Sh(Cart) as a classifying topos

$\mathbf{Cart} = \{\mathbb{R}^n \mid n \in \mathbb{N}\} \subseteq \mathbf{Top}$ site with open cover topology.

Model for real cohesion : $\Pi \dashv \Delta \dashv \Gamma \dashv \nabla : \mathbf{Sh}(\mathbf{Cart}) \to \mathbf{Set}$ has nice properties:

- dedekind reals are representables
- covering spaces
- vector bundles, Grassmanians
- modal type theory / cohesion

What does the topos classify?

Observation: $\mathbf{Cart}$ is a Lawvere theory. It's a *super theory* of known Lawvere theories:

$\mathrm{L}(\text{Ring}) \subseteq \mathrm{L}(\mathbb{R}\text{-Alg}) \subseteq \mathrm{L}(C^\infty\text{-Ring}) \subseteq \mathbf{Cart} =: \mathrm{L}(\text{RC-Alg})$

Call its models *real-continuous algebras*. So a real-continuous algebra is in particular an $\mathbb{R}$-algebra, but besides the polynomials it has n-ary operations for all continuous $f : \mathbb{R}^n \to \mathbb{R}$.

We write $\mathbf{RCL} = \mathbf{RC\text{-}Alg^{op}}$ for the formal dual of $\mathbf{RC\text{-}Alg}$ and call its objects *real-continuous loci*.

The inclusion $\mathbf{Cart} \subseteq \mathbf{Top}$ gives rise to a Galois connection

$$\mathbf{RC\text{-}Alg} \leftrightarrows \mathbf{Top^{op}}$$

which we formally decompose as follows:

$$\begin{gathered}\mathbf{RC\text{-}Alg^{op}} \\ C(-) \uparrow \simeq \downarrow \text{spec} \\ \mathbf{RCL} \\ L \uparrow \dashv \downarrow P \\ \mathbf{Top}\end{gathered}$$

The functor L is fully faithful on *realcompact* spaces.

- A realcompact space is a completely regular space which contains all the "real" points of its SC compactification (it is possible to associate some kind of residue fields to elements of the SC compactification).
- A space is realcompact iff it's a closed subspace of some $\mathbb{R}^I$.
- All "small" metric spaces are realcompact: specifically, (X, d) is realcompact whenever it has no closed discrete subspace of measurable cardinal.

The classifying topos for RC-algs is the presheaf-topos on $\mathbf{RC\text{-}Alg}_{fp}$. So let's try to understand those.

- The fg-free rc-algs are precisely the algs $C(R^n)$.
- The finpres rc-algs are of the form $C(R^n)/I$ for finitely generated RC ideals L.

Def:

- For $f \in C(\mathbb{R}^n)$ write $Z(f)={\vec x ;|; f(\vec x)0 }
- For $A \subseteq \mathbb{R}^n$ write $I(A) = \{f \in C(\mathbb{R}^n) \mid f|_A = 0\}$. This is a rcidl.

Not every rcidl is of this form. A counterexample is given by $L(0) = \bigcup_{n \in \mathbb{N}} I([-1/n, 1/n])$

However, every *finitely generated* rcidl is of this form! This is a consecquence of the following.

Proposition: We have $\langle f\rangle = I(Z(f))$.

Cor: We have $I(Z(a)) = a$ for all fgidls a.

Cor: The functor $\mathbf{Cart} \to \mathbf{Top} \to \mathbf{RCL}$ preserves equalizers.

Def: Write $\overline{\mathbf{Cart}}$ for the category of closed subspaces of cartesian spaces.

Cor: $\overline{\mathbf{Cart}}$ is the fl-completion of $\mathbf{Cart}$ and thus the fl-theory of rcalgs.

Classifying toposes

We have geom inclusions

$$\mathbf{Sh}(\mathbf{Cart}) \hookrightarrow \widehat{\mathbf{Cart}} \hookrightarrow \widehat{\overline{\mathbf{Cart}}}$$

where the last one classifies rcalgs

The presheaves on the l-theory classify *flat* rcalgs. These are rc-algs validating the geometric sequent

$$f(\vec{x}) = 0 \vdash_{\vec{x}} \bigvee_m \bigvee_{\substack{\vec{h} \in C(\mathbb{R}^m, \mathbb{R})^k \\ f \circ \vec{h} = 0}} \exists \vec{y} \,.\, \vec{h}(\vec{y}) = \vec{x} \,.$$

for all f. Thinking about elts of A as functions on spec(A) we can draw the following picture ...

Examples:

- $C(R^n)$ flat
- $C(1+1)$ not flat
- $C(s1)$ not flat

That looks like a contractibility condition!

Def:

- Call rcalg A local, if it is local as a ring.
- Call rcalg A *archimedean*, if it validates

  $$f : A \vdash \exists n f \text{ factors through } B_n$$

  (expressible since all cont functions are ops)

Prop:

- finite open cover topology on $\overline{\mathbf{Cart}}$ corresponds to local rcalgs
- open cover topology on $\overline{\mathbf{Cart}}$ corresponds to local archimedean rcalgs.
- However, in presence of flatness, local algs are automatically archimedean (since open covers generate all covers)

Examples:

- nonstandard reals are local but not archimedean or flat
- bounded nonstandard reals are local and archimedean but not flat
- $C(\mathbb{R})/L(0)$ is flat, local, and thus archimedean
- local neighbourhood of closed interval is flat and archimedean but not local

Lemma: spec(A) has a unique point for all flat local A.

Thus, spec(A) topologically a very small, pointed, and contractible space.