$Characterizing\ clan-algebraic\ categories$

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Overview

Context

- In talks at HoTT/UF 2020 and at CT 2021 I presented a conjecture concerning categories of models of a clan.
- In this talk I will give/outline a proof of this conjecture.

Structure

• Part I: Recall Functorial semantics

• Part II : Duality for clans

Part III: Models in higher types

• Part IV : Syntax

• Part V : Monadicity?

Part I – Classical Functorial Semantics

Algebraic Theories

Definition

A single-sorted algebraic theory (SSAT) is a pair (Σ, E) consisting of

- a family $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$, of sets of *n*-ary **operations**
- a set of equations E whose elements are pairs of open terms over Σ

Definition

The syntactic category $C(\Sigma, E)$ of a SSAT is given as follows:

- 1. For each natural number $n \in \mathbb{N}$ there is an **object** [n]
- 2. morphisms $\sigma: [n] \to [m]$ are m-tuples of terms in n variables modulo E-provable equality
- 3. identities are lists of variables, composition is given by substitution

Proposition

Given a SSAT (Σ, E) :

- 1. $\mathcal{C}(\Sigma, E)$ has finite products given by $[n] \times [m] = [n+m]$
- 2. Set-Mod(Σ , E) \simeq FP($\mathcal{C}(\Sigma, E)$, Set)

Finite-product theories

Definition

- A FP-theory is just a small FP-category C.
- **Models** of \mathcal{C} are FP-functors $A : \mathcal{C} \to \mathbf{Set}$ (or into another FP-category).

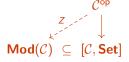
Denote the category of models by

$$\mathsf{Mod}(\mathcal{C}) := \mathsf{FP}(\mathcal{C},\mathsf{Set}) \overset{\mathrm{full}}{\subseteq} [\mathcal{C},\mathsf{Set}].$$

For every object $\Gamma \in \mathcal{C}$ of an FP-theory, the co-representable functor

$$\mathcal{C}(\Gamma,-)\ :\ \mathcal{C} o \mathsf{Set}$$

is a model. Thus, the dual Yoneda embedding co-restricts to Mod(C).



Finite-limit theories

Definition

- A FL-theory is a small finite-limit category ∠.
- A **model** of \mathcal{L} is a finite-limit preserving functor $A: \mathcal{L} \to \mathbf{Set}$.

FL-theories are more expressive than FP-theories – structures definable by finite-limit theories include

• categories, posets, 2-categories, monoidal categories, categories with families . . .

Again $\mathcal{L}(\Gamma, -)$ is a model for every $\Gamma \in \mathcal{L}$ and we get an embedding

$$Z \,:\, \mathcal{L}^{\mathsf{op}} \, o\, \mathsf{Mod}(\mathcal{L}) := \mathsf{FL}(\mathcal{L},\mathsf{Set}) \overset{\mathrm{full}}{\subseteq} [\mathcal{L},\mathsf{Set}].$$

Moreover, we can characterize the essential image of Z in $Mod(\mathcal{L})$.

Locally finitely presentable categories

Definition

• An object C of a cocomplete locally small category \mathfrak{X} is called **compact**^a, if

$$\mathfrak{X}(\mathcal{C},-):\mathfrak{X} o \mathbf{Set}$$

preserves filtered colimits.

- A category \mathfrak{X} is called **locally finitely presentable**, if
 - X is locally small and cocomplete
 - the full subcategory $comp(\mathfrak{X}) \subseteq \mathfrak{X}$ on compact objects is essentially small and dense.

Theorem

- $\mathsf{Mod}(\mathcal{L})$ is locally finitely presentable for all finite-limit theories \mathcal{L} .
- The essential image of $Z: \mathcal{L}^{op} \to \mathsf{Mod}(\mathcal{L})$ comprises precisely the compact objects.

^aMore traditionally: 'finitely presentable'

Gabriel- $Ulmer\ duality^1$

Theorem

There is a bi-equivalence of 2-categories

$$\mathsf{FL} \quad \xleftarrow{\operatorname{comp}(\mathfrak{X})^{\mathsf{op}} \, \longleftrightarrow \, \mathfrak{X}} \qquad \mathsf{LFP}^{\mathsf{op}}$$

where

- FL is the 2-category of small FL-categories and FL-functors
- LFP is the 2-category of locally finitely presentable categories and functors preserving small limits and filtered colimits ('forgetful functors').

¹P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Springer-Verlag, 1971.

Duality for finite-product theories²

There's a 'restriction' of G–U duality to finite-product theories:

$$\begin{array}{c} \textbf{FP}_{cc} \xleftarrow{\hspace{0.2cm} \mathcal{C} \mapsto \textbf{FP}(\mathcal{C}, \textbf{Set})} & \textbf{ALG}^{op} \\ F \swarrow \swarrow \mathcal{U} & & \swarrow \mathcal{L} \mapsto \textbf{FL}(\mathcal{L}, \textbf{Set}) \\ \textbf{FL} & & & \swarrow \mathcal{L} \mapsto \textbf{FL}(\mathcal{L}, \textbf{Set}) \\ & & & & \downarrow \mathcal{L} \mapsto \textbf{FP}^{op} \end{array}$$

- FP_{cc} is the 2-category of Cauchy-complete finite-product categories
- ALG is the 2-category of algebraic categories and algebraic functors
 - An algebraic category is an I.f.p. category which is Barr-exact and where the compact (regular) projective objects are dense
 - An algebraic functor is a functor that preserves small limits, filtered colimits, and regular epimorphisms.
- There's also a formulation in terms of sifted colimits, but we don't need it.

²J. Adámek, J. Rosický, and E.M. Vitale. *Algebraic theories: a categorical introduction to general algebra*. Cambridge University Press, 2010.

Part II – Gabriel Ulmer Duality for Clans

Toward clans

- Finite-limit theories have a nice duality theory but seem far from syntax
- Syntactic counterparts are given by
 - Freyd's essentially algebraic theories³
 - Cartmell's **generalized algebraic theories**⁴ (or 'dependent algebraic theories')
 - Johnstone's cartesian theories⁵
 - Palmgren and Vickers' quasi-equational theories⁶
 - and probably others
- Clans can be viewed as a categorical representation of generalized algebraic theories
- They're as expressive as FL-theories, but 'finer', i.e. closer to syntax

³P. Freyd. "Aspects of topoi". In: Bulletin of the Australian Mathematical Society (1972).

⁴J. Cartmell. "Generalised algebraic theories and contextual categories". In: *Annals of Pure and Applied Logic* (1986).

⁵P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 2.* Oxford: Oxford University Press, 2002.

⁶E. Palmgren and S. J. Vickers. "Partial horn logic and Cartesian categories". In: *Annals of Pure and Applied Logic* (2007).

Definition

A **clan** is a small category \mathcal{T} with terminal object 1, equipped with a class $\mathcal{T}_{\dagger} \subseteq \operatorname{mor}(\mathcal{T})$ of morphisms – called **display maps** and written \rightarrow – such that

- 1. pullbacks of display maps along all maps exist and are display maps $\begin{array}{ccc} \Delta^+ & \xrightarrow{s^+} & \Gamma^+ \\ q^{\downarrow}_{\downarrow} & \neg & \downarrow_p \\ & & \Delta & \xrightarrow{s} & \Gamma \\ \end{array}$
- 2. display maps are closed under composition, and
- 3. isomorphisms and terminal projections $\Gamma \to 1$ are display maps.
- Definition due to Taylor⁷, name due to Joyal⁸ ('a clan is a collection of families')
- Relation to semantics of dependent type theory: display maps represent type families.
- Observation: clans have finite products (as pullbacks over 1).

⁷P. Taylor. "Recursive domains, indexed category theory and polymorphism". PhD thesis. University of Cambridge, 1987, § 4.3.2.

⁸A. Joyal. "Notes on clans and tribes". In: arXiv preprint arXiv:1710.10238 (2017).

Examples

- Finite-product categories $\mathcal C$ can be viewed as clans with $\mathcal C_\dagger = \{ \text{product projections} \}$
- Finite-limit categories \mathcal{L} can be viewed as clans with $\mathcal{L}_{\dagger} = \operatorname{mor}(\mathcal{L})$

We call such clans **FP-clans**, and **FL-clans**, respectively.

- The syntactic category of every Cartmell-style generalized algebraic theory is a clan.
- Clan for categories:

```
\mathcal{K} = \{\text{categories free on finite graphs}\}^{\text{op}} \subseteq \text{Cat}^{\text{op}}
\mathcal{K}_{\dagger} = \{\text{functors induced by graph inclusions}\}^{\text{op}}
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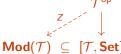
 \mathcal{K} can be viewed as syntactic category of a generalized algebraic theory of categories with a sort \mathcal{O} of objects, and a dependent sort $x,y:\mathcal{O} \vdash M(x,y)$ of morphisms – vertices of a finite graph are object variables and edges are morphism variables in a context. Graph inclusions are dual to context extensions.

Models

Definition

A **model** of a clan \mathcal{T} is a functor $A: \mathcal{T} \to \mathbf{Set}$ which preserves 1 and pullbacks of display-maps.

- The category $Mod(\mathcal{T}) \subseteq [\mathcal{T}, \mathbf{Set}]$ of models is l.f.p. and contains \mathcal{T}^{op} .
- For FP-clans $(\mathcal{C}, \mathcal{C}_{\dagger})$ we have $\mathbf{Mod}(\mathcal{C}, \mathcal{C}_{\dagger}) = \mathbf{FP}(\mathcal{C}, \mathbf{Set})$.
- For FL-clans $(\mathcal{L}, \mathcal{L}_{\dagger})$ we have $\mathbf{Mod}(\mathcal{L}, \mathcal{L}_{\dagger}) = \mathbf{FL}(\mathcal{L}, \mathbf{Set})$.
- $\mathsf{Mod}(\mathcal{K}, \mathcal{K}_\dagger) = \mathsf{Cat}$.



Observation

The same category of models may be represented by different clans.

For example, SSATs can be represented by FP-clans as well as FL-clans.

The weak factorization system

- Would like duality between clans and their categories of models.
- Since the same l.f.p. category can be represented by different clans, we cannot hope to reconstruct the clan from the models alone.
- Solution: equip the models with additional structure in form of a **weak factorization system**.

Definition

Let \mathcal{T} be a clan. Define w.f.s. $(\mathcal{E}, \mathcal{F})$ on $\mathsf{Mod}(\mathcal{T})$ by

- $\mathcal{F} := \mathsf{RLP}(\{Z(p) \mid p \in \mathcal{T}_{\dagger}\})$ class of **full maps**
- $\mathcal{E} := \mathsf{LLP}(\mathcal{F})$ class of **extensions**

I.e. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by the image of \mathcal{T}_{\dagger} under $Z : \mathcal{T}^{op} \to \mathbf{Mod}(\mathcal{T})$.

- Call $A \in \mathbf{Mod}(\mathcal{T})$ a 0-extension, if $(0 \to A) \in \mathcal{E}$
- E.g. corepresentables $Z(\Gamma)$ are 0-extensions since terminal projections $\Gamma \to 1$ are display maps.
- The same weak factorization system was also introduced by S. Henry in a HoTTEST talk⁹, see also¹⁰.

⁹S. Henry, *The language of a model category*, HoTTEST seminar, Jan. 2020, https://youtu.be/7_X0qbSX1fk ¹⁰S. Henry. "Algebraic models of homotopy types and the homotopy hypothesis". In: arXiv preprint arXiv:1609.04622 (2016).

Full maps

• $f: A \to B$ in $\operatorname{\mathsf{Mod}}(\mathcal{T})$ is full iff it has the RLP with respect to all Z(p) for display maps $p: \Delta \to \Gamma$.

$$\begin{array}{cccc}
\mathcal{T}(\Gamma,-) & \longrightarrow & A & & A(\Delta) & \xrightarrow{f_{\Delta}} & B(\Delta) \\
Z(p)=\mathcal{T}(p,-)\downarrow & & \downarrow f & & A(p)\downarrow & & \downarrow B(p) \\
\mathcal{T}(\Delta,-) & \longrightarrow & B & & A(\Gamma) & \xrightarrow{f_{\Gamma}} & B(\Gamma)
\end{array}$$

- This is equivalent to display-naturality-squares being weak pullbacks.
- Considering $p: \Delta \to 1$ we see that full maps are surjective and hence regular epis.

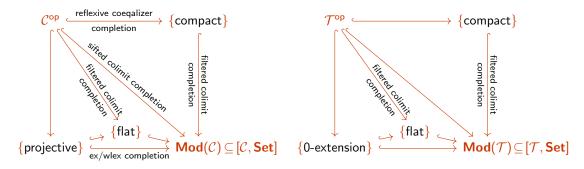
$$\begin{array}{ccccc} A(\Delta) & \xrightarrow{f_{\Delta}} & B(\Delta) & & & A(\Delta) & \xrightarrow{f_{\Delta}} & B(\Delta) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 & & A(\Delta) \times A(\Delta) & \xrightarrow{f_{\Delta} \times f_{\Delta}} & B(\Delta) \times B(\Delta) \end{array}$$

- For FL-clans, only isos are full (consider naturality square for diagonal $\Delta \to \Delta \times \Delta$)
- For FP-clans we have

full map = regular epimorphism extension = coproduct inclusion $A \hookrightarrow P + A$ with P projective O-extension = projective object

The fat small object argument

Motivation: subcategories of models for FP-theory $\mathcal C$ and clan $\mathcal T$.



- Flat algebras are filtered colimits of corepresentables, computed freely in the functor categories.
- For SSATs we have $\{projective\} \subseteq \{flat\}$ since
 - arbitrary free objects are filtered colimits of free objects over finite sets
 - projective objects are retracts of free objects
- In the general clan case, $\{0\text{-extension}\}\subseteq \{\text{flat}\}\$ by the **fat small object argument**¹¹.

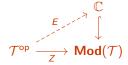
¹¹ M. Makkai, J. Rosicky, and L. Vokrinek. "On a fat small object argument". In: Advances in Mathematics (2014).

Reconstructing the clan

Definition

Given a clan \mathcal{T} , let $\mathbb{C} \subseteq \mathsf{Mod}(\mathcal{T})$ be the full subcategory on **compact** 0-extensions.

• $Z : \mathcal{T}^{op} \to \mathbf{Mod}(\mathcal{T})$ factors through $\mathbb C$ since corepresentables $Z(\Gamma)$ are compact and 0-extensions.



- $0 \in \mathbb{C}$ and if $\begin{array}{c} C \longrightarrow D \\ \downarrow^e & \Gamma \downarrow \\ E \longrightarrow F \end{array}$ is a pushout with $F \in \mathbb{C}$ and $e \in \mathcal{E}$ then $F \in \mathbb{C}$.

Reconstructing the clan

Theorem

The full inclusion $E: \mathcal{T}^{op} \hookrightarrow \mathbb{C}$ exhibits \mathbb{C} as *Cauchy-completion* of \mathcal{T}^{op} , i.e. every compact 0-extension is a retract of a corepresentable.

Proof.

- Let $C \in \mathbb{C}$.
- Since 0-extensions are flat, $\int C$ is filtered, thus C is a filtered colimit of corepresentables.
- Since *C* is compact, id_{*C*} factors through a colimit inclusion map.



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Clan-algebraic categories

Definition

A clan-algebraic category is a category \mathfrak{X} with a w.f.s. $(\mathcal{E}, \mathcal{F})$ that arises as category of models of a clan.

With this definition we get a contravariant bi-equivalence

$$\begin{array}{ccc} \text{Clan}_{\text{cc}} & \xleftarrow{& \operatorname{comp}(\mathfrak{X})^{\text{op}} \ \leftarrow \ \mathfrak{X}} & \text{cAlg}^{\text{op}} \end{array}$$

between

- the 2-category Clan_{cc} of Cauchy-complete clans and functors preserving 1, display maps, and pullbacks of display maps, and
- the 2-category cAlg of clan-algebraic categories and functors preserving small limits, filtered colimits, and full maps.

Can we characterize clan-algebraic categories more abstractly?

Characterizing clan-algebraic categories

Assume \mathfrak{X} is clan-algebraic with w.f.s. $(\mathcal{E}, \mathcal{F})$. Then

- 1. \mathfrak{X} is cocomplete,
- 2. \mathfrak{X} has a small dense family of compact 0-extensions, and
- 3. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between compact 0-extensions.

Now assume we have a category \mathfrak{X} with w.f.s. $(\mathcal{E}, \mathcal{F})$ satisfying 1–3.

Then the subcategory $\mathbb{C} \subseteq \mathfrak{X}$ of compact 0-extensions is a coclan.

We get a nerve/realization adjunction

$$\mathbb{C} \xrightarrow{J} \mathfrak{X}$$

$$\downarrow \qquad \qquad L(A) = \operatorname{colim}(\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})$$

$$N(X) = \mathfrak{X}(J(-), X)$$

$$\mathsf{Mod}(\mathbb{C}^{\operatorname{op}})$$

However, this adjunction is not an equivalence in general:

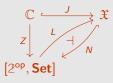
Characterizing clan-algebraic categories

Counter example

Consider

- $\mathfrak{X} \subseteq [2^{op}, \mathbf{Set}]$ full subcategory on injections
- $(\mathcal{E}, \mathcal{F})$ w.f.s. on \mathfrak{X} cofib. generated by $\{(0 \to Y0), (0 \to Y1)\}$

Then $Mod(\{compact \ 0\text{-extensions}\}^{op}) \simeq [2^{op}, \mathbf{Set}]$ and N is the subcategory inclusion.



Conclusion: We're missing an 'exactness condition' analogous to 'Barr-exactness' in the characterization of algebraic categories!

Quotients of componentwise-full equivalence relations

- Recall that a FL-category ∠ is called Barr-exact, if all equivalence relations in ∠ have stable effective quotients.
- This can't be the case for clan algebraic categories in general. However, we have:

Lemma

For any clan \mathcal{T} , $\mathsf{Mod}(\mathcal{T})$ has full and effective quotients of componentwise-full equivalence relations.

Proof.

Given equivalence relation $r: R \rightarrow A \times A$ with $r_0, r_1: R \rightarrow A$ full, show that component-wise quotient is a model again.

Characterizing clan-algebraic categories

Definition

An **adequate category** is a category \mathfrak{X} with a with a w.f.s. $(\mathcal{E}, \mathcal{F})$ (whose maps we call extensions and full, respectively), s.th.

- 1. \mathfrak{X} is cocomplete,
- 2. \mathfrak{X} has a small dense family of compact 0-extensions (in particular \mathfrak{X} is l.f.p.),
- 3. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between compact 0-extensions, and
- 4. \mathfrak{X} has full and effective quotients of componentwise-full equivalence relations.

Lemma

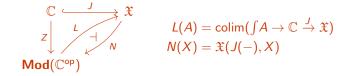
Assume \mathfrak{X} is adequate and $F: \mathfrak{X} \to \mathbf{Set}$ preserves finite limits and sends full maps to surjections. Then F preserves quotients of componentwise-full equivalence relations.

Proof.

Let $R \xrightarrow[r_1]{r_0} A \xrightarrow{f} B$ be a **full exact sequence** in \mathfrak{X} , i.e. all arrows are full, f is the coequalizer of r_0 , r_1 , and r_0 , r_1 is the kernel pair of f. Then Ff is a surjection with kernel pair Fr_0 , Fr_1 . But surjections are always coequalizers of their kernel pair.

Idea of proof

- Assume that X is adequate.
- To show that it is clan-algebraic, we want to show that its nerve/realization adjunction



is an equivalence.

- ullet By density the right adjoint ${\it N}$ is fully faithful, i.e. the counit is an isomorphism.
- It remains to show that the unit of the adjunction is an isomorphism, i.e.

$$A(C) \xrightarrow{\cong} \mathfrak{X}(C, \operatorname{colim}(\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})).$$

for all $A \in \mathbf{Mod}(\mathbb{C}^{\mathrm{op}})$ and $C \in \mathbb{C}$.

- We know that $\mathcal{X}(C,-)$ preserves filtered colimits and quotients of componentwise-full equivalence relations, so we'd like to decompose $\operatorname{colim}(\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})$ in terms of these constructions.
- This is essentially what we're doing in the following.

Jointly full cones

- Let $D: \mathcal{I} \to \mathfrak{X}$ be a diagram in an adequate category.
- A cone (A, ϕ) over D is called **jointly full**, if for every cone (C, γ) , extension $e : B \to C$ and map $g : B \to A$ constituting a cone morphism $g : (B, \gamma \circ e) \to (A, \phi)$, there exists a map $h : C \to A$ such that

$$B \xrightarrow{g} A$$

$$e \downarrow \xrightarrow{h} \stackrel{\gamma}{\longrightarrow} D_{i}$$

commutes for all $i \in \mathcal{I}$.

• **Observation:** The cone (A, ϕ) is jointly full iff the canonical map to the limit is full.

Definition

A **nice diagram** in an adequate category \mathfrak{X} is a truncated simplicial diagram

$$A_2 \stackrel{\overline{\downarrow} d_0}{\underset{d_1}{\longleftarrow} s_0} \xrightarrow{s_0} A_1 \stackrel{\overline{\downarrow} d_0}{\underset{d_1}{\longleftarrow} s_0} \xrightarrow{s_0} A_0$$

where

- 1. A_0 , A_1 , and A_2 are 0-extensions,
- 2. the maps $d_0, d_1 : A_1 \rightarrow A_0$ are full,
- 3. in the square $A_1 \longrightarrow A_1 \longrightarrow A_1$ $A_1 \longrightarrow A_0$ the span constitutes a jointly full diagram over the cospan,
- 4. there exists a symmetry map $A_1 \xrightarrow{d_1} A_0 \\ A_0 \xleftarrow{d_1} A_1$ making the triangles commute, and
- 5. there exists a 0-extension \tilde{A} and full maps $f,g:\tilde{A} \to A_1$ constituting a jointly full cone over the diagram

$$\begin{array}{cccc}
A_1 & & A_1 \\
d_0 \downarrow & & \downarrow d_1 \\
A_0 & & A_0
\end{array}$$

Nice diagrams

Lemma

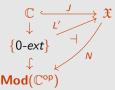
For any nice diagram, the pairing $A_1 \xrightarrow{\langle d_0, d_1 \rangle} A_0 \times A_0$ admits a decomposition $A_1 \twoheadrightarrow R \xrightarrow{\langle r_0, r_1 \rangle} A_0 \times A_0$ into a full map and a monomorphism, and $\langle r_0, r_1 \rangle$ is a componentwise-full equivalence relation.

Lemma

Assume \mathfrak{X} is adequate and $F: \mathfrak{X} \to \mathbf{Set}$ preserves finite limits and sends full maps to surjections. Then for every nice diagram, F preserves coequalizers of the arrows $d_0, d_1: A_1 \to A_0$.

Lemma

The restriction L' of L in the nerve/realization adjunction



to 0-extensions is fully faithful and preserves full maps and nice diagrams.

Nice diagrams

Lemma

For every object A of an adequate category $\mathfrak X$ there exists a nice diagram

$$A_2 \xleftarrow{\stackrel{-}{\downarrow} \stackrel{-}{d_0}} \xrightarrow{s_0} A_1 \xleftarrow{\stackrel{-}{\downarrow} \stackrel{-}{d_0}} \xrightarrow{s_0} A_0$$

such that A is the coequalizer of $d_0, d_1 : A_1 \rightarrow A_0$.

Proof.

- A_0 is given by covering A by a 0-extension, i.e. factoring $0 \to A$ as $0 \hookrightarrow A_0 \stackrel{e}{\to} A$.
- A_1 is given by covering the kernel of $A_0 woheadrightarrow A$ by a 0-extension $0 woheadrightarrow A_1 woheadrightarrow A_1 woheadrightarrow A_2 woheadrightarrow A_1 woheadrightarrow A_2 woheadrightarrow A_2 woheadrightarrow A_1 woheadrightarrow A_2 woheadrightarrow A_2 woheadrightarrow A_2 woheadrightarrow A_2 woheadrightarrow A_3 woheadrightarrow A_4 woheadrightarrow A_5 woheadrightarrow A_$
- A_2 is given by covering the following pullback: $\begin{matrix} 0 \hookrightarrow A_2 \longrightarrow \bullet \longrightarrow A_1 \\ \downarrow & \downarrow d_0 \\ A_1 \stackrel{d_1}{\longrightarrow} A_0 \end{matrix}$

The theorem

Theorem

Adequate categories are clan-algebraic.

Proof.

Let \mathfrak{X} be adequate and let $\mathbb{C} \subseteq \mathfrak{X}$ be the co-clan of compact 0-extensions. It remains to show that

$$AC \cong \mathfrak{X}(C, LA).$$

for all $A \in \mathbf{Mod}(\mathbb{C}^{op})$ and $C \in \mathbb{C}$. Let A_{\bullet} be a nice diagram with coequalizer A. We have

$$\mathfrak{X}(C, LA) = \mathfrak{X}(C, L(\mathsf{coeq}(A_1 \Rightarrow A_0)))$$

$$\cong \mathfrak{X}(C, \mathsf{coeq}(LA_1 \Rightarrow LA_0))$$

$$\cong \mathsf{coeq}(\mathfrak{X}(C, LA_1) \Rightarrow \mathfrak{X}(C, LA_0))$$

$$\cong \mathsf{coeq}(A_1 C \Rightarrow A_0 C)$$

$$\cong \mathsf{coeq}(\mathsf{Mod}(ZC, A_1) \Rightarrow \mathsf{Mod}(ZC, A_0))$$

$$\cong \mathsf{Mod}(ZC, \mathsf{coeq}(A_1 \Rightarrow A_0))$$

$$\cong \mathsf{Mod}(ZC, A_0)$$

$$\cong \mathsf{AC}$$

since
$$L$$
 preserves colimits since $\mathfrak{X}(C,-)$ preserves coeqs of nice diags

since $A = coeq(A_1 \Rightarrow A_0)$

since
$$\mathit{LA}_i = \mathsf{colim}(\int \! A_i \to \mathbb{C} \to \mathfrak{X})$$
 filtered

Part III – Models in Higher Types

Models in higher types

Let \mathcal{S} be the ∞ -topos of spaces/types.

Let \mathcal{C}_{Mon} be the finite-product theory of monoids, and let \mathcal{L}_{Mon} be the finite-limit theory of monoids. Then

$$\mathsf{FP}(\mathcal{C}_\mathsf{Mon},\mathsf{Set}) \simeq \mathsf{FL}(\mathcal{L}_\mathsf{Mon},\mathsf{Set})$$

but $\mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ and $\mathsf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$ are different:

- $FL(\mathcal{L}_{Mon}, \mathcal{S})$ is just the category of monoids
- $\mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ is the ∞ -category ' A_{∞} -algebras', i.e. homotopy-coherent monoids.

Moral

By being 'slimmer', finite-product theories leave room for higher coherences when interpreted in higher types.

This phenomenon has been discussed under the name 'animation' in:

• K. Cesnavicius and P. Scholze. "Purity for flat cohomology". In: arXiv preprint arXiv:1912.10932 (2019)

Four clans for categories

Cat admits several clan-algebraic weak factorization systems:

```
• (\mathcal{E}_1,\mathcal{F}_1) is cofib. generated by \{(0 \to 1),(2 \to 2)\}

• (\mathcal{E}_2,\mathcal{F}_2) is cofib. generated by \{(0 \to 1),(2 \to 2),(2 \to 1)\}

• (\mathcal{E}_3,\mathcal{F}_3) is cofib. generated by \{(0 \to 1),(2 \to 2),(\mathbb{P} \to 2)\}

• (\mathcal{E}_4,\mathcal{F}_4) is cofib. generated by \{(0 \to 1),(2 \to 2),(\mathbb{P} \to 2),(2 \to 1)\}

where \mathbb{P}=(\bullet \rightrightarrows \bullet).
```

The right classes are:

```
      \mathcal{F}_1 = \{ \text{full and surjective-on-objects functors} \} 
      \mathcal{F}_2 = \{ \text{full and bijective-on-objects functors} \} 
      \mathcal{F}_3 = \{ \text{fully faithful and surjective-on-objects functors} \} 
      \mathcal{F}_4 = \{ \text{isos} \}
```

Note that \mathcal{F}_3 is the class of trivial fibrations for the canonical model structure on Cat.

Four clans for categories

These correspond to the following clans:

Models in higher types:

$\mathsf{Part}\ \mathsf{IV}-\mathsf{Syntax}$

From clans to theories

Duality between clans and clan-algebraic categories is a theory/model duality, where the
theories themselves are of a categorical nature. po-

$$\textbf{Clan}_{cc} \quad \xleftarrow{\quad \operatorname{comp}(\mathfrak{X})^{op} \ \leftarrow \mathfrak{X} \quad} \quad \textbf{cAlg}^{op}$$

- There's also a correspondence between categorical theories (clans) and syntactic theories (GATs)
 - The syntactic category of every GAT is a clan
 - Moreover, (I think that) every clan is equivalent to the syntactic category of a GAT, giving rise to an essentially surjective map as below.

$$\begin{array}{ccc} & & \text{Clan}_{cc} \longleftarrow & \cong & \rightarrow & \text{cAlg}^{op} \\ & & & \downarrow & & \\ & & & \text{Clan} & & & \end{array}$$

 This map can be enhanced to an equivalence by defining 1- and 2-cells between GATs to be 1and 2-cells between the corresponding clans.

Four GATs for categories

GAT for \mathcal{T}_1

- $\vdash O$ type
- $x : O \vdash 1 : A(x,x)$
- $xy: O \vdash A(x,y)$ type $xyz: O, f: A(x,y), g: A(y,z) \vdash g \circ f: A(x,z)$
- $w \times y \times z : O$, e : A(w,x), f : A(x,y), $g : A(y,z) \vdash (g \circ f) \circ e = g \circ (f \circ e) : A(w,z)$
- $xy : O, f \in A(x,y) \vdash 1 \circ f = f = f \circ 1 : A(x,y)$
- \mathcal{T}_2 should have an equivalent syntactic category but more display maps, including the diagonal

$$\delta_O = (x, x) : [x : O] \rightarrow [x y : O].$$

• This is not a context projection, but we can make it isomorphic to one by introducing a new type over $[x \ y : O]$ and forcing it to be isomorphic to [x : O]

Four GATs for categories

Additional axioms for \mathcal{T}_2

- $xy: O \vdash E(x,y)$ type
- $x : O \vdash r : E(x,x)$

- $xy : O, p : E(x,y) \vdash x = y$
- $xy : O, pq : E(x,y) \vdash p = q$
- In other words, we add an extensional equality type for O 'by hand'
- With this we can show the isomorphism of contexts $[x:O] \cong [xy:O,p:E(x,y)]$
- Similarly, add extensional equality for morphisms to get \mathcal{T}_e :

Additional axioms for \mathcal{T}_3

- $xy : O, fg : A(x,y) \vdash F(f,g)$ type
 - e
- $xy : O, fg : A(x,y), p : F(f,g) \vdash f = g$

• $xy : O, f : A(x,y) \vdash s : F(f,f)$

• $xy : O, fg : A(x,y), pq : F(f,g) \vdash p = q$

• Adding both sets of axioms yields \mathcal{T}_4

Part V – Monadicity over Toposes?

Seeking a forgetful functor

- Models of SSATs are monadic over Set
- Models of MSATs are monadic over Set¹
- For models of abstract FP-theories of FL-theories we don't have a forgetful functor
- Moreover, only FP-theories can be finitary monadic over powers of Set
- But models of FL-theories can be monadic over presheaf toposes:
 - Trivially since they're Ifp and thus reflective in presheaves
 - Less trivially in specific examples, e.g. categories over graphs
- For a clan, can we reconstruct a codomain for a forgetful functor from algebras?
- Idea: Consider functors

$$A: \mathcal{T}^{\dagger} \rightarrow \mathbf{Set}$$

on the wide subcategory of display maps that preserve 1 and pullbacks of display maps

- This yields a topos in interesting cases, but not always
- Connection to 'cd-structures' (see nlab)

Thanks for your attention!