Realizability toposes as homotopy categories

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Bures-sur-Yvette, 25 November 2015

History, related work

- Initial inspiration: Jaap van Oosten's talk

  A notion of homotopy for the effective topos

  at "Réalisabilité à Chambéry" in 2010 (meanwhile published in MSCS)
- discussions with Zhen Lin Low, Rasmus Møgelberg, Benno van den Berg
- result quite different from van Oosten's approach
  - topos is the *homotopy category*, not the underlying category
  - applies to larger class of structures
- yesterday on arxiv:
  P. Rosolini, "The category of equilogical spaces and the effective topos as homotopical quotients" (to appear in JHRS)

Overview

In this talk: For any **tripos**[1]

$$\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Ord}$$

define a **category of fibrant objects**[2]

$$\mathbb{C}\langle\mathcal{P}\rangle$$

such that the homotopy category is isomorphic to the **topos**

$$\mathbb{C}[\mathcal{P}]$$

obtained by the **tripos-to-topos construction**.

[1] J.M.E. Hyland, P.T. Johnstone, and A.M. Pitts. “Tripos theory”. In: *Math. Proc. Cambridge Philos. Soc.* 88.2 (1980), pp. 205–231.

[2] K.S. Brown. “Abstract homotopy theory and generalized sheaf cohomology”. In: *Transactions of the American Mathematical Society* 186 (1973), pp. 419–458.

More generally: For any **regular hyperdoctrine**

$$\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Ord}$$

define a **category of fibrant objects**

$$\mathbb{C}\langle\mathcal{P}\rangle$$

such that the homotopy category is isomorphic to the **exact category**

$$\mathbb{C}[\mathcal{P}]$$

obtained by the **???-construction**.

Regular hyperdoctrines

Definition

A **regular hyperdoctrine**[3] is a (pseudo)functor

$\mathcal{P} : \mathbb{C}^{op} \to \mathbf{Ord}$ (**Ord** category of preorders and monot. maps)

such that

- $\mathbb{C}$ has finite limits
- all $\mathcal{P}(A)$ (for $A \in \mathbb{C}$) have finite meets
- for $f : A \to B$, the **reindexing map** $f^* = \mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$ has a left adjoint $\exists_f = f_! : \mathcal{P}(A) \to \mathcal{P}(B)$
- given $f : A \to B$ and **predicates** $\varphi \in \mathcal{P}(A)$, $\psi \in \mathcal{P}(B)$ we have

$$(\exists_f \varphi) \wedge \psi \cong \exists_f(\varphi \wedge f^* \psi)$$

- for all pullbacks $\begin{array}{ccc} D & \xrightarrow{k} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C \end{array}$ we have $\exists_k h^* \cong g^* \exists_f$

[3] F.W. Lawvere. "Adjointness in foundations". In: *Dialectica* 23.3-4 (1969), pp. 281–296, F.W. Lawvere. "Equality in hyperdoctrines and the comprehension schema as an adjoint functor". In: *Applications of Categorical Algebra* 17 (1970), pp. 1–14

Examples of regular hyperdoctrines

- For X a **locale**, define $\mathcal{P}_A : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ by

$$\mathcal{P}_X(A) \;=\; (X^A, \leq) \qquad \text{(pointwise ordering)}$$

- Define the **effective tripos** $\mathbf{eff} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ by

$$\mathbf{eff}(A) = (P(\mathbb{N})^A, \leq)$$

  with $\varphi \leq \psi$ if there exists a *partial recursive* $f : \mathbb{N} \rightharpoonup \mathbb{N}$ such that

$$\forall a \in A\, \forall n \in \varphi(a) \,.\, f(n) \in \psi(a).$$

- Define the **primitive recursive hyperdoctrine** $\mathbf{prim} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ by

$$\mathbf{prim}(A) = (P(\mathbb{N})^A, \leq)$$

  with $\varphi \leq \psi$ if there exists a *primitive recursive* $f : \mathbb{N} \to \mathbb{N}$ such that

$$\forall a \in A\, \forall n \in \varphi(a) \,.\, f(n) \in \psi(a).$$

Internal logic

From now on $\mathcal{P} : \mathbb{C}^{\mathrm{op}} \to \mathbf{Ord}$ is a fixed regular hyperdoctrine

Use **regular logic** ($\wedge, \top, \exists, =$, Olivia's talk) as notation for constructions in $\mathcal{P}$.

E.g. for $\varphi \in \mathcal{P}(A \times B), \psi \in \mathcal{P}(B \times C)$, write

$$\theta(x, z) \;\equiv\; \exists y \,.\, \varphi(x, y) \wedge \psi(y, z)$$

instead of

$$\theta \;=\; \exists_{\partial_1}(\partial_2{}^*\varphi \wedge \partial_0{}^*\psi).$$

$$\begin{tikzcd} A \times B & \\ A \times B \times C \arrow[u, "\partial_2"'] \arrow[r, "\partial_1"] \arrow[d, "\partial_0"] & A \times C \\ B \times C & \end{tikzcd}$$

Given **predicates** $\varphi_1, \ldots, \varphi_n, \psi \in \mathcal{P}(A_1 \times \ldots \times A_k)$, say that the **judgment**

$$\varphi_1(\vec{x}), \ldots, \varphi_n(\vec{x}) \vdash_{\vec{x}} \psi(\vec{x})$$

is **valid**, if

$$\varphi_1 \wedge \cdots \wedge \varphi_n \leq \psi \quad \text{in} \quad \mathcal{P}(A_1 \times \ldots \times A_k).$$

More generally, $\varphi_1 \ldots \varphi_n, \psi$ can be **formulas** instead of (atomic) predicates.

Validity relation closed under deduction rules for regular logic.

The category $\mathbb{C}[\mathcal{P}]$

Definition

$\mathbb{C}[\mathcal{P}]$ is the category where

- **objects** are pairs $(A \in \mathbb{C}, \rho \in \mathcal{P}(A \times A))$ such that

$$
\begin{aligned}
\textit{(sym)} \quad & \rho(x,y) \vdash \rho(y,x) \\
\textit{(trans)} \quad & \rho(x,y), \rho(y,z) \vdash \rho(x,z)
\end{aligned}
$$

- **morphisms** $(A,\rho) \to (B,\sigma)$ are (equivalence classes of) predicates $\phi \in \mathcal{P}(A \times B)$ such that

$$
\begin{aligned}
\textit{(strict)} \quad & \phi(x,y) \vdash \rho x \wedge \sigma y \qquad \text{[short for } \rho(x,x) \wedge \sigma(y,y)\text{]} \\
\textit{(cong)} \quad & \rho(x,x'), \phi(x',y), \sigma(y,y') \vdash \phi(x,y') \\
\textit{(sv)} \quad & \phi(x,y), \phi(x,y') \vdash \sigma(y,y') \\
\textit{(tot)} \quad & \rho x \vdash \exists y \,.\, \phi(x,y)
\end{aligned}
$$

- $\phi, \phi' \in \mathcal{P}(A \times B)$ are identified as morphisms, if $\phi \cong \phi'$
- composition is relational composition

Lemma

*$\mathbb{C}[\mathcal{P}]$ is a **Barr-exact** category (and a topos, if $\mathcal{P}$ is a tripos).*

Examples

- $\mathbf{Set}[\mathcal{P}_X] \simeq \mathbf{Sh}(X)$ for any locale X
- $\mathbf{Set}[\mathbf{eff}]$ is the **effective topos**[4] (the best-known *realizablity topos*)
- $\mathbf{Set}[\mathbf{prim}]$ is a **list-arithmetic pretopos**[5]

[4] J.M.E. Hyland. “The effective topos”. In: *The L.E.J. Brouwer Centenary Symposium (Noordwijkerhout, 1981)*. Vol. 110. Stud. Logic Foundations Math. Amsterdam: North-Holland, 1982, pp. 165–216.

[5] M. Maietti. “Joyal’s arithmetic universe as list-arithmetic pretopos”. In: *Theory and Applications of Categories* 24.3 (2010), pp. 39–83.

The category $\mathbb{C}\langle\mathcal{P}\rangle$

Definition

$\mathbb{C}\langle\mathcal{P}\rangle$ is the category where

- **objects** are pairs $(A \in \mathbb{C}, \rho \in \mathcal{P}(A \times A))$ such that

$$\begin{array}{rl} (sym) & \rho(x,y) \vdash \rho(y,x) \\ (trans) & \rho(x,y), \rho(y,z) \vdash \rho(x,z) \end{array}$$

- **morphisms** $(A,\rho) \to (B,\sigma)$ are morphisms $f : A \to B$ in $\mathbb{C}$ such that

$$\rho(x,y) \vdash \sigma(fx, fy)$$

- composition and identities are inherited from $\mathbb{C}$

Lemma

$\mathbb{C}\langle\mathcal{P}\rangle$ *has finite limits.*

Proof.

$(1,\top)$ terminal, pullbacks given by
$$\begin{tikzcd} (D, \rho \bowtie_C \sigma) \arrow[r, "k"] \arrow[d, "h"'] & (B,\sigma) \arrow[d, "g"] \\ (A,\rho) \arrow[r, "f"'] & (C,\tau) \end{tikzcd}$$
, where
$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$
pullback in $\mathbb{C}$, and $(\rho \bowtie_C \sigma)(p,q) \equiv \rho(hp,hq) \wedge \sigma(kp,kq)$.

Definition (Kenneth Brown)

A *category of fibrant objects* is a category $\mathbb{C}$ with finite products, and two classes $\mathcal{F}, \mathcal{W} \subseteq \mathrm{mor}(\mathbb{C})$ of morphisms (called *fibrations* and *weak equivalences*), subject to the following axioms.

(A) For any composable pair $A \xrightarrow{f} B \xrightarrow{g} C$, if either two of the three morphisms f, g, and gf are in $\mathcal{W}$, then so is the third.

(B) $\mathcal{F}$ contains all isomorphisms and is closed under composition.

(C) Pullbacks of fibrations along arbitrary maps exist and are fibrations. Pullbacks of trivial fibrations (ie. elements of $\mathcal{F} \cap \mathcal{W}$) are trivial fibrations.

(D) For any $X \in \mathbb{C}$ there exists a *path object*, i.e. a factorization

$$X \xrightarrow{s} X^I \xrightarrow{d=\langle d_0, d_1 \rangle} X \times X$$

of the diagonal, where $s \in \mathcal{W}$ and $d \in \mathcal{F}$.

(E) For any $X \in \mathbb{C}$, the map $X \to 1$ is a fibration.

$\mathbb{C}\langle\mathcal{P}\rangle$ *as a category of fibrant objects*

Definition

A morphism $f : (A, \rho) \to (B, \sigma)$ in $\mathbb{C}\langle\mathcal{P}\rangle$ is a **fibration**, if

$$\textit{(fib)} \quad \rho x, \sigma(fx, u) \vdash \exists y \,.\, \rho(x, y) \wedge fy = u$$

holds. It is a **weak equivalence**, if

$$\textit{(inj)} \quad \rho x, \sigma(fx, fy), \rho y \vdash \rho(x, y) \text{ and}$$

$$\textit{(esurj)} \quad \sigma u \vdash \exists x \,.\, \rho x \wedge \sigma(fx, u)$$

hold.

Lemma

$f : (A, \rho) \to (B, \sigma)$ is a trivial fibration if and only if (inj) and

$$\textit{(surj)} \quad \sigma u \vdash \exists x \,.\, \rho x \wedge fx = u$$

hold.

Theorem

$\mathbb{C}\langle\mathcal{P}\rangle$ with the above classes of fibrations and weak equivalences is a category of fibrant objects.

The homotopy category

- Homotopy category is solution to the problem of freely inverting weak equivalences
- Want to show that $\mathbb{C}[\mathcal{P}]$ is the homotopy category of $\mathbb{C}\langle\mathcal{P}\rangle$
- direct description of homotopy category of a category of fibrant objects fairly complicated
- easier to verify universal property

Definition

Define the functor

$$E : \mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{C}[\mathcal{P}]$$

by

$$\begin{array}{ccc} (A,\rho) & \mapsto & (A,\rho) \\ \Big\downarrow f & \mapsto & \Big\downarrow [\phi] \\ (B,\sigma) & \mapsto & (B,\sigma) \end{array} \quad \text{where} \quad \phi(x,y) \equiv \rho x \wedge \sigma(fx,y)$$

The homotopy category

Lemma

$[\phi] : (A, \rho) \to (B, \sigma)$ *is iso in* $\mathbb{C}[\mathcal{P}]$ *iff the judgments*

$$\text{(inj*)} \quad \phi(x, u), \phi(y, u) \vdash \rho(x, y)$$

$$\text{(esurj*)} \quad \sigma u \vdash \exists x \,.\, \phi(x, u)$$

hold in $\mathcal{P}$.

Lemma

$f : (A, \rho) \to (B, \sigma)$ *is a weak equivalence in* $\mathbb{C}\langle\mathcal{P}\rangle$ *iff* $E(f)$ *is an iso in* $\mathbb{C}[\mathcal{P}]$.

The homotopy category

Theorem

$E : \mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{C}[\mathcal{P}]$ *is universal among functors inverting weak equivalences in* $\mathbb{C}\langle\mathcal{P}\rangle$*, i.e. for every* $F : \mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{D}$ *inverting weak equivalences, there exists a unique* $\tilde{F} : \mathbb{C}[\mathcal{P}] \to \mathbb{D}$ *with* $\tilde{F} \circ E = F$.

$$\begin{tikzcd} \mathbb{C}\langle\mathcal{P}\rangle \arrow[d, "E"'] \arrow[dr, "F"] & \\ \mathbb{C}[\mathcal{P}] \arrow[r, dashed] & \mathbb{D} \end{tikzcd}$$

Proof (sketch).

$\tilde{F}$ coincides with F on objects. For $[\phi] : (A, \rho) \to (B, \sigma)$ construct the span

$$(A, \rho) \xleftarrow{\phi_l} (A \times B, (\rho \bowtie \sigma)|_\phi) \xrightarrow{\phi_r} (B, \sigma)$$

where the underlying maps are projections, and

$$(\rho \bowtie \sigma)|_\phi(a, b, a', b') \;\equiv\; \rho(a, a') \wedge \sigma(b, b') \wedge \phi(a, b).$$

Then ϕ_l is a weak equivalence, and $\tilde{F}([\phi])$ is given by

$$\tilde{F}([\phi]) = F(\phi_r) \circ F(\phi_l)^{-1}$$

Conclusion

- new description of categories $\mathbb{C}[\mathcal{P}]$ (up to iso), and of localic Grothendieck toposes (up to equivalence)
- homotopy theory in $\mathbb{C}\langle\mathcal{P}\rangle$ degenerate since $A^I \to A \times A$ monic
- In the construction of the homotopy category of a category $\mathcal{C}$ of fibrant objects, Brown considers an intermediate category $\pi(\mathcal{C})$. If $\mathcal{P}$ is a tripos, then $\pi(\mathbb{C}\langle\mathcal{P}\rangle)$ is a *q-topos*[6]

[6] J. Frey. "Triposes, q-toposes and toposes". In: *Annals of pure and applied logic* 166.2 (2015), pp. 232–259.

Thanks for your attention!