Basic Combinatory Objects, Uniform Preorders and Partial Combinatory Algebras

Jonas Frey

Category Theory Octoberfest

29 October 2022

Dedicated to the Memory of Pieter Hofstra

Remembering Pieter (CMU Pittsburgh, 2016)

Overview

Two parts:

- Hofstra's Basic Combinatory Objects
- Two variations: DCOs and Uniform Preorders

Pieter Hofstra's Basic Combinatory Objects

Basic Combinatory Objects

In his 2006 paper *"All realizability is relative"*[1], Pieter Hofstra introduced the notion of *basic combinatory object* (building on jww van Oosten on *ordered PCAs*).

Definition

A **basic combinatory object (BCO)** is a set A equipped with a partial order $\leq$ and a set $\mathcal{F}_A$ of partial endofunctions called 'computable', which have down-closed domain, s.t.

1. $\exists i \in \mathcal{F}\ \forall a \in A\,.\, i(a) \leq a$
2. $\forall f, g \in \mathcal{F}\ \exists h \in \mathcal{F}\ \forall a \in \mathrm{dom}(g \circ f)\,.\, h(a) \leq g(f(a))$

BCOs form a locally ordered category **BCO** admitting a full and order-reflecting embedding

$$\mathrm{fam} : \mathbf{BCO} \hookrightarrow \mathbf{IOrd}$$

into the locally ordered category $\mathbf{IOrd} = [\mathbf{Set}^{\mathrm{op}}, \mathbf{Ord}]$ of **Set-indexed preorders**, given by $\mathrm{fam}(A)(J) = (A^J, \leq)$ where

$$\varphi \leq \psi \quad :\Leftrightarrow \quad \exists f \in \mathcal{F}_A\ \forall j \in J\,.\, f(\varphi(j)) \leq \psi(j)$$

for $\varphi, \psi : J \to A$.

[1] Hofstra, Pieter JW. "All realizability is relative." *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 141. No. 2. Cambridge University Press, 2006.

Basic Combinatory objects – finite meets

BCOs are closed under products in **IOrd**, thus $\mathrm{fam}(A)$ is an indexed meet-semilattice iff

$$A \to A \times A \qquad \text{and} \qquad A \to 1$$

have right adjoints

$$(- \wedge -) : A \to A \times A \qquad \text{and} \qquad \top : A \to 1$$

in **BCO**. We call such BCOs **cartesian.**

Basic Combinatory objects – existential quantification

- Say that an indexed preorder $P : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ **admits existential quantification**, if the reindexing maps $f^* : P(I) \to P(J)$ have left adjoints $\exists_f : P(J) \to P(I)$ for all $f : J \to I$, subject to the **Beck–Chevalley condition.**
- Denote by $\exists$-$\mathbf{IOrd}$ the subcategory of $\mathbf{IOrd}$ on indexed preorders admitting $\exists$ and indexed monotone maps preserving $\exists$.
- Pieter Hofstra showed that
  1. the forgetful functor $\exists$-$\mathbf{IOrd} \to \mathbf{IOrd}$ is 2-monadic, and
  2. the induced '**$\exists$-completion**' 2-monad $D : \mathbf{IOrd} \to \mathbf{IOrd}$ restricts to $\mathbf{BCO}$.

$$\begin{tikzcd} \mathbf{BCO} \arrow[r, dashed, "D"] \arrow[d, hook, "\mathrm{fam}"'] & \mathbf{BCO} \arrow[d, hook, "\mathrm{fam}"] \\ \mathbf{IOrd} \arrow[r, "D"] & \mathbf{IOrd} \end{tikzcd}$$

     For a BCO A, the carrier of $D(A)$ is the set of **downsets**.
  3. Furthermore, D plays well with finite meets: if $\mathcal{H}$ has finite meets then $D(\mathcal{H})$ has finite meets and moreover it satisfies the **Frobenius condition**.

Examples: BCOs from posets and (O)PCAs

- Every poset can be viewed as BCO where only the identity function is computable, which gives a full embedding
$$\mathbf{Pos} \hookrightarrow \mathbf{BCO}.$$

- Every **PCA** $\mathcal{A}$ can be viewed as a *cartesian* BCO where the ordering is trivial and
$$\mathcal{F}_{\mathcal{A}} = \{e\cdot(-) : \mathcal{A} \rightharpoonup \mathcal{A} \mid e \in \mathcal{A}\}.$$

- More generally, **filtered ordered PCAs** $\mathcal{A}$ can be viewed as cartesian BCO with
$$\mathcal{F}_{\mathcal{A}} = \{e\cdot(-) : \mathcal{A} \rightharpoonup \mathcal{A} \mid e \in \Phi_{\mathcal{A}}\}.$$

- Pieter observed that in both cases the associated **realizability tripos** $\mathrm{rt}(\mathcal{A}) : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}$ is given by
$$\mathrm{rt}(\mathcal{A}) \;=\; D(\mathsf{fam}(\mathcal{A})) \;=\; \mathsf{fam}(D(\mathcal{A}))$$

- In particular this means that **realizability triposes are freely generated under existential quantification!**
- This is related to the fact that realizability toposes are ex/lex-completions.

Characterizing filtered OPCAs among BCOs

Theorem (Hofstra)

TFAE for a cartesian BCO A:

1. A is (induced by) a filtered OPCA.
2. $\mathrm{fam}(D(A))$ is a tripos.
3. The fibers of $\mathrm{fam}(D(A))$ are Heyting algebras.

Proof.

The implications $1 \Rightarrow 2 \Rightarrow 3$ are clear.

For $3 \Rightarrow 1$, the application is informally given by the 'universal realizer of $\varphi \Rightarrow \psi, \varphi \vdash \psi$'. Specifically, let $\iota \in \mathrm{fam}(D(A))(A)$ be the function sending every a to its principal downset, and let $\varepsilon \in \mathcal{F}_A$ be a witness of the inequality

$$(\pi_1(\iota) \Rightarrow \pi_2(\iota)) \wedge \pi_1(\iota) \;\leq\; \pi_2(\iota)$$

in $\mathrm{fam}(D(A))(A \times A)$. Then the application operation of the OPCA is given by $\varepsilon \circ \wedge$.

The filter Φ_A is given by the **designated truth values**, i.e. the $a \in A$ that are equivalent to $\top$ in $\mathrm{fam}(A)(1)$.

□

Two variations : DCOs and uniform preorders

Overview

- Pieter's paper formed the starting point for my PhD thesis in which I gave characterizations of realizability triposes and toposes over PCAs.
- In hindsight, the only missing piece in the BCO-approach is that the image of **BCO** $\hookrightarrow$ **IOrd** does not have an easy characterization — if we could characterize (O)PCAs among BCOs and BCOs among indexed preorders then we could characterize (O)PCAs among indexed preorders.
- In the following I introduce a **sub- and a super-category** of **BCO** which *do* have simple characterizations in **IOrd**, and explain how to adapt Pieter's techniques.

DCO $\hookrightarrow$ **BCO** $\hookrightarrow$ **UOrd** $\hookrightarrow$ **IOrd**

Discrete combinatory objects

Definition

A **discrete combinatory object (DCO)** is simply a BCO whose partial order structure is trivial. We write $\mathbf{DCO} \subseteq \mathbf{BCO}$ for the full subcategory of DCOs.

Definition

Given an indexed preorder $\mathcal{H} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$, we call $\delta \in \mathcal{H}(J)$ **discrete**, if it is right orthogonal to all cartesian maps over surjections in the total category $\int \mathcal{H}$ of $\mathcal{H}$.

Lemma

An indexed preorder $\mathcal{H} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ is equivalent to one of the form $\mathrm{fam}(A)$ for a DCO A, iff it has a discrete generic predicate.

Proof.

Given a discrete predicate $\delta \in \mathcal{A}$, define DCO structure on A by taking as computable those partial functions $f : A \rightharpoonup A$ satisfying $\iota|_{\mathrm{dom}(f)} \leq f^*(\iota)$ in $\mathcal{H}(\mathrm{dom}(f))$. $\square$

Characterizing $\mathsf{fam}(\mathcal{A})$

We immediately get the following.

Lemma

An indexed meet-semilattice $\mathcal{H} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ *comes from a filtered PCA* $\mathcal{A}$ *iff it has a discrete generic predicate and* $D(\mathcal{H})$ *is a tripos. The filter is trivial iff* $\mathcal{H}(1) \simeq 1$.

- Filtered PCAs are better known as **inclusions of PCAs**, their realizability toposes are called **relative realizability toposes**.
- To be able to characterize (relative) realizability *triposes*, we have to reconstruct $\mathcal{H}$ from $D(\mathcal{H})$. This is what we do next.

$\exists$-prime predicates

As motivation consider non-indexed case:

- Given a poset P, the lattice of $D(P)$ of **downsets** in P is the **join-completion**, i.e. the **free sup-lattice** on P.
- The **principal downsets** $\downarrow x = \{y \in P \mid y \leq x\}$ can be characterized as **completely join-prime elements** in $D(P)$ – an element x of a lattice L is called completely join-prime if we have

$$x \leq \bigvee_{j \in J} y_j \quad \Rightarrow \quad \exists j \in J \,.\, x \leq y_j$$

  for all families $(y_j)_{j \in J}$ of elements.

Proposition

A complete lattice L is a join-completion iff it has **enough** completely join-prime elements, i.e. if every element is a join of completely-join-primes.
In this case L the join-completion of its completely join-prime elements.

- We can do something analogous, with $\exists$ instead of $\bigvee$.

$\exists$-prime predicates

Definition

Given an indexed preorder $\mathcal{H}$ which admits existential quantification, a predicate $\pi \in \mathcal{H}(I)$ is called **$\exists$-prime** if for all functions $I \xleftarrow{u} J \xleftarrow{v} K$ and predicates $\theta \in \mathcal{H}(K)$ such that $u^*\pi \leq \exists_v \theta$, there exists a section s of v such that $u^*\pi \leq s^*\theta$.

Proposition

An indexed preorder $\mathcal{H}$ is an $\exists$-completion iff it has enough $\exists$-prime predicates, i.e. if for every predicate $\varphi \in \mathcal{H}(I)$ there exists a function $u : J \to I$ and an $\exists$-prime predicate $\pi \in \mathcal{H}(J)$ with $\varphi \cong \exists_u \pi$.
In this case, we have $\mathcal{H} \simeq D(\mathcal{P})$ where $\mathcal{P} \subseteq \mathcal{H}$ is the indexed sub-preorder on $\exists$-prime predicates.

With this we can characterize (relative) realizability triposes!

Characterizing realizability triposes

Theorem

A tripos $\mathcal{H}$ is a relative realizability tripos over an inclusion of PCAs, iff

1. *$\mathcal{H}$ has enough $\exists$-prime predicates, and*
2. *the indexed sup-preorder $\mathcal{P} \subseteq \mathcal{H}$ on $\exists$-prime predicates is closed under finite meets and has a discrete generic predicate δ.*

- The discreteness condition on δ can be stated in $\mathcal{H}$ rather than $\mathcal{P}$, which is a slight strengthening.
- We get ordinary (non-relative) realizability if the tripos is 2-*valued*, i.e. $\mathcal{H}(1) \simeq \mathrm{Bool}$.

Uniform preorders

Rather than a subcategory, uniform preorders form a **super-category** of **BCO** inside **IOrd**.

$$\mathbf{DCO} \hookrightarrow \mathbf{BCO} \hookrightarrow \mathbf{UOrd} \hookrightarrow \mathbf{IOrd}$$

Definition

A **uniform preorder** is a set A with a set $\mathcal{R}_A \subseteq P(A \times A)$ of **binary relations** such that:

1. $r \in \mathcal{R}_A,\ s \subseteq r \implies s \in \mathcal{R}_a$
2. $r, s \in \mathcal{R}_A \implies s \circ r \in \mathcal{R}_A$
3. $\mathrm{id} \in \mathcal{R}_A$

- Uniform preorders form a locally ordered category **UOrd** which admits a full embedding $\mathrm{fam} : \mathbf{UOrd} \hookrightarrow \mathbf{IOrd}$ into indexed preorders, where $\mathrm{fam}(A)(J) = (A^J, \leq)$ with the ordering defined by

$$\varphi \leq \psi \quad :\Leftrightarrow \quad \{(\varphi(j), \psi(j)) \mid j \in J\} \in \mathcal{R}_A$$

  for $\varphi, \psi : J \to A$.
- To transform a BCO A into a uniform preorder, we take the smallest uniform preorder structure containing $\mathcal{F}_A \cup \{\leq\}$.
  Ordered structure and computable functions are both subsumed in the relational structure!

Indexed preorders arising from uniform preorders

The characterization of the image of **UOrd** $\hookrightarrow$ **IOrd** is very easy:

Lemma

An indexed preorder $\mathcal{H}$ can be represented by a uniform preorder iff it has a generic predicate.

Proof.

Given a generic predicate $\iota \in \mathcal{H}(A)$, define a uniform preorder structure on A by $\mathcal{R}_A = \{r \subseteq A \times A \mid p^*\iota \leq q^*\iota\}$ where $p, q : r \to A$ are the projections. $\square$

Finite meets and existential quantification

- Just as **BCO**, **UOrd** is closed under products in **IOrd** and the $\exists$-completion monad lifts to $D : \mathbf{UOrd} \to \mathbf{UOrd}$ (the latter is not true for **DCO**).
- Obvious question: given a cartesian uniform preorder A, when is $D(\mathrm{fam}(A))$ a tripos?

Theorem

*For A cartesian, $D(\mathrm{fam}(A))$ is a tripos iff there exists an $@ \in \mathcal{R}_A$ such that for all relations $r \in \mathcal{R}_A$ there exists a **total function** $\tilde{r} \in \mathcal{R}_A$ such that*

$$\forall a, b, c \in A \,.\, r(a \wedge b, c) \implies @(\tilde{r}(a) \wedge b, c).$$

I call uniform preorders satisfying this condition **relationally complete**. Examples are:

1. Uniform preorders induced by filtered OPCAs are relationally complete
2. For every tripos $\mathcal{K}$, the associated uniform preorder is relationally complete (since $D(\mathcal{K})$ is also a tripos).

Open question

- **Open Question**: Are there relationally complete uniform preorders that do not come from filtered OPCAs?
- I would think so but I haven't been able to come up with any examples!
- The paper

  Liron Cohen, Sofia Abreu Faro, and Ross Tate. "The effects of effects on constructivism." Electronic Notes in Theoretical Computer Science 347 (2019): 87-120.

  introduces a notion of **relational combinatory algebra**, but that doesn't seem to fit.

Thank you for your attention