Moens' theorem and fibered toposes

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Plan of talk

- Elementary toposes and Grothendieck toposes
- Realizability toposes
- Fibered categories
- Characterizing realizability toposes

Elementary toposes and Grothendieck toposes

Elementary toposes

Definition (Lawvere, ca. 1970)

An **elementary topos** is a category $\mathcal{E}$ with

- finite limits
- exponential objects B^A for $A, B \in \mathcal{E}$ (cartesian closed)
- a subobject classifier, i.e. a morphism $\mathrm{t} : 1 \to \Omega$ such that for every monomorphism $m : U \rightarrowtail A$ there exists $\chi : A \to \Omega$ making

$$\begin{tikzcd} U \arrow[r] \arrow[d, tail, "m"'] & 1 \arrow[d, "\mathrm{t}"] \\ A \arrow[r, "\chi"'] & \Omega \end{tikzcd}$$

  a pullback.

Grothendieck toposes

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Grothendieck toposes can equivalently be defined in the following ways:

1. Introduced around 1960 by G. as categories of *sheaves* on a *site*
2. Characterized 1963 by Giraud as *locally small* ∞*-pretoposes* with a *separating set of objects*
3. Equivalently: elementary topos $\mathcal{E}$ admitting a (necessarily unique) *bounded geometric morphism* $\mathcal{E} \to$ **Set**
4. Inspired by 3, define a Grothendieck topos over an (elementary) base topos $\mathcal{S}$ as a bounded geometric morphism $\mathcal{E} \to \mathcal{S}$

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What do all these words mean??

Locally small, separating set

- $\mathbb{C}$ is called **locally small**, if the 'homsets' $\mathbb{C}(A, B)$ are really sets, as opposed to proper classes
- A **separating set of objects** in $\mathbb{C}$ is a family $(C_i)_{i \in I}$ of objects indexed by a set I such that for all parallel pairs $f, g : A \to B$ we have

$$(\forall i \in I \, \forall h : C_i \to A \, . \, fh = gh) \Rightarrow f = g.$$

$$\begin{aligned} \infty\text{-pretopos} &= \text{exact } \infty\text{-extensive category} \\ &= \text{effective regular } \infty\text{-extensive category} \end{aligned}$$

Definition

A **regular category** is a category with finite limits and pullback-stable regular-epi/mono factorizations.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & B \\ & U \arrow[ur, tail, "m"'] & \end{tikzcd}$$

- An *equivalence relation* in a f.l. category $\mathbb{C}$ is a jointly monic pair $r_1, r_2 : R \to A$ such that for all $X \in \mathbb{C}$, the set

$$\{(r_1 x, r_2 x) \mid x : X \to R\}$$

is an equivalence relation on $\mathbb{C}(X, A)$

- The *kernel pair* of any morphism $f : A \to B$ – given by the pullback

$$\begin{tikzcd} X \arrow[r, "r_1"] \arrow[d, "r_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is always an equivalence relation

> *Definition*
>
> An **exact** (or **effective regular**) category is a regular category in which every equivalence relation is a kernel pair.

Assume $\mathbb{C}$ has finite limits and small coproducts

- Coproducts in $\mathbb{C}$ are called **disjoint**, if the squares

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A_i \arrow[d] \\ A_j \arrow[r] & \coprod_{i \in I} A_i \end{tikzcd} \quad (i \neq j) \qquad \text{and} \qquad \begin{tikzcd} A_i \arrow[r] \arrow[d] & A_i \arrow[d] \\ A_i \arrow[r] & \coprod_{i \in I} A_i \end{tikzcd}$$

  are always pullbacks

- Coproducts in $\mathbb{C}$ are called **stable**, if for any $f : B \to \coprod_{i \in I} A_i$, the family

$$(B_i \xrightarrow{\sigma_i} B)_{i \in I} \quad \text{given by pullbacks} \quad \begin{tikzcd} B_i \arrow[r, "\sigma_i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ A_i \arrow[r] & \coprod_{i \in I} A_i \end{tikzcd}$$

  represents B as coproduct of the B_i

Definition

An ∞**-(l)extensive category** is a category $\mathbb{C}$ with finite limits and disjoint and stable small coproducts.

∞-*Pretoposes*

Examples

- Complete lattices $(A, \leq)$ viewed as categories have finite limits and small coproducts, but these are not disjoint – coproducts are stable precisely for *complete Heyting algebras*
- **Top** (topological spaces) and **Cat** (small categories) are ∞-extensive but not regular
- Monadic categories over **Set** are always exact and have small coproducts, but are rarely extensive

Definition

An **∞-pretopos** is a category which is exact and ∞-extensive.

Examples

- Grothendieck toposes
- the category of *small* presheaves on **Set**

Geometric morphisms

- A **geometric morphism** $\mathcal{E} \to \mathcal{S}$ between toposes $\mathcal{E}$ and $\mathcal{S}$ is an adjunction
$$(\Delta : \mathcal{S} \to \mathcal{E}) \dashv (\Gamma : \mathcal{E} \to \mathcal{S})$$
of f.l.p. functors (Δ is the 'inverse image part'; Γ the 'direct image part')
- $(\Delta \dashv \Gamma)$ is called **bounded**, if there exists $B \in \mathcal{E}$ such that for every $E \in \mathcal{E}$ there exists a subquotient span $B \times \Delta(S) \leftarrowtail \bullet \twoheadrightarrow E$
- It is called **localic** if it is *bounded by* 1
- If $\Delta \dashv \Gamma : \mathcal{E} \to \mathbf{Set}$, then we necessarily have
$$\Delta(J) = \sum_{j \in J} 1 \qquad \text{and} \qquad \Gamma(A) = \mathcal{E}(1, A)$$
for $J \in \mathbf{Set}$ and $A \in \mathcal{E}$

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Remark

Without the bound in 3, $\mathcal{E}$ need not be cocomplete. Example: subcategory of $\widehat{\mathbb{Z}}$ on actions with uniform bound on the size of orbits.

Realizability toposes

Realizability toposes

- Were introduced in 1980 by Hyland, Johnstone, and Pitts
- Not Grothendieck toposes
- Most well known: Hyland's **effective topos** $\mathcal{E}\!f\!f$ – 'Universe of constructive recursive mathematics'
- usually constructed via *triposes*

Partial combinatory algebras

Definition

A **PCA** is a set $\mathcal{A}$ with a partial binary operation

$$(- \cdot -) : \mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A}$$

having elements $k, s \in \mathcal{A}$ such that

$$(i)\ k{\cdot}x{\cdot}y = x \qquad (ii)\ s{\cdot}x{\cdot}y{\downarrow} \qquad (iii)\ s{\cdot}x{\cdot}y{\cdot}z \preceq x{\cdot}z{\cdot}(y{\cdot}z)$$

for all $x, y, z \in \mathcal{A}$.

Example

First Kleene algebra: $(\mathbb{N}, \cdot)$ with

$$n{\cdot}m \simeq \phi_n(m) \qquad \text{for } n, m \in \mathbb{N},$$

where $(\phi_n)_{n\in\mathbb{N}}$ is an effective enumeration of partial recursive functions.

Fibrations from PCAs

PCA $\mathcal{A}$ gives rise to indexed preorders $\mathrm{fam}(\mathcal{A}), \mathrm{rt}(\mathcal{A}) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$.

- **Family fibration**: $\mathrm{fam}(\mathcal{A})(J) = (\mathcal{A}^J, \leq)$, with

$$\varphi \leq \psi \quad :\Leftrightarrow \quad \exists e \in \mathcal{A}\, \forall j \in J \,.\, e{\cdot}\varphi(j) = \psi(i)$$

  for $\varphi, \psi : J \to \mathcal{A}$.

- **Realizability tripos**: $\mathrm{rt}(\mathcal{A})(J) = ((P\mathcal{A})^J, \leq)$, with

$$\varphi \leq \psi \quad :\Leftrightarrow \quad \exists e \in \mathcal{A}\, \forall j \in J\, \forall a \in \varphi(j) \,.\, e{\cdot}a \in \psi(i)$$

  for $\varphi, \psi : J \to P\mathcal{A}$.

Observations

- $\mathrm{fam}(\mathcal{A})$ has indexed finite meets
- $\mathrm{rt}(\mathcal{A})$ models full 1st order logic
- both have *generic predicates*
- $\mathrm{rt}(\mathcal{A})$ is free cocompletion of $\mathrm{fam}(\mathcal{A})$ under $\exists$ (Hofstra 2006)

Realizability toposes

Definition

- The **realizability topos** $\mathbf{RT}(\mathcal{A})$ over $\mathcal{A}$ is the category of partial equivalence relations and compatible functional relations in $\mathcal{A}$ (details omitted)
- The **constant objects functor** $\Delta : \mathbf{Set} \to \mathbf{RT}(\mathcal{A})$ maps $J \in \mathbf{Set}$ to (J, δ_J) (discrete/diagonal equivalence relation)

- $\mathbf{RT}(\mathcal{A})$ is never a Grothendieck topos (except for the trivial pca)
- Δ is bounded by 1, but not the inverse image part of a geometric morphism
- it makes sense to compare constant objects functors and inverse image functors, since both are instances of the same construction in the context of triposes

Fibered Categories

Δ *and gluing fibrations*

Goal: Understand inverse image functors

$$(\Delta : \mathbf{Set} \to \mathcal{E}) \dashv \Gamma$$

and constant objects functors

$$\Delta : \mathbf{Set} \to \mathbf{RT}(\mathcal{A})$$

better by looking at their *gluing fibrations*, defined by the pullback

$$\begin{tikzcd}
\mathrm{Gl}_\Delta(\mathcal{E}) \arrow[r] \arrow[d, "\mathrm{gl}_\Delta(\mathcal{E})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}{\downarrow}\mathcal{E} \arrow[d, "\mathrm{cod}(\mathcal{E})"] \\
\mathbf{Set} \arrow[r, "\Delta"] & \mathcal{E}
\end{tikzcd}$$

Fibered category theory

References

- Jean Bénabou, *Fibered categories and the foundations of naive category theory*, 1985
- Thomas Streicher, *Fibred categories à la Jean Bénabou*, unpublished, 1999-2012
- Peter Johnstone, *Sketches of an Elephant*, 2003

Idea/Philosophy

- *Elementary category theory*: finitary conditions, first order axiomatizable, no size conditions, avoid ZFC (f.l. category, elementary topos)
- *Naive category theory*: not concerned about formal, foundational aspects, use size conditions and make reference to **Set** freely
- Bénabou proposes fibrations to reconcile both, fibrations allow to express 'non-finitary conditions' in an elementary manner
- generalize and form analogies from *family fibrations*

Family fibrations

Definition

Let $\mathbb{C}$ be a category.

- The category $\mathrm{Fam}(\mathbb{C})$ has families $(C_i)_{i \in I}$ of objects of $\mathbb{C}$ as objects; a morphism $(C_i)_{i \in I} \to (D_j)_{j \in J}$ is a pair

$$(u : I \to J, (f_i : C_i \to D_{ui})_{i \in I}.$$

- The **family fibration** of $\mathbb{C}$ is the functor

$$\begin{array}{rcccc} \mathrm{fam}(\mathbb{C}) & : & \mathrm{Fam}(\mathbb{C}) & \to & \mathbf{Set} \\ & & (C_i)_{i \in I} & \mapsto & I \\ & & (u, (f_i)_{i \in I}) & \mapsto & u \end{array}$$

  mapping $(C_i)_{i \in I}$ $\mathrm{fam}(\mathbb{C}) : \mathrm{Fam}(\mathbb{C}) \to \mathbf{Set}$ of a category $\mathbb{C}$ is the fibration having

Local smallness

Definition

Let $P : \mathbb{X} \to \mathbb{B}$ be a fibration, $I \in \mathbb{B}$, $X, Y \in P(I)$. A **family of morphisms** from X to Y is a span $X \overset{c}{\leftsquigarrow} \bullet \xrightarrow{f} Y$ where $P(c) = P(f)$ and c is cartesian. P is called **locally small**, if for every pair $X, Y \in P(I)$ there exists a *universal* family of morphisms (terminal among such spans).

$$\begin{tikzcd}
\bullet \arrow[dr, dashed] \arrow[drr, squiggly, bend left] \arrow[ddrr, bend right] & & \\
& \bullet \arrow[r, squiggly] \arrow[dr] & X \\
J \arrow[d, dashed] \arrow[drr] & & Y \\
\mathrm{hom}(X, Y) \arrow[rr] & & I
\end{tikzcd}$$

Lemma

A category $\mathbb{C}$ is locally small, iff $\mathrm{fam}(\mathbb{C})$ is locally small in the above sense.

Finite limit fibrations

... towards extensive fibratiions and Moens' theorem

Definition

Let $\mathbb{B}$ be a f.l. category. A **finite limit fibration** on $\mathbb{B}$ is a fibration $P : \mathbb{X} \to \mathbb{B}$ satisfying either of the following equivalent definitions.

- $\mathbb{X}$ has finite limits and P preserves them
- All fibers $P(I)$ have finite limits, and they are preserved under reindexing

Lemma

A category $\mathbb{C}$ has finite limits iff $\mathrm{fam}(\mathbb{C})$ is a finite limit fibration.

Extensive fibrations

Let $P : \mathbb{X} \to \mathbb{C}$ be a finite limit fibration.

- P is said to have **internal sums**, if it is also an *opfibration* ($P^{\mathrm{op}} : \mathbb{X}^{\mathrm{op}} \to \mathbb{C}^{\mathrm{op}}$ is a fibration), and cocartesian maps in $\mathbb{X}$ are stable under pullback along cartesian maps ('Beck-Chevalley condition')
- P is said to have **stable internal sums**, if cocartesian maps are stable under pullback along *arbitrary* maps in $\mathbb{X}$
- Internal sums are called **disjoint**, if the mediating arrow m in the diagram

$$\begin{tikzcd}
 & & A \arrow[dr, "\sigma"] & \\
A \arrow[r, dashed, "m"] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "\mathrm{id}"'] & R \arrow[ur] \arrow[dr] & & S \\
 & & A \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

  is cocartesian for every cocartesian map $\sigma : A \to S$ in $\mathbb{X}$

- An **extensive fibration** is a finite-limit fibration with stable disjoint internal sums.

Lemma

A category $\mathbb{C}$ is ∞-extensive iff $\mathrm{fam}(\mathbb{C})$ is extensive.

Moens' theorem

- Fundamental fib's $\mathrm{cod}(\mathbb{D}) : \mathbb{D}{\downarrow}\mathbb{D} \to \mathbb{D}$ of f.l. cat's are extensive
- Extensive fib's are stable under pullback along f.l.p. functors $\Delta : \mathbb{C} \to \mathbb{D}$
- Thus, gluing fibrations $\mathrm{gl}_\Delta(\mathbb{D}) : \mathrm{Gl}_\Delta(\mathbb{D}) \to \mathbb{C}$ are extensive

Theorem (Moens' theorem)

The assignment $\Delta \mapsto \mathrm{gl}_\Delta(\mathbb{D}) = \Delta^*\mathrm{cod}(\mathbb{D})$ gives rise to a biequivalence

$$\mathbf{ExtFib}(\mathbb{C}) \simeq \mathbb{C} /\!\!/ \mathbf{Lex}$$

between the 2-category $\mathbf{ExtFib}(\mathbb{C})$ of extensive fibrations on $\mathbb{C}$ and the pseudo-co-slice 2-category $\mathbb{C} /\!\!/ \mathbf{Lex}$ of f.l. categories under $\mathbb{C}$.

$\mathbf{ExtFib}(\mathbb{C}) \to \mathbb{C} /\!\!/ \mathbf{Lex}$

The functor corresponding to a fibration $P : \mathbb{X} \to \mathbb{C}$ is given by

$$\begin{array}{rcl} \Delta : \mathbb{C} & \to & \mathbb{X}(1) \\ C & \mapsto & \sum_C 1 \end{array} \qquad \begin{array}{c} 1 \rightarrowtriangle \sum_C 1 \\ \\ C \longrightarrow 1 \end{array}$$

- For Grothendieck toposes $\mathcal{E}$ with geometric morphism $\Delta \dashv \Gamma : \mathcal{E} \to \textbf{Set}$, we have

$$\mathrm{gl}_{\Delta}(\mathcal{E}) \simeq \mathrm{fam}(\mathcal{E})$$

- Thus, when studying Grothendieck toposes $\Delta \dashv \Gamma : \mathcal{E} \to \textbf{Set}$ relative to a base topos $\mathcal{S}$, the fibration $\mathrm{gl}_{\Delta}(\mathcal{E})$ is an adequate substitute for the family fibration
- For realizability toposes with c.o.f. $\Delta : \textbf{Set} \to \textbf{RT}(\mathcal{A})$, the fibrations $\mathrm{gl}_{\Delta}(\textbf{RT}(\mathcal{A}))$ and $\mathrm{fam}(\textbf{RT}(\mathcal{A}))$ are different
- We will see just how different

Gluing and local smallness

Theorem

If $\Delta : \mathcal{S} \to \mathcal{E}$ is a f.l.p. functor between toposes, then $\mathrm{gl}_\Delta(\mathcal{E})$ is a locally small fibration iff Δ has a right adjoint

- Thus, gluing fibrations $\mathrm{gl}_\Delta(\mathbf{RT}(\mathcal{A}))$ of realizability toposes are not locally small

We have two ways of looking at realizability toposes

- From the point of view of ordinary CT, toposes $\mathbf{RT}(\mathcal{A})$ are locally small, but not cocomplete
- Viewed as gluing fibrations, they have small sums, but are not locally small

Characterizing Realizability Toposes

Motivation

- Peter Johnstone pointed out the lack of a 'Giraud style' theorem for realizability toposes
- It seemed easier to characterize the gluing fibrations $\mathrm{gl}_\Delta(\mathbf{RT}(\mathcal{A}))$ (or equivalently the functors $\Delta : \mathbf{Set} \to \mathbf{RT}(\mathcal{A})$) than the 'bare' toposes
- Fibrationally realizability toposes resemble presheaf toposes

Moens' theorem for fibered pretoposes

- A **pre-stack** is a fibration $P : \mathbb{X} \to \mathbb{R}$ on a regular category $\mathbb{R}$ where the reindexing functors $e^* : P(I) \to P(J)$ are full and faithful for all regular epis $e : J \twoheadrightarrow I$
- All fibrations on **Set** are pre-stacks with AC, and without still most
- A **fibered pretopos** is an extensive pre-stack $P : \mathbb{X} \to \mathbb{R}$ with exact fibers
- $\mathrm{fam}(\mathcal{E})$ is a fibered pretopos iff $\mathcal{E}$ is an ∞-pretopos

Theorem (Moens' theorem for fibered pretoposes)

The assignment $\Delta \mapsto \mathrm{gl}(\Delta)$ gives rise to a biequivalence

$$\mathbf{PretopFib}(\mathbb{R}) \simeq \mathbb{R} /\!\!/ \mathbf{Ex}$$

between the 2-category $\mathbf{PretopFib}(\mathbb{R})$ of fibered pretoposes on $\mathbb{R}$ and the pseudo-co-slice 2-category $\mathbb{R} /\!\!/ \mathbf{Ex}$ of exact categories under $\mathbb{C}$.

Fibered presheaf construction

Theorem

Let $\mathbb{R}$ be a regular category The forgetful functor

$$\mathbf{PretopFib}(\mathbb{R}) \to \mathbf{Lex}(\mathbb{R}),$$

*where $\mathbf{Lex}(\mathbb{R})$ is the category of finite-limit pre-stacks on $\mathbb{R}$, has a left biadjoint $\mathscr{C} \mapsto \widehat{\mathscr{C}}$, called **fibered presheaf construction**.*

- If $\mathbb{C}$ is a small category with finite limits, then $\widehat{\mathrm{fam}(\mathbb{C})} = \mathrm{fam}(\mathbf{Set}^{\mathbb{C}^{\mathrm{op}}})$
- For any PCA $\mathcal{A}$ we have $\widehat{\mathrm{fam}(\mathcal{A})} = \mathrm{gl}_{\Delta}(\mathbf{RT}(\mathcal{A}))$

Characterization of fibrations of presheaves

Which fibered pretoposes $P : \mathbb{X} \to \mathbb{R}$ are of the form $\mathscr{X} \simeq \widehat{\mathscr{C}}$?

Theorem (Bunge 77)

A locally small ∞-pretopos $\mathcal{E}$ is a presheaf topos iff it has a separating family of ***indecomposable projective*** *objects.*

In a similar way, we can show:

Theorem

A fibered pretopos $\mathscr{X} : |\mathscr{X}| \to \mathbb{R}$ is a fibration of presheaves iff

- the subfibration of $\mathscr{X}$ on **indecomposable projectives** is closed under finite limits, and
- Every $X \in |\mathscr{X}|$ can be covered by an internal sum of **indecomposable projectives**.

... where indecomposable projectives in fibrations are defined on the next slide

Indecomposables and projectives

Let $\mathscr{X} : |\mathscr{X}| \to \mathbb{R}$ be a fibered pretopos.

Definition

- Call $P \in |\mathscr{X}|$ **projective** , if given c, e, f as in the diagram

$$\begin{tikzcd}
 & \bullet \arrow[r, dotted, "g"'] \arrow[d, squiggly, "d"'] & Y \arrow[d, two heads, "e"] \\
P & \bullet \arrow[l, squiggly, "c"'] \arrow[r, "f"] & X
\end{tikzcd}$$

  where c is cartesian and e is vertical and a regular epimorphism in its fiber, we can fill in d, g with d epicartesian such that the square commutes.

- Call $X \in |\mathscr{X}|$ **indecomposable**, if for every diagram

$$\begin{tikzcd}
X^* \arrow[r, squiggly, "c"] \arrow[d, dotted, "m"'] \arrow[dr] & X \\
Y \arrow[r, "d"'] & \sum Y
\end{tikzcd}$$

  in $|\mathscr{X}|$ where c is cartesian and d is cocartesian, there exists a *unique* mediating arrow m.

With a bit of work one can prove the following

Theorem

Gluing fibrations $\mathrm{gl}_\Delta(\mathbf{RT}(\mathcal{A}))$ of realizability toposes can be characterized as fibered pretoposes $P : \mathbb{X} \to \mathbf{Set}$ such that

- P is a fibered cocompletion (previous theorem)
- the fibers of P are lccc
- The subfibration $Q \subseteq P$ on indecomposable projectives is posetal, has a **discrete** generic predicate, and $Q(1) \simeq 1$

[*discrete* means right orthogonal to cartesian maps over surjective functions]

Characterizing realizability toposes

In realizability toposes, we have $(\mathbf{RT}(\mathcal{A})(1, -) : \mathbf{RT}(\mathcal{A}) \to \mathbf{Set}) \dashv \Delta$, thus the global sections functor is uniquely determined and does not contain additional information. Thus, our analysis yields a characterization of 'bare' toposes after all:

Theorem

A locally small category $\mathcal{E}$ is equivalent to a realizability topos $\mathbf{RT}(\mathcal{A})$ over a PCA $\mathcal{A}$, if and only if

1. $\mathcal{E}$ is exact and locally cartesian closed,
2. $\mathcal{E}$ has enough projectives, and the subcategory $\mathbf{Proj}(\mathcal{E})$ of projectives is closed under finite limits,
3. the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$ has a right adjoint Δ factoring through $\mathbf{Proj}(\mathcal{E})$, and
4. there exists a separated and discrete projective $D \in \mathcal{E}$ such that for all projectives $P \in \mathcal{E}$ there exists a closed $u : P \to D$.