Factorizations on clans and monadicity

Jonas Frey

Abstract

We discuss work in progress (joint with Mathieu Anel) on weak and orthogonal factorization systems on the 2-category of clans, and the question when clan morphisms induce monadic adjunctions between categories of models.

Clans [Joy17, Fre23] are categorical representations of generalized algebraic theories (GATs) which contain more information than *finite-limit theories*, but are less strict than Cartmell's contextual categories [Car78, Car86]. Formally, a clan is a (typically small) category $\mathcal{C}$ with a terminal object 1 and a designated class $\mathcal{C}^\dagger$ of *display maps* which contains all isomorphisms and terminal projections, and is closed under composition and pullback along arbitrary maps; in particular, pullbacks along of display maps along arbitrary maps are assumed to exist.

The syntactic category of any GAT (i.e. the category of contexts and substitutions) is a clan, where the display maps are (the isomorphism closure of) projections $(\Gamma, \Delta) \to (\Gamma)$ to initial segments of contexts, and up to equivalence, every clan is of this form.

The category of *models* of a clan $\mathcal{C}$ is the full subcategory $\mathsf{Mod}(\mathcal{C}) \subseteq [\mathcal{C}, \mathsf{Set}]$ of the category of Set-valued functors on functors which preserve 1 and pullbacks of display maps. If $\mathcal{C}$ is the syntactic clan of a GAT, then this is equivalent to the category of models of the GAT.

Clan morphisms from $\mathcal{D}$ to $\mathcal{C}$ are functors which preserve 1 and pullbacks of display maps[1], and syntactically they can be viewed as extensions of a GAT by additional axioms. On the level of models, clan morphisms $\phi : \mathcal{C} \to \mathcal{D}$ give rise to adjunctions: the 'forgetful' functor $\phi^* : \mathsf{Mod}(\mathcal{D}) \to \mathsf{Mod}(\mathcal{C})$ given by precomposition always has a left adjoint 'free' functor $\phi_! : \mathsf{Mod}(\mathcal{C}) \to \mathsf{Mod}(\mathcal{D})$. A natural question to ask is when this adjunction is monadic (in which case we will call the clan-morphism monadic), and whether the clan-perspective can shed light on the issue of (non)composability of monadic adjunctions.

1 Finite-product theories and finite-limit theories

The issue of composability of monadic adjunctions is easy in the case of ordinary, *non-dependent*, algebraic theories, which correspond to *finite-product categories* [ARV10]. Note that every finite-product category admits a clan structure whose display maps are the product projections, and this gives rise of a full embedding $\mathsf{FP} \hookrightarrow \mathsf{Clan}$ from the 2-category of finite-product categories to that of clans. We identify finite-product categories with their associated clans and call them FP-clans.

1.1 Proposition *A morphism $\phi : \mathcal{C} \to \mathcal{D}$ of FP-clans is monadic if and only if ϕ is* Cauchy-surjective, *i.e. every $D \in \mathcal{D}$ is a retract of some $\phi(C)$. In particular, monadic morphisms of FP-clans are closed under composition.*

Proof. If ϕ is Cauchy-surjective then ϕ^* is easily seen to be conservative, and since reflexive coequalizers—being sifted colimits—are pointwise in categories of models of algebraic theories, they are preserved by ϕ^*. Thus, monadicity follows from the crude monadicity theorem.

Conversely, if ϕ is not Cauchy-surjective then it factors through an FP-clan $\mathcal{U}$ as $\phi = (\mathcal{C} \xrightarrow{\psi} \mathcal{U} \xrightarrow{\theta} \mathcal{D})$ with ψ Cauchy-surjective and θ fully faithful. The functor $\theta_! : \mathsf{Mod}(\mathcal{U}) \to \mathsf{Mod}(\mathcal{D})$ is given by

$$\theta_!(A) = \mathsf{col}\Big(\mathsf{elts}(A) \to \mathcal{U}^{\mathsf{op}} \xrightarrow{\theta} \mathcal{D}^{\mathsf{op}} \xrightarrow{\text{よ}} \mathsf{Mod}(\mathcal{D})\Big)$$

where $\mathsf{elts}(A)$ is sifted (having finite coproducts), and sifted colimits are computed pointwise in $\mathsf{Mod}(\mathcal{D})$, i.e. preserved by $\mathsf{Mod}(\mathcal{D}) \hookrightarrow [\mathcal{D}, \mathsf{Set}]$[2]. It follows that $\theta_! : \mathsf{Mod}(\mathcal{U}) \to \mathsf{Mod}(\mathcal{D})$ is fully faithful since $\theta_! : [\mathcal{U}, \mathsf{Set}] \to [\mathcal{D}, \mathsf{Set}]$, whence the adjunction is a coreflection, and this implies that ψ and $\phi = \theta\psi$ induce the same monad. ∎

The factorization $\phi = \theta\psi$ given in the proof is part of a bicategorical *orthogonal factorization system* (OFS) on the 2-category FP. A similar OFS also exists on the sub-2-category $\mathsf{FL} \subseteq \mathsf{Clan}$ of finite-limit categories (which can be identified with clans in which all maps are display maps, and we refer to as FL-clans from now on), but here the left class is not monadic anymore. However, there is a *weak* factorization system (WFS) on FL whose *right* class is

[1] Thus, a model of a clan is the same thing as a clan morphism to Set with the maximal clan structure where every map is a display map.

[2] In the terminology of [BMW12] this means that $\theta_! : \mathsf{Mod}(\mathcal{U}) \to \mathsf{Mod}(\mathcal{D})$ is a *functor with arities* w.r.t. the dense inclusions $\mathcal{U}^{\mathsf{op}} \hookrightarrow \mathsf{Mod}(\mathcal{U})$ and $\mathcal{D}^{\mathsf{op}} \hookrightarrow \mathsf{Mod}(\mathcal{D})$.

induces monadic adjunctions. This WFS is cofibrantly generated by the free FL-functor on the endpoint inclusion functor $\mathsf{cod} : 1 \to 2$ of the arrow category, and its right class consists of FL-functors $\phi : \mathcal{C} \to \mathcal{D}$ all of whose slice functors $\phi/C : \mathcal{C}/C \to \mathcal{D}/\phi(C)$ are essentially surjective. This class contains in particular all Grothendieck fibrations, whence we will refer to its elements as *weak fibrations*. Monadicity of weak fibrations between FL-clans will follow from a criterion that we give for clans below, and of course as a right class of a WFS they are closed under composition. However, there are monadic FL-clan maps which are not weak fibrations, and not closed under composition. An example is given by the FL-clan morphisms corresponding via Gabriel–Ulmer duality to the finitary adjunctions

$$\begin{tikzcd} \mathsf{Set} \arrow[r, shift right] \arrow[r, phantom, "\top"] & \mathsf{rGrph} \arrow[l, shift right] \arrow[r, shift right] \arrow[r, phantom, "\top"] & \mathsf{Cat} \arrow[l, shift right] \end{tikzcd}$$

between sets, reflexive graphs, and categories (note that rGph is monadic over Set via the forgetful functor which sends a graph to its set of edges).

2 Monadic clan morphisms

Syntactically, Proposition 1.1 can be interpreted as saying that adding new operations and/or equations to an algebraic theory yields monadic extensions, while adding new sorts does not. In GATs, there is an additional complication, since adding a new operation to a theory can yield new substitution instances of sorts, if the codomain of the new operation indexes another dependent sort. In this case, the extension by the new operation will be monadic, but further extensions where the substitution instance appears in the context will typically not be over the original theory. This insight has been formulated in similar forms by several participants in a recent discussion on the *Category Theory Zulip Server*[3].

In the following we will isolate a monadicity criterion on clan morphisms which cicrcumvents this issue and is closed under composition.

2.1 Proposition *A clan morphism $\phi : \mathcal{C} \to \mathcal{D}$ is monadic whenever the square*
$$\begin{tikzcd} \mathsf{Mod}(\mathcal{D}) \arrow[r, "\phi^*"] \arrow[d, hook] & \mathsf{Mod}(\mathcal{C}) \arrow[d, hook] \\ {[\mathcal{D}, \mathsf{Set}]} \arrow[r, "\phi^*"] & {[\mathcal{C}, \mathsf{Set}]} \end{tikzcd}$$
is a (bi)pullback in Cat, and the upper ϕ^* is conservative.

Proof. By Beck's theorem, it is sufficient to show that $\phi^* : \mathsf{Mod}(\mathcal{D}) \to \mathsf{Mod}(\mathcal{C})$ preserves ϕ^*-split coequalizers. Consider a ϕ^*-split parallel pair $f, g : A \rightrightarrows B$ in $\mathsf{Mod}(\mathcal{D})$, let $A \rightrightarrows B \overset{c}{\twoheadrightarrow} C$ be its coequalizer in $[\mathcal{D}, \mathsf{Set}]$, and let $\phi^* A \rightrightarrows \phi^* B \overset{d}{\twoheadrightarrow} D$ be the split coequalizer in $\mathsf{Mod}(\mathcal{C})$. As a split coequalizer the latter is absolute, and is therefore preserved by $\mathsf{Mod}(\mathcal{C}) \hookrightarrow [\mathcal{C}, \mathsf{Set}]$. The lower ϕ^* preserves colimits, and we conclude that $\phi^* C \cong D$. It follows that C is a model since the square is a bipullback. ∎

Note that this monadicity criterion can be viewed as a kind of converse to [BMW12, Theorem 1.10], which states that a similar square constructed from a 'monad with arities' is a pullback.

The talk will discuss this parallel, and present examples of clan morphisms fulfilling the above criterion, including weak fibrations between FL-clans, and clan morphisms presenting categories over non-reflexive graphs, and 2-categories over 2-polygraphs.

Moreover, we will discuss factorization systems on Clan in analogy to those preseented on FP and FL above.

References

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[3] https://categorytheory.zulipchat.com/#narrow/stream/229199-learning.3A-questions/topic/distributive.20laws.20and.20monadic.20functors