Uniform Preorders and Partial Combinatory Algebras

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Abstract

Uniform preorders are a class of combinatory representations of Set-indexed preorders that generalize Hofstra's *basic relational objects* [Hof06]. An indexed preorder is representable by a uniform preorder if and only if it has as generic predicate. We study the ∃-completion of indexed preorders on the level of uniform preorders, and identify a combinatory condition (called 'relational completeness') which characterizes those uniform preorders with finite meets whose ∃-completions are triposes. The class of triposes obtained this way contains *relative realizability triposes*, for which we derive a characterization as a fibrational analogue of the characterization of realizability toposes given in earlier work [Fre19].

Besides relative partial combinatory algebras, the class of relationally complete uniform preorders contains *filtered ordered* partial combinatory algebras, and it is unclear if there are any others.

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Introduction

In his seminal article [Hof06], Pieter Hofstra gave an analysis of *filtered ordered combinatory algebras* (filtered OPCAs) [HvO03] in terms of the more primitive notion of *basic combinatory objects* (BCOs). These are combinatory representations $(A, \leq, \mathcal{F})$ of certain Set-indexed preorders by partial orders equipped with a class of partial endomaps, and Hofstra showed that a BCO $(A, \leq, \mathcal{F})$ arises from a filtered OPCA if and only if

(1) it is *cartesian* in the sense that the associated indexed preorder $\mathsf{fam}(A, \leq, \mathcal{F})$ is an meet-semilattice, and

(2) the free completion of $\mathsf{fam}(A, \leq, \mathcal{F})$ under existential quantification ('$\exists$-completion') is a tripos.

The present work gives two variations on this theme, replacing BCOs by the more general notion of *uniform preorder* on the one hand, and by the more restrictive notion of *discrete combinatory object* on the other hand, together fitting into a sequence

$$\mathsf{DCO} \to \mathsf{BCO} \to \mathsf{UOrd} \to \mathsf{IOrd}$$

of embeddings of *locally ordered categories*. A uniform preorder is a set equipped with a monoid of *binary relations* (Definition 1.1), and a DCO is a set with a monoid of *partial functions* (Definition 8.1(i)), and the locally ordered categories DCO and UOrd have the advantage over BCO that their bi-essential images in the locally ordered category IOrd of Set-indexed preorders admit straightforward characterizations: an indexed preorder is representable by an uniform preorder iff it has a generic predicate (Lemma 1.6), and it is representable by a DCO iff it has a *discrete* generic predicate (Corollary 8.4).

After developing the basic theory of uniform preorders in Sections 1–5, we give a combinatorial criterion for the $\exists$-completion of a cartesian uniform preorder to be a tripos in Definition 6.3 and Theorem 6.5, which we call *relational completeness*. In Example 6.7(b), relational completeness is used to show that the $\exists$-completion of a tripos is again a tripos, and Remark 6.6(b) gives a characterization of the triposes that arise as $\exists$-completions of (the indexed preorders associated to) relationally complete uniform preorders, building on a prior characterization of $\exists$-completions in terms of $\exists$-*prime predicates* (Proposition 4.3). This characterization is augmented by a discreteness condition in Theorem 9.5 to obtain a characterization of relative realizability triposes:

> *A tripos $\mathcal{P}$ is a relative realizability tripos if and only if it has enough $\exists$-prime predicates, and the indexed sub-preorder* $\mathsf{prim}(\mathcal{P})$ *of prime predicates has finite meets and a discrete generic predicate.*

In light of the close analogy between Theorem 9.5 and Remark 6.6(b), relationally complete uniform preorders might be viewed as (relative/filtered) *relational PCAs*.

A central question remains open: every filtered OPCA gives rise to a relationally complete uniform preorder, but are there any others?

Most of the work presented here is already contained in the author's PhD thesis [Fre13], where the theory of uniform preorders is developed in greater generality, including *many-sorted* uniform preorders, and without the use of the axiom of choice. To get a more accessible presentation, we have left out the subtleties of a choice-free development here, and focused on the single-sorted case.

1 The locally ordered category of uniform preorders

Uniform preorders were introduced in [Fre13] as representations of certain Set-indexed preorders that generalize Hofstra's *basic combinatorial objects (BCOs)* [Hof06].

Contrary to BCOs, for uniform preorders there exists a straightforward characterization of the induced class of indexed preorders, which makes the notion both conceptually very clear and somewhat tautological. In this section we reconstruct the definition of uniform preorders from this characterization, after fixing terminology and notation on locally ordered categories and indexed preorders, which constitute the central formalisms in this article.

A Set-*indexed preorder* is a pseudofunctor $\mathsf{Set}^{\mathsf{op}} \to \mathsf{Ord}$ where Ord is the locally ordered category of preorders and monotone maps. We view locally ordered categories as degenerate 2-categories, and use 2-categorical concepts and terminology. As we only consider indexed preorders on Set in this paper, we omit the prefix. Given an indexed preorder $\mathcal{P}$ and a set A, we call $\mathcal{P}(A)$ the *fiber* of $\mathcal{P}$ over A, and refer to its elements as *predicates* on A. Given a function $f : A \to B$, the monotone map $\mathcal{P}(f)$ is called *reindexing along* f and abbreviated f^*. We write IOrd for the locally ordered category of indexed preorders and pseudo-natural transformations.

Strict indexed preorders and transformations form a non-full locally ordered subcategory $[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]$ of IOrd, which by a well known argument about models of geometric theories in presheaf categories[1] is isomorphic to the locally ordered category $\mathsf{Ord}([\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}])$ of internal preorders in $[\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}]$.

The locally ordered category UOrd of uniform preorders is now characterized as fitting into the following strict pullback of locally ordered categories, where U sends internal preorders to underlying presheaves, the categories in the lower line are viewed as having codiscretely ordered hom-sets (to make U well-defined), よ is the Yoneda embedding, and fam is the indicated composition.

$$\begin{tikzcd}
\mathsf{UOrd} \arrow[r, "J"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "\mathsf{fam}"] & \mathsf{Ord}([\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}]) \arrow[r, "\cong"'] \arrow[d, "U"] & {[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]} \arrow[r, hook] & \mathsf{IOrd} \\
\mathsf{Set} \arrow[r, "\text{よ}"] & {[\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}]} & &
\end{tikzcd}$$

The 2-functor J is 2-fully faithful since よ is, which means that UOrd can be identified with the 2-full subcategory of $\mathsf{Ord}([\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}])$ on internal preorders whose underlying presheaves are representable. In other words, a uniform preorder is a set A together with an internal preorder structure on $\text{よ}(A)$. Such a preorder structure is given by a subfunctor of $\text{よ}(A) \times \text{よ}(A) \cong \text{よ}(A \times A)$, i.e. a sieve on $A \times A$, subject to reflexivity and transitivity conditions.

Since surjections split in Set, sieves are completely determined by their monomorphisms, or equivalently subset-inclusions, which means that a sieve on $A \times A$ is equivalently represented as a down-closed subset of the powerset $P(A \times A)$. We leave it to the reader to verify that unwinding the meaning of reflexivity, transitivity, monotonicity, and the hom-set ordering in terms of this representation of sieves yields the following concrete descriptions of the locally ordered category UOrd and the 2-functor fam.

Definition 1.1 The locally ordered category UOrd of uniform preorders and monotone maps is defined as follows.

(i) A *uniform preorder* is a pair (A, R) of a set A and a set $R \subseteq P(A \times A)$ of binary relations on A, such that

[1][Joh02, Corollary D1.2.14(i)] gives a statement for small index categories, but smallness is not essential.

– $\mathrm{id}_A \in R$,
– $s \circ r \in R$ whenever $r \in R$ and $s \in R$, and
– $s \in R$ whenever $r \in R$ and $s \subseteq r$.

(ii) A *monotone map* between uniform preorders (A, R) and (B, S) is a function $f : A \to B$ such that for all $r \in R$, the set

$$(f \times f)[r] \;=\; f \circ r \circ f^{\circ} \;=\; \{(fa, fa') \mid (a, a') \in r\}$$

is in S.

(iii) The ordering relation $\leq$ on monotone maps $f, g : (A, R) \to (B, S)$ is defined by $f \leq g$ iff the set

$$\mathsf{im}\langle f, g\rangle \;=\; \{(fa, ga) \mid a \in A\}$$

is in S. $\diamondsuit$

Definition 1.2 The 2-functor $\mathsf{fam} : \mathsf{UOrd} \to \mathsf{IOrd}$ is defined as follows.

(i) For every uniform preorder (A, R), the indexed preorder $\mathsf{fam}(A, R)$ maps

– sets I to preorders $(A^I, \leq)$, where $\varphi \leq \psi : I \to A$ iff

$$\mathsf{im}\langle \varphi, \psi\rangle \;=\; \{(\varphi i, \psi i) \mid i \in I\} \tag{1.1}$$

is in R, and

– functions $f : J \to I$ to monotone maps $f^* : (A^J, \leq) \to (A^I, \leq)$ given by precomposition.

(ii) For every monotone map $f : (A, R) \to (B, S)$ between indexed preorders, the components of the indexed monotone map $\mathsf{fam}(f) : \mathsf{fam}(A, R) \to \mathsf{fam}(B, S)$ are given by postcomposition. $\diamondsuit$

Remarks 1.3 – Given a uniform preorder (A, R) and predicates, $\varphi, \psi : I \to A$, we say that a relation $r \in R$ *realizes* an inequality $\varphi \leq \psi$ if $\mathsf{im}\langle \varphi, \psi\rangle \subseteq r$ (and thus $\mathsf{im}\langle \varphi, \psi\rangle \in R$). This is stable under reindexing: if r realizes $\varphi \leq \psi$ and $u : J \to I$ then r realizes $u^*\varphi \leq u^*\psi$.

– The ordering on monotone maps $f, g : (A, R) \to (B, S)$ defined in 1.1(iii) is the restriction of the ordering on $\mathsf{fam}(B, S)(A)$ as defined in 1.2(i). $\diamondsuit$

Definition 1.4 A *basis* for a uniform preorder (A, R) is a subset $R_0 \subseteq R$ of binary relations whose *down-closure* $\downarrow R_0$ in $P(A \times A)$ is R, i.e. R and R_0 generate the same sieve on $A \times A$. In other words, $R_0 \subseteq R$ is a basis of R if for every $r \in R$ there is an $r_0 \in R_0$ with $r \subseteq r_0$. $\diamondsuit$

Remark 1.5 Given a set A and a set $R_0 \subseteq P(A \times A)$ of binary relations, its down-closure $R = \downarrow R_0$ is a uniform preorder structure on A iff

(a) there exists an $r \in R_0$ with $\mathrm{id}_A \subseteq r$, and

(b) for all $r, s \in R_0$ there exists a $t \in R_0$ with $s \circ r \in t$.

Just like continuity of functions between topological spaces, monotonicity of functions between uniform preorders can be expressed in terms of bases. Specifically, given uniform preorders (A, R) and (B, S) with bases R_0 and S_0, a function $f : A \to B$ is monotone iff for all $r \in R_0$ there exists an $s \in S_0$ with $(f \times f)[r] \subseteq s$, and given $\varphi, \psi : I \to A$ we have $\varphi \leq \psi$ in $\mathsf{fam}(A, R)(I)$ iff there exists an $r \in R_0$ with $\mathsf{im}\langle\varphi, \psi\rangle \subseteq r$. $\diamondsuit$

The following lemma gives a better understanding of the combined embedding from UOrd to IOrd. Recall that a *generic predicate* in an indexed preorder $\mathcal{A}$ is a predicate $\iota \in \mathcal{A}(A)$ for some A, such that for every other set B and predicate $\varphi \in \mathcal{A}(B)$ there exists a function $f : B \to A$ with $f^*\iota \cong \varphi$.

Lemma 1.6 *The 2-functor* $\mathsf{fam} : \mathsf{UOrd} \to \mathsf{IOrd}$ *is a local equivalence, and its bi-essential image consists of the indexed preorders which admit a generic predicate.*

Concretely, if $\mathcal{H}$ *is an indexed preorder with generic predicate* $\iota \in \mathcal{H}(A)$*, then the corresponding uniform preorder is given by* (A, R) *with*

$$R = \{r \subseteq A \times A \mid p^*\iota \leq q^*\iota\}$$

$$\begin{tikzcd} & r \arrow[dl, "p"'] \arrow[d, hook] \arrow[dr, "q"] & \\ A & A \times A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & A \end{tikzcd}$$

where $p, q : r \to A$ *are the first and second projections as in the diagram.*

Proof. For the first claim — since $\mathsf{UOrd} \to [\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]$ is an isomorphism on hom-preorders, and $[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}] \to \mathsf{IOrd}$ is locally order reflecting — it is sufficient to show that for every uniform preorder (A, R), strict indexed preorder $\mathcal{K}$, and pseudonatural $f : \mathsf{fam}(A, R) \to \mathcal{K}$ there exists a strict transformation $\bar{f} : \mathsf{fam}(A, R) \to \mathcal{K}$ with $\bar{f} \cong f$. The transformation $\bar{f}$ is given by $\bar{f}_I(\varphi : I \to A) = \varphi^*(f_A(\mathrm{id}_A))$[2].

For the second claim it is clear that indexed preorders $\mathsf{fam}(A, R)$ have generic predicates (the identity), and that this property is stable under equivalence. Conversely, it was stated earlier that uniform preorders can be identified with strict indexed preorders whose underlying presheaf of sets is representable, and every indexed preorder $\mathcal{H}$ with generic predicate $\iota \in \mathcal{H}(A)$ is equivalent to the strict indexed preorder with underlying presheaf $\mathsf{Set}(-, A)$, and ordering on $\mathsf{Set}(I, A)$ given by $f \leq g$ iff $f^*\iota \leq g^*\iota$. $\square$

Examples 1.7 (a) The *canonical indexing* of a preorder $(A, \leq)$ is the strict indexed preorder whose underlying presheaf is the representable presheaf $\mathsf{Set}(-, A)$, and whose fibers are ordered pointwise, i.e. $(\varphi : I \to A) \leq (\psi : I \to A)$ iff $\forall i \in I \,.\, \varphi(i) \leq \psi(i)$.

The corresponding uniform preorder is $(A, R_{\leq})$ where $R_{\leq} = {\downarrow}\{\leq\} \subseteq P(A \times A)$.

(b) Hofstra's *basic combinatory objects* (BCOs) [Hof06, pg. 241] can be embedded into uniform preorders: recall that a BCO is a triple $(A, \leq, \mathcal{F})$ where $(A, \leq)$ is a partial order and $\mathcal{F}$ is a set of monotone partial endofunction with down-closed domains, which is weakly closed under composition in the sense that

(i) there exists an $i \in \mathcal{F}_A$ such that $i(a) \leq a$ for all $a \in A$, and

(ii) for all $f, g \in \mathcal{F}$ there exists $h \in \mathcal{F}$ such that $h(a) \leq g(f(a))$ whenever the right side is defined.

[2] More generally, this argument works for pseudonatural transformations $f : \mathcal{H} \to \mathcal{K}$ between strict indexed preorders where $\mathcal{H}$'s underlying presheaf of sets is *projective*, i.e. a coproduct of representables. Such indexed preorders $\mathcal{H}$ correspond to the 'many-sorted uniform preorders' studied in [Fre13].

Given a BCO $(A, \leq, \mathcal{F})$, we get an indexed preorder structure on $\mathsf{Set}(-, A)$ by setting

$$(\varphi : I \to A) \leq (\psi : I \to A) \qquad \text{iff} \qquad \exists f \in \mathcal{F}\, \forall i \in I \,.\, f(\varphi(i)) \leq \psi(i).$$

Just as for the indexed preorders associated to ordinary preorders and uniform preorders, we write $\mathsf{fam}(A, \leq, \mathcal{F})$ for this indexed preorder.

The corresponding uniform preorder structure $R_{\mathcal{F}}$ on A is generated by the relations $\{r_f \subseteq A \times A \mid f \in \mathcal{F}\}$, where $r_f = \{(a, b) \mid f(a) \leq b\}$ for $f \in \mathcal{F}$. The axioms (i), (ii) ensure that the relations r_f form a basis in the sense of Definition 1.4. $\diamondsuit$

Hofstra defined a locally ordered category BCO of BCOs whose notion of morphism is a bit subtle, but is justified and fully explained by the fact that it extends the mapping $(A, \leq, \mathcal{F}) \mapsto \mathsf{fam}(A, \leq, \mathcal{F})$ to a 2-functor $\mathsf{fam} : \mathsf{BCO} \to [\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]$ into strict indexed preorders which is *2-fully faithful*, i.e. a local isomorphism. Since the embeddings of Ord and UOrd into $[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]$ are also local isomorphisms, we obtain a sequence

$$\mathsf{Ord} \to \mathsf{BCO} \to \mathsf{UOrd} \to [\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}] \tag{1.2}$$

of 2-full embeddings of locally ordered categories.

2 Adjunctions of uniform preorders

An adjunction in a locally ordered category $\mathfrak{A}$ is a pair of arrows $f : A \to B$, $g : B \to A$, such that $\mathrm{id}_A \leq g \circ f$ and $f \circ g \leq \mathrm{id}_B$. Since $\mathsf{UOrd} \to \mathsf{IOrd}$ is a local equivalence, a monotone map $f : (A, R) \to (B, S)$ has a right adjoint in UOrd precisely if $\mathsf{fam}(f)$ has a right adjoint in IOrd. The following lemma gives a criterion for the existence of right adjoints in which monotonicity does not have to be checked explicitly.

Lemma 2.1 *The following are equivalent for uniform preorders (A, R), (B, S), a monotone map $f : (A, R) \to (B, S)$, and a function $g : B \to A$.*

(i) *The function g is a monotone map from (B, S) to (A, R), and right adjoint to f.*

(ii) (1) *The relation $\mathsf{im}\langle f \circ g, \mathrm{id}_B\rangle = \{(f(g(b)), b) \mid b \in B\}$ is in S, and*

(2) *for all $s \in S$, the relation $s^* = \{(a, gb) \mid (fa, b) \in s\}$ is in R.*

If (B, S) is given by a basis, then it is sufficient to verify (2) on the elements of the basis.

Proof. First assume (i). Condition (1) is equivalent to $f \circ g \leq \mathrm{id}_B$ by (1.1). For condition (2), let $I = \{(a, b) \in A \times B \mid (fa, b) \in s\}$, and let $p : I \to A$ and $q : I \to B$ be the projections. Then we have $f \circ p \leq q$ in $\mathsf{fam}(B, S)(I)$ by direct verification, and therefore $p \leq g \circ q$ in $\mathsf{fam}(A, R)(I)$ by exponential transpose. the latter is equivalent to the claim.

Conversely, assume (ii). To see that postcomposition with g induces a left adjoint to $\mathsf{fam}(f) : \mathsf{fam}(A, R) \to \mathsf{fam}(B, S)$, it is enough to check that for all sets I and $h : I \to B$, the function $g \circ h$ is a greatest element of

$$\Phi = \{k : I \to A \mid f \circ k \leq h\} \subseteq \mathsf{fam}(A, R)(I).$$

We have $g \circ h \in \Phi$ by (1). To show that it is a greatest element we have to show that $f \circ k \leq h$ implies $k \leq g \circ h$, which follows from (2) since

$$\mathsf{im}\langle k, g \circ h\rangle \subseteq \mathsf{im}\langle f \circ k, h\rangle^*$$

and R is down-closed. $\square$

3 Cartesian uniform preorders

The full subcategory of IOrd on indexed preorders admitting a generic predicate is closed under small 2-products: if $(\mathcal{H}_k)_{k \in K}$ is a family of indexed preorders with generic predicates $(\iota_k \in \mathcal{H}_k(A_k))_{k \in K}$, then a generic predicate of the (pointwise) product $\prod_{k \in K} \mathcal{H}_k$ is given by the family

$$(\pi_k^* \iota_k)_{k \in K} \in \prod_{k \in K} \mathcal{H}_k(\prod_{k \in K} A_k).$$

Thus, UOrd has products which are preserved by $\mathsf{fam} : \mathsf{UOrd} \to \mathsf{IOrd}$. Concretely, the terminal uniform preorder is the singleton set with the unique uniform preorder structure, and a product of (A, R) and (B, S) is given by $(A \times B, R \otimes S)$, where $R \otimes S$ is the uniform preorder structure generated by the basis $\{r \times s \mid r \in R, s \in S\}$.

Definition 3.1 An object A of a locally ordered category $\mathfrak{A}$ with finite 2-products is called *cartesian* if the terminal projection $A \to 1$ and the diagonal $A \to A \times A$ have right adjoints $\top : 1 \to A$ and $\wedge : A \to A \times A$.

Given cartesian objects A, B, a *morphism* $f : A \to B$ is called cartesian if the diagrams

$$\begin{tikzcd} A \times A \arrow[r, "f \times f"'] \arrow[d, "\wedge"'] & B \times B \arrow[d, "\wedge"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[d, "\top"'] \arrow[dr, "\top"] & \\ A \arrow[r, "f"] & B \end{tikzcd}$$

commute up to isomorphism. ♢

Since $\mathsf{UOrd} \to \mathsf{IOrd}$ is a local equivalence and preserves (finite) 2-products, a uniform preorder (A, R) is cartesian if and only if $\mathsf{fam}(A, R)$ is cartesian, and the latter is easily seen to be equivalent to $\mathsf{fam}(A, R)$ being an *indexed meet-semilattice*, i.e. an indexed preorder whose fibers have finite meets, which are preserved by reindexing. Instantiating Lemma 2.1 we get the following characterization.

Lemma 3.2 *A uniform preorder (A, R) is cartesian if and only if there exists a function $\wedge : A \times A \to A$ and an element $\top \in A$ such that the relations*

$$\tau = \{(a, \top) \mid a \in A\} \qquad \lambda = \{(a \wedge b, a) \mid a, b \in A\} \qquad \rho = \{(a \wedge b, b) \mid a, b \in A\}$$

are in R, and for all $r, s \in R$ the relation

$$\langle\!\langle r, s \rangle\!\rangle \;:=\; \wedge \circ (r \times s) \circ \delta_A \;=\; \{(a, b \wedge c) \mid (a, b) \in r, (a, c) \in s\}$$

is in R. □

Examples 3.3 (a) The canonical indexing of a preorder $(A, \leq)$ is an indexed meet-semilattice if and only if $(A, \leq)$ is an meet-semilattice if and only if the uniform preorder $(A, \downarrow\{\leq\})$ is cartesian. This follows since $\mathsf{Ord} \to \mathsf{UOrd}$ is 2-fully faithful and preserves finite 2-products.

(b) The primitive recursive functions $f : \mathbb{N} \to \mathbb{N}$ form a basis (Definition 1.4) for a uniform preorder structure on $\mathbb{N}$ which is cartesian: $\top$ is given by 0 (or any other number), and a meet operation $\wedge : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is given by any primitive recursive pairing function.

(c) Instead of primitive recursive function, we can use total recursive, or even partial recursive functions in the previous example. The last option gives an instance of the concept of *partial combinatory algebra*, to which we will come back later. ♢

Remark 3.4 The forgetful functor from cartesian uniform preorders to uniform preorders does not have a left biadjoint. This is because the meet-completion of an indexed preorder with generic predicate does generally not have a generic predicate. The situation is different for existential quantification, which we treat next. $\diamondsuit$

4 Existential quantification

Definition 4.1 (i) We say that an indexed preorder $\mathcal{H}$ *has existential quantification*, if for every function $u : J \to I$, the monotone map $u^* : \mathcal{H}(I) \to \mathcal{H}(J)$ has a left adjoint $\exists_u : \mathcal{H}(J) \to \mathcal{H}(I)$, and the *Beck–Chevalley condition* holds: for every pullback

$$\begin{array}{ccc} L & \xrightarrow{\bar{u}} & K \\ {\scriptstyle \bar{v}}\downarrow & \lrcorner & \downarrow{\scriptstyle v} \\ J & \xrightarrow{u} & I \end{array}$$

in Set we have $u^* \circ \exists_v \cong \exists_{\bar{v}} \circ \bar{u}^*$.

(ii) We say that an indexed monotone map $f : \mathcal{H} \to \mathcal{K}$ *commutes with existential quantification*, if $f_I \circ \exists_u \cong \exists_u \circ f_J$ for all $u : J \to I$.

We write $\exists$-IOrd for the sub-2-category of IOrd on indexed preorders with existential quantification, and indexed monotone maps commuting with existential quantification, and we write $\exists$-UOrd for the corresponding sub-2-category of UOrd, given by the following pullback.

$$\begin{array}{ccc} \exists\text{-}\mathsf{UOrd} & \longrightarrow & \exists\text{-}\mathsf{IOrd} \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{UOrd} & \xrightarrow{\mathsf{fam}} & \mathsf{IOrd} \end{array}$$

(iii) An indexed monotone map $f : \mathcal{A} \to \mathcal{H}$ from an indexed preorder $\mathcal{A}$ to an indexed preorder $\mathcal{H}$ with existential quantification is called an $\exists$-*completion*, if for all indexed preorders $\mathcal{K}$ with existential quantification, the precomposition map

$$(- \circ f) \;:\; \exists\text{-}\mathsf{IOrd}(\mathcal{H}, \mathcal{K}) \;\to\; \mathsf{IOrd}(\mathcal{A}, \mathcal{K})$$

is an equivalence of preorders.

(iv) Given a uniform preorder $\mathcal{H}$ with existential quantification, a predicate $\pi \in \mathcal{H}(I)$ is called $\exists$-*prime* if for all functions $I \xleftarrow{u} J \xleftarrow{v} K$ and predicates $\varphi \in \mathcal{H}(K)$ with $u^*\pi \leq \exists_v \varphi$, there exists a function $s : J \to K$ such that $v \circ s = \mathrm{id}_J$ and $u^*\pi \leq s^*\varphi$.

We write $\mathsf{prim}(\mathcal{H})$ for the indexed sub-preorder of $\mathcal{H}$ on $\exists$-prime predicates.

We say that $\mathcal{H}$ has *enough* $\exists$-prime predicates if for every set I and $\varphi \in \mathcal{H}(I)$ there exists a $u : J \to I$ and a $\pi \in \mathsf{prim}(\mathcal{H})(J)$ such that $\exists_u \pi \cong \varphi$. $\diamondsuit$

Remark 4.2 Using the fibrational—rather than the indexed—point of view, we can give the following characterization of $\exists$-prime predicates: $\pi \in \mathcal{H}(I)$ is $\exists$-prime iff for all $f : J \to I$, the object $(J, f^*\pi)$ of the total category $\int\mathcal{H}$ has the left lifting property w.r.t. cocartesian arrows. $\diamondsuit$

The notion of $\exists$-prime predicate gives rise to a sufficient criterion for an indexed preorder with existential quantification to be a $\exists$-completion.

Proposition 4.3 *Let $\mathcal{H}$ be an indexed preorder with existential quantification, and assume that $\mathcal{A} \subseteq \mathcal{H}$ is an indexed sub-preorder such that*

(i) *all predicates in $\mathcal{A}$ are $\exists$-prime in $\mathcal{H}$, and*

(ii) *for every set I and predicate $\varphi \in \mathcal{H}(I)$ there exists a function $u : J \to I$ and a predicate $\pi \in \mathcal{A}(J)$ such that $\varphi \cong \exists_u \pi$.*

Then the inclusion $\mathcal{A} \hookrightarrow \mathcal{H}$ is an $\exists$-completion, and moreover $\mathcal{A} \hookrightarrow \mathsf{prim}(\mathcal{H})$ is an equivalence, i.e. every $\exists$-prime predicate in $\mathcal{H}$ is isomorphic to one in $\mathcal{A}$. In particular, if $\mathcal{H}$ has enough $\exists$-prime predicates, then $\mathsf{prim}(\mathcal{H}) \hookrightarrow \mathcal{H}$ is an $\exists$-completion.

Proof. Given an indexed preorder $\mathcal{K}$ with existential quantification and an indexed monotone map $f : \mathcal{A} \to \mathcal{K}$, define $\tilde{f} : \mathcal{H} \to \mathcal{K}$ by $\tilde{f}_I(\varphi) = \exists_u f(\pi)$ for a choice of function $u : J \to I$ and predicate $\pi \in \mathcal{A}(J)$ with $\exists_u \pi \cong \varphi$. It is straightforward to verify that $\tilde{f}$ gives a well defined indexed monotone map commuting with existential quantification, and the assignment $f \mapsto \tilde{f}$ gives a pseudoinverse to the restriction map $\exists\text{-}\mathsf{IOrd}(\mathcal{H}, \mathcal{K}) \to \mathsf{IOrd}(\mathcal{A}, \mathcal{K})$.

Now assume that $\pi \in \mathsf{prim}(\mathcal{H})(I)$, and choose $u : J \to I$ and $\sigma \in \mathcal{A}(J)$ with $\exists_u \sigma \cong \pi$. Then from $\pi \leq \exists_u \sigma$ it follows that there exists a section s of u with $\pi \leq s^*\sigma$. On the other hand, the inequality $\exists_u \sigma \leq \pi$ is equivalent to $\sigma \leq u^*\pi$, which implies $s^*\sigma \leq \pi$ by applyinng s^* on both sides, and we conclude that $s^*\sigma \cong \pi$. $\square$

Definition 4.4 A *primal $\exists$-completion* is an $\exists$-completion $e : \mathcal{A} \to \mathcal{H}$ fitting the hypotheses of Proposition 4.3, i.e. $\mathcal{H}$ has enough $\exists$-primes and e is equivalent to $\mathsf{prim}(\mathcal{H}) \hookrightarrow \mathcal{H}$. $\diamondsuit$

It is well known that indexed preorders on *small* index categories $\mathbb{C}$ always admit primal $\exists$-completions[3]: given an indexed preorder $\mathcal{A} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Ord}$, predicates on $I \in \mathbb{C}$ in its $\exists$-completion $D\mathcal{A} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Ord}$ are given by pairs $(J \overset{u}{\to} I, \varphi \in \mathcal{A}(J))$, where $(J \overset{u}{\to} I, \phi) \leq (K \overset{v}{\to} I, \psi)$ iff there exists a $w : J \to K$ such that $v \circ w = u$ and $\varphi \leq w^*\psi$. However, for indexed preorders on Set this construction may not be well-defined, since the resulting indexed preorder may have large fibers. In the following we show that indexed preorders arising from uniform preorders *do* always admit primal $\exists$-completions, which are again representable by uniform preorders (the question if there are non-primal $\exists$-completions over Set remains open).

Definition 4.5 For (A, R) a uniform preorder, we define the uniform preorder

$$D(A, R) \;=\; (PA, DR)$$

where PA is the powerset of A, and DR is the uniform preorder structure on PA generated by the basis of relations

$$[r] \;=\; \{(U, V) \in PA \times PA \mid \forall a \in U\, \exists b \in V \,.\, (a, b) \in r\}$$

for $r \in R$. $\diamondsuit$

Remarks 4.6 (a) The relations $[r]$ do indeed constitute a basis since $\mathrm{id}_{PA} \subseteq [\mathrm{id}_A]$ and $[s] \circ [r] \subseteq [s \circ r]$ for $r, s \in R$.

[3] For accounts of closely related constructions see e.g. [Fre13, Definition 3.4.5] for the $\exists$-completion of fibered preorders satisfying a stack-condition, [Tro20, Section 4] for $\exists$-completion of indexed meet-semilattices, and Hofstra [Hof11, Section 3.2] for the analogous construction for non-posetal fibrations.

(b) Unwinding the definition of $D(A,R)$ we see that for $\varphi, \psi : I \to PA$ we have $\varphi \leq \psi$ in $\mathsf{fam}(D(A,R))(I)$ if and only if there exists an $r \in R$ such that

$$\forall i \in I\ \forall a \in \varphi(i)\ \exists b \in \psi(i)\,.\,(a,b) \in r. \qquad \diamondsuit$$

Proposition 4.7 *For every uniform preorder (A,R), the indexed preorder $\mathsf{fam}(D(A,R))$ has existential quantification and the singleton map $\eta : A \to PA$ is monotone from (A,R) to $D(A,R)$. The induced indexed monotone map $\mathsf{fam}(\eta) : \mathsf{fam}(A,R) \to \mathsf{fam}(D(A,R))$ is a primal $\exists$-completion.*

Proof. Existential quantification in $\mathsf{fam}(D(A,R))$ is given by union, i.e.

$$(\exists_u \varphi)(i) = \textstyle\bigcup_{u(j)=i} \varphi(j)$$

for $u : J \to I$ and $\varphi : J \to PA$, and η is monotone since for every $r \in R$ we have

$$\{(\{a\},\{a'\}) \mid (a,a') \in r\} \subseteq [r].$$

To show that $\mathsf{fam}(\eta)$ is a primal $\exists$-completion it remains to show that it is fiberwise order reflecting, and its image in $\mathsf{fam}(D(A,R))$—the indexed sub-preorder of *singleton-valued predicates*, i.e. predicates factoring through $\eta : A \to PA$—satisfies the hypotheses of Proposition 4.3.

The fact that $\mathsf{fam}(\eta)$ is order reflecting follows immediately from the explicit description of the fiberwise ordering in $\mathsf{fam}(D(A,R))$ in Remark 4.6(b).

To see that singleton-valued predicates are $\exists$-prime in $\mathsf{fam}(D(A,R))$, assume $\varphi : I \to A$, $\psi : J \to PA$, and $u : J \to I$ such that $\eta \circ \varphi \leq \exists_u \psi$. Unwinding definitions this means that there exists an $r \in R$ such that

$$\forall i \in I\ \forall a \in \{\varphi(i)\}\ \exists b \in \textstyle\bigcup_{u(j)=i} \psi(j)\ .\ (a,b) \in r\,,$$

i.e.

$$\forall i \in I\ \exists j \in J\ .\ u(j) = i \wedge \exists b \in \psi(j)\ .\ (\varphi(i),b) \in r\,,$$

and the required section of u is given by a Skolem function for the first two quantifiers.

Finally, $\mathsf{fam}(D(A,R))$ has 'enough' singleton-valued predicates, since every predicate $\varphi : I \to PA$ can be written as $\varphi = \exists_u \sigma$ for $J = \coprod_{i \in I} \varphi I$, u the first projection, and $\sigma = (J \xrightarrow{\pi_2} A \xrightarrow{\eta} PA)$. $\square$

Remark 4.8 The assignment $(A,R) \mapsto D(A,R)$ gives rise to a left 2-adjoint to the inclusion $\exists$-$\mathsf{UOrd} \to \mathsf{UOrd}$, and the unit η and multiplication μ of the induced 2-monad $D : \mathsf{UOrd} \to \mathsf{UOrd}$ are componentwise given by singleton map and union. The 2-monad is *lax idempotent*[4] in the sense that $D\eta_{(A,R)} \dashv \mu_{(A,R)} \dashv \eta_{D(A,R)}$ for all uniform preorders (A,R). In particular, a uniform preorder (A,R) is a D-algebra iff $\eta_{(A,R)}$ has a left adjoint (the adjunction is then automatically a reflection, since $\mathsf{fam}(\eta_{(A,R)})$ is fiberwise order-reflecting). Finally, the adjunction is monadic, since reflective indexed sub-preorders of indexed preorders with existential quantification have existential quantification. $\diamondsuit$

[4] Lax idempotent monads were introduced in [Zöb76, Koc95] and are also known as *Kock-Zöberlein monads*. (The articles were published 19 years apart, but Kock's preprint seems to have been contemporaneous with Zöberlein's thesis, on which his article is based. The name *lax idempotent* is due to Zöberlein and was later picked up by Kelly and Lack [KL97].)

5 Indexed frames

We recall the definition of indexed frames from [Fre23].

Definition 5.1 An *indexed frame* is an indexed meet-semilattice $\mathcal{H}$ which has existential quantification and moreover satisfies the *Frobenius condition*: for all functions $u : J \to I$, and predicates $\varphi \in \mathcal{H}_I$ and $\psi \in \mathcal{H}_J$ we have $\varphi \wedge \exists_u \psi \cong \exists_u(u^*\varphi \wedge \psi)$. $\diamondsuit$

Examples 5.2 (a) The canonical indexing of a poset $(A, \leq)$ is an indexed frame if and only if A is a frame [PP12], i.e. a complete lattice satisfying the infinitary distributive law $a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$.

(b) If $(L, \leq)$ is a frame and M is a monoid of frame-endomorphisms (i.e. monotone maps preserving finite meets and arbitrary joins), we obtain an indexed frame structure on the representable functor $\mathsf{Set}(-, L)$ by setting

$$\varphi \leq \psi \qquad \text{if and only if} \qquad \exists m \in M \; \forall i \in I \; . \; m(\varphi(i)) \leq \psi(i)$$

for $\varphi, \psi : I \to L$. This indexed frame structure is only representable by an ordinary frame if M has a least element (which is then an 'interior operator' i.e. a posetal comonad). A non-trivial example is the *Lipschitz hyperdoctrine* which has been recently proposed by Barton and Comelin, and is obtained by taking $L = ([0, \infty], \geq)$ and $M = \mathbb{R}_{>0}$ acting by multiplication. See also [FvdB22] for similar constructions of non-Set-based indexed preorders. $\diamondsuit$

Another way of producing indexed frames is given by the following.

Proposition 5.3 *If (A, R) is cartesian then so are $D(A, R)$ and $\eta : (A, R) \to D(A, R)$, and moreover $\mathsf{fam}(D(A, R))$ is an indexed frame.*

Proof. To show that $D(A, R)$ is cartesian we use Lemma 3.2 and define $\wedge : PA \times PA \to PA$ and $\top \in PA$ by $U \wedge V = \{a \wedge b \mid a \in U, b \in V\}$ and $\top = \{\top\}$. Then the verification of the conditions is straightforward. $\square$

6 Relational completeness

Definition 6.1 (i) We say that an indexed preorder *has universal quantification* if it satisfies the dual condition of Definition 4.1(i).

(ii) A *Heyting preorder*[5] is a meet-semilattice $(H, \leq)$ which is cartesian closed, i.e. for all $a \in H$ the monotone map $(- \wedge a)$ has a right adjoint $(a \Rightarrow -)$ called *Heyting implication*.

(iii) An *indexed meet-semilattice* $\mathcal{H}$ is said to *have implication* if its fibers are Heyting preorders, and this structure is preserved up to isomorphism by reindexing.

(iv) A *tripos* is an indexed meet-semilattice $\mathcal{P}$ which has universal quantification, implication, and a generic predicate.

Remarks 6.2 (a) Since they're assumed to have generic predicates, all triposes are representable by uniform preorders.

[5]Contrary to the better-known *Heyting algebras*, Heyting preorders need not have finite joins—those will turn out to exist in the cases we're interested in, but we don't have to assume them.

(b) As explained in [HJP80, Theorem 1.4], using Prawitz-style second order encodings [Pra65, page 67] one can show that triposes have existential quantification and fiberwise finite joins which are stable under reindexing. In other words, triposes are models of full first order logic. ◇

Definition 6.3 A cartesian uniform preorder (A, R) is called *relationally complete* if there exists a relation $@ \in R$ (called 'universal relation'), such that for every relation $r \in R$ there exists a *function* (i.e. a single-valued and entire relation) $\tilde{r} \in R$ with

$$r \circ \wedge \;\subseteq\; @ \circ \wedge \circ (\tilde{r} \times \mathrm{id}_A),$$

in other words

$$\forall a\, b\, c \in A \,.\, (a \wedge b, c) \in r \;\Rightarrow\; (\tilde{r}(a) \wedge b, c) \in @. \tag{6.1}$$ ◇

Remarks 6.4 (a) Relational completeness is called 'relational completeness' in [Fre13], and 'functional completeness' in [Fre19] for the special case of *DCOs* (Section 8). The present work uses the neutral 'combinatory completeness', to avoid having to change terminology when switching to a special case.

(b) Relational completeness can be viewed as a generalization of the functional completeness property of recursive functions expressed by the *s-m-n theorem*, which in its most basic form (see e.g. [Cut80, Theorem 4.4.1]) says that for every partial recursive function $f(x, y)$ in two arguments there exists a *total recursive* function $\tilde{f}(x)$ in one argument such that the partial functions $f(x, y)$ and $\phi_{\tilde{f}(x)}(y)$ are equal, where $(\phi_n)_{n\in\mathbb{N}}$ is a effective enumeration of partial recursive functions.

Note that besides using relations instead of partial functions, the statement above is somewhat weaker than that of the s-m-n theorem since *equality* of partial functions is replaced by *inclusion* of relations. See also Remark 9.3(b). ◇

Theorem 6.5 *The following are equivalent for a cartesian uniform preorder (A, R).*

(i) (A, R) *is relationally complete.*

(ii) $\mathsf{fam}(D(A, R))$ *is a tripos.*

Proof. Assume first that $\mathsf{fam}(D(A, R))$ is a tripos, and assume w.l.o.g. that conjunction is given 'on the nose' by the pointwise construction $U \wedge V = \{u \wedge v \mid u \in U, v \in V\}$ from the proof of Proposition 5.3. Let $E \hookrightarrow A \times A \times P(A \times A)$ be the membership relation, and define $u : E \to P(A \times A)$ and $\varphi, \psi : E \to PA$ by

$$u(b, c, s) = s \qquad \varphi(b, c, s) = \{b\} \qquad \psi(b, c, s) = \{c\}.$$

We set $\theta = \forall_u(\varphi \Rightarrow \psi) : P(A \times A) \to PA$ and let $@ \in R$ such that $[@]$ is a realizer of $u^*\theta \wedge \varphi \leq \psi$. Now for every $r \in R$ we construct a pullback

$$\begin{tikzcd} M \arrow[r, "x"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "u"] \\ A \arrow[r, "w"'] & P(A \times A) \end{tikzcd}
\qquad
\begin{aligned} M &= \{(a, b, c) \mid (a \wedge b, c) \in r\} \\ v(a, b, c) &= a \\ x(a, b, c) &= (r, b, c) \\ w(a) &= \{(b, c) \mid (a \wedge b, c) \in r\} \end{aligned}$$

and a simple argument using the Beck–Chevalley condition gives $\eta \leq w^*\theta$, where $\eta : A \to PA$ is the singleton map. Any $s \in R$ such that $[s]$ realizes this inequality is total, and using choice we pick $\tilde{r}$ to be a subfunction, so that $\forall a \in A \,.\, \tilde{r}(a) \in \theta(w(a))$. Implication (6.1) follows since $[@]$ is a realizer of the inequality $v^*w^*\theta \wedge x^*\varphi \leq x^*\psi$.

Conversely, assume that (A, R) is relationally complete. Instead of constructing implication and universal quantification separately, we show how to define the 'synthetic' connective $\forall_u(\varphi \Rightarrow \psi)$ for $u : J \to I$ and $\varphi, \psi \in \mathsf{fam}(D(A,R))(I)$. Implication and universal quantification can then be recovered by either replacing u by the identity, or φ by the true predicate. For $\varphi, \psi : J \to PA$ define $\forall_u(\varphi \Rightarrow \psi) : I \to PA$ by

$$\forall_u(\varphi \Rightarrow \psi)(i) = \bigcap_{uj=i} \{a \in A \mid \forall b \in \varphi(j)\, \exists c \in \psi(j) \,.\, @(a \wedge b, c)\}.$$

It is then easy to see that the inequality $u^*\forall_u(\varphi \Rightarrow \psi) \wedge \varphi \leq \psi$ is realized by $@$; and if $\zeta : I \to PA$ such that the inequality $u^*\xi \wedge \varphi \leq \psi$ is realized by $r \in R$, then $\tilde{r}$ realizes $\xi \leq \forall_u(\varphi \Rightarrow \psi)$. □

Remarks 6.6 (a) The list of equivalent statements in Theorem 6.5 can be extended by the following, where $\mathsf{Set}[\mathsf{fam}(D(A,R))]$ is the *category of partial equivalence relations and compatible functional relations*[6] in the fibered frame $\mathsf{fam}(D(A,R))$, and $\mathsf{PAsm}(A,R) = \int(\mathsf{fam}(A,R))$ is the total category of the indexed preorder $\mathsf{fam}(A,R)$ (which is the classical category of *partitioned assemblies* if (A,R) comes from a PCA):

(iii) $\mathsf{Set}[\mathsf{fam}(D(A,R))]$ is a topos.

(iv) $\mathsf{Set}[\mathsf{fam}(D(A,R))]$ is locally cartesian closed.

(v) $\mathsf{PAsm}(A,R)$ is weakly locally cartesian closed.

It is well known that (iii) follows from (ii): this is the reason for the term 'tripos-to-topos construction'. Clearly (iii) implies (iv). Next, (iv) implies (ii) since since every fibered frame $\mathcal{H}$ can be presented as

$$\mathcal{H} \;\simeq\; \left(\mathsf{Set}^{\mathsf{op}} \xrightarrow{\Delta^{\mathsf{op}}} \mathsf{Set}[\mathcal{H}]^{\mathsf{op}} \xrightarrow{\mathsf{sub}} \mathsf{Ord}\right)$$

where Δ is the constant-objects-functor and sub is the indexed preorder of subobjects. If $\mathsf{Set}[\mathcal{H}]$ is locally cartesian closed then $\mathcal{H}$ is an indexed Heyting algebra with $\forall$ and $\exists$ since sub is and this property is stable under precomposition with the finite-limit preserving functor Δ. In the case $\mathcal{H} = \mathsf{fam}(D(A,R))$ we furthermore have a generic predicate, so that $\mathsf{fam}(D(A,R))$ is a tripos.

Finally, the equivalence between (iv) and (v) follows from Carboni–Rosolini's characterization of locally cartesian closed exact completions ('the exact completion of a finite-limit category $\mathcal{C}$ is locally cartesian closed iff $\mathcal{C}$ is weakly locally cartesian closed', [CR00]), since $\mathsf{Set}[\mathsf{fam}(A,R)]$ is an ex/lex completion of $\mathsf{PAsm}(A,R)$ by means of the functor

$$\mathsf{PAsm}(A,R) \;\to\; \mathsf{Set}[\mathsf{fam}(D(A,R))]$$

[6] The construction of $\mathsf{Set}[\mathsf{fam}(D(A,R))]$ from $\mathsf{fam}(D(A,R))$ is called *exact completion of the 'existential elementary doctrine'* $\mathsf{fam}(D(A,R))$ e.g. in [MR12]. If $\mathsf{fam}(D(A,R))$ is a tripos, the construction is the well known *tripos-to-topos construction* [HJP80].

which sends $\varphi : I \to A$ to the sub-diagonal p.e.r. on I with support $I \xrightarrow{\varphi} A \hookrightarrow PA$: to verify this fact observe that the functor is fully faithful, and the objects in its image are projective and cover all other objects.

Analogous reformulations of relational completeness for *many-sorted* uniform preorders are given in [Fre13, Theorem 4.10.3].

(b) Theorem 6.5 gives rise to an correspondence between

- relational complete uniform preorders (A, R), and
- triposes $\mathcal{P}$ with enough $\exists$-primes, such that $\mathsf{prim}(\mathcal{P})$ has finite meets.

If (A, R) is relationally complete then $\mathsf{fam}(D(A, R))$ is such a tripos, and conversely if $\mathcal{P}$ is such a tripos, then any $\exists$-prime predicate which covers the generic predicate of $\mathcal{P}$ is generic in $\mathsf{prim}(\mathcal{P})$, whence the latter is representable by an uniform preorder (A, R), which is cartesian by assumption, and relationally complete by the theorem.

In type theoretic, 'univalent' language [Uni13] one would state this correspondence as an *equivalence* the *type* of relationally complete preorders and the *type* of the specified triposes. In classical foundations this translates into an equivalence of two 1-groupoids, which can both be realized as sub-groupoids of the *core*[7] of the hom-wise poset reflection IOrd. $\diamondsuit$

Examples 6.7 (a) For every meet-semilattice $(A, \leq)$, the uniform preorder $(A, R_{\leq})$ corresponding to its canonical indexing $\mathsf{fam}(A, \leq)$ is relationally complete. This is because $\mathsf{fam}(D(A, R_{\leq}))$ is equivalent to the canonical indexing of the frame of down-sets in $(A, \leq)$, and the latter is known to be a tripos.

(b) If (A, R) is an uniform preorder such that $\mathsf{fam}(A, R)$ is a tripos, then (A, R) is relationally complete, and thus $\mathsf{fam}(D(A, R))$ is a tripos as well. This is shown using variant of the construction in the proof of Theorem 6.5: we take $E \hookrightarrow A \times A \times P(A \times A)$ to be the membership relation as above, let $\varphi, \psi : E \to A$ and $u : E \to P(A \times A)$ be three projections, set $\theta = \forall_u(\varphi \Rightarrow \psi)$, and take @ to be a realizer of $u^*\theta \wedge \varphi \leq \psi$. Given $r \in R$ we again we construct the pullback

$$\begin{tikzcd} M \arrow[r, "x"] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "u"] \\ A \arrow[r, "w"'] & P(A \times A) \end{tikzcd} \qquad \begin{aligned} M &= \{(a, b, c) \mid (a \wedge b, c) \in r\} \\ v(a, b, c) &= a \\ x(a, b, c) &= (r, b, c) \\ w(a) &= \{(b, c) \mid (a \wedge b, c) \in r\}\,, \end{aligned}$$

and chasing around it we get $\mathrm{id}_A \leq w^*\theta$, i.e. $\theta \circ w \in R$, and we take this function to be $\tilde{r}$. The implication (6.1) follows since @ realizes $v^*w^*\theta \wedge x^*\varphi \leq x^*\psi$.

Since $\mathsf{fam}(A, R)$ is a tripos by assumption, (A, R) is a D-algebra, i.e. $\eta : (A, R) \to D(A, R)$ has a left adjoint $\alpha : D(A, R) \to (A, R)$ (see Remark 4.8), and it is easy to verify by hand that this left adjoint is cartesian. In other words, $\mathsf{fam}(A, R)$ is a *geometric subtripos* of $\mathsf{fam}(D(A, R))$, and this subtripos inclusion gives rise to a geometric *subtopos* inclusion $\mathsf{Set}[\mathsf{fam}(A, R)] \hookrightarrow \mathsf{Set}[\mathsf{fam}(D(A, R))]$ via the tripos-to-topos construction. The intermediate quasitopos of separated objects is the *q-topos* $\boldsymbol{Q}(\mathsf{fam}(A, R))$ associated to the tripos via the construction described in [Fre15, Definition 5.1]. We recall that the notion of q-topos is slightly weaker than that of quasitopos

[7] i.e. the subgroupoid of all isos

(not requiring coproducts or local cartesian closure), and was introduced in [Fre15] since the construction of $\boldsymbol{Q}(\mathcal{P})$ does not seem to produce a quasitopos over arbitrary base categories. However, the argument above shows that the construction *does* produce quasitoposes for Set-based triposes. $\diamondsuit$

Another large class of examples of relationally complete uniform preorders is given in the next section.

7 Ordered partially combinatory algebras

We recall the relevant definitions from [vO08, Section 2.6.5].

Definition 7.1 An *ordered applicative structrure* (OPAS) is a triple $(A, \leq, \cdot)$ where $(A, \leq)$ is a poset and $(- \cdot -) : A \times A \rightharpoonup A$ is a partial binary operation. $\diamondsuit$

Remarks 7.2 (a) Application associates to the left, i.e. $a{\cdot}b{\cdot}c$ is a shorthand for $(a{\cdot}b){\cdot}c$.

(b) A *polynomial* over an OPAS $(A, \leq, \cdot)$ is a term built up from variables, constants from A, and application $(- \cdot -)$. We write $p[x_1, \ldots, x_n]$ for a polynomial which may (but is not required to) contain the variables $x_1, \ldots, x_n$, and if $a_1, \ldots, a_n \in A$ we write $p[a_1, \ldots, a_n]$ for the possibly undefined result of substituting and evaluating.

(c) When reasoning with partial terms, $t{\downarrow}$ means that t is defined, and the statement of an equality $s = t$ or inequality $s \leq t$ contains the implicit assertion that both sides are defined. $\diamondsuit$

Proposition 7.3 *The following are equivalent for an OPAS $(A, \leq, \cdot)$.*

(i) *For all polynomials $p[x_1, \ldots, x_n, y]$ over A there exists an element $e \in A$ such that for all $a_1, \ldots, a_n, b \in A$:*

- $e{\cdot}a_1{\cdot}\ldots{\cdot}a_n{\downarrow}$
- $p[a_1, \ldots, a_n, b]{\downarrow}$ *implies* $e{\cdot}a_1{\cdot}\ldots{\cdot}a_n{\cdot}b{\downarrow}$ *and* $e{\cdot}a_1{\cdot}\ldots{\cdot}a_n{\cdot}b \leq p[a_1, \ldots, a_n, b]$

(ii) *there exist elements $\mathsf{k}, \mathsf{s} \in A$ such that for all $a, b, c \in A$:*

- $\mathsf{k}{\cdot}a{\cdot}b \leq a$
- $\mathsf{s}{\cdot}a{\cdot}b{\downarrow}$
- $a{\cdot}c{\cdot}(b{\cdot}c){\downarrow}$ *implies* $\mathsf{s}{\cdot}a{\cdot}b{\cdot}c{\downarrow}$ *and* $\mathsf{s}{\cdot}a{\cdot}b{\cdot}c \leq a{\cdot}c{\cdot}(b{\cdot}c)$

Proof. [vO08, Theorem 1.8.4] $\square$

Definition 7.4 (i) An *ordered combinatory algebra* (OPCA) is an OPAS satisfying the equivalent conditions of Proposition 7.3.

(ii) A *filter* on an OPCA is a subset $\Phi \subseteq A$ which is upward closed, closed under application, and contains choices of elements k, s as in Proposition 7.3(ii). A *filtered OPCA* is a quadruple $(A, \leq, \cdot, \Phi)$ where $(A, \leq, \cdot)$ is an OPCA and Φ is a filter on A. $\diamondsuit$

Given a filtered OPCA $(A, \leq, \cdot, \Phi)$ we define a strict indexed preorder structure on the representable presheaf $\mathsf{Set}(-, A)$ by setting

$$(\varphi : I \to A) \leq (\psi : I \to A) \qquad :\Leftrightarrow \qquad \exists e \in \Phi \, \forall i \in I \, . \, e{\cdot}\phi(i) \leq \psi(i)\,,$$

It follows from standard arguments in combinatory logic that this indexed preorder is well defined (i.e. reflexive and transitive), and actually an *indexed meet-semilattice*, and as Hofstra explains in [Hof06, p. 252], its $\exists$-completion is an ordered variant of a relative realizability construction and in particular a tripos. Thus, the corresponding uniform preorder (A, R_Φ) is relationally complete by Theorem 6.5.

The mapping from filtered OPCAs to uniform preorders factors through BCOs: the BCO corresponding to $(A, \leq, \cdot, \Phi)$ is given by $(A, \leq, \mathcal{F}_\Phi)$, where

$$\mathcal{F}_\Phi \;=\; \{(e{\cdot}-) : A \rightharpoonup A \mid e \in \Phi\}.$$

Thus, a basis for the uniform preorder structure R_Φ is given by $\{r_e \subseteq A \times A \mid e \in \Phi\}$, with $r_e = \{(a, b) \in A \times A \mid e{\cdot}a \leq b\}$.

In the following we describe the *discretely ordered* special case of this correspondence, which identifies filtered (better known as 'relative') PCAs with relationally complete *discrete combinatory objects*. Since discrete combinatory objects admit an easy characterization among indexed preorders, this enables us to give a characterization of (relative) realizability triposes.

8 Discreteness

Definition 8.1 (i) A *discrete combinatory object (DCO)* is a uniform preorder where all relations $r \in R$ are *single-valued*, i.e. partial functions. We write DCO for the full locally ordered subcategory of UOrd on DCOs.

(ii) A predicate $\delta \in \mathcal{A}(I)$ of an indexed preorder $\mathcal{A}$ is called *discrete* if for every surjection $e : K \twoheadrightarrow J$, function $f : K \to I$, and predicate $\varphi \in \mathcal{A}(J)$ such that $e^*\varphi \leq f^*\delta$, there exists a (necessarily unique) $g : J \to I$ with $g \circ e = f$ (and therefore $\varphi \leq g^*\delta$ since reindexing along split epis is order-reflecting). $\diamondsuit$

Remarks 8.2 (a) DCOs were introduced in [Fre19, Definition 2.2] in terms of bases, i.e. as sets A equipped with a set $\mathcal{F}$ of partial endofunctions containing the identify and weakly closed under composition in the sense that for all $f, g \in \mathcal{F}$ there exists an $h \in \mathcal{F}$ such that $g \circ f \subseteq h$. Down-closure in $P(A \times A)$ of such a structure yields a DCO $(A, {\downarrow}\mathcal{F})$ in the above sense inducing the same indexed preorder and the two definitions give rise to equivalent locally ordered categories, the principal difference being that for the above, 'saturated' definition, the 2-functor $\mathsf{DCO} \to \mathsf{IOrd}$ is injective on objects.

(b) In fibrational language, discreteness of $\delta \in \mathcal{A}(A)$ says that (A, δ) has the right lifting property in the total category $\int \mathcal{A}$ w.r.t. all cartesian maps over surjections.

(c) It is easy to see that reindexings of discrete predicates along injections are discrete again. Reindexings along surjections, on the other hand, are discrete only in the trivial case that the surjection is a bijection.

(d) DCOs embed into BCOs: modulo the issue of bases vs. saturated presentations discussed in (a), they correspond precisely to BCOs whose order structure is trivial. Thus, we can extend the sequence (1.2) of embeddings to the following diagram.

$$\begin{tikzcd}
\mathsf{Set} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{DCO} \arrow[d] & & & \\
\mathsf{Ord} \arrow[r] & \mathsf{BCO} \arrow[r] & \mathsf{UOrd} \arrow[r] & {[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]} \arrow[r] & \mathsf{IOrd}
\end{tikzcd}$$

The intersection of Ord and DCO is trivial, in the sense that it only contains discretely ordered representable presheaves: this is because indexed preorders representable by ordinary preorders are stacks for the canonical topology, and if $\mathsf{fam}(A, R)$ is such a stack for a DCO (A, R), then R contains only subfunctions of id_A (otherwise, the stack condition would give $(a, a) \le (a, f(a))$ over 2). ♢

The following clarifies the relationship between the two notions of discreteness introduced in Definition 8.1.

Proposition 8.3 *A uniform preorder (A, R) is a DCO if and only if the generic predicate $\mathrm{id}_A \in \mathsf{fam}(A, R)(A)$ is discrete.*

Proof. Assume first that (A, R) is a DCO and consider a span $J \xleftarrow{e} K \xrightarrow{f} A$ with e surjective, and a predicate $\varphi : J \to A$ with $e^*\varphi \le f^*\mathrm{id}_A$. Form the image factorization (1) of $\langle \varphi \circ e, f\rangle$.

$$(1)\quad \begin{tikzcd}
K \arrow[r, two heads, "h"] \arrow[dr, "\langle \varphi\circ e, f\rangle"'] & r \arrow[d, hook, "\langle p,q\rangle"] \\
& A \times A
\end{tikzcd}
\qquad\qquad
(2)\quad \begin{tikzcd}
K \arrow[r, two heads, "h"] \arrow[d, two heads, "e"'] & r \arrow[d, tail, "p"] \\
J \arrow[r, "\varphi"'] \arrow[ur, dashed, "k"] & A
\end{tikzcd}$$

Then $r \in R$ and therefore p is injective since (A, R) is a DCO. Since e is surjective we obtain a lifting k in the square (2) and the desired map is $q \circ k$.

Conversely assume that id_A is discrete, let $r \in R$, write $\langle p, q\rangle : r \hookrightarrow A \times A$ for the inclusion, and let $r \overset{e}{\twoheadrightarrow} U \overset{m}{\hookrightarrow} A$ be an image factorization of p. We have $p^*(\mathrm{id}_A) = e^*(m^*(\mathrm{id}_A)) \le q^*(\mathrm{id}_A)$, and discreteness of id_A implies that there exists $g : U \to A$ with $g \circ e = q$. We obtain a factorization $\langle p, q\rangle = \langle m, g\rangle \circ e$, and since $\langle p, q\rangle$ is injective we conclude that e is bijective and thus r is single-valued. □

Corollary 8.4 *An indexed preorder $\mathcal{A}$ is representable by a DCO if and only if it has a discrete generic predicate.*

Proof. This follows from Proposition 8.3 together with Lemma 1.6. A direct proof is given in [Fre19, Theorem 2.4]. □

Remark 8.5 It is possible that the same indexed preorder has discrete and non-discrete generic predicates: if $\mathcal{A}$ is an indexed preorder with discrete generic predicate $\iota \in \mathcal{A}(A)$ and $f : B \twoheadrightarrow A$ is a surjection, then f^*A is a generic predicate which is discrete only if f is a bijection. If f is not a bijection, we obtain a DCO-representation of $\mathcal{A}$ with underlying set A, and a representation as a non-discrete uniform preorder with underlying set B. ♢

Remark 8.6 (Cartesian DCOs) If a cartesian uniform preorder (A, R) is a DCO, then the relations $\lambda, \rho \in R$ from Lemma 2.1 are partial functions, and jointly form a retraction $\langle \lambda, \rho\rangle : A \rightharpoonup A \times A$ of $\wedge : A \times A \to A$, i.e. we have $\langle \lambda, \rho\rangle \circ \wedge = \mathrm{id}_{A\times A}$. Moveover, although we don't have $\wedge \circ \langle \lambda, \rho\rangle = \mathrm{id}_A$, we have an inclusion $\wedge \circ \langle \lambda, \rho\rangle \subseteq \mathrm{id}_A$ of partial functions, since by construction λ an ρ are only defined on the range of $\wedge$.

More generally we define n-ary versions

$$\wedge^{(n)} : A^n \to A \quad \text{for} \quad n \in \mathbb{N} \qquad \text{and} \qquad \pi_i^{(n)} \in R \quad \text{for} \quad 1 \leq i \leq n$$

by $\wedge^{(0)}(*) = \top$, $\wedge^{(n+1)}(\vec{a}, b) = \wedge^{(n)}(\vec{a}) \wedge b$, and $\pi_i^{(n)} = \rho \circ \lambda_i^n$, so that we have

$$\langle \pi_1^{(n)}, \ldots, \pi_n^{(n)}\rangle \circ \wedge^{(n)} = \mathrm{id}_{A^n} \qquad \text{and} \qquad \wedge^{(n)} \circ \langle \pi_1^{(n)}, \ldots, \pi_n^{(n)}\rangle \subseteq \mathrm{id}_A$$

for all $n \in \mathbb{N}$. Loosely following Hofstra [Hof06, pg. 254], we introduce the notation

$$\begin{aligned} R^{(n)} &= \{r \subseteq A^n \times A \mid \exists s \in R \,.\, r = s \circ \wedge^{(n)}\} \\ &= \{r \subseteq A^n \times A \mid r \circ \langle \pi_1^{(n)}, \ldots, \pi_n^{(n)}\rangle \in R\} \end{aligned}$$

for 'n-ary computable' functions, which can be viewed as representing 'multi-inequalities' $\varphi_1, \ldots, \varphi_n \leq \psi$ matching the form of intuitionistic sequents. A paradigmatic example is given by the DCO of subrecursive functions (Example 3.3(c)): here $R^{(n)}$ contains precisely the *n-ary partial sub-recursive functions*, i.e. sub-functions of n-ary partial recursive functions in the usual sense. $\diamondsuit$

9 Partial combinatory algebras

Partial combinatory algebras can be viewed as trivially ordered OPCAs, but there is a slight mismatch with the traditional definition of PCA which we address — following Streicher [Str17] — by introducing the term of *weak* PCA.

Definition 9.1 (i) A *weak partial combinatory algebra* (weak PCA) is a discretely ordered OPCA, i.e. a pair $(A, \cdot)$ such that $(A, =, \cdot)$ is an OPCA.

(ii) A *partial combinatory algebra* (PCA) is a weak PCA in which the element s from Proposition 7.3(ii) can be chosen such that $\mathsf{s}{\cdot}a{\cdot}b{\cdot}c{\downarrow}$ (if and) only if $a{\cdot}c{\cdot}(b{\cdot}c){\downarrow}$. $\diamondsuit$

There are obvious 'filtered' versions of these definitions, for which we use the adjecrive 'relative' as is more common in the unordered case.

Definition 9.2 (i) A *weak relative PCA* is a triple $(A, \cdot, A_\#)$ where $(A, \cdot)$ is a PCA and $A_\# \subseteq A$ is a filter in the sense of Definition 7.4(ii).

(ii) A *relative PCA* is a weak relative PCA in which the $\mathsf{s} \in A_\#$ can be chosen to satisfy the stronger condition of Definition 9.1(ii). $\diamondsuit$

Remarks 9.3 (a) Relative PCAs are called *elementary inclusions of PCAs* in [vO08, Sections 2.6.9 and 4.5]

(b) Faber and van Oosten showed that for every weak PCA $(A, \cdot)$ there is a PCA $(A, *)$ such that inducing the same indexed preorder structure on $\mathsf{Set}(-, A)$ and thus the same uniform preorder structure A (strictly speaking their result is phrased in terms of *applicative morphisms*, but the statement about indexed preorders is an easy consequence) [FvO16, Theorem 5.1]. Their argument generalizes easily to relative PCAs.

Specializing the constructions from Section 7, every relative (weak) PCA $(A, \cdot, A_\#)$ gives rise to a relationally complete DCO $(A, R_{A_\#})$ with a basis given by $\{(e\cdot-) : A \rightharpoonup A \mid e \in A_\#\}$. Thus, the fiberwise ordering of the $\exists$-completion $\mathsf{fam}(D(A, R_{A_\#}))$ is given by

$$(\varphi : I \to A) \leq (\psi : I \to A) \qquad \text{iff} \qquad \exists e \in A_\# \; \forall i \in I \; \forall a \in \varphi(i) \; . \; e\cdot a \in \psi(i)$$

and we recognize at once that this is the *relative realizability tripos* over $(A, \cdot, A_\#)$ [vO08, Section 2.6.9].

In the following we sketch the argument that *every* relationally complete DCO arises from a relative PCA this way. To start, given a relationally complete DCO (A, R) with @ (which we call *generic function* in the discrete case), we define $(- \cdot -) : A \times A \rightharpoonup A$ by $a\cdot b = @(a \wedge b)$ and $A_\# \subseteq A$ by

$$A_\# \;:=\; \{a \in A \mid \{(\top, a)\} \in R\} \;=\; \{a \in A \mid \top \leq a \text{ in } \mathsf{fam}(A, R)(1)\}.$$

Note that the elements of $A_\#$ correspond to Hofstra's *designated truth values* [Hof06, pg. 244]. If $a, b \in A_\#$ such that $a\cdot b = @(a \wedge b)$ is defined, then $a\cdot b \in A_\#$ since $\top \leq a$ and $\top \leq b$ implies $\top \leq a \wedge b$; and $a \wedge b \leq @(a \wedge b)$, i.e. $A_\#$ is closed under application in A.

Proposition 9.4 *Let (A, R) be a relationally complete cartesian DCO.*

(i) *For every n-ary polynomial $p[x_1, \ldots, x_n]$ over the partial applicative structure $(A, \cdot, A_\#)$ with coefficients in $A_\#$, the partial evaluation function $\vec{a} \mapsto p[\vec{a}]$ is in $R^{(n)}$ (see Remark 8.6).*

(ii) *For all $n \in \mathbb{N}$ and $r \in R^{(n+1)}$ there exists an $e \in A_\#$ such that for all $a_1, \ldots, a_n, b \in A$,*

- *$e\cdot a_1\cdot \ldots \cdot a_n\downarrow$, and*
- *$r(a_1, \ldots, a_n, b) = e\cdot a_1\cdot \ldots \cdot a_n\cdot b$ whenever $r(a_1, \ldots, a_n, b)\downarrow$.*

(iii) *$(A, \cdot, A_\#)$ is a weak relative PCA, and the induced relationally complete DCO $(A, \downarrow\mathcal{F}_{A_\#})$ is equal to (A, R).*

Proof. This is proved in [Fre19, Lemma 2.14] for the non-relative case, and the generalization to the relative case is straightforward. Hofstra proved analogous statements for BCOs and filtered OPCAs in [Hof06, Section 6]. $\square$

Theorem 9.5 *The following are equivalent for a tripos $\mathcal{P}$.*

(i) *$\mathcal{P}$ is equivalent to a relative realizability tripos over a relative PCA.*

(ii) *$\mathcal{P}$ has enough $\exists$-prime predicates, and $\mathsf{prim}(\mathcal{P})$ has finite meets and a discrete generic predicate.*

Proof. Assume first that $\mathcal{P} = \mathsf{fam}(D(A, R_{A_\#}))$ for a relative PCA $(A, \cdot, A_\#)$. Then Proposition 4.7 implies that $\mathcal{P}$ has enough $\exists$-primes and $\mathcal{P} \simeq \mathsf{fam}(A, \downarrow\mathcal{F}_{A_\#})$. We have established in Section 7 that $\mathsf{fam}(A, \downarrow\mathcal{F}_{A_\#})$ is an indexed meet-semilattice. and, it has a discrete generic predicate by Proposition 8.3.

Conversely, assume (ii). Then $\mathsf{prim}(\mathcal{P}) \hookrightarrow \mathcal{P}$ is an $\exists$-completion by Proposition 4.3, and $\mathsf{prim}(\mathcal{P})$ is representable by a relative DCO (A, R) by Corollary 8.4. The DCO (A, R) is cartesian since $\mathsf{prim}(\mathcal{P})$ has finite meets, and relationally complete since its $\exists$-completion is a tripos. Thus, it comes from a weak relative PCA by Proposition 9.4, and from a relative PCA by Remark 9.3(b). $\square$

Remark 9.6 Theorem 9.5 specializes to a characterization of *non-relative* realizability triposes by adding the condition that $\mathcal{P}$ is *two-valued*, i.e. $\mathcal{P}(1) \simeq \{\perp < \top\}$. This is equivalent to, $\mathsf{prim}(\mathcal{P})(1) \simeq 1$, a property that is called 'shallow' in [Fre19]. $\diamondsuit$

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