Categorical Semantics and Synthetic Topology

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(these are the slides of the first part, by Jonas Frey)

Logic and Formal Epistemology Summer School

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Overview of the week

Broad Outline

- Part I : Categorical Semantics of First Order Logic (Jonas)

Setting up for:

- Part II : Synthetic Topology (Reid)

Schedule (subject to change!)

- Day 1 : intuitionistic propositional logic, Heyting algebras, intuitionistic first order logic
- Day 2 : category theory
- Day 3 : Semantics of 1st order logic in Hyperdoctrines
- Day 4 : Synthetic Topology
- Day 5 : Synthetic Topology

Part I : Categorical Models of First Order Logic

References / Further Reading

Books:

- B. Jacobs. *Categorical logic and type theory*. Elsevier Science Ltd, 2001
- A.M. Pitts. "Categorical logic". In: *Handbook of logic in computer science, Vol. 5*. Oxford Univ. Press, New York, 2000
- M. Makkai and G.E. Reyes. *First order categorical logic. Model-theoretical methods in the theory of topoi and related categories*. English. Springer, Cham, 1977

Lecture Notes:

- Steve Awodey's **Categorical Logic** course at CMU (specifically part 3): https://awodey.github.io/catlog/notes/
- Thomas Streicher's **Categorical Models of Constructive Logic** (shorter): https://www2.mathematik.tu-darmstadt.de/~streicher/cmcl.pdf

Bill Lawvere's original works:

- F.W. Lawvere. "Adjointness in foundations". In: *Dialectica* (1969)
- F.W. Lawvere. "Equality in hyperdoctrines and the comprehension schema as an adjoint functor". In: *Applications of Categorical Algebra* (1970)
- F.W. Lawvere. "Quantifiers and sheaves". In: *Actes du congres international des mathematiciens, Nice*. 1970

building on his thesis:

- F.W. Lawvere. "Functorial semantics of algebraic theories". In: *Proceedings of the National Academy of Sciences of the United States of America* (1963)

Propositional logic

Syntax of propositional logic

- **Propositional logic** is the logic of **propositional formulas**, built up from
  - **propositional variables**[1] $X, Y, Z, \dots$, and
  - **propositional connectives**

$\wedge$	conjunction	$\top$	truth
$\vee$	disjunction	$\bot$	falsity
$\Rightarrow$	implication	$\neg$	negation

- Examples of **well-formed formulas**

$$\neg(X \wedge \bot) \qquad (X \wedge Y) \vee Z \qquad (\neg X \Rightarrow X) \Rightarrow X$$

- Examples of **ill-formed formulas**

$$X\neg X \qquad X \wedge \vee Y \qquad Z((\neg$$

- We omit parentheses, such that e.g.
  - $A \wedge B \wedge C$ is short for $(A \wedge B) \wedge C$ (fill in parens from the left)
  - $A \Rightarrow B \Rightarrow C$ is short for $A \Rightarrow (B \Rightarrow C)$ (fill in parens from the right)
  - $A \wedge B \Rightarrow C$ is short for $(A \wedge B) \Rightarrow C$ ($\wedge$ binds stronger than $\Rightarrow$)

[1] For consistency with the upcoming treatment of first order logic, these could also be called **propositional constants**.

Boolean truth value semantics

The meaning of the logical connectives is customarily given by **truth value semantics**

A	$\neg A$
t	f
f	t

A	B	$A \wedge B$
t	t	t
t	f	f
f	t	f
f	f	f

A	B	$A \vee B$
t	t	t
t	f	t
f	t	t
f	f	f

A	B	$A \Rightarrow B$
t	t	t
t	f	f
f	t	t
f	f	t

The values of $\top$ and $\bot$ are t and f, respectively.

- A formula is called a **tautology**, if it is true for all values of the propositional variables:
  - $X \Rightarrow X$ is a tautology
  - $X \wedge \top$ is not a tautology
- A formula is called **satisfiable**, if there exists a valuation — i.e. an assignment of truth values to propositional variables — making the formula true:
  - $X \wedge \neg Y$ is satisfiable ($X = t, Y = f$)
  - $X \wedge \neg X$ is not satisfiable

Quiz

Which of the following formulas are satisfiable?

- $X \wedge \neg X$
- $X \Rightarrow \neg X$
- $(X \vee Y) \wedge (\neg X \vee \neg Y)$

Which of the following formulas are tautologies?

1. $\perp \Rightarrow X$
2. $X \Rightarrow \perp$
3. $X \wedge (Y \vee Z) \Rightarrow (X \wedge Y) \vee (X \wedge Z)$
4. $X \vee (Y \wedge Z) \Rightarrow (X \vee Y) \wedge (X \vee Z)$
5. $X \Rightarrow \neg\neg X$
6. $(X \wedge Y) \vee (X \wedge Z) \Rightarrow X \wedge (Y \vee Z)$
7. $(X \vee Y) \wedge (X \vee Z) \Rightarrow X \vee (Y \wedge Z)$
8. $\neg\neg X \Rightarrow X$
9. $X \vee \neg X$

Quiz

Which of the following formulas are satisfiable?

- $X \wedge \neg X$
- $X \Rightarrow \neg X$
- $(X \vee Y) \wedge (\neg X \vee \neg Y)$

Which of the following formulas are tautologies?

1. $\bot \Rightarrow X$
2. $X \Rightarrow \bot$
3. $X \wedge (Y \vee Z) \Rightarrow (X \wedge Y) \vee (X \wedge Z)$
4. $X \vee (Y \wedge Z) \Rightarrow (X \vee Y) \wedge (X \vee Z)$
5. $X \Rightarrow \neg\neg X$
6. $(X \wedge Y) \vee (X \wedge Z) \Rightarrow X \wedge (Y \vee Z)$
7. $(X \vee Y) \wedge (X \vee Z) \Rightarrow X \vee (Y \wedge Z)$
8. $\neg\neg X \Rightarrow X$
9. $X \vee \neg X$

The last two formulas are classical tautologies, but **not** intuitionistic/constructive ones!

Classical vs Constructive Logic

Broadly speaking, **Constructive Logic** is obtained from **Classical Logic**, but omitting the **Law of the Excluded Middle (LEM)**

$$\phi \vee \neg\phi\,,$$

or equivalently the **double negation elimination**

$$\neg\neg\phi \Rightarrow \phi$$

justifying proof by contradiction.

Classical vs Constructive Logic — History

Constructivism arose in mathematics in the early 20th century in opposition to perceived overly abstract tendencies in set theory, and concerns about consistency.

A central role was played by **L. E. J. Brouwer** and his school of **intuitionism** (and later by **Per Martin-Löf**).

We treat **constructive** and **intuitionistic** as synonyms, ignoring philosophical subtleties.

Disclaimer: We are not interested in constructive logic and mathematics out of any doubts in the correctness / consistency of classical math, but because of

- its links to **computability theory** and **theoretical computer science** ('Curry Howard Isomorphism'), and
- its use in **Category Theory** as an **Internal Language**.

L.E.J. Brouwer

Per Martin-Löf

Classical vs intuitionistic logic — basic connectives

- In classical logic the system of connectives $\wedge, \vee, \neg, \top \bot, \Rightarrow$ is redundant, since some of them can be defined in terms of others:
  - $\phi \wedge \psi$ is equivalent to $\neg(\neg\phi \vee \neg\psi)$
  - $\phi \vee \psi$ is equivalent to $\neg(\neg\phi \wedge \neg\psi)$
  - $\phi \Rightarrow \psi$ is equivalent to $\neg\phi \vee \psi$
  - $\top$ is equivalent to $\neg\bot$
  - $\bot$ is equivalent to $\neg\top$
  - $\neg\phi$ is equivalent to $\phi \Rightarrow \bot$
- For this reason, classical logic is often presented using the sets of connectives $\{\neg, \wedge, \top\}$ or $\{\neg, \vee, \bot\}$.
- Intuitionistically, only the last two are valid. In particular, we have to use $\Rightarrow$ as a primitive, and instead define negation as:

$$\neg\phi \;\equiv\; \phi \Rightarrow \bot$$

Deduction systems for propositional logic

- To know if a propositional formula ϕ is a classical tautology, we can try all possible assignments of t, f for the variables.
- This takes a long time for a high number of variables, and doesn't work for intuitionistic logic and first order logic
- Another possibility is to derive tautologies using a **deduction system**, such as
  - natural deduction
  - Hilbert style
  - Fitch style
  - sequent calculus
- We'll use a **natural deduction system** in **sequent notation**!

Natural deduction for intuitionistic propositional logic

A **sequent** or **judgment** is an expression

$$\phi_1, \ldots, \phi_n \vdash \psi$$

where $\phi_1, \ldots, \phi_n, \psi$ are formulas.

- The symbol $\vdash$ is called **turnstile**.
- Read: **"**$\phi_1, \ldots, \phi_n$ **entails** ψ**"** or **"**ψ **follows from** $\phi_1, \ldots, \phi_n$**"**.
- Lists of formulas are also called **contexts**, and denoted by capital Greek letters $\Gamma, \Delta, \Theta, \ldots$.
- E.g. we may set $\Gamma \equiv \phi_1, \ldots, \phi_n$ and write

  $$\Gamma \vdash \phi$$

  for the above judgment.

Natural deduction rules for intuitionistic propositional logic

Axiom rules:

$$\frac{}{\phi_1, \ldots, \phi_n \vdash \phi_i}\,(Ax) \quad \text{for} \quad 1 \leq i \leq n$$

Conjunction rules:

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\,(\wedge\text{-}I) \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}\,(\wedge\text{-}E_1) \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}\,(\wedge\text{-}E_2)$$

Disjunction rules:

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}\,(\vee\text{-}I_1) \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}\,(\vee\text{-}I_2) \qquad \frac{\Gamma \vdash \phi \vee \psi \qquad \Gamma, \phi \vdash \theta \qquad \Gamma, \psi \vdash \theta}{\Gamma \vdash \theta}\,(\vee\text{-}E)$$

Implication rules:

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}\,(\Rightarrow\text{-}I) \qquad \frac{\Gamma \vdash \phi \Rightarrow \psi \qquad \Gamma \vdash \phi}{\Gamma \vdash \psi}\,(\Rightarrow\text{-}E)$$

Truth and falsity rules:

$$\frac{}{\Gamma \vdash \top}\,(\top\text{-}I) \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}\,(\bot\text{-}E)$$

- Truth and falsity can be viewed as nullary versions of conjunction and disjunction.
- Each connective comes with associated *I*ntroduction and *E*limination rules, except that there is no elim-rule for $\top$, and no intro-rule for $\bot$.

Structural rules

The following ‘structural rules’ are **admissible** in natural deduction in the sense that one can show that the set of derivable judgments is closed under them.

weakening: $\dfrac{\Gamma \ \vdash \ \psi}{\Gamma, \phi \ \vdash \ \psi}\,(\mathit{Weak})$ exchange: $\dfrac{\Gamma, \phi, \psi \ \vdash \ \theta}{\Gamma, \psi, \phi \ \vdash \ \theta}\,(\mathit{Ex})$

contraction: $\dfrac{\Gamma, \phi, \phi \ \vdash \ \psi}{\Gamma, \phi \ \vdash \ \psi}\,(\mathit{Contr})$ cut: $\dfrac{\Gamma \ \vdash \ \phi \qquad \Gamma, \phi \ \vdash \ \psi}{\Gamma \ \vdash \ \psi}\,(\mathit{Cut})$

- **Substructural logics** — such as **linear logic** or **relevant logic** — explicitly exclude some structural rules. Thus, they have to be formulated in a way not making them admissible.
- Warning: when adding axioms, admissibility is generally only preserved when the axioms are closed under admissible rules as well!

Classical propositional logic

We obtain **classical** propositional logic by adding either of the rules

$$\frac{\Gamma \vdash \neg\neg\phi}{\Gamma \vdash \phi} \qquad\qquad \frac{}{\Gamma \vdash \phi \vee \neg\phi}$$

where $\neg\phi$ is a shorthand for $\phi \Rightarrow \bot$.

Completeness of classical natural deduction w.r.t. interpretation in $\{t, f\}$

Theorem

A propositional formula ϕ is a tautology if and only if the judgment $\vdash \phi$ is derivable in classical propositional logic.

- In other words, $\vdash \phi$ is derivable if and only if it is true for all valuations of the propositional variables with boolean values t, f.
- This is **not** the case for intuitionistic derivability!
- Rather, here we have a completeness result w.r.t. interpretations in **Heyting algebras** — the intuitionistic counterpart of **Boolean algebras**.

Distributive lattices and Heyting algebras

Definition

A **distributive lattice** is a partially ordered set (poset) $(A, \leq)$ which has:

1. a greatest element $\top$
2. a least element $\bot$
3. for every pair of elements $a, b \in A$:
   - a **greatest lower bound** / infimum / meet $a \wedge b$, and
   - a **least upper bound** / supremum / join $a \vee b$.

... such that for all $a, b, c \in A$, the **distributivity law** $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds.

Definition

A **Heyting algebra** is a distributive lattice $(H, \leq)$ where for all $a, b \in H$ there exists a (necessarily unique) element $a \Rightarrow b$ such that for all $c \in H$:

$$c \wedge a \leq b \quad \text{if and only if} \quad c \leq a \Rightarrow b$$

Remark: A Heyting algebra H is a Boolean algebra iff we have $x \vee (x \Rightarrow \bot) = \top$ for all $x \in H$.

Interpretation of propositional formulas in Heyting algebras

Heyting algebras have structure corresponding to all propositional connectives:

symbol	propositional logic	Heyting algebras
$\wedge$	conjunction	(binary) meet / inf
$\vee$	disjunction	(binary) join / sup
$\Rightarrow$	implication	Heyting implication[2]
$\top$	truth	greatest element
$\bot$	falsity	least element

This allows us to define interpretations of propositional formulas in Heyting algebras: given a function ('valuation')

$$\rho : \mathsf{Var} \to H$$

from the set Var of propositional variables to a Heyting algebra H, we define the interpretation $[\![\phi]\!]_\rho$ of propositional formulas ϕ **inductively** as follows:

$$[\![X]\!]_\rho = \rho(X) \qquad [\![\phi \wedge \psi]\!]_\rho = [\![\phi]\!]_\rho \wedge [\![\psi]\!]_\rho$$

$$[\![\top]\!]_\rho = \top \qquad [\![\phi \vee \psi]\!]_\rho = [\![\phi]\!]_\rho \vee [\![\psi]\!]_\rho$$

$$[\![\bot]\!]_\rho = \bot \qquad [\![\phi \Rightarrow \psi]\!]_\rho = [\![\phi]\!]_\rho \Rightarrow [\![\psi]\!]_\rho$$

[2] sometimes called 'pseudocomplement'

Soundness of intuitionistic logic w.r.t. Heyting algebras

Theorem

If a judgment $\phi_1, \ldots, \phi_n \ \vdash \ \phi$ is derivable in intuitionistic logic, then we have

$$[\![\phi_1]\!]_\rho \wedge \cdots \wedge [\![\phi_n]\!]_\rho \ \leq \ [\![\psi]\!]_\rho$$

for all Heyting algebras H and valuations $\rho : \mathsf{Var} \to H$.

Proof.

Structural induction. $\square$

Completeness of intuitionistic logic w.r.t. Heyting algebras

Theorem

A judgment $\phi_1, \ldots, \phi_n \vdash \psi$ is derivable whenever we have

$$[\![\phi_1]\!]_\rho \wedge \cdots \wedge [\![\phi_n]\!]_\rho \leq [\![\psi]\!]_\rho$$

for all H and ρ.

Proof.

Take H to be the Heyting algebra whose elements are equivalence classes of propositional formulas modulo logical equivalence.
(This is known as the **Lindenbaum-Tarski algebra**!) □

Corollary

A propositional formula ϕ is derivable in intuitionistic propositional logic if and only if we have

$$[\![\phi]\!]_\rho = \top$$

for all valuations ρ.

Examples of Heyting algebras I: open sets

Given a topological space (X, τ) — for example the real line $\mathbb{R}$ with the **euclidean topology** — the poset τ of open sets (ordered by inclusion) forms a Heyting algebra:

- finite meets and joins are given by set-theoretic intersections and unions, and
- the Heyting implication $U \Rightarrow V$ of two open sets is given by

$$U \Rightarrow V \ = \ (X \backslash U \ \cup \ V)^{\circ}$$

- in particular, the negation $\neg U \ \equiv \ U \Rightarrow \bot$ of an open set U is given by the interior of the complement.

Easy non-derivability results

Proposition

The judgment $\neg\neg X \vdash X$ is not derivable in intuitionistic propositional logic.

Proof.

Interpret X by the open set $U = \mathbb{R} \backslash \{0\} \subseteq \mathbb{R}$ in the real line.

Then $[\![\neg\neg X]\!] = \top = \mathbb{R}$, which is not contained in U. $\square$

Proposition

The judgment $\vdash X \vee \neg X$ is not derivable.

Proof.

Interpret X by $V = (0, \infty) \subseteq \mathbb{R}$. Then $[\![\neg X]\!] = (-\infty, 0)$ and $[\![X \vee \neg X]\!] = \mathbb{R} \backslash \{0\} \neq \mathbb{R}$. $\square$

Kripke semantics

Another example of Heyting algebras is given as follows: let $(P, \leq)$ be a poset. A **downset** in P is a subset $U \subseteq P$ such that

$$x \in U, y \leq x \quad \text{implies} \quad y \in U\,.$$

The collection $\mathsf{down}(P)$ of all downsets in $(P, \leq)$ is a Heyting algebra: finite meets and joins are given by set-theoretic intersections and unions, and Heyting implication is given by

$$U \Rightarrow V \;=\; \{p \in P \mid \text{for all } q \leq p,\ \text{if } q \in U \text{ then } q \in V\}.$$

Interpretation in Heyting algebras of this form is known as **Kripke semantics**: writing $p \Vdash_\rho \phi$ for $p \in [\![\phi]\!]_\rho$ (where $\rho : \mathsf{Var} \to \mathsf{down}(P)$), the inductive clauses can be presented as follows:

$$\begin{array}{ll}
p \Vdash_\rho X & \text{if } p \in \rho(X) \\
p \Vdash_\rho \phi \wedge \psi & \text{if } p \Vdash_\rho \phi \text{ and } p \Vdash_\rho \psi \\
p \Vdash_\rho \phi \vee \psi & \text{if } p \Vdash_\rho \phi \text{ or } p \Vdash_\rho \psi \\
p \Vdash_\rho \phi \Rightarrow \psi & \text{if for all } q \leq p, \quad q \Vdash_\rho \phi \text{ implies } q \Vdash_\rho \psi \\
p \Vdash_\rho \top & \text{always} \\
p \Vdash_\rho \bot & \text{never}
\end{array}$$

Example

Setting $P = \{0 < 1\}$ and $[\![X]\!] = \{0\}$, we get $[\![\neg X]\!] = \varnothing$ and $[\![\neg\neg X]\!] = \{0, 1\}$. Thus, we have another countermodel for the judgment $\neg\neg X \vdash X$.

First order logic

Syntax of first order logic

Definition

A **(single-sorted) first order signature** Σ is given by

- a collection $f, g, h, \ldots$ of **function symbols**, and
- a collection $R, S, T, \ldots$ of **relation symbols**.

Each function and each relation symbol has an **arity**, which is a natural number ≥ 0.

Remark

Nullary function symbols, i.e. function symbols of arity 0, are also known as **constants**.

Nullary relation symbols will be called **propositional constants**[3].

.

[3] In a slight mismatch, these correspond to the propositional **variables** in the first part

Syntax of first order logic: formulas and terms in context

The **first order language** $\mathcal{L}(\Sigma)$ over a first order signature $\Sigma = \{f, g, h, \dots, R, S, T, \dots\}$ comprises **terms in context** and **formulas in context**:

- A **context** is a list of first order variables $x_1, \dots, x_n$
- **terms** in context $\vec{x} \equiv x_1, \dots, x_n$ are given as follows:
  - every **variable** x_i (for $1 \leq i \leq n$) is a term in context $\vec{x}$
  - if $t_1, \dots, t_k$ are terms in context $\vec{x}$, and f is a k-ary function symbol, then $f(t_1, \dots, t_k)$ is a term in context $\vec{x}$.
- **formulas** are given as follows:
  - if $t_1, \dots, t_k$ are terms in context $\vec{x}$, and R is a k-ary relation symbol, then $R(t_1, \dots, t_k)$ is a formula in context $\vec{x}$.
  - if s, t are terms in context $\vec{x}$, then $s = t$ is a formula in context $\vec{x}$.
  - if ϕ and ψ are formulas in context $\vec{x}$, then $\phi \wedge \psi$, $\phi \vee \psi$, and $\phi \Rightarrow \psi$ are formulas in context $\vec{x}$.
  - $\top$ and $\bot$ are formulas in any context $\vec{x}$.
  - if ϕ is a formula in context $\vec{x}, y$, then $\forall y \,.\, \phi$ and $\exists y \,.\, \phi$ are formulas in context $\vec{x}$.

Many sorted languages

With a bit more bureaucracy one can generalize all this to **many sorted signatures** and **many sorted languages**: A many-sorted signature has a comes with a collection of sorts $A, B, C, \ldots$ and arities are no longer natural numbers. Instead, the arity of a relation symbol is a list of sorts, and for function symbols we moreover have to specify an output sort:

$$R : A_1, \ldots, A_n$$
$$f : A_1, \ldots, A_n \to B$$

In the definition of terms and atomic formulas we have to add a side condition saying that the sorts match.

First order natural deduction

We use a natural deduction system with **explicit contexts**: a **judgment** is an expression

$$\phi_1, \dots, \phi_n \vdash_{\vec{x}} \psi$$

where $\phi_1, \dots, \phi_n, \psi$ are formulas in context $\vec{x}$.

The rules are the rules for propositional logic (with turnstiles annotated by $\vec{x}$), plus the following:

- Equality rules:

$$\frac{}{\Gamma \vdash_{\vec{x}} s = s}\ (=\text{-}I) \qquad \frac{\Gamma \vdash_{\vec{x}} \phi[s/y] \qquad \Gamma \vdash_{\vec{x}} s = t}{\Gamma \vdash_{\vec{x}} \phi[t/y]}\ (=\text{-}E)$$

  Here, s and t are terms in context $\vec{x}$, the formulas in Γ are in context $\vec{x}$, and ϕ and ψ are formulas in context $\vec{x}, y$.

- Universal quantification $\forall$:

$$\frac{\Gamma \vdash_{\vec{x},y} \phi}{\Gamma \vdash_{\vec{x}} \forall y \,.\, \phi}\ (\forall\text{-}I) \qquad \frac{\Gamma \vdash_{\vec{x}} \forall y \,.\, \phi}{\Gamma \vdash_{\vec{x}} \phi[t/y]}\ (\forall\text{-}E)$$

  Here, t is a term in context $\vec{x}$, and ϕ is a formula in context $\vec{x}, y$.

- Existential quantification:

$$\frac{\Gamma \vdash_{\vec{x}} \phi[t/y]}{\Gamma \vdash_{\vec{x}} \exists y \,.\, \phi}\ (\exists\text{-}I) \qquad \frac{\Gamma \vdash_{\vec{x}} \exists y \,.\, \phi \qquad \Gamma, \phi \vdash_{\vec{x},y} \psi}{\Gamma \vdash_{\vec{x}} \psi}\ (\exists\text{-}E)$$

  Here, ϕ is a formula in context $\vec{x}, y$, and ψ is a formula in context $\vec{x}$.

Structural rules in first order logic

The structural rules of propositional logic are admissible also in first order natural deduction once we include variable contexts:

$$\frac{\Gamma \vdash_{\vec{x}} \psi}{\Gamma, \phi \vdash_{\vec{x}} \psi}\,(\mathit{Weak}) \qquad \frac{\Gamma, \phi, \phi \vdash_{\vec{x}} \psi}{\Gamma, \phi \vdash_{\vec{x}} \psi}\,(\mathit{Contr})$$

$$\frac{\Gamma, \phi, \psi \vdash_{\vec{x}} \theta}{\Gamma, \psi, \phi \vdash_{\vec{x}} \theta}\,(\mathit{Ex}) \qquad \frac{\Gamma \vdash_{\vec{x}} \phi \qquad \Gamma, \phi \vdash_{\vec{x}} \psi}{\Gamma \vdash_{\vec{x}} \psi}\,(\mathit{Cut})$$

Moreover, we have the following admissible structural rules that are specific to first order logic:

$$\frac{\Gamma \vdash_{\vec{x}} \psi}{\Gamma \vdash_{\vec{x},y} \psi}\,(\mathit{Weak}') \qquad \frac{\Gamma \vdash_{\vec{x},y} \psi}{\Gamma[t/y] \vdash_{\vec{x}} \psi[t/y]}\,(\mathit{Subst})$$

In the (Subst) rule, t is a term in context $\vec{x}$.

Categorical semantics of first order logic

- How can we generalize the Heyting algebra semantics of propositional logic to first order logic?
- One possibility is to use **complete** Heyting algebras, and interpret quantifiers by infinite meets and joins.
- We'll do something more general.
- Motivating question: what algebraic structure does the syntax naturally form?
- For this we need:

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🎉 Category Theory 🎉

A Whirlwind Introduction to Category Theory

Motivation: The Rising Level of Abstraction in the History of Mathematics

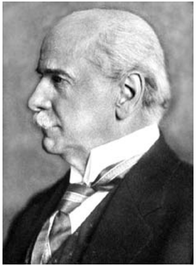
- Classical Antiquity: numbers, triangles
- Scientific Revolution (Leibniz, Newton): functions and calculus
- late 19th and early 20th century:
  - set theory (Cantor 1874)
  - symbolic logic
  - axiomatically defined **structures**, such as **groups** and **partial orders**

Groups

- Group theory arose from the study of **symmetries**, which is an ancient topic.
- The beginning of group theory as a field is commonly pinpointed to **Évariste Galois'** work on solvability of polynomial equations ($\sim$1830). (here, the 'symmetries' were automorphisms of field extensions)
- The modern, abstract notion of group, independently of an object that the group elements are symmetries of, was explicitly formulated first by **Walther von Dyck** in 1882:

Évariste Galois

> *Definition*
>
> A **group** is a set G equipped with a binary operation
>
> $$G \times G \to G, \qquad (x, y) \mapsto x \cdot y$$
>
> such that
>
> - the operation is **associative**: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in G$.
> - there is a **unit element** $e \in G$ satisfying $e \cdot x = x \cdot e = x$ for all $x \in G$, and
> - every $x \in G$ has an **inverse**: an element x^{-1} satisfying $x \cdot x^{-1} = x^{-1} \cdot x = e$.



Walther von Dyck

Partial orders

The notion of partial order axiomatizes common features of the classical ordering relation on real numbers, and the inclusion ordering of subsets of a given set.

Definition

A **partially ordered set** (poset) is a set P equipped with a binary relation $(\leq) \subseteq P \times P$ satisfying the following axioms:

- **reflexivity**: for all $x \in P$ we have $x \leq x$
- **transitivity**: for all $x, y, z \in P$ we have $x \leq z$ whenever $x \leq y$ and $y \leq z$
- **antisymmetry**: for all $x, y \in P$, if $x \leq y$ and $y \leq x$ then $x = y$

Remarks

- A **linear order** is a poset $(A, \leq)$ satisfying the additional axiom
  - for all $x, y \in P$, either $x \leq y$ or $y \leq x$
- On the other hand, a **preorder** is a set P with a relation $\leq$ satisfying the reflexivity and transitivity, but not necessarily the antisymmetry axiom.

More examples of axiomatically defined structures

Do you know other axiomatically defined structures?

More examples of axiomatically defined structures

Do you know other axiomatically defined structures?

Here are some more examples:

- distributive lattices
- Heyting algebras (a kind of poset)
- algebraic structures
  - rings
  - fields
  - Lie algebras
  - modules / vector spaces
  - ...
- topological spaces
- measurable spaces, i.e. sets equipped with a σ-algebra

Structure preserving maps

Axiomatically defined structures typically come with a associated class of **structure preserving map**:

- A **group homomorphism** from $(G, \cdot)$ to $(H, \cdot)$ is a function $f : G \to H$ satisfying $f(e) = e$ and $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in G$.
- A **monotone map** between posets $(P, \leq)$ and $(Q, \leq)$ is a function $f : P \to Q$ such that $f(x) \leq f(y)$ whenever $x \leq y$.
- A **morphism of distributive lattices** from $(A, \leq)$ to $(B, \leq)$ is a monotone map $f : A \to B$ such that

$$f(\top_A) = \top_B \qquad f(x \wedge_A y) = f(x) \wedge_B f(y)$$
$$f(\bot_A) = \bot_B \qquad f(x \vee_A y) = f(x) \vee_B f(y)$$

  for all $x, y \in A$.

These structure preserving maps are usually closed under composition, and contain identities!

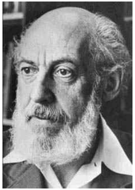
This leads to the definition of category.

Definition of category[4]

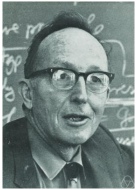
A **category** consists of the following data:

- **Objects** $A, B, C, \ldots$
- **Arrows** $f, g, h, \ldots$
- For each arrow, f, there are given objects $\mathrm{dom}(f)$, $\mathrm{cod}(f)$ called the **domain** and **codomain** of f.

  We write $f : A \to B$ to indicate that $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$.
- Given arrows $f : A \to B$ and $g : B \to C$, that is, with $\mathrm{cod}(f) = \mathrm{dom}(g)$, there is given an arrow
  $$g \circ f : A \to C$$
  called the **composite** of f and g.
- For each object A, there is given an arrow
  $$1_A : A \to A$$
  called the **identity arrow** of A.



Samuel Eilenberg



Saunders Mac Lane

These data are required to satisfy the following laws:

- Associativity:
  $$h \circ (g \circ f) = (h \circ g) \circ f$$
  for all $f : A \to B$, $g : B \to C$, $h : C \to D$.
- Unit:
  $$f \circ 1_A = f = 1_B \circ f$$
  for all $f : A \to B$.

[3] S. Eilenberg and S. MacLane. "General theory of natural equivalences". In: *Transactions of the American Mathematical Society* (1945)

Definition of Category: Remarks

- The previous definition is taken literally from Steve Awodey's book [5]
- The original definition is equivalent but less readable
- Another variant is in terms of **hom-sets**

Steve Awodey

[5] S. Awodey. *Category theory*. Second Edition. Oxford University Press, Oxford, 2010

Hom-sets

Given objects A, B in a category $\mathbb{C}$, we write

$$\mathrm{hom}(A, B) \; = \; \{f \mid \mathsf{dom}(f) = A \text{ and } \mathsf{cod}(f) = B\}$$

for the set of arrows from A to B. We call these sets **hom-sets**.

It is possible to state the definition of category directly in terms of hom-sets:

Definition 2

A category $\mathbb{C}$ is given by:

- A collection $\mathbb{C}_0$ of **objects**
- For each pair A, B of objects, a set $\mathrm{hom}(A, B)$ of arrows from A to B
- For each triple of objects A, B, C, a composition function

$$\mathrm{hom}(B, C) \times \mathrm{hom}(A, B) \to \mathrm{hom}(A, C)$$

- ...
- ...

Examples of categories

For each notion of mathematical structure there is typically an associated category, whose objects are the structures, and whose arrows are structure-preserving maps.

For example:

- Grp is the category of **groups** and **group homomorphisms**
- Pos is the category of **posets** and **monotone maps**
- Dist is the category of **distributive lattices** and **morphisms of distributive lattices**
- Heyt is the category of **Heyting algebras**, and **Heyting algebra morphisms**
- Ring is the category of **rings** and **ring homomorphisms**
- Field is its subcategory of **fields**
- Top is the category of **topological spaces** and **continuous functions**
- Meas is the category of **measurable spaces** and **measurable maps**

It turns out that the most trivial example is of special importance:

- Set is the category of **sets** and **functions**

Large and small

- Category theory pushes set theory to its limits, since categories of structures and structure preserving maps are typically **large** collections.
- For example, Set_0 is the collection of **all sets**, which is not a set itself (or it would contain itself, which is ruled out by the foundation axiom).
- We call a category **small**, if its collections of objects and morphisms **are** sets.

Concrete vs abstract categories

In extracting the definition of category from the practice of structuralist mathematics, we made an important abstraction step:

While the motivating examples of categories are categories of structured sets and structure preserving maps — and composition is function composition and therefore automatically associative — this is **not required by the definition**! (otherwise the axioms would not be necessary)

We will later see examples of non-concrete categories for which this is not the case, notably **syntactic categories**.

The power of category theory: Isomorphisms

Category theory allows to capture core concepts of structuralist thinking purely arrow theoretically. The most important such concept is the concept of **isomorphism**:

Definition

Let $\mathbb{C}$ be a category.

- An **inverse** of an arrow $f : A \to B$ in $\mathbb{C}$ is an arrow $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.
- An **isomorphism** is an arrow which has an inverse.

Exercises

- Show that inverses are unique, i.e. if g and h are both inverses of f, then $g = h$.
- $g : B \to A$ is called a **left inverse** of $f : A \to B$ if $g \circ f = \mathrm{id}_A$. Dually, g is called a **right inverse** if $f \circ g = \mathrm{id}_B$.

  Show that if f has both a left and a right inverse, then it is an isomorphism.

The power of category theory: reformulating set-theoretic concepts

- One core technique in CT is to reformulate set-theoretic concepts using **arrows instead of elements**.
- A basic example is the notion of **monomorphism**, generalizing the notion of **injective function**.

Definition

An arrow $m : A \to B$ in a category $\mathbb{C}$ is called **monomorphism** (short **mono**), if

$(\star)$ for all objects C and arrows $g, h : C \to B$, we have $g = h$ whenever $m \circ g = m \circ h$.

- In other words, m is a mono if we can cancel it on the left of equations!
- In yet other words, m is a mono if for all C, the **postcomposition function**

$$\hom(C, A) \to \hom(C, B), \qquad f \mapsto m \circ f$$

  is injective.
- The monos in Set are precisely the **injecive functions**.
- More generally, in the categories on the 'Examples' slide, the monic structure preserving maps coincide with the injective ones.
  The same is true for **all** categories on the examples slide!

The power of category theory: dualization

- For every category theoretic concept, there is a **dual** concept obtained by 'turning the arrows around' in the definition.
- The dual definition of monomorphism is **epimorphism**:

Definition

An arrow $e : A \to B$ in a category $\mathbb{C}$ is an **epimorphism** (short **epi**), if

- for all objects C and arrows $g, h : B \to C$, we have $g = h$ whenever $g \circ e = h \circ e$.

- In other words, e is an epi if we can cancel it on the right!
- In yet other words, m is a mono if for all C, the **precomposition function**

$$\mathrm{hom}(B, C) \to \mathrm{hom}(A, C), \qquad f \mapsto f \circ e$$

  is injective.
- Epis in Set are precisely **surjective** functions.
- This generalizes to Top and Pos, but **not** to e.g. Ring.

The power of category theory: opposite categories

- Dualization can be phrased in terms of **opposite categories**!

Definition

The **opposite category** $\mathbb{C}^{\text{op}}$ of a category $\mathbb{C}$ is given as follows:

- $\mathbb{C}^{\text{op}}$ has the same objects as $\mathbb{C}$.
- Arrows of $\mathbb{C}^{\text{op}}$ are 'formal duals' of arrows in $\mathbb{C}$, i.e. for every arrow $f : A \to B$ in $\mathbb{C}$ there is an arrow $f^{\text{op}} : B \to A$ in $\mathbb{C}^{\text{op}}$.
- Composition in $\mathbb{C}^{\text{op}}$ is given by $f^{\text{op}} \circ g^{\text{op}} = (g \circ f)^{\text{op}}$.

Example: An arrow f is an epimorphism in a category $\mathbb{C}$ iff its dual arrow f^{op} is a monomorphism in $\mathbb{C}^{\text{op}}$.

The power of category theory: Universal Properties

- Category theory is about identifying and studying common patterns in different categories
- A core technique here is the notion of **universal mapping property**, which characterizes certain constructions purely arrow theoretically.
- For example, many kinds of structure come with a notion of **product** $A \times B$, whose underlying set is typically the cartesian product of its factors.
- Category theory allows to give a characterization of these products purely in terms of **arrows** (rather than elements):

> *Definition*
>
> A **product** of objects A, B in a category $\mathbb{C}$ is an object P equipped with arrows $\pi_1 : P \to A$, $\pi_2 : P \to B$ such that:
>
> $(\star)$ for every object X and pair of arrows $f : X \to A$, $g : X \to B$ there exists a **unique** arrow $h : X \to P$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.
>
> $$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ A & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

The power of category theory: Universal properties

- Most of the categories in our list admit products of arbitrary pairs of objects (except fields).
- These are typically given by the cartesian product of underlying sets, structured in a suitable manner.
- The situation is more interesting for **coproducts** — the dual concept to products. But we don't have time to go into details now.
- Another — arguably more fundamental — universal property is that of **terminal objects**

Definition

A **terminal object** in a category $\mathbb{C}$ is an object 1 such that for all objects A there exits a **unique** arrow $A \to 1$

- This looks pretty boring, but is fundamental since **every universal property can be reformulated in terms of terminality** (or **initiality** — its formal dual).

Functors

- Functors are **morphisms of categories**. Formally:

Definition

Given categories $\mathbb{C}$ and $\mathbb{D}$, a **functor** $F : \mathbb{C} \to \mathbb{D}$ is a mapping which sends

- Objects in A in $\mathbb{C}$ to objects $F(A)$ in $\mathbb{D}$, and
- Morphisms in $f : A \to B$ in $\mathbb{C}$ to morphisms $F(f) : F(A) \to F(B)$ in $\mathbb{D}$,

such that

- $F(g \circ f) = F(g) \circ F(f)$, and
- $F(\mathrm{id}_A) = \mathrm{id}_{FA}$ for all objects A in $\mathbb{C}$.

Examples

- forgetful functors
- discrete poset functor $D : \mathsf{Set} \to \mathsf{Pos}$
- powerset functor $P : \mathsf{Set} \to \mathsf{Pos}$

Contravariant functors

- A **contravariant functor** is a functor which reverses the direction of arrows.
- More formally, a contravariant functor from $\mathbb{C}$ to $\mathbb{D}$ is a functor from $\mathbb{C}^{\mathsf{op}}$ to $\mathbb{D}$
- Example: contravariant powerset functor.

The category of categories

- functors compose and there are identity functors
- therefore, categories form a category!
- For size reasons, we have to restrict to small categories

Definition

Cat is the category of **small categories** and **functors** between them.

Embedding Posets into Categories

- Let $(P, \leq)$ be a poset.
- We define a small category $\mathcal{C}(P, \leq)$ as follows:
  - the objects of $\mathcal{C}(P, \leq)$ are the elements of P
  - the morphisms of $\mathcal{C}(P, \leq)$ are the pairs $(x, y) \in P \times P$ such that $x \leq y$, where $\mathsf{dom}(x, y) = x$ and $\mathsf{cod}(x, y) = y$.
  - composition is given by $(y, z) \circ (x, y) = (x, z)$, and identities by $\mathrm{id}_x = (x, x)$.
- in other words, there is a morphism in $\mathcal{C}(P, \leq)$ from x to y iff we have $x \leq y$, and this morphism is unique.
- We call a category $\mathbb{C}$ **thin**, if $\hom(A, B)$ has at most one element for all objects A, B.
- Thus, categories $\mathcal{C}(P, \leq)$ are thin.
- Conversely, every **small thin category** is isomorphic to one of the form $\mathcal{C}(P, \leq)$ for $(P, \leq)$ a **preorder** (generalization of poset without antisymmetry).

Embedding Posets into Categories — Functoriality

-
- Every monotone map $f : (P, \leq)$ to $(Q, \leq)$ gives rise a functor $\mathcal{C}(f) : \mathcal{C}(P, \leq) \to \mathcal{C}(Q, \leq)$.
- This means that the construction $(P, \leq) \mapsto \mathcal{C}(P, \leq)$ gives rise to a functor

$$\mathcal{C} : \mathsf{Pos} \to \mathsf{Cat}$$

- one can show that for all posets $(P, \leq)$ and $(Q, \leq)$, the induced function

$$(\star) \qquad \hom((P, \leq), (Q, \leq)) \to \hom(\mathcal{C}(P, \leq), \mathcal{C}(Q, \leq)), \qquad f \mapsto \mathcal{C}(f)$$

  is a bijection.
- In this case, we call the functor **fully faithful**
- More generally, we call a functor **faithful** if $(*)$ is injective for all pairs of objects, and **full** if $(*)$ is surjective for all pairs of objects.

Order-Adjoints

- **Adjoint functors** are one of the most important concepts in category theory.
- Unfortunately, for time reasons, we can only talk about the 'poset version':

Definition

Let $(P, \leq)$ and $(Q, \leq)$ be posets, and $f : (P, \leq) \to (Q, \leq)$ and $g : (Q, \leq) \to (P, \leq)$ be monotone maps. We say that f **is left adjoint to** g, or that g **is right adjoint to** f — and write $f \dashv g$ — if we have

$$f(x) \leq y \quad \text{iff} \quad x \leq g(y)$$

for all $x \in P$ and $y \in Q$.

Examples:

- for every function $f : S \to T$, the inverse image construction $f^{-1} : P(B) \to P(A)$ is left adjoint to the direct image construction $f[-] : P(A) \to P(B)$.

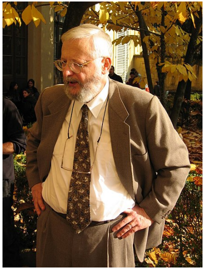
Hyperdoctrines: Categorical Models of First Order Logic

Motivation

- The categorical semantics of first order logic was pioneered by **William Lawvere**[6]
- Idea:
  - While the formulas of intuitionistic propositional logic naturally form a Heyting algebra, in the case of first order logic we have a Heyting algebra of formulas in context $\vec{x}$ for each context $\vec{x}$.
  - We can move between these Heyting algebras using **weakening**, (**substitution**,) and **quantification**:

$$
\begin{array}{lcl}
(\vec{x} \mid \phi) & \xmapsto{\text{weaken}} & (\vec{x}, y \mid \phi) \\
(\vec{x}, y \mid \psi) & \xmapsto{\text{quantify}} & (\vec{x} \mid \forall y \,.\, \psi) \\
(\vec{x}, y \mid \psi) & \xmapsto{\text{quantify}} & (\vec{x} \mid \exists y \,.\, \psi)
\end{array}
$$

  - Lawvere's insight was that the rules for $\forall$ and $\exists$ can be read as saying that $\exists$ and $\forall$ are respectively **left** and **right** adjoint to weakening!



William Lawvere

[6] F.W. Lawvere. "Adjointness in foundations". In: *Dialectica* (1969)
F.W. Lawvere. "Equality in hyperdoctrines and the comprehension schema as an adjoint functor". In: *Applications of Categorical Algebra* (1970)

Quantifiers as adjoints

The natural deduction rules for $\forall$ and $\exists$

$$\frac{\Gamma \ \vdash_{\vec{x},y} \ \phi}{\Gamma \ \vdash_{\vec{x}} \ \forall y \,.\, \phi} \ (\forall\text{-}I) \qquad \frac{\Gamma \ \vdash_{\vec{x}} \ \forall y \,.\, \phi}{\Gamma \ \vdash_{\vec{x}} \ \phi[t/y]} \ (\forall\text{-}E)$$

$$\frac{\Gamma \ \vdash_{\vec{x}} \ \phi[t/y]}{\Gamma \ \vdash_{\vec{x}} \ \exists y \,.\, \phi} \ (\exists\text{-}I) \qquad \frac{\Gamma \ \vdash_{\vec{x}} \ \exists y \,.\, \phi \qquad \Gamma, \phi \ \vdash_{\vec{x},y} \ \psi}{\Gamma \ \vdash_{\vec{x}} \ \psi} \ (\exists\text{-}E)$$

are equivalent to the following **'adjoint-style' bidirectional rules**, together with the **structural rules**.

$$\frac{\Gamma \ \vdash_{\vec{x},y} \ \phi}{\overline{\Gamma \ \vdash_{\vec{x}} \ \forall y \,.\, \phi}} \ (\forall\text{-}A) \qquad \frac{\Gamma, \phi \ \vdash_{\vec{x},y} \ \psi}{\overline{\Gamma, \exists y \,.\, \phi \ \vdash_{\vec{x}} \ \psi}} \ (\exists\text{-}A)$$

The first one exactly matches the adjunction pattern

$$\frac{f(a) \leq b}{\overline{a \leq u(b)}}$$

where f is weakening (invisible in the syntax), and u is $\forall$.

The second one matches the pattern only for empty Γ — but in presence of implication, this is not a real restriction.

Indexed posets

Definition

- An **indexed poset** on a category $\mathbb{C}$ is a contravariant functor
$$\mathcal{H} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Pos}.$$
Given $f : B \to A$ in $\mathbb{C}$, we usually write f^* for the induced monotone map
$$\mathcal{H}(f) : \mathcal{H}(A) \to \mathcal{H}(B)$$
and call it the **reindexing map** along f.
- Similarly, an **indexed Heyting algebra** on $\mathbb{C}$ is a contravariant functor
$$\mathcal{H} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Heyt}.$$

Terminology

We call the Heyting algebras $\mathcal{H}(A)$ (for A in $\mathbb{C}$) the **fibers** of $\mathcal{H}$, and refer to their elements as **predicates**: they'll be used to interpret first-order formulas.

First Order Hyperdoctrines

Definition

A **first-order hyperdoctrine** is an indexed Heyting algebra

$$\mathcal{H} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Heyt}$$

on a category $\mathbb{C}$ with finite products, such that for all $f : A \to B$ in $\mathbb{C}$, the reindexing map f^* has left and right adjoints $\exists_f$ and $\forall_f$.

$$\begin{tikzcd} A \arrow[d, "f"'] & \mathcal{H}(A) \arrow[d, bend right, "\exists_f"'] \arrow[d, bend left, "\forall_f"] \\ B & \mathcal{H}(B) \arrow[u, "f^*"] \end{tikzcd} \quad \exists_f \dashv f^* \dashv \forall_f .$$

These have to satisfy the **Beck–Chevalley condition:** for $f : A \to B$, $g : X \to Y$ and $\phi \in \mathcal{H}(A \times Y)$, we have:

$$\exists_{A\times g}((f \times X)^*(\varphi)) = (f \times Y)^*(\exists_{B\times g}(\varphi))$$
$$\forall_{A\times g}((f \times X)^*(\varphi)) = (f \times Y)^*(\forall_{B\times g}(\varphi))$$

$$\begin{tikzcd} A \times X \arrow[r, "A\times g"] \arrow[d, "f\times X"'] & A \times Y \arrow[d, "f\times Y"] \\ B \times X \arrow[r, "B\times g"'] & B \times Y \end{tikzcd}$$

First Order Hyperdoctrines

Remarks on the definition

- For $f : A \to B$ in $\mathbb{C}$, the left and right adjoints $\exists_f$ and $\forall_f$ of the reindexing map $f^* : \mathcal{H}(B) \to \mathcal{H}(A)$ are only required to be monotone, not preserve Heyting algebra structure.
- However, one can show that $\exists_f$ preserves finite joins since it is a left adjoint, and $\forall_f$ preserves finite meets since it is a right adjoint.
  (More generally, left adjoints preserve colimits, and right adjoints preserve limits.)
- It is sufficient to require/verify the Beck–Chevalley condition for either $\forall$ or $\exists$, the other variant then follows.

Interpreting First-Order Logic in First-Order Hyperdoctrines

- Let $\Sigma = \{f, g, h, \ldots, R, S, T, \ldots\}$ be a first-order signature and $\mathcal{H} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Heyt}$ be a first-order hyperdoctrine.
- To define an interpretation of the language $\mathcal{L}(\Sigma)$ in $\mathcal{H}$ we fix:
  - an object A in $\mathbb{C}$ which serves as carrier[7]
  - for each n-ary function symbol f, an arrow $[\![f]\!] : A^n \to A$ in $\mathbb{C}$
  - for each n-ary relation symbol R, a predicate $[\![R]\!] \in \mathcal{H}(A^n)$
- The interpretation $[\![\vec{x} \mid t]\!] : A^n \to A$ of terms t in context $\vec{x} = x_1, \ldots, x_n$ is defined inductively as follows:
  - $[\![\vec{x} \mid x_i]\!] = \pi_i$ (i-th projection)
  - $[\![\vec{x} \mid f(t_1, \ldots, t_k)]\!] = [\![f]\!] \circ [\![\vec{x} \mid t_1, \ldots, t_k]\!]$, where $[\![\vec{x} \mid t_1, \ldots, t_k]\!]$ is the unique arrow satisfying

    $$\pi_i \circ [\![\vec{x} \mid t_1, \ldots, t_k]\!] = [\![t_i]\!]$$

    by the universal property of products

[7]For many sorted languages fix one object A_S for each sort S.

The interpretation $[\![\vec{x} \mid \phi]\!] \in \mathcal{H}(A^n)$ of formulas φ in context $\vec{x} = x_1, \dots, x_n$ is defined inductively as follows:

- $[\![\vec{x} \mid R(t_1, \dots, t_k)]\!] = [\![\vec{x} \mid t_1, \dots, t_k]\!]^* [\![R]\!]$
- $[\![s = t]\!] = [\![\vec{x} \mid s, t]\!]^* (\mathsf{eq}_A)$ where $\mathsf{eq}_A = \exists_{\delta_A} \top$ with $\delta_A : A \to A \times A$ the 'diagonal'[8]
- $[\![\vec{x} \mid \exists y \,.\, \phi]\!] = \exists_\pi [\![\vec{x}, y \mid \phi]\!]$ where $\pi : A^{n+1} \to A^n$ is the projection omitting the last factor
- $[\![\vec{x} \mid \forall y \,.\, \phi]\!] = \forall_\pi [\![\vec{x}, y \mid \phi]\!]$ where $\pi : A^{n+1} \to A^n$ is the projection omitting the last factor
- Propositional connectives are interpreted by the Heyting algebra structure in the fibers of $\mathcal{H}$ as on Monday.

[8] In this line we have the problem that the symbol $=$ appears both at the **object level** and at the **meta level**. However, the Scott brackets $[\![-]\!]$ clear up possible confusion.

To show a **Soundness Theorem** for the interpretation of first-order logic in first-order hyperdoctrines we first need the following **Substitution Lemma**:

Substitution lemma

Let Σ be a first-order signatrure, let $\vec{x} = x_1, \ldots, x_n$ and $\vec{y} = y_1, \ldots, y_k$ be variable contexts, and let $s_1, \ldots, s_n$ be a list of Σ-terms in context $\vec{x}$.

1. For every Σ-term t in context $\vec{y}$ we have

$$[\![\vec{x} \mid t[s_1, \ldots, s_k/y_1, \ldots, y_k]]\!] = [\![y_1, \ldots, y_k \mid t]\!] \circ [\![\vec{x} \mid s_1, \ldots, s_k]\!].$$

2. For every Σ-formula ϕ in context $\vec{y}$ we have

$$[\![\vec{x} \mid \phi[s_1, \ldots, s_k/y_1, \ldots, y_k]]\!] = [\![\vec{x} \mid s_1, \ldots, s_k]\!]^* [\![y_1, \ldots, y_k \mid \phi]\!].$$

Proof.

Inductions on the structure of t and ϕ.
For 2 we need the Beck–Chevalley condition for formulas of the forms $\exists y \,.\, \phi$, and $\forall y \,.\, \phi$. □

Soundness Theorem

Let $\phi_1, \dots, \phi_n, \psi$ be formulas in context $\vec{x} = x_1, \dots, x_n$ over a signature Σ, and consider an interpretation $[\![-]\!]$ in a first-order hyperdoctrine $\mathcal{H}$ as two slides back.
If $\phi_1, \dots, \phi_n \vdash_{\vec{x}} \psi$ is derivable in intuitionistic first-order logic, then we have

$$[\![\vec{x} \mid \phi_1]\!] \wedge \cdots \wedge [\![\vec{x} \mid \phi_n]\!] \leq [\![\vec{x} \mid \psi]\!]$$

in $\mathcal{H}(A^n)$.

Proof.

By induction on the derivation of $\phi_1, \dots, \phi_n \vdash_{\vec{x}} \psi$, where we need the Beck-Chevalley condition in the case of $(=\text{-}E)$. $\square$

Completeness Theorem

Let $\phi_1, \ldots, \phi_n, \psi$ be formulas in context $\vec{x} = x_1, \ldots, x_n$ over a signature Σ.
If we have

$$[\![\vec{x} \mid \phi_1]\!] \wedge \cdots \wedge [\![\vec{x} \mid \phi_n]\!] \leq [\![\vec{x} \mid \psi]\!]$$

in all interpretations in hyperdoctrines $\mathcal{H}$, then $\phi_1, \ldots, \phi_n \vdash_{\vec{x}} \psi$ is derivable in intuitionistic first-order logic.

Proof sketch on next slide.

We construct an interpretation in a **syntactic Hyperdoctrine** $\mathcal{H}_\Sigma : \mathbb{C}_\Sigma^{\mathsf{op}} \to \mathsf{Pos}$, similar to the completeness proof for propositional intuitionistic logic using the Lindenbaum-Tarski algebra.

The category $\mathbb{C}_\Sigma$ is given as follows:

- Objects of are variable contexts $(x_1, \ldots, x_n)$.
- Arrows from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_k)$ are tuples $(x_1, \ldots, x_n \mid s_1, \ldots, s_k)$ of terms in context $(x_1, \ldots, x_n)$.
- Composition is given by simultaneous substitution:

$$(y_1, \ldots, y_k \mid t_1, \ldots, t_l) \circ (x_1, \ldots, x_n \mid s_1, \ldots, s_k) \;=\; (x_1, \ldots, x_n \mid \vec{t}[\vec{s}/\vec{y}])$$

- Identities are given by $(\vec{x} \mid \vec{x})$.
- Products in $\mathbb{C}_\Sigma$ are given by concatenation of lists of variables:

$$(x_1, \ldots, x_n) \times (y_1, \ldots, y_k) = (x_1, \ldots, x_n, y_1, \ldots, y_k)$$

  Since the variables in a context are required to be distinct, this may require renaming of variable.

Interpreting First-Order Logic in First-Order Hyperdoctrines

Completeness theorem — Proof ctd.

$\mathcal{H}_\Sigma$ is given as follows:

- For each context $\vec{x}$, the Heyting algebra $\mathcal{H}_\Sigma(\vec{x})$ consists of Σ-formulas in context $\vec{x}$, quotiented by logical equivalence and ordered by logical entailment entailment — i.e.

$$\phi \leq \psi \text{ in } \mathcal{H}_\Sigma(\vec{x}) \quad \text{iff} \quad \phi \vdash_{\vec{x}} \psi \text{ is derivable}$$

- For a morphism $\vec{t} : \vec{x} \to \vec{y}$, the reindexing map

$$\vec{t}^* : \mathcal{H}(\vec{y}) \to \mathcal{H}(\vec{x})$$

  is given by simultaneous substitution, i.e. $\vec{t}^*(\phi) = \phi(\vec{t}/\vec{y})$.

- A **first order theory** $\mathbb{T}$ is given by a signature Σ, and a set of **sentences** (formulas without free variables) in $\mathcal{L}(\Sigma)$ — the axioms of the theory.
- A **model** of $\mathbb{T}$ is an interpretation of $\mathcal{L}(\Sigma)$ such that

$$[\![\phi]\!] = \top \in \mathcal{H}(1)$$

  for all axioms.

Examples of first-order hyperdoctrines

The subset hyperdoctrine

The **contravariant powerset functor**

$$P : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Pos}$$

is a hyperdoctrine which we also call the **subset hyperdoctrine**, since its predicates are subsets. To see this, note that

- the powerset $P(S)$ of every set S is a **Boolean algebra**, and therefore a Heyting algebra
- it is easy to see that the inverse image functions $f^{-1} : P(T) \to P(S)$ for $f : S \to T$ preserve all Heyting algebra structure
- we have already seen that f^{-1} has a left adjoint given by **direct image**, and it is easy to see that the BC condition is satisfied
- right adjoint is left as an exercise

Models of a first order theory $\mathbb{T}$ in the subset hyperdoctrine are precisely the same as models in the sense of classical model theory!

Examples of first-order hyperdoctrines

Hyperdoctrines from complete Heyting algebras

Definition

A **complete Heyting algebra** is a Heyting algebra H in which every (possibly infinite) subset has a supremum.

Example

The Heyting algebra $\mathcal{O}(X)$ of open sets in a topological space X is complete.

- Given a complete Heyting algebra H, we can define a first-order hyperdoctrine

  $$\mathsf{fam}(H) : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Heyt}, \qquad S \mapsto (H^S, \leq)$$

  where the ordering on H^S is **pointwise**, i.e.

  $$(\phi : S \to H) \leq (\psi : H \to S) \qquad \text{iff} \qquad \forall s \in S \,.\, \phi(s) \leq \psi(s).$$

- Reindexing is given by precomposition, and $\forall$ and $\exists$ are given by

  $$\exists_f(\phi)(t) = \bigvee_{f(s)=t} \phi(s) \qquad \text{and} \qquad \forall_f(\phi)(t) = \bigwedge_{f(s)=t} \phi(s)$$

  for $f : S \to T$ and $\phi \in \mathsf{fam}(H)(S)$, i.e. $\phi : S \to H$.

Kleene's number realizability

- The first order theory of **Heyting Arithmetic** is given by $\Sigma = \{0, 1, +, \cdot\}$ and axioms including $0 \neq 1$ and **induction axioms**.
- Kleene's number realizability[9][10] is a construction which associates sets of natural numbers $[\![\phi]\!] \subseteq \mathbb{N}$ to Σ-sentences as follows (we also write $n \Vdash \phi$ and say 'n **realizes** ϕ' for $n \in [\![\phi]\!]$):
  - $[\![\bot]\!] = \varnothing$
  - $[\![\top]\!] = \mathbb{N}$
  - $[\![s = t]\!] = \{n \in \mathbb{N} \mid [\![s]\!] = [\![t]\!]\}$
  - $[\![\phi \wedge \psi]\!] = \{n \in \mathbb{N} \mid \mathrm{fst}(n) \in [\![\phi]\!] \;\&\; \mathrm{snd}(n) \in [\![\psi]\!]\}$
  - $[\![\phi \vee \psi]\!] = \{\langle 0, n\rangle \mid n \in [\![\phi]\!]\} \cup \{\langle 1, n\rangle \mid n \in [\![\psi]\!]\}$
  - $[\![\phi \Rightarrow \psi]\!] = \{n \in \mathbb{N} \mid \forall k \in [\![\phi]\!] \,.\, \{n\}(k) \in [\![\psi]\!]\}$
  - $[\![\forall x \,.\, \phi]\!] = \{n \in \mathbb{N} \mid \forall k \in \mathbb{N} \,.\, \{n\}(k) \in [\![\phi[\underline{k}/x]]\!]\}$
  - $[\![\exists x \,.\, \phi]\!] = \{\langle n, k\rangle \mid k \in [\![\phi[\underline{n}/x]]\!]\}$

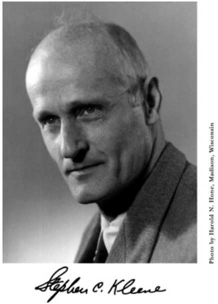
  where
  - $[\![s]\!] \in \mathbb{N}$ is the 'standard' interpretation of a closed Σ-term s.
  - $\langle -, -\rangle$ is a **primitive recursive pairing function** with **projections** fst, snd (such that $\mathrm{fst}\langle n, k\rangle = n$ and $\mathrm{snd}\langle n, k\rangle = k$)
  - $\{n\}(k)$ is the **Kleene bracket** notation for the possibly undefined application of the n-th partial recursive function (with respect to an **effective enumeration**) to argument k.
  - $\underline{n}$ is the **numeral** of n, represented e.g. as the **term** $1 + 1 + \cdots + 1$.

[9] S.C. Kleene. "On the interpretation of intuitionistic number theory". In: *J. Symb. Log.* (1945).

[10] A.S. Troelstra and D. van Dalen. *Constructivism in mathematics. Vol. I.* North-Holland Publishing Co., Amsterdam, 1988.

Kleene's number realizability

- These definitions represent an encoding of the interpretation of constructive proofs given by the **Brouwer-Heyting-Kolmogorov** interpretation, which says e.g. that a proof of an implication $\phi \Rightarrow \psi$ should be a construction transforming proofs of ϕ into proofs of ψ.
- call ϕ **realizable** if $[\![\phi]\!]$ is inhabited.
- one can show that provable sentences are realizable
- the realizability interpretation is useful to establish metamathematical properties of Heyting arithmetic such as the **existence property** and the **disjunction property**.
- around 1980, Martin Hyland gave a reformulation of number realizability in terms of hyperdoctrines (and toposes)



Stephen Cole Kleene


Martin Hyland

The effective tripos — a hyperdoctrine from computable functions

- The **effective tripos** is given by

$$\mathsf{eff} \ : \ \mathsf{Set}^{\mathsf{op}} \to \mathsf{Heyt}\,, \qquad\qquad S \mapsto (P(\mathbb{N})^S, \leq)$$

  where the ordering on $P(\mathbb{N})^S$ is given by

  $(\phi : S \to P(\mathbb{N})) \leq (\psi : S \to P(\mathbb{N}))$
  iff there exists a **partial recursive** $\alpha : \mathbb{N} \rightharpoonup \mathbb{N}$ such that $\forall n \in \phi(n) \,.\, \alpha(n) \in \psi(n)$.

- Using elementary recursion theory one can show that $(P(\mathbb{N})^S, \leq)$ is indeed a Heyting algebra for each S, (strictly speaking it's a **pre-Heyting algebra** since it's not antisymmetric — but we can quotient to get a genuine Heyting algebra), and it's easy to see that the reindexing maps (given by precomposition) have left and right adjoints satisfying the BC condition.
- This means that eff is a first-order hyperdoctrine, and it is turns out that it's even a **tripos** (and so are the earlier examples):

The effective tripos — ctd.

Definition

A **tripos** is a first-order hyperdoctrine $\mathcal{P} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Heyt}$ such that for every object A in $\mathbb{C}$ there exists an object $\mathcal{P}(A)$ and a predicate $(\in_A) \in A \times \mathcal{P}(A)$ such that for every object B in $\mathbb{C}$ and predicate $\phi \in \mathcal{P}(A \times B)$ there exists an arrow $f : B \to \mathcal{P}(A)$ such that $(1_A \times f)^*(\in_A) = \phi$.

- Triposes are not only models of first order logic, but of **non-extensional higher order logic** (with power types).
- Triposes were introduced by Hyland, Johnstone, and Pitts as an auxiliary device to exhibit **realizability toposes** via the **tripo-to-topos construction**[11].
- The realizability topos constructed from the effective tripos eff is Hyland's **effective topos**[12].

[11] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. "Tripos theory". In: *Math. Proc. Cambridge Philos. Soc.* (1980).

[12] J.M.E. Hyland. "The effective topos". In: *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*. Amsterdam: North-Holland, 1982.

The internal language of a hyperdoctrine

The **internal language** of a hyperdoctrine $\mathcal{H} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Heyt}$ is the first-order language $\mathcal{L}(\Sigma)$ where Σ is the **maximal many-sorted signature**:

- every object $A \in \Sigma$ is a **sort**
- there are **relation symbols** of arity $A_1, \ldots, A_n$ for all predicates $\phi \in \mathcal{H}(A_1 \times \ldots \times A_n)$, and there are **function symbols** of arity $A_1, \ldots, A_n \to B$ for every morphism $f : A_1 \times \ldots \times A_n \to B$.

The use of the internal language is demonstrated by the following definition of the **tripos-to-topos construction**.

The tripos-to-topos construction

For a tripos $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Heyt}$, the category $\mathbb{C}[\mathcal{P}]$ is given as follows:

- objects are pairs (A, ρ) of an object A in $\mathbb{C}$ and a predicate $\rho \in \mathcal{P}(A \times A)$ such that
  - (sym) $\rho(x, y) \ \vdash_{x,y} \ \rho(y, x)$
  - (trans) $\rho(x, y), \rho(y, z) \ \vdash_{x,y,z} \ \rho(x, z)$, and
- arrows from (A, ρ) to (B, σ) are given by predicates $\phi \in \mathcal{P}(A \times B)$ satisfying
  - (strict) $\phi(x, y) \ \vdash_{x,y} \ \rho(x, x) \wedge \sigma(y, y)$
  - (cong) $\phi(x, y), \rho(x, x'), \sigma(y, y') \ \vdash_{x,x',y,y'} \ \phi(x', y')$
  - (singval) $\phi(x, y), \phi(x, y') \ \vdash_{x,y,y'} \ \sigma(y, y')$
  - (tot) $\rho(x, x) \ \vdash_{x} \ \exists_{y} \,.\, \phi(x, y)$

Thus, objects are **partial equivalence relations** in $\mathcal{P}$, and arrows are **compatible functional relations**.

The effective topos

- Applying the tripos-to-topos construction to the effective tripos yields the **effective topos** $\mathcal{E}\mathsf{ff} = \mathsf{Set}[\mathsf{eff}]$[13].
- Improving over Kleene's number realizability, it admits a model of **intuitionistic higher order arithmetic**.
- This model validates interesting interesting classically constradictory statements, e.g. that all functions from $\mathbb{N}$ to $\mathbb{N}$ are computable.

[13] J.M.E. Hyland. "The effective topos". In: *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*. Amsterdam: North-Holland, 1982.

Fragments of first order logic

- **Fragments of first order logic** are subsets of the set $\{\top, \wedge, \bot, \vee, \Rightarrow, \forall, \exists, =\}$ of connectives of first order logic
- for example
  - **regular logic** is given by $\{\top, \wedge, \exists, =\}$, and
  - **coherent logic** is given by $\{\top, \wedge, \bot, \vee, \exists, =\}$.
- For technical reason we always want to keep $\top$ and $\wedge$.
- Sometimes we are interested in **weakenings** of the notion of first-order hyperdoctrine which are only able to interpret a fragment of first order logic.

Regular and coherent hyperdoctrines

Definition

A **regular hyperdoctrine** is an indexed poset

$$\mathcal{R} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Pos}$$

such that

- $\mathbb{C}$ has finite products
- all $\mathcal{R}(A)$ have finite meets which are preserved by reindexing
- for every $f : A \to B$, the reindexing map $f^* : \mathcal{R}(B) \to \mathcal{R}(A)$ has a left adjoint $\exists_f : \mathcal{R}(A) \to \mathcal{R}(B)$ which satisfies the Beck-Chevalley condition (as before), and the **Frobenius condition**: for all $\phi \in \mathcal{R}(B)$ and $\psi \in \mathcal{R}(A)$ we have

$$\phi \wedge (\exists_f \psi) = \exists_f (f^* \phi \wedge \psi).$$

A **coherent hyperdoctrine** is a regular hyperdoctrine $\mathcal{R}$ where

- all fibers $\mathcal{H}(A)$ are distributive lattices, and
- reindexing preserves the structure of distributive lattices.

Regular and coherent hyperdoctrines

- The tripos-to-topos construction only uses the structure of a regular hyperdoctrine!
- However, if we apply it to a regular hyperdoctrine, we won't get a topos, but only an **exact category**.
- An example of a coherent hyperdoctrine is obtained by replacing partial recursive functions in the definition of the effective tripos eff by **prmitive recursive** functions.
- In this case, the 'tripos-to-topos construction' produces a **pretopos**!
- Other examples of regular hyperdoctrines arise from topology.