A Fibrational Analysis of Realizability Toposes

Jonas Frey

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Introduction
Constructive Mathematics, Realizability and Toposes

- Appeared in the beginning of the 20th century
- Response to unease about abstract methods and inconsistencies
- Asserts that it is necessary to find (or "construct") a mathematical object to prove that it exists.
- Major difference to classical logic/mathematics: rejection of the **principle of the excluded middle**

$$\vdash A \vee \neg A$$

- What is a mathematical proof?
- What counts as a constructive proof?

The *Brouwer–Heyting–Kolmogorov (BHK) interpretation states*:

1. A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.
2. A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B.
3. A proof of $A \Rightarrow B$ is a construction which transforms any proof of A into a proof of B.
4. Absurdity $\perp$ ('the contradiction') has no proof.

- Does not validate excluded middle $A \vee \neg A$

- Formalization of the BHK interpretation

$$
\begin{array}{lll}
n \Vdash s = t & \text{iff} & s = t \ (s, t \text{ are closed terms}) \\
n \Vdash \varphi \wedge \psi & \text{iff} & (p_0(n) \Vdash \varphi) \wedge (p_1(n) \Vdash \psi) \\
n \Vdash \varphi \Rightarrow \psi & \text{iff} & \forall m\,.\,(m \Vdash \varphi) \Rightarrow (\theta_n(m) \Vdash \psi) \\
n \Vdash \bot & \text{never} & \\
n \Vdash \varphi \vee \psi & \text{iff} & (p_0(n) = 0 \wedge (p_1(n) \Vdash \varphi)) \vee (p_0(n) = 1 \wedge (p_1(n) \Vdash \psi)) \\
n \Vdash \forall x\,.\,\varphi(x) & \text{iff} & \forall m \in \mathbb{N}\,.\,\theta_n(m) \Vdash \varphi(m) \\
n \Vdash \exists x\,.\,\varphi(x) & \text{iff} & p_1(n) \Vdash \varphi(p_0(n))
\end{array}
$$

$(\theta_n)_{n\in\mathbb{N}}$ Gödel numbering of partial recursive functions

Algebraic geometry	**Constructive mathematics**
Grothendieck 1960	
Giraud theorem 1963	
	Lawvere 1970
Bunge 77 Characterizes presheaf toposes	
	Hyland, Johnstone, Pitts 1980 Realizability toposes

- Theory of Grothendieck toposes highly developed
- In contrast, Peter Johnstone famously compared the study of realizability toposes to 'stamp collecting', writing

  > *... we have lots of exotic and colourful examples, which we can proudly display in our albums, but what we lack is a general theory which would indicate just where the boundaries of the subject lie.*

- Motivation of my work : 'Giraud theorem for realizability'

Overview

Overview

Main result

- Characterization of **realizability toposes** $\mathbf{RT}(\mathcal{A})$ over **partial combinatory algebras** $\mathcal{A}$
- Inspired by Giraud's theorem, but really analogous to Bunge's result

Sketch of proof

1. Identify $\mathcal{A}$ with a certain **posetal fibration**
2. Identify $\mathbf{RT}(\mathcal{A})$ with a certain **categorical fibration**
3. Characterize the second fibration as a **cocompletion** of the first

Overview

Outline

1. Posetal fibrations
2. Uniform preorders
3. Moens' theorem
4. The fibered presheaf construction
5. The characterization
6. Further results, future work

Part I – Posetal fibrations

Outline

Truth value semantics vs posetal fibrations

	truth values	predicates on set S
Classical model	true, false	subsets of S
Model with values in Heyting algebra A	elements of A	functions $\varphi : S \to A$
Model in posetal fibration $\mathcal{P}$	elements of $\mathcal{P}(1)$	elements of $\mathcal{P}(S)$

Posetal fibrations

Definition

An posetal fibration is a **functor**

$$\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$$

- For each set J, $\mathcal{P}(J)$ is the **order of predicates on** J
- For $f : K \to J$, the map

  $$f^* : \mathcal{P}(J) \to \mathcal{P}(K)$$

  is called **reindexing map** and **models substitution**

Remark

The same data can be encoded in a functor

$$P : \int \mathcal{P} \to \mathbf{Set}$$

Interpreting the logical connectives

- Bare posetal fibrations : not enough structure to interpret connectives
- **Propositional logic** : Heyting algebras as fibers
- **Quantifiers and equality** : adjoints to reindexing maps

Fragments of predicate logic

Definition

- **Existential fibration** :
  posetal fibration interpreting **regular logic** $\{\exists, =, \wedge, \top\}$
- **Hyperdoctrine** :
  interprets **first order constructive logic** $\{\exists, =, \wedge, \top, \vee, \bot, \Rightarrow, \forall\}$

Definition (Generic predicate)

A **generic predicate** is a predicate

$$\mathsf{tr} \in \mathcal{P}(\mathsf{Prop})$$

on a set Prop such that for any other set J and predicate $\varphi \in \mathcal{P}(J)$, there exists a function $f : J \to \mathsf{Prop}$ with

$$\varphi \cong f^*\mathsf{tr}.$$

Definition

A **tripos** is a hyperdoctrine with a generic predicate.

Kleene realizability as a tripos

Definition

The **effective tripos eff** : $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$

- **Predicates** on J: functions $\varphi : J \to P(\mathbb{N})$
- **Ordering**: $\varphi \leq \psi$ iff there exists a *partial recursive* f such that

$$\forall j \in J\ \forall n \in \varphi(j)\,.\quad f(n) \text{ is defined and } f(n) \in \psi(j).$$

The construction of the effective tripos can be generalized:

Definition

Partial combinatory algebra (PCA) : set $\mathcal{A}$ with a partial binary operation $(- \cdot -) : \mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A}$ and $k, s \in \mathcal{A}$ such that

- $\forall x, y \,.\; k{\cdot}x{\cdot}y = y$
- $\forall x, y \,.\; s{\cdot}x{\cdot}y{\downarrow}$
- $\forall x, y, z \,.\; x{\cdot}z{\cdot}(y{\cdot}z){\downarrow} \Rightarrow s{\cdot}x{\cdot}y{\cdot}z = x{\cdot}z{\cdot}(y{\cdot}z)$

Definition (Realizability tripos)

The **realizability tripos** $\mathbf{rt}(\mathcal{A}) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ for a PCA $\mathcal{A}$:

- **Predicates** on the set J : functions $\varphi : J \to P\mathcal{A}$
- **Ordering** $\varphi \le \psi$ iff there exists $e \in \mathcal{A}$

$$\forall j{:}J, a \in \varphi(j) \,.\; e{\cdot}a \text{ is defined and } e{\cdot}a \in \psi(j)$$

Definition (The category of partial equivalence relations)

For an tripos $\mathcal{P}$, the category $\mathbf{Set}[\mathcal{P}]$ is defined as follows.

- **Objects** : pairs (C, ρ) where C is a set and $\rho \in \mathcal{P}(C \times C)$ such that

$$\begin{array}{ll} \text{(trans)} & \rho(x,y), \rho(y,z) \vdash \rho(x,z) \\ \text{(sym)} & \rho(x,y) \vdash \rho(y,x) \end{array}$$

- **Morphisms** from (C, ρ) to (D, σ) : predicates $\phi \in \mathcal{P}(C \times D)$ satisfying

$$\begin{array}{ll} \text{(strict)} & \phi(x,y) \vdash \rho(x,x) \wedge \sigma(y,y) \\ \text{(cong)} & \rho(x',x), \phi(x,y), \sigma(y,y') \vdash \rho(x',y') \\ \text{(singval)} & \phi(x,y), \phi(x,y') \vdash \sigma(y,y') \\ \text{(tot)} & \rho(x,x) \vdash \exists y\,.\,\phi(x,y) \end{array}$$

The tripos-to-topos construction

Theorem (Hyland, Johnstone, Pitts 1981)

*For any tripos $\mathcal{P}$, **Set**$[\mathcal{P}]$ is a topos.*

Definition

- $\mathbf{RT}(\mathcal{A}) = \mathbf{Set}[\mathbf{rt}(\mathcal{A})]$ is called the **realizability topos over $\mathcal{A}$**
- $\mathcal{E}\!f\!f = \mathbf{Set}[\mathbf{eff}]$ is the **effective topos**

Set$[\mathcal{F}]$ *for existential fibrations*

Observation

- The category **Set**$[\mathcal{F}]$ is definable for any **existential fibration** $\mathcal{F}$
- In this case, **Set**$[\mathcal{F}]$ is an **exact category**

Part II – Uniform preorders

Outline

Uniform preorders

- Idea: representations of posetal fibrations
- Generalization of Hofstra's **basic combinatory objects**
- Related to Longley's **computability structures**

Definition

A **uniform preorder** is a pair

$$(A, R)$$

of a set A and a set

$$R \subseteq P(A \times A)$$

of binary relations, subject to the following axioms.

1. $r \in R,\ s \subseteq r \implies s \in R$
2. $\mathrm{id} \in R$
3. $r, s \in R \implies s \circ r \in R$

The elements of R are called **primitive entailments**

Uniform preorders as posetal fibrations

Definition

The **posetal fibration** $\mathrm{fam}(A, R)$ associated to a uniform preorder (A, R) has functions $\varphi : J \to A$ as predicates; the ordering relation is defined by

$$\varphi \le \psi \quad \text{iff} \quad \{(\varphi(j), \psi(j)) \mid j{:}J\} \in R \quad \text{for} \quad \varphi, \psi : J \to A.$$

Observation

The fibration $\mathrm{fam}(A, R)$ has a generic predicate,
given by $\mathsf{Prop} = A$ and $\mathrm{id}_A : A \to A$.

Representation Lemma

An posetal fibration is representable by a uniform preorder iff it has a generic predicate.

Remark

The proof of the lemma depends on AC, without choice uniform preorders correspond to posetal **pre-stacks** with generic predicate.

PCAs as uniform preorders

To each pca $\mathcal{A}$, we associate the uniform preorder $(\mathcal{A}, R_{\mathcal{A}})$, where

$$R_{\mathcal{A}} = \{r \subset \mathcal{A} \times \mathcal{A} \mid \exists e{:}\mathcal{A}\, \forall (a, b) \in r \,.\, e{\cdot}a = b\}$$

is the set of 'sub-computable' partial functions.

The 2-category **UOrd**

Definition

- A **monotone map** between uniform preorders (A, R), (B, S) is a function $f : A \to B$ such that

$$r \in R \quad \text{implies} \quad \{(fa, fa') \mid (a, a') \in r\} \in S.$$

- For monotone maps $f, g : (A, R) \to (B, S)$, define

$$f \le g \quad \text{iff} \quad \{(fa, ga) \mid a \in A\} \in S.$$

- This defines a **locally ordered 2-category UOrd**.

Tower of full and order-reflecting inclusions of locally ordered 2-categories:

$$\mathbf{Ord} \hookrightarrow \mathbf{BCO} \hookrightarrow \mathbf{UOrd} \hookrightarrow \mathbf{PFib}$$

- **Ord** : preorders
- **BCO** : Hofstra's basic combinatory objects
- **UOrd** : uniform preorders
- **PFib** : posetal fibrations

UOrd is a 2-category and **has finite products**. Therefore, we can define

Definition (Following Hofstra 2006)

A uniform preorder (A, R) is called **finitely complete**, if the maps

$$\delta : (A, R) \to (A, R) \times (A, R) \qquad \text{and} \qquad ! : (A, R) \to 1$$

have right adjoints

$$\wedge : (A, R) \times (A, R) \to (A, R) \qquad \text{and} \qquad \top : 1 \to (A, R).$$

Observation

The uniform preorder $(\mathcal{A}, R_{\mathcal{A}})$ is finitely complete for any PCA $\mathcal{A}$.

Existential quantification

- The set $\mathrm{dcl}(A)$ of **downsets** is the **join-cocompletion** of a preorder A
- An analogous construction **freely adds** $\exists$ to a uniform preorder (A, R)

Definition (Following Hofstra, van Oosten)

The uniform preorder $D(A, R)$ is defined as follows:

- **Underlying set**: PA
- **Primitive entailments**:
  subrelations of relations $[r] \subseteq PA \times PA$ for $r \in R$, defined by

$$[r](M, N) :\Leftrightarrow \forall m\, \exists n \,.\, r(m, n)$$

The mapping $(A, R) \mapsto D(A, R)$ is a **lax idempotent monad** on **UOrd**

Key observation

For a pca $\mathcal{A}$, we have $\mathrm{fam}(D(\mathcal{A}, R_{\mathcal{A}})) = \mathbf{rt}(\mathcal{A})$

Let (A, R) be a finitely complete uniform preorder.

- $\mathrm{fam}(D(A, R))$ is always an **existential fibration**.
- When is $\mathrm{fam}(D(A, R))$ a **tripos**?

Definition

A finitely complete uniform preorder (A, R) is called **relationally complete**, if there exists $@ \in R$ such that for all $r \in R$ there exists $\tilde{r} \in R$ such that

$$\forall a \in A\, \exists h \in A\,.\, \tilde{r}(a, h) \wedge r(a \wedge -, -) \subseteq @(h \wedge -, -)$$

Theorem (Following Hofstra 2006)

$\mathrm{fam}(D(A, R))$ *is a tripos iff* (A, R) *is relationally complete.*

Towards a characterization of PCAs

Definition

A uniform preorder (A, R) is called **functional**, if all elements of R are functional relations.

Observation

For any PCA $\mathcal{A}$, $(\mathcal{A}, R_{\mathcal{A}})$ is functional

Definition (Pitts 81)

Let (A, R) be a finitely complete uniform preorder.
A **designated truth value** is an element $a \in A$ such that $\{(\top, a)\} \in R$.

Observation

In uniform preorders $(\mathcal{A}, R_{\mathcal{A}})$, all elements are designated truth values.

PCAs

Theorem

PCAs can be identified with finitely complete uniform preorders which are

- *relationally complete,*
- *functional, and*
- *all elements are designated truth values*

Part III – Moens' theorem

Outline

The gluing construction

$$\mathbf{Set} \xrightarrow[\Delta]{\qquad\qquad} \mathbf{RT}(\mathcal{A})$$

- Constant objects functor $\Delta : J \mapsto (J, =)$ $\quad [= \in \mathbf{rt}(\mathcal{A})(J \times J)]$

$$\begin{tikzcd} & \mathbf{RT}(\mathcal{A}){\downarrow}\mathbf{RT}(\mathcal{A}) \arrow[d, "\mathrm{cod}"] \\ \mathbf{Set} \arrow[r, "\Delta"'] & \mathbf{RT}(\mathcal{A}) \end{tikzcd}$$

- Constant objects functor $\Delta : J \mapsto (J, =)$ $\qquad [= \in \mathbf{rt}(\mathcal{A})(J \times J)]$

The gluing construction

$$
\begin{tikzcd}
\mathrm{Glue}_\Delta(\mathbf{RT}(\mathcal{A})) \arrow[r] \arrow[d, "\mathrm{glue}_\Delta(\mathbf{RT}(\mathcal{A}))"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{RT}(\mathcal{A}){\downarrow}\mathbf{RT}(\mathcal{A}) \arrow[d, "\mathrm{cod}"] \\
\mathbf{Set} \arrow[r, "\Delta"'] & \mathbf{RT}(\mathcal{A})
\end{tikzcd}
$$

- Constant objects functor $\Delta : J \mapsto (J, =)$ $\qquad [= \in \mathbf{rt}(\mathcal{A})(J \times J)]$

The gluing construction

$$\begin{tikzcd}
\mathrm{Glue}_\Delta(\mathbf{RT}(\mathcal{A})) \arrow[r] \arrow[d, "\mathrm{glue}_\Delta(\mathbf{RT}(\mathcal{A}))"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{RT}(\mathcal{A})\downarrow\mathbf{RT}(\mathcal{A}) \arrow[d, "\mathrm{cod}"] \\
\mathbf{Set} \arrow[r, "\Delta"'] & \mathbf{RT}(\mathcal{A})
\end{tikzcd}$$

- Constant objects functor $\Delta : J \mapsto (J, =)$ $[= \in \mathbf{rt}(\mathcal{A})(J \times J)]$
- Δ, $\mathbf{RT}(\mathcal{A})$ can be recovered from $\mathrm{glue}_\Delta(\mathbf{RT}(\mathcal{A}))$

Moens' theorem for fibered pretoposes

Variant: let $\mathbb{R}$ be a regular category.

> *Moens' theorem for fibered pretoposes*
>
> There is a biequivalence
>
> $$\mathbb{R}{\Downarrow}\mathbf{Ex} \simeq \mathbf{Pretop}(\mathbb{R})$$
>
> between
>
> - regular functors into exact categories
> - pre-stacks of exact categories with extensive internal sums
>
> We call such fibrations **fibered pretoposes**.

$$\begin{tikzcd}
\mathrm{Glue}_{\Delta}(\mathbb{X}) \arrow[r] \arrow[d, "\mathrm{glue}_{\Delta}(\mathbb{X})"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{X}{\downarrow}\mathbb{X} \arrow[d, "\mathrm{cod}"] \\
\mathbb{R} \arrow[r, "\Delta"'] & \mathbb{X}
\end{tikzcd}$$

Part V – The fibered presheaf construction

Outline

The fibered presheaf construction

Presheaf categories

- $\mathbb{C}$ small finite limit category
- Category $\widehat{\mathbb{C}}$ of presheaves is a Grothendieck topos
- Characterized by equivalence

$$\mathbf{Lex}(\mathbb{C}, \mathcal{X}) \simeq \mathbf{Geo}(\widehat{\mathbb{C}}, \mathcal{X}) \qquad (\mathcal{X}\ \infty\text{-pretopos})$$

of functor categories

Fibrations of presheaves

- $\mathscr{C} : \int \mathscr{C} \to \mathbb{R}$ **finite-limit pre-stack**
- **Fibration of presheaves** $\widehat{\mathscr{C}} : \int \widehat{\mathscr{C}} \to \mathbb{R}$ is a fibered pretopos
- Characterized by equivalence

$$\mathbf{Lex}(\mathbb{R})(\mathscr{C}, \mathscr{X}) \simeq \mathbf{Geo}(\mathbb{R})(\widehat{\mathscr{C}}, \mathscr{X}) \qquad (\mathscr{X}\ \text{fibered pretopos})$$

of fibered functor categories

Characterization of fibrations of presheaves

Which fibered pretoposes $\mathscr{X} : \int \mathscr{X} \to \mathbb{R}$ are of the form $\mathscr{X} \simeq \widehat{\mathscr{C}}$?

Theorem (Bunge 77)

*A Grothendieck topos $\mathcal{E}$ is a presheaf topos iff it has a generating family of **indecomposable projective** objects.*

In a similar way, we can show:

Theorem

A fibered pretopos $\mathscr{X} : |\mathscr{X}| \to \mathbb{R}$ is a fibration of presheaves iff

- the subfibration of $\mathscr{X}$ on **indecomposable projectives** is closed under finite limits, and
- Every $X \in |\mathscr{X}|$ can be covered by an internal sum of **indecomposable projectives**.

Using this, we can show:

Theorem

For any PCA $\mathcal{A}$, we have $\mathrm{glue}_{\Delta}(\mathbf{RT}(\mathcal{A})) \simeq \mathrm{fam}\widehat{(\mathcal{A}, R_{\mathcal{A}})}$.

Outline

Main result

Therem

Realizability toposes $\mathbf{RT}(\mathcal{A})$ can be characterized as categories $\mathbb{X}$ such that

- $\mathbb{X}$ is exact and locally cartesian closed
- $\Gamma = \mathbb{X}(1, -) : \mathbb{X} \to \mathbf{Set}$ has a regular right adjoint $\Delta : \mathbf{Set} \to \mathbb{X}$
- There exists a monomorphism $\phi : M \rightarrowtail \Delta A$ such that
  - ϕ is indecomposable and projective in $\mathrm{glue}_\Delta(\mathbb{X})$
  - M is discrete with respect to Δ
  - The subfibration of $\mathrm{glue}_\Delta(\mathbb{X})$ generated by ϕ is closed under finite meets
  - Every $X \in \mathbb{X}$ can be covered like

$$\begin{tikzcd} B \arrow[d] & & \Delta B \arrow[d] & \bullet \arrow[l, tail] \arrow[r, two heads] \arrow[d] \arrow[dl, phantom, very near start, "\llcorner"] & X \\ A & & \Delta A & M \arrow[l, tail] & \end{tikzcd}$$

Main result

Therem

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- $\mathbb{X}$ is exact and locally cartesian closed
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Discreteness?

- $X \in \mathbb{X}$ is **discrete**, if it is right orthogonal to all constant objects $\Delta(J)$
- Used to assure that the reconstructed uniform preorder is **functional**

Part VI – Further results, future work

Outline

Further results

- Characterization of realizability categories over **typed pcas**
- Charaterization of fibered **relative** realizability categories
- Framework of fibrational cocompletions for pre-stacks on regular base categories
- Characterization of pre-stacks $\mathcal{A}$ of meet-semilattices such that $\widehat{\mathcal{A}}$ is fiberwise locally cartesian closed
- Uniform preorders on arbirary bases
- Factorization theorem for constant objects functors

Directions for further research

- Many sorted uniform preorders as preorders internal to small sheaves on **Set** for the regular topology
- ‘Geometric theory of realizability toposes’
- Iteration (in the sense of Pitts, analogous to composition of bounded geometric morphisms)
- Unrelated: better understanding of relationally complete uniform preorders
- ‘relational clones’

Thank you for your attention

Appendix

Indecomposables and projectives

Let $\mathscr{X} : |\mathscr{X}| \to \mathbb{R}$ be a fibered pretopos.

Definition

- Call $P \in |\mathscr{X}|$ **projective** , if given c, e, f as in the diagram

$$
\begin{tikzcd}
 & \bullet \arrow[r, dotted, "g"'] \arrow[d, squiggly, "d"'] & Y \arrow[d, twoheadrightarrow, "e"] \\
P & \bullet \arrow[l, squiggly, "c"'] \arrow[r, "f"] & X
\end{tikzcd}
$$

  where c is cartesian and e is vertical and a regular epimorphism in its fiber, we can fill in d, g with d epicartesian such that the square commutes.

- Call $X \in |\mathscr{X}|$ **indecomposable**, if for every diagram

$$
\begin{tikzcd}
X^* \arrow[r, squiggly, "c"] \arrow[d, dotted, "m"'] \arrow[dr] & X \\
Y \arrow[r, "d"'] & \sum Y
\end{tikzcd}
$$

  in $|\mathscr{X}|$ where c is cartesian and d is cocartesian, there exists a *unique* mediating arrow m.

Discreteness

Definition

Let $\Delta : \mathbb{R} \to \mathbb{X}$ be a regular functor into an exact category.
Call $X \in \mathbb{X}$ **discrete** (with respect to Δ), if in any diagram of shape

$$\begin{tikzcd}
L \arrow[d, two heads] & \Delta L \arrow[d, two heads] & \bullet \arrow[l, tail] \arrow[d, two heads] \arrow[dr] \arrow[dl, phantom, "\llcorner", very near start] & \\
K & \Delta K & \bullet \arrow[l, tail] \arrow[r, dashed] & X
\end{tikzcd}$$

there exists a mediating arrow.

- Given an posetal fibration $\mathcal{A} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ with generic predicate $\iota \in \mathcal{A}(A)$, define uniform preorder (A, R) by

$$R = \{r \subseteq A \times A \mid \pi_l^* \iota \leq \pi_r^* \iota \text{ in } \mathcal{A}_r\},$$

  where $\pi_l, \pi_r : r \to A$ are the projections.
- Definition of uniform preorder can be internalized in any topos
- However, the above construction only works on set – it gives a set of binary relations, not an object of binary relations on other base categories
- Solution: For base topos $\mathcal{E}$, fiber not over $\mathcal{E}$, but over $\mathcal{E}{\downarrow}\mathcal{E}$

Definition

Let (A, R) be a uniform preorder internal to $\mathcal{E}$. The posetal fibration $\mathrm{fam}(A, R) : \mathcal{E}{\downarrow}\mathcal{E} \to \mathbf{Ord}$ is defined as follows:

- A predicate on $f : X \to I$ is a function $\varphi : X \to A$
- $\varphi \leq \psi$ iff
$$\forall i \; \{(\varphi(x), \psi(x)) \mid x \in X_i\} \in R$$
holds in $\mathcal{E}$.

- Slogan: pointwise in I, uniform in the fibers X_i
- We call $\mathrm{fam}(A, R)$ a **fibered fibration**, since its base is the total category of another fibration (the codomain fibration of $\mathcal{E}$).

Arbitrary base toposes

We can characterize the fibrations of the form $\mathrm{fam}(A, R)$ for internal (A, R).

Definition

Let $\mathcal{A} : \mathcal{S}{\downarrow}\mathcal{S}$ be an posetal fibration.

- We say that $\leq$ is **horizontally definable** in $\mathcal{A}$, if for every $\varphi, \psi \in \mathcal{A}_{M \xrightarrow{m} L}$ there exists a greatest subobject $m : U \rightarrowtail L$ of L such that $\varphi|_U \leq \psi|_U$.

$$\begin{tikzcd} \psi|_U \arrow[r, rightsquigarrow] & \psi \\ \varphi|_U \arrow[u, phantom, "\vee\!\mid"] \arrow[r, rightsquigarrow] & \varphi \end{tikzcd}$$

$$\begin{tikzcd} \cdot \arrow[r, rightarrowtail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "m"] \\ U \arrow[r, rightarrowtail] & L \end{tikzcd}$$

- We call $\mathcal{A}$ a *fibered posetal pre-stack*, if $e^*\varphi \leq e^*\psi$ implies $\varphi \leq \psi$ for every *vertical* epimorphism e in $\mathcal{S}{\downarrow}\mathcal{S}$.

Arbitrary base toposes

Lemma

The locally ordered category $\mathbf{UOrd}(\mathcal{S})$ *of uniform preorders internal to* $\mathcal{S}$ *is biequivalent to the locally ordered category of fibered posetal pre-stacks on* $\mathcal{S}$ *with horizontally definable* $\leq$ *and a generic predicate* ι *over* $1 \to 1$*.*