# Coproducts in $\infty$ -LCCCs with subobject classifier

HoTT-UF 2021

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July 17, 2021

#### Context

Lawvere and Tierney explicitly required finite colimits in the original definition of elementary topos.

F.W. Lawvere. "Quantifiers and sheaves". In: Actes du congres international des mathematiciens, Nice. Vol. 1. 1970

By a topos we mean a category  $\underline{\mathbb{E}}$  which has finite limits and finite colimits, which is (a) cartesian closed and which (b) has a subobject classifier T.

M. Tierney. "Sheaf theory and the continuum hypothesis". In: *Toposes, algebraic geometry and logic*. Springer, 1972

Axiom 1. All finite limits and colimits exist.

[...]

Axiom 2. S is cartesian closed.

[...]

Axiom 3. Subobjects in S are representable.

#### Context

#### The colimit axiom was soon found to be redundant:

- 1. C.J. Mikkelsen. "Finite colimits in toposes". In: Talk at the conference on category theory at Oberwolfach. 1972
- 2. R. Paré. "Colimits in topoi". In: *Bulletin of the American Mathematical Society* 80.3 (1974)
- 3. C.J. Mikkelsen. Lattice theoretic and logical aspects of elementary topoi. 25. Aarhus Universitet, Matematisk Institut, 1976

## *Monadicity*

Paré's proof proceeds by showing that for every topos  ${\mathcal E},$  the autoadjunction

$$\Omega^{(-)} \dashv \Omega^{(-)} : \mathcal{E}^{\mathsf{op}} \to \mathcal{E}$$

is monadic.

Then  $\mathcal{E}^{op}$  – as a category of algebras over a category with finite limits – has finite limits itself.

There's also an 'internal-language' proof – I don't know if it's Mikkelsen's:

Initial object and binary coproducts are given by

$$0 = \{x \in 1 \mid \bot\}$$

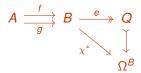
$$A + B = \{(U, V) \in \Omega^A \times \Omega^B \mid (\#U = 0 \land \#V = 1) \land (\#U = 1 \land \#V = 0)\}.$$

• To get a coequalizer of  $f, g : A \to B$  let  $U \mapsto B \times B$  be the image of  $\langle f, g \rangle$ .



Let  $R \rightarrow B \times B$  the 'least equivalence relation containing U, and let  $\chi : B \times B \to \Omega$  be its characteristic map.

The coequalizer e is then the image of the exponential transpose of  $\chi$ .



## 1st order logic in toposes

To make sense of these encodings, need 1st order connectives.

These all be encoded in terms of *equality*, *conjunction*, and *subset* abstraction alone<sup>1</sup>:

$$p \Rightarrow q \equiv (p \land q) = p$$

$$\forall x : A \cdot p[x] \equiv \{x \mid p[x]\} = \{x \mid \top\}$$

$$\perp \equiv \forall z : \Omega \cdot z$$

$$p \lor q \equiv \forall z : \Omega \cdot (p \Rightarrow z) \land (q \Rightarrow z) \Rightarrow z$$

$$\exists x : A \cdot p[x] \equiv \forall z : \Omega \cdot (\forall x : A \cdot p[x] \Rightarrow z) \Rightarrow z$$

<sup>&</sup>lt;sup>1</sup>A. Boileau and A. Joyal. "La logique des topos". In: *The Journal of Symbolic Logic* 46.1 (1981).

## *Elementary* **∞**-*toposes*

In 2017, Shulman<sup>2</sup> proposed a definition of elementary ∞-topos.

#### Definition

An elementary  $\infty$ -topos is a locally cartesian closed  $\infty$ -category  $\mathcal{E}$  with finite colimits, a subobject classifier  $\Omega$ , and enough universes.

Is the requirement of finite colimits redundant again?

We give a partial answer:

Theorem (F, Rasekh)

∴LCCCs with SOC have disjoint finite coproducts.

In the following I'll discuss the construction and proof, comparing it to the proof for 1-toposes.

<sup>&</sup>lt;sup>2</sup>M. Shulman. *Elementary* (∞, 1)-*topoi*. 2017. URL: https://golem.ph.utexas.edu/category/2017/04/elementary\_1topoi.html.

## *Logic of subobjects in* $\infty$ -*LCCCs*

Let  $\mathcal{E}$  be an locally cartesian closed  $\infty$ -category.

#### Definition

Given  $A \in \mathcal{E}$ , let sub(A) be the full subcategory of  $\mathcal{E}/A$  on *embeddings*, i.e. (-1)-truncated maps.

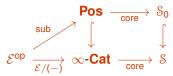
- sub(A) is a poset in the sense that all homs are proposions.
- Local cartesian closure of ε implies that sub(A) is cartesian closed, i.e. a Heyting algebra.
- Pullback and pushforward  $f^* \dashv \Pi_f$  along arbitrary  $f : B \to A$  restrict to subobjects.

# The subobject classifier

With the pullback action,  $A \mapsto \text{sub}(A)$  is contravariantly functorial.

$$\mathsf{sub}: \mathcal{E}^\mathsf{op} \to \mathbf{Pos}$$

A **subobject classifier** is by definition an object  $\Omega$  representing the presheaf core  $\circ$  sub of 0-types.



In particular  $\Omega$  is itself a 0-type.

The representability condition means that there's a **generic subobject** (tt:  $U \rightarrow \Omega$ ) such that for all objects A, the induced map

$$hom(A, \Omega) \rightarrow core(sub(A)), \qquad f \mapsto f^*(tt)$$

is an equivalence.

As in the 1-topos case, the domain U of the generic element is terminal, since it classifies maximal subobjects.

# Joins of subobjects

#### Lemma

Let  $A \in \mathcal{E}$  be an object in an  $\infty$ -LCCC with SOC. Then  $\operatorname{sub}(A)$  has finite joins.

#### Proof.

The smallest subobject is given by

$$\Pi_{(A\times\Omega\to A)}(A\times tt)$$

and the join of  $m: U \rightarrow A$  and  $n: V \rightarrow A$  is given by

$$\Pi_{(A \times \Omega \to A)} \left[ \left( m \times \Omega \Rightarrow A \times tt \right) \Rightarrow \left( n \times \Omega \Rightarrow A \times tt \right) \Rightarrow A \times tt \right],$$

by the same argument as in 1-toposes.

#### Remark

Image factorization, i.e. existential quantification also works by the same encoding as in 1-toposes. But we won't need that today.

# The initial object

#### Lemma

TFAE for an object J of an  $\infty$ -LCCC  $\mathcal{E}$ .

(1) 
$$sub(J) \simeq 1$$
 (2)  $\mathcal{E}/J \simeq 1$  (3)  $J$  is initial

(2) 
$$\mathcal{E}/J \simeq 1$$

#### Proof.

- $3 \Rightarrow 1$ : Initiality / implies that all subobjects of / have sections.
- 1 ⇒ 2: An object A of an  $\infty$ -LCCC is contractible iff the **proposition**

$$\mathsf{isContr}(A) \ \equiv \ \Sigma_{(A \to 1)} \Pi_{(\pi:A \times A \to A)} (A \xrightarrow{\delta} A \times A)$$

has a section.

 $2 \Rightarrow 3$ : Given  $A \in \mathcal{E}$ , it's enough to show that  $A^J = \prod_J (A \times J \to J)$  is contractible. But  $(A \times J \rightarrow J)$  is so by assumption and  $\Pi$  preserves contractibility as a right adjoint.

#### Corollary

Any  $\infty$ -LCCC with SOC has an initial object. (Take least subobject of 1.)

# Analyzing the coproduct construction in 1-toposes The construction of binary coproducts in 1-toposes

$$A + B \equiv \{(U, V) \in \Omega^A \times \Omega^B \mid (\#U = 0 \land \#V = 1) \lor (\#U = 1 \land \#V = 0)\}$$

can be understood as follows. Every A admits a **singleton embedding** 

$$\{-\}: A \rightarrowtail \Omega^A, \qquad a \mapsto \{a\} = \{b:A \mid b=a\}$$

into  $\Omega^A$ , whose image is disjoint from  $\emptyset = \{x : | \bot \}$ .

$$\begin{array}{ccc}
0 & \longrightarrow & 1 \\
\downarrow & & \downarrow \varnothing \\
A & \stackrel{\{-\}}{\longrightarrow} & \Omega^A
\end{array}$$

Given a second object B, we form the product of pullback squares

$$\begin{pmatrix} 0 & \longrightarrow & 1 \\ \downarrow & & \downarrow & \downarrow \\ A & \longrightarrow & \Omega^A \end{pmatrix} \times \begin{pmatrix} 0 & \longrightarrow & B \\ \downarrow & & \downarrow & \downarrow \\ 1 & \longrightarrow & \Omega^B \end{pmatrix} = \begin{pmatrix} 0 & \longleftarrow & B \\ \downarrow & & \downarrow & \downarrow \\ A & \rightarrowtail & \Omega^A \times \Omega^B \end{pmatrix}$$

to exhibit disjoint embeddings of A and B into  $\Omega^A \times \Omega^B$ .

We then get a coproduct by forming the join of A and B in  $sub(\Omega^A \times \Omega^B)$ .

This can only work if all objects are 0-truncated since  $\Omega^A$  always is.

## Interlude: Sums and product along inclusions

$$\begin{array}{ccc}
B & \mathcal{E}/B \\
\downarrow^{m} & \Sigma_{m} \left( \begin{matrix} \uparrow \\ - \mid m^{*} \end{matrix} \right) \Pi_{m} \\
A & \mathcal{E}/A
\end{array}$$

#### Lemma

If  $m:A \rightarrow B$  is an embedding in an  $\infty$ -LCCC  $\mathcal{E}$  then  $\Sigma_m$  and  $\Pi_m$  are fully faithful.

#### Proof.

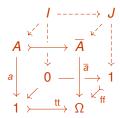
Given  $h, k \in \mathcal{E}/B$ , it is immediate that  $\hom_{\mathcal{E}/B}(h, k) \simeq \hom_{\mathcal{E}/A(fh, fk)}$ . The claim about  $\Pi$  follows by calculus of mates.

In particular we have  $m^* \circ m_* \simeq \operatorname{id}$  in this case, i.e. every  $h: X \to A$  is a pullback of its own pushforward.

$$\begin{array}{ccc}
X & \longrightarrow & \overline{X} \\
h \downarrow & & & \downarrow \Pi_m h \\
A & \stackrel{h}{\longrightarrow} & B
\end{array}$$

## Coproducts in $\infty$ -LCCCs with SOC

Given an object A in an  $\infty$ -LCCC with  $\Omega$ , consider the following diagram.



- Bottom is classifying square for the least subobj. 0 → 1
- $\overline{a} = \prod_{t \neq a}$ , so that  $a = tt^* \overline{a}$  ( $\overline{A}$  is known as partial map representer)
- remainder constitutes pullback in arrow category, in particular right and left side squares – and therefore all squares are pullbacks
- / is initial by strictness, / is terminal by Beck-Chevalley
- again we have embedded A into a 'larger' object with a disjointly embedded point.

 $\downarrow \qquad \downarrow \\
A & \longmapsto \overline{A}$ 

Given two object A, B we can again form the transposed product of pullbacks

$$\left(\begin{array}{cc} 0 \rightarrowtail 1 \\ \updownarrow & \updownarrow \\ A \rightarrowtail \overline{A} \end{array}\right) \times \left(\begin{array}{cc} 0 \rightarrowtail B \\ \updownarrow & \updownarrow \\ 1 \rightarrowtail \overline{B} \end{array}\right) = \left(\begin{array}{cc} 0 \rightarrowtail B \\ \updownarrow & \updownarrow \\ A \rightarrowtail \overline{A} \times \overline{B} \end{array}\right)$$

to obtain an object that disjointly embeds A and BForming the join of A and B in  $\overline{A} \times \overline{B}$  yields a cospan

$$A \stackrel{i}{\hookrightarrow} C \stackrel{j}{\hookleftarrow} B$$

such that  $A \wedge B = \bot$  and  $A \vee B = \top$  in sub(C). It remains to show:

#### Lemma

Let  $A \stackrel{i}{\hookrightarrow} C \stackrel{j}{\hookleftarrow} B$  be embeddings in an  $\infty$ -LCCC  $\mathcal{E}$ , such that  $A \wedge B = \bot$  and  $A \vee B = \top$  in sub(C). Then i and j exhibit C as a coproduct of A and B.

#### Proof.

Since  $\Sigma_C : \mathcal{E}/C \to \mathcal{E}$  preserves colimits, we may wlog assume C = 1. For  $X \in \mathcal{E}$  and  $f : A \to X$ ,  $g : B \to X$ , have to show that the pullback of

$$1 \xrightarrow{\langle f,g \rangle} \mathcal{E}(A,X) \times \mathcal{E}(B,X) \leftarrow \mathcal{E}(1,X)$$

is contractible in spaces. This is the image of

$$1 \xrightarrow{\langle f,g \rangle} X^A \times X^B \xleftarrow{\langle X^i,X^j \rangle} X$$

under  $\mathcal{E}(1,-)$  so it suffices to show that the pullback of the latter is terminal in  $\mathcal{E}$ . This pullback can be written type theoretically as

$$\Sigma x \cdot (\forall a \cdot f \ a = x) \times (\forall b \cdot g \ b = x)$$

so it's sufficient to show

$$\vdash$$
 isContr( $\Sigma x . (\forall a . f a = x) \times (\forall b . g b = x)$ ).

... continued on next slide ...

#### Proof ct'd

Since  $A \lor B = \top$  in sub(1) it's sufficient to show the judgments

$$a_0$$
: $A \vdash \mathsf{isContr}(\Sigma x . (\forall a . f \ a = x) \times (\forall b . g \ b = x))$   
 $b_0$ : $B \vdash \mathsf{isContr}(\Sigma x . (\forall a . f \ a = x) \times (\forall b . g \ b = x)).$ 

We show the first one.

It's easy to see that in context  $a_0$ : A, the first factor is equivalent to  $f a_0 = x$  and the second is equivalent to T. Thus, the claim reduces to

$$a_0$$
:  $A \vdash \mathsf{isContr}(\Sigma x \, . \, f \, a_0 = x)$ 

Thanks for your attention!