# $Characterizing\ clan-algebraic\ categories$

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## **Overview**

#### Context

- In talks at HoTT/UF 2020 and at CT 2021 I presented a conjecture concerning categories of models of a clan.
- In this talk I will give/outline a proof of this conjecture.

#### Three Parts

- Recall functorial semantics of (essentially) algebraic theories
- Clans as generalized algebraic theories
- If there's time: Examples and models in higher (homotopy) types

# Part I

## Algebraic Theories

### Definition

A single-sorted algebraic theory (SSAT) is a pair  $(\Sigma, E)$  consisting of

- a family  $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ , of sets of *n*-ary **operations**
- a set of equations E whose elements are pairs of open terms over  $\Sigma$

### Definition

The syntactic category  $C(\Sigma, E)$  of a SSAT is given as follows:

- 1. For each natural number  $n \in \mathbb{N}$  there is an **object** [n]
- 2. morphisms  $\sigma: [n] \to [m]$  are m-tuples of terms in n variables modulo E-provable equality
- 3. identities are lists of variables, composition is given by substitution

### Proposition

Given a SSAT  $(\Sigma, E)$ :

- 1.  $\mathcal{C}(\Sigma, E)$  has finite products given by  $[n] \times [m] = [n+m]$
- 2. Set-Mod( $\Sigma$ , E)  $\simeq$  FP( $\mathcal{C}(\Sigma, E)$ , Set)

## Finite-product theories

### Definition

- A FP-theory is just a small FP-category C.
- **Models** of  $\mathcal{C}$  are FP-functors  $A : \mathcal{C} \to \mathbf{Set}$  (or into another FP-category).

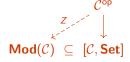
Denote the category of models by

$$\mathsf{Mod}(\mathcal{C}) := \mathsf{FP}(\mathcal{C},\mathsf{Set}) \overset{\mathrm{full}}{\subseteq} [\mathcal{C},\mathsf{Set}].$$

For every object  $\Gamma \in \mathcal{C}$  of an FP-theory, the co-representable functor

$$\mathcal{C}(\Gamma,-)\ :\ \mathcal{C} \to \textbf{Set}$$

is a model. Thus, the dual Yoneda embedding co-restricts to Mod(C).



### Finite-limit theories

### Definition

- A FL-theory is a small finite-limit category ∠.
- A **model** of  $\mathcal{L}$  is a finite-limit preserving functor  $A: \mathcal{L} \to \mathbf{Set}$ .

FL-theories are more expressive than FP-theories – structures definable by finite-limit theories include

• categories, posets, 2-categories, monoidal categories, categories with families . . .

Again  $\mathcal{L}(\Gamma, -)$  is a model for every  $\Gamma \in \mathcal{L}$  and we get an embedding

$$Z: \mathcal{L}^{\mathsf{op}} o \mathsf{Mod}(\mathcal{L}) := \mathsf{FL}(\mathcal{L}, \mathsf{Set}) \overset{\mathrm{full}}{\subseteq} [\mathcal{L}, \mathsf{Set}].$$

Moreover, we can characterize the essential image of Z in  $Mod(\mathcal{L})$ .

## Locally finitely presentable categories

### Definition

• An object C of a cocomplete locally small category  $\mathfrak{X}$  is called **compact**<sup>a</sup>, if

$$\mathfrak{X}(C,-):\mathfrak{X} o \mathbf{Set}$$

preserves filtered colimits.

- A category X is called locally finitely presentable, if
  - X is locally small and cocomplete
  - the full subcategory  $comp(\mathfrak{X}) \subseteq \mathfrak{X}$  on compact objects is essentially small and dense.

#### Theorem

- $Mod(\mathcal{L})$  is locally finitely presentable for all finite-limit theories  $\mathcal{L}$ .
- The essential image of  $Z: \mathcal{L}^{op} \to \mathsf{Mod}(\mathcal{L})$  comprises precisely the compact objects.

<sup>&</sup>lt;sup>a</sup>More traditionally: 'finitely presentable'

## Gabriel- $Ulmer\ duality^1$

#### Theorem

There is a bi-equivalence of 2-categories

$$\mathsf{FL} \quad \xleftarrow{\operatorname{comp}(\mathfrak{X})^{\mathsf{op}} \, \hookleftarrow \, \mathfrak{X}} \qquad \mathsf{LFP}^{\mathsf{op}}$$

#### where

- FL is the 2-category of small FL-categories and FL-functors
- LFP is the 2-category of locally finitely presentable categories and functors preserving small limits and filtered colimits ('forgetful functors').

<sup>&</sup>lt;sup>1</sup>P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Springer-Verlag, 1971.

# Duality for finite-product theories<sup>2</sup>

There's a 'restriction' of G–U duality to finite-product theories:

$$\begin{array}{c} \textbf{FP}_{cc} \xleftarrow{\hspace{0.2cm} \mathcal{C} \mapsto \textbf{FP}(\mathcal{C}, \textbf{Set})} & \textbf{ALG}^{op} \\ F \swarrow \swarrow \mathcal{U} & & \swarrow \mathcal{L} \mapsto \textbf{FL}(\mathcal{L}, \textbf{Set}) \\ \textbf{FL} & & & \swarrow \mathcal{L} \mapsto \textbf{FL}(\mathcal{L}, \textbf{Set}) \\ & & & & \downarrow \mathcal{L} \mapsto \textbf{FP}^{op} \end{array}$$

- FP<sub>cc</sub> is the 2-category of Cauchy-complete finite-product categories
- ALG is the 2-category of algebraic categories and algebraic functors
  - An algebraic category is an I.f.p. category which is Barr-exact and where the compact (regular) projective objects are dense
  - An algebraic functor is a functor that preserves small limits, filtered colimits, and regular epimorphisms.
- There's also a formulation in terms of sifted colimits, but we don't need it.

<sup>&</sup>lt;sup>2</sup>J. Adámek, J. Rosický, and E.M. Vitale. *Algebraic theories: a categorical introduction to general algebra*. Cambridge University Press, 2010.

# Part II

### Toward clans

- Finite-limit theories have a nice duality theory but seem far from syntax
- Syntactic counterparts are given by
  - Freyd's essentially algebraic theories<sup>3</sup>
  - Cartmell's generalized algebraic theories<sup>4</sup> (or 'dependent algebraic theories')
  - Johnstone's cartesian theories<sup>5</sup>
  - Palmgren and Vickers' quasi-equational theories<sup>6</sup>
  - and probably others
- Clans can be viewed as a categorical representation of generalized algebraic theories
- They're as expressive as FL-theories, but 'finer', i.e. closer to syntax

<sup>&</sup>lt;sup>3</sup>P. Freyd. "Aspects of topoi". In: Bulletin of the Australian Mathematical Society (1972).

<sup>&</sup>lt;sup>4</sup>J. Cartmell. "Generalised algebraic theories and contextual categories". In: *Annals of Pure and Applied Logic* (1986).

<sup>&</sup>lt;sup>5</sup>P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 2.* Oxford: Oxford University Press, 2002.

<sup>&</sup>lt;sup>6</sup>E. Palmgren and S. J. Vickers. "Partial horn logic and Cartesian categories". In: *Annals of Pure and Applied Logic* (2007).

#### Definition

A **clan** is a small category  $\mathcal{T}$  with terminal object 1, equipped with a class  $\mathcal{T}_{\dagger} \subseteq \operatorname{mor}(\mathcal{T})$  of morphisms – called **display maps** and written  $\rightarrow$  – such that

- 1. pullbacks of display maps along all maps exist and are display maps  $\begin{array}{ccc} \Delta^+ & \xrightarrow{s^+} & \Gamma^+ \\ q^{\downarrow} & \neg & \downarrow p \\ & & \Delta & \xrightarrow{s} & \Gamma \end{array}$
- 2. display maps are closed under composition, and
- 3. isomorphisms and terminal projections  $\Gamma \rightarrow 1$  are display maps.
- Definition due to Taylor<sup>7</sup>, name due to Joyal<sup>8</sup> ('a clan is a collection of families')
- Relation to semantics of dependent type theory: display maps represent type families.
- Observation: clans have finite products (as pullbacks over 1).

<sup>&</sup>lt;sup>7</sup>P. Taylor. "Recursive domains, indexed category theory and polymorphism". PhD thesis. University of Cambridge, 1987, § 4.3.2.

<sup>&</sup>lt;sup>8</sup>A. Joyal. "Notes on clans and tribes". In: arXiv preprint arXiv:1710.10238 (2017).

## Examples

- Finite-product categories  $\mathcal C$  can be viewed as clans with  $\mathcal C_\dagger = \{ \text{product projections} \}$
- Finite-limit categories  $\mathcal{L}$  can be viewed as clans with  $\mathcal{L}_{\dagger} = \operatorname{mor}(\mathcal{L})$

We call such clans **FP-clans**, and **FL-clans**, respectively.

- The syntactic category of every Cartmell-style generalized algebraic theory is a clan.
- Clan for categories:

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\mathcal{K} = \{\text{categories free on finite graphs}\}^{\text{op}} \subseteq \text{Cat}^{\text{op}}
\mathcal{K}_{t} = \{\text{functors induced by graph inclusions}\}^{\text{op}}
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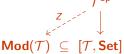
 $\mathcal{K}$  can be viewed as syntactic category of a generalized algebraic theory of categories with a sort  $\mathcal{O}$  of objects, and a dependent sort  $x,y:\mathcal{O} \vdash M(x,y)$  of morphisms – vertices of a finite graph are object variables and edges are morphism variables in a context. Graph inclusions are dual to context extensions.

## Models

### **Definition**

A model of a clan  $\mathcal{T}$  is a functor  $A: \mathcal{T} \to \mathbf{Set}$  which preserves 1 and pullbacks of display-maps.

- The category  $Mod(\mathcal{T}) \subseteq [\mathcal{T}, Set]$  of models is l.f.p. and contains  $\mathcal{T}^{op}$ .
- For FP-clans  $(\mathcal{C}, \mathcal{C}_{\dagger})$  we have  $\mathbf{Mod}(\mathcal{C}, \mathcal{C}_{\dagger}) = \mathbf{FP}(\mathcal{C}, \mathbf{Set})$ .
- For FL-clans  $(\mathcal{L}, \mathcal{L}_{\dagger})$  we have  $Mod(\mathcal{L}, \mathcal{L}_{\dagger}) = FL(\mathcal{L}, Set)$ .
- $\mathsf{Mod}(\mathcal{K}, \mathcal{K}_{\dagger}) = \mathsf{Cat}$ .



#### Observation

The same category of models may be represented by different clans.

For example, SSATs can be represented by FP-clans as well as FL-clans.

## The weak factorization system

- Would like duality between clans and their categories of models.
- Since the same l.f.p. category can be represented by different clans, we cannot hope to reconstruct the clan from the models alone.
- Solution: equip the models with additional structure in form of a **weak factorization system**.

### Definition

Let  $\mathcal{T}$  be a clan. Define w.f.s.  $(\mathcal{E}, \mathcal{F})$  on  $\mathsf{Mod}(\mathcal{T})$  by

- $\mathcal{F} := \mathsf{RLP}(\{Z(p) \mid p \in \mathcal{T}_{\dagger}\})$  class of **full maps**
- $\mathcal{E} := \mathsf{LLP}(\mathcal{F})$  class of **extensions**

I.e.  $(\mathcal{E}, \mathcal{F})$  is cofibrantly generated by the image of  $\mathcal{T}_{\dagger}$  under  $Z : \mathcal{T}^{op} \to \mathbf{Mod}(\mathcal{T})$ .

- Call  $A \in \mathbf{Mod}(\mathcal{T})$  a 0-extension, if  $(0 \to A) \in \mathcal{E}$
- E.g. corepresentables  $Z(\Gamma)$  are 0-extensions since terminal projections  $\Gamma \to 1$  are display maps.
- The same weak factorization system was also introduced by S. Henry in a HoTTEST talk<sup>9</sup>, see also<sup>10</sup>.

<sup>&</sup>lt;sup>9</sup>S. Henry, *The language of a model category*, HoTTEST seminar, Jan. 2020, https://youtu.be/7\_X0qbSX1fk <sup>10</sup>S. Henry. "Algebraic models of homotopy types and the homotopy hypothesis". In: *arXiv preprint arXiv:1609.04622* (2016).

## Full maps

•  $f: A \to B$  in  $Mod(\mathcal{T})$  is full iff it has the RLP with respect to all Z(p) for display maps  $p: \Delta \to \Gamma$ .

$$\begin{array}{cccc}
\mathcal{T}(\Gamma,-) & \longrightarrow & A & & A(\Delta) & \xrightarrow{f_{\Delta}} & B(\Delta) \\
Z(p)=\mathcal{T}(p,-)\downarrow & & \downarrow f & & A(p)\downarrow & & \downarrow B(p) \\
\mathcal{T}(\Delta,-) & \longrightarrow & B & & A(\Gamma) & \xrightarrow{f_{\Gamma}} & B(\Gamma)
\end{array}$$

- This is equivalent to display-naturality-squares being weak pullbacks.
- Considering  $p: \Delta \to 1$  we see that full maps are surjective and hence regular epis.

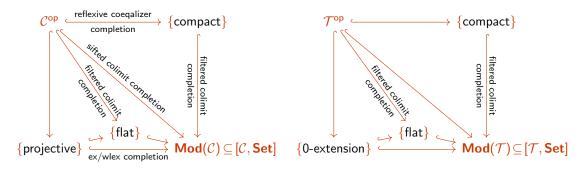
$$\begin{array}{ccccc} A(\Delta) & \xrightarrow{f_{\Delta}} & B(\Delta) & & & A(\Delta) & \xrightarrow{f_{\Delta}} & B(\Delta) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 & & A(\Delta) \times A(\Delta) & \xrightarrow{f_{\Delta} \times f_{\Delta}} & B(\Delta) \times B(\Delta) \end{array}$$

- For FL-clans, only isos are full (consider naturality square for diagonal  $\Delta \to \Delta \times \Delta$ )
- For FP-clans we have

full map = regular epimorphism extension = coproduct inclusion  $A \hookrightarrow P + A$  with P projective O-extension = projective object

## The fat small object argument

Motivation: subcategories of models for FP-theory  $\mathcal C$  and clan  $\mathcal T$ .



- Flat algebras are filtered colimits of corepresentables, computed *freely* in the functor categories.
- For SSATs we have  $\{projective\} \subseteq \{flat\}$  since
  - arbitrary free objects are filtered colimits of free objects over finite sets
  - projective objects are retracts of free objects
- In the general clan case,  $\{0\text{-extension}\}\subseteq \{\text{flat}\}\$ by the **fat small object argument**<sup>11</sup>.

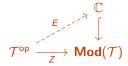
<sup>11</sup> M. Makkai, J. Rosicky, and L. Vokrinek. "On a fat small object argument". In: Advances in Mathematics (2014).

## Reconstructing the clan

### Definition

Given a clan  $\mathcal{T}$ , let  $\mathbb{C} \subseteq \mathsf{Mod}(\mathcal{T})$  be the full subcategory on **compact** 0-extensions.

•  $Z : \mathcal{T}^{op} \to \mathbf{Mod}(\mathcal{T})$  factors through  $\mathbb C$  since corepresentables  $Z(\Gamma)$  are compact and 0-extensions.



- $0 \in \mathbb{C}$  and if  $\begin{array}{c} C \longrightarrow D \\ \downarrow_e & \downarrow \downarrow \\ E \longrightarrow F \end{array}$  is a pushout with  $F \in \mathbb{C}$  and  $e \in \mathcal{E}$  then  $F \in \mathbb{C}$ .
- Therefore  $\mathbb C$  is a **coclan** with extensions as "co-display maps".

## Reconstructing the clan

#### **Theorem**

The full inclusion  $E: \mathcal{T}^{op} \hookrightarrow \mathbb{C}$  exhibits  $\mathbb{C}$  as *Cauchy-completion* of  $\mathcal{T}^{op}$ , i.e. every compact 0-extension is a retract of a corepresentable.

## Proof.

- Let  $C \in \mathbb{C}$ .
- Since 0-extensions are flat,  $\int C$  is filtered, thus C is a filtered colimit of corepresentables.
- Since *C* is compact, id<sub>*C*</sub> factors through a colimit inclusion map.



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## Clan-algebraic categories

### Definition

A clan-algebraic category is a category  $\mathfrak{X}$  with a w.f.s.  $(\mathcal{E}, \mathcal{F})$  that arises as category of models of a clan.

With this definition we get a contravariant bi-equivalence

$$\begin{array}{ccc} \text{Clan}_{cc} & \xleftarrow{& \operatorname{comp}(\mathfrak{X})^{\mathsf{op}} \ \leftarrow \ \mathfrak{X}} & \text{cAlg}^{\mathsf{op}} \end{array}$$

#### between

- the 2-category Clan<sub>cc</sub> of Cauchy-complete clans and functors preserving 1, display maps, and pullbacks of display maps, and
- the 2-category cAlg of clan-algebraic categories and functors preserving small limits, filtered colimits, and full maps.

Can we characterize clan-algebraic categories more abstractly?

## Characterizing clan-algebraic categories

Assume  $\mathfrak{X}$  is clan-algebraic with w.f.s.  $(\mathcal{E}, \mathcal{F})$ . Then

- 1.  $\mathfrak{X}$  is cocomplete,
- 2.  $\mathfrak{X}$  has a small dense family of compact 0-extensions, and
- 3.  $(\mathcal{E}, \mathcal{F})$  is cofibrantly generated by maps between compact 0-extensions.

Now assume we have a category  $\mathfrak{X}$  with w.f.s.  $(\mathcal{E}, \mathcal{F})$  satisfying 1–3.

Then the subcategory  $\mathbb{C} \subseteq \mathfrak{X}$  of compact 0-extensions is a coclan.

We get a nerve/realization adjunction

$$\mathbb{C} \xrightarrow{J} \mathfrak{X}$$

$$\downarrow \qquad \qquad L(A) = \operatorname{colim}(\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})$$

$$N(X) = \mathfrak{X}(J(-), X)$$

$$\mathsf{Mod}(\mathbb{C}^{\operatorname{op}})$$

However, this adjunction is not an equivalence in general:

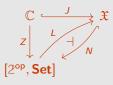
## Characterizing clan-algebraic categories

### Counter example

#### Consider

- $\mathfrak{X} \subseteq [2^{op}, \mathbf{Set}]$  full subcategory on injections
- $(\mathcal{E}, \mathcal{F})$  w.f.s. on  $\mathfrak{X}$  cofib. generated by  $\{(0 \to Y0), (0 \to Y1)\}$

Then  $Mod(\{compact \ 0\text{-extensions}\}^{op}) \simeq [2^{op}, \mathbf{Set}]$  and N is the subcategory inclusion.



Conclusion: We're missing an 'exactness condition' analogous to 'Barr-exactness' in the characterization of algebraic categories!

## Quotients of componentwise-full equivalence relations

- Recall that a FL-category ∠ is called Barr-exact, if all equivalence relations in ∠ have stable effective quotients.
- This can't be the case for clan algebraic categories in general. However, we have:

#### Lemma

For any clan  $\mathcal{T}$ ,  $\mathsf{Mod}(\mathcal{T})$  has full and effective quotients of componentwise-full equivalence relations.

### Proof.

Given equivalence relation  $r: R \rightarrow A \times A$  with  $r_0, r_1: R \rightarrow A$  full, show that component-wise quotient is a model again.

## Characterizing clan-algebraic categories

## Definition

An **adequate category** is a category  $\mathfrak{X}$  with a with a w.f.s.  $(\mathcal{E}, \mathcal{F})$  (whose maps we call extensions and full, respectively), s.th.

- 1.  $\mathfrak{X}$  is cocomplete,
- 2.  $\mathfrak{X}$  has a small dense family of compact 0-extensions (in particular  $\mathfrak{X}$  is l.f.p.),
- 3.  $(\mathcal{E}, \mathcal{F})$  is cofibrantly generated by maps between compact 0-extensions, and
- 4.  $\mathfrak{X}$  has full and effective quotients of componentwise-full equivalence relations.

#### Lemma

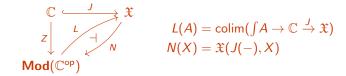
Assume  $\mathfrak{X}$  is adequate and  $F: \mathfrak{X} \to \mathbf{Set}$  preserves finite limits and sends full maps to surjections. Then F preserves quotients of componentwise-full equivalence relations.

### Proof.

Let  $R \xrightarrow[r_1]{r_0} A \xrightarrow{f} B$  be a **full exact sequence** in  $\mathfrak{X}$ , i.e. all arrows are full, f is the coequalizer of  $r_0$ ,  $r_1$ , and  $r_0$ ,  $r_1$  is the kernel pair of f. Then Ff is a surjection with kernel pair  $Fr_0$ ,  $Fr_1$ . But surjections are always coequalizers of their kernel pair.

## Idea of proof

- Assume that X is adequate.
- To show that it is clan-algebraic, we want to show that its nerve/realization adjunction



is an equivalence.

- ullet By density the right adjoint  ${\it N}$  is fully faithful, i.e. the counit is an isomorphism.
- It remains to show that the unit of the adjunction is an isomorphism, i.e.

$$A(C) \xrightarrow{\cong} \mathfrak{X}(C, \operatorname{colim}(\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})).$$

for all  $A \in \mathbf{Mod}(\mathbb{C}^{\mathrm{op}})$  and  $C \in \mathbb{C}$ .

- We know that  $\mathcal{X}(C,-)$  preserves filtered colimits and quotients of componentwise-full equivalence relations, so we'd like to decompose  $\operatorname{colim}(\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})$  in terms of these constructions.
- This is essentially what we're doing in the following.

## Jointly full cones

- Let  $D: \mathcal{I} \to \mathfrak{X}$  be a diagram in an adequate category.
- A cone  $(A, \phi)$  over D is called **jointly full**, if for every cone  $(C, \gamma)$ , extension  $e : B \to C$  and map  $g : B \to A$  constituting a cone morphism  $g : (B, \gamma \circ e) \to (A, \phi)$ , there exists a map  $h : C \to A$  such that

$$B \xrightarrow{g} A$$

$$e \downarrow \xrightarrow{h} \stackrel{\gamma}{\longrightarrow} D_{i}$$

commutes for all  $i \in \mathcal{I}$ .

• **Observation:** The cone  $(A, \phi)$  is jointly full iff the canonical map to the limit is full.

### Definition

A **nice diagram** in an adequate category  $\mathfrak{X}$  is a truncated simplicial diagram

$$A_2 \stackrel{\overline{\downarrow} d_0}{\underset{d_1}{\longleftarrow} s_0} \xrightarrow{s_0} A_1 \stackrel{\overline{\downarrow} d_0}{\underset{d_1}{\longleftarrow} s_0} \xrightarrow{s_0} A_0$$

#### where

- 1.  $A_0$ ,  $A_1$ , and  $A_2$  are 0-extensions,
- 2. the maps  $d_0, d_1: A_1 \rightarrow A_0$  are full,
- 3. in the square  $A_1 \longrightarrow A_1 \longrightarrow A_1$  $A_1 \longrightarrow A_0$  the span constitutes a jointly full diagram over the cospan,
- 4. there exists a symmetry map  $A_1 \xrightarrow{d_1} A_0 \\ A_0 \xleftarrow{d_1} A_1$  making the triangles commute, and
- 5. there exists a 0-extension  $\tilde{A}$  and full maps  $f,g:\tilde{A} \to A_1$  constituting a jointly full cone over the diagram

$$\begin{array}{ccccc}
A_1 & & & A_1 \\
d_0 \downarrow & & & \downarrow d_1 \\
A_0 & & & & A_0
\end{array}$$

## Nice diagrams

#### Lemma

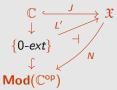
For any nice diagram, the pairing  $A_1 \xrightarrow{\langle d_0, d_1 \rangle} A_0 \times A_0$  admits a decomposition  $A_1 \twoheadrightarrow R \xrightarrow{\langle r_0, r_1 \rangle} A_0 \times A_0$  into a full map and a monomorphism, and  $\langle r_0, r_1 \rangle$  is a componentwise-full equivalence relation.

#### Lemma

Assume  $\mathfrak{X}$  is adequate and  $F: \mathfrak{X} \to \mathbf{Set}$  preserves finite limits and sends full maps to surjections. Then for every nice diagram, F preserves coequalizers of the arrows  $d_0, d_1: A_1 \to A_0$ .

#### Lemma

The restriction L' of L in the nerve/realization adjunction



to 0-extensions is fully faithful and preserves full maps and nice diagrams.

## Nice diagrams

#### Lemma

For every object A of an adequate category  $\mathfrak{X}$  there exists a nice diagram

$$A_2 \xrightarrow[d_1]{\stackrel{s_0}{\xrightarrow{d_1}}} A_1 \xrightarrow[d_1]{\stackrel{s_0}{\xrightarrow{d_1}}} A_0$$

such that A is the coequalizer of  $d_0, d_1 : A_1 \rightarrow A_0$ .

## Proof.

- $A_0$  is given by covering A by a 0-extension, i.e. factoring  $0 \to A$  as  $0 \hookrightarrow A_0 \stackrel{e}{\to} A$ .
- $A_1$  is given by covering the kernel of  $A_0 woheadrightarrow A$  by a 0-extension  $O \hookrightarrow A_1 woheadrightarrow R woheadrightarrow A_0 woheadrightarrow A_0 woheadrightarrow A$   $O \hookrightarrow A_1 woheadrightarrow R woheadrightarrow A_0 woheadrig$
- $A_2$  is given by covering the following pullback:  $\begin{matrix} 0 \hookrightarrow A_2 \longrightarrow \bullet \longrightarrow A_1 \\ \downarrow & \downarrow d_0 \\ A_1 \stackrel{d_1}{\longrightarrow} A_0 \end{matrix}$

## The theorem

#### Theorem.

Adequate categories are clan-algebraic.

### Proof.

Let  $\mathfrak{X}$  be adequate and let  $\mathbb{C} \subseteq \mathfrak{X}$  be the co-clan of compact 0-extensions. It remains to show that

$$AC \cong \mathfrak{X}(C, LA).$$

for all  $A \in \mathbf{Mod}(\mathbb{C}^{op})$  and  $C \in \mathbb{C}$ . Let  $A_{\bullet}$  be a nice diagram with coequalizer A. We have

$$\mathfrak{X}(C, LA) = \mathfrak{X}(C, L(\mathsf{coeq}(A_1 \rightrightarrows A_0)))$$

$$\cong \mathfrak{X}(C, \mathsf{coeq}(LA_1 \rightrightarrows LA_0))$$

$$\cong \mathsf{coeq}(\mathfrak{X}(C, LA_1) \rightrightarrows \mathfrak{X}(C, LA_0))$$

$$\cong \mathsf{coeq}(A_1 C \rightrightarrows A_0 C)$$

$$\cong \mathsf{coeq}(\mathsf{Mod}(ZC, A_1) \rightrightarrows \mathsf{Mod}(ZC, A_0))$$

$$\cong \mathsf{Mod}(ZC, \mathsf{coeq}(A_1 \rightrightarrows A_0))$$

$$\cong \mathsf{Mod}(ZC, A))$$

$$\cong AC$$

since 
$$\mathcal{X}(\mathcal{C},-)$$
 preserves coeqs of nice diags

since  $A = coeg(A_1 \Rightarrow A_0)$ 

since 
$$LA_i = \operatorname{colim}(\int A_i \to \mathbb{C} \to \mathfrak{X})$$
 filtered

# Part III

## Models in higher types

Let  $\mathcal{S}$  be the  $\infty$ -topos of spaces/types.

Let  $\mathcal{C}_{\mathsf{Mon}}$  be the finite-product theory of monoids, and let  $\mathcal{L}_{\mathsf{Mon}}$  be the finite-limit theory of monoids. Then

$$\mathsf{FP}(\mathcal{C}_\mathsf{Mon},\mathsf{Set}) \simeq \mathsf{FL}(\mathcal{L}_\mathsf{Mon},\mathsf{Set})$$

but  $\mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$  and  $\mathsf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$  are different:

- $FL(\mathcal{L}_{Mon}, \mathcal{S})$  is just the category of monoids
- $\mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$  is the  $\infty$ -category ' $A_{\infty}$ -algebras', i.e. homotopy-coherent monoids.

#### Moral

By being 'slimmer', finite-product theories leave room for higher coherences when interpreted in higher types.

This phenomenon has been discussed under the name 'animation' in:

• K. Cesnavicius and P. Scholze. "Purity for flat cohomology". In: arXiv preprint arXiv:1912.10932 (2019)

## Four clans for categories

Cat admits several clan-algebraic weak factorization systems:

```
• (\mathcal{E}_1,\mathcal{F}_1) is cofib. generated by \{(0 \to 1),(2 \to 2)\}

• (\mathcal{E}_2,\mathcal{F}_2) is cofib. generated by \{(0 \to 1),(2 \to 2),(2 \to 1)\}

• (\mathcal{E}_3,\mathcal{F}_3) is cofib. generated by \{(0 \to 1),(2 \to 2),(\mathbb{P} \to 2)\}

• (\mathcal{E}_4,\mathcal{F}_4) is cofib. generated by \{(0 \to 1),(2 \to 2),(\mathbb{P} \to 2),(2 \to 1)\}

where \mathbb{P}=(\bullet \rightrightarrows \bullet).
```

The right classes are:

```
\mathcal{F}_1 = \{\text{full and surjective-on-objects functors}\}\
\mathcal{F}_2 = \{\text{full and bijective-on-objects functors}\}\
\mathcal{F}_3 = \{\text{fully faithful and surjective-on-objects functors}\}\
\mathcal{F}_4 = \{\text{isos}\}\
```

Note that  $\mathcal{F}_3$  is the class of trivial fibrations for the canonical model structure on Cat.

## Four class for categories

These correspond to the following clans:

### Models in higher types:

Thanks for your attention!