

Modest sets and equilogical spaces as mono-fibrational cocompletions

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In memoriam: Thomas Streicher (1958-2025)



Triposes

Definition

A **Set**-tripos¹ is an **indexed poset**

$$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \rightarrow \mathbf{Pos}$$

such that:

- For all sets I , the poset $\mathcal{P}(I)$ is a **Heyting algebra**.
- For all functions $f : I \rightarrow J$, the **reindexing map** $f^* : \mathcal{P}(J) \rightarrow \mathcal{P}(I)$ is a **Heyting algebra morphism** and has left and right adjoints $\exists_f \dashv f^* \dashv \forall_f$ satisfying the **Beck-Chevalley condition**:

(BCC) For all pullback squares
$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ k \downarrow & \lrcorner & \downarrow g \\ C & \xrightarrow{f} & D \end{array}$$
 in **Set**, we have $g^* \circ \exists_f = \exists_h \circ k^*$ and $g^* \circ \forall_f = \forall_h \circ k^*$.

- There exists a **generic predicate**, i.e. a set Σ and a predicate $\mathrm{tr} \in \mathcal{P}(\Sigma)$ such that for all sets A and elements $\phi \in \mathcal{P}(A)$ there exists an $f : A \rightarrow \Sigma$ with $f^*(\mathrm{tr}) = \phi$.

Triposes were introduced as an auxiliary tool in the construction of **realizability toposes** from **partial combinatory algebras** (PCAs), notably Hyland's **effective topos**².

² Hyland. "The effective topos". In: *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*. Amsterdam: North-Holland, 1982.

² Hyland, Johnstone, and Pitts. "Tripos theory". In: *Math. Proc. Cambridge Philos. Soc.* (1980)

Realizability triposes

Definition

The **effective tripos** $\mathbf{eff} : \mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Preord}$ is given by

$$\mathbf{eff}(I) = (P(\mathbb{N})^I, \leq)$$

where

$$(\phi : I \rightarrow P(\mathbb{N})) \leq (\psi : I \rightarrow P(\mathbb{N})) \quad \text{iff} \quad \exists (f : \mathbb{N} \xrightarrow{\text{part. rec.}} \mathbb{N}) \forall (i \in I) \forall (n \in \phi(i)) . f(n) \in \psi(i)$$

More generally:

Definition

Let \mathcal{A} be a **partial combinatory algebra (PCA)**. The **realizability tripos** $\mathbf{rt}(\mathcal{A}) : \mathbf{Set}^{\mathbf{op}} \rightarrow \mathbf{Preord}$ is given by

$$\mathbf{rt}(I) = (P(\mathcal{A})^I, \leq)$$

where

$$(\phi : I \rightarrow P(\mathcal{A})) \leq (\psi : I \rightarrow P(\mathcal{A})) \quad \text{iff} \quad \exists (e \in \mathcal{A}) \forall (i \in I) \forall (a \in \phi(i)) . e \cdot a \in \psi(i)$$

Characterization of realizability triposes over PCAs

Theorem (F)³

A tripos $\mathcal{P} : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Pos}$ is a realizability tripos over a PCA, iff :

- (1) \mathcal{P} has enough \exists -**prime predicates**.
- (2) The full indexed sub-poset $\mathcal{A} = \mathbf{prim}(\mathcal{P}) \subseteq \mathcal{P}$ of \exists -prime predicates has finite meets.
- (3) \mathcal{A} has a **discrete** generic predicate.
- (4) \mathcal{A} is **shallow**, i.e. $\mathcal{A}(1) = 1$

Here:

- A predicate $\pi \in \mathcal{P}(I)$ is called \exists -**prime** if all its reindexings have the **left lifting property** (LLP) w.r.t. **cocartesian maps** in the total category $\int \mathcal{P}$.
- A predicate $\delta \in \mathcal{A}(I)$ is called **discrete**, if it has the **right lifting property** (RLP) w.r.t. **cartesian maps over surjections** in the total category $\int \mathcal{A} (= \mathbf{PAsm}(\mathcal{A}))$.
- (1) means that \mathcal{P} is a **cocompletion**, and (3) means that \mathcal{A} is a **completion**.
- Thus, realizability triposes are **cocompletions of completions** (combined via a distributive law), which we'll analyze in this talk.

³ Frey. "A fibrational study of realizability toposes". PhD thesis. Paris 7 University, 2013
Frey. *Uniform Preorders and Partial Combinatory Algebras*. arxiv 2024, accepted in TAC

Fibrations vs indexed categories

Definition

A functor $p : \mathbb{E} \rightarrow \mathbb{B}$ is a **Grothendieck fibration**, if for all $E \in \mathbb{E}$, the functor $\mathbb{E} \downarrow E \rightarrow \mathbb{B} \downarrow p(E)$ is a **strict reflection**, i.e. it has a right adjoint section.

- For categories \mathbb{C} in a fixed universe (i.e. 'small') we have a biequivalence

$$\mathbf{Fib}(\mathbb{C}) \simeq [\mathbb{C}^{\text{op}}, \mathbf{Cat}]$$

where $\mathbf{Fib}(\mathbb{C})$ is the 2-category of Grothendieck fibrations over \mathbb{C} with small domain, and $[\mathbb{C}^{\text{op}}, \mathbf{Cat}]$ the 2-category of pseudofunctors, pseudo-natural transformations, and modifications.

- This restricts to a biequivalence of locally ordered categories

$$\mathbf{Fib}_{\text{ff}}(\mathbb{C}) \simeq [\mathbb{C}^{\text{op}}, \mathbf{Pos}]$$

between **faithful fibrations** and **indexed posets**.

- In the following we use faithful fibrations as analogues of indexed posets over **Set**, but there's a **size mismatch**: in general the fibers won't be small (but they will if the fibration has a generic predicate, such as a tripos).
- As a basis for our analysis, we introduce a more basic locally ordered category: **FIFib** is the category of **faithful isofibrations** (a.k.a. **concrete categories**) over **Set**.
- Notation: instead of $(U : \mathbb{C} \rightarrow \mathbf{Set}) \in \mathbf{FIFib}$ write $\mathbb{C} \in \mathbf{FIFib}$ and always write U for the functor.

Four monads

We consider four monads on **FIFib**

- $T_{\text{inj}} : \mathbf{FIFib} \rightarrow \mathbf{FIFib}$ freely adds cartesian lifts along injections.
- $T_{\text{surj}} : \mathbf{FIFib} \rightarrow \mathbf{FIFib}$ freely adds cartesian lifts along surjections.
- $S_{\text{inj}} : \mathbf{FIFib} \rightarrow \mathbf{FIFib}$ freely adds cocartesian lifts along injections.
- $S_{\text{surj}} : \mathbf{FIFib} \rightarrow \mathbf{FIFib}$ freely adds cocartesian lifts along surjections.

All these are given by similar constructions. For example, for $\mathbb{C} \in \mathbf{FIFib}$, the category $T_{\text{inj}}\mathbb{C}$ has pairs $(C \in \mathbb{C}, m : S \rightrightarrows UC)$ as objects, and morphisms $(C, m : X \rightrightarrows UC) \rightarrow (D, n : Y \rightrightarrows UD)$ are given by functions $f : X \rightarrow Y$ such that there exists a $g : C \rightarrow D$ with $Ug \circ m = n \circ f$.

$$\begin{array}{ccc} X & \xrightarrow{m} & UC \\ \downarrow f & & \downarrow Ug \\ Y & \xrightarrow{n} & UD \end{array} \qquad \begin{array}{c} C \\ \downarrow g \\ D \end{array}$$

The ‘underlying set’ functor is given by $U(C, m : X \rightrightarrows UC) = X$

Remarks:

- We only require that g ‘exists’ since contrary to Quentin yesterday, we’re freely generating **faithful** fibrations.
- For T_{surj} and S_{inj} this doesn’t make a difference by cancellation properties.
- S_{inj} and S_{surj} are **lax idempotent**, and S_{inj} and S_{surj} are **colax idempotent**.

Distributive laws

Definition

Given monads $S, T : \mathbb{C} \rightarrow \mathbb{C}$ on a category \mathbb{C} , a **distributive law** is a natural transformation $\delta : TS \rightarrow ST$ satisfying certain axioms.

Proposition (Beck, ?)

TFAE:

- distributive laws $\delta : TS \rightarrow ST$
- monad structures on ST satisfying certain conditions

- 'liftings' of S to the category \mathbb{C}^T of T -algebras

$$\begin{array}{ccc} \mathbb{C}^T & \dashrightarrow & \mathbb{C}^T \\ \downarrow U & & \downarrow U \\ \mathbb{C} & \xrightarrow{S} & \mathbb{C} \end{array}$$

- 'extensions' of T to the Kleisli category \mathbb{C}_S of S

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{T} & \mathbb{C} \\ \downarrow F & & \downarrow F \\ \mathbb{C}_S & \dashrightarrow & \mathbb{C}_S \end{array}$$

Claim: In general, there may be many distributive laws between two monads S, T . However, if T is 'property-like' (e.g. lax idempotent or colax idempotent), then there is at most one, and it exists iff S maps T -algebras to T -algebras.

Monadic lifting

Given a distributive law $\delta : TS \rightarrow ST$ we get a square of categories and forgetful functors where three sides (and the diagonal) are monadic.

$$\begin{array}{ccc} \mathbb{C}^{ST} & \xrightleftharpoons{T} & \mathbb{C}^T \\ \uparrow \dashv \downarrow & & \uparrow \dashv \downarrow \\ \mathbb{C}^S & \xrightleftharpoons{T} & \mathbb{C} \end{array}$$

By adjoint lifting and monadic cancellation⁴, the fourth side is monadic whenever the \mathbb{C}^{ST} has reflexive coequalizers, which is very often the case. Examples:

$$\begin{array}{ccc} \text{Sup} & \xrightleftharpoons{T} & \text{jSLat} \\ \uparrow \dashv \downarrow & & \uparrow \dashv \downarrow \\ \text{DCPO} & \xrightleftharpoons{T} & \text{Pos} \end{array} \qquad \begin{array}{ccc} \text{Frm} & \xrightleftharpoons{T} & \text{mSLat} \\ \uparrow \dashv \downarrow & & \uparrow \dashv \downarrow \\ \text{Sup} & \xrightleftharpoons{T} & \text{Pos} \end{array}$$

Empirical observation: If the RHS adjunction is (co)lax idempotent then the LHS is as well, but typically not **mnemonic** (cf. Quentin's talk). Source of interesting LNL adjunctions.

⁴See [this Zulip discussion](#), thanks to Tom Hirschowitz and Mathieu Anel.

Many distributive laws

I claim that the monads T_{inj} , T_{surj} , S_{inj} , S_{surj} , admit distributive laws **for any distinct pair in both directions**. We're interested specifically in

- $T_{\text{inj}} \circ T_{\text{surj}} \rightarrow T_{\text{surj}} \circ T_{\text{inj}} =: T_{\text{all}}$
(free faithful fibration monad, arising from epi-mono factorization system)
- $S_{\text{surj}} \circ S_{\text{inj}} \rightarrow S_{\text{inj}} \circ S_{\text{surj}} =: S_{\text{all}}$
(free faithful opfibration monad, arising from epi-mono factorization system)
- $T_j \circ S_i \rightarrow S_i \circ T_j =: B_i^j$ for $i, j \in \{\text{surj}, \text{inj}, \text{all}\}$, (free faithful BC-bifibrations, arising from pullbacks)

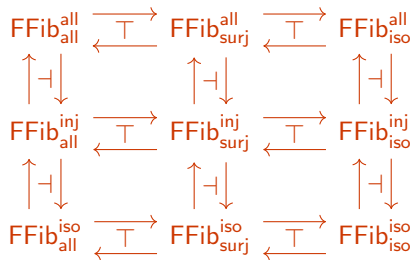
Characterization of free faithful fibrations

Proposition

- A faithful mono-fibration is free over a faithful iso-fibration iff it has enough **injective objects** (RLP w.r.t. cartesian maps over injections)
- A faithful fibration is free over a faithful mono-fibration iff it has enough **discrete objects** (RLP w.r.t. cartesian maps over surjections)
- A faithful fibration is free over a faithful iso-fibration iff it has enough **discrete injective objects** (RLP w.r.t. all cartesian maps)

Grid of monadic functors

Together, we get the following grid of locally ordered categories of faithful BCC-bifibrations over **Set**, and monadic (co)lax idempotent adjunctions between them.



The superscript i in \mathbf{FFib}_j^i denotes along which functions there are cartesian liftings, and the subscript j corresponds to co-cartesian liftings.

E.g., $\mathbf{FFib}_{\text{surj}}^{\text{inj}}$ is the locally ordered category of faithful functors $U : \mathbb{C} \rightarrow \mathbf{Set}$ admitting cartesian liftings along injections, and cocartesian liftings along surjections, subject to BCC for all suitable squares.

Realizability triposes over a pca \mathcal{A} are freely generated by the category $\mathbf{MPAsm}(\mathcal{A})$ of **modest partitioned assemblies** in $\mathbf{FFib}_{\text{iso}}^{\text{inj}}$. Many of the intermediate ‘partial’ completions are also known categories:

$\int \text{rt}(\mathcal{A})$	$\mathbf{Asm}(\mathcal{A})$	$\mathbf{PAsm}(\mathcal{A})$
?	$\mathbf{Mod}(\mathcal{A})$	$\mathbf{MPAsm}(\mathcal{A})$

The grid cells correspond to the positions in the diagram above.

The claim is that all the stated categories are (co)completions of $\mathbf{MPAsm}(\mathcal{A})$ along the suitable left adjoints.

If the PCA \mathcal{A} is **total**, we can even fill the bottom row.

Equiological Spaces

Scott's category **Equ** of **equiological spaces**⁵ fits into a similar grid:

?	$\mathbf{Top}_{\text{reg/lex}}$	\mathbf{Top}
?	\mathbf{Equ}	\mathbf{Top}_{T_0}
?	?	$\mathbf{ContLat}$

Here, **ContLat** is the full subcategory of **Top** on continuous lattices with the Scott topology.

Relevant observations:

- $\mathbf{Top} \rightarrow \mathbf{Set}$ is a faithful fibration, $\mathbf{Top}_{T_0} \rightarrow \mathbf{Set}$ is a faithful mono-fibration.
- T_0 -spaces have the r.l.p. w.r.t. cartesian maps over surjections, and every space is a cartesian lifting of a T_0 space along a surjection.
- Continuous lattices (with Scott topology) are injective w.r.t. subspace inclusions of T_0 spaces, and every T_0 space embeds into a continuous lattice (even into an algebraic lattice).
- Claim: going from the top right to the top middle grid cell is always a reg/lex completion.
- Observation: **Equ** is locally cartesian closed just like **Mod**, as observed by Rosolini⁶.

⁵ Bauer, Birkedal, and Scott. "Equiological spaces". In: *Theoretical Computer Science* (2004).

⁶ Rosolini. "The category of equiological spaces and the effective topos as homotopical quotients". In: *Journal of Homotopy and Related Structures* (2016).

Posets

And another variation:

?	?	Preord
?	?	Pos
?	?	CompLat

Here, **CompLat** is the full subcategory of **Pos** on complete lattices.

Relevant facts:

- **Preord** \rightarrow **Set** is a fibration
- Posets have the r.l.p. w.r.t. surjective cartesian maps between preorders
- complete lattices are injective w.r.t. embeddings of posets, and every posets embeds into a complete lattice.

Claim: In the middle we get an locally cartesian closed category again, since **CompLat** is CCC.

Thank you for your attention!

Announcement:

MathStic Category Theory Workshop

- https://lipn.univ-paris13.fr/~breuvert/journee_mathstic_axe_categories
- June 16, 2025 at LIPN / Paris 13
- Speakers
 - Sacha Ikonicoff
 - Jad Koleilat
 - Samuel Mimram
 - Paige North
 - Paula Verdugo