A refinement of Gabriel-Ulmer duality

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Cartmell's generalized algebraic theories [Car86] – which extend algebra by type dependency – and Freyd's essentially algebraic theories [Fre72] – which permit a controlled form of partiality – are commonly recognized as being equally expressive, and are both subsumed by the less syntactic and more abstract *finite-limit theories*, which are simply small finite-limit categories.

More specifically, for each generalized/essentially algebraic theory $\mathcal{T}$ there exists a small finite-limit category $\mathbb{C}$ such that

$$\mathrm{Mod}(\mathcal{T}) \simeq \mathbf{Lex}(\mathbb{C}, \mathbf{Set}),$$

i.e. the category of models of $\mathcal{T}$ is equivalent to the category of finite-limit-preserving ('lex') functors from $\mathbb{C}$ to the category of sets.

Classical Gabriel-Ulmer duality [GU71] states that the categories of models of such theories are precisely the *locally finitely presentable categories*, giving rise to a contravariant biequivalence

$$\mathbf{Lex} \simeq \mathbf{LFP}^{\mathsf{op}}$$

between the 2-categories of finite-limit categories and locally finitely presentable categories.

We propose a refinement of this duality based on the notion of *clan*, which was introduced by Taylor under the name 'category with display maps' [Tay87, 4.3.2], and later renamed by Joyal [Joy17, 1.1.1].

Definition 1 *A* clan *is a small category $\mathcal{T}$ with terminal object* 1 *equipped with a class $\mathcal{D}$ of* display maps *such that pullbacks of display maps along arbitrary maps exist and are again display maps, and display maps contain isomorphisms and terminal projections and are closed under composition.*

A model *of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathbf{Set}$ which preserves* 1 *and pullbacks of display maps. We denote the full subcategory of $[\mathcal{T}, \mathbf{Set}]$ on models by $\mathrm{Mod}(\mathcal{T})$.*

Since corepresentable functors $\mathcal{T}(\Gamma, -)$ preserve all limits, the dual Yoneda embedding co-restricts to a fully faithful functor

$$Z \;:\; \mathcal{T}^{\mathsf{op}} \to \mathrm{Mod}(\mathcal{T}).$$

Clans refine finite-limit theories in that the 'same' finite-limit theory can be represented by different clans, thus we cannot expect to reconstruct a clan from its category of models alone.

To recover a duality we equip the category $\mathrm{Mod}(\mathcal{T})$ of models of $\mathcal{T}$ with additional information in form of a weak factorization system $(\mathcal{E}, \mathcal{F})$ which is cofibrantly generated by the set

$$\mathcal{E}_0 = \{Z(f) \mid f \in \mathcal{D}\},$$

of morphisms[1]. We call maps in $\mathcal{F}$ *full maps*, maps in $\mathcal{E}$ *extensions*, and models A such that $(0 \to A) \in \mathcal{E}$ 0-*extensions*.

Using an instance of the *fat small object argument* [MRV14] we show that (the opposite of) $\mathcal{T}$ can be recovered up to Cauchy completion from $\mathrm{Mod}(\mathcal{T})$ and $(\mathcal{E}, \mathcal{F})$ as the full subcategory of $\mathrm{Mod}(\mathcal{T})$ on *finitely presentable* 0-*extensions*, i.e. finitely presentable models A such that $(0 \to A) \in \mathcal{E}$.

Moreover, we give a characterization of those locally finitely presentable categories equipped with weak factorization systems that arise as categories of models of clans:

Proposition 1 *A lfp category $\mathfrak{A}$ with wfs $(\mathcal{E}, \mathcal{F})$ is clan-algebraic, if*

[1] The same w.f.s. construction was independently proposed by Simon Henry in a HoTTEST seminar.

1. *the full subcategory* $\mathbb{C}$ *on* $\{A \in \mathfrak{A} \mid A$ *finitely presented* 0*-extension*$\}$ *is dense in* $\mathfrak{A}$*,*

2. $(\mathcal{E}, \mathcal{F})$ *is cofibrantly generated by* $\mathcal{E} \cap \mathsf{mor}(\mathbb{C})$*, and*

3. *for* $C \in \mathbb{C}$ *and* $A \in \mathrm{Mod}(\mathbb{C}^{\mathsf{op}})$*, the functor* $\mathfrak{A}(C, -) : \mathfrak{A} \to \mathbf{Set}$ *preserves* $\mathsf{colim}(\int A \to \mathbb{C} \to \mathfrak{A})$*.*

We conjecture a simplification of condition 3 which refines the characterization of algebraic categories in [ARV10, Corollary 18.4]:

Conjecture 1 *Condition 3 of the theorem is implied by* $\mathfrak{A}$ *having full and effective quotients of componentwise full equivalence relations.*

Finally we showcase a major advantage of clans over finite-limit theories, which is that different clanic representations of the same finite-limit theory when interpreted in higher types give rise to different 'homotopy-coherent generalizations' of the set-level algebraic structure.

For example, the finite-limit theory of categories admits (at least) 4 clanic refinements, whose models in higher types are respectively *Segal spaces*, *Segal categories*, *pre-categories* (in the sense of [Uni13]), and *'discrete'* 1*-categories*. This can be seen as a refinement of Cesnavicius and Scholze's concept of *animated* model of an algebraic theory [CS19].

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