Notes on 2-categorical limits

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Abstract

These are some notes that I made for my talk in the "Groupe de travail catégories supérieures, polygraphes et homotopie" on February 12, 2010 at PPS.

The right notion of limit for 2-categories is *weighted limit*, which comes from the theory of enriched categories. We start by having a look at the general case.

1 Weighted limits in enriched categories

The standard reference on weighted limits in enriched categories is Kelly's [Kel05].

First of all, we recall the definition of 'ordinary limits' in 'ordinary categories' The limit of a functor $G : \mathbb{I} \to \mathbb{C}$ is — whenever it exists — defined to be the object that represents the presheaf

$$\mathbb{C}^{\mathbb{I}}(\Delta-, G), \tag{1.1}$$

where $\Delta : \mathbb{C} \to \mathbb{C}^{\mathbb{I}}$ is the functor that sends $C \in \mathbb{C}$ to the functor that is constant C. In other words, the limit of G is characterized by

$$\mathbb{C}(X, \lim G) \cong \mathbb{C}^{\mathbb{I}}(\Delta X, G) \qquad \text{natural in } X. \tag{1.2}$$

We could try to transport this approach directly to enriched category theory, but *this does not work, since the functor Δ is generally not definable for enriched $\mathbb{I}$ and $\mathbb{C}$!* (It is the exponential transpose of $\pi : \mathbb{C} \times \mathbb{I} \to \mathbb{C}$, which exists only if the base category is cartesian. So for **Cat** it would work, but I want to make a general point here.) The solution is to rephrase (1.2) in a way that works in the enriched case. Observe that a natural transformation of type

$$\Delta X \to G : \mathbb{I} \to \mathbb{C}$$

is 'the same thing' as a transformation

$$\Delta 1 \to \mathbb{C}(X, G-) : \mathbb{I} \to \mathbf{Set}, \tag{1.3}$$

whence

$$\mathbb{C}^{\mathbb{I}}(\Delta X, G) \cong \mathbf{Set}^{\mathbb{I}}(\Delta 1, \mathbb{C}(X, G-)). \tag{1.4}$$

The use of the dash here is not very fortunate, since it is not clear at which point the free variable represented by '$-$' gets bound in the expression. Even worse, we would like to replace the X by a second dash get an analogue expression to (1.1).

One way to make this more precise and in particular to avoid the double dash is to replace the $\mathbb{C}(X, G-)$ by $Y_{\mathbb{C}}X \circ G$, where Y is the contravariant Yoneda embedding. Another possibility is to write an explicit lambda abstraction. This is never done in the literature, but perfectly clean, since **Cat** and categories of enriched categories are symmetric monoidal closed[1]. Thus, we can write

$$\mathbb{C}^{\mathbb{I}}(\Delta-, G) \cong \mathbf{Set}^{\mathbb{I}}(\Delta 1, \lambda I \,.\, \mathbb{C}(-, GI)). \tag{1.5}$$

The dashes here are unproblematic, since the corresponding variable is meant to be bound at the outermost level, as one would expect. The idea of a *weighted limit* is now to replace the $\Delta 1$ (which might still not exist in enriched context) by an arbitrary presheaf $F : \mathbb{I} \to \mathbf{Set}$, or an enriched functor $F : \mathbb{I} \to \mathscr{V}$ in the $\mathscr{V}$-enriched case — the weight. In the **Set** case, this gives nothing new, every weighted limit may be replaced by an ordinary limit (This is related to the fact that every set is a small coproduct of the one-element set).

For the enriched case, we define

Definition 1.1 Let $\mathscr{V}$ be a complete and cocomplete symmetric monoidal category, $\mathbb{I}$ a small $\mathscr{V}$-category, and $\mathbb{C}$ a $\mathscr{V}$-category. Furthermore, let $F : \mathbb{I} \to \mathscr{V}$ and $G : \mathbb{I} \to \mathbb{C}$ be enriched functors. A limit of G weighted by F is given by a pair $(\{F, G\}, \xi)$ where $\{F, G\}$ is an object of $\mathbb{C}$ and

$$\xi \;:\; F \to \lambda I \,.\, \mathbb{C}(\{F, G\}, GI) \;:\; \mathbb{I} \to \mathscr{V}$$

is a natural transformation such that the induced transformation (via Yoneda)

$$\hat{\xi} \;:\; \mathbb{C}(-, \{F, G\}) \to [\mathbb{I}, \mathscr{V}](F, \lambda I \,.\, \mathbb{C}(-, GI)) \;:\; \mathbb{C}^{\mathsf{op}} \to \mathscr{V}$$

is an isomorphism.

Here, $[\mathbb{I}, \mathscr{V}]$ denotes the enriched functor category. $\diamondsuit$

2 Weighted limits in 2-categories

The first(?) paper on this was [Str76], a collection of basic facts can be found in [Kel89], and [BKPS89] describes the class of *flexible limits*, which are especially important in the context of algebras for 2-monads [BKP89, Lac05].

Since 2-categories are **Cat**-enriched categories, the weights take values in **Cat**. More precisely, given a small 2-category $\mathscr{I}$, a 2-category $\mathscr{B}$ and 2-functors $F : \mathscr{I} \to \mathbf{Cat}, G : \mathscr{I} \to \mathscr{B}$, the *2-limit of G weighted by F* is given by an object $\{F, G\}$ of $\mathscr{B}$ together with a 2-natural transformation

$$\xi \;:\; F \to \lambda I \,.\, \mathscr{B}(\{F, G\}, GI) \;:\; \mathscr{I} \to \mathbf{Cat}.$$

such that the induced 2-natural transformtion

$$\hat{\xi} \;:\; \mathscr{B}(-, \{F, G\}) \to [\mathscr{I}, \mathbf{Cat}](F, \lambda I \,.\, \mathscr{B}(-, GI)) \;:\; \mathscr{B}^{\mathsf{op}} \to \mathbf{Cat} \tag{2.1}$$

is an isomorphism. ($[\mathscr{I}, \mathbf{Cat}]$ denotes the 2-category of 2-functors, 2-natural transformations and modifications, i.e., the exponential in the category of 2-categories.)

[1] Provided that the category that we enrich in is symmetric monoidal closed and has small limits

Name	$\mathscr{I}$	F
inserter	$\bullet \rightrightarrows \bullet$	$1 \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{2}$
iso-inserter	$\bullet \rightrightarrows \bullet$	$1 \underset{1}{\overset{0}{\rightrightarrows}} \nabla 2$
comma-object	$\bullet \longrightarrow \bullet \longleftarrow \bullet$	$1 \xrightarrow{0} \mathbb{2} \xleftarrow{1} 1$
iso-comma-object	$\bullet \longrightarrow \bullet \longleftarrow \bullet$	$1 \xrightarrow{0} \nabla 2 \xleftarrow{1} 1$
equifier	$\bullet \overset{\Downarrow\ \Downarrow}{\rightrightarrows} \bullet$	$1 \underset{1}{\overset{0}{\overset{\Downarrow\ \Downarrow}{\rightrightarrows}}} \mathbb{2}$
'lax limit'	$\bullet \longrightarrow \bullet$	$1 \xrightarrow{0} \mathbb{2}$
inverter	$\bullet \overset{\Downarrow}{\rightrightarrows} \bullet$	$1 \underset{1}{\overset{0}{\overset{\Downarrow}{\rightrightarrows}}} \nabla 2$
identifier	$\bullet \overset{\Downarrow}{\rightrightarrows} \bullet$	$1 \overset{\Downarrow}{\rightrightarrows} 1$
cotensor with $\mathbb{C}$	$\bullet$	$\mathbb{C}$

$\mathbb{2}$ denotes the category with two objects 0, 1 and one nontrivial arrow $0 \to 1$, and $\nabla 2$ denotes the indiscrete category with two objects.

Table 1: Some weights for 2-limits

The most familiar phrasing of 1-dimensional limits is in terms of 'terminal cones' on diagrams, and the role of the terminal cone is here taken by ξ. We also have a universal lifting property of ξ with respect to other 'cones', but this property is not sufficient to exhibit a given cone as 2-limit. The lifting property corresponds only to the bijection on objects of the 2-natural isomorphism (2.1), and to show that a given cone is a 2-limit, we furthermore have to show that modifications between parallel cones can uniquely be lifted.

Table 1 lists some important types of weighted limits.

3 Pseudolimits and lax limits

Enrichment does not capture everything that is going on in 2-dimensional category theory, since it only allows to talk about 'strict' constructions such as 2-functors and 2-transformations. It is also possible to define pseudolimits and lax limits, and this gives rise to some interesting examples, such as the category of Eilenberg-Moore algebras of a monad as a lax limit of the lax functor that corresponds to the monad. Formally, a pseudo or lax limit of a pseudo or lax functor is what we get wen whe replace the $[\mathscr{I}, \mathbf{Cat}]$ in (2.1) by one of the 2-categories $\mathbf{StrictPseudo}(\mathscr{I}, \mathbf{Cat})$, $\mathbf{StrictLax}(\mathscr{I}, \mathbf{Cat})$, $\mathbf{PseudoPseudo}(\mathscr{I}, \mathbf{Cat})$, $\mathbf{PseudoLax}(\mathscr{I}, \mathbf{Cat})$, $\mathbf{LaxLax}(\mathscr{I}, \mathbf{Cat})$ of strict/pseudo/lax functors and pseudo/lax transformations, and correspondingly allow ξ to be pseudo/lax. In connection with pseudo/lax functors, it would also make sense to allow the weights to be pseudo/lax, but this is not done in the literature; lax limits are normally only treated for the *conical* case, i.e. for constant weight 1, where we have the important case of monads. Thus, for example the lax limit of a strict 2-functor $G : \mathscr{I} \to \mathscr{B}$ weighted by a strict presheaf $F : \mathscr{I} \to \mathbf{Cat}$ is given by by a *lax transformation*

$$\xi \ : \ F \to \lambda I \,.\mathscr{B}(\{F, G\}, GI) \ : \ \mathscr{I} \to \mathbf{Cat}.$$

such that the induced *2-natural transformtion*

$$\hat{\xi} \ : \ \mathscr{B}(-, \{F, G\}) \to \mathbf{StrictLax}(\mathscr{I}, \mathbf{Cat})(F, \lambda I \,.\mathscr{B}(-, GI)) \ : \ \mathscr{B}^{\mathsf{op}} \to \mathbf{Cat}$$

is an isomorphism.

Although pseudolimits and lax limits are interesting, they don't really give a greater generality then strict weighted limits, since for any given lax diagram with lax weight, we can find a strict diagram an a strict weight (on another indexing category), such that the lax limit of the lax diagram weighted by the lax weight is eqivalent to the strict limit of the strict diagram weighed by the strict weight. This transfer has to be done in two steps.

First, we replace the indexing category $\mathscr{I}$ by an indexing category $\tilde{\mathscr{I}}$ such that $\mathbf{LaxLax}(\mathscr{I}, \mathscr{B}) \cong \mathbf{StrictLax}(\tilde{\mathscr{I}}, \mathscr{B})$ and lift the diagram $G : \mathscr{I} \to \mathscr{B}$ and the weight $F : \mathscr{I} \to \mathbf{Cat}$ to a strict diagram $\hat{G} : \tilde{\mathscr{I}} \to \mathscr{B}$ and a strict weight $\hat{F} : \tilde{\mathscr{I}} \to \mathbf{Cat}$. It is an old observation by Bénabou that this is always possible.

The second step has to lead us from the world of lax transformations to the world of strict transformations. For this, we make use of the fact that the canonical embedding

$$\mathbf{StrictStrict}(\mathscr{I}, \mathbf{Cat}) \to \mathbf{StrictLax}(\mathscr{I}, \mathbf{Cat})$$

has a left 2-adjoint $(-)^{\dagger}$. Now $\{\hat{F}^{\dagger}, \hat{G}\}$ is a lax limit for G weighted by F.

The second step of this construction is described in [Kel89], but for the first step I only found some vague remarks in [Joh02].

References

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