Impredicative encodings in $(1, 2)$-toposes

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joint work with Colin Zwanziger

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Overview

Context

- directed ~~type theory~~ first-order logic
- synthetic category theory
- …

Directed logic

'Categorifying' 1st order logic

set A	Category $\mathbb{A}$
function $f : A \to B$	functor $F : \mathbb{A} \to \mathbb{B}$
relation $R \subseteq A \times B$ $\varphi : A \times B \to \{0, 1\}$	'relator' $\mathbb{A} \leftarrow \mathbb{R} \to \mathbb{B}$ (two sided discrete fibration) $\varphi : \mathbb{B}^{\mathrm{op}} \times \mathbb{A} \to \mathbf{Set}$ (profunctor/distributor/bimodule)
truth values: $\{0, 1\}$	**Set**
conjunction $p \wedge q$	cartesian product $A \times B$
disjunction $p \vee q$	coproduct $A + B$
implication $p \Rightarrow q$	set of functions B^A
existential quant. $\exists x$	coend $\int^X$
universal quant. $\forall x$	end $\int_X$
equality $a = b$	hom-set $\mathrm{hom}(A, B)$

- last one is a directed version of groupoid-model of type theory
- naive attempts to devise directed 1st order logic calculus for cats fails since **dinatural transformations don't compose**
- but there's a nice bicategory **Dist** of distributors incorporating many of the above constructions

The bicategory **Dist**

- categorification of the category **Rel** of sets and relations
- objects: small categories $\mathbb{C}, \mathbb{D}, \mathbb{E}, \ldots$
- morphisms from $\mathbb{C}$ to $\mathbb{D}$: distributors $\mathbb{D}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$
- composition: Given distributors $\mathbb{E} \xleftarrow{\psi} \mathbb{D} \xleftarrow{\varphi} \mathbb{C}$, composition is given by

$$(\psi \otimes \varphi)(E, C) = \int^{D} \psi(E, D) \times \varphi(D, C)$$

  categorifying composition of relations (given $R \subseteq C \times D$ and $S \subseteq D \times E$, composite is given by $S \circ R = \{(c, e) \mid \exists d \,.\, (c, d) \in R \wedge (d, e) \in S\}$)
- identity 1-cell on $\mathbb{C}$ is given by $\mathsf{hom}_{\mathbb{C}} : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$
- See e.g.:
  - J. Bénabou. “Distributors at work”. In: (2000). Lecture notes written by T. Streicher, `https://www2.mathematik.tu-darmstadt.de/~streicher/FIBR/DiWo.pdf`

Closed structure

Dist is **closed**, meaning that pre- and composition functors have right adjoints:

$$\begin{array}{ccc} \psi & \to & \varphi \multimap \theta \\ \hline\hline \varphi \otimes \psi & \to & \theta \\ \hline\hline \varphi & \to & \theta \multimapinv \psi \end{array}$$

for $\mathbb{A} \overset{\varphi}{\leftharpoondown\!\!\rightharpoonup} \mathbb{B} \overset{\psi}{\leftharpoondown\!\!\rightharpoonup} \mathbb{C}$ and $\mathbb{A} \overset{\theta}{\leftharpoondown\!\!\rightharpoonup} \mathbb{C}$.

Formula for $\varphi \multimap \theta$:

$$(\varphi \multimap \theta)(B, C) \;=\; \int_A \theta(A, C)^{\varphi(A,B)}$$

In logical notation:

$$(\varphi \multimap \theta)(B, C) \;=\; \forall A \,.\, \varphi(A, B) \Rightarrow \theta(A, C)$$

'bounded quantification'

Elementary toposes

Definition

An elementary topos is a category $\mathcal{E}$ with **finite limits** and **power objects**, where a power object of $A \in \mathcal{E}$ is an object PA representing the presheaf

$$\mathsf{Sub}(- \times A) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}.$$

(For $B \in \mathcal{E}$, $\mathsf{Sub}(B)$ is the set of subobjects of B, i.e. isomorphism classes of monos into A.)

1st order logic in toposes

We can interpret 1st order logic in elementary toposes using the following encodings of logical connectives in terms of equality and power objects alone.

$$
\begin{array}{rcl}
p \Rightarrow q & \equiv & (p \wedge q) = p \\
\forall x{:}A\,.\,p[x] & \equiv & \{x \mid p[x]\} = \{x \mid \top\} \\
\bot & \equiv & \forall z{:}\Omega\,.\,z \\
p \vee q & \equiv & \forall z{:}\Omega\,.\,(p \Rightarrow z) \wedge (q \Rightarrow z) \Rightarrow z \\
\exists x{:}A\,.\,p[x] & \equiv & \forall z{:}\Omega\,.\,(\forall x{:}A\,.\,p[x] \Rightarrow z) \Rightarrow z
\end{array}
$$

- T. Streicher. “Introduction to Category Theory and Categorical Logic”. Lecture notes, www.mathematik.tu-darmstadt.de/~streicher
- J. Lambek and P.J. Scott. **Introduction to higher order categorical logic**. Vol. 7. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 1986, pp. x+293. ISBN: 0-521-24665-2
- A. Boileau and A. Joyal. “La logique des topos”. In: **The Journal of Symbolic Logic** 46.1 (1981), pp. 6–16. ISSN: 0022-4812

Toward 2-toposes

- 2-toposes should abstract categories of sheaves of categories, in the same way toposes abstract categories of sheaves of sets
- 2-toposes should admit an internal calculus of distributors
- Mark Weber proposed an elementary axiomatization:
  - M. Weber. "Yoneda structures from 2-toposes". In: **Applied Categorical Structures** 15.3 (2007), pp. 259–323
- Colin pointed out a remark by Shulman, saying that 2-toposes in general don't have an $(-)^{\mathrm{op}}$ operation
- specifically, if $\mathfrak{A}$ is a genuine 2-category then taking fiberwise opposites in a presheaf

  $$F : \mathfrak{A}^{\mathrm{op}} \to \mathbf{Cat}$$

  yields a functor

  $$F^{\mathrm{op}} : \mathfrak{A}^{\mathrm{coop}} \to \mathbf{Cat}$$

- This means that we can't encode distributors using presheaves, have to axiomatize 2-sided fibrations directly
- **Removing symmetries clarifies the situation, makes structure more canonical** – compare linear logic vs tensor logic

Toward $(1, 2)$*-toposes*

- Colin brought up the notion of $(1, 2)$-topos
- $(1, 2)$-toposes are half-way between toposes and 2-toposes.
- A $(1, 2)$-category is a category where the hom-sets are posets and composition is monotone
- Easier since we don't have to worry about coherences, Cauchy completeness, Rezk completeness
- Moreover there is the **possibility of impredicativity**
- Advantage over 1-toposes: can represent **posets** – rather than sets of subobjects.

The enrichment table[1]

Definition

- An $(n, 0)$-category is an n-groupoid
- An $(n+1, k+1)$-category is enriched in (n, k)-categories

n \ k	0	1	2	...
-1	(-1)-groupoids propositions			
0	0-groupoids sets	$(0, 1)$-categories posets		
1	1-groupoids groupoids	$(1, 1)$-categories categories	$(1, 2)$-categories **Pos**-categories	
2	2-groupoids	$(2, 1)$-categories **Gpd**-categories	$(2, 2)$-categories 2-categories	...
...	...	...	...	...

[1] J.C. Baez and M. Shulman. "Lectures on n-categories and cohomology". In: **Towards higher categories**. Springer, 2010, pp. 1–68, Section-5.1.

Comparisons

Definition

A **comparison** between posets $(A, \leq)$ and $(B, \leq)$ is a binary relation $\phi \subseteq A \times B$ which is upward closed in A and downward closed in B, i.e. $(a', b') \in \phi$ whenever $(a, b) \in \phi$, $a \leq a'$, and $b' \leq b$.

- The char. function of the ϕ is a monotone map $(B, \leq)^{\mathrm{op}} \times (A, \leq) \to \mathbb{2}$.

Definition

Let $\mathcal{X}$ be a locally ordered category. A **comparison** in $\mathcal{X}$ is a span $A \xleftarrow{p} U \xrightarrow{q} B$ s.t. for every $X \in \mathcal{X}$, the monotone function

$$(f \mapsto (p \circ f, q \circ f)) \; : \; \mathcal{X}(X, U) \to \mathcal{X}(X, A) \times \mathcal{X}(X, B)$$

is order-reflecting, and its image is a comparison between $\mathcal{X}(X, A)$ and $\times \mathcal{X}(X, B)$.

The term **comparison** was suggested by Lambek in

- J. Lambek. “Bilinear logic in algebra and linguistics”. In: **London Mathematical Society Lecture Note Series** (1995), pp. 43–60

Functoriality

Definition

Write $(B \rightharpoonup A)$ for the poset of comparisons from A to B.

If $\mathcal{X}$ has pullbacks then $(B \rightharpoonup A)$ is **Pos**-functorial in A and B, of variance

$$(- \rightharpoonup -) \ : \ \mathcal{X}^{\mathrm{coop}} \times \mathcal{X}^{\mathrm{op}} \to \mathbf{Pos} \ .$$

Given a comparison $\varphi : B \rightharpoonup A$ and maps $g : B' \to B$, $f : A' \to A$, denote the induced comparison by

$$\varphi[g, f] : B' \rightharpoonup A' \ .$$

$(1, 2)$-*toposes*

(Working) Definition

An (elementary) $(1, 2)$-topos is a locally ordered category $\mathcal{E}$ with finite limits (including cotensors with $\mathbb{2}$) s.t. for all $A \in \mathcal{E}$, the presheaves of posets $(A \rightharpoonup -)$ and $(- \rightharpoonup A)^{\mathrm{op}}$ are representable.

Representability of $(A \rightharpoonup -)$ means that there are $P_{\downarrow}A \in \mathcal{E}$ and $\varepsilon : A \rightharpoonup P_{\downarrow}A$ such that for all B, the monotone map

$$\varepsilon[1, -] \quad : \quad \mathcal{E}(B, P_{\downarrow}A) \to (A \rightharpoonup B)$$

is an isomorphism of posets. Denoting its inverse by $(-)^{\downarrow}$, we have

$$\phi = \varepsilon[1, \phi^{\downarrow}] \qquad f = \varepsilon[1, f]^{\downarrow} \qquad \phi^{\downarrow} \circ h = \phi[1, h]^{\downarrow}$$

for $\phi : A \rightharpoonup B$ and $f : B \to P_{\downarrow}A$ and $h : B' \to B$.

Similar for representability of $(- \rightharpoonup A)^{\mathrm{op}}$.

Unit

Given $A \in \mathcal{E}$, the **unit comarison** $I : A \rightharpoonup A$ is given by the cotensor $I^{\mathbb{2}}$ together with the two projections.

Entailment

Definition

Given

$$A_0 \xleftrightarrow{\varphi_1} A_1 \xleftrightarrow{\varphi_2} \ldots A_{n-1} \xleftrightarrow{\varphi_n} A_n \quad \text{and} \quad \psi : A_0 \leftrightharpoons A_n$$

write

$$\varphi_1, \ldots, \varphi_n \vdash \psi$$

if the multi-pullback of the spans φ_i factors through the span ψ.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d] & \cdots & \varphi_n \arrow[r] \arrow[d] & A_n \\
\bullet \arrow[r] & \bullet & \cdots & A_{n-1} & \\
\vdots & \vdots & \iddots & & \\
\varphi_1 \arrow[r] \arrow[d] & A_1 & & & \\
A_0 & & & &
\end{tikzcd}$$

Note that the multi-pullback is in general not itself a comparison.

Equipments

The structure of entailment together with substitution $\phi[-,-]$ is an instance of what Shulman calls a **virtual equipment**.

Entailment reformulation

Definition

1. Given $\varphi : A \rightharpoonup A$, write

$$\vdash \varphi$$

   if φ contains the diagonal.

2. Given

$$A_0 \xleftharpoondown{\varphi_1} A_1 \xleftharpoondown{\varphi_2} \ldots A_{n-1} \xleftharpoondown{\varphi_n} A_n \quad \text{and} \quad \psi : A_0 \rightharpoonup A_n$$

   write

$$\varphi_1, \ldots, \varphi_n \vdash \psi$$

   if for all X and $(a_i : X \to A_i \mid 0 \leq i \leq n)$ we have

$$(\vdash \varphi_1[a_0, a_1]) \ \wedge \ \ldots \ \wedge \ (\vdash \varphi_n[a_{n-1}, a_n]) \ \Rightarrow \ (\vdash \psi[a_0, a_n]) \ .$$

Note that there's no ambiguity, nullary case of 2 coincides with 1.

Remarks on entailment relation

1. Given $\varphi : A \rightharpoonup B$ and $A \xleftarrow{a} X \xrightarrow{b} B$, we have $\vdash \varphi[a, b]$ iff the span (a, b) factors through the span φ.
2. For example, given $f, g : X \to A$ we have $\vdash \mathsf{hom}[f, g]$ iff $f \leq g$ in $\mathcal{E}(X, A)$.
3. For the case $n = 1$, it is easy to see that $\varphi \vdash \psi$ iff $\varphi \leq \psi$.

Some valid rules

$$\frac{\varphi_1, \ldots, \varphi_n \vdash \psi}{\varphi_1[f_0, f_1], \ldots, \varphi_n[f_{n-1}, f_n] \vdash \psi[f_0, f_n]}$$

$$\frac{\Delta \vdash \varphi \qquad \Gamma, \varphi, \Lambda \vdash \psi}{\Gamma, \Delta, \Lambda \vdash \psi}$$

$$\frac{\Gamma, \Delta \vdash \psi}{\Gamma, \mathsf{hom}, \Delta \vdash \psi}$$

Toward impredicative encodings

Let's have another look at the 1-topos encodings:

$$
\begin{array}{rcl}
p \Rightarrow q & \equiv & (p \wedge q) = p \\
\forall x{:}A\,.\,p[x] & \equiv & \{x \mid p[x]\} = \{x \mid \top\} \\
\bot & \equiv & \forall z{:}\Omega\,.\,z \\
p \vee q & \equiv & \forall z{:}\Omega\,.\,(p \Rightarrow z) \wedge (q \Rightarrow z) \Rightarrow z \\
\exists x{:}A\,.\,p[x] & \equiv & \forall z{:}\Omega\,.\,(\forall x{:}A\,.\,p[x] \Rightarrow z) \Rightarrow z
\end{array}
$$

- Can we do something similar in $(1,2)$-toposes?
- Have to construct $\Rightarrow$, $\forall$ first, the other connectives depend on it
- Construct a combined 'synthetic' connective implementing closed structure in dist **Dist**:

$$(\varphi \multimap \theta)(B, C) \;=\; \forall A\,.\,\varphi(A, B) \Rightarrow \theta(A, C)$$

Rephrase RHS:

$$
\dfrac{\dfrac{\forall A\,.\,\varphi(A, B) \Rightarrow \theta(A, C)}{\{A \mid \varphi(A, B)\} \subseteq \{A \mid \theta(A, C)\}}}{\varphi^{\downarrow}(B) \leq \theta^{\downarrow}(D)}
$$

- This suggests to define $(\varphi \multimap \theta) := I[\varphi^{\downarrow}, \theta^{\downarrow}]$

Implification

Definition

For comparisons $A \overset{\varphi}{\leftharpoondown\!\rightharpoondown} B \overset{\psi}{\leftharpoondown\!\rightharpoondown}$ and $A \overset{\theta}{\leftharpoondown\!\rightharpoondown} C$ in a $(1,2)$-toposes $\mathcal{E}$ define

$$(\varphi \multimap \theta) := I[\varphi^{\downarrow}, \theta^{\downarrow}] \qquad\qquad (\theta \circ\!\!- \psi) := I[\theta^{\uparrow}, \psi^{\uparrow}]$$

Theorem

$$\dfrac{\dfrac{\psi \vdash \varphi \multimap \theta}{\varphi, \psi \vdash \theta}}{\varphi \vdash \theta \circ\!\!- \psi}$$

Proof.

First equivalence:

$$
\begin{aligned}
&\psi \vdash \mathrm{hom}[\varphi^{\downarrow}, \theta^{\downarrow}] \\
&\quad \text{iff} \quad \forall X\, b\, c\,.\ (\vdash \psi[b, c]) \Rightarrow (\vdash \mathrm{hom}[\varphi^{\downarrow} \circ b, \theta^{\downarrow} \circ c]) \\
&\quad \text{iff} \quad \forall X\, b\, c\,.\ (\vdash \psi[b, c]) \Rightarrow \varphi^{\downarrow} \circ b \leq \theta^{\downarrow} \circ c \\
&\quad \text{iff} \quad \forall X\, b\, c\,.\ (\vdash \psi[b, c]) \Rightarrow \varphi[1, b]^{\downarrow} \leq \theta[1, c]^{\downarrow} \\
&\quad \text{iff} \quad \forall X\, b\, c\,.\ (\vdash \psi[b, c]) \Rightarrow \varphi[1, b] \leq \theta[1, c] \\
&\quad \text{iff} \quad \forall X\, b\, c\,.\ (\vdash \psi[b, c]) \Rightarrow (\varphi[1, b] \vdash \theta[1, c]) \\
&\quad \text{iff} \quad \forall X\, b\, c\,.\ (\vdash \psi[b, c]) \Rightarrow \forall Y\, a\, x\,.\ (\vdash \varphi[a, b \circ x]) \Rightarrow (\vdash \theta[a, c \circ x]) \\
&\quad \text{iff} \quad \forall X\, b\, c\, Y\, a\, x\,.\ (\vdash \varphi[a, b \circ x]) \wedge (\vdash \psi[b, c]) \Rightarrow (\vdash \theta[a, c \circ x]) \\
&\quad \text{iff} \quad \forall X\, a\, b\, c\,.\ (\vdash \varphi[a, b]) \wedge (\vdash \psi[b, c]) \Rightarrow (\vdash \theta[a, c]) \\
&\quad \text{iff} \quad \varphi, \psi \vdash \theta
\end{aligned}
$$

□

'Kripke-Joyal style'

Existensor

Next we want to give an encoding of a tensor/composition operation satisfying

$$\frac{\varphi, \psi \vdash \theta}{\varphi \otimes \psi \vdash \theta}$$

- How to do it?
- In higher order logic, encodings of positive connectives all depend on the equivalence

  $$p \dashv\vdash \forall q{:}\Omega\,.\,(p \Rightarrow q) \Rightarrow q$$

  for $p : \Omega$.
- It turns out that we can do something similar in $(1, 2)$-toposes!

Double negation elimination

Theorem

Given $\varphi : A \rightharpoonup B$ *we have* $\varphi = \varepsilon \multimapinv (\varphi \multimap \varepsilon)$.

Proof.

We show $\varphi \vdash \varepsilon \multimapinv (\varphi \multimap \varepsilon)$ and $\varepsilon \multimapinv (\varphi \multimap \varepsilon) \vdash \varphi$.
First:

$$\cfrac{\cfrac{\cfrac{}{\varphi \multimap \varepsilon \vdash \varphi \multimap \varepsilon}}{\varphi, \varphi \multimap \varepsilon \vdash \varepsilon}}{\varphi \vdash \varepsilon \multimapinv (\varphi \multimap \varepsilon)}$$

Second:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\varepsilon \multimapinv (\varphi \multimap \varepsilon) \vdash \varepsilon \multimapinv (\varphi \multimap \varepsilon)}}{\varepsilon \multimapinv (\varphi \multimap \varepsilon), \varphi \multimap \varepsilon \vdash \varepsilon}}{\varepsilon \multimapinv (\varphi \multimap \varepsilon), (\varphi \multimap \varepsilon)[1, \varphi^{\downarrow}] \vdash \varepsilon[1, \varphi^{\downarrow}]}}{\varepsilon \multimapinv (\varphi \multimap \varepsilon), \mathsf{hom}[\varphi^{\downarrow}, \varphi^{\downarrow}] \vdash \varphi}\ \text{rewrite} \qquad \cfrac{\cfrac{\cfrac{}{\mathsf{hom} \vdash \mathsf{hom}}}{\vdash \mathsf{hom}}}{\vdash \mathsf{hom}[\varphi^{\downarrow}, \varphi^{\downarrow}]}}{\varepsilon \multimapinv (\varphi \multimap \varepsilon) \vdash \varphi}$$

$\square$

1. Now it's easy to derive an encoding for $\otimes$:

$$\varphi \otimes \psi \;=\; \varepsilon \circ\!\!- (\varphi \otimes \psi \multimap \varepsilon) \;=\; \varepsilon \circ\!\!- (\psi \multimap \varphi \multimap \varepsilon)$$

2. It's also easy to derive $\varphi, \psi \vdash \varphi \otimes \psi$.
3. But the left intro is harder – we need global operations on the context

Negation rules

Lemma

The following rules are admissible

$$
\begin{array}{c}
\Gamma \vdash \varphi \\
\hline\hline
\Gamma, (\varphi \multimap \varepsilon) \vdash \varepsilon
\end{array}
\qquad
\begin{array}{c}
\Gamma \vdash \varphi \\
\hline
(\varepsilon \multimapinv \varphi), \Gamma \vdash \varepsilon
\end{array}
$$

Proof.

$$
\begin{array}{c}
\Gamma \vdash \varphi \\
\hline\hline
\Gamma \vdash \varepsilon \multimapinv (\varphi \multimap \varepsilon) \\
\hline\hline
\Gamma, (\varphi \multimap \varepsilon) \vdash \varepsilon
\end{array}
\qquad
\dfrac{\Gamma \vdash \varphi \qquad \dfrac{\dfrac{}{\varepsilon \multimapinv \varphi \vdash \varepsilon \multimapinv \varphi}}{\varepsilon \multimapinv \varphi, \varphi \vdash \varepsilon}}{(\varepsilon \multimapinv \varphi), \Gamma \vdash \varepsilon}
$$

□

Left $\otimes$-intro

Lemma

$$\frac{\Gamma, \varphi, \psi, \Delta \vdash \theta}{\Gamma, \varphi \otimes \psi, \Delta \vdash \theta}$$

Proof.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Gamma, \varphi, \psi, \Delta \vdash \theta}{\varphi, \psi, \Delta \vdash \Gamma \multimap \theta}}{\varphi, \psi, \Delta, (\Gamma \multimap \theta) \multimap \varepsilon \vdash \varepsilon}}{\Delta, (\Gamma \multimap \theta) \multimap \varepsilon \vdash \psi \multimap \varphi \multimap \varepsilon}}{(\varepsilon \multimapinv (\psi \multimap \varphi \multimap \varepsilon)), \Delta, (\Gamma \multimap \theta) \multimap \varepsilon \vdash \varepsilon}}{\varphi \otimes \psi, \Delta, (\Gamma \multimap \theta) \multimap \varepsilon \vdash \varepsilon}}{\varphi \otimes \psi, \Delta, \vdash \Gamma \multimap \theta}}{\Gamma, \varphi \otimes \psi, \Delta \vdash \theta}$$

$\square$

Conclusion

- Probably we can show that comparisons in any $(1, 2)$-toposes form a closed cartesian bicategory.
- Future work: colimits, exactness?
- Possibility of binding syntax – see my CT19 slides

Thanks for your attention!