An introduction to cohesive homotopy type theory

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Overview

Three Parts

1. Type theory
2. Homotopy Type Theory
3. Cohesive Homotopy Type Theory

Part I – Type Theory

Pre-history: Russell, Simple Types, Lambda Calculus

- According to the *Stanford Encyclopedia of Philosopy*[1]:

  «The theory of types was introduced by Russell in order to cope with some contradictions he found in his account of set theory and was introduced in Appendix B: The Doctrine of Types of Russell 1903[2]*.»*

- Detailed presentation in *Principia Mathematica*, using
  - a type i of individuals, and
  - types $P(A_1, \ldots, A_n)$ of n-ary relations ranging over previously defined types
- Functions are defined as *functional relations*
- Church 1940[3] gives a reformulation using λ-calculus

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1. T. Coquand. "Type Theory". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2018. Metaphysics Research Lab, Stanford University, 2018.
2. B. Russell. *The principles of Mathematics. Vol I.*. English. Cambridge: University Press. XXIX u. 534 S. 8° (1903). 1903.
3. A. Church. "A formulation of the simple theory of types". In: *The Journal of Symbolic Logic* (1940).

Higher order logic in simply typed λ-calculus

Here's a variation of Church's system:

- We have *base types* ι (individuals) and o (propositions), and *arrow types* $A \to B$ for all types A and B
- Well-formed terms are presented using *typing judgments*: i.e. expressions of the form

$$\Gamma \vdash t : B\,.$$

  Here $\Gamma \;\equiv\; x_1{:}A_1, \ldots, x_n{:}A_n$ is a list of 'typed' variables called the **context** of the judgments and the judgment asserts that t is a well-formed term of type B possibly containing the variables of the specified types
- Well-formed terms-in-context are formed inductively by the rules

$$\frac{}{x_1{:}A_1, \ldots, x_n{:}A_n \vdash x_i : A_i} \qquad \frac{\Gamma \vdash t : B \to C \qquad \Gamma \vdash u : B}{\Gamma \vdash (t\,u) : C} \qquad \frac{\Gamma, x{:}B \vdash t : C}{\Gamma \vdash (\lambda x{:}B\,.\,t) : B \to C}$$

  from constants for logical operations (including e.g. $\forall_A : (A \to o) \to o$ for all types A).
- 'Logic' is given by a Hilbert-style system on terms of type o representing propositions and predicates

Another point of view on the λ*-calculus*

- Erasing terms from the typing rules gives a natural deduction system for the implicational fragment of intuitionistic propositional logic (IPC)

$$\frac{}{A_1, \ldots, A_n \vdash A_i} \qquad \frac{\Gamma \vdash B \to C \qquad \Gamma \vdash B}{\Gamma \vdash C} \qquad \frac{\Gamma, B \vdash C}{\Gamma \vdash B \to C}$$

- Thus, in absence of constants a type A is inhabited by a closed term iff it is a propositional tautology
- This is an instance of the *propositions as types* paradigm (a.k.a. Curry Howard isomorphism)
- Observe the difference
    - In Church's system types represent collections ('sets') of mathematical objects, and terms represent elements
    - Under the propositons-as-types reading, types are propositions and λ-terms are encodings of proof trees!
- Per Martin-Löf's **dependent type theory** unifies both points of view, and types can be used to represent both propositions and collections of mathematical objects ('sets').

Martin-Löf's Dependent Type Theory

Dependent type theory extends the language of simple type theory by introducing **type families** ('dependent types') written

$$x{:}A \vdash B[x] \ \text{type}.$$

- Depending on the point of view, dependent types can represent both families of sets, and predicates ('families of propositions')
- We can consider types depending on several variables:

  $$x{:}A, y{:}B[x] \vdash C[x, y] \ \text{type}$$

  and more generally

  $$x_1{:}A_1, \ldots, x_n{:}A_n[x_1, \ldots, x_{n-1}] \vdash B[x_1, \ldots, x_n] \ \text{type}$$

- Observe that types in the context may depend on preceding types.
- Well-typed terms $x{:}A \vdash t[x] : B[x]$ can represent
  - either families of elements ('choice functions') of a family of sets, or
  - proofs of a predicate.

Type formers in dependent type theory

Besides ‘simple’ type formers

$$\frac{\Gamma \vdash A \ \text{type} \quad \Gamma \vdash B \ \text{type}}{\Gamma \vdash A \times B \ \text{type}} \qquad \frac{\Gamma \vdash A \ \text{type} \quad \Gamma \vdash B \ \text{type}}{\Gamma \vdash A + B \ \text{type}} \qquad \frac{\Gamma \vdash A \ \text{type} \quad \Gamma \vdash B \ \text{type}}{\Gamma \vdash A \to B \ \text{type}}$$

and constant types $0, 1, N$, we have dependent sum and dependent product type formers:

$$\frac{\Gamma, x{:}A \vdash B \ \text{type}}{\Gamma \vdash \Pi x{:}A \,.\, B \ \text{type}} \qquad \frac{\Gamma, x{:}A \vdash B \ \text{type}}{\Gamma \vdash \Sigma x{:}A \,.\, B \ \text{type}}$$

These type formers admit different readings, depending on whether we see types as sets or propositions

	types as sets	types as propositions
$A \times B$	cartesian product	conjunction $\wedge$
$A + B$	disjoint union	disjunction $\vee$
$A \to B$	set of functions	implication $\Rightarrow$
1	singleton	‘true’
0	empty set	‘false’
$\Pi x{:}A \,.\, B$	product of a family of sets	universal quantification $\forall$
$\Sigma x{:}A \,.\, B$	disjoint union of a family of sets	existential quantification $\exists$

Introduction and elimination rules

All types come with ‘introduction’ and ‘elimination’ rules. For example:

Type former	Intro	Elim
$(- \times -)$	$\dfrac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash (t, u) : A \times B}$	$\dfrac{\Gamma \vdash p : A \times B}{\Gamma \vdash \pi_1(p) : A} \qquad \dfrac{\Gamma \vdash p : A \times B}{\Gamma \vdash \pi_2(p) : B}$
$(- + -)$	$\dfrac{\Gamma \vdash t : A}{\Gamma \vdash \sigma_1(t) : A + B} \qquad \dfrac{\Gamma \vdash u : B}{\Gamma \vdash \sigma_2(u) : A + B}$	$\dfrac{\Gamma \vdash t : A{+}B \quad \Gamma, x{:}A \vdash u : C \quad \Gamma, y{:}B \vdash v : C}{\Gamma \vdash \mathrm{match}(t, \sigma_1(x).u, \sigma_2(y).v) : C}$
Π	$\dfrac{\Gamma, x{:}A \vdash t : B}{\Gamma \vdash (\lambda x{:}A \,.\, t) : \Pi x{:}A \,.\, B}$	$\dfrac{\Gamma \vdash t : \Pi x{:}A \,.\, B \quad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B[u/x]}$
Σ	$\dfrac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[a/x]}{\Gamma \vdash (a, b) : \Sigma x{:}A \,.\, B}$	$\dfrac{\Gamma \vdash p : \Sigma x{:}A \,.\, B}{\Gamma \vdash \pi_1(p) : A} \qquad \dfrac{\Gamma \vdash p : \Sigma x{:}A \,.\, B}{\Gamma \vdash \pi_2(p) : B}$

Again, these rules admit a set theoretic and a logical reading.

Definitional equality

- **Definitional equality** identifies terms of the same type.
- For each type former there's a 'β-rule' and possibly an η-rule, e.g.:
  - If $\Gamma, x{:}A \vdash t : B$ and $\Gamma \vdash u : A$ then $\Gamma \vdash (\lambda x{:}A\,.\,t)u \equiv t[u/x] : B$ (β-equality)
  - If $\Gamma \vdash f : \Pi x{:}A\,.\,B$ then $\Gamma \vdash f \equiv (\lambda x{:}A\,.\,f\,x) : \Pi x{:}A\,.\,b$ (η-equality)
  - If $\Gamma \vdash t : A$ and $\Gamma \vdash u : B$ then $\Gamma \vdash \pi_1(t, u) \equiv t : A$ and $\Gamma \vdash \pi_2(t, u) \equiv u : B$ (β-equality)
  - If $\Gamma \vdash p : A \times B$ then $\Gamma \vdash p \equiv (\pi_1(p), \pi_2(p)) : A \times B$ (η-equality)
  - ...
- Definitional equality is the **congruence relation** generated by these equations, i.e. we close up under reflexivity, transitivity, symmetry, and term formers.
- Because we can substitute types into terms, we have to extend definitional equality to types:

$$\frac{\Gamma, x{:}A \vdash B \ \text{type} \qquad \Gamma \vdash t \equiv u : A}{\Gamma \vdash B[t/x] \equiv B[u/x]}$$

- This forces us to introduce the dreaded **conversion rule**:

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash A \equiv B}{\Gamma \vdash t : B}$$

Propositional equality: Identity types

- Definitional equality lives on the 'meta-level' of type theory, cannot be used to formalize mathematical 'hypothetical' reasoning in type theory where equalities are hypotheses.
- This is because definitional equality cannot occur 'on the left of the turnstile' in a judgment.
- An 'object-level' notion of equality is given by **identity types** which in accordance with the propositions-as-types principle represent the equality predicate on a type A as a type family in two arguments of type A:

Id-*types*

Formation
$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{Id}_A(t, u) \ \text{type}}$$

Introduction
$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}(a) : \mathsf{Id}_A(a, a)}$$

Elimination
$$\frac{\Gamma, x{:}A, z : \mathsf{Id}_A(a, x) \vdash C[x, z] \ \text{type} \qquad \Gamma \vdash t : C[a, \mathsf{refl}_a] \qquad \Gamma \vdash p : \mathsf{Id}_A(a, b)}{\Gamma \vdash t \triangleright_{C[-,-]} p : C[b, p]}$$

Def. equality
$$\Gamma \vdash (t \triangleright_{C[-,-]} \mathsf{refl}(a)) \equiv_\beta t : C[a, \mathsf{refl}_a]$$

Extensional vs intensional type theory

- We can also formulate an η-**rule for identity types**:

$$\frac{\Gamma, x{:}A, z : \mathsf{Id}_A(a, x) \vdash t[x, z] : C[x, z]}{\Gamma, x{:}A, z : \mathsf{Id}_A(a, x) \vdash t[x, z] \equiv (t[a, \mathsf{refl}(a)] \triangleright z) : C[x, z]}$$

- This rule implies the **equality reflection** rule

$$\frac{\Gamma \vdash p : \mathsf{Id}_A(a, b)}{\Gamma \vdash a \equiv b : A}$$

  and leads us to the system of **extensional type theory**, which Martin-Löf originally considered[4].
- Unfortunately, type checking in extensional type theory is not decidable, and thus the system is not suited for *computer implementations*.
- Therefore computer scientists turned their attention to *intensional type theory* (without η-for identity types).

[4] P. Martin-Löf and G. Sambin. *Intuitionistic type theory*. Bibliopolis Napoli, 1984.

(Non-)Uniqueness of identity proofs

- The absence of equality reflection in intensional type theory has a curious consequence, which was long seen as a weakness or 'incompleteness phenomenon':
- Although
  - the only 'canonical elements' (those given by intro rules) of identity types are reflexivity terms, and
  - it is possible to show *externally* that whenever we have $\vdash t : \mathsf{Id}_A(a, b)$ in *empty context*, then $\vdash a \equiv b : A$ and $\vdash t \equiv \mathsf{refl}(a) : \mathsf{Id}_A(a, b)$

  it is impossible to show *internally* that identity proofs are unique, i.e. the type

$$\Pi(a\, b : A)\Pi(p\, q : \mathsf{Id}_A(a, b))\, .\, \mathsf{Id}_{\mathsf{Id}_A(a,b)}(p, q)$$

  is not inhabited for a generic type A.
- The latter was shown by Hofmann and Streicher[5] by giving a countermodel in the category of **groupoids**, where
  - types are interpreted as groupoids,
  - terms are interpreted as objects of a groupoid, and
  - $\mathsf{Id}_A(a, b)$ is interpreted as the set of morphisms between a and b.

[5] M. Hofmann and T. Streicher. "The Groupoid Model Refutes Uniqueness of Identity Proofs". In: *Proceedings of the Ninth Annual Symposium on Logic in Computer Science (LICS '94), Paris, France, July 4-7, 1994*. IEEE Computer Society, 1994.

The groupoid model

In[6], Hofmann and Streicher write

> *«In this paper we have shown that the current formulation of intensional constructive type theory is incomplete w.r.t. the intuitive understanding of identity types. Namely, it is possible to construct a model where not all proof objects of an identity type are necessarily propositionally equal.»*

In[7] they write:

> *«... allows for the addition of axioms (inconsistent with UIP) expressing a view of propositional equality as a generalised notion of isomorphism. Intuitively, these axioms state that for a universe U and $A, B : U$ the identity set $\mathrm{Id}_U(A, B)$ corresponds to the set of isomorphisms between A and B.»*

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[6] M. Hofmann and T. Streicher. "The Groupoid Model Refutes Uniqueness of Identity Proofs". In: *Proceedings of the Ninth Annual Symposium on Logic in Computer Science (LICS '94), Paris, France, July 4-7, 1994*. IEEE Computer Society, 1994.

[7] M. Hofmann and T. Streicher. "The groupoid interpretation of type theory". In: *Twenty-five years of constructive type theory (Venice, 1995)* (1998).

Part II – Homotopy Type Theory

Homotoy type theory

- **Homotopy type theory** extends Hofmann and Streicher's ideas radically: instead of groupoids, it proposes an interpretation of type theory in ∞-**groupoids**, which by **Grothendieck's homotopy hypothesis** are equivalent to **homotopy types**.

$$\{\text{Homotopy types}\} \simeq \{\infty\text{-groupoids}\}$$

- Under this interpretation, elements of an identity type $\mathsf{Id}_A(a, b)$ can be thought of as **paths** between points a and b in a topological space representing the homotopy type A.
- *Higher homotopies* between paths can be accessed through iterated identity types.
- This idea can be traced back to
    - the work of Awodey and Warren[8] (first presented at FMCS 2006) which makes the idea of the homotopical interpretation precise using **path objects** in **Quillen model categories** to interpret identity types, and
    - the work of Voevodsky, who proposed a system called *homotopy lambda calculus*[9] as a system for formalization of mathematics.

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[8] S. Awodey and M. Warren. "Homotopy theoretic models of identity types". In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Cambridge University Press. 2009.

[9] Vladimir Voevodsky. "A very short note on the homotopy λ-calculus". In: *Unpublished note* (2006).

h-levels and univalence

One of Voevodsky's early core insights was that the homotopy theoretic notion of *truncation level* can be expressed using identity types.

Truncation levels

Let U be a type theoretic universe. We define the predicates $\mathsf{isTrunc}_n$ on U by induction on $n \geq -2$:

- $\mathsf{isContr}(A) \equiv \mathsf{isTrunc}_{-2}(A) \equiv \Sigma a{:}A\, \Pi b{:}A \,.\, \mathsf{Id}_A(a, b)$
- $\mathsf{isTrunc}_{(n+1)}(A) \equiv \Pi(a\, b : A) \,.\, \mathsf{isTrunc}_n(\mathsf{Id}_A(a, b))$

- (-1)-types are also called **propositions**, and 0-types **sets**

Equivalences and univalence

- Given $f : A \to B$ we define $\mathsf{isEquiv}(f) \equiv \Pi b \,.\, \mathsf{isContr}(\Sigma a \,.\, \mathsf{Id}(f\ a, b))$
- Given types A, B we define $\mathsf{Equiv}(A, B) \equiv \Sigma(f : A \to B) \,.\, \mathsf{isEquiv}(f)$
- Given types A, B we define $\mathsf{IdtoEquiv}(A, B) : \mathsf{Id}_U(A, B) \to \mathsf{Equiv}(A, B)$ by $\mathsf{IdtoEquiv}\, p\, a \equiv (a \triangleright_{\mathsf{el}} p)$, where el is the **universal type family** over U.
- The **univalence axiom** asserts that the map $\mathsf{IdtoEquiv}(A, B)$ itself is an equivalence.
  In other words, the notions of equality and equivalence coincide for types in U, or '*equivalent types are equal*'.

Synthetic homotopy theory in homotopy type theory

- The term **homotopy type theory (HoTT)** refers to a system of Martin-Löf type theory with *universes*, certain *(higher) inductive types*, and a constant for the *univalence axiom*.
- This system is powerful enough to express the core concepts, and to prove central results in homotopy theory *synthetically*, including
  - the *Freudenthal suspension theorem*[10],
  - the *Blakers–Massey theorem*[11],
  - the notion of *Eilenberg-MacLane spaces*[12], and
  - the *Serre spectral sequence*[13].
- The majority of this work has been **computer formalized** in one or more of the proof assistants *Agda*, *Coq*, and *Lean* that implement variants of MLTT.

[10] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[11] K. Hou (Favonia), E. Finster, D. Licata, and P. Lumsdaine. "A mechanization of the Blakers-Massey connectivity theorem in homotopy type theory". In: *Proceedings of the 31st Annual ACM-IEEE Symposium on Logic in Computer Science (LICS 2016)*. ACM, New York, 2016.

[12] D. Licata and E. Finster. "Eilenberg-MacLane spaces in homotopy type theory". In: *Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. ACM, New York, 2014.

[13] F. van Doorn. "On the formalization of higher inductive types and synthetic homotopy theory". In: *PhD thesis arXiv:1808.10690* (2018).

Models of homotopy type theory

- The 'standard' model of HoTT is the model in simplicial sets[14].
- Recently, Shulman showed[15] that HoTT admits models in arbitrary ∞-**toposes**[16].
- Being the ∞-categorical generalization of Grothendieck's toposes, and ∞-toposes are ∞-categories of **sheaves** of homotopy types.
- Shulman's result means that HoTT can be used as an **internal language** in ∞-toposes, in the same way that higher order logic can be used as internal language in toposes[17].
- Results in pure HoTT hold in arbitrary ∞-toposes, to talk about specific ∞-toposes we need additional axioms and type theoretic principles.

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14 K. Kapulkin and P. Lumsdaine. "The simplicial model of univalent foundations (after Voevodsky)". In: *Journal of the European Mathematical Society (JEMS)* (2021).

15 M. Shulman. "All $(\infty, 1)$-toposes have strict univalent universes". In: *arXiv preprint arXiv:1904.07004* (2019).

16 J. Lurie. *Higher topos theory*. Princeton University Press, 2009.

17 J. Lambek and P.J. Scott. *Introduction to higher order categorical logic*. Cambridge: Cambridge University Press, 1986.

Part III – Cohesive Homotopy Type Theory

Lawvere's axiomatic cohesion

- Lawvere introduced **axiomatic cohesion**[18] as a category theoretic axiomatization of the "Dialectic between continuous and discrete"
- Paradigmatic example: chain of adjoint functors between **sets** and **locally connected topological spaces**:

$$\mathsf{lcTop} \quad \downarrow \Pi_0 \;\dashv\; \uparrow \Delta \;\dashv\; \downarrow \Gamma \;\dashv\; \uparrow \nabla \quad \mathbf{Set}$$

Π_0 : connected components

Δ : discrete space on set

Γ : points

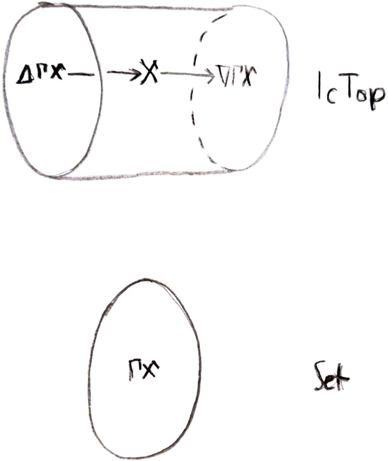
∇ : codiscrete space on set

- The following **axioms of cohesion** hold:
  1. Δ and ∇ are fully faithful
  2. Π_0 preserves finite products
  3. The **point-to-pieces** map $\Gamma X \overset{\cong}{\to} \Pi_z(\Delta(\Gamma X)) \to \Pi X$ is surjective for all $X \in \mathsf{lcTop}$ ('Nullstellensatz')
  4. The terminal projection $\Pi_0(\nabla S) \to 1$ is injective for all sets S ('connected codiscreteness')

[18] F.W. Lawvere. "Cohesive toposes and Cantor's "lauter Einsen"". In: Categories in the foundations of mathematics and language. 1994.

Adjoint cylinder

- The rightmost three adjoints $\Delta \dashv \Gamma \dashv \nabla$ constitute an **adjoint cylinder**
- The functors Δ and ∇ embed **Set** into lcTop in two extremal ways: discrete and codiscrete
- Every space X is 'suspended' between its 'discretization' and its 'codiscretization'



Cohesive toposes

Definition

A **cohesive topos** is a Grothendieck topos $\mathcal{E}$ whose global sections functor $\Gamma = \mathcal{E}(1, -) : \mathcal{E} \to \mathbf{Set}$ fits into a sequence $\Pi_0 \dashv \Delta \dashv \Gamma \dashv \nabla$ of adjoints satisfying the axioms of cohesion 1–4.

Examples

- The topos $\mathbf{SSet} = [\Delta^{\mathrm{op}}, \mathbf{Set}]$ of simplicial sets
- The topos $\mathbf{Sh}(\mathsf{Man})$ of sheaves on the category Man of second-countable topological manifolds with the open-cover Grothendieck topology

- The second example is very closely analogous to the example of locally connected topological spaces. In particular, $\mathbf{Sh}(\mathsf{Man})$ contains the category of Δ-generated spaces as a full subcategory.

Cohesive ∞-toposes

Around 2010 (??), Urs Schreiber and Michael Shulman realized that the cohesive framework works even better in the ∞-categorical setting:

Definition

A **cohesive ∞-topos** is an ∞-topos $\mathcal{E}$ whose global sections functor $\Gamma = \mathcal{E}(1, -) : \mathcal{E} \to \mathcal{S}$ fits into a sequence $\Pi \dashv \Delta \dashv \Gamma \dashv \nabla$ of adjoints satisfying the axioms of cohesion 1–4.

Remark

(Here $\mathcal{S}$ is the ∞-topos of homotopy types, a.k.a. 'spaces'.)

Examples

- The ∞-topos $[\Delta^{\mathrm{op}}, \mathcal{S}]$ of simplicial types
- The topos ∞-$\mathbf{Sh}(\mathsf{Man})$ of 'topological stacks', i.e. ∞-sheaves on Man

- Note that we dropped the subscript 0 on the left-most adjoint Π. This is because in the higher world, Π sends a cohesive/spatial objects $X \in \mathcal{E}$ not to their connected components, but to their **homotopy types**.

The cohesive ∞*-topos* $\infty\text{-}\mathbf{Sh}(\mathrm{Man})$ *of 'topological stacks'*

- Topological spaces embed into $\infty\text{-}\mathbf{Sh}(\mathrm{Man})$ in three ways:
  - Every space X gives a stack $\mathbf{Top}(J-, X)$ (its 'nerve'), where $J : \mathrm{Man} \to \mathbf{Top}$ is the inclusion
  - Every space X represents a homotopy type, i.e. an object of $\mathcal{S}$, which can be mapped into $\infty\text{-}\mathbf{Sh}(\mathrm{Man})$ via Δ and ∇
- We reserve the word 'space' for the 'cohesive' stacks obtained by the nerve construction, and therefore refer to the objects of $\mathcal{S}$ as *(homotopy) types*.
- There's also potential for confusion around the word *discrete*.
  For us, 'discrete' means 'cohesively discrete', i.e. in the image of Δ, rather than 0-truncated.

The modalities

- Schreiber and Shulman introduced names for the (co)monads induced by the adjunctions $\Pi \dashv \Delta \dashv \Gamma \dashv \nabla$:
    - $\int = \Delta \circ \Pi$ is the **shape modality**
    - $\flat = \Delta \circ \Gamma$ is the **flat modality**
    - $\sharp = \nabla \circ \Gamma$ is the **sharp modality**
    - $\int$ and $\sharp$ are idempotent *monads*, $\flat$ is an idempotent *comonad*
- Would like to use these modalities in type theoretic reasoning, i.e. use HoTT with modal operators as internal language of cohesive ∞-toposes.
- Ideally, one would like to introduce new type formers:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \int A \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \flat A \text{ type}} \qquad \frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash \sharp A \text{ type}}$$

- However, there's a problem: it's not clear what the meaning of $\flat$ should be in arbitrary context Γ: Given a map $A \to \Gamma$ in $\mathcal{E}$, there is no canonical way to 'vertically discretize' while keeping the same points.
- This means we can only allow to use $\flat$ in *discrete contexts*

Flat type theory

- To accommodate $\flat$ we introduce split contexts and consider judgments of the forms

$$\Delta \mid \Gamma \vdash A \ \mathsf{type} \qquad\qquad \Delta \mid \Gamma \vdash t : A,$$

  where Δ is the *discrete context* and Γ the *cohesive context*.
- Then the rules for the flat modality are then the following

$$\frac{\Delta \mid \cdot \vdash A \ \mathsf{type}}{\Delta \mid \cdot \vdash \flat A \ \mathsf{type}} \qquad\qquad \frac{\Delta \mid \cdot \vdash t : A}{\Delta \mid \cdot \vdash t^\flat : \flat A}$$

$$\frac{\Delta \mid x : \flat A, \Gamma \vdash C \ \mathsf{type} \qquad \Delta \mid \Gamma \vdash t : \flat A \qquad \Delta, y :: A \mid \Gamma \vdash u : C[y^\flat / x]}{\Delta \mid \Gamma \vdash (\mathsf{let}\ y := t \ \mathsf{in}\ u) : C[t/x]}$$

  i.e. we can only form $\flat A$ in a purely discrete context.
- Introduced by Shulman in[19], based on ideas in[20].

[19] M. Shulman. "Brouwer's fixed-point theorem in real-cohesive homotopy type theory". In: *Mathematical Structures in Computer Science* (2018).

[20] F. Pfenning and R. Davies. "A judgmental reconstruction of modal logic". English. In: *MSCS. Mathematical Structures in Computer Science* (2001).

Brouwer's fixed point theorem in cohesive HoTT

- Shulman introduced flat type theory (and more generally *spatial* type theory) in[21], to give an analysis of **Brouwer's fixed point theorem**.

Theorem (Brouwer)

Let $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Every continuous $f : D^2 \to D^2$ has a fixed point.

- It's known that the fixed point can't be chosen 'effectively' in a way continuously depending on f.
- This non-effectivity can be made explicit in $\mathbf{Sh}(\mathsf{Man})$ as follows:

Theorem (Shulman)

The judgment

$$f :: D^2 \to D^2 \mid \cdot \vdash \exists x \,.\, f(x) = x$$

holds in $\mathbf{Sh}(\mathsf{Man})$, but the judgment

$$\cdot \mid f : D^2 \to D^2 \vdash \exists x \,.\, f(x) = x$$

does *not* hold in $\mathbf{Sh}(\mathsf{Man})$,

[21] M. Shulman. "Brouwer's fixed-point theorem in real-cohesive homotopy type theory". In: *Mathematical Structures in Computer Science* (2018).

Covering spaces in cohesive HoTT

- Given an object $A \in \infty\text{-}\mathbf{Sh}(\mathrm{Man})$, we can form its **fundamental groupoid**

$$\Pi_1(A) = \|\Pi(A)\|_1$$

  as 1-**truncation** of its homotopy type $\Pi(A) \in \mathcal{S}$.
- By the **fundamental theorem of covering spaces**, covering spaces on A correspond to maps

$$\Pi_1(A) \to \mathbf{Set} = \{S \in U_{\mathcal{S}} \mid S \text{ 0-truncated}\}.$$

- Since $\mathbf{Set}$ is 1-truncated we can drop the truncation and transpose to get an equivalent presentation as

$$A \to \Delta(\mathbf{Set}) = \flat\{B \in U \mid B \text{ discrete and 0-truncated}\}.$$

- Ideas along these lines are developed in[22], see also[23].

[22] F. Cherubini and E. Rijke. "Modal descent". English. In: *MSCS. Mathematical Structures in Computer Science* (2021).

[23] https://youtu.be/ACGjJDarEc4?t=2788

Outlook

- Differential cohesion introduces even more modalities
- Need for new type theories[24] and more modular/flexible proof assistants

[24] D. Gratzer, G.A. Kavvos, A. Nuyts, and L. Birkedal. "Multimodal dependent type theory". English. In: *Logical Methods in Computer Science* (2021). Id/No 11. URL: lmcs.episciences.org/7713.

Thanks for your attention!