Duality for Clans: a Refinement of Gabriel–Ulmer Duality

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Abstract

We exhibit an idempotent biadjunction between a 2-category of clans and a 2-category of locally finitely presentable categories equipped with a weak factorization system, and characterize the stable subcategories.

Contents

1 Introduction

Gabriel–Ulmer duality [GU71] is a contravariant biequivalence

$$\mathsf{FL} \overset{\mathsf{op}}{\simeq} \mathsf{LFP}$$

between the 2-category FL of *small finite-limit categories*, and the 2-category LFP of *locally finitely presentable categories*, i.e. locally small cocomplete categories admitting a dense set of compact (a.k.a. finitely presentable) objects. The duality assigns to every small finite-limit category $\mathcal{C}$ the category $\mathsf{FL}(\mathcal{C}, \mathsf{Set})$ of finite-limit preserving functors into Set, and conversely it associates to every locally finitely presentable $\mathfrak{X}$ the opposite of its full subcategory $\mathsf{comp}(\mathfrak{X}) \subseteq \mathfrak{X}$ of compact objects[1].

We view Gabriel–Ulmer duality as a *theory-model duality*: small finite-limit categories $\mathcal{C}$ are viewed as theories (which we call 'finite-limit theories'), and—in the spirit of Lawverian functorial semantics—the functor category $\mathsf{FL}(\mathcal{C}, \mathsf{Set})$ is viewed as the category of *models* of the finite-limit theory $\mathcal{C}$.

It is well known that finite-limit theories are equally expressive as various syntactically defined classes of theories, including

(1) Freyd's *essentially algebraic theories* [Fre72],

(2) Cartmell's *generalized algebraic theories (GATs)* [Car78, Car86],

(3) Johnstone's *cartesian theories* [Joh02, Definition D1.3.4], and

(4) Palmgren–Vickers's *quasi-equational theories* [PV07],

in the sense that for any theory $\mathbb{T}$ from either of these classes, the category $\mathsf{Mod}(\mathbb{T})$ of models is locally finitely presentable, and conversely for every locally finitely presentable category $\mathfrak{X}$ there exits a theory from either of the classes whose category of models is equivalent to $\mathfrak{X}$. While from a high level this means that the classes (1)–(4) of theories are all equivalent to finite-limit theories, strictly speaking this isn't entirely accurate since the syntactic representations might contain additional information that is not reflected in the categories of models. The present article argues that this is the case for

The present article is specifically concerned with Cartmell's GATs, which generalize 'ordinary' algebraic theories (like the theories of groups or rings) by admitting *dependent sorts*, the prime example being the sort

$$x, y : O \;\vdash\; A(x, y)$$

of arrows depending on two variables of sort O ('object') in the standard GAT presentation of the theory of categories.

, which do indeed contain additional information not captured by the corresponding lex theory, and this motivates the *refinement* of the duality announced in the title. That the additional information in GATs goes beyond 'syntactic details' and is meaningful from the point of view of categorical logic is is suggested by the fact that Cartmell gives a *different* categorical presentation of GATs, replacing small finite-limit categories by what he calls *contextual categories*: a contextual category is a small category $\mathbb{C}$ with a *grading* $\mathsf{deg} : \mathbb{C}_0 \to \mathbb{N}$ on its set of objects, such that

[1]Strictly speaking we have to choose a small category which is *equivalent* to $\mathsf{comp}(\mathfrak{X})^{\mathsf{op}}$, since the latter is only *essentially small* in general.

(1) there's a unique object $*$ of degree 0, which is terminal,

(2) every object Γ of degree $n \neq 0$ comes with a *projection* $p_\Gamma : \Gamma \twoheadrightarrow \Gamma^-$ into an object of degree $n-1$, and

(3) $\mathbb{C}$ admits pullbacks of projections along arbitrary maps, and is in fact equipped with a *choice* of such pullbacks which satisfies a strictness condition[2].

Of this data, we do indeed consider the grading and the choice ofllbacks as 'syntactic details', and dropping this from the definition and closing the display maps under composition and isomorphisms leads us to to the concept of *clan*, which was introduced by Taylor [Tay87, § 4.3.2] under the name of *category with a class of display maps*, and later renamed by Joyal [Joy17]. Thus, a clan is a category[3] $\mathcal{T}$ equipped with a class $\mathcal{T}_\dagger \subseteq \mathcal{T}_1$ of *display maps* which contains terminal projections and is closed under composition and pullback along arbitrary maps (Definition 2.1), and a *model* is a functor $A : \mathcal{T} \to \mathsf{Set}$ preserving 1 and pullbacks of display maps.

2 Clans

Definition 2.1 A *clan* is a small category $\mathcal{T}$ with a distinguished class $\mathcal{T}_\dagger$ of arrows called *display maps*, such that:

(i) Arbitrary pullbacks of display maps exist and are again display maps.

(ii) Isomorphisms and compositions of display maps are display maps.

(iii) $\mathcal{T}$ has a terminal object, and terminal projections are display maps.

A *clan morphism* is a functor between clans which preserves display maps, pullbacks of display maps, and the terminal object. We write Clan for the 2-category of clans, clan-morphisms, and natural transformations. $\diamondsuit$

Since it seems to lead to a more readable exposition, we introduce explicit notation and terminology for the dual notion.

Definition 2.2 A *coclan* is a small category $\mathcal{C}$ with a distinguished class $\mathcal{C}_\dagger$ of arrows called *codisplay maps* satisfying the dual axioms of clans. The 2-category CoClan of coclans is defined dually to that of clans, i.e.

$$\mathsf{CoClan}(\mathcal{C}, \mathcal{D}) = \mathsf{Clan}(\mathcal{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}})^{\mathsf{op}}$$

for coclans $\mathcal{C}, \mathcal{D}$. $\diamondsuit$

Remarks 2.3 (a) Definition 2.1 (apart from the smallness condition), and the term 'display map', are due to Taylor [Tay87, 4.3.2]. The name 'clan' was suggested by Joyal [Joy17, Definition 1.1.1].

(b) We have defined clans to be *small* by default, since this fits with our point of view of clans as theories, and makes the duality theory work.

However, it is also reasonable to consider non-small, 'semantic' clans, and we will mention them occasionally, using the term *large clan* in this case. $\diamondsuit$

[2]For the full definition of contextual category see Definition B.3

[3]We drop the smallness condition from the general definition since we want the definition to comprise not only 'theories' but also large categories in which to study models.

Examples 2.4 (a) Finite-product categories can be viewed as clans where the display maps are the morphisms that are (isomorphic to) product projections. We call such clans *finite-product clans.*

(b) Finite-limit categories can be viewed as clans where *all* morphimsms are display maps. We call such clans *finite-limit clans.*

(c) Kan is the *large* clan whose underlying category is the full subcategory of the category $[\boldsymbol{\Delta}^{\mathsf{op}}, \mathsf{Set}]$ of simplicial sets on *Kan complexes*, and whose display maps are the *Kan fibrations.*

(d) The syntactic category of every *generalized algebraic theory* in the sense of Cartmell [Car78, Car86] is a clan. We describe this In more detail in the following subsection. ♢

3 Algebras

Definition 3.1 Given a clan $\mathcal{T}$, a $\mathcal{T}$*-algebra* is a functor $\mathcal{T} \to \mathsf{Set}$ which preserves the terminal object and pullbacks of display maps. We write $\mathcal{T}\text{-}\mathsf{Alg}$ for the category of $\mathcal{T}$-algebras, viewed as a full subcategory of the functor category $[\mathcal{T}, \mathsf{Set}]$. ♢

Remark 3.2 In the spirit of functorial semantics, it is possible to consider algebras of clans in other categories than sets, and even in other (possibly large) clans. However, the duality theory presented here is about models in Set and we don't consider any other kind. ♢

Remarks 3.3 (a) As category of models of a finite-limit sketch, $\mathcal{T}\text{-}\mathsf{Alg}$ is reflective (and therefore closed under arbitrary limits) in $[\mathcal{T}, \mathsf{Set}]$, and moreover it is closed under filtered colimits [AR94, Section 1.C]. In particular, $\mathcal{T}\text{-}\mathsf{Alg}$ is locally finitely presentable.

(b) The hom-functors $\mathcal{T}(\Gamma, -) : \mathcal{T} \to \mathsf{Set}$ are $\mathcal{T}$-algebras for all $\Gamma \in \mathcal{T}$ (we'll refer to them as hom-*algebras*), i.e. the Yoneda embedding $よ : \mathcal{T}^{\mathsf{op}} \to [\mathcal{T}, \mathsf{Set}]$ lifts along the inclusion $\mathcal{T}\text{-}\mathsf{Alg} \hookrightarrow [\mathcal{T}, \mathsf{Set}]$ to a fully faithful functor $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Alg}$.

$$\begin{tikzcd} & \mathcal{T}\text{-}\mathsf{Alg} \arrow[d, hook] \\ \mathcal{T}^{\mathsf{op}} \arrow[ur, dashed, "H"] \arrow[r, "よ"'] & {[\mathcal{T}, \mathsf{Set}]} \end{tikzcd}$$

(c) For $\Gamma \in \mathcal{T}$, the hom-functor

$$\mathcal{T}\text{-}\mathsf{Alg}(H(\Gamma), -) : \mathcal{T}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

is isomorphic to the evaluation functor $A \mapsto A(\Gamma)$, hence it preserves filtered colimits as those are computed in $[\mathcal{T}, \mathsf{Set}]$ and therefore pointwise. This means that $H(\Gamma)$ is *compact*[4] in $\mathcal{T}\text{-}\mathsf{Alg}$.

[4] Following Lurie [Lur09] we use the shorter term 'compact' instead of the more traditional 'finitely presented' for objects whose covariant hom-functor preserves filtered colimits.

3.1 Flat algebras

Convention 3.4 One finds opposing definitions for the *category of elements* of a Set-valued functor in the literature, depending on whether the functors at hand are covariant or contravariant. Since in this work we don't want to regard variance as an intrinsic property of a functor (a covariant functor out of a clan is the same thing as a contravariant functor out of a coclan), we write $\underrightarrow{\mathsf{elts}}(F)$ for the *covariant category of elements* of a Set-valued functor $F : \mathbb{C} \to \mathsf{Set}$, and $\underleftarrow{\mathsf{elts}}(F)$ for the *contravariant category of elements*. Note that we have $\underleftarrow{\mathsf{elts}}(F) = \underrightarrow{\mathsf{elts}}(F)^{\mathsf{op}}$, and the forgetful functors $\underrightarrow{\mathsf{elts}}(F) \to \mathbb{C}$ and $\underleftarrow{\mathsf{elts}}(F) \to \mathbb{C}^{\mathsf{op}}$ are discrete opfibrations and discrete fibrations, respectively.

Recall that for small $\mathbb{C}$, a functor $F : \mathbb{C} \to \mathsf{Set}$ is called *flat* if $\underleftarrow{\mathsf{elts}}(F)$ is filtered, or equivalently if the left Kan extension $F_! : [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}] \to \mathsf{Set}$ of F along $\text{よ} : \mathbb{C} \to [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$ preserves finite limits [Bor94, Definition 6.3.1 and Proposition 6.3.8]. From the second characterization it follows that flat functors preserve all finite limits that exist in $\mathbb{C}$, thus for the case of a clan $\mathcal{T}$, flat functors $F : \mathcal{T} \to \mathsf{Set}$ are always algebras. We refer to them as *flat* algebras.

Definition 3.5 A $\mathcal{T}$-algebra $A : \mathcal{T} \to \mathsf{Set}$ is called *flat*, if $\underleftarrow{\mathsf{elts}}(F)$ is filtered. $\diamondsuit$

Lemma 3.6 *An algebra A over a clan $\mathcal{T}$ is flat iff it is a filtered colimit of* hom*-algebras.*

Proof. We always have $A = \mathsf{colim}(\underleftarrow{\mathsf{elts}}(A) \to \mathcal{T}^{\mathsf{op}} \xrightarrow{H} \mathcal{T}\text{-}\mathsf{Alg})$, thus if F is flat then it is a filtered colimit of hom-algebras. The other direction follows since hom-algebras are flat, and flat functors are closed under filtered colimits in $[\mathcal{T}, \mathsf{Set}]$ [Bor94, Proposition 6.3.6]. $\square$

3.2 The weak factorization system on algebras

Next we introduce the *extension–full weak factorization system* on the category of algebras over a clan. We assume familiarity with lifting properties and weak factorization systems (w.f.s.'s) and refer to [Rie14, Section 11.2] for a modern reference.

Definition 3.7 Let $\mathcal{T}$ be a clan.

(i) We call a map $f : A \to B$ in $\mathcal{T}\text{-}\mathsf{Alg}$ *full*, if it has the *right lifting property* (r.l.p.) with respect to all maps $H(p)$ for p a display map.

(ii) We call $f : A \to B$ an *extension*, if it has the *left lifting property* l.l.p. w.r.t. all full maps.

(iii) We call $A \in \mathcal{T}\text{-}\mathsf{Alg}$ a 0-*extension*, if $0 \to A$ is an extension. $\diamondsuit$

Remarks 3.8 (a) We use the arrow symbols '$\hookrightarrow$' of extensions, and '$\twoheadrightarrow$' for full maps. We write $\mathcal{E}$ and $\mathcal{F}$ for the classes of extensions and full maps in $\mathcal{T}\text{-}\mathsf{Alg}$, respectively. By the *small object argument* ([Rie14, Theorem 12.2.2], [Lur09, Proposition A.1.2.5]), extensions and full maps form a w.f.s. $(\mathcal{E}, \mathcal{F})$ on $\mathcal{T}\text{-}\mathsf{Alg}$.

(b) A map $f : A \to B$ in $\mathcal{T}\text{-}\mathsf{Alg}$ is full if and only if the naturality square

$$\begin{tikzcd} A(\Delta) \arrow[r, "A(p)"] \arrow[d, "f_\Gamma"'] & A(\Gamma) \arrow[d, "f_\Delta"] \\ B(\Delta) \arrow[r, "B(p)"] & B(\Gamma) \end{tikzcd}$$

is a weak pullback for all display maps $p : \Delta \twoheadrightarrow \Gamma$. Setting $\Gamma = 1$ we see that full maps are pointwise surjective and therefore regular epis (the pointwise kernel pair $p, q : R \to A$ of f is in $\mathcal{T}\text{-}\mathsf{Alg}$ since $\mathcal{T}\text{-}\mathsf{Alg} \hookrightarrow [\mathcal{T}, \mathsf{Set}]$ creates limits, and pointwise surjective maps are coequalizers of their kernel pairs in $[\mathcal{T}, \mathsf{Set}]$, hence all the more so in $\mathcal{T}\text{-}\mathsf{Alg}$).

(c) For every display map $p : \Delta \twoheadrightarrow \Gamma$ in $\mathcal{T}$, the arrow $H(p) : H(\Gamma) \hookrightarrow H(\Delta)$ is an extension—these are precisely the generators of the w.f.s. In particular, all hom-algebras $H(\Gamma)$ are 0-extensions, since all terminal projections $\Gamma \twoheadrightarrow 1$ are display maps in $\mathcal{T}$.

(d) If $\mathcal{T}$ is a finite-product clan, then $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by initial inclusions $0 \cong H(1) \hookrightarrow H(\Gamma)$ alone, since for every display map $p : \Delta \times \Gamma \to \Delta$ the generator $H(p)$ is a pushout

$$\begin{tikzcd} H(1) \arrow[r, hook] \arrow[d, hook] & H(\Delta) \arrow[d, hook, "H(p)"] \\ H(\Gamma) \arrow[r, hook] & H(\Gamma \times \Delta) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in $\mathcal{T}\text{-}\mathsf{Alg}$ of an initial inclusion, and left classes of w.f.s.'s are closed under pushout. It follows that the full maps are *precisely* the pointwise surjective maps, which in this case also coincide with the regular epis, since finite-product preserving functors are closed under image factorization in $[\mathcal{T}, \mathsf{Set}]$ (and thus every non-surjective arrow factors through a strict subobject). Thus, the 0-extensions are precisely the regular-projective objects in the finite-product case, which also play a central role in [ARV10].

(e) If $\mathcal{T}$ is a finite-limit clan then *all* naturality squares of full maps $f : A \twoheadrightarrow B$ are weak pullbacks, including the naturality squares

$$\begin{tikzcd} A(\Gamma) \arrow[r] \arrow[d] & A(\Gamma \times \Gamma) \cong A(\Gamma) \times A(\Gamma) \arrow[d] \\ B(\Gamma) \arrow[r] & B(\Gamma \times \Gamma) \cong B(\Gamma) \times B(\Gamma) \end{tikzcd}$$

of diagonals $\Gamma \to \Gamma \times \Gamma$. From this it follows easily that f_Γ is injective, and since we have shown that it is surjective above, we conclude that only isomorphisms are full in the finite-limit case. ♢

We call a map $f : A \to B$ of algebras *full* By Quillen's small object argument, the full maps form the right class of a cofibrantly generated WFS

$$(\mathcal{E}, \mathcal{F})$$

on $\mathcal{T}\text{-}\mathsf{Alg}$ whose left maps we call *extensions.*

4 Comodels and the universal property of $\mathcal{T}$-Alg

It is well known that for $\mathbb{C}$ a small category, the presheaf category $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$ is the completion of $\mathbb{C}$ under small colimits, in the sense that for every cocomplete category $\mathfrak{X}$, precomposition with the Yoneda embedding $よ : \mathbb{C} \to \widehat{\mathbb{C}}$ induces an equivalence

$$\mathsf{CoCont}(\widehat{\mathbb{C}}, \mathfrak{X}) \xrightarrow{\simeq} [\mathbb{C}, \mathfrak{X}]$$

between the categories of cocontinuous functors $\widehat{\mathbb{C}} \to \mathfrak{X}$, and of functors $\mathbb{C} \to \mathfrak{X}$. Specifically, the cocontinuous functor $F_! : \widehat{\mathbb{C}} \to \mathfrak{X}$ corresponding to a functor $F : \mathbb{C} \to \mathfrak{X}$ is the left Kan extension of F along $よ : \mathbb{C} \to \widehat{\mathbb{C}}$ and can be written as

$$F_!(A) = \mathsf{colim}(\underleftarrow{\mathsf{elts}}(A) \to \mathbb{C} \xrightarrow{F} \mathfrak{X}).$$

If $\mathfrak{X}$ is locally small then $F_!$ has a right adjoint $F^* : \mathfrak{X} \to \widehat{\mathbb{C}}$ given by $F^*(X) = \mathfrak{X}(F(-), X)$. We call F^* and $F_!$ the *nerve* and *realization* functors of F, respectively, and $F_! \dashv F^*$ the nerve–realization adjunction.

The universal property of $\mathcal{T}\text{-}\mathsf{Alg}$ is an equivalence between cocontinuous functors out of $\mathcal{T}\text{-}\mathsf{Alg}$ and coclan morphisms out of $\mathcal{T}^{\mathsf{op}}$. Following a suggestion by Mathieu Anel, we refer to the latter as *comodels* of the clan (rather than co*algebras* since the term has other connotations). We will only use this term for morphisms with cocomplete codomain.

Definition 4.1 A *comodel* of a clan $\mathcal{T}$ in a cocomplete category $\mathfrak{X}$ is a coclan morphism $F : \mathcal{T}^{\mathsf{op}} \to \mathfrak{X}$, where $\mathfrak{X}$ is equipped with the maximal coclan structure. We write $\mathcal{T}\text{-}\mathsf{CoMod}(\mathfrak{X})$ for the category of comodels of $\mathcal{T}$ in $\mathfrak{X}$, as a full subcategory of the functor category. ♢

Theorem 4.2 (The universal property of $\mathcal{T}$-Alg) *Let $\mathcal{T}$ be a clan.*

(i) *The functor $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Alg}$ is a comodel.*

(ii) *For every cocomplete $\mathfrak{X}$ and comodel $F : \mathcal{T}^{\mathsf{op}} \to \mathfrak{X}$, the restriction of $F_! : [\mathbb{C}, \mathsf{Set}] \to \mathfrak{X}$ to $\mathcal{T}\text{-}\mathsf{Alg}$ is cocontinuous. Thus, precomposition with H gives rise to an equivalence*

$$\mathsf{CoCont}(\mathcal{T}\text{-}\mathsf{Alg}, \mathfrak{X}) \xrightarrow{\simeq} \mathcal{T}\text{-}\mathsf{CoMod}(\mathfrak{X})$$

between categories of continuous functors and of comodels.

(iii) *If $F : \mathcal{T}^{\mathsf{op}} \to \mathfrak{X}$ is a comodel and $\mathfrak{X}$ is locally small, then the nerve functor $F^* : \mathfrak{X} \to [\mathbb{C}, \mathsf{Set}]$ factors through the inclusion $\mathcal{T}\text{-}\mathsf{Alg} \hookrightarrow [\mathbb{C}, \mathsf{Set}]$, giving rise to a 'restricted nerve realization adjunction' $F_! : \mathcal{T}\text{-}\mathsf{Alg} \leftrightarrows \mathfrak{X} : F^*$.*

$$\begin{tikzcd}
\mathcal{T}^{\mathsf{op}} \arrow[r, hook, "H"] \arrow[dr, "F"'] & \mathcal{T}\text{-}\mathsf{Alg} \arrow[r, hook] \arrow[d, shift right=2, "F_!"'] \arrow[d, phantom, "\dashv"] & {[\mathcal{T}^{\mathsf{op}}, \mathsf{Set}]} \arrow[dl, shift right=2, "F_!"'] \arrow[dl, phantom, "\dashv"] \\
& \mathfrak{X} \arrow[u, shift right=2, "F^*"'] \arrow[ur, shift right=2, "F^*"'] &
\end{tikzcd}$$

Proof. Analogous statements to (i), (ii) hold more generally for arbitrary small realized[5] limit sketches. As Brandenburg points out on MathOverflow [Bra], the earliest reference for this seems to be [Pul70, Theorem 2.5]. See also [Bra21] which gives a careful account of an even more general statement for non-small sketches.

For claim (iii), it's easy to see that for $X \in \mathfrak{X}$, the functor $F^*(X) = \mathfrak{X}(F(-), X)$ is an algebra since F is a comodel. □

[5]A sketch is called 'realized' if all its designated cones are limiting.

4.1 Slicing and co-slicing

Definition 4.3 For $\mathcal{T}$ a clan and $\Gamma \in \mathcal{T}$, we write $\mathcal{T}(\Gamma)$ for the full subcategory of $\mathcal{T}/\Gamma$ on display maps. Then $\mathcal{T}(\Gamma)$ is a clan where an arrow in $\mathcal{T}(\Gamma)$ is a display map if its underlying map is one in $\mathcal{T}$. Compare [Joy17, Proposition 1.1.6]. $\diamondsuit$

Proposition 4.4 *Let $\mathcal{T}$ be a clan, and $\Gamma \in \mathcal{T}$ an object.*

(i) *The functor*

$$H_\Gamma : \mathcal{T}(\Gamma)^{\mathsf{op}} \to H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg}$$

which sends $d : \Delta \twoheadrightarrow \Gamma$ to $H(d) : H(\Gamma) \rightarrowtail H(\Delta)$ is a comodel.

(ii) *The induced nerve–realization adjunction*

$$(H_\Gamma)_! \;:\; \mathcal{T}(\Gamma)\text{-}\mathsf{Alg} \leftrightarrows H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg} \;:\; (H_\Gamma)^*$$

is an equivalence, and identifies the w.f.s. $(\mathcal{E}, \mathcal{F})$ on $\mathcal{T}(\Gamma)\text{-}\mathsf{Alg}$ with the coslice w.f.s. on $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg}$.

Proof. The first claim is straightforward.

For the second claim, since arrows $H(\Gamma) \to A$ correspond to elements of $A(\Gamma)$, we can identify the coslice category $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg}$ with the category of 'Γ-pointed $\mathcal{T}$-algebras', i.e. pairs (A, x) of a $\mathcal{T}$-algebra A and an element $x \in A(\Gamma)$, and morphisms preserving chosen elements.

Under this identification, we first verify that the functor $(H_\Gamma)^*$ is given by

$$(H_\Gamma)^*(A, x)(\Delta \overset{d}{\twoheadrightarrow} \Gamma) = \{y \in A(\Delta) \mid d \cdot y = x\},$$

and then that it is an equivalence with inverse $\Phi : \mathcal{T}(\Gamma)\text{-}\mathsf{Alg} \to H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg}$ given by

$$\Phi(B) = (B(- \times \Gamma \overset{\pi_2}{\twoheadrightarrow} \Gamma), \delta \cdot \star)$$

where $\star$ is the unique element of $B(\mathrm{id}_\Gamma)$ and $\delta : \Gamma \to \Gamma \times \Gamma$ is the diagonal map viewed as global element of $\pi_2 : \Gamma \times \Gamma \twoheadrightarrow \Gamma$ in $\mathcal{T}(\Gamma)$. Thus, $(H_\Gamma)_! = \Phi$.

For the second claim we first note that the coslice WFS on $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg}$ is cofibrantly generated by commutative triangles

$$\begin{tikzcd} H(\Gamma) \arrow[d, "H(\pi_2)"'] \arrow[dr, "H(\pi_2)"] & \\ H(\Delta \times \Gamma) \arrow[r, "H(d \times \Gamma)"'] & H(\Theta \times \Gamma) \end{tikzcd} \tag{4.1}$$

for display maps $d : \Theta \twoheadrightarrow \Gamma$ [Hir21, Theorem 2.7]. On the other hand, since $(H_\Gamma)_! \circ H = H_\Gamma$, the functor $(H_\Gamma)_!$ sends the generators of the extension/full WFS on $\mathcal{T}(\Gamma)\text{-}\mathsf{Alg}$ to triangles

$$\begin{tikzcd} H(\Gamma) \arrow[d, "H(e)"'] \arrow[dr, "H(f)"] & \\ H(\Delta) \arrow[r, "H(d)"'] & H(\Theta) \end{tikzcd} \tag{4.2}$$

for arbitrary display maps d, e, f in $\mathcal{T}$. Now the triangles of shape (4.2) contain the triangles of shape (4.1), but are contained in their saturation, which is the left class of the coslice WFS. Thus, the two WFSs are equal. $\square$

Proposition 4.5 *Let $\mathcal{T}$ be a clan, and A a $\mathcal{T}$-algebra. Then the forgetful functor $U : \underrightarrow{\mathsf{elts}}(A) \to \mathcal{T}$ creates a clan structure on $\underrightarrow{\mathsf{elts}}(A)$, i.e. $\underrightarrow{\mathsf{elts}}(A)$ is a clan with display maps those arrows that are mapped to display maps in $\mathcal{T}$ by U. Moreover, the canonical functor*

$$H_A : \underrightarrow{\mathsf{elts}}(A)^{\mathsf{op}} = \underleftarrow{\mathsf{elts}}(A) \simeq \mathcal{T}^{\mathsf{op}}/A \to \mathcal{T}\text{-}\mathsf{Alg}/A$$

is a coclan morphism, and its restricted nerve–realization adjunction

$$(H_A)_! \dashv (H_A)^* : \mathcal{T}\text{-}\mathsf{Alg}/A \to \underrightarrow{\mathsf{elts}}(A)\text{-}\mathsf{Alg}$$

is an equivalence which identifies the extension/full WFS on $\underrightarrow{\mathsf{elts}}(A)\text{-}\mathsf{Alg}$ and the slice WFS on $\mathcal{T}\text{-}\mathsf{Alg}/A$.

Proof. The verification that $\underrightarrow{\mathsf{elts}}(A)$ is a clan and H_A is a coclan morphism is straightforward. The equivalence is a restriction of the well-known equivalence $\widehat{\mathcal{T}^{\mathsf{op}}}/A \simeq \widehat{\mathcal{T}^{\mathsf{op}}/A}$. The WFSs coincide since — again by $(H_A)_! \circ H = H_A$ — the functor $(H_A)_!$ sends the generators of the WFS on $\underrightarrow{\mathsf{elts}}(A)$ to commutative triangles

$$\begin{tikzcd} H(\Gamma) \arrow[r, "d", rightarrowtriangle] \arrow[dr, "\hat{x}"'] & H(\Delta) \arrow[d, "\hat{y}"] \\ & A \end{tikzcd}$$

in $\mathcal{T}\text{-}\mathsf{Alg}/A$, where $d : \Delta \rightarrowtriangle \Gamma$ is a display map in $\mathcal{T}$ and $x \in A(\Gamma)$ and $y \in A(\Delta)$ are elements with $d \cdot y = x$. By [Hir21, Theorem 1.5], these form a set of generators for the slice WFS on $\mathcal{T}\text{-}\mathsf{Alg}/A$. □

5 $(\mathcal{E}, \mathcal{F})$-categories

Definition 5.1 An $(\mathcal{E}, \mathcal{F})$-category is a locally finitely presentable category $\mathcal{L}$ with a w.f.s. $(\mathcal{E}, \mathcal{F})$ whose maps we call *extensions* and *full maps*. A *morphism of $(\mathcal{E}, \mathcal{F})$-categories* is a functor $F : \mathcal{L} \to \mathcal{M}$ preserving small limits, filtered colimits, and full maps. We write EFCat for the 2-category of $(\mathcal{E}, \mathcal{F})$-categories, morphisms of $(\mathcal{E}, \mathcal{F})$-categories, and natural transformations. ◇

Lemma 5.2 *If $F : \mathcal{L} \to \mathcal{M}$ is a morphism of $(\mathcal{E}, \mathcal{F})$-categories, then it has a left adjoint $L : \mathcal{M} \to \mathcal{L}$ which preserves compact objects and extensions. Conversely, if $L : \mathcal{M} \to \mathcal{L}$ is a cocontinuous functor preserving compact objects and extensions, then it has a right adjoint $F : \mathcal{L} \to \mathcal{M}$ which is a morphism of $(\mathcal{E}, \mathcal{F})$-categories. Writing $\mathsf{EFCat}_L(\mathcal{M}, \mathcal{L})$ for the category of cocontinuous functors $\mathcal{M} \to \mathcal{L}$ preserving extensions and compact objects, we thus have $\mathsf{EFCat}_L(\mathcal{M}, \mathcal{L}) \simeq \mathsf{EFCat}(\mathcal{L}, \mathcal{M})^{\mathsf{op}}$.*

Proof. That morphisms of $(\mathcal{E}, \mathcal{F})$-categories have left adjoints follows from the *adjoint functor theorem for presentable categories* [AR94, Theorem 1.66], and conversely the *special adjoint functor theorem* [Mac98, Section V-8] implies that cocontinuous functors between l.f.p. categories have right adjoints. It follows from standard arguments that the left adjoint preserves compact objects iff the right adjoint preserves filtered colimits, and that the left adjoint preserves extensions iff the right adjoint preserves full maps. □

Lemma 5.3 *For any morphism $F : \mathcal{S} \to \mathcal{T}$ of clans, the precomposition functor $F^* : \mathcal{T}\text{-}\mathsf{Alg} \to \mathcal{S}\text{-}\mathsf{Alg}$ is a morphism of $(\mathcal{E}, \mathcal{F})$-categories. Thus, the assignment $\mathcal{T} \mapsto \mathcal{T}\text{-}\mathsf{Alg}$ extends to a contravariant 2-functor*

$$(-)\text{-}\mathsf{Alg} \;:\; \mathsf{Clan}^{\mathsf{op}} \;\to\; \mathsf{EFCat}$$

from clans to $(\mathcal{E}, \mathcal{F})$-categories.

Proof. The preservation of small limits and filtered colimits is obvious since they are computed pointwise (Remark 3.3-(a)). To show that F^* preserves full maps, let $f : A \to B$ be full in $\mathcal{T}\text{-}\mathsf{Alg}$. It is sufficient to show that the $(f \circ F)$-naturality squares are weak pullbacks at all display maps $p :$ in $\mathcal{S}\text{-}\mathsf{Alg}$. But the $(f \circ F)$-naturality square at p is the same as the f-naturality square at $F(p)$ so the claim follows since f is full and F preserves display maps. $\square$

Definition 5.4 Given an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$, write $\mathfrak{C}(\mathcal{L}) \subseteq \mathcal{L}$ for the full subcategory on compact 0-extensions. $\diamondsuit$

The following lemma is straightforward.

Lemma 5.5 *$\mathfrak{C}(\mathcal{L})$ is a coclan with extensions as codisplay maps.* $\square$

Proposition 5.6 *The functor $(-)\text{-}\mathsf{Alg} : \mathsf{Clan}^{\mathsf{op}} \to \mathsf{EFCat}$ has a left biadjoint, which sends $(\mathcal{E}, \mathcal{F})$-categories $\mathcal{L}$ to clans $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$.*

Proof. Given a clan $\mathcal{T}$ and an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$, it is easy to see that the equivalence

$$\mathsf{CoCont}(\mathcal{T}\text{-}\mathsf{Alg}, \mathcal{L}) \;\simeq\; \mathcal{T}\text{-}\mathsf{CoMod}(\mathcal{L})$$

from Theorem 4.2 (where $|\mathcal{L}|$ is the underlying category of $\mathcal{L}$) restricts to an equivalence

$$\mathsf{EFCat}_L(\mathcal{T}\text{-}\mathsf{Alg}, \mathcal{L}) \;\simeq\; \mathsf{CoClan}(\mathcal{T}^{\mathsf{op}}, \mathfrak{C}(\mathcal{L})).$$

Taking opposite categories on both sides we get

$$\mathsf{EFCat}(\mathcal{L}, \mathcal{T}\text{-}\mathsf{Alg}) \;\simeq\; \mathsf{Clan}(\mathcal{T}, \mathfrak{C}(\mathcal{L})^{\mathsf{op}}), \tag{5.1}$$

which shows that $\mathsf{EFCat}(\mathcal{L}, (-)\text{-}\mathsf{Alg}) : \mathsf{Clan}^{\mathsf{op}} \to \mathsf{Cat}$ is birepresented by $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$. $\square$

Remark 5.7 From the construction of the natural equivalence (5.1) we can extract explicit descriptions of the components

$$\Theta_{\mathcal{L}} : \mathcal{L} \to \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg} \qquad \text{and} \qquad E_{\mathcal{T}} : \mathcal{T} \to \mathfrak{C}(\mathcal{T}\text{-}\mathsf{Alg})^{\mathsf{op}}$$

of the unit Θ and the counit E of the biadjunction

$$\mathfrak{C}(-)^{\mathsf{op}} : \mathsf{EFCat} \leftrightarrows \mathsf{Clan}^{\mathsf{op}} : (-)\text{-}\mathsf{Alg} \tag{5.2}$$

at an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ and a clan $\mathcal{T}$ respectively. Specifically, $\Theta_{\mathcal{L}}$ is the nerve of the inclusion $J : \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L}$ (which is obviously a comodel), and the $E_{\mathcal{T}}$ is $(-)^{\mathsf{op}}$ of the evident corestriction of $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Alg}$. $\diamondsuit$

In the following we will show that the biadjunction between clans and $(\mathcal{E}, \mathcal{F})$-categories is *idempotent* (in the sense that the associated monad and comonad are), and characterize the fixed-points on both sides.

6 Cauchy complete clans and the fat small object argument

Definition 6.1 A clan $\mathcal{T}$ is called *Cauchy complete* if its underlying category is Cauchy complete (i.e. idempotents split), and retracts of display maps are display maps. $\diamondsuit$

Clearly, every clan of the form $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$ is Cauchy complete, thus Cauchy completeness is a necessary condition for the counit $E_{\mathcal{T}} : \mathcal{T} \to \mathfrak{C}(\mathcal{T}\text{-}\mathsf{Alg})^{\mathsf{op}}$ of the biadjunction (5.2) to be an equivalence. We will show that it is also sufficient, but for this we need the *fat small object argument*:

Proposition 6.2 *For any clan $\mathcal{T}$, the 0-extensions in $\mathcal{T}$-Alg are flat (Definition 3.5).*

Proof. This is a special case of [MRV14, Corollary 5.1], but we give a direct proof in the appendix (Corollary C.10), which simplifies onsiderably in the case of clans. $\square$

Definition 6.3 Let $\mathfrak{X}$ be a cocomplete locally small category.

(i) We say that an arrow $f : A \to B$ is *left orthogonal* to a small diagram $D : \mathbb{J} \to \mathfrak{X}$, and write $f \perp D$, if the square

$$\begin{tikzcd} \mathsf{colim}_{j\in\mathbb{J}}\, \mathfrak{X}(B, D_j) \arrow[r] \arrow[d] & \mathfrak{X}(B, \mathsf{colim}(D)) \arrow[d] \\ \mathsf{colim}_{j\in\mathbb{J}}\, \mathfrak{X}(A, D_j) \arrow[r] & \mathfrak{X}(A, \mathsf{colim}(D)) \end{tikzcd}$$

is a pullback in Set.

(ii) We call f *compact* if it is left orthogonal to all small filtered diagrams. $\diamondsuit$

Lemma 6.4 *Let $\mathfrak{X}$ be a locally small cocomplete category.*

(i) *An object $A \in \mathfrak{X}$ is compact in the usual sense that $\mathfrak{X}(A, -)$ preserves filtered colimits, if and only if the arrow $0 \to A$ is compact in the sense of Definition 6.3.*

(ii) *If the arrow g in a commutative triangle* $\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "f"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$ *is compact, then f is compact if and only if h is compact. In other words, compact arrow are closed under composition and have the right cancellation property.*

(iii) *If $f : A \to B$ is compact as an arrow in $\mathfrak{X}$, then it is compact as an object in $A/\mathfrak{X}$.*

(iv) *If $h : B \to C$ is an arrow between compact objects in $\mathfrak{X}$, then h is compact.*

Proof. (i) is obvious, and (ii) follows from the pullback lemma.

For (iii) assume that f is compact as an arrow in $\mathfrak{X}$ and consider a filtered diagram in $A/\mathfrak{X}$, given by a filtered diagram $D : \mathbb{I} \to \mathfrak{X}$ and a cocone $\gamma = (\gamma_i : A \to D_i)_{i\in\mathbb{I}}$. Note that since the forgetful functor $A/\mathfrak{X} \to \mathfrak{X}$ creates connected colimits, we have $\mathsf{colim}(\gamma) : A \to \mathsf{colim}(D)$. Also because $\mathbb{I}$ is connected, all γ_i are in the same equivalence class in

$\mathsf{colim}_{i\in\mathbb{I}}\,\mathfrak{X}(A, D_i)$, which we denote by $\overline{\gamma} : 1 \to \mathsf{colim}_{i\in\mathbb{I}}\,\mathfrak{X}(A, D_i)$. We have to show that the canonical map

$$\mathsf{colim}_i(A/\mathfrak{X})(f, \gamma_i) \quad \longrightarrow \quad (A/\mathfrak{X})(f, \mathsf{colim}(\gamma))$$

is a bijection. This follows because this function can be presented as a pullback in Set^2 as in the following diagram.

$$\begin{tikzcd}
& \mathsf{colim}_i\,\mathfrak{X}(B, D_i) \arrow[rr] \arrow[dd] & & \mathfrak{X}(B, \mathsf{colim}(D)) \arrow[dd] \\
\mathsf{colim}_i(A/\mathfrak{X})(f, \gamma_i) \arrow[ur] \arrow[rr] \arrow[dd] & & (A/\mathfrak{X})(f, \mathsf{colim}(\gamma)) \arrow[ur] \arrow[dd] & \\
& \mathsf{colim}_i\,\mathfrak{X}(A, D_i) \arrow[rr] & & \mathfrak{X}(A, \mathsf{colim}(D)) \\
1 \arrow[ur, "\overline{\gamma}"] \arrow[rr] & & 1 \arrow[ur, "\mathsf{colim}(\gamma)"'] &
\end{tikzcd}$$

The front square is a pullback since the back one is by compactness of f as an arrow, and the side ones are pullbacks by construction; thus the gray horizontal arrow is a bijection since $1 \to 1$ is.

Finally, claim (iv) now follows directly from (i), (ii), and (iii). □

Remark 6.5 One can show the implication of Lemma 6.4(iii) is actually an equivalence, i.e. $f : A \to B$ is compact as an arrow if and only if it is so as an object of the coslice category, but the other direction is more awkward to write down and we don't need it. ♢

Proposition 6.6 *If $\mathcal{T}$ is a Cauchy complete clan, then*

(i) *the functor $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Alg}$ co-restricts to an equivalence between $\mathcal{T}^{\mathsf{op}}$ and the coclan $\mathfrak{C}(\mathcal{T}\text{-}\mathsf{Alg}) \subseteq \mathcal{T}\text{-}\mathsf{Alg}$ of compact 0-extensions, and*

(ii) *an arrow $f : \Delta \to \Gamma$ is a display map in $\mathcal{T}$ if and only if $H(f) : H(\Gamma) \to H(\Delta)$ is an extension in $\mathcal{T}\text{-}\mathsf{Alg}$.*

Proof. Let $C \in \mathcal{T}\text{-}\mathsf{Alg}$ be a compact 0-extension. Then by Proposition 6.2, C is a filtered colimit of hom-algebras, and since C is compact the identity id_C factors through one of the colimit inclusions, whence C is a retract of a hom-algebra. By Cauchy completeness, C is thus itself representable, i.e. we have an equivalence of categories.

For the second claim, we know that $H(f)$ is an extension whenever f is a display map. Conversely, assume that $f : \Delta \to \Gamma$ is an arrow in $\mathcal{T}$ such that $H(f) : H(\Gamma) \to H(\Delta)$ is an extension. Then $H(f)$ is compact in $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg}$ by Lemma 6.4(iv) and $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg} \simeq \mathcal{T}(\Gamma)\text{-}\mathsf{Alg}$ by Proposition 4.4. This means that the object corresponding to $H(f)$ in $\mathcal{T}(\Gamma)\text{-}\mathsf{Alg}$ is a compact 0-extension, and thus it is isomorphic to a hom-algebra $\mathcal{T}(\Gamma)(d, -)$ for a display map $d : \Theta \twoheadrightarrow \Gamma$ by the first claim. This means that f is isomorphic to d over Γ, and therefore a display map. □

The preceding proposition together with the remarks at the beginning of the section shows that the monad on Clan induced by the biadjunction (5.2) is *idempotent*: applying the monad once produces a Cauchy complete clan, and applying it again gives something equivalent.

By general facts about adjunctions, the induced monad on EFCat is also idempotent. In the following section we characterize its fixed-points as being *clan-algebraic categories*.

7 Clan-algebraic categories

Definition 7.1 An $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ is called *clan-algebraic* if

(D) the inclusion $J : \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L}$ is dense,

(CG) $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by $\mathcal{E} \cap \mathrm{mor}(\mathfrak{C}(\mathcal{L}))$, and

(FQ) quotients of componentwise full equivalence relations are effective and have full quotient maps. ♢

Theorem 7.2 *For every clan $\mathcal{T}$, the category $\mathcal{T}$-Alg is clan-algebraic.*

Proof. $\mathfrak{C}(\mathcal{L})$ is dense since it contains the representables. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between representables, and and therefore a fortiori by maps between compact 0-extensions.

For the third condition, let

$$r = \langle r_1, r_2 \rangle : R \rightarrowtail A \times A$$

be an equivalence relation such that r_1 and r_2 are full maps. This means that we have an equivalence relation $\sim$ on each $A(\Gamma)$, such that

(1) for all arrows $s : \Delta \to \Gamma$, the function $A(s) = s \cdot (-) : A(\Delta) \to A(\Gamma)$ preserves this relation, and

(2) for every display map $p : \Gamma^+ \twoheadrightarrow \Gamma$ and all $a, b \in A(\Gamma)$ and $c \in A(\Gamma^+)$ such that $a \sim b$ and $p \cdot c = a$, there exists a $d \in A(\Gamma^+)$ with $c \sim d$ and $p \cdot d = b$.

We show first that the pointwise quotient A/R is an algebra. Clearly $(A/R)(1) = 1$, and it remains to show that given a pullback

$$\begin{tikzcd} \Delta^+ \arrow[r, "t"] \arrow[d, two heads, "q"] & \Gamma^+ \arrow[d, two heads, "p"] \\ \Delta \arrow[r, "s"] & \Gamma \end{tikzcd}$$

with p and q display maps, and elements $a \in A(\Delta)$, $b \in A(\Gamma^+)$ with $s \cdot a \sim p \cdot b$, there exists a unique-up-to-$\sim$ $c \in A(\Delta^+)$ with $q \cdot c \sim a$ and $t \cdot c \sim b$. Since p is a display map, there exists a b' with $b \sim b'$ and $p \cdot b' = s \cdot a$, and since A is an algebra there exists therefore a c with $q \cdot c = a$ and $t \cdot c = b'$. For uniqueness assume that $c, c' \in A(\Delta^+)$ with $q \cdot c \sim q \cdot c'$ and $t \cdot c \sim t \cdot c'$. Then $c \sim c'$ follows from the fact that R is an algebra. This shows that A/R is an algebra, and also that the quotient is effective, since the kernel pair is computed pointwise. The fact that $A \to A/R$ is full is similarly easy to see. □

The following lemma is a kind of converse to (FQ).

Lemma 7.3 *Full maps in clan-algebraic categories are regular epimorphisms.*

Proof. Given a full map in a clan-algebraic category $\mathcal{L}$, the lifting property against (compact) 0-extensions implies that $\Theta_{\mathcal{L}}(f) = J^*(f)$ is componentwise surjective in $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$-$\mathsf{Alg}$, and therefore the coequalizer of its kernel pair. Since left adjoints preserve regular epis, we deduce that $J_!(J^*(f))$ is regular epic in $\mathcal{L}$ and the claim follows since $J_! \circ J^* \cong \mathrm{id}$ by (D). □

Remark 7.4 Observe that we only used property (D) in the proof, no exactness. $\diamondsuit$

Lemma 7.5 *The class $\mathcal{F}$ of full maps in a clan-algebraic category $\mathcal{L}$ has the* right cancellation property*, i.e. we have $g \in \mathcal{F}$ whenever $gf \in \mathcal{F}$ and $f \in \mathcal{F}$ for composable pairs $f : A \to B$, $g : B \to C$.*

Proof. By (CG) it suffices to show that g has the r.l.p. with respect to extensions $e : I \hookrightarrow J$ between compact 0-extensions I, J. Let

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[d, hook, "e"] & B \arrow[d, "g"] \\ J \arrow[r, "k"] & C \end{tikzcd}$$

be a filling problem. Since I is a 0-extension and f is full, there exists a map $h' : I \to A$ with $fh' = h$. We obtain a new filling problem

$$\begin{tikzcd} I \arrow[r, "h'"] \arrow[d, hook, "e"] & A \arrow[d, two heads, "gf"] \\ J \arrow[r, "k"] & C \end{tikzcd}$$

which can be filled by a map $m : J \to A$ since gf is full. Then fm is a filler for the original problem. $\square$

Lemma 7.6 *Let $\mathcal{L}$ be a clan-algebraic category, let $f : A \to B$ be an arrow in $\mathcal{L}$ with componentwise full kernel pair $p, q : R \twoheadrightarrow A$, and let $e : A \twoheadrightarrow C$ be the coequalizer of p and q. Then the unique $m : C \to B$ with $me = f$ is monic.*

Proof. By (D) it is sufficient to test monicity of m on maps out of compact 0-extensions E. Let $h, k : E \to C$ such that $mh = mk$. Since e is full by (FQ), there exist $h', k' : E \to A$ with $eh' = h$ and $ek' = k$. In particular we have $fh' = fk'$ and therefore there is an $u : E \to R$ with $pu = h'$ and $qu = k'$. Thus we can argue

$$h = eh' = epu = equ = ek' = k$$

which shows that m is monic. $\square$

Lemma 7.7 *If $A \in \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg}$ is flat, then $A \to J^*(J_!(A))$ is an isomorphism, thus $J_!$ restricted to flat algebras is fully faithful.*

Proof. We have

$$\begin{aligned} J^*(J_!(A))(C) &= \mathcal{L}(C, \mathsf{colim}(\underleftarrow{\mathsf{elts}}(A) \to \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L})) \\ &\cong \mathsf{colim}(\underleftarrow{\mathsf{elts}}(A) \to \mathfrak{C}(\mathcal{L}) \xrightarrow{よ(C)} \mathsf{Set}) \qquad \text{since } \underleftarrow{\mathsf{elts}}(A) \text{ is filtered} \\ &\cong A \otimes よ(C) \cong A(C). \end{aligned}$$

The second claim follows since for flat B, the mapping

$$(\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg})(A, B) \to \mathcal{L}(J_!A, J_!B)$$

can be decomposed as

$$(\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg})(A, B) \to (\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg})(A, J^*J_!B) \to \mathcal{L}(J_!A, J_!B).$$

$\square$

Lemma 7.8 *The following are equivalent for a cone $\phi : \Delta C \to D$ on a diagram $D : \mathbb{J} \to \mathcal{L}$ in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$.*

(i) *Given an extension $e : A \to B$, an arrow $h : A \to C$, and a cone $\kappa : \Delta B \to D$ such that $\phi_j \circ h = \kappa_j e$ for all $j \in \mathbb{J}$, there exists $l : B \to C$ such that $le = h$ and $\phi_j l = \kappa_j$ for all $j \in \mathbb{J}$.*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "e"'] & C \arrow[d, "\phi_j"] \\ B \arrow[r, "\kappa_j"] \arrow[ur, dashed, "l"] & D_j \end{tikzcd}$$

(ii) *The mediating arrow $: C \to \mathsf{lim}(D)$ is full.*

Proof. The data of e, h, κ is equivalent to e, h, and $k : B \to \mathsf{lim}(D)$ such that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "e"'] & C \arrow[d, "f"] \\ B \arrow[r, "k"] & \mathsf{lim}(D) \end{tikzcd}$$

commutes, and $l : B \to C$ fills the latter square iff it fills all the squares with the D_j. □

Definition 7.9 We call a cone $\phi : \Delta C \to D$ satisfying the conditions of the lemma *jointly full.* ♢

Remark 7.10 The interest of this is that it allows us to talk about full 'covers' of limits without actually computing the limits, which is useful when talking about cones and diagrams in the full subcategory of a clan-algebraic category on 0-extensions, which does not admit limits. ♢

Definition 7.11 A *nice diagram* in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ is a 2-truncated semi-simplicial diagram

$$A_\bullet = \left(\begin{tikzcd} A_2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] & A_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & A_0 \end{tikzcd} \right)$$

where

(i) A_0, A_1, and A_2 are 0-extensions,

(ii) the maps $d_0, d_1 : A_1 \to A_0$ are full,

(iii) in the commutative square $\begin{tikzcd} A_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] & A_1 \arrow[d, "d_1"] \\ A_1 \arrow[r, "d_0"] & A_0 \end{tikzcd}$ the span constitutes a jointly full cone over the cospan,

(iv) there exists a 'symmetry' map $\begin{tikzcd} A_1 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, "\sigma"] & A_0 \\ A_0 & A_1 \arrow[l, "d_1"] \arrow[u, "d_0"'] \end{tikzcd}$ making the triangles commute, and

(v) there exists a 0-extension $\tilde{A}$ and full maps $f, g : \tilde{A} \twoheadrightarrow A_1$ constituting a jointly full cone over the diagram

$$\begin{tikzcd} A_1 \arrow[d, "d_0"'] \arrow[dr, "d_1" near start] & A_1 \arrow[d, "d_1"] \arrow[dl, "d_0" near end] \\ A_0 & A_0 \end{tikzcd} .$$

$\diamondsuit$

Lemma 7.12 *If $A_\bullet$ is a nice diagram in a clan-algebraic category $\mathcal{L}$, the pairing $\langle d_0, d_1 \rangle : A_1 \to A_0 \times A_0$ factors as $A_1 \xrightarrow{f} R \xrightarrow{\langle r_0, r_1 \rangle} A_0 \times A_0$, where f is full and $r = \langle r_0, r_1 \rangle$ is monic and a componentwise full equivalence relation.*

Proof. Condition (v) of the preceding definition gives us the following diagram

$$\begin{tikzcd} \tilde{A} \arrow[dr, two heads, "h"] \arrow[drr, two heads, bend left, "g"] \arrow[ddr, two heads, bend right, "f"'] & & \\ & S \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner" very near start] & A_1 \arrow[d, "\langle d_0, d_1 \rangle"] \\ & A_1 \arrow[r, "\langle d_0, d_1 \rangle"] & A_0 \times A_0 \end{tikzcd} ,$$

i.e. S is the kernel of $\langle d_0, d_1 \rangle$ with projections p, q, $\tilde{A}$ is a 0-extension, and f, g, h are full. By right cancellation we deduce that p and q are full, and the existence of the factorization follows from Lemma 7.6. Fullness of r_0, r_1 follows again from right cancellation because f, d_0, and d_1 are full.

It remains to show that r is an equivalence relation. This is easy: condition 4 gives symmetry, and condition 3 gives transitivity, and reflexivity follows from the fact that r_0 admits a section as a full map into a 0-extension, together with symmetry (we internalize the argument that if in a symmetric and transitive relation everything is related to *something*, then it is reflexive.) $\square$

Definition 7.13 A 0-*extension replacement* of an object A in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ is a full map $f : \overline{A} \twoheadrightarrow A$ from a 0-extension $\overline{A}$ to A. $\diamondsuit$

0-extension replacements can always be obtained as $(\mathcal{E}, \mathcal{F})$-factorizations of $0 \to A$.

Lemma 7.14 *For every object A in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ there exists a nice diagram $A_\bullet$ with colimit A.*

Proof. A_0 is constructed as a 0-extension replacement $f : A_0 \twoheadrightarrow A$ of A. Similarly, A_1 is given by a 0-extension replacement $f_1 : A_1 \twoheadrightarrow A_0 \times_A A_0$ of $A_0 \times_A A_0$, and A_2 is a 0-extension replacement $f_2 : A_2 \twoheadrightarrow P$ of the pullback

$$\begin{tikzcd} P \arrow[r, "p_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner" very near start] & A_1 \arrow[d, "d_0"] \\ A_1 \arrow[r, "d_1"] & A_0 \end{tikzcd} ,$$

with $d_0, d_1, d_2 : A_2 \to A_1$ given by $d_0 = p_0 \circ f$, $d_2 = p_1 \circ f$, and d_1 a lifting of $\langle d_0 \circ d_0, d_1 \circ d_2 \rangle$ along f_1. The map σ is constructed as a lifting of the symmetry of $A_0 \times_A A_0$ along f_1. The object $\tilde{A}$ is a 0-extension replacement of the kernel of f_1. $\square$

Lemma 7.15 *For any clan-algebraic category $\mathcal{L}$, the realization functor $J_!$ preserves jointly full cones in flat algebras, and nice diagrams.*

Proof. The first claim follows since $J_!$ is fully faithful on 0-extensions by Lemma 7.7 and in both sides the weak factorization system determined by the same generators. Thus there's a one-to-one correspondence between lifting problems. The second claim follows since $J_!$ preserves 0-extensions and 0-extensions are flat by the fat small object argument. $\square$

Lemma 7.16 *For any clan-algebraic category $\mathcal{L}$, the nerve functor $J^* : \mathcal{L} \to \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg}$ preserves quotients of nice diagrams.*

Proof. Given a nice diagram $A_\bullet$ in $\mathcal{L}$, its colimit is the coequalizer of $d_0, d_1 : A_1 \to A_0$. By Lemma 7.12, $\langle d_0, d_1 \rangle$ factors as $\langle r_0, r_1 \rangle \circ f$ with f full and $r = \langle r_0, r_1 \rangle$ an equivalence relation. The pairs d_0, d_1 and r_0, r_1 have the same coequalizer (since f is epic), and J^* preserves the coequalizer of r_0, r_1 since it preserves full maps and kernel pairs. Finally, the coequalizer of $J^*(r_0), J^*(r_1)$ is also the coequalizer of $J^*(d_0), J^*(d_1)$ since $J^*(f)$ is full and therefore epic. $\square$

Theorem 7.17 *If $\mathcal{L}$ is clan-algebraic, then $J^* : \mathcal{L} \to \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg}$ is an equivalence in* EFCat.

Proof. By density, ε^* is fully faithful. It remains to verify that it is essentially surjective, and to this end we show that the unit map $\eta_A : A \to J^*(J_!(A))$ is an isomorphism for all $A \in \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg}$. Let $A_\bullet$ be a nice diagram with colimit A. We have:

$$\begin{aligned} J^*(J_!(A)) &= J^*(J_!(\mathsf{colim}(A_\bullet))) \\ &= \mathsf{colim}(J^*(J_!(A_\bullet))) \\ &= \mathsf{colim}(A_\bullet) \\ &= A \end{aligned}$$

$\square$

8 A counterexample

(1) Does every compact object admit a full map from a compact 0-extension?

(2) Does the weak factorization system restrict to compact objects?

In the following we give a counterexample for both questions. Consider the following generalized algebraic theory with infinitely many sorts and operations:

$$\begin{array}{rcll} & \vdash & X & \\ & \vdash & Y & \\ y : Y & \vdash & Z_n(y) & \qquad n \in \mathbb{N} \\ x : X & \vdash & f(x) : Y & \\ x : X & \vdash & g_n(x) : Z_n(f(x)) & \qquad n \in \mathbb{N} \end{array}$$

Its category of models is (equivalent to) the set-valued functors on the posetal category

$$\mathbb{C} = \left(\begin{tikzcd} & X \arrow[d, "g_0"] \arrow[dr, "g_1"] \arrow[drrr, "g_n"] \arrow[dd, bend right, "f"'] & & & & \\ & Z_0 \arrow[d, tail, "z_0"] & Z_1 \arrow[dl, tail, "z_1"] & \dots & Z_n \arrow[dlll, tail, "z_n"] & \dots \\ & Y & & & & \end{tikzcd} \right)$$

and the extension/full weak factorization system on $[\mathbb{C}, \mathsf{Set}]$ is cofibrantly generated by the arrows $(\varnothing \rightarrowtail \text{よ}(X))$, $(\varnothing \rightarrowtail \text{よ}(Y))$, and $(\text{よ}(Y) \rightarrowtail \text{よ}(Z_n))$ for $n \in \mathbb{N}$, reflecting the idea that algebras $A : \mathbb{C} \to \mathsf{Set}$ can be built up by successively adding elements to $A(X)$ or $A(Y)$, and to $A(Z)$ over a given element x of $A(Y)$, as in the following pushouts.

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, tail] & A \arrow[d, tail] \\ 1 = \text{よ}(X) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & 1 + A \end{tikzcd} \qquad \begin{tikzcd} \varnothing \arrow[r] \arrow[d, tail] & A \arrow[d, tail] \\ \text{よ}(Y) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \text{よ}(Y) + A \end{tikzcd} \qquad \begin{tikzcd} \text{よ}(Y) \arrow[r, "\hat{x}"] \arrow[d, tail] & A \arrow[d, tail] \\ \text{よ}(Z_n) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & B \end{tikzcd}$$

The following lemma gives explicit descriptions of the weak factorization system and the compact objects in $[\mathbb{C}, \mathsf{Set}]$.

Lemma 8.1 *Let $f : A \to B$ in $[\mathbb{C}, \mathsf{Set}]$.*

(i) f is full if and only if it is componentwise surjective and the naturality squares for z_n are weak pullbacks for all $n \in \mathbb{N}$.

(ii) f is an extension if an only if $f_X : A(X) \to B(X)$ is injective, and the squares

$$\begin{tikzcd} A(X) \arrow[r] \arrow[d] & B(X) \arrow[d] \\ A(Y) \arrow[r] & B(Y) \end{tikzcd} \qquad \begin{tikzcd} A(X) \arrow[r] \arrow[d] & B(X) \arrow[d] \\ A(Z_n) \arrow[r] & B(Z_n) \end{tikzcd}$$

are quasi-pushouts*, in the sense that the gap maps $A(Y) +_{A(X)} B(X) \to B(Y)$ and $A(Z_n) +_{A(X)} B(X) \to B(Z_n)$ are injective. (This implies that the components f_Y and f_{Z_n} are also injective).*

(iii) A is a 0-extension if an only if $A(f)$ and all $A(g_n)$ are injective.

(iv) A is compact if an only if it is componentwise finite, and $A(f_n) : A(X) \to A(Z_n)$ is a bijection for all but finitely many $n \in \mathbb{N}$. □

Using this lemma, we can give negative answers to the two question at the beginning of the section.

Proposition 8.2 (i) *The object P in the pushout*

$$\begin{tikzcd} \text{よ}(Y) + \text{よ}(Y) \arrow[r] \arrow[d] & \text{よ}(X) + \text{よ}(X) \arrow[d] \\ \text{よ}(Y) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & P \end{tikzcd}$$

is compact, but does not admit a full map from a compact 0-extension.

(ii) *The map $よ(Y) \to よ(X)$ does not admit an extension/full factorization through a compact object.*

Proof. For the first claim, P is compact as a finite colimit of representables. Let $f : E \twoheadrightarrow P$ be a full map with E a 0-extension. For each $n \in \mathbb{N}$ we get a diagram

$$\begin{tikzcd}
E(X) \arrow[rr, two heads, "f_X"] \arrow[dr, tail, "E(g_n)"] \arrow[ddr, tail, "E(f)"'] & & 2 \arrow[dr] \arrow[ddr] & \\
& E(Z_n) \arrow[rr, two heads, "f_{Z_n}"] \arrow[d, "E(z_n)"] & & 2 \arrow[d] \\
& E(Y) \arrow[rr, two heads, "f_Y"] & & 1
\end{tikzcd}$$

where $E(f)$ and $E(g_n)$ are injective because E is a 0-extension, and the components of f are surjective and the z_n-naturality square is a weak pullback since because f is full. In particular $E(Y)$ is inhabited and the fibers of $E(z_n)$ have at least two elements. Since the fibers of $E(f)$ have at most one element, this means that $E(g_n)$ can't be surjective for any n, and it follows from Lemma 8.1(iv) that E is not compact.

For the second claim consider an extension $e : よ(Y) \to A$ such that $A \to よ(X) = 1$ is full. Then $A(Y)$ is inhabited, all $A(Z_n) \to A(Y)$ are surjective, and $1 + A(X) \to A(Y)$ is injective. From this we can again deduce that none of the $A(g_n)$ are surjective and thus A is not compact. □

9 Outlook: Models in higher types

One practical use of having inequivalent clans $\mathcal{S}, \mathcal{T}$ with equivalent categories of models in Set is that they might have inequivalent ∞-categories of models in the ∞-category $\mathcal{S}$ of *homotopy types* (a.k.a. 'spaces'). We leave this issue for future work and content ourselves here with outlining some main ideas.

The phenomenon that finite-product theories and finite-limit theories of the same algebraic gadget have equivalent categories of models is well known. Consider for example the finite-product theory $\mathcal{C}_{\mathsf{Mon}}$ and the finite-limit theory $\mathcal{L}_{\mathsf{Mon}}$ of monoids. Then $\mathsf{Mod}(\mathcal{C}_{\mathsf{Mon}}, \mathsf{Set}) = \mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathsf{Set})$ and $\mathsf{Mod}(\mathcal{L}_{\mathsf{Mon}}, \mathsf{Set}) = \mathsf{FL}(\mathcal{C}_{\mathsf{Mon}}, \mathsf{Set})$ are both equivalent to the category of monoids, but the categories

$$\mathsf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S}) \qquad \text{and} \qquad \mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathcal{S})$$

of models in $\mathcal{S}$ are different: since finite-limit preserving functors preserve truncation, every functor in $\mathsf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S})$ factors through $\mathsf{Set} \hookrightarrow \mathcal{S}$, hence we have $\mathsf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathcal{S}) \simeq \mathsf{FL}(\mathcal{L}_{\mathsf{Mon}}, \mathsf{Set})$. The higher models of the finite product theory, on the other hand, are not restricted in such a way: the objects of $\mathsf{FP}(\mathcal{C}_{\mathsf{Mon}}, \mathsf{Set})$ are equivalent to models of the associative operad. Variants of this phenomenon are discussed under the name 'animation' in [CS19], the general pattern being that finite-product theories admit non-trivial higher models, while the models of 'strict' finite-limit theories are necessarily 0-dimensional.

Now the nice thing about clans is that we can have finer graduations of 'levels of strictness' ... talk about different clans for Cat here.

A Locally finitely presentable categories and Quillen's small object argument

In this appendix we recall the definition of locally finitely presentable categories, weak factorization systems, and Quillen's small object argument.

Definition A.1 A category $\mathcal{C}$ is called *filtered*, if every diagram $D : \mathbb{J} \to \mathbb{C}$ with finite domain admits a cocone. A *filtered colimit* is a colimit of a diagram indexed by a filtered category. $\diamondsuit$

Definition A.2 Let $\mathfrak{X}$ be a cocomplete locally small category.

(i) An object object $C \in \mathfrak{X}$ is called *compact*, if the covariant hom-functor

$$\mathfrak{X}(C, -) : \mathfrak{X} \to \mathsf{Set}$$

preserves small filtered colimits.

(ii) $\mathfrak{X}$ is called *locally finitely presentable* (l.f.p.) if it admits a *small dense family of compact objects*, i.e. a family $(C_i)_{i \in I}$ of compact objects indexed by a small set I, such that the nerve functor

$$J^* : \mathfrak{X} \to \widehat{\mathbb{C}}$$

of the inclusion $J : \mathbb{C} \hookrightarrow \mathfrak{X}$ of the full subcategory on the $(C_i)_{i \in I}$ is fully faithful. $\diamondsuit$

Remarks A.3 (a) Compact objects are more commonly known as *finitely presentable objects*, e.g. in [GU71, AR94]. We adopted the term compact from [Lur09, Definition A.1.1.1] since it is more concise, and in particular since *compact* 0*-extension* sounds less awkward than *finitely presented* 0*-extension*. Moreover I think the fact that objects of algebraic categories (such as groups, rings, modules ...) are compact iff they admit a presentation by finitely many generators and relations is an important theorem, and one which is difficult to state if one uses the same terminology for the syntactic and the categorical notion.

(b) The density condition in the definition is equivalent to saying that the family $(C_i)_{i \in I}$ is a *strong generator*, in the sense that the canonical arrow

$$\coprod_{i \in I, f : C_i \to A} C_i \to A$$

is an extremal epimorphism for all $A \in \mathfrak{X}$. We stated the definition in terms of density here, since nerve functors play a central role in this work.

(c) The notion of l.f.p. category is a special case of the notion of *locally α-presentable category* for a regular cardinal α. In this work, only the case $\alpha = \omega$ plays a role. $\diamondsuit$

Definition A.4 Let $\mathcal{C}$ be a category.

(i) Given two arrows $f : A \to B$, $g : X \to Y$ in $\mathcal{C}$, we say that f has the *left lifting property* (l.l.p.) w.r.t. g (or equivalently that g has the *right lifting property* (r.l.p.)

w.r.t. f), and write $f \pitchfork g$, if in each commutative square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "k"] \arrow[ur, dashed, "m"] & Y \end{tikzcd}$$

there exists a diagonal arrow h making the two triangles commute.

(ii) Given a class $\mathcal{E} \subseteq \mathrm{mor}(\mathcal{C})$ of arrows in $\mathcal{C}$, we define:

$$\begin{aligned} {}^{\pitchfork}\mathcal{E} &= \{f \in \mathrm{mor}(\mathcal{C}) \mid \forall g \in \mathcal{E} \,.\, f \pitchfork g\} \\ \mathcal{E}^{\pitchfork} &= \{g \in \mathrm{mor}(\mathcal{C}) \mid \forall f \in \mathcal{E} \,.\, f \pitchfork g\} \end{aligned}$$

(iii) A *weak factorization system* (w.f.s.) on $\mathcal{C}$ is a pair $\mathcal{L}, \mathcal{R} \subseteq \mathrm{mor}(\mathcal{C})$ of classes of morphisms such that $\mathcal{L}^{\pitchfork} = \mathcal{R}$, $\mathcal{R}^{\pitchfork} = \mathcal{L}$, and every $f : A \to B$ in $\mathcal{C}$ admits a factorization $f = r \circ l$ with $l \in \mathcal{L}$ and $r \in \mathcal{R}$. $\diamondsuit$

We call $\mathcal{L}$ the *left class*, and $\mathcal{R}$ the *right class* of the w.f.s. One can show that Left classes of w.f.s.'s contain all isomorphisms, and are closed under composition and pushouts, i.e. if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & B \arrow[d, "m"] \\ C \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout in $\mathcal{C}$ and is a left map, then so is m. Dually, right maps are closed under (isomorphisms, composition, and) pullbacks. With this, we have the prerequesites to state Quillen's *small object argument.*

Theorem A.5 (Small object argument for l.f.p. categories) *Let $\mathcal{E} \subseteq \mathrm{mor}(\mathfrak{X})$ be a small set of morphisms in a l.f.p. category. Then $({}^{\pitchfork}(\mathcal{E}^{\pitchfork}), \mathcal{E}^{\pitchfork})$ is a w.f.s. on $\mathfrak{X}$.*

B Generalized algebraic theories

Cartmells *generalized algebraic theories* extend the notion of algebraic theory (which can be 'single sorted', such as the theories of *groups* or *rings*, or 'many sorted', such as the theories of *reflexive graphs*, *chain complexes of abelian groups*, or *modules over a non-fixed base ring*) by introducing *dependent sorts* (a.k.a. dependent 'types'), which represent families of sets and can be used e.g. to axiomatize the notion of a (small) category $\mathbb{C}$ as a structure with a set $\mathbb{C}_0$ of objects, and a family $(\mathbb{C}(A, B))_{A,B \in \mathbb{C}_0}$ of hom-sets[6].

Compared to ordinary algebraic theories—whose specification in terms of sorts, operations, and equations is fairly straightforward—the syntactic description of generalized algebraic theories is complicated by the fact that the domains of definition of operations and dependent sorts, and the codomains of operations, may themselves be compound expressions involving previously declared operations and sorts, whose well-formedness has to be ensured and may even depend on the equations of the theory. This means that we

[6] An alternative would be to formalize the notion of category as a two-sorted structure with a sort of objects and a sort of arrows, which would avoid type dependency and remain in the realm of first order logic, but at the cost of having to consider composition as a *partial* operation. This approach leads to Freyd's notion of *essentially algebraic theory* [Fre72]

$$
\begin{array}{rl}
& \vdash \; M \\
u : M & \vdash \; R_u \\
& \vdash \; e : M \\
u\,v : M & \vdash \; u{\cdot}v : M \\
u : M & \vdash \; 0_u : R_u \\
u : M \,,\, x\,y : R_u & \vdash \; x + y : R_u \\
u : M \,,\, x : R_u & \vdash \; -y : R_x \\
& \vdash \; 1 : R_e \\
u\,v : M \,,\, x : R_u \,,\, y : R_v & \vdash \; x{\cdot}y : R_{u\cdot v} \\
u : M & \vdash \; e{\cdot}u = u = u{\cdot}e \\
u\,v\,w : M & \vdash \; (u{\cdot}v){\cdot}w = u{\cdot}(v{\cdot}w) \\
u : M \,,\, x\,y : R_u & \vdash \; x + y = y + x \\
u : M \,,\, x\,y : R_u & \vdash \; x + 0_u = x \\
u : M \,,\, x\,y : R_u & \vdash \; x + (-x) = 0_u \\
u : M \,,\, x : R_u & \vdash \; 1{\cdot}x = x = x{\cdot}1 \\
u\,v\,w : M \,,\, x : R_u \,,\, y : R_v \,,\, z : R_w & \vdash \; (x{\cdot}y){\cdot}z = x{\cdot}(y{\cdot}z) \\
u\,v : M \,,\, x : R_u \,,\, y\,z \,,\, R_v & \vdash \; x{\cdot}(y + z) = x{\cdot}y + x{\cdot}z \\
u\,v : M \,,\, x\,y : R_u \,,\, z \,,\, R_v & \vdash \; (x + y){\cdot}z = x{\cdot}z + y{\cdot}z
\end{array}
$$

Figure 1: The generalized algebraic theory of monoid-graded rings

have to state the declarations of *sorts* and of *operations*, and the *equations* (which we collecively refer to as *axioms* of the theory) in an ordered way, where the later axioms have to be well-formed on the basis of the earlier axioms. This looks as follows in the case of the generalized algebraic theory of categories:

$$
\begin{array}{rl}
& \vdash \; O \\
x\,y : O & \vdash \; A_{x,y} \\
x : O & \vdash \; \mathrm{id}_x : A_{x,x} \\
x\,y\,z : O \,,\, f : A_{x,y} \,,\, g : A_{y,z} & \vdash \; g{\circ}f : A_{x,z} \\
x\,y : O \,,\, f : A_{x,y} & \vdash \; \mathrm{id}_y{\circ}f = f \\
x\,y : O \,,\, f : A_{x,y} & \vdash \; f{\circ}\mathrm{id}_x = f \\
w\,x\,y\,z : O \,,\, e : A_{w,x} \,,\, f : A_{x,y} \,,\, g : A_{y,z} & \vdash \; (g{\circ}f){\circ}e = g{\circ}(f{\circ}e)
\end{array}
$$

Each line contains one axiom, the first two declaring the sort O of objects and the dependent sort $A_{x,y}$ of arrows, the third and the fourth declaring the identity and composition operations, and the last three stating the identity and associativity axioms.

Each axiom is of the form $\Gamma \vdash \mathcal{J}$, where the $\mathcal{J}$ on the right of the 'turnstile' symbol '$\vdash$' is the actual declaration or equation, and the part Γ on the left—called 'context'—specifies the sorts of the variables occuring in $\mathcal{J}$. Note that the ordering of these 'variable declarations' is not arbitrary, since the sorts of variables may themselves contain variables which have to be declared further left in the context. An example is the context $(x\,y\,z : O \,,\, f : A_{x,y} \,,\, g : A_{y,z})$ of the composition operation, where the sorts of the 'arrow' variables f, g depend on the 'object' variables x, y, z. See Figure 1 for another example generalized algebraic theory: the generalized algebraic theory of *rings graded over monoids*.

The dependent stucture of contexts and the well-formedness requirement of axioms on the basis of other axioms makes the formulation of a general notion of generalized algebraic theory somewhat subtle and technical. We refer to Cartmell's [Car78, Car78] for the authoritative account and to Section 2 of [Gar15] for a rigorous and concise summary.

The good news is that to understand specific examples, these technicalities may safely be ignored: all we have to know is that for every generalized algebraic theory $\mathbb{T}$ there is a notion of 'derivable judgment' which includes the axioms and is closed under various *rules* expressing that the set of derivable judgments is closed under operations like like substitutions and weakening, and that equality is reflexive, symmetric, and transitive.

Besides the forms of judgments

$\Gamma \vdash S$	'S is a sort in context Γ'
$\Gamma \vdash t : S$	't is term of sort S in context Γ'
$\Gamma \vdash s = t : S$	's and t are equal terms in context Γ'

that we have already encountered, it is useful to consider the following additional forms of judgments:

$\Gamma \vdash S = T$	'S and T are equal types in context Γ'
$\Gamma \vdash$	'Γ is a well-formed context'
$\Gamma \vdash \sigma : \Delta$	'σ is a substitution from Γ to Δ'
$\Gamma \vdash \sigma = \tau : \Delta$	'σ and τ are equal substitutions from Γ to Δ'

The last two of these introduce a novel kind of expression called *substitution*: a substitution $\Gamma \vdash \sigma : \Delta$ is a list of terms that is suitable to be simultaneously substituted for the variables in a judgment in context Δ (in particular σ and Δ must have the same length), to produce a new judgment in context Γ, as expressed by the following *substitution* rule.

$$\text{(Subst)} \qquad \frac{\Gamma \;\vdash\; \sigma : \Delta \qquad \Delta \;\vdash\; \mathcal{J}}{\Gamma \;\vdash\; \mathcal{J}[\sigma]}$$

Here, $\mathcal{J}$ is the result of simultaneous substitution of the terms in σ for the variables in $\mathcal{J}$, replacing each occurrence of the ith variable declared in Δ with the ith term in σ. This operation of simultaneous substitution also appears in the derivaton rules for substitutions themselves, which we present in the following table together with the rules for the formation of well-formed contexts:

$$\text{(B.1)} \qquad \begin{array}{cc} \dfrac{}{* \;\vdash} & \dfrac{}{\Gamma \;\vdash\; () : *} \\[2ex] \dfrac{\Gamma \;\vdash\; A}{\Gamma, y : A \vdash} & \dfrac{\Gamma \;\vdash\; \sigma : \Delta \qquad \Delta \;\vdash\; A \qquad \Gamma \;\vdash\; t : A[\sigma]}{\Gamma \;\vdash\; (\sigma, t) : (\Delta, x : A)} \end{array}$$

The two rules in the first line say respectively that the *empty context* $\varnothing$ is a context, and that for any context Γ, the *empty substitution* $()$ is a substitution to the empty context. The first rule in the second line is known as *context extension*, since it says that we can extend any context by a well-formed sort in this context (here y has to be a 'fresh' variable, i.e. a variable not appearing in Γ). The last rule says that a substitution to an extended context is a pair of a substitution into the original context and a term whose sort is a *substitution instance* of the extending sort — it wouldn't make sense to ask for t to be of sort A since A is only well-formed in context Δ, and we want something in context Γ.

B.1 The syntactic category of a generalized algebraic theory

For any generalized algebraic theory $\mathbb{T}$ one can define a *syntactic category* $\mathcal{C}[\mathbb{T}]$, whose objects are the *contexts* of the theory, and whose morphisms are the *substitutions*. Let's state that as a formal definition.

Definition B.1 The *syntactic category* $\mathcal{C}[\mathbb{T}]$ of a generalized algebraic theory $\mathbb{T}$ is given as follows.

- The objects are the contexts of Γ.
- Morphisms from Γ to Δ are substitutions $\Gamma \vdash \sigma : \Delta$ *modulo derivable equality*, i.e. $\Gamma \vdash \sigma : \Delta$ and $\Gamma \vdash \tau : \Delta$ are identified if the judgment

$$\Gamma \vdash \sigma = \tau : \Delta$$

  is derivable.
- Composition is given by substitution of representatives, and identities are given by lists of variables:
  - $(\Delta \vdash \tau : \Theta) \circ (\Gamma \vdash \sigma : \Delta) = (\Gamma \ \vdash \ \tau[\sigma] : \Theta)$
  - $\mathrm{id}_\Gamma = (\Gamma \vdash (\vec{x}) : \Gamma)$ where $\vec{x}$ is the list of variables declared in Γ. ♢

Remark B.2 If we want to reason about generalized algebraic theories *up to isomorphism*, then we have to quotient contexts by equality as well. However, this is not relevant for the present work. ♢

B.2 $\mathcal{C}[\mathbb{T}]$ as a contextual category and as a clan

The syntactic category $\mathcal{C}[\mathbb{T}]$ of a GAT $\mathbb{T}$ has the structure of a *contextual category*:

Definition B.3 A *contextual category* consists of

(1) a small category $\mathcal{C}$ with a grading function $\mathsf{deg} : \mathcal{C}_0 \to \mathbb{N}$ on its objects, and

(2) a presheaf $\mathsf{Ty} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$, together with

- an arrow $p_A : \Gamma.A \to \Gamma$ for each $\Gamma \in \mathcal{C}$ and $A \in \mathsf{Ty}(\Gamma)$, and
- an arrow $\sigma.A : \Delta.A\sigma \to \Gamma.A$ for each $\Gamma \in \mathcal{C}$, $A \in \mathsf{Ty}(\Gamma)$, and $\sigma : \Delta \to \Gamma$,

such that:

(i) The square
$$\begin{tikzcd} \Delta.A\sigma \arrow[r, "\sigma.A"] \arrow[d, "p_{A\sigma}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "p_A"] \\ \Delta \arrow[r, "\sigma"] & \Gamma \end{tikzcd}$$
is a pullback for all $A \in \mathsf{Ty}(\Gamma)$ and $\sigma : \Delta \to \Gamma$.

(ii) The mappings $(\Gamma, A) \mapsto \Gamma.A$ and $(\sigma, A) \mapsto \sigma.A$ constitute a functor $\underleftarrow{\mathsf{elts}}(\mathsf{Ty}) \to \mathcal{C}$.

(iii) We have $\mathsf{deg}(\Gamma.A) = \mathsf{deg}(\Gamma) + 1$ for all $\Gamma \in \mathcal{C}$ and $A \in \mathsf{Ty}(\Gamma)$.

(iv) There is a unique object $*$ of degree 0, and $*$ is terminal.

(v) For all Γ with $\mathsf{deg}(\Gamma) > 0$ there is a unique $(\Gamma^-, A) \in \underleftarrow{\mathsf{elts}}(\mathsf{Ty})$ with $\Gamma = \Gamma^-.A$. ♢

In the case of the syntactic category $\mathcal{C}[\mathbb{T}]$ of a GAT $\mathbb{T}$, the grading assigns to each context its length, and $\mathsf{Ty}(\Gamma)$ is the set of 'types in context Γ', i.e. type expressions A such that $\Gamma \vdash A$ is derivable, modulo the equivalence relation of derivable equality. The presheaf action is given by substitution. Given a type $A \in \mathsf{Ty}(\Gamma)$, the extended context $\Gamma.A$ is given by $\Gamma, y{:}A$ obtained via the context formation rule in (B.1), p_A is the substitution

$$\Gamma, y{:}A \vdash (\vec{x}) : \Gamma$$

where $\vec{x}$ is the list of variables declared in Γ, and for $\sigma : \Gamma \to \Delta$ and $A \in \mathsf{Ty}(\Delta)$, the substitution $\sigma.A$ is given by

$$\Gamma, y{:}A[\sigma] \vdash (\sigma, x) : \Delta, y{:}A\,.$$

Then the fact that the square in Definition B.3(i) is a pullback follows from the substitution formation rule in (B.1) together with the equality rules for substitutions that we didn't state, but can be found in the cited references.

Finally, given any contextual category $\mathbb{C}$, we obtain a *clan* by taking as display maps the projection arrows $p_A : \Gamma.A \to \Gamma$ and closing off under composition and isomorphisms.

C The fat small object argument

C.1 Colimit decomposition formula and pushouts of sieves

In this subsection we discuss two results that we need in our proof of the fat small object argument.

Theorem C.1 (Colimit decomposition formula (CDF)) *Let $\mathbb{C} : \mathbb{J} \to \mathsf{Cat}$ be a small diagram in the 1-category of small categories, let $D : \mathsf{colim}(\mathbb{C}) \to \mathfrak{X}$ be a diagram in a category $\mathfrak{X}$ such that*

(i) *for each $j \in \mathbb{J}$, the colimit of $\mathsf{colim}_{c \in \mathbb{C}_j} D_{\sigma_j c}$ exists, and*

(ii) *the iterated colimit $\mathsf{colim}_{j \in \mathbb{J}} \mathsf{colim}_{c \in \mathbb{C}_j} D_{\sigma_j c}$ exists.*

Then $\mathsf{colim}_{j \in \mathbb{J}} \mathsf{colim}_{c \in \mathbb{C}_j} D_{\sigma_j c}$ is a colimit of D.

Proof. Peschke and Tholen [PT20] give three proofs of this under the additional assumption that $\mathfrak{X}$ is cocomplete. The third proof (Section 5.3, 'via Fubini') easily generalizes to the situation where only the necessary colimits are assumed to exist. We sketch a slightly simplified argument here. Let $\int \mathbb{C}$ be the covariant Grothendieck construction of $\mathbb{C}$, whose projection $\int \mathbb{C} \to \mathbb{J}$ is a split opfibration. Then $\mathsf{colim}(\mathbb{C})$ is the 'joint coidentifier' of the splitting, i.e. there is a functor $E : \int \mathbb{C} \to \mathsf{colim}(\mathbb{C})$ such that for every category $\mathfrak{X}$, the precomposition functor

$$(- \circ E) : [\mathsf{colim}(\mathbb{C}), \mathfrak{X}] \to [\textstyle\int \mathbb{C}, \mathfrak{X}]$$

restricts to an isomorphism between the functor category $[\mathsf{colim}(\mathbb{C}), \mathfrak{X}]$ and the full subcategory of $[\int \mathbb{C}, \mathfrak{X}]$ on functors which send the arrows of the splitting to identities. In particular, $(- \circ E)$ is fully faithful and thus it induces an isomorphism

$$(\mathsf{colim}(\mathbb{C}))(D, \Delta -) \overset{\cong}{\to} (\textstyle\int \mathbb{C})(D \circ E, \Delta -) : \mathfrak{X} \to \mathsf{Set}$$

of co-presheaves of cocones for every diagram $D : \mathsf{colim}(\mathbb{C}) \to \mathfrak{X}$. In other words, E is *final*, which is the crucial point of the argument, and for which Peschke and Tholen give a more complicated proof in [PT20, Theorem 5.8].

Finality of E means that D has a colimit iff $D \circ E$ has a colimit, and the existence of the latter follows if successive left Kan extensions along the composite $\int \mathbb{C} \to \mathbb{J} \to 1$ exist. The first of these can be computed as fiberwise colimit since $\int \mathbb{C} \to \mathbb{J}$ is a split cofibration [PT20, Theorem 4.6], which yields the inner term in the double colimit in the proposition. $\square$

In the following we use the CDF specifically for pushouts of sieve inclusions of posets. Recall that a *sieve* (a.k.a. *downset* or *lower set*) in a poset P is a subset $U \subseteq P$ satisfying

$$x \in U \wedge y \leq x \implies y \in U$$

for all $x, y \in P$. A monotone map $f : P \to Q$ is called a *sieve inclusion* if it is order-reflecting and its image $\mathsf{im}(f) = f[P]$ is a sieve in Q. The proof of the following lemma is straightforward, but we state it explicitly since it will play a crucial role.

Lemma C.2 (i) *If $f : P \to Q$ and $g : P \to R$ are sieve inclusions of posets, a pushout of f and g in the 1-category* Cat *of small categories is given by*

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "f"'] & R \arrow[d, "\sigma_2"] \\ Q \arrow[r, "\sigma_1"] & Q +_P R \end{tikzcd}$$

where $Q +_P R$ is the set-theoretic pushout, ordered by

$$\begin{array}{ll} \sigma_1(x) \leq \sigma_1(y) \text{ iff } x \leq y & \quad \sigma_1(x) \leq \sigma_2(y) \text{ iff } \exists z \,.\, x = f(z) \wedge g(z) \leq y \\ \sigma_2(x) \leq \sigma_2(y) \text{ iff } x \leq y & \quad \sigma_2(x) \leq \sigma_1(y) \text{ iff } \exists z \,.\, x = g(z) \wedge f(z) \leq y. \end{array}$$

In particular, the maps σ_1 and σ_2 are also sieve inclusions.

(ii) *If U and V are sieves in a poset P then the square*

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \cup V \end{tikzcd}$$

is a pushout in Cat*, where the sieves are equipped with the induced ordering.* $\square$

C.2 The fat small object argument

Throughout this subsection let $\mathcal{C}$ be a coclan.

We start by establishing some notation. Given a poset P and an element $x \in P$, we write $P_{\leq x} = \{y \in P \mid y \leq x\}$ for the principal sieve generated by x, and $P_{<x} = \{y \in P \mid y < x\}$ for its subset on elements that are strictly smaller than x. If x is a maximal element of P, we write $P \backslash x$ for the sub-poset obtained by removing x. Given a diagram $D : P \to \mathcal{C}$, we write $D_{\leq x}$, $D_{<x}$, and $D \backslash x$ for the restrictions of D to $P_{\leq x}$, $P_{<x}$, and $P \backslash x$, respectively. More generally we write D_U for the restriction of D to arbitrary sieves $U \subseteq P$.

Note that we have $P_{\leq x} = P_{<x} \star 1$, where $\star$ is the *join* or *ordinal sum*, thus diagrams $D : P_{\leq x} \to \mathcal{C}$ are in correspondence with cocones on $D_{<x}$ with vertex D_x, and with arrows $\mathsf{colim}(D_{<x}) \to D_x$ whenever the colimit exists.

Definition C.3 A *finite* $\mathcal{C}$*-complex* is a pair (P, D) of a finite poset P and a diagram $D : P \to \mathcal{C}$, such that:

(i) $\mathsf{colim}(D_{<x})$ exists for all $x \in P$, and the induced $\alpha_x : \mathsf{colim}(D_{<x}) \to D_x$ is co-display.

(ii) For $x, y \in P$ we have $x = y$ whenever $P_{<x} = P_{<y}$, $D_x = D_y$, and $\alpha_x = \alpha_y$.

An *inclusion of finite* $\mathcal{C}$*-complexes* $f : (P, D) \to (Q, E)$ is a sieve inclusion $f : P \to Q$ such that $D = E \circ f$. We write $\mathsf{FC}(\mathcal{C})$ for the category of finite $\mathcal{C}$-complexes and inclusions. $\diamondsuit$

Remark C.4 We view a finite $\mathcal{C}$-complex as a construction of an object by a finite (though not necessarily linearly ordered) number of 'cell attachments', represented by the co-display maps $\alpha_x : \mathsf{colim}(D_{<x}) \rightarrowtriangle D_x$. Condition (ii) should be read as saying that 'every cell can only be attached once at the same stage'. This is needed in Lemma C.7 to show that $\mathsf{FC}(\mathcal{C})$ is a preorder. $\diamondsuit$

Lemma C.5 (i) $\mathsf{colim}(D)$ *exists for every finite* $\mathcal{C}$*-complex* (P, D).

(ii) *The induced functor*

$$\mathsf{Colim} : \mathsf{FC}(\mathcal{C}) \to \mathcal{C} \tag{C.1}$$

sends inclusions of finite $\mathcal{C}$*-complexes to co-display maps.*

Proof. The first claim is shown by induction on $|P|$. For empty P the statement is true since coclans have initial objects. For $|P| = n+1$ assume that $x \in P$ is a maximal element. Then the quare

$$\begin{tikzcd} P_{<x} \arrow[r] \arrow[d] & P \backslash x \arrow[d] \\ P_{\leq x} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & P \end{tikzcd}$$

is a pushout in Cat by Lemma C.2, which by the colimit decomposition formula C.1 means that the pushout of the span

$$\begin{tikzcd} \mathsf{colim}(D_{<x}) \arrow[r] \arrow[d, rightarrowtail] & \mathsf{colim}(D \backslash x) \arrow[d, dashed, rightarrowtail] \\ D_x \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner"] & \mathsf{colim}(D) \end{tikzcd} \tag{C.2}$$

– which exists since the left arrow is a co-display map by C.3-(i) – is a colimit of D in $\mathcal{C}$.

For the second claim let $f : (E, Q) \to (D, P)$ be an inclusion of finite $\mathcal{C}$-complexes. Since co-display maps compose and every inclusion of finite $\mathcal{C}$-complexes can be decomposed into 'atomic' inclusions with $|P \backslash f[Q]| = 1$ by successively removing maximal elements from the codomain, we may assume without loss of generality that $Q = P \backslash x$ for some maximal element $x \in P$. Then the image of f under colim is the right dashed arrow in (C.2), which is a co-display map since co-displays are stable under pushout. $\square$

Remark C.6 Lemma C.5 implies that the assumption '$\mathsf{colim}(D_{<x})$ exists' in Definition C.3-(i) is redundant, since the colimits in question are colimits of finite subcomplexes. $\diamondsuit$

Lemma C.7 *The category* $\mathsf{FC}(\mathcal{C})$ *is an essentially small preorder with finite joins.*

Proof. $\mathsf{FC}(\mathcal{C})$ is essentially small as a collection of finite diagrams in a small category. To see that it is a preorder let $f, g : (P, D) \to (Q, E)$ be inclusions of finite $\mathcal{C}$-complexes. We show that $f(x) = g(x)$ by well-founded induction on $x \in P$. Let $x \in P$ and assume that $f(y) = g(y)$ for all $y < x$. Then since f and g are sieve inclusions we have $Q_{<f(x)} = Q_{<g(x)}$ and since $E\, f = D = E\, g$ we have the equalities

$$\big(E_y \to E_{f(x)}\big)_{y<f(x)} = (D_y \to D_x)_{y<x} = \big(E_y \to E_{g(x)}\big)_{y<g(x)}$$

of cocones, whence $f(x) = g(x)$ by Definition C.3-(ii).

It remains to show that $\mathsf{FC}(\mathcal{C})$ has finite suprema. The empty complex is clearly initial. We show that a supremum of (P, D) and (Q, E) exists by induction on $|P|$. The empty case is trivial, so assume that P is inhabited and let x be a maximal element. Let (R, F) be a supremum of $(P\backslash x, D\backslash x)$ and (Q, E), with inclusion maps $f : (P\backslash x, D\backslash x) \to (R, F)$ and $g : (Q, E) \longrightarrow (R, F)$. If there exists a $y \in R$ such that $R_{<y} = f[P_{<x}]$ and $(D_z \to D_x)_{z<x} = \big(R_{f(z)} \to R_y\big)_{z<x}$ then 'the cell-attachment corresponding to x is already contained in (R, F)', i.e. f extends to an inclusion $f' : (P, D) \to (R, F)$ of finite complexes with $f'(x) = y$, whence (R, F) is a supremum of (P, D) and (Q, E).

If no such y exists then a supremum of (P, D) and (R, F) is given by $(P +_{P\backslash x} R, [D, F])$, as in the pushout diagram

$$\begin{tikzcd}
P\backslash x \arrow[r, "f"'] \arrow[d, hook] & R \arrow[d] \arrow[ddr, bend left, "F"] & \\
P \arrow[r] \arrow[drr, bend right, "D"'] & P +_{P\backslash x} R \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "{[D,F]}"] & \\
& & \mathcal{C}
\end{tikzcd}$$

constructed as in Lemma C.2. $\square$

Definition C.8 A *smooth diagram* in a $\diamondsuit$

Lemma C.9 *The object* $C = \mathsf{colim}_{(P,D)\in\mathsf{FC}(\mathcal{C})}\, H(\mathsf{colim}(D))$ *is a* 0*-extension in* $\mathcal{C}^{\mathsf{op}}\text{-}\mathsf{Alg}$ *and* $C \to 1$ *is full.*

Proof. To see that $C \to 1$ is full, let $e : I \rightarrowtriangle J$ be co-display in $\mathcal{C}$ and let $f : H(I) \to C$. Since $\mathsf{FC}(\mathcal{C})$ is filtered and $H(I)$ is compact, f factors through a colimit inclusion as

$$f \;=\; \big(H(I) \xrightarrow{H(g)} H(\mathsf{colim}(D)) \xrightarrow{\sigma_{(P,D)}} C\big)$$

for some finite complex (P, D). We form the pushout

$$\begin{tikzcd}
I \arrow[r, "g"] \arrow[d, "e"'] & \mathsf{colim}(D) \arrow[d, "k"] \\
J \arrow[r] & K \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

and extend the finite complex (P, D) to $(P\star 1, D\star k)$ where $P\star 1$ is the join of P and 1, and $D\star k : P\star 1 \to \mathcal{C}$ is the diagram extending D with the cell-attachment $k : \mathsf{colim}(D) \rightarrowtriangle K$.

Then $K = \mathsf{colim}(D \star k)$ and k is the image of the inclusion $(P, D) \hookrightarrow (P \star 1, D \star k)$ of finite complexes under the colimit functor (C.1), thus we obtain an extension of f along $H(e)$ as in the following diagram.

$$\begin{tikzcd}
H(I) \arrow[r, "H(g)"'] \arrow[rr, bend left, "f"] \arrow[d, hook, "H(e)"'] & H(\mathsf{colim}(D)) \arrow[r, "\sigma_{(P,D)}"'] \arrow[d, hook, "H(k)"] & C \\
H(J) \arrow[r] & H(K) \arrow[ur, bend right, "\sigma_{(P\star 1, D\star k)}"'] \arrow[ul, phantom, very near start, "\ulcorner"] &
\end{tikzcd}$$

To see that C is a 0-extension, consider a full map $f : Y \twoheadrightarrow X$ in $\mathcal{C}^{\mathsf{op}}\text{-}\mathsf{Alg}$ and an arrow $h : C \to X$. To show that h lifts along f we construct a lift of the cocone

$$\left(H(\mathsf{colim}(D)) \xrightarrow{\sigma_{(P,D)}} C \xrightarrow{h} X \right)_{(P,D) \in \mathsf{FC}(\mathcal{C})}$$

by induction over the preorder $\mathsf{FC}(\mathcal{C})$ which is well-founded since every finite $\mathcal{C}$-complex has only finitely many subcomplexes. Given a finite complex (D, P) it is sufficient to exhibit a lift $\kappa_{(P,D)} : H(\mathsf{colim}(D)) \to Y$ satisfying

(C.3) $\quad f \circ \kappa_{(P,D)} = h \circ \sigma_{(P,D)} \quad$ and

(C.4) $\quad \kappa_{(P,D)} \circ H(\mathsf{colim}\, j) = \kappa_{(Q,E)} \quad$ for all subcomplexes $j : (Q, E) \to (P, D)$,

where we may assume that the $\kappa_{(Q,E)}$ satisfy the analogous equations by induction hypothesis. We distinguish two cases:

1. If P has a greatest element x then we can take $\kappa_{(P,D)}$ to be a lift in the square

$$\begin{tikzcd}
H(\mathsf{colim}(D_{<x})) \arrow[rr, "\kappa_{(P_{<x}, D_{<x})}"] \arrow[d] & & Y \arrow[d, two heads, "f"] \\
H(D_x) \arrow[r, "\sigma_{(P,D)}"'] \arrow[urr, dashed, "\kappa_{(P,D)}"] & C \arrow[r, "h"'] & X
\end{tikzcd}$$

whose left side is an extension by Lemma C.5 and whose right side is full by assumption. Then (C.3) holds by construction, and (C.4) holds for all subcomplexes since it holds for the largest strict subcomplex $(P_{<x}, D_{<x}) \to (P, D)$.

2. If P *doesn't* have a greatest element we can write $P = U \cup V$ as union of two strict sub-sieves, wence we have pushouts

$$\begin{tikzcd}
U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\
U \arrow[r] & P \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathsf{colim}(D_{U \cap V}) \arrow[r] \arrow[d] & \mathsf{colim}(D_V) \arrow[d] \\
\mathsf{colim}(D_U) \arrow[r] & \mathsf{colim}(D) \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

by Lemma C.2 and the CDF. This means that condition (C.4) forces us to define $\kappa_{(P,D)}$

to be the unique arrow fitting into

$$
\begin{tikzcd}
H(\mathsf{colim}(D_{U\cap V})) \arrow[r, "\phi_V^{U\cap V}"'] \arrow[d, "\phi_U^{U\cap V}"] & H(\mathsf{colim}(D_V)) \arrow[d, "\phi_P^V"] \arrow[ddr, bend left, "\kappa_{(V,D_V)}"] & \\
H(\mathsf{colim}(D_U)) \arrow[r, "\phi_P^U"] \arrow[drr, bend right, "\kappa_{(U,D_U)}"] & H(\mathsf{colim}(D)) \arrow[dr, dashed, "\kappa_{(P,D)}"] & \\
 & & Y
\end{tikzcd}
\tag{C.5}
$$

where for the remainder of the proof we write $\phi_W^X : H(\mathsf{colim}(D_X)) \to H(\mathsf{colim}(D_W))$ for the canonical arrows induced by successive sieve inclusions $X \subseteq W \subseteq P$. Using the fact that the ϕ_P^U and ϕ_P^V are jointly epic it is easy to see that the $\kappa_{(P,D)}$ defined in this way satisfies condition (C.3), and it remains to show that (C.4) is satisfied for arbitrary sieves $W \subseteq P$, i.e. $\kappa_{(P,D)} \circ \phi_P^W = \kappa_{(W,D_W)} : H(\mathsf{colim}(D_W)) \to Y$. Since

$$
\begin{tikzcd}
H(\mathsf{colim}(D_{U\cap V\cap W})) \arrow[r, "\phi_{V\cap W}^{U\cap V\cap W}"] \arrow[d, "\phi_{U\cap W}^{U\cap V\cap W}"'] & H(\mathsf{colim}(D_{V\cap W})) \arrow[d, "\phi_W^{V\cap W}"] \\
H(\mathsf{colim}(D_{U\cap W})) \arrow[r, "\phi_W^{U\cap W}"] & H(\mathsf{colim}(D_W))
\end{tikzcd}
$$

is a pushout it is enough to verify this equation after precomposing with $\phi_W^{U\cap W}$ and $\phi_W^{V\cap W}$. We have

$$
\begin{aligned}
\kappa_{(P,D)} \circ \phi_P^W \circ \phi_W^{U\cap W} &= \kappa_{(P,D)} \circ \phi_P^U \circ \phi_U^{U\cap W} && \text{by functoriality} \\
&= \kappa_{(U,D_U)} \circ \phi_U^{U\cap W} && \text{by (C.5)} \\
&= \kappa_{(U\cap W, D_{U\cap W})} && \text{by (C.4)} \\
&= \kappa_{(W,D_W)} \circ \phi_W^{U\cap W} && \text{by (C.4)}
\end{aligned}
$$

and the case with $\phi_W^{V\cap W}$ is analogous. □

Corollary C.10 *For any clan $\mathcal{T}$, the 0-extensions in $\mathcal{T}$-Alg are flat algebras.*

Proof. Let $E \in \mathcal{T}$-Alg be a 0-extension. By applying Lemma C.9 in $\mathcal{T}$-Alg/E (using Proposition 4.5), we obtain full map $f : F \twoheadrightarrow E$ where F is a 0-extension f is a filtered colimit of arrows $H(\Gamma) \to E$ in $\mathcal{T}$-Alg/A. Since $\mathcal{T}$-$\mathsf{Alg}/A \to \mathcal{T}$-$\mathsf{Alg}$ creates colimits this means that F is a filtered colimit of hom-algebras in $\mathcal{T}$-Alg, and therefore flat (Lemma 3.6). Since f is a full map into a 0-extension it has a section, thus E is a retract of F and therefore flat as well. □

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