Characterizing clan-algebraic categories

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Overview

Context

- In talks at HoTT/UF 2020 and at CT 2021 I presented a conjecture concerning categories of models of a *clan*.
- In this talk I will give/outline a proof of this conjecture.

Three Parts

- Recall functorial semantics of (essentially) algebraic theories
- Clans as generalized algebraic theories
- If there's time: Examples and models in higher (homotopy) types

Part I

Algebraic Theories

Definition

A **single-sorted algebraic theory** (SSAT) is a pair (Σ, E) consisting of

- a family $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$, of sets of n-ary **operations**
- a set of **equations** E whose elements are pairs of open terms over Σ

Definition

The **syntactic category** $\mathcal{C}(\Sigma, E)$ of a SSAT is given as follows:

1. For each natural number $n \in \mathbb{N}$ there is an **object** $[n]$
2. **morphisms** $\sigma : [n] \to [m]$ are m-tuples of terms in n variables modulo E-provable equality
3. **identities** are lists of variables, **composition** is given by substitution

Proposition

Given a SSAT (Σ, E):

1. $\mathcal{C}(\Sigma, E)$ has finite products given by $[n] \times [m] = [n + m]$
2. $\mathbf{Set}\text{-}\mathbf{Mod}(\Sigma, E) \simeq \mathbf{FP}(\mathcal{C}(\Sigma, E), \mathbf{Set})$

Finite-product theories

Definition

- A **FP-theory** is just a small FP-category $\mathcal{C}$.
- **Models** of $\mathcal{C}$ are FP-functors $A : \mathcal{C} \to \mathbf{Set}$ (or into another FP-category).

Denote the category of models by

$$\mathbf{Mod}(\mathcal{C}) \;:=\; \mathbf{FP}(\mathcal{C}, \mathbf{Set}) \;\overset{\text{full}}{\subseteq}\; [\mathcal{C}, \mathbf{Set}].$$

For every object $\Gamma \in \mathcal{C}$ of an FP-theory, the co-representable functor

$$\mathcal{C}(\Gamma, -) \;:\; \mathcal{C} \to \mathbf{Set}$$

is a model. Thus, the dual Yoneda embedding co-restricts to $\mathbf{Mod}(\mathcal{C})$.

$$\begin{tikzcd} & \mathcal{C}^{\mathrm{op}} \arrow[dl, dashed, "Z"'] \arrow[d] \\ \mathbf{Mod}(\mathcal{C}) \arrow[r, phantom, "\subseteq"] & {[\mathcal{C}, \mathbf{Set}]} \end{tikzcd}$$

Finite-limit theories

Definition

- A **FL-theory** is a small finite-limit category $\mathcal{L}$.
- A **model** of $\mathcal{L}$ is a finite-limit preserving functor $A : \mathcal{L} \to \mathbf{Set}$.

FL-theories are more expressive than FP-theories – structures definable by finite-limit theories include

- categories, posets, 2-categories, monoidal categories, categories with families ...

Again $\mathcal{L}(\Gamma, -)$ is a model for every $\Gamma \in \mathcal{L}$ and we get an embedding

$$Z \ : \ \mathcal{L}^{\mathsf{op}} \ \to \ \mathbf{Mod}(\mathcal{L}) := \mathbf{FL}(\mathcal{L}, \mathbf{Set}) \overset{\text{full}}{\subseteq} [\mathcal{L}, \mathbf{Set}].$$

Moreover, we can characterize the essential image of Z in $\mathbf{Mod}(\mathcal{L})$.

Locally finitely presentable categories

Definition

- An object C of a cocomplete locally small category $\mathfrak{X}$ is called **compact**[a], if

$$\mathfrak{X}(C, -) : \mathfrak{X} \to \mathbf{Set}$$

  preserves filtered colimits.
- A category $\mathfrak{X}$ is called **locally finitely presentable**, if
  - $\mathfrak{X}$ is locally small and cocomplete
  - the full subcategory $\mathrm{comp}(\mathfrak{X}) \subseteq \mathfrak{X}$ on compact objects is essentially small and dense.

[a] More traditionally: 'finitely presentable'

Theorem

- $\mathbf{Mod}(\mathcal{L})$ *is locally finitely presentable for all finite-limit theories* $\mathcal{L}$.
- *The essential image of* $Z : \mathcal{L}^{\mathrm{op}} \to \mathbf{Mod}(\mathcal{L})$ *comprises precisely the compact objects.*

Gabriel-Ulmer duality[1]

Theorem

There is a bi-equivalence of 2-categories

$$\mathbf{FL} \underset{\mathcal{L} \mapsto \mathbf{Mod}(\mathcal{L})}{\overset{\mathrm{comp}(\mathfrak{X})^{\mathrm{op}} \leftarrow\!\shortmid\, \mathfrak{X}}{\longleftrightarrow}} \mathbf{LFP}^{\mathrm{op}}$$

where

- **FL** is the 2-category of **small** FL-categories and FL-functors
- **LFP** is the 2-category of locally finitely presentable categories and functors preserving small limits and filtered colimits (‘forgetful functors’).

[1] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Springer-Verlag, 1971.

Duality for finite-product theories[2]

There's a 'restriction' of G–U duality to finite-product theories:

$$\begin{tikzcd}
\mathbf{FP}_{\mathsf{cc}} \arrow[rr, leftrightarrow, "\mathcal{C} \mapsto \mathbf{FP}(\mathcal{C}{,}\mathbf{Set})", "\{\text{compact projectives}\}^{\mathsf{op}} \mapsfrom \mathfrak{X}"'] \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] & & \mathbf{ALG}^{\mathsf{op}} \arrow[d, hook, "J"] \\
\mathbf{FL} \arrow[u, bend right, "U"'] \arrow[rr, leftrightarrow, "\mathcal{L} \mapsto \mathbf{FL}(\mathcal{L}{,}\mathbf{Set})", "\{\text{compact objects}\}^{\mathsf{op}} \mapsfrom \mathfrak{X}"'] & & \mathbf{LFP}^{\mathsf{op}}
\end{tikzcd}$$

- $\mathbf{FP}_{\mathsf{cc}}$ is the 2-category of Cauchy-complete finite-product categories
- **ALG** is the 2-category of **algebraic categories** and **algebraic functors**
  - An **algebraic category** is an l.f.p. category which is Barr-exact and where the compact (regular) projective objects are dense
  - An **algebraic functor** is a functor that preserves small limits, filtered colimits, and regular epimorphisms.
- There's also a formulation in terms of sifted colimits, but we don't need it.

[2] J. Adámek, J. Rosický, and E.M. Vitale. *Algebraic theories: a categorical introduction to general algebra*. Cambridge University Press, 2010.

Part II

Toward clans

- Finite-limit theories have a nice duality theory but seem far from syntax
- Syntactic counterparts are given by
  - Freyd's **essentially algebraic theories**[3]
  - Cartmell's **generalized algebraic theories**[4] (or 'dependent algebraic theories')
  - Johnstone's **cartesian theories**[5]
  - Palmgren and Vickers' **quasi-equational theories**[6]
  - and probably others
- Clans can be viewed as a categorical representation of generalized algebraic theories
- They're as expressive as FL-theories, but 'finer', i.e. closer to syntax

[3] P. Freyd. "Aspects of topoi". In: *Bulletin of the Australian Mathematical Society* (1972).

[4] J. Cartmell. "Generalised algebraic theories and contextual categories". In: *Annals of Pure and Applied Logic* (1986).

[5] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 2*. Oxford: Oxford University Press, 2002.

[6] E. Palmgren and S. J. Vickers. "Partial horn logic and Cartesian categories". In: *Annals of Pure and Applied Logic* (2007).

Definition

A **clan** is a small category $\mathcal{T}$ with terminal object 1, equipped with a class $\mathcal{T}_\dagger \subseteq \mathrm{mor}(\mathcal{T})$ of morphisms – called **display maps** and written $\twoheadrightarrow$ – such that

1. pullbacks of display maps along all maps exist and are display maps
$$\begin{tikzcd} \Delta^+ \arrow[r, "s^+"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma^+ \arrow[d, two heads, "p"] \\ \Delta \arrow[r, "s"] & \Gamma \end{tikzcd},$$
2. display maps are closed under composition, and
3. isomorphisms and terminal projections $\Gamma \twoheadrightarrow 1$ are display maps.

- Definition due to Taylor[7], name due to Joyal[8] ('a clan is a collection of families')
- Relation to semantics of dependent type theory: display maps represent **type families**.
- Observation: clans have finite products (as pullbacks over 1).

[7] P. Taylor. "Recursive domains, indexed category theory and polymorphism". PhD thesis. University of Cambridge, 1987, § 4.3.2.

[8] A. Joyal. "Notes on clans and tribes". In: *arXiv preprint arXiv:1710.10238* (2017).

Examples

- Finite-product categories $\mathcal{C}$ can be viewed as clans with $\mathcal{C}_\dagger = \{\text{product projections}\}$
- Finite-limit categories $\mathcal{L}$ can be viewed as clans with $\mathcal{L}_\dagger = \mathrm{mor}(\mathcal{L})$

We call such clans **FP-clans**, and **FL-clans**, respectively.

- The syntactic category of every Cartmell-style **generalized algebraic theory** is a clan.
- Clan for categories:

$$\begin{aligned} \mathcal{K} &= \{\text{categories free on finite graphs}\}^{\mathsf{op}} \subseteq \mathbf{Cat}^{\mathsf{op}} \\ \mathcal{K}_\dagger &= \{\text{functors induced by graph inclusions}\}^{\mathsf{op}} \end{aligned}$$

  $\mathcal{K}$ can be viewed as syntactic category of a generalized algebraic theory of categories with a sort O of objects, and a dependent sort $x, y{:}O \vdash M(x, y)$ of morphisms – vertices of a finite graph are object variables and edges are morphism variables in a context. Graph inclusions are dual to context extensions.

Models

Definition

A **model** of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathbf{Set}$ which preserves 1 and pullbacks of display-maps.

- The category $\mathbf{Mod}(\mathcal{T}) \subseteq [\mathcal{T}, \mathbf{Set}]$ of models is l.f.p. and contains $\mathcal{T}^{\mathsf{op}}$.
- For FP-clans $(\mathcal{C}, \mathcal{C}_\dagger)$ we have $\mathbf{Mod}(\mathcal{C}, \mathcal{C}_\dagger) = \mathbf{FP}(\mathcal{C}, \mathbf{Set})$.
- For FL-clans $(\mathcal{L}, \mathcal{L}_\dagger)$ we have $\mathbf{Mod}(\mathcal{L}, \mathcal{L}_\dagger) = \mathbf{FL}(\mathcal{L}, \mathbf{Set})$.
- $\mathbf{Mod}(\mathcal{K}, \mathcal{K}_\dagger) = \mathbf{Cat}$.

$$
\begin{tikzcd}
 & \mathcal{T}^{\mathsf{op}} \arrow[dl, dashed, "z"'] \arrow[d] \\
\mathbf{Mod}(\mathcal{T}) \arrow[r, phantom, "\subseteq"] & {[\mathcal{T}, \mathbf{Set}]}
\end{tikzcd}
$$

Observation

The same category of models may be represented by different clans.
For example, SSATs can be represented by FP-clans as well as FL-clans.

The weak factorization system

- Would like duality between clans and their categories of models.
- Since the same l.f.p. category can be represented by different clans, we cannot hope to reconstruct the clan from the models alone.
- Solution: equip the models with additional structure in form of a **weak factorization system**.

Definition

Let $\mathcal{T}$ be a clan. Define w.f.s. $(\mathcal{E}, \mathcal{F})$ on $\mathbf{Mod}(\mathcal{T})$ by

- $\mathcal{F} := \mathbf{RLP}(\{Z(p) \mid p \in \mathcal{T}_\dagger\})$ class of **full maps**
- $\mathcal{E} := \mathbf{LLP}(\mathcal{F})$ class of **extensions**

I.e. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by the image of $\mathcal{T}_\dagger$ under $Z : \mathcal{T}^{\mathrm{op}} \to \mathbf{Mod}(\mathcal{T})$.

- Call $A \in \mathbf{Mod}(\mathcal{T})$ a 0-**extension**, if $(0 \to A) \in \mathcal{E}$

- E.g. corepresentables $Z(\Gamma)$ are 0-extensions since terminal projections $\Gamma \twoheadrightarrow 1$ are display maps.
- The same weak factorization system was also introduced by S. Henry in a HoTTEST talk[9], see also[10].

[9] S. Henry, *The language of a model category*, HoTTEST seminar, Jan. 2020, https://youtu.be/7_X0qbSXlfk

[10] S. Henry. "Algebraic models of homotopy types and the homotopy hypothesis". In: *arXiv preprint arXiv:1609.04622* (2016).

Full maps

- $f : A \to B$ in $\mathbf{Mod}(\mathcal{T})$ is full iff it has the RLP with respect to all $Z(p)$ for display maps $p : \Delta \rightarrowtriangle \Gamma$.

$$\begin{tikzcd}
\mathcal{T}(\Gamma,-) \arrow[r] \arrow[d, "Z(p)=\mathcal{T}(p,-)"'] & A \arrow[d, "f"] \\
\mathcal{T}(\Delta,-) \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A(\Delta) \arrow[r, "f_\Delta"] \arrow[d, "A(p)"'] & B(\Delta) \arrow[d, "B(p)"] \\
A(\Gamma) \arrow[r, "f_\Gamma"] & B(\Gamma)
\end{tikzcd}$$

- This is equivalent to display-naturality-squares being weak pullbacks.
- Considering $p : \Delta \rightarrowtriangle 1$ we see that full maps are surjective and hence regular epis.

$$\begin{tikzcd}
A(\Delta) \arrow[r, "f_\Delta"] \arrow[d] & B(\Delta) \arrow[d] \\
1 \arrow[r] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
A(\Delta) \arrow[r, "f_\Delta"] \arrow[d] & B(\Delta) \arrow[d] \\
A(\Delta) \times A(\Delta) \arrow[r, "f_\Delta \times f_\Delta"] & B(\Delta) \times B(\Delta)
\end{tikzcd}$$

- For FL-clans, only isos are full (consider naturality square for diagonal $\Delta \to \Delta \times \Delta$)
- For FP-clans we have

$$\begin{array}{rcl}
\text{full map} & = & \text{regular epimorphism} \\
\text{extension} & = & \text{coproduct inclusion } A \hookrightarrow P + A \text{ with } P \text{ projective} \\
0\text{-extension} & = & \text{projective object}
\end{array}$$

The fat small object argument

Motivation: subcategories of models for FP-theory $\mathcal{C}$ and clan $\mathcal{T}$.

$$
\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[rr, hook, "\text{reflexive coeqalizer}", "\text{completion}"'] \arrow[dd, hook] \arrow[dr, hook, "\text{filtered colimit}", "\text{completion}"'] \arrow[ddrr, hook, "\text{sifted colimit completion}"] & & \{\text{compact}\} \arrow[dd, hook, "\text{filtered colimit}", "\text{completion}"'] \\
& \{\text{flat}\} \arrow[dr, hook] & \\
\{\text{projective}\} \arrow[ur, hook] \arrow[rr, hook, "\text{ex/wlex completion}"'] & & \mathbf{Mod}(\mathcal{C}) \subseteq [\mathcal{C}, \mathbf{Set}]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}^{\mathrm{op}} \arrow[rr, hook] \arrow[dd, hook] \arrow[dr, hook, "\text{filtered colimit}", "\text{completion}"'] \arrow[ddrr, hook] & & \{\text{compact}\} \arrow[dd, hook, "\text{filtered colimit}", "\text{completion}"'] \\
& \{\text{flat}\} \arrow[dr, hook] & \\
\{\text{0-extension}\} \arrow[ur, hook] \arrow[rr, hook] & & \mathbf{Mod}(\mathcal{T}) \subseteq [\mathcal{T}, \mathbf{Set}]
\end{tikzcd}
$$

- Flat algebras are filtered colimits of corepresentables, computed *freely* in the functor categories.
- For SSATs we have $\{\text{projective}\} \subseteq \{\text{flat}\}$ since
  - arbitrary free objects are filtered colimits of free objects over finite sets
  - projective objects are retracts of free objects
- In the general clan case, $\{\text{0-extension}\} \subseteq \{\text{flat}\}$ by the **fat small object argument**[11].

[11] M. Makkai, J. Rosicky, and L. Vokrinek. "On a fat small object argument". In: *Advances in Mathematics* (2014).

Reconstructing the clan

Definition

Given a clan $\mathcal{T}$, let $\mathbb{C} \subseteq \mathbf{Mod}(\mathcal{T})$ be the full subcategory on **compact** 0**-extensions**.

- $Z : \mathcal{T}^{\mathsf{op}} \to \mathbf{Mod}(\mathcal{T})$ factors through $\mathbb{C}$ since corepresentables $Z(\Gamma)$ are compact and 0-extensions.

$$\begin{tikzcd} & \mathbb{C} \arrow[d, hook] \\ \mathcal{T}^{\mathsf{op}} \arrow[ur, dashed, "E"] \arrow[r, "Z"'] & \mathbf{Mod}(\mathcal{T}) \end{tikzcd}$$

- $0 \in \mathbb{C}$ and if $\begin{tikzcd} C \arrow[r] \arrow[d, hook, "e"] & D \arrow[d, hook] \\ E \arrow[r] \arrow[ur, phantom, "\ulcorner"] & F \end{tikzcd}$ is a pushout with $F \in \mathbb{C}$ and $e \in \mathcal{E}$ then $F \in \mathbb{C}$.
- Therefore $\mathbb{C}$ is a **coclan** with extensions as "co-display maps".

Reconstructing the clan

Theorem

The full inclusion $E : \mathcal{T}^{\mathrm{op}} \hookrightarrow \mathbb{C}$ exhibits $\mathbb{C}$ as *Cauchy-completion* of $\mathcal{T}^{\mathrm{op}}$, i.e. every compact 0-extension is a retract of a corepresentable.

Proof.

- Let $C \in \mathbb{C}$.
- Since 0-extensions are flat, $\int C$ is filtered, thus C is a filtered colimit of corepresentables.
- Since C is compact, id_C factors through a colimit inclusion map.

$$\begin{tikzcd} & C \arrow[dl, dashed] \arrow[d, "\mathrm{id}"] \\ Z(\Gamma) \arrow[r, "\sigma_{(\Gamma,x)}"'] & C \end{tikzcd}$$

□

Clan-algebraic categories

Definition

A **clan-algebraic category** is a category $\mathfrak{X}$ with a w.f.s. $(\mathcal{E}, \mathcal{F})$ that arises as category of models of a clan.

With this definition we get a contravariant bi-equivalence

$$\mathbf{Clan}_{\mathsf{cc}} \quad \underset{\mathcal{T} \mapsto \mathbf{Mod}(\mathcal{T})}{\overset{\mathrm{comp}(\mathfrak{X})^{\mathsf{op}} \leftarrow\!\shortmid\, \mathfrak{X}}{\longleftarrow\!\!\!\longrightarrow}} \quad \mathbf{cAlg}^{\mathsf{op}}$$

between

- the 2-category $\mathbf{Clan}_{\mathsf{cc}}$ of Cauchy-complete clans and functors preserving 1, display maps, and pullbacks of display maps, and
- the 2-category $\mathbf{cAlg}$ of clan-algebraic categories and functors preserving small limits, filtered colimits, and full maps.

Can we characterize clan-algebraic categories more abstractly?

Characterizing clan-algebraic categories

Assume $\mathfrak{X}$ is clan-algebraic with w.f.s. $(\mathcal{E}, \mathcal{F})$. Then

1. $\mathfrak{X}$ is cocomplete,
2. $\mathfrak{X}$ has a small dense family of compact 0-extensions, and
3. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between compact 0-extensions.

Now assume we have a category $\mathfrak{X}$ with w.f.s. $(\mathcal{E}, \mathcal{F})$ satisfying 1–3.

Then the subcategory $\mathbb{C} \subseteq \mathfrak{X}$ of compact 0-extensions is a coclan.

We get a nerve/realization adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook, "J"] \arrow[d, "Z"'] & \mathfrak{X} \arrow[dl, bend left, "N"] \\ \mathbf{Mod}(\mathbb{C}^{\mathrm{op}}) \arrow[ur, bend left, "L"] \arrow[ur, phantom, "\dashv"] & \end{tikzcd}$$

$$L(A) = \mathrm{colim}(\textstyle\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})$$
$$N(X) = \mathfrak{X}(J(-), X)$$

However, this adjunction is not an equivalence in general:

Characterizing clan-algebraic categories

Counterexample

Consider

- $\mathfrak{X} \subseteq [2^{\mathrm{op}}, \mathbf{Set}]$ full subcategory on injections
- $(\mathcal{E}, \mathcal{F})$ w.f.s. on $\mathfrak{X}$ cofib. generated by $\{(0 \to Y0), (0 \to Y1)\}$

Then $\mathbf{Mod}(\{\text{compact 0-extensions}\}^{\mathrm{op}}) \simeq [2^{\mathrm{op}}, \mathbf{Set}]$ and N is the subcategory inclusion.

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook, "J"] \arrow[d, "Z"'] & \mathfrak{X} \arrow[dl, hook, bend left, "N"] \\ {[2^{\mathrm{op}}, \mathbf{Set}]} \arrow[ur, bend left, "L"] \arrow[ur, phantom, "\dashv"] & \end{tikzcd}$$

Conclusion: We're missing an 'exactness condition' analogous to 'Barr-exactness' in the characterization of algebraic categories!

Quotients of componentwise-full equivalence relations

- Recall that a FL-category $\mathcal{L}$ is called *Barr-exact*, if all equivalence relations in $\mathcal{L}$ have stable effective quotients.
- This can't be the case for clan algebraic categories in general. However, we have:

Lemma

For any clan $\mathcal{T}$, **$\mathbf{Mod}(\mathcal{T})$** *has* **full and effective quotients** *of* **componentwise-full equivalence relations***.*

Proof.

Given equivalence relation $r : R \rightarrowtail A \times A$ with $r_0, r_1 : R \twoheadrightarrow A$ full, show that component-wise quotient is a model again. $\square$

Characterizing clan-algebraic categories

Definition

An **adequate category** is a category $\mathfrak{X}$ with a with a w.f.s. $(\mathcal{E}, \mathcal{F})$ (whose maps we call extensions and full, respectively), s.th.

1. $\mathfrak{X}$ is cocomplete,
2. $\mathfrak{X}$ has a small dense family of compact 0-extensions (in particular $\mathfrak{X}$ is l.f.p.),
3. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between compact 0-extensions, and
4. $\mathfrak{X}$ has full and effective quotients of componentwise-full equivalence relations.

Lemma

Assume $\mathfrak{X}$ is adequate and $F : \mathfrak{X} \to \mathbf{Set}$ preserves finite limits and sends full maps to surjections. Then F preserves quotients of componentwise-full equivalence relations.

Proof.

Let $R \underset{r_1}{\overset{r_0}{\rightrightarrows}} A \xrightarrow{f} B$ be a **full exact sequence** in $\mathfrak{X}$, i.e. all arrows are full, f is the coequalizer of r_0, r_1, and r_0, r_1 is the kernel pair of f. Then Ff is a surjection with kernel pair Fr_0, Fr_1. But surjections are always coequalizers of their kernel pair. $\square$

Idea of proof

- Assume that $\mathfrak{X}$ is adequate.
- To show that it is clan-algebraic, we want to show that its nerve/realization adjunction

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook, "J"] \arrow[d, "Z"'] & \mathfrak{X} \arrow[dl, bend left, "N"] \\ \mathbf{Mod}(\mathbb{C}^{\mathrm{op}}) \arrow[ur, bend left, "L"] \arrow[ur, phantom, "\dashv"] & \end{tikzcd} \qquad \begin{aligned} L(A) &= \mathrm{colim}(\textstyle\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X}) \\ N(X) &= \mathfrak{X}(J(-), X) \end{aligned}$$

  is an equivalence.
- By density the right adjoint N is fully faithful, i.e. the counit is an isomorphism.
- It remains to show that the unit of the adjunction is an isomorphism, i.e.

$$A(C) \xrightarrow{\cong} \mathfrak{X}(C, \mathrm{colim}(\textstyle\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})).$$

  for all $A \in \mathbf{Mod}(\mathbb{C}^{\mathrm{op}})$ and $C \in \mathbb{C}$.
- We know that $\mathcal{X}(C, -)$ preserves filtered colimits and quotients of componentwise-full equivalence relations, so we'd like to decompose $\mathrm{colim}(\int A \to \mathbb{C} \xrightarrow{J} \mathfrak{X})$ in terms of these constructions.
- This is essentially what we're doing in the following.

Jointly full cones

- Let $D : \mathcal{I} \to \mathfrak{X}$ be a diagram in an adequate category.
- A cone (A, ϕ) over D is called **jointly full**, if for every cone (C, γ), extension $e : B \to C$ and map $g : B \to A$ constituting a cone morphism $g : (B, \gamma \circ e) \to (A, \phi)$, there exists a map $h : C \to A$ such that

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "e"'] & A \arrow[d, "\phi_i"] \\ C \arrow[r, "\gamma_i"] \arrow[ur, dashed, "h"] & D_i \end{tikzcd}$$

  commutes for all $i \in \mathcal{I}$.
- **Observation:** The cone (A, ϕ) is jointly full iff the canonical map to the limit is full.

Definition

A **nice diagram** in an adequate category $\mathfrak{X}$ is a truncated simplicial diagram

$$\begin{tikzcd} A_2 \arrow[r, shift left=4, "d_0"] \arrow[r, shift left=0, "d_1" description] \arrow[r, shift right=4, "d_2"'] & A_1 \arrow[l, shift right=2, "s_0"'] \arrow[l, shift left=2, "s_1"] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & A_0 \arrow[l, "s_0" description] \end{tikzcd}$$

where

1. A_0, A_1, and A_2 are 0-extensions,
2. the maps $d_0, d_1 : A_1 \to A_0$ are full,
3. in the square $\begin{tikzcd} A_2 \arrow[r, "d_0"'] \arrow[d, "d_2"'] & A_1 \arrow[d, "d_1"] \\ A_1 \arrow[r, "d_0"] & A_0 \end{tikzcd}$ the span constitutes a jointly full diagram over the cospan,
4. there exists a symmetry map $\begin{tikzcd} A_1 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, "\sigma"] & A_0 \\ A_0 & A_1 \arrow[l, "d_1"] \arrow[u, "d_0"'] \end{tikzcd}$ making the triangles commute, and
5. there exists a 0-extension $\tilde{A}$ and full maps $f, g : \tilde{A} \twoheadrightarrow A_1$ constituting a jointly full cone over the diagram

$$\begin{tikzcd} A_1 \arrow[d, "d_0"'] \arrow[dr, "d_1"] & A_1 \arrow[d, "d_1"] \arrow[dl, "d_0"'] \\ A_0 & A_0 \end{tikzcd}.$$

Nice diagrams

Lemma

For any nice diagram, the pairing $A_1 \xrightarrow{\langle d_0, d_1 \rangle} A_0 \times A_0$ *admits a decomposition* $A_1 \twoheadrightarrow R \xrightarrow{\langle r_0, r_1 \rangle} A_0 \times A_0$ *into a full map and a monomorphism, and* $\langle r_0, r_1 \rangle$ *is a componentwise-full equivalence relation.*

Lemma

Assume $\mathfrak{X}$ *is adequate and* $F : \mathfrak{X} \to \mathbf{Set}$ *preserves finite limits and sends full maps to surjections. Then for every nice diagram,* F *preserves coequalizers of the arrows* $d_0, d_1 : A_1 \to A_0$.

Lemma

The restriction L' *of* L *in the nerve/realization adjunction*

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, hook, "J"] \arrow[d] & & \mathfrak{X} \arrow[ddll, bend left, "N"] \\
\{0\text{-}ext\} \arrow[urr, "L'"] \arrow[d, hook] & \dashv & \\
\mathbf{Mod}(\mathbb{C}^{op}) & &
\end{tikzcd}$$

to 0*-extensions is fully faithful and preserves full maps and nice diagrams.*

Nice diagrams

Lemma

For every object A of an adequate category $\mathfrak{X}$ there exists a nice diagram

$$\begin{tikzcd} A_2 \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=3, "d_2"'] & A_1 \arrow[l, shift right=1.5, "s_0"'] \arrow[l, shift left=1.5, "s_1"] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & A_0 \arrow[l, "s_0"'] \end{tikzcd}$$

such that A is the coequalizer of $d_0, d_1 : A_1 \to A_0$.

Proof.

- A_0 is given by covering A by a 0-extension, i.e. factoring $0 \to A$ as $0 \hookrightarrow A_0 \overset{e}{\twoheadrightarrow} A$.
- A_1 is given by covering the kernel of $A_0 \twoheadrightarrow A$ by a 0-extension

$$\begin{tikzcd} 0 \arrow[r, hook] & A_1 \arrow[r, two heads] & R \arrow[r, "r_0"] \arrow[d, "r_1"'] \arrow[dr, phantom, "\lrcorner"] & A_0 \arrow[d, "e"] \\ & & A_0 \arrow[r, "e"] & A \end{tikzcd}$$

- A_2 is given by covering the following pullback:

$$\begin{tikzcd} 0 \arrow[r, hook] & A_2 \arrow[r, two heads] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_1 \arrow[d, "d_0"] \\ & & A_1 \arrow[r, "d_1"] & A_0 \end{tikzcd}$$

$\square$

The theorem

Theorem

Adequate categories are clan-algebraic.

Proof.

Let $\mathfrak{X}$ be adequate and let $\mathbb{C} \subseteq \mathfrak{X}$ be the co-clan of compact 0-extensions. It remains to show that

$$AC \cong \mathfrak{X}(C, LA).$$

for all $A \in \mathbf{Mod}(\mathbb{C}^{\mathrm{op}})$ and $C \in \mathbb{C}$. Let $A_\bullet$ be a nice diagram with coequalizer A. We have

$$\begin{aligned}
\mathfrak{X}(C, LA) &= \mathfrak{X}(C, L(\mathrm{coeq}(A_1 \rightrightarrows A_0))) && \text{since } A = \mathrm{coeq}(A_1 \rightrightarrows A_0) \\
&\cong \mathfrak{X}(C, \mathrm{coeq}(LA_1 \rightrightarrows LA_0)) && \text{since } L \text{ preserves colimits} \\
&\cong \mathrm{coeq}(\mathfrak{X}(C, LA_1) \rightrightarrows \mathfrak{X}(C, LA_0)) && \text{since } \mathfrak{X}(C, -) \text{ preserves coeqs of nice diags} \\
&\cong \mathrm{coeq}(A_1 C \rightrightarrows A_0 C) && \text{since } LA_i = \mathrm{colim}(\textstyle\int A_i \to \mathbb{C} \to \mathfrak{X}) \text{ filtered} \\
&\cong \mathrm{coeq}(\mathbf{Mod}(ZC, A_1) \rightrightarrows \mathbf{Mod}(ZC, A_0)) \\
&\cong \mathbf{Mod}(ZC, \mathrm{coeq}(A_1 \rightrightarrows A_0)) \\
&\cong \mathbf{Mod}(ZC, A)) \\
&\cong AC
\end{aligned}$$

□

Part III

Models in higher types

Let $\mathcal{S}$ be the ∞-topos of spaces/types.

Let $\mathcal{C}_{\mathsf{Mon}}$ be the finite-product theory of monoids, and let $\mathcal{L}_{\mathsf{Mon}}$ be the finite-limit theory of monoids. Then

$$\mathbf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathbf{Set}) \simeq \mathbf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathbf{Set})$$

but $\mathbf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ and $\mathbf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$ are different:

- $\mathbf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$ is just the category of monoids
- $\mathbf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$ is the ∞-category 'A_∞-algebras', i.e. homotopy-coherent monoids.

Moral

By being 'slimmer', finite-product theories leave room for higher coherences when interpreted in higher types.

This phenomenon has been discussed under the name 'animation' in:

- K. Cesnavicius and P. Scholze. "Purity for flat cohomology". In: *arXiv preprint arXiv:1912.10932* (2019)

Four clans for categories

Cat admits several clan-algebraic weak factorization systems:

- $(\mathcal{E}_1, \mathcal{F}_1)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2})\}$
- $(\mathcal{E}_2, \mathcal{F}_2)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (2 \to 1)\}$
- $(\mathcal{E}_3, \mathcal{F}_3)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2})\}$
- $(\mathcal{E}_4, \mathcal{F}_4)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2}), (2 \to 1)\}$

where $\mathbb{P} = (\bullet \rightrightarrows \bullet)$.

The right classes are:

$$
\begin{aligned}
\mathcal{F}_1 &= \{\text{full and surjective-on-objects functors}\} \\
\mathcal{F}_2 &= \{\text{full and bijective-on-objects functors}\} \\
\mathcal{F}_3 &= \{\text{fully faithful and surjective-on-objects functors}\} \\
\mathcal{F}_4 &= \{\text{isos}\}
\end{aligned}
$$

Note that $\mathcal{F}_3$ is the class of trivial fibrations for the canonical model structure on **Cat**.

Four clans for categories

These correspond to the following clans:

$$\mathcal{T}_1 = \{\text{free cats on fin. graphs}\}^{\mathsf{op}} \qquad \mathcal{T}_1^\dagger = \{\text{graph inclusions}\}$$

$$\mathcal{T}_2 = \{\text{free cats on fin. graphs}\}^{\mathsf{op}} \qquad \mathcal{T}_2^\dagger = \{\text{injective-on-edges maps}\}$$

$$\mathcal{T}_3 = \{\text{f.p. cats}\}^{\mathsf{op}} \qquad \mathcal{T}_3^\dagger = \{\text{injective-on-objects functors}\}$$

$$\mathcal{T}_4 = \{\text{f.p. cats}\}^{\mathsf{op}} \qquad \mathcal{T}_4^\dagger = \{\text{all functors}\}$$

Models in higher types:

$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_1) = \{\text{Segal spaces}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_2) = \{\text{Segal categories}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_3) = \{\text{pre-categories}\}$$
$$\infty\text{-}\mathbf{Mod}(\mathcal{T}_4) = \{\text{discrete 1-categories}\}$$

Thanks for your attention!