A Fibrational Analysis of Realizability Toposes

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Introduction Constructive Mathematics, Realizability and Toposes

Introduction

Constructive logic and mathematics

- Appeared in the beginning of the 20th century
- Response to unease about abstract methods and inconsistencies
- Asserts that it is necessary to find (or "construct") a mathematical object to prove that it exists.
- Major difference to classical logic/mathematics: rejection of the principle of the excluded middle



Introduction

Brouwer-Heyting-Kolmogorov interpretation

- What is a mathematical proof?
- What counts as a constructive proof?

The Brouwer–Heyting–Kolmogorov (BHK) interpretation states:

- ② A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.
- ② A proof of $A \lor B$ is given by presenting either a proof of A or a proof of B.
- A proof of A ⇒ B is a construction which transforms any proof of A into a proof of B.
- Does not validate excluded middle A ∨ ¬A

Formalization of the BHK interpretation

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n \Vdash s = t \qquad \text{iff} \qquad s = t \, (s, \, t \, \text{are closed terms})
n \Vdash \varphi \wedge \psi \qquad \text{iff} \qquad (p_0(n) \Vdash \varphi) \wedge (p_1(n) \Vdash \psi)
n \Vdash \varphi \Rightarrow \psi \qquad \text{iff} \qquad \forall m . (m \Vdash \varphi) \Rightarrow (\theta_n(m) \Vdash \psi)
n \Vdash \bot \qquad \text{never}
n \Vdash \varphi \vee \psi \qquad \text{iff} \qquad (p_0(n) = 0 \wedge (p_1(n) \Vdash \varphi)) \vee (p_0(n) = 1 \wedge (p_1(n) \Vdash \psi))
n \vdash \forall x . \varphi(x) \qquad \text{iff} \qquad \forall m \in \mathbb{N} . \theta_n(m) \vdash \varphi(m)
n \vdash \exists x . \varphi(x) \qquad \text{iff} \qquad p_1(n) \vdash \varphi(p_0(n))
(\theta_n)_{n \in \mathbb{N}} \text{ G\"{o}del numbering of partial recursive functions}
```

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Toposes – Historical overview

Algebraic geometry	Constructive mathematics
Grothendieck 1960	
Giraud theorem 1963	
	Lawvere 1970
Bunge 77 Characterizes presheaf toposes	
	Hyland, Johnstone, Pitts 1980 Realizability toposes

Introduction

Motivation

- Theory of Grothendieck toposes highly developed
- In contrast, Peter Johnstone famously compared the study of realizability toposes to 'stamp collecting', writing

... we have lots of exotic and colourful examples, which we can proudly display in our albums, but what we lack is a general theory which would indicate just where the boundaries of the subject lie.

Motivation of my work: 'Giraud theorem for realizability'

Overview

Overview

Main result

- Characterization of realizability toposes RT(A) over partial combinatory algebras A
- Inspired by Giraud's theorem, but really analogous to Bunge's result

Sketch of proof

- Identify A with a certain posetal fibration
- ② Identify RT(A) with a certain categorical fibration
- Oharacterize the second fibration as a cocompletion of the first

Overview

Outline

- Posetal fibrations
- ② Uniform preorders
- Moens' theorem
- The fibered presheaf construction
- The characterization
- Further results, future work

Part I – Posetal fibrations

Outline

- Posetal fibrations
- ② Uniform preorders
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Truth value semantics vs posetal fibrations

	truth values	predicates on set S
Classical model	true, false	subsets of S
Model with values in Heyting algebra A	elements of A	functions $\varphi: \mathcal{S} \to \mathcal{A}$
Model in posetal fibration ${\mathfrak P}$	elements of $\mathcal{P}(1)$	elements of $\mathcal{P}(S)$

Posetal fibrations

Definition

An posetal fibration is a functor

$$\mathfrak{P}: \textbf{Set}^{\text{op}} \to \textbf{Ord}$$

- For each set J, $\mathcal{P}(J)$ is the **order of predicates on** J
- For $f: K \to J$, the map

$$f^*: \mathcal{P}(J) \to \mathcal{P}(K)$$

is called reindexing map and models substitution

Remark

The same data can be encoded in a functor

$$P: \int \mathcal{P} \to \mathbf{Set}$$

Interpreting the logical connectives

- Bare posetal fibrations : not enough structure to interpret connectives
- Propositional logic : Heyting algebras as fibers
- Quantifiers and equality: adjoints to reindexing maps

Fragments of predicate logic

Definition

- Existential fibration:
 posetal fibration interpreting regular logic {∃,=, ∧, ⊤}
- Hyperdoctrine:
 interprets first order constructive logic {∃, =, ∧, ⊤, ∨, ⊥, ⇒, ∀}

Definition (Generic predicate)

A generic predicate is a predicate

$$tr \in \mathcal{P}(\mathsf{Prop})$$

on a set Prop such that for any other set J and predicate $\varphi \in \mathcal{P}(J)$, there exists a function $f: J \to \text{Prop}$ with

$$\varphi \cong f^*$$
tr.

Definition

A **tripos** is a hyperdoctrine with a generic predicate.

Kleene realizability as a tripos

Definition

The effective tripos eff : $Set^{op} \rightarrow Ord$

- **Predicates** on J: functions $\varphi: J \to P(\mathbb{N})$
- Ordering: $\varphi \leq \psi$ iff there exists a partial recursive f such that

```
\forall j \in J \ \forall n \in \varphi(j). f(n) is defined and f(n) \in \psi(j).
```

PCAs and realizability triposes

The construction of the effective tripos can be generalized:

Definition

Partial combinatory algebra (PCA): set \mathcal{A} with a partial binary operation $(-\cdot-): \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ and $k, s \in \mathcal{A}$ such that

- $\forall x, y . k \cdot x \cdot y = y$
- $\forall x, y . s \cdot x \cdot y \downarrow$
- $\forall x, y, z . x \cdot z \cdot (y \cdot z) \downarrow \Rightarrow s \cdot x \cdot y \cdot z = x \cdot z \cdot (y \cdot z)$

Definition (Realizability tripos)

The realizability tripos rt(A): Set^{op} \rightarrow Ord for a PCA A:

- **Predicates** on the set J: functions $\varphi: J \to PA$
- Ordering $\varphi \leq \psi$ iff there exists $e \in A$

 $\forall j: J, a \in \varphi(j)$. $e \cdot a$ is defined and $e \cdot a \in \psi(j)$

The tripos-to-topos construction

Definition (The category of partial equivalence relations)

For an tripos \mathcal{P} , the category **Set**[\mathcal{P}] is defined as follows.

• **Objects**: pairs (C, ρ) where C is a set and $\rho \in \mathcal{P}(C \times C)$ such that

$$\begin{array}{c} (\text{trans}) & \rho(x,y), \rho(y,z) \vdash \rho(x,z) \\ (\text{sym}) & \rho(x,y) \vdash \rho(y,x) \end{array}$$

• Morphisms from (C, ρ) to (D, σ) : predicates $\phi \in \mathcal{P}(C \times D)$ satisfying

$$\begin{array}{ll} \text{(strict)} & \phi(x,y) \vdash \rho(x,x) \land \sigma(y,y) \\ \text{(cong)} & \rho(x',x), \phi(x,y), \sigma(y,y') \vdash \rho(x',y') \\ \text{(singval)} & \phi(x,y), \phi(x,y') \vdash \sigma(y,y') \\ \text{(tot)} & \rho(x,x) \vdash \exists y \, . \, \phi(x,y) \\ \end{array}$$

The tripos-to-topos construction

Theorem (Hyland, Johnstone, Pitts 1981)

For any tripos \mathcal{P} , $\mathbf{Set}[\mathcal{P}]$ is a topos.

Definition

- RT(A) = Set[rt(A)] is called the realizability topos over A
- *Eff* = **Set**[**eff**] is the **effective topos**

 $\mathbf{Set}[\mathfrak{F}]$ for existential fibrations

Observation

- \bullet The category $\textbf{Set}[\mathfrak{F}]$ is definable for any existential fibration \mathfrak{F}
- In this case, Set[𝔻] is an exact category

Part II – Uniform preorders

Outline

- Posetal fibrations
- 2 Uniform preorders
- Moens' theorem
- The fibered presheaf construction
- The characterization
- Further results, future work

Uniform preorders

- Idea: representations of posetal fibrations
- Generalization of Hofstra's basic combinatory objects
- Related to Longley's computability structures

Definition

A uniform preorder is a pair

of a set A and a set

$$R \subseteq P(A \times A)$$

of binary relations, subject to the following axioms.

- $oldsymbol{0}$ $r \in R$, $s \subseteq r \implies s \in R$
- \bigcirc id $\in R$

The elements of *R* are called **primitive entailments**

Uniform preorders as posetal fibrations

Definition

The **posetal fibration** fam(A, R) associated to a uniform preorder (A, R) has functions $\varphi : J \to A$ as predicates; the ordering relation is defined by

$$\varphi \le \psi$$
 iff $\{(\varphi(j), \psi(j)) \mid j:J\} \in R$ for $\varphi, \psi: J \to A$.

Observation

The fibration fam(A, R) has a generic predicate, given by Prop = A and $id_A : A \rightarrow A$.

Representation Lemma

An posetal fibration is representable by a uniform preorder iff it has a generic predicate.

Remark

The proof of the lemma depends on AC, without choice uniform preorders correspond to posetal **pre-stacks** with generic predicate.

PCAs as uniform preorders

To each pca \mathcal{A} , we associate the uniform preorder $(\mathcal{A}, \mathcal{R}_{\mathcal{A}})$, where

$$R_{\mathcal{A}} = \{ r \subset \mathcal{A} \times \mathcal{A} \mid \exists e : \mathcal{A} \ \forall (a, b) \in r . e \cdot a = b \}$$

is the set of 'sub-computable' partial functions.

The 2-category **UOrd**

Definition

A monotone map between uniform preorders (A, R), (B, S) is a function
 f: A → B such that

$$r \in R$$
 implies $\{(fa, fa') \mid (a, a') \in r\} \in S$.

• For monotone maps $f, g: (A, R) \rightarrow (B, S)$, define

$$f \leq g$$
 iff $\{(fa, ga) \mid a \in A\} \in S$.

This defines a locally ordered 2-category UOrd.

Representations of posetal fibrations

Tower of full and order-reflecting inclusions of locally ordered 2-categories:

$$\textbf{Ord} \hookrightarrow \textbf{BCO} \hookrightarrow \textbf{UOrd} \hookrightarrow \textbf{PFib}$$

• Ord : preorders

BCO: Hofstra's basic combinatory objects

• **UOrd** : uniform preorders

• PFib : posetal fibrations

UOrd is a 2-category and has finite products. Therefore, we can define

Definition (Following Hofstra 2006)

A uniform preorder (A, R) is called **finitely complete**, if the maps

$$\delta: (A, R) \rightarrow (A, R) \times (A, R)$$

and

 $!:(A,R)\rightarrow 1$

have right adjoints

$$\wedge: (A,R) \times (A,R) \rightarrow (A,R)$$

and
$$T: 1 \rightarrow (A, R)$$
.

Observation

The uniform preorder (A, R_A) is finitely complete for any PCA A.

Existential quantification

- The set dcl(A) of downsets is the join-cocompletion of a preorder A
- An analogous construction freely adds \exists to a uniform preorder (A, R)

Definition (Following Hofstra, van Oosten)

The uniform preorder D(A, R) is defined as follows:

- Underlying set: PA
- Primitive entailments: subrelations of relations [r] ⊆ PA × PA for r ∈ R, defined by

$$[r](M,N):\Leftrightarrow \forall m\; \exists n\,.\, r(m,n)$$

The mapping $(A, R) \mapsto D(A, R)$ is a lax idempotent monad on UOrd

Key observation

For a pca \mathcal{A} , we have $fam(D(\mathcal{A}, R_{\mathcal{A}})) = rt(\mathcal{A})$

Relational completeness

Let (A, R) be a finitely complete uniform preorder.

- fam(D(A, R)) is always an existential fibration.
- When is fam(D(A, R)) a tripos?

Definition

A finitely complete uniform preorder (A, R) is called **relationally complete**, if there exists $0 \in R$ such that for all $r \in R$ there exists $\tilde{r} \in R$ such that

$$\forall a \in A \ \exists h \in A . \tilde{r}(a,h) \land r(a \land -,-) \subseteq @(h \land -,-)$$

Theorem (Following Hofstra 2006)

fam(D(A, R)) is a tripos iff (A, R) is relationally complete.

Towards a characterization of PCAs

Definition

A uniform preorder (A, R) is called **functional**, if all elements of R are functional relations.

Observation

For any PCA \mathcal{A} , $(\mathcal{A}, \mathcal{R}_{\mathcal{A}})$ is functional

Definition (Pitts 81)

Let (A, R) be a finitely complete uniform preorder.

A designated truth value is an element $a \in A$ such that $\{(\top, a)\} \in R$.

Observation

In uniform preorders (A, R_A) , all elements are designated truth values.

PCAs

Theorem

PCAs can be identified with finitely complete uniform preorders which are

- relationally complete,
- functional, and
- all elements are designated truth values

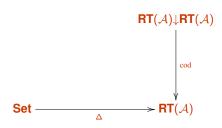
Part III – Moens' theorem

Outline

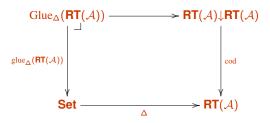
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$$\mathbf{Set} \xrightarrow{\quad \quad } \mathbf{RT}(\mathcal{A})$$

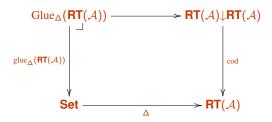
• Constant objects functor $\Delta: J \mapsto (J, =)$ $[= \in \mathsf{rt}(\mathcal{A})(J \times J)]$



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- Constant objects functor $\Delta: J \mapsto (J, =)$ $[= \in \mathsf{rt}(\mathcal{A})(J \times J)]$
- Δ , $\mathsf{RT}(\mathcal{A})$ can be recovered from $\mathsf{glue}_{\Delta}(\mathsf{RT}(\mathcal{A})$

Moens' theorem for fibered pretoposes

Variant: let \mathbb{R} be a regular category.

Moens' theorem for fibered pretoposes

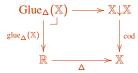
There is a biequivalence

$$\mathbb{R} \Downarrow \mathsf{Ex} \simeq \mathsf{Pretop}(\mathbb{R})$$

between

- regular functors into exact categories
- pre-stacks of exact categories with extensive internal sums

We call such fibrations fibered pretoposes.



Part V – The fibered presheaf construction

Outline

- Posetal fibrations
- ② Uniform preorders
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The fibered presheaf construction

Presheaf categories

- C small finite limit category
- · Characterized by equivalence

$$\mathsf{Lex}(\mathbb{C},\mathcal{X}) \simeq \mathsf{Geo}(\widehat{\mathbb{C}},\mathcal{X})$$
 $(\mathcal{X} \infty\text{-pretopos})$

of functor categories

Fibrations of presheaves

- $\mathscr{C}: \int \mathscr{C} \to \mathbb{R}$ finite-limit pre-stack
- Fibration of presheaves $\widehat{\mathscr{C}}: \int \widehat{\mathscr{C}} \to \mathbb{R}$ is a fibered pretopos
- · Characterized by equivalence

$$\mathsf{Lex}(\mathbb{R})(\mathscr{C},\mathscr{X}) \simeq \mathsf{Geo}(\mathbb{R})(\widehat{\mathscr{C}},\mathscr{X}) \qquad \qquad (\mathscr{X} \text{ fibered pretopos})$$

of fibered functor categories

Characterization of fibrations of presheaves

Which fibered pretoposes $\mathscr{X}:\int\mathscr{X}\to\mathbb{R}$ are of the form $\mathscr{X}\simeq\widehat{\mathscr{C}}$?

Theorem (Bunge 77)

A Grothendieck topos \mathcal{E} is a presheaf topos iff it has a generating family of **indecomposable projective** objects.

In a similar way, we can show:

Theorem

A fibered pretopos $\mathscr{X}: |\mathscr{X}| \to \mathbb{R}$ is a fibration of presheaves iff

- the subfibration of X on indecomposable projectives is closed under finite limits, and
- Every X ∈ |X| can be covered by an internal sum of indecomposable projectives.

Realizability toposes as fibrations of presheaves

Using this, we can show:

Theorem

For any PCA \mathcal{A} , we have $\operatorname{glue}_{\Delta}(\operatorname{RT}(\mathcal{A})) \simeq \operatorname{fam}(\mathcal{A}, \widehat{R}_{\mathcal{A}})$.

Part V – The characterization

Outline

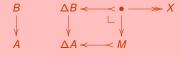
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Main result

Therem

Realizability toposes RT(A) can be characterized as categories X such that

- X is exact and locally cartesian closed
- $\Gamma = \mathbb{X}(1, -) : \mathbb{X} \to \mathbf{Set}$ has a regular right adjoint $\Delta : \mathbf{Set} \to \mathbb{X}$
- There exists a monomorphism $\phi: M \rightarrow \Delta A$ such that
 - \circ ϕ is indecomposable and projective in $\operatorname{glue}_{\Delta}(\mathbb{X})$
 - M is discrete with respect to Δ
 - The subfibration of $\operatorname{glue}_{\Delta}(\mathbb{X})$ generated by ϕ is closed under finite meets
 - Every $X \in \mathbb{X}$ can be covered like

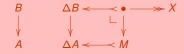


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Discreteness?

- $X \in \mathbb{X}$ is **discrete**, if it is right orthogonal to all constant objects $\Delta(J)$
- Used to assure that the reconstructed uniform preorder is functional

Part VI – Further results, future work

Outline

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Further results

- Characterization of realizability categories over typed pcas
- Charaterization of fibered **relative** realizability categories
- Framework of fibrational cocompletions for pre-stacks on regular base categories
- Characterization of pre-stacks A of meet-semilattices such that is fiberwise locally cartesian closed
- Uniform preorders on arbirary bases
- Factorization theorem for constant objects functors

Directions for further research

- Many sorted uniform preorders as preorders internal to small sheaves on Set for the regular topology
- · 'Geometric theory of realizability toposes'
- Iteration (in the sense of Pitts, analogous to composition of bounded geometric morphisms)
- Unrelated: better understanding of relationally complete uniform preorders
- · 'relational clones'



Appendix

Indecomposables and projectives

Let $\mathcal{X}: |\mathcal{X}| \to \mathbb{R}$ be a fibered pretopos.

Definition

• Call $P \in |\mathcal{X}|$ projective, if given c, e, f as in the diagram



where c is cartesian and e is vertical and a regular epimorphism in its fiber, we can fill in d, g with d epicartesian such that the square commutes.

• Call $X \in |\mathcal{X}|$ indecomposable, if for every diagram

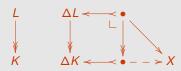


in $|\mathcal{X}|$ where c is cartesian and d is cocartesian, there exists a *unique* mediating arrow m.

Discreteness

Definition

Let $\Delta : \mathbb{R} \to \mathbb{X}$ be a regular functor into an exact category. Call $X \in \mathbb{X}$ discrete (with respect to Δ), if in any diagram of shape



there exists a mediating arrow.

$$R = \{ r \subseteq A \times A \mid \pi_I^* \iota \leq \pi_r^* \iota \text{ in } A_r \},$$

where $\pi_l, \pi_r : r \to A$ are the projections.

- Definition of uniform preorder can be internalized in any topos
- However, the above construction only works on set it gives a set of binary relations, not an object of binary relations on other base categories
- Solution: For base topos \mathcal{E} , fiber not over \mathcal{E} , but over $\mathcal{E} \downarrow \mathcal{E}$

Definition

Let (A, R) be a uniform preorder internal to \mathcal{E} . The posetal fibration $fam(A, R) : \mathcal{E} \downarrow \mathcal{E} \rightarrow \mathbf{Ord}$ is defined as follows:

- A predicate on $f: X \to I$ is a function $\varphi: X \to A$
- $\varphi < \psi$ iff

$$\forall i \{ (\varphi(x), \psi(x)) \mid x \in X_i \} \in R$$

holds in \mathcal{E} .

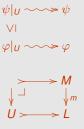
- Slogan: pointwise in I, uniform in the fibers Xi
- We call fam(A, R) a fibered fibration, since its base is the total category
 of another fibration (the codomain fibration of ε).

We can characterize the fibrations of the form fam(A, R) for internal (A, R).

Definition

Let $A: S \downarrow S$ be an posetal fibration.

• We say that \leq is **horizontally definable** in \mathcal{A} , if for every $\varphi, \psi \in \mathcal{A}_{M} \xrightarrow{m} L$ there exists a greatest subobject $m : U \mapsto L$ of L such that $\varphi|_{U} \leq \psi|_{U}$.



We call A a fibered posetal pre-stack, if e*φ ≤ e*ψ implies φ ≤ ψ for every vertical epimorphism e in S↓S.

Lemma

The locally ordered category UOrd(S) of uniform preorders internal to S is biequivalent to the locally ordered category of fibered posetal pre-stacks on S with horizontally definable \leq and a generic predicate ι over $1 \to 1$.