Triposes, q-Toposes and Toposes

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Abstract

We characterize the tripos-to-topos construction of Hyland, Johnstone and Pitts as a biadjunction in a 2-category enriched category of equipment-like structures. These abstract concepts are necessary to handle the presence of oplax constructs — the construction is only *oplax* functorial on a certain class of tripos morphisms.

A by-product of our analysis is the decomposition of the tripos-to-topos construction into two steps, the intermediate step being a generalization of quasitoposes.

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1. Introduction

The motivation of the presented work is to get a better understanding and in particular a universal characterization of Hyland, Johnstone and Pitts' *tripos-to-topos construction* [1].

Triposes and toposes are two classes of categorical models of higher order intuitionistic logic, of slightly different kinds: triposes are *indexed preorders* $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Ord}$ where logical formulas are interpreted as elements $\varphi \in \mathcal{P}(I)$ (called *predicates*) of some fiber, whereas toposes are *categories* and formulas

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are interpreted by monomorphisms. The class of triposes is more general since monomorphisms in any topos give rise to a tripos.

For any tripos $\mathcal{P}$, the tripos-to-topos construction produces a topos $\boldsymbol{T}(\mathcal{P})$ whose objects are partial equivalence relations internal to $\mathcal{P}$, and whose morphisms are (external) equivalence classes of internal functional relations.

A way of understanding the construction is that it 'freely' internalizes the abstract predicates of a tripos as monomorphisms in a topos, and it is the word 'freely' that a universal characterization has to make precise.

To formulate a universal property for the tripos-to-topos construction, we have to place triposes and toposes as objects in a categorical framework, and the natural choice is to consider 2-categories of toposes and triposes where 1-cells of toposes are functors, and the 1-cells of triposes are natural transformations relative to a 'change of base' functor between the base categories.

A logical intuition on these 1-cells is that they map the interpretation of types, terms, and formulas from one model into another, in a way that commutes with the interpretation of logical connectives. However it turns out that the requirement to preserve *all* logical connectives is too restrictive and we rather want to focus on some fragment of first order logic whose interpretation we want to preserve. A natural choice here is the $(\wedge, \top, =, \exists)$-fragment (called *regular logic*) encompassing the connectives necessary to define the partial equivalence relations and functional relations occurring in the tripos-to-topos construction, and indeed we can obtain a universal characterization this way, as a restriction

$$\begin{tikzcd}
\mathbf{Top}_r \arrow[r, bend left] \arrow[r, phantom, "\top"] \arrow[d, hook] & \mathbf{Trip}_r \arrow[l, bend left] \arrow[d, hook] \\
\mathbf{Ext} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{EED} \arrow[l, bend left]
\end{tikzcd}$$ [1]

of a biadjunction between the 2-categories **EED** of *elementary existential hyperdoctrines* (i.e. indexed preorders modeling regular logic) and **Ext** of *exact categories*. This biadjunction is the subject of Maietti and Rosolini's [2].

While this biadjunction arising from regular logic is the appropriate framework for general elementary existential doctrines, it does not account for a phenomenon that is particular to the higher order nature of triposes: using a construction related to sheafification, the tripos-to-topos construction can be extended to morphisms which do not have to preserve regular logic, but only finite meets. This is important in the context of geometric morphisms, and for this reason already explained in the original paper [1] on triposes.

Functors from finite-meet-preserving tripos morphisms

The basic idea (as already described in [1]) is the following: when trying to construct a functor $\boldsymbol{T}(\mathfrak{F}) : \boldsymbol{T}(\mathcal{P}) \to \boldsymbol{T}(\mathcal{R})$ from a tripos morphism $\mathfrak{F} : \mathcal{P} \to \mathcal{R}$ which only preserves finite meets, one runs into problems with the morphism part since $\mathfrak{F}$ need not preserve totality of functional relations (totality being expressed by a judgment involving existential quantification). The solution is to consider only morphisms in $\boldsymbol{T}(\mathcal{P})$ whose totality is witnessed (or 'tracked') by a morphism in the base category of $\mathcal{P}$ – we can do this without loss of

[1] $\mathbf{Top}_r$ and $\mathbf{Trip}_r$ are the 2-categories of toposes and triposes, with maps preserving *regular* logic.

generality since by an argument related to sheafification, every object in $\boldsymbol{T}(\mathcal{P})$ is isomorphic to a 'weakly complete object', which is an object with the property that *all* morphisms having it as codomain are tracked by morphisms in the base.

Using this technique, we can construct a functor from the subcategory of $\boldsymbol{T}(\mathcal{P})$ on weakly complete objects to $\boldsymbol{T}(\mathcal{R})$, and precomposing by an equivalence of categories, we obtain the desired functor $\boldsymbol{T}(\mathfrak{F}) : \boldsymbol{T}(\mathcal{P}) \to \boldsymbol{T}(\mathcal{R})$. This is easy enough to describe, but the weak completion process causes problems with functoriality: given tripos morphisms $\mathcal{P} \xrightarrow{\mathfrak{F}} \mathcal{R} \xrightarrow{\mathfrak{G}} \mathcal{S}$, there is an intermediate weak completion step in $\boldsymbol{T}(\mathfrak{G}) \circ \boldsymbol{T}(\mathfrak{F})$ but not in $\boldsymbol{T}(\mathfrak{G} \circ \mathfrak{F})$. This results into a non-invertible natural transformation $\boldsymbol{T}(\mathfrak{G} \circ \mathfrak{F}) \to \boldsymbol{T}(\mathfrak{G}) \circ \boldsymbol{T}(\mathfrak{F})$, making $\boldsymbol{T}$ an *oplax* functor.

For a simple example of this phenomenon, let $\mathbb{B} = \{\mathsf{false} \leq \mathsf{true}\}$ be the locale of booleans. The indexed posets $\mathbf{Set}(-, \mathbb{B})$ and $\mathbf{Set}(-, \mathbb{B} \times \mathbb{B})$ are triposes, and the induced toposes are equivalent to $\mathbf{Set}$ and $\mathbf{Set} \times \mathbf{Set}$, respectively. The meet-preserving maps

$$\delta = \langle \mathrm{id}, \mathrm{id} \rangle : \mathbb{B} \to \mathbb{B} \times \mathbb{B} \quad \text{and} \quad \wedge : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$

give rise to tripos morphisms

$$\mathbf{Set}(-, \mathbb{B}) \xrightarrow{\mathbf{Set}(-,\delta)} \mathbf{Set}(-, \mathbb{B} \times \mathbb{B}) \quad \text{and} \quad \mathbf{Set}(-, \mathbb{B} \times \mathbb{B}) \xrightarrow{\mathbf{Set}(-,\wedge)} \mathbf{Set}(-, \mathbb{B}),$$

and the induced functors turn out to be the familiar

$$\mathbf{Set} \xrightarrow{\Delta = \langle \mathrm{id}, \mathrm{id} \rangle} \mathbf{Set} \times \mathbf{Set} \quad \text{and} \quad \mathbf{Set} \times \mathbf{Set} \xrightarrow{(- \times -)} \mathbf{Set}.$$

Since the composition of the maps $\wedge \circ \delta = \mathrm{id}_{\mathbb{B}}$ gives rise to $\mathrm{id}_{\mathbf{Set}}$, we obtain a *non-invertible* constraint cell $\mathrm{id}_{\mathbf{Set}} \to \times \circ \Delta$ (the unit of $\Delta \dashv \times$).

Pre-equipments

One objective of this paper is to explain how this oplax nature of the tripos-to-topos construction can be understood in a 2-categorical framework, namely as a biadjunction of *pre-equipments*. Pre-equipments are a variant of *(proarrow) equipments* introduced by Wood [3, 4] and examined by Verity and others [5, 6, 7, 8]. Formally, they are 2-categories $\mathbf{C}$ with a designated subclass $\mathbf{C}_r$ of *regular* 1-cells[2], and we consider *special functors* and *special transformations* between them, which are oplax functors and transformations where some of the constraint cells are invertible, and furthermore the components of the transformations are regular.

The nice thing about special functors and transformations is that contrary to general oplax functors and transformations between 2-categories they form a well behaved three dimensional structure, which has been formalized as a *2-category enriched category* by Verity [5].

An overview of 2-category enriched categories is given in Section 2.1, where we also introduce the concept of biadjunction in a 2-category enriched category. The 2-category enriched category $\mathfrak{Spec}$ of pre-equipments is defined in Section 2.2. Taking inspiration from [9], we refer to 1-cells, 2-cells, and biadjunctions in $\mathfrak{Spec}$ as *special functors, special transformations, and special biadjunctions.*

[2] In our case these are the 1-cells commuting with regular logic, hence the name.

The pre-equipments **Top** of toposes and **Trip** of triposes are introduced in Definitions 2.6 and 3.4, respectively. Section 2.3 contains a criterion for the existence of special left biadjoints that is used later to prove Theorems 4.18 and Theorem 5.12.

Q-toposes

The second objective of the paper is to describe a decomposition of the tripos-to-topos construction which arises from a careful examination of weakly complete objects and which fits nicely into the framework of pre-equipments.

A possible starting point to the relevant ideas is the observation that the concept of 'weakly complete object' in a topos $\boldsymbol{T}(\mathcal{P})$ is not a good categorical concept, since it is not invariant under isomorphism – as remarked earlier, *every* object in $\boldsymbol{T}(\mathcal{P})$ is isomorphic to a weakly complete one. If we want weakly complete objects to have a categorical characterization, we have to change the morphisms and consider the wide subcategory $\boldsymbol{Q}(\mathcal{P}) \subseteq \boldsymbol{T}(\mathcal{P})$ on those functional relations in $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Ord}$ that have tracking morphisms in $\mathbb{C}$. In $\boldsymbol{Q}(\mathcal{P})$, weakly complete objects can be characterized as *coarse objects*, which is a concept known from quasitoposes.

This observation sheds new light on weakly complete objects, by analogy to a classical situation: given a Lawvere-Tierney topology $j : \Omega \to \Omega$ on a topos $\mathcal{E}$, the category of j-separated objects in $\mathcal{E}$ is a quasitopos, and the coarse objects in this quasitopos are precisely the j-sheaves in $\mathcal{E}$. In the context of triposes we can recover this situation for (pre)sheaves on locales: if $\mathcal{P} = \mathbf{Set}(-, A) : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}$ is a tripos induced by a locale A, the category $\boldsymbol{Q}(\mathcal{P})$ is equivalent to the quasitopos $\mathrm{Sep}(A)$ of *separated presheaves* on A, and the subcategory of weakly complete objects is equivalent to sheaves on A.

For general triposes $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Ord}$ we cannot show that $\boldsymbol{Q}(\mathcal{P})$ is a quasitopos, for lack of exponential objects. Therefore we axiomatize the weaker concept of *q-topos*, which instead requires only the quasitopos version of power objects. The notion of q-topos is examined in Section 4, where we show in particular that the fibration of strong monomorphisms in a q-topos is a tripos (Lemma 4.13), and the subcategory of coarse objects is a topos (Lemma 4.16). This allows us to define special functors $\boldsymbol{S} : \mathbf{QTop} \to \mathbf{Trip}$ and $\boldsymbol{T} : \mathbf{QTop} \to \mathbf{Top}$ from the pre-equipment **QTop** of q-toposes to the pre-equipments of triposes and toposes, where the $\boldsymbol{T}$ is special left biadjoint to the inclusion $\boldsymbol{U} : \mathbf{Top} \to \mathbf{QTop}$ (Theorem 4.18).

The tripos-to-q-topos construction

The last section of the paper is devoted to the proof that the category $\boldsymbol{Q}(\mathcal{P})$ is a q-topos for every tripos $\mathcal{P}$, and that this construction constitutes a special left biadjoint to $\boldsymbol{S} : \mathbf{QTop} \to \mathbf{Trip}$.

This special left biadjoint, together with the special left biadjoint $\boldsymbol{T}$ to $\boldsymbol{U} : \mathbf{QTop} \to \mathbf{Top}$ constitute our decomposition and characterization of the tripos-to-topos construction.

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend left, "\boldsymbol{U}"] \arrow[r, phantom, "\top"] & \mathbf{QTop} \arrow[l, bend left, "\boldsymbol{T}"] \arrow[r, bend left, "\boldsymbol{S}"] \arrow[r, phantom, "\top"] & \mathbf{Trip} \arrow[l, bend left, "\boldsymbol{Q}"] \end{tikzcd}$$

Terminology, conventions

We consider different kinds of functors and natural transformations between 2-categories, and we will use the adjectives *strict, strong, lax and oplax* to specify whether they have identity-, isomorphic or directed constraint cells.

For an oplax functor $F : \mathbf{A} \to \mathbf{B}$ and $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{A}$, the direction of constraint cells is $\phi_{f,g} : F(gf) \to Fg\, Ff$ and $\phi_A : F(\mathrm{id}_A) \to \mathrm{id}_{FA}$. We generally denote composition and identity constraints by $\phi_{f,g}$ and ϕ_A, or simply by ϕ for more compact notation. For an oplax transformation $\eta : F \to G : \mathbf{A} \to \mathbf{B}$, the direction of constraints is $\eta_f : \eta_B \circ Ff \to Gf \circ \eta_A$.

When taking about categories with structure (like e.g. finite limits), we adopt the established practice that categories are always equipped with a choice of this structure, but structure preserving functors are not required to preserve the choices.

2. Pre-equipments

2.1. 2-category enriched categories

2-category enriched categories are the 2-category version of Verity's bicategory enriched categories [5]: they are categories enriched in a symmetric multicategory of 2-categories and a certain kind of multilinear functors, called 'n-homomorphisms'.

Definition 2.1 (n-homomorphism) Let $\mathbf{A}_1, \ldots, \mathbf{A}_n, \mathbf{B}$ be 2-categories. An *n-homomorphism* $F : \mathbf{A}_1, \ldots, \mathbf{A}_n \to \mathbf{B}$ is given by

- An object $F(A_1, \ldots, A_n) \in \mathbf{B}$ for each n-tuple $(A_1, \ldots, A_n)$ of objects with $A_i \in \mathbf{A}_i$.
- For each $1 \leq i \leq n$ and each $(n-1)$-tuple $(A_l)_{l \neq i}$ of objects with $A_i \in \mathbf{A}_i$ a *strong functor* $F(A_1, \ldots, A_{i-1}, -, A_{i+1}, \ldots, A_n) : \mathbf{A}_i \to \mathbf{B}$ enriching the mapping on objects.
  In the following we leave constant arguments implicit in the notation, and write e.g. $F(-_i)$ for this functor.
- For all $1 \leq i < j \leq n$, all appropriate $(n-2)$-tuples of objects (suppressed in the notation), and all $f_i : A_i \to A'_i, f_j : A_j \to A'_j$ isomorphic 2-cells

$$\begin{tikzcd} F(A_i, A_j) \arrow[r, "F(A_i,f_j)"] \arrow[d, "F(f_i,A_j)"'] \arrow[dr, phantom, "\Downarrow F(f_i,f_j)"] & F(A_i, A'_j) \arrow[d, "F(f_i,A'_j)"] \\ F(A'_i, A_j) \arrow[r, "F(A'_i,f_j)"'] & F(A'_i, A'_j) \end{tikzcd}$$

  such that
  - The 1-cells $F(f_i)$ together with the 2-cells $F(f_i, f_j)$ give rise to strong transformations of type $F(A_i, -_j) \to F(A'_i, -_j)$,
  - The 1-cells $F(f_j)$ together with the 2-cells $F(f_i, f_j)$ give rise to strong transformations of type $F(-_i, A_j) \to F(-_i, A'_j)$,
  - For each triple $1 \leq i < j < k \leq n$, for all $f_i : A_i \to A'_i, f_j : A_j \to A'_j, f_k : A_k \to A'_k$ (and for all implicit $(n-3)$-tuples of objects), we

have

$$
\begin{tikzcd}[column sep=tiny]
 & FA_iA_jA_k \arrow[dl] \arrow[dr] & \\
FA'_iA_jA_k \arrow[dr] \arrow[dd] \arrow[rr, phantom, "\Rightarrow"] & & FA_iA_jA'_k \arrow[dl] \arrow[dd] \\
 & FA'_iA_jA'_k \arrow[dd] \arrow[dr, Rightarrow] & \\
FA'_iA'_jA_k \arrow[ur, Rightarrow] \arrow[dr] & & FA_iA'_jA'_k \arrow[dl] \\
 & FA'_iA'_jA'_k &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=tiny]
 & FA_iA_jA_k \arrow[dl] \arrow[dr] \arrow[dd] & \\
FA'_iA_jA_k \arrow[dr, Rightarrow] \arrow[dd] & & FA_iA_jA'_k \arrow[dd] \\
 & FA_iA'_jA_k \arrow[ur, Rightarrow] \arrow[dl] \arrow[dr] & \\
FA'_iA'_jA_k \arrow[dr] \arrow[rr, phantom, "\Rightarrow"] & & FA_iA'_jA'_k \arrow[dl] \\
 & FA'_iA'_jA'_k &
\end{tikzcd}
.
$$

$\diamondsuit$

Observe that a 0-homomorphism is just an object of $\mathbf{B}$, and a 1-homomorphism is a strong functor.

We skip the definition of composition of n-homomorphisms and the verification of the axioms of a symmetric multicategory, and refer to Verity's thesis [5]. The following lemma justifies and characterizes n-homomorphisms by relating them to strong higher dimensional structure.

Lemma 2.2 *Let $\mathbf{A}_1, \ldots, \mathbf{A}_{n+1}, \mathbf{B}$ be 2-categories. There are natural bijections*

$$\mathbf{hom}(\mathbf{A}_1, \ldots, \mathbf{A}_{n+1}; \mathbf{B}) \cong \mathbf{hom}(\mathbf{A}_1, \ldots, \mathbf{A}_n; [\![\mathbf{A}_{n+1}, \mathbf{B}]\!]),$$

where **hom** *denotes sets of n-homomorphisms, and $[\![-,-]\!]$ is the 2-category of strong functors, strong transformations and modifications.* $\blacksquare$

A 2-category enriched category is now given by a set $\mathbf{X}_0$ of objects, for each pair X, Y of objects a 2-category $\mathbf{X}(X,Y)$, identity 0-homomorphisms id_X (which are just objects of $\mathbf{X}(X,X)$), and composition 2-homomorphisms

$$\mathrm{comp}_{\mathrm{X,Y,Z}} : \mathbf{X}(X,Y), \mathbf{X}(Y,Z) \longrightarrow \mathbf{X}(X,Z),$$

subject to *strict* associativity and identity axioms. In 2-category enriched categories, we call the 0-, 1-, and 2-cells of the 2-categories $\mathbf{X}(X,Y)$ *1-, 2-, and 3-cells* of the 2-category enriched category, respectively, and we denote horizontal composition of 1-, 2- and 3-cells by juxtaposition (i.e. $\mathrm{comp}_{X,Y,Z}(f,g) = gf$), and vertical composition of 2- and 3-cells by $(- \circ -)$. For $\eta : f \to f'$ in $\mathbf{X}(X,Y)$ and $\theta : g \to g'$ in $\mathbf{X}(Y,Z)$ we write the exchange isomorphism for horizontal composition as $\theta\eta : \theta f' \circ g\eta \xrightarrow{\cong} g'\eta \circ \theta f$ – in a diagram this looks as follows:

$$
\begin{tikzcd}
gf \arrow[r, "g\eta"] \arrow[d, "\theta f"'] \arrow[dr, phantom, "\swarrow\theta\eta"] & gf' \arrow[d, "\theta f'"] \\
g'f \arrow[r, "g'\eta"'] & g'f'
\end{tikzcd}
$$

Now we come to the concept of *biadjunction*, which we will use to characterize the tripos-to-topos construction.

Definition 2.3 Let $\mathbf{X}$ be a 2-category enriched category, and let A, B be objects of $\mathbf{X}$. A biadjunction between A and B is given by

- 1-cells $\quad f : A \to B \quad g : B \to A,$
- 2-cells $\quad \eta : \mathrm{id}_A \to gf \quad \varepsilon : fg \to \mathrm{id}_B$
- *invertible* 3-cells $\quad \mu : \mathrm{id}_g \xrightarrow{\cong} g\varepsilon \circ \eta g \quad \nu : \varepsilon f \circ f\eta \xrightarrow{\cong} \mathrm{id}_f$

such that the diagrams

$$
\begin{tikzcd}
 & \mathrm{id}_A \arrow[dl, "\eta"'] \arrow[dr, "\eta"] \arrow[d, phantom, "\overset{\eta\eta}{\Rightarrow}"] & \\
gf \arrow[r, "\eta gf"] \arrow[dr, "\mathrm{id}"'] \arrow[dr, phantom, "\mu f \Nearrow", near start, shift left=3] & gfgf \arrow[d, "g\varepsilon f"] & gf \arrow[l, "gf\eta"'] \arrow[dl, "\mathrm{id}"] \arrow[dl, phantom, "\Swarrow g\nu", near start, shift right=3] \\
 & gf &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & fg \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{id}"] \arrow[d, "f\eta g"] \arrow[dl, phantom, "\Swarrow f\mu", near end, shift right=3] \arrow[dr, phantom, "\nu g \Nearrow", near end, shift left=3] & \\
fg \arrow[dr, "\varepsilon"'] & fgfg \arrow[l, "fg\varepsilon"'] \arrow[r, "\varepsilon fg"] \arrow[d, phantom, "\overset{\varepsilon\varepsilon}{\Rightarrow}"] & fg \arrow[dl, "\varepsilon"] \\
 & \mathrm{id}_B &
\end{tikzcd}
$$

of isomorphic 3-cells compose to identities in $\mathbf{X}(A,A)$ and $\mathbf{X}(B,B)$, respectively. $\diamondsuit$

The following lemma, which is a categorification of the fact that adjoints are unique up to isomorphism, is important to us since we want to use biadjunctions to characterize the tripos-to-topos construction up to equivalence of categories.

Lemma 2.4 *Let* $\mathbf{X}$ *be a 2-category enriched category, and let*

$$(f \dashv g : B \to A, \eta, \varepsilon, \mu, \nu) \quad \textit{and} \quad (f' \dashv g : B \to A, \eta', \varepsilon', \mu', \nu')$$

be two biadjunctions in $\mathbf{X}$ *sharing the same right adjoint* g*. Then* f *and* f' *are equivalent.*

Proof. The 2-cells between f and f' are given by $\varepsilon f' \circ f\eta' : f \to f'$ and $\varepsilon' f \circ f'\eta : f' \to f$. We leave it to the reader to construct isomorphic 3-cells $\alpha : \mathrm{id}_f \xrightarrow{\cong} \varepsilon' f \circ f'\eta \circ \varepsilon f' \circ f\eta'$ and $\beta : \varepsilon f' \circ f\eta' \circ \varepsilon' f \circ f'\eta \xrightarrow{\cong} \mathrm{id}'_f$ and to verify that these constitute an adjoint equivalence between f and f'. $\blacksquare$

2.2. Pre-equipments

We now introduce the 2-category enriched category $\mathfrak{Spec}$ of pre-equipments and special functors and have a close look at its biadjunctions, which we call *special* biadjunctions. Special biadjunctions are the goal of our higher dimensional digressions – we will use them to characterize the tripos-to-topos construction.

Pre-equipments are very similar to what Verity calls *weak proarrow equipments* – Verity's definition is a bit more general since he does not have the closedness condition under vertical isomorphisms and furthermore considers bicategories, not 2-categories. For morphisms, the difference is more substantial: the 1-cells in $\mathfrak{Spec}$ are certain oplax functors between the 2-categories underlying pre-equipments, whereas Verity considers double functors between double categories induced by weak proarrow equipments. Every 1-cell in $\mathfrak{Spec}$ gives rise to a morphism of weak proarrow equipments, but not the embedding is not full.

Definition 2.5 1. A *pre-equipment* is given by a 2-category $\mathbf{C}$ together with a designated subcategory $\mathbf{C}_r$ of the 1-cells which is closed under vertical isomorphisms.

We think of the 1-cells in $\mathbf{C}_r$ as particularly 'nice' arrows and we call them *regular 1-cells.*

2. A *special functor* between pre-equipments $\mathbf{C}$ and $\mathbf{D}$ is an oplax functor $F : \mathbf{C} \to \mathbf{D}$ such that Ff is a regular 1-cell whenever f is a regular 1-cell, all identity constraints $F\mathrm{id}_A \to \mathrm{id}_{FA}$ are invertible, and the composition constraints $F(gf) \to Fg\,Ff$ are invertible whenever g is a regular 1-cell.
3. A *special transformation* between special functors F, G is an oplax transformation $\eta : F \to G$ such that all η_A are regular 1-cells and the naturality constraint $\eta_B\,Ff \to Gf\,\eta_A$ is invertible whenever f is a regular 1-cell.

It is well known that oplax functors, oplax transformations, and modifications between 2-categories $\mathbf{A}, \mathbf{B}$ form a 2-category $\mathfrak{Oplax}(\mathbf{A}, \mathbf{B})$ (see e.g. [10, Section 2.0]). Moreover, special transformations are closed under composition, and thus for pre-equipments $\mathbf{C}, \mathbf{D}$, there is a locally full sub-2-category $\mathfrak{Spec}(\mathbf{C}, \mathbf{D})$ of $\mathfrak{Oplax}(\mathbf{C}, \mathbf{D})$ which consists of special functors, special transformations and modifications.

Strong functors, *strong* transformations, and modifications form a locally full sub-2-category of $\mathfrak{Spec}(\mathbf{A}, \mathbf{B})$, which we denote by $\mathfrak{Strong}(\mathbf{A}, \mathbf{B})$. It is well known that strong functors and transformations can be can be horizontally composed, giving rise to a 2-homomorphism

$$\mathrm{comp}_{\mathbf{C},\mathbf{D},\mathbf{E}} : \mathfrak{Strong}(\mathbf{C}, \mathbf{D}), \mathfrak{Strong}(\mathbf{D}, \mathbf{E}) \longrightarrow \mathfrak{Strong}(\mathbf{C}, \mathbf{E}). \tag{2.1}$$

for every triple $\mathbf{C}, \mathbf{D}, \mathbf{E}$ of 2-categories. This allows to define a 2-category enriched category $\mathfrak{Strong}$ of 2-categories, strong functors, strong transformations, and modifications.

However, it is *not* possible to extend the composition 2-homomorphism (2.1) to general oplax functors and transformations, since the directions of 2-cells do not match in certain situations when we pass to non-invertible constraint cells.

Now the interest of special functors and transformations is that they impose just enough invertibility of constraint cells to avoid this problem: for pre-equipments $\mathbf{C}, \mathbf{D}, \mathbf{E}$, the composition 2-homomorphism (2.1) extends directly to a 2-homomorphism

$$\mathrm{comp}_{\mathbf{C},\mathbf{D},\mathbf{E}} : \mathfrak{Spec}(\mathbf{C}, \mathbf{D}), \mathfrak{Spec}(\mathbf{D}, \mathbf{E}) \longrightarrow \mathfrak{Spec}(\mathbf{C}, \mathbf{E}).$$

Using this composition, we can define a 2-category enriched category $\mathfrak{Spec}$. We omit the details, which are routine to fill in and can be found in [5] in more generality.

As a first example, we define the pre-equipment of toposes. In Sections 3 and 4, we will furthermore introduce the pre-equipments **Trip** and **QTop** of triposes and q-toposes, respectively.

Definition 2.6 The pre-equipment **Top** of toposes has the 2-category of toposes, finite limit preserving functors and arbitrary natural transformations as underlying 2-category, and regular (i.e. epimorphism preserving) functors as regular 1-cells. ◇

We call a biadjunction in $\mathfrak{Spec}$ a *special biadjunction*. Special biadjunctions have the following interesting property, which we state without proof.

Lemma 2.7 *Let $\boldsymbol{F} \dashv \boldsymbol{U} : \mathbf{D} \to \mathbf{C}$ be a special biadjunction. Then $\boldsymbol{U}$ is a strong functor.* ■

2.3. Constructing special left biadjoints

As stated in the introduction, the objective of this article is to characterize the tripos-to-topos construction as the composition of two special left biadjoints to forgetful functors

$$\mathbf{Top} \to \mathbf{QTop} \to \mathbf{Trip}$$

between pre-equipments of toposes, q-toposes and triposes. The explicit construction of these left adjoints is cumbersome, but it is possible to give simple conditions for the *existence* of special left biadjoints, inspired by the description of ordinary left adjoints in terms of 'universal arrows' [11, Section III.1].

Concretely, the characterization given below in Lemma 2.9 can be viewed as a relaxation of the characterization of (strong) left biadjoints in terms of birepresentability (see e.g. [12, Section 5]), in a way that leaves room for 'oplaxness'. The characterization makes use of the notion of 'oplax comma category'[3]:

Definition 2.8 For a strong functor $\boldsymbol{U} : \mathbf{B} \to \mathbf{A}$ between 2-categories $\mathbf{A}, \mathbf{B}$ and an object $A \in \mathbf{A}$, the *oplax comma category* $(A \swarrow \boldsymbol{U})$ is the 2-category defined as follows.

- *Objects* are pairs (B, f) where $B \in \mathbf{B}$ and $f : A \to \boldsymbol{U}B$
- *1-cells* from (B, f) to (C, g) are pairs $(h : B \to C, \alpha : g \to \boldsymbol{U}h \circ f)$
- *2-cells* between parallel 1-cells (h, α), $(k, \beta) : (B, f) \to (C, g)$ are 2-cells $\xi : h \to k$ such that $(\boldsymbol{U}\xi \circ f) \cdot \alpha = \beta$.

$$
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, "g"] \arrow[dr, phantom, shift right=3, "\Downarrow\alpha"] & \\
\boldsymbol{U}B \arrow[r, "\boldsymbol{U}h" description] \arrow[r, bend right=40, "\boldsymbol{U}k" description] \arrow[r, phantom, shift right=2, "\Downarrow\boldsymbol{U}\xi"] & \boldsymbol{U}C \\
B \arrow[r, "h" description] \arrow[r, bend right=40, "k" description] \arrow[r, phantom, shift right=2, "\Downarrow\xi"] & C
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, "g"] & \\
\boldsymbol{U}B \arrow[r, bend right=40, "\boldsymbol{U}k" description] \arrow[r, phantom, "\Downarrow\beta"] & \boldsymbol{U}C \\
B \arrow[r, bend right=40, "k" description] & C
\end{tikzcd}
$$

(Here and in the following, we often depict constructions in oplax comma categories $(A \swarrow \boldsymbol{U})$ by drawing the data in $\mathbf{B}$ under that in $\mathbf{A}$, and aligning the part in $\mathbf{B}$ with its image under $\boldsymbol{U}$ in $\mathbf{A}$.)

Compositions and identities are given in the evident way. ◇

Lemma 2.9 *Let* $\mathbf{A}, \mathbf{B}$ *be pre-equipments, and let* $\boldsymbol{U} : \mathbf{B} \to \mathbf{A}$ *be a* strong *special functor. Then* $\boldsymbol{U}$ *has a special left biadjoint if and only if*

1. *For each* $A \in \mathbf{A}$ *there is an object* $(\boldsymbol{F}A, \eta_A) \in (A \swarrow \boldsymbol{U})$ *such that* η_A *is a* regular *1-cell, and for all objects* $(B, f) \in (A \swarrow \boldsymbol{U})$*, the category* $(A \swarrow \boldsymbol{U})(\eta_A, f)$ *has an initial object* $(\hat{f}, \alpha_f)$*.*
2. *If* $f : A \to UB$ *is regular then so is* $\hat{f}$*, and* α_f *is invertible.*

[3] The definition of oplax comma categories can be traced back to Gray [13, Section I.2.5], who called them '2-comma categories'.

3. $(\mathrm{id}_{\boldsymbol{F}A}, \phi^{-1}_{\boldsymbol{F}A} \circ \eta_A)$ *is initial in* $(A\swarrow\boldsymbol{U})(\eta_A, \eta_A)$.

$$
\begin{tikzcd}
A \arrow[d, "\eta_A"'] \arrow[dr, "\eta_A"] \arrow[d, phantom, shift left=4ex, "="] & \\
\boldsymbol{U}\boldsymbol{F}A \arrow[r, "\mathrm{id}_{\boldsymbol{U}\boldsymbol{F}A}"] \arrow[r, bend right=50, "\boldsymbol{U}\mathrm{id}_{\boldsymbol{F}A}"'] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\phi^{-1}"] & \boldsymbol{U}\boldsymbol{F}A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "\eta_A"'] \arrow[dr, "f"] \arrow[d, phantom, shift left=4ex, "\Downarrow\alpha_f"] & & \\
\boldsymbol{U}\boldsymbol{F}A \arrow[r] \arrow[rr, bend right=30, "\boldsymbol{U}(g\hat{f})"'] \arrow[rr, phantom, shift right=2.5ex, "\Downarrow\phi^{-1}"] & \boldsymbol{U}B \arrow[r, "\boldsymbol{U}g"] & \boldsymbol{U}C
\end{tikzcd}
$$

$$
\boldsymbol{F}A \xrightarrow{\mathrm{id}_{\boldsymbol{F}A}} \boldsymbol{F}A \qquad\qquad \boldsymbol{F}A \xrightarrow{\hat{f}} B \xrightarrow{g} C
$$

4. *For all* $f : A \to \boldsymbol{U}B$ *in* $\mathbf{A}$ *and all* regular $g : B \to C$ *in* $\mathbf{B}$*, the pair* $(g \circ \hat{f}, (\phi^{-1}_{(\hat{f},g)} \circ \eta_A) \cdot (\boldsymbol{U}g \circ \alpha_f))$ *is initial in* $(A\swarrow\boldsymbol{U})(\eta_A, \boldsymbol{U}g \circ f)$.

Proof. Given a special biadjunction $(\boldsymbol{F} \dashv \boldsymbol{U} : \mathbf{B} \to \mathbf{A}, \eta, \varepsilon, \mu, \nu)$ and $f : A \to \boldsymbol{U}B$, the initial object in $(A\swarrow\boldsymbol{U})(\eta_A, f)$ is given by

$$
(\hat{f},\ \alpha_f) = (\varepsilon_B \circ \boldsymbol{F}f,\ \ (\phi^{-1}_{(\boldsymbol{F}f,\varepsilon_B)} \circ \eta_A) \cdot (\boldsymbol{U}\varepsilon_B \circ \eta_f) \cdot (\mu_B \circ f))
$$

as in the diagram

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & \boldsymbol{U}B \arrow[d, "\eta_{\boldsymbol{U}B}"] \arrow[dr, "\mathrm{id}"] \arrow[dr, phantom, shift right=2ex, "\Downarrow\mu_B"] & \\
\boldsymbol{U}\boldsymbol{F}A \arrow[r] \arrow[rr, bend right=25, "\boldsymbol{U}(\varepsilon_B\boldsymbol{F}f)"'] \arrow[rr, phantom, shift right=3ex, "\Downarrow\phi^{-1}"] & \boldsymbol{U}\boldsymbol{F}\boldsymbol{U}B \arrow[r] & \boldsymbol{U}B
\end{tikzcd},
$$

$$
\boldsymbol{F}A \xrightarrow{\boldsymbol{F}f} \boldsymbol{F}\boldsymbol{U}B \xrightarrow{\varepsilon_B} B
$$

and conditions 1–4 are straightforward to verify.

In the converse direction, assume that $\boldsymbol{U} : \mathbf{B} \to \mathbf{A}$ is a strong special functor satisfying conditions 1–4. We want to construct a special biadjunction $(\boldsymbol{F} \dashv \boldsymbol{U} : \mathbf{B} \to \mathbf{A}, \eta, \varepsilon, \mu, \nu)$, and to this end we fix choices of $\boldsymbol{F}A$, η_A, $\hat{f}$ and α_f for each $A \in \mathbf{A}$, $B \in \mathbf{B}$ and $f : A \to \boldsymbol{U}B$.

The object part of the special functor $\boldsymbol{F} : \mathbf{A} \to \mathbf{B}$ and the components of the special transformation $\eta : \mathrm{id}_{\mathbf{A}} \to \boldsymbol{U} \circ \boldsymbol{F}$ are already anticipated in the naming. The action of $\boldsymbol{F}$ on 1-cells is given by $\boldsymbol{F}f = \widehat{\eta_{A'} \circ f}$ for $f : A \to A'$, and the identity and composition constraints are induced by initiality. The constraint cells of η are given by $\eta_f = \alpha_{(\eta_{A'} \circ f)}$.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & A' \arrow[d, "\eta_{A'}"] \\
\boldsymbol{U}\boldsymbol{F}A \arrow[r] & \boldsymbol{U}\boldsymbol{F}A'
\end{tikzcd}
\qquad
\begin{aligned}
\boldsymbol{F}f &= \widehat{\eta_{A'} \circ f} \\
\eta_f &= \alpha_{(\eta_{A'} \circ f)}
\end{aligned}
$$

$$
\boldsymbol{F}A \xrightarrow{\boldsymbol{F}f} \boldsymbol{F}A'
$$

It now follows from the initiality condition alone that $\boldsymbol{F}$ is an oplax functor and that η is an oplax transformation. Conditions 2-4 imply that $\boldsymbol{F}$ and η are special.

The components of $\varepsilon : \boldsymbol{F}\boldsymbol{U} \to \mathrm{id}_{\mathbf{B}}$ are defined by $\varepsilon_B = \widehat{\mathrm{id}_{\boldsymbol{U}B}}$ for $B \in \mathbf{B}$. For the construction of the constraint cell ε_g for $g : B \to B'$, consider the objects of

$(\boldsymbol{U}B \swarrow \boldsymbol{U})(\eta_{\boldsymbol{U}B}, \boldsymbol{U}g)$ defined by the following diagrams.

$$
\begin{tikzcd}
 & \boldsymbol{U}B' \arrow[d, "\eta_{\boldsymbol{U}B'}"] \arrow[dr, bend left, "\mathrm{id}_{\boldsymbol{U}B'}"] \arrow[dr, phantom, "\Downarrow\alpha_{\mathrm{id}_{\boldsymbol{U}B'}}"'] & \\
\boldsymbol{U}B \arrow[ur, "\boldsymbol{U}g"] \arrow[d, "\eta_{\boldsymbol{U}B}"'] \arrow[r, phantom, "\Downarrow\eta_{\boldsymbol{U}g}"] & \boldsymbol{U}\boldsymbol{F}\boldsymbol{U}B' \arrow[r] & \boldsymbol{U}B' \\
\boldsymbol{U}\boldsymbol{F}\boldsymbol{U}B \arrow[ur] & \boldsymbol{F}\boldsymbol{U}B' \arrow[r, "\varepsilon_{B'}"'] & B' \\
\boldsymbol{F}\boldsymbol{U}B \arrow[ur, "\boldsymbol{F}\boldsymbol{U}g"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \boldsymbol{U}B' \arrow[dr, bend left, "\mathrm{id}_{\boldsymbol{U}B'}"] & \\
\boldsymbol{U}B \arrow[ur, "\boldsymbol{U}g"] \arrow[d, "\eta_{\boldsymbol{U}B}"'] \arrow[dr, bend left, "\mathrm{id}_{\boldsymbol{U}B}"] \arrow[dr, phantom, "\Downarrow\alpha_{\mathrm{id}_{\boldsymbol{U}B}}"'] & = & \boldsymbol{U}B' \\
\boldsymbol{U}\boldsymbol{F}\boldsymbol{U}B \arrow[r] & \boldsymbol{U}B \arrow[ur] & B' \\
\boldsymbol{F}\boldsymbol{U}B \arrow[r, "\varepsilon_B"] & B \arrow[ur, "g"'] &
\end{tikzcd}
$$

Since $(\widehat{\boldsymbol{U}g}, \alpha_{\boldsymbol{U}g})$ is initial in $(\boldsymbol{U}B \swarrow \boldsymbol{U})(\eta_{\boldsymbol{U}B}, \boldsymbol{U}g)$, we obtain 2-cells $\zeta : \widehat{\boldsymbol{U}g} \to \varepsilon_{B'} \circ \boldsymbol{F}\boldsymbol{U}g$ and $\xi : \widehat{\boldsymbol{U}g} \to g \circ \varepsilon_B$. Of these 2-cells, ζ is invertible by 3, and we define $\varepsilon_g = \xi \cdot \zeta^{-1}$.

The modification $\mu : \mathrm{id}_{\boldsymbol{U}} \to \boldsymbol{U}\varepsilon \circ \eta\boldsymbol{U}$ is defined by $\mu_B = \alpha_{\mathrm{id}_{\boldsymbol{U}B}}$ for $B \in \mathbf{B}$ (which is invertible by 2), and $\nu : \varepsilon\boldsymbol{F} \circ \boldsymbol{F}\eta \to \mathrm{id}_{\boldsymbol{F}}$ at $A \in \mathbf{A}$ is the unique arrow between the objects of $(A \swarrow \boldsymbol{U})(\eta_A, \eta_A)$ defined in the following diagrams.

$$
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\Downarrow\eta_{\eta_A}"] & \boldsymbol{U}\boldsymbol{F}A \arrow[d, "\eta_{\boldsymbol{U}\boldsymbol{F}A}"] \arrow[dr, bend left, "\mathrm{id}_{\boldsymbol{U}\boldsymbol{F}A}"] \arrow[dr, phantom, "\Downarrow\alpha_{\mathrm{id}_{\boldsymbol{U}\boldsymbol{F}A}}"'] & \\
\boldsymbol{U}\boldsymbol{F}A \arrow[r] \arrow[rr, bend right, "\boldsymbol{U}(\varepsilon_{\boldsymbol{F}A} \circ \boldsymbol{F}\eta_A)"'] & \boldsymbol{U}\boldsymbol{F}\boldsymbol{U}\boldsymbol{F}A \arrow[r] \arrow[d, phantom, "\Downarrow\phi^{-1}"] & \boldsymbol{U}\boldsymbol{F}A \\
\boldsymbol{F}A \arrow[r, "\boldsymbol{F}\eta_A"'] & \boldsymbol{F}\boldsymbol{U}\boldsymbol{F}A \arrow[r, "\varepsilon_{\boldsymbol{F}A}"'] & \boldsymbol{F}A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \arrow[d, "\eta_A"'] \arrow[dr, "\eta_A"] & \\
\boldsymbol{U}\boldsymbol{F}A \arrow[r, "\mathrm{id}_{\boldsymbol{U}\boldsymbol{F}A}"] \arrow[r, bend right, "\boldsymbol{U}\mathrm{id}_{\boldsymbol{F}A}"'] \arrow[r, phantom, "="', shift left=4] \arrow[r, phantom, "\Downarrow\phi^{-1}"', shift right=3] & \boldsymbol{U}\boldsymbol{F}A \\
\boldsymbol{F}A \arrow[r, "\mathrm{id}_{\boldsymbol{F}A}"'] & \boldsymbol{F}A
\end{tikzcd}
.
$$

These objects are both initial – the first one by 2 and 4 and stability under composition with the isomorphic 2-cell $\alpha_{\mathrm{id}_{\boldsymbol{U}\boldsymbol{F}A}}$, and the second one by property 3 –, and thus they are canonically isomorphic.

It is now straightforward to verify that μ and ν are indeed modifications satisfying the axioms of Definition 2.3, and we leave this to the reader. ■

3. Triposes

We now recall the definition of triposes and describe how they give rise to a pre-equipment **Trip**.

Triposes were introduced in [1] by Hyland, Johnstone and Pitts as indexed preorders on **Set**[4]; Pitts [14] considered them on arbitrary finite limit categories, and van Oosten [15] on categories with finite products only. In this work, we closely follow van Oosten [15, Section 2.1], but we use a version of the Beck-Chevalley condition that is slightly weaker (similar to Pitts [16] and Jacobs [17]).

A *Heyting prealgebra* is a preorder which is bicartesian closed as a category. Equivalently, it is a preorder whose poset reflection is a Heyting algebra. We denote by **HA** the locally ordered category of Heyting prealgebras and monotone maps which preserve all structure (in the weak sense that e.g. the image of a binary meet is a binary meet of the images – we do *not* require the strict preservation of any particular choice of structure).

Definition 3.1 Let $\mathbb{C}$ be a category with finite products. A *tripos on* $\mathbb{C}$ is a strong functor $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$ such that

[4]An *indexed preorder* on **Set** is a strong functor from $\mathbf{Set}^{\mathsf{op}}$ into the locally ordered category **Ord** of preorders and monotone maps.

1. For each $f : A \to B$ in $\mathbb{C}$, there are *monotone*[5] maps $\exists_f, \forall_f : \mathcal{P}(A) \to \mathcal{P}(B)$ which are left and right adjoint to $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$, respectively.
2. For all $f : A \to B,\ g : X \to Y$ in $\mathbb{C}$ and $\varphi \in \mathcal{P}(B \times X)$, we have
$$Q_{A\times g}\left(\mathcal{P}(f \times X)(\varphi)\right) \cong \mathcal{P}(f \times Y)(Q_{B\times g}(\varphi)),$$
where Q is either $\forall$ or $\exists$ (this is the *Beck-Chevalley condition*).
3. $\mathcal{P}$ has *weak power objects*, i.e. for every $A \in \mathbb{C}$ there exist $\mathfrak{P}A \in \mathbb{C}$ and $\in_A \in \mathcal{P}(A\times\mathfrak{P}A)$ such that for all $\varphi \in \mathcal{P}(A \times C)$ there exists a $\chi : C \to \mathfrak{P}A$ with $\varphi \cong (A \times \chi)^*(\varepsilon_A)$. $\diamondsuit$

We call $\mathbb{C}$ the *base category* of the tripos $\mathcal{P}$, and for $f : A \to B$ in $\mathbb{C}$, we denote the map $\mathcal{P}(f)$ by f^* and call it *reindexing map along* f. The maps $\exists_f, \forall_f$ are called *existential* and *universal quantification along* f. We refer to the elements of $\mathcal{P}(A)$ for $A \in \mathbb{C}$ as *predicates* (on A), since they are the semantic counterparts of formulas when interpreting predicate logic.

3.1. The internal logic of a tripos

When doing calculations in triposes, we make extensive use of their *internal logic*, which is a many sorted intuitionistic predicate logic. For an outline of the internal logic of triposes, see [15, Section 2.1.3]. For a more detailed introduction, see Jacobs' [17], in particular Section 4.3[6].

In the following, we recall the basic features of the internal logic, and state notational conventions, in which we mostly follow Jacobs.

Terms, formulas and judgments of the internal logic always come with an explicit context $\Delta \equiv x_1{:}A_1 \ldots x_n{:}A_n$ of typed variables, where the types A_i are objects of the base category $\mathbb{C}$ (in practice types are often omitted if they can easily be inferred).

Terms in context (which are built up from morphisms in $\mathbb{C}$ serving as typed function symbols) are denoted by $(\Delta \mid t : A)$. Formulas in context (built up from terms, predicates in $\mathcal{P}$ and first order connectives) are denoted by $(\Delta \mid P)$. Judgments are of the form $(\Delta \mid \Gamma \vdash Q)$ with $\Gamma \equiv P_1 \ldots P_n$, where $P_1 \ldots P_n, Q$ are formulas in variable context Δ.

The *denotation* (or interpretation) of a formula $(\Delta \mid P)$ is a predicate $[\![x_1{:}A_1 \ldots x_n{:}A_n \mid P]\!] \in \mathcal{P}(A_1 \times \cdots \times A_n)$, defined by induction on the structure of P, where propositional connectives are interpreted by Heyting algebra operations, and quantifiers are interpreted by $\exists_\pi$ or $\forall_\pi$ for π an appropriate projection. Equality on an object A is interpreted by $\exists_{\delta_A}(\top) \in \mathcal{P}(A \times A)$ where $\delta_A : A \to A \times A$ is the diagonal, and we will generally denote this predicate by '$=_A$' or simply '$=$' in the following.

We say that $(\Delta \mid \Gamma \vdash Q)$ *holds* (or *is valid*) in $\mathcal{P}$, if
$$[\![\Delta \mid P_1]\!] \wedge \cdots \wedge [\![\Delta \mid P_n]\!] \leq [\![\Delta \mid Q]\!] \quad \text{in} \quad \mathcal{P}(A_1 \times \cdots \times A_n). \tag{3.1}$$

The internal logic is sound w.r.t intuitionistic first order logic, in the sense that if a judgment $(\Delta \mid \Gamma \vdash Q)$ is derivable from judgments that are valid in $\mathcal{P}$

---

[5] $\exists_f$ and $\forall_f$ are not required to preserve Heyting algebra structure.

[6] Jacobs treats the interpretation of predicate logic not in indexed preorders, but in *posetal fibrations*, which is equivalent. For the precise connection between the two presentations, see [17, Section 1.10].

using the rules of intuitionistic first order logic alone, then it is itself valid in $\mathcal{P}$ (see [17, Lemma 4.3.3]).

The interpretation of first order logic relies only on clauses 1 and 2 of Definition 3.1. Clause 3 is about *higher order logic*: given an object $A \in \mathbb{C}$, the weak power object $\mathfrak{P}A$ validates the axiom scheme

$$(\Delta \mid \vdash \exists m{:}\mathfrak{P}A\, \forall x{:}A\,.\, x \in m \Leftrightarrow P[x])$$

for subset comprehension, where the symbol '$\in$' is interpreted by the predicate $\in_A$ from Definition 3.1-3, and $(\Delta, x{:}A \mid P[x])$ is an arbitrary formula.

3.2. *Tripos morphisms and transformations*

We now introduce the pre-equipment **Trip** of triposes, starting with the definition of the 1-cells.

Definition 3.2 Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$ and $\mathcal{R} : \mathbb{D}^{\mathsf{op}} \to \mathbf{HA}$ be two triposes.

- A *tripos morphism* is a pair $\mathfrak{F} = (F, \Phi)$[7] of
  1. a finite product preserving functor $F : \mathbb{C} \to \mathbb{D}$
  2. a strong natural transformation

$$\Phi \;:\; \boldsymbol{G} \circ \mathcal{P} \to \boldsymbol{G} \circ \mathcal{R} \circ F \;:\; \mathbb{C}^{\mathsf{op}} \to \mathbf{SLat}$$

  where $\boldsymbol{G} : \mathbf{HA} \to \mathbf{SLat}$ is the forgetful 2-functor from Heyting prealgebras to pre-meet-semilattices.
- (F, Φ) is called *regular*, if for all $f : A \to B$ in $\mathbb{C}$ and all $\psi \in \mathcal{P}(A)$ we have

$$\exists_{Ff}(\Phi_A \psi) \cong \Phi_B(\exists_f \psi). \qquad \diamondsuit$$

Given a term $(\Delta \mid t : A)$ in the internal language of $\mathcal{P}$, with $\Delta \equiv x_1{:}A_1 \ldots x_n{:}A_n$, and a tripos morphism $\mathfrak{F} = (F, \Phi) : \mathcal{P} \to \mathcal{R}$, we can construct a term $(F(\Delta) \mid F(t) : F(A))$ where $F(\Delta) = x_1{:}F(A_1) \ldots x_n{:}F(A_n)$, by applying F to all the function symbols in t (strictly speaking, we have to insert some coherence isomorphisms since we do not require F to preserve products on the nose, but we will ignore these details). Similarly, given a formula $(\Delta \mid P)$ in the language of $\mathcal{P}$, we can construct a formula $(F(\Delta) \mid \mathfrak{F}(P))$ in $\mathcal{R}$ by applying F to all the function symbols and Φ to all the predicate symbols in P. This operation commutes with interpretation of terms and formulas containing only conjunction, in the sense that

$$[\![F(\Delta) \mid \mathfrak{F}(P)]\!] \cong \Phi_{[\![\Delta]\!]}([\![\Delta \mid P]\!]),$$

provided that P contains no logical connectives except conjunction. If $\mathfrak{F}$ is regular, then P may also contain equality and existential quantification, i.e., it may be a formula of regular logic.

The commutation with interpretation implies the preservation of validity of restricted judgments, i.e.

$$\big(\Delta \mid \Gamma \vdash P\big) \text{ in } \mathcal{P} \quad \text{implies} \quad \big(F(\Delta) \mid \mathfrak{F}(\Gamma) \vdash \mathfrak{F}(P)\big) \text{ in } \mathcal{R} \tag{3.2}$$

if Γ and ψ are built up only from atomic formulas with conjunction – and with existential quantification and equality if $\mathfrak{F}$ is regular.

The next definition gives the 2-cells in the 2-category of triposes.

[7] We generally denote the pairs by uppercase gothic letters, and the first and second components by uppercase latin and greek, respectively, but we deviate from this convention in (5.2) to denote the 'constant objects functor' by greek Δ, in accordance with parts of the literature.

Definition 3.3 Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$ and $\mathcal{R} : \mathbb{D}^{\mathsf{op}} \to \mathbf{HA}$ be triposes. A *tripos transformation* between tripos morphisms $(F, \Phi), (G, \Gamma) : \mathcal{P} \to \mathcal{R}$ is a natural transformation $\eta : F \to G$ such that for all $A \in \mathbb{C}$ and $\psi \in \mathcal{P}(A)$ we have

$$\Phi_A(\psi) \leq \eta_A^*(\Gamma_A(\psi)) \tag{3.3}$$

in $\mathcal{R}_{F(A)}$. $\diamondsuit$

Definition 3.4 **Trip** is the pre-equipment of triposes, tripos morhpisms and tripos transformations, where regular 1-cells are regular tripos morphisms. $\diamondsuit$

4. Q-toposes

Q-toposes are a generalization of quasitoposes [18, 19]. Like quasitoposes they have a classifier for strong monomorphisms, but they are not required to be locally cartesian closed (or even cartesian closed), nor do they need to have all finite colimits.

It turns out that neither of these features is needed to get a working higher order logic, and it suffices to postulate the quasitopos version of power objects. For the construction of the special biadjunction between triposes and q-toposes in Section 5 we furthermore need a minimal amount of 'exactness', namely stable effective quotients of strong equivalence relations.

Strong monomorphisms (which, following Johnstone, we also call *cocovers*) play the same central role in q-toposes as they do in quasitoposes. We recall their definition, and that of *covers*, from [20, Definitions 5.4.1 and 4.3.5].

Definition 4.1 Let $\mathbb{C}$ be a category.

- Let $f : A \to B$, $g : X \to Y$ in $\mathcal{C}$. We say that f is *left orthogonal* to g (or alternatively that g is *right orthogonal* to f), if for any commuting square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

  there exists a *unique* $h : B \to X$ such that the two triangles commute.
- An epimorphism $e : A \to B$ is called a *cover*, or a *strong epimorphism*, if it is left orthogonal to all monomorphisms.
- A monomorphism $m : X \to Y$ is called a *cocover*, or a *strong monomorphism*, if it is right orthogonal to all epimorphisms. $\diamondsuit$

We denote covers by the arrow '$\twoheadrightarrow$', and cocovers by the arrow '$\rightarrowtail$'.

Remark 4.2 If $\mathbb{C}$ has equalizers, we can drop the condition that e is an epimorphism in the definition of cover, as an arrow that is left orthogonal to all monomorphisms is then epic (see [20, Lemma 4.3.7]).

Similarly, but not dually, we can drop the condition that m is a mono from the definition of cocover whenever $\mathbb{C}$ has all kernel pairs and coequalizers of kernel pairs.

In q-toposes, both these requirements are met, thus we can use the simplified definitions. However, a large part of the theory (in particular Sections 4.1 and 4.2) can be developed independently of clauses 2 and 3 in Definition 4.3 below, in which case we need the above definition of cocover.

Even without clauses 2 and 3 it is possible to show *a posteriori* that co-covers coincide with the class of morphisms which are right orthogonal to all epimorphisms. This is a consequence of Lemma 4.11. $\diamondsuit$

Given an object A of a category $\mathbb{C}$ with finite limits, strong monomorphisms into A are ordered by inclusion, forming a preorder $\boldsymbol{S}(\mathbb{C})(A)$. Since they are stable under arbitrary pullbacks, we obtain an indexed preorder

$$\boldsymbol{S}(\mathbb{C}) : \mathbb{C}^{\mathsf{op}} \to \mathbf{Ord}. \tag{4.1}$$

By associating to each fiber of $\boldsymbol{S}(\mathbb{C})$ its set of isomorphism classes, we obtain the presheaf

$$\mathrm{ssub} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Set}.$$

of *strong subobjects*. Thus, the elements of $\mathrm{ssub}(A)$ are equivalence classes of strong monomorphisms into A modulo mutual inclusion.

Now the definition of q-topos is the following.

Definition 4.3 A *q-topos* is a finite limit category $\mathcal{E}$ such that

1. all presheaves $\mathrm{ssub}(A \times -) : \mathcal{E}^{\mathsf{op}} \to \mathbf{Set}$ are representable,
2. strong equivalence relations have effective quotients[8], and
3. regular epimorphisms are stable under pullback. $\diamondsuit$

Given $A \in \mathcal{E}$, we call an object representing $\mathrm{ssub}(A \times -)$ a *power object* of A, and denote it by PA. The corresponding strong monomorphism $\in_A : E_A \rightarrowtail A \times PA$ is called the *membership relation*, and we usually omit the subscript for brevity.

Lemma 4.4 *Any q-topos is a regular category. Therefore in a q-topos regular epimorphisms coincide with covers.*

Proof. Regular categories are categories with finite limits and coequalizers of kernel pairs, where regular epimorphisms are stable under pullbacks (see e.g. [21, Definition 2.1.1]). Kernel pairs have quotients in q-toposes since they can be constructed as pullbacks of the split monomorphism $\delta_B : B \rightarrowtail B \times B$ along $f \times f$ and therefore are strong equivalence relations. The other conditions are literally contained in Definition 4.3.

The fact that regular epimorphisms coincide with covers in regular categories is proved e.g. in [21, Proposition 2.1.4]. ■

The following lemma shows how to rephrase the first condition of Definition 4.3 in a way that is closer to the internal language which will be introduced in the next section.

Lemma 4.5 *In a finite limit category $\mathcal{E}$,* $\mathrm{ssub}(- \times A) : \mathcal{E}^{\mathsf{op}} \to \mathbf{Set}$ *is representable for all $A \in \mathcal{E}$ if and only if* $\mathrm{ssub}(-) : \mathcal{E}^{\mathsf{op}} \to \mathbf{Set}$ *is representable by an exponentiating object.* ■

We denote the object representing $\mathrm{ssub}(-)$ by Ω and the associated universal element of $\mathrm{ssub}(\Omega)$ by $\mathtt{t}$.

Lemma 4.6 *Let $\mathcal{E}$ be a q-topos.*

[8] i.e. they are effective [19, Definition 1.3.6] and their components have a coequalizer

1. *The domain of* $\mathtt{t} : U \rightarrowtriangle \Omega$ *is terminal.*
2. *The class of cocovers coincides with the class of regular monomorphisms.*

Proof. *Ad 1.* The postcomposition map $\mathtt{t} \circ - : \mathcal{E}(A, U) \to \mathcal{E}(A, \Omega)$ induces a bijection between $\mathcal{E}(A, U)$ and arrows $f : A \to \Omega$ such that $f^*\mathtt{t}$ is an isomorphism. For each A, there is exactly one such arrow.

Ad 2. It is well known that in every category, regular monomorphisms are strong. Conversely, every cocover $m : U \rightarrowtriangle A$ is the equalizer of its classifying map $\chi_m : A \to \Omega$ and $\mathtt{t} \circ !_A$. ■

The pre-equipment **QTop** of q-toposes is defined in a similar way as the pre-equipment **Top** of toposes (Definition 2.6):

Definition 4.7 A *q-topos morphism* is a finite-limit preserving functor. A *regular q-topos morphism* is a q-topos morphism that moreover preserves epimorphisms and covers.

QTop is the pre-equipment of q-toposes, (regular) q-topos morphisms, and natural transformations. ♢

Observe that q-topos morphisms preserve cocovers, since they coincide with regular monomorphisms by Lemma 4.6.

4.1. The logic of q-toposes

We now explain how to interpret higher order logic in the indexed preorder of cocovers on a q-topos. Our approach is analogous to the presentation of the internal logic of toposes by Boileau-Joyal [22] and Lambek-Scott [23].

Concretely, for a q-topos $\mathcal{E}$, we define a minimal internal language (called *core calculus*), whose term constructors are pairing, projections singleton, subset comprehension, membership relation and equality. We view terms of type Ω of the core calculus as formulas, and give a intuitionistic sequent calculus style system of inference rules for them. The definitions of the core calculus and the deduction system are given in Table 1. Just as for the internal logic of triposes, we will often suppress types in the notation if they can easily be inferred.

The interpretation of the language in $\mathcal{E}$ is given as follows. Types are inductively interpreted in the obvious way, where $[\![PA]\!] = \Omega^{[\![A]\!]}$. Contexts are interpreted as cartesian products of their constituents, and terms are interpreted by suitably typed morphisms as follows:

$$\begin{aligned}
[\![\Delta \mid x_i]\!] &= \pi_i \\
[\![\Delta \mid *]\!] &= !_{[\![\Delta]\!]} \\
[\![\Delta \mid \{x|\varphi[x]\}]\!] &= \Lambda([\![\Delta, x \mid \varphi[x]]\!]) \\
[\![\Delta \mid a \in M]\!] &= \varepsilon_{[\![A]\!]} \circ \langle [\![\Delta \mid M]\!], [\![\Delta \mid a]\!] \rangle \\
[\![\Delta \mid (a,b)]\!] &= \langle [\![\Delta \mid a]\!], [\![\Delta \mid b]\!] \rangle \\
[\![\Delta \mid s = t]\!] &= \chi_{\delta_A} \circ \langle [\![\Delta \mid s]\!], [\![\Delta \mid t]\!] \rangle \\
[\![\Delta \mid f(a)]\!] &= f \circ [\![\Delta \mid a]\!]
\end{aligned}$$

Here π_i denotes the suitable projection, $!_{[\![\Delta]\!]}$ denotes the terminal projection, $\Lambda : \mathcal{E}([\![\Delta, x{:}A]\!], \Omega) \to \mathcal{E}([\![\Delta]\!], \Omega^{[\![A]\!]})$ is the exponential transpose operation, $\varepsilon_{[\![A]\!]} : \Omega^{[\![A]\!]} \times [\![A]\!] \to \Omega$ is evaluation, and $\chi_{\delta_A} : A \times A \to \Omega$ is the classifying map of the diagonal $\delta_A : A \to A \times A$.

Types:

$$A := X \mid 1 \mid \Omega \mid PA \mid A \times A \qquad X \in \mathcal{E}$$

Terms:

We use Δ to denote a context $x_1{:}A_1 \dots x_n{:}A_n$ of typed variables.

$$\frac{}{\Delta \mid x_i : A_i}\ {\scriptstyle (i=1,\dots,n)} \qquad \frac{}{\Delta \mid * : 1}$$

$$\frac{\Delta, x{:}A \mid \varphi[x] : \Omega}{\Delta \mid \{x|\varphi[x]\} : PA} \qquad \frac{\Delta \mid a : A \qquad \Delta \mid M : PA}{\Delta \mid a \in M : \Omega}$$

$$\frac{\Delta \mid a : A \qquad \Delta \mid b : B}{\Delta \mid (a,b) : A \times B} \qquad \frac{\Delta \mid c : A_1 \times A_2}{\Delta \mid \pi_i(c) : A_i}\ i = 1,2$$

$$\frac{\Delta \mid a : A \qquad \Delta \mid a' : A}{\Delta \mid a = a' : \Omega} \qquad \frac{\Delta \mid a : A}{\Delta \mid f(a) : B}\ f \in \mathcal{E}(\llbracket A \rrbracket, \llbracket B \rrbracket)$$

Deduction rules:

$p_1, \dots, p_n, p, q$ are terms of type Ω, and Γ is a list of such terms.

$$\frac{}{\Delta \mid p_1, \dots, p_n \vdash p_i}\ \mathrm{Ax}\ {\scriptstyle (i=1,\dots,n)} \qquad \frac{\Delta \mid \Gamma \vdash p \qquad \Delta \mid \Gamma, p \vdash q}{\Delta \mid \Gamma \vdash q}\ \mathrm{Cut}$$

$$\frac{}{\Delta \mid \Gamma \vdash a = a}\ {=}\mathrm{I} \qquad \frac{\Delta \mid \Gamma \vdash p[a] \qquad \Delta \mid \Gamma \vdash a = a'}{\Delta \mid \Gamma \vdash p[a']}\ {=}\mathrm{E}$$

$$\frac{\Delta, x{:}A \mid \Gamma \vdash p[x] = (x \in M)}{\Delta \mid \Gamma \vdash \{x|p[x]\} = M}\ \mathrm{P}\text{-}\eta \qquad \frac{}{\Delta \mid \Gamma \vdash (a \in \{x|p[x]\}) = p[a]}\ \mathrm{P}\text{-}\beta$$

$$\frac{}{\Delta \mid \Gamma \vdash t = *}\ 1\text{-}\eta \qquad \frac{\Delta \mid \Gamma, p \vdash q \qquad \Delta \mid \Gamma, q \vdash p}{\Delta \mid \Gamma \vdash p = q}\ \mathrm{Ext}$$

$$\frac{}{\Delta \mid \Gamma \vdash \pi_i(a_1, a_2) = a_i}\ \times\text{-}\beta\ {\scriptstyle (i=1,2)} \qquad \frac{}{\Delta \mid \Gamma \vdash (\pi_1(c), \pi_2(c)) = c}\ \times\text{-}\eta$$

$$\frac{}{\Delta \mid \Gamma \vdash \mathrm{id}_{\llbracket A \rrbracket}(a) = a}\ \mathrm{Id} \qquad \frac{}{\Delta \mid \Gamma \vdash g(f(a)) = (g \circ f)(a)}\ \mathrm{Comp}$$

Table 1: The core calculus

In analogy to (3.1) we say that a judgment $\big(\Delta \mid p_1, \dots, p_n \vdash q\big)$ is *valid in* $\mathcal{E}$, or *holds in* $\mathcal{E}$, if the inequality

$$\llbracket \Delta \mid p_1 \rrbracket^* \mathsf{t} \wedge \dots \wedge \llbracket \Delta \mid p_n \rrbracket^* \mathsf{t} \leq \llbracket \Delta \mid q \rrbracket^* \mathsf{t}$$

holds in $\boldsymbol{S}(\mathcal{E})(\llbracket \Delta \rrbracket)$.

The following substitution lemma and a soundness theorem are stated without proofs, since the proofs are standard and consist mainly of unfolding of definitions.

Lemma 4.8 (Substitution lemma) *For terms* $(\Delta \mid s_i : B_i), 1 \leq i \leq n$, *and* $(x_1{:}B_1 \dots x_n{:}B_n \mid t : C)$ *of the core calculus, we have* $\llbracket \Delta \mid t[s_1/x_1 \dots s_n/x_n] \rrbracket = \llbracket x_1 \dots x_n \mid t \rrbracket \circ \langle \llbracket \Delta \mid s_1 \rrbracket \dots \llbracket \Delta \mid s_n \rrbracket \rangle : \llbracket \Delta \rrbracket \to \llbracket C \rrbracket$. ∎

Theorem 4.9 (Soundness theorem) *The deduction rules of the core calculus preserve the validity of judgments, i.e. if a judgment* $\big(\Delta \mid \Gamma \vdash q\big)$ *is derivable from judgments that hold in* $\mathcal{E}$, *then it holds in* $\mathcal{E}$. ∎

The connectives of predicate logic can be encoded in the core calculus as follows:

$$\begin{array}{rcl} \top & \equiv & * = * \\ p \wedge q & \equiv & (p, q) = (\top, \top) \\ p \Rightarrow q & \equiv & p \wedge q = p \\ \forall x{:}A \,.\, p[x] & \equiv & \{x \mid p[x]\} = \{x \mid \top\} \\ \bot & \equiv & \forall z{:}\Omega \,.\, z \\ p \vee q & \equiv & \forall z{:}\Omega \,.\, (p \Rightarrow z) \wedge (q \Rightarrow z) \Rightarrow z \\ \exists x{:}A \,.\, p[x] & \equiv & \forall z{:}\Omega \,.\, (\forall x{:}A \,.\, p[x] \Rightarrow z) \Rightarrow z \end{array}$$

We use '$\equiv$' to denote syntactic equality of formulas, to avoid confusion with the equality sign that is part of the core calculus. It can now be derived from the rules of the core calculus that the logical connectives defined in this way validate the corresponding rules of intuitionistic predicate logic.

Some reasoning principles that can be transferred directly from toposes to q-toposes are proved in the following lemma.

Lemma 4.10 *Let $\mathcal{E}$ be a q-topos.*

1. *$f, g : A \to B$ are equal if and only if $\big(x{:}A \mid\, \vdash f(x) = g(x)\big)$ holds in $\mathcal{E}$.*
2. *$f : A \to B$ is monic if and only if $\big(x{:}A, y{:}A \mid f(x) = f(y) \vdash x = y\big)$ holds.*
3. *$f : A \to B$ is epic if and only if $\big(y{:}B \mid\, \vdash \exists x \,.\, f(x) = y\big)$ holds.*

Proof. Ad 1. Straightforward, since the interpretation of $[\![x{:}A \mid f(x) = g(x)]\!]$ is the equalizer of f and g.

Ad 2. The interpretation of $[\![x{:}A, y{:}A \mid f(x) = f(y)]\!]$ is the kernel pair of f, and a morphism is monic if and only if its kernel pair is included in the diagonal.

Ad 3. Assume that $\big(y{:}B \mid\, \vdash \exists x \,.\, f(x) = y\big)$ holds. Let $h, k : B \to C$ be arbitrary and assume that $hf = kf$. Then the deduction

$$\dfrac{y \mid\, \vdash \exists x \,.\, fx = y \qquad \dfrac{x \mid\, \vdash h(f(x)) = k(f(x))}{x, y \mid fx = y \vdash h(y) = k(y)}}{y \mid\, \vdash h(y) = k(y)}$$

establishes the claim.

Conversely, assume that f is an epimorphism. f obviously equalizes the classifying maps of the predicates $[\![y{:}B \mid \top]\!]$ and $[\![y{:}B \mid \exists x \,.\, fx = y]\!]$, and since it is epic they are both equal, whence the second is valid in $\mathcal{E}$. ∎

Note that statement 3 stands in contrast to regular categories, where the judgment $\big(y{:}B \mid\, \vdash \exists x \,.\, fx = y\big)$ characterizes *regular* epimorphisms. This discrepancy is due to the fact that the internal logic of a regular category is based on the indexed preorders of *all* monomorphisms, whereas we only consider strong monomorphisms in q-toposes.

Next we show that the class of cocovers is part of a *factorization system* [20, Definition 5.5.1], which gives a demonstration of the power of the internal logic.

Lemma 4.11 *Let $f : A \to B$ in $\mathcal{E}$. The cocover $m : U \rightarrowtriangle A$ that is classified by the predicate $[\![y{:}B \mid \exists x \,.\, f(x) = y]\!]$ is the minimal cocover through which f factors, and in the corresponding factorization $f = me$, e is an epimorphism. In particular, the class of epimorphisms and the class of cocovers together form a factorization system on $\mathcal{E}$.*

Proof. The minimality condition is just a rephrasing of the universal property of existential quantification.

To see that $e : A \to U$ is epic, it suffices by Lemma 4.10-3 to show that the strong monomorphism $n : V \rightarrowtail U$ classified by $[\![y{:}B \mid \exists x\,.\,e(x) = y]\!]$ is an isomorphism. This follows from the minimality of m, since f factors through mn.

Finally, cocovers are right orthogonal to epimorphisms by definition. ∎

The epi/cocover factorization system is even a *stable* factorization system, as the following lemma shows.

Lemma 4.12 *Epimorphisms are stable under pullback in q-toposes.*

Proof. Consider a pullback square

$$\begin{tikzcd} X \arrow[r, "q"'] \arrow[d, "p"'] & B \arrow[d, twoheadrightarrow, "e"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

where e is an epimorphism. Then $(c \mid\ \vdash \exists b\,.\,eb = c)$ holds by Lemma 4.10.3, and we want to show the same for p. This is an easy exercise using the fact that $[\![a, b \mid fa = eb]\!]$ as well as $[\![a, b \mid \exists x\,.\,(px, qx) = (a, b)]\!]$ classify the cocover $X \rightarrowtail A \times B$, and are therefore equivalent. ∎

Lemma 4.13 *The indexed preorder $\boldsymbol{S}(\mathcal{E})$ of cocovers on a q-topos $\mathcal{E}$ is a tripos.*

Proof. The internal language gives us the propositional structure, equality, and quantification along projections. Quantification along arbitrary morphisms $f : A \to B$ can be encoded as $\forall_f(\varphi) = [\![y{:}B \mid \forall x{:}A\,.\,f(x) = y \Rightarrow \varphi(x)]\!]$ and $\exists_f(\varphi) = [\![y{:}B \mid (\exists x{:}A\,.\,f(x) = y \wedge \varphi(x)]\!]$ for $f : A \to B$ and $\varphi : A \to \Omega$.

The Beck-Chevalley condition for quantification along projections follows from the Substitution Lemma 4.8, and the general case follows from the instance for projections by unfolding the previously given encodings.

Finally, it is easy to see that the power objects of the q-topos give power objects in the tripos sense. ∎

The assignment $\mathcal{E} \mapsto \boldsymbol{S}(\mathcal{E})$ extends to a *strict* special functor

$$\boldsymbol{S} : \mathbf{QTop} \to \mathbf{Trip} \tag{4.2}$$

as follows: given a q-topos morphism $F : \mathcal{E} \to \mathcal{F}$, we have

$$\boldsymbol{S}(F) = (F, \widetilde{F} : \boldsymbol{S}(\mathcal{E}) \to \boldsymbol{S}(\mathcal{F}) \circ F)$$

with $\widetilde{F}_C(m) = Fm$ for $m : U \rightarrowtail C$ in $\mathcal{E}$. $\boldsymbol{S}$ preserves regular 1-cells since existential quantification in $\boldsymbol{S}(\mathcal{E})$ and $\boldsymbol{S}(\mathcal{F})$ can be expressed in terms of epi/cocover factorization, and regular q-topos morphisms preserve both epimorphisms and cocovers.

For the action of $\boldsymbol{S}$ on 2-cells, it is easy to see that every natural transformation $\eta : F \to G : \mathcal{E} \to \mathcal{F}$ is automatically a tripos transformation of type $\boldsymbol{S}(F) \to \boldsymbol{S}(G) : \boldsymbol{S}(\mathcal{E}) \to \boldsymbol{S}(\mathcal{F})$.

In Section 5 we construct a special left biadjoint to $\boldsymbol{S}$.

In Lemma 4.10 we saw that the internal logic of a q-topos is powerful enough to detect equality of arrows. However, the internal logic lacks another important feature: it cannot distinguish isomorphisms from maps which are monomorphisms and epimorphisms at the same time. This follows from the following proposition.

Proposition 4.14 *Let $f : A \rightarrowtail\!\!\!\!\!\twoheadrightarrow B$ be monic as well as epic. Then the induced map $f^* : \boldsymbol{S}(\mathcal{E})(B) \to \boldsymbol{S}(\mathcal{E})(A)$ is an equivalence of preorders.*

Proof. In the internal language f^* can be written as $\varphi \mapsto [\![a{:}A \mid \varphi(fa)]\!]$, and a map in the converse direction is given by $\psi \mapsto [\![b{:}B \mid \exists a \,.\, fa = b \wedge \psi(a)]\!]$. Using the characterizations of monomorphisms and epimorphisms of Lemma 4.10, it is easy to verify that these two maps are inverse to each other. ■

So in a sense the arrows which are monic and epic at the same time disclose a mismatch between the category and the internal logic. This can be seen as a motivation for the following definition of *coarse objects*, which are just as undiscerning as the internal logic, so that the correspondence between category and internal logic is restored if we restrict to the full subcategory on the coarse objects.

Coarse objects are also considered for quasitoposes, and the treatment here is a variation of the presentation in [19] for quasitoposes.

Definition 4.15 An object C of a q-topos $\mathcal{E}$ is called *coarse*, if for each morphism $f : A \rightarrowtail\!\!\!\!\!\twoheadrightarrow B$ which is both monic and epic and all morphisms $g : A \to C$, there exists a morphism $h : B \to C$ such that $hf = g$. ♢

Because the arrow f in the previous definition is an epimorphism, the mediating arrow h is automatically unique.

Lemma 4.16 *Let $\mathcal{E}$ be a q-topos.*

1. *If $C \in \mathcal{E}$ is coarse and $f : C \rightarrowtail\!\!\!\!\!\twoheadrightarrow A$ is monic and epic, then it is iso.*
2. *If C is coarse and $m : C \rightarrowtail A$ is monic, then it is a cocover.*
3. *If C is coarse and $m : U \mathrel{\triangleright\!\!\!\rightarrow} C$ is a cocover, then U is coarse.*
4. *Finite products of coarse objects are coarse.*
5. *For every object $A \in \mathcal{E}$, its power object PA is coarse.*
6. *The full subcategory $\boldsymbol{T}(\mathcal{E})$ of $\mathcal{E}$ on coarse objects is a topos.*

Proof. Ad 1. By coarseness of C, there exists an arrow $g : A \to C$ such that $gf = \mathrm{id}_C$. Because f is an epimorphism, it follows that $fg = \mathrm{id}_A$.

Ad 2. Let $m = \tilde{m}e$ be the epi/cocover factorization of m. As a first factor of a monomorphism e is monic, and the conclusion follows from 1.

Ad 3. Let $f : A \to B$ be monic and epic, and suppose $g : A \to U$. Then by coarseness of C, there exists a map $h : B \to C$ with $hf = mg$, and by orthogonality there exists a map $k : B \to U$ such that $kf = g$ and $mk = h$.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, tail, two heads, "f"'] & U \arrow[d, "m"] \\ B \arrow[r, dotted, "h"'] \arrow[ur, dotted, "k"] & C \end{tikzcd}$$

Ad 4. The terminal object is trivially coarse. For binary products, let $C_1, C_2 \in \mathcal{E}$ be coarse, and suppose that $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ is monic and epic. To extend an arrow $g = \langle g_1, g_2\rangle : A \to C_1 \times C_2$ along f, extend g_1 and g_2 individually.

Ad 5. Let $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ be monic and epic and let $g : A \to PD$. The lifting of g along f is given by $[\![b \mid \{d \mid \exists a \,.\, f(a) = b \wedge d \in g(a)\}]\!]$.

Ad 6. It follows from 3 and 4 that $\boldsymbol{T}(\mathcal{E})$ has finite limits, because equalizers are cocovers. The powersets are coarse by 5, and from 2 and 3 it follows that the maps that are classified by arrows of type $A \to PB$ are precisely the monomorphisms $m : U \rightarrowtail A \times B$ with coarse U. ∎

Next, we show that the subcategory of coarse objects of a q-topos is reflective, and establish properties of this reflection.

Lemma 4.17 *Let $\mathcal{E}$ be a q-topos.*

1. *The inclusion functor $I_{\mathcal{E}} : \boldsymbol{T}(\mathcal{E}) \to \mathcal{E}$ has a left adjoint $J_{\mathcal{E}} : \mathcal{E} \to \boldsymbol{T}(\mathcal{E})$.*
2. *$J_{\mathcal{E}} : \mathcal{E} \to \boldsymbol{T}(\mathcal{E})$ preserves finite limits and epimorphisms.*
3. *$I_{\mathcal{E}} : \boldsymbol{T}(\mathcal{E}) \to \mathcal{E}$ preserves epimorphisms.*

Proof. Ad 1. It is sufficient to find a monic epic morphism $\eta_A : A \rightarrowtail\!\!\!\!\twoheadrightarrow J_{\mathcal{E}}A$ into a coarse object $J_{\mathcal{E}}A$ for every $A \in \mathcal{E}$. From this, the required universal property follows easily.

Such a map is given by the epi/cocover factorization $A \rightarrowtail\!\!\!\!\twoheadrightarrow J_{\mathcal{E}}A \rhd\!\!\!\to PA$ of the singleton map $\iota = [\![a \mid \{b \mid a = b\}]\!] : A \rightarrowtail PA$ – the object $J_{\mathcal{E}}A$ is coarse by Lemma 4.16-3 and -5, and using Lemma 4.10-2 it can be shown that ι, and thus η_A, is monic.

Ad 2. Epimorphisms are preserved because $J_{\mathcal{E}}$ is a left adjoint.

For the finite limits, we show that $J_{\mathcal{E}}$ preserves the terminal object, binary products and equalizers.

We have already seen that the terminal object is coarse, hence $J_{\mathcal{E}}(1) = 1$.

To show that $J_{\mathcal{E}}$ preserves products, it is sufficient to show that for all $A, B \in \mathcal{E}$, the arrow $\eta_A \times \eta_B : A \times B \to J_{\mathcal{E}}(A) \times J_{\mathcal{E}}(B)$ constitutes a coarse reflection of $A \times B$. $J_{\mathcal{E}}(A) \times J_{\mathcal{E}}(B)$ is coarse by Lemma 4.16-4, thus it remains to show that $\eta_A \times \eta_B$ is monic and epic. This follows from the decomposition $\eta_A \times \eta_B = (\eta_A \times J_{\mathcal{E}}B) \circ (A \times \eta_B)$ since $\eta_A \times J_B$ and $A \times \eta_B$ are monic and epic as pullbacks of η_A and η_B (epimorphisms are stable under pullback by Lemma 4.12).

Finally, consider a pair $f, g : A \to B$ of parallel arrows, with equalizer $m : U \rhd\!\!\!\to A$. To show that $J_{\mathcal{E}}$ preserves equalizers, consider the diagram

$$\begin{tikzcd}
U \arrow[r, "m"] \arrow[d, dotted, "h"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\
V \arrow[r, "n"'] & J_{\mathcal{E}}(A) \arrow[r, shift left, "J_{\mathcal{E}}(f)"] \arrow[r, shift right, "J_{\mathcal{E}}(g)"'] & J_{\mathcal{E}}(B)
\end{tikzcd}$$

where n is the equalizer of $J_{\mathcal{E}}(f)$ and $J_{\mathcal{E}}(g)$. It is sufficient to show that the left square is a pullback, since this implies that h is monic and epic.

n is classified by $[\![x{:}J_{\mathcal{E}}A \mid (J_{\mathcal{E}}f)(x) = (J_{\mathcal{E}}g)(x)]\!]$, and its pullback along η_A is classified by $[\![x{:}A \mid (J_{\mathcal{E}}f)(\eta_A x) = (J_{\mathcal{E}}g)(\eta_A x)]\!]$. Thus, it remains to show the equivalence of the latter predicate to $[\![x{:}A \mid fx = gx]\!]$ which is straightforward using naturality of η and the fact that η_B is monic.

Ad 3. Let $e : A \to B$ be an epimorphism in $\boldsymbol{T}(\mathcal{E})$. We take its epi/cocover factorization $A \twoheadrightarrow D \rightarrowtail B$ in $\mathcal{E}$. Then by Lemma 4.16-3 D is coarse, and since $J_{\mathcal{E}}$ preserves monomorphisms and epimorphisms by 2, the factorization is also an epi/mono factorization in $\boldsymbol{T}(\mathcal{E})$. Since e is an epimorphism and $\boldsymbol{T}(\mathcal{E})$ is balanced (as a topos), the map $D \rightarrowtail B$ is an isomorphism. Thus, e is an epimorphism in $\mathcal{E}$. ∎

There is an evident special inclusion 2-functor $\boldsymbol{U} : \mathbf{Top} \to \mathbf{QTop}$ from the pre-equipment of toposes (Definition 2.6) into the pre-equipment of q-toposes. We can now show the following theorem.

Theorem 4.18 *$\boldsymbol{U}$ has a special left biadjoint $\boldsymbol{T} : \mathbf{QTop} \to \mathbf{Top}$.*

Proof. We proceed by verifying the conditions of Lemma 2.9.

Given a q-topos $\mathcal{E}$, its image under $\boldsymbol{T}$ is the subtopos $\boldsymbol{T}(\mathcal{E})$ of coarse objects introduced in Lemma 4.16; the unit $\boldsymbol{\eta}_{\mathcal{E}} : \mathcal{E} \to \boldsymbol{T}(\mathcal{E})$ is given by the reflector $J_{\mathcal{E}}$, which is regular by Lemma 4.17-2.

For $F : \mathcal{E} \to \mathcal{F}$ (where $\mathcal{F}$ is a topos), the initial object of $(\mathcal{E}\swarrow\boldsymbol{U})(J_{\mathcal{E}}, F)$ is given by $(FI_{\mathcal{E}}, F\eta)$, where $\eta : \mathrm{id}_{\mathcal{E}} \to I_{\mathcal{E}}J_{\mathcal{E}}$ is the unit of $J_{\mathcal{E}} \dashv I_{\mathcal{E}}$. To see that this is indeed initial, note that for $G : \boldsymbol{T}(\mathcal{E}) \to \mathcal{F}$, we have a bijection between natural transformations $\beta : F \to GJ_{\mathcal{E}}$ and natural transformations $\iota : FI_{\mathcal{E}} \to G$ induced by the adjunction – concretely, this bijection maps $\beta : F \to GJ_{\mathcal{E}}$ to $G\varepsilon \circ \beta I : \widehat{F} \to G$.

For condition 2.9-2, we have to verify that $FI_{\mathcal{E}}$ is regular and $F\eta$ is invertible whenever F is a regular q-topos morphism. The first claim follows because $I_{\mathcal{E}}$ preserves epimorphisms by Lemma 4.17-3. The second claim follows since the components of η are monic and epic by the proof of Lemma 4.17-1, F preserves both monomorphisms and epimorphisms, and $\mathcal{F}$ is balanced.

Conditions 2.9-3 and 4 follow by comparing the asserted initial objects with the ones obtained by the previously described construction and observing that the mediators are iso. ∎

5. The tripos-to-q-topos construction

To complete our analysis of the tripos-to-topos construction, we want to show that the forgetful functor $\boldsymbol{S} : \mathbf{QTop} \to \mathbf{Trip}$ defined after Lemma 4.13 has a special left biadjoint $\boldsymbol{Q} : \mathbf{Trip} \to \mathbf{QTop}$.

We start by constructing the q-topos $\boldsymbol{Q}(\mathcal{P})$ associated to a tripos $\mathcal{P}$.

5.1. The topos $\boldsymbol{Q}(\mathcal{P})$

Throughout this section let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$ be a fixed tripos.

Definition 5.1 The category $\boldsymbol{Q}(\mathcal{P})$ is defined as follows.

- Objects of $\boldsymbol{Q}(\mathcal{P})$ are pairs (C, ρ), where $C \in \mathbb{C}$ and $\rho \in \mathcal{P}(C \times C)$ is a *partial equivalence relation* (i.e. symmetric and transitive, abbreviated p.e.r.) in the logic of $\mathcal{P}$.
- Morphisms $[f] : (C, \rho) \to (D, \sigma)$ are equivalence classes of morphisms $f : C \to D$ in $\mathbb{C}$ which satisfy $\big(x, y \mid \rho(x, y) \vdash \sigma(fx, fy)\big)$ in $\mathcal{P}$ (if this judgement holds, we say that f is *compatible* with ρ and σ). Two such morphisms $f, g : C \to D$ are identified as morphisms from (C, ρ) to (D, σ), if $\big(x \mid \rho(x, x) \vdash \sigma(fx, gx)\big)$ holds.

- Composition and identities are inherited from $\mathbb{C}$. $\diamondsuit$

Remark 5.2 An analogous construction to that of $\boldsymbol{Q}(\mathcal{P})$ with total equivalence relations instead of partial ones has been used by Maietti and Rosolini [24] to construct 'quotient completions' of indexed preorders interpreting $\{\top, \wedge, =\}$ and satisfying certain comprehension principles. $\diamondsuit$

It is easy to see that $\boldsymbol{Q}(\mathcal{P})$ is well-defined, but it takes some effort to show that it is a q-topos. The next three lemmas are devoted to this.

Lemma 5.3 *For any tripos* $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$*, the category* $\boldsymbol{Q}(\mathcal{P})$ *has finite limits.*

Proof. Binary products of (C, ρ) and (D, σ) are given by $(C \times D, \rho \bowtie \sigma)$ with $\rho \bowtie \sigma = [\![c, d, c', d' \mid \rho(c, c') \wedge \sigma(d, d')]\!]$, and $(1, \top)$ is a terminal object.

An equalizer of $[f], [g] : (C, \rho) \to (D, \sigma)$ is given by $[\mathrm{id}] : (C, \tau) \to (C, \rho)$, where $\tau = [\![c, c' \mid \rho(c, c') \wedge \sigma(fc, gc)]\!]$. $\blacksquare$

The construction of the equalizer in the proof can be understood as 'restricting the p.e.r. ρ to a smaller support'. In the next lemma, we show that all strong subobjects in $\boldsymbol{Q}(\mathcal{P})$ can be represented this way. The supports in this representation are predicates which are compatible with the p.e.r., in a sense made precise in the following definition.

Definition 5.4 Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$ be a tripos.

1. Given $(C, \rho) \in \boldsymbol{Q}(\mathcal{P})$, we call $\varphi \in \mathcal{P}(C)$ *strict w.r.t* ρ, if the judgments
$$\big(x{:}C \mid \varphi(x) \vdash \rho(x, x)\big) \quad \text{and} \quad \big(x{:}C, y{:}C \mid \varphi(x), \rho(x, y) \vdash \varphi(y)\big)$$
hold in $\mathcal{P}$.
2. The indexed preorder $\overline{\mathcal{P}} : \boldsymbol{Q}(\mathcal{P})^{\mathsf{op}} \to \mathbf{Ord}$ is defined as follows:
   - $\overline{\mathcal{P}}(C, \rho)$ is the sub-preorder of $\mathcal{P}(C)$ on predicates which are strict w.r.t. ρ.
   - Given $[f] : (C, \rho) \to (D, \sigma)$ and $\varphi \in \overline{\mathcal{P}}(D, \sigma)$, $[f]^*(\varphi) \in \overline{\mathcal{P}}(C, \rho)$ is the predicate $[\![x{:}C \mid \rho(x, x) \wedge \varphi(fx)]\!]$. $\diamondsuit$

The notation $\overline{\mathcal{P}}$ is adopted from [24] where an analogous construction is defined with total instead of partial equivalence relations.

We show now that $\overline{\mathcal{P}}$ is equivalent to the indexed preorder of strong monomorphisms on $\boldsymbol{Q}(\mathcal{P})$.

Lemma 5.5 *Let* $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$ *be a tripos.*

1. *The predicate* $\mathsf{eqv} = [\![p,q : \mathfrak{P}1 \mid * \in p \Leftrightarrow * \in q]\!]$[9] *is a p.e.r. on* $\mathfrak{P}1 \in \mathbb{C}$.
2. $[e] : (C, \rho) \to (D, \sigma)$ *is epic if and only if* $\big(d \mid \sigma(d, d) \vdash \exists c \,.\, \rho(c, c) \wedge \sigma(ec, d)\big)$ *holds in* $\mathcal{P}$.
3. *If* $\psi \in \overline{\mathcal{P}}(D, \sigma)$*, then* $\sigma|_\psi = [\![d, d' \mid \sigma(d, d') \wedge \varphi(d)]\!]$ *is a p.e.r. on* D*, and* $[\mathrm{id}] : (D, \sigma|_\psi) \rightarrowtail (D, \sigma)$ *is a cocover.*
4. *For* $[f] : (C, \rho) \to (D, \sigma)$*, the predicate* $\psi = [\![d{:}D \mid \exists c \,.\, \rho(c, c) \wedge \sigma(fc, d)]\!]$ *is strict w.r.t.* σ*. Furthermore,* $[f]$ *factors through* $[\mathrm{id}] : (D, \sigma|_\psi) \rightarrowtail (D, \sigma)$*, giving rise to a epi/cocover factorization.*

[9] As in the core calculus, $*$ denotes the unique global element of 1.

5. *The mapping*

$$\overline{\mathcal{P}}(D,\sigma) \ni \psi \quad \mapsto \quad \big((D,\sigma|_{\psi}) \overset{[\mathrm{id}]}{\rightarrowtail} (D,\sigma)\big) \in \mathrm{ssub}(\boldsymbol{Q}(\mathcal{P}))(D,\sigma)$$

gives an equivalence between the indexed preorders $\overline{\mathcal{P}}$ *and* $\mathrm{ssub}(\boldsymbol{Q}(\mathcal{P}))$.

Proof. Ad 1. This follows from symmetry and transitivity of logical equivalence.

Ad 2. Assume first that the judgment holds, and that $[f],[g] : (D,\sigma) \to (E,\tau)$ such that $[fe] = [ge]$. Then by the definition of equality of morphisms we have $\big(c \mid \rho(c,c) \vdash \tau(f(ec), g(ec))\big)$, and from this judgment and the hypothesis we can derive $\big(d \mid \sigma(d,d) \vdash \tau(fd, gd)\big)$.

Conversely assume that $[e]$ is epic, and let $f,g : D \to \mathfrak{P}1$ be maps such that $[\![d \mid * \in fd]\!] \cong [\![d \mid \sigma(d,d)]\!]$ and $[\![d \mid * \in gd]\!] \cong [\![d \mid \exists c\,.\,\rho(c,c) \wedge \sigma(ec,d)]\!]$ in $\mathcal{P}(D)$. Then f and g are both compatible with σ and eqv, and furthermore $[fe] = [ge]$ as morphisms from (C,ρ) to $(\mathfrak{P}1, \mathsf{eqv})$. Since $[e]$ is epic we can deduce $[f] = [g]$, and from this the claim follows.

Ad 3. It is easy to see that $[\mathrm{id}] : (D,\sigma|_{\psi}) \to (D,\sigma)$ is well-defined and monic. To see that it is a cocover, consider a commuting square

$$\begin{tikzcd}
(C,\rho) \arrow[r, "{[f]}"'] \arrow[d, two heads, "{[e]}"'] & (D,\sigma|_{\psi}) \arrow[d, "{[\mathrm{id}]}"] \\
(E,\eta) \arrow[r, "{[g]}"] & (D,\sigma)
\end{tikzcd}$$

with epic $e : (C,\rho) \twoheadrightarrow (E,\eta)$. Using the previous characterization of epimorphisms, one can show that g is compatible with η and $\sigma|_{\psi}$, which gives the desired mediator.

Ad 4. Straightforward.

Ad 5. Clearly the construction is monotone. Furthermore, it is easy to show that the square

$$\begin{tikzcd}
(C,\rho|_{[f]^*\psi}) \arrow[r, "{[f]}"'] \arrow[d, tail, "{[\mathrm{id}]}"'] & (D,\sigma|_{\psi}) \arrow[d, tail, "{[\mathrm{id}]}"] \\
(C,\rho) \arrow[r, "{[f]}"] & (D,\sigma)
\end{tikzcd}$$

is a pullback for all $[f] : (C,\rho) \to (D,\sigma)$ and $\psi \in \overline{\mathcal{P}}(D,\sigma)$, which shows compatibility with reindexing. Finally, the construction from 4 gives a pointwise monotone inverse. ■

Lemma 5.6 1. $\boldsymbol{Q}(\mathcal{P})$ *has effective quotients of strong equivalence relations, and up to isomorphism the regular epimorphisms coincide with the morphisms of the form* $[\mathrm{id}] : (C,\rho) \to (C,\tau)$ *with* $\big(x \mid \tau(x,x) \vdash \rho(x,x)\big)$.
In the following we refer to maps of this form as regular epimorphisms in normalized presentation.

2. *Regular epimorphisms in* $\boldsymbol{Q}(\mathcal{P})$ *are stable under pullback.*
3. *The presheaves* $\mathrm{ssub}(- \times (C,\rho)) : \boldsymbol{Q}(\mathcal{P})^{\mathsf{op}} \to \mathbf{Set}$ *are representable.*
4. $\boldsymbol{Q}(\mathcal{P})$ *is a q-topos.*

Proof. Ad 1. Via the equivalence established in Lemma 5.5-5, strong equivalence relations on $(C,\rho) \in \boldsymbol{Q}(\mathcal{P})$ correspond to equivalence relations on (C,ρ) in the indexed preorder $\overline{\mathcal{P}}$ of strict predicates. Using elementary logic and the construction of binary products given in the proof of Lemma 5.3, one can show that those are exactly the predicates $\tau \in \mathcal{P}(C \times C)$ that are *partial* equivalence relations in $\mathcal{P}$ and satisfy the judgments

$$\big(x \mid \tau(x,x) \vdash \rho(x,x)\big) \quad \text{and} \quad \big(x,y \mid \rho(x,y) \vdash \tau(x,y)\big). \tag{5.1}$$

Given such a τ, we want to show that $[\mathrm{id}] : (C,\rho) \to (C,\tau)$ is a quotient map, i.e. that

$$[\pi_0],[\pi_1] : (C \times C, (\rho \bowtie \rho)|_\tau) \rightrightarrows (C,\rho) \xrightarrow{[\mathrm{id}]} (C,\tau)$$

is a coequalizer diagram. This is straightforward and left to the reader.

To see that the quotient is effective, note that the strict predicate in $\overline{\mathcal{P}}(C \times C, \tau \bowtie \tau)$ corresponding to the cocover $\delta : (C,\tau) \rightarrowtail (C,\tau) \times (C,\tau)$ is τ itself, and reindexing along $[\mathrm{id}] \times [\mathrm{id}] : (C,\rho) \times (C,\rho) \to (C,\tau) \times (C,\tau)$ in $\overline{\mathcal{P}}$ gives τ viewed as a strict predicate on $(C \times C, \rho \bowtie \rho)$.

Ad 2. It is sufficient to show stability under pullback of regular epimorphisms in normalized presentation. A pullback of such a morphism $[\mathrm{id}] : (D,\sigma) \twoheadrightarrow (D,\tau)$ along $[f] : (C,\rho) \to (D,\tau)$, is given by the square

$$\begin{array}{ccc} (C \times D, \theta) & \xrightarrow{[\pi_1]} & (D,\sigma) \\ {\scriptstyle [\pi_0]}\downarrow & & \downarrow{\scriptstyle [\mathrm{id}]} \\ (C,\rho) & \xrightarrow[{[f]}]{} & (D,\tau) \end{array} \quad \text{with} \quad \theta = [\![c,d,c',d' \mid \rho(c,c') \wedge \sigma(d,d') \wedge \tau(fc,d)]\!].$$

Postcomposition of $[\pi_0]$ with the isomorphism $[\langle \mathrm{id}, f\rangle] : (C,\rho) \to (C \times D, \xi)$ where $\xi = [\![c,d,c',d' \mid \rho(c,c') \wedge \tau(d,d') \wedge \tau(fc,d)]\!]$ and whose inverse is given by $[\pi_0]$ gives a regular epimorphism in normalized presentation again.

Ad 3. For $(C,\rho) \in \boldsymbol{Q}(\mathcal{P})$, we define its power object as $(\mathfrak{P}C, \mathfrak{P}\rho)$ with

$$\begin{aligned} \mathfrak{P}\rho = [\![m,n \mid & (\forall x \,.\, x \in m \Rightarrow \rho(x,x)) \wedge (\forall x,y \,.\, \rho(x,y) \Rightarrow x \in m \Rightarrow y \in m) \\ & \wedge (\forall x \,.\, x \in m \Leftrightarrow x \in n)]\!]. \end{aligned}$$

The predicate $[\![x{:}C, m{:}\mathfrak{P}C \mid x \in m \wedge (\mathfrak{P}\rho)(m,m)]\!]$ in $\mathcal{P}$ is strict w.r.t $\rho \bowtie \mathfrak{P}\rho$ and gives the membership relation. The verification that this gives a representation of $\mathrm{ssub}(- \times (C,\rho))$ is lengthy but straightforward.

Ad 4. This follows from Lemma 5.3, and items 1 and 3. ∎

We can embed $\mathbb{C}$ into $\boldsymbol{Q}(\mathcal{P})$ via the *constant objects functor*

$$\Delta : \mathbb{C} \to \boldsymbol{Q}(\mathcal{P}) \quad \text{defined by} \quad C \mapsto (C,=), \quad f \mapsto [f].$$

It is easy to see that Δ preserves finite products.

Since all predicates are strict w.r.t equality, predicates $\varphi \in \mathcal{P}(C)$ for $C \in \mathbb{C}$ correspond to strong subobjects of ΔC, which allows us to define a 2-natural transformation $\Xi : \mathcal{P} \to \boldsymbol{S}(\boldsymbol{Q}(\mathcal{P})) \circ \Delta$ giving rise to a tripos transformation

$$\mathfrak{D} = (\Delta, \Xi) : \mathcal{P} \to \boldsymbol{S}(\boldsymbol{Q}(\mathcal{P})) \tag{5.2}$$

which preserves all first order logical structure.

Remark 5.7 In general, Δ does not preserve equalizers. Consider for example the equalizer diagram $0 \rightarrowtail 1 \rightrightarrows 2$ in **Set**. Its image under $\Delta : \mathbf{Set} \to \boldsymbol{Q}(\boldsymbol{S}(\mathbf{Set}))$ has empty carrier, but the equalizer of $\Delta(1) \rightrightarrows \Delta(2)$ has carrier 1. ◇

5.2. The special left biadjoint

Using Lemma 2.9, we now show that $\boldsymbol{S} : \mathbf{QTop} \to \mathbf{Trip}$ has a special left biadjoint $\boldsymbol{Q} : \mathbf{Trip} \to \mathbf{QTop}$. For fixed $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$, the tripos morphism (5.2) serves as unit $\eta_{\mathcal{P}} : \mathcal{P} \to \boldsymbol{S}(\boldsymbol{Q}(\mathcal{P}))$. Now let $\mathcal{E}$ be a q-topos, and $\mathfrak{F} = (F, \Phi) : \mathcal{P} \to \boldsymbol{S}(\mathcal{E})$ a tripos morphism. To construct the initial object $(\widehat{\mathfrak{F}}, \alpha)$ of $(\mathcal{P} \swarrow \boldsymbol{S})(\mathfrak{D}, \mathfrak{F})$, we fix some notation and terminology.

Given $C \in \mathcal{E}$ and a p.e.r. $r \in \boldsymbol{S}(\mathcal{E})(C \times C)$, we construct a *subquotient span*

$$C \xleftarrow{m} |r| \xrightarrow{e} C/r. \tag{5.3}$$

Here $|r|$ – called the *support of* r – is the domain of the predicate $m = [\![c \mid r(c,c)]\!]$, and C/r – the *quotient of* C *by* r – is the quotient of the strong equivalence relation $[\![x, y{:}|r| \mid r(mx, my)]\!]$ obtained by restricting r to its support. Given the subquotient span (5.3), we can reconstruct r up to equivalence as the predicate

$$[\![c, c' \mid \exists x, y{:}|r| \,.\, ex = ey \wedge mx = c \wedge my = c']\!]. \tag{5.4}$$

The functor $\widehat{\mathfrak{F}} : \boldsymbol{Q}(\mathcal{P}) \to \mathcal{E}$ is now defined by mapping objects (C, ρ) to $FC/\Phi\rho$, and morphisms $[f] : (C, \rho) \to (D, \sigma)$ to the unique mediator on the right in the following diagram.

$$\begin{tikzcd} FC & {|\Phi\rho|} \arrow[l] \arrow[r] \arrow[d, dotted] & FC/\Phi\rho \arrow[d, dotted, "\widehat{\mathfrak{F}}([f])"] \\ FD \arrow[from=u, "Ff"'] & {|\Phi\sigma|} \arrow[l] \arrow[r] & FD/\Phi\sigma \end{tikzcd} \tag{5.5}$$

The existence of the mediators follows from compatibility of f with ρ and σ, and the right mediator is independent of the choice of representative of $[f]$. Functoriality follows from uniqueness.

Lemma 5.8 *The functor $\widehat{\mathfrak{F}} : \boldsymbol{Q}(\mathcal{P}) \to \mathcal{E}$ preserves finite limits and covers. If $\mathfrak{F}$ commutes with existential quantification along projections, then $\widehat{\mathfrak{F}}$ also preserves epimorphisms.*

Proof. We show first that $\widehat{\mathfrak{F}}$ preserves finite products. Given $(C, \rho), (D, \sigma) \in \boldsymbol{Q}(\mathcal{P})$, the product

$$FC \times FD \xleftarrow{m \times m'} |\Phi\rho| \times |\Phi\sigma| \xrightarrow{e \times e'} FC/\Phi\rho \times FD/\Phi\sigma$$

of the subquotient spans associated to $\widehat{\mathfrak{F}}(C, \rho)$ and $\widehat{\mathfrak{F}}(D, \sigma)$ is a subquotient span again, and the associated p.e.r. given by (5.4) is easily seen to be equivalent to $\rho \bowtie \sigma$. The preservation of terminal objects is straightforward, and the preservation of equalizers follows in a similar way using the representation of equalizers given in the proof of Lemma 5.3.

For the preservation of covers, it is sufficient to consider those in normalized presentation. If we apply the construction (5.5) to such a morphism $[\mathrm{id}] : (C, \rho) \twoheadrightarrow (C, \sigma)$, we obtain

$$\begin{tikzcd} FC \arrow[d, "\mathrm{id}"'] & {|\Phi\rho|} \arrow[l] \arrow[r, "e"'] \arrow[d, "\cong"] & FC/\Phi\rho \arrow[d, "\widehat{\mathfrak{F}}([\mathrm{id}])"] \\ FC & {|\Phi\sigma|} \arrow[l] \arrow[r, "e'"] & FC/\Phi\sigma \end{tikzcd} .$$

The middle mediator is an isomorphism since $\big(x{:}FC \mid \Phi\sigma(x,x) \vdash \Phi\rho(x,x)\big)$ holds in $\boldsymbol{S}(\mathcal{E})$, and $\widehat{\mathfrak{F}}([\mathrm{id}])$ is a cover since e and e' are.

Finally, assume that $\mathfrak{F}$ is regular and let $[e] : (C, \rho) \twoheadrightarrow (D, \sigma)$ be an epimorphism. By Lemma 5.5-2, we have $\big(y{:}D \mid \sigma(y, y) \vdash \exists x{:}C \,.\, \rho(x, x) \wedge \sigma(ex, y)\big)$, and thus $\big(y{:}FD \mid \Phi\sigma(y, y) \vdash \exists x{:}FC \,.\, \Phi\rho(x, x) \wedge \Phi\sigma(ex, y)\big)$. From this we can infer that $\widehat{\mathfrak{F}}([e])$ is epic using the characterization of Lemma 4.10-3. ∎

Next we define the tripos transformation $\alpha_{\mathfrak{F}} : \mathfrak{F} \to \boldsymbol{S}(\widehat{\mathfrak{F}}) \circ \mathfrak{D}$. If we apply $\widehat{\mathfrak{F}}$ to a constant object $(C, =)$, we obtain $FC/\Phi(=)$, and we define the transformation

$$\alpha_{\mathfrak{F}} : F \to \widehat{\mathfrak{F}} \circ \Delta \tag{5.6}$$

at $C \in \mathbb{C}$ to be the quotient map $FC \twoheadrightarrow FC/\Phi(=)$. To show that $\alpha_{\mathfrak{F}}$ is a 2-cell between the tripos morphisms $\mathfrak{F}$ and $\boldsymbol{S}(\widehat{\mathfrak{F}}) \circ \mathfrak{D}$, we have to check inequality (3.3), which amounts to demonstrating the existence of a mediator h in

$$\begin{tikzcd} \cdot \arrow[r, dotted, "h"] \arrow[d, rightarrowtail, "\Phi\psi"'] & FC/\Phi(=_\psi)^{12} \arrow[d, rightarrowtail] \\ FC \arrow[r, twoheadrightarrow] & FC/\Phi(=) \end{tikzcd} .$$

Both the domain of $\Phi\psi$ and $FC/\Phi(=_\psi)$ are subquotients of FC, corresponding to the p.e.r.s $[\![x, y \mid x = y \wedge (\Phi\psi)(x)]\!]$ and $[\![x, y \mid \Phi(x = y) \wedge (\Phi\psi)(x)]\!]$. The existence of h follows since the former p.e.r. is contained in the latter.

Lemma 5.9 *Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$ be a tripos, $\mathcal{E}$ a q-topos and $\mathfrak{F} = (F, \Phi) : \mathcal{P} \to \boldsymbol{S}(\mathcal{E})$ a tripos morphism. The pair $(\widehat{\mathfrak{F}}, \alpha_{\mathfrak{F}})$ is initial in $(\mathcal{P} \swarrow \boldsymbol{S})(\mathfrak{D}, \mathfrak{F})$.*

If Φ preserves equality predicates, then $\alpha_{\mathfrak{F}}$ is invertible.

Proof. To show that $(\widehat{\mathfrak{F}}, \alpha_{\mathfrak{F}})$ is initial in $(\mathcal{P} \swarrow \boldsymbol{S})(\mathfrak{D}, \mathfrak{F})$, consider a second object (G, β) with $G : \boldsymbol{Q}(\mathcal{P}) \to \mathcal{E}$ finite limit preserving and $\beta : \mathfrak{F} \to \boldsymbol{S}(G) \circ \mathfrak{D}$. To construct a natural transformation $\iota : \widehat{\mathfrak{F}} \to G$, let $(C, \rho) \in \boldsymbol{Q}(\mathcal{P})$, and consider the subquotient span

$$\Delta C \xleftarrow{\;m\;} (C, =_{|\rho|}) \xrightarrow{\;e\;} (C, \rho) \; . \tag{5.7}$$

We construct the following four spans in $\mathcal{E}$.

1. $FC \leftarrowtail |\Phi\rho| \twoheadrightarrow FC/\Phi\rho$
2. $G\Delta C \leftarrowtail |G\Xi\rho| \twoheadrightarrow G\Delta C/G\Xi\rho$
3. $FC/\Phi(=) \leftarrowtail FC/\Phi(=_{|\rho|}) \twoheadrightarrow FC/\Phi\rho$
4. $G\Delta C \leftarrowtail G(C, =_{|\rho|}) \to G(C, \rho)$

Spans 1 and 2 are subquotient spans associated to the p.e.r.s $\Phi\rho$ and $G\Xi\rho$, and 3 and 4 are the images of (5.7) under $\widehat{\mathfrak{F}}$ and G, respectively. Note that the right leg of 3 is a cover by Lemma 5.8.

Unless G preserves covers, spans 2 and 4 need not be isomorphic, but they have isomorphic left legs, and their right legs have isomorphic kernels, which implies that the right leg of 2 is a factor of the right leg of 4. Since β is a tripos transformation, the judgment

$$\big(x, y{:}FC \mid \Phi\rho(x, y) \vdash G\Xi\rho(\beta_C x, \beta_C y)\big)$$

[12] We write $=_\psi$ instead of $=|_\psi$ for the restriction $[\![x{:}C, y{:}C \mid x = y \wedge \psi(x)]\!]$ of the equality predicate to ψ.

holds in $\boldsymbol{S}(\mathcal{E})$, which is equivalent to the existence of mediators between the spans 1 and 2. Hence, we can combine spans 1, 2 and 4 into one diagram:

$$\begin{tikzcd}
FC & {|\Phi\rho|} \arrow[l] \arrow[r] \arrow[d] & FC/\Phi\rho \arrow[d] \arrow[dd, bend left, dashed, "\iota_{(C,\rho)}"] \\
G\Delta C \arrow[from=1-1, "\beta_C"'] & {|G\Xi\rho|} \arrow[l] \arrow[r] \arrow[d, "\cong"] & G\Delta C/G\Xi\rho \arrow[d] \\
& G(C, =_{|\rho|}) \arrow[r, "Ge"] \arrow[ul, "Gm"] & G(C,\rho)
\end{tikzcd},$$

and we can define $\iota_{(C,\rho)}$ as the composition on the right.

It is routine to check that this construction is natural in (C,ρ); to see that it is a morphism in $(\mathcal{P}\swarrow\boldsymbol{S})(\mathfrak{D},\mathfrak{F})$ we apply the construction to ΔC, which gives

$$\begin{tikzcd}
FC \arrow[r, equal] \arrow[d, "\beta_C"'] & FC \arrow[r, "\alpha_{\mathfrak{F},C}"] \arrow[d, "\beta_C"'] & FC/\Phi(=) \arrow[d] \arrow[dd, bend left, dashed, "\iota_{\Delta C}"] \\
G\Delta C \arrow[r, equal] \arrow[dr, equal] & G\Delta C \arrow[r, equal] \arrow[d, equal] & G\Delta C \arrow[d, equal] \\
& G\Delta C \arrow[r, equal] & G\Delta C
\end{tikzcd}$$

and we can read off the required equality $\iota\Delta \circ \alpha_{\mathfrak{F}} = \beta$[13].

To see that ι is uniquely determined by this condition, consider the following diagram relating spans 3 and 4.

$$\begin{tikzcd}
FC \arrow[r, "\alpha_{\mathfrak{F},C}"] \arrow[dr, "\beta_C"'] & \widehat{\mathfrak{F}}\Delta C \arrow[d, "\iota_{\Delta C}"] & \widehat{\mathfrak{F}}(C, =_{|\rho|}) \arrow[l, "\widehat{\mathfrak{F}}m"'] \arrow[r, "\widehat{\mathfrak{F}}e"] \arrow[d, "\iota_{(C,=_{|\rho|})}"] & \widehat{\mathfrak{F}}(C,\rho) \arrow[d, "\iota_{(C,\rho)}"] \\
& G\Delta C & G(C, =_{|\rho|}) \arrow[l, "Gm"'] \arrow[r, "Ge"] & G(C,\rho)
\end{tikzcd}$$

∎

The unicity of $\iota_{(C,\rho)}$ follows since Gm is monic, and $\alpha_{\mathfrak{F},C}$ and $\widehat{\mathfrak{F}}e$ are epic ($\widehat{\mathfrak{F}}e$ by Lemma 5.8).

The final claim about invertibility of $\alpha_{\mathfrak{F}}$ follows directly from its definition after equation (5.6).

Lemma 5.10 *Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$ be a tripos. The pair $(\mathrm{id}_{\boldsymbol{Q}(\mathcal{P})}, \mathrm{id}_{\mathfrak{D}})$ is initial in $(\mathcal{P}\swarrow\boldsymbol{S})(\mathfrak{D},\mathfrak{D})$.*

Proof. It suffices to show that the unique 2-cell $\iota : (\widehat{\mathfrak{D}}, \alpha_{\mathfrak{D}}) \to (\mathrm{id}_{\boldsymbol{Q}(\mathcal{P})}, \mathrm{id}_{\mathfrak{D}})$ in $(\mathcal{P}\swarrow\boldsymbol{S})(\mathfrak{D},\mathfrak{D})$ is an isomorphism. This follows since for $(C,\rho) \in \boldsymbol{Q}(\mathcal{P})$ we have $\widehat{\mathfrak{D}}(C,\rho) = \Delta C/\Xi\rho$ where $\Xi\rho$ is the cocover $(D \times D, =_\rho) \rightarrowtail (D \times D, =)$, and (C,ρ) is a quotient for this partial equivalence relation by the argument in the proof of Lemma 5.6-1. ∎

Lemma 5.11 *Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{HA}$ be a tripos, let $\mathcal{E}, \mathcal{F}$ be q-toposes, let $\mathfrak{F} : \mathcal{P} \to \boldsymbol{S}(\mathcal{E})$ be a tripos morphism and let $H : \mathcal{E} \to \mathcal{F}$ be a regular morphism of*

[13] The construction does not depend on particular choices of subquotient spans, so we can choose the spans in a way which gives us all the identity morphisms in the diagram.

q-toposes. The pair $(H \circ \widehat{\mathfrak{F}}, \boldsymbol{S}(H) \circ \alpha_{\mathfrak{F}})$ *is initial in* $(\mathcal{P}\swarrow\boldsymbol{S})(\mathfrak{D}, \boldsymbol{S}(H) \circ \mathfrak{F})$.

$$\begin{tikzcd}
\mathcal{P} \arrow[d, "\mathfrak{D}"'] \arrow[dr, bend left, "\mathfrak{F}"] & & \\
\boldsymbol{S}(\boldsymbol{Q}(\mathcal{P})) \arrow[r] \arrow[r, phantom, shift left=3, "\Downarrow\alpha_{\mathfrak{F}}"] & \boldsymbol{S}(\mathcal{E}) \arrow[r, "\boldsymbol{S}(H)"] & \boldsymbol{S}(\mathcal{F}) \\
\boldsymbol{Q}(\mathcal{P}) \arrow[r, "\widehat{\mathfrak{F}}"] & \mathcal{E} \arrow[r, "H"] & \mathcal{F}
\end{tikzcd}$$

Proof. Similarly to the previous lemma, we argue by comparing $H \circ \widehat{\mathfrak{F}}$ with $\widehat{\boldsymbol{S}(H) \circ \mathfrak{F}}$. The latter maps $(C, \rho) \in \boldsymbol{Q}(\mathcal{P})$ to $HFC/H\Phi\rho$, whereas the former maps (C, ρ) to $H(FC/\Phi\rho)$. The claim follows since H is assumed regular, and thus preserves quotients of strong partial equivalence relations. ■

Now we can state the theorem.

Theorem 5.12 *The forgetful functor* $\boldsymbol{S}$: **QTop** $\to$ **Trip** *has a special left biadjoint.*

Proof. This follows from Lemma 2.9, whose hypotheses we have verified in Lemmas 5.8, 5.9, 5.10, and 5.11. ■

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