Elementary $(1,2)$-cosmoses and labeled linear logic

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July 14, 2021

Abstract

We introduce the notion of *elementary* $(1,2)$*-cosmos* as a simplified poset-enriched version of Street's *elementary cosmoses*, and show that every elementary $(1,2)$-cosmos gives rise to a *virtual double category* which is closed and admits compositions.

As a computational tool we use a sequent calculus for Lambek-style *labeled linear logic* extended by a notion of substitution, which admits a natural interpretation in virtual double categories.

Let $\mathcal{K}$ be a poset-enriched category with finite 2-limits, i.e. finite products, equalizers, and cotensors with $\mathbb{2}$. Following [Lam95] we call a span

$$A \xleftarrow{f} P \xrightarrow{g} B$$

in $\mathcal{K}$ a *comparison* between A and B if for all $X \in \mathcal{K}$, the induced map

$$\mathcal{K}(X,P) \to \mathcal{K}(X,A) \times \mathcal{K}(X,B)$$

is an order embedding, and its image – viewed as a binary relation – is downward closed in the first argument and upward closed in the second. Comparisons should be thought of as 'posetal profunctors', in particular in the case $\mathcal{K} = \mathbf{Pos}$ comparisons between A and B correspond to monotone maps $A^{\mathsf{op}} \times B \to \mathbb{2}$.

We denote the poset of comparisons between A and B (ordered by inclusion, quotiented by equivalence) by $\mathsf{Comp}(A,B)$. The assignment $(A,B) \mapsto \mathsf{Comp}(A,B)$ extends to a 2-functor

$$\mathsf{Comp} : \mathcal{K}^{\mathsf{coop}} \times \mathcal{K}^{\mathsf{op}} \to \mathbf{Pos},$$

the functorial action being given by pullback.

Comparisons form the horizontal arrows of a *virtual double category* [CS10] (a.k.a. *fc-multicategory* [Lei04]) $\mathbf{Comp}(\mathcal{K})$ whose vertical arrows are the morphisms of $\mathcal{K}$, and where there is a unique 2-cell in the square

$$\begin{tikzcd}
B_0 \arrow[rrrrr, "\psi"] & & & & & B_1 \\
A_0 \arrow[u, "f"] \arrow[r, "\phi_1"] & A_1 \arrow[r, "\phi_2"] & A_2 & \cdots & A_{n-1} \arrow[r, "\phi_n"] & A_n \arrow[u, "g"']
\end{tikzcd}$$

whenever the pullback of the comparisons $\phi_1, \ldots, \phi_n$ (which is in general not itself a comparison) factors through the comparison span $\mathsf{Comp}(f,g)(\psi)$.

We call $\mathcal{K}$ an *elementary* $(1,2)$*-cosmos* if $\mathsf{Comp}(-,-)$ is representable in both arguments, i.e. for all $A \in \mathcal{K}$ there exist objects $P_{\downarrow}A$ and $P_{\uparrow}A$ and natural isomorphisms

$$\mathcal{K}(-, P_{\downarrow}A) \cong \mathsf{Comp}(A,-) \qquad \text{and} \qquad \mathcal{K}(-, P_{\uparrow}A) \cong \mathsf{Comp}(-,A),$$

the latter being componentwise antitone. Elementary $(1,2)$-cosmoses can be viewed as the **Pos**-enriched version of Street's 2-categorical elementary cosmoses [Str74], with the important simplification that in our axiomatization the representability assumption is 'absolute', not relative to a postulated notion of size ('admissibility'). Examples of elementary $(1,2)$-cosmoses are given by categories of presheaves of posets over locally ordered index categories, and by internal posets in elementary toposes.

Assuming that $\mathcal{K}$ is an elementary $(1,2)$-cosmos we show that

(1) the virtual double category $\mathbf{Comp}(\mathcal{K})$ is closed,

and then, using an impredicative argument formalized in a sequent calculus for 'labeled linear logic' in the style of [Lam95], that

(2) the virtual double category $\mathbf{Comp}(\mathcal{K})$ has compositions[1].

Using the same techniques, we further show that

(3) for every $A \in \mathcal{K}$, the posets $\mathsf{Comp}(A,1)$ and $\mathsf{Comp}(1,A)$ are Heyting algebras.

In any elementary $(1,2)$-cosmos, the assignments $A \mapsto P_{\downarrow}A$ and $A \mapsto P_{\uparrow}A$ extend to mutually adjoint functors

$$P_{\downarrow}, P_{\uparrow} : \mathcal{K}^{\mathsf{coop}} \to \mathcal{K}.$$

We conclude by discussing the failure of monadicity of this adjunction in the case $\mathcal{K} = \mathbf{Pos}$.

References

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[1] For details of this proof see [Fre21].