Coproducts in ∞-LCCCs with subobject classifier

HoTT-UF 2021

Jonas Frey, Nima Rasekh

July 17, 2021

Context

Lawvere and Tierney explicitly required finite colimits in the original definition of elementary topos.

F.W. Lawvere. “Quantifiers and sheaves”. In: *Actes du congres international des mathematiciens, Nice*. Vol. 1. 1970

> *By a topos we mean a category* $\underline{E}$ *which has finite limits and finite colimits, which is (a) cartesian closed and which (b) has a subobject classifier* T*.*

M. Tierney. “Sheaf theory and the continuum hypothesis”. In: *Toposes, algebraic geometry and logic*. Springer, 1972

> *Axiom 1. All finite limits and colimits exist.*
> *[...]*
> *Axiom 2.* $\underline{S}$ *is cartesian closed.*
> *[...]*
> *Axiom 3. Subobjects in* $\underline{S}$ *are representable.*

Context

The colimit axiom was soon found to be redundant:

1. C.J. Mikkelsen. “Finite colimits in toposes”. In: *Talk at the conference on category theory at Oberwolfach*. 1972
2. R. Paré. “Colimits in topoi”. In: *Bulletin of the American Mathematical Society* 80.3 (1974)
3. C.J. Mikkelsen. *Lattice theoretic and logical aspects of elementary topoi*. 25. Aarhus Universitet, Matematisk Institut, 1976

Monadicity

Paré's proof proceeds by showing that for every topos $\mathcal{E}$, the autoadjunction

$$\Omega^{(-)} \dashv \Omega^{(-)} : \mathcal{E}^{\mathsf{op}} \to \mathcal{E}$$

is *monadic*.

Then $\mathcal{E}^{\mathsf{op}}$ – as a category of algebras over a category with finite limits – has finite limits itself.

There's also an 'internal-language' proof – I don't know if it's Mikkelsen's:

- Initial object and binary coproducts are given by

$$\begin{aligned} 0 &= \{x \in 1 \mid \bot\} \\ A + B &= \{(U, V) \in \Omega^A \times \Omega^B \mid (\#U = 0 \wedge \#V = 1) \\ &\qquad \vee (\#U = 1 \wedge \#V = 0)\}. \end{aligned}$$

- To get a coequalizer of $f, g : A \to B$ let $U \rightarrowtail B \times B$ be the image of $\langle f, g\rangle$.

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[dr, "\langle f,g\rangle"'] & U \arrow[d, tail] \\ & B \times B \end{tikzcd}$$

Let $R \rightarrowtail B \times B$ the 'least equivalence relation containing U, and let $\chi : B \times B \to \Omega$ be its characteristic map.

The coequalizer e is then the image of the exponential transpose of χ.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, two heads, "e"] \arrow[dr, "\chi^*"'] & Q \arrow[d, tail] \\ & & \Omega^B \end{tikzcd}$$

1st order logic in toposes

To make sense of these encodings, need 1st order connectives.

These all be encoded in terms of *equality*, *conjunction*, and *subset abstraction* alone[1]:

$$
\begin{aligned}
p \Rightarrow q \quad &\equiv \quad (p \wedge q) = p \\
\forall x{:}A \,.\, p[x] \quad &\equiv \quad \{x \mid p[x]\} = \{x \mid \top\} \\
\bot \quad &\equiv \quad \forall z{:}\Omega \,.\, z \\
p \vee q \quad &\equiv \quad \forall z{:}\Omega \,.\, (p \Rightarrow z) \wedge (q \Rightarrow z) \Rightarrow z \\
\exists x{:}A \,.\, p[x] \quad &\equiv \quad \forall z{:}\Omega \,.\, (\forall x{:}A \,.\, p[x] \Rightarrow z) \Rightarrow z
\end{aligned}
$$

[1] A. Boileau and A. Joyal. “La logique des topos”. In: *The Journal of Symbolic Logic* 46.1 (1981).

Elementary ∞-toposes

In 2017, Shulman[2] proposed a definition of elementary ∞-topos.

Definition

An elementary ∞-topos is a locally cartesian closed ∞-category $\mathcal{E}$ with finite colimits, a subobject classifier Ω, and enough universes.

Is the requirement of finite colimits redundant again?

We give a partial answer:

Theorem (F, Rasekh)

∞-LCCCs with SOC have disjoint finite coproducts.

In the following I'll discuss the construction and proof, comparing it to the proof for 1-toposes.

[2] M. Shulman. *Elementary $(\infty, 1)$-topoi*. 2017. URL: https://golem.ph.utexas.edu/category/2017/04/elementary_1topoi.html.

Logic of subobjects in ∞-LCCCs

Let $\mathcal{E}$ be an locally cartesian closed ∞-category.

Definition

Given $A \in \mathcal{E}$, let $\mathsf{sub}(A)$ be the full subcategory of $\mathcal{E}/A$ on *embeddings*, i.e. (-1)-truncated maps.

- $\mathsf{sub}(A)$ is a *poset* in the sense that all homs are proposions.
- Local cartesian closure of $\mathcal{E}$ implies that $\mathsf{sub}(A)$ is cartesian closed, i.e. a *Heyting algebra*.
- Pullback and pushforward $f^* \dashv \Pi_f$ along arbitrary $f : B \to A$ restrict to subobjects.

$$
\begin{tikzcd}
B \arrow[d, "f"] \\
A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{sub}(B) \arrow[rr, hook] \arrow[d, bend left, "\forall_f"] & & \mathcal{E}/B \arrow[d, bend right, "\Sigma_f"'] \arrow[d, bend left, "\Pi_f"] \\
\mathsf{sub}(A) \arrow[rr, hook] \arrow[u, "f^*"] \arrow[u, phantom, shift left=3, "\dashv"] & & \mathcal{E}/A \arrow[u, "f^*" description] \arrow[u, phantom, shift right=3, "\dashv"] \arrow[u, phantom, shift left=3, "\dashv"]
\end{tikzcd}
$$

The subobject classifier

With the pullback action, $A \mapsto \mathrm{sub}(A)$ is contravariantly functorial.

$$\mathrm{sub} : \mathcal{E}^{\mathrm{op}} \to \mathbf{Pos}$$

A **subobject classifier** is by definition an object Ω representing the presheaf $\mathrm{core} \circ \mathrm{sub}$ of 0-types.

$$\begin{tikzcd} & \mathbf{Pos} \arrow[r, "\mathrm{core}"'] \arrow[d, hook] & \mathcal{S}_0 \arrow[d, hook] \\ \mathcal{E}^{\mathrm{op}} \arrow[ur, "\mathrm{sub}"] \arrow[r, "\mathcal{E}/(-)"'] & \infty\text{-}\mathbf{Cat} \arrow[r, "\mathrm{core}"'] & \mathcal{S} \end{tikzcd}$$

In particular Ω is itself a 0-type.

The representability condition means that there's a **generic subobject** $(\mathrm{tt} : U \rightarrowtail \Omega)$ such that for all objects A, the induced map

$$\hom(A, \Omega) \to \mathrm{core}(\mathrm{sub}(A)), \qquad f \mapsto f^*(\mathrm{tt})$$

is an equivalence.

As in the 1-topos case, the domain U of the generic element is terminal, since it classifies maximal subobjects.

Joins of subobjects

Lemma

Let $A \in \mathcal{E}$ be an object in an ∞-LCCC with SOC. Then $\mathrm{sub}(A)$ has finite joins.

Proof.

The smallest subobject is given by

$$\Pi_{(A\times\Omega\to A)}(A \times \mathrm{tt})$$

and the join of $m : U \rightarrowtail A$ and $n : V \rightarrowtail A$ is given by

$$\Pi_{(A\times\Omega\to A)}\left[\left(m\times\Omega \Rightarrow A\times\mathrm{tt}\right) \Rightarrow \left(n\times\Omega \Rightarrow A\times\mathrm{tt}\right) \Rightarrow A\times\mathrm{tt}\right],$$

by the same argument as in 1-toposes. □

Remark

Image factorization, i.e. existential quantification also works by the same encoding as in 1-toposes. But we won't need that today.

The initial object

Lemma

TFAE for an object J of an ∞-LCCC $\mathcal{E}$.

$$(1)\ \mathrm{sub}(J) \simeq 1 \qquad (2)\ \mathcal{E}/J \simeq 1 \qquad (3)\ J \textit{ is initial}$$

Proof.

$3 \Rightarrow 1$: Initiality I implies that all subobjects of I have sections.

$1 \Rightarrow 2$: An object A of an ∞-LCCC is contractible iff the **proposition**

$$\mathrm{isContr}(A) \equiv \Sigma_{(A \to 1)} \Pi_{(\pi : A \times A \to A)} (A \xrightarrow{\delta} A \times A)$$

has a section.

$2 \Rightarrow 3$: Given $A \in \mathcal{E}$, it's enough to show that $A^J = \Pi_J(A \times J \to J)$ is contractible. But $(A \times J \to J)$ is so by assumption and Π preserves contractibility as a right adjoint. □

Corollary

Any ∞-LCCC with SOC has an initial object. (Take least subobject of 1.)

Analyzing the coproduct construction in 1-toposes

The construction of binary coproducts in 1-toposes

$$A + B \;\equiv\; \{(U, V) \in \Omega^A \times \Omega^B \mid (\#U = 0 \wedge \#V = 1) \vee (\#U = 1 \wedge \#V = 0)\}$$

can be understood as follows. Every A admits a **singleton embedding**

$$\{-\} : A \rightarrowtail \Omega^A, \qquad a \mapsto \{a\} = \{b{:}A \mid b = a\}$$

into Ω^A, whose image is disjoint from $\varnothing = \{x{:} \mid \bot\}$.

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\varnothing"] \\
A \arrow[r, tail, "\{-\}"] & \Omega^A
\end{tikzcd}$$

Given a second object B, we form the product of pullback squares

$$\left(\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail] \\
A \arrow[r, tail] & \Omega^A
\end{tikzcd}\right) \times \left(\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail] \\
1 \arrow[r, tail] & \Omega^B
\end{tikzcd}\right) = \left(\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail] \\
A \arrow[r, tail] & \Omega^A \times \Omega^B
\end{tikzcd}\right)$$

to exhibit disjoint embeddings of A and B into $\Omega^A \times \Omega^B$.

We then get a coproduct by forming the join of A and B in $\mathrm{sub}(\Omega^A \times \Omega^B)$.

This can only work if all objects are 0-truncated since Ω^A always is.

Interlude: Sums and product along inclusions

$$\begin{tikzcd} B \arrow[d, tail, "m"] \\ A \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}/B \arrow[d, bend right=60, "\Sigma_m"'] \arrow[d, bend left=60, "\Pi_m"] \\ \mathcal{E}/A \arrow[u, "m^*"] \end{tikzcd} \quad \Sigma_m \dashv m^* \dashv \Pi_m$$

Lemma

If $m : A \rightarrowtail B$ is an embedding in an ∞-LCCC $\mathcal{E}$ then Σ_m and Π_m are fully faithful.

Proof.

Given $h, k \in \mathcal{E}/B$, it is immediate that $\hom_{\mathcal{E}/B}(h, k) \simeq \hom_{\mathcal{E}/A(fh,fk)}$. The claim about Π follows by calculus of mates. □

In particular we have $m^* \circ m_* \simeq \mathrm{id}$ in this case, i.e. every $h : X \to A$ is a pullback of its own pushforward.

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{X} \arrow[d, "\Pi_m h"] \\ A \arrow[r, tail, "h"] & B \end{tikzcd}$$

Coproducts in ∞-LCCCs with SOC

Given an object A in an ∞-LCCC with Ω, consider the following diagram.

$$
\begin{tikzcd}
 & I \arrow[rr, dashed] \arrow[dl, dashed] \arrow[dd, dashed] & & J \arrow[dl, dashed] \arrow[dd, dashed] \\
A \arrow[rr, tail] \arrow[dd, "a"'] & & \overline{A} \arrow[dd, "\overline{a}"] & \\
 & 0 \arrow[rr] \arrow[dl] & & 1 \arrow[dl, tail, "\mathrm{ff}"] \\
1 \arrow[rr, tail, "\mathrm{tt}"] & & \Omega &
\end{tikzcd}
$$

- Bottom is classifying square for the least subobj. $0 \rightarrowtail 1$
- $\overline{a} = \Pi_{\mathrm{tt}} a$, so that $a = \mathrm{tt}^* \overline{a}$ ($\overline{A}$ is known as *partial map representer*)
- remainder constitutes pullback in arrow category, in particular right and left side squares – and therefore all squares are pullbacks
- I is initial by strictness, J is terminal by Beck-Chevalley
- again we have embedded A into a ‘larger’ object with a disjointly embedded point.

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail] \\
A \arrow[r, tail] & \overline{A}
\end{tikzcd}
$$

Given two object A, B we can again form the transposed product of pullbacks

$$\left(\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail] \\ A \arrow[r, tail] & \overline{A} \end{tikzcd}\right) \times \left(\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail] \\ 1 \arrow[r, tail] & \overline{B} \end{tikzcd}\right) = \left(\begin{tikzcd} 0 \arrow[rr] \arrow[d, tail] \arrow[drr, phantom, "\lrcorner", very near start] & & B \arrow[d, tail] \\ A \arrow[r, tail] & \overline{A} \times \overline{B} & \end{tikzcd}\right)$$

to obtain an object that disjointly embeds A and B
Forming the join of A and B in $\overline{A} \times \overline{B}$ yields a cospan

$$A \overset{i}{\hookrightarrow} C \overset{j}{\hookleftarrow} B$$

such that $A \wedge B = \bot$ and $A \vee B = \top$ in $\mathrm{sub}(C)$.

It remains to show:

Lemma

Let $A \overset{i}{\hookrightarrow} C \overset{j}{\hookleftarrow} B$ be embeddings in an ∞-LCCC $\mathcal{E}$, such that $A \wedge B = \bot$ and $A \vee B = \top$ in $\mathrm{sub}(C)$. Then i and j exhibit C as a coproduct of A and B.

Proof.

Since $\Sigma_C : \mathcal{E}/C \to \mathcal{E}$ preserves colimits, we may wlog assume $C = 1$. For $X \in \mathcal{E}$ and $f : A \to X$, $g : B \to X$, have to show that the pullback of

$$1 \xrightarrow{\langle f,g \rangle} \mathcal{E}(A, X) \times \mathcal{E}(B, X) \leftarrow \mathcal{E}(1, X)$$

is contractible in spaces. This is the image of

$$1 \xrightarrow{\langle f,g \rangle} X^A \times X^B \xleftarrow{\langle X^i, X^j \rangle} X$$

under $\mathcal{E}(1, -)$ so it suffices to show that the pullback of the latter is terminal in $\mathcal{E}$. This pullback can be written type theoretically as

$$\Sigma x \,.\, (\forall a \,.\, f\, a = x) \times (\forall b \,.\, g\, b = x)$$

so it's sufficient to show

$$\vdash \mathrm{isContr}(\Sigma x \,.\, (\forall a \,.\, f\, a = x) \times (\forall b \,.\, g\, b = x)).$$

... continued on next slide ... □

Proof ct'd

Since $A \vee B = \top$ in $\mathrm{sub}(1)$ it's sufficient to show the judgments

$$a_0{:}A \vdash \mathrm{isContr}(\Sigma x\,.\,(\forall a\,.\,f\,a = x) \times (\forall b\,.\,g\,b = x))$$
$$b_0{:}B \vdash \mathrm{isContr}(\Sigma x\,.\,(\forall a\,.\,f\,a = x) \times (\forall b\,.\,g\,b = x)).$$

We show the first one.

It's easy to see that in context $a_0{:}A$, the first factor is equivalent to $f\,a_0 = x$ and the second is equivalent to $\top$. Thus, the claim reduces to

$$a_0{:}A \vdash \mathrm{isContr}(\Sigma x\,.\,f\,a_0 = x)$$

Thanks for your attention!