A refinement of Gabriel-Ulmer duality

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Category Theory 2021 Genoa

Overview

Three Parts

- Recall functorial semantics of (essentially) algebraic theories
- Clans as generalized algebraic theories
- Models in higher (homotopy) types

Part I

Algebraic Theories / Lawvere Theories

Definition

A single-sorted algebraic theory (SSAT) is a pair (Σ, E) consisting of

- a family $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$, of sets of *n*-ary **operations**
- ullet a set of **equations** ${\color{red} E}$ whose elements are pairs of open terms over ${\color{red} \Sigma}$

Definition

The syntactic category $C(\Sigma, E)$ of a SSAT is given as follows:

- 1. For each natural number $n \in \mathbb{N}$ there is an **object** [n]
- 2. **morphisms** $\sigma: [n] \to [m]$ are m-tuples of terms in n variables modulo E-provable equality
- 3. identities are lists of variables, composition is given by substitution

Proposition

Given a SSAT (Σ, E) :

- 1. $C(\Sigma, E)$ has finite products given by $[n] \times [m] = [n + m]$
- 2. Set-Mod $(\Sigma, E) \simeq \mathsf{FP}(\mathcal{C}(\Sigma, E), \mathsf{Set})$

Finite-product theories

Definition

- A fp-theory is just a small fp-category C.
- **Models** of \mathcal{C} are fp-functors $A : \mathcal{C} \to \mathsf{Set}$ (or into another fp-category).

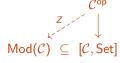
Denote the category of models by

$$\mathsf{Mod}(\mathcal{C}) \,:=\, \mathsf{FP}(\mathcal{C},\mathsf{Set}) \,\stackrel{\mathrm{full}}{\subseteq} \, [\mathcal{C},\mathsf{Set}].$$

For every object $\Gamma \in \mathcal{C}$ of an fp-theory, the co-representable functor

$$\mathcal{C}(\Gamma,-)\ :\ \mathcal{C} o \mathsf{Set}$$

is a model. Thus, the dual Yoneda embedding co-restricts to Mod(C).



Finite-limit theories

Definition

- An **fl-theory** is a small finite-limit category \mathcal{L} .
- A **model** of \mathcal{L} is a finite-limit preserving functor $A: \mathcal{L} \to \mathsf{Set}$.

Finite-limit theories are more expressive than finite-product theories – structures definable by finite-limit theories include

• categories, posets, 2-categories, monoidal categories, categories with families . . .

Again $\mathcal{L}(\Gamma, -)$ is a model for every $\Gamma \in \mathcal{L}$ and we get an embedding

$$Z: \mathcal{L}^{\mathsf{op}} o \mathsf{Mod}(\mathcal{L}) := \mathsf{FL}(\mathcal{L},\mathsf{Set}) \overset{\mathrm{full}}{\subseteq} [\mathcal{L},\mathsf{Set}].$$

Moreover, we can characterize the essential image of Z in $Mod(\mathcal{L})$.

Locally finitely presentable categories

Definition

• An object C of a cocomplete locally small category $\mathfrak X$ is called **compact**^a, if

$$\mathfrak{X}(C,-):\mathfrak{X} o\mathsf{Set}$$

preserves filtered colimits.

- A category **X** is called **locally finitely presentable**, if
 - X is locally small and cocomplete
 - the full subcategory $comp(\mathfrak{X}) \subseteq \mathfrak{X}$ on compact objects is essentially small and dense.

Theorem

- $\mathsf{Mod}(\mathcal{L})$ is locally finitely presentable for all finite-limit theories \mathcal{L} .
- The essential image of $Z: \mathcal{L}^{op} \to \mathsf{Mod}(\mathcal{L})$ comprises precisely the compact objects.

^aMore traditionally: 'finitely presentable'

Gabriel- $Ulmer\ duality^1$

Theorem

There is a bi-equivalence of 2-categories

$$\mathsf{FL} \quad \xleftarrow{\operatorname{comp}(\mathfrak{X})^{\mathsf{op}} \, \longleftrightarrow \, \mathfrak{X}} \quad \mathsf{LFP}^{\mathsf{op}}$$

where

- FL is the 2-category of small fl-categories and fl-functors
- LFP is the 2-category of locally finitely presentable categories and functors preserving small limits and filtered colimits ('forgetful functors').

¹P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Vol. 221. Lecture Notes in Math. Springer-Verlag, 1971.

Duality for finite-product theories²

There's a 'restriction' of G–U duality to finite-product theories:

$$\begin{array}{c} \mathsf{FP}_{\mathsf{cc}} \xleftarrow{\qquad \mathcal{C} \mapsto \mathsf{FP}(\mathcal{C},\mathsf{Set})} & \mathsf{ALG}^{\mathsf{op}} \\ \mathsf{F} \swarrow \mathsf{J} & & & & \downarrow \mathsf{J} \\ \mathsf{FL} \xleftarrow{\qquad \mathcal{L} \mapsto \mathsf{FL}(\mathcal{L},\mathsf{Set})} & \mathsf{LFP}^{\mathsf{op}} \end{array}$$

- FP_{cc} is the 2-category of Cauchy-complete finite-product categories
- ALG is the 2-category of algebraic categories and algebraic functors
 - An algebraic category is an Ifp category which is Barr exact and where the compact (regular) projective objects are dense
 - An algebraic functor is a functor that preserves small limits, filtered colimits, and regular epimorphisms.
- There's also a formulation in terms of sifted colimits, but we don't need it.

²J. Adámek, J. Rosický, and E.M. Vitale. *Algebraic theories: a categorical introduction to general algebra*. Vol. 184. Cambridge University Press, 2010.

Part II

Toward clans

- Finite-limit theories have a nice duality theory but seem far from syntax
- Syntactic counterparts are given by
 - Freyd's essentially algebraic theories³
 - Cartmell's generalized algebraic theories⁴ (or 'dependent algebraic theories')
 - Palmgren and Vickers' quasi-equational theories⁵
 - and probably others
- Clans can be viewed as a categorical representation of generalized algebraic theories
- They're as expressive as fl-theories, but 'finer', i.e. closer to syntax

³P. Freyd. "Aspects of topoi". In: Bulletin of the Australian Mathematical Society 7.1 (1972).

⁴J. Cartmell. "Generalised algebraic theories and contextual categories". In: *Annals of Pure and Applied Logic* 32 (1986).

⁵E. Palmgren and S. J. Vickers. "Partial horn logic and Cartesian categories". In: *Annals of Pure and Applied Logic* 145.3 (2007).

Definition

A **clan** is a small category \mathcal{T} with terminal object 1, equipped with a class $\mathcal{T}_{\dagger} \subseteq \operatorname{mor}(\mathcal{T})$ of morphisms – called **display maps** and written \rightarrow – such that

- 1. pullbacks of display maps along all maps exist and are display maps q_{\downarrow}
- $\Delta^{+} \xrightarrow{s^{+}} \Gamma^{+}$ $\downarrow p$ $\Lambda \xrightarrow{s} \Gamma$

- 2. display maps are closed under composition, and
- 3. isomorphisms and terminal projections $\Gamma \to 1$ are display maps.
 - Definition due to Taylor⁶, name due to Joyal⁷.
- Relation to semantics of dependent type theory: display maps represent type families.
- Observation: clans have finite products (as pullbacks over 1).

⁶P. Taylor. "Recursive domains, indexed category theory and polymorphism". PhD thesis. University of Cambridge, 1987. § 4.3.2.

⁷A. Joyal. "Notes on clans and tribes". In: arXiv preprint arXiv:1710.10238 (2017).

Examples

- Finite-product categories \mathcal{C} can be viewed as clans with $\mathcal{C}_{\dagger} = \{\text{product projections}\}$
- Finite-limit categories \mathcal{L} can be viewed as clans with $\mathcal{L}_{\dagger} = \operatorname{mor}(\mathcal{L})$

We call such clans **fp-clans**, and **fl-clans**, respectively.

- The syntactic category of every Cartmell-style generalized algebraic theory is a clan.
- Clan for categories:

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\mathcal{K} = \{\text{categories free on finite graphs}\}^{\text{op}} \subseteq \text{Cat}^{\text{op}}
\mathcal{K}_{\dagger} = \{\text{functors induced by graph inclusions}\}^{\text{op}}
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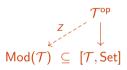
 \mathcal{K} can be viewed as syntactic category of a generalized algebraic theory of categories with a sort O of objects, and a dependent sort $x,y:O \vdash M(x,y)$ of morphisms – vertices of a finite graph are object variables and edges are morphism variables in a context. Graph inclusions are dual to context extensions.

Models

Definition

A **model** of a clan \mathcal{T} is a functor $A: \mathcal{T} \to \mathsf{Set}$ which preserves 1 and pullbacks of display-maps.

- The category $Mod(\mathcal{T}) \subseteq [\mathcal{T}, Set]$ of models is Ifp and contains \mathcal{T}^{op} .
- For fp-clans $(\mathcal{C}, \mathcal{C}_{\dagger})$ we have $\mathsf{Mod}(\mathcal{C}, \mathcal{C}_{\dagger}) = \mathsf{FP}(\mathcal{C}, \mathsf{Set})$.
- For fl-clans $(\mathcal{L}, \mathcal{L}_{\dagger})$ we have $\mathsf{Mod}(\mathcal{L}, \mathcal{L}_{\dagger}) = \mathsf{FL}(\mathcal{L}, \mathsf{Set})$.
- $\mathsf{Mod}(\mathcal{K},\mathcal{K}_\dagger) = \mathsf{Cat}$.



Observation

The same category of models may be represented by different clans. For example, SSATs can be represented by fp-clans as well as fl-clans.

The weak factorization system

- We would like a duality between clans and their categories of models.
- Since the same Ifp category can be represented by different clans, we cannot hope to reconstruct the clan from the models alone.
- Solution: equip the models with additional structure in form of a weak factorization system.

Definition

Let \mathcal{T} be a clan. Define wfs $(\mathcal{E}, \mathcal{F})$ on $\mathsf{Mod}(\mathcal{T})$ by

- $\mathcal{F} := \mathsf{RLP}(\{Z(p) \mid p \in \mathcal{T}_{\dagger}\})$ class of **full maps**
- $\mathcal{E} := \text{Cell}(\{Z(p) \mid p \in \mathcal{T}_{\dagger}\}) = \text{LLP}(\mathcal{F})$ class of extensions

I.e. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by the image of \mathcal{T}_{t} under $Z : \mathcal{T}^{op} \to \mathsf{Mod}(\mathcal{T})$.

- Call $A \in \mathsf{Mod}(\mathcal{T})$ a 0-extension, if $(0 \to A) \in \mathcal{E}$
- The same weak factorization system was also introduced by Simon Henry in a video seminar in January 2020⁸.

⁸Simon Henry, The language of a model category, HoTTEST seminar, 2020, https://youtu.be/7_X0qbSX1fk

Full maps

• $f: A \to B$ in $Mod(\mathcal{T})$ is full iff has lhp with respect to all Z(p) for display maps $p: \Delta \to \Gamma$.

$$\begin{array}{cccc} \mathcal{T}(\Gamma,-) & \longrightarrow & A & & A(\Delta) & \xrightarrow{f_{\Delta}} & B(\Delta) \\ Z(p) = \mathcal{T}(p,-) \downarrow & & \downarrow f & & A(p) \downarrow & & \downarrow B(p) \\ \mathcal{T}(\Delta,-) & \longrightarrow & B & & A(\Gamma) & \xrightarrow{f_{\Gamma}} & B(\Gamma) \end{array}$$

- This is equivalent to display-naturality-squares being weak pullbacks.
- Considering $p: \Delta \to 1$ we see that full maps are surjective and hence regular epis.

$$\begin{array}{ccccc} A(\Delta) & \xrightarrow{f_{\Delta}} & B(\Delta) & & & A(\Delta) & \xrightarrow{f_{\Delta}} & B(\Delta) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 & & A(\Delta) \times A(\Delta) & \xrightarrow{f_{\Delta} \times f_{\Delta}} & B(\Delta) \times B(\Delta) \end{array}$$

- For fl-clans, only isos are full (consider naturality square for diagonal $\Delta \to \Delta \times \Delta$)
- For fp-clans, the full maps are *precisely* the regular epis, and therefore

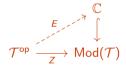
0-extension = projective object

Reconstructing the clan

Definition

Given a clan \mathcal{T} , let $\mathbb{C} \subseteq \mathsf{Mod}(\mathcal{T})$ be the full subcategory on **compact** 0-extensions.

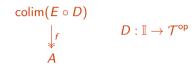
Observation: $Z: \mathcal{T}^{op} \to \mathsf{Mod}(\mathcal{T})$ factors through \mathbb{C} .



Theorem

 \mathbb{C}^{op} is a coclan (with extensions as display maps), and E exhibits \mathbb{C} as Cauchy-completion of \mathcal{T}^{op} .

Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in \mathcal{T} .

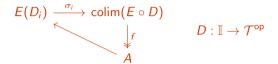


Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in \mathcal{T} .

$$\begin{array}{ccc} \operatorname{colim}(E \circ D) \\
 & s \uparrow \downarrow_f \\
 & A \end{array} \qquad D : \mathbb{I} \to \mathcal{T}^{\operatorname{op}}$$

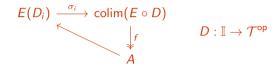
• f splits since A is a 0-extension

Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in \mathcal{T} .



- f splits since A is a 0-extension
- s factors through one of the colimit-injections, since A is compact

Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in \mathcal{T} .



- f splits since A is a 0-extension
- s factors through one of the colimit-injections, since A is compact
- D exists by the fat small object argument:
 - M. Makkai, J. Rosicky, and L. Vokrinek. "On a fat small object argument". In: Advances in Mathematics 254 (2014)

Clan-algebraic categories

Definition

A clan-algebraic category is an Ifp category \mathfrak{A} with an wfs $(\mathcal{E}, \mathcal{F})$ that arises as category of models of a clan.

With this definition we get a contravariant bi-equivalence

$$\mathsf{Clan}_\mathsf{cc} \quad \xleftarrow{\operatorname{comp}(\mathfrak{X})^\mathsf{op} \, \hookleftarrow \, \mathfrak{X}} \quad \mathsf{cAlg}^\mathsf{op}$$

between

- the 2-category Clancc of Cauchy-complete clans and functors preserving 1, display maps, and pullbacks of display maps, and
- the 2-category cAlg of clan-algebraic categories and functors preserving small limits, filtered colimits, and full maps.

Can we characterize clan-algebraic categories more abstractly?

Characterization of clan-algebraic categories

Theorem

A Ifp category \mathfrak{A} with wfs $(\mathcal{E}, \mathcal{F})$ is clan-algebraic, if

- 1. $\mathbb{C} = \{\text{compact 0-extensions}\}\$ is dense in \mathfrak{A} ,
- 2. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by $\mathcal{E} \cap \operatorname{mor}(\mathbb{C})$, and
- 3. for $C \in \mathbb{C}$ and $A \in \mathsf{Mod}(\mathbb{C}^\mathsf{op})$, the functor $\mathfrak{A}(C,-) : \mathfrak{A} \to \mathsf{Set}$ preserves the colimit of the diagram $\int A \to \mathbb{C} \to \mathfrak{A}$.
- Condition 3 is not very nice
- Can we find an 'exactness condition' similar to the one given by Adámek, Rosický and Vitale for algebraic categories?
- ... there is at least a *necessary* exactness condition

Quotients of componentwise-full equivalence relations

- In algebraic categories, all equivalence relations have effective quotients (they are 'Barr exact')
- This can't be true for clan algebraic categories in general. However, we have:

Lemma

For any clan \mathcal{T} , $\mathsf{Mod}(\mathcal{T})$ has full and effective quotients of **componentwise-full equivalence** relations.

Conjecture

Condition 3 of the theorem is implied by \mathfrak{A} having full and effective quotients of componentwise-full equivalence relations.

Part III

Models in higher types

Let \mathcal{S} be the ∞ -topos of spaces/types.

Let $\mathcal{C}_{\mathsf{Mon}}$ be the finite-product theory of monoids, and let $\mathcal{L}_{\mathsf{Mon}}$ be the finite-limit theory of monoids. Then

$$\mathsf{FP}(\mathcal{C}_\mathsf{Mon},\mathsf{Set})\simeq \mathsf{FL}(\mathcal{L}_\mathsf{Mon},\mathsf{Set})$$

but $FP(\mathcal{C}_{Mon}, \mathcal{S})$ and $FL(\mathcal{L}_{Mon}, \mathcal{S})$ are different:

- $FL(\mathcal{L}_{Mon}, \mathcal{S})$ is just the category of monoids
- $\mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ is the ∞ -category ' A_{∞} -algebras', i.e. homotopy-coherent monoids.

Moral

By being 'slimmer', finite-product theories leave room for higher coherences when interpreted in higher types.

This phenomenon has been discussed under the name 'animation' in:

• K. Cesnavicius and P. Scholze. "Purity for flat cohomology". In: arXiv preprint arXiv:1912.10932 (2019)

Four clans for categories

Cat admits several clan-algebraic weak factorization systems:

```
• (\mathcal{E}_1, \mathcal{F}_1) is cofib. generated by \{(0 \to 1), (2 \to 2) \}

• (\mathcal{E}_2, \mathcal{F}_2) is cofib. generated by \{(0 \to 1), (2 \to 2), (2 \to 1)\}

• (\mathcal{E}_3, \mathcal{F}_3) is cofib. generated by \{(0 \to 1), (2 \to 2), (\mathbb{P} \to 2) \}

• (\mathcal{E}_4, \mathcal{F}_4) is cofib. generated by \{(0 \to 1), (2 \to 2), (\mathbb{P} \to 2), (2 \to 1)\}

where \mathbb{P} = (\bullet \Rightarrow \bullet).
```

The right classes are:

```
\mathcal{F}_1 = \{ \text{full and surjective-on-objects functors} \}
\mathcal{F}_2 = \{ \text{full and bijective-on-objects functors} \}
\mathcal{F}_3 = \{ \text{fully faithful and surjective-on-objects functors} \}
\mathcal{F}_4 = \{ \text{isos} \}
```

Note that \mathcal{F}_3 is the class of trivial fibrations for the canonical model structure on Cat.

Four clans for categories

These correspond to the following clans:

Models in higher types:

```
\infty-Mod(\mathcal{T}_1) = \{ \text{Segal spaces} \}
\infty-Mod(\mathcal{T}_2) = \{ \text{Segal categories} \}
\infty-Mod(\mathcal{T}_3) = \{ \text{pre-categories} \}
\infty-Mod(\mathcal{T}_4) = \{ \text{discrete 1-categories} \}
```

Thanks for your attention!