Categories of partial equivalence relations as localizations

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Abstract

We construct a category of fibrant objects $\mathbb{C}\langle\mathcal{P}\rangle$ in the sense of K. Brown from any *indexed frame* (a kind of indexed poset generalizing triposes) $\mathcal{P}$, and show that its homotopy category is the Barr-exact category $\mathbb{C}[\mathcal{P}]$ of partial equivalence relations and compatible functional relations.

We give criteria for the existence of left and right derived functors to functors $\mathbb{C}\langle\Phi\rangle : \mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{C}\langle\mathcal{Q}\rangle$ induced by finite-meet-preserving transformations $\Phi : \mathcal{P} \to \mathcal{Q}$.

Introduction

Hyland, Johnstone and Pitts introduced the notion of *tripos*, together with the *tripos-to-topos construction*, as a powerful tool to describe elementary toposes, in particular the novel family of *realizability toposes* [HJP80]. A tripos is an indexed poset $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ with sufficient structure to interpret intuitionistic higher order logic, and the associated topos $\mathbb{C}[\mathcal{P}]$ is defined as a category of partial equivalence relations and compatible functional relations in the internal language of $\mathcal{P}$. The proof that $\mathbb{C}[\mathcal{P}]$ is a topos [HJP80, Theorem 2.13] relies on the higher-order structure of $\mathcal{P}$, but the construction of $\mathbb{C}[\mathcal{P}]$ is well-defined and produces an *exact category*[1] as soon as soon as $\mathcal{P}$ models the $(\exists, \wedge, \top)$-fragment of first order logic. In the terminology of [MR12a, MR12b, MPR17] such a $\mathcal{P}$ is called an *existential doctrine*.

If $\mathcal{P}$ furthermore models *equality* — thus the $(\exists, \wedge, \top, =)$-fragment of first order logic known as *regular logic* [Joh02b, D1.1] — then it is called an *elementary* existential doctrine[2]. In [MR12b], Maietti and Rosolini show that for elementary existential doctrines $\mathcal{P}$, the construction of $\mathbb{C}[\mathcal{P}]$ can be characterized as left biadjoint to a forgetful functor $U : \mathbf{Ex} \to \mathbf{EED}$ sending exact categories to their indexed poset of subobjects, see Example 1.3(a).

This characterization implies immediately that the assignment sending elementary existential doctrines $\mathcal{P}$ to exact categories $\mathbb{C}[\mathcal{P}]$ is functorial w.r.t. the morphisms of **EED**, which are indexed monotone maps preserving the connectives of regular logic. However, remarkably, it turns out that between indexed posets with *more* logical structure (specifically triposes) it is possible to construct functors from indexed monotone maps $\Phi : \mathcal{P} \to \mathcal{Q}$ preserving *less* logical structure (specifically only finite meets); a phenomenon that was exploited in [HJP80] to construct geometric morphisms between toposes from suitable adjunctions of indexed monotone maps between triposes, and has been analyzed in a 2-categorical framework in [Fre15].

In the present work we show that for a subclass of elementary existential doctrines which we call *indexed frames*, and which satisfy a stronger Beck–Chevalley condition, the category

[1] I.e. a regular category with effective equivalence relations; called *effective regular* category in [Joh02a].

[2] The term 'elementary existential doctrine' was introduced by Lawvere [Law70] for a slightly different — in particular non-posetal — notion, and was adapted by Maietti and Rosolini to the posetal setting.

$\mathbb{C}[\mathcal{P}]$ of partial equivalence relations $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ can be presented as localization of a *category of fibrant objects* $\mathbb{C}\langle\mathcal{P}\rangle$ (which is homotopically trivial, see Remark 4.8), and that the functors arising from finite-meet-preserving transformations $\Phi : \mathcal{P} \to \mathcal{Q}$ between triposes mentioned above can be understood as *right derived functors* of functors $\mathbb{C}\langle\Phi\rangle : \mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{C}\langle\mathcal{Q}\rangle$ between categories of fibrant objects (Theorem 8.4(iii)). We also show that $\mathbb{C}\langle\Phi\rangle$ admits a *left* derived functor whenever $\mathcal{P}$ has 'enough $\exists$-prime predicates' (Theorem 8.4(iv)), and to set this up we develop in Section 6 the notion of $\exists$-*prime predicate* and its relation to indexed frames that are obtained by 'freely adding' existential quantification to more primitive indexed posets.

It should be pointed out that the fibrations in the category-of-fibrant-objects structure on $\mathbb{C}\langle\mathcal{P}\rangle$ play a somewhat curious role in our analysis: the fact that $\mathbb{C}[\mathcal{P}]$ is the homotopy category can be proven directly without ever mentioning them, and to prove existence of right derived functors we use a fibrant replacement-style argument (Lemma 7.3) using the alternative notion of *proto-fibrant object* (Definition 7.1), ordinary fibrant replacement being trivial and unhelpful in a setting where everything is already fibrant. However, the fibrations play an indirect role in the existence proof of left derived functors (via the notion of *cofibrant object* in a category of fibrant objects), and are used in Remark 4.8 to show that the localizations are homotopically trivial.

Related work and acknowledgements.

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1 Indexed posets

An *indexed poset* is a functor $\mathcal{A} : \mathbb{B}^{\mathsf{op}} \to \mathbf{Pos}$ from the opposite of an arbitrary category $\mathbb{B}$ — called *base category* — to the category $\mathbf{Pos}$ of posets and monotone maps[3]. For $I \in \mathbb{B}$, the poset $\mathcal{A}(I)$ is called the *fiber* of $\mathcal{A}$ over I and its elements are called *predicates* on I. The monotone maps $\mathcal{P}(f) : \mathcal{P}(I) \to \mathcal{P}(J)$ for $f : J \to I$ are called *reindexing maps* and are abbreviated f^* when $\mathcal{P}$ is clear from the context. The *total category* (a.k.a. *Grothendieck construction*) of $\mathcal{A}$ is the category $\int\mathcal{A}$ whose objects are pairs $(I \in \mathbb{B}, \varphi \in \mathcal{A}(I))$ and whose morphisms from (I, φ) to (J, ψ) are morphisms $f : I \to J$ in $\mathbb{B}$ satisfying $\varphi \leq f^*\psi$. The total category admits a forgetful functor $\int\mathcal{A} \to \mathbb{B}$ which is a Grothendieck fibration.

Given indexed posets $\mathcal{A}, \mathcal{B} : \mathbb{B}^{\mathsf{op}} \to \mathbf{Pos}$, an *indexed monotone map* is a natural transformation $\Phi : \mathcal{A} \to \mathcal{B}$.

An *indexed meet-semilattice* is an indexed poset $\mathcal{A} : \mathbb{B}^{\mathsf{op}} \to \mathbf{Pos}$ where all fibers are meet-semilattices (which we always assume to be 'bounded', i.e. to have a greatest element $\top$) and

[3] Different assumptions on the relative sizes of $\mathbb{B}$ and the posets in $\mathbf{Pos}$ are possible here, we view as basic the case where $\mathbb{B}$ is small and $\mathbf{Pos}$ is the category of small posets. Then e.g. the case of small posets over locally small categories can be recovered by postulating a universe and adding assumptions.

all reindexing maps preserve finite meets. An indexed monotone map $\Phi : \mathcal{A} \to \mathcal{B} : \mathbb{B}^{\mathsf{op}} \to \mathbf{Pos}$ between indexed meet-semilattices is called *cartesian* if it preserves fiberwise finite meets.

A *indexed frame* is an indexed meet-semilattice $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ on a finite-limit category $\mathbb{C}$, satisfying the following three conditions.

(Ex) All reindexing maps f^* (for $f : J \to I$) have left adjoints $\exists_f : \mathcal{P}(J) \to \mathcal{P}(I)$.

(Fr) We have $(\exists_f \varphi) \wedge \psi = \exists_f(\varphi \wedge f^*\psi)$ for all $f : J \to I$, $\varphi \in \mathcal{P}(J)$, $\psi \in \mathcal{P}(I)$.

(BC) We have $\exists_h \circ k^* = f^* \circ \exists_g$ for all pullback squares $\begin{array}{ccc} D & \xrightarrow{k} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array}$ in $\mathbb{C}$.

Condition (Fr) is known as the *Frobenius law*, and (BC) as the *Beck–Chevalley* condition.

A *morphism of indexed frames* is an indexed monotone map that is cartesian and commutes with $\exists$ along all maps in the base.

Remark 1.1 Notions similar to indexed frames appear under various names in the literature — compare e.g. to *regular fibrations* in [Jac01, Definition 4.2.1] and the already mentioned *elementary existential doctrines*, which only require finite *products* in the base category and use a weaker version of the Beck–Chevalley condition.

Stekelenburg's *fibered locales* [Ste13] require (BC) for all existing pullback squares, but the definition is stated for arbitrary base categories.

These weaker notions are not adequate for the present paper, since the proof of Theorem 4.7 relies — via Lemma 2.5 — on the fact that arbitrary pullbacks exist in the base category and satisfy (BC). ♢

A *tripos* is an indexed frame $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ satisfying moreover the following three conditions.

(HA) All fibers $\mathcal{P}(I)$ are Heyting algebras.

(U) Besides left adjoints $\exists_f$, all reindexing maps f^* have right adjoints $\forall_f$.

(PO) For every object $I \in \mathbb{C}$ there exists a *power object*, i.e. an object $\mathfrak{P}(I) \in \mathbb{C}$ together with a predicate $\varepsilon_I \in \mathcal{P}(I \times \mathfrak{P}(I))$ such that for all $J \in \mathbb{C}$ and $\varphi \in \mathcal{P}(I \times J)$ there exists a (not necessarily unique) morphism $\ulcorner\varphi\urcorner : J \to \mathfrak{P}(I)$ satisfying $(I \times \ulcorner\varphi\urcorner)^*(\varepsilon_I) = \varphi$.

Remarks 1.2 (*a*) Whereas we were careful to distinguish between elementary existential doctrines and indexed frames because of different assumptions on the base category and phrasings of the Beck–Chevalley condition, there is some ambiguity around these issues for the notion of tripos, and different variants of the definition exist in the literature. The above version is given by Pitts in [Pit81].

(*b*) It follows from the presence of the right adjoint and from (Fr) that the reindexing maps in a tripos preserve the Heyting algebra structure of the fibers.

Moreover, the Beck–Chevalley condition for $\forall$ follows from (BC) for $\exists$ by an argument involving 2-categorical *mates*, because the class of squares for which we postulate (BC) is closed under transpose (which is not the case e.g. for existential doctrines). ♢

Examples 1.3 (*a*) The *canonical indexing*

$$\mathsf{fam}(P) : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Pos}$$

of a poset P is the **Set**-indexed poset defined by $\mathsf{fam}(P)(I) = P^I$ with the pointwise ordering. Monotone maps $f : P \to Q$ are in bijective correspondence with indexed monotone maps $\mathsf{fam}(f) : \mathsf{fam}(P) \to \mathsf{fam}(Q)$ between canonical indexings. The indexed poset $\mathsf{fam}(P)$ is an indexed meet-semilattice if and only if P is a meet-semilattice, and it is a tripos if and only if it is an indexed frame if and only if P is a frame. Provided domain and codomain have the appropriate structure, $\mathsf{fam}(f)$ is cartesian if and only if f preserves finite meets, and $\mathsf{fam}(f)$ is a morphism of indexed frames if and only if f is a frame morphism.

(*b*) For every finite-limit category $\mathbb{C}$ there is an indexed meet-semilattice

$$\mathsf{sub}(\mathbb{C}) : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$$

where for each $A \in \mathbb{C}$, $\mathsf{sub}(\mathbb{C})(A)$ is the poset reflection of the full subcategory of $\mathbb{C}/A$ on monomorphism, and reindexing is given by pullbacks.

The category $\mathbb{C}$ is a *regular category* [Joh02a, A1.3] if and only if $\mathsf{sub}(\mathbb{C})$ is an indexed frame. ◇

2 The internal language

The *internal language* of an indexed frame $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ is a many-sorted first-order language in the sense of [Joh02b, Section D1.1]. It is generated from a signature whose sorts are the objects of $\mathbb{C}$, whose function symbols of arity $A_1, \dots, A_n \to B$ are the morphisms of type $A_1 \times \cdots \times A_n \to B$ in $\mathbb{C}$, and whose relation symbols of arity $A_1, \dots, A_n$ are the elements of $\mathcal{P}(A_1 \times \cdots \times A_n)$[4].

Over this signature we consider *terms* — which are built up from sorted variables[5] and function symbols, subject to matching arities and sorts — and *regular formulas*, which are generated from *atomic formulas* $\varphi(\vec{t})$ (where φ is a relation symbol and $\vec{t}$ is a list of terms matching its arity) and $s = t$ (where s and t are terms of the same sort), using the connectives of conjunction $\wedge$, truth $\top$, and existential quantification $\exists$.

A *context* is a list $\vec{x}$ of distinct variables. We say that a term t or formula P is *in context* $\vec{x}$, if all of its free variables are contained in $\vec{x}$.

Instead of writing that a term t or formula P is in context $\vec{x}$, we also write $(\vec{x} \mid t)$ is a *term-in-context* and $(\vec{x} \mid P)$ is a *formula-in-context*.

We write $\mathfrak{s}(x)$ and $\mathfrak{s}(t)$ for the sort of a variable and a term, respectively, and we use the shorthand $\mathfrak{s}(x_1, \dots, x_n) = \mathfrak{s}(x_1) \times \cdots \times \mathfrak{s}(x_n)$ for contexts.

The *interpretation* of terms-in-context and formlas-in-context is defined by structural induction by the clauses in Table 1. In general, the interpretation of a term-in-context $(\vec{x} \mid t)$ is a morphism $[\![t]\!]_{\vec{x}} : \mathfrak{s}(\vec{x}) \to \mathfrak{s}(t)$ in $\mathbb{C}$, and the interpretation of a formula-in-context $(\vec{x} \mid P)$ is a predicate $[\![P]\!]_{\vec{x}} \in \mathcal{P}(\mathfrak{s}(\vec{x}))$.

When defining a predicate in an indexed frame by a formula in the internal language, we normally write $\varphi(\vec{x}) \equiv P$ instead of $\varphi = [\![P]\!]_{\vec{x}}$.

The following standard lemmas are verified by structural induction.

[4] Note that the same predicate in $\mathcal{P}$ can give rise to different relation symbols, e.g. $\rho \in \mathcal{P}(A \times B)$ can both be viewed as a binary relation symbol of arity A, B and as unary relation symbol of arity $A \times B$. The same is true for morphisms and function symbols. The intended interpretation will always be clear from the context.

[5] We don't annotate variables with sorts as the sorts are always clear from the context.

$$
\begin{aligned}
[\![x_i]\!]_{\vec{x}} &= \pi_i \\
[\![f(t_1,\dots,t_n)]\!]_{\vec{x}} &= f \circ \langle [\![t_1]\!]_{\vec{x}}, \dots, [\![t_n]\!]_{\vec{x}} \rangle \\
[\![\varphi(t_1,\dots,t_n)]\!]_{\vec{x}} &= \langle [\![t_1]\!]_{\vec{x}}, \dots, [\![t_n]\!]_{\vec{x}} \rangle^*(\varphi) \\
[\![t = u]\!]_{\vec{x}} &= \langle [\![t]\!]_{\vec{x}}, [\![u]\!]_{\vec{x}} \rangle^*(\exists_\delta \top) \\
[\![P \wedge Q]\!]_{\vec{x}} &= [\![P]\!]_{\vec{x}} \wedge [\![Q]\!]_{\vec{x}} \\
[\![\top]\!]_{\vec{x}} &= \top \\
[\![\exists y \,.\, P]\!]_{\vec{x}} &= \exists_\pi([\![P]\!]_{\vec{x},y})
\end{aligned}
$$

In the fourth clause δ is the diagonal map $\mathfrak{s}(t) \to \mathfrak{s}(t) \times \mathfrak{s}(t)$, and in the last clause π is the projection $\mathfrak{s}(\vec{x}) \times \mathfrak{s}(y) \to \mathfrak{s}(\vec{x})$.

Table 1: Interpretation of the internal language

Lemma 2.1 (Weakening) *We have*

- $[\![t]\!]_{\vec{x},y,\vec{z}} = [\![t]\!]_{\vec{x},\vec{z}} \circ \pi$
- $[\![P]\!]_{\vec{x},y,\vec{z}} = \pi^*([\![P]\!]_{\vec{x},\vec{z}})$

for all terms-in-context $(\vec{x},\vec{z} \mid t)$ *and formulas-in-context* $(\vec{x},\vec{z} \mid P)$*, where* $\pi : \mathfrak{s}(\vec{x},y,\vec{z}) \to \mathfrak{s}(\vec{x},\vec{z})$ *is the obvious projection.* ■

Lemma 2.2 (Substitution) *We have*

- $[\![t[u/y]]\!]_{\vec{x}} = [\![t]\!]_{\vec{x},y} \circ \langle \mathrm{id}_{\mathfrak{s}(\vec{x})}, [\![u]\!]_{\vec{x}} \rangle$
- $[\![P[u/y]]\!]_{\vec{x}} = \langle \mathrm{id}_{\mathfrak{s}(\vec{x})}, [\![u]\!]_{\vec{x}} \rangle^*([\![P]\!]_{\vec{x},y})$

for all formulas-in-context $(\vec{x},y \mid P)$ *and terms-in-context* $(\vec{x},y \mid t)$, $(\vec{x} \mid u)$ *such that* $\mathfrak{s}(y) = \mathfrak{s}(u)$. ■

We call terms-in-context $(\vec{x} \mid t)$ and $(\vec{x} \mid u)$ (or formulas-in-context $(\vec{x} \mid P)$ and $(\vec{x} \mid Q)$) *semantically equal*, if $[\![t]\!]_{\vec{x}} = [\![u]\!]_{\vec{x}}$ (or $[\![P]\!]_{\vec{x}} = [\![Q]\!]_{\vec{x}}$).

The following lemma justifies *local rewriting*. Because of the presence of quantifiers it cannot be deduced from the substitution lemma, but it can be proven straightforwardly by structural induction.

Lemma 2.3 (Congruence) *Semantic equality of terms and formulas in-context is preserved when replacing subterms-in-context (or subformulas-in-context) of a formula-in-context* $(\vec{x} \mid P)$ *by semantically equal ones.*

A *judgment* in the internal language is an expression of the form $\Gamma \vdash_{\vec{x}} Q$, where $\Gamma \equiv P_1,\dots,P_n$ is a list of formulas in context $\vec{x}$, and Q is a formula in context $\vec{x}$. We say that the judgment is *valid* (or *holds*), if

$$[\![P_1]\!]_{\vec{x}} \wedge \cdots \wedge [\![P_n]\!]_{\vec{x}} \leq [\![Q]\!]_{\vec{x}} \quad \text{in} \quad \mathcal{P}(\mathfrak{s}(\vec{x})) \ .$$

For convenience introduce the abbreviation $[\![\Gamma]\!]_{\vec{x}} = [\![P_1]\!]_{\vec{x}} \wedge \cdots \wedge [\![P_n]\!]_{\vec{x}}$ for the left hand side of this inequality.

$$
\begin{array}{ccc}
\dfrac{}{P_1,\dots,P_n \vdash_{\vec{x}} P_i} & \dfrac{\Gamma \vdash_{\vec{x},y} R[t/y]}{\Gamma \vdash_{\vec{x}} \exists y\,.\,R} & \dfrac{\Gamma \vdash_{\vec{x}} \exists y\,.\,R \qquad \Gamma, R \vdash_{\vec{x},y} P}{\Gamma \vdash_{\vec{x}} P} \\[2ex]
\dfrac{}{\Gamma \vdash_{\vec{x}} \top} & \dfrac{}{\Gamma \vdash_{\vec{x}} t = t} & \dfrac{\Gamma \vdash_{\vec{x}} R[s/y] \qquad \Gamma \vdash_{\vec{x}} s = t}{\Gamma \vdash_{\vec{x}} R[t/y]} \\[2ex]
\dfrac{\Gamma \vdash_{\vec{x}} P \wedge Q}{\Gamma \vdash_{\vec{x}} P} & \dfrac{\Gamma \vdash_{\vec{x}} P \wedge Q}{\Gamma \vdash_{\vec{x}} Q} & \dfrac{\Gamma \vdash_{\vec{x}} P \qquad \Gamma \vdash_{\vec{x}} Q}{\Gamma \vdash_{\vec{x}} P \wedge Q}
\end{array}
$$

Table 2: The rules of regular logic

A *deduction* is an $(n+1)$-tuple of judgments $(\mathcal{H}_1, \dots, \mathcal{H}_n; \mathcal{K})$ for some $n \geq 0$, where the $\mathcal{H}_i$ are called *hypotheses* and $\mathcal{K}$ the *conclusion*. The deduction is called *admissible* if the validity of the hypotheses implies the validity of the conclusion. As is customary, we often write deductions with a horizontal line as

$$\frac{\mathcal{H}_1 \quad \dots \quad \mathcal{H}_n}{\mathcal{K}} \; .$$

A *rule* is a 'deduction scheme', i.e. a deduction containing placeholders for terms and formulas and contexts. A rule is called admissible if all syntactically-correct instantiations of these placeholders yield admissible deductions.

The following *soundness theorem* is straightforward and holds in fact in the internal language of any elementary existential doctrine.

Theorem 2.4 (Soundness) *The rules of regular logic in Table 2 are admissible.* ■

On the other hand, the next lemma relies on the stronger version of the Beck–Chevalley condition postulated in indexed frames.

Lemma 2.5 *Assume that*

$$
\begin{tikzcd}
D \arrow[r, "k"'] \arrow[d, "h"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"] & C
\end{tikzcd}
\tag{2.1}
$$

is a pullback square in $\mathbb{C}$, and Γ and Q are respectively a list of formulas and a formula in the same context $\vec{x}, y, z$, with $\mathfrak{s}(y) = A$ and $\mathfrak{s}(z) = B$. Then the deduction

$$\frac{\Gamma[hp/y, kp/z] \vdash_{\vec{x},p} Q[hp/y, kp/z]}{\Gamma, fy = gz \vdash_{\vec{x},y,z} Q} \tag{2.2}$$

is admissible.

Proof. Let $X = \mathfrak{s}(\vec{x})$. The assumption that (2.1) is a pullback implies that

$$
\begin{tikzcd}
X \times D \arrow[r, "v"'] \arrow[d, "m"'] & C \arrow[d, "\delta"] \\
X \times A \times B \arrow[r, "u"] & C \times C
\end{tikzcd}
\qquad
\begin{array}{c}
m = X \times \langle h, k \rangle \\
u = \langle f \circ \pi_2, g \circ \pi_3 \rangle \\
v = g \circ k \circ \pi_2 = f \circ h \circ \pi_2
\end{array}
$$

is a pullback as well. The hypothesis of the deduction (2.2) unfolds into the inequality

$$m^* [\![\Gamma]\!]_{\vec{x},y,z} \leq m^* [\![Q]\!]_{\vec{x},y,z}. \tag{2.3}$$

For the interpretation of the left hand side of the conclusion we have

$$\begin{aligned}
[\![\Gamma]\!]_{\vec{x},y,z} \wedge [\![fy = gz]\!]_{\vec{x},y,z} &= [\![\Gamma]\!]_{\vec{x},y,z} \wedge u^* \exists_\delta \top \\
&= [\![\Gamma]\!]_{\vec{x},y,z} \wedge \exists_m \top && \text{by (BC)} \\
&= \exists_m m^* [\![\Gamma]\!]_{\vec{x},y,z} && \text{by (Fr)}
\end{aligned}$$

and we see that the last term is less-or-equal $[\![Q]\!]_{\vec{x},y,z}$ if and only if the inequality (2.3) holds. ■

2.6 Interpreting non-regular connectives in triposes. Triposes admit a richer internal language than general indexed frames, since they can model *all* connectives of intuitionistic first-order logic, not only the regular fragment. The *non-regular* connectives $\forall, \vee, \bot, \Rightarrow$ are interpreted by the clauses

$$\begin{aligned}
[\![\forall y \,.\, P]\!]_{\vec{x}} &= \forall_\pi [\![P]\!]_{\vec{x},y} & \qquad [\![P \vee Q]\!]_{\vec{x}} &= [\![P]\!]_{\vec{x}} \vee [\![Q]\!]_{\vec{x}} \\
[\![P \Rightarrow Q]\!]_{\vec{x}} &= [\![P]\!]_{\vec{x}} \Rightarrow [\![Q]\!]_{\vec{x}} & [\![\bot]\!]_{\vec{x}} &= \bot
\end{aligned}$$

where again, π is the appropriate projection and '$\vee$', '$\bot$', and '$\Rightarrow$' on the right hand sides denote binary join, least element, and Heyting implication in the Heyting algebra $\mathcal{P}(\mathfrak{s}(\vec{x}))$. The augmented language satisfies the *Weakening*, *Substitution*, and *Congruence Lemmas* 2.1, 2.2 and 2.3 without modification, and the *Soundness Theorem* 2.4 for the set of rules in Table 2 and the following rules for the new connectives.

$$\frac{\Gamma \vdash_{\vec{x},y} R}{\Gamma \vdash_{\vec{x}} \forall y \,.\, R} \qquad \frac{\Gamma, P \vdash_{\vec{x}} Q}{\Gamma \vdash_{\vec{x}} P \Rightarrow Q} \qquad \frac{\Gamma \vdash_{\vec{x}} P}{\Gamma \vdash_{\vec{x}} P \vee Q}$$

$$\frac{\Gamma \vdash_{\vec{x}} \forall y \,.\, R}{\Gamma \vdash_{\vec{x}} R[t/y]} \qquad \frac{\Gamma \vdash_{\vec{x}} P \Rightarrow Q \qquad \Gamma \vdash_{\vec{x}} P}{\Gamma \vdash_{\vec{x}} Q} \qquad \frac{\Gamma \vdash_{\vec{x}} Q}{\Gamma \vdash_{\vec{x}} P \vee Q}$$

$$\frac{\Gamma \vdash_{\vec{x}} \bot}{\Gamma \vdash_{\vec{x}} P} \qquad \frac{\Gamma \vdash_{\vec{x}} P \vee Q \qquad \Gamma, P \vdash_{\vec{x}} S \qquad \Gamma, Q \vdash_{\vec{x}} S}{\Gamma \vdash_{\vec{x}} S}$$

Finally, to account for the power objects in triposes we could extend the language by a higher order term former for subset comprehension, or alternatively express comprehension as an axiom scheme [Pit02, Axiom 4.1]. We don't need this here — our only use of internal language for triposes is in the proof of Theorem 8.1 and there the first order language is sufficient. ♢

3 The category of partial equivalence relations and compatible maps

Definition 3.1 Let $\mathcal{A} : \mathbb{B}^{\mathsf{op}} \to \mathbf{Pos}$ be an indexed meet-semilattice on a finite-product category. The category $\mathbb{B}\langle\mathcal{A}\rangle$ is defined as follows.

- Objects are pairs $(A \in \mathbb{B}, \rho \in \mathcal{A}(A \times A))$ such that the judgments

  (sym) $\rho(x,y) \vdash_{x,y} \rho(y,x)$

  (trans) $\rho(x,y), \rho(y,z) \vdash_{x,y,z} \rho(x,z)$

  hold in $\mathcal{A}$.

- Morphisms from (A,ρ) to (B,σ) are maps $f : A \to B$ in $\mathbb{B}$ such that (compat) $\rho(x, y) \vdash_{x,y} \sigma(fx, fy)$

holds in $\mathcal{A}$.

- Composition and identities are inherited from $\mathbb{B}$. $\diamondsuit$

Notation and Terminology 3.2 (a) If $\rho \in \mathcal{A}(A \times A)$ satisfies (sym) and (trans), we call it a *partial equivalence relation*. If $f : A \to B$ satisfies (compat), we say that it is *compatible* with ρ and σ.

(b) We write $\rho_0 := \delta_A^* \rho$ for the reindexing of a partial equivalence relation along the diagonal $\delta_A : A \to A \times A$, and call it the *support* of ρ.

(c) In the internal language we simply write ρx instead of $\rho_0 x$ or $\rho(x, x)$. $\diamondsuit$

Lemma 3.3 *Let $\mathcal{A} : \mathbb{C}^{\mathsf{op}} \to$ **Pos** be an indexed meet-semilattice on a finite-limit category $\mathbb{C}$.*

(i) *The forgetful functor $U : \mathbb{C}\langle\mathcal{A}\rangle \to \mathbb{C}$ has a right adjoint ∇.*

(ii) *The category $\mathbb{C}\langle\mathcal{A}\rangle$ has finite limits and U preserves them.*

(iii) *A morphism $f : (A, \rho) \to (B, \sigma)$ is an isomorphism in $\mathbb{C}\langle\mathcal{A}\rangle$ if and only if f is an isomorphism in $\mathbb{C}$ and $(f \times f)^*\sigma = \rho$.*

Proof. The right adjoint is given by $\nabla(A) = (A, \top)$. The terminal object in $\mathbb{C}\langle\mathcal{A}\rangle$ is $(1, \top)$. A pullback of $(A, \rho) \xrightarrow{f} (C, \tau) \xleftarrow{g} (B, \sigma)$ is given by

$$\begin{array}{ccc} (D, \rho \bowtie_C \sigma) & \xrightarrow[k]{} & (B, \sigma) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ (A, \rho) & \xrightarrow{f} & (C, \tau) \end{array}$$

where $\begin{array}{ccc} D & \xrightarrow[k]{} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array}$ is a pullback in $\mathbb{C}$ and $(\rho \bowtie_C \sigma)(p, q) \equiv \rho(hp, hq) \wedge \sigma(kp, kq)$. For the third claim, the necessity of the conditions becomes obvious by considering an inverse to $f : (A, \rho) \to (B, \sigma)$. Conversely, the conditions also allow to construct this inverse. $\blacksquare$

Remark 3.4 Setting $(C, \tau) = (1, \top)$ in the construction of the pullback, we get a representation of binary products as $(A, \rho) \times (B, \sigma) = (A \times B, \rho \bowtie \sigma)$ with $(\rho \bowtie \sigma)(u, v) \equiv \rho(\pi_1 u, \pi_1 v) \wedge \sigma(\pi_2 u, \pi_2 v)$. $\diamondsuit$

Definition 3.5 Given an indexed meet-semilattice $\mathcal{A} : \mathbb{B}^{\mathsf{op}} \to$ **Pos** on a finite-product category and an object $(A, \rho) \in \mathbb{B}\langle\mathcal{A}\rangle$, we say that $\varphi \in \mathcal{A}(A)$ is a *ρ-descent predicate* if the judgments

$$\varphi x \vdash_x \rho x \qquad \text{and} \qquad \varphi x, \rho(x, y) \vdash_{x,y} \varphi y$$

hold in $\mathcal{A}$. $\diamondsuit$

Remarks 3.6 (a) For every indexed meet-semilattice $\mathcal{A} : \mathbb{B}^{\mathsf{op}} \to$ **Pos** on a finite-product category, we can define an indexed meet-semilattice $\widetilde{\mathcal{A}} : \mathbb{B}\langle\mathcal{A}\rangle^{\mathsf{op}} \to$ **Pos** where the elements of $\widetilde{\mathcal{A}}(A, \rho)$ are the ρ-descent predicates. In [MR12a, Theorem 4.8], $\widetilde{\mathcal{A}}$ is characterized as a free completion of $\mathcal{A}$ with *comprehension* and *descent quotients*.

(b) In [vO08, pg. 67] van Oosten writes 'strict relation on (A, ρ)' for what we call ρ-descent predicate. In [Fre13], the terminology 'predicate compatible with ρ' is used. $\diamondsuit$

4 The category $\mathbb{C}\langle\mathcal{P}\rangle$ as a category of fibrant objects

In this section we show that for every indexed frame $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$, the category $\mathbb{C}\langle\mathcal{P}\rangle$ can be equipped with the structure of a 'category of fibrant objects' in a natural way. We start by recalling the definition from [Bro73].

Definition 4.1 A *category of fibrant objects* is a category $\mathcal{C}$ with finite products, together with two distinguished classes $\mathcal{F}, \mathcal{W} \subseteq \mathrm{mor}(\mathcal{C})$ of morphisms whose elements are called *fibrations* and *weak equivalences*, respectively. Morphisms in $\mathcal{F} \cap \mathcal{W}$ are called *trivial fibrations*. These classes are subject to the following axioms.

(A) All isomorphisms are weak equivalences, and for any composable pair $A \xrightarrow{f} B \xrightarrow{g} C$, if either two of the three morphisms f, g, and gf are weak equivalences, then so is the third.

(B) The class of fibrations contains all isomorphisms and is closed under composition.

(C) Pullbacks of fibrations along arbitrary maps exist and are fibrations. Pullbacks of trivial fibrations are trivial fibrations.

(D) For any $X \in \mathcal{C}$ there exists a *path object*, i.e. a factorization

$$X \xrightarrow{s} P(X) \xrightarrow{d=\langle d_0,d_1\rangle} X \times X$$

of the diagonal with $s \in \mathcal{W}$ and $d \in \mathcal{F}$.

(E) For any $X \in \mathcal{C}$, the map $X \to 1$ is a fibration. $\diamondsuit$

For the remainder of the section let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ be a fixed indexed frame.

Definition 4.2 A morphism $f : (A, \rho) \to (B, \sigma)$ in $\mathbb{C}\langle\mathcal{P}\rangle$ is a *fibration* if the judgment

(fib) $\rho x, \sigma(fx, u) \vdash_{x,u} \exists y \,.\, \rho(x, y) \wedge fy = u$

holds in $\mathcal{P}$. It is a *weak equivalence* if the judgments

(inj) $\rho x, \sigma(fx, fy), \rho y \vdash_{x,y} \rho(x, y)$

(esurj) $\sigma u \vdash_u \exists x \,.\, \rho x \wedge \sigma(fx, u)$

hold in $\mathcal{P}$. $\diamondsuit$

Lemma 4.3 *$f : (A, \rho) \to (B, \sigma)$ is a trivial fibration if and only if (inj) and*

(surj) $\sigma u \vdash_u \exists x \,.\, \rho x \wedge fx = u$

hold in $\mathcal{P}$.

Proof. We show the following implications.

- (surj) $\implies$ (esurj)
- (inj), (surj) $\implies$ (fib)
- (esurj), (fib) $\implies$ (surj)

The proof is carried out in the internal language. In the following enumerated lists, each line represents an admissible deduction, with hypotheses and conclusion separated by the arrow '$\implies$'. In each line, the hypotheses are known to be valid either because they are assumptions, or because they were established in the previous lines. The admissibility of the individual deductions follows from the rules of regular logic (Table 2) by elementary logical reasoning.

Implication (inj), (surj) $\implies$ (fib):

(1) (sym), (trans) $\implies$ $\sigma(fx, u) \vdash_{x,u} \sigma u$

(2) (1), (surj) $\implies$ $\sigma(fx, u) \vdash_{x,u} \exists y \,.\, \rho y \wedge fy = u$

(3) (inj) $\implies$ $\rho x, \sigma(fx, u), \rho y, fy = u \vdash_{x,y,u} \rho(x, y)$

(4) (3) $\implies$ $\rho x, \sigma(fx, u), \rho y, fy = u \vdash_{x,y,u} \rho(x, y) \wedge fy = u$

(5) (4) $\implies$ $\rho x, \sigma(fx, u), \rho y, fy = u \vdash_{x,y,u} \exists y \,.\, \rho(x, y) \wedge fy = u$

(6) (2), (5) $\implies$ $\rho x, \sigma(fx, u) \vdash_{x,u} \exists y \,.\, \rho(x, y) \wedge fy = u$

Implication (surj) $\implies$ (esurj):

(1) (compat) $\implies$ $\rho x \vdash_{x} \sigma(fx)$

(2) (1) $\implies$ $\rho x, fx = u \vdash_{x,u} \sigma(fx, u)$

(3) (2) $\implies$ $\rho x, fx = u \vdash_{x,u} \rho x \wedge \sigma(fx, u)$

(4) (3) $\implies$ $\exists x \,.\, \rho x \wedge fx = u \vdash_{u} \exists x \,.\, \rho x \wedge \sigma(fx, u)$

(5) (4), (surj) $\implies$ $\sigma u \vdash_{u} \exists x \,.\, \rho x \wedge \sigma(fx, u)$

Implication (esurj), (fib) $\implies$ (surj):

(1) (sym), (trans) $\implies$ $\rho(x, y) \vdash_{x,y,u} \rho y$

(2) (1) $\implies$ $\rho(x, y), fy = u \vdash_{x,y,u} \rho y \wedge fy = u$

(3) (2) $\implies$ $\exists y \,.\, \rho(x, y) \wedge fy = u \vdash_{x,u} \exists y \,.\, \rho y \wedge fy = u$

(4) (3), (fib) $\implies$ $\rho x, \sigma(fx, u) \vdash_{x,u} \exists y \,.\, \rho y \wedge fy = u$

(5) (4), (esurj) $\implies$ $\sigma u \vdash_{u} \exists y \,.\, \rho y \wedge fy = u$ ∎

Remark 4.4 Stated variable-freely, the condition (surj) reduces to the inequality $\sigma_0 \leq \exists_f \rho_0$, and since the reverse inequality follows from (compat) this is equivalent to the equality $\sigma_0 = \exists_f \rho_0$. ♢

Definition 4.5 Given an object $(A, \rho) \in \mathbb{C}\langle \mathcal{P} \rangle$ and a ρ-descent predicate $\varphi \in \mathcal{P}(A)$, we define the *restriction* $\rho|_{\varphi}$ of ρ to φ by $(\rho|_{\varphi})(x, y) \;\equiv\; \rho(x, y) \wedge \varphi(x)$. ♢

The following is easy to see.

Lemma 4.6 *If $\rho \in \mathcal{P}(A \times A)$ is a partial equivalence relation and $\varphi \in \mathcal{P}(A)$ is a ρ-descent predicate, then the restriction $\rho|_\varphi$ is a partial equivalence relation, and the identity* $\mathrm{id} : A \to A$ *in $\mathbb{C}$ induces a monomorphism*

$$(A, \rho|_\varphi) \to (A, \rho)$$

in $\mathbb{C}\langle \mathcal{P} \rangle$ which is a fibration. ■

Theorem 4.7 *$\mathbb{C}\langle \mathcal{P} \rangle$ with the classes of fibrations and weak equivalences from Definition 4.2 is a category of fibrant objects.*

Proof. It is easy to see that (fib), (inj), and (esurj) hold for isos (using Lemma 3.3(iii)), and are stable under composition. Given a composable pair $(A, \rho) \xrightarrow{f} (B, \sigma) \xrightarrow{g} (C, \tau)$, if (inj) holds for gf, then it holds for f, and if (esurj) holds for gf, then it holds for g; for the same reason that initial segments of injective functions are injective and end segments of surjective functions are surjective. Furthermore it is easy to show that (esurj) for gf and (inj) for g implies (esurj) for f, and that (inj) for gf and (esurj) for f implies (inj) for g, again formalizing set theoretic arguments. This shows conditions (A) and (B).

For condition (C) consider a pullback square

$$\begin{tikzcd}
(D, \rho \bowtie_C \sigma) \arrow[r, "k"'] \arrow[d, "h"'] & (B, \sigma) \arrow[d, "g"] \\
(A, \rho) \arrow[r, "f"] & (C, \tau)
\end{tikzcd}$$

and assume that g is a fibration. To show the validity of (fib) for h (abbreviated (fib)(h)) we use internal language with the notation introduced in the proof of Lemma 4.3. In the step from (5) to (6) we use the admissible rule from Lemma 2.5.

(1) (compat) $\implies$ $\rho(hp, x) \vdash_{p,x} \tau(g(kp), fx)$

(2) (fib)(g) $\implies$ $\sigma(kp), \tau(g(kp), fx) \vdash_{p,x} \exists v \,.\, \sigma(kp, v) \wedge gv = fx$

(3) (1), (2) $\implies$ $\sigma(kp), \rho(hp, x) \vdash_{p,x} \exists v \,.\, \sigma(kp, v) \wedge gv = fx$

(4) $\implies$ $\rho(hp, hq^*), \sigma(kp, kq^*) \vdash_{p,q^*} \rho(hp, hq^*) \wedge \sigma(kp, kq^*) \wedge hq^* = hq^*$

(5) (4) $\implies$ $\rho(hp, hq^*), \sigma(kp, kq^*) \vdash_{p,q^*} \exists q \,.\, \rho(hp, hq) \wedge \sigma(kp, kq) \wedge hq = hq^*$

(6) (5) $\implies$ $\rho(hp, x), \sigma(kp, v), gv = fx \vdash_{p,x,v} \exists q \,.\, \rho(hp, hq) \wedge \sigma(kp, kq) \wedge hq = x$

(7) (3), (6) $\implies$ $\sigma(kp), \rho(hp, x) \vdash_{p,x} \exists q \,.\, \rho(hp, hq) \wedge \sigma(kp, kq), hq = x$

(8) (7) $\implies$ (fib)(h)

This shows that fibrations are stable under pullback. To show that *trivial* fibrations are stable under pullback, we show pullback stability of conditions (inj) and (surj) separately.

Pullback stability of (surj) is shown as follows.

(1) (surj)(g), (compat)(f) $\implies$ $\rho x \vdash_x \exists u \,.\, \sigma u \wedge gu = fx$

(2) $\implies$ $\rho(hp^*), \sigma(kp^*) \vdash_{p^*} \rho(hp^*) \wedge \sigma(kp^*) \wedge hp^* = hp^*$

(3) (2) $\implies$ $\rho(hp^*), \sigma(kp^*) \vdash_{p^*} \exists p \,.\, \rho(hp) \wedge \sigma(kp) \wedge hp = hp^*$

(4) (3) $\implies$ $\rho x, \sigma u, gu = fx \vdash_{x,u} \exists p \,.\, \rho(hp) \wedge \sigma(kp) \wedge hp = x$

(5) (1), (3) $\implies$ $\rho x \vdash_x \exists p\,.\,\rho(hp) \wedge \sigma(kp) \wedge hp = x$

Pullback stability of (inj) is shown as follows.

(1) (inj)(g) $\implies$ $\sigma(kp), \sigma(kq), \tau(gkp, gkq) \vdash_{p,q} \sigma(kp, kq)$

(2) (compat)(f), $fh = gk$ $\implies$ $\rho(hp, hq) \vdash_{p,q} \tau(gkp, gkq)$

(3) (1), (2) $\implies$ $\sigma(kp), \rho(hp, hq), \sigma(kq) \vdash_{p,q} \sigma(kp, kq)$

(4) (3) $\implies$ $\rho(hp), \sigma(kp), \rho(hp, hq), \rho(hq), \sigma(hq) \vdash_{p,q} \rho(hp, hq) \wedge \sigma(kp, kq)$

A path object for (A, ρ) is given by

$$(A, \rho) \xrightarrow{s} (A \times A, \tilde{\rho}) \xrightarrow{d} (A, \rho) \times (A, \rho) = (A \times A, \rho \bowtie \rho) \tag{4.1}$$

with $\tilde{\rho}(u, v) \equiv (\rho \bowtie \rho)|_\rho$ and where the underlying maps of s and d are δ and id, respectively. Then d is a fibration by Lemma 4.6, and it is easy to see that s is compatible with ρ and $\tilde{\rho}$ and a weak equivalence.

Finally, it is straightforward to check that terminal projections $(A, \rho) \to 1$ are fibrations. ■

Remark 4.8 The fibration part of all path object factorizations (4.1) is monic, since the underlying map is iso and the forgetful functor reflects monomorphisms. This implies that the *∞-localization of* $\mathbb{C}\langle\mathcal{P}\rangle$ — i.e. the finitely complete ∞-category obtained by weakly inverting weak equivalences, see e.g. [KS17] — has the property that all of its hom-spaces are discrete, or equivalently that all of its objects are 0-truncated. Indeed, if the second factor of a path object factorization $X \to PX \to X \times X$ is monic, then

$$\begin{array}{ccc} PX & \longrightarrow & PX \\ \downarrow & & \downarrow \\ PX & \longrightarrow & X \times X \end{array}$$

is a pullback of fibrations and therefore a homotopy-pullback, which means that $PX \to X \times X$ is a homotopy embedding, and therefore so is the diagonal $X \to X \times X$. ♢

5 The homotopy category

Throughout this section let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ be a fixed indexed frame. In the following we show that the homotopy category of $\mathbb{C}\langle\mathcal{P}\rangle$ is the category $\mathbb{C}[\mathcal{P}]$ of partial equivalence relations and *functional relations* in $\mathcal{P}$. We start by recalling the definition from [Pit02, Def. 3.1].

Definition 5.1 The category $\mathbb{C}[\mathcal{P}]$ has the same objects as $\mathbb{C}\langle\mathcal{P}\rangle$, and morphisms from (A, ρ) to (B, σ) are predicates $\phi \in \mathcal{P}(A \times B)$ satisfying the judgments

(strict) $\phi(x, u) \vdash_{x,u} \rho x \wedge \sigma u$

(cong) $\rho(y, x), \phi(x, u), \sigma(u, v) \vdash_{x,y,u,v} \phi(y, v)$

(singval) $\phi(x, u), \phi(x, v) \vdash_{x,u,v} \sigma(u, v)$

(tot) $\rho x \vdash_x \exists u\,.\,\phi(x, u)$.

Composition of morphisms $(A,\rho) \xrightarrow{\phi} (B,\sigma) \xrightarrow{\gamma} (C,\tau)$ is given by

$$(\gamma \circ \phi)(x,r) \;\equiv\; \exists u \,.\, \phi(x,u) \wedge \gamma(u,r).$$

The identity morphism on (A,ρ) is given by the predicate ρ itself. $\diamondsuit$

Remarks 5.2 (*a*) As already mentioned, $\mathbb{C}[\mathcal{P}]$ is an exact category, and Maietti and Rosolini have shown that it is in fact the 'free' one on $\mathcal{P}$ [MR12b, Corollary 3.4].

(*b*) Besides the presentation as homotopy category of $\mathbb{C}\langle\mathcal{P}\rangle$ (which depends on the weak equivalences as additional data), the category $\mathbb{C}[\mathcal{P}]$ can also be presented as category of functional relations in the sense of [MPR17, Section 3] in the indexed poset $\widetilde{\mathcal{P}} : \mathbb{C}\langle\mathcal{P}\rangle^{\mathsf{op}} \to \mathbf{Pos}$ of descent predicates from Remark 3.6(*a*). This works because $\widetilde{\mathcal{P}}$ is an indexed frame whenever $\mathcal{P}$ is.

(*c*) If $\mathcal{P}$ is the canonical indexing of a frame H, then $\mathbf{Set}[\mathcal{P}] = \mathbf{Set}[\mathsf{fam}(H)]$ is the category $\mathbf{Sh}(H)$ of sheaves on H [Pit81, Section 2.8]. $\diamondsuit$

The following lemma recalls constructions of finite limits and characterizations of monomorphisms and covers in $\mathbb{C}[\mathcal{P}]$.

Lemma 5.3 (i) *Finite products in* $\mathbb{C}[\mathcal{P}]$ *are given by* $1 = (1,\top)$ *and* $(A,\rho) \times (B,\sigma) = (A\times B, \rho \bowtie \sigma)$ *as in* $\mathbb{C}\langle\mathcal{P}\rangle$. *An equalizer of* $\phi,\gamma : (A,\rho) \to (B,\sigma)$ *is given by*

$$(A, \rho|_\upsilon) \xrightarrow{\rho|_\upsilon} (A,\rho) \quad \textit{where} \quad \upsilon(a) \equiv \exists b \,.\, \phi(a,b) \wedge \gamma(a,b).$$

(ii) *A map* $\phi : (A,\rho) \to (B,\sigma)$ *in* $\mathbb{C}[\mathcal{P}]$ *is a monomorphism if and only if the judgment*

(inj*) $\phi(x,u), \phi(y,u) \vdash_{x,y,u} \rho(x,y)$

holds in $\mathcal{P}$. *It is a* cover[6] *if and only if*

(esurj*) $\sigma u \vdash_u \exists x \,.\, \phi(x,u)$

holds in $\mathcal{P}$. *In particular,* ϕ *is an isomorphism if and only if both (inj*) and (esurj*) hold.*

Proof. These statements can be found in [Pit81, Sections 2.4, 2.5] and [vO08, Section 2.2] for triposes. The latter reference explicitly uses the word 'cover' whereas the former speaks about general epimorphisms, which are known to coincide with regular epimorphisms — i.e. covers — in toposes. In [Fre13, Section 2.2.1] the same statements can be found for 'existential fibrations', which correspond to indexed frames under the equivalence of indexed posets and posetal fibrations. In all cases the statements are given without proofs, which are considered straightforward. ■

Definition 5.4 The functor

$$E : \mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{C}[\mathcal{P}]$$

is the identity on objects, and is defined by

$$E(f)(x,u) \;\equiv\; \rho(x) \wedge \sigma(fx,u)$$

for morphisms $f : (A,\rho) \to (B,\sigma)$. $\diamondsuit$

[6] Covers are maps that do not factor through any proper subobject of their codomain. In regular categories they coincide with regular epimorphisms. See [Joh02a, A1.3].

Theorem 5.5 (i) *A morphism $f : (A, \rho) \to (B, \sigma)$ in $\mathbb{C}\langle \mathcal{P} \rangle$ is a weak equivalence if and only if $E(f)$ is an isomorphism in $\mathbb{C}[\mathcal{P}]$.*

(ii) *For any category $\mathbb{D}$ and any functor $F : \mathbb{C}\langle \mathcal{P} \rangle \to \mathbb{D}$ sending weak equivalences to isomorphisms there exists a unique $\widetilde{F} : \mathbb{C}[\mathcal{P}] \to \mathbb{D}$ satisfying $\widetilde{F} \circ E = F$.*

Proof. The first claim follows from Lemma 5.3 and the facts that (inj) holds for f if and only if (inj*) holds for $E(f)$, and that (esurj) holds for f if and only if (esurj*) holds for $E(f)$, as is easily verified.

For the second claim assume that $F : \mathbb{C}\langle \mathcal{P} \rangle \to \mathbb{D}$ inverts weak equivalences. Since E is identity-on-objects, we only have to define $\widetilde{F}$ on morphisms. Let $\phi : (A, \rho) \to (B, \sigma)$ in $\mathbb{C}[\mathcal{P}]$. We construct the span

$$(A, \rho) \xleftarrow{\phi_l} (A \times B, (\rho \bowtie \sigma)|_{\phi}) \xrightarrow{\phi_r} (B, \sigma)$$

in $\mathbb{C}\langle \mathcal{P} \rangle$, where the underlying functions of ϕ_l and ϕ_r are the projections, and $(\rho \bowtie \sigma)|_{\phi}$ is as in Definition 4.5. Then one easily verifies that ϕ_l is a trivial fibration, and that

$$\phi \circ E(\phi_l) = E(\phi_r)$$

in $\mathbb{C}[\mathcal{P}]$. For any $\widetilde{F}$ satisfying $\widetilde{F} \circ E = F$ we therefore must have

$$\widetilde{F}(\phi) \circ F(\phi_l) = F(\phi_r)$$

and since F is assumed to invert weak equivalences we can deduce

$$\widetilde{F}(\phi) = F(\phi_r) \circ F(\phi_l)^{-1}. \tag{5.1}$$

It remains to show that (5.1) defines a functor satisfying $\widetilde{F} \circ E = F$.

To see that the construction commutes with composition, let

$$(A, \rho) \xrightarrow{\phi} (B, \sigma) \xrightarrow{\gamma} (C, \tau)$$

in $\mathbb{C}[\mathcal{P}]$, and define $\xi \in \mathcal{P}(A \times B \times C)$ and $\theta \in \mathcal{P}(A \times C)$ by

$$\begin{aligned} \xi(x, u, r) &\equiv \phi(x, u) \wedge \gamma(u, r) \quad \text{and} \\ \theta(x, r) &\equiv \exists u \,.\, \xi(x, u, r), \end{aligned}$$

in other words $\theta = \gamma \circ \phi$. Consider the following diagram.

$$\begin{tikzcd}
 & (A \times C, (\rho \bowtie \tau)|_{\theta}) \arrow[dl, "\theta_l"', "\sim"] \arrow[dr, "\theta_r"] & \\
(A, \rho) & & (C, \tau) \\
 & (A \times B \times C, (\rho \bowtie \sigma \bowtie \tau)|_{\xi}) \arrow[uu, "\partial_1", "\sim"'] \arrow[dl, "\partial_2", "\sim"'] \arrow[dr, "\partial_0"] & \\
(A \times B, (\rho \bowtie \sigma)|_{\phi}) \arrow[uu, "\phi_l", "\sim"'] \arrow[dr, "\phi_r"'] & & (B \times C, (\sigma \bowtie \tau)|_{\gamma}) \arrow[uu, "\gamma_r"'] \arrow[dl, "\gamma_l", "\sim"'] \\
 & (B, \sigma) &
\end{tikzcd}$$

The three squares commute since the underlying maps are simply projections, ϕ_l, γ_l, and θ_l are trivial fibrations as remarked earlier, and moreover the upper left square is a pullback, whence ∂_1 and ∂_2 are trivial fibrations, in particular weak equivalences, as well[7]. Applying F we can argue

$$\begin{aligned}\widetilde{F}(\gamma) \circ \widetilde{F}(\phi) &= F(\gamma_r) \circ F(\gamma_l)^{-1} \circ F(\phi_r) \circ F(\phi_l)^{-1} \\ &= F(\gamma_r) \circ F(\partial_0) \circ F(\partial_2)^{-1} \circ F(\phi_l)^{-1} \\ &= F(\theta_r) \circ F(\partial_1) \circ F(\partial_2)^{-1} \circ F(\phi_l)^{-1} \\ &= F(\theta_r) \circ F(\theta_l)^{-1} = \widetilde{F}(\theta) = \widetilde{F}(\gamma \circ \phi).\end{aligned}$$

Preservation of identities is straightforward and thus $\tilde{F}$ is functorial.

To see that $\widetilde{F} \circ E = F$, let $f : (A, \rho) \to (B, \sigma)$ in $\mathbb{C}\langle\mathcal{P}\rangle$ and consider the diagram

$$\begin{tikzcd} (A \times B, (\rho \bowtie \sigma)|_{E(f)}) \arrow[d, bend right, "E(f)_l"'] \arrow[dr, "E(f)_r"] & \\ (A, \rho) \arrow[u, bend right, "s"'] \arrow[r, "f"] & (B, \sigma) \end{tikzcd} \tag{5.2}$$

in $\mathbb{C}\langle\mathcal{P}\rangle$, where s has underlying map $\langle \mathrm{id}_A, f\rangle$. Then $E(f)_r \circ s = f$ and s is a section of the weak equivalence $E(f)_l$, which means that $F(s)$ is an inverse of $F(E(f)_l)$ and we can argue

$$\widetilde{F}(E(f)) = F(E(f)_r) \circ F(E(f)_l)^{-1} = F(E(f)_r) \circ F(s) = F(f)$$

as required. ■

Remark 5.6 (*a*) Theorem 5.5(ii) says that the functor $E : \mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{C}[\mathcal{P}]$ exhibits $\mathbb{C}[\mathcal{P}]$ as the *localization* $\mathbb{C}\langle\mathcal{P}\rangle[\mathcal{W}^{-1}]$ of $\mathbb{C}\langle\mathcal{P}\rangle$ by the class $\mathcal{W}$ of weak equivalences. See [GZ67, Section I.1], [DHKS04, I.2.2(ii)]. Following [DHKS04, I.2.3(iv)] we also use the term *homotopy category* instead of 'localization'.

(*b*) The factorization of f displayed in (5.2) is an instance of the standard *mapping cocylinder factorization*, which factors morphisms $h : X \to Y$ in a category of fibrant objects into a section s of a trivial fibration followed by a fibration $d_1 \circ h'$, as in the following diagram where the square is a pullback [Bro73, Factorization Lemma pg. 421].

$$\begin{tikzcd} \mathsf{cocyl}(h) \arrow[r, "h'"'] \arrow[d] & PY \arrow[r, "d_1"'] \arrow[d, "d_0"] & Y \\ X \arrow[u, bend left, "s"] \arrow[r, "h"] & Y & \end{tikzcd}$$

In particular we have $(A \times B, (\rho \bowtie \sigma)|_{E(f)}) = (A, \rho) \times_{(B,\sigma)} P(B, \sigma)$. ♢

The following lemma characterizes the *kernel* of the localization functor E.

Lemma 5.7 *For parallel arrows $f, g : (A, \rho) \to (B, \sigma)$ in $\mathbb{C}\langle\mathcal{P}\rangle$, the following are equivalent:*

(i) $E(f) = E(g)$

[7] Alternatively one can verify by hand that ∂_1 and ∂_2 are weak equivalences and skip the pullback argument, to obtain a proof that does not depend on the fibrations in the category-of-fibrant-objects-structure on $\mathbb{C}\langle\mathcal{P}\rangle$ (cf. discussion in the introduction).

(ii) *The judgment* $\rho x \vdash_x \sigma(fx, gx)$ *holds.*

(iii) $\langle f, g\rangle$ *factors through the path object from* (4.1).

$$\begin{tikzcd} & (B\times B,\tilde{\sigma}) \arrow[d, "d"] \\ (A,\rho) \arrow[ur, dashed] \arrow[r, "\langle f,g\rangle"] & (B,\sigma)\times(B,\sigma) \end{tikzcd}$$

Proof. Easy. ∎

Remarks 5.8 (a) In general categories of fibrant objects, the equivalence between conditions (i) and (iii) of Lemma 5.7 is replaced by the more complicated statement that parallel arrows $f, g : A \to B$ are identified in the homotopy category whenever there exists an equivalence $e : A' \to A$ such that $\langle f, g\rangle \circ e$ factors through a path object (see [Bro73, Theorem 1-(ii)]).

(b) The construction of the homotopy category of a category of fibrant objects $\mathcal{C}$ given in [Bro73] proceeds in two steps: first one defines a category $\pi(\mathcal{C})$ by quotienting the morphisms of $\mathcal{C}$ by the relation described in item 1, and then $\mathsf{ho}(\mathcal{C})$ is obtained by localizing $\pi(\mathcal{C})$ by a calculus of fractions. This two-step construction gives rise to a factorization

$$\mathcal{C} \to \pi(\mathcal{C}) \to \mathsf{ho}(\mathcal{C})$$

of the localization functor into a full functor followed by a faithful functor (both identity-on-objects).

When applying this factorization to the functor $\mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{C}[\mathcal{P}]$ in the case where $\mathcal{P}$ is a tripos — i.e. we quotient $\mathbb{C}\langle\mathcal{P}\rangle$ by the congruence relation analyzed in Lemma 5.7 — we recover in the middle the *q-topos* $\mathbf{Q}(\mathcal{P})$ described in [Fre15, Definition 5.1].

$$\mathbb{C}\langle\mathcal{P}\rangle \to \mathbf{Q}(\mathcal{P}) \to \mathbb{C}[\mathcal{P}]$$

In particular if $\mathcal{P}$ is the canonical indexing of a frame A then the middle category is equivalent to the quasitopos of separated presheaves.

$$\mathbf{Set}\langle\mathsf{fam}(A)\rangle \to \mathbf{Sep}(A) \to \mathbf{Sh}(A) \qquad \diamondsuit$$

6 Cofibrant objects and ∃-completions

Definition 6.1 Let $\mathbb{C}$ be a finite-limit category.

(a) Let $\mathcal{D} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ be an indexed poset, let $I \in \mathbb{C}$, and let $\varphi_1, \dots, \varphi_n \in \mathcal{D}(I)$ $(n \geq 0)$ be a list of predicates.

We define $\mathcal{D}/_I(\varphi_1, \dots, \varphi_n)$ to be the category whose objects are pairs $(f : J \to I, \psi \in \mathcal{D}(J))$ satisfying $\psi \leq f^*\varphi_i$ for $1 \leq i \leq n$, and whose morphisms from $(f : J \to I, \psi)$ to $(g : K \to I, \theta)$ are arrows $h : J \to K$ such that $g \circ h = f$ and $\psi \leq h^*\theta$.

(b) An indexed poset $\mathcal{D} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ is said to satisfy the *solution set condition for finite meets (s.s.c.)*, if for all $I \in \mathbb{C}$ and $\varphi_1, \dots, \varphi_n \in \mathcal{D}(I)$ the category $\mathcal{D}/_I(\varphi_1, \dots, \varphi_n)$ has a weakly terminal object.

(*c*) An indexed monotone map $\Phi : \mathcal{D} \to \mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ from an indexed poset $\mathcal{D}$ satisfying the s.s.c. to an indexed frame $\mathcal{P}$ is called *flat*, if

$$\exists_f \Phi_J \psi \geq \Phi_I \varphi_1 \wedge \cdots \wedge \Phi_I \varphi_n$$

whenever $(f : J \to I, \psi)$ is weakly terminal in $\mathcal{D}/_I(\varphi_1, \ldots, \varphi_n)$. (The converse inequality always holds.)

(*d*) A flat map $\Phi : \mathcal{D} \to \mathcal{P}$ is called an $\exists$-*completion of* $\mathcal{D}$ if precomposition induces an order isomorphism

$$\mathbf{IFrm}(\mathcal{P}, \mathcal{Q}) \xrightarrow{\cong} \mathbf{Flat}(\mathcal{D}, \mathcal{Q})$$

between morphisms of indexed frames $\mathcal{P} \to \mathcal{Q}$ and flat maps $\mathcal{D} \to \mathcal{Q}$.

(*e*) A predicate $\varpi \in \mathcal{P}(I)$ of an indexed frame $\mathcal{P}$ is called $\exists$-*prime*, if for every composable pair $I \xleftarrow{u} J \xleftarrow{v} K$ of maps and every $\psi \in \mathcal{P}(K)$ satisfying $u^*\varpi \leq \exists_v \psi$, there exists a section s of v such that $u^*\varpi \leq s^*\psi$.

(*f*) We say that an indexed frame $\mathcal{P}$ has *enough* $\exists$-prime predicates, if for every predicate $\varphi \in \mathcal{P}(I)$ there is an $\exists$-prime predicate $\varpi \in \mathcal{P}(J)$ and a map $f : J \to I$ such that $\varphi = \exists_f \varpi$. $\diamondsuit$

It is clear that $\exists$-completions of a given $\mathcal{D}$ are unique up to isomorphism whenever they exist, and that $\exists$-prime predicates are stable under reindexing in any indexed frame. The term 'flat' is justified by the following lemma.

Lemma 6.2 (i) *Every indexed meet-semilattice satisfies the s.s.c.*

(ii) *An indexed monotone map $\Phi : \mathcal{D} \to \mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ from an indexed meet-semilattice to an indexed frame is flat if and only if it is cartesian.*

Proof. If $\mathcal{D}$ is an indexed meet-semilattice then $(\mathrm{id}_I, \varphi_1 \wedge \cdots \wedge \varphi_n)$ is terminal in $\mathcal{D}/_I(\varphi_1, \ldots, \varphi_n)$, and with this it is immediate that flat maps are cartesian.

Conversely assume that Φ is cartesian and that $(f : J \to I, \psi)$ is weakly terminal in $\mathcal{D}/_I(\varphi_1, \ldots, \varphi_n)$. Then the terminal projection

$$(f, \psi) \to (\mathrm{id}_I, \varphi_1 \wedge \cdots \wedge \varphi_n)$$

has a section s and we can argue

$$\begin{aligned} \Phi_I\, \varphi_1 \wedge \cdots \wedge \Phi_I\, \varphi_n &\leq s^* \Phi_J\, \psi && \text{since } \varphi_1 \wedge \cdots \wedge \varphi_n \leq s^*\, \psi \\ &= \exists_f\, \exists_s\, s^* \Phi_J\, \psi && \text{since } f \circ s = \mathrm{id}_I \\ &\leq \exists_f\, \Phi_J\, \psi && \text{since } \exists_s \dashv s^*. \end{aligned}$$

■

Theorem 6.3 (i) *Let $\mathcal{P}$ be an indexed frame, and $\mathcal{D} \subseteq \mathcal{P}$ an indexed subposet such that*

- *the predicates in $\mathcal{D}$ are $\exists$-prime in $\mathcal{P}$, and*
- *for every $\varphi \in \mathcal{P}(I)$ there are $f : J \to I$ and $\varpi \in \mathcal{D}(J)$ with $\exists_f\, \varpi = \varphi$.*

Then $\mathcal{D}$ satisfies the s.s.c. and the inclusion $\mathcal{D} \to \mathcal{P}$ is an $\exists$-completion.

In particular, the inclusion of the indexed subposet of $\exists$-prime predicates is an $\exists$-completion whenever $\mathcal{P}$ has enough $\exists$-prime predicates.

(ii) *Every indexed poset $\mathcal{D}$ satisfying the s.s.c. has an $\exists$-completion.*

(iii) *If $\Phi : \mathcal{D} \to \mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ is an $\exists$-completion, then Φ is fiberwise order-reflecting and its image consists precisely of the $\exists$-prime predicates in $\mathcal{P}$.*

Proof. Ad (i). By assumption, given $\varpi_1, \dots, \varpi_n \in \mathcal{D}(I)$ there exist $g : K \to I$ and $\upsilon \in \mathcal{D}(K)$ such that $\exists_g \upsilon = \varpi_1 \wedge \cdots \wedge \varpi_n$, and we claim that (g, υ) is weakly terminal in $\mathcal{D}/_I(\varpi_1, \dots, \varpi_n)$. To see this let $f : J \to I$ and $\pi \in \mathcal{D}(J)$ with $\pi \le f^*(\varpi_i)$ for $1 \le i \le n$. Let $J \xleftarrow{\tilde{g}} L \xrightarrow{\tilde{f}} K$ be a pullback of $J \xrightarrow{j} I \xleftarrow{g} K$. Then we can argue

$$\pi \le f^*(\varpi_1 \wedge \cdots \wedge \varpi_n) = f^* \exists_g \upsilon = \exists_{\tilde{g}} \tilde{f}^* \upsilon$$

and since π is $\exists$-prime, $\tilde{g}$ has a section s such that $\pi \le s^* \tilde{f}^* \upsilon$. It follows that $\tilde{f} \circ s$ constitutes the required morphism from (f, π) to (g, υ).

To see that $\mathcal{D} \to \mathcal{P}$ is an $\exists$-completion consider a flat map $\Phi : \mathcal{D} \to \mathcal{Q}$ and define $\Psi : \mathcal{P} \to \mathcal{Q}$ by

$$\Psi_I \varphi = \exists_f \Phi_J \pi \tag{6.1}$$

where $\pi \in \mathcal{D}(J)$ with $\exists_f \pi = \varphi$. To show that Ψ is fiberwise monotone—and in particular well defined—let $\varphi \le \psi \in \mathcal{P}(I)$ and let $\pi \in \mathcal{D}(J)$ and $\upsilon \in \mathcal{D}(K)$ such that $\exists_f \pi = \varphi$ and $\exists_g \upsilon = \psi$. Again let $J \xleftarrow{\tilde{g}} L \xrightarrow{\tilde{f}} K$ be a pullback of the span $J \xrightarrow{f} I \xleftarrow{g} K$. We have

$$\exists_f \pi \le \exists_g \upsilon \quad \text{which implies} \quad \pi \le f^* \exists_g \upsilon = \exists_{\tilde{g}} \tilde{f}^* \upsilon,$$

and since π is $\exists$-prime, $\tilde{g}$ has a section s with $\pi \le s^* \tilde{f}^* \upsilon$, whence we can deduce

$$\exists_f \Phi_J \pi = \exists_{g \tilde{f} s} \Phi_J \pi \le \exists_g \exists_{\tilde{f} s} \Phi_J (\tilde{f} s)^* \upsilon \le \exists_g \Phi_K \upsilon.$$

Naturality of Ψ follows from (BC) and preservation of $\exists$ is straightforward.

To see that Ψ preserves binary meets, let $\varphi, \psi \in \mathcal{P}(I)$ and consider maps $f : J \to I$, $g : K \to I$ and predicates $\pi \in \mathcal{D}(J)$, $\upsilon \in \mathcal{D}(K)$ such that

$$\exists_f \pi = \varphi \quad \text{and} \quad \exists_g \upsilon = \psi. \tag{6.2}$$

Let $J \xleftarrow{\tilde{g}} L \xrightarrow{\tilde{f}} K$ be a pullback of the span $J \xrightarrow{f} I \xleftarrow{g} K$, and let $h : M \to L$ and $\mu \in \mathcal{D}(M)$ such that

$$\exists_h \mu = \tilde{g}^* \pi \wedge \tilde{f}^* \upsilon. \tag{6.3}$$

Then (h, μ) is weakly terminal in $\mathcal{D}/_L(\tilde{g}^*\pi, \tilde{f}^*\upsilon)$ and since Φ is flat we have

$$\exists_h \Phi_M \mu = \tilde{g}^* \Phi_J \pi \wedge \tilde{f}^* \Phi_K \upsilon. \tag{6.4}$$

Moreover we have

$$\begin{aligned}
\exists_f \exists_{\tilde{g}} \exists_h \mu &= \exists_f \exists_{\tilde{g}}(\tilde{g}^* \pi \wedge \tilde{f}^* \upsilon) && \text{by (6.3)} \\
&= \exists_f (\pi \wedge \exists_{\tilde{g}} \tilde{f}^* \upsilon) && \text{by (Fr)} \\
&= \exists_f (\pi \wedge f^* \exists_g \upsilon) && \text{by (BC)} \\
&= \exists_f \pi \wedge \exists_g \upsilon && \text{by (Fr)} \\
&= \varphi \wedge \psi && \text{by (6.2)}
\end{aligned} \tag{6.5}$$

and thus we can argue

$$
\begin{aligned}
\Psi_I(\varphi \wedge \psi) &= \Psi_I \,\exists_f \,\exists_{\tilde{g}} \,\exists_h \,\mu && \text{by (6.5)}\\
&= \exists_f \,\exists_{\tilde{g}} \,\exists_h \,\Psi_M \,\mu && \text{since } \Psi \text{ commutes with } \exists\\
&= \exists_f \,\exists_{\tilde{g}} (\tilde{g}^* \Phi_J \,\pi \wedge \tilde{f}^* \,\Phi_K \,\upsilon) && \text{by (6.4) and since } \Phi = \Psi \text{ on } \mathcal{D}\\
&= \exists_f (\Phi_J \,\pi \wedge \exists_{\tilde{g}} \,\tilde{f}^* \,\Phi_K \,\upsilon) && \text{by (Fr)}\\
&= \exists_f (\Phi_J \,\pi \wedge f^* \,\exists_g \,\Phi_K \,\upsilon) && \text{by (BC)}\\
&= \exists_f \,\Phi_J \,\pi \wedge \exists_g \,\Phi_K \,\upsilon && \text{by (Fr)}\\
&= \Psi_I \,\varphi \wedge \Psi_I \,\psi && \text{by (6.1).}
\end{aligned}
$$

The fact that Ψ preserves $\top$ is shown along the same lines.

Ad (ii). Given $\mathcal{D} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ satisfying the s.s.c., define $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ to be the indexed poset whose predicates on I are equivalence classes of pairs $(f : J \to I, \varphi \in \mathcal{D}(J))$, ordered by

$$(J \xrightarrow{f} I, \varphi) \leq (K \xrightarrow{g} I, \psi) \text{ iff there exists an } (J \xrightarrow{h} K) \text{ with } gh = f \text{ and } \varphi \leq h^* \,\psi\,,$$

and quotiented by the symmetric part of this relation. Reindexing is defined by $g^*[(f, \varphi)] = [(\tilde{f}, \tilde{g}^*\varphi)]$ where $\cdot \xleftarrow{\tilde{f}} \cdot \xrightarrow{\tilde{g}} \cdot$ is a pullback of $\cdot \xrightarrow{g} \cdot \xleftarrow{f} \cdot$, and existential quantification is given by $\exists_g[(f, \varphi)] = [(g \circ f, \varphi)]$. A binary meet of $[(f : J \to I, \varphi)]$ and $[(g : K \to I, \psi)]$ is given by $[(f \circ \tilde{g} \circ h, \theta)]$ where $(h : L \to J \times_I K, \theta)$ is weakly terminal in $\mathcal{D}/_{J\times_I K}(\tilde{g}^*\varphi, \tilde{f}^*\psi)$, and the greatest element of $\mathcal{P}(I)$ consists of the weakly terminal objects of $\mathcal{D}/_I()$. The verifications of (Fr) and (BC) are straightforward. Thus $\mathcal{P}$ is an indexed frame, and it is easy to see that the assignment $\varphi \mapsto (\mathrm{id}, \varphi)$ defines an order-reflecting indexed monotone map $\mathcal{D} \to \mathcal{P}$ satisfying the conditions of (i).

Ad (iii). From the construction in the proof of (ii) we know that up to isomorphism every $\exists$-completion $\mathcal{D} \to \mathcal{P}$ is of the form described in (i), in particular it is order-reflecting and its image consists of $\exists$-prime predicates. Moreover, for all $\varphi \in \mathcal{P}(I)$ there exist $f : J \to I$ and $\varpi \in \mathcal{D}(J)$ such that

$$\exists_f \varpi = \varphi, \tag{6.6}$$

and if φ is $\exists$-prime then f has a section with $\varphi \leq s^*\varpi$ (by $\exists$-primality) and $s^*\varpi \leq \varphi$ (follows from (6.6)) which shows that all $\exists$-prime predicates are contained in $\mathcal{D}$. ■

Examples 6.4 (*a*) Realizability triposes [vO08] have enough $\exists$-prime predicates. Specifically, the $\exists$-prime predicates on a set I in the tripos over a PCA $\mathcal{A}$ are precisely the (equivalence classes of) families $\varphi : I \to P(\mathcal{A})$ of subsets of $\mathcal{A}$ which consist only of singletons. Analogous statements are true for *relative realizability* and realizability over *ordered* PCAs [Hof06].

(*b*) The canonical indexing $\mathsf{fam}(P) : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Pos}$ of a small poset P always satisfies the s.s.c., its $\exists$-completion is given by

$$\mathsf{fam}(P) \to \mathsf{fam}(\mathsf{low}(P))$$

where $\mathsf{low}(P)$ is the frame of lower sets in P.

(*c*) The canonical indexing of a frame A has enough $\exists$-prime predicates precisely if A is of the form $\mathsf{low}(P)$ for some poset P, which in turn is equivalent to A having enough *completely join prime elements* [Dav79, Proposition 1.1]. This observation is the reason for the choice of terminology.

(*d*) The restriction of the canonical indexing of a poset P to the category of finite sets satisfies the s.s.c. precisely if every finite intersection of principal ideals in P is also a finite union of principal ideals. In this case, the $\exists$-completion of the restriction is the restriction of the canonical indexing of the completion of P under finite suprema, i.e. the subposet of $\mathsf{low}(P)$ on finite unions of principal ideals.

(*e*) Analogously to (*d*), the canonical indexing of a *large* poset P satisfies the s.s.c. if and only if the small-join completion of P is closed under intersections. $\diamondsuit$

Remarks 6.5 (*a*) An indexed preorder $\mathcal{D} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ satisfies the s.s.c. if and only if its total category $\int \mathcal{D}$ has *weak finite limits*.

If this is the case and $\mathcal{D} \to \mathcal{P}$ is an $\exists$-completion, then the functor

$$\textstyle\int \mathcal{D} \to \mathbb{C}[\mathcal{P}], \qquad (I, \varpi) \mapsto (I, =|_{\varpi})$$

is an exact completion in the sense of [CV98, Theorem 29].

(*b*) $\exists$-completions and $\exists$-prime predicates are treated in [Fre13, Section 3.4.2.1] for indexed posets that are pre-stacks for the regular topology on a regular category. In this case, the primality condition and the construction of the $\exists$-completion have to be stated slightly differently to take the topology into account.

In [Tro20] Trotta introduces a notion of $\exists$-completion relative to a class of maps that is closed under composition and pullbacks.

(*c*) $\exists$-primality can be viewed as a kind of 'choice principle'. In particular, an indexed frame $\mathcal{P}$ satisfies the 'rule of choice' from [MPR17, Definition 5.5] if and only if $\top \in \mathcal{P}(1)$ is $\exists$-prime. $\diamondsuit$

Following Baues [Bau89, Section I.1], we call an object C of a category of fibrant objects $\mathcal{C}$ *cofibrant*, if every trivial fibration $f : B \to C$ admits a section. $\mathcal{C}$ is said to have *enough* cofibrant objects, if every $A \in \mathcal{C}$ admits a *cofibrant replacement*, i.e. a cofibrant object C and a trivial fibration $f : C \to A$.

Proposition 6.6 *Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ be an indexed frame.*

(i) *If $(C, \tau) \in \mathbb{C}\langle\mathcal{P}\rangle$ such that the support τ_0 of τ is $\exists$-prime, then (C, τ) is cofibrant.*

(ii) *If $\mathcal{P}$ has enough $\exists$-prime predicates, then $\mathbb{C}\langle\mathcal{P}\rangle$ has enough cofibrant objects.*

Proof. To see that (C, τ) is cofibrant, let $f : (B, \sigma) \to (C, \tau)$ be a trivial fibration. From Lemma 4.3 and Remark 4.4 we know that $\tau_0 \le \exists_f \sigma_0$, and since τ_0 is $\exists$-prime f has a section $s : C \to B$ such that $\varpi \le s^*\sigma_0$, i.e. the judgment $\tau(c) \vdash_c \sigma(sc)$ holds. The judgment (compat) for s then follows from this and (inj) for f. Thus, s constitutes a morphism of type $(C, \tau) \to (B, \sigma)$ in $\mathbb{C}\langle\mathcal{P}\rangle$, which gives the required section.

For the second claim, if $(A, \rho) \in \mathbb{C}\langle\mathcal{P}\rangle$ and $\mathcal{P}$ has enough $\exists$-prime predicates then there exists an object $C \in \mathbb{C}$, an $\exists$-prime predicate $\varpi \in \mathcal{P}(C)$, and a morphism $e : C \to A$ such that $\exists_e \varpi = \rho_0$. Setting

$$\tau(c, c') \;\equiv\; \varpi(c) \wedge \rho(ec, ec')$$

it is easy to see that τ is a partial equivalence relation with support $\tau_0 = \varpi$, and again using Remark 4.4 that e constitutes a trivial fibration from (C, τ) to (A, ρ). $\blacksquare$

7 Kan extensions along localization functors

Recall from [DHKS04, Section I-2.3] that a *category with weak equivalences* (or *we-category*) is a category $\mathcal{C}$ equipped with a class $\mathcal{W}$ of arrows, called *weak equivalences*, which satisfies axiom (A) of the definition of categories of fibrant objects.

The *homotopy category* of a we-category is the category $\mathsf{ho}(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$ obtained by freely inverting all weakly equivalence, meaning that it comes with a *localization functor* $E : \mathcal{C} \to \mathsf{ho}(\mathcal{C})$ which inverts all weak equivalences and is initial among functors doing so, see the references in Remark 5.6(a).

Definition 7.1 Let $\mathcal{C}$ be a we-category with localization functor $E : \mathcal{C} \to \mathsf{ho}(\mathcal{C})$.

(a) We call $X \in \mathcal{C}$ *proto-fibrant*, if $\mathcal{C}(A, X) \xrightarrow{E_{A,X}} \mathsf{ho}(\mathcal{C})(EA, EX)$ is surjective for all $A \in \mathcal{C}$.

(b) We say that $\mathcal{C}$ has *enough* proto-fibrant objects, if every $A \in \mathcal{C}$ admits a weak equivalence $\iota : A \to \overline{A}$ into a proto-fibrant object (called its *proto-fibrant replacement*).

Proto-cofibrant objects, and we-categories with enough proto-cofibrant objects are defined dually. ♢

Examples 7.2 If in a model category $\mathcal{M}$ all objects are cofibrant, then all fibrant objects are proto-fibrant. This is because morphisms from cofibrant to fibrant objects in $\mathsf{ho}(\mathcal{M})$ can be represented as equivalence classes of morphisms in $\mathcal{M}$ modulo homotopy.

Similarly, cofibrant objects in categories of fibrant objects $\mathcal{C}$ are proto-cofibrant, since morphisms $f : A \to B$ in $\mathsf{ho}(\mathcal{C})$ can be represented as right fractions $f = E(g) \circ E(t)^{-1}$ with t a trivial fibration [Bro73, 2nd remark after Theorem 1], and t splits whenever A is cofibrant. ♢

Lemma 7.3 *Assume that $\mathcal{C}$ is a we-category with enough proto-fibrant objects, $E : \mathcal{C} \to \mathsf{ho}(\mathcal{C})$ is its localization functor, and $F : \mathcal{C} \to \mathbb{D}$ is a functor into an arbitrary category such that for all parallel pairs of arrows $f, g : A \to X$ into a proto-fibrant object X we have*

$$Ef = Eg \quad \text{implies} \quad Ff = Fg. \tag{7.1}$$

Then F admits a left Kan extension along E.

Proof. Using the dual of [Mac98, Theorem X.3.1] and the fact that E is identity-on-objects, it is sufficient to show that for each $A \in \mathcal{C}$ the functor

$$(E/EA) \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathbb{D}$$

has a colimit. Let $\iota : A \to \overline{A}$ be a proto-fibrant replacement. Then for each $f : EB \to EA$ there exists $f^\dagger : B \to \overline{A}$ such that $Ef^\dagger = E\iota \circ f$. We define a cocone $\eta : F \circ U \dot{\to} F\overline{A}$ by letting $\eta_f = F(f^\dagger)$, which does not depend on the choice of $f^\dagger$ and is natural by (7.1). We define $\kappa = (E\iota)^{-1}$ and note that $\eta_\kappa = \mathrm{id}_{F\overline{A}}$ since $\mathrm{id}_{\overline{A}}$ is a choice for $\kappa^\dagger$. Given another cocone $\theta : F \circ U \dot{\to} D$ the map θ_κ constitutes a cocone morphism $\eta \to \theta$, and it can be seen to be the only one by inspecting the naturality condition for this cocone morphism at κ. ∎

7.4 Comparing left and right Kan extensions. From the **Cat**-enriched universal property of localizations [GZ67, Lemma 1.2] it follows that precomposition with localization functors $E : \mathcal{C} \to \mathsf{ho}(\mathcal{C})$ is fully faithful for arbitrary target categories $\mathbb{D}$. From this it is immediate that every functor $G : \mathsf{ho}(\mathcal{C}) \to \mathbb{D}$ is both a left and a right Kan extension of $G \circ E$ along E.

$$G = E_*(G \circ E) = E_!(G \circ E)$$

In particular, if $F : \mathcal{C} \to \mathbb{D}$ inverts all weak equivalences in $\mathcal{C}$, then the unique $\tilde{F} : \mathsf{ho}(\mathcal{C}) \to \mathbb{D}$ with $\tilde{F} \circ E = F$ is both a left and a right Kan extension.

Now if $F : \mathcal{C} \to \mathbb{D}$ is such that both Kan extensions $E_!F$ and E_*F exist, the counit map $\varepsilon : (E_*F) \circ E \to F$ gives rise by functoriality to a transformation

$$\xi : E_*F = E_!(E_*F \circ E) \xrightarrow{E_*\varepsilon} E_!F \tag{7.2}$$

from the *right* to the *left* Kan extension. This is called the *'points-to-pieces' transform* by Lawvere in the setting of axiomatic cohesion [Law07]. If F inverts weak equivalences, then ε and therefore ξ becomes invertible by the remarks above.

Given a we-category $\mathcal{C}$ with enough proto-fibrant and enough proto-cofibrant objects, and a functor $F : \mathcal{C} \to \mathbb{D}$ such that $E(f) = E(g)$ implies $F(f) = F(g)$ for all parallel f, g with either cofibrant domain or fibrant codomain — such that both Kan extensions exist by Lemma 7.3 — there is an easy direct description of the transformation (7.2): Given $A \in \mathcal{C}$ with proto-cofibrant replacement $\kappa : \underline{A} \to A$ and proto-fibrant replacement $\iota : A \to \overline{A}$, we have $E_*FA = F\underline{A}$, $E_!FA = F\overline{A}$ and

$$\xi_A \;=\; E(\underline{A} \xrightarrow{\kappa} A \xrightarrow{\iota} \overline{A}) \;:\; E_*FA \to E_!FA. \qquad \diamondsuit$$

8 Deriving cartesian maps between indexed frames

Theorem 8.1 *Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ be an indexed frame.*

(i) *If $\mathcal{P}$ is a tripos, then $\mathbb{C}\langle\mathcal{P}\rangle$ has enough proto-fibrant objects.*

(ii) *If $\mathcal{P}$ has enough $\exists$-prime predicates, then $\mathbb{C}\langle\mathcal{P}\rangle$ has enough proto-cofibrant objects.*

Proof. For partial equivalence relations in triposes the property of being proto-fibrant is equivalent to what is called *weakly complete* in [HJP80, Definition 3.2], and from the proofs of Lemma 3.1 and Proposition 3.3 of loc. cit. one can extract a proof that $\mathbb{C}\langle\mathcal{P}\rangle$ has enough proto-fibrant objects. Specifically, a proto-fibrant replacement of (A, ρ) is given by

$$\ulcorner\rho\urcorner : (A, \rho) \to (\mathfrak{P}(A), \overline{\rho})$$

where $\ulcorner\rho\urcorner$ is as in (PO) and

$$\begin{aligned}\overline{\rho}(m, n) \;\equiv\; & (\forall x \,.\, \varepsilon_A(x, m) \Leftrightarrow \varepsilon_A(x, n)) \\ & \wedge (\forall x\, y \,.\, \varepsilon_A(x, m) \wedge \varepsilon_A(y, m) \Rightarrow \rho(x, y)) \\ & \wedge (\exists x \,.\, \varepsilon_A(x, m)).\end{aligned}$$

The second claim follows from Proposition 6.6 and Examples 7.2. ■

Definition 8.2 Let $\Phi : \mathcal{P} \to \mathcal{Q} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ be a cartesian map between indexed frames. The functor

$$\mathbb{C}\langle\Phi\rangle \;:\; \mathbb{C}\langle\mathcal{P}\rangle \;\to\; \mathbb{C}\langle\mathcal{Q}\rangle$$

sends objects $(A, \rho) \in \mathbb{C}\langle\mathcal{P}\rangle$ to objects $(A, \Phi_{A\times A}(\rho)) \in \mathbb{C}\langle\mathcal{Q}\rangle$, and morphisms in $\mathbb{C}\langle\mathcal{P}\rangle$ to morphisms in $\mathbb{C}\langle\mathcal{Q}\rangle$ having the same underlying map in $\mathbb{C}$. $\diamondsuit$

For the statement of the next theorem we use the terminology of 'derived functor', which we recall from [DHKS04, I-2.3 (v)].

Definition 8.3 Let $\mathcal{C}$ and $\mathcal{D}$ be we-categories with localization functors $E_{\mathcal{C}} : \mathcal{C} \to \mathsf{ho}(\mathcal{C})$ and $E_{\mathcal{D}} : \mathcal{D} \to \mathsf{ho}(\mathcal{D})$, and let $F : \mathcal{C} \to \mathcal{D}$ be an arbitrary functor.

(a) A *left derived functor* of F is a *right* Kan extension of $E_{\mathcal{D}} \circ F$ along $E_{\mathcal{C}}$.

(b) A *right derived functor* of F is a *left* Kan extension of $E_{\mathcal{D}} \circ F$ along $E_{\mathcal{C}}$. ♢

Theorem 8.4 *Let $\Phi : \mathcal{P} \to \mathcal{Q} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ be a cartesian map between indexed frames.*

(i) *The functor $\mathbb{C}\langle\Phi\rangle$ preserves finite limits.*

(ii) *We have $E_{\mathcal{Q}}(\mathbb{C}\langle\Phi\rangle(f)) = E_{\mathcal{Q}}(\mathbb{C}\langle\Phi\rangle(g))$ for all parallel maps $f, g : A \to B$ in $\mathbb{C}\langle\mathcal{P}\rangle$ with $E_{\mathcal{P}}(f) = E_{\mathcal{P}}(g)$, where $E_{\mathcal{P}}$ an $E_{\mathcal{Q}}$ are the respective localization functors.*

(iii) *If $\mathcal{P}$ is a tripos then $\mathbb{C}\langle\Phi\rangle$ admits a right derived functor.*

(iv) *If $\mathcal{P}$ has enough $\exists$-prime predicates then $\mathbb{C}\langle\Phi\rangle$ admits a left derived functor.*

Proof. The first two claims are straightforward (for the second one can use Lemma 5.7). The third and fourth claim follow from Lemma 7.3 and its dual, respectively, whose hypotheses are satisfied by the second claim and the two parts of Theorem 8.1. ∎

Notation 8.5 In the situation of Theorem 8.4, whenever they exist, we denote the left and right derived functors of $\mathbb{C}\langle\Phi\rangle$ by $L_\Phi, R_\Phi : \mathbb{C}[\mathcal{P}] \to \mathbb{C}[\mathcal{Q}]$, respectively. ♢

Lemma 8.6 *Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Pos}$ be a tripos with enough $\exists$-prime predicates and let $\Phi : \mathcal{P} \to \mathcal{Q}$ be a cartesian map into an indexed frame. Then the comparison natural transformation $\xi : L_\Phi \to R_\Phi$ is componentwise monic.*

Proof. Let $(A, \rho) \in \mathbb{C}\langle\mathcal{P}\rangle$ and let

$$(\underline{A}, \underline{\rho}) \xrightarrow{\kappa} (A, \rho) \xrightarrow{\iota} (\overline{A}, \overline{\rho})$$

be proto-cofibrant and proto-fibrant replacements. Then $\iota \circ \kappa$ is a weak equivalence in $\mathbb{C}\langle\mathcal{P}\rangle$, in particular we have

(inj) $\underline{\rho}x, \underline{\rho}y, \overline{\rho}(\iota(\kappa x), \iota(\kappa y)) \vdash_{x,y} \underline{\rho}(x, y)$.

The validity of this judgment is preserved by $\mathbb{C}\langle\Phi\rangle$, which implies that the image under $E_{\mathcal{Q}}$ is monic in $\mathbb{C}[\mathcal{Q}]$, as in the proof of Theorem 5.5. ∎

Remark 8.7 As is elaborated in [Fre15] using different terminology, the universal property of right derived functors implies that the assignment $\Phi \mapsto R_\Phi$ is *oplax functorial* in the sense that there is a canonical natural transformation $R_{\Psi\circ\Phi} \to R_\Psi \circ R_\Phi$ whenever the respective derived functors exist. Dually, the assignment $\Phi \mapsto L_\Phi$ is lax functorial. ♢

We conclude by giving some examples.

Examples 8.8 (a) If $\Phi : \mathcal{P} \to \mathcal{Q}$ is a morphism of indexed frames then $\mathbb{C}\langle\Phi\rangle$ preserves weak equivalences and therefore $E_{\mathcal{Q}} \circ \mathbb{C}\langle\mathcal{Q}\rangle$ inverts them. By the remarks in 7.4 we conclude that L_Φ and R_Φ can be chosen to be equal and then the comparison ξ is an identity.

(b) As mentioned in the introduction, for cartesian maps $\Phi : \mathcal{P} \to \mathcal{Q}$ between *triposes* the functors R_Φ were used (using different terminology) in [HJP80, Section 3] to construct geometric morphisms between toposes from adjunctions of cartesian indexed monotone maps between triposes.

(c) For every **Set**-based tripos $\mathcal{P} : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Pos}$ we can define a cartesian transformation

$$\gamma : \mathcal{P} \to \mathsf{sub}(\mathbf{Set}), \qquad \gamma_I(\varphi) = \{i \in I \mid \top \leq (1 \xrightarrow{i} I)^*(\varphi)\}$$

— 'i.e. $\gamma_I(\varphi)$ is the subset of I where φ is 'pointwise true'. We have $\mathbf{Set}[\mathsf{sub}(\mathbf{Set})] \simeq \mathbf{Set}$, and modulo this equivalence R_γ coincides with the global sections functor

$$\Gamma = \mathbf{Set}[\mathcal{P}](1, -) : \mathbf{Set}[\mathcal{P}] \to \mathbf{Set}\,.$$

If $\mathcal{P}$ has enough $\exists$-prime predicates then $\mathbb{C}\langle\Phi\rangle$ also admits a left derived functor by Theorem 8.4. In this case we have to distinguish two cases.

- If $\top \in \mathcal{P}(1)$ is $\exists$-prime then γ is a morphism of indexed frames and L_γ and R_γ coincide.
- Otherwise L_γ is constant 0 since γ sends all $\exists$-prime predicates to $\bot$.

(d) Let $j : \mathsf{eff} \to \mathsf{eff}$ be the local operator on the effective tripos corresponding to Lifschitz realizability [vO08, Section 4.4]. Its right derived functor $R_j : \mathcal{E}\!f\!f \to \mathcal{E}\!f\!f$ is the cartesian reflector corresponding to the Lifschitz-subtopos of the effective topos $\mathcal{E}\!f\!f = \mathbf{Set}[\mathsf{eff}]$.

To understand L_j, let $(C, \tau) \in \mathbf{Set}\langle\mathsf{eff}\rangle$ be a partial equivalence relation with $\exists$-prime support and fibrant replacement $\iota : (C, \tau) \to (\overline{C}, \overline{\tau})$. The composite

$$(C, \tau) \to (C, j\,\tau) \xrightarrow{\iota} (\overline{C}, j\,\overline{\tau})$$

in $\mathbf{Set}\langle\mathsf{eff}\rangle$ is mapped by $E : \mathbf{Set}\langle\mathsf{eff}\rangle \to \mathcal{E}\!f\!f$ to

$$\begin{tikzcd}
(C,\tau) \arrow[r, "f"'] \arrow[dr, "\eta"'] & L_j(C,\tau) \arrow[d, tail, "\xi"] \\
& R_j(C,\tau)
\end{tikzcd}$$

where η is the unit of the reflector. We've shown in Lemma 8.6 that ξ is monic. Moreover it can be seen relatively easily from the definition in [vO08] that $j(\varpi) = \varpi$ for any $\exists$-prime predicate ϖ in eff, which means by Remark 4.4 that the judgment (surj) holds for the map $(C, \tau) \to (C, j\,\tau)$. From this we can deduce using Lemma 5.3(ii) that its image f under the localization functor is a cover, and we get a characterization of $L_j(C, \tau)$ as fitting in the middle of the cover-mono factorization of $(C, \tau) \to R_j(C, \tau)$.

In other words, L_j is the reflection onto the j-separated objects in $\mathcal{E}\!f\!f$.

(e) Consider the poset $P = \{l \leq \top \geq r\}$ and let $f : \mathsf{low}(P) \to \{0 \leq 1\}$ be the unique map with $f^{-1}(\{1\}) = \{\{l, r\}, \{l, \top, r\}\}$. Then f preserves finite meets. Moreover we have

$$\mathbf{Set}[\mathsf{fam}(\mathsf{low}(P))] \simeq \mathbf{Set}^{P^{\mathsf{op}}} \qquad \text{and} \qquad \mathbf{Set}[\mathsf{fam}(\{0 \leq 1\})] \simeq \mathbf{Set}\,,$$

and modulo these equivalences the left and right derived functors of $\mathbf{Set}\langle\mathsf{fam}(f)\rangle$ are given by

$$\begin{aligned}
L_f, R_f \ &: \ \mathbf{Set}^{P^{\mathsf{op}}} \to \mathbf{Set} \\
L_f(A \leftarrow C \to B) &= \mathsf{im}(C \to A \times B) \\
R_f(A \leftarrow C \to B) &= A \times B.
\end{aligned}$$

$\diamondsuit$

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