Uniform Preorders and Partial Combinatory Algebras

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Abstract

We give a reconstruction of partial combinatory algebras from Set-indexed preorders.

1 The Ord-category of uniform preorders

Uniform preorders were introduced in [Fre13] as a class of representations of Set-*indexed preorders* that generalize Hofstra's *basic combinatorial objects (BCOs)* [Hof06].

Contrary to BCOs, for uniform preorders there exists a straightforward characterization of the induced class of indexed preorders, which makes the notion both conceptually very clear and somewhat tautological. In this section we reconstruct the definition of uniform preorders from this characterization, after fixing terminology and notation on locally ordered categories and indexed preorders, which constitute the central formalisms in this work.

A Set-*indexed preorder* is a pseudofunctor $\mathsf{Set}^{\mathsf{op}} \to \mathsf{Ord}$ where Ord is the locally ordered category of preorders and monotone maps. As we only consider indexed preorders on Set in this paper, we omit the prefix. Given an indexed preorder $\mathcal{P}$ and a set A, we call $\mathcal{P}(A)$ the *fiber* of $\mathcal{P}$ over A, and refer to its elements as *predicates* on A. Given a function $f : A \to B$, the monotone map $\mathcal{P}(f)$ is called *reindexing along* f and abbreviated by f^*. We denote by IOrd the locally ordered category of indexed preorders and pseudo-natural transformations.

Strict indexed preorders and *strict* transformations form a non-full sub-2-category $[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]$ of IOrd, which by a well known argument about essentially algebraic structures in presheaf categories is isomorphic to the locally ordered category $\mathsf{Ord}([\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}])$ of internal preorders in $[\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}]$.

The locally ordered category UOrd of uniform preorders is now characterized as fitting into the following strict pullback of locally ordered categories, where the categories in the lower line are viewed as having codiscretely ordered hom-sets, U sends internal preorders to underlying presheaves, よ is the Yoneda embedding, and fam is defined to be the evident composition.

$$\begin{tikzcd}
\mathsf{UOrd} \arrow[r, "J"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rrr, bend left, "\mathsf{fam}"] & \mathsf{Ord}([\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}]) \arrow[r, "\cong"'] \arrow[d, "U"] & {[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]} \arrow[r, hook] & \mathsf{IOrd} \\
\mathsf{Set} \arrow[r, "\text{よ}"] & {[\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}]} & &
\end{tikzcd}$$

The 2-functor J is 2-fully faithful since よ is, which means that UOrd can be identified with the 2-full subcategory of $\mathsf{Ord}([\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}])$ on internal preorders whose underlying presheaves are representable. In other words, a uniform preorder is a set A together with an internal preorder structure on $\text{よ}(A)$. Such a preorder structure is given by a subfunctor of $\text{よ}(A) \times \text{よ}(A) \cong \text{よ}(A \times A)$, i.e. a sieve on $A \times A$, subject to reflexivity and transitivity conditions.

Since surjections split in Set, sieves are completely determined by their monomorphisms, or equivalently subset-inclusions, which means that a sieve on $A \times A$ is equivalently represented as a down-closed subset of the powerset $P(A \times A)$. We leave it to the reader to verify that unwinding the meaning of reflexivity, transitivity, monotonicity, and the hom-set ordering in terms of this representation of sieves yields the following concrete descriptions of the locally ordered category UOrd and the 2-functor fam.

Definition 1.1 The locally ordered category UOrd of uniform preorders and monotone maps is defined as follows.

(i) A *uniform preorder* is a pair (A, R) of a set A and a set $R \subseteq P(A \times A)$ of binary relations on A, such that

- $\mathrm{id}_A \in R$,
- $s \circ r \in R$ whenever $r \in R$ and $s \in R$, and
- $s \in R$ whenever $r \in R$ and $s \subseteq r$.

(ii) A *monotone map* between uniform preorders (A, R) and (B, S) is a function $f : A \to B$ such that for all $r \in R$, the set

$$(f \times f)[r] \;=\; f \circ r \circ f^\circ \;=\; \{(fa, fa') \mid (a, a') \in r\}$$

is in S.

(iii) The ordering relation $\leq$ on monotone maps $f, g : (A, R) \to (B, S)$ is defined by $f \leq g$ if and only if the set

$$\mathsf{im}\langle f, g\rangle \;=\; \{(fa, ga) \mid a \in A\}$$

is in S. $\diamondsuit$

Definition 1.2 The 2-functor $\mathsf{fam} : \mathsf{UOrd} \to \mathsf{IOrd}$ (called *family construction*) is defined as follows.

(i) For every uniform preorder (A, R), the indexed preorder $\mathsf{fam}(A, R)$ maps

- sets I to preorders $(A^I, \leq)$, where $\varphi \leq \psi : I \to A$ if and only if

$$\mathsf{im}\langle \varphi, \psi\rangle \;=\; \{(\varphi i, \psi i) \mid i \in I\} \tag{1.1}$$

  is in R, and
- functions $f : J \to I$ to monotone maps $f^* : (A^J, \leq) \to (A^I, \leq)$ given by precomposition.

(ii) For every monotone map $f : (A, R) \to (B, S)$ between indexed preorders, the components of the indexed monotone map $\mathsf{fam}(f) : \mathsf{fam}(A, R) \to \mathsf{fam}(B, S)$ are given by postcomposition. $\diamondsuit$

Remark 1.3 The ordering on monotone maps $f, g : (A, R) \to (B, S)$ defined in 1.1(iii) is the restriction of the ordering on $\mathsf{fam}(B, S)(A)$ as defined in 1.2(i). $\diamondsuit$

Definition 1.4 A *base* for a uniform preorder (A, R) is a subset $R_0 \subseteq R$ of binary relations whose down-closure $\downarrow R_0$ in $P(A \times A)$ is equal to R, i.e. R and R_0 generate the same sieve on $A \times A$. In other words, $R_0 \subseteq R$ is a base of R if for every $r \in R$ there is an $r_0 \in R_0$ with $r \subseteq r_0$. $\diamondsuit$

Remark 1.5 Given a set A and a set $R_0 \subseteq P(A \times A)$ of binary relations, its down-closure $\downarrow R_0$ is a uniform preorder structure on A iff

(a) there exists an $r \in R_0$ with $\mathrm{id}_A \subseteq r$, and

(b) for all $r, s \in R_0$ there exists a $t \in R_0$ with $s \circ r \in t$.

Just like continuity of functions between topological spaces, monotonicity of functions between uniform preorders can be expressed in terms of bases. Specifically, given uniform preorders (A, R) and (B, S) with bases R_0 and S_0 respectively, a function $f : A \to B$ is monotone iff for all $r \in R_0$ there exists an $s \in S_0$ with $(f \times f)[r] \subseteq s$, and given $\varphi, \psi : I \to A$ we have $\varphi \leq \psi$ in $\mathsf{fam}(A, R)(I)$ iff there exists an $r \in R_0$ with $\mathsf{im}\langle\varphi, \psi\rangle \subseteq r$. $\diamondsuit$

The following lemma gives a better understanding of the combined embedding from UOrd to IOrd. Recall that a *generic predicate* in an indexed preorder $\mathcal{A}$ is a predicate $\iota \in \mathcal{A}(A)$ for some A, such that for every other set B and predicate $\varphi \in \mathcal{A}(B)$ there exists a function $f : B \to A$ with $f^*\iota \cong \varphi$.

Lemma 1.6 *The pseudofunctor* $\mathsf{fam} : \mathsf{UOrd} \to \mathsf{IOrd}$ *is a local equivalence, and its essential image consists of the indexed preorders which admit a generic predicate.*

Proof. For the first claim it is sufficient to show that for every uniform preorder (A, R), strict indexed preorder $\mathcal{K}$, and pseudonatural transformation $f : \mathsf{fam}(A, R) \to \mathcal{K}$ there exists a strict transformation $\bar{f} : \mathsf{fam}(A, R) \to \mathcal{K}$ with $\bar{f} \cong f$. The transformation $\bar{f}$ is given by $\bar{f}_I(\varphi : I \to A) = \varphi^*(f_A(\mathrm{id}_A))$. More generally, this argument works for pseudonatural transformations $f : \mathcal{H} \to \mathcal{K}$ between strict indexed preorders where the underlying presheaf of $\mathcal{H}$ is a coproduct of representables.

For the second claim it is clear that indexed preorders $\mathsf{fam}(A, R)$ have generic predicates (the identity), and that this property is stable under equivalence. Given an indexed preoder $\mathcal{H}$ with generic predicate $\iota \in \mathcal{H}(A)$, a uniform preorder (A, R) whose family construction is equivalent to $\mathcal{H}$ is given by (A, R), where $R = \{r \subseteq A \times A \mid p^*\iota \leq q^*\iota\}$. Here, $p, q : R \to A$ are the compositions of the inclusion with the projections, as in the following diagram.

$$\begin{tikzcd}
 & R \arrow[dl, "p"'] \arrow[d, hook] \arrow[dr, "q"] & \\
A & A \times A \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & A
\end{tikzcd}$$

$\square$

Remark 1.7 Note that the fact that $\mathsf{UOrd} \to \mathsf{IOrd}$ is a local equivalence is not completely obvious, since $[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}] \to \mathsf{IOrd}$ is *not* a local equivalence. $\diamondsuit$

1.1 Adjunctions

An adjunction in a locally ordered category $\mathfrak{A}$ is a pair of arrows $f : A \to B$, $g : B \to A$, such that $\mathrm{id}_A \leq g \circ f$ and $f \circ g \leq \mathrm{id}_B$. Since $\mathsf{UOrd} \to \mathsf{IOrd}$ is a local equivalence, a monotone map $f : (A, R) \to (B, S)$ has a right adjoint in UOrd precisely if $\mathsf{fam}(f)$ has a right adjoint in IOrd. The following lemma gives a criterion for the existence of right adjoints in which monotonicity does not have to be checked explicitly.

Lemma 1.8 *The following are equivalent for uniform preorders* (A, R), (B, S), *a monotone function* $f : (A, R) \to (B, S)$, *and a function* $g : B \to A$.

(i) *The function g is monotone from (B,S) to (A,R), and a right adjoint of f.*

(ii) (1) $\mathsf{im}\langle f \circ g, \mathrm{id}_B\rangle \in S$, *and*

(2) *for all $s \in S$, the set $s^* = \{(a, gb) \mid (fa, b) \in s\}$ is in R.*

If (B,S) is given by a base, then it's sufficient to verify (2) on the elements of the base.

Proof. First assume (i). Condition (1) is equivalent to $f \circ g \le \mathrm{id}_B$ by (1.1). For condition (2), let $I = \{(a,b) \in A \times B \mid (fa, b) \in s\}$, and let $p : I \to A$ and $q : I \to B$ be the projections. Then we have $f \circ p \le q$ in $\mathsf{fam}(B,S)(I)$ by direct verification, and therefore $p \le g \circ q$ in $\mathsf{fam}(A,R)(I)$ by exponential transpose. the latter is equivalent to the claim.

Conversely, assume (ii). To see that postcomposition with g induces a left adjoint to $\mathsf{fam}(f) : \mathsf{fam}(A,R) \to \mathsf{fam}(B,S)$, it is enough to check that for all sets I and $h : I \to B$, the function $g \circ h$ is a greatest element of

$$\Phi = \{k : I \to A \mid f \circ k \le h\} \subseteq \mathsf{fam}(A,R)(I).$$

We have $g \circ h \in \Phi$ by (1). To show that it is a greatest element we have to show that $f \circ k \le h$ implies $k \le g \circ h$, which follows from (2) since

$$\mathsf{im}\langle k, g \circ h\rangle \subseteq \mathsf{im}\langle f \circ k, h\rangle^*$$

and R is down-closed. □

1.2 Cartesian uniform preorders

It is easy to see that the full subcategory of IOrd on indexed preorders admitting a generic predicate is closed under products. If $(\mathcal{H}_k)_{k\in K}$ is a family of indexed preorders with generic predicates $(\iota_k \in \mathcal{H}_k(A_k))_{k\in K}$, then a generic predicate of the (pointwise) product $\prod_{k\in K} \mathcal{H}_k$ is given by $(\pi_k^* \iota_k)_{k\in K} \in (\prod_{k\in K} \mathcal{H}_k)(\prod_{k\in K} A_k)$. Thus, UOrd has products which are preserved by fam. Concretely, the terminal uniform preorder is the singleton set with the unique uniform preorder structure, and a product of (A,R) and (B,S) is given by $(A \times B, R \otimes S)$, where $R \otimes S$ is the uniform preorder structure generated by the base $\{r \times s \mid r \in R, s \in S\}$.

Definition 1.9 A uniform prerder (A,R) is called *cartesian*, if the diagonal $(A,R) \to (A,R) \times (A,R)$ and the terminal projection $(A,R) \to 1$ have right adjoints. ♢

Instantiating Lemma 1.8 we get the following.

Lemma 1.10 *A uniform preorder (A,R) is cartesian if and only if there exists a function $(-\wedge-) : A \times A \to A$ and an element $\top \in A$ such that*

(i) $\{(a, \top) \mid a \in A\} \in R$,

(ii) $\{(a \wedge b, a) \mid a, b \in A\}, \{(a \wedge b, b) \mid a, b \in A\} \in R$, *and*

(iii) *for all $r, s \in R$, the relation*

$$\langle\!\langle r, s\rangle\!\rangle \;:=\; \wedge \circ (r \times s) \circ \delta_A \;=\; \{(a, b \wedge c) \mid (a,b) \in r, (a,c) \in s\}$$

is in R. □

1.3 Existential quantification

Definition 1.11 (i) We say that an indexed preorder $\mathcal{H}$ *has existential quantification*, if for every function $u : J \to I$, the monotone function $u^* : \mathcal{H}(I) \to \mathcal{H}(J)$ has a left adjoint $\exists_u : \mathcal{H}(J) \to \mathcal{H}(I)$, and for every pullback square

$$\begin{array}{ccc} L & \xrightarrow{\bar{u}} & K \\ {\scriptstyle \bar{v}}\downarrow & \lrcorner & \downarrow{\scriptstyle v} \\ J & \xrightarrow{u} & I \end{array}$$

of sets and functions we have $u^* \circ \exists_v \cong \exists_{\bar{v}} \circ \bar{u}^*$.

(ii) We say that an indexed monotone map $f : \mathcal{H} \to \mathcal{K}$ *commutes with exists quantification*, if $f_J \circ u^* \cong u^* f_I$ for all $f : J \to I$.

We write $\exists$-IOrd for the subcategory of UOrd on indexed preorders with existential quantification and indexed monotone maps commuting with existential quantification. ♢

Definition 1.12 An indexed monotone map $f : \mathcal{A} \to \mathcal{H}$ from an indexed preorder $\mathcal{A}$ to an indexed preorder $\mathcal{H}$ with existential quantification is called an $\exists$-*completion*, if for all indexed preorders $\mathcal{K}$ with existential quantification, the precomposition map

$$(- \circ f) \ : \ \exists\text{-}\mathsf{IOrd}(\mathcal{H}, \mathcal{K}) \ \to \ \mathsf{IOrd}(\mathcal{A}, \mathcal{K})$$

is an equivalence of preorders. ♢

Definition 1.13 Given a uniform preorder $\mathcal{H}$ with existential quantification, a predicate $\pi \in \mathcal{H}(I)$ is called $\exists$-*prime* if for all functions $I \xleftarrow{u} J \xleftarrow{v} K$ and predicates $\varphi \in \mathcal{H}(K)$ with $u^*\pi \le \exists_v \varphi$, there exists an $s : J \to K$ such that $v \circ s = \mathrm{id}_J$ and $u^*\pi \le s^*\varphi$. ♢

Proposition 1.14 (i) *An indexed monotone map $f : \mathcal{A} \to \mathcal{H}$ into an indexed preorder with existential quantification is an* $\exists$-completion *iff it is order-reflecting, and its essential image is the indexed sub-preorder of $\mathcal{H}$ on $\exists$-prime predicates.*

(ii) *For every indexed preorder $\mathcal{A}$ there exists an $\exists$-completion $f : \mathcal{A} \to \mathcal{H}$.*

(iii) *The forgetful functor $U : \exists\text{-}\mathsf{IOrd} \to \mathsf{IOrd}$ is* Ord*-monadic, and the induced* Ord*-monad $D : \mathsf{IOrd} \to \mathsf{IOrd}$ is lax idempotent.*

Proposition 1.15 *If $\mathcal{A}$ is an indexed preorder with a generic predicate, then its $\exists$-completion $D(A)$ has a generic predicate as well. Thus, the* Ord*-monad $D : \mathsf{IOrd} \to \mathsf{IOrd}$ restricts to* UOrd.

A concrete description of D on the level of uniform preorders is given by

$$D(A, R) = (PA, {\downarrow}\{[r] \mid r \in R\})$$

where for $r \in R$, the relation $[r] \subseteq PA \times PA$ is given by

$$[r] \ = \ \{(U, V) \in PA \times PA \mid \forall a \in U \ \exists b \in V \, . \, (a, b) \in r\}.$$

Remark 1.16 Unwinding the definition of $D(A, R)$ we see that given $\varphi, \psi : I \to PA$ we have $\varphi \le \psi$ in $\mathsf{fam}(D(A, R))_I$ iff there exists an $r \in R$ such that

$$\forall i \in I, \, a \in \varphi(i) \ \exists b \in \psi(i) \, . \, (a, b) \in r.$$

In this case we call r a *realizer* of the inequality $\varphi \le \psi$. ♢

Proposition 1.17 (i) *If $\mathcal{H}$ is a cartesian indexed preorder with existential quantification, then $D(\mathcal{H})$ is cartesian as well, and it satisfies the* Frobenius condition *which says that $\varphi \wedge \exists_u \psi \cong \exists_u(u^*\varphi \wedge \psi)$ for $u : J \to I$, $\varphi \in \mathcal{H}_I$, and $\psi \in \mathcal{H}_J$.*

(ii) *If $(A, R) \in \mathsf{UOrd}$ is cartesian then right adjoints to $D(A, R) \to D(A, R) \times D(A, R)$ and $D(A, R) \to 1$ are given by $(U, V) \mapsto \{a \wedge b \mid a \in U, b \in V\}$ and $* \mapsto \{\top\}$.*

1.4 Relational completeness

Definition 1.18 We say that an indexed preorder $\mathcal{A}$ *has universal quantification* if it satisfies the dual condition of Definition 1.11(i). If (A, R) is cartesian, we say that it *has implication* if its fibers are Heyting preorders, i.e. cartesian closed, and this structure is preserved up to isomorphism by reindexing. ♢

Theorem 1.19 *The following are equivalent for a cartesian uniform preorder (A, R).*

(i) *The indexed preorder $\mathsf{fam}(D(A, R)) \cong D(\mathsf{fam}(A, R))$ has universal quantification and implication, and is therefore a tripos.*

(ii) *(A, R) is* relationally complete, *i.e. there exists a relation $@ \in R$ such that for every $r \in R$ there exists a* function *$\tilde{r} \in R$ with $r \circ \wedge \subseteq @ \circ \wedge \circ (\tilde{r} \times \mathrm{id}_A)$, i.e.*

$$\forall a\, b\, c \in A \,.\, (a \wedge b, c) \in r \Rightarrow (\tilde{r}(a) \wedge b, c) \in @. \tag{1.2}$$

Proof. Assume first that $\mathsf{fam}(D(A, R))$ has implication and universal quantification. Let $E \hookrightarrow A \times A \times P(A \times A)$ be the membership predicate, define

$$\begin{aligned} u &: E \to P(A \times A) & u(b, c, s) &= s \\ \varphi, \psi &: E \to PA & \varphi(b, c, s) &= \{b\} & \psi(b, c, s) &= \{c\}, \end{aligned}$$

and set $\theta = \forall_u(\varphi \Rightarrow \psi)$. Then we have $u^*\theta \wedge \varphi \leq \psi$, and we define @ to be a realizer of this inequality, such that

$$\forall s \in P(A \times A),\, (b, c) \in s,\, a \in \theta(s) \,.\, (a \wedge b, c) \in @. \tag{1.3}$$

Now given $r \in R$ set $M = \{(a, b, c) \mid (a \wedge b, c) \in r\}$ and define

$$\begin{aligned} v &: M \to A & v(a, b, c) &= a \\ \beta, \gamma &: M \to PA & \beta(a, b, c) &= \{b\} & \gamma(a, b, c) &= \{c\} \\ \iota &: A \to PA & \iota(a) &= \{a\}. \end{aligned}$$

Then we have $v^*\iota \wedge \beta \leq \gamma$, and therefore $\iota \leq \forall_v(\beta \Rightarrow \gamma)$. Defining $w : A \to P(A \times A)$ by $w(a) = \{(b, c) \mid (a \wedge b, c) \in r\}$ we get a pullback square

$$\begin{tikzcd} M \arrow[r, "v"'] \arrow[d, "x"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "w"] \\ E \arrow[r, "u"'] & P(A \times A) \end{tikzcd}$$

and moreover $x^*\varphi = \beta$ and $x^*\psi = \gamma$, which implies using the Beck–Chevalley condition that $\forall_v(\beta \Rightarrow \gamma) \cong w^*\theta$, and thus $\iota \leq w^*\theta$. Taking t to be a realizer of this inequality we get that

$$\forall a \in A\; \exists b \in \theta(w(a)) \,.\, (a, b) \in t,$$

in particular, t is total, and we choose $\tilde{r}$ to be a subfunction of t satisfying

$$\forall a \in A \,.\, \tilde{r}(a) \in \theta(w(a)).$$

Finally, the implication (1.2) follows from (1.3) by instantiating s with $w(a)$.

Conversely, assume that (A, R) is relationally complete. Instead of constructing implication and universal quantification separately, we show how to define the 'synthetic' connective $\forall_u(\varphi \Rightarrow \psi)$ for $u : J \to I$ and $\varphi, \psi \in \mathsf{fam}(D(A, R))_I$. Implication and universal quantification can then be recovered by either replacing u by the identity, or φ by the true predicate. For $\varphi, \psi : J \to PA$ define $\forall_u(\varphi \Rightarrow \psi) : I \to PA$ by

$$\forall_u(\varphi \Rightarrow \psi)(i) = \bigcap_{uj=i} \{a \in A \mid \forall b \in \varphi(j)\, \exists c \in \psi(j) \,.\, @(a \wedge b, c)\}.$$

It is then easy to see that the inequality $u^*\forall_u(\varphi \Rightarrow \psi) \wedge \varphi \leq \psi$ is realized by $@$; and if $\zeta : I \to PA$ such that the inequality $u^*\xi \wedge \varphi \leq \psi$ is realized by $r \in R$, then $\tilde{r}$ realizes $\xi \leq \forall_u(\varphi \Rightarrow \psi)$. □

1.5 Discreteness

Definition 1.20 A *discrete combinatory object (DCO)* is a uniform preorder where all relations $r \in R$ are *single-valued*. We write DCO for the subcategory of UOrd on DCOs. ♢

To obtain a characterization of indexed preorders which are presentable by DCOs, we need the notion of *discreteness*.

Definition 1.21 A predicate $\delta \in \mathcal{A}(I)$ of an indexed preorder $\mathcal{A}$ is called *discrete*, if for every surjection $e : K \twoheadrightarrow J$, function $f : K \to I$, and predicate $\varphi \in \mathcal{A}(J)$ such that $e^*\varphi \leq f^*\delta$, there exists a (necessarily unique) $g : J \to I$ with $g \circ e = f$ (and therefore $\varphi \leq g^*\delta$ since reindexing along split epis is order-reflecting). ♢

Proposition 1.22 *An indexed preorder $\mathcal{A}$ is in the essential image of $\mathsf{DCO} \to \mathsf{IOrd}$ if and only if it has a discrete generic predicate.*

Proof. See [Fre19, Theorem 2.4]. □

Proposition 1.23 *If $\mathcal{H}$ is an indexed preorder and $\iota \in \mathcal{H}(A)$ is a discrete predicate, then ι is also discrete in $D(\mathcal{H})$.*

2 Relative partial combinatory algebras

Definition 2.1 A *relative partial applicative structure (relative PAS)* is a triple $(A, A_\#, \cdot)$ consisting of a set A, a subset $A_\# \subseteq A$, and a partial binary 'application' operation

$$(- \cdot -) : A \times A \rightharpoonup A$$

such that $a{\cdot}b \in A_\#$ whenever $a, b \in A_\#$ and $a{\cdot}b$ is defined. ♢

Remarks 2.2 (a) Application associates to the left, i.e. $a{\cdot}b{\cdot}c$ is a shorthand for $(a{\cdot}b){\cdot}c$.

(b) A *polynomial* over a relative PAS $(A, A_\#, \cdot)$ is a term built up from variables, constants from A, and application $(- \cdot -)$.

(c) When reasoning with partial terms, $t{\downarrow}$ means that t is defined, $t = u$ means that t and u are defined and equal, $t \preceq u$ means that u is defined whenever t is defined, and in this case they're equal, and $t \simeq u$ means $t \preceq u$ and $u \preceq t$. The $\preceq$ convention is in conflict with some of the literature, but used here since it is consistent with subset inclusion. $\diamondsuit$

Definition 2.3 A *relative partial combinatory algebra (relative PCA)* is a relative PAS satisfying one of the following equivalent conditions.

(i) There are $\mathsf{k}, \mathsf{s} \in A_\#$ such that $\mathsf{k}{\cdot}x{\cdot}y = x$, $\mathsf{s}{\cdot}x{\cdot}y{\downarrow}$ and $\mathsf{s}{\cdot}x{\cdot}y{\cdot}z \succeq x{\cdot}z{\cdot}(y{\cdot}z)$.

(ii) For every polyonmial $p[x_1, \dots, x_{n+1}]$ with coefficients in $A_\#$ there is an $e \in A_\#$ with $e{\cdot}x_1{\cdot}\dots{\cdot}x_n{\downarrow}$ and $e{\cdot}x_1{\cdot}\dots{\cdot}x_{n+1} \succeq t[x_1, \dots, x_{n+1}]$. $\diamondsuit$

Remarks 2.4 (a) When setting $A = A_\#$ in the definition of relative PCA, we *almost* recover the classical notion of PCA; the difference being that the latter uses the stronger condition $\mathsf{s}{\cdot}x{\cdot}y{\cdot}z \simeq x{\cdot}z{\cdot}(y{\cdot}z)$ instead of our $\mathsf{s}{\cdot}x{\cdot}y{\cdot}z \succeq x{\cdot}z{\cdot}(y{\cdot}z)$. See the discussion in the introduction of [Fre19]. $\diamondsuit$

Proposition 2.5 *For every relative PCA $(A, A_\#, \cdot)$, the set of 'computable' partial functions*

$$\mathcal{F}_A \;=\; \{(a \cdot -) : A \rightharpoonup A \mid a \in A\}$$

form a base for a cartesian and relationally complete DCO structure on A.

Proof. $\mathcal{F}_A$ satisfies the condition of Remark 1.5 since (relative) PCAs admit identity and composition operators, which are traditionally called i and b and satisfy the relations $\mathsf{i}{\cdot}x = x$ and $\mathsf{s}{\cdot}x{\cdot}y{\cdot}z \succeq x{\cdot}(y{\cdot}z)$. Using k and s, the combinators i and b can be defined as $\mathsf{i} = \mathsf{s}{\cdot}\mathsf{k}{\cdot}\mathsf{k}$ and $\mathsf{b} = \mathsf{s}{\cdot}(\mathsf{k}{\cdot}\mathsf{s}){\cdot}\mathsf{k}$. The indexed preorder $\mathsf{fam}(D(A, \mathcal{F}_A))$ is easily recognized to be the *relative realizability tripos* associated to the relative PCA [vO08, Section 2.6.9], thus $(A, \mathcal{F}_A)$ is relationally complete by Theorem 1.19. $\square$

Proposition 2.6 *Every relationally complete cartesian DCO arises from a relative PCA by the construction in Proposition 2.5.*

Proof. Easy generalization of Lemma [Fre19, Lemma 2.12] $\square$

Theorem 2.7 *The following are equivalent for a tripos $\mathcal{P}$ with generic predicate $\mathsf{tr} \in \mathcal{P}(\Sigma)$.*

(i) *$\mathcal{P}$ is a relative realizability tripos.*

(ii) *There exists a discrete $\exists$-prime predicate $\iota \in \mathcal{P}(A)$ and a function $f : A \to \Sigma$ such that $\mathsf{tr} \cong \exists_f \iota$, and the indexed sub-poset of $\mathcal{P}$ generated by ι is closed under finite limits.*

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