

Moens' theorem and fibered toposes

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Plan of talk

- Elementary toposes and Grothendieck toposes
- Realizability toposes
- Fibered categories
- Characterizing realizability toposes

Elementary toposes and Grothendieck toposes

Elementary toposes

Definition (Lawvere, ca. 1970)

An **elementary topos** is a category \mathcal{E} with

- finite limits
- exponential objects B^A for $A, B \in \mathcal{E}$ (cartesian closed)
- a subobject classifier, i.e. a morphism $t : 1 \rightarrow \Omega$ such that for every monomorphism $m : U \rightarrowtail A$ there exists $\chi : A \rightarrow \Omega$ making

$$\begin{array}{ccc} U & \xrightarrow{\quad} & 1 \\ m \downarrow & & \downarrow t \\ A & \xrightarrow{\quad \chi \quad} & \Omega \end{array}$$

a pullback.

Grothendieck toposes

Grothendieck toposes can equivalently be defined in the following ways:

- ① Introduced around 1960 by G. as categories of *sheaves* on a *site*
- ② Characterized 1963 by Giraud as *locally small* ∞ -pretoposes with a *separating set of objects*
- ③ Equivalently: elementary topos \mathcal{E} admitting a (necessarily unique) *bounded geometric morphism* $\mathcal{E} \rightarrow \mathbf{Set}$
- ④ Inspired by 3, define a Grothendieck topos over an (elementary) base topos \mathcal{S} as a bounded geometric morphism $\mathcal{E} \rightarrow \mathcal{S}$

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What do all these words mean??

Locally small, separating set

- \mathbb{C} is called **locally small**, if the ‘homsets’ $\mathbb{C}(A, B)$ are really sets, as opposed to proper classes
- A **separating set of objects** in \mathbb{C} is a family $(C_i)_{i \in I}$ of objects indexed by a set I such that for all parallel pairs $f, g : A \rightarrow B$ we have

$$(\forall i \in I \forall h : C_i \rightarrow A. fh = gh) \Rightarrow f = g.$$

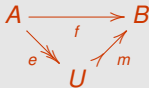
∞ -Pretoposes

Regular categories

∞ -pretopos = exact ∞ -extensive category
= effective regular ∞ -extensive category

Definition

A **regular category** is a category with finite limits and pullback-stable regular-epi/mono factorizations.



∞ -Pretoposes

Exact categories

- An *equivalence relation* in a f.l. category \mathbb{C} is a jointly monic pair $r_1, r_2 : R \rightarrow A$ such that for all $X \in \mathbb{C}$, the set

$$\{(r_1 x, r_2 x) \mid x : X \rightarrow R\}$$

is an equivalence relation on $\mathbb{C}(X, A)$

- The *kernel pair* of any morphism $f : A \rightarrow B$ – given by the pullback

$$\begin{array}{ccc} X & \longrightarrow & A \\ r_2 \downarrow & \lrcorner^{r_1} & \downarrow f \\ A & \longrightarrow & B \end{array}$$

is always an equivalence relation

Definition

An **exact** (or **effective regular**) category is a regular category in which every equivalence relation is a kernel pair.

∞ -Pretoposes

Extensive categories

Assume \mathbb{C} has finite limits and small coproducts

- Coproducts in \mathbb{C} are called **disjoint**, if the squares

$$\begin{array}{ccc} 0 & \longrightarrow & A_i \\ \downarrow & & \downarrow \\ A_j & \twoheadrightarrow & \coprod_{i \in I} A_i \end{array} \quad (i \neq j) \quad \text{and} \quad \begin{array}{ccc} A_i & \longrightarrow & A_i \\ \downarrow & & \downarrow \\ A_i & \twoheadrightarrow & \coprod_{i \in I} A_i \end{array}$$

are always pullbacks

- Coproducts in \mathbb{C} are called **stable**, if for any $f : B \rightarrow \coprod_{i \in I} A_i$, the family

$$(B_i \xrightarrow{\sigma_i} B)_{i \in I} \quad \text{given by pullbacks} \quad \begin{array}{ccc} B_i & \xrightarrow{\sigma_i} & B \\ \downarrow \lrcorner & & \downarrow f \\ A_i & \twoheadrightarrow & \coprod_{i \in I} A_i \end{array}$$

represents B as coproduct of the B_i

Definition

An ∞ -(l)extensive category is a category \mathbb{C} with finite limits and disjoint and stable small coproducts.

∞ -Pretoposes

Examples

- Complete lattices (A, \leq) viewed as categories have finite limits and small coproducts, but these are not disjoint – coproducts are stable precisely for *complete Heyting algebras*
- **Top** (topological spaces) and **Cat** (small categories) are ∞ -extensive but not regular
- Monadic categories over **Set** are always exact and have small coproducts, but are rarely extensive

Definition

An ∞ -**pretopos** is a category which is exact and ∞ -extensive.

Examples

- Grothendieck toposes
- the category of *small* presheaves on **Set**

Geometric morphisms

- A **geometric morphism** $\mathcal{E} \rightarrow \mathcal{S}$ between toposes \mathcal{E} and \mathcal{S} is an adjunction

$$(\Delta : \mathcal{S} \rightarrow \mathcal{E}) \dashv (\Gamma : \mathcal{E} \rightarrow \mathcal{S})$$

of f.l.p. functors (Δ is the ‘inverse image part’; Γ the ‘direct image part’)

- $(\Delta \dashv \Gamma)$ is called **bounded**, if there exists $B \in \mathcal{E}$ such that for every $E \in \mathcal{E}$ there exists a subquotient span $B \times \Delta(S) \leftarrow \bullet \rightarrow E$
- It is called **localic** if it is *bounded by 1*
- If $\Delta \dashv \Gamma : \mathcal{E} \rightarrow \mathbf{Set}$, then we necessarily have

$$\Delta(J) = \sum_{j \in J} 1 \quad \text{and} \quad \Gamma(A) = \mathcal{E}(1, A)$$

for $J \in \mathbf{Set}$ and $A \in \mathcal{E}$

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Remark

Without the bound in 3, \mathcal{E} need not be cocomplete. Example: subcategory of $\widehat{\mathbb{Z}}$ on actions with uniform bound on the size of orbits.

Realizability toposes

Realizability toposes

- Were introduced in 1980 by Hyland, Johnstone, and Pitts
- Not Grothendieck toposes
- Most well known: Hyland's **effective topos** *Eff* – ‘Universe of constructive recursive mathematics’
- usually constructed via *triposes*

Partial combinatory algebras

Definition

A **PCA** is a set \mathcal{A} with a partial binary operation

$$(- \cdot -) : \mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A}$$

having elements $k, s \in \mathcal{A}$ such that

$$(i) \ k \cdot x \cdot y = x \quad (ii) \ s \cdot x \cdot y \downarrow \quad (iii) \ s \cdot x \cdot y \cdot z \preceq x \cdot z \cdot (y \cdot z)$$

for all $x, y, z \in \mathcal{A}$.

Example

First Kleene algebra: (\mathbb{N}, \cdot) with

$$n \cdot m \simeq \phi_n(m) \quad \text{for } n, m \in \mathbb{N},$$

where $(\phi_n)_{n \in \mathbb{N}}$ is an effective enumeration of partial recursive functions.

Fibrations from PCAs

PCA \mathcal{A} gives rise to indexed preorders $\mathbf{fam}(\mathcal{A}), \mathbf{rt}(\mathcal{A}) : \mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Ord}$.

- **Family fibration:** $\mathbf{fam}(\mathcal{A})(J) = (\mathcal{A}^J, \leq)$, with

$$\varphi \leq \psi \quad :\Leftrightarrow \quad \exists \mathbf{e} \in \mathcal{A} \, \forall j \in J. \, \mathbf{e} \cdot \varphi(j) = \psi(j)$$

for $\varphi, \psi : J \rightarrow \mathcal{A}$.

- **Realizability tripos:** $\mathbf{rt}(\mathcal{A})(J) = ((P\mathcal{A})^J, \leq)$, with

$$\varphi \leq \psi \quad :\Leftrightarrow \quad \exists \mathbf{e} \in \mathcal{A} \, \forall j \in J \, \forall a \in \varphi(j). \, \mathbf{e} \cdot a \in \psi(j)$$

for $\varphi, \psi : J \rightarrow P\mathcal{A}$.

Observations

- $\mathbf{fam}(\mathcal{A})$ has indexed finite meets
- $\mathbf{rt}(\mathcal{A})$ models full 1st order logic
- both have *generic predicates*
- $\mathbf{rt}(\mathcal{A})$ is free cocompletion of $\mathbf{fam}(\mathcal{A})$ under \exists (Hofstra 2006)

Realizability toposes

Definition

- The **realizability topos** $\mathbf{RT}(\mathcal{A})$ over \mathcal{A} is the category of partial equivalence relations and compatible functional relations in \mathcal{A} (details omitted)
 - The **constant objects functor** $\Delta : \mathbf{Set} \rightarrow \mathbf{RT}(\mathcal{A})$ maps $J \in \mathbf{Set}$ to (J, δ_J) (discrete/diagonal equivalence relation)
-
- $\mathbf{RT}(\mathcal{A})$ is never a Grothendieck topos (except for the trivial pca)
 - Δ is bounded by 1 , but not the inverse image part of a geometric morphism
 - it makes sense to compare constant objects functors and inverse image functors, since both are instances of the same construction in the context of triposes

Fibered Categories

Δ and gluing fibrations

Goal: Understand inverse image functors

$$(\Delta : \mathbf{Set} \rightarrow \mathcal{E}) \vdash \Gamma$$

and constant objects functors

$$\Delta : \mathbf{Set} \rightarrow \mathbf{RT}(\mathcal{A})$$

better by looking at their *gluing fibrations*, defined by the pullback

$$\begin{array}{ccc} \mathbf{Gl}_{\Delta}(\mathcal{E}) & \longrightarrow & \mathcal{E} \downarrow \mathcal{E} \\ \text{gl}_{\Delta}(\mathcal{E}) \downarrow & \lrcorner & \downarrow \text{cod}(\mathcal{E}) \\ \mathbf{Set} & \xrightarrow{\Delta} & \mathcal{E} \end{array}$$

Fibered category theory

References

- Jean Bénabou, *Fibered categories and the foundations of naive category theory*, 1985
- Thomas Streicher, *Fibred categories à la Jean Bénabou*, unpublished, 1999-2012
- Peter Johnstone, *Sketches of an Elephant*, 2003

Idea/Philosophy

- *Elementary category theory*: finitary conditions, first order axiomatizable, no size conditions, avoid ZFC (f.l. category, elementary topos)
- *Naive category theory*: not concerned about formal, foundational aspects, use size conditions and make reference to **Set** freely
- Bénabou proposes fibrations to reconcile both, fibrations allow to express 'non-finitary conditions' in an elementary manner
- generalize and form analogies from *family fibrations*

Family fibrations

Definition

Let \mathbb{C} be a category.

- The category $\mathbf{Fam}(\mathbb{C})$ has families $(C_i)_{i \in I}$ of objects of \mathbb{C} as objects; a morphism $(C_i)_{i \in I} \rightarrow (D_j)_{j \in J}$ is a pair

$$(u : I \rightarrow J, (f_i : C_i \rightarrow D_{ui})_{i \in I}).$$

- The **family fibration** of \mathbb{C} is the functor

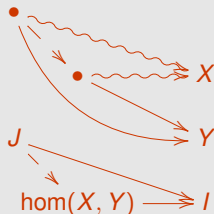
$$\begin{array}{rclcl} \mathbf{fam}(\mathbb{C}) & : & \mathbf{Fam}(\mathbb{C}) & \rightarrow & \mathbf{Set} \\ & & (C_i)_{i \in I} & \mapsto & I \\ & & (u, (f_i)_{i \in I}) & \mapsto & u \end{array}$$

mapping $(C_i)_{i \in I} \mathbf{fam}(\mathbb{C}) : \mathbf{Fam}(\mathbb{C}) \rightarrow \mathbf{Set}$ of a category \mathbb{C} is the fibration having

Local smallness

Definition

Let $P : \mathbb{X} \rightarrow \mathbb{B}$ be a fibration, $I \in \mathbb{B}$, $X, Y \in P(I)$. A **family of morphisms** from X to Y is a span $X \xleftarrow{c} \bullet \xrightarrow{f} Y$ where $P(c) = P(f)$ and c is cartesian. P is called **locally small**, if for every pair $X, Y \in P(I)$ there exists a *universal* family of morphisms (terminal among such spans).



Lemma

A category \mathbb{C} is locally small, iff $\text{fam}(\mathbb{C})$ is locally small in the above sense.

Finite limit fibrations

... towards extensive fibrations and Moens' theorem

Definition

Let \mathbb{B} be a f.l. category. A **finite limit fibration** on \mathbb{B} is a fibration $P : \mathbb{X} \rightarrow \mathbb{B}$ satisfying either of the following equivalent definitions.

- \mathbb{X} has finite limits and P preserves them
- All fibers $P(I)$ have finite limits, and they are preserved under reindexing

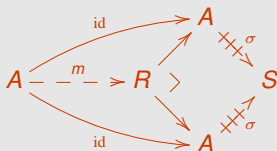
Lemma

A category \mathbb{C} has finite limits iff $\mathbf{fam}(\mathbb{C})$ is a finite limit fibration.

Extensive fibrations

Let $P : \mathbb{X} \rightarrow \mathbb{C}$ be a finite limit fibration.

- P is said to have **internal sums**, if it is also an *opfibration* ($P^{\text{op}} : \mathbb{X}^{\text{op}} \rightarrow \mathbb{C}^{\text{op}}$ is a fibration), and cocartesian maps in \mathbb{X} are stable under pullback along cartesian maps ('Beck-Chevalley condition')
- P is said to have **stable internal sums**, if cocartesian maps are stable under pullback along *arbitrary* maps in \mathbb{X}
- Internal sums are called **disjoint**, if the mediating arrow m in the diagram



is cocartesian for every cocartesian map $\sigma : A \rightarrow S$ in \mathbb{X}

- An **extensive fibration** is a finite-limit fibration with stable disjoint internal sums.

Lemma

A category \mathbb{C} is ∞ -extensive iff $\text{fam}(\mathbb{C})$ is extensive.

Moens' theorem

- Fundamental fib's $\text{cod}(\mathbb{D}) : \mathbb{D} \downarrow \mathbb{D} \rightarrow \mathbb{D}$ of f.l. cat's are extensive
- Extensive fib's are stable under pullback along f.l.p. functors $\Delta : \mathbb{C} \rightarrow \mathbb{D}$
- Thus, gluing fibrations $\text{gl}_\Delta(\mathbb{D}) : \text{Gl}_\Delta(\mathbb{D}) \rightarrow \mathbb{C}$ are extensive

Theorem (Moens' theorem)

The assignment $\Delta \mapsto \text{gl}_\Delta(\mathbb{D}) = \Delta^* \text{cod}(\mathbb{D})$ gives rise to a biequivalence

$$\mathbf{ExtFib}(\mathbb{C}) \simeq \mathbb{C} // \mathbf{Lex}$$

between the 2-category $\mathbf{ExtFib}(\mathbb{C})$ of extensive fibrations on \mathbb{C} and the pseudo-co-slice 2-category $\mathbb{C} // \mathbf{Lex}$ of f.l. categories under \mathbb{C} .

$\mathbf{ExtFib}(\mathbb{C}) \rightarrow \mathbb{C} // \mathbf{Lex}$

The functor corresponding to a fibration $P : \mathbb{X} \rightarrow \mathbb{C}$ is given by

$$\begin{array}{ccc} \Delta : \mathbb{C} & \rightarrow & \mathbb{X}(1) \\ \mathbb{C} & \mapsto & \sum_c 1 \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{++++} & \sum_c 1 \\ \mathbb{C} & \xrightarrow{\quad\quad} & 1 \end{array}$$

Gluing fibrations for Grothendieck toposes and realizability toposes

- For Grothendieck toposes \mathcal{E} with geometric morphism $\Delta \dashv \Gamma : \mathcal{E} \rightarrow \mathbf{Set}$, we have

$$\mathrm{gl}_{\Delta}(\mathcal{E}) \simeq \mathrm{fam}(\mathcal{E})$$

- Thus, when studying Grothendieck toposes $\Delta \dashv \Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ relative to a base topos \mathcal{S} , the fibration $\mathrm{gl}_{\Delta}(\mathcal{E})$ is an adequate substitute for the family fibration
- For realizability toposes with c.o.f. $\Delta : \mathbf{Set} \rightarrow \mathbf{RT}(\mathcal{A})$, the fibrations $\mathrm{gl}_{\Delta}(\mathbf{RT}(\mathcal{A}))$ and $\mathrm{fam}(\mathbf{RT}(\mathcal{A}))$ are different
- We will see just how different

Gluing and local smallness

Theorem

If $\Delta : \mathcal{S} \rightarrow \mathcal{E}$ is a f.l.p. functor between toposes, then $\text{gl}_\Delta(\mathcal{E})$ is a locally small fibration iff Δ has a right adjoint

- Thus, gluing fibrations $\text{gl}_\Delta(\mathbf{RT}(\mathcal{A}))$ of realizability toposes are not locally small

We have two ways of looking at realizability toposes

- From the point of view of ordinary CT, toposes $\mathbf{RT}(\mathcal{A})$ are locally small, but not cocomplete
- Viewed as gluing fibrations, they have small sums, but are not locally small

Characterizing Realizability Toposes

Motivation

- Peter Johnstone pointed out the lack of a ‘Giraud style’ theorem for realizability toposes
- It seemed easier to characterize the gluing fibrations $\mathbf{gl}_{\Delta}(\mathbf{RT}(\mathcal{A}))$ (or equivalently the functors $\Delta : \mathbf{Set} \rightarrow \mathbf{RT}(\mathcal{A})$) than the ‘bare’ toposes
- Fibrationally realizability toposes resemble presheaf toposes

Moens' theorem for fibered pretoposes

- A **pre-stack** is a fibration $P : \mathbb{X} \rightarrow \mathbb{R}$ on a regular category \mathbb{R} where the reindexing functors $e^* : P(I) \rightarrow P(J)$ are full and faithful for all regular epis $e : J \rightarrow I$
- All fibrations on **Set** are pre-stacks with AC, and without still most
- A **fibered pretopos** is an extensive pre-stack $P : \mathbb{X} \rightarrow \mathbb{R}$ with exact fibers
- $\text{fam}(\mathcal{E})$ is a fibered pretopos iff \mathcal{E} is an ∞ -pretopos

Theorem (Moens' theorem for fibered pretoposes)

The assignment $\Delta \mapsto \text{gl}(\Delta)$ gives rise to a biequivalence

$$\mathbf{PretopFib}(\mathbb{R}) \simeq \mathbb{R} // \mathbf{Ex}$$

between the 2-category $\mathbf{PretopFib}(\mathbb{R})$ of fibered pretoposes on \mathbb{R} and the pseudo-co-slice 2-category $\mathbb{R} // \mathbf{Ex}$ of exact categories under \mathbb{C} .

Fibered presheaf construction

Theorem

Let \mathbb{R} be a regular category The forgetful functor

$$\mathbf{PretopFib}(\mathbb{R}) \rightarrow \mathbf{Lex}(\mathbb{R}),$$

where $\mathbf{Lex}(\mathbb{R})$ is the category of finite-limit pre-stacks on \mathbb{R} , has a left biadjoint $\mathcal{C} \mapsto \widehat{\mathcal{C}}$, called **fibered presheaf construction**.

- If \mathbb{C} is a small category with finite limits, then $\widehat{\mathbf{fam}(\mathbb{C})} = \mathbf{fam}(\mathbf{Set}^{\mathbb{C}^{\text{op}}})$
- For any PCA \mathcal{A} we have $\widehat{\mathbf{fam}(\mathcal{A})} = \mathbf{gl}_{\Delta}(\mathbf{RT}(\mathcal{A}))$

Characterization of fibrations of presheaves

Which fibered pretoposes $P : \mathbb{X} \rightarrow \mathbb{R}$ are of the form $\mathcal{X} \simeq \hat{\mathcal{C}}$?

Theorem (Bunge 77)

A locally small ∞ -pretopos \mathcal{E} is a presheaf topos iff it has a separating family of **indecomposable projective** objects.

In a similar way, we can show:

Theorem

A fibered pretopos $\mathcal{X} : |\mathcal{X}| \rightarrow \mathbb{R}$ is a fibration of presheaves iff

- the subfibration of \mathcal{X} on **indecomposable projectives** is closed under finite limits, and
- Every $X \in |\mathcal{X}|$ can be covered by an internal sum of **indecomposable projectives**.

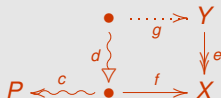
... where indecomposable projectives in fibrations are defined on the next slide

Indecomposables and projectives

Let $\mathcal{X} : |\mathcal{X}| \rightarrow \mathbb{R}$ be a fibered pretopos.

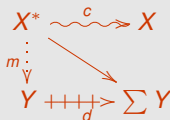
Definition

- Call $P \in |\mathcal{X}|$ **projective**, if given c, e, f as in the diagram



where c is cartesian and e is vertical and a regular epimorphism in its fiber, we can fill in d, g with d epicartesian such that the square commutes.

- Call $X \in |\mathcal{X}|$ **indecomposable**, if for every diagram



in $|\mathcal{X}|$ where c is cartesian and d is cocartesian, there exists a *unique* mediating arrow m .

Characterizing fibered realizability toposes

With a bit of work one can prove the following

Theorem

Gluing fibrations $\mathbf{gl}_{\Delta}(\mathbf{RT}(\mathcal{A}))$ of realizability toposes can be characterized as fibered pretoposes $P : \mathbb{X} \rightarrow \mathbf{Set}$ such that

- P is a fibered cocompletion (previous theorem)
- the fibers of P are lccc
- The subfibration $Q \subseteq P$ on indecomposable projectives is posetal, has a **discrete** generic predicate, and $Q(1) \simeq 1$

[*discrete* means right orthogonal to cartesian maps over surjective functions]

Characterizing realizability toposes

In realizability toposes, we have $(\mathbf{RT}(\mathcal{A})(1, -) : \mathbf{RT}(\mathcal{A}) \rightarrow \mathbf{Set}) \dashv \Delta$, thus the global sections functor is uniquely determined and does not contain additional information. Thus, our analysis yields a characterization of ‘bare’ toposes after all:

Theorem

A locally small category \mathcal{E} is equivalent to a realizability topos $\mathbf{RT}(\mathcal{A})$ over a PCA \mathcal{A} , if and only if

- ① \mathcal{E} is exact and locally cartesian closed,
- ② \mathcal{E} has enough projectives, and the subcategory $\mathbf{Proj}(\mathcal{E})$ of projectives is closed under finite limits,
- ③ the global sections functor $\Gamma : \mathcal{E} \rightarrow \mathbf{Set}$ has a right adjoint Δ factoring through $\mathbf{Proj}(\mathcal{E})$, and
- ④ there exists a separated and discrete projective $D \in \mathcal{E}$ such that for all projectives $P \in \mathcal{E}$ there exists a closed $u : P \rightarrow D$.