Characterizing Realizability Toposes

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Main result

Theorem

A **locally small** category $\mathcal{E}$ is equivalent to a realizability topos $\mathbf{RT}(\mathcal{A})$ over a PCA $\mathcal{A}$, if and only if

1. $\mathcal{E}$ is **exact** and **locally cartesian closed**,
2. $\mathcal{E}$ has enough **projectives**, and the subcategory $\mathbf{Proj}(\mathcal{E})$ of projectives is closed under **finite limits**,
3. the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$ has a **right adjoint** Δ factoring through $\mathbf{Proj}(\mathcal{E})$, and
4. there exists a **separated** and **discrete projective** $D \in \mathcal{E}$ such that for all projectives $P \in \mathcal{E}$ there exists a **closed** $u : P \to D$.

Here,

- $P \in \mathcal{E}$ is called *projective*, if $\mathcal{E}(P, e)$ is surjective for all regular epis e
- $A \in \mathcal{E}$ is called *separated*, if $\eta_A : A \to \Delta\Gamma A$ is monic
- $f : X \to Y$ is called *closed*, if $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \Delta\Gamma X & \xrightarrow{\Delta\Gamma f} & \Delta\Gamma Y \end{array}$ is a pullback
- $D \in \mathcal{E}$ is called *discrete* if it is right orthogonal to all closed regular epis

Part I
Realizability toposes

Realizability toposes $=$ categorical incarnation of 'Kleene realizability'

Genesis

- **Kleene – *On the interpretation of intuitionistic number theory*, 1945**
- Higgs – *A category approach to Boolean-valued set theory*, 1973
- Fourman, Scott – *Sheaves and logic*, 1979
- **Hyland, Johnstone, Pitts – *Tripos theory*, 1980**
- Pitts – *The theory of triposes*, 1982
- Hyland – *The effective topos*, 1982

- sheaves on a complete Heyting algebra A are equivalent to 'A- sets'
- triposes generalize cHa's while admitting the same construction
- Kleene realizability (and more generally realizability over **partial combinatory algebras**) gives rise to triposes

Partial combinatory algebras

Definition

A **PCA** is a set $\mathcal{A}$ with a partial binary operation

$$(- \cdot -) : \mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A}$$

having elements $k, s \in \mathcal{A}$ such that

$$(i)\ k{\cdot}x{\cdot}y = x \qquad (ii)\ s{\cdot}x{\cdot}y{\downarrow} \qquad (iii)\ s{\cdot}x{\cdot}y{\cdot}z \preceq x{\cdot}z{\cdot}(y{\cdot}z)$$

for all $x, y, z \in \mathcal{A}$.

Example

First Kleene algebra: $(\mathbb{N}, \cdot)$ with

$$n{\cdot}m \simeq \phi_n(m) \qquad \text{for } n, m \in \mathbb{N},$$

where $(\phi_n)_{n\in\mathbb{N}}$ is an effective enumeration of partial recursive functions.

Triposes

Definition

A **tripos** is an indexed preorder $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ such that

- for all $J \in \mathbf{Set}$, $\mathcal{P}(J)$ is a **Heyting pre-algebra**
- for $f : J \to I$, the **reindexing map** $f^* : \mathcal{P}(I) \to \mathcal{P}(J)$ preserves Hpa structure, and has adjoints $\exists_f \dashv f^* \dashv \forall_f$ satisfying **BC conditions**
- $\mathcal{P}$ has a **generic predicate** i.e. there is a $\mathrm{tr} \in \mathcal{P}(\mathrm{Prop})$ such that for all $J \in \mathbf{Set}$, $\varphi \in \mathcal{P}(J)$ there exists n $f : J \to \mathrm{Prop}$ with $\varphi \cong f^*(\mathrm{tr})$

Localic triposes

For any cHa A, the representable functor $\mathbf{Set}(-, A) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ with pointwise ordering on the fibers is a tripos

Realizability triposes

The **realizability tripos** $\mathrm{rt}(\mathcal{A})$ over a PCA $\mathcal{A}$ is given by the representable functor $\mathbf{Set}(-, P\mathcal{A})$, with fibers ordered by

$$\varphi \leq \psi \quad :\Leftrightarrow \quad \exists e \in \mathcal{A}\, \forall j \in J\, \forall a \in \varphi(j)\,.\; e{\cdot}a \in \psi(i) \quad \text{for} \quad \varphi, \psi : J \to P\mathcal{A}.$$

Toposes from triposes

Definition

Any tripos $\mathcal{P}$ gives rise to a category $\mathbf{Set}[\mathcal{P}]$ of $\mathcal{P}$-sets .

- **Objects:** partial equivalence relations in $\mathcal{P}$
- **Morphisms:** Equivalence classes of compatible functional relations

- $\mathbf{Set}[\mathcal{P}]$ is a topos for any tripos $\mathcal{P}$
- Toposes constructed from localic triposes $\mathbf{Set}(-, A)$ for cHa's A are equivalent to sheaf toposes $\mathbf{Sh}(A)$
- Realizability triposes $\mathrm{rt}(\mathcal{A})$ give rise to **realizability toposes** $\mathbf{RT}(\mathcal{A})$
- The realizability topos over the first Kleene algebra $(\mathbb{N}, \cdot)$ is Hyland's **effective topos**

The constant objects functor

Definition

Given a tripos $\mathcal{P}$, the **constant objects functor**

$$\begin{aligned} \Delta : \mathbf{Set} &\to \mathbf{Set}[\mathcal{P}] \\ J &\mapsto (J, \delta_J) \end{aligned}$$

maps any set J to the discrete/diagonal equivalence relation δ_J in $\mathcal{P}$

- Δ is **bounded by** 1 – every $A \in \mathbf{Set}[\mathcal{P}]$ is a **subquotient**

$$A \twoheadleftarrow \bullet \rightarrowtail \Delta(J)$$

- in localic case, Δ is *left* adjoint to the **global sections functor** $\Gamma = \mathbf{Sh}(A)(1, -)$, giving a **localic geometric morphism**

$$(\Delta : \mathbf{Set} \to \mathbf{Sh}(A)) \dashv (\Gamma : \mathbf{Sh}(A) \to \mathbf{Set})$$

- for realizability, Δ is *right* adjoint to Γ

$$(\Gamma : \mathbf{RT}(\mathcal{A}) \to \mathbf{Set}) \dashv (\Delta : \mathbf{Set} \to \mathbf{RT}(\mathcal{A}))$$

The family fibration of a PCA

The **family fibration** $\mathrm{fam}(\mathcal{A}) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ of a PCA $\mathcal{A}$ is defined by $\mathrm{fam}(\mathcal{A})(J) = (\mathcal{A}^J, \leq)$, with

$$\varphi \leq \psi \quad :\Leftrightarrow$$

$$\exists e \in \mathcal{A}\ \forall j \in J \,.\ e{\cdot}\varphi(j) = \psi(j)$$

$$\begin{tikzcd} & J \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathcal{A} \arrow[rr, "e\cdot(-)"'] & & \mathcal{A} \end{tikzcd}$$

for $\varphi, \psi : J \to \mathcal{A}$.

- $\mathrm{fam}(\mathcal{A})$ has indexed finite meets
- $\mathrm{rt}(\mathcal{A})$ is free cocompletion of $\mathrm{fam}(\mathcal{A})$ under $\exists$ (Hofstra [1])

[1] Pieter Hofstra – *All realizability is relative*, 2006

Realizability toposes by exact completion

Definition (Partitioned assemblies)

For PCA $\mathcal{A}$, the category $\mathbf{PAsm}(\mathcal{A})$ is the Grothendieck construction $\int(\mathrm{fam}(\mathcal{A}))$ of the family fibration $\mathrm{fam}(\mathcal{A})$

- **Objects:** pairs (I, φ) with $I \in \mathbf{Set}$ and $\varphi : I \to \mathcal{A}$
- **Morphisms** from (I, φ) to (J, ψ): functions $f : I \to J$ such that $\exists e \in \mathcal{A}\, \forall i \in I\, .\, e{\cdot}\varphi(i) = \psi(fi)$

$$\begin{tikzcd} I \arrow[r, "f"'] \arrow[d, "\varphi"'] & J \arrow[d, "\psi"] \\ \mathcal{A} \arrow[r, "e\cdot(-)"', harpoon] & \mathcal{A} \end{tikzcd}$$

Theorem (Robinson, Rosolini 1990)

$\mathbf{RT}(\mathcal{A})$ *is equivalent to the ex/lex completion* $\mathbf{PAsm}(\mathcal{A})_{\mathrm{ex}}$

- Depends on the axiom of choice

General facts about exact completion imply that

- $\mathbf{PAsm}(\mathcal{A})$ can be identified the $\mathbf{Proj}(\mathbf{RT}(\mathcal{A})) \subseteq \mathbf{RT}(\mathcal{A})$
- $\mathbf{RT}(\mathcal{A})$ has enough projectives

Part II
The Characterization

Main result

Theorem

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1. $\mathcal{E}$ is **exact** and **locally cartesian closed**,
2. $\mathcal{E}$ has enough **projectives**, and the subcategory $\mathbf{Proj}(\mathcal{E})$ of projectives is closed under **finite limits**,
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4. there exists a **separated** and **discrete projective** $D \in \mathcal{E}$ such that for all projectives $P \in \mathcal{E}$ there exists a **closed** $u : P \to D$.

Here,

- $P \in \mathcal{E}$ is called *projective*, if $\mathcal{E}(P, e)$ is surjective for all regular epis e
- $A \in \mathcal{E}$ is called *separated*, if $\eta_A : A \to \Delta\Gamma A$ is monic
- $f : X \to Y$ is called *closed*, if $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \Delta\Gamma X & \xrightarrow{\Delta\Gamma f} & \Delta\Gamma Y \end{array}$ is a pullback
- $D \in \mathcal{E}$ is called *discrete* if it is right orthogonal to all closed regular epis

Necessity of the conditions

- Conditions 1–3 are easy
- For condition 4, D is the partitioned assembly $(\mathcal{A}, \mathrm{id})$
- Projectives in realizability toposes are separated, and for any partitioned assembly (J, φ), the map $\varphi : (J, \varphi) \to (\mathcal{A}, \mathrm{id})$ is closed in $\mathbf{RT}(\mathcal{A})$

$$
\begin{tikzcd}
J \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & \mathcal{A} \arrow[d, equal] \\
\mathcal{A} \arrow[r, equal] & \mathcal{A}
\end{tikzcd}
$$

- To see that $(\mathcal{A}, \mathrm{id})$ is discrete, we show right orthogonality to maps $g : (I, \varphi \circ g) \twoheadrightarrow (J, \varphi)$ with surjective g

$$
\begin{tikzcd}
I \arrow[rr, bend left, "f"] \arrow[d, "\varphi\circ g"'] \arrow[dr, two heads, "g"] & & \mathcal{A} \arrow[d, equal] \\
\mathcal{A} \arrow[rr, bend left, "e"] \arrow[dr, equal] & J \arrow[ur, dashed, "e\circ\varphi"] \arrow[d, "\varphi"] & \mathcal{A} \\
& \mathcal{A} \arrow[ur, dashed, "e"'] &
\end{tikzcd}
$$

- Orthogonality to general closed epis follows for general reasons

Sufficiency of the conditions

- Assume $\mathcal{E}$ satisfies 1–4
- 2 implies that $\mathcal{E}$ is an exact completion
- we have to construct a PCA $\mathcal{A}$ such that **PAsm**$(\mathcal{A}) \simeq$ **Proj**$(\mathcal{E})$
- use **discrete combinatory objects**, a special case of Hofstra's **basic combinatory objects** (Hofstra 2006)

Discrete combinatory objects

Definition

A **discrete combinatory object** (DCO) is a pair $(A, \mathcal{F}_A)$ with $A \in \textbf{Set}$ and $\mathcal{F}_A \subseteq (A \rightharpoonup A)$ a set of partial functions such that

- $\mathrm{id}_A \in \mathcal{F}_A$
- $\forall \alpha, \beta \in \mathcal{F}_A \, \exists \gamma \in \mathcal{F}_A \, . \, \beta \circ \alpha \subseteq \gamma$

DCOs from PCAs

Any PCA $\mathcal{A}$ gives rise to a BCO $(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$ with

$$\mathcal{F}_{\mathcal{A}} = \{\phi_e \mid e \in \mathcal{A}\}$$

$$\text{where} \quad \phi_e(a) = e \cdot a$$

DCOs as indexed preorders

Definition

Let $(A, \mathcal{F}_A)$ be a DCO

- The **family fibration** $\mathrm{fam}(A, \mathcal{F}_A) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ is given by $\mathrm{fam}(A, \mathcal{F}_A)(J) = (A^J, \leq)$, with

$$\begin{aligned} &\varphi \leq \psi \quad :\Leftrightarrow \\ &\exists \alpha \in \mathcal{F}_A \,.\, \alpha \circ \varphi = \psi \\ &\text{for } \varphi, \psi : J \to A \end{aligned}$$

$$\begin{tikzcd} & J \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ A \arrow[rr, "\alpha"] & & A \end{tikzcd}$$

- $\mathbf{PAsm}(A, \mathcal{F}_A)$ is the Grothendieck construction of $\mathrm{fam}(A, \mathcal{F}_A)$. Objects are pairs $(I \in \mathbf{Set}, \varphi : I \to A)$; morphisms $(I, \varphi) \to (J, \psi)$ are functions $f : I \to J$ such that $\exists \alpha \in \mathcal{F}_A \,.\, \alpha \circ \varphi = \psi \circ f$.

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[d, "\varphi"'] & J \arrow[d, "\psi"] \\ A \arrow[r, "\alpha"] & A \end{tikzcd}$$

- For PCAs we have $\mathrm{fam}(\mathcal{A}) = \mathrm{fam}(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$ and $\mathbf{PAsm}(\mathcal{A}) = \mathbf{PAsm}(\mathcal{A}, \mathcal{F}_{\mathcal{A}})$

Indexed preorders representable by DCOs

Definition

Let $\mathcal{D} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ be an indexed preorder.

$\mu \in \mathcal{D}(A)$ is called **discrete**, if for any span $I \overset{e}{\twoheadleftarrow} J \overset{f}{\to} A$ with surjective e, and any $\varphi \in \mathcal{D}(I)$ with $e^*(\varphi) \leq f^*(\mu)$, there exists $h : I \to A$ with $he = f$.

$$\begin{tikzcd}
 & e^*(\varphi) \leq f^*(\mu) \arrow[d, dotted, no head] & \\
\varphi \arrow[d, dotted, no head] & J \arrow[dl, "e"'] \arrow[dr, "f"] & \mu \arrow[d, dotted, no head] \\
I \arrow[rr, dashed, "h"] & & A
\end{tikzcd}$$

In other words, μ is discrete, if (A, μ) is right orthogonal to all epicartesian maps in $\int \mathcal{D} \to \mathbf{Set}$.

Lemma

An indexed preorder $\mathcal{D}$ is representable by a DCO iff it has a discrete generic predicate.

- To reconstruct a DCO structure from an indexed preorder $\mathscr{D}$ with discrete generic predicate $\mu \in \mathcal{D}(A)$, take all the partial functions given by spans

$$A \overset{m}{\hookleftarrow} U \overset{f}{\to} A \quad \text{with} \quad m^*(\mu) \leq f^*(\mu)$$

Functional completeness

Lemma

$\mathrm{fam}(A, \mathcal{F}_A)$ *has stable finite meets iff there exist* $\top \in A$, $\wedge : A \times A \to A$, *and* $\lambda, \rho \in \mathcal{F}_A$ *such that*

1. *the constant function* $\mathsf{c}_\top$ *with value* $\top$ *is in* $\mathcal{F}_A$,
2. *for all* $a, b \in A$ *we have* $\lambda(a \wedge b) = a$ *and* $\rho(a \wedge b) = b$, *and*
3. *for all* $\alpha, \beta \in \mathcal{F}_A$ *there exists* $\gamma \in \mathcal{F}_A$ *with* $\gamma \supseteq (\alpha \times \beta) \circ \delta_A$.

- DCOs such that $\mathrm{fam}(A, \mathcal{F}_A)$ has stable finite meets are called **cartesian**.

Theorem (following [Hofstra 2006])

Let $(A, \mathcal{F}_A)$ be a cartesian DCO. The exact completion $\mathbf{PAsm}(A, \mathcal{F}_A)_{\mathrm{ex}}$ is locally cartesian closed iff there exists a **universal function** $@ \in \mathcal{F}_A$ such that for all $\alpha \in (A, \mathcal{F}_A)$ there exists a *total* $\tilde{\alpha} \in (A, \mathcal{F}_A)$ with

$$\alpha(a \wedge b) \preceq @(\tilde{\alpha}(a) \wedge b) \quad \text{for all} \quad a, b \in A.$$

- We call cartesian DCOs with universal functions **functionally complete.**

Functional completeness and PCAs

- On DCOs $(A, \mathcal{F}_A)$ with universal function, define a partial application by

$$a{\cdot}b := @(a \wedge b) \quad \text{for} \quad a, b \in A.$$

This makes $(A, \cdot)$ into a PCA. Moreover we can show:

Lemma

An indexed preorder $\mathcal{D}$ is equivalent to $\mathrm{fam}(\mathcal{A})$ for some PCA $\mathcal{A}$ iff

1. *$\mathcal{D}$ has stable finite meets*
2. *$\mathcal{D}$ has a discrete generic predicate*
3. *$(\int \mathcal{D})_{\mathrm{ex}}$ is locally cartesian closed*
4. *$\mathcal{D}(1)$ is equivalent to the terminal preorder*

- Omitting condition 4 characterizes $\mathrm{fam}(\mathcal{A}, \mathcal{A}_\#)$ for inclusions of PCAs (relative realizability)

Finishing the proof

- Assume that $\mathcal{E}$ satisfies conditions 1–4 of the theorem.
- To finish the proof we have to find a PCA $\mathcal{A}$ with $\mathbf{PAsm}(\mathcal{A}) \simeq \mathbf{Proj}(\mathcal{E})$
- Equivalently, find indexed preorder $\mathcal{D}$ satisfying the conditions of the lemma, such that $\int \mathcal{D} \simeq \mathbf{Proj}(\mathcal{E})$
- As finite limit preserving reflector, $\Gamma : \mathbf{Proj}(\mathcal{E}) \to \mathbf{Set}$ is a *Street fibration*
- In the corresponding indexed category $\mathcal{D}$, predicates on $J \in \mathbf{Set}$ are maps $f : P \to \Delta J$ in $\mathbf{Proj}(\mathcal{E})$ which are inverted by Γ
- $\eta_D : D \rightarrowtail \Delta\Gamma D$ is a generic predicate, and $\mathcal{D}$ is posetal since D is (and thus all projectives are) separated
- Discreteness of η_D in $\mathcal{D}$ follows from discreteness of D in $\mathcal{E}$
- $\mathcal{D}(1)$ consists of monos into 1 with 1 global element. Thus $\mathcal{D}(1) \simeq 1$.

Part III
Without Choice and Generalizations

f-projectives

The theorem depends on the axiom of choice. This can be avoided using *f-projectives*

Definition

Let $\mathbb{R}$ be regular and locally small, such that $\Gamma : \mathbb{R} \to \mathbf{Set}$ has a regular right adjoint Δ. $P \in \mathbb{R}$ is called **f-projective**, if every cospan $f : P \to X \twoheadleftarrow Y : e$ with e regular epic can be closed to

$$\begin{tikzcd} Q \arrow[r, dotted, two heads, "c"'] \arrow[d, dotted, "g"'] & P \arrow[d, "f"] \\ Y \arrow[r, two heads, "e"] & X \end{tikzcd}$$

with c closed regular epic.

Theorem

A locally small category $\mathcal{E}$ is equivalent to a realizability topos $\mathbf{RT}(\mathcal{A})$ over a PCA $\mathcal{A}$, if and only if

1. $\mathcal{E}$ is exact and locally cartesian closed,
2. the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$ has a **regular** right adjoint Δ
3. the subcategory $\mathbf{fProj}(\mathcal{E})$ of **f-projectives** is closed under finite limits, and contains the image of Δ
4. there exists a separated and discrete **f-projective** $D \in \mathcal{E}$ such that for all $A \in \mathcal{E}$ there exists a span $D \xleftarrow{c} \bullet \xrightarrow{e} A$ with c closed and e epic

Generalizations

Uniform preorders

- Dropping the discreteness condition gives a characterization of a class of toposes arising from a generalization of PCAs based on relations instead of partial functions: **shallow relationally complete uniform preorders**
- Examples are ordered PCAs, e.g. $P_+(\mathcal{A})$ – nonempty subsets of a PCA

Realizability-like triposes

- Dropping furthermore all conditions relating to projectives gives a characterization of toposes induced by **realizability-like triposes** – a tripos $\mathcal{P}$ is called *realizability-like* if we have $\gamma \dashv \delta : \mathrm{sub}(\mathbf{Set}) \to \mathcal{P}$, where δ and γ are tripos transformations definable for every tripos $\mathcal{P}$ on $\mathbf{Set}$.

Thank you for your attention