \mathbf{IOrd} \to \mathbf{IOrd}$ restricts to **BCO**.

$$\begin{tikzcd} \mathbf{BCO} \arrow[r, dashed, "D"] \arrow[d, hook, "\mathsf{fam}"'] & \mathbf{BCO} \arrow[d, hook, "\mathsf{fam}"] \\ \mathbf{IOrd} \arrow[r, "D"] & \mathbf{IOrd} \end{tikzcd}$$

  For a BCO A, the carrier of $D(A)$ is the set of **downsets**.

  3. Furthermore, D plays well with finite meets: if P has finite meets then $D(P)$ has finite meets and moreover it satisfies the **Frobenius condition**.

Examples: BCOs from posets and (O)PCAs

- Every poset can be viewed as BCO where only the identity function is computable, which gives a full embedding
$$\mathbf{Pos} \to \mathbf{BCO}$$
- Every **PCA** $\mathcal{A}$ can be viewed as a *cartesian* BCO where the ordering is trivial and
$$\mathcal{F}_{\mathcal{A}} = \{e\cdot(-) : \mathcal{A} \rightharpoonup \mathcal{A} \mid e \in \mathcal{A}\}.$$
- More generally, **filtered ordered PCAs** $\mathcal{A}$ can be viewed as cartesian BCO with
$$\mathcal{F}_{\mathcal{A}} = \{e\cdot(-) : \mathcal{A} \rightharpoonup \mathcal{A} \mid e \in \Phi_{\mathcal{A}}\}.$$
- Pieter observed that in both cases the associated **realizability tripos** $\mathrm{rt}(\mathcal{A}) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ is given by
$$\mathrm{rt}(\mathcal{A}) \;=\; D(\mathrm{fam}(\mathcal{A})) \;=\; \mathrm{fam}(D(\mathcal{A}))$$
- In particular this means that **realizability triposes are freely generated under existential quantification!**
- This is related to the fact that realizability toposes are ex/lex-completions.

Characterizing filtered OPCAs among BCOs

Theorem (Hofstra)

TFAE for a cartesian BCO A:

1. A is (induced by) a filtered OPCA.
2. $\mathrm{fam}(D(A))$ is a tripos.
3. The fibers of $D\,\mathrm{fam}(A)$ are Heyting algebras.

Proof.

The implications $1 \Rightarrow 2 \Rightarrow 3$ are clear.

For $3 \Rightarrow 1$ let $\iota \in \mathrm{fam}(D(A))(A)$ be the function sending every a to its principal downset, and let $\varepsilon \in \mathcal{F}_A$ be a witness of the inequality

$$(\pi_1(\iota) \Rightarrow \pi_2(\iota)) \wedge \pi_1(\iota) \;\leq\; \pi_2(\iota)$$

in $\mathrm{fam}(D(A))(A \times A)$. Then the application operation of the OPCA is given by $\varepsilon \circ \wedge$.

The filter Φ_A is given by the **designated truth values**, i.e. the $a \in A$ that are equivalent to $\top$ in $\mathrm{fam}(A)(1)$.

□

Two variations : DCOs and uniform preorders

Overview

- Pieter's paper formed the starting point for my PhD thesis in which I gave characterizations of realizability triposes and toposes over PCAs.
- In hindsight, the only missing piece in the BCO-approach is that the image of **BCO** $\hookrightarrow$ **IOrd** does not have an easy characterization — if we could characterize (O)PCAs among BCOs and BCOs among indexed posets then we could characterize (O)PCAs among indexed preorders.
- In the following I introduce a sub- and a super-category of **BCO**, which *do* have simple characterizations as subcategories of **IOrd**, and explain how to adapt Pieter's techniques.

Discrete combinatory objects

Definition

A **discrete combinatory object (DCO)** is simply a BCO whose pratial order structure is trivial. We write $\mathbf{DCO} \subseteq \mathbf{BCO}$ for the full subcategory of DCOs.

Definition

Given an indexed preorder $P : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$, we call $\delta \in P(J)$ **discrete**, if it is right orthogonal to all cartesian maps over surjections in the total category $\int P$ of P.

Lemma

An indexed preorder $P : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ is equivalent to one of the form $\mathrm{fam}(A)$ for a DCO A, iff it has a discrete generic predicate.

Proof.

Given a discrete predicate $\delta \in \mathcal{A}$, define DCO structure on A by taking as computable those partial functions $f : A \rightharpoonup A$ satisfying $\iota|_{\mathrm{dom}(f)} \leq f^*(\iota)$ in $P(\mathrm{dom}(f))$. □

Characterizing $\mathsf{fam}(\mathcal{A})$

We immediately get the following.

Lemma

An indexed meet-semilattice $\mathcal{H} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}$ *is equivalent to one of the form* $\mathsf{fam}(\mathcal{A})$ *for a filtered PCA* $\mathcal{A}$ *iff it has a discrete generic predicate and* $D(\mathcal{H})$ *is a tripos. The filter is trivial iff* $\mathcal{H}(1) \simeq 1$.

- Filtered PCAs are better known as **inclusions of PCAs**, their realizability toposes are called **relative realizability toposes**.
- To be able to characterize realizability *triposes*, we have reconstruct $\mathcal{H}$ from $D(\mathcal{H})$. This is what we do next.

$\exists$-*prime predicates*

Motivation/Analogy:

- Given a poset P, the set of $D(P)$ of **downsets** in P is a complete lattice under inclusion, and is in fact the *free sup-lattice* on P.
- The **principal downsets** $\downarrow x = \{y \in P \mid y \leq x\}$ can be characterized as **completely join-prime elements** in D – an element x of a sup-lattice L is called completely join-prime if we have
$$x \leq \bigvee_{j \in J} y_j \quad \Rightarrow \quad \exists j \in J \,.\, x \leq y_j$$
for all families $(y_j)_{j \in J}$ of elements.
- L is a free join-completion iff it has **enough** completely join-prime elements, i.e. if every element is a join of completely-join-primes.

$\exists$-prime predicates

Definition

Given an indexed preorder $\mathcal{H}$ which admits existential quantification, a predicate $\pi \in \mathcal{H}(I)$ is called **$\exists$-prime** if for all functions $I \xleftarrow{u} J \xleftarrow{v} K$ and predicates $\theta \in \mathcal{H}(K)$ such that $u^*\pi \leq \exists_v \theta$, there exists a section s of v such that $u^*\pi \leq s^*\theta$.

Lemma

An indexed preorder $\mathcal{H}$ is an $\exists$-completion iff it has enough $\exists$-prime predicates, i.e. if for every predicate $\varphi \in \mathcal{H}(I)$ there exists a function $u : J \to I$ and an $\exists$-prime predicate $\pi \in \mathcal{H}(J)$ with $\varphi \cong \exists_u \pi$.
In this case, we have $\mathcal{H} \simeq D(\mathcal{P})$ where $\mathcal{P} \subseteq \mathcal{H}$ is the indexed sub-preorder on $\exists$-prime predicates.

With this we can characterize (relative) realizability triposes!

Characterizing realizability triposes

Theorem

A tripos $\mathcal{H} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ is a relative realizability tripos over an inclusion of PCAs, iff

1. *$\mathcal{H}$ has enough $\exists$-prime predicates, and*
2. *the full indexed sup-preorder $\mathcal{P} \subseteq \mathcal{H}$ on $\exists$-prime predicates is closed under finite meets and has a discrete generic predicate δ.*

- The discreteness condition on δ can be stated in $\mathcal{H}$ rather than $\mathcal{P}$, which is a slight strengthening.
- We get ordinary (non-relative) realizability if the tripos is 2-*valued*, i.e. $\mathcal{H}(1) \simeq \mathsf{Bool}$.