A refinement of Gabriel-Ulmer duality

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Genoa

Overview

Three Parts

- Recall functorial semantics of (essentially) algebraic theories
- Clans as generalized algebraic theories
- Models in higher (homotopy) types

Part I

Algebraic Theories / Lawvere Theories

Definition

A **single-sorted algebraic theory** (SSAT) is a pair (Σ, E) consisting of

- a family $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$, of sets of n-ary **operations**
- a set of **equations** E whose elements are pairs of open terms over Σ

Definition

The **syntactic category** $\mathcal{C}(\Sigma, E)$ of a SSAT is given as follows:

1. For each natural number $n \in \mathbb{N}$ there is an **object** $[n]$
2. **morphisms** $\sigma : [n] \to [m]$ are m-tuples of terms in n variables modulo E-provable equality
3. **identities** are lists of variables, **composition** is given by substitution

Proposition

Given a SSAT (Σ, E):

1. $\mathcal{C}(\Sigma, E)$ has finite products given by $[n] \times [m] = [n + m]$
2. $\mathsf{Set}\text{-}\mathsf{Mod}(\Sigma, E) \simeq \mathsf{FP}(\mathcal{C}(\Sigma, E), \mathsf{Set})$

Finite-product theories

Definition

- A **fp-theory** is just a small fp-category $\mathcal{C}$.
- **Models** of $\mathcal{C}$ are fp-functors $A : \mathcal{C} \to \mathsf{Set}$ (or into another fp-category).

Denote the category of models by

$$\mathsf{Mod}(\mathcal{C}) \;:=\; \mathsf{FP}(\mathcal{C}, \mathsf{Set}) \;\overset{\text{full}}{\subseteq}\; [\mathcal{C}, \mathsf{Set}].$$

For every object $\Gamma \in \mathcal{C}$ of an fp-theory, the co-representable functor

$$\mathcal{C}(\Gamma, -) \;:\; \mathcal{C} \to \mathsf{Set}$$

is a model. Thus, the dual Yoneda embedding co-restricts to $\mathsf{Mod}(\mathcal{C})$.

$$\begin{tikzcd} & \mathcal{C}^{\mathsf{op}} \arrow[dl, dashed, "Z"'] \arrow[d] \\ \mathsf{Mod}(\mathcal{C}) \arrow[r, phantom, "\subseteq"] & {[\mathcal{C}, \mathsf{Set}]} \end{tikzcd}$$

Finite-limit theories

Definition

- An **fl-theory** is a small finite-limit category $\mathcal{L}$.
- A **model** of $\mathcal{L}$ is a finite-limit preserving functor $A : \mathcal{L} \to \mathsf{Set}$.

Finite-limit theories are more expressive than finite-product theories – structures definable by finite-limit theories include

- categories, posets, 2-categories, monoidal categories, categories with families . . .

Again $\mathcal{L}(\Gamma, -)$ is a model for every $\Gamma \in \mathcal{L}$ and we get an embedding

$$Z \;:\; \mathcal{L}^{\mathsf{op}} \;\to\; \mathsf{Mod}(\mathcal{L}) := \mathsf{FL}(\mathcal{L}, \mathsf{Set}) \overset{\text{full}}{\subseteq} [\mathcal{L}, \mathsf{Set}].$$

Moreover, we can characterize the essential image of Z in $\mathsf{Mod}(\mathcal{L})$.

Locally finitely presentable categories

Definition

- An object C of a cocomplete locally small category $\mathfrak{X}$ is called **compact**[a], if

$$\mathfrak{X}(C, -) : \mathfrak{X} \to \mathsf{Set}$$

  preserves filtered colimits.
- A category $\mathfrak{X}$ is called **locally finitely presentable**, if
  - $\mathfrak{X}$ is locally small and cocomplete
  - the full subcategory $\mathrm{comp}(\mathfrak{X}) \subseteq \mathfrak{X}$ on compact objects is essentially small and dense.

[a] More traditionally: 'finitely presentable'

Theorem

- $\mathsf{Mod}(\mathcal{L})$ *is locally finitely presentable for all finite-limit theories* $\mathcal{L}$.
- *The essential image of* $Z : \mathcal{L}^{\mathsf{op}} \to \mathsf{Mod}(\mathcal{L})$ *comprises precisely the compact objects.*

Gabriel-Ulmer duality[1]

Theorem

There is a bi-equivalence of 2-categories

$$\mathsf{FL} \quad \underset{\mathcal{L} \mapsto \mathsf{Mod}(\mathcal{L})}{\overset{\mathrm{comp}(\mathfrak{X})^{\mathsf{op}} \leftarrow\!\shortmid \mathfrak{X}}{\longleftrightarrow}} \quad \mathsf{LFP}^{\mathsf{op}}$$

where

- FL is the 2-category of **small** fl-categories and fl-functors
- LFP is the 2-category of locally finitely presentable categories and functors preserving small limits and filtered colimits ('forgetful functors').

[1] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Vol. 221. Lecture Notes in Math. Springer-Verlag, 1971.

Duality for finite-product theories[2]

There's a 'restriction' of G–U duality to finite-product theories:

$$\begin{tikzcd}
\mathsf{FP}_{\mathsf{cc}} \arrow[rr, shift left, "\mathcal{C} \mapsto \mathsf{FP}(\mathcal{C},\mathsf{Set})"] \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] & & \mathsf{ALG}^{\mathsf{op}} \arrow[ll, shift left, "\{\text{compact projectives}\}^{\mathsf{op}} \mapsfrom \mathfrak{A}"] \arrow[d, hook, "J"] \\
\mathsf{FL} \arrow[rr, shift left, "\mathcal{L} \mapsto \mathsf{FL}(\mathcal{L},\mathsf{Set})"] \arrow[u, bend right, "U"'] & & \mathsf{LFP}^{\mathsf{op}} \arrow[ll, shift left, "\{\text{compact objects}\}^{\mathsf{op}} \mapsfrom \mathfrak{X}"]
\end{tikzcd}$$

- $\mathsf{FP}_{\mathsf{cc}}$ is the 2-category of Cauchy-complete finite-product categories
- ALG is the 2-category of **algebraic categories** and **algebraic functors**
  - An **algebraic category** is an lfp category which is Barr exact and where the compact (regular) projective objects are dense
  - An **algebraic functor** is a functor that preserves small limits, filtered colimits, and regular epimorphisms.
- There's also a formulation in terms of sifted colimits, but we don't need it.

[2] J. Adámek, J. Rosický, and E.M. Vitale. *Algebraic theories: a categorical introduction to general algebra*. Vol. 184. Cambridge University Press, 2010.

Part II

Toward clans

- Finite-limit theories have a nice duality theory but seem far from syntax
- Syntactic counterparts are given by
  - Freyd's **essentially algebraic theories**[3]
  - Cartmell's **generalized algebraic theories**[4] (or 'dependent algebraic theories')
  - Palmgren and Vickers' **quasi-equational theories**[5]
  - and probably others
- Clans can be viewed as a categorical representation of generalized algebraic theories
- They're as expressive as fl-theories, but 'finer', i.e. closer to syntax

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[3] P. Freyd. "Aspects of topoi". In: *Bulletin of the Australian Mathematical Society* 7.1 (1972).

[4] J. Cartmell. "Generalised algebraic theories and contextual categories". In: *Annals of Pure and Applied Logic* 32 (1986).

[5] E. Palmgren and S. J. Vickers. "Partial horn logic and Cartesian categories". In: *Annals of Pure and Applied Logic* 145.3 (2007).

Definition

A **clan** is a small category $\mathcal{T}$ with terminal object 1, equipped with a class $\mathcal{T}_\dagger \subseteq \mathrm{mor}(\mathcal{T})$ of morphisms – called **display maps** and written $\twoheadrightarrow$ – such that

1. pullbacks of display maps along all maps exist and are display maps
$$\begin{tikzcd} \Delta^+ \arrow[r, "s^+"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma^+ \arrow[d, two heads, "p"] \\ \Delta \arrow[r, "s"] & \Gamma \end{tikzcd},$$
2. display maps are closed under composition, and
3. isomorphisms and terminal projections $\Gamma \twoheadrightarrow 1$ are display maps.

- Definition due to Taylor[6], name due to Joyal[7].
- Relation to semantics of dependent type theory: display maps represent **type families**.
- Observation: clans have finite products (as pullbacks over 1).

[6] P. Taylor. "Recursive domains, indexed category theory and polymorphism". PhD thesis. University of Cambridge, 1987, § 4.3.2.

[7] A. Joyal. "Notes on clans and tribes". In: *arXiv preprint arXiv:1710.10238* (2017).

Examples

- Finite-product categories $\mathcal{C}$ can be viewed as clans with $\mathcal{C}_\dagger = \{$product projections$\}$
- Finite-limit categories $\mathcal{L}$ can be viewed as clans with $\mathcal{L}_\dagger = \mathrm{mor}(\mathcal{L})$

We call such clans **fp-clans**, and **fl-clans**, respectively.

- The syntactic category of every Cartmell-style **generalized algebraic theory** is a clan.
- Clan for categories:

$$
\begin{aligned}
\mathcal{K} \; &= \{\text{categories free on finite graphs}\}^{\mathsf{op}} \subseteq \mathsf{Cat}^{\mathsf{op}} \\
\mathcal{K}_\dagger &= \{\text{functors induced by graph inclusions}\}^{\mathsf{op}}
\end{aligned}
$$

$\mathcal{K}$ can be viewed as syntactic category of a generalized algebraic theory of categories with a sort O of objects, and a dependent sort $x, y{:}O \vdash M(x, y)$ of morphisms – vertices of a finite graph are object variables and edges are morphism variables in a context. Graph inclusions are dual to context extensions.

Models

Definition

A **model** of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathsf{Set}$ which preserves 1 and pullbacks of display-maps.

- The category $\mathsf{Mod}(\mathcal{T}) \subseteq [\mathcal{T}, \mathsf{Set}]$ of models is lfp and contains $\mathcal{T}^{\mathsf{op}}$.
- For fp-clans $(\mathcal{C}, \mathcal{C}_\dagger)$ we have $\mathsf{Mod}(\mathcal{C}, \mathcal{C}_\dagger) = \mathsf{FP}(\mathcal{C}, \mathsf{Set})$.
- For fl-clans $(\mathcal{L}, \mathcal{L}_\dagger)$ we have $\mathsf{Mod}(\mathcal{L}, \mathcal{L}_\dagger) = \mathsf{FL}(\mathcal{L}, \mathsf{Set})$.
- $\mathsf{Mod}(\mathcal{K}, \mathcal{K}_\dagger) = \mathsf{Cat}$.

$$\begin{tikzcd} & \mathcal{T}^{\mathsf{op}} \arrow[dl, dashed, "z"'] \arrow[d] \\ \mathsf{Mod}(\mathcal{T}) \arrow[r, phantom, "\subseteq"] & {[\mathcal{T}, \mathsf{Set}]} \end{tikzcd}$$

Observation

The same category of models may be represented by different clans.
For example, SSATs can be represented by fp-clans as well as fl-clans.

The weak factorization system

- We would like a duality between clans and their categories of models.
- Since the same lfp category can be represented by different clans, we cannot hope to reconstruct the clan from the models alone.
- Solution: equip the models with additional structure in form of a **weak factorization system**.

Definition

Let $\mathcal{T}$ be a clan. Define wfs $(\mathcal{E}, \mathcal{F})$ on $\mathsf{Mod}(\mathcal{T})$ by

- $\mathcal{F} := \mathsf{RLP}(\{Z(p) \mid p \in \mathcal{T}_\dagger\})$ class of **full maps**
- $\mathcal{E} := \mathsf{Cell}(\{Z(p) \mid p \in \mathcal{T}_\dagger\}) = \mathsf{LLP}(\mathcal{F})$ class of **extensions**

I.e. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by the image of $\mathcal{T}_\dagger$ under $Z : \mathcal{T}^{\mathsf{op}} \to \mathsf{Mod}(\mathcal{T})$.

- Call $A \in \mathsf{Mod}(\mathcal{T})$ a 0-**extension**, if $(0 \to A) \in \mathcal{E}$

- The same weak factorization system was also introduced by Simon Henry in a video seminar in January 2020[8].

[8] Simon Henry, *The language of a model category*, HoTTEST seminar, 2020, https://youtu.be/7_X0qbSXlfk

Full maps

- $f : A \to B$ in $\mathrm{Mod}(\mathcal{T})$ is full iff has lhp with respect to all $Z(p)$ for display maps $p : \Delta \twoheadrightarrow \Gamma$.

$$\begin{tikzcd} \mathcal{T}(\Gamma,-) \arrow[r] \arrow[d, "Z(p)=\mathcal{T}(p,-)"'] & A \arrow[d, "f"] \\ \mathcal{T}(\Delta,-) \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \qquad \begin{tikzcd} A(\Delta) \arrow[r, "f_\Delta"] \arrow[d, "A(p)"'] & B(\Delta) \arrow[d, "B(p)"] \\ A(\Gamma) \arrow[r, "f_\Gamma"] & B(\Gamma) \end{tikzcd}$$

- This is equivalent to display-naturality-squares being weak pullbacks.
- Considering $p : \Delta \twoheadrightarrow 1$ we see that full maps are surjective and hence regular epis.

$$\begin{tikzcd} A(\Delta) \arrow[r, "f_\Delta"] \arrow[d] & B(\Delta) \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} A(\Delta) \arrow[r, "f_\Delta"] \arrow[d] & B(\Delta) \arrow[d] \\ A(\Delta) \times A(\Delta) \arrow[r, "f_\Delta \times f_\Delta"] & B(\Delta) \times B(\Delta) \end{tikzcd}$$

- For fl-clans, only isos are full (consider naturality square for diagonal $\Delta \to \Delta \times \Delta$)
- For fp-clans, the full maps are *precisely* the regular epis, and therefore

$$0\text{-extension} = \text{projective object}$$

Reconstructing the clan

Definition

Given a clan $\mathcal{T}$, let $\mathbb{C} \subseteq \mathsf{Mod}(\mathcal{T})$ be the full subcategory on **compact** 0**-extensions**.

Observation: $Z : \mathcal{T}^{\mathsf{op}} \to \mathsf{Mod}(\mathcal{T})$ factors through $\mathbb{C}$.

$$\begin{tikzcd} & \mathbb{C} \arrow[d, hook] \\ \mathcal{T}^{\mathsf{op}} \arrow[ur, dashed, "E"] \arrow[r, "Z"'] & \mathsf{Mod}(\mathcal{T}) \end{tikzcd}$$

Theorem

$\mathbb{C}^{\mathsf{op}}$ is a coclan (with extensions as display maps), and E exhibits $\mathbb{C}$ as Cauchy-completion of $\mathcal{T}^{\mathsf{op}}$.

Proof idea

Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in $\mathcal{T}$.

$$\begin{tikzcd} \mathrm{colim}(E \circ D) \arrow[d, two heads, "f"] \\ A \end{tikzcd} \qquad D : \mathbb{I} \to \mathcal{T}^{\mathsf{op}}$$

Proof idea

Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in $\mathcal{T}$.

$$
\begin{tikzcd}
\operatorname{colim}(E \circ D) \arrow[d, two heads, "f"] \\
A \arrow[u, bend left, "s"]
\end{tikzcd}
\qquad D : \mathbb{I} \to \mathcal{T}^{\mathsf{op}}
$$

- f splits since A is a 0-extension

Proof idea

Given $A \in \mathbb{C}$, fully cover it by a filtered colimit of objects in $\mathcal{T}$.

$$
\begin{tikzcd}
E(D_i) \arrow[r, "\sigma_i"] & \mathrm{colim}(E \circ D) \arrow[d, two heads, "f"] \\
& A \arrow[ul]
\end{tikzcd}
\qquad D : \mathbb{I} \to \mathcal{T}^{\mathsf{op}}
$$

- f splits since A is a 0-extension
- s factors through one of the colimit-injections, since A is compact

Proof idea

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- f splits since A is a 0-extension
- s factors through one of the colimit-injections, since A is compact
- D exists by the **fat small object argument**:
  - M. Makkai, J. Rosicky, and L. Vokrinek. "On a fat small object argument". In: *Advances in Mathematics* 254 (2014)

Clan-algebraic categories

Definition

A **clan-algebraic category** is an lfp category $\mathfrak{A}$ with an wfs $(\mathcal{E}, \mathcal{F})$ that arises as category of models of a clan.

With this definition we get a contravariant bi-equivalence

$$\mathsf{Clan}_{\mathsf{cc}} \quad \underset{\mathcal{T} \mapsto \mathsf{Mod}(\mathcal{T})}{\overset{\mathrm{comp}(\mathfrak{X})^{\mathsf{op}} \leftarrow\!\shortmid \mathfrak{X}}{\longleftarrow\!\!\!-\!\!\!\longrightarrow}} \quad \mathsf{cAlg}^{\mathsf{op}}$$

between

- the 2-category $\mathsf{Clan}_{\mathsf{cc}}$ of Cauchy-complete clans and functors preserving 1, display maps, and pullbacks of display maps, and
- the 2-category cAlg of clan-algebraic categories and functors preserving small limits, filtered colimits, and full maps.

Can we characterize clan-algebraic categories more abstractly?

Characterization of clan-algebraic categories

Theorem

A lfp category $\mathfrak{A}$ with wfs $(\mathcal{E}, \mathcal{F})$ is clan-algebraic, if

1. $\mathbb{C} = \{\textit{compact } 0\textit{-extensions}\}$ *is dense in* $\mathfrak{A}$,
2. $(\mathcal{E}, \mathcal{F})$ *is cofibrantly generated by* $\mathcal{E} \cap \mathrm{mor}(\mathbb{C})$, *and*
3. *for* $C \in \mathbb{C}$ *and* $A \in \mathsf{Mod}(\mathbb{C}^{\mathsf{op}})$, *the functor* $\mathfrak{A}(C, -) : \mathfrak{A} \to \mathsf{Set}$ *preserves the colimit of the diagram* $\int A \to \mathbb{C} \to \mathfrak{A}$.

- Condition 3 is not very nice
- Can we find an 'exactness condition' similar to the one given by Adámek, Rosický and Vitale for algebraic categories?
- . . . there is at least a *necessary* exactness condition

Quotients of componentwise-full equivalence relations

- In algebraic categories, all equivalence relations have effective quotients (they are ‘Barr exact’)
- This can’t be true for clan algebraic categories in general. However, we have:

Lemma

For any clan $\mathcal{T}$, $\mathrm{Mod}(\mathcal{T})$ *has* full and effective quotients *of* **componentwise-full equivalence relations**.

Conjecture

Condition 3 of the theorem is implied by $\mathfrak{A}$ having full and effective quotients of componentwise-full equivalence relations.

Part III

Models in higher types

Let $\mathcal{S}$ be the ∞-topos of spaces/types.

Let $\mathcal{C}_{\mathsf{Mon}}$ be the finite-product theory of monoids, and let $\mathcal{L}_{\mathsf{Mon}}$ be the finite-limit theory of monoids. Then

$$\mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathsf{Set}) \simeq \mathsf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathsf{Set})$$

but $\mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ and $\mathsf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$ are different:

- $\mathsf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$ is just the category of monoids
- $\mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ is the ∞-category 'A_∞-algebras', i.e. homotopy-coherent monoids.

Moral

By being 'slimmer', finite-product theories leave room for higher coherences when interpreted in higher types.

This phenomenon has been discussed under the name 'animation' in:

- K. Cesnavicius and P. Scholze. "Purity for flat cohomology". In: *arXiv preprint arXiv:1912.10932* (2019)

Four clans for categories

Cat admits several clan-algebraic weak factorization systems:

- $(\mathcal{E}_1, \mathcal{F}_1)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2})\}$
- $(\mathcal{E}_2, \mathcal{F}_2)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (2 \to 1)\}$
- $(\mathcal{E}_3, \mathcal{F}_3)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2})\}$
- $(\mathcal{E}_4, \mathcal{F}_4)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2}), (2 \to 1)\}$

where $\mathbb{P} = (\bullet \rightrightarrows \bullet)$.

The right classes are:

$$\begin{aligned}
\mathcal{F}_1 &= \{\text{full and surjective-on-objects functors}\} \\
\mathcal{F}_2 &= \{\text{full and bijective-on-objects functors}\} \\
\mathcal{F}_3 &= \{\text{fully faithful and surjective-on-objects functors}\} \\
\mathcal{F}_4 &= \{\text{isos}\}
\end{aligned}$$

Note that $\mathcal{F}_3$ is the class of trivial fibrations for the canonical model structure on Cat.

Four clans for categories

These correspond to the following clans:

$$\mathcal{T}_1 = \{\text{free cats on fin. graphs}\}^{\mathsf{op}} \qquad \mathcal{T}_1^\dagger = \{\text{graph inclusions}\}$$

$$\mathcal{T}_2 = \{\text{free cats on fin. graphs}\}^{\mathsf{op}} \qquad \mathcal{T}_2^\dagger = \{\text{injective-on-edges maps}\}$$

$$\mathcal{T}_3 = \{\text{f.p. cats}\}^{\mathsf{op}} \qquad \mathcal{T}_3^\dagger = \{\text{injective-on-objects functors}\}$$

$$\mathcal{T}_4 = \{\text{f.p. cats}\}^{\mathsf{op}} \qquad \mathcal{T}_4^\dagger = \{\text{all functors}\}$$

Models in higher types:

$$\infty\text{-Mod}(\mathcal{T}_1) = \{\text{Segal spaces}\}$$
$$\infty\text{-Mod}(\mathcal{T}_2) = \{\text{Segal categories}\}$$
$$\infty\text{-Mod}(\mathcal{T}_3) = \{\text{pre-categories}\}$$
$$\infty\text{-Mod}(\mathcal{T}_4) = \{\text{discrete 1-categories}\}$$

Thanks for your attention!