Doctrines and bilocalizations

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What is a doctrine? Lawvere says[1]:

> Here I use the term *doctrine* — due to Jon Beck[2] — which in general is used to signify 'something like a theory, but higher' …

He then goes on to mention hyperdoctrines. In my opinion that's a misnomer: Hyperdoctrines are just theories, not 'higher theories'.

James Dolan also has some comments on doctrines[3]:

> "doctrines" in the beck/lawvere category-theoretic sense: […] a doctrine is associated with (or perhaps simply is) a 2-category of "theories", […] a hom-category in the 2-category represents a category of "models" of the domain theory …

Doctrines are like theories … so what's a theory again?

Single sorted algebraic theories (SSATs) can be represented equivalently

- by a Lawvere theory
- by a finitary monad on Set, or
- by a finitary monadic functor $U : \mathbb{A} \to \mathsf{Set}$

General algebraic theories can be equivalently represented

- by a smal FP category $\mathbb{C}$,
- by an *algebraic category* $\mathfrak{A}$ (that's a Barr exact LFP category where the full subcategory of compact projectives is dense, see [ARV10, Fre22b])

The more abstract *lex theories* can be equivalently represented

- by a small lex category $\mathcal{C}$, or
- by a LFP category $\mathcal{X}$.

FP and LEX are examples of *doctrines*. Let's list a few more:

[1] https://www.youtube.com/watch?v=ZYGyEPXu8as, around 4:50.

[2] https://en.wikipedia.org/wiki/Jonathan_Mock_Beck

[3] https://ncatlab.org/jamesdolan/published/Algebraic+Geometry, see also https://www.youtube.com/watch?v=rx0amwt_tj0&list=PLuAO-1XXEh0bDbVapKJm5JVKaIVI05mwy&index=1, https://ncatlab.org/johnbaez/show/Doctrines+of+algebraic+geometry

- Lex – the doctrine of finite-limit theories
- FP – the doctrine of finite-product theories
- Mon – the doctrine of monoidal categories
- SMon – the doctrine of symmetric monoidal categories
- Clan – the doctrine of clans
- DDCat – the doctrine of democratic display categories
- Top – the doctrine of toposes/geometric theories

So what is a doctrine? Again, there are different ways of materializing the idea. Classically, the word was often use for (finitary) 2-monads or pseudomonads on Cat.

Let's look at a few. $S, M, C, L : \mathsf{Cat} \to \mathsf{Cat}$ are the monads for free monoidal categories, free symmetric monoidal categories ...

- $M(\mathbb{C})$ is the monoidal category whose objects are finite lists of objects, and whose morphisms are lists of arrows (always between lists of the same length)
- $S(\mathbb{C})$ is the symmetric monoidal category whose objects are lists, and whose arrows are pairs of a permutation and a list of arrows
- $C(\mathbb{C})$ is the fp-cat whose objects are lists of objects, and where a morphism from $(A_1 \dots A_n)$ to $(B_1 \dots B_k)$ consists of a function $f : \{1, \dots, k\} \to \{1, \dots, n\}$ and a family of arrows $h_i : A_{f(i)} \to B_i$.
- The free lex category $L(\mathbb{C})$ on a small category $\mathbb{C}$ can be described as the full subcategory of $[\mathbb{C}, \mathsf{Set}]^{\mathsf{op}}$ on finite lims of representables. If $\mathbb{C}$ is finite this is equivalent to $[\mathbb{C}, \mathsf{Fin}]^{\mathsf{op}}$.

Remarks:

- Lex is not a 2-monad but just a pseudomonad
- Lex and FP are *colax idempotent*:

Definition 1 ([Koc95])

- A 2-monad is *lax idempotent* if we have $T_\eta \dashv \mu \dashv \eta_T$
- A 2-monad is *colax idempotent* if we have $\eta_T \dashv \mu \dashv T\eta$ ♢

Algebras for (co)lax idempotent monads are left (right) adjoint to units and therefore being an algebra is a proerty rather than an structure. Eg, being lex is a property of a category, but monoidal not.

It sounds like 'finitary pseudomonad' is a good approximation for the notion of 'doctrine', in the same way that finitary monads are equivalent to SSATs.

However, clans and toposes don't fit into the picture.

For now we follow Dolan and just say doctrine = 2-category, but we keep in mind that that's somewhat weird since it's like saying that an AT *is* its category of models.

Anyway, now lets talk about strictification.

The goal is to talk about strict vs non-strict notions of type theories and arrows between type theories, but to warm up we look at monoidal categories (following Pietro).

The 2-category Mon of monoidal categories and monoidal functors has a non-full subcategory Mon_s whose objects are *strict monoidal categories*, and whose 1-cells are *strict* monoidal functors (the distinction strict/strong algebra/1-cell makes sense for arbitrary 2-monads).

$U : \mathsf{Mon}_s \to \mathsf{Mon}$ has left and right biadjoints:

- $L(M)$ is constructed by freely readding monoidal products (taking lists), and then identifying them with the old ones by adding isos.
- $R(M)$ is the category of endomorphisms of the action of M on itself by left multiplication. (special case of strictification of bicat by taking image of Yoneda)

Looking at the units and counits of the adjunctions:

- $LUS \to S \to RUS$ in Mon_s and
- $URM \to M \to ULM$ in Mon

The second two are equivalences in Mon, thus we're dealing with a reflection and a coreflection!

Thus, Mon embeds reflectively and coreflectively into Mon_s, we have an adjoint cylinder.

Let's say it again: even though Mon is 'literally' a *supercategory* of Mon_s, it's better to think of it as a (co)reflective *subcategory*!

Morever, U is a *bilocalization* (just like any (co)reflection):

Definition 2 Given a class W of 1-cells in a 2-category $\mathcal{A}$, a *bilocalization* of $\mathcal{A}$ is a pseudofunctor $E : \mathcal{A} \to \mathcal{A}[W^{-1}]$ such that

- E sends all arrows in W to weak equivalences, and
- for every 2-cat $\mathcal{X}$, precomposition with E induces an biequivalence between $[\mathcal{A}[W^{-1}], \mathcal{X}]$[4] and the full sub-2-category of $[\mathcal{A}, \mathcal{X}]$ on pseudofunctors which send all W-maps to equivalences. $\diamondsuit$

> Claim/Slogan/Hypothesis:
> Non-strict doctrines are *bilocalizations* of strict doctrines.

In the language of homotopy theory, it makes sense to think of left and right strictifications as *(co)fibrant* replacements.

We have strict versions of doctrines in the list above, with forgetful functors to the weak ones:

[4]That's the 2-category of pseudofunctors, pseudo-natural transformations, and modifications.

non-strict	strict
Lex	D-Σ-Eq-NM
FP	full sub-2-category of D-Σ-NM natural models where $\mathsf{Ty}(1) \to \mathsf{Ty}(\Gamma)$ is surjective for all Γ
Mon	Mon_s
SMon	SMon_s
Clan	D-Σ-NM
DDCat	D-CwF / D-CwA / D-NM / CxtCat

In general we might not have both strictification adjoints. In the last row they are there, but for the other ones you'll have to ask Fernando.

Homotopy theory of localizations

Let's spell this out with 1-cats for simplicity.

Definition 3 Given a localization $E : \mathbb{C} \to \mathbb{D}$, call $A \in \mathbb{C}$ *proto-fibrant*, if $\mathbb{C}(X, A) \to \mathbb{D}(EX, EA)$ is surjective for all X. Say that $\mathbb{C}$ has *enough* proto-fibrant objects, if all objects admit a weak equivalence (arrow inverted by E) into a proto-fibrant object. $\diamondsuit$

Theorem 4 *Assume $E : \mathbb{C} \to \mathbb{D}$ localization with enough proto-fibrant objects, and $F : \mathbb{C} \to \mathbb{X}$ a functor such that for all parallel $f, g : X \to A$ into a proto-fibrant object we have $Ff = Fg$ whenever $Ef = Eg$. Then F admits a left Kan extension along E.*

Proof. See [Fre22a, Lemma 7.3]. ∎

Thoughts on strictness

The above only makes sense in a setting where 2-categories have *sets* – i.e. 0-types – of objects, and so do their hom-categories.

In general it might seem intuitive to think about a setting where the 'strict' doctrines are of this kind, but the non-strict ones are univalent. In this case I think we lose the strictification adjoints, but the non-strict doctrines could still be bilocalizations (combined with Rezk completion) of the strict ones. This can be viewed as another argument in favor of the 'bilocalization' point of view.

References

[ARV10] J. Adámek, J. Rosický, and E.M. Vitale. *Algebraic theories: a categorical introduction to general algebra*, volume 184. Cambridge University Press, 2010.

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[Koc95] A. Kock. Monads for which structures are adjoint to units. *Journal of Pure and Applied Algebra*, 104(1):41–59, 1995.