Duality for Clans: a Refinement of Gabriel–Ulmer Dualiy

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Interactions of Proof Assistants and Mathematics

Regensburg, September 2023

https://arxiv.org/abs/2308.11967

Functorial Semantics: Lawvere Theories

Theorem (Lawvere[1])

For every **algebraic theory** $\mathbb{T}$ (i.e. theory of groups, rings, ...), there exists a small category $\mathcal{C}[\mathbb{T}]$ with finite products such that

$$\mathsf{Mod}(\mathbb{T}) \simeq \mathsf{FP}(\mathcal{C}[\mathbb{T}], \mathsf{Set})$$

- $\mathcal{C}[\mathbb{T}]$ is is known as the **Lawvere theory** of $\mathbb{T}$
- **Syntactic description** of $\mathcal{C}[\mathbb{T}]$:
  - objects: natural numbers $n, k, \dots \in \mathbb{N}$
  - $\mathcal{C}[\mathbb{T}](n,k) = \{\mathbb{T}\text{-terms in vars } x_1, \dots, x_n\}^k$
- **Semantic description**: Yoneda よ : $\mathcal{C}[\mathbb{T}]^{op} \to \mathsf{FP}(\mathcal{C}[\mathbb{T}], \mathsf{Set})$ identifies $\mathcal{C}[\mathbb{T}]^{op}$ with **finitely generated free models**

Principle of functorial semantics:

- **theories** are identified with structured categories, and
- **models** correspond to structure-preserving functors into Set (or another semantic 'background category')

[1] F.W. Lawvere. "Functorial semantics of algebraic theories". In: *Proceedings of the National Academy of Sciences of the United States of America* (1963)

Finite product theories

Moving away from syntax, we define:

Definition

- A **finite-product theory** is a small category with finite products.
- a **model** of a finite-product theory $\mathcal{C}$ is a functor $A : \mathcal{C} \to \mathsf{Set}$ which preserves finite products.

Finite-product theories correspond to **many-sorted algebraic theories**, such as

- the theory of reflexive graphs
- the theory of graded rings/modules
- the theory of modules over non-constant base ring
- ...

but there are algebraic gadgets that cannot be represented by finite-product theories, notably **categories**!

How to include the theory of categories ?

Syntactic theories		**Categorical theories**
Single sorted algebraic theories	$\simeq$	Lawvere theories
$\hookdownarrow$		$\hookdownarrow$
Many-sorted algebraic theories	$\simeq$	Finite-product theories
$\hookdownarrow$		$\hookdownarrow$
Essentially algebraic theories (Freyd[2]) Generalized algebraic theories (Cartmell[3])	$\overset{?}{\simeq}$	Finite-limit theories

[2] P. Freyd. "Aspects of topoi". In: *Bulletin of the Australian Mathematical Society* (1972).

[3] J. Cartmell. "Generalised algebraic theories and contextual categories". In: *Annals of Pure and Applied Logic* (1986).

Finite-limit theories

Definition

- A **finite-limit theory** is a small category with finite limits
- a **model** of a finite-limit theory is a finite-limit preserving functor $A : \mathcal{C} \to \mathsf{Set}$

Finite-limit theories can be **reconstructed** from their categories of models, which gives a nice duality theory:

Proposition

Let $\mathcal{C}$ be a finite-limit theory.

1. For every $\Gamma \in \mathcal{C}$, the representable functor $\mathcal{C}(\Gamma, -) : \mathcal{C} \to \mathsf{Set}$ is a model.
2. A model $A \in \mathsf{Mod}(\mathcal{C})$ is representable by an object of $\mathcal{C}$ iff it is **compact**, i.e. $\mathsf{Mod}(\mathcal{C})(A, -)$ preserves filtered colimits.
3. The category $\mathsf{Mod}(\mathcal{C}) = \mathsf{FP}(\mathcal{C}, \mathsf{Set})$ is **locally finitely presentable**, i.e. cocomplete with a dense set of compact objects.

Duality for finite-limit theories (Gabriel-Ulmer duality[4])

Theorem

There is a contravariant bi-equivalence of 2-categories

$$\mathbf{FL} \quad \xrightleftharpoons[\mathcal{L} \mapsto \mathrm{Mod}(\mathcal{L}) := \mathbf{FL}(\mathcal{L}, \mathrm{Set})]{\{\text{compact objects}\}^{\mathrm{op}} \leftarrow\!\shortmid \mathfrak{X}} \quad \mathbf{LFP}^{\mathrm{op}}.$$

between the 2-category **FL** of small finite-limit theories, and the 2-category **LFP** of locally finitely presentable categories.

- Categories are representable by a finite-limit theory since Cat is locally finitely presentable.

[4] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Springer-Verlag, 1971.

Generalized algebraic theories (GATs)

GATs generalize many-sorted algebraic theories by introducing **sort dependency**. Best explained with an example:

The GAT $\mathbb{T}_{\mathsf{Cat}}$ *of categories*

$$
\begin{aligned}
&\vdash O \\
x\, y : O &\vdash A(x, y) \\
x : O &\vdash \mathrm{id}(x) : A(x, x) \\
x\, y\, z : O\,,\ f : A(x, y)\,,\ g : A(y, z) &\vdash g \circ f : A(x, z) \\
x\, y : O\,,\ f : A(x, y) &\vdash \mathrm{id}(y) \circ f = f \\
x\, y : O\,,\ f : A(x, y) &\vdash f \circ \mathrm{id}(x) = f \\
w\, x\, y\, z : O\,,\ e : A(w, x)\,,\ f : A(x, y)\,,\ g : A(y, z) &\vdash (g \circ f) \circ e = g \circ (f \circ e)
\end{aligned}
$$

GATs vs finite-limit theories, clans

GATs (and ess. alg. theories) are equally expressive as finite-limit theories w.r.t. models in Set:

- For every GAT $\mathbb{T}$, the category $\mathsf{Mod}(\mathbb{T})$ is locally finitely presentable, and
- For every locally finitely presentable category $\mathfrak{X}$ there exists a GAT $\mathbb{T}$ with $\mathsf{Mod}(\mathbb{T}) \cong \mathfrak{X}$

However there is a mismatch, since the syntactic category (category of contexts) of a GAT is generally not a finite-limit category, but only a clan!

Definition

A **clan** is a small category $\mathcal{T}$ with terminal object 1, equipped with a class $\mathcal{T}_\dagger \subseteq \mathsf{mor}(\mathcal{T})$ of morphisms – called **display maps** and written $\twoheadrightarrow$ – such that

1. pullbacks of display maps along all maps exist and are display maps
$$\begin{tikzcd} \Delta^+ \arrow[r, "s^+"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma^+ \arrow[d, two heads, "p"] \\ \Delta \arrow[r, "s"] & \Gamma \end{tikzcd},$$
2. display maps are closed under composition, and
3. isomorphisms and terminal projections $\Gamma \twoheadrightarrow 1$ are display maps.

- Definition due to Taylor[5], name due to Joyal[6]

[5] P. Taylor. “Recursive domains, indexed category theory and polymorphism”. PhD thesis. University of Cambridge, 1987, § 4.3.2.

[6] A. Joyal. “Notes on clans and tribes”. In: *arXiv preprint arXiv:1710.10238* (2017).

Examples

- Syntactic category $\mathcal{C}[\mathbb{T}]$ of a GAT $\mathbb{T}$ is a clan:
  - Objects: type-theoretic contexts
  - Morphisms: substitutions (modulo definitional equality)
  - Terminal object empty context
  - Display maps: context projections $(\Gamma, \Delta) \to \Gamma$
- Finite-product theories $\mathcal{C}$ can be viewed as clans with $\mathcal{C}_\dagger = \{\text{product projections}\}$ (*'FP-clans'*)
- Finite-limit theories $\mathcal{L}$ can be viewed as clans with $\mathcal{L}_\dagger = \mathrm{mor}(\mathcal{L})$ (*'FL-clans'*)

Models

> *Definition*
>
> A **model** of a clan $\mathcal{T}$ is a functor $A : \mathcal{T} \to \mathsf{Set}$ which preserves 1 and pullbacks of display-maps.

- The category $\mathsf{Mod}(\mathcal{T}) \subseteq [\mathcal{T}, \mathsf{Set}]$ of models is l.f.p. and contains $\mathcal{T}^{\mathsf{op}}$.
- For FP-clans $(\mathcal{C}, \mathcal{C}_\dagger)$ we have $\mathsf{Mod}(\mathcal{C}, \mathcal{C}_\dagger) = \mathbf{FP}(\mathcal{C}, \mathsf{Set})$.
- For FL-clans $(\mathcal{L}, \mathcal{L}_\dagger)$ we have $\mathsf{Mod}(\mathcal{L}, \mathcal{L}_\dagger) = \mathbf{FL}(\mathcal{L}, \mathsf{Set})$.

$$\begin{tikzcd}
 & \mathcal{T}^{\mathsf{op}} \arrow[dl, dashed] \arrow[d, "\text{よ}"] \\
\mathsf{Mod}(\mathcal{T}) \arrow[r, phantom, "\subseteq"] & {[\mathcal{T}, \mathsf{Set}]}
\end{tikzcd}$$

The clan of categories

- The syntactic category $\mathcal{C}[\mathbb{T}_{\mathsf{Cat}}]$ of the GAT $\mathbb{T}_{\mathsf{Cat}}$ has contexts

$$(x_1 \ldots x_n : O, f_1 : A(x_{i_1}, x_{j_1}), \ldots f_k : A(x_{i_k}, x_{j_k}))$$

  as objects, and substitutions as morphisms.
- As for any clan, we have the Yoneda embedding

$$よ \ : \mathcal{C}[\mathbb{T}_{\mathsf{Cat}}]^{\mathsf{op}} \longrightarrow \mathsf{Mod}(\mathcal{C}[\mathbb{T}_{\mathsf{Cat}}]) \simeq \mathsf{Cat}.$$

- Its image is the full subcategory of Cat on **free categories on finite graphs**
- Display maps correspond (contravariantly) to **graph inclusions**

Towards duality for clans

- Note that the different clans can have the same category of Set-models
- For example, algebraic theories give rise to clans either as finite-product theories or as finite-limit theories
- To get a duality theory for clans, have to **refine** Gabriel–Ulmer duality.
- We do this by equipping the categories of models with additional data in form of a **weak factorization system**

The extension–full weak factorization system

Definition

Let $\mathcal{T}$ be a clan and よ $: \mathcal{T}^{\mathrm{op}} \to \mathrm{Mod}(\mathcal{T})$. Define w.f.s. $(\mathcal{E}, \mathcal{F})$ on $\mathrm{Mod}(\mathcal{T})$:

$$\mathcal{F} = \mathbf{RLP}(よ(\mathcal{T}^\dagger)) \qquad \textbf{‘full maps’}$$

$$\mathcal{E} = \mathbf{LLP}(\mathcal{F}) \qquad \textbf{‘extensions’}$$

Call $A \in \mathrm{Mod}(\mathcal{T})$ a 0-**extension**, if $(0 \to A) \in \mathcal{E}$.

- Representable models よ$(\Gamma) = \mathcal{T}(\Gamma, -)$ are 0-extensions since all $\Gamma \twoheadrightarrow 1$ are display maps.
- The same weak factorization system was also introduced by S. Henry[7].

[7] S. Henry. “Algebraic models of homotopy types and the homotopy hypothesis”. In: *arXiv preprint arXiv:1609.04622* (2016).

Examples

- If $\mathcal{T}$ is a FL-clan, then
    - only isos are full in $\mathsf{Mod}(\mathcal{T})$, and
    - all maps are extensions.
- If $\mathcal{T}$ is a FP-clan, then
    - $\mathsf{Mod}(\mathcal{T})$ is Barr-exact,
    - the full maps are the **regular epis**, and
    - the 0-extensions are the **projective objects**.
- In $\mathsf{Cat} = \mathsf{Mod}(\mathbb{T}_{\mathsf{Cat}})$:
    - full maps are functors that are **full and surjective on objects**,
    - and 0-extensions are **free categories**.

Duality for clans

Theorem

There is a contravariant bi-equivalence of 2-categories

$$\mathbf{Clan}_{\mathrm{cc}} \underset{\mathcal{T} \mapsto \mathrm{Mod}(\mathcal{T})}{\overset{\mathrm{CZE}(\mathfrak{X})^{\mathrm{op}} \hookleftarrow \mathfrak{X}}{\longleftrightarrow}} \mathbf{cAlg}^{\mathrm{op}}$$

where

- $\mathbf{Clan}_{\mathrm{cc}}$ is the 2-category of **Cauchy-complete**[8] clans,
- $\mathbf{cAlg}$ is the 2-category of **clan-algebraic categories**, i.e. l.f.p. categories $\mathfrak{X}$ equipped with an 'extension/full' WFS $(\mathcal{E}, \mathcal{F})$ such that
  1. the **full subcategory** $\mathrm{CZE}(\mathfrak{X}) \subseteq \mathfrak{X}$ **on compact** 0**-extensions** is dense in $\mathfrak{X}$,
  2. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps in $\mathrm{CZE}(\mathfrak{X})$, and
  3. $\mathfrak{X}$ has **full and effective quotients of componentwise-full equivalence relations**.

As special cases for FL-clans and FP-clans we recover

- Gabriel–Ulmer duality, and
- Adamek–Rosicky–Vitale's characterization of **algebraic categories** as Barr-exact LFP categories which are generated by compact projectives[9].

[8] A clan $\mathcal{T}$ is Cauchy-complete if idempotents split in $\mathcal{T}$, and retracts of display maps are display maps.

[9] Theorem 9.15 in J. Adámek, J. Rosický, and E.M. Vitale. *Algebraic theories: a categorical introduction to general algebra*. Cambridge University Press, 2010

Proof sketch

- Have to show that:
  1. $\mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}$ is a clan for all clan-algebraic categories $\mathfrak{X}$ (with extensions as display maps).
  2. $\mathsf{Mod}(\mathcal{T})$ is clan-algebraic for all clans $\mathcal{T}$.
  3. $\mathsf{CZE}(\mathfrak{X})^{\mathsf{op}}\text{-}\mathsf{Mod} \simeq \mathfrak{X}$ for all clan-algebraic categories $\mathfrak{X}$.
  4. $\mathcal{T} \simeq \mathsf{CZE}(\mathsf{Mod}(\mathcal{T}))^{\mathsf{op}}$ for all Cauchy-complete clans $\mathcal{T}$.
- 1 and 2 are easy
- For 3 we use a Reedy factorization on 2-truncated semi-simplicial models
- For 4 we use the **fat small object argument**[10], which implies that 0-extensions are filtered colimits of representable algebras.

[10] M. Makkai, J. Rosicky, and L. Vokrinek. “On a fat small object argument”. In: *Advances in Mathematics* (2014).

Models in Higher Types

Models in higher types

- The following is known to experts:
- Let $\mathbb{T}$ be an algebraic theory (e.g. monoids), let $\mathcal{C}[\mathbb{T}]$ and $\mathcal{L}[\mathbb{T}]$ be the associated finite-product theory and finite-limit theory, and let $\mathcal{S}$ be the ∞-category of spaces / homotopy types. Then
  - $\mathsf{FL}(\mathcal{L}[\mathbb{T}], \mathcal{S}) \simeq \mathsf{FL}(\mathcal{L}[\mathbb{T}], \mathsf{Set})$ (since FL-functors preserve truncation levels), but
  - $\mathsf{FP}(\mathcal{C}[\mathbb{T}], \mathcal{S}) \supsetneq \mathsf{FP}(\mathcal{C}[\mathbb{T}], \mathsf{Set})$ — e.g. $\mathsf{FP}(\mathcal{C}[\mathbb{T}_{\mathsf{Mon}}], \mathcal{S})$ is the ∞-category of A_∞-algebras.
- In[11], Cesnavicius and Scholze refer to higher models of a Lawvere theory as 'animated models'.
- Moral: By being 'slimmer', finite-product theories leave room for higher coherences when interpreted in higher types.
- With clans, we can **interpolate** between FP-theories and FL-theories, and thus define higher models of varying levels of strictness for the same classical algebraic structure.

[11] K. Cesnavicius and P. Scholze. "Purity for flat cohomology". In: *arXiv preprint arXiv:1912.10932* (2019).

Four clan-algebraic weak factorization systems on **Cat**

Cat admits several clan-algebraic weak factorization systems:

- $(\mathcal{E}_1, \mathcal{F}_1)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2})\}$
- $(\mathcal{E}_2, \mathcal{F}_2)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (2 \to 1)\}$
- $(\mathcal{E}_3, \mathcal{F}_3)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2})\}$
- $(\mathcal{E}_4, \mathcal{F}_4)$ is cofib. generated by $\{(0 \to 1), (2 \to \mathbb{2}), (\mathbb{P} \to \mathbb{2}), (2 \to 1)\}$

where $\mathbb{P} = (\bullet \rightrightarrows \bullet)$.

The right classes are:

$$\mathcal{F}_1 = \{\text{full and surjective-on-objects functors}\}$$
$$\mathcal{F}_2 = \{\text{full and bijective-on-objects functors}\}$$
$$\mathcal{F}_3 = \{\text{fully faithful and surjective-on-objects functors}\}$$
$$\mathcal{F}_4 = \{\text{isos}\}$$

Note that $\mathcal{F}_3$ is the class of trivial fibrations for the canonical model structure on **Cat**.

Four clans for categories

These correspond to the following clans:

$$\mathcal{T}_1 = \{\text{free cats on fin. graphs}\}^{\mathsf{op}} \qquad \mathcal{T}_1^\dagger = \{\text{graph inclusions}\}$$

$$\mathcal{T}_2 = \{\text{free cats on fin. graphs}\}^{\mathsf{op}} \qquad \mathcal{T}_2^\dagger = \{\text{injective-on-edges maps}\}$$

$$\mathcal{T}_3 = \{\text{f.p. cats}\}^{\mathsf{op}} \qquad \mathcal{T}_3^\dagger = \{\text{injective-on-objects functors}\}$$

$$\mathcal{T}_4 = \{\text{f.p. cats}\}^{\mathsf{op}} \qquad \mathcal{T}_4^\dagger = \{\text{all functors}\}$$

Syntax: four GATs for categories

- Syntactically, adding $(2 \to 1)$ to the generators turns the diagonal of the type $\vdash\ O$ of objects into a display map. This corresponds to adding an extensional identity type with rules

  - $x\,y : O \vdash E(x,y)$ type
  - $x : O \vdash r : E(x,x)$
  - $x\,y : O\,,\ p : E(x,y) \vdash x = y$
  - $x\,y : O\,,\ p\,q : E(x,y) \vdash p = q$

  to the GAT.
- Similarly, adding $(\mathbb{P} \to \mathbb{2})$ corresponds to adding an extensional identity type with rules

  - $x\,y : O\,,\ f\,g : A(x,y) \vdash F(f,g)$ type
  - $x\,y : O\,,\ f : A(x,y) \vdash s : F(f,f)$
  - $x\,y : O\,,\ f\,g : A(x,y)\,,\ p : F(f,g) \vdash f = g$
  - $x\,y : O\,,\ f\,g : A(x,y)\,,\ p\,q : F(f,g) \vdash p = q$

  to the dependent type $x\,y : O \vdash A(x,y)$ of arrows.

Models in higher types

Models of $\mathcal{T}_1$ in $\mathcal{S}$ are **Segal spaces**, and adding extensional identity types to $\vdash\ O$ or to $x\,y : O\ \vdash\ A(x,y)$ forces the respective types to be 0-truncated. Thus:

$$\infty\text{-}\mathrm{Mod}(\mathcal{T}_1) = \{\text{Segal spaces}\}$$
$$\infty\text{-}\mathrm{Mod}(\mathcal{T}_2) = \{\text{Segal categories}\}$$
$$\infty\text{-}\mathrm{Mod}(\mathcal{T}_3) = \{\text{pre-categories}\}$$
$$\infty\text{-}\mathrm{Mod}(\mathcal{T}_4) = \{\text{discrete 1-categories}\}$$

Comparison with Benedikt's talk

- Similarity: look at higher models of set-sevel theories
- Clans are more abstract than the FOLDS-theories that Benedikt mentioned.
- Missing structure of type dependency comes back through FSOA, which in particular requires clans to be strict categories

Thank you for your attention!