# Representable and definable fibered functors

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**Abstract**

This note discusses the relationship between Swan's theory of *definability* [Swa22] (which in turns relates Bénabou's *definability* [Bé85, §7], Johnstone's *comprehensibility* [Joh02, page 272], and Shulman's *local representability* [Shu19, Definition 3.10]) and the notion of *representable discrete opfibration* which is central in Zwanziger's take [Zwa22, Definition 3.2.2] on Awodey's notion of *natural model of type theory* [Awo18], in a fibrational framework.

## 1 Introduction

We start by reviewing Zwanziger's and Swan's definitions.

### 1.1 Full natural models

Zwanziger's notion of *full natural model*[1] is a variation of Awodey's notion of natural model which is based not on a $\mathsf{Set}$-valued presheaf of types, but on a $\mathsf{Cat}$-valued presheaf which fully embeds into the codomain functor, in analogy to the distinction between CwA and 'full split comprehension category'.

| natural model | full natural model |
|---|---|
| CwA | full split comprehension category |

**Definition 1** A *full natural model* on a category $\mathbb{C}$ is a 2-natural transformation

$$p \;:\; \mathsf{Tm} \to \mathsf{Ty} \;:\; \mathbb{C}^{\mathsf{op}} \to \mathsf{Cat}$$

between strict presheaves $\mathsf{Ty}, \mathsf{Tm}$ of categories, such that

1. for all $\Gamma \in \mathbb{C}$, the functor $p_\Gamma : \mathsf{Tm}(\Gamma) \to \mathsf{Ty}(\Gamma)$ is an opfibration (so that we may call $p$ a *strict indexed opfibration*), and
2. for all $\Gamma \in \mathbb{C}$ and $A \in \mathsf{Ty}(\Gamma)$, the left upper corner in the 2-pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\hat{A}^* p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "p"] \\ \text{よ}(\Gamma) \arrow[r, "\hat{A}"] & \mathsf{Ty} \end{tikzcd}$$

   is representable, where $\hat{A} : \text{よ}(\Gamma) \to \mathsf{Ty}$ is the unique 2-natural transformation with $\hat{A}_\Gamma(\mathrm{id}_\Gamma) = A$. ◇

[1] In his thesis he just calls them 'natural models', but I prefer the term 'full natural model' that he used earlier.

**Terminology 2** We call 2-natural transformation satisfying condition 1 a *strict indexed opfibration*, and we call it *representable* if it also satisfies condition 2. Thus, a full natural model is the same thing as a representable strict indexed opfibration. $\diamondsuit$

**Remarks 3**
1. Everything in the definition is 'as strict as possible', in particular we're talking about 2-functors, 2-natural transformations, and 2-pullbacks as opposed to the pseudo-versions.

2. To get an actual model of type theory, we have to fix for every $\Gamma$ and $A$ a *representation* $p_A : \Gamma.A \to \Gamma$ of $\hat{A}^*p$.

3. Zwanziger formulated his definition [Zwa22, Definition 3.2.2] relative to a class of *display maps*, which we omit here. (However, every (full) natural model gives rise to a class of display maps by repletion of the class of context projections $p_A : \Gamma.A \to \Gamma$.) $\diamondsuit$

**Example 4 (Full natural models vs FSCCs)** A full split comprehension category (FSCC) consists of a strict presheaf $\mathsf{Ty} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Cat}$ of categories and a $\mathbb{C}$-fibered (a.k.a. cartesian) functor $p : \int \mathsf{Ty} \to \mathbb{C}^2$

$$\begin{tikzcd} \int \mathsf{Ty} \arrow[r, "p"] \arrow[dr] & \mathbb{C}^2 \arrow[d, "\mathsf{cod}"] \\ & \mathbb{C} \end{tikzcd}$$

mapping $A \in \mathsf{Ty}(\Gamma)$ to $p_A : \Gamma.A \to \Gamma$, where we call $\Gamma.A$ the *context extension* (of $\Gamma$ by $A$) and $p_A$ the *context projection*.

To get a full natural model from a FSCC we define $\mathsf{Tm} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Cat}$ by taking for $\mathsf{Tm}(\Gamma)$ the category whose objects are 'pointed types', i.e. pairs $(A \in \mathsf{Ty}(\Gamma), s : \Gamma \to \Gamma.A)$ with $p_A \circ s = \mathrm{id}_\Gamma$, and whose morphisms of type $(A, s) \to (B, t)$ are arrows $f : A \to B$ such that $p_B \circ \Gamma_f = p_A$ and $\Gamma_f \circ s = t$.

Conversely, given a full natural model, it is easy to see that any choice of representations of pullbacks as in Remark 3.2 determines a context extension functor $p : \int \mathsf{Ty} \to \mathbb{C}^2$. $\diamondsuit$

### 1.2 Definable notions of structure

Definable notions of structure are formulated in terms of Grothendieck fibrations. We start by fixing notation and terminology.

**Notation 5** We write $\mathsf{Fib}(\mathbb{B})$ for the category of fibrations over $\mathbb{B}$. For $\mathcal{C} \in \mathsf{Fib}(\mathbb{B})$ we write $\int \mathcal{C}$ for its *total category*, i.e. its domain. Furthermore, we write $\mathcal{C}_I$ for the fiber of $\mathcal{C}$ over $I \in \mathbb{B}$, and $f^* : \mathcal{C}_I \to \mathcal{C}_J$ for 'the' reindexing functor along $f : I \to J$ (given a choice of cartesian lifts). We use the arrow symbol $\rightsquigarrow$ for cartesian arrows, and $\mapsto$ for vertical arrows. Given a fibration $\mathcal{C}$ we write $\mathsf{core}(\mathcal{C})$ for the wide subfibration on cartesian arrows, which makes sense since the fibers of $\mathsf{core}(\mathcal{C})$ are given by 'category-theoretic' core of the fibers, i.e. the groupoids of isomorphisms. $\diamondsuit$

**Definition 6 ([Swa22, 3.4])** A *definable notion of structure* is a fibered functor $F : \mathcal{C} \to \mathcal{D}$ between Grothendieck fibrations, such that

1. $\mathsf{core}(F) : \mathsf{core}(\mathcal{C}) \to \mathsf{core}(\mathcal{D})$ is a discrete fibration, and
2. $\int \mathsf{core}(F) : \int \mathsf{core}(\mathcal{C}) \to \int \mathsf{core}(\mathcal{D})$ has a (non-fibered right adjoint). $\diamondsuit$

**Remarks 7** 1. $\mathsf{core}(F)$ is a discrete fibration iff for every object $C \in \int \mathcal{C}$ and cartesian arrow $d : D \rightsquigarrow FC$ there exists a unique cartesian arrow $c : d^*(C) \to C$ with $F(c) = d$. Swan refers to this property by saying that $F$ 'creates cartesian lifts'. $\diamondsuit$

### 1.3 Comparing and generalizing

In the following we examine non-strict generalizations of the above definitions, and show that

1. every full natural model is a representable fibered notion of structure (this is relatively easy to see and was obvious to Zwanziger), and
2. a fibered discrete opfibration is representable iff its total functor has a right adjoint (without taking cores).

## Representable fibered functors

**Proposition 8** *(Compare [Str18, Theorem 9.1]) Let $\mathcal{C} \in \mathsf{Fib}(\mathbb{B})$.*

1. *$\mathcal{C}$ is faithful iff its fibers $\mathcal{C}_I$ are preorders. In this case we call $\mathcal{C}$ a* fibered preorder*.*
2. *$\mathcal{C}$ is conservative iff its fibers are groupoids iff all arrows in $\int \mathcal{C}$ are cartesian. ('fibered groupoid').*
3. *$\mathcal{C}$ is faithful and conservative iff its fibers are equivalence relations ('elementary fibration in [Str18]').*
4. *$\mathcal{C}$ is amnestic iff its fibers are 'gaunt', i.e. the only isos are identities.*
5. *$\mathcal{C}$ is amnestic and faithful iff its fibers are posets ('fibered poset').*
6. *$\mathcal{C}$ is amnestic and conservative iff its fibers are sets ('discrete fibration').* ■

**Remarks 9** 1. The distinction between elementary and discrete fibrations, and between fibered preorders and fibered posets, should be viewed as purely technical. Morally they represent the same concepts. In particular there is no difference in a univalent metatheory.

2. Note that every amnestic and conservative fibration is automatically faithful, since every gaunt groupoid is a set and therefore a preorder.
3. Amnestic functors were introduced in [AHS90], their relevance to homotopy type theory was recognized in [AL19, Prop. 7.12]: if $\mathbb{C}$ is univalent and $F : \mathbb{C} \to \mathbb{D}$ is a functor then $\mathbb{C}$ is univalent iff $F$ is amnestic. (This is a simplification, look in the article for the precise statement).

4. It seems like we're missing an invariant version of amnestic, which would be "faithful on isos". Does a name exist for this? $\diamondsuit$

**Proposition 10** *Let* $F : \mathcal{C} \to \mathcal{D}$ *in* $\mathsf{Fib}(\mathbb{B})$.

1. *The functor* $F : \int\mathcal{C} \to \int\mathcal{D}$ *is a fibration iff all* $F_I : \mathcal{C}_I \to \mathcal{D}_I$ *are fibrations. In this case call* $F$ *a* fibered fibration.

2. $F$ *admits liftings of cartesian arrows, i.e. for every* $C \in \int\mathcal{C}$ *and cartesian* $d : D \to FC$ *there exists a cartesian* $c : C' \to C$ *with* $F(c) = d$, *iff all its fibers are isofibrations. In this case call* $F$ *a* fibered isofibration.

3. $F$ *is fiberwise conservative iff it is conservative iff it reflects cartesian arrows.*

4. $F$ *is faithful iff it is fiberwise faithful.* $\blacksquare$

**Example 11** If $\mathcal{D}$ is amnestic (e.g. a fibered poset or discrete fibration) then every fibered functor $F : \mathcal{C} \to \mathcal{D}$ is a fibered isofibration – and thus admits liftings of cartesian arrows – since every functor into a gaunt category is an isofibration. $\diamondsuit$

**Definition 12** (i) A fibration is called representable if it is equivalent to one of the form $よ(I) = (\mathbb{B}/I \to \mathbb{B})$ in $\mathsf{Fib}(\mathbb{B})$.

(ii) A fibered functor $F : P \to Q$ is called representable if in every pseudopullback

$$\begin{tikzcd} \mathcal{I} \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{J} \arrow[r] & \mathcal{D} \end{tikzcd}$$

in $\mathsf{Fib}(\mathbb{B})$[2], $\mathcal{I}$ is representable whenever $\mathcal{J}$ is representable. $\diamondsuit$

A first intuition about representable fibered functors is that they have representable – and thus in particular discrete fibers. However, the kind of fibers we're looking at are not sufficient to get a good understanding of $F$ – for example, representable fibered functors need not be faithful: the functor $F : (\bullet \rightrightarrows \bullet) \to (\bullet \to \bullet)$ is representable as a fibered functor over 1, but not faithful. Are they always conservative?

**Remarks 13** 1. A fibration is representable if and only if it is conservative and its total category has a terminal object.

2. In general, representability of a fibration is *not* equivalent to representability of its terminal projection as a fibered functor. However, if $\mathbb{B}$ has binary products then the former implies the latter and if $\mathbb{B}$ has a terminal object then the formal implies the latter. Thus, they *are* equivalent if $\mathbb{B}$ has finite products.

3. If $F$ is a fibered isofibration, then the definition of representability can equivalently be stated using strict pullbacks. This is in particular $\diamondsuit$

[2] Note that 'pseudopullback in $\mathsf{Fib}(\mathbb{B})$' means 'fiberwise pseudopullback', which is different from pseudopullback of total categories.

The equivalence $\mathcal{D}_I \cong \mathsf{Fib}(\mathbb{B})(よ(I), \mathcal{D})$ involves a choice of cartesian lifts, but it turns out that we can find a representation of the fibration $\mathcal{P}$ in the pullback

$$\begin{tikzcd} F_D \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathcal{C} \arrow[d, "F"] \\ よ(I) \arrow[r, "D"'] & \mathcal{D} \end{tikzcd}$$

without making such choices.

**Proposition 14** *Let $G : よ(I) \to \mathcal{D}$ be a fibered functor with $G(1_I) = D \in \mathcal{D}_I$. Then the fibration $F_D$ in the diagram above is equivalent to the fibration whose objects over $J$ are pairs $(C \in \mathcal{C}_J, f : FC \rightsquigarrow D)$ and whose objects are arrows in $C$ making the triangle commute. In other words, it's the full subcategory of the non-fibered comma category $F/D$ on cartesian arrows.*

**Proposition 15** *Representable fibered functors are conservative.*

*Proof.* Assume that $f : C' \to C$ in $\mathcal{C}$ such that $F(f)$ is cartesian. Then $f$ constitutes a morphism from $Ff$ to $\mathrm{id}_{FC}$ in $F_{FC}$, which is cartesian by representability of $F_D$.

**Definition 16** ([Joh02, Swa22]) A fibered functor $F : \mathcal{C} \to \mathcal{D}$ is called *definable*, if the fiberwise core $\mathsf{Core}(F) : \mathsf{Core}(\mathcal{C}) \to \mathsf{Core}(\mathcal{D})$ has a (non-fibered) right adjoint. This is eqivalent to the fibers of the core being representable. ♢

**Lemma 17** *$F$ is representable iff it is definable and conservative.*

*Proof.* Because for a conservative functor, the fibers are the same as the core-fibers. ∎

### Fibered elementary opfibrations

The following is a variant of the *indexed discrete opfibrations* considered by Zwanziger in his analysis of models of type theory [Zwa22]. See also [CMMV20].

**Definition 18** A *fibered elementary opfibration* is a fibered functor whose fibers are elementary opfibrations. ♢

Zwanziger calls a discrete

**Proposition 19** *Fibered elementary opfibrations are definable if and only if the total functor has a right adjoint.*

*Proof.* Assume that $F : \mathcal{C} \to \mathcal{D}$ is a definable fibered elementary opfibration. We have to show that for every $D \in \int\mathcal{D}$, the comma category $F/D$ has a terminal object, and by definability and conservativity of $F$ we know that its full subcategory $F_D$ on cartesian arrows has a terminal object $(T, t : FT \rightsquigarrow D)$. We show that this is also terminal in the larger category.

Consider an object $(C, f : FC \to D) \in F/D$, and let $FC \overset{v}{\rightarrowtriangle} D' \overset{c}{\rightsquigarrow} D$ be a vertical/cartesian factorization of $f$. Then $C$ lifts along $v$ since $F$ is a fiberwise opfibration, i.e. there's an arrow $w : C \to C'$ in $\int\mathcal{C}$ with $F(w) = v$. The pair $(C', c)$ constitutes an object in $F_D$ and since $(T, t)$ is terminal, there exists a unique (cartesian) $d : C' \to T$ with $t \circ d = c$.

$$\begin{tikzcd}
C \arrow[d, "w"'] & FC \arrow[d, "v"'] \arrow[dr, "f"] & \\
C' \arrow[d, squiggly, "d"'] & FC' \arrow[r, squiggly, "c"] \arrow[d, squiggly] & D \\
T & FT \arrow[ur, squiggly, "t"'] &
\end{tikzcd}$$

Then $dw$ constitutes a map from $(C, f)$ to $(T, t)$ in $F/D$.

It remains to show that this is uniqueness. For this, assume $h, k : C \to T$ with $t \circ F(h) = t \circ F(h) = f$, i.e. $h$ and $k$ are both morphisms from $(C, f)$ to $(T, t)$ in $F/D$. We form vertical/cartesian factorizations $h = (C \overset{v}{\rightarrowtriangle} C' \overset{c}{\rightsquigarrow} T)$ and $k = (C \overset{w}{\rightarrowtriangle} C'' \overset{d}{\rightsquigarrow} T)$. Then

$$\begin{aligned}
\mathcal{D}(t) \circ \mathcal{D}(F(c)) &= \mathcal{D}(t) \circ \mathcal{D}(F(c)) \circ \mathcal{D}(F(v)) \\
&= \mathcal{D}(f) \\
&= \mathcal{D}(t) \circ \mathcal{D}(F(d)) \circ \mathcal{D}(F(w)) \\
&= \mathcal{D}(t) \circ \mathcal{D}(F(d))
\end{aligned}$$

i.e. $t \circ F(c)$ and $t \circ F(d)$ are 'codirectional'[3] cartesian arrows in $\int \mathcal{D}$ with the same codomain. Thus, there's a vertical isomorphism $j : FC' \overset{\cong}{\to} FC''$ with $t \circ F(d) \circ j = t \circ F(c)$.

$$\begin{tikzcd}
& C \arrow[dl, "v"'] \arrow[dr, "w"] & & & FC \arrow[dl] \arrow[dr] \arrow[drrr, "f"] & & & \\
C' \arrow[dr, squiggly, "c"'] & & C'' \arrow[dl, squiggly, "d"] & FC' \arrow[rr, "j", "\cong"'] \arrow[dr, squiggly] & & FC'' \arrow[dl, squiggly] & & D \\
& T & & & FT \arrow[urrr, squiggly, "t"'] & & &
\end{tikzcd}$$

Now we can argue

$$\begin{aligned}
t \circ F(d) \circ j \circ F(v) &= t \circ F(c) \circ F(v) \\
&= t \circ F(d) \circ F(w)
\end{aligned}$$

i.e. the two vertical maps $j \circ F(v)$ and $F(w)$ are equal when postcomposed with the same cartesian map, and thus $j \circ F(v) = F(w)$. By the universal property of cocartesian arrows in the elementary opfibration $F_{\mathcal{C}(C)} : \mathcal{C}_{\mathcal{C}(C)} \to \mathcal{D}_{\mathcal{C}(C)}$ there now exists a unique $x : C' \to C''$ with $Fx = j$ and $x \circ v = w$. Both $c$ and $c \circ x$ constitute arrows from $(C', t \circ F(c))$ to $(T, t)$ in $F_D$, thus $d \circ x = c$. We have shown that both triangles in the left diamond in the diagram

$$\begin{tikzcd}
& C \arrow[dl, "v"'] \arrow[dr, "w"] & & & FC \arrow[dl] \arrow[dr] \arrow[drrr, "f"] & & & \\
C' \arrow[rr, "x", "\cong"'] \arrow[dr, squiggly, "c"'] & & C'' \arrow[dl, squiggly, "d"] & FC' \arrow[rr, "j", "\cong"'] \arrow[dr, squiggly] & & FC'' \arrow[dl, squiggly] & & D \\
& T & & & FT \arrow[urrr, squiggly, "t"'] & & &
\end{tikzcd}$$

[3] i.e. over the same arrow in the base

commute, which implies $h = k$ as required.

Conversely, assume that $F : \int \mathcal{C} \to \int \mathcal{D}$ has a non-fibered right adjoint, i.e. all $F/D$ have terminal objects. It is enough to show that $t$ is cartesian in $\mathcal{D}$ whenever $(T, t)$ is terminal in $F/D$. Let $t = cv$ be a vertical/cartesian factorization, and let $w : T \to U$ be a lift of $T$ along $v$. Then $w$ is a morphism from $(T, t)$ to $(U, c)$ in $F/D$, which has a retraction since $(T, t)$ is terminal. Thus $t$ is a retract of a cartesian arrow, and therefore cartesian. ∎

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