Characterizing realizability toposes

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Trouville, 6 September 2011

The category **Asm** *of assemblies*

Objects

- Assembly : Pair $(X, \|\cdot\|)$ with $\|\cdot\| : X \to P^+\mathbb{N}$
- $\|x\| \subseteq \mathbb{N}$: set of *codes*, or *realizers* of $x \in X$

Morphisms

A morphism $f : (X, \|\cdot\|) \to (Y, \|\cdot\|)$ is a function $f : X \to Y$ such that there exists a partial recursive $\rho : \mathbb{N} \rightharpoonup \mathbb{N}$ such that $\forall x \, \forall n \in \|x\| \, . \, \rho(x) \in \|fx\|$.

Properties of **Asm**

- **Asm** locally cartesian closed with *natural numbers object*
- $\Rightarrow$ models *Martin Löf Type Theory*
- **Asm** even models *Calculus of Constructions* with polymorphic Set!
- Δ *right* adjoint to Γ with $\Gamma = \mathbf{Asm}(1, -)$

Uniform, modest and projective assemblies

An assembly $(X, \|\cdot\|)$ is called

- **uniform**, if $\bigcap_{x \in X} \|x\| \neq \varnothing$ (sets)
- **modest**, if $x \neq y \Rightarrow \|x\| \cap \|y\| = \varnothing$ (data types)
- **projective**, if there exists $(Y, \|\cdot\|) \cong (X, \|\cdot\|)$ such that every element of $(Y, \|\cdot\|)$ has a unique realizer

- If $P = (X, \|\cdot\|)$ is projective, then

  $$\forall p{:}P\, \exists a{:}A\, .\, \varphi(p, a) \vdash \exists f{:}P \to A\, \forall p{:}P\, .\, \varphi(p, fp)$$

  holds for arbitrary A and φ (axiom of choice on P)
- If U is uniform, then we have

  $$\forall u{:}U\, .\, \varphi \vee \psi \vdash (\forall u\, .\, \varphi) \vee (\forall u\, .\, \psi)$$

- If U is uniform and M is modest, we have

  $$\forall u{:}U\, \exists m{:}M\, .\, \varphi(u, m) \vdash \exists m{:}M\, \forall u{:}U\, .\, \varphi(u, m)$$

The functor Δ

- For X a set, let $\Delta X = (X, \|\cdot\|)$ with $\|x\| = \mathbb{N}$ for all $x \in X$
- This gives functor $\Delta : \mathbf{Set} \to \mathbf{Asm}$
- Objects of the form ΔX are uniform
- Δ is *right* adjoint to $\Gamma = \mathbf{Asm}(1, -)$

$$\Gamma \dashv \Delta$$

ex/reg completion

- **Asm** does not have well behaved quotients
- **Exact completion** (ex/reg completion) formally adds nice quotients
- $\mathbf{ex}/\mathbf{reg}(\mathbf{Asm})$ is the **effective topos** $\mathcal{E}\!f\!f$
- **Asm** is the subcategory of $\neg\neg$-separated objects of $\mathcal{E}\!f\!f$

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\top"] & \mathbf{Asm} \arrow[l, bend left, "\Gamma"] \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathcal{E}\!f\!f \arrow[l, bend left] \end{tikzcd}$$

- $\neg\neg$-sheaves are the image of Δ

ex/lex completion

- $\mathcal{E}\mathit{ff}$ is *also* the ex/lex completion of **Proj**, the subcategory of **Asm** on the projective assemblies

$$
\begin{tikzcd}
\mathbf{Proj} \in \mathbf{Lex} \arrow[rr, bend left, "\mathbf{ex/lex}"] \arrow[rr, phantom, "\perp"] \arrow[dr, bend left, "\mathbf{reg/lex}"] \arrow[dr, phantom, "\vdash"] & & \mathbf{Ex} \ni \mathcal{E}\mathit{ff} \arrow[ll, bend left] \arrow[dl, bend left] \\
& \mathbf{Asm} \in \mathbf{Reg} \arrow[ul, bend left] \arrow[ur, bend left, "\mathbf{ex/reg}"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}
$$

Table: Biadjunctions of 2-categories

The effective tripos **eff**

- effective predictate: $\varphi : X \to P\mathbb{N}$
- ordering: $\varphi \vdash \psi :\Leftrightarrow \exists \rho$ part. rec. $\forall x{:}X\, \forall n \in \varphi(x)\,.\, \rho(n) \in \psi(x)$
- reindexing: $\varphi : X \to P\mathbb{N}, f : Y \to X \implies \varphi \circ f : Y \to P\mathbb{N}$
  reindexing of φ along f
- This gives a functor **eff** : $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ — the **effective tripos**
- **eff** is model of intuitionistic logic
- We can construct a category of partial equivalence relations and compatible functional relations with respect to the logic of **eff** — this category is again $\mathcal{E}\mathit{ff}$

Triposes

Definition (Tripos)

A **tripos** (on **Set**) is a functor $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ such that

- all $\mathcal{P}(I)$ are pre-Heyting algebras
- all maps $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ have left and right adjoints $\exists_f \dashv \mathcal{P}(f) \dashv \forall_f$ subject to the *Beck-Chevalley condition*
- There exist $\mathrm{Prop} \in \mathbf{Set}$ and $\mathrm{tr} \in \mathcal{P}(\mathrm{Prop})$ such that for all $\varphi \in \mathcal{P}(I)$ there exists $h : I \to \mathrm{Prop}$ with $\mathcal{P}(h)(\mathrm{tr}) \cong \varphi$ (generic predicate)

Partial combinatory algebras

- Other kinds of realizers possible, e.g. λ-terms
- More generally, elements of a *partial combinatory algebra*

Definition

A partial combinatory algebra (pca) is a set $\mathcal{A}$ together with a binary partial operation $\cdot : \mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A}$ such that there exist $\mathsf{k}, \mathsf{s} \in \mathcal{A}$ such that the equations

$$k \cdot x \cdot y = x \tag{1}$$

$$s \cdot x \cdot y \cdot z = x \cdot z \cdot (y \cdot z) \tag{2}$$

hold whenever the right side is defined.

- Previously presented constructions possible for arbitrary pcas $\mathcal{A}$, giving rise to toposes $\mathbf{RT}(\mathcal{A})$
- Can we characterize the toposes obtained this way?
- Not quite, need some additional information

The Giraud theorem

- Analogy: Giraud's theorem
- Giraud's theorem characterizes **Grothendieck toposes** as ∞-pretopses having a small generating family
- Grothendieck topos $\mathcal{G}$ cocomplete, thus can define functor

$$\Delta : \mathbf{Set} \to \mathcal{G}, \qquad I \mapsto \coprod_{i \in I} 1$$

- Realizability toposes not cocomplete $\Rightarrow$ postulate a functor Δ explicitely!

- Can characterize pairs $(\mathcal{E}, \Delta)$ of toposes and functors $\Delta : \mathbf{Set} \to \mathcal{E}$ which arise from the realizability construction!

Pitts functors

- Pitts characterizes functors $\Delta : \mathcal{S} \to \mathcal{E}$ arising from the tripos-to-topos construction as functors such that
  - Δ preserves finite limits
  - for every $A \in \mathcal{E}$ there exists $\Delta I \leftarrowtail U \twoheadrightarrow A$ such that for for every $\Delta J \leftarrowtail V \to A$ there exists $h : J \to I$ such that

$$\begin{tikzcd}
\Delta I & U \arrow[l, tail] \arrow[r, two heads] & A \\
\Delta J \arrow[u, "\Delta h"] & V \arrow[l, tail] \arrow[u] \arrow[ur] \arrow[ul, phantom, "\ulcorner", very near start] &
\end{tikzcd}$$

  - (in fact, we only want to consider regular Pitts functors)

- Suffices to characterize the triposes arising from pcas!

Totally connected triposes and abstract assemblies

- Assemblies can be defined for realizability, but not for arbitrary triposes
- For a tripos $\mathcal{P} : \textbf{Set}^{\text{op}} \to \textbf{Ord}$ we have well behaved assemblies iff the tripos is **totally connected**, i.e., $\delta : \text{sub}(\textbf{Set}) \to \mathcal{P}$ has a finite-meet-preserving left adjoint $\pi_0 : \mathcal{P} \to \text{sub}(\textbf{Set})$ exhibiting $\text{sub}(\textbf{Set})$ as a localization of $\mathcal{P}$ (i.e., $\pi_0 \circ \delta \cong \mathrm{id}$)
- it remains to isolate realizability triposes among totally connected triposes

Hofstra's basic combinatorial objects

- Can we reconstruct the pca from a realizability tripos $\mathbf{rt}(\mathcal{A})$?
- difficult to reconstruct the application map $\cdot : \mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A}$
- success thanks to Hofstra's theory

Definition

A **basic combinatory object** (BCO) (A, Σ) is a set A together with a set Σ of partial endofunctions ('the computable functions') which are closed under composition and inclusion and contain the identity.

- every pca $\mathcal{A}$ carries canonical BCO structure, the computable maps being the subsets of maps of the form $a \cdot -$
- BCO A induces indexed preorder $\mathcal{P}_A : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ (not necessarily a tripos), analogous to PCAs
- Hofstra characterizes BCOs which arise from pcas, roughly as those such that P_A is a tripos (some details are missing here)

Reconstructing the BCO

- **rt**$(\mathcal{A})$ contains a special predicate $\phi : \mathcal{A} \to P\mathcal{A}$ with $\phi(a) = \{a\}$
- can reconstruct BCO structure by defining computable maps to be partial maps $\mathcal{A} \supseteq U \xrightarrow{f} \mathcal{A}$ such that $\phi|_U \vdash f^*\phi$
- how to characterize ϕ among the predicates of **rt**$(\mathcal{A})$?
- the assembly induced by ϕ is modest and projective, and ϕ is weakly universal among projective predicates

The characterization

Theorem

A tripos $\mathcal{P}$ is of the form $\mathbf{rt}(\mathcal{A})$ for some pca $\mathcal{A}$ iff

- *$\mathcal{P}$ is totally connected*
- *for the left adjoint π_0 of $\delta : \mathrm{sub}(\mathbf{Set}) \to \mathcal{P}$, we have*

  $$\top \vdash \pi_0\psi \quad \Longrightarrow \quad \top \vdash \psi \quad \text{for } \psi \in \mathcal{P}(1)$$

  (π_0 reflects truth in the empty context)
- *the induced topos $\boldsymbol{T}\mathcal{P}$ has enough projectives, projectives in $\boldsymbol{T}\mathcal{P}$ are closed under finite limits, and $\Delta : \mathbf{Set} \to \boldsymbol{T}\mathcal{P}$ preserves projectives*
- *there exists a modest projective dense predicate ϕ which is weakly universal among projectives*

Concluding remarks

- This all works on arbitrary base toposes, as long as the base has *enough internal projectives*, and internal projectives are closed under finite limits
- Parallel between Pitts functors and geometric morphisms — would like factorization theorems for Pitts functors