Characterizing realizability triposes over PCAs

Jonas Frey (LIPN, Paris 13)

Luminy

6 April, 2025

In memoriam: Thomas Streicher (1958-2025)

- Memorial colloquium in Darmstadt organized by Kohlenbach: April 23 2025, with talks by van Oosten (on Krivine realizability) and Hyland (on Dialectica)
  https://www.mathematik.tu-darmstadt.de/fb/aktuelles/veranstaltungen/veranstaltung_details_194629.en.jsp

Characterization of realizability triposes over PCAs

Theorem (F)[1]

A tripos $\mathcal{P} : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Pos}$ is a realizability tripos over a PCA, iff :

1. $\mathcal{P}$ has enough $\exists$-**prime predicates**.
2. The full indexed sub-poset $\mathcal{A} = \mathsf{prim}(\mathcal{P}) \subseteq \mathcal{P}$ of $\exists$-prime predicates has finite meets.
3. $\mathcal{A}$ has a **discrete** generic predicate.
4. $\mathcal{A}$ is **shallow**, i.e. $\mathcal{A}(1) = 1$

In the following we explain what the words in the statement mean.

[1] Frey. "A fibrational study of realizability toposes". PhD thesis. Paris 7 University, 2013
Frey. *Uniform Preorders and Partial Combinatory Algebras*. arxiv 2024, accepted in TAC

Triposes

Definition

A Set-tripos is an indexed poset $\mathcal{P} : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Pos}$ such that:

- For all sets I, the poset $\mathcal{P}(I)$ is a **Heyting algebra**.
- For all functions $f : I \to J$, the **reindexing map** $f^* : \mathcal{P}(J) \to \mathcal{P}(I)$ is a **Heyting algebra morphism** and has left and right adjoints $\exists_f \dashv f^* \dashv \forall_f$ satisfying the **Beck-Chevalley condition**:

  (BCC) For all pullback squares $\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D \end{array}$ in Set, we have $g^* \circ \exists_f = \exists_h \circ k^*$ and $g^* \circ \forall_f = \forall_h \circ k^*$.

- There exists a **generic predicate**, i.e. a set Σ and a predicate $\mathrm{tr} \in \mathcal{P}(\Sigma)$ such that for all sets A and elements $\phi \in \mathcal{P}(A)$ there exists an $f : A \to \Sigma$ with $f^*(\mathrm{tr}) = \phi$.

Remarks

- Triposes where introduced in 1980 by Hyland, Johnstone and Pitts to construct **realizability toposes**, notably **the effective topos**.
- HJP used indexed **preorders** instead of indexed posets. I'm being sloppy about the distinction.
  - On the one hand, definitions are easier to state for indexed posets.
  - On the other hand, examples are typically indexed preorders.

  Fortunately, we can always quotient out indexed preorders to get equivalent indexed posets.

Realizability triposes

Definition

The **effective tripos** $\mathbf{eff} : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Preord}$ is given by

$$\mathbf{eff}(I) = (P(\mathbb{N})^I, \leq)$$

where

$$(\phi : I \to P(\mathbb{N})) \leq (\psi : I \to P(\mathbb{N})) \quad \text{iff} \quad \exists (f : \mathbb{N} \xhookrightarrow{\text{part. rec.}} \mathbb{N})\ \forall (i \in I)\ \forall (n \in \phi(i))\ .\ f(n) \in \psi(i)$$

More generally:

Definition

Let $\mathcal{A}$ be a **partial combinatory algebra (PCA)**. The **realizability tripos** $\mathsf{rt}(\mathcal{A}) : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Preord}$ is given by

$$\mathsf{rt}(I) = (P(\mathcal{A})^I, \leq)$$

where

$$(\phi : I \to P(\mathcal{A})) \leq (\psi : I \to P(\mathcal{A})) \quad \text{iff} \quad \exists (e \in \mathcal{A})\ \forall (i \in I)\ \forall (a \in \phi(i))\ .\ e{\cdot}a \in \psi(i)$$

Remark: There are also tripos accounts of modified realizability and dialectica. A good source is van Oosten's book (except for dialectica).

Characterization of realizability triposes

Theorem

A tripos $\mathcal{P} : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Pos}$ is a realizability tripos over a PCA, iff :

1. $\mathcal{P}$ has enough $\exists$-**prime predicates**.
2. The full indexed sub-poset $\mathcal{A} = \mathsf{prim}(\mathcal{P}) \subseteq \mathcal{P}$ of $\exists$-prime predicates has finite meets.
3. $\mathcal{A}$ has a **discrete** generic predicate.
4. $\mathcal{A}$ is **shallow**, i.e. $\mathcal{A}(1) = 1$

$\exists$-*prime predicates*

Definition

Let $\mathbb{C}$ be a category with finite limits.

1. $\mathsf{IPos}(\mathbb{C}) = [\mathbb{C}^{\mathsf{op}}, \mathsf{Pos}]$ is the locally ordered category of indexed posets on $\mathbb{C}$.
2. Say that $\mathcal{A} \in \mathsf{IPos}(\mathbb{C})$ **has existential quantification**, if all reindexing maps f^* have left adjoints subject to the Beck–Chevalley condition.
3. $\exists$-$\mathsf{IPos}(\mathbb{C}) \subseteq \mathsf{IPos}(\mathbb{C})$ is the category of indexed posets having existential quantification, and indexed monotone maps preserving existential quantification.
4. For $\mathcal{H} \in \exists$-$\mathsf{IPos}(\mathbb{C})$, a predicate $\pi \in \mathcal{H}(I)$ is called $\exists$-**prime** if for all maps $K \xrightarrow{g} J \xrightarrow{f} I$ and objects $\phi \in \mathcal{H}(K)$ we have
$$f^*\pi \leq \exists_g \phi \quad \Rightarrow \quad \text{there exists } s : J \to K \text{ with } gs = \mathrm{id}_J \text{ and } f^*\pi \leq s^*\psi.$$
5. We say that $\mathcal{H} \in \exists$-$\mathsf{IPos}(\mathbb{C})$ **has enough $\exists$-prime predicates**, if for all predicates $\phi \in \mathcal{H}(I)$ there exists a map $f : J \to I$ and an $\exists$-prime $\pi \in \mathcal{H}(J)$ with $\phi = \exists_f \pi$.

$\exists$-completion

Theorem

Let $\mathbb{C}$ be a **small** category with finite limits.

1. The inclusion functor $\exists\text{-}\mathsf{IPos}(\mathbb{C}) \hookrightarrow \mathsf{IPos}(\mathbb{C})$ has a a left adjoint $D : \mathsf{IPos}(\mathbb{C}) \to \exists\text{-}\mathsf{IPos}(\mathbb{C})$.
2. $\mathcal{H} \in \exists\text{-}\mathsf{IPos}(\mathbb{C})$ is **an $\exists$-completion**, i.e. of the form $D(\mathcal{A})$ for some $\mathcal{A} \in \mathsf{IPos}(\mathbb{C})$ **iff** it has enough $\exists$-prime predicates.

   In this case we have $\mathcal{A} \cong \mathsf{prim}(\mathcal{H})$.

Remarks

1. Analogy: a sup-lattice L is a free cocompletion iff it has enough **completely join prime elements**.
2. Free $\exists$-completion of indexed posets as well as free join-completion of posets are instances of **lax idempotent monads** – for such such monads it is often possible to reconstruct 'starting data' from the cocompletion by some kind of atomicity/primality/compactness condition.
3. All this over a small base category, i.e. not over Set.
4. Over Set, we have to impose an additional smallness condition, e.g. existence of a generic predicate. This brings us to **uniform preorders**.

Uniform preorders

Definition

A **uniform preorder** is a pair (A, R) where A is a set and $R \subseteq P(A \times A)$ is a set of binary relations such that:

1. R contains 1_A and is closed under composition.
2. R is downward closed, i.e. $r \in R$ and $s \subseteq r$ implies $s \in R$.

A **morphism of uniform preorders** between uniform preorders (A, R) and (B, S) is a function $f : A \to B$ such that $(f \times f)(r) \in S$ for all $r \in R$.

UOrd is the category of uniform preorders and their morphisms. This category is locally ordered: given morphisms of uniform preorders $f, g : (A, R) \to (B, S)$, we set $f \leq g$ iff $(f \times g)(\mathrm{id}_A) \in B$.

Remark: uniform preorders are related to **evidenced frames**[2].

[2] Cohen, Miquey, and Tate: "Evidenced frames: A unifying framework broadening realizability models" (LICS 2021)

Uniform preorders vs indexed preorders

- Every uniform preorder (A, R) induces an indexed preorder $\mathsf{fam}(A, R) : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Preord}$ given by $\mathsf{fam}(A, R)(I) = (A^I, \leq)$ with $(\phi : I \to A) \leq (\psi : I \to A)$ iff $(\phi \times \psi)(\mathrm{id}_I) \in P(A \times A)$.
- Taking fiberwise poset reflections gives a functor

$$\mathsf{fam} : \mathsf{UOrd} \to \mathsf{IPos}(\mathsf{Set}).$$

Proposition

The functor fam is a local equivalence. Its essential image comprises precisely the indexed posetes with generic predicates.

- The uniform preorder corresponding to an indexed poset $\mathcal{A} : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Pos}$ with generic predicate $\mathsf{tr} \in \mathcal{A}(A)$ is given by (A, R) with

$$R = \{r \subseteq R \times R \mid p^*\mathsf{tr} \leq q^*\mathsf{tr}\} \qquad \begin{tikzcd} & r \arrow[dl, "p"'] \arrow[d, hook] \arrow[dr, "q"] & \\ A & A \times A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & A \end{tikzcd} .$$

$\exists$-*completion of uniform preorders*

- Problem: $\exists$-completion of Set-indexed preorders does not exist in general.
- However, $\exists$-completion of Set-indexed preorders with generic predicate **does** exist and admits a nice representation on the level of uniform preorders.
- Concretely, the $\exists$-completion of a uniform preorder (A, R) is given by (PA, DR), where DR is the uniform preorder structure on PA generated by relations

$$[r] = \{(U, V) \in PA \times PA \mid \forall a \in U\, \exists b \in V\,.\,(a, b) \in R\}$$

  for $r \in R$.
- More generally, $\exists$-completions exist of **many-sorted uniform preorders**, representing Set-indexed preorders with a generic family of predicates.
- The category of many-sorted uniform preorders has the advantage that it's cartesian closed.

Let's revisit the theorem:

Theorem

A tripos $\mathcal{P} : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Pos}$ is a realizability tripos over a PCA, iff :

1. $\mathcal{P}$ has enough $\exists$-**prime predicates**.
2. The full indexed sub-poset $\mathcal{A} = \mathsf{prim}(\mathcal{P}) \subseteq \mathcal{P}$ of $\exists$-prime predicates has finite meets.
3. $\mathcal{A}$ has a **discrete** generic predicate.
4. $\mathcal{A}$ is **shallow**, i.e. $\mathcal{A}(1) = 1$

- For $\mathsf{rt}(\mathcal{A}) : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Ord}$, prime predicates are **singleton valued predicates**.
- Finite meets in $\mathsf{sing}(\mathcal{A}) \cong \mathsf{prim}(\mathsf{rt}(\mathcal{A}))$ come from pairing and projection operators.
- Applying **Grothendieck construction** gives $\int \mathsf{sing}(\mathcal{A}) = \mathsf{PAsm}(\mathcal{A})$ (cat. of **partitioned assemblies**).
- Equation $D(\mathsf{sing}(\mathcal{A})) = \mathsf{rt}(\mathcal{A})$ is analogous to $\mathsf{PAsm}(\mathcal{A})_{\mathsf{ex/lex}} = \mathsf{RT}(\mathcal{A})$.
- The observation that realizability triposes are $\exists$-completions of their indexed sub-preorders of singletons is originally due to Pieter Hofstra[3].

[3] Hofstra. "Relative completions". In: *Journal of Pure and Applied Algebra* (2004).

Conditions 3 and 4

- Omitting condition 4 gives a characterization of **relative realizability triposes** — these are defined w.r.t. an **inclusion** $\mathcal{A}_\# \subseteq \mathcal{A}$ **of PCAs**[4].
- So what's the discreteness in condition 3 about? — It is related to **functionality**.

Definition

Let $\mathcal{A} : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Pos}$ be an indexed poset. A predicate $\delta \in \mathcal{A}(A)$ is called **discrete**, if for all spans $I \xleftarrow{e} J \xrightarrow{f} A$ with e surjective and predicates $\phi \in \mathcal{A}(I)$ with $e^*\phi \leq f^*\delta$, there exists $g : I \to A$ with $eg = f$ (and $g^*\pi \leq \psi$).

- Exercise: given an indexed poset $\mathcal{A}$ with generic predicate $\mathsf{tr} \in \mathcal{A}(A)$, tr is discrete iff for the associated uniform preorder (A, R), all the relations $r \in R$ are functional.
- Omitting discreteness condition from the theorem characterizes a class of triposes corresponding to **relationally complete uniform preorders**.
- Before introducing those, we have to introduce **cartesian** uniform preorders

[4] Birkedal and Oosten. "Relative and modified relative realizability". In: *Ann. Pure Appl. Logic* (2002).

Cartesian uniform preorders

Definition / Lemma

A uniform preorder (A, R) is called **cartesian**, if one/any of the following equivalent conditions are satisfied:

1. $\mathrm{fam}(A, R) : \mathsf{Set}^{\mathsf{op}} \to \mathsf{Pos}$ is an indexed meet-semilattice
2. $(A, R) \to 1$ and $(A, R) \to (A, R) \times (A, R)$ have right adjoints in UOrd.
3. There exists a function $\wedge : A \times A \to A$ and an element $\top \in A$ such that the relations

$$\tau = \{(a, \top) \mid a \in A\} \qquad \lambda = \{(a \wedge b, a) \mid a, b \in A\} \qquad \rho = \{(a \wedge b, b) \mid a, b \in A\}$$

   are in R, and for all $r, s \in R$ the relation

$$\langle r, s\rangle \;:=\; \wedge \circ (r \times s) \circ \delta_A \;=\; \{(a, b \wedge c) \mid (a, b) \in r, (a, c) \in s\}$$

   is in R.

Relationally complete uniform preorders

Definition

A cartesian uniform preorder (A, R) is called **relationally complete**, if there exists a relation $@ \in R$ (called 'universal relation'), such that for every relation $r \in R$ there exists a *function* (i.e. a single-valued and entire relation) $\tilde{r} \in R$ with

$$r \circ \wedge \;\subseteq\; @ \circ \wedge \circ (\tilde{r} \times \mathrm{id}_A),$$

in other words

$$\forall a\, b\, c \in A\,.\,(a \wedge b, c) \in r \;\Rightarrow\; (\tilde{r}(a) \wedge b, c) \in @.$$

Remark: Relationally complete uniform preorders can be viewed as a kind of **relational PCAs**.

Theorem

TFAE for a cartesian uniform preorder (A, R).

1. (A, R) is relationally complete.
2. $\mathsf{fam}(D(A, R))$ is a tripos.

Besides PCAs, relationally complete uniform preorders comprise **ordered PCAs** (with filters).
Open question: are there any others?

Mono-fibered concrete categories

Observation

Let $\mathcal{A} \in \exists\text{-}\mathsf{IPos}(\mathbb{C})$ for $\mathbb{C}$ with finite limits, and $\phi \in \mathcal{A}(I)$.

- ϕ is **$\exists$-prime** iff all its reindexings have the **left lifting property** w.r.t. **cocartesian arrows**.
- ϕ is **discrete**, if it has the **right lifting property** w.r.t. **cartesian arrows over surjections**.

We saw that $\exists$-primality is related to $\exists$-completion. It turns out that discreteness is also related to a completion operation!

Definition

For $\mathbb{C}$ a category, a **mono-fibered concrete category** over $\mathbb{C}$ is a faithful functor $\mathbb{X} \to \mathbb{C}$ which admits **cartesian liftings along monomorphisms**.

Proposition

For small $\mathbb{C}$, the functor $\int : \mathsf{IPos}(\mathbb{C}) \to \mathsf{MFConc}(\mathbb{C})$ sending indexed posets to their Grothendieck construction has a left adjoint.

The indexed posets in the image of this left adjoint are precisely those with **enough discrete predicates**.

Thank you for your attention!