DUALITY FOR CLANS: A REFINEMENT OF GABRIEL–ULMER DUALITY

JONAS FREY

Abstract. We exhibit an idempotent contravariant biadjunction between 2-categories of clans, and of locally finitely presentable categories equipped with a weak factorization system, and characterize the stable subcategories.

Contents

§1. Introduction. *Gabriel–Ulmer duality* [14] is a contravariant biequivalence

$$\mathsf{FL} \overset{\mathsf{op}}{\simeq} \mathsf{LFP}$$

between the 2-category FL of *small finite-limit categories*, and the 2-category LFP of *locally finitely presentable categories*, i.e. locally small cocomplete categories admitting a dense set of compact (a.k.a. finitely presentable) objects. The duality assigns to every small finite-limit category $\mathcal{C}$ the category $\mathsf{FL}(\mathcal{C}, \mathsf{Set})$ of finite-limit preserving functors to Set, and conversely it associates to every locally finitely presentable $\mathfrak{X}$ the opposite of its full subcategory $\mathsf{comp}(\mathfrak{X}) \subseteq \mathfrak{X}$ of compact objects[1].

We view Gabriel–Ulmer duality as a *theory-model duality*: small finite-limit categories $\mathcal{C}$ are viewed as theories (which we call 'finite-limit theories'), and—in the spirit of Lawverian functorial semantics—the functor category $\mathsf{FL}(\mathcal{C}, \mathsf{Set})$ is viewed as the category of *models* of the finite-limit theory $\mathcal{C}$.

[1]Strictly speaking we have to choose a small category which is *equivalent* to $\mathsf{comp}(\mathfrak{X})^{\mathsf{op}}$, since the latter is only *essentially small* in general.

It is well known that finite-limit theories are equally expressive as various syntactically defined classes of theories, including

- Freyd's *essentially algebraic theories* [13], which permit a controlled form of partiality,
- Cartmell's *generalized algebraic theories (GATs)* [10, 11], which extend algebra by 'dependent sorts', and
- Johnstone's *cartesian theories* [23, Definition D1.3.4], which permit a limited form of existential quantification,

in the sense that for any theory $\mathbb{T}$ from either of these classes, the category $\mathbb{T}$-Mod of models is locally finitely presentable, and conversely for every locally finitely presentable category $\mathfrak{X}$ there exits a theory from either class whose category of models is equivalent to $\mathfrak{X}$. While from a high-level perspective this means that the classes (1)–(4) of theories are all equivalent to finite-limit theories, the syntactic representations of theories contain additional information that is not reflected in the categories of models, and in the finite-limit theories. This 'abstracting away' of syntactic details is typically viewed as a *strength* of the categorical/functorial approach, and indeed in mathematical practice we do no more want to distinguish between the classical axiomatization of groups and Higman–Neumann's [18] axiomatization in terms of one operation and one equation, than we want to distinguish between the symmetric group S_3 and the dihedral group D_3.

However, it turns out that specifically in the case of GATs, the theories contain additional information which is not reflected in the corresponding finite-limit category, but goes beyond mere syntactic details. This information is related to the structure of *sort dependency* in the theories, and is reflected by certain weak factorization systems on the l.f.p. categories of models. The present work presents a duality

$$\mathsf{Clan}_{\mathsf{cc}} \overset{\mathsf{op}}{\simeq} \mathsf{ClanAlg} \tag{1.1}$$

of 2-categories which refines Gabriel–Ulmer duality by incorporating this additional information. On the right we have the 2-category of *clan-algebraic categories*, which are locally finitely presentable categories equipped with a well-behaved kind of weak factorization system (Definition 6.1), while on the left we have a 2-category of Cauchy-complete *clans* (Definition 2.1). Clans are categorical representations of GATs which can be viewed as a non-strict variant of Cartmell's *contextual categories* (Definition B.2), and are given by small categories equipped with a class of 'display maps' representing type families, admitting certain (but not all) finite limits.

Besides being a refinement of Gabriel–Ulmer duality, the duality (1.1) contains Adamek–Rosicky–Vitale's duality between *algebraic theories* and *algebraic categories* [2, Theorem 9.15] as a special case, and the latter duality was in fact inspirational for the present work.

1.1. Structure of the paper. Section 2 introduces clans (Definition 2.1), the category $\mathcal{T}$-Mod of models of a clan (Definition 2.6), and the extension–full weak factorization system on models (2.10).

Section 3 gives a characterization of $\mathcal{T}$-Mod as a kind of cocompletion of $\mathcal{T}^{\mathsf{op}}$ (Theorem 3.3), and uses this to give presentations of slice categories $\mathcal{T}$-Mod/A, and of coslice categories $H(\Gamma)/\mathcal{T}$-Mod under representable models, as categories of models of derived clans (Propositions 3.5 and 3.6).

Section 4 introduces the auxiliary notion of $(\mathcal{E},\mathcal{F})$*-category* (a l.f.p. category with a w.f.s. $(\mathcal{E},\mathcal{F})$) in Definition 4.1, and shows that the mapping $\mathcal{T} \mapsto \mathcal{T}$-$\mathsf{Mod}$ gives rise to a contravariant 2-functor from clans to $(\mathcal{E}, \mathcal{F})$-categories which admits a left biadjoint (Proposition 4.6).

Section 5 shows that this biadjunction is idempotent, and that its fixed points in clans are precisely the *Cauchy complete* clans (Definition 5.1). For this we use the notion of *flat model* (Definition 5.3), and the *fat small object argument*, a Corollary of which we state in Corollary 5.5, but whose sytematic treatement we defer to Appendix C. Lemma 5.7 is an argument about compact objects in coslice categories which I was not able to find in the literature.

Section 6 characterizes the fixed-points of the biadjunction among $(\mathcal{E}, \mathcal{F})$-categories as *clan-algebraic categories*, which are $(\mathcal{E}, \mathcal{F})$-categories satisfying a density and an exactness condition (Definition 6.1). The characterization is given by Theorems 6.2 and 6.18, where the proof of the latter requires a lot of machinery including a Reedy-like resolution argument. This finishes the proof of our refinement of Gabriel–Ulmer duality (1.1). Subsection 6.1 gives additional clan-algebraic w.f.s.s on Cat, which by the duality correspond to additional clan-representations of Cat.

Section 7 contains a common counterexample to two natural questions about clan-algebraic w.f.s.s, and Section 8 discusses ∞-models of clans in higher types.

Appendix A contains basic facts about locally finitely presentable categories, weak factorization systems, and Quillen's small-object argument, and Appendix B is an informal introduction to Cartmell's generalized algebraic theories.

Finally, Appendix C contains a careful development of the *fat* small object argument for clans.

1.2. Related work. [36, 3, 4, 25]

1.3. Acknowledgements. Thanks to Steve Awodey, Andrew Swan, and especially to Mathieu Anel for many discussions on the topic of this paper. Thanks to Reid Barton for telling me about the fat small object argument. Thanks to Benjamin Steinberg for locating the reference [16] for me after I asked about it on MathOverflow[2].

§2. Clans.

DEFINITION 2.1. A *clan* is a small category $\mathcal{T}$ with a distinguished class $\mathcal{T}_\dagger$ of arrows called *display maps*, such that:

(i) Pullbacks of display maps along arbitrary maps exist and are display maps, i.e. if $p : \Gamma' \twoheadrightarrow \Gamma$ is a display map and $s : \Delta \to \Gamma$ is arbitrary, then there exists a pullback square

$$\begin{tikzcd} \Delta' \arrow[r, "s'"] \arrow[d, "q"', two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma' \arrow[d, "p", two heads] \\ \Delta \arrow[r, "s"] & \Gamma \end{tikzcd} \tag{2.1}$$

where q is a display map.

(ii) Isomorphisms and compositions of display maps are display maps.

(iii) $\mathcal{T}$ has a terminal object, and terminal projections are display maps.

A *clan morphism* is a functor between clans which preserves display maps, pullbacks of display maps, and the terminal object. We write Clan for the 2-category of clans, clan-morphisms, and natural transformations.

[2] https://mathoverflow.net/a/90747/51432

REMARKS 2.2. (a) Definition 2.1 (apart from the smallness condition), and the term 'display map', were introduced by Taylor in his thesis [37, 4.3.2], the explicit link to Cartmell's work was made in his textbook [38, Chapter VIII]. The name 'clan' was suggested by Joyal [24, Definition 1.1.1].

(b) Following Cartmell, we use the arrow symbol $\twoheadrightarrow$ for display maps.

(c) We have defined clans to be *small* by default, since this fits with our point of view of clans as theories, and makes the duality theory work.

However, it is also reasonable to consider non-small, 'semantic' clans, and we will mention them occasionally (e.g. in Example 2.3(c) below), using the term *large clan* in this case.

EXAMPLES 2.3. (a) Small finite-limit categories can be viewed as clans where *all* morphimsms are display maps. We call such clans *finite-limit clans*.

(b) Small finite-product categories can be viewed as clans where the display maps are the morphisms that are (isomorphic to) product projections. We call such clans *finite-product clans*.

(c) Kan is the *large* clan whose underlying category is the full subcategory of the category $[\Delta^{\mathsf{op}}, \mathsf{Set}]$ of simplicial sets on *Kan complexes*, and whose display maps are the *Kan fibrations*.

(d) The syntactic category of every *generalized algebraic theory* in the sense of Cartmell [10, 11] is a clan. This is explained in Section B, and we discuss the example of the clan for categories in greater detail in Subsection 2.2 below.

Since it seems to lead to a more readable exposition, we introduce explicit notation and terminology for the dual notion.

DEFINITION 2.4. A *coclan* is a small category $\mathbb{C}$ with a distinguished class $\mathbb{C}_\dagger$ of arrows called *co-display maps* satisfying the dual axioms of clans. The 2-category CoClan of coclans is defined dually to that of clans, i.e.

$$\mathsf{CoClan}(\mathbb{C}, \mathbb{D}) = \mathsf{Clan}(\mathbb{C}^{\mathsf{op}}, \mathbb{D}^{\mathsf{op}})^{\mathsf{op}}$$

for coclans $\mathbb{C}, \mathbb{D}$.

REMARK 2.5. Coclans are called *cofibration categories* in [17, Def 2.1.5]. This is however in conflict with Baues' notion of cofibration category, which also includes a notion of weak equivalence [5, Section I.1].

2.1. Models.

DEFINITION 2.6. A *model* of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathsf{Set}$ which preserves the terminal object and pullbacks of display maps. We write $\mathcal{T}\text{-}\mathsf{Mod}$ for the category of models of $\mathcal{T}$, viewed as a full subcategory of the functor category $[\mathcal{T}, \mathsf{Set}]$.

REMARK 2.7. In other words, a model of a clan $\mathcal{T}$ is a clan morphism into the large clan with underlying category Set set and the maximal (i.e. finite-limit) clan structure.

In the spirit of functorial semantics, it is possible to consider models of clans in other categories than sets, and even in other (typically large) clans. However, the duality theory presented here is about models in Set and we don't consider any other kind (apart from some speculations about ∞-categorical models in Section 8).

EXAMPLES 2.8. (a) If $\mathcal{C}$ is a finite-limit clan (Example 2.3(a)) then $\mathsf{Mod}(\mathcal{C})$ coincides with the category $\mathsf{FL}(\mathcal{C}, \mathsf{Set})$ of finite-limit preserving functors into Set, which we also view as category of models of $\mathcal{C}$ qua finite-limit theory. This means that it makes sense to view finite-limit theories as a special case of clans.

(b) If $\mathcal{C}$ is a finite-product clan, then $\mathsf{Mod}(\mathcal{C})$ is the category $\mathsf{FP}(\mathcal{C}, \mathsf{Set})$ of finite-product preserving functors into Set.

In Adámek, Rosický and Vitale's textbook [2, Def. 1.1], small finite-product categories are called *algebraic theories*, and models of algebraic theories (which they not unreasonably call *algebras*) are defined to be finite-product preserving functors into Set. Thus, we recover their notions as a special case, i.e. finite-product clans correspond to algebraic theories, and models correspond to algebras. To emphasize the analogy to the finite-limit case, we refer to algebraic theories also as *finite-product theories*.

(c) If $\mathbb{T}$ is a GAT, then the category of models of its syntactic category $\mathcal{C}[\mathbb{T}]$ (with the clan structure described in Subsection B.1) is equivalent to the models $\mathbb{T}$, which Cartmell defines[3] to be $\mathsf{ConFunc}(\mathcal{C}[\mathbb{T}], \mathsf{Fam})$ of contextual functors and natural transformations into the contextual category Fam of iterated families of sets.

The equivalence $\mathsf{ConFunc}(\mathcal{C}[\mathbb{T}], \mathsf{Fam}) \simeq \mathsf{Mod}(\mathcal{C})$ holds because Fam is equivalent in the 2-category of contextual categories, contextual functors, and natural transformations to the cofree contextual category on the large clan Set with the finite-limit structure.

The following remarks discuss some categorical properties of the category $\mathcal{T}\text{-}\mathsf{Mod}$ of models of a clan, establishing in particular that it is locally finitely presentable. We refer to Section A for the relevant definitions.

REMARKS 2.9. (a) As category of models of a finite-limit sketch, $\mathcal{T}\text{-}\mathsf{Mod}$ is reflective (and therefore closed under arbitrary limits) in $[\mathcal{T}, \mathsf{Set}]$, and moreover it is closed under filtered colimits [1, Section 1.C]. In particular, $\mathcal{T}\text{-}\mathsf{Mod}$ is locally finitely presentable.

(b) The hom-functors $\mathcal{T}(\Gamma, -) : \mathcal{T} \to \mathsf{Set}$ are models of $\mathcal{T}$ for all $\Gamma \in \mathcal{T}$ (we'll refer to them as hom-*models*), i.e. the Yoneda embedding $よ : \mathcal{T}^{\mathsf{op}} \to [\mathcal{T}, \mathsf{Set}]$ lifts along the inclusion $\mathcal{T}\text{-}\mathsf{Mod} \hookrightarrow [\mathcal{T}, \mathsf{Set}]$ to a fully faithful functor $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Mod}$.

$$\begin{tikzcd} & \mathcal{T}\text{-}\mathsf{Mod} \arrow[d, hook] \\ \mathcal{T}^{\mathsf{op}} \arrow[ur, dashed, "H"] \arrow[r, "よ"'] & {[\mathcal{T}, \mathsf{Set}]} \end{tikzcd}$$

(c) For $\Gamma \in \mathcal{T}$, the hom-functor

$$\mathcal{T}\text{-}\mathsf{Mod}(H(\Gamma), -) : \mathcal{T}\text{-}\mathsf{Mod} \to \mathsf{Set}$$

[3] Strictly speaking, Cartmell does not 'define' the models of $\mathbb{T}\text{-}\mathsf{Mod}$ to be $\mathsf{ConFunc}(\mathcal{C}[\mathbb{T}], \mathsf{Fam})$ but 'asserts' that the categories are equivalent [10, pg. 2.77]. But since he refrains from giving a formal definition of $\mathbb{T}\text{-}\mathsf{Mod}$—writing only 'It should be quite clear what we mean by model' [10, pg. 1.45]—we take the assertion as a definition.

is isomorphic to the evaluation functor $A \mapsto A(\Gamma)$, hence it preserves filtered colimits as those are computed in $[\mathcal{T}, \mathsf{Set}]$ and therefore pointwise. This means that $H(\Gamma)$ is *compact*[4] in $\mathcal{T}$-Mod.

2.2. The clan for categories. Subsection B.1 describes how the syntactic category $\mathcal{C}[\mathbb{T}]$ of every GAT $\mathbb{T}$ can be viewed as a clan. The present section elaborates this for the specific case of the GAT $\mathbb{T}_{\mathsf{Cat}}$ of categories (B.1). We will use this clan and variations as a running example throughout the article.

Recall from Definition B.1 that the objects of $\mathcal{T}_{\mathsf{Cat}} := \mathcal{C}[\mathbb{T}_{\mathsf{Cat}}]$ are equivalence classes of contexts, and the arrows are equivalence classes of substitutions. By inspection of the axioms we see that sorts in $\mathbb{T}_{\mathsf{Cat}}$ cannot depend on non-variable terms, since the only non-constant sort symbol is $x\,y : O \vdash A(x, y)$ and there are no function symbols of type O. This means that up to reordering, all contexts are of the form

$$(x_1 \dots x_n : O, y_1 : A(x_{s_1}, x_{t_1}), \dots, y_k : A(x_{s_k}, x_{t_k})) \tag{2.2}$$

where $n, k \geq 0$ (such that $n > 0$ whenever $k > 0$) and $1 \leq s_l, t_l \leq n$ for $1 \leq l \leq k$; declaring first a list of object variables and then a list of arrow variables, each depending on a pair of the object variables. Given another context $(u_1 \dots u_m, v_1 \dots v_h)$, a substitution

$$u_1 \dots u_m, v_1 \dots v_h \;\vdash\; \sigma \;:\; x_1 \dots x_n, y_1 \dots y_k$$

is a tuple $\sigma = (u_{i_1} \dots u_{i_n}, f_1 \dots f_k)$ where $1 \leq i_1, \dots, i_n \leq m$ and the f_l are terms

$$u_1 \dots u_m, v_1 \dots v_h \;\vdash\; f_l \;:\; A(u_{i_{s_l}}, u_{i_{t_l}}).$$

Some reflection shows that $\mathcal{C}[\mathbb{T}_{\mathsf{Cat}}]$ is dual to the full subcategory of Cat on free categories on finite graphs: the data of a context (2.2) is that of finite sets $V = \{x_1, \dots, x_n\}$, $E = \{y_1, \dots, y_k\}$ of of *vertices* and *edges*, and source and target functions $s, t : E \to V$, and a substitution σ as above consists of a mapping from $\{x_1, \dots, x_n\}$ to $\{u_1, \dots, u_m\}$ and a mapping from $\{y_1, \dots, y_k\}$ to suitable paths in the graph represented by the domain. This is not surprising, since every clan embeds contravariantly into its category of models by Remark 2.9(b). Finally, the display maps in $\mathcal{T}_{\mathsf{Cat}}$, which syntactically correspond to projections 'from longer contexts to shorter ones', correspond to functors $G^* \hookrightarrow H^*$ between free categories induced by inclusions (i.e. monomorphisms) $G \hookrightarrow H$ of finite graphs in the dual presentation.

2.3. The weak factorization system on models. Next we introduce the *extension–full weak factorization system* on the category of models of a clan. We refer to Section A for basic facts about lifting properties and weak factorization systems (w.f.s.s) as well as pointers to the literature.

Definition 2.10. Let $\mathcal{T}$ be a clan.

(i) We call a map $f : A \to B$ in $\mathcal{T}$-Mod *full*, if it has the *right lifting property* (r.l.p., see Definition A.4(i)) with respect to all maps $H(p)$ for p a display map.
(ii) We call $f : A \to B$ an *extension*, if it has the *left lifting property* (l.l.p.) w.r.t. all full maps.
(iii) We call $A \in \mathcal{T}$-Mod a 0-*extension*, if $0 \to A$ is an extension.

[4] Following Lurie [26] we use the shorter term 'compact' instead of the more traditional 'finitely presented' for objects whose covariant hom-functor preserves filtered colimits.

REMARKS 2.11. (a) We use the arrow symbols '$\rightarrowtail$' for extensions (just as for co-display maps), and '$\twoheadrightarrow$' for full maps. We write $\mathcal{E}$ and $\mathcal{F}$ for the classes of extensions and full maps in $\mathcal{T}$-Mod, respectively. By the *small object argument* (Theorem A.5), extensions and full maps form a w.f.s. $(\mathcal{E}, \mathcal{F})$ on $\mathcal{T}$-Mod.

(b) A map $f : A \to B$ in $\mathcal{T}$-Mod is full if and only if the naturality square

$$\begin{tikzcd} A(\Delta) \arrow[r, "A(p)"] \arrow[d, "f_\Gamma"'] & A(\Gamma) \arrow[d, "f_\Delta"] \\ B(\Delta) \arrow[r, "B(p)"] & B(\Gamma) \end{tikzcd}$$

is a *weak pullback*[5] in Set for all display maps $p : \Delta \twoheadrightarrow \Gamma$. Setting $\Gamma = 1$ we see that full maps are pointwise surjective and therefore regular epimorphisms (the pointwise kernel pair $p, q : R \to A$ of f is in $\mathcal{T}$-Mod since $\mathcal{T}$-$\mathsf{Mod} \hookrightarrow [\mathcal{T}, \mathsf{Set}]$ creates limits, and pointwise surjective maps are coequalizers of their kernel pairs in $[\mathcal{T}, \mathsf{Set}]$, hence all the more so in $\mathcal{T}$-Mod).

(c) For every display map $p : \Delta \twoheadrightarrow \Gamma$ in $\mathcal{T}$, the arrow $H(p) : H(\Gamma) \rightarrowtail H(\Delta)$ is an extension—these are precisely the generators of the w.f.s. In particular, all hom-models $H(\Gamma)$ are 0-extensions, since all terminal projections $\Gamma \twoheadrightarrow 1$ are display maps in $\mathcal{T}$.

(d) The same w.f.s. was already defined in [17, Definition 2.4.2], using the terminology of cofibration categories mentioned in Remark 2.5. There, extensions are called cofibrations, and full maps trivial fibrations. We have not used this homotopical terminology here since we don't want to think about full maps as being 'trivial' in any way.

EXAMPLES 2.12. (a) If $\mathcal{T}$ is a finite-product clan, then $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by initial inclusions $0 \cong H(1) \rightarrowtail H(\Gamma)$ alone, since for every display map $p : \Delta \times \Gamma \to \Delta$ the generator $H(p)$ is a pushout

$$\begin{tikzcd} H(1) \arrow[r, hook] \arrow[d, hook] & H(\Delta) \arrow[d, hook, "H(p)"] \\ H(\Gamma) \arrow[r, hook] & H(\Gamma \times \Delta) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in $\mathcal{T}$-Mod of an initial inclusion, and left classes of w.f.s.'s are closed under pushout. It follows that the full maps are *precisely* the pointwise surjective maps, which in this case also coincide with the regular epis, since finite-product preserving functors are closed under image factorization in $[\mathcal{T}, \mathsf{Set}]$ (and thus every non-surjective arrow factors through a strict subobject). Thus, the 0-extensions are precisely the regular-projective objects in the finite-product case, which also play a central role in [2].

[5] Meaning that the comparison map to the actual pullback is a surjection.

(b) If $\mathcal{T}$ is a finite-limit clan then *all* naturality squares of full maps $f : A \twoheadrightarrow B$ are weak pullbacks, including the naturality squares

$$\begin{tikzcd} A(\Gamma) \arrow[r] \arrow[d] & A(\Gamma \times \Gamma) \cong A(\Gamma) \times A(\Gamma) \arrow[d] \\ B(\Gamma) \arrow[r] & B(\Gamma \times \Gamma) \cong B(\Gamma) \times B(\Gamma) \end{tikzcd}$$

of diagonals $\Gamma \to \Gamma \times \Gamma$. From this it follows easily that f_Γ is injective, and since we have shown that it is surjective above, we conclude that only isomorphisms are full in the finite-limit case.

(c) The w.f.s. on Cat induced by the presentation $\mathsf{Cat} = \mathcal{T}_{\mathsf{Cat}}\text{-}\mathsf{Mod}$ (see Subsection 2.2) has as full maps functors $F : \mathbb{C} \to \mathbb{D}$ which have the r.l.p. w.r.t. all functor $G^* \hookrightarrow H^*$ for inclusions $G \hookrightarrow H$ of finite graphs. It is not difficult to see that these are precisely the functors which are full in the classical sense and moreover surjective on objects, and that the w.f.s. is already generated by the functors $(0 \hookrightarrow 1)$ and $(2 \hookrightarrow \mathbb{2})$, where 2 is the discrete category with two objects and $\mathbb{2}$ is the interval category.

§3. Comodels and the universal property of $\mathcal{T}$-Mod.

3.1. Nerve–realization adjunctions. We recall basic facts about *nerve–realization* adjunctions, to establish notation and conventions. Recall that for small $\mathbb{C}$ the presheaf category $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$ is the *small-colimit completion* of $\mathbb{C}$, in the sense that for every cocomplete category $\mathfrak{X}$, precomposition with the Yoneda embedding $よ : \mathbb{C} \to \widehat{\mathbb{C}}$ induces an equivalence

$$\mathsf{CoCont}(\widehat{\mathbb{C}}, \mathfrak{X}) \xrightarrow{\simeq} [\mathbb{C}, \mathfrak{X}] \tag{3.1}$$

between the categories of cocontinuous functors $\widehat{\mathbb{C}} \to \mathfrak{X}$, and of functors $\mathbb{C} \to \mathfrak{X}$. Specifically, the cocontinuous functor $F_\otimes : \widehat{\mathbb{C}} \to \mathfrak{X}$ corresponding to $F : \mathbb{C} \to \mathfrak{X}$ is the left Kan extension of F along $よ : \mathbb{C} \to \widehat{\mathbb{C}}$, whose value at $A \in \widehat{\mathbb{C}}$ admits alternative representations

$$\begin{aligned} F_\otimes(A) = F \otimes A &= \int^{C \in \mathbb{C}} F(C) \times A(C) \\ &= \mathsf{colim}(\mathsf{El}(A) \to \mathbb{C} \xrightarrow{F} \mathfrak{X}) \end{aligned}$$

as a coend and as a colimit indexed by the category $\mathsf{El}(A)$ of elements of A. If $\mathfrak{X}$ is locally small then $F_\otimes$ has a right adjoint $F_N : \mathfrak{X} \to \widehat{\mathbb{C}}$ given by $F_N(X) = \mathfrak{X}(F(-), X)$. We call F_N and $F_\otimes$ the *nerve* and *realization* functors of F, respectively, and $F_\otimes \dashv F_N$ the *nerve–realization adjunction* of F.

3.2. Comodels and the universal property of $\mathcal{T}$-Mod. The universal property of $\mathcal{T}$-Mod is an equivalence between cocontinuous functors out of $\mathcal{T}$-Mod and coclan morphisms out of $\mathcal{T}^{\mathsf{op}}$. Following a suggestion by Mathieu Anel, we refer to the latter as *comodels* of the clan. We will only use this term for coclan morphisms with cocomplete codomain.

Definition 3.1. A *comodel* of a clan $\mathcal{T}$ in a cocomplete category $\mathfrak{X}$ is a functor $F : \mathcal{T}^{\mathsf{op}} \to \mathfrak{X}$ which sends 1 to 0, and display-pullbacks to pushouts. We write $\mathcal{T}\text{-}\mathsf{CoMod}(\mathfrak{X})$ for the category of comodels of $\mathcal{T}$ in $\mathfrak{X}$, as a full subcategory of the functor category.

REMARK 3.2. In other words, a comodel of $\mathcal{T}$ in $\mathfrak{X}$ is a coclan morphism from $\mathcal{T}^{\mathsf{op}}$ to the large coclan with underlying category $\mathfrak{X}$ and the maximal coclan structure.

THEOREM 3.3 (The universal property of $\mathcal{T}$-Mod). *Let $\mathcal{T}$ be a clan.*

(i) *The functor $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Mod}$ from Remark 2.9(b) is a comodel.*

(ii) *For every cocomplete $\mathfrak{X}$ and comodel $F : \mathcal{T}^{\mathsf{op}} \to \mathfrak{X}$, the restriction of $F_\otimes : [\mathbb{C}, \mathsf{Set}] \to \mathfrak{X}$ to $\mathcal{T}$-Mod is cocontinuous. Thus, precomposition with H gives rise to an equivalence*

$$\mathsf{CoCont}(\mathcal{T}\text{-}\mathsf{Mod}, \mathfrak{X}) \xrightarrow{\simeq} \mathcal{T}\text{-}\mathsf{CoMod}(\mathfrak{X}) \tag{3.2}$$

between categories of continuous functors and of comodels.

(iii) *If $F : \mathcal{T}^{\mathsf{op}} \to \mathfrak{X}$ is a comodel and $\mathfrak{X}$ is locally small, then the nerve functor $F_N : \mathfrak{X} \to [\mathbb{C}, \mathsf{Set}]$ factors through the inclusion $\mathcal{T}\text{-}\mathsf{Mod} \hookrightarrow [\mathbb{C}, \mathsf{Set}]$, giving rise to a* restricted nerve realization adjunction $F_\otimes : \mathcal{T}\text{-}\mathsf{Mod} \leftrightarrows \mathfrak{X} : F_N$.

$$\begin{tikzcd}
\mathcal{T}^{\mathsf{op}} \arrow[r, hook, "H"] \arrow[dr, "F"'] & \mathcal{T}\text{-}\mathsf{Mod} \arrow[r, hook] \arrow[d, shift right=2, "F_\otimes"'] \arrow[d, phantom, "\dashv"] & {[\mathcal{T}^{\mathsf{op}}, \mathsf{Set}]} \arrow[dl, shift right=2, "F_\otimes"'] \arrow[dl, phantom, "\dashv"] \\
& \mathfrak{X} \arrow[u, shift right=2, "F_N"'] \arrow[ur, shift right=2, "F_N"'] &
\end{tikzcd}$$

PROOF. Analogous statements to (i) and (ii) hold more generally for arbitrary small realized[6] limit sketches. As Brandenburg points out on MathOverflow [8], the earliest reference for this seems to be [32, Theorem 2.5]. See also [9] which gives a careful account of an even more general statement for non-small sketches.

For claim (iii), it's easy to see that for $X \in \mathfrak{X}$, the functor $F_N(X) = \mathfrak{X}(F(-), X)$ is a model since F is a comodel. $\square$

3.3. Slicing and coslicing. As an application of Theorem 3.3, this subsection gives statements about clan presentations of slice categories $\mathcal{T}$-Mod/A of categories of models (Proposition 3.6), and of coslice categories $H(A)/\mathcal{T}$-Mod under representable models (Proposition 3.5).

DEFINITION 3.4. For $\mathcal{T}$ a clan and $\Gamma \in \mathcal{T}$, we write $\mathcal{T}_\Gamma$ for the full subcategory of $\mathcal{T}/\Gamma$ on display maps. Then $\mathcal{T}_\Gamma$ is a clan where an arrow in $\mathcal{T}_\Gamma$ is a display map if its underlying map is one in $\mathcal{T}$. Compare [24, Proposition 1.1.6].

PROPOSITION 3.5. *Let Γ be an object of a clan $\mathcal{T}$. Then the functor*

$$\Gamma/H : (\mathcal{T}_\Gamma)^{\mathsf{op}} \to H(\Gamma)/\mathcal{T}\text{-}\mathsf{Mod} \tag{3.3}$$

which sends $d : \Delta \twoheadrightarrow \Gamma$ to $H(d) : H(\Gamma) \rightarrowtail H(\Delta)$ is a comodel. Moreover, its restricted nerve–realization adjunction (in the sense of Theorem 3.3(iii))

$$\begin{tikzcd}
(\mathcal{T}_\Gamma)^{\mathsf{op}} \arrow[r, hook, "H"] \arrow[dr, "\Gamma/H"'] & \mathcal{T}_\Gamma\text{-}\mathsf{Mod} \arrow[d, shift right=2, "(\Gamma/H)_\otimes"'] \arrow[d, phantom, "\dashv"] \\
& H(\Gamma)/\mathcal{T}\text{-}\mathsf{Mod} \arrow[u, shift right=2, "(\Gamma/H)_N"']
\end{tikzcd} \tag{3.4}$$

is an equivalence and identifies the extension–full w.f.s. on $\mathcal{T}_\Gamma$-Mod with the coslice w.f.s. on $H(\Gamma)/\mathcal{T}$-Mod.

[6] A sketch is called 'realized' if all its designated cones are limiting.

Proof. It is easy to see that Γ/H is a comodel. For the second claim, since arrows $H(\Gamma) \to A$ correspond to elements of $A(\Gamma)$, we can identify the coslice category $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Mod}$ with the category of 'Γ-pointed models of $\mathcal{T}$', i.e. pairs (A, x) of a model A and an element $x \in A(\Gamma)$, and morphisms preserving chosen elements.

Under this identification, we first verify that the functor $(\Gamma/H)_N$ is given by

$$(\Gamma/H)_N(A, x)(\Delta \overset{d}{\twoheadrightarrow} \Gamma) = \{y \in A(\Delta) \mid d \cdot y = x\},$$

and then that it is an equivalence with inverse $\Phi : \mathcal{T}_\Gamma\text{-}\mathsf{Mod} \to H(\Gamma)/\mathcal{T}\text{-}\mathsf{Mod}$ given by

$$\Phi(B) = (B(- \times \Gamma \overset{\pi_2}{\twoheadrightarrow} \Gamma), \delta \cdot \star)$$

where $\star$ is the unique element of $B(\mathrm{id}_\Gamma)$ and $\delta : \Gamma \to \Gamma \times \Gamma$ is the diagonal map viewed as global element of $\pi_2 : \Gamma \times \Gamma \twoheadrightarrow \Gamma$ in $\mathcal{T}_\Gamma$. Thus, $(\Gamma/H)_\otimes = \Phi$.

Finally we note that the w.f.s. on $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Mod}$ is cofibrantly generated by commutative triangles

$$\begin{tikzcd} H(\Gamma) \arrow[d, "H(\pi_2)"'] \arrow[dr, "H(\pi_2)"] & \\ H(\Delta \times \Gamma) \arrow[r, "H(d\times\Gamma)"'] & H(\Theta \times \Gamma) \end{tikzcd} \tag{3.5}$$

for display maps $d : \Theta \twoheadrightarrow \Gamma$ [19, Theorem 2.7]. On the other hand, since $(\Gamma/H)_\otimes \circ H = \Gamma/H$ (see (3.4)), the functor $(\Gamma/H)_\otimes$ sends the generators of the extension–full w.f.s. on $\mathcal{T}_\Gamma\text{-}\mathsf{Mod}$ to triangles

$$\begin{tikzcd} H(\Gamma) \arrow[d, "H(e)"'] \arrow[dr, "H(f)"] & \\ H(\Delta) \arrow[r, "H(d)"'] & H(\Theta) \end{tikzcd} \tag{3.6}$$

for arbitrary display maps d, e, f in $\mathcal{T}$. Now the triangles of shape (3.6) contain the triangles of shape (3.5), but are contained in their saturation, which is the left class of the coslice w.f.s. Thus, the two w.f.s.s are equal. $\square$

Proposition 3.6. *Let A be a model of a clan $\mathcal{T}$. Then the projection functor $\mathsf{El}(A) \to \mathcal{T}^{\mathsf{op}}$ creates a coclan structure on $\mathsf{El}(A)$, i.e. $\mathsf{El}(A)$ is a coclan with co-display maps those arrows that are mapped to display maps in $\mathcal{T}$. Moreover, the canonical functor*

$$H/A : \mathsf{El}(A) \simeq \mathcal{T}^{\mathsf{op}}/A \to \mathcal{T}\text{-}\mathsf{Mod}/A$$

is a coclan morphism, and its restricted nerve–realization adjunction

$$\begin{tikzcd} \mathsf{El}(A) \arrow[r, hook, "H"] \arrow[dr, "H/A"'] & \mathsf{El}(A)^{\mathsf{op}}\text{-}\mathsf{Mod} \arrow[d, shift right=2, "(H/A)_\otimes"'] \arrow[d, phantom, "\dashv"] \\ & \mathcal{T}\text{-}\mathsf{Mod}/A \arrow[u, shift right=2, "(H/A)_N"'] \end{tikzcd}$$

is an equivalence which identifies the extension–full w.f.s. on $\mathsf{El}(A)^{\mathsf{op}}\text{-}\mathsf{Mod}$ and the slice w.f.s. on $\mathcal{T}\text{-}\mathsf{Mod}/A$.

PROOF. The verification that $\mathsf{El}(A)^{\mathsf{op}}$ is a clan and H/A is a coclan morphism is straightforward. The equivalence is a restriction of the well-known equivalence $\widehat{\mathcal{T}^{\mathsf{op}}}/A \simeq \widehat{\mathcal{T}^{\mathsf{op}}/A}$. The w.f.s.s coincide since—again by $(H/A)_\otimes \circ H = H/A$—the functor $(H/A)_\otimes$ sends the generators of the w.f.s. on $\mathsf{El}(A)^{\mathsf{op}}\text{-}\mathsf{Mod}$ to commutative triangles

$$\begin{tikzcd} H(\Gamma) \arrow[r, "d"] \arrow[dr, "\hat{x}"'] & H(\Delta) \arrow[d, "\hat{y}"] \\ & A \end{tikzcd}$$

in $\mathcal{T}\text{-}\mathsf{Mod}/A$, where $d : \Delta \twoheadrightarrow \Gamma$ is a display map in $\mathcal{T}$ and $x \in A(\Gamma)$ and $y \in A(\Delta)$ are elements with $d \cdot y = x$. By [19, Theorem 1.5], these form a set of generators for the slice w.f.s. on $\mathcal{T}\text{-}\mathsf{Mod}/A$. $\square$

§4. $(\mathcal{E}, \mathcal{F})$-categories and the biadjunction.

DEFINITION 4.1. An $(\mathcal{E}, \mathcal{F})$-category is a l.f.p. category $\mathcal{L}$ with a w.f.s. $(\mathcal{E}, \mathcal{F})$ whose maps we call *extensions* and *full maps*. A *morphism of* $(\mathcal{E}, \mathcal{F})$*-categories* is a functor $F : \mathcal{L} \to \mathcal{M}$ preserving small limits, filtered colimits, and full maps. We write EFCat for the 2-category of $(\mathcal{E}, \mathcal{F})$-categories, morphisms of $(\mathcal{E}, \mathcal{F})$-categories, and natural transformations.

LEMMA 4.2. *If $F : \mathcal{L} \to \mathcal{M}$ is a morphism of $(\mathcal{E}, \mathcal{F})$-categories, then it has a left adjoint $L : \mathcal{M} \to \mathcal{L}$ which preserves compact objects and extensions. Conversely, if $L : \mathcal{M} \to \mathcal{L}$ is a cocontinuous functor preserving compact objects and extensions, then it has a right adjoint $F : \mathcal{L} \to \mathcal{M}$ which is a morphism of $(\mathcal{E}, \mathcal{F})$-categories. Writing $\mathsf{EFCat}_L(\mathcal{M}, \mathcal{L})$ for the category of cocontinuous functors $\mathcal{M} \to \mathcal{L}$ preserving extensions and compact objects, we thus have $\mathsf{EFCat}_L(\mathcal{M}, \mathcal{L}) \simeq \mathsf{EFCat}(\mathcal{L}, \mathcal{M})^{\mathsf{op}}$.*

PROOF. That morphisms of $(\mathcal{E}, \mathcal{F})$-categories have left adjoints follows from the *adjoint functor theorem for presentable categories* [1, Theorem 1.66], and conversely the *special adjoint functor theorem* [28, Section V-8] implies that cocontinuous functors between l.f.p. categories have right adjoints. It follows from standard arguments that the left adjoint preserves compact objects iff the right adjoint preserves filtered colimits, and that the left adjoint preserves extensions iff the right adjoint preserves full maps. $\square$

LEMMA 4.3. *For any morphism $F : \mathcal{S} \to \mathcal{T}$ of clans, the precomposition functor*

$$(-) \circ F \ : \ \mathcal{T}\text{-}\mathsf{Mod} \ \to \ \mathcal{S}\text{-}\mathsf{Mod}$$

is a morphism of $(\mathcal{E}, \mathcal{F})$-categories. Thus, the assignment $\mathcal{T} \mapsto \mathcal{T}\text{-}\mathsf{Mod}$ extends to a contravariant 2-functor

$$(-)\text{-}\mathsf{Mod} \ : \ \mathsf{Clan}^{\mathsf{op}} \ \to \ \mathsf{EFCat}$$

from clans to $(\mathcal{E}, \mathcal{F})$-categories.

PROOF. The preservation of small limits and filtered colimits is obvious since they are computed pointwise (Remark 2.9(a)). To show that F_N preserves full maps, let $f : A \to B$ be full in $\mathcal{T}\text{-}\mathsf{Mod}$. It is sufficient to show that the $(f \circ F)$-naturality squares are weak pullbacks at all display maps $p :$ in $\mathcal{S}\text{-}\mathsf{Mod}$. But the $(f \circ F)$-naturality square at p is the same as the f-naturality square at $F(p)$ so the claim follows since f is full and F preserves display maps. $\square$

Definition 4.4. Given an $(\mathcal{E},\mathcal{F})$-category $\mathcal{L}$, write $\mathfrak{C}(\mathcal{L}) \subseteq \mathcal{L}$ for the full subcategory on compact 0-extensions.

The following is straightforward.

Lemma 4.5. *$\mathfrak{C}(\mathcal{L})$ is a coclan with extensions as co-display maps.* □

Proposition 4.6. *The assignment $\mathcal{L} \mapsto \mathfrak{C}(\mathcal{L})^{\mathsf{op}}$ extends to a pseudofunctor*

$$\mathfrak{C}(-)^{\mathsf{op}} \ : \ \mathsf{EFCat} \to \mathsf{Clan}^{\mathsf{op}}$$

which is left biadjoint to $(-)\text{-}\mathsf{Mod} : \mathsf{Clan}^{\mathsf{op}} \to \mathsf{EFCat}$.

Proof. We show that for every $(\mathcal{E},\mathcal{F})$-category $\mathcal{L}$, the 2-functor

$$\mathsf{EFCat}(\mathcal{L},(-)\text{-}\mathsf{Mod}) : \mathsf{Clan}^{\mathsf{op}} \to \mathsf{Cat}$$

is birepresented by $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$. Given a clan $\mathcal{T}$ it is easy to see that the equivalence

$$\mathsf{CoCont}(\mathcal{T}\text{-}\mathsf{Mod},\mathcal{L}) \ \simeq \ \mathcal{T}\text{-}\mathsf{CoMod}(\mathcal{L})$$

from Theorem 3.3 restricts to an equivalence

$$\mathsf{EFCat}_L(\mathcal{T}\text{-}\mathsf{Mod},\mathcal{L}) \ \simeq \ \mathsf{CoClan}(\mathcal{T}^{\mathsf{op}},\mathfrak{C}(\mathcal{L})).$$

Taking opposite categories on both sides we get

$$\mathsf{EFCat}(\mathcal{L},\mathcal{T}\text{-}\mathsf{Mod}) \ \simeq \ \mathsf{Clan}(\mathcal{T},\mathfrak{C}(\mathcal{L})^{\mathsf{op}}) \tag{4.1}$$

as required. □

Remark 4.7. From the construction of the natural equivalence (4.1) we can extract explicit descriptions of the components

$$\Theta_{\mathcal{L}} : \mathcal{L} \to \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Mod} \qquad \text{and} \qquad E_{\mathcal{T}} : \mathcal{T} \to \mathfrak{C}(\mathcal{T}\text{-}\mathsf{Mod})^{\mathsf{op}}$$

of the unit Θ and the counit E of the biadjunction

$$\mathfrak{C}(-)^{\mathsf{op}} \ : \ \mathsf{EFCat} \leftrightarrows \mathsf{Clan}^{\mathsf{op}} \ : \ (-)\text{-}\mathsf{Mod} \tag{4.2}$$

at an $(\mathcal{E},\mathcal{F})$-category $\mathcal{L}$ and a clan $\mathcal{T}$ respectively. Specifically, $\Theta_{\mathcal{L}}$ is the nerve of the inclusion $J : \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L}$ (which is obviously a comodel), and $E_{\mathcal{T}}$ is $(-)^{\mathsf{op}}$ of the evident corestriction of $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Mod}$.

In Sections 5 and 6 will show that the biadjunction (4.2) is *idempotent* (in the sense that the associated monad and comonad are), and characterize the fixed-points on both sides (Theorems5.9 and 6.18).

§5. Cauchy complete clans and the fat small object argument.

Definition 5.1. A clan $\mathcal{T}$ is called *Cauchy complete* if its underlying category is Cauchy complete (i.e. idempotents split), and retracts of display maps are display maps.

Examples 5.2. (a) Finite-limit clans are always Cauchy complete, since finite-limit categories are and all arrows are display maps in finite-limit clans.

(b) A finite-product clan is Cauchy complete if and only if idempotents split in the underlying finite-product category, which may or may not be the case for the presentation of a single-sorted algebraic theory $\mathbb{T}$ as *Lawvere theory* (i.e. the opposite of the full subcategory of $\mathsf{Mod}(\mathbb{T})$ on *finitely generated free models*). For example the Lawvere theory of abelian groups is Cauchy complete since all finitely presented projective abelian groups are free, whereas the Lawvere theory of distributive lattices is *not* Cauchy complete. A non-free retract of a finitely generated free distributive lattice may be obtained by starting with a section–retraction pair $s : \{0 < 1 < 2\} \leftrightarrows \{0 < 1\}^2 : r$ in posets, and then taking the distributive lattice of upper sets on both sides, i.e. applying the functor $\mathsf{Pos}(-, \{0 < 1\}) : \mathsf{Pos}^{\mathsf{op}} \to \mathsf{DLat}$. Then $\mathsf{Pos}(\{0 < 1\}^2, \{0 < 1\})$ is the free distributive lattice on 2 generators, but $\mathsf{Pos}(\{0 < 1 < 1\}, \{0 < 1\})$ is not free.

Further details on the question of Cauchy-completeness of finite-limit theories, including a discussion of how the classical theory of *Morita equivalence of rings* fits into the picture, can be found in [2, Sections 8, 15].

(c) The clan $\mathcal{T}_{\mathsf{Cat}}$ of categories is Cauchy complete. To see this assume that G is a finite graph and that $\mathbb{D}$ is a retract of the free category G^* on G. Then we know that $\mathbb{D}$ is a compact 0-extension and we have to show that $\mathbb{D}$ is free on a finite graph. Call an arrow f in $\mathbb{D}$ *irreducible* if it is not an identity and in any decomposition $f = gh$, either g or h is an identity. Since the factors of every non-trivial decomposition have shorter length in G^*, every arrow in $\mathbb{D}$ admits a decomposition into irreducible factors. Let H be the graph of irreducible arrows in $\mathbb{D}$, and let $F : H^* \to \mathbb{D}$ be the canonical functor. Then F is full since all arrows in D are composites of irreducibles, and it admits a section $K : \mathbb{D} \to H^*$ since $\mathbb{D}$ is a 0-extension. As a section, K sends arrows in $\mathbb{D}$ to decompositions into irreducibles, thus it sends irreducible arrows to themselves. It follows that $K(F(j)) = j$ for generators j in H, and from this we can deduce that $K \circ F = \mathrm{id}_{H^*}$. Thus, $\mathbb{D} \cong H^*$. Finiteness of H follows from compactness.

This argument is an adaption of a similar argument for monoids [16]

(d) For every $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$, the clan $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$ (Definition 4.4) is Cauchy complete, since compact objects and extensions are closed under retracts.

By Example 5.2(d), Cauchy completeness is a necessary condition for the counit $E_{\mathcal{T}} : \mathcal{T} \to \mathfrak{C}(\mathcal{T}\text{-}\mathsf{Mod})^{\mathsf{op}}$ of the biadjunction (4.2) to be an equivalence. We will show that it is also sufficient, but for this we need the notion of *flat model*, and the *fat small object argument*.

Recall that for small $\mathbb{C}$, a functor $F : \mathbb{C} \to \mathsf{Set}$ is called *flat* if $\mathsf{El}(F)$ is filtered, or equivalently if $F_{\otimes} : [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}] \to \mathsf{Set}$ preserves finite limits [7, Definition 6.3.1 and Proposition 6.3.8]. From the second characterization it follows that flat functors preserve all finite limits that exist in $\mathbb{C}$, thus for the case of a clan $\mathcal{T}$, flat functors $F : \mathcal{T} \to \mathsf{Set}$ are always models. We refer to them as *flat* models:

Definition 5.3. A model $A : \mathcal{T} \to \mathsf{Set}$ of a clan $\mathcal{T}$ is called *flat*, if $\mathsf{El}(F)$ is filtered.

Lemma 5.4. *A $\mathcal{T}$-model A is flat iff it is a filtered colimit of* hom*-models.*

Proof. We always have $A = \mathsf{colim}(\mathsf{El}(A) \to \mathcal{T}^{\mathsf{op}} \xrightarrow{H} \mathcal{T}\text{-}\mathsf{Mod})$, thus if A is flat then it is a filtered colimit of hom-models. The other direction follows since hom-models are flat, and flat functors are closed under filtered colimits in $[\mathcal{T}, \mathsf{Set}]$ [7, Proposition 6.3.6]. □

Corollary 5.5. *For any clan $\mathcal{T}$, the* 0*-extensions in* $\mathcal{T}$-Mod *are flat.*

Proof. This follows from the *fat small object argument* and can be seen as a special case of [29, Corollary 5.1], but we give a direct proof in Appendix C (specifically Corollary C.9), which simplifies considerably in the case of clans. □

Definition 5.6. Let $\mathfrak{X}$ be a cocomplete locally small category.

(i) We say that an arrow $f : A \to B$ is *orthogonal* to a small diagram $D : \mathbb{J} \to \mathfrak{X}$, and write $f \perp D$, if the square

$$\begin{tikzcd}
\mathsf{colim}_{j\in\mathbb{J}}\,\mathfrak{X}(B,D_j) \arrow[r] \arrow[d] & \mathfrak{X}(B,\mathsf{colim}(D)) \arrow[d] \\
\mathsf{colim}_{j\in\mathbb{J}}\,\mathfrak{X}(A,D_j) \arrow[r] & \mathfrak{X}(A,\mathsf{colim}(D))
\end{tikzcd}$$

is a pullback in Set.

(ii) We call f *compact* if it is orthogonal to all small filtered diagrams.

Lemma 5.7. *Let $\mathfrak{X}$ be a locally small cocomplete category.*

(i) *An object $A \in \mathfrak{X}$ is compact in the usual sense that $\mathfrak{X}(A,-)$ preserves filtered colimits, if and only if the arrow $0 \to A$ is compact in the sense of Definition 5.6.*

(ii) *If the arrow g in a commutative triangle* $\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "f"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$ *is compact, then f is compact if and only if h is compact. In other words, compact arrow are closed under composition and have the right cancellation property.*

(iii) *If $f : A \to B$ is compact as an arrow in $\mathfrak{X}$, then it is compact as an object in $A/\mathfrak{X}$.*

(iv) *If $h : B \to C$ is an arrow between compact objects in $\mathfrak{X}$, then h is compact as an object in $B/\mathfrak{X}$.*

Proof. (i) is obvious, and (ii) follows from the pullback lemma.

For (iii) assume that f is compact as an arrow in $\mathfrak{X}$ and consider a filtered diagram in $A/\mathfrak{X}$, given by a filtered diagram $D : \mathbb{I} \to \mathfrak{X}$ and a cocone $\gamma = (\gamma_i : A \to D_i)_{i\in\mathbb{I}}$. Note that since the forgetful functor $A/\mathfrak{X} \to \mathfrak{X}$ creates connected colimits, we have $\mathsf{colim}(\gamma) : A \to \mathsf{colim}(D)$. Also because $\mathbb{I}$ is connected, all γ_i are in the same equivalence class in $\mathsf{colim}_{i\in\mathbb{I}}\,\mathfrak{X}(A,D_i)$, which we denote by $\overline{\gamma} : 1 \to \mathsf{colim}_{i\in\mathbb{I}}\,\mathfrak{X}(A,D_i)$. We have to show that the canonical map

$$\mathsf{colim}_i (A/\mathfrak{X})(f,\gamma_i) \;\longrightarrow\; (A/\mathfrak{X})(f,\mathsf{colim}(\gamma))$$

is a bijection. This follows because this function can be presented by a pullback in $\mathsf{Set}^{\mathbb{2}}$ as in the following diagram.

$$\begin{tikzcd}
& \mathsf{colim}_i\,\mathfrak{X}(B,D_i) \arrow[rr] \arrow[dd] & & \mathfrak{X}(B,\mathsf{colim}(D)) \arrow[dd] \\
\mathsf{colim}_i(A/\mathfrak{X})(f,\gamma_i) \arrow[ur] \arrow[rr] \arrow[dd] & & (A/\mathfrak{X})(f,\mathsf{colim}(\gamma)) \arrow[ur] \arrow[dd] & \\
& \mathsf{colim}_i\,\mathfrak{X}(A,D_i) \arrow[rr] & & \mathfrak{X}(A,\mathsf{colim}(D)) \\
1 \arrow[ur, "\overline{\gamma}"] \arrow[rr] & & 1 \arrow[ur, "\mathsf{colim}(\gamma)"'] &
\end{tikzcd}$$

The front square is a pullback since the back one is by compactness of f as an arrow, and since the side ones are pullbacks by construction. Thus the gray horizontal arrow is a bijection since $1 \to 1$ is.

Finally, claim (iv) now follows directly from (i), (ii), and (iii). $\square$

Remark 5.8. One can show the implication of Lemma 5.7(iii) is actually an equivalence, i.e. $f : A \to B$ is compact as an arrow if and only if it is so as an object of the coslice category, but the other direction is more awkward to write down and we don't need it.

Theorem 5.9. *If $\mathcal{T}$ is a Cauchy complete clan, then $E_{\mathcal{T}} : \mathcal{T} \to \mathfrak{C}(\mathcal{T}\text{-}\mathsf{Mod})^{\mathsf{op}}$ is an equivalence of clans.*

Proof. Let $C \in \mathcal{T}\text{-}\mathsf{Mod}$ be a compact 0-extension. Then by Corollary 5.5, C is a filtered colimit of hom-models, and since C is compact the identity id_C factors through one of the colimit inclusions, whence C is a retract of a hom-model. By Cauchy completeness, C is thus itself representable, i.e. we have an equivalence of categories.

It remains to show that $E_{\mathcal{T}}$ reflects display maps. Assume that $f : \Delta \to \Gamma$ in $\mathcal{T}$ such that $H(f) : H(\Gamma) \to H(\Delta)$ is an extension. Then $H(f)$ is compact in $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Mod}$ by Lemma 5.7(iv) and $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Mod} \simeq \mathcal{T}_\Gamma\text{-}\mathsf{Mod}$ by Proposition 3.5. This means that the object corresponding to $H(f)$ in $\mathcal{T}_\Gamma\text{-}\mathsf{Mod}$ is a compact 0-extension, and thus it is isomorphic to a hom-model $\mathcal{T}_\Gamma(d, -)$ for a display map $d : \Theta \twoheadrightarrow \Gamma$ by the argument in the first part of the proof. This means that f is isomorphic to d over Γ, and therefore a display map. $\square$

The preceding proposition together with Example 5.2(d) shows that the pseudomonad on Clan induced by the biadjunction (4.2) is *idempotent*: applying the pseudomonad once produces a Cauchy complete clan, and applying it again gives something equivalent. By general facts about (bi)adjunctions, the induced pseudomonad on EFCat is also idempotent. In the following section we characterize its fixed-points as being *clan-algebraic categories*.

§6. Clan-algebraic categories.

Definition 6.1. An $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ is called *clan-algebraic* if

(D) the inclusion $J : \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L}$ is dense,

(CG) the w.f.s. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by $\mathcal{E} \cap \mathfrak{C}(\mathcal{L})$, and

(FQ) equivalence relations $\langle p, q \rangle : R \rightarrowtail A \times A$ in $\mathcal{L}$ with *full components* p, q are effective, and have full coequalizers.

A *clan-algebraic weak factorization system* is a w.f.s. on a l.f.p. category $\mathcal{L}$ making $\mathcal{L}$ into a clan-algebraic category.

Theorem 6.2. *The category $\mathcal{T}\text{-}\mathsf{Mod}$ is clan-algebraic for every clan $\mathcal{T}$.*

Proof. Conditions (D) and (CG) are straightforward. For for (FQ) let $\langle p, q \rangle : R \rightarrowtail A \times A$ be an equivalence relation with full components. This means that we have an equivalence relation $\sim$ on each $A(\Gamma)$, such that

- for all arrows $s : \Delta \to \Gamma$, the function $A(s) = s \cdot (-) : A(\Delta) \to A(\Gamma)$ preserves this relation, and
- for every display map $p : \Gamma^+ \twoheadrightarrow \Gamma$ and all $a, b \in A(\Gamma)$ and $c \in A(\Gamma^+)$ such that $a \sim b$ and $p \cdot c = a$, there exists a $d \in A(\Gamma^+)$ with $c \sim d$ and $p \cdot d = b$.

We show first that the pointwise quotient A/R is a model. Clearly $(A/R)(1) = 1$, and it remains to show that given a pullback

$$\begin{tikzcd} \Delta^+ \arrow[r, "t"] \arrow[d, "q"] & \Gamma^+ \arrow[d, "p"] \\ \Delta \arrow[r, "s"] & \Gamma \end{tikzcd}$$

with p and q display maps, and elements $a \in A(\Delta)$, $b \in A(\Gamma^+)$ with $s \cdot a \sim p \cdot b$, there exists a unique-up-to-$\sim$ element $c \in A(\Delta^+)$ with $q \cdot c \sim a$ and $t \cdot c \sim b$. Since p is a display map, there exists a b' with $b \sim b'$ and $p \cdot b' = s \cdot a$, and since A is a model there exists therefore a c with $q \cdot c = a$ and $t \cdot c = b'$. For uniqueness assume that $c, c' \in A(\Delta^+)$ with $q \cdot c \sim q \cdot c'$ and $t \cdot c \sim t \cdot c'$. Then $c \sim c'$ follows from the fact that R is a model. This shows that A/R is a model, and also that the quotient is effective, since the kernel pair is computed pointwise. The fact that $A \to A/R$ is full is similarly easy to see. $\square$

The following counterexample shows that conditions (D) and (CG) alone are not sufficient to characterize categories $\mathcal{T}$-Mod.

Example 6.3. Let Inj be the full subcategory of $\widehat{2}$ on injections, and let $(\mathcal{E}, \mathcal{F})$ be the w.f.s. generated by $0 \to よ(0)$ and $0 \to よ(1)$. Then $(\mathcal{E}, \mathcal{F})$ satisfies (D) and (CG), and $\mathcal{F}$ consists precisely of the pointwise surjective maps, in particular it contains all split epis. However, the equivalence relation on $2 \to 2$ which is discrete on the domain and codiscrete on the codomain is not effective.

The following is a restatement of Remark 2.11(b) for clan-algebraic categories.

Lemma 6.4. *Full maps in clan-algebraic categories are regular epimorphisms.*

Proof. Given a full map in a clan-algebraic category $\mathcal{L}$, the lifting property against 0-extensions implies that $\Theta_{\mathcal{L}}(f) = J_N(f)$ is componentwise surjective in $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$-$\mathsf{Mod}$, and therefore the coequalizer of its kernel pair. Since left adjoints preserve regular epis, we deduce that $J_\otimes(J_N(f))$ is regular epic in $\mathcal{L}$ and the claim follows since $J_\otimes \circ J_N \cong \mathrm{id}$ by (D). $\square$

Remark 6.5. Observe that we only used property (D) in the proof, no exactness.

Lemma 6.6. *The class $\mathcal{F}$ of full maps in a clan-algebraic category $\mathcal{L}$ has the* right cancellation property*, i.e. we have $g \in \mathcal{F}$ whenever $gf \in \mathcal{F}$ and $f \in \mathcal{F}$ for composable pairs $f : A \to B$, $g : B \to C$.*

Proof. By (CG) it suffices to show that g has the r.l.p. with respect to extensions $e : I \rightarrowtail J$ between compact 0-extensions I, J. Let

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[d, "e"] & B \arrow[d, "g"] \\ J \arrow[r, "k"] & C \end{tikzcd}$$

be a filling problem. Since I is a 0-extension and f is full, there exists a map $h' : I \to A$ with $fh' = h$. We obtain a new filling problem

$$\begin{tikzcd} I \arrow[r, "h'"] \arrow[d, tail, "e"] & A \arrow[d, two heads, "gf"] \\ J \arrow[r, "k"] & C \end{tikzcd}$$

which can be filled by a map $m : J \to A$ since gf is full. Then fm is a filler for the original problem. □

LEMMA 6.7. *Let $\mathcal{L}$ be a clan-algebraic category, let $f : A \to B$ be an arrow in $\mathcal{L}$ with componentwise full kernel pair $p, q : R \twoheadrightarrow A$, and let $e : A \twoheadrightarrow C$ be the coequalizer of p and q. Then the unique $m : C \to B$ with $me = f$ is monic.*

PROOF. By (D) it is sufficient to test monicity of m on maps out of compact 0-extensions E. Let $h, k : E \to C$ such that $mh = mk$. Since e is full by (FQ), there exist $h', k' : E \to A$ with $eh' = h$ and $ek' = k$. In particular we have $fh' = fk'$ and therefore there is an $u : E \to R$ with $pu = h'$ and $qu = k'$. Thus we can argue

$$h = eh' = epu = equ = ek' = k$$

which shows that m is monic. □

LEMMA 6.8. *If $A \in \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Mod}$ is flat, then $A \to J_N(J_\otimes(A))$ is an isomorphism, thus $J_\otimes$ restricted to flat models is fully faithful.*

PROOF. For the fist claim we have

$$\begin{aligned} J_N(J_\otimes(A))(C) &= \mathcal{L}(C, \mathsf{colim}(\mathsf{El}(A) \to \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L})) \\ &\cong \mathsf{colim}(\mathsf{El}(A) \to \mathfrak{C}(\mathcal{L}) \xrightarrow{よ(C)} \mathsf{Set}) \qquad \text{since } \mathsf{El}(A) \text{ is filtered} \\ &\cong よ(C) \otimes A \cong A(C). \end{aligned}$$

The second claim follows since for flat B, the mapping

$$(\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Mod})(A, B) \to \mathcal{L}(J_\otimes A, J_\otimes B)$$

can be decomposed as

$$(\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Mod})(A, B) \to (\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Mod})(A, J_N J_\otimes B) \to \mathcal{L}(J_\otimes A, J_\otimes B).$$

□

LEMMA 6.9. *The following are equivalent for a cone $\phi : \Delta C \to D$ on a diagram $D : \mathbb{J} \to \mathcal{L}$ in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$.*

(i) *Given an extension $e : A \to B$, an arrow $h : A \to C$, and a cone $\kappa : \Delta B \to D$ such that $\phi_j h = \kappa_j e$ for all $j \in \mathbb{J}$, there exists $l : B \to C$ such that $le = h$ and $\phi_j l = \kappa_j$ for all $j \in \mathbb{J}$.*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "e"'] & C \arrow[d, "\phi_j"] \\ B \arrow[r, "\kappa_j"'] \arrow[ur, dashed, "l"] & D_j \end{tikzcd}$$

(ii) *The mediating arrow $: C \to \mathsf{lim}(D)$ is full.*

Proof. The data of e, h, κ is equivalent to e, h, and $k : B \to \mathsf{lim}(D)$ such that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "e"'] & C \arrow[d, "f"] \\ B \arrow[r, "k"] & \mathsf{lim}(D) \end{tikzcd}$$

commutes, and $l : B \to C$ fills the latter square iff it fills all the squares with the D_j. $\square$

Definition 6.10. A cone ϕ satisfying the conditions of the lemma is called *jointly full.*

Remark 6.11. The interest of this is that it allows us to talk about full 'covers' of limits without actually computing the limits, which is useful when talking about cones and diagrams in the full subcategory of a clan-algebraic category on 0-extensions, which does not admit limits.

Definition 6.12. A *nice diagram* in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ is a 2-truncated semi-simplicial diagram

$$A_\bullet \;=\; \left(\begin{tikzcd} A_2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_2"'] & A_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & A_0 \end{tikzcd} \right)$$

where

(i) A_0, A_1, and A_2 are 0-extensions,

(ii) the maps $d_0, d_1 : A_1 \to A_0$ are full,

(iii) in the commutative square $\begin{tikzcd} A_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] & A_1 \arrow[d, "d_1"] \\ A_1 \arrow[r, "d_0"] & A_0 \end{tikzcd}$ the span constitutes a jointly full cone over the cospan,

(iv) there exists a 'symmetry' map $\begin{tikzcd} A_1 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, "\sigma"] & A_0 \\ A_0 & A_1 \arrow[l, "d_1"] \arrow[u, "d_0"'] \end{tikzcd}$ making the triangles commute, and

(v) there exists a 0-extension $\tilde{A}$ and full maps $f, g : \tilde{A} \twoheadrightarrow A_1$ constituting a jointly full cone over the diagram

$$\begin{tikzcd} A_1 \arrow[d, "d_0"'] \arrow[dr, "d_1"] & A_1 \arrow[d, "d_1"] \arrow[dl, "d_0"] \\ A_0 & A_0 \end{tikzcd}.$$

Lemma 6.13. *If $A_\bullet$ is a nice diagram in a clan-algebraic category $\mathcal{L}$, the pairing $\langle d_0, d_1\rangle$: $A_1 \to A_0 \times A_0$ factors as $A_1 \xrightarrow{f} R \xrightarrow{\langle r_0, r_1\rangle} A_0 \times A_0$, where f is full and $r = \langle r_0, r_1\rangle$ is monic and a componentwise full equivalence relation.*

Proof. Condition (v) of the preceding definition gives us the following diagram

$$\begin{tikzcd} \tilde{A} \arrow[dr, two heads, "h"] \arrow[drr, two heads, bend left, "g"] \arrow[ddr, two heads, bend right, "f"'] & & \\ & S \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner" very near start] & A_1 \arrow[d, "\langle d_0, d_1\rangle"] \\ & A_1 \arrow[r, "\langle d_0, d_1\rangle"] & A_0 \times A_0 \end{tikzcd},$$

i.e. S is the kernel of $\langle d_0, d_1\rangle$ with projections p, q, $\tilde{A}$ is a 0-extension, and f, g, h are full. By right cancellation we deduce that p and q are full, and the existence of the factorization follows from Lemma 6.7. Fullness of r_0, r_1 follows again from right cancellation because f, d_0, and d_1 are full.

It remains to show that r is an equivalence relation. This is easy: condition 4 gives symmetry, and condition 3 gives transitivity, and reflexivity follows from the fact that r_0 admits a section as a full map into a 0-extension, together with symmetry (we internalize the argument that if in a symmetric and transitive relation everything is related to *something*, then it is reflexive.) □

Definition 6.14. A 0-*extension replacement* of an object A in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ is a full map $f : \overline{A} \twoheadrightarrow A$ from a 0-extension $\overline{A}$ to A.

0-extension replacements can always be obtained as $(\mathcal{E}, \mathcal{F})$-factorizations of $0 \to A$.

Lemma 6.15. *For every object A in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ there exists a nice diagram $A_\bullet$ with colimit A.*

Proof. A_0 is constructed as a 0-extension replacement $f : A_0 \twoheadrightarrow A$ of A. Similarly, A_1 is given by a 0-extension replacement $f_1 : A_1 \twoheadrightarrow A_0 \times_A A_0$ of $A_0 \times_A A_0$, and A_2 is a 0-extension replacement $f_2 : A_2 \twoheadrightarrow P$ of the pullback

$$\begin{tikzcd} P \arrow[r, "p_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_1 \arrow[d, "d_0"] \\ A_1 \arrow[r, "d_1"] & A_0 \end{tikzcd},$$

with $d_0, d_1, d_2 : A_2 \to A_1$ given by $d_0 = p_0 \circ f$, $d_2 = p_1 \circ f$, and d_1 a lifting of $\langle d_0 \circ d_0, d_1 \circ d_2\rangle$ along f_1. The map σ is constructed as a lifting of the symmetry of $A_0 \times_A A_0$ along f_1. The object $\tilde{A}$ is a 0-extension replacement of the kernel of f_1. □

Lemma 6.16. *For any clan-algebraic category $\mathcal{L}$, the realization functor $J_\otimes$ preserves jointly full cones in flat models, and nice diagrams.*

Proof. The first claim follows since $J_\otimes$ is fully faithful on 0-extensions by Lemma 6.8 and in both sides the weak factorization system determined by the same generators. Thus there's a one-to-one correspondence between lifting problems. The second claim follows since $J_\otimes$ preserves 0-extensions and 0-extensions are flat by the fat small object argument. □

Lemma 6.17. *For any clan-algebraic category $\mathcal{L}$, the functor $J_N : \mathcal{L} \to \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Mod}$ preserves quotients of nice diagrams.*

Proof. Given a nice diagram $A_\bullet$ in $\mathcal{L}$, its colimit is the coequalizer of $d_0, d_1 : A_1 \to A_0$. By Lemma 6.13, $\langle d_0, d_1\rangle$ factors as $\langle r_0, r_1\rangle \circ f$ with f full and $r = \langle r_0, r_1\rangle$ an equivalence relation. The pairs d_0, d_1 and r_0, r_1 have the same coequalizer (since f is epic), and J_N preserves the coequalizer of r_0, r_1 since it preserves full maps and kernel pairs. Finally, the coequalizer of $J_N(r_0), J_N(r_1)$ is also the coequalizer of $J_N(d_0), J_N(d_1)$ since $J_N(f)$ is full and therefore epic. □

Theorem 6.18. *If $\mathcal{L}$ is clan-algebraic, then $J_N : \mathcal{L} \to \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Mod}$ is an equivalence.*

Proof. By density, J_N is fully faithful. It remains to verify that it is essentially surjective, and to this end we show that the unit map $\eta_A : A \to J_N(J_\otimes(A))$ is an isomorphism for all $A \in \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Mod}$. Let $A_\bullet$ be a nice diagram with colimit A. We have:

$$\begin{aligned} J_N(J_\otimes(A)) &= J_N(J_\otimes(\mathsf{colim}(A_\bullet))) \\ &= \mathsf{colim}(J_N(J_\otimes(A_\bullet))) && \text{by Lemmas 6.16 and 6.17} \\ &= \mathsf{colim}(A_\bullet) && \text{by Lemma 6.8} \\ &= A \end{aligned}$$

□

6.1. Clan-algebraic weak factorization systems on Cat. The chacterization of $(\mathcal{E},\mathcal{F})$-categories of models of clans as clan-algebraic categories allows us to exhibit new clans by defining suitable w.f.s.s on l.f.p. categories. In this subsection we demonstrate this by defining three more clan-algebraic w.f.s.s on Cat.

We start with the clan-algebraic w.f.s. from Example 2.12(c), corresponding to the 'standard' clan-presentation $\mathcal{T}_{\mathsf{Cat}}$ from Subsection 2.2. We observed that this w.f.s. is cofibrantly generated by the functors $0 \hookrightarrow 1$ and $2 \hookrightarrow \mathbb{2}$. Our strategy to define new clan-algebraic w.f.s.s is to add additional generators. If we make sure that the domain and codomain of these are compact 0-extensions, we only have to verify condition (FQ) when verifying that the new w.f.s. is still clan-algebraic. The additional generators we consider are the arrows $\mathbb{P} \to \mathbb{2}$ and $2 \to 1$, where $\mathbb{P}$ is the 'parallel pair category' $\bullet \rightrightarrows \bullet$. By adding either one or both of the additional generators we obtain three additional w.f.s.s $(\mathcal{E}_{\mathsf{O}},\mathcal{F}_{\mathsf{O}})$, $(\mathcal{E}_{\mathsf{A}},\mathcal{F}_{\mathsf{A}})$, and $(\mathcal{E}_{\mathsf{OA}},\mathcal{F}_{\mathsf{OA}})$, where:

$$\begin{aligned} \mathcal{F} &= \{(0 \to 1), (2 \to \mathbb{2})\}^{\pitchfork} \\ \mathcal{F}_{\mathsf{O}} &= \{(0 \to 1), (2 \to \mathbb{2}), (2 \to 1)\}^{\pitchfork} \\ \mathcal{F}_{\mathsf{A}} &= \{(0 \to 1), (2 \to \mathbb{2}), \qquad\quad (\mathbb{P} \to \mathbb{2})\}^{\pitchfork} \\ \mathcal{F}_{\mathsf{OA}} &= \{(0 \to 1), (2 \to \mathbb{2}), (2 \to 1), (\mathbb{P} \to \mathbb{2})\}^{\pitchfork} \end{aligned}$$

We have already observed that $\mathcal{F}$ consists of the functors that are full and surjective on objects, and it is easy to see that $\mathcal{F}_{\mathsf{O}}$ contains only those functors which are full and *bijective* on objects, whereas $\mathcal{F}_{\mathsf{A}}$ consists of functors which are *fully faithful* and surjective on objects. Finally, $\mathcal{F}_{\mathsf{OA}}$ only contains functors which are fully faithful and bijective on objects, i.e. isomorphisms of categories.

To convince ourselves that the new w.f.s.s are indeed clan-algebraic we only have to verify that for every equivalence relation $\langle p, q\rangle : \mathbb{R} \to \mathbb{A} \times \mathbb{A}$ in Cat, the coequalizer is in either of $\mathcal{F}_{\mathsf{O}}, \mathcal{F}_{\mathsf{A}}, \mathcal{F}_{\mathsf{OA}}$ whenever p and q are, since effectity has already been established for equivalence relations with components in $\mathcal{F}$. This is not difficult to see for $\mathcal{F}_{\mathsf{O}}$ and $\mathcal{F}_{\mathsf{A}}$, and trivial for $\mathcal{F}_{\mathsf{OA}}$.

The coclans corresponding to the new w.f.s.s are:

- $\mathcal{T}^{\mathsf{op}}_{\mathsf{Cat}_{\mathsf{O}}} = \{$categories free on finite graphs$\}$, with functors $G^* \to H^*$ arising from faithful graph morphisms as codisplays,
- $\mathcal{T}^{\mathsf{op}}_{\mathsf{Cat}_{\mathsf{A}}} = \{$finitely presented categories$\}$, with injective-on-objects functors as codisplays, and
- $\mathcal{T}^{\mathsf{op}}_{\mathsf{Cat}_{\mathsf{OA}}} = \{$finitely presented categories$\}$, with arbitrary functors as codisplays.

We note the clan $\mathcal{T}_{\mathsf{Cat}_{\mathsf{OA}}}$ is simply the finite-limit theory of categories. One may ask whether the clans $\mathcal{T}_{\mathsf{Cat}_{\mathsf{O}}}$, $\mathcal{T}_{\mathsf{Cat}_{\mathsf{A}}}$, and $\mathcal{T}_{\mathsf{Cat}_{\mathsf{OA}}}$ admit simple syntactic presentations by GATs, and indeed they do. To obtain such a presentation e.g. for $\mathcal{T}_{\mathsf{Cat}_{\mathsf{O}}}$, we have to modify the GAT $\mathbb{T}_{\mathsf{Cat}}$ in such a way that the syntactic category stays the same, but acquires additional display maps, such as the diagonal $(x : O) \to (x\,y : O)$ corresponding to the new generator $2 \to 1$. Display maps in the syntactic category of a GAT are always of the form $p \circ i$ where p is a projection omitting a finite number of variables and i is an isomorphism, so to turn $(x : O) \to (x\,y : O)$ into a display map we have to make $(x : O)$ isomorphic to an *extension* of $(x\,y : O)$. To achieve this we postulate a new type family $x\,y : O \vdash E_O(x, y)$ and add axioms forcing the projection $(x\,y : A\,,\, z : E_O(x, y)) \to (x : A)$ to become an isomorphism:

$$\begin{aligned} x\,y : O \;&\vdash\; E_O(x, y) \\ x : O \;&\vdash\; r_O(x) : E_O(x, x) \\ x\,y : O\,,\, p : E_O(x, y) \;&\vdash\; x = y \\ x : O\,,\, p : E_O(x, x) \;&\vdash\; r_O(x) = p \end{aligned} \tag{6.1}$$

The function symbol r_O gives a section for the projection, and the two last axioms force the retract to be an isomorphism. We recognize at once that these axioms make E_O an *extensional identity type* of O [21, Section 3.2]: the term r_O is reflexivity, and the third and fourth rule give equality reflection and uniqueness of identity proofs. We write $\mathbb{T}_{\mathsf{Cat}_{\mathsf{O}}}$ for the extension of the GAT $\mathbb{T}_{\mathsf{Cat}}$ by the axioms (6.1), and $\mathsf{Cat}_{\mathsf{O}}$ for the corresponding clan-algebraic category.

Similarly, we obtain an GAT-representation $\mathbb{T}_{\mathsf{Cat}_{\mathsf{A}}}$ of the clan $\mathcal{T}_{\mathsf{Cat}_{\mathsf{A}}}$ by augmenting $\mathbb{T}_{\mathsf{Cat}}$ by a type family E_A with the following rules:

$$\begin{aligned} x\,y : O\,,\, f\,g : A(x, y) \;&\vdash\; E_A(f, g) \\ x\,y : O\,,\, f : A(x, y) \;&\vdash\; r_A(f) : E_A(f, f) \\ x\,y : O\,,\, f\,g : A(x, y)\,,\, p : E_A(f, g) \;&\vdash\; f = g \\ x\,y : O\,,\, f : A(x, y)\,,\, p : E_A(f, f) \;&\vdash\; r_A(f) = p \end{aligned} \tag{6.2}$$

Adding both sets of axioms (6.1) and (6.2) to $\mathbb{T}_{\mathsf{Cat}}$ yields a GAT for the clan $\mathcal{T}_{\mathsf{Cat}_{\mathsf{OA}}}$, i.e. the finite-limit theory of categories.

§7. A counterexample. This section gives a common counterexample to two related natural questions about the extension–full w.f.s. on a clan-algebraic category $\mathcal{L}$:

(1) Does every compact object admit a full map from a compact 0-extension?
(2) Does the weak factorization system always restrict to compact objects?

The counterexample to both question is given by the category of models of the following GAT with infinitely many sorts and operations:

$$\begin{aligned} &\vdash\; X \\ &\vdash\; Y \\ y{:}Y \;&\vdash\; Z_n(y) && n \in \mathbb{N} \\ x{:}X \;&\vdash\; f(x) : Y \\ x{:}X \;&\vdash\; g_n(x) : Z_n(f(x)) && n \in \mathbb{N} \end{aligned}$$

Its category of models is equivalent to the set-valued functors on the posetal category

$$\mathbb{C} = \left(\begin{tikzcd} X \arrow[d, "g_0"] \arrow[dr, "g_1"] \arrow[drrr, "g_n"] \arrow[dd, bend right=60, "f"'] & & & & \\ Z_0 \arrow[d, "z_0"] & Z_1 \arrow[dl, "z_1"] & \dots & Z_n \arrow[dlll, "z_n"] & \dots \\ Y & & & & \end{tikzcd} \right)$$

and the w.f.s. on $[\mathbb{C}, \mathsf{Set}]$ is generated by the arrows $(\varnothing \rightarrowtail \text{よ}(X))$, $(\varnothing \rightarrowtail \text{よ}(Y))$, and $(\text{よ}(Y) \rightarrowtail \text{よ}(Z_n))$ for $n \in \mathbb{N}$, reflecting the idea that models $A : \mathbb{C} \to \mathsf{Set}$ can be built up by successively adding elements to $A(X)$ or $A(Y)$, and to $A(Z_n)$ over a given element x of $A(Y)$, as in the following pushouts.

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, tail] & A \arrow[d, tail] \\ 1 = \text{よ}(X) \arrow[r] & 1 + A \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d, tail] & A \arrow[d, tail] \\ \text{よ}(Y) \arrow[r] & \text{よ}(Y) + A \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \text{よ}(Y) \arrow[r, "\hat{x}"] \arrow[d, tail] & A \arrow[d, tail] \\ \text{よ}(Z_n) \arrow[r] & B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The following lemma gives explicit descriptions of the w.f.s. and the compact objects in $[\mathbb{C}, \mathsf{Set}]$.

LEMMA 7.1. *Let $f : A \to B$ in $[\mathbb{C}, \mathsf{Set}]$.*

(i) f is full if and only if it is componentwise surjective and the naturality squares for z_n are weak pullbacks for all $n \in \mathbb{N}$.

(ii) f is an extension if an only if $f_X : A(X) \to B(X)$ is injective, and the squares

$$\begin{tikzcd} A(X) \arrow[r] \arrow[d] & B(X) \arrow[d] \\ A(Y) \arrow[r] & B(Y) \end{tikzcd} \qquad \begin{tikzcd} A(X) \arrow[r] \arrow[d] & B(X) \arrow[d] \\ A(Z_n) \arrow[r] & B(Z_n) \end{tikzcd}$$

are quasi-pushouts*, in the sense that the gap maps $A(Y) +_{A(X)} B(X) \to B(Y)$ and $A(Z_n) +_{A(X)} B(X) \to B(Z_n)$ are injective. (This implies that the components f_Y and f_{Z_n} are also injective).*

(iii) A is a 0-extension if an only if $A(f)$ and all $A(g_n)$ are injective.

(iv) A is compact if an only if (a) it is componentwise finite, and (b) $A(f_n) : A(X) \to A(Z_n)$ is a bijection for all but finitely many $n \in \mathbb{N}$.

□

Using this lemma, we can give negative answers to the two question at the beginning of the section.

PROPOSITION 7.2. (i) *The object P in the pushout*

$$\begin{tikzcd} \text{よ}(Y) + \text{よ}(Y) \arrow[r] \arrow[d] & \text{よ}(X) + \text{よ}(X) \arrow[d] \\ \text{よ}(Y) \arrow[r] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is compact, but does not admit a full map from a compact 0-extension.

(ii) *The map* $よ(Y) \to よ(X)$ *does not admit an extension–full factorization through a compact object.*

PROOF. For the first claim, P is compact as a finite colimit of representables. Let $f : E \twoheadrightarrow P$ be a full map with E a 0-extension. For each $n \in \mathbb{N}$ we get a diagram

$$\begin{tikzcd}
E(X) \arrow[rr, two heads, "f_X"] \arrow[dr, tail, "E(g_n)"] \arrow[ddr, tail, "E(f)"'] & & 2 \arrow[dr] \arrow[ddr] & \\
& E(Z_n) \arrow[rr, two heads, "f_{Z_n}"] \arrow[d, "E(z_n)"] & & 2 \arrow[d] \\
& E(Y) \arrow[rr, two heads, "f_Y"] & & 1
\end{tikzcd}$$

where $E(f)$ and $E(g_n)$ are injective because E is a 0-extension, and the components of f are surjective and the z_n-naturality square is a weak pullback since because f is full. In particular $E(Y)$ is inhabited and the fibers of $E(z_n)$ have at least two elements. Since the fibers of $E(f)$ have at most one element, this means that $E(g_n)$ can't be surjective for any n, and it follows from Lemma 7.1(iv) that E is not compact.

For the second claim consider an extension $e : よ(Y) \to A$ such that $A \to よ(X) = 1$ is full. Then $A(Y)$ is inhabited, all $A(Z_n) \to A(Y)$ are surjective, and $1 + A(X) \to A(Y)$ is injective. From this we can again deduce that none of the $A(g_n)$ are surjective and thus A is not compact. □

§8. Outlook: Models in higher types. One practical use of having inequivalent clans with equivalent categories of Set-models is that they can have inequivalent ∞-categories of models in the ∞-category $\mathcal{S}$ of *homotopy types* (a.k.a. 'spaces'). We leave this issue for future work and content ourselves here with outlining some main ideas.

The first observation is that for every finite-limit theory $\mathcal{L}$, the ∞-category $\infty\text{-}\mathsf{Mod}(\mathcal{L}) = \mathsf{FL}(\mathcal{L}, \mathcal{S})$ is in fact a 1-category and equivalent to $\mathsf{Mod}(\mathcal{L})$, since finite-limit preserving functors preserve truncation levels and thus every finite-limit preserving $F : \mathcal{L} \to \mathcal{S}$ must factor through the inclusion of 0-types $\mathsf{Set} \hookrightarrow \mathcal{S}$.

For finite-product theories, on the other hand, there is no such restriction. The ∞-models of the finite-product theory $\mathcal{C}_{\mathsf{Mon}}$ of monoids, for example, are the models of the *associative ∞-operad* [27, Section 4.11], whereas the ∞-models of the finite-product theory of abelian groups are related to the Dold–Kan correspondence. Variants of this phenomenon are discussed under the name 'animation' in [12], Rosicky's [35] contains an earlier account.

Now the nice thing about clans is that they admit finer graduations of 'levels of strictness' (or truncation levels). Among the clans $\mathcal{T}$, $\mathcal{T}_{\mathsf{Cat_O}}$, $\mathcal{T}_{\mathsf{Cat_A}}$, and $\mathcal{T}_{\mathsf{Cat_{OA}}}$ from Section 6.1, for example, we know that the ∞-models of the finite-limit clan $\mathcal{T}_{\mathsf{Cat_{OA}}}$ are precisely the strict 1-categories. The presence of the extensional identity type on O in $\mathcal{T}_{\mathsf{Cat_O}}$ behaves like a kind of 'partial finite-limit completion', and has the effect that the sort O is interpreted by a 0-type in every model $\mathcal{T}_{\mathsf{Cat_O}} \to \mathcal{S}$, whereas the presence of extensional identities on A in $\mathcal{T}_{\mathsf{Cat_A}}$ has the effect that the projection $(x\,y : O\,,\ f : A(x, y)) \to (x\,y : O)$ is mapped to a function with 0-truncated fibers by every ∞-model $C : \mathcal{T}_{\mathsf{Cat_A}} \to \mathcal{S}$. This means that ∞-models of $\mathcal{T}_{\mathsf{Cat_A}}$ are *pre-categories* in the sense of *Homotopy Type Theory* [39, Definition 9.1.1], whereas ∞-models of $\mathcal{T}_{\mathsf{Cat_O}}$ seem to correspond to *Segal-categories* [20, Section 2], [6, Section 5]. Finally, the clan $\mathcal{T}_{\mathsf{Cat}}$ does not impose *any* truncation conditions, which makes its ∞-models resemble *Segal spaces* (not necessarily complete), in the sense of [33, Section 4].

Appendix A. Locally finitely presentable categories, weak factorization systems, and Quillen's small object argument. This appendix recalls basic definitions and facts about the concepts mentioned in the title.

Definition A.1. A category $\mathcal{C}$ is called *filtered*, if every diagram $D : \mathbb{J} \to \mathbb{C}$ with finite domain admits a cocone. A *filtered colimit* is a colimit of a diagram indexed by a filtered category.

Definition A.2. Let $\mathfrak{X}$ be a cocomplete locally small category.

(i) An object object $C \in \mathfrak{X}$ is called *compact*, if the covariant hom-functor

$$\mathfrak{X}(C, -) : \mathfrak{X} \to \mathsf{Set}$$

preserves small filtered colimits.

(ii) $\mathfrak{X}$ is called *locally finitely presentable* (l.f.p.) if it admits a *small dense family of compact objects*, i.e. a family $(C_i)_{i\in I}$ of compact objects indexed by a small set I, such that the nerve functor

$$J_N : \mathfrak{X} \to \widehat{\mathbb{C}}$$

of the inclusion $J : \mathbb{C} \hookrightarrow \mathfrak{X}$ of the full subcategory on the $(C_i)_{i\in I}$ is fully faithful.

Remarks A.3. (a) Compact objects are also known as *finitely presentable objects*, e.g. in [14, 1]. We adopted the term compact from [26, Definition A.1.1.1] since it is more concise, and in particular since *compact* 0*-extension* sounds less awkward than *finitely presented* 0*-extension*. Moreover I think the fact that objects of algebraic categories (such as groups, rings, modules ...) are compact if and only if they admit a presentation by finitely many generators and relations is an important theorem, which is difficult to state if one uses the same terminology for the syntactic and the categorical notion.

(b) The density condition in the definition is equivalent to saying that the family $(C_i)_{i\in I}$ is a *strong generator*, in the sense that the canonical arrow

$$\coprod_{i\in I, f:C_i\to A} C_i \to A$$

is an extremal epimorphism for all $A \in \mathfrak{X}$. We stated the definition in terms of density here, since nerve functors play a central role in this work, contrary to strong generation.

(c) The notion of l.f.p. category is a special case of the notion of *locally α-presentable category* for a regular cardinal α [14, 1]. In this work, only the case $\alpha = \omega$ plays a role.

Definition A.4. Let $\mathcal{C}$ be a category.

(i) Given two arrows $f : A \to B$, $g : X \to Y$ in $\mathcal{C}$, we say that f has the *left lifting property* (l.l.p.) w.r.t. g (or equivalently that g has the *right lifting property* (r.l.p.)

w.r.t. f), and write $f \pitchfork g$, if in each commutative square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "k"] \arrow[ur, dashed, "m"] & Y \end{tikzcd}$$

there exists a diagonal arrow h making the two triangles commute.

(ii) Given a class $\mathcal{E} \subseteq \mathsf{mor}(\mathcal{C})$ of arrows in $\mathcal{C}$, we define:

$$^{\pitchfork}\mathcal{E} = \{f \in \mathsf{mor}(\mathcal{C}) \mid \forall g \in \mathcal{E} \,.\, f \pitchfork g\}$$
$$\mathcal{E}^{\pitchfork} = \{g \in \mathsf{mor}(\mathcal{C}) \mid \forall f \in \mathcal{E} \,.\, f \pitchfork g\}$$

(iii) A *weak factorization system* (w.f.s.) on $\mathcal{C}$ is a pair $\mathcal{L}, \mathcal{R} \subseteq \mathsf{mor}(\mathcal{C})$ of classes of morphisms such that $\mathcal{L}^{\pitchfork} = \mathcal{R}$, $\mathcal{R}^{\pitchfork} = \mathcal{L}$, and every $f : A \to B$ in $\mathcal{C}$ admits a factorization $f = l \circ r$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

We call $\mathcal{L}$ the *left class*, and $\mathcal{R}$ the *right class* of the w.f.s. One can show that left classes of w.f.s.'s contain all isomorphisms, and are closed under composition and pushouts, i.e. if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & B \arrow[d, "m"] \\ C \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout in $\mathcal{C}$ and is a left map, then so is m. Dually, right maps are closed under (isomorphisms, composition, and) pullbacks. With this, we have the prerequesites to state Quillen's *small object argument*.

THEOREM A.5 (Small object argument for l.f.p. categories). *Let $\mathcal{E} \subseteq \mathsf{mor}(\mathfrak{X})$ be a small set of morphisms in a l.f.p. category. Then $({}^{\pitchfork}(\mathcal{E}^{\pitchfork}), \mathcal{E}^{\pitchfork})$ is a w.f.s. on $\mathfrak{X}$.*

PROOF. Hovey [22, Thm. 2.1.14] and Riehl [34, Thm. 12.2.2] prove stronger statements in a more general setting. □

Appendix B. Generalized algebraic theories. Cartmell's *generalized algebraic theories* extend the notion of algebraic theory (which can be 'single sorted', such as the theories of *groups* or *rings*, or 'many sorted', such as the theories of *reflexive graphs*, *chain complexes of abelian groups*, or *modules over a non-fixed base ring*) by introducing *dependent sorts* (a.k.a. dependent 'types'), which represent families of sets and can be used e.g. to axiomatize the notion of a (small) category $\mathbb{C}$ as a structure with a set $\mathbb{C}_0$ of objects, and a family $(\mathbb{C}(A,B))_{A,B\in\mathbb{C}_0}$ of hom-sets (see (B.1) below).

Compared to ordinary algebraic theories—whose specification in terms of sorts, operations, and equations is fairly straightforward—the syntactic description of generalized algebraic theories is complicated by the fact that the domains of definition of operations and dependent sorts, and the codomains of operations, may themselves be compound expressions involving previously declared operations and sorts, whose well-formedness has to be ensured and may even depend on the equations of the theory. This means that we have to state the declarations of *sorts* and of *operations*, and the *equations* (which we collecively refer to as *axioms* of the theory) in an ordered way, where the later axioms

$$
\begin{array}{rl}
 & \vdash\ M \\
u:M & \vdash\ R(u) \\
 & \vdash\ e:M \\
u\,v:M & \vdash\ u{\cdot}v:M \\
u:M & \vdash\ 0(u):R(u) \\
u:M\,,\ x\,y:R(u) & \vdash\ x+y:R(u) \\
u:M\,,\ x:R(u) & \vdash\ -y:R(x) \\
 & \vdash\ 1:R(e) \\
u\,v:M\,,\ x:R(u)\,,\ y:R(v) & \vdash\ x{\cdot}y:R(u{\cdot}v) \\
u:M & \vdash\ e{\cdot}u=u=u{\cdot}e \\
u\,v\,w:M & \vdash\ (u{\cdot}v){\cdot}w=u{\cdot}(v{\cdot}w) \\
u:M\,,\ x\,y:R(u) & \vdash\ x+y=y+x \\
u:M\,,\ x\,y:R(u) & \vdash\ x+0(u)=x \\
u:M\,,\ x\,y:R(u) & \vdash\ x+(-x)=0(u) \\
u:M\,,\ x:R(u) & \vdash\ 1{\cdot}x=x=x{\cdot}1 \\
u\,v\,w:M\,,\ x:R(u)\,,\ y:R(v)\,,\ z:R(w) & \vdash\ (x{\cdot}y){\cdot}z=x{\cdot}(y{\cdot}z) \\
u\,v:M\,,\ x:R(u)\,,\ y\,z:R(v) & \vdash\ x{\cdot}(y+z)=x{\cdot}y+x{\cdot}z \\
u\,v:M\,,\ x\,y:R(u)\,,\ z:R(v) & \vdash\ (x+y){\cdot}z=x{\cdot}z+y{\cdot}z
\end{array}
$$

FIGURE 1. The generalized algebraic theory of monoid-graded rings

have to be well-formed on the basis of the earlier axioms. This looks as follows in the case of the generalized algebraic theory $\mathbb{T}_{\mathsf{Cat}}$ of categories:

$$
\begin{array}{rl}
 & \vdash\ O \\
x\,y:O & \vdash\ A(x,y) \\
x:O & \vdash\ \mathrm{id}(x):A(x,x) \\
x\,y\,z:O\,,\ f:A(x,y)\,,\ g:A(y,z) & \vdash\ g{\circ}f:A(x,z) \\
x\,y:O\,,\ f:A(x,y) & \vdash\ \mathrm{id}(y){\circ}f=f \\
x\,y:O\,,\ f:A(x,y) & \vdash\ f{\circ}\mathrm{id}(x)=f \\
w\,x\,y\,z:O\,,\ e:A(w,x)\,,\ f:A(x,y)\,,\ g:A(y,z) & \vdash\ (g{\circ}f){\circ}e=g{\circ}(f{\circ}e)
\end{array}
\tag{B.1}
$$

Each line contains one axiom, the first two declaring the sort O of objects and the dependent sort $A(x,y)$ of arrows, the third and the fourth declaring the identity and composition operations, and the last three stating the identity and associativity axioms.

Each axiom is of the form $\Gamma \vdash \mathcal{J}$, where the $\mathcal{J}$ on the right of the 'turnstile' symbol '$\vdash$' is the actual declaration or equation, and the part Γ on the left—called 'context'—specifies the sorts of the variables occuring in $\mathcal{J}$. Note that the ordering of these 'variable declarations' is not arbitrary, since the sorts of variables may themselves contain variables which have to be declared further left in the context. An example is the context $(x\,y\,z : O\,,\ f : A(x,y)\,,\ g : A(y,z))$ of the composition operation, where the sorts of the 'arrow' variables f,g depend on the 'object' variables x,y,z. See Figure 1 for another example generalized algebraic theory: the generalized algebraic theory of *rings graded over monoids*.

The dependent stucture of contexts and the well-formedness requirement of axioms on the basis of other axioms makes the formulation of a general notion of generalized algebraic theory somewhat subtle and technical. We refer to [10, 11] for the authoritative account and to [31, Section 6] and [15, Section 2] for rigorous and concise summaries. The good news is that to understand specific examples of GATs, these technicalities may safely be ignored: all we have to know is that for every generalized algebraic theory $\mathbb{T}$ there is a notion of 'derivable judgment' which includes the axioms and is closed under various *rules* expressing that the set of derivable judgments is closed under operations like substitutions and weakening, and that equality is reflexive, symmetric, and transitive.

Besides the forms of judgments

$\Gamma \vdash S$	'S is a sort in context Γ'
$\Gamma \vdash t : S$	't is term of sort S in context Γ'
$\Gamma \vdash s = t : S$	's and t are equal terms in context Γ'

that we have already encountered, we consider the following additional forms of judgments:

$\Gamma \vdash S = T$	'S and T are equal sorts in context Γ'
$\Gamma \vdash$	'Γ is a context'
$\Gamma = \Delta \vdash$	'Γ and Δ are equal contexts'
$\Gamma \vdash \sigma : \Delta$	'σ is a substitution from Γ to Δ'
$\Gamma \vdash \sigma = \tau : \Delta$	'σ and τ are equal substitutions from Γ to Δ'

The last two of these introduce a novel kind of expression called *substitution*: a substitution $\Gamma \vdash \sigma : \Delta$ is a list of terms that is suitable to be simultaneously substituted for the variables in a judgment in context Δ (in particular σ and Δ must have the same length), to produce a new judgment in context Γ, as expressed by the following *substitution* rule.

$$\frac{\Gamma \ \vdash \ \sigma : \Delta \qquad \Delta \ \vdash \ \mathcal{J}}{\Gamma \ \vdash \ \mathcal{J}[\sigma]} \tag{Subst}$$

Here, $\mathcal{J}[\sigma]$ is the result of simultaneous substitution of the terms in σ for the variables in $\mathcal{J}$, replacing each occurrence of the ith variable declared in Δ with the ith term in σ. This operation of simultaneous substitution also appears in the derivaton rules for substitutions themselves, which we present in the following table together with the rules for the formation of well-formed contexts:

$$\begin{array}{cc} \dfrac{}{* \ \vdash} & \dfrac{}{\Gamma \ \vdash \ () : *} \\[2ex] \dfrac{\Gamma \ \vdash \ A}{\Gamma, y : A \vdash} & \dfrac{\Gamma \ \vdash \ \sigma : \Delta \qquad \Delta \ \vdash \ A \qquad \Gamma \ \vdash \ t : A[\sigma]}{\Gamma \ \vdash \ (\sigma, t) : (\Delta, x : A)} \end{array} \tag{B.2}$$

The two rules in the first line say respectively that the *empty context* $\varnothing$ is a context, and that for any context Γ, the *empty substitution* $()$ is a substitution to the empty context. The first rule in the second line is known as *context extension*, since it says that we can extend any context by a well-formed sort in this context (here y has to be a 'fresh' variable, i.e. a variable not appearing in Γ). The last rule says that a substitution to an extended context is a pair of a substitution into the original context and a term whose sort is a *substitution instance* of the extending sort — it wouldn't make sense to ask for t to be of sort A since A is only well-formed in context Δ, and we want something in context Γ.

B.1. The syntactic category of a generalized algebraic theory.

Definition B.1. The *syntactic category* $\mathcal{C}[\mathbb{T}]$ of a generalized algebraic theory $\mathbb{T}$ is given as follows.

- The objects are the contexts of Γ *modulo derivable equality*, i.e. contexts Γ and Δ are identified if the judgment $\Gamma = \Delta \vdash$ is derivable.
- Similarly, morphisms $[\Gamma] \to [\Delta]$ from the equivalence class of Γ to the equivalence class of Δ are substitutions $\Gamma \vdash \sigma : \Delta$ modulo derivable equality. (The closure conditions on the set of derivable judgments ensure independence of representatives, e.g. that $\Gamma' \vdash \sigma : \Delta'$ whenever $\Gamma \vdash \sigma : \Delta$ and $\Gamma = \Gamma' \vdash$ and $\Delta = \Delta' \vdash$.)
- Composition is given by substitution of representatives, and identities are given by lists of variables:
  - $[\Delta \vdash \tau : \Theta] \circ [\Gamma \vdash \sigma : \Delta] = [\Gamma \ \vdash \ \tau[\sigma] : \Theta]$
  - $\mathrm{id}_\Gamma = (\Gamma \vdash (\vec{x}) : \Gamma)$ where $\vec{x}$ is the list of variables declared in Γ.

The syntactic category $\mathcal{C}[\mathbb{T}]$ of a GAT $\mathbb{T}$ has the structure of a *contextual category*:

Definition B.2. A *contextual category* consists of

(1) a small category $\mathcal{C}$ with a grading function $\mathsf{deg} : \mathcal{C}_0 \to \mathbb{N}$ on its objects, and
(2) a presheaf $\mathsf{Ty} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$, together with
  - an arrow $p_A : \Gamma.A \to \Gamma$ for each $\Gamma \in \mathcal{C}$ and $A \in \mathsf{Ty}(\Gamma)$, and
  - an arrow $\sigma.A : \Delta.A\sigma \to \Gamma.A$ for each $\Gamma \in \mathcal{C}$, $A \in \mathsf{Ty}(\Gamma)$, and $\sigma : \Delta \to \Gamma$,

such that:

(i) The square $\begin{tikzcd} \Delta.A\sigma \arrow[r, "\sigma.A"] \arrow[d, "p_{A\sigma}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "p_A"] \\ \Delta \arrow[r, "\sigma"] & \Gamma \end{tikzcd}$ is a pullback for all $A \in \mathsf{Ty}(\Gamma)$ and $\sigma : \Delta \to \Gamma$.
(ii) The mappings $(\Gamma, A) \mapsto \Gamma.A$ and $(\sigma, A) \mapsto \sigma.A$ constitute a functor $\mathsf{El}(\mathsf{Ty}) \to \mathcal{C}$.
(iii) We have $\mathsf{deg}(\Gamma.A) = \mathsf{deg}(\Gamma) + 1$ for all $\Gamma \in \mathcal{C}$ and $A \in \mathsf{Ty}(\Gamma)$.
(iv) There is a unique object $*$ of degree 0, and $*$ is terminal.
(v) For all Γ with $\mathsf{deg}(\Gamma) > 0$ there is a unique $(\Gamma_0, A) \in \mathsf{El}(\mathsf{Ty})$ with $\Gamma = \Gamma_0.A$.

In the case of the syntactic category $\mathcal{C}[\mathbb{T}]$ of a GAT $\mathbb{T}$, the grading assigns to each context its length, and $\mathsf{Ty}(\Gamma)$ is the set of 'types in context Γ', i.e. equivalence classes of type expressions A such that $\Gamma \vdash A$ is derivable, modulo the equivalence relation of derivable equality. The presheaf action is given by substitution. Given a type $A \in \mathsf{Ty}(\Gamma)$, the extended context $\Gamma.A$ is given by $\Gamma, y{:}A$ obtained via the context formation rule in (B.2), and p_A is the substitution

$$\Gamma, y{:}A \ \vdash \ (\vec{x}) \ : \ \Gamma$$

where $\vec{x}$ is the list of variables declared in Γ. For $\sigma : \Gamma \to \Delta$ and $A \in \mathsf{Ty}(\Delta)$, the substitution $\sigma.A$ is given by

$$\Gamma, y{:}A[\sigma] \ \vdash \ (\sigma, x) \ : \ \Delta, y{:}A\,.$$

Then the fact that the square in Definition B.2(i) is a pullback follows from the substitution formation rule in (B.2) together with the equality rules for substitutions that can be found in the cited references.

Finally, we have:

PROPOSITION B.3. *Every contextual category $\mathcal{C}$ admits a clan structure, where the display maps are the closure of the class of projection arrows $p_A : \Gamma.A \to \Gamma$ under composition and isomorphisms.* $\square$

Appendix C. The fat small object argument for clans.

C.1. Colimit decomposition formula and pushouts of sieves. In this subsection we discuss two results that are needed in the proof of the fat small object argument.

THEOREM C.1 (Colimit decomposition formula (CDF)). *Let $\mathbb{C} : \mathbb{J} \to \mathsf{Cat}$ be a small diagram in the 1-category of small categories, let $D : \mathsf{colim}(\mathbb{C}) \to \mathfrak{X}$ be a diagram in a category $\mathfrak{X}$ such that*

(i) *for each $j \in \mathbb{J}$, the colimit of $\mathsf{colim}_{c\in\mathbb{C}_j} D_{\sigma_j c}$ exists, and*
(ii) *the iterated colimit $\mathsf{colim}_{j\in\mathbb{J}} \mathsf{colim}_{c\in\mathbb{C}_j} D_{\sigma_j c}$ exists.*

Then $\mathsf{colim}_{j\in\mathbb{J}} \mathsf{colim}_{c\in\mathbb{C}_j} D_{\sigma_j c}$ is a colimit of D.

PROOF. Peschke and Tholen [30] give three proofs of this under the additional assumption that $\mathfrak{X}$ is cocomplete. The third proof (Section 5.3, 'via Fubini') easily generalizes to the situation where only the necessary colimits are assumed to exist. We sketch a slightly simplified argument here. Let $\int\mathbb{C}$ be the covariant Grothendieck construction of $\mathbb{C}$, whose projection $\int\mathbb{C} \to \mathbb{J}$ is a split opfibration. Then $\mathsf{colim}(\mathbb{C})$ is the 'joint coidentifier' of the splitting, i.e. there is a functor $E : \int\mathbb{C} \to \mathsf{colim}(\mathbb{C})$ such that for every category $\mathfrak{X}$, the precomposition functor

$$(- \circ E) : [\mathsf{colim}(\mathbb{C}), \mathfrak{X}] \to [\textstyle\int\mathbb{C}, \mathfrak{X}]$$

restricts to an isomorphism between the functor category $[\mathsf{colim}(\mathbb{C}), \mathfrak{X}]$ and the full subcategory of $[\int\mathbb{C}, \mathfrak{X}]$ on functors which send the arrows of the splitting to identities. In particular, $(- \circ E)$ is fully faithful and thus it induces an isomorphism

$$(\mathsf{colim}(\mathbb{C}))(D, \Delta-) \xrightarrow{\cong} (\textstyle\int\mathbb{C})(D \circ E, \Delta-) : \mathfrak{X} \to \mathsf{Set}$$

of co-presheaves of cocones for every diagram $D : \mathsf{colim}(\mathbb{C}) \to \mathfrak{X}$. In other words, E is *final*, which is the crucial point of the argument, and for which Peschke and Tholen give a more complicated proof in [30, Theorem 5.8].

Finality of E implies that D has a colimit if and only if $D \circ E$ has a colimit, and the existence of the latter follows if successive left Kan extensions along the composite $\int\mathbb{C} \to \mathbb{J} \to 1$ exist. The first of these can be computed as fiberwise colimit since $\int\mathbb{C} \to \mathbb{J}$ is a split cofibration [30, Theorem 4.6], which yields the inner term in the double colimit in the proposition. $\square$

In the following we use the CDF specifically for pushouts of sieve inclusions of posets. Recall that a *sieve* (a.k.a. *downset* or *lower set*) in a poset P is a subset $U \subseteq P$ satisfying

$$x \in U \wedge y \leq x \implies y \in U$$

for all $x, y \in P$. A monotone map $f : P \to Q$ is called a *sieve inclusion* if it is order-reflecting and its image $\mathsf{im}(f) = f[P]$ is a sieve in Q. The proof of the following lemma is straightforward, but we state it explicitly since it will play a crucial role.

Lemma C.2. (i) *If $f : P \to Q$ and $g : P \to R$ are sieve inclusions of posets, a pushout of f and g in the 1-category* Cat *of small categories is given by*

$$\begin{array}{ccc} P & \xrightarrow{g} & R \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\ Q & \xrightarrow{\sigma_1} & Q +_P R \end{array}$$

where $Q +_P R$ is the set-theoretic pushout, ordered by

$$\begin{array}{ll} \sigma_1(x) \le \sigma_1(y) \ \text{ iff } \ x \le y & \quad \sigma_1(x) \le \sigma_2(y) \ \text{ iff } \ \exists z\,.\, x = f(z) \wedge g(z) \le y \\ \sigma_2(x) \le \sigma_2(y) \ \text{ iff } \ x \le y & \quad \sigma_2(x) \le \sigma_1(y) \ \text{ iff } \ \exists z\,.\, x = g(z) \wedge f(z) \le y. \end{array}$$

In particular, the maps σ_1 and σ_2 are also sieve inclusions.

(ii) *If U and V are sieves in a poset P then the square*

$$\begin{array}{ccc} U \cap V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & U \cup V \end{array}$$

is a pushout in Cat*, where the sieves are equipped with the induced ordering.*

□

C.2. The fat small object argument for clans. Throughout this subsection let $\mathbb{C}$ be a coclan.

We start by establishing some notation. Given a poset P and an element $x \in P$, we write $P_{\le x} = \{y \in P \mid y \le x\}$ for the principal sieve generated by x, and $P_{<x} = \{y \in P \mid y < x\}$ for its subset on elements that are strictly smaller than x. If x is a maximal element of P, we write $P \backslash x$ for the sub-poset obtained by removing x. Given a diagram $D : P \to \mathbb{C}$, we write $D_{\le x}$, $D_{<x}$, and $D\backslash x$ for the restrictions of D to $P_{\le x}$, $P_{<x}$, and $P\backslash x$, respectively. More generally we write D_U for the restriction of D to arbitrary sieves $U \subseteq P$.

Note that we have $P_{\le x} = P_{<x} \star 1$, where $\star$ is the *join* or *ordinal sum*, thus diagrams $D : P_{\le x} \to \mathbb{C}$ are in correspondence with cocones on $D_{<x}$ with vertex D_x, and with arrows $\mathsf{colim}(D_{<x}) \to D_x$ whenever the colimit exists.

Definition C.3. A *finite $\mathbb{C}$-complex* is a pair (P, D) of a finite poset P and a diagram $D : P \to \mathbb{C}$, such that:

(i) $\mathsf{colim}(D_{<x})$ exists for all $x \in P$, and the induced $\alpha_x : \mathsf{colim}(D_{<x}) \to D_x$ is co-display.
(ii) For $x, y \in P$ we have $x = y$ whenever $P_{<x} = P_{<y}$, $D_x = D_y$, and $\alpha_x = \alpha_y$.

An *inclusion of finite $\mathbb{C}$-complexes* $f : (P, D) \to (Q, E)$ is a sieve inclusion $f : P \to Q$ such that $D = E \circ f$. We write $\mathsf{FC}(\mathbb{C})$ for the category of finite $\mathbb{C}$-complexes and inclusions.

Remark C.4. We view a finite $\mathbb{C}$-complex as a construction of an object by a finite (though not necessarily linearly ordered) number of 'cell attachments', represented by the co-display maps $\alpha_x : \mathsf{colim}(D_{<x}) \rightarrowtriangle D_x$. Condition (ii) should be read as saying that 'every cell can only be attached once at the same stage'. This is needed in Lemma C.7 to show that $\mathsf{FC}(\mathbb{C})$ is a preorder.

Lemma C.5. (i) $\mathsf{colim}(D)$ *exists for every finite $\mathbb{C}$-complex (P, D).*

(ii) *The induced functor*

$$\mathsf{Colim} : \mathsf{FC}(\mathbb{C}) \to \mathbb{C} \tag{C.1}$$

sends inclusions of finite $\mathbb{C}$*-complexes to co-display maps.*

Proof. The first claim is shown by induction on $|P|$. For empty P the statement is true since coclans have initial objects. For $|P| = n+1$ assume that $x \in P$ is a maximal element. Then the quare

$$\begin{tikzcd} P_{<x} \arrow[r] \arrow[d] & P\backslash x \arrow[d] \\ P_{\le x} \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & P \end{tikzcd}$$

is a pushout in Cat by Lemma C.2, which by the colimit decomposition formula C.1 means that the pushout of the span

$$\begin{tikzcd} \mathsf{colim}(D_{<x}) \arrow[r] \arrow[d, rightarrowtail] & \mathsf{colim}(D\backslash x) \arrow[d, dashed, rightarrowtail] \\ D_x \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner" very near end] & \mathsf{colim}(D) \end{tikzcd} \tag{C.2}$$

—which exists since the left arrow is a co-display map by C.3-(i)—is a colimit of D in $\mathbb{C}$.

For the second claim let $f : (E, Q) \to (D, P)$ be an inclusion of finite $\mathbb{C}$-complexes. Since every inclusion of finite $\mathbb{C}$-complexes can be decomposed into 'atomic' inclusions with $|P\backslash f[Q]| = 1$, we may assume without loss of generality that $Q = P\backslash x$ for some maximal $x \in P$. Then the image of f under Colim is the right dashed arrow in (C.2), which is co-display since co-display maps are stable under pushout. □

Remark C.6. Lemma C.5 implies that the assumption '$\mathsf{colim}(D_{<x})$ exists' in Definition C.3-(i) is redundant, since the colimits in question are colimits of finite subcomplexes.

Lemma C.7. *The category* $\mathsf{FC}(\mathbb{C})$ *is an essentially small preorder with finite joins.*

Proof. $\mathsf{FC}(\mathbb{C})$ is essentially small as a collection of finite diagrams in a small category. To see that it is a preorder let $f, g : (P, D) \to (Q, E)$ be inclusions of finite $\mathbb{C}$-complexes. We show that $f(x) = g(x)$ by well-founded induction on $x \in P$. Let $x \in P$ and assume that $f(y) = g(y)$ for all $y < x$. Then since f and g are sieve inclusions we have $Q_{<f(x)} = Q_{<g(x)}$ and since $E\,f = D = E\,g$ we have the equalities

$$\big(E_y \to E_{f(x)}\big)_{y<f(x)} = (D_y \to D_x)_{y<x} = \big(E_y \to E_{g(x)}\big)_{y<g(x)}$$

of cocones, whence $f(x) = g(x)$ by Definition C.3-(ii).

It remains to show that $\mathsf{FC}(\mathbb{C})$ has finite suprema. The empty complex is clearly initial. We show that a supremum of (P, D) and (Q, E) exists by induction on $|P|$. The empty case is trivial, so assume that P is inhabited and let x be a maximal element. Let (R, F) be a supremum of $(P\backslash x, D\backslash x)$ and (Q, E), with inclusion maps $f : (P\backslash x, D\backslash x) \to (R, F)$ and $g : (Q, E) \longrightarrow (R, F)$. If there exists a $y \in R$ such that $R_{<y} = f[P_{<x}]$ and $(D_z \to D_x)_{z<x} = \big(R_{f(z)} \to R_y\big)_{z<x}$ then 'the cell-attachment corresponding to x is already contained in (R, F)', i.e. f extends to an inclusion $f' : (P, D) \to (R, F)$ of finite complexes with $f'(x) = y$, whence (R, F) is a supremum of (P, D) and (Q, E).

If no such y exists then a supremum of (P, D) and (R, F) is given by $(P +_{P\backslash x} R, [D, F])$, as in the pushout diagram

$$\begin{tikzcd}
P\backslash x \arrow[r, "f"'] \arrow[d, hook] & R \arrow[d] \arrow[drr, bend left, "F"] & & \\
P \arrow[r] \arrow[drrr, bend right, "D"'] & P +_{P\backslash x} R \arrow[ul, phantom, "\ulcorner"] \arrow[rr, dashed, "{[D,F]}"] & & \mathbb{C}
\end{tikzcd}$$

constructed as in Lemma C.2. □

THEOREM C.8. *The object* $C = \mathsf{colim}_{(P,D)\in\mathsf{FC}(\mathbb{C})} H(\mathsf{colim}(D))$ *is a* 0*-extension in* $\mathbb{C}^{\mathsf{op}}$*-*Alg *and* $C \to 1$ *is full.*

PROOF. To see that $C \to 1$ is full, let $e : I \rightarrowtail J$ be co-display in $\mathbb{C}$ and let $f : H(I) \to C$. Since $\mathsf{FC}(\mathbb{C})$ is filtered and $H(I)$ is compact, f factors through a colimit inclusion as

$$f \;=\; \big(H(I) \xrightarrow{H(g)} H(\mathsf{colim}(D)) \xrightarrow{\sigma_{(P,D)}} C\big)$$

for some finite complex (P, D). We form the pushout

$$\begin{tikzcd}
I \arrow[r, "g"] \arrow[d, tail, "e"'] & \mathsf{colim}(D) \arrow[d, tail, "k"] \\
J \arrow[r] & K \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

and extend the finite complex (P, D) to $(P\star 1, D\star k)$ where $P\star 1$ is the join of P and 1, and $D\star k : P\star 1 \to \mathbb{C}$ is the diagram extending D with the cell-attachment $k : \mathsf{colim}(D) \rightarrowtail K$. Then $K = \mathsf{colim}(D\star k)$ and k is the image of the inclusion $(P, D) \hookrightarrow (P\star 1, D\star k)$ of finite complexes under the colimit functor (C.1), thus we obtain an extension of f along $H(e)$ as in the following diagram.

$$\begin{tikzcd}
H(I) \arrow[r, "H(g)"'] \arrow[d, hook, "H(e)"'] \arrow[rr, bend left, "f"] & H(\mathsf{colim}(D)) \arrow[r, "\sigma_{(P,D)}"'] \arrow[d, hook, "H(k)"] & C \\
H(J) \arrow[r] & H(K) \arrow[ul, phantom, "\ulcorner"] \arrow[ur, bend right, "\sigma_{(P\star 1, D\star k)}"'] &
\end{tikzcd}$$

To see that C is a 0-extension, consider a full map $f : Y \twoheadrightarrow X$ in $\mathbb{C}^{\mathsf{op}}$-Alg and an arrow $h : C \to X$. To show that h lifts along f we construct a lift of the cocone

$$\Big(H(\mathsf{colim}(D)) \xrightarrow{\sigma_{(P,D)}} C \xrightarrow{h} X\Big)_{(P,D)\in\mathsf{FC}(\mathbb{C})}$$

by induction over the preorder $\mathsf{FC}(\mathbb{C})$ which is well-founded since every finite $\mathbb{C}$-complex has only finitely many subcomplexes. Given a finite complex (D, P) it is sufficient to exhibit a lift $\kappa_{(P,D)} : H(\mathsf{colim}(D)) \to Y$ satisfying

$$f \circ \kappa_{(P,D)} = h \circ \sigma_{(P,D)} \qquad \text{and} \tag{C.3}$$

$$\kappa_{(P,D)} \circ H(\mathsf{colim}\, j) = \kappa_{(Q,E)} \qquad \text{for all subcomplexes } j : (Q, E) \to (P, D), \tag{C.4}$$

where we may assume that the $\kappa_{(Q,E)}$ satisfy the analogous equations by induction hypothesis. We distinguish two cases:

1. If P has a greatest element x then we can take $\kappa_{(P,D)}$ to be a lift in the square

$$\begin{tikzcd} H(\mathsf{colim}(D_{<x})) \arrow[rr, "\kappa_{(P_{<x},D_{<x})}"] \arrow[d] & & Y \arrow[d, "f"] \\ H(D_x) \arrow[r, "\sigma_{(P,D)}"'] \arrow[urr, dashed, "\kappa_{(P,D)}"] & C \arrow[r, "h"'] & X \end{tikzcd}$$

whose left side is an extension by Lemma C.5 and whose right side is full by assumption. Then (C.3) holds by construction, and (C.4) holds for all subcomplexes since it holds for the largest strict subcomplex $(P_{<x}, D_{<x}) \to (P,D)$.

2. If P *doesn't* have a greatest element we can write $P = U \cup V$ as union of two strict sub-sieves, wence we have pushouts

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & P \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{colim}(D_{U\cap V}) \arrow[r] \arrow[d] & \mathsf{colim}(D_V) \arrow[d] \\ \mathsf{colim}(D_U) \arrow[r] & \mathsf{colim}(D) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

by Lemma C.2 and the CDF. This means that condition (C.4) forces us to define $\kappa_{(P,D)}$ to be the unique arrow fitting into

$$\begin{tikzcd} H(\mathsf{colim}(D_{U\cap V})) \arrow[r, "\phi_V^{U\cap V}"'] \arrow[d, "\phi_U^{U\cap V}"] & H(\mathsf{colim}(D_V)) \arrow[d, "\phi_P^V"] \arrow[ddr, bend left, "\kappa_{(V,D_V)}"] & \\ H(\mathsf{colim}(D_U)) \arrow[r, "\phi_P^U"] \arrow[drr, bend right, "\kappa_{(U,D_U)}"'] & H(\mathsf{colim}(D)) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\kappa_{(P,D)}"] & \\ & & Y \end{tikzcd}, \tag{C.5}$$

where for the remainder of the proof we write $\phi_W^X : H(\mathsf{colim}(D_X)) \to H(\mathsf{colim}(D_W))$ for the canonical arrows induced by successive sieve inclusions $X \subseteq W \subseteq P$. Using the fact that the ϕ_P^U and ϕ_P^V are jointly epic it is easy to see that the $\kappa_{(P,D)}$ defined in this way satisfies condition (C.3), and it remains to show that (C.4) is satisfied for arbitrary sieves $W \subseteq P$, i.e. $\kappa_{(P,D)} \circ \phi_P^W = \kappa_{(W,D_W)} : H(\mathsf{colim}(D_W)) \to Y$. Since

$$\begin{tikzcd} H(\mathsf{colim}(D_{U\cap V\cap W})) \arrow[r, "\phi_{V\cap W}^{U\cap V\cap W}"] \arrow[d, "\phi_{U\cap W}^{U\cap V\cap W}"'] & H(\mathsf{colim}(D_{V\cap W})) \arrow[d, "\phi_W^{V\cap W}"] \\ H(\mathsf{colim}(D_{U\cap W})) \arrow[r, "\phi_W^{U\cap W}"] & H(\mathsf{colim}(D_W)) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout it is enough to verify this equation after precomposing with $\phi_W^{U\cap W}$ and $\phi_W^{V\cap W}$. We have

$$\begin{aligned} \kappa_{(P,D)} \circ \phi_P^W \circ \phi_W^{U\cap W} &= \kappa_{(P,D)} \circ \phi_P^U \circ \phi_U^{U\cap W} && \text{by functoriality} \\ &= \kappa_{(U,D_U)} \circ \phi_U^{U\cap W} && \text{by (C.5)} \\ &= \kappa_{(U\cap W, D_{U\cap W})} && \text{by (C.4)} \\ &= \kappa_{(W,D_W)} \circ \phi_W^{U\cap W} && \text{by (C.4)} \end{aligned}$$

and the case with $\phi_W^{V\cap W}$ is analogous. □

Corollary C.9. *For any clan $\mathcal{T}$, the 0-extensions in $\mathcal{T}$-*Mod *are flat.*

Proof. Let $E \in \mathcal{T}\text{-}\mathsf{Mod}$ be a 0-extension. By applying Theorem C.8 in $\mathcal{T}\text{-}\mathsf{Mod}/E$ (using Proposition 3.6), we obtain a full map $f : F \twoheadrightarrow E$ where F is a 0-extension and f is a filtered colimit of arrows $H(\Gamma) \to E$ in $\mathcal{T}\text{-}\mathsf{Mod}/A$. Since $\mathcal{T}\text{-}\mathsf{Mod}/A \to \mathcal{T}\text{-}\mathsf{Mod}$ creates colimits this means that F is a filtered colimit of hom-models in $\mathcal{T}\text{-}\mathsf{Mod}$, and therefore flat (Lemma 5.4). Since f is a full map into a 0-extension it has a section, thus E is a retract of F and therefore flat as well. $\square$

REFERENCES

[1] J. Adámek and J. Rosicky, ***Locally presentable and accessible categories***, vol. 189, Cambridge University Press, 1994.

[2] J. Adámek, J. Rosický, and E.M. Vitale, ***Algebraic theories: a categorical introduction to general algebra***, vol. 184, Cambridge University Press, 2010.

[3] B. Ahrens, P. North, M. Shulman, and D. Tsementzis, *A higher structure identity principle*, ***Proceedings of the 2020 35th annual acm/ieee symposium on logic in computer science, lics 2020, virtual event, july 8–11, 2020***, New York, NY: Association for Computing Machinery (ACM), 2020, pp. 53–66 (English).

[4] ———, *The univalence principle*, (2021).

[5] H.J. Baues, ***Algebraic homotopy***, Cambridge Studies in Advanced Mathematics, vol. 15, Cambridge University Press, Cambridge, 1989.

[6] J. Bergner, *A survey of* $(\infty, 1)$*-categories*, ***Towards higher categories***, IMA Vol. Math. Appl., vol. 152, Springer, New York, 2010, pp. 69–83.

[7] F. Borceux, ***Handbook of categorical algebra. 1***, Encyclopedia of Mathematics and its Applications, vol. 50, Cambridge University Press, Cambridge, 1994.

[8] M. Brandenburg, *Universal property of the cocomplete category of models of a limit sketch*, MathOverflow, https://mathoverflow.net/q/403653 (version: 2021-09-10).

[9] M. Brandenburg, *Large limit sketches and topological space objects*, ***arXiv preprint arXiv:2106.11115***, (2021).

[10] J. Cartmell, *Generalised algebraic theories and contextual categories*, ***Ph.D. thesis***, Oxford University, 1978, available at https://ncatlab.org/nlab/files/Cartmell-Thesis.pdf.

[11] ———, *Generalised algebraic theories and contextual categories*, ***Annals of Pure and Applied Logic***, vol. 32 (1986), pp. 209–243.

[12] K. Cesnavicius and P. Scholze, *Purity for flat cohomology*, ***arXiv preprint arXiv:1912.10932***, (2019).

[13] P. Freyd, *Aspects of topoi*, ***Bulletin of the Australian Mathematical Society***, vol. 7 (1972), no. 1, pp. 1–76.

[14] P. Gabriel and F. Ulmer, ***Lokal präsentierbare kategorien***, Lecture Notes in Math., vol. 221, Springer-Verlag, 1971.

[15] R. Garner, *Combinatorial structure of type dependency*, ***Journal of Pure and Applied Algebra***, vol. 219 (2015), no. 6, pp. 1885–1914.

[16] T. Head, *Expanded subalphabets in the theories of languages and semigroups*, ***International Journal of Computer Mathematics. Section A. Programming Theory and Methods. Section B. Computational Methods***, vol. 12 (1982/83), no. 2, pp. 113–123.

[17] S. Henry, *Algebraic models of homotopy types and the homotopy hypothesis*, ***arXiv preprint arXiv:1609.04622***, (2016).

[18] G. Higman and B. H. Neumann, *Groups as groupoids with one law*, ***Publicationes Mathematicae Debrecen***, vol. 2 (1952), pp. 215–221.

[19] P.S. Hirschhorn, *Overcategories and undercategories of cofibrantly generated model categories*, ***Journal of Homotopy and Related Structures***, vol. 16 (2021), no. 4, pp. 753–768.

[20] A. Hirschowitz and C. Simpson, *Descente pour les n-champs (descent for n-stacks)*, (1998).

[21] M. Hofmann, ***Extensional constructs in intensional type theory***, CPHC/BCS Distinguished Dissertations, Springer-Verlag London, Ltd., London, 1997.

[22] Mark Hovey, ***Model categories***, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[23] P.T. Johnstone, ***Sketches of an elephant: a topos theory compendium. Vol. 2***, Oxford Logic Guides, vol. 44, Oxford University Press, Oxford, 2002.

[24] A. Joyal, *Notes on clans and tribes*, ***arXiv preprint arXiv:1710.10238***, (2017).

[25] I. Di Liberti and J. Rosický, *Enriched locally generated categories*, (2020).

[26] J. Lurie, ***Higher topos theory***, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[27] J. Lurie, *Higher algebra*, 2017, Unpublished. Available at https://www.math.ias.edu/~lurie/.

[28] S. Mac Lane, ***Categories for the working mathematician***, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[29] M. Makkai, J. Rosicky, and L. Vokrinek, *On a fat small object argument*, ***Advances in Mathematics***, vol. 254 (2014), pp. 49–68.

[30] G. Peschke and W. Tholen, *Diagrams, fibrations, and the decomposition of colimits*, ***arXiv preprint arXiv:2006.10890***, (2020).

[31] A.M. Pitts, *Categorical logic*, ***Handbook of logic in computer science, Vol. 5***, Handb. Log. Comput. Sci., vol. 5, Oxford Univ. Press, New York, 2000, p. 39–128.

[32] A. Pultr, *The right adjoints into the categories of relational systems*, ***Reports of the Midwest Category Seminar, IV***, Lecture Notes in Mathematics, Vol. 137, Springer, Berlin, 1970, pp. 100–113.

[33] C. Rezk, *A model for the homotopy theory of homotopy theory*, ***Transactions of the American Mathematical Society***, vol. 353 (2001), no. 3, pp. 973–1007.

[34] E. Riehl, ***Categorical homotopy theory***, New Mathematical Monographs, vol. 24, Cambridge University Press, Cambridge, 2014.

[35] J. Rosický, *On homotopy varieties*, ***Advances in Mathematics***, vol. 214 (2007), no. 2, pp. 525–550.

[36] C.L. Subramaniam, *From dependent type theory to higher algebraic structures*, (2021).

[37] P. Taylor, *Recursive domains, indexed category theory and polymorphism*, ***Ph.D. thesis***, University of Cambridge, 1987.

[38] ———, ***Practical foundations of mathematics***, vol. 59, Cambridge University Press, 1999.

[39] The Univalent Foundations Program, ***Homotopy type theory: Univalent foundations of mathematics***, https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.