Computability and Krivine realizability

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Abstract

This is a note on a construction in realizability that J.-L. Krivine showed me in PPS on November 20, 2015.

1 Detecting computability

We use the formalism of Krivine realizability with λ-terms (not combinators), without stack constants (only the symbol ε for the empty stack), and with a constant end representing termination. Thus the syntax is given by the following grammar.

Terms:	$t ::= x \mid \lambda x.t \mid t\,t \mid \mathsf{cc} \mid \mathsf{k}_\pi \mid \mathsf{end}$	
Stacks:	$\pi ::= \varepsilon \mid t{\cdot}\pi$	t closed
Processes:	$p ::= t \star \pi$	t closed

On processes we define the usual reduction relation.

$$\begin{array}{cccc}
\text{(push)} & tu \star \pi & \succ & t \star u{\cdot}\pi \\
\text{(pop)} & (\lambda x\,.\,t[x]) \star u{\cdot}\pi & \succ & t[u] \star \pi \\
\text{(save)} & \mathsf{cc} \star t{\cdot}\pi & \succ & t \star \mathsf{k}_\pi{\cdot}\pi \\
\text{(restore)} & \mathsf{k}_\pi \star t{\cdot}\rho & \succ & t \star \pi
\end{array}$$

The set of closed terms is denoted by Λ, Π is the set of stacks, and QP is the set of *quasi-proofs*, i.e. terms not containing 'end' (contrary to the traditional approach we allow continuation terms k_π – this difference is not a big deal, and is discussed in Section 5.2 of [1]). Furthermore, we use the notation $(t \star \pi)\!\downarrow$ to say that the process $t \star \pi$ terminates, i.e.

$$(t \star \pi)\!\downarrow \quad :\Leftrightarrow \quad \exists \rho\,.\, t \star \pi \succ^* \mathsf{end} \star \rho.$$

The set of terminating processes gives a *pole* $\perp\!\!\!\perp$.

$$\perp\!\!\!\perp = \{t \star \pi \in \Lambda \times \Pi \mid (t \star \pi)\!\downarrow\}$$

Using this pole, we model higher order classical logic in the usual way, where *truth values* are subsets of Π and more generally *predicates* are families of truth values $\varphi, \psi : I \to P(\Pi)$. In particular, truth $\top$ and falsity $\bot$ are given by $\varnothing$ and Π, respectively, and (using the $\|-\|$ and $|-|$ notation as e.g. in [3]) the inequality predicate on a set I can be defined as

$$\|i \neq j\| = \begin{cases} \bot & i = j \\ \top & i \neq j, \end{cases}$$

which is equivalent to the negation of *Leibniz equality* (or *Lawvere equality* [5])[1]. Moreover, it is easy to see that

$$|i \neq j| = \|i \neq j\|^{\perp\!\!\!\perp} = \begin{cases} \{t \mid (t \star \varepsilon)\downarrow\} & i = j \\ \Lambda & i \neq j. \end{cases}$$

Now let $f : \mathbb{N} \to \{0, 1\}$ be a function, and consider the formula

$$\Phi \quad \equiv \quad \forall x \,.\, \mathrm{nat}(x) \Rightarrow f(x) \neq 0 \Rightarrow f(x) \neq 1 \Rightarrow \bot.$$

Assume this formula is *valid*, i.e. there exists a quasi-proof t realizing it. Assume that ω is a diverging term, and that for $n \in \mathbb{N}$, $\overline{n}$ denotes the n-th church numeral. Let $n \in \mathbb{N}$ and assume that $f(n) = 0$. Then we have $\overline{n} \in |\mathrm{nat}(n)|$, $\mathsf{end} \in |n \neq 0|$, $\omega \in |n \neq 1|$, and $\varepsilon \in \|\bot\|$. Since t is a realizer of Φ, we can therefore deduce that the process

$$t \; \overline{n} \; \mathsf{end} \; \omega \star \varepsilon$$

terminates, and similarly, if $f(n) = 1$, we can deduce that

$$t \; \overline{n} \; \omega \; \mathsf{end} \star \varepsilon$$

terminates. Since we assumed that t is a quasi-proof, t has to bring the 'end' supplied as argument in head position in both cases to achieve termination. The argument ω, on the other hand, may never come in head position since this would mean divergence. This means that t has to compute the value of $f(n)$ in order to decide which argument to put in head position, *thus f has to be computable.*

Conversely, I assume it shouldn't be too hard to prove that Φ is valid whenever f is computable (but how?).

Next, assume that f is *not* computable. It turns out that in this case the *negation* of Φ is valid! To show this, it suffices to find a quasi-proof u such that $ut \star \varepsilon$ terminates for all realizers t of Φ. For $t \in \Phi^{\perp\!\!\!\perp}$ and $n \in \mathbb{N}$ we know that

$$t\overline{n} \Vdash f(n) \neq 0 \Rightarrow f(n) \neq 1 \Rightarrow \bot,$$

i.e.

$$t\overline{n} \Vdash \bot \Rightarrow \top \Rightarrow \bot \quad \text{or} \quad t\overline{n} \Vdash \top \Rightarrow \bot \Rightarrow \bot$$

since $f(n) = 0$ or $f(n) = 1$ for all $n \in \mathbb{N}$.

Moreover, since f is not computable, there must exist an $n \in \mathbb{N}$ such that

$$t\overline{n} \Vdash \top \Rightarrow \top \Rightarrow \bot$$

(as before, if t were able to always pick the useful one among the two arguments, then f would be computable).

Using this observation, we can construct a realizer of $\neg\Phi$ by means of a fixed point construction. Concretely, let while be a term such that

$$\mathsf{while} \; t \; n \succ^* t \; n \; (\mathsf{while} \; t \; (Sn))$$

[1]It is not obvious that this 'pointwise' definition works, but it does.

Then the realizer of $\neg\Phi$ is given by

$$u = \lambda t\,.\,\mathsf{while}\ (\lambda n x\,.\,t n x x)\ \overline{0}.$$ [2]

Thus, for non-computable f, the model validates

$$\exists n\,.\ \mathrm{nat}(n) \wedge f(n) \neq 0 \wedge f(n) \neq 1.$$

Since we have $f(n) = 0$ or $f(n) = 1$ for all 'named' integers, Krivine interprets this as saying that the model has *non-standard integers*.

Topos theoretically, a question that arises in this context is whether the *natural numbers object* has global sections that are not 'named'. This does not necessarily follow from the established existence property, since it is not a *unique* existence statement.

2 Topos theoretic interpretation

In the topos $\mathbf{Set}[\mathcal{K}_{⫫}]$ associated to the pole $⫫$, the above can be interpreted as follows. The function $f : \mathbb{N} \to 2$ (where $2 = \{0, 1\}$) gives rise to a morphism $\Delta(f) : \Delta(\mathbb{N}) \to \Delta(2)$[3]. The natural numbers object N of $\mathbf{Set}[\mathcal{K}_{⫫}]$ embeds into $\Delta(\mathbb{N})$, and the object $2 = 1+1$ embeds into $\Delta(2)$. The validity of Φ is equivalent to a map $h : \mathbf{N} \to \mathbf{2}$ making the diagram

$$\begin{tikzcd}
\mathbf{N} \arrow[r, dashed, "h"] \arrow[d, tail] & \mathbf{2} \arrow[d, tail] \\
\Delta(\mathbb{N}) \arrow[r, "\Delta f"'] & \Delta(2)
\end{tikzcd}$$

commute.

This observation is curious, for the following reason. We have just shown that such an h exists if and only if f is computable. However, higher order classical logic allows to prove the existence of non-computable functions from the integers to a 2-element set – in particular any definable predicate on the integers has a characteristic function. So if we choose f to be the characteristic function of such a definable predicate, then interpreting the same predicate in the model gives rise to a function $h : \mathbf{N} \to \mathbf{2}$ in $\mathbf{Set}[\mathcal{K}_{⫫}]$. However, this h will not make the above diagram commute, and in particular will not be 'tracked' by f.

A The while loop

Let $⫫$ be an arbitrary pole, let $S = \lambda n f x\,.\,f(n f x)$, let $\overline{n} = S^n(\lambda f x\,.\,x)$ for $n \in \mathbb{N}$ (then the $\overline{n}$ are β-equivalent to Church numerals), and let 'while' be a term with

$$\mathsf{while}\ x y \star \pi \succ^* x y \star (\mathsf{while}\ x(Sy))\cdot\pi.$$

[2] A systematic exposition of 'while' is in Appendix A.

[3] Δ is the 'constant objects functor' [2, Def. 3.7] and it seems to correspond to Krivine's $\gimel$-function [4, pg. 14].

Theorem 1 *Let t be a term such that*

$$\forall n \in \mathbb{N} \,.\, t\,\overline{n} \Vdash \bot \Rightarrow \bot \quad \textit{and} \quad \exists n \in \mathbb{N} \,.\, t\,\overline{n} \Vdash \top \Rightarrow \bot.$$

Then $\mathsf{while}\ t\ \overline{0} \Vdash \bot$.

Proof. We show that the set $\{n \in \mathbb{N} \mid \mathsf{while}\ t\ \overline{n} \Vdash \bot\}$ is inhabited and downward closed. By assumption there exists n_0 with $t\ \overline{n_0} \Vdash \top \Rightarrow \bot$. Let $\pi \in \|\bot\| = \mathsf{\Pi}$. Then we have

$$\mathsf{while}\ t\ \overline{n_0} \star \pi \succ^* t\ \overline{n_0}\ \star (\mathsf{while}\ t\ (S\ \overline{n_0}))\cdot\pi \in \perp\!\!\!\perp.$$

Now let $n \in \mathbb{N}$ such that $\mathsf{while}\ t\ \overline{n+1} \Vdash \bot$ and let $\pi \in \|\bot\|$. Then we have

$$\mathsf{while}\ t\ \overline{n} \star \pi \ \succ^* \ t\ \overline{n} \star (\mathsf{while}\ t\ (S\ \overline{n}))\cdot\pi \ = \ t\ \overline{n} \star (\mathsf{while}\ t\ \overline{n+1})\cdot\pi$$

which is in $\perp\!\!\!\perp$ since $t\overline{n} \Vdash \bot \Rightarrow \bot$. ∎

References

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