Enriched triposes and enriched Pitts functors

in parts joint work with Richard Garner

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Motivation/Introduction

- Hyland, Johnstone, Pitts 1980: tripos-to-topos construction generalizing toposes of sheaves on a locale
- Pitts 1982: Triposes on other base toposes/categories, *iteration*
- Can characterize realizability toposes over **Set**, but only *enriched* realizability toposes over other base toposes $\mathcal{S}$
- Goal: a framework of 'enriched' triposes and 'enriched' Pitts functors generalizing localic geometric morphisms

Part 0
Locallic geometric morphisms

Localic geometric morphisms

Definition

A localic geometric morphism between toposes $\mathcal{S}$, $\mathcal{E}$ is a pair of functors $\Delta : \mathcal{S} \to \mathcal{E}$, $\Gamma : \mathcal{E} \to \mathcal{S}$ such that

1. $\Delta \dashv \Gamma$
2. Δ preserves finite limits
3. Δ is bounded by 1, i.e. every $A \in \mathcal{E}$ can be represented as subquotient $\Delta(J) \leftarrowtail U \twoheadrightarrow A$

- Using Δ, $\mathcal{E}$ can be fibered over $\mathcal{S}$:
$$\begin{tikzcd} \mathrm{Gl}_\Delta(\mathcal{E}) \arrow[r] \arrow[d, "\mathrm{gl}_\Delta(\mathcal{E})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}{\downarrow}\mathcal{E} \arrow[d, "\mathrm{cod}(\mathcal{E})"] \\ \mathcal{S} \arrow[r, "\Delta"'] & \mathcal{E} \end{tikzcd}$$
- Using Γ, we can enrich $\mathcal{E}$ in $\mathcal{S}$: $\hom_{\mathcal{S}}(A, B) = \Gamma(B^A)$
- Since $\Delta \dashv \Gamma$, fibration and enrichment are related – $\mathrm{Gl}_\Delta(\mathcal{S})$ is *locally small*

Part I
Triposes

Triposes

Definition

Let $\mathcal{S}$ be a topos. A **tripos** on $\mathcal{S}$ is an indexed preorder

$$\mathcal{P} : \mathcal{S}^{\mathrm{op}} \to \textbf{Ord}$$

such that

1. the fibers $\mathcal{P}(I)$ are **cartesian closed**
2. the **reindexing maps** $u^* : \mathcal{P}(I) \to \mathcal{P}(J)$ induced by $u : J \to I$ preserve cartesian closed structure, and have **right adjoints** $\forall_u : \mathcal{P}(J) \to \mathcal{P}(I)$ satisfying the **Beck-Chevalley condition**
3. $\mathcal{P}$ has a **generic predicate**, i.e. an element $\mathrm{tr} \in \mathcal{P}(\mathrm{Prop})$ such that for every $\varphi \in \mathcal{P}(J)$ there exists an $u : J \to \mathrm{Prop}$ with $u^*\mathrm{tr} \cong \varphi$
4. reindexing along epis is order-reflecting (not in the original definition)

Remark

Triposes also have fiberwise **finite joins**, and **left adjoints** $\exists_u : \mathcal{P}(J) \to \mathcal{P}(I)$ to reindexing, all constructed using 2nd order encodings.

The tripos-to-topos construction

Any **tripos** $\mathcal{P} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Ord}$ gives rise to a **topos** $\mathcal{S}[\mathcal{P}]$ and a functor $\Delta_{\mathcal{P}} : \mathcal{S} \to \mathcal{S}[\mathcal{P}]$

Properties of $\Delta_{\mathcal{P}}$

1. $\Delta_{\mathcal{P}}$ is **regular** (preserves finite limits and epimorphisms)
2. $\Delta_{\mathcal{P}}$ is **bounded by** 1
3. $\mathcal{P}$ can be reconstructed from $\Delta_{\mathcal{P}}$:

$$\begin{tikzcd}
\mathcal{S}^{\mathrm{op}} \arrow[r, "\Delta_{\mathcal{P}}"] \arrow[dr, "\mathcal{P}"'] & \mathcal{S}[\mathcal{P}]^{\mathrm{op}} \arrow[d, "\mathrm{sub}"] \arrow[dl, phantom, "\cong"] \\
 & \mathbf{Ord}
\end{tikzcd}$$

[For $\mathbb{C}$ with finite limits, $\mathrm{sub} : \mathbb{C}^{\mathrm{op}} \to \mathbf{Ord}$ is its **subobject fibration** – $\mathrm{sub}(C)$ is the preorder of monomorphisms into C]

Pitts functors

Theorem (Pitts 81)

Functors $\Delta_{\mathcal{P}} : \mathcal{S} \to \mathcal{S}[\mathcal{P}]$ can be characterized as functors $\Delta : \mathcal{S} \to \mathcal{E}$ where

1. *$\mathcal{S}$ and $\mathcal{E}$ are toposes*
2. *Δ is regular*
3. *for every $A \in \mathcal{E}$ there exists a* ***generic covering****, i.e. a subquotient span $\Delta(J) \leftarrowtail U \twoheadrightarrow A$ such that for any span $\Delta(K) \leftarrowtail V \to A$ there exists an $h : K \to J$ giving rise to a pullback diagram:*

$$
\begin{tikzcd}
K \arrow[d, "h"'] & \Delta(K) \arrow[d] & V \arrow[l, tail] \arrow[d] \arrow[dr] \arrow[dl, phantom, "\lrcorner", very near start] & \\
J & \Delta(J) & U \arrow[l, tail] \arrow[r, two heads] & A
\end{tikzcd}
$$

Definition

We call such functors **Pitts functors**.

Iteration

Theorem (Pitts 81)

Pitts functors compose.

This means that we can 'compose' triposes – given $\mathcal{P} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Ord}$ and $\mathcal{Q} : \mathcal{S}[\mathcal{P}]^{\mathrm{op}} \to \mathbf{Ord}$, we obtain $\mathcal{Q} \ltimes \mathcal{P} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Ord}$.

$$\begin{tikzcd}
\mathcal{S} \arrow[rr, "\Delta_{\mathcal{P}}"] \arrow[dr, "\Delta_{\mathcal{Q} \ltimes \mathcal{P}}"'] & & \mathcal{S}[\mathcal{P}] \arrow[d, "\Delta_{\mathcal{Q}}"] \\
& \mathcal{S}[\mathcal{Q} \ltimes \mathcal{P}] \simeq & \mathcal{S}[\mathcal{P}][\mathcal{Q}]
\end{tikzcd}$$

Tripos transformations

Definition

Let $\mathcal{S}$ be a topos.

- A **tripos morphism** $f : \mathcal{P} \to \mathcal{Q}$ is an indexed monotone map preserving fiberwise finite meets.
- a **regular** tripos morphism is a tripos morphism which preserves $\exists$
- **Trip**$(\mathcal{S})$ is the category of triposes on $\mathcal{S}$ and regular tripos morphisms

- Regular morphisms $f : \mathcal{P} \to \mathcal{Q}$ induce **regular functors**
$$\begin{tikzcd} \mathcal{S} \arrow[r, "\Delta_{\mathcal{P}}"] \arrow[dr, "\Delta_{\mathcal{Q}}"'] & \mathcal{S}[\mathcal{P}] \arrow[d, "{\mathcal{S}[f]}"] \arrow[dl, phantom, "\simeq"] \\ & \mathcal{S}[\mathcal{Q}] \end{tikzcd}$$
- Every regular functor
$$\begin{tikzcd} \mathcal{S} \arrow[r, "\Delta_{\mathcal{P}}"] \arrow[dr, "\Delta_{\mathcal{Q}}"'] & \mathcal{S}[\mathcal{P}] \arrow[d, "F"] \arrow[dl, phantom, "\simeq"] \\ & \mathcal{S}[\mathcal{Q}] \end{tikzcd}$$
arises this way!
- Moreover, every such functor is automatically a Pitts functor.

Lemma

Trip$(\mathcal{S}[\mathcal{P}]) \simeq \mathcal{P}\backslash$**Trip**$(\mathcal{S})$

Tripos transformations 2

- General tripos morphisms $f : \mathcal{P} \to \mathcal{Q}$ give rise to flp functors and natural transformations

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\Delta_{\mathcal{P}}"] \arrow[dr, "\Delta_{\mathcal{Q}}"'] & \mathcal{S}[\mathcal{P}] \arrow[d, "{\mathcal{S}[f]}"] \arrow[dl, phantom, "\overset{\eta}{\Rightarrow}"] \\ & \mathcal{S}[\mathcal{Q}] \end{tikzcd}$$

- *Not* every diagram $\begin{tikzcd} \mathcal{S} \arrow[r, "\Delta_{\mathcal{P}}"] \arrow[dr, "\Delta_{\mathcal{Q}}"'] & \mathcal{S}[\mathcal{P}] \arrow[d, "F"] \arrow[dl, phantom, "\overset{\eta}{\Rightarrow}"] \\ & \mathcal{S}[\mathcal{Q}] \end{tikzcd}$ arises this way!

- Necessary and sufficient condition: For every generic covering $\Delta_{\mathcal{P}}(J) \leftarrowtail U \twoheadrightarrow A$, the map h in the diagram below is epic.

$$\begin{tikzcd} \Delta_{\mathcal{Q}}(J) \arrow[d, "\eta_J"'] & V \arrow[l, tail] \arrow[d] \arrow[dr, "h"] \arrow[dl, phantom, "\lrcorner"] & \\ F(\Delta_{\mathcal{P}}(J)) & F(U) \arrow[l, tail] \arrow[r] & F(A) \end{tikzcd}$$

- Tripos morphisms $f : \mathcal{P} \to \mathcal{Q}$ are **coreflective** in diagrams $\begin{tikzcd} \mathcal{S} \arrow[r, "\Delta_{\mathcal{P}}"] \arrow[dr, "\Delta_{\mathcal{Q}}"'] & \mathcal{S}[\mathcal{P}] \arrow[d, "F"] \arrow[dl, phantom, "\overset{\eta}{\Rightarrow}"] \\ & \mathcal{S}[\mathcal{Q}] \end{tikzcd}$

Part II
Enriched triposes

Motivation

- $\Delta_{\mathcal{P}} : \mathcal{S} \to \mathcal{S}[\mathcal{P}]$ gives fibering of $\mathcal{S}[\mathcal{P}]$ over $\mathcal{S}$, but no enrichment
- In standard examples like realizability over an internal PCA, such an enrichment is natural
- enrichment and fibering can not be as tightly linked as in the localic case

Enriched triposes

Definition

An enriched tripos on $\mathcal{S}$ is a tripos $\mathcal{P} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Ord}$ together with a tripos morphism $\gamma : \mathcal{P} \to \mathsf{sub}$ satisfying

$$(*) \qquad \top \leq \gamma_1(\varphi) \quad \Rightarrow \quad \top \leq \varphi \quad \text{for} \quad \varphi \in \mathcal{P}(1).$$

- The data of an enriched tripos give rise to

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\Delta_{\mathcal{P}}"] \arrow[dr, "\mathrm{id}"'] & \mathcal{S}[\mathcal{P}] \arrow[d, "\Gamma"] \arrow[dl, phantom, "\overset{\eta}{\Rightarrow}"] \\ & \mathcal{S} \simeq \mathcal{S}[\mathsf{sub}] \end{tikzcd}$$

- Γ induces an $\mathcal{S}$-enrichment of $\mathcal{S}[\mathcal{P}]$ via $\mathsf{hom}(A, B) = \Gamma(B^A)$
- By $(*)$, the enrichment is **standard**, i.e. $\mathcal{S}(1, \mathsf{hom}(A, B)) \cong \mathcal{S}[\mathcal{P}](A, B)$
- η makes $\Delta_{\mathcal{P}}$ into an $\mathcal{S}$-enriched functor:

$$\dfrac{K^J \xrightarrow{\eta} \Gamma(\Delta(K^J)) \qquad \dfrac{\dfrac{\Delta(K^J) \times \Delta(J) \to \Delta(K)}{\Delta(K^J) \to \Delta(K)^{\Delta(J)}}}{\Gamma(\Delta(K^J)) \to \Gamma(\Delta(K)^{\Delta(J)})}}{K^J \to \Gamma(\Delta(K)^{\Delta(J)})}$$

Enriched Pitts-functors

Definition

An **enriched Pitts functor** is a Pitts functor $\Delta : \mathcal{S} \to \mathcal{E}$ together with a flp functor $\Gamma : \mathcal{E} \to \mathcal{S}$ and a natural transformation $\eta : \mathrm{id}_{\mathcal{S}} \to \Gamma \circ \Delta$ such that

1. For every generic covering $\Delta(J) \leftarrowtail U \twoheadrightarrow A$, the map h in the diagram

$$\begin{tikzcd} J \arrow[d, "\eta_J"'] & V \arrow[l, tail] \arrow[d] \arrow[dr, "h"] \arrow[dl, phantom, "\lrcorner"] & \\ \Gamma(\Delta(J)) & \Gamma(U) \arrow[l, tail] \arrow[r] & \Gamma(A) \end{tikzcd}$$

   is epic
2. For every $A \in \mathcal{E}$, we have a bijection $\mathcal{E}(1, A) \xrightarrow{\cong} \mathcal{S}(1, \Gamma(A))$

- enriched Pitts functors $(\Delta, \Gamma, \eta) : \mathcal{S} \to \mathcal{E}$ are equivalent to enriched triposes on $\mathcal{S}$
- Every tripos $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ and every Pitts functor $\Delta : \mathbf{Set} \to \mathcal{E}$ has a unique canonical enrichment
- Attention: enriched Pitts functors do not compose 'directly' – the generic covering condition is not stable under composition –> use coreflection

Fibrations from enriched Pitts functors

- Using η, we can extend Γ to a fibered functor

$$\begin{tikzcd} \mathrm{Gl}_\Delta(\mathcal{E}) \arrow[rr, "\Gamma"] \arrow[dr, "\mathrm{gl}_\Delta(\mathcal{E})"'] & & \mathcal{S}{\downarrow}\mathcal{S} \arrow[dl, "\mathrm{cod}(\mathcal{S})"] \\ & \mathcal{S} & \end{tikzcd}$$

$$\mathcal{E}/\Delta(J) \to \mathcal{S}/\Gamma(\Delta(J)) \to \mathcal{S}/J$$

- Enrichment of the fibers of $\mathrm{gl}_\Delta(\mathcal{E})$ in the slices of $\mathcal{S}$ – locally internal category!
- underlying fibration locally small and different from $\mathrm{gl}_\Delta(\mathcal{E})$ – think 'family fibration'

$$\begin{tikzcd} \mathcal{S}{\downarrow}\mathcal{S} \arrow[rr, "\Delta"] \arrow[dr, "\mathrm{cod}(\mathcal{S})"'] & & \mathrm{Fam}(\mathcal{E}) \arrow[dl, "\mathrm{fam}(\mathcal{E})"] \\ & \mathcal{S} & \end{tikzcd}$$

- fibers are still toposes (not obvious)

Enriched triposes on $\mathcal{S}[\mathcal{P}]$

- Recall that triposes on $\mathcal{S}[\mathcal{P}]$ correspond to regular tripos transformations $f : \mathcal{P} \to \mathcal{Q}$
- **Enriched** triposes on $\mathcal{S}[\mathcal{P}]$ correspond to pairs $f^\bullet : \mathcal{P} \to \mathcal{Q}, \quad f_\bullet : \mathcal{Q} \to \mathcal{P}$ of tripos morphisms where
  1. $f^\bullet$ is regular
  2. $\top \leq f_\bullet(\varphi) \Rightarrow \top \leq \varphi$ for $\varphi \in \mathcal{Q}(1)$
  3. $\mathrm{id}_{\mathcal{P}} \leq f_\bullet \circ f^\bullet$
- This suggests the definition of the following category

Definition

ETrip$(\mathcal{S})$ is the category of enriched triposes on $\mathcal{S}$, where morphisms $f : (\mathcal{P}, \gamma) \to (\mathcal{Q}, \gamma')$ are pairs $f^\bullet : \mathcal{P} \to \mathcal{Q}$, $f_\bullet : \mathcal{Q} \to \mathcal{P}$ of tripos morphisms with

1. $f^\bullet$ regular
2. $\gamma \circ f_\bullet \cong \gamma'$
3. $\mathrm{id}_{\mathcal{P}} \leq f_\bullet \circ f^\bullet$

Lemma

ETrip$(\mathcal{S}[\mathcal{P}]) \cong (\mathcal{P}, \gamma) \backslash$**ETrip**$(\mathcal{S})$ *for any enriched tripos* $(\mathcal{P}, \gamma)$ *on* $\mathcal{S}$.

Factorizing enriched Pitts functors

Definition

An enriched Pitts functor $(\Delta, \Gamma, \eta) : \mathcal{S} \to \mathcal{E}$ is called

- **localic**, if $\Delta \dashv \Gamma$
- **realizability-like**, if $\Gamma \dashv \Delta$

- There is a factorization of enriched Pitts functors into a localic and a realizability-like part
- Analyze this factorization relative to $\mathcal{S}$, by doing calculations in **ETrip**$(\mathcal{S})$

The realizability-like/localic factorization

- Given a morphism $f = (f^\bullet, f_\bullet) : \mathcal{P} \to \mathcal{Q}$ of enriched triposes on $\mathcal{S}$, define an indexed preorder $\mathcal{U} : \mathcal{S}^{\mathrm{op}} \to \mathbf{Ord}$ by

$$\mathcal{U}(J) = \{(\mu, \varphi) \in \mathcal{P}(J) \times \mathcal{Q}(J) \mid \mu \leq f_\bullet(\varphi), \varphi \leq f^\bullet(\mu)\}$$

Lemma

$\mathcal{U}$ *is a tripos.*

- E.g. implication is given by

$$\begin{aligned}(\mu, \varphi) \Rightarrow (\nu, \psi) \; = & \\ & ((\mu \Rightarrow \nu) \wedge f_\bullet(\varphi \Rightarrow \psi), f^\bullet(\mu \Rightarrow \nu) \wedge (\varphi \Rightarrow \psi) \wedge f^\bullet f_\bullet(\varphi \Rightarrow \psi))\end{aligned}$$

- There is a decomposition $f = (\mathcal{P} \xrightarrow{l} \mathcal{U} \xrightarrow{r} \mathcal{Q})$ with l realizability-like and r localic:

$$l^\bullet(\mu) = (\mu, f^\bullet \mu) \qquad\qquad r^\bullet(\mu, \varphi) = \varphi$$
$$l_\bullet(\mu, \varphi) = \mu \qquad\qquad r_\bullet(\varphi) = (f_\bullet \varphi, \varphi \wedge f^\bullet f_\bullet \varphi)$$

- Problem: factorization is only lax functorial and has bad orthogonality properties

Thank you for your attention!