# Basic combinatory objects, uniform preorders and partial combinatory algebras

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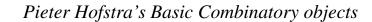
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Dedicated to the Memory of Pieter Hofstra

# Remembering Pieter



## Overview



## Basic Combinatory Objects

In his 2006 paper "All realizability is relative", Pieter Hofstra introduced the notion of basic combinatory object (building on work with van Oosten on ordered combinatory algebras).

#### Definition

A **basic combinatory object (BCO)** is a set A equipped with a partial order  $\leq$  and a set  $\mathcal{F}_A$  of 'computable' partial endofunctions with down-closed domain, such that

- 1.  $\exists i \in \mathcal{F} \ \forall a \in A . i(a) \leq a$
- 2.  $\forall f, g \in \mathcal{F} \exists h \in \mathcal{F} \forall a \in \text{dom}(g \circ f) . h(a) \leq g(f(a))$

BCOs form a locally ordered category BCO which admits a full embedding

fam : 
$$BCO \hookrightarrow IOrd$$

into the locally ordered category  $IOrd = [Set^{op}, Ord]$  of Set-indexed preorders, given by  $fam(A)(J) = (A^J, \leq)$  where

$$\varphi \leq \psi \quad :\Leftrightarrow \quad \exists f \in \mathcal{F}_A \ \forall j \in J \ . \ f(\varphi(j)) \leq \psi(j)$$

for  $\varphi, \psi : \mathbf{J} \to \mathbf{A}$ .

<sup>&</sup>lt;sup>1</sup>Hofstra, Pieter JW. "All realizability is relative." *Mathematical Proceedings of the Cambridge Philosophical Society.* Vol. 141. No. 2. Cambridge University Press, 2006.

## Basic Combinatory objects II – finite meets

BCOs are closed under products in [ $Set^{op}$ , Ord], thus fam(A) is an indexed meet-semilattice iff

$$A \rightarrow A \times A$$
 and  $A \rightarrow 1$ 

have right adjoints

$$(- \wedge -) : A \to A \times A$$
 and  $\top : A \to 1$ 

in BCO. We call such BCOs cartesian.

## Basic Combinatory objects III – existential quantification

- Say that an indexed preorder  $P: \mathbf{Set}^\mathsf{op} \to \mathbf{Ord}$  admits existential quantification, if the reindexing maps  $f^*: P(I) \to P(J)$  have left adjoints  $\exists_f: P(J) \to P(I)$  for all  $f: J \to$ , subject to the **Beck–Chevalley condition**.
- Denote by ∃-IOrd the subcategory of IOrd on indexed preorders admitting ∃ and indexed monotone maps preserving ∃.
- Pieter Hofstra showed that
  - 1. the forgetful functor  $\exists$ -IOrd  $\rightarrow$  IOrd is 2-monadic, and
  - 2. the induced (lax idempotent) 2-monad  $D: IOrd \rightarrow IOrd$  restricts to BCO.

$$\begin{array}{ccc} \mathbf{BCO} & -\stackrel{D}{\longrightarrow} & \mathbf{BCO} \\ \mathsf{fam} & & & & & \mathsf{fam} \\ \mathbf{IOrd} & \stackrel{D}{\longrightarrow} & \mathbf{IOrd} \end{array}$$

For a BCO A, the carrier of D(A) is the set of **downsets**.

3. Furthermore,  $\frac{D}{P}$  plays well with finite meets: if  $\frac{P}{P}$  has finite meets then  $\frac{D(P)}{P}$  has finite meets and moreover it satisfies the **Frobenius condition**.

## Examples: BCOs from posets and (O)PCAs

 Every poset can be viewed as BCO where only the identity function is computable, which gives a full embedding

$$Pos \rightarrow BCO$$

• Every **PCA** A can be viewed as a *cartesian* BCO where the ordering is trivial and

$$\mathcal{F}_{\mathcal{A}} = \{ e \cdot (-) : \mathcal{A} \rightharpoonup \mathcal{A} \mid e \in \mathcal{A} \}.$$

More generally, filtered ordered PCAs A can be viewed as cartesian BCO with

$$\mathcal{F}_{\mathcal{A}} = \{ e \cdot (-) : \mathcal{A} \rightharpoonup \mathcal{A} \mid e \in \Phi_{\mathcal{A}} \}.$$

Pieter observed that in both cases the associated realizability tripos
rt(A): Set<sup>op</sup> → Ord is given by

$$\operatorname{rt}(\mathcal{A}) = D(\operatorname{fam}(\mathcal{A})) = \operatorname{fam}(D(\mathcal{A}))$$

- In particular this means that realizability triposes are freely generated under existential quantification!
- This is related to the fact that realizability toposes are ex/lex-completions.

## Characterizing filtered OPCAs among BCOs

### Theorem (Hofstra)

TFAE for a cartesian BCO A:

- 1. A is (induced by) a filtered OPCA.
- 2. fam(D(A)) is a tripos.
- 3. The fibers of  $D \operatorname{fam}(A)$  are Heyting algebras.

### Proof.

The implications  $1 \Rightarrow 2 \Rightarrow 3$  are clear.

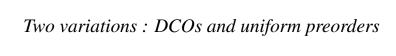
For  $3 \Rightarrow 1$  let  $\iota \in \text{fam}(D(A))(A)$  be the function sending every a to its principal downset, and let  $\varepsilon \in \mathcal{F}_A$  be a witness of the inequality

$$(\pi_1(\iota) \Rightarrow \pi_2(\iota)) \wedge \pi_1(\iota) \leq \pi_2(\iota)$$

in  $fam(D(A))(A \times A)$ . Then the application operation of the OPCA is given by  $\varepsilon \circ \wedge$ .

The filter  $\Phi_A$  is given by the **designated truth values**, i.e. the  $a \in A$  that are equivalent to  $\top$  in fam(A)(1).

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## **Overview**

- Pieter's paper formed the starting point for my PhD thesis in which I gave characterizations of realizability triposes and toposes over PCAs.
- In hindsight, the only missing piece in the BCO-approach is that the image of BCO → IOrd does not have an easy characterization — if we could characterize (O)PCAs among BCOs and BCOs among indexed posets then we could characterize (O)PCAs among indexed preorders.
- In the following I introduce a sub- and a super-category of BCO, which do have simple characterizations as subcategories of IOrd, and explain how to adapt Pieter's techniques.

## Discrete combinatory objects

#### Definition

A discrete combinatory object (DCO) is simply a BCO whose pratial order structure is trivial. We write  $DCO \subseteq BCO$  for the full subcategory of DCOs.

## Definition

Given an indexed preorder  $P : \mathbf{Set}^{\mathsf{op}} \to \mathbf{Ord}$ , we call  $\delta \in P(J)$  **discrete**, if it is right orthogonal to all cartesian maps over surjections in the total category  $\int P$  of P.

#### Lemma

An indexed preorder  $P: \mathbf{Set}^{op} \to \mathbf{Ord}$  is equivalent to one of the form fam(A) for a DCO A, iff it has a discrete generic predicate.

#### Proof.

Given a discrete predicate  $\delta \in \mathcal{A}$ , define DCO structure on A by taking as computable those partial functions  $f: A \longrightarrow A$  satisfying  $\iota|_{\mathsf{dom}(f)} \leq f^*(\iota)$  in  $P(\mathsf{dom}(f))$ .

# Characterizing fam(A)

We immediately get the following.

#### Lemma

An indexed meet-semilattice  $\mathfrak{H}: \mathbf{Set}^{op} \to \mathbf{Ord}$  is equivalent to one of the form  $\mathsf{fam}(\mathcal{A})$  for a filtered PCA  $\mathcal{A}$  iff it has a discrete generic predicate and  $D(\mathfrak{H})$  is a tripos. The filter is trivial iff  $\mathfrak{H}(1) \simeq 1$ .

- Filtered PCAs are better known as inclusions of PCAs, their realizability toposes are called relative realizability toposes.
- To be able to characterize realizability *triposes*, we have reconstruct  $\mathfrak{H}$  from  $\mathcal{D}(\mathfrak{H})$ . This is what we do next.

# **∃**-*prime predicates*

## Motivation/Analogy:

- Given a poset P, the set of D(P) of downsets in P is a complete lattice under inclusion, and is in fact the free sup-lattice on P.
- The principal downsets  $\downarrow x = \{y \in P \mid y \le x\}$  can be characterized as **completely join-prime elements** in D an element x of a sup-lattice L is called completely join-prime if we have

$$x \leq \bigvee_{j \in J} y_j \quad \Rightarrow \quad \exists j \in J \,.\, x \leq y_j$$

for all families  $(y_i)_{i \in J}$  of elements.

• L is a free join-completion iff it has **enough** completely join-prime elements, i.e. if every element is a join of completely-join-primes.

## **∃**-*prime predicates*

#### Definition

Given an indexed preorder  $\mathcal H$  which admits existential quantification, a predicate  $\pi \in \mathcal H(I)$  is called  $\exists$ -**prime** if for all functions  $I \overset{u}{\leftarrow} J \overset{v}{\leftarrow} K$  and predicates  $\theta \in \mathcal H(K)$  such that  $u^*\pi \leq \exists_v \theta$ , there exists a section s of v such that  $u^*\pi \leq s^*\theta$ .

#### Lemma

An indexed preorder  $\mathfrak H$  is an  $\exists$ -completion iff it has enough  $\exists$ -prime predicates, i.e. if for every predicate  $\varphi \in \mathfrak H(I)$  there exists a function  $u: J \to I$  and an  $\exists$ -prime predicate  $\pi \in \mathfrak H(J)$  with  $\varphi \cong \exists_u \pi$ .

In this case, we have  $\mathbb{H} \simeq D(\mathbb{P})$  where  $\mathbb{P} \subseteq \mathbb{H}$  is the indexed sub-preorder on  $\exists$ -prime predicates.

With this we can characterize (relative) realizability triposes!

# Characterizing realizability triposes

#### Theorem

A tripos  $\mathfrak{H}: \mathbf{Set}^{op} \to \mathbf{Ord}$  is a relative realizability tripos over an inclusion of PCAs, iff

- 1. ℍ has enough ∃-prime predicates, and
- 2. the full indexed sup-preorder  $\mathcal{P} \subseteq \mathcal{H}$  on  $\exists$ -prime predicates is closed under finite meets and has a discrete generic predicate  $\delta$ .
- The discreteness condition on  $\delta$  can be stated in  $\mathcal H$  rather than  $\mathcal P$ , which is a slight strengthening.
- We get ordinary (non-relative) realizability if the tripos is 2-valued, i.e.  $\mathfrak{H}(1) \simeq \mathsf{Bool}$ .