The shape of contexts

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Context

- Recent work on clans, see
  - https://arxiv.org/abs/2308.11967
  - https://www.youtube.com/watch?v=0SAQSx0vPmU (HoTTEST seminar talk)

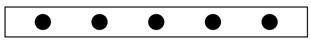
  Clans arise as syntactic categories of *generalized algebraic theories* (GATs) [Car86], e.g. the theory of categories:

$$\begin{array}{rl}
 & \vdash\ O \\
x\,y : O & \vdash\ A(x,y) \\
x : O & \vdash\ \mathrm{id}(x) : A(x,x) \\
x\,y\,z : O\,,\ f : A(x,y)\,,\ g : A(y,z) & \vdash\ g{\circ}f : A(x,z) \\
x\,y : O\,,\ f : A(x,y) & \vdash\ \mathrm{id}(y){\circ}f = f \\
x\,y : O\,,\ f : A(x,y) & \vdash\ f{\circ}\mathrm{id}(x) = f \\
w\,x\,y\,z : O\,,\ e : A(w,x)\,,\ f : A(x,y)\,,\ g : A(y,z) & \vdash\ (g{\circ}f){\circ}e = g{\circ}(f{\circ}e)
\end{array}$$

- Discussions with Mathieu Anel, Carlo Angiuli, and Chaitanya Leena Subramaniam

Motivation

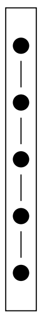
Contexts in simple type theory / STLC can be viewed as 'flat', since the variable declarations don't depend on each other and can be permuted:



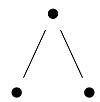
In dependent type theory, the general form of contexts is

$$(x_1 : A_1, x_2 : A_2(x_1), \ldots, x_n : A_n(x_1, \ldots, x_{n-1}))$$

with a linear dependency chain between the variable declarations:



However, the specific contexts occuring in practice are not always linearly ordered. For example, the context $(x\,y : O\,,\, f : A(x,y))$ of the id-equations of the GAT above has the following shape.



So maybe we should allow finite posets as shapes of contexts? It turns out to be better to use an even more general class of structures: *finite direct categories*.

FDCs

Definition 1 (1) A category $\mathbb{C}$ is called *direct*, if there are no infinite inverse paths

$$A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \dots$$

of non-identity arrows.

(2) A category is called *inverse*, if its opposite category is direct.

(3) A category is called *one-way*, if the only endomorphisms are identities. $\diamondsuit$

Lemma 2 • *Direct categories are one-way and skeletal.*

- *A finite category is direct iff it is one-way and skeletal.*

Remark 3 According to [Mak95]:

> One-way categories were isolated by F. W. Lawvere in [Law91] [...] Lawvere observed that one-way categories are intimately related to the sketch-based syntax of [Mak97].

Shulman [Shu15] connected inverse categories with Homotopy Type Theory. $\diamondsuit$

4 Analogy. Consider the syntactic category $\mathcal{C}[\mathbb{T}]$ of a many-sorted algebraic theory $\mathbb{T}$. Its objects are contexts / lists of sorts, and a morphism from $(S_1 \dots S_n)$ to $(T_1 \dots T_k)$ is a k-tuple of terms of sorts T_i, with variables of types S_j. The category $\mathcal{C}[\mathbb{T}]$ has important subclasses of arrows:

- projections $(\vec{S}, \vec{T}) \to (\vec{S})$ (corresponding to display maps in clans), and
- *variable renamings*, generated by projections and diagonals $(\vec{S}, T) \to (\vec{S}, T, T)$.

The wide subcategory $\mathcal{C}_v[\mathbb{T}] \subseteq \mathcal{C}[\mathbb{T}]$ of variable renamings (which depends only on the sorts of $\mathbb{T}$) admits a forgetful functor $\mathcal{C}_v[\mathbb{T}]^{\mathsf{op}} \to \mathsf{Fin}$ which is a discrete fibration. Intuition:

- finite sets: shape of simply type contexts
- function between finite sets: (opposite to) shape of variable renamings. $\diamondsuit$

We propose to extend this dictionary to dependently typed contexts as follows:

- FDCs: shapes of dependently typed contexts

- discrete fibrations: correspond contravariantly to variable renamings between dependent contexts
- *projections* correspond to a subclass of discrete fibrations: the *sieve inclusions*.

Definition 5 FDC is the category of FDCs and discrete fibrations. $\diamondsuit$

Proposition 6 *Consider a cospan $D \overset{u}{\hookleftarrow} C \overset{f}{\to} E$ in $\mathbb{Q}$, where u is a sieve inclusion. Then the pushout*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, hook, "u"'] & E \arrow[d, hook, "v"] \\ D \arrow[r, "g"] & F \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

exists in $\mathbb{Q}$ and is also a pullback, v is a sieve inclusion, and the pushout and pullback is preserved by $\mathbb{Q} \hookrightarrow \mathsf{Cat}$. (Compare [Joh02, A2.4.3])

Proof. We know that discrete fibrations admit surjection/injection factorizations, and that pushouts along injections exist, so it's sufficient to consider the case where f is surjective.

Form the pushout

$$\begin{tikzcd} C_0 \arrow[r, two heads, "f"] \arrow[d, hook, "u"'] & E_0 \arrow[d, hook, "v"] \\ D_0 \arrow[r, two heads, "g"] & F_0 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

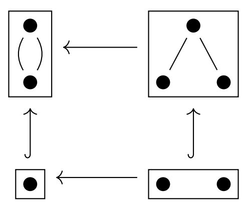
of underlying sets and define $F(e,d) = \bigcup_{fc=e} D(c,d)$ whenever e is in the domain of v and d is not. $\blacksquare$

Remark 7 FDC has coproducts but not all coequalizers: e.g. he two maps $\mathbb{2} \to \mathbb{P}$ don't have a coequalizer. Thus, FDC can't have all pushouts either. However, pushouts of sieve inclusions along arbitrary maps *do exist*, making FDC into a *coclan*. $\diamondsuit$

Example 8 Consider the following pullback of contexts and renamings in the syntactic category of the GAT above:

$$\begin{tikzcd} (x : O\,,\ f : A(x,x)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (x : O) \arrow[d] \\ (x\,y : O\,,\ f : A(x,y)) \arrow[r] & (x\,y : O) \end{tikzcd}$$

This corresponds contravariantly to the *pushout*



in the category FDC of of FDCs and discrete fibrations. $\diamondsuit$

Idea: Can we can we specify GATs (or something suitably equivalent) by first specifying the structure of sorts, and only then the operations and equations? The system of types of a DAT should have the following structure:

- For every FDC $\mathbb{F}$ there is a set $T(\mathbb{F})$ of 'contexts of shape $\mathbb{F}$'
- contexts should vary contravariantly over FDCs
- pushouts of sieve inclusions should give pullbacks of sets of contexts.
- there is a unique context of shape $\varnothing$

Definition 9 A *context structrue* is a functor $C : \mathsf{FDC}^{\mathsf{op}} \to \mathsf{Set}$ with the properties stated above. In other words, a context structure is a model of the coclan FDC. $\diamondsuit$

Next analogy: a Lawvere theory is a monoid in $[\mathsf{Fin}, \mathsf{Set}]$ with the *substitution tensor product*. More generally, an S-sorted algebra algebraic theory can be represented as a profunctor $\mathsf{Fin}^S \times S \to \mathsf{Set}$ which with a monad structure w.r.t. a suitable tensor product [FGHW08, FGHW18]. Here, Fin^I is the category of I-contexts. Such functors are equivalent to functors $\mathsf{Fin}^I \to \mathsf{Fin}^I \to \mathsf{Set}$ which preserve finite products in the second argument, and under this correspondence the substitution tensor product becomes ordinary profunctor composition.

We try to generalize this as follows: given a context structure $C : \mathsf{FDC}^{\mathsf{op}} \to \mathsf{Set}$, set $\mathbb{C} = \mathsf{elts}(C)^{\mathsf{op}}$. This is the category of contexts and renamings associated to the context structure $\mathbb{C}$, and it is a clan. Now a C-theory should be something like a monad $T : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathsf{Set}$ in profunctors such that that $T(X, -)$ is a clan model for each X.

Chaitanya raised an issue with this. If T has the clan model property, does $T \circ T$ also have the clan model property? This is not clear yet, maybe we have the wrong condition.

DLFCs and LFDCs

Definition 10 A category is called *locally finite*, or said to have *finite fan-out*, if $\mathbb{C}/C$ is finite for all $\mathbb{C}$. $\diamondsuit$

Definition 11 LFDC is the category of direct locally finite categories and *Street fibrations*, modulo (necessarily unique) iso. DLFC is its full subcategory of direct categories. $\diamondsuit$

Example 12 The full subcategory FDC_0 of FDC on FDCs with terminal object is LFD but not DLF. LFDC itself is $\diamondsuit$

Proposition 13 *We have* $\mathsf{Mod}(\mathsf{FDC}) = \mathsf{DLFC}$*, the full subcategory of* 0*-extensions is* LFDC.

Proof. The DLFC corresponding to $C : \mathsf{FDC}^{\mathsf{op}} \to \mathsf{Set}$ is given by $\mathsf{elts}(C_0)$, where C_0 is the restriction of C to FDC_0 — the full subcategory of FDC on FDCs with a terminal object.

Conversely, the context structure corresponding to a DLFC $\mathbb{D}$ is given by $\mathsf{DLFC}(-, \mathbb{D})$. ∎

Thus, the category DLFC is l.f.p., in particular it is complete and cocomplete. However, colimits in DLFC are not preserved by $\mathsf{DLFC} \to \mathsf{Cat}$ (if this functor is even well defined). Counterexample: coequalizer of $\mathbb{P} \rightrightarrows \mathbb{P}$ is the index category of 'dagger graphs'.

What's the terminal object? It's FDC_0! Every dlfc $\mathbb{D}$ admits a canonical map to FDC_0, defined by $d \mapsto \mathbb{D}/d$. Conversely, the elements of every presheaf over FDC_0 form a DLFC $\Rightarrow$ LFDC is a presheaf category !

Analogy with Chaitanya's thesis: for DLFCs $\mathbb{D}$, Chaitanya [LS21] considers $\mathbb{D}$-*sorted theories*, which are finitary monads

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