Uniform Preorders and Partial Combinatory Algebras

Jonas Frey

February 11, 2024

Abstract

We introduce *uniform preorders*: a class of combinatory representations of Set-indexed preorders that generalize Hofstra's *basic relational objects* [Hof06]. An indexed preorder is representable by a uniform preorder iff it has as generic predicate. We study the $\exists$-completion of indexed preorders on the level of uniform preorders, and identify a combinatory condition ('relational completeness') characterizing those uniform preorders with finite meets whose $\exists$-completions are triposes. The class of triposes obtained this way contains *relative realizability triposes*, for which we derive a characterization as a fibrational analogue of the characterization of realizability toposes given in earlier work [Fre19].

Besides relative partial combinatory algebrass, the class of closed uniform preorders contains (relative) *ordered* partial combinatory algebras, and it is unclear if there are any others.

1 The locally ordered category of uniform preorders

Uniform preorders were introduced in [Fre13] as representations of certain Set-indexed preorders that generalize Hofstra's *basic combinatorial objects (BCOs)* [Hof06].

Contrary to BCOs, for uniform preorders there exists a straightforward characterization of the induced class of indexed preorders, which makes the notion both conceptually very clear and somewhat tautological. In this section we reconstruct the definition of uniform preorders from this characterization, after fixing terminology and notation on locally ordered categories and indexed preorders, which constitute the central formalisms in this article.

A Set-*indexed preorder* is a pseudofunctor $\mathsf{Set}^{\mathsf{op}} \to \mathsf{Ord}$ where Ord is the locally ordered category of preorders and monotone maps. We view locally ordered categories as degenerate 2-categories, and use 2-categorical concepts and terminology. As we only consider indexed preorders on Set in this paper (apart from Remark **????**), we omit the prefix. Given an indexed preorder $\mathcal{P}$ and a set A, we call $\mathcal{P}(A)$ the *fiber* of $\mathcal{P}$ over A, and refer to its elements as *predicates* on A. Given a function $f : A \to B$, the monotone map $\mathcal{P}(f)$ is called *reindexing along* f and abbreviated by f^*. We write IOrd for the locally ordered category of indexed preorders and pseudo-natural transformations.

Strict indexed preorders and transformations form a non-full sub-2-category $[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]$ of IOrd, which by a well known argument about models of geometric theories in presheaf categories[1] is isomorphic to the locally ordered category $\mathsf{Ord}([\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}])$ of internal preorders in $[\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}]$.

[1] [Joh02, Corollary D1.2.14(i)] gives a statement for small index categories, but smallness is not essential.

The locally ordered category UOrd of uniform preorders is now characterized as fitting into the following strict pullback of locally ordered categories, where U sends internal preorders to underlying presheaves, the categories in the lower line are viewed as having codiscretely ordered hom-sets (to make U well-defined), よ is the Yoneda embedding, and fam is defined to be the evident composition.

{eq:uord-pb}

$$\begin{tikzcd}
\mathsf{UOrd} \arrow[r, "J"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rrr, bend left, "\mathsf{fam}"] & \mathsf{Ord}([\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}]) \arrow[r, "\cong"'] \arrow[d, "U"] & {[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]} \arrow[r, hook] & \mathsf{IOrd} \\
\mathsf{Set} \arrow[r, "\text{よ}"] & {[\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}]} & &
\end{tikzcd}$$

The 2-functor J is 2-fully faithful since よ is, which means that UOrd can be identified with the 2-full subcategory of $\mathsf{Ord}([\mathsf{Set}^{\mathsf{op}}, \mathsf{Set}])$ on internal preorders whose underlying presheaves are representable. In other words, a uniform preorder is a set A together with an internal preorder structure on $\text{よ}(A)$. Such a preorder structure is given by a subfunctor of $\text{よ}(A) \times \text{よ}(A) \cong \text{よ}(A \times A)$, i.e. a sieve on $A \times A$, subject to reflexivity and transitivity conditions.

Since surjections split in Set, sieves are completely determined by their monomorphisms, or equivalently subset-inclusions, which means that a sieve on $A \times A$ is equivalently represented as a down-closed subset of the powerset $P(A \times A)$. We leave it to the reader to verify that unwinding the meaning of reflexivity, transitivity, monotonicity, and the hom-set ordering in terms of this representation of sieves yields the following concrete descriptions of the locally ordered category UOrd and the 2-functor fam.

def:uord

Definition 1.1 The locally ordered category UOrd of uniform preorders and monotone maps is defined as follows.

def:uord-uord

(i) A *uniform preorder* is a pair (A, R) of a set A and a set $R \subseteq P(A \times A)$ of binary relations on A, such that

- $\mathrm{id}_A \in R$,
- $s \circ r \in R$ whenever $r \in R$ and $s \in R$, and
- $s \in R$ whenever $r \in R$ and $s \subseteq r$.

def:uord-mmap

(ii) A *monotone map* between uniform preorders (A, R) and (B, S) is a function $f : A \to B$ such that for all $r \in R$, the set

$$(f \times f)[r] \;=\; f \circ r \circ f^{\circ} \;=\; \{(fa, fa') \mid (a, a') \in r\}$$

is in S.

def:uord-hom˙ord

(iii) The ordering relation $\leq$ on monotone maps $f, g : (A, R) \to (B, S)$ is defined by $f \leq g$ iff the set

$$\mathsf{im}\langle f, g\rangle \;=\; \{(fa, ga) \mid a \in A\}$$

is in S. $\diamondsuit$

def:fam

Definition 1.2 The 2-functor $\mathsf{fam} : \mathsf{UOrd} \to \mathsf{IOrd}$ (called *family construction*) is defined as follows.

def:fam-obj

(i) For every uniform preorder (A, R), the indexed preorder $\mathsf{fam}(A, R)$ maps *def:fam-mor*

– sets I to preorders $(A^I, \leq)$, where $\varphi \leq \psi : I \to A$ iff

(1.1) *{eq:unif-order}* $\quad \mathsf{im}\langle\varphi, \psi\rangle \;=\; \{(\varphi i, \psi i) \mid i \in I\}$

is in R, and

– functions $f : J \to I$ to monotone maps $f^* : (A^J, \leq) \to (A^I, \leq)$ given by precomposition.

def:uord-fam-mor

(ii) For every monotone map $f : (A, R) \to (B, S)$ between indexed preorders, the components of the indexed monotone map $\mathsf{fam}(f) : \mathsf{fam}(A, R) \to \mathsf{fam}(B, S)$ are given by postcomposition. $\diamondsuit$

Remark 1.3 The ordering on monotone maps $f, g : (A, R) \to (B, S)$ defined in 1.1(iii) is the restriction of the ordering on $\mathsf{fam}(B, S)(A)$ as defined in 1.2(i). $\diamondsuit$

def:basis

Definition 1.4 A *basis* for a uniform preorder (A, R) is a subset $R_0 \subseteq R$ of binary relations whose *down-closure* $\downarrow\!R_0$ in $P(A \times A)$ is R, i.e. R and R_0 generate the same sieve on $A \times A$. In other words, $R_0 \subseteq R$ is a basis of R if for every $r \in R$ there is an $r_0 \in R_0$ with $r \subseteq r_0$. $\diamondsuit$

rem:basis

Remark 1.5 Given a set A and a set $R_0 \subseteq P(A \times A)$ of binary relations, its down-closure $\downarrow\!R_0$ is a uniform preorder structure on A iff

(a) there exists an $r \in R_0$ with $\mathrm{id}_A \subseteq r$, and

(b) for all $r, s \in R_0$ there exists a $t \in R_0$ with $s \circ r \in t$.

Just like continuity of functions between topological spaces, monotonicity of functions between uniform preorders can be expressed in terms of bases. Specifically, given uniform preorders (A, R) and (B, S) with bases R_0 and S_0 respectively, a function $f : A \to B$ is monotone iff for all $r \in R_0$ there exists an $s \in S_0$ with $(f \times f)[r] \subseteq s$, and given $\varphi, \psi : I \to A$ we have $\varphi \leq \psi$ in $\mathsf{fam}(A, R)(I)$ iff there exists an $r \in R_0$ with $\mathsf{im}\langle\varphi, \psi\rangle \subseteq r$. $\diamondsuit$

The following lemma gives a better understanding of the combined embedding from UOrd to IOrd. Recall that a *generic predicate* in an indexed preorder $\mathcal{A}$ is a predicate $\iota \in \mathcal{A}(A)$ for some A, such that for every other set B and predicate $\varphi \in \mathcal{A}(B)$ there exists a function $f : B \to A$ with $f^*\iota \cong \varphi$.

lem:genpred-image

Lemma 1.6 *The 2-functor $\mathsf{fam} : \mathsf{UOrd} \to \mathsf{IOrd}$ is a local equivalence, and its bi-essential image consists of the indexed preorders which admit a generic predicate.*

Concretely, if $\mathcal{H}$ is an indexed preorder with generic predicate $\iota \in \mathcal{H}(A)$, then the corresponding uniform preorder is given by (A, R) with

{eq:pq} $\quad R = \{r \subseteq A \times A \mid p^*\iota \leq q^*\iota\}$

$$\begin{tikzcd} & & r \arrow[dll, "p"'] \arrow[d, hook] \arrow[drr, "q"] & & \\ A & & A \times A \arrow[ll, "\pi_1"'] \arrow[rr, "\pi_2"] & & A \end{tikzcd}$$

where $p, q : r \to A$ are the first and second projections.

Proof. For the first claim — since $\mathsf{UOrd} \to [\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}]$ is an isomorphism on hom-preorders, and $[\mathsf{Set}^{\mathsf{op}}, \mathsf{Ord}] \to \mathsf{IOrd}$ is locally order reflecting — it is sufficient to show that for every uniform preorder (A, R), strict indexed preorder $\mathcal{K}$, and pseudonatural $f : \mathsf{fam}(A, R) \to \mathcal{K}$ there exists a strict transformation $\bar{f} : \mathsf{fam}(A, R) \to \mathcal{K}$ with $\bar{f} \cong f$. The transformation $\bar{f}$ is given by $\bar{f}_I(\varphi : I \to A) = \varphi^*(f_A(\mathrm{id}_A))$[2].

For the second claim it is clear that indexed preorders $\mathsf{fam}(A, R)$ have generic predicates (the identity), and that this property is stable under equivalence. Conversely, it was stated earlier that uniform preorders can be identified with strict indexed preorders whose underlying presheaf of sets is representable, and every indexed preorder $\mathcal{H}$ with generic predicate $\iota \in \mathcal{H}(A)$ is equivalent to the strict indexed preorder with underlying presheaf $\mathsf{Set}(-, A)$, and ordering on $\mathsf{Set}(I, A)$ given by $f \leq g$ iff $f^*\iota \leq g^*\iota$. □

exists:uords:canonical-indexing ex:uords

Examples 1.7 (a) For every preorder $(A, \leq)$, we can make $\mathsf{Set}(-, A)$ into an indexed preorder via the *pointwise ordering*, i.e. $\varphi \leq \psi$ if $\forall i \in I \,.\, \varphi(i) \leq \psi(i)$ for $i \in I$. We will refer to this indexed preorder as the *canonical indexing* of $(A, \leq)$. The corresponding uniform preorder is given by $(A, R_{\leq})$ where $R_{\leq} = \downarrow\{\leq\} \subseteq P(A \times A)$.

ex:uords:bco

(b) Hofstra's *basic combinatory objects* (BCOs) [Hof06, pg. 241] can be embedded into uniform preorders: a BCO is a triple $(A, \leq, \mathcal{F})$ where $(A, \leq)$ is a partial order and $\mathcal{F}$ is a set of monotone partial endofunction with down-closed domains, which is weakly closed under composition in the sense that

(i) there exists an $i \in \mathcal{F}_A$ such that $i(a) \leq a$ for all $a \in A$, and

(ii) for all $f, g \in \mathcal{F}$ there exists $h \in \mathcal{F}$ such that $h(a) \leq g(f(a))$ whenever the right side is defined.

Every poset $(A, \leq)$ gives rise to a BCO $(A, \leq, \{\mathrm{id}_A\})$.

For a given BCO $(A, \leq, \mathcal{F})$, Hofstra defined an indexed poset structure on $\mathsf{Set}(-, A)$ by setting

$$\varphi \leq \psi \quad \text{iff} \quad \exists f \in \mathcal{F}\, \forall i \in I \,.\, f(\varphi(i)) \leq \psi(i)$$

for $\varphi, \psi : I \to A$, and the corresponding uniform preorder structure $R_{\mathcal{F}}$ on A is generated by the relations $\{r_f f \subseteq A \times A \mid f \in \mathcal{F}\}$, where $r_f = \{(a, b) \mid f(a) \leq b\}$ for $f \in \mathcal{F}$. The axioms (i), (ii) ensure that the relations r_f form a basis in the sense of Definition 1.4. ♢

The two preceding examples give rise to a sequence of embeddings

$$\{\text{eq:ord-bco-uord-iord}\} \qquad \mathsf{Ord} \to \mathsf{BCO} \to \mathsf{UOrd} \to \mathsf{IOrd}$$

of locally ordered categories, where we have already seen that $\mathsf{UOrd} \to \mathsf{IOrd}$ is a local equivalence, and Hofstra gave definitions of BCO morphisms and an ordering on them which entail that the embeddings $\mathsf{Ord} \to \mathsf{BCO}$ and $\mathsf{BCO} \to \mathsf{UOrd}$ are local *isomorphisms*. This is most naturally verified by observing that the canonical embeddings of Ord, BCO, and UOrd into the locally ordered category $\mathsf{Ord}([\mathsf{Set}, \mathsf{Set}])$ of strict indexed preorders and strict transformations are local isomorphisms, which the reader also may also take as a definition of arrows and their ordering in the case of BCOs.

[2] More generally, this argument works for pseudonatural transformations $f : \mathcal{H} \to \mathcal{K}$ between strict indexed preorders where $\mathcal{H}$'s underlying presheaf of sets is *projective*, i.e. a coproduct of representables. Such indexed preorders $\mathcal{H}$ correspond to the 'many-sorted uniform preorders' studied in [Fre13].

1.1 Adjunctions

An adjunction in a locally ordered category $\mathfrak{A}$ is a pair of arrows $f : A \to B$, $g : B \to A$, such that $\mathrm{id}_A \leq g \circ f$ and $f \circ g \leq \mathrm{id}_B$. Since $\mathsf{UOrd} \to \mathsf{IOrd}$ is a local equivalence, a monotone map $f : (A, R) \to (B, S)$ has a right adjoint in UOrd precisely if $\mathsf{fam}(f)$ has a right adjoint in IOrd. The following lemma gives a criterion for the existence of right adjoints in which monotonicity does not have to be checked explicitly.

lem:adj-cond

Lemma 1.8 *The following are equivalent for uniform preorders (A, R), (B, S), a monotone function $f : (A, R) \to (B, S)$, and a function $g : B \to A$.*

lem:adj-cond-uord

(i) *The function g is monotone from (B, S) to (A, R), and right adjoint to f.*

lem:adj-cond-cond lem:adj-cond-cond-counit

(ii) (1) *The relation $\mathsf{im}\langle f \circ g, \mathrm{id}_B\rangle = \{(f(g(b)), b) \mid b \in B\}$ is in S, and* lem:adj-cond-cond-trans

(2) *for all $s \in S$, the relation $s^* = \{(a, gb) \mid (fa, b) \in s\}$ is in R.*

If (B, S) is given by a basis, then it's sufficient to verify (2) on the elements of the basis.

Proof. First assume (i). Condition (1) is equivalent to $f \circ g \leq \mathrm{id}_B$ by (1.1). For condition (2), let $I = \{(a, b) \in A \times B \mid (fa, b) \in s\}$, and let $p : I \to A$ and $q : I \to B$ be the projections. Then we have $f \circ p \leq q$ in $\mathsf{fam}(B, S)(I)$ by direct verification, and therefore $p \leq g \circ q$ in $\mathsf{fam}(A, R)(I)$ by exponential transpose. the latter is equivalent to the claim.

Conversely, assume (ii). To see that postcomposition with g induces a left adjoint to $\mathsf{fam}(f) : \mathsf{fam}(A, R) \to \mathsf{fam}(B, S)$, it is enough to check that for all sets I and $h : I \to B$, the function $g \circ h$ is a greatest element of

$$\Phi = \{k : I \to A \mid f \circ k \leq h\} \subseteq \mathsf{fam}(A, R)(I).$$

We have $g \circ h \in \Phi$ by (1). To show that it is a greatest element we have to show that $f \circ k \leq h$ implies $k \leq g \circ h$, which follows from (2) since

$$\mathsf{im}\langle k, g \circ h\rangle \subseteq \mathsf{im}\langle f \circ k, h\rangle^*$$

and R is down-closed. □

1.2 Cartesian uniform preorders

The full subcategory of IOrd on indexed preorders admitting a generic predicate is closed under small 2-products: if $(\mathcal{H}_k)_{k \in K}$ is a family of indexed preorders with generic predicates $(\iota_k \in \mathcal{H}_k(A_k))_{k \in K}$, then a generic predicate of the (pointwise) product $\prod_{k \in K} \mathcal{H}_k$ is given by the family

$$(\pi_k^* \iota_k)_{k \in K} \in \textstyle\prod_{k \in K} \mathcal{H}_k(\prod_{k \in K} A_k).$$

Thus, UOrd has products which are preserved by $\mathsf{fam} : \mathsf{UOrd} \to \mathsf{IOrd}$. Concretely, the terminal uniform preorder is the singleton set with the unique uniform preorder structure, and a product of (A, R) and (B, S) is given by $(A \times B, R \otimes S)$, where $R \otimes S$ is the uniform preorder structure generated by the basis $\{r \times s \mid r \in R, s \in S\}$.

Definition 1.9 An object A of a locally ordered category $\mathfrak{A}$ with finite 2-products is called *cartesian* if the terminal projection $A \to 1$ and the diagonal $A \to A \times A$ have right adjoints $\top : 1 \to A$ and $\wedge : A \to A \times A$.

Given cartesian objects A, B, a *morphism* $f : A \to B$ is called cartesian, if the diagrams

$$\begin{tikzcd} A \times A \arrow[r, "f \times f"'] \arrow[d, "\wedge"'] & B \times B \arrow[d, "\wedge"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[d, "\top"'] \arrow[dr, "\top"] & \\ A \arrow[r, "f"] & B \end{tikzcd}$$

commute up to isomorphism. ♢

Since $\mathsf{UOrd} \to \mathsf{IOrd}$ is a local equivalence and preserves (finite) 2-products, a uniform preorder (A, R) is cartesian if and only if $\mathsf{fam}(A, R)$ is cartesian, and the latter is easily seen to be equivalent to $\mathsf{fam}(A, R)$ being an *indexed meet-semilattice*, i.e. an indexed preorder whose fibers have finite meets, which are preserved by reindexing. Instantiating Lemma 1.8 we get the following characterization.

lem:cart-cond

Lemma 1.10 *A uniform preorder (A, R) is cartesian if and only if there exists a function $\wedge : A \times A \to A$ and an element $\top \in A$ such that the relations*

$$\tau = \{(a, \top) \mid a \in A\} \qquad \lambda = \{(a \wedge b, a) \mid a, b \in A\} \qquad \rho = \{(a \wedge b, b) \mid a, b \in A\}$$

are in R, and for all $r, s \in R$ the relation

$$\langle\!\langle r, s \rangle\!\rangle \;:=\; \wedge \circ (r \times s) \circ \delta_A \;=\; \{(a, b \wedge c) \mid (a, b) \in r, (a, c) \in s\}$$

is in R. □

ex:cuords:ord
ex:cuords

Examples 1.11 (a) The canonical indexing of a preorder $(A, \leq)$ is an indexed meet-semilattice if and only if $(A, \leq)$ is an meet-semilattice if and only if the uniform preorder $(A, \downarrow\{\leq\})$ is cartesian. This follows since all 2-functors in diagram (**??**) are local equivalences and preserve finite 2-products.

ex:cuord:prim

(b) The primitive recursive functions $f : \mathbb{N} \to \mathbb{N}$ form a basis (Definition 1.4) for a uniform preorder structure on $\mathbb{N}$ which is cartesian: $\top$ is given by 0 (or any other number), and a meet operation $\wedge : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is given by any primitive recursive pairing function.

ex:cuords:rec

(c) Instead of primitive recursive function, we can use total recursive, or even partial recursive functions in the previous example. The last option gives an instance of the concept of *partial combinatory algebra*, to which we will come back later. ♢

Remark 1.12 The forgetful functor from cartesian uniform preorders to uniform preorders does not have a left biadjoint. This is because the meet-completion of an indexed preorder with generic predicate does generally not have a generic predicate. The situation is different for existential quantification, which we treat next. ♢

1.3 Existential quantification

def:ex-iord:has-ex
def:ex-iord

Definition 1.13 (i) We say that an indexed preorder $\mathcal{H}$ *has existential quantification*, if for every function $u : J \to I$, the monotone map $u^* : \mathcal{H}(I) \to \mathcal{H}(J)$ has a left adjoint $\exists_u : \mathcal{H}(J) \to \mathcal{H}(I)$, and the *Beck–Chevalley condition* holds: for every pullback

$$\begin{tikzcd} L \arrow[r, "\bar{u}"] \arrow[d, "\bar{v}"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "v"] \\ J \arrow[r, "u"] & I \end{tikzcd}$$

in Set we have $u^* \circ \exists_v \cong \exists_{\bar{v}} \circ \bar{u}^*$.

def:ex-iord:commutes-ex

(ii) We say that an indexed monotone map $f : \mathcal{H} \to \mathcal{K}$ *commutes with existential quantification*, if $f_I \circ \exists_u \cong \exists_u \circ f_J$ for all $u : J \to I$.

We write $\exists$-IOrd for the sub-2-category of IOrd on indexed preorders with existential quantification, and indexed monotone maps commuting with existential quantification, and we write $\exists$-UOrd for the corresponding sub-2-category of UOrd, given by the following pullback.

$$\begin{tikzcd} \exists\text{-}\mathsf{UOrd} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \exists\text{-}\mathsf{IOrd} \arrow[d, hook] \\ \mathsf{UOrd} \arrow[r, "\mathsf{fam}"] & \mathsf{IOrd} \end{tikzcd}$$

def:ex-iord:ex-compl

(iii) An indexed monotone map $f : \mathcal{A} \to \mathcal{H}$ from an indexed preorder $\mathcal{A}$ to an indexed preorder $\mathcal{H}$ with existential quantification is called an $\exists$-*completion*, if for all indexed preorders $\mathcal{K}$ with existential quantification, the precomposition map

$$(- \circ f) \ : \ \exists\text{-}\mathsf{IOrd}(\mathcal{H}, \mathcal{K}) \to \mathsf{IOrd}(\mathcal{A}, \mathcal{K})$$

is an equivalence of preorders.

def:ex-iord:ex-prime

(iv) Given a uniform preorder $\mathcal{H}$ with existential quantification, a predicate $\pi \in \mathcal{H}(I)$ is called $\exists$-*prime* if for all functions $I \xleftarrow{u} J \xleftarrow{v} K$ and predicates $\varphi \in \mathcal{H}(K)$ with $u^*\pi \leq \exists_v \varphi$, there exists a function $s : J \to K$ such that $v \circ s = \mathrm{id}_J$ and $u^*\pi \leq s^*\varphi$.

We write $\mathsf{prim}(\mathcal{H})$ for the indexed sub-preorder of $\mathcal{H}$ on $\exists$-prime predicates.

We say that $\mathcal{H}$ has *enough* $\exists$-prime predicates if for every set I and $\varphi \in \mathcal{H}(I)$ there exists a $u : J \to I$ and a $\pi \in \mathsf{prim}(\mathcal{H})(J)$ such that $\exists_u \pi \cong \varphi$. $\diamondsuit$

rem:prime-fib

Remark 1.14 Using the fibrational—rather than the indexed—point of view, we can give the following characterization of $\exists$-prime predicates: $\pi \in \mathcal{H}(I)$ is $\exists$-prime iff for all $f : J \to I$, the object $(J, f^*\pi)$ of the total category $\int \mathcal{H}$ has the left lifting property w.r.t. cocartesian arrows. $\diamondsuit$

The notion of $\exists$-prime predicate gives rise to a sufficient criterion for an indexed preorder with existential quantification to be a $\exists$-completion.

prop:ecomp-if-enough-primes

Proposition 1.15 *Let $\mathcal{H}$ be an indexed preorder with existential quantification, and assume that $\mathcal{A} \subseteq \mathcal{H}$ is an indexed sub-preorder such that*

(i) *all predicates in $\mathcal{A}$ are $\exists$-prime in $\mathcal{H}$, and*

(ii) *for every set I and predicate $\varphi \in \mathcal{H}(I)$ there exists a function $u : J \to I$ and a predicate $\pi \in \mathcal{A}(J)$ such that $\varphi \cong \exists_u \pi$.*

Then the inclusion $\mathcal{A} \hookrightarrow \mathcal{H}$ is an $\exists$-completion, and moreover $\mathcal{A} \hookrightarrow \mathsf{prim}(\mathcal{H})$ is an equivalence, i.e. every $\exists$-prime predicate in $\mathcal{H}$ is isomorphic to one in $\mathcal{A}$. In particular, if $\mathcal{H}$ has enough $\exists$-prime predicates, then $\mathsf{prim}(\mathcal{H}) \hookrightarrow \mathcal{H}$ is an $\exists$-completion.

Proof. Given an indexed preorder $\mathcal{K}$ with existential quantification and an indexed monotone map $f : \mathcal{A} \to \mathcal{K}$, define $\tilde{f} : \mathcal{H} \to \mathcal{K}$ by $\tilde{f}_I(\varphi) = \exists_u f(\pi)$ for a choice of function $u : J \to I$ and predicate $\pi \in \mathcal{A}(J)$ with $\exists_u \pi \cong \varphi$. It is straightforward to verify that $\tilde{f}$ gives a well defined indexed monotone map commuting with existential quantification, and the assignment $f \mapsto \tilde{f}$ gives a pseudoinverse to the restriction map $\exists\text{-}\mathsf{IOrd}(\mathcal{H}, \mathcal{K}) \to \mathsf{IOrd}(\mathcal{A}, \mathcal{K})$.

Now assume that $\pi \in \mathsf{prim}(\mathcal{H})(I)$, and choose $u : J \to I$ and $\sigma \in \mathcal{A}(J)$ with $\exists_u \sigma \cong \pi$. Then from $\pi \le \exists_u \sigma$ it follows that there exists a section s of u with $\pi \le s^*\sigma$. On the other hand, the inequality $\exists_u \sigma \le \pi$ is equivalent to $\sigma \le u^*\pi$, which implies $s^*\sigma \le \pi$ by applyinng s^* on both sides, and we conclude that $s^*\sigma \cong \pi$. □

Definition 1.16 A *primal* $\exists$*-completion* is an $\exists$-completion $e : \mathcal{A} \to \mathcal{H}$ fitting the hypotheses of Proposition 1.15, i.e. $\mathcal{H}$ has enough $\exists$-primes and e is fiberwise order reflecting, and its essential image is $\mathsf{prim}(\mathcal{H})$. ◊

It is well known that indexed preorders on *small* index categories $\mathbb{C}$ always admit primal $\exists$-completions[3]: given an indexed preorder $\mathcal{A} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Ord}$, predicates on $I \in \mathbb{C}$ in its $\exists$-completion $D\mathcal{A} : \mathbb{C}^{\mathsf{op}} \to \mathsf{Ord}$ are given by pairs $(J \xrightarrow{u} I, \varphi \in \mathcal{A}(J))$, where $(J \xrightarrow{u} I, \phi) \le (K \xrightarrow{v} I, \psi)$ iff there exists a $w : J \to K$ such that $v \circ w = u$ and $\varphi \le w^*\psi$. However, for indexed preorders on Set this construction may not be well-defined, since the resulting indexed preorder may have large fibers. In the following we show that indexed preorders arising from uniform preorders *do* always admit primal $\exists$-completions, which are again representable by uniform preorders (the question if there are non-primal $\exists$-completions over Set remains open).

def:dar

Definition 1.17 For (A, R) a uniform preorder, we define the uniform preorder

$$D(A, R) \;=\; (PA, DR)$$

where PA is the powerset of A, and DR is the uniform preorder structure on PA generated by the basis of relations

$$[r] \;=\; \{(U, V) \in PA \times PA \mid \forall a \in U\ \exists b \in V \,.\, (a, b) \in r\}$$

for $r \in R$. ◊

rem:dar:is-basis *rem:dar*

Remarks 1.18 (a) The relations $[r]$ do indeed constitute a basis since $\mathrm{id}_{PA} \subseteq [\mathrm{id}_A]$ and $[s] \circ [r] \subseteq [s \circ r]$ for $r, s \in R$.

rem:dar:fiberwise-order

(b) Unwinding the definition of $D(A, R)$ we see that for $\varphi, \psi : I \to PA$ we have $\varphi \le \psi$ in $\mathsf{fam}(D(A, R))(I)$ if and only if there exists an $r \in R$ such that

$$\forall i \in I\ \forall a \in \varphi(i)\ \exists b \in \psi(i) \,.\, (a, b) \in r.$$

In this case we call r a *realizer* of the inequality $\varphi \le \psi$. ◊

prop:dar-ecomp

Proposition 1.19 *For every uniform preorder* (A, R)*, the indexed preorder* $\mathsf{fam}(D(A, R))$ *has existential quantification and the singleton map* $\eta : A \to PA$ *is monotone from* (A, R) *to* $D(A, R)$*. The induced indexed monotone map* $\mathsf{fam}(\eta) : \mathsf{fam}(A, R) \to \mathsf{fam}(D(A, R))$ *is a primal* $\exists$*-completion.*

[3] For accounts of closely related constructions see e.g. [Fre13, Definition 3.4.5] for the $\exists$-completion of fibered preorders satisfying a stack-condition, [Tro20, Section 4] for $\exists$-completion of indexed meet-semilattices, and Hofstra [Hof11, Section 3.2] for the analogous construction for non-posetal fibrations.

Proof. Existential quantification in $\mathsf{fam}(D(A,R))$ is given by union, i.e.

$$(\exists_u\varphi)(i) = \bigcup_{u(j)=i}\varphi(j)$$

for $u : J \to I$ and $\varphi : J \to PA$, and η is monotone since for every $r \in R$ we have

$$\{(\{a\},\{a'\}) \mid (a,a') \in r\} \subseteq [r].$$

To show that $\mathsf{fam}(\eta)$ is a primal $\exists$-completion, it remains to show that it is fiberwise order reflecting, and its image in $\mathsf{fam}(D(A,R))$—the indexed sub-preorder of *singleton-valued predicates*, i.e. predicates factoring through $\eta : A \to PA$—satisfies the hypotheses of Proposition 1.15.

The fact that $\mathsf{fam}(\eta)$ is order reflecting follows immediately from the explicit description of the fiberwise ordering in $\mathsf{fam}(D(A,R))$ in Remark 1.18(b).

To see that singleton-valued predicates are $\exists$-prime in $\mathsf{fam}(D(A,R))$, assume $\varphi : I \to A$, $\psi : J \to PA$, and $u : J \to I$ such that $\eta \circ \varphi \le \exists_u\psi$. Unwinding definitions this means that there exists an $r \in R$ such that

$$\forall i \in I\ \forall a \in \{\varphi(i)\}\ \exists b \in \bigcup_{u(j)=i}\psi(j)\,.\,(a,b) \in r\,,$$

i.e.

$$\forall i \in I\ \exists j \in J\,.\,u(j) = i \wedge \exists b \in \psi(j)\,.\,(\varphi(i),b) \in r\,,$$

and the required section of u is given by a Skolem function for the first two quantifiers.

Finally, $\mathsf{fam}(D(A,R))$ has 'enough' singleton-valued predicates, since every predicate $\varphi : I \to PA$ can be written as $\varphi = \exists_u\sigma$ for $J = \coprod_{i\in I}\varphi I$, u the first projection, and $\sigma = (J \xrightarrow{\pi_2} A \xrightarrow{\eta} PA)$. $\square$

rem:d-lax-idempotent

Remark 1.20 The assignment $(A,R) \mapsto D(A,R)$ gives rise to a left 2-adjoint to the inclusion $\exists\text{-}\mathsf{UOrd} \to \mathsf{UOrd}$, and the unit η and multiplication μ of the induced 2-monad $D : \mathsf{UOrd} \to \mathsf{UOrd}$ are componentwise given by singleton map and union. The 2-monad is *lax idempotent*[4] in the sense that $D\eta_{(A,R)} \dashv \mu_{(A,R)} \dashv \eta_{D(A,R)}$ for all uniform preorders (A,R). In particular, a uniform preorder (A,R) is a D-algebra iff $\eta_{(A,R)}$ has a left adjoint (the adjunction is then automatically a reflection, since $\mathsf{fam}(\eta_{(A,R)})$ is fiberwise order-reflecting). Finally, the adjunction is monadic, since reflective indexed sub-preorders of indexed preorders with existential quantification have existential quantification. $\diamondsuit$

1.4 Indexed frames

We recall the definition of indexed frames from [Fre22].

Definition 1.21 An *indexed frame* is an indexed meet-semilattice $\mathcal{H}$ which has existential quantification and moreover satisfies the *Frobenius condition*: for all functions $u : J \to I$, and predicates $\varphi \in \mathcal{H}_I$ $\psi \in \mathcal{H}_J$ we have $\varphi \wedge \exists_u\psi \cong \exists_u(u^*\varphi \wedge \psi)$. $\diamondsuit$

Examples 1.22 (a) The canonical indexing of a poset $(A,\le)$ is an indexed frame if and only if A is a *frame*, i.e. a complete lattice satisfying the infinitary distributive law $a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$.

[4] Lax idempotent monads were introduced in [Koc95] as *Kock-Zöberlein monads.*

(b) If $(L, \leq)$ is a frame and M is a monoid of frame-endomorphisms (i.e. monotone maps preserving finite meets and arbitrary joins), we obtain an indexed frame structure on the representable functor $\mathsf{Set}(-, L)$ by setting

$$\varphi \leq \psi \qquad \text{if and only if} \qquad \exists m \in M\, \forall i \in I \,.\, m(\varphi(i)) \leq \psi(i)$$

for $\varphi, \psi : I \to L$. This indexed frame structure is only representable by an ordinary frame if M has a least element (which is then an 'interior operator' i.e. a posetal comonad). A non-trivial example is the *Lipschitz hyperdoctrine* which has been recently proposed by Barton and Comelin, and is obtained by taking $L = ([0, \infty], \geq)$ and $M = \mathbb{R}_{>0}$ acting by multiplication. See also [FvdB22] for similar constructions of non-Set-based indexed preorders. ♢

Another way of producing indexed frames is given by the following.

prop:dar-ifrm

Proposition 1.23 *If (A, R) is cartesian then so are $D(A, R)$ and $\eta : (A, R) \to D(A, R)$, and moreover $\mathsf{fam}(D(A, R))$ is an indexed frame.*

Proof. To show that $D(A, R)$ is cartesian we use Lemma 1.10 and define $\wedge : PA \times PA \to PA$ and $\top \in PA$ by $U \wedge V = \{a \wedge b \mid a \in U, b \in V\}$ and $\top = \{\top\}$. Then the verification of the conditions is straightforward. □

1.5 Relational completeness

def:uquant

Definition 1.24 (i) We say that an indexed preorder *has universal quantification* if it satisfies the dual condition of Definition 1.13(i).

(ii) A *Heyting preorder* is a meet-semilattice $(H, \leq)$ which is cartesian closed, i.e. for all $a \in H$ the monotone map $(- \wedge a)$ has a right adjoint $(a \Rightarrow -)$ called *Heyting implication*[5].

(iii) An *indexed meet-semilattice* $\mathcal{H}$ is said to *have implication* if its fibers are Heyting preorders, and this structure is preserved up to isomorphism by reindexing.

(iv) A *tripos* $\mathcal{P}$ is an indexed meet-semilattice $\mathcal{H}$ which has universal quantification and implication, and a generic predicate.

Remarks 1.25 (a) As explained in [HJP80, Theorem 1.4], using Prawitz-style second order encodings [Pra65, page 67] one can show that triposes have existential quantification and fiberwise finite joins which are stable under reindexing. In other words, triposes are models of full first order logic. ♢

def:rcomp

Definition 1.26 A cartesian uniform preorder (A, R) is called *relationally complete*, if there exists a relation $@ \in R$ (called 'universal relation'), such that for every relation $r \in R$ there exists a *function* (i.e. a single-valued and entire relation) $\tilde{r} \in R$ with

$$r \circ \wedge \ \subseteq \ @ \circ \wedge \circ (\tilde{r} \times \mathrm{id}_A),$$

in other words

(1.2) {eq:impl-rel-comp} $\forall a\, b\, c \in A \,.\, (a \wedge b, c) \in r \ \Rightarrow \ (\tilde{r}(a) \wedge b, c) \in @.$ ♢

[5]Contrary to the better-known *Heyting algebras*, Heyting preorders need not have finite joins.

Remark 1.27 The notion of *relational completeness* should be viewed as a relational analogue of the functional completeness property of recursive functions expressed by the *s-m-n Theorem*, which in its most basic form (see e.g. [Cut80, Theorem 4.4.1]) says that for every partial recursive function $f(x, y)$ in two arguments there exists a *total recursive* function $\tilde{f}(x)$ in one argument such that the partial functions $f(x, y)$ and $\phi_{\tilde{f}(x)}(y)$ are equal, where $(\phi_n)_{n\in\mathbb{N}}$ is a effective enumeration of partial recursive functions.

Note that besides using relations instead of partial functions, the statement above is somewhat weaker than that of the s-m-n-theorem since equality of partial functions is replaced by inclusion of relations. See also Remark 2.4(a). ♢

thm:rel-compl

Theorem 1.28 *The following are equivalent for a cartesian uniform preorder* (A, R)*.*

(i) (A, R) *is relationally complete.*

(ii) $\mathsf{fam}(D(A, R))$ *is a tripos.*

Proof. Assume first that $D(A, R)$ has implication and universal quantification. Let

$$E \hookrightarrow A \times A \times P(A \times A)$$

be the membership relation, define $u : E \to P(A \times A)$ and $\varphi, \psi : E \to PA$ by

$$u(b, c, s) = s \qquad \varphi(b, c, s) = \{b\} \qquad \psi(b, c, s) = \{c\},$$

and set $\theta = \forall_u(\varphi \Rightarrow \psi) : P(A \times A) \to PA$. Then we have $u^*\theta \wedge \varphi \le \psi$, and we define @ to be a realizer of this inequality, such that

$$\forall s \in P(A \times A)\ \forall (b, c) \in s\ \forall a \in \theta(s) \,.\, (a \wedge b, c) \in @ \,. \tag{1.3}$$ {eq:at-char}

Now for every $r \in R$ we set $M = \{(a, b, c) \mid (a \wedge b, c) \in r\}$ and define

$$\begin{array}{rcc} v : M \to A & v(a,b,c) = a & \\ \beta, \gamma : M \to PA & \beta(a,b,c) = \{b\} & \gamma(a,b,c) = \{c\} \\ \eta : A \to PA & \eta(a) = \{a\}. & \end{array}$$

Then we have $v^*\eta \wedge \beta \le \gamma$, and therefore $\eta \le \forall_v(\beta \Rightarrow \gamma)$. Defining $w : A \to P(A \times A)$ by $w(a) = \{(b, c) \mid (a \wedge b, c) \in r\}$ we get a pullback square

$$\begin{tikzcd} M \arrow[r, "v"] \arrow[d, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "w"] \\ E \arrow[r, "u"'] & P(A \times A) \end{tikzcd}$$

and moreover $x^*\varphi = \beta$ and $x^*\psi = \gamma$, which using the Beck–Chevalley condition implies that $\forall_v(\beta \Rightarrow \gamma) \cong w^*\theta$, and thus $\eta \le w^*\theta$. Taking t to be a realizer of this inequality we get that

$$\forall a \in A\ \exists b \in \theta(w(a)) \,.\, (a, b) \in t,$$

in particular, t is total, and we choose $\tilde{r}$ to be a subfunction of t satisfying

$$\forall a \in A \,.\, \tilde{r}(a) \in \theta(w(a)).$$

Finally, the implication (1.2) follows from (1.3) by instantiating s with $w(a)$.

Conversely, assume that (A, R) is relationally complete. Instead of constructing implication and universal quantification separately, we show how to define the 'synthetic' connective $\forall_u(\varphi \Rightarrow \psi)$ for $u : J \to I$ and $\varphi, \psi \in \mathsf{fam}(D(A, R))(I)$. Implication and universal quantification can then be recovered by either replacing u by the identity, or φ by the true predicate. For $\varphi, \psi : J \to PA$ define $\forall_u(\varphi \Rightarrow \psi) : I \to PA$ by

$$\forall_u(\varphi \Rightarrow \psi)(i) = \bigcap_{uj=i} \{a \in A \mid \forall b \in \varphi(j)\, \exists c \in \psi(j)\, .\, @(a \wedge b, c)\}.$$

It is then easy to see that the inequality $u^*\forall_u(\varphi \Rightarrow \psi) \wedge \varphi \leq \psi$ is realized by $@$; and if $\zeta : I \to PA$ such that the inequality $u^*\xi \wedge \varphi \leq \psi$ is realized by $r \in R$, then $\tilde{r}$ realizes $\xi \leq \forall_u(\varphi \Rightarrow \psi)$. $\square$

Corollary 1.29 *The following are equivalent for a tripos $\mathcal{P}$.*

(i) *There exists a relationally complete uniform preorder (A, R) with $\mathcal{P} \simeq \mathsf{fam}(D(A, R))$.*

(ii) *$\mathcal{P}$ has enough $\exists$-prime predicates, and $\mathsf{prim}(\mathcal{P})$ is closed under finite meets in $\mathcal{P}$.*

Proof. First assume that $\mathcal{P} \simeq \mathsf{fam}(D(A, R))$ with (A, R) relationally complete. Then $\mathcal{P}$ has enough $\exists$-prime predicates by **????**, and $\mathsf{prim}(\mathsf{fam}(D(A, R)))$ is equivalent $\mathsf{fam}(A, R)$. The latter has finite meets since relationally complete uniform preorders are in particular cartesian, and the inclusion $\mathsf{fam}(A, R) \to \mathsf{fam}(D(A, R))$ preserves finite meets by Proposition 1.23.

Conversely assume that $\mathcal{P}$ has enough $\exists$-prime predicates, and $\mathsf{prim}(\mathcal{P})$ is closed under finite meets in $\mathcal{P}$. Let $\mathsf{tr} \in \mathcal{P}(\Sigma)$ be a generic predicate. Since $\mathcal{P}$ has enough $\exists$-prime predicates, there exists a prime $\pi \in \mathcal{P}(A)$ and a function $u : A \to \Sigma$ with $\mathsf{tr} \cong \exists_u \pi$. We claim that π is generic in $\mathsf{prim}(\mathcal{P})$. Let $\xi \in \mathsf{prim}(\mathcal{P})(B)$. By genericity of tr there exists a $v : B \to \Sigma$ with $\xi \cong v^*\mathsf{tr} \cong v^*(\exists_u \pi)$, and forming the pullback $\begin{array}{ccc} P & \xrightarrow{\overline{v}} & A \\ {\scriptstyle\overline{u}}\downarrow & & \downarrow{\scriptstyle u} \\ B & \xrightarrow{v} & \Sigma \end{array}$ and applying the Beck–Chevalley condition, we get $\xi \cong \exists_{\overline{u}}(\overline{v}^*\pi)$. Since ξ is prime, $\overline{u}$ has a section $s : A \to P$ with $\xi \cong s^*(\overline{v}^*\pi) \cong (vs)^*\pi$, thus π is generic in $\mathsf{prim}(\mathcal{P})$. This means that $\mathsf{prim}(\mathcal{P})$ is representable by a uniform preorder (A, R), which is cartesian since $\mathsf{prim}(\mathcal{P})$ is, and relationally complete since its $\exists$-completion is a tripos. $\square$

1.6 Discreteness

def:discrete:dco *ssec:discrete*

Definition 1.30 (i) A *discrete combinatory object (DCO)* is a uniform preorder where all relations $r \in R$ are *single-valued*, i.e. partial functions. We write DCO for the full locally ordered subcategory of UOrd on DCOs.

def:discrete:predicate

(ii) A predicate $\delta \in \mathcal{A}(I)$ of an indexed preorder $\mathcal{A}$ is called *discrete*, if for every surjection $e : K \twoheadrightarrow J$, function $f : K \to I$, and predicate $\varphi \in \mathcal{A}(J)$ such that $e^*\varphi \leq f^*\delta$, there exists a (necessarily unique) $g : J \to I$ with $g \circ e = f$ (and therefore $\varphi \leq g^*\delta$ since reindexing along split epis is order-reflecting). $\diamondsuit$

rem:discrete:ipaa-ref *rem:discrete*

Remarks 1.31 (a) DCOs were introduced in [Fre19, Definition 2.2] in terms of bases, i.e. as sets A equipped with a set $\mathcal{F}$ of partial endofunctions containing the identify and weakly closed under composition in the sense that for all $f, g \in \mathcal{F}$ there exists an $h \in \mathcal{F}$ such that $g \circ f \subseteq h$. Down-closure in $P(A \times A)$ of such a structure yields a DCO

$(A, \downarrow\mathcal{F})$ in the above sense inducing the same indexed preorder and the two definitions give rise to equivalent locally ordered categories, the principal difference being that for the above, 'saturated' definition, the 2-functor $\mathsf{DCO} \to \mathsf{lOrd}$ is injective on objects.
rem:discrete:total-cat

(b) In fibrational language, discreteness of $\delta \in \mathcal{A}(A)$ can be expressed by saying that (A, δ) has the right lifting property w.r.t. (or equivalently is right orthogonal to) all *cover-cartesian maps*—i.e. cartesian maps over surjections—in the total category $\int \mathcal{A}$.
rem:discrete:reind-inj-surj

(c) It is easy to see that reindexings of discrete predicates along injections are discrete again. Reindexings along surjections, on the other hand, are discrete only in the trivial case that the surjection is a bijection. ♢

The following clarifies the relationship between the two notions of discreteness introduced in Definition 1.30.

prop:dco-iff-discrete

Proposition 1.32 *A uniform preorder (A, R) is a DCO if and only if the generic predicate $\mathrm{id}_A \in \mathsf{fam}(A, R)(A)$ is discrete.*

Proof. Assume first that (A, R) is a DCO and consider a span $J \overset{e}{\twoheadleftarrow} K \overset{f}{\to} A$ with e surjective, and a predicate $\varphi : J \to A$ such that $e^*\varphi \leq f^*\mathrm{id}_A$. Form the image factorization

$$\begin{tikzcd} K \arrow[r, two heads, "h"] \arrow[dr, "\langle \varphi\circ e, f\rangle"'] & r \arrow[d, hook, "\langle p,q\rangle"] \\ & A \times A \end{tikzcd}$$

of $\langle \varphi \circ e, f\rangle$. Then $r \in R$ and therefore p is injective since (A, R) is a DCO. Since e is surjective we obtain a lifting k in the square $\begin{tikzcd} K \arrow[r, two heads, "h"] \arrow[d, two heads, "e"'] & r \arrow[d, tail, "p"] \\ J \arrow[r, "\varphi"'] \arrow[ur, dashed, "k"] & A \end{tikzcd}$ and the desired map is $q \circ k$.

Conversely assume that id_A is discrete, let $r \in R$, write $\langle p, q\rangle : r \hookrightarrow A \times A$ for the inclusion, and let $r \overset{e}{\twoheadrightarrow} U \overset{m}{\hookrightarrow} A$ be an image-factorization of p. We have $p^*(\mathrm{id}_A) = e^*(m^*(\mathrm{id}_A)) \leq q^*(\mathrm{id}_A)$, and discreteness of id_A implies that there exists $g : U \to A$ with $g \circ e = q$. We obtain a factorization $\langle p, q\rangle = \langle m, g\rangle \circ e$, and since $\langle p, q\rangle$ is injective we conclude that e is bijective and thus r is single-valued. □

cor:dco-discgen

Corollary 1.33 *An indexed preorder $\mathcal{A}$ is representable by a DCO if and only if it has a discrete generic predicate.*

Proof. This follows from Proposition 1.32 together with Lemma 1.6. A direct proof is given in [Fre19, Theorem 2.4]. □

Remark 1.34 It is possible that the same indexed preorder has discrete and non-discrete generic predicates: if $\mathcal{A}$ is an indexed preorder with discrete generic predicate $\iota \in \mathcal{A}(A)$ and $f : B \twoheadrightarrow A$ is a surjection, then f^*A is a generic predicate which is only discrete if f is a bijection. If f is not a bijection, we obtain a DCO-representation of $\mathcal{A}$ with underlying set A, and a representation as a non-discrete uniform preorder with underlying set B. ♢

rem:cart-dco

Remark 1.35 (Cartesian DCOs) If a cartesian uniform preorder (A, R) is a DCO, then the relations $\lambda, \rho \in R$ from Lemma 1.8 are partial functions, and jointly form a retraction $\langle \lambda, \rho\rangle : A \rightharpoonup A \times A$ of $\wedge : A \times A \to A$, i.e. we have $\langle \lambda, \rho\rangle \circ \wedge = \mathrm{id}_{A\times A}$. Moveover, although we don't have $\wedge \circ \langle \lambda, \rho \rangle = \mathrm{id}_A$, we have the inclusion $\wedge \circ \langle \lambda, \rho \rangle \subseteq \mathrm{id}_A$ of partial functions, since by construction, λ an ρ are only defined on the range of $\wedge$.

More generally we define n-ary versions

$$\wedge^{(n)} : A^n \to A \quad \text{for} \quad n \in \mathbb{N} \qquad \text{and} \qquad \pi_i^{(n)} \in R \quad \text{for} \quad 1 \leq i \leq n$$

by $\wedge^{(0)}(*) = \top$, $\wedge^{(n+1)}(\vec{a}, b) = \wedge^{(n)}(\vec{a}) \wedge b$, and $\pi_i^{(n)} = \rho \circ \lambda_i^n$, so that we have

$$\langle \pi_1^{(n)}, \ldots, \pi_n^{(n)} \rangle \circ \wedge^{(n)} = \mathrm{id}_{A^n} \qquad \text{and} \qquad \wedge^{(n)} \circ \langle \pi_1^{(n)}, \ldots, \pi_n^{(n)} \rangle \subseteq \mathrm{id}_A$$

for all $n \in \mathbb{N}$. Loosely following Hofstra [Hof06, pg. 254], we introduce the notation

$$\begin{aligned} R^{(n)} &= \{ r \subseteq A^n \times A \mid \exists s \in R \,.\, r = s \circ \wedge^{(n)} \} \\ &= \{ r \subseteq A^n \times A \mid r \circ \langle \pi_1^{(n)}, \ldots, \pi_n^{(n)} \rangle \in R \} \end{aligned}$$

for 'n-ary computable' functions, which can be viewed as representing realizers of inequalities $\varphi_1, \ldots, \varphi_n \leq \psi$ in the indexed preorder. A paradigmatic example is given by the DCO of subrecursive functions (Example 1.11(c)): here $R^{(n)}$ contains precisely the *n-ary partial sub-recursive functions*, i.e. sub-functions of n-ary partial recursive functions in the usual sense. $\diamondsuit$

2 Relative partial combinatory algebras

se:rpcas

In the following we recall the definitions of relative PCAs and relative realizability triposes from [vO08, Section 2.6.9]. Deviating slightly from the treatment there, we introduce the auxiliary notion of *relative partial applicative structure*, which is a straightforward generalization of the non-relative version given in [vO08, Section 1.1].

Definition 2.1 A *relative partial applicative structure (relative PAS)* is a triple $(A, A_\#, \cdot)$ consisting of a set A, a subset $A_\# \subseteq A$, and a partial binary 'application' operation

$$(- \cdot -) : A \times A \rightharpoonup A$$

such that $a \cdot b \in A_\#$ whenever $a, b \in A_\#$ and $a \cdot b$ is defined. $\diamondsuit$

Remarks 2.2 (a) Application associates to the left, i.e. $a \cdot b \cdot c$ is a shorthand for $(a \cdot b) \cdot c$.

(b) A *polynomial* over a relative PAS $(A, A_\#, \cdot)$ is a term built up from variables, constants from A, and application $(- \cdot -)$.

(c) When reasoning with partial terms, $t{\downarrow}$ means that t is defined, $t = u$ means that t and u are defined and equal, $t \preceq u$ means that u is defined whenever t is defined, and in this case they're equal, and $t \simeq u$ means $t \preceq u$ and $u \preceq t$. The $\preceq$ convention is in conflict with some of the literature, but used here since it is consistent with subset inclusion. Note that subterms of defined terms are necessarily defined. $\diamondsuit$

def:rpca

Definition 2.3 A *relative partial combinatory algebra (relative PCA)* is a relative PAS that is *combinatory complete*, in the sense that for every polynomial $p[x_1, \ldots, x_n, y]$ with coefficients in $A_\#$ there is a (non-unique) element $e \in A_\#$ such that for all $a_1, \ldots, a_n, b \in A$ we have

(2.1) {eq:combice} $e \cdot a_1 \cdot \ldots \cdot a_n{\downarrow} \qquad \text{and} \qquad e \cdot a_1 \cdot \ldots \cdot a_n \cdot b \succeq t[a_1, \ldots, a_n, b]\,.$ $\diamondsuit$

rem:rpca:trad *rem:rpca*

Remarks 2.4 (a) Relative PCAs are called *elementary inclusions of PCAs* in [vO08, Sections 2.6.9 and 4.5]—or rather the relative PCAs presented here are a slight generalization of van Oosten's elementary inclusions, the reason being that the traditional definition of PCA [Fef75] requires that both sides of (2.1) are *equidefined*, whereas we only require that the left side is defined whenever the right side is defined. This difference does not change the class of induced indexed preorders and realizability models, see [FvO16, Theorem 5.8] and the discussion in the introduction of [Fre19].

(b) We sometimes use λ-calculus notation and write $\lambda x_1 \dots x_n y \,.\, p[x_1, \dots, x_n, y]$ for the element e associated to the polynomial $p[x_1, \dots, x_n, y]$ in Definition 2.3. Note that this does not give rise to an interpretation of arbitrary untyped λ-terms, since λ has to 'abstract' *all* of the free variables of the polynomial in the body.

rem:rpca:combs

(c) It follows from the definition that for every relative PCA $(A, A_\#, \cdot)$ there are elements $\mathsf{k}, \mathsf{s}, \mathsf{i}, \mathsf{b}, \mathsf{p}, \mathsf{p}_0, \mathsf{p}_1 \in A_\#$ satisfying the statements

$$\begin{array}{cccc} & \mathsf{k}{\cdot}a{\cdot}b = a & \mathsf{i}{\cdot}a = a & \mathsf{p}_0{\cdot}(\mathsf{p}{\cdot}a{\cdot}b) = a \\ \mathsf{s}{\cdot}a{\cdot}b{\downarrow} & \mathsf{s}{\cdot}a{\cdot}b{\cdot}c \succeq a{\cdot}c{\cdot}(b{\cdot}c) & \mathsf{b}{\cdot}a{\cdot}b{\cdot}c = a{\cdot}(b{\cdot}c) & \mathsf{p}_1{\cdot}(\mathsf{p}{\cdot}a{\cdot}b) = b \end{array}$$

for all $a, b, c \in A$. These elements are given by the usual combinator encodings in untyped λ-calculus, specifically:

$$\begin{array}{llll} \mathsf{k} = \lambda xy \,.\, x & \mathsf{s} = \lambda xyz \,.\, x{\cdot}z{\cdot}(y{\cdot}z) & \mathsf{i} = \lambda x \,.\, x & \mathsf{b} = \lambda xyz \,.\, x{\cdot}(y{\cdot}z) \\ \mathsf{p} = \lambda xyz \,.\, z{\cdot}x{\cdot}y & \mathsf{p}_0 = \lambda x \,.\, x{\cdot}(\lambda yz \,.\, y) & \mathsf{p}_1 = \lambda x \,.\, x{\cdot}(\lambda yz \,.\, z) & \end{array}$$

rem:rpca:ks

(d) Conversely, if a relative PAS $(A, A_\#, \cdot)$ contains elements $\mathsf{s}, \mathsf{k} \in A_\#$ satisfying

$$\mathsf{k}{\cdot}a{\cdot}b = a \qquad \mathsf{s}{\cdot}a{\cdot}b{\downarrow} \qquad \mathsf{s}{\cdot}a{\cdot}b{\cdot}c \succeq a{\cdot}c(b{\cdot}c)\,,$$

then it is a relative PCA. This is shown by a well known argument by induction on the structure of the polynomial $p[x_1, \dots, x_n, y]$ (see [vO08, Theorem 1.1.3]), and inspection of the proof shows that the argument also works for polynomials with coefficients in A (rather than in $A_\#$), producing in this case an element $e \in A$. $\diamondsuit$

Definition 2.5 The *relative realizability tripos* associated to a relative PCA is the indexed preorder $\mathsf{rt}(A, A_\#, \cdot)$ given by $\mathsf{rt}(A, A_\#, \cdot)(I) = (A^I, \leq)$ with the order defined by

$$\varphi \leq \psi \quad \text{iff} \quad \exists e \in A_\# \; \forall i \in I \; \forall a \in \varphi(i) \,.\, e{\cdot}a{\downarrow} \;\&\; e{\cdot}a \in \psi(i)$$

for $\varphi, \psi : I \to A$ [vO08, Section 2.6.9]. $\diamondsuit$

As observed by Hofstra [Hof06] in his more general setting of relative *ordered* PCAs and *basic combinatory objects* (BCOs), relative realizability triposes are $\exists$-completions of indexed meet-semilattices arising from cartesian DCOs:

prop:relative-pca-relcomp

Proposition 2.6 *For every relative PCA $(A, A_\#, \cdot)$, the set of 'subcomputable' partial functions*

$$\mathcal{F}_{A_\#} \;=\; \{\phi_a = (a{\cdot}-) : A \rightharpoonup A \mid a \in A_\#\}$$

form a basis for a cartesian relationally complete DCO structure on A, and $\mathsf{fam}(D(A, {\downarrow}\mathcal{F}_{A_\#}))$ *is equal the* relative realizability tripos $\mathsf{rt}(A, A_\#, \cdot)$.

Proof. To see that $\mathcal{F}_{A_\#}$ is a basis, we verify the conditions of Remark 1.5: we have $\mathrm{id}_A \subseteq \phi_{\mathsf{i}}$ and $\phi_a \circ \phi_b \subseteq \phi_{\mathsf{b}\cdot a\cdot b}$ for $a, b \in A_\#$, where i and b are as in Remark 2.4(c).

To show that the resulting uniform preorder is cartesian, we verify the conditions of Lemma 1.10: defining $\top = \mathsf{k}$ and $a \wedge b = \mathsf{p}\cdot a\cdot b$, we have $\tau = \mathsf{k}\cdot\mathsf{k}$, $\lambda \subseteq \phi_{\mathsf{p}_0}$, and $\rho \subseteq \phi_{\mathsf{p}_1}$. For the last condition, combinatory completeness implies that for all $a, b \in A_\#$ there exists a $c \in A_\#$ such that $\langle\!\langle \phi_a, \phi_b \rangle\!\rangle \subseteq \phi_c$.

The fact that $\mathsf{fam}(D(A, \downarrow\mathcal{F}_{A_\#})) = \mathsf{rt}(A, A_\#, \cdot)$ is immediate from the definitions, and relational completeness of $(A, \downarrow\mathcal{F}_{A_\#})$ follows from Theorem 1.28 since $\mathsf{rt}(A, A_\#, \cdot)$ is a tripos.

Alternatively, relational completeness can be shown directly: a universal relation is given by $\phi_{(\lambda x\,.\,\mathsf{p}_0\cdot x(\mathsf{p}_1\cdot x))}$, and for $a \in A$, the 'transpose' $\widetilde{\phi_a}$ of ϕ_a is given by $\phi_{\tilde{a}}$, with $\tilde{a} = \lambda xy\,.\,a\cdot(\mathsf{p}\cdot x\cdot y)$. $\square$

In the following we show that *every* relationally complete DCO arises via the construction of Proposition 2.6. To start, given a relationally complete DCO (A, R), we define a relative PAS with underlying set A by

{eq:rpas-from-rcdco}
$$a\cdot b = @(a \wedge b) \quad \text{and} \qquad \begin{aligned} A_\# &:= \{a \in A \mid \{(\top, a)\} \in R\} \\ &= \{a \in A \mid \top \leq a \text{ in } \mathsf{fam}(A, R)(1)\}\,. \end{aligned}$$

Note that the elements of $A_\#$ correspond to Hofstra's *designated truth values* [Hof06, pg. 244]. If $a, b \in A_\#$ such that $a\cdot b = @(a \wedge b)$ is defined, then $a\cdot b \in A_\#$ since $\top \leq a$ and $\top \leq b$ implies $\top \leq a \wedge b$; and $a \wedge b \leq @(a \wedge b)$. Thus, $(A, A_\#, \cdot)$ is indeed relative PAs, and moreover we can show the following.

prop:rcdcos-are-relpcas

Proposition 2.7 *Let (A, R) be a relationally complete cartesian DCO.*

(i) *For every n-ary polynomial $p[x_1, \dots, x_n]$ over the relative PAS $(A, A_\#, \cdot)$ with coefficients in $A_\#$, the partial 'evaluation function' $\vec{a} \mapsto p[\vec{a}]$ is in $R^{(n)}$ (see Remark 1.35).*

(ii) *For all $n \in \mathbb{N}$ and $r \in R^{(n+1)}$ there exists an $e \in A_\#$ such that for all $a_1, \dots, a_n, b \in A$,*
$$e\cdot a_1\cdot \ldots \cdot a_n\downarrow \qquad \text{and} \qquad r(a_1, \dots, a_n, b) \succeq e\cdot a_1\cdot \ldots \cdot a_n\cdot b.$$

(iii) *$(A, A_\#, \cdot)$ is a relative PCA, and the induced relationally complete DCO $(A, \downarrow\mathcal{F}_{A_\#})$ from Proposition 2.6 is equal to (A, R).*

Proof. This was shown in [Fre19, Lemma 2.12] for the non-relative case, and with the modified definitions and statements given above, the generalzation of the proof to the relative case is straightforward.

Theorem 2.8 *The following are equivalent for a tripos $\mathcal{P}$.*

(i) *$\mathcal{P}$ is equivalent to a relative realizability tripos $\mathsf{rt}(A, A_\#, \cdot)$.*

(ii) *$\mathcal{P}$ has enough $\exists$-prime predicates, and $\mathsf{prim}(\mathcal{P})$ has finite meets and a discrete generic predicate.*

Proof. Assume first that $\mathcal{P}$ is equivalent to a relative realizability tripos $\mathsf{rt}(A, A_\#, \cdot)$. By Proposition 2.6 we have $\mathsf{rt}(A, A_\#, \cdot) = \mathsf{fam}(D(A, \downarrow\mathcal{F}_{A_\#}))$ for every relative combinatory algebra $(A, A_\#, \cdot)$, thus Proposition 1.19 implies that $\mathsf{rt}(A, A_\#, \cdot)$ has enough $\exists$-primes and $\mathsf{prim}(\mathsf{rt}(A, A_\#, \cdot)) \simeq \mathsf{fam}(A, \downarrow\mathcal{F}_{A_\#})$. The latter is an indexed meet-semilattice by Proposition 2.6, and has a discrete generic predicate by Proposition 1.32.

Conversely, assume (ii). Then $\mathsf{prim}(\mathcal{P}) \hookrightarrow \mathcal{P}$ is an $\exists$-completion by Prop 1.15, and $\mathsf{prim}(\mathcal{P})$ is representable by a relative DCO (A, R) by Corollary 1.33. The DCO (A, R) is cartesian since $\mathsf{prim}(\mathcal{P})$ has finite meets, and relationally complete since its $\exists$-completion is a tripos. Thus, it comes from a relative PCA by Proposition 2.7. $\square$

References

[Cut80] N. Cutland. *Computability.* Cambridge University Press, Cambridge-New York, 1980. An introduction to recursive function theory.

[Fef75] S. Feferman. A language and axioms for explicit mathematics. In *Algebra and logic (Fourteenth Summer Res. Inst., Austral. Math. Soc., Monash Univ., Clayton, 1974)*, Lecture Notes in Math., Vol. 450, pages 87–139. Springer, Berlin-New York, 1975.

[Fre13] J. Frey. *A fibrational study of realizability toposes.* PhD thesis, Paris 7 University, 2013.

[Fre19] J. Frey. Characterizing partitioned assemblies and realizability toposes. *Journal of Pure and Applied Algebra*, 223(5):2000–2014, 2019.

[Fre22] J. Frey. Categories of partial equivalence relations as localizations. *Journal of Pure and Applied Algebra*, 2022. To appear.

[FvdB22] D. Figueroa and B. van den Berg. A topos for continuous logic. *Theory and Applications of Categories*, 38:1108–1135, 2022.

[FvO16] E. Faber and J. van Oosten. Effective operations of type 2 in pcas. *Computability*, 5(2):127–146, 2016.

[HJP80] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. Tripos theory. *Math. Proc. Cambridge Philos. Soc.*, 88(2):205–232, 1980.

[Hof06] P.J.W. Hofstra. All realizability is relative. *Math. Proc. Cambridge Philos. Soc.*, 141(02):239–264, 2006.

[Hof11] Pieter J. W. Hofstra. The Dialectica monad and its cousins. In *Models, logics, and higher-dimensional categories: A tribute to the work of Mihály Makkai. Proceedings of a conference, CRM, Montréal, Canada, June 18–20, 2009*, pages 107–137. Providence, RI: American Mathematical Society (AMS), 2011.

[Joh02] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 2*, volume 44 of *Oxford Logic Guides.* Oxford University Press, Oxford, 2002.

[Koc95] A. Kock. Monads for which structures are adjoint to units. *Journal of Pure and Applied Algebra*, 104(1):41–59, 1995.

[Pra65] D. Prawitz. *Natural deduction. A proof-theoretical study*, volume 3 of *Acta Universitatis Stockholmiensis. Stockholm Studies in Philosophy.* Almqvist & Wiksell, Stockholm, 1965.

[Tro20] D. Trotta. The existential completion. *Theory and Applications of Categories*, 35:Paper No. 43, 1576–1607, 2020.

[vO08] J. van Oosten. *Realizability: An Introduction to its Categorical Side.* Elsevier Science Ltd, 2008.