Duality for generalized algebraic theories

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Abstract

We exhibit an idempotent biadjunction between a 2-category of small clans and a 2-category of locally finitely presentable categories equipped with a weak factorization system, and characterize the stable subcategories.

1 Clans

Definition 1.1 A *clan* is a category $\mathcal{T}$ with a distinguished class of morphisms called *display maps*, such that:

1. Arbitrary pullbacks of display maps exist and are again display maps.
2. Isomorphisms and compositions of display maps are display maps.
3. $\mathcal{T}$ has a terminal object, and terminal projections are display maps.

A *clan morphism* is a functor between clas which preserves display maps, pullbacks of display maps, and the terminal object. We write Clan for the 2-category of clans, clan-morphisms, and natural transformations. $\diamondsuit$

Remarks 1.2 1. Clans can be viewed as 'non-strict' version of Cartmell's *contextual categories* [Car78, Car86].

2. The above definition and the term 'display map' are due to Taylor [Tay87, §4.3.2], the name 'clan' was suggested by Joyal [Joy17, Definition 1.1.1].

Examples 1.3 1. Finite-product categories can be viewed as clans where the display maps are the morphisms that are (isomorphic to) product projections. We call such clans *finite-product clans*.

2. Finite-limit categories can be viewed as clans where *all* morphimsms are display maps. We call such clans *finite-limit clans*.

3. The base category of every *category with attributes* in the original sense of Cartmell [Car78, Section 3.2] is a clan with display maps the arrows that are isomorphic to projection maps $\rho_B : \Sigma B \to A$.

4. **Kan** is the clan whose underlying category is the full subcategory of the category **SSet** of simplicial sets on *Kan complexes*, and whose display maps are the *Kan fibrations*.

Since it seems to lead to a more readable exposition, we introduce explicit notation and terminology for the dual notion.

Definition 1.4 A *coclan* is a category $\mathcal{C}$ with a distinguished class of morphisms called *codisplay maps* satisfying the dual axioms of clans.

The 2-category CoClan of coclans is defined dually to that of clans, i.e. $\mathsf{CoClan}(\mathcal{C}, \mathcal{D}) = \mathsf{Clan}(\mathcal{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}})^{\mathsf{op}}$. $\diamondsuit$

2 Algebras

Definition 2.1 Given a small clan $\mathcal{T}$ and a complete category $\mathfrak{A}$, a $\mathcal{T}$-*algebra in* $\mathfrak{A}$ is a clan morphism $A : \mathcal{T} \to \mathfrak{A}$ from where $\mathfrak{A}$ is equipped with the finite-limit clan structure. We write $\mathcal{T}\text{-}\mathsf{Alg}(\mathfrak{A})$ for the category of algebras and natural transformations, i.e. $\mathcal{T}\text{-}\mathsf{Alg}(\mathfrak{A}) = \mathsf{Clan}(\mathcal{T}, \mathfrak{A})$.

If $\mathfrak{A} = \mathsf{Set}$ we simply speak of $\mathcal{T}$-algebras and write $\mathcal{T}\text{-}\mathsf{Alg}$ for $\mathcal{T}\text{-}\mathsf{Alg}(\mathsf{Set})$. This is the case we will mainly be concerned with.

Remarks 2.2

1. As category of models of a finite-limit sketch, $\mathcal{T}\text{-}\mathsf{Alg}$ is reflective (and therefore closed under arbitrary limits) in $[\mathcal{T}, \mathsf{Set}]$, and moreover it is closed under filtered colimits [AR94, Section 1.C]. In particular, $\mathcal{T}\text{-}\mathsf{Alg}$ is locally finitely presentable.

2. The hom-functors $\mathcal{T}(\Gamma, -) : \mathcal{T} \to \mathsf{Set}$ are $\mathcal{T}$-algebras for all $\Gamma \in \mathcal{T}$ (we'll refer to them as hom-*algebras*), i.e. the Yoneda embedding $よ : \mathcal{T}^{\mathsf{op}} \to [\mathcal{T}, \mathsf{Set}]$ lifts along the inclusion $\mathcal{T}\text{-}\mathsf{Alg} \hookrightarrow [\mathcal{T}, \mathsf{Set}]$ to a fully faithful functor $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Alg}$.

$$\begin{tikzcd} & \mathcal{T}\text{-}\mathsf{Alg} \arrow[d, hook] \\ \mathcal{T}^{\mathsf{op}} \arrow[ur, dashed, "H"] \arrow[r, "よ"'] & {[\mathcal{T}, \mathsf{Set}]} \end{tikzcd}$$

3. For $\Gamma \in \mathcal{T}$, the hom-functor

$$\mathcal{T}\text{-}\mathsf{Alg}(H(\Gamma), -) : \mathcal{T}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

is isomorphic to the evaluation functor $A \mapsto A(\Gamma)$, hence it preserves filtered colimits as those are computed in $[\mathcal{T}, \mathsf{Set}]$ and therefore pointwise. This means that $H(\Gamma)$ is *compact*[1] in $\mathcal{T}\text{-}\mathsf{Alg}$.

We call a map $f : A \to B$ of algebras *full* if it has the r.l.p. with respect to all maps $H(p)$ for $p \in \mathcal{D}$, i.e. if the naturality square

$$\begin{tikzcd} A(\Gamma) \arrow[r, "A(p)"] \arrow[d, "f_\Gamma"] & A(\Delta) \arrow[d, "f_\Delta"] \\ B(\Gamma) \arrow[r, "B(p)"] & B(\Delta) \end{tikzcd}$$

[1]Following Lurie [Lur09] we use the shorter term 'compact' instead of the more traditional 'finitely presented'.

is a weak pullback for each display map $p : \Gamma \to \Delta$. By the small object argument, the full maps form the right class of a cofibrantly generated w.f.s.

$$(\mathcal{E}, \mathcal{F})$$

on $\mathcal{T}$-Alg whose left maps we call *extensions*.

We call $A \in \mathcal{T}$-Alg a *0-extension*, if $(0 \to A) \in \mathcal{E}$. In particular, all objects $H(\Gamma)$ are 0-extensions, since all terminal projections in $\mathcal{T}$ are display maps and Z sends terminal projections to initial inclusions.

3 Coalgebras and the universal property of $\mathcal{T}$-Alg

Dually to algebras of a clan, we have *coalgebras of a coclan.* These allow us to formulate the universal property of $\mathcal{T}$-Alg.

Definition 3.1 Given a small coclan $\mathcal{C}$ and a cocomplete category $\mathfrak{X}$, a $\mathcal{C}$-coalgebra in $\mathfrak{X}$ is a coclan morphism $C : \mathcal{C} \to \mathfrak{X}$ from $\mathcal{C}$ into $\mathfrak{X}$ equipped with the finite-colimit coclan structure. We write $\mathcal{C}$-$\mathsf{CoAlg}(\mathfrak{A})$ for the category of coalgebras and natural transformations. $\diamondsuit$

Theorem 3.2 (The universal property of $\mathcal{T}$-Alg) *For a small clan $\mathcal{T}$, the functor $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}$-$\mathsf{Alg}$ is the* universal $\mathcal{T}^{\mathsf{op}}$-coalgebra*, in the sense that for every cocomplete category, precomposition with H induces an equivalence*

$$\mathsf{CoCont}(\mathcal{T}\text{-}\mathsf{Alg}, \mathfrak{X}) \xrightarrow{\cong} \mathsf{CoAlg}(\mathcal{T}^{\mathsf{op}}, \mathfrak{X})$$

between the category of cocontinuous functors from $\mathcal{T}$-Alg *to $\mathfrak{X}$ and the category of $\mathcal{T}^{\mathsf{op}}$-coalgebras in $\mathfrak{X}$.*

Proof. See `https://mathoverflow.net/questions/349409/,universal-property-of-the-cocomplete-`

TODO: discuss Brandenburg's article. $\square$

4 $(\mathcal{E}, \mathcal{F})$-categories

Definition 4.1 An $(\mathcal{E}, \mathcal{F})$-category is a locally finitely presentable category $\mathcal{L}$ equipped with a weak factorization system $(\mathcal{E}, \mathcal{F})$ whose maps we call *extensions* and *full maps* respectively.

A *morphism of $(\mathcal{E}, \mathcal{F})$-categories* from $\mathcal{L}$ to $\mathcal{M}$ is an adjunction $F_! \dashv F^*$ where

1. the *direct image part* $F_! : \mathcal{L} \to \mathcal{M}$ preserves compact objects and extensions, and

2. the *inverse image part* $F^* : \mathcal{M} \to \mathcal{L}$ preserves filtered colimits and full maps.

A 2-*cell* $\eta : F \to G$ between morphisms of $(\mathcal{E}, \mathcal{F})$-categories is a natural transformation $\eta : F^* \to G^*$ between the *inverse image parts.* We write EFCat for the 2-category of $(\mathcal{E}, \mathcal{F})$-categories. $\diamondsuit$

Remark 4.2 It follows from standard arguments that conditions 1 and 2 in Definition 4.1 are equivalent. Moreover, by the special adjoint functctor theorem [Mac98, Section V-8] and the *adjoint functor theorem for presentable categories* [AR94, Theorem 1.66] respectively, the two adjoints can be reconstructed from each other. This means that a morphism from $\mathcal{L}$ to $\mathcal{M}$ of $(\mathcal{E}, \mathcal{F})$-categories is determined equivalently by

- a *cocontinuous* functor $F_! : \mathcal{L} \to \mathcal{M}$ preserving extensions and compact objects, and
- a *continuous* functor $F^* : \mathcal{M} \to \mathcal{L}$ preserving full maps and filtered colimits.

Lemma 4.3 *For any morphism $F : \mathcal{S} \to \mathcal{T}$ between small clans, the precomposition functor*

$$F^* \ : \ \mathcal{T}\text{-}\mathsf{Alg} \to \mathcal{S}\text{-}\mathsf{Alg}$$

is the image inverse part of a morphism of $(\mathcal{E}, \mathcal{F})$-categories.

Proof. The preservation of small limits and filtered colimits is obvious since they are computed pointwise (Remark 2.2-1). To show that F^* preserves full maps, let $f : A \to B$ be full in $\mathcal{T}\text{-}\mathsf{Alg}$. It is sufficient to show that the $(f \circ F)$-naturality squares are weak pullbacks at all display maps $p :$ in $\mathcal{S}\text{-}\mathsf{Alg}$. But the $(f \circ F)$-naturality square at p is the same as the f-naturality square at Fp so the claim follows since f is full and F preserves display maps. □

From Lemma 4.3 it is immediate that the assignment $\mathcal{T} \mapsto \mathcal{T}\text{-}\mathsf{Alg}$ extends to a pseudofunctor

$$(-)\text{-}\mathsf{Alg} \ : \ \mathsf{Clan}_{\mathsf{sm}} \ \to \ \mathsf{EFCat} \tag{4.1}$$

from the 2-category $\mathsf{Clan}_{\mathsf{sm}}$ of small clans to the 2-category of $(\mathcal{E}, \mathcal{F})$-categories.

Proposition 4.4 *The pseudofunctor* (4.1) *has a right biadjoint.*

Proof. Given a small clan $\mathcal{T}$ and an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$, it is easy to see that the natural equivalence

$$\mathsf{CoCont}(\mathcal{T}\text{-}\mathsf{Alg}, |\mathcal{L}|) \ \simeq \ \mathsf{CoAlg}(\mathcal{T}^{\mathsf{op}}, |\mathcal{L}|)$$

from Theorem 3.2 (where $|\mathcal{L}|$ is the underlying category of $\mathcal{L}$) restricts to an equivalence

$$\mathsf{EFCat}(\mathcal{T}\text{-}\mathsf{Alg}, \mathcal{L})^{\mathsf{op}} \ \simeq \ \mathsf{CoClan}(\mathcal{T}^{\mathsf{op}}, \mathfrak{C}(\mathcal{L}))$$

where $\mathfrak{C}(\mathcal{L})$ is the coclan whose underlying category is the full subcategory of $\mathcal{L}$ on compact 0-extensions, and whose co-display maps are the extensions. Taking opposite categories on both sides we get

$$\mathsf{EFCat}(\mathcal{T}\text{-}\mathsf{Alg}, \mathcal{L}) \ \simeq \ \mathsf{Clan}(\mathcal{T}, \mathfrak{C}(\mathcal{L})^{\mathsf{op}}),$$

which shows that the presheaf $\mathsf{EFCat}((-)\text{-}\mathsf{Alg}, \mathcal{L})$ of categories is birepresented by $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$. □

In the following we will show that this biadjunction between small clans and $(\mathcal{E}, \mathcal{F})$-categories is idempotent, and characterize the fixed-points on both sides.

5 Cauchy-complete clans and the fat small object argument

Definition 5.1 A clan $\mathcal{T}$ is called *Cauchy-complete*, if its underlying category is Cauchy-complete, and retracts of display maps are display maps. $\diamondsuit$

Clearly, every clan of the form $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$ is Cauchy-complete, thus Cauchy-completeness is a necessary condition for the unit $\eta_{\mathcal{T}} : \mathcal{T} \to \mathfrak{C}(\mathcal{T}\text{-}\mathsf{Alg})^{\mathsf{op}}$ of the biadjunction to be an equivalence. We will show that it is also sufficient, but for this we need the *fat small object argument*.

Proposition 5.2 (Fat small object argument) *For any clan $\mathcal{T}$, the 0-extensions in $\mathcal{T}$-Alg are* flat*, i.e. filtered colimits of* hom*-algebras.*

Proof. This is a special case of [MRV14, Corollary 5.1], but we give a direct proof in the appendix which simplifies considerably in the finitary case. $\square$

Corollary 5.3 *If $\mathcal{T}$ is small and Cauchy-complete then*

- *the functor $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Alg}$ co-restricts to an equivalence between $\mathcal{T}$-Alg and the full subcategory $\mathfrak{C}(\mathcal{T}\text{-}\mathsf{Alg})$ of $\mathcal{T}$-Alg on compact 0-extensions, and*
- *$f : \Delta \to \Gamma$ in $\mathcal{T}$ is a display map if and only if $H(f)$ is an extension.*

Proof. Let $C \in \mathcal{T}\text{-}\mathsf{Alg}$ be a compact 0-extension. By Proposition 5.2 there exists a filtered diagram $D : \mathbb{J} \to \mathcal{T}^{\mathsf{op}}$ and a limiting cocone $\sigma : \text{よ} \circ D \to \Delta(C)$. Since C is compact, the identity arrow id_C factors through one of the cocone maps σ_j, i.e. C is a retract of $\hom(D_j, -)$. By Cauchy-completeness, C is itself corepresentable. Thus, we have an equivalence of categories.

TODO: part about display maps. $\square$

6 Clan-algebraic categories

Given an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$, the counit $E : \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg} \to \mathcal{L}$ of the biadjunction is given by the *nerve-realization adjunction*

$$\begin{tikzcd}
\mathfrak{C}(\mathcal{L}) \arrow[r, hook, "J"] \arrow[d, "H"'] & \mathcal{L} \arrow[dl, bend left, "E^*"] \arrow[dl, phantom, "\dashv"] \\
\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg} \arrow[ur, bend left, "E_!"] &
\end{tikzcd}$$

where E^* is the nerve of J given by $E^*(L) = \mathcal{L}(J(-), L)$, and its left adjoint $E_!$ is the Kan extension of H along J, given by

$$E_!(A) = \mathsf{colim}(\mathsf{elts}(A) \to \mathfrak{C}(\mathcal{L}) \xrightarrow{J} \mathcal{L}),$$

where $\mathsf{elts}(A)$ is the (contravariant) category of elements. In this section we show that E is an equivalence in EFCat if and only if $\mathcal{L}$ is *clan-algebraic* in the sense of the following definition.

Definition 6.1 An $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ is called *clan-algebraic* if

(D) the subcategory $\mathfrak{C}(\mathcal{L})$ is dense,

(CG) $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by $\mathcal{E} \cap \mathrm{mor}(\mathfrak{C}(\mathcal{L}))$, and

(FQ) quotients of componentwise-full equivalence relations are effective and have full quotient maps. $\diamondsuit$

Theorem 6.2 *For every clan $\mathcal{T}$, $\mathcal{T}$-Alg is clan-algebraic.*

Proof. $\mathfrak{C}(\mathcal{L})$ is dense since it contains the corepresentables. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between corepresentables, and and therefore a fortiori by maps between compact 0-extensions.

For the third condition, let

$$r = \langle r_1, r_2 \rangle : R \hookrightarrow A \times A$$

be an equivalence relation such that r_1 and r_2 are full maps. This means that we have an equivalence relation $\sim$ on each $A(\Gamma)$, such that

1. for each $s : \Delta \to \Gamma$, the function $A(s) = s \cdot (-) : A(\Delta) \to A(\Gamma)$ preserves this relation, and

2. for every display map $p : \Gamma^+ \to \Gamma$ and all $a, b \in A(\Gamma)$ and $c \in A(\Gamma^+)$ such that $a \sim b$ and $p \cdot c = a$, there exists a $d \in A(\Gamma^+)$ with $c \sim d$ and $p \cdot d = b$.

We show first that the pointwise quotient A/R is a model. Clearly $(A/R)(1) = 1$, and it remains to show that given a pullback

$$\begin{array}{ccc} \Delta^+ & \xrightarrow{t} & \Gamma^+ \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \Delta & \xrightarrow{s} & \Gamma \end{array}$$

with p and q display maps, and elements $a \in A(\Delta)$, $b \in A(\Gamma^+)$ with $s \cdot a \sim p \cdot b$, there exists a unique-up-to-$\sim$ $c \in A(\Delta^+)$ with $q \cdot c \sim a$ and $t \cdot c \sim b$. Since p is a display map, there exists a b' with $b \sim b'$ and $p \cdot b' = s \cdot a$, and since A is a model there exists therefore a c with $q \cdot c = a$ and $t \cdot c = b'$. For uniqueness assume that $c, c' \in A(\Delta^+)$ with $q \cdot c \sim q \cdot c'$ and $t \cdot c \sim t \cdot c'$. Then $c \sim c'$ follows from the fact that R is a model. This shows that A/R is an algebra, and also that the quotient is effective, since the kernel pair is computed pointwise. The fact that $A \to A/R$ is full is similarly easy to see. $\square$

The following lemma is a kind of converse to (FQ).

Lemma 6.3 *Full maps in clan-algebraic categories are regular epimorphisms.*

Proof. Given a full map in a clan-algebraic category $\mathcal{L}$, the lifting property against (compact) 0-extensions implies that $E^*(f)$ is componentwise surjective in $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$-$\mathsf{Alg}$, and therefore the coequalizer of its kernel pair. Since left adjoints preserve regular epis, we deduce that $E_!(E^*(f))$ is regular epic in $\mathcal{L}$ and the claim follows since $E_! \circ E^* \cong \mathrm{id}$ by (D). $\square$

Remark 6.4 Observe that we only used property (D) in the proof, no exactness.

Lemma 6.5 *The class $\mathcal{F}$ of full maps in a clan-algebraic category $\mathfrak{L}$ has the* right cancellation property, *i.e. we have $g \in \mathcal{F}$ whenever $gf \in \mathcal{F}$ and $f \in \mathcal{F}$ for composable pairs $f : A \to B$, $g : B \to C$.*

Proof. By (CG) it suffices to show that g has the r.l.p. with respect to extensions $e : I \hookrightarrow J$ between compact 0-extensions I, J. Let

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[d, hook, "e"] & B \arrow[d, "g"] \\ J \arrow[r, "k"] & C \end{tikzcd}$$

be a filling problem. Since I is a 0-extension and f is full, there exists a map $h' : I \to A$ with $fh' = h$. We obtain a new filling problem

$$\begin{tikzcd} I \arrow[r, "h'"] \arrow[d, hook, "e"] & A \arrow[d, two heads, "gf"] \\ J \arrow[r, "k"] & C \end{tikzcd}$$

which can be filled by a map $m : J \to A$ since gf is full. Then fm is a filler for the original problem. $\square$

Lemma 6.6 *Let $\mathfrak{L}$ be a clan-algebraic category, let $f : A \to B$ be an arrow in $\mathfrak{L}$ with componentwise full kernel pair $p, q : R \twoheadrightarrow A$, and let $e : A \twoheadrightarrow C$ be the coequalizer of p and q. Then the unique $m : C \to B$ with $me = f$ is monic.*

Proof. By (D) it is sufficient to test monicity of m on maps out of compact 0-extensions E. Let $h, k : E \to C$ such that $mh = mk$. Since e is full by (FQ), there exist $h', k' : E \to A$ with $eh' = h$ and $ek' = k$. In particular we have $fh' = fk'$ and therefore there is an $u : E \to R$ with $pu = h'$ and $qu = k'$. Thus we can argue

$$h = eh' = epu = equ = ek' = k$$

which shows that m is monic. $\square$

Lemma 6.7 *I $A \in \mathfrak{C}(\mathfrak{L})^{\mathsf{op}}\text{-}\mathsf{Alg}$ is flat, then $A \to E^*(E_!(A))$ is an isomorphism.*

Proof. We have

$$\begin{aligned} E^*(E_!(A))(C) &= \mathfrak{L}(C, \mathsf{colim}(\mathsf{elts}(A) \to \mathfrak{C}(\mathfrak{L}) \hookrightarrow \mathfrak{L})) \\ &\cong \mathsf{colim}(\mathsf{elts}(A) \to \mathfrak{C}(\mathfrak{L}) \xrightarrow{\text{よ}(C)} \mathsf{Set}) \qquad \text{since } \mathsf{elts}(A) \text{ is filtered} \\ &\cong A \otimes \text{よ}(C) \cong A(C). \end{aligned}$$

$\square$

Lemma 6.8 *The following are equivalent for a cone $\phi : \Delta C \to D$ on a diagram $D : \mathbb{J} \to \mathfrak{L}$ in an $(\mathcal{E}, \mathcal{F})$-category $\mathfrak{L}$.*

1. *Given an extension $e : A \to B$, an arrow $h : A \to C$, and a cone $\kappa : \Delta B \to D$ such that $\phi_j \circ h = \kappa_j e$ for all $j \in \mathbb{J}$, there exists $l : B \to C$ such that $le = h$ and $\phi_j l = \kappa_j$ for all $j \in \mathbb{J}$.*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "e"'] & C \arrow[d, "\phi_j"] \\ B \arrow[r, "\kappa_j"] \arrow[ur, dashed, "l"] & D_j \end{tikzcd}$$

2. *The mediating arrow* $: C \to \mathsf{lim}(D)$ *is full.*

Proof. The data of e, h, κ is equivalent to e, h, and $k : B \to \mathsf{lim}(D)$ such that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "e"'] & C \arrow[d, "f"] \\ B \arrow[r, "k"] & \mathsf{lim}(D) \end{tikzcd}$$

commutes, and $l : B \to C$ fills the latter square iff it fills all the squares with the D_j. ◻

Definition 6.9 We call a cone $\phi : \Delta C \to D$ satisfying the conditions of the lemma *jointly full.* ♢

Remark 6.10 The interest of this is that it allows us to talk about full 'covers' of limits without actually computing the limits, which is useful when talking about cones and diagrams in the full subcategory of a clan-algebraic category on 0-extensions, which does not admit limits.

Definition 6.11 A *nice diagram* in a clan-algebraic category $\mathcal{L}$ is a 2-truncated simplicial diagram

$$\begin{tikzcd} A_2 \arrow[r, shift left=4, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=4, "d_2"'] & A_1 \arrow[l, shift left=2, "s_0"] \arrow[l, shift right=2, "s_1"'] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & A_0 \arrow[l, "s_0"] \end{tikzcd}$$

where

1. A_0, A_1, and A_2 are 0-extensions,
2. the maps $d_0, d_1 : A_1 \to A_0$ are full,
3. in the square $\begin{tikzcd} A_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] & A_1 \arrow[d, "d_1"] \\ A_1 \arrow[r, "d_0"] & A_0 \end{tikzcd}$ the span constitutes a jointly full diagram over the cospan,
4. there exists a symmetry map $\begin{tikzcd} A_1 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, "\sigma"] & A_0 \\ A_0 & A_1 \arrow[l, "d_1"] \arrow[u, "d_0"'] \end{tikzcd}$ making the triangles commute, and
5. there exists a 0-extension $\tilde{A}$ and full maps $f, g : \tilde{A} \twoheadrightarrow A_1$ constituting a jointly full cone over the diagram

$$\begin{tikzcd} A_1 \arrow[d, "d_0"'] \arrow[dr, "d_1"] & A_1 \arrow[d, "d_1"] \arrow[dl, "d_0"] \\ A_0 & A_0 \end{tikzcd} .$$

Lemma 6.12 *In any nice diagram $A_\bullet$, the pairing $\langle d_0, d_1 \rangle : A_1 \to A_0 \times A_0$ factors as $A_1 \xrightarrow{f} R \xrightarrow{\langle r_0, r_1 \rangle}$ where f is full, and $r = \langle r_0, r_1 \rangle$ is a (monic) equivalence relation.*

Proof. Condition 5 of the preceding definition gives us the following diagram

$$\begin{tikzcd}
\tilde{A} \arrow[dr, two heads, "h"'] \arrow[drr, two heads, bend left, "g"] \arrow[ddr, two heads, bend right, "f"'] & & \\
& S \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, phantom, very near start, "\lrcorner"] & A_1 \arrow[d, "\langle d_0, d_1 \rangle"] \\
& A_1 \arrow[r, "\langle d_0, d_1 \rangle"] & A_0 \times A0
\end{tikzcd}$$

i.e. S is the kernel of $\langle d_0, d_1 \rangle$ with projections p, q, $\tilde{A}$ is a 0-extension, and f, g, h are full. By right cancellation we deduce that p and q are full, and the existence of the factorization follows from Lemma 6.6. It remains to show that r is an equivalence relation. This is easy: reflexivity is witnessed by $s_0 : A_0 \to A_1$, condition 4 gives symmetry, and condition 3 transitivity. $\square$

Theorem 6.13 *If $\mathcal{L}$ is clan-algebraic, then $\varepsilon : \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg} \to \mathcal{L}$ is an equivalence in* EFCat.

Proof. By density, ε^* is fully faithful. It remains to show that it is essentially surjective. It remains to show that the unit map $\eta_A : A \to E^*(E_! A)$ is an isomorphism for all $A \in \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg}$, i.e. that $A(C) \cong \mathcal{L}(C, E_! A)$ for all $C \in \mathfrak{C}(\mathcal{L})$.

We have

$$\begin{aligned}
\mathcal{L}(C, E_! A) &= \mathcal{L}(C, E_!(\mathsf{colim}(A_\bullet))) && \text{since } A = \mathsf{colim}(A_\bullet) \\
&\cong \mathcal{L}(C, \mathsf{colim}(E_! A_\bullet)) && \text{since } E_! \text{ preserves colims} \\
&\cong \mathsf{colim}(\mathcal{L}(C, E_! A_\bullet)) && \text{since } E_! A_\bullet \text{ is nice and } \mathcal{L}(C, -) \text{ preserves nice colims} \\
&\cong \mathsf{colim}(A_\bullet(C)) && \text{since } E_! A_i = \mathsf{colim}(\textstyle\int A_i \to \mathbb{C} \to \mathcal{L}) \text{ filtered} \\
&\cong \mathsf{colim}(\mathrm{Mod}(ZC, A_\bullet)) && \text{by Yoneda} \\
&\cong \mathrm{Mod}(ZC, \mathsf{colim}(A_\bullet)) && \text{since } A_\bullet \text{ is nice and } \mathrm{Mod}(ZC, -) \text{ preserves nice colims} \\
&\cong \mathrm{Mod}(ZC, A) && \text{since } A = \mathsf{colim}(A_\bullet) \\
&\cong AC && \text{by Yoneda}
\end{aligned}$$

$\blacksquare$

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