Enriched triposes and enriched Pitts functors in parts joint work with Richard Garner

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Motivation/Introduction

- Hyland, Johnstone, Pitts 1980: tripos-to-topos construction generalizing toposes of sheaves on a locale
- Pitts 1982: Triposes on other base toposes/categories, iteration
- Can characterize realizability toposes over Set, but only enriched realizability toposes over other base toposes S
- Goal: a framework of 'enriched' triposes and 'enriched' Pitts functors generalizing localic geometric morphisms

Part 0 Localic geometric morphisms

Localic geometric morphisms

Definition

A localic geometric morphism between toposes \mathcal{S} , \mathcal{E} is a pair of functors $\Delta: \mathcal{S} \to \mathcal{E}$, $\Gamma: \mathcal{E} \to \mathcal{S}$ such that

- \bigcirc $\triangle \dashv \Gamma$
- ② A preserves finite limits
- ③ Δ is bounded by 1, i.e. every $A \in \mathcal{E}$ can be represented as subquotient $\Delta(J) \leftarrow U \twoheadrightarrow A$
- $\begin{array}{c} \operatorname{Gl}_{\Delta}(\mathcal{E}) \longrightarrow \mathcal{E} \downarrow \mathcal{E} \\ \bullet \text{ Using } \Delta, \, \mathcal{E} \text{ can be fibered over } \mathcal{S} \colon \left. \begin{smallmatrix} \operatorname{gl}_{\Delta}(\mathcal{E}) \\ \downarrow \end{smallmatrix} \right. & \left. \begin{smallmatrix} \operatorname{cod}(\mathcal{E}) \\ \downarrow \end{smallmatrix} \right. \\ \mathcal{S} \xrightarrow{\Delta} \mathcal{E} \end{array}$
- Using Γ , we can enrich \mathcal{E} in \mathcal{S} : $hom_{\mathcal{S}}(A,B) = \Gamma(B^A)$
- Since △ ⊢ Γ, fibration and enrichment are related Gl_△(S) is locally small

Part I Triposes

Triposes

Definition

Let S be a topos. A **tripos** on S is an indexed preorder

$$\mathcal{P}:\mathcal{S}^{\mathsf{op}} o \mathsf{Ord}$$

such that

- a the fibers P(I) are cartesian closed
- ② the **reindexing maps** $u^*: \mathcal{P}(I) \to \mathcal{P}(J)$ induced by $u: J \to I$ preserve cartesian closed structure, and have **right adjoints** $\forall u: \mathcal{P}(J) \to \mathcal{P}(I)$ satisfying the **Beck-Chevalley condition**
- ③ \mathcal{P} has a **generic predicate**, i.e. an element $\mathsf{tr} \in \mathcal{P}(\mathsf{Prop})$ such that for every $\varphi \in \mathcal{P}(J)$ there exists an $u : J \to \mathsf{Prop}$ with $u^*\mathsf{tr} \cong \varphi$
- Teindexing along epis is order-reflecting (not in the original definition)

Remark

Triposes also have fiberwise **finite joins**, and **left adjoints** $\exists_u : \mathcal{P}(J) \to \mathcal{P}(I)$ to reindexing, all constructed using 2nd order encodings.

The tripos-to-topos construction

Any **tripos** $\mathcal{P}: \mathcal{S}^{op} \to \textbf{Ord}$ gives rise to a **topos** $\mathcal{S}[\mathcal{P}]$ and a functor $\Delta_{\mathcal{P}}: \mathcal{S} \to \mathcal{S}[\mathcal{P}]$

Properties of $\Delta_{\mathcal{P}}$

- Δ_P is regular (preserves finite limits and epimorphisms)
- ② $\Delta_{\mathcal{P}}$ is bounded by 1



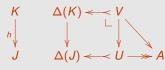
[For $\mathbb C$ with finite limits, sub: $\mathbb C^{op} \to \mathbf{Ord}$ is its subobject fibration – sub($\mathcal C$) is the preorder of monomorphisms into $\mathcal C$]

Pitts functors

Theorem (Pitts 81)

Functors $\Delta_{\mathbb{P}}: \mathcal{S} \to \mathcal{S}[\mathbb{P}]$ can be characterized as functors $\Delta: \mathcal{S} \to \mathcal{E}$ where

- \bigcirc S and \mathcal{E} are toposes
- \(\Delta \) is regular
- ③ for every $A \in \mathcal{E}$ there exists a **generic covering**, i.e. a subquotient span $\Delta(J) \leftarrow U \twoheadrightarrow A$ such that for any span $\Delta(K) \leftarrow V \rightarrow A$ there exists an $h : K \rightarrow J$ giving rise to a pullback diagram:



Definition

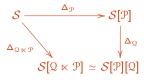
We call such functors Pitts functors.

Iteration

Theorem (Pitts 81)

Pitts functors compose.

This means that we can 'compose' triposes – given $\mathcal{P}:\mathcal{S}^{op}\to \mathbf{Ord}$ and $\mathcal{Q}:\mathcal{S}[\mathcal{P}]^{op}\to \mathbf{Ord}$, we obtain $\mathcal{Q}\ltimes\mathcal{P}:\mathcal{S}^{op}\to \mathbf{Ord}$.



Tripos transformations

Definition

Let S be a topos.

- A **tripos morphism** $f: \mathcal{P} \to \mathcal{Q}$ is an indexed monotone map preserving fiberwise finite meets.
- a regular tripos morphism is a tripos morphism which preserves ∃
- Trip(S) is the category of triposes on S and regular tripos morphisms

• Regular morphisms $f: \mathcal{P} \to \mathcal{Q}$ induce regular functors

$$\mathcal{S} \xrightarrow{\Delta_{\mathcal{P}}} \mathcal{S}[\mathcal{P}]$$

$$\stackrel{\simeq}{\longrightarrow} \mathcal{S}[g]$$

$$\mathcal{S}[g]$$

• Every regular functor
$$S \xrightarrow{\Delta_{\mathcal{P}}} S[\mathcal{P}]$$

$$\simeq \bigvee_{F} \text{ arises this way!}$$

$$S[0]$$

Moreover, every such functor is automatically a Pitts functor.

Lemma

$$\mathsf{Trip}(\mathcal{S}[\mathbb{P}]) \simeq \mathbb{P} \backslash \mathsf{Trip}(\mathcal{S})$$

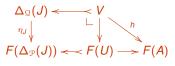
Tripos transformations 2

 General tripos morphisms f: P→Q give rise to flp functors and natural transformations

 $\mathcal{S} \xrightarrow{\Delta_{\mathcal{P}}} \mathcal{S}[\mathcal{P}]$ $\downarrow^{\eta} \qquad \downarrow^{\mathcal{S}[f]}$ $\mathcal{S}[\mathcal{Q}]$

• *Not* every diagram $\begin{array}{c} \mathcal{S} \stackrel{\Delta_{\mathcal{P}}}{\longrightarrow} \mathcal{S}[\mathcal{P}] \\ & \stackrel{\wedge}{\longrightarrow} \mathcal{S}[\mathcal{Q}] \end{array}$ arises this way!

Necessary and sufficient condition: For every generic covering
 [∆]_T(J) ← U → A, the map h in the diagram below is epic.



• Tripos morphisms $f: \mathcal{P} \to \mathcal{Q}$ are **coreflective** in diagrams



Part II Enriched triposes

Motivation

- $\Delta_{\mathbb{P}}: \mathcal{S} \to \mathcal{S}[\mathbb{P}]$ gives fibering of $\mathcal{S}[\mathbb{P}]$ over \mathcal{S} , but no enrichment
- In standard examples like realizability over an internal PCA, such an enrichment is natural
- enrichment and fibering can not be as tightly linked as in the localic case

Enriched triposes

Definition

An enriched tripos on $\mathcal S$ is a tripos $\mathcal P:\mathcal S^{\mathsf{op}}\to \mathbf{Ord}$ together with a tripos morphism $\gamma:\mathcal P\to \mathsf{sub}$ satisfying

(*)
$$T \leq \gamma_1(\varphi) \quad \Rightarrow \quad T \leq \varphi \quad \text{for} \quad \varphi \in \mathcal{P}(1).$$

• The data of an enriched tripos give rise to



- Γ induces an S-enrichment of S[P] via $hom(A, B) = \Gamma(B^A)$
- By (*), the enrichment is **standard**, i.e. $S(1, hom(A, B)) \cong S[P](A, B)$
- η makes $\Delta_{\mathcal{P}}$ into an \mathcal{S} -enriched functor:

$$\frac{\frac{\Delta(K^J) \times \Delta(J) \to \Delta(K)}{\Delta(K^J) \to \Delta(K)^{\Delta(J)}}}{\frac{\Gamma(\Delta(K^J)) \to \Gamma(\Delta(K)^{\Delta(J)})}{K^J \to \Gamma(\Delta(K)^{\Delta(J)})}}$$

Enriched Pitts-functors

Definition

An **enriched Pitts functor** is a Pitts functor $\Delta: \mathcal{S} \to \mathcal{E}$ together with a flp functor $\Gamma: \mathcal{E} \to \mathcal{S}$ and a natural transformation $\eta: \mathrm{id}_{\mathcal{S}} \to \Gamma \circ \Delta$ such that

① For every generic covering $\Delta(J) \leftarrow U \rightarrow A$, the map h in the diagram

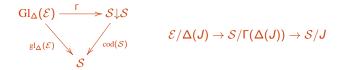
$$\begin{array}{ccc}
J & & & V \\
\downarrow^{\eta_J} & & \downarrow^{h} & \\
\Gamma(\Delta(J)) & & & \Gamma(U) \longrightarrow \Gamma(A)
\end{array}$$

is epic

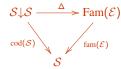
- ② For every $A \in \mathcal{E}$, we have a bijection $\mathcal{E}(1,A) \stackrel{\cong}{\longrightarrow} \mathcal{S}(1,\Gamma(A))$
- enriched Pitts functors $(\Delta, \Gamma, \eta) : \mathcal{S} \to \mathcal{E}$ are equivalent to enriched triposes on \mathcal{S}
- Every tripos $\mathcal P: \textbf{Set}^{op} o \textbf{Ord}$ and every Pitts functor $\Delta: \textbf{Set} o \mathcal E$ has a unique canonical enrichment
- Attention: enriched Pitts functors do not compose 'directly' the generic covering condition is not stable under composition –> use coreflection

Fibrations from enriched Pitts functors

• Using η , we can extend Γ to a fibered functor



- Enrichment of the fibers of gl_Δ(ε) in the slices of ε locally internal category!
- underlying fibration locally small and different from gl_△(E) think 'family fibration'



fibers are still toposes (not obvious)

Enriched triposes on $\mathcal{S}[\mathbb{P}]$

- Recall that triposes on S[P] correspond to regular tripos transformations
 f: P → Q
- Enriched triposes on S[P] correspond to pairs f[•]: P → Q,
 f_•: Q → P of tripos morphisms where
 - f
 is regular

 - $3 \operatorname{id}_{\mathcal{P}} \leq f_{\bullet} \circ f^{\bullet}$
- This suggests the definition of the following category

Definition

ETrip(\mathcal{S}) is the category of enriched triposes on \mathcal{S} , where morphisms $f: (\mathcal{P}, \gamma) \to (\mathcal{Q}, \gamma')$ are pairs $f^{\bullet}: \mathcal{P} \to \mathcal{Q}$, $f_{\bullet}: \mathcal{Q} \to \mathcal{P}$ of tripos morphisms with

- fegular

Lemma

 $\mathsf{ETrip}(\mathcal{S}[\mathbb{P}]) \cong (\mathbb{P}, \gamma) \backslash \mathsf{ETrip}(\mathcal{S})$ for any enriched tripos (\mathbb{P}, γ) on \mathcal{S} .

Factorizing enriched Pitts functors

Definition

An enriched Pitts functor $(\Delta, \Gamma, \eta) : \mathcal{S} \to \mathcal{E}$ is called

- localic, if △ ⊢ Γ
- realizability-like, if Γ ⊢ △
- There is a factorization of enriched Pitts functors into a localic and a realizability-like part
- Analyze this factorization relative to \mathcal{S} , by doing calculations in $\mathsf{ETrip}(\mathcal{S})$

The realizability-like/localic factorization

 Given a morphism f = (f[•], f_•): P → Q of enriched triposes on S, define an indexed preorder U: S^{op} → Ord by

$$\mathcal{U}(J) = \{ (\mu, \varphi) \in \mathcal{P}(J) \times \mathcal{Q}(J) \mid \mu \leq f_{\bullet}(\varphi), \varphi \leq f^{\bullet}(\mu) \}$$

Lemma

u is a tripos.

. E.g. implication is given by

$$(\mu, \varphi) \Rightarrow (\nu, \psi) = ((\mu \Rightarrow \nu) \land f_{\bullet}(\varphi \Rightarrow \psi), f^{\bullet}(\mu \Rightarrow \nu) \land (\varphi \Rightarrow \psi) \land f^{\bullet}f_{\bullet}(\varphi \Rightarrow \psi))$$

There is a decomposition f = (P → U → Q) with I realizability-like and r localic:

$$l^{\bullet}(\mu) = (\mu, f^{\bullet}\mu) \qquad \qquad r^{\bullet}(\mu, \varphi) = \varphi$$
$$l_{\bullet}(\mu, \varphi) = \mu \qquad \qquad r_{\bullet}(\varphi) = (f_{\bullet}\varphi, \varphi \wedge f^{\bullet}f_{\bullet}\varphi)$$

 Problem: factorization is only lax functorial and has bad orthogonality properties

