Duality for generalized algebraic theories

Jonas Frey

January 9, 2023

Abstract

We exhibit an idempotent biadjunction between a 2-category of small clans and a 2-category of locally finitely presentable categories equipped with a weak factorization system, and characterize the stable subcategories.

1 Clans

Definition 1.1 A *clan* is a category $\mathcal{T}$ with a distinguished class $\mathcal{T}_\dagger$ of arrows called *display maps*, such that:

1. Arbitrary pullbacks of display maps exist and are again display maps.
2. Isomorphisms and compositions of display maps are display maps.
3. $\mathcal{T}$ has a terminal object, and terminal projections are display maps.

A *clan morphism* is a functor between clans which preserves display maps, pullbacks of display maps, and the terminal object. We write Clan for the 2-category of clans, clan-morphisms, and natural transformations. $\diamondsuit$

Remarks 1.2 1. Clans can be viewed as 'non-strict' version of Cartmell's *contextual categories* [Car78, Car86].

2. The above definition and the term 'display map' are due to Taylor [Tay87, §4.3.2], the name 'clan' was suggested by Joyal [Joy17, Definition 1.1.1].

Examples 1.3 1. Finite-product categories can be viewed as clans where the display maps are the morphisms that are (isomorphic to) product projections. We call such clans *finite-product clans.*

2. Finite-limit categories can be viewed as clans where *all* morphimsms are display maps. We call such clans *finite-limit clans.*

3. The base category of every *category with attributes* in the original sense of Cartmell [Car78, Section 3.2] is a clan with display maps the arrows that are isomorphic to projection maps $\rho_B : \Sigma B \to A$.

4. **Kan** is the clan whose underlying category is the full subcategory of the category **SSet** of simplicial sets on *Kan complexes*, and whose display maps are the *Kan fibrations.*

Since it seems to lead to a more readable exposition, we introduce explicit notation and terminology for the dual notion.

Definition 1.4 A *coclan* is a category $\mathcal{C}$ with a distinguished class $\mathcal{C}_\dagger$ of arrows called *codisplay maps* satisfying the dual axioms of clans. The 2-category CoClan of coclans is defined dually to that of clans, i.e.

$$\mathsf{CoClan}(\mathcal{C}, \mathcal{D}) = \mathsf{Clan}(\mathcal{C}^{\mathsf{op}}, \mathcal{D}^{\mathsf{op}})^{\mathsf{op}}$$

for coclans $\mathcal{C}, \mathcal{D}$. $\diamondsuit$

2 Algebras

In the spirit of functorial semantics we can think of clan morphisms $A : \mathcal{S} \to \mathcal{T}$ as '$\mathcal{S}$-algebras in $\mathcal{T}$', which makes sense in particular when $\mathcal{S}$ is a small clan (e.g. the syntactic category of a generalized algebraic theory), and $\mathcal{T}$ is a large clan like a topos, or the clan **Kan** from Example 1.3-4. Of particular interest in this work are algebras in Set, for which we introduce a special notation.

Definition 2.1 If $\mathcal{T}$ is a small clan, we write $\mathcal{T}$-Alg for the category $\mathsf{Clan}(\mathcal{T}, \mathsf{Set})$ of clan morphism from $\mathcal{T}$ to Set with the finite-limit clan structure, and refer to its objects simply as '$\mathcal{T}$-algebras'. $\diamondsuit$

Remarks 2.2
1. As category of models of a finite-limit sketch, $\mathcal{T}$-Alg is reflective (and therefore closed under arbitrary limits) in $[\mathcal{T}, \mathsf{Set}]$, and moreover it is closed under filtered colimits [AR94, Section 1.C]. In particular, $\mathcal{T}$-Alg is locally finitely presentable.

2. The hom-functors $\mathcal{T}(\Gamma, -) : \mathcal{T} \to \mathsf{Set}$ are $\mathcal{T}$-algebras for all $\Gamma \in \mathcal{T}$ (we'll refer to them as hom-*algebras*), i.e. the Yoneda embedding $よ : \mathcal{T}^{\mathsf{op}} \to [\mathcal{T}, \mathsf{Set}]$ lifts along the inclusion $\mathcal{T}$-$\mathsf{Alg} \hookrightarrow [\mathcal{T}, \mathsf{Set}]$ to a fully faithful functor $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}$-$\mathsf{Alg}$.

$$\begin{tikzcd} & \mathcal{T}\text{-}\mathsf{Alg} \arrow[d, hook] \\ \mathcal{T}^{\mathsf{op}} \arrow[ur, dashed, "H"] \arrow[r, "よ"'] & {[\mathcal{T}, \mathsf{Set}]} \end{tikzcd}$$

3. For $\Gamma \in \mathcal{T}$, the hom-functor

$$\mathcal{T}\text{-}\mathsf{Alg}(H(\Gamma), -) : \mathcal{T}\text{-}\mathsf{Alg} \to \mathsf{Set}$$

is isomorphic to the evaluation functor $A \mapsto A(\Gamma)$, hence it preserves filtered colimits as those are computed in $[\mathcal{T}, \mathsf{Set}]$ and therefore pointwise. This means that $H(\Gamma)$ is *compact*[1] in $\mathcal{T}$-Alg.

[1]Following Lurie [Lur09] we use the shorter term 'compact' instead of the more traditional 'finitely presented' for objects whose covariant hom-functor preserves filtered colimits.

2.1 Flat algebras

Convention 2.3 One finds opposing definitions for the *category of elements* of a Set-valued functor in the literature, depending on whether the functors at hand are covariant or contravariant. Since in this work we don't want to regard variance as an intrinsic property of a functor (a covariant functor out of a clan is the same thing as a contravariant functor out of a coclan), and we don't want to take sides as to what is the 'correct' convention, we write $\underrightarrow{\mathsf{elts}}(F)$ for the *covariant category of elements* of a Set-valued functor $F : \mathbb{C} \to \mathsf{Set}$ on a small category $\mathbb{C}$, and $\underleftarrow{\mathsf{elts}}(F)$ for the *contravariant category of elements.* Note that we have $\underleftarrow{\mathsf{elts}}(F) = \underrightarrow{\mathsf{elts}}(F)^{\mathsf{op}}$, and the forgetful functors $\underrightarrow{\mathsf{elts}}(F) \to \mathbb{C}$ and $\underleftarrow{\mathsf{elts}}(F) \to \mathbb{C}^{\mathsf{op}}$ are discrete opfibrations and discrete fibrations, respectively.

Recall that for small $\mathbb{C}$, a functor $F : \mathbb{C} \to \mathsf{Set}$ is called *flat* if $\underleftarrow{\mathsf{elts}}(F)$ is filtered, or equivalently if the left Kan extension $F_! : [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}] \to \mathsf{Set}$ of F along $よ : \mathbb{C} \to [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$ preserves finite limits [Bor94, Definition 6.3.1 and Proposition 6.3.8]. From the second characterization it follows that flat functors preserve all finite limits that exist in $\mathbb{C}$, thus for the case of a small clan $\mathcal{T}$, flat functors $F : \mathcal{T} \to \mathsf{Set}$ are always algebras. We refer to them as *flat* algebras.

Lemma 2.4 *An algebra A over a small clan $\mathcal{T}$ is flat iff it is a filtered colimit of* hom*-algebras.*

Proof. We always have $A = \mathsf{colim}(\underleftarrow{\mathsf{elts}}(A) \to \mathcal{T}^{\mathsf{op}} \xrightarrow{よ} \mathcal{T}\text{-}\mathsf{Alg})$, thus if F is flat then it is a filtered colimit of hom-algebras. The other direction follows since hom-algebras are flat, and flat functors are closed under filtered colimits in $[\mathcal{T}, \mathsf{Set}]$ [Bor94, Proposition 6.3.6]. $\square$

2.2 The weak factorization system on algebras

We call a map $f : A \to B$ of algebras *full* if it has the r.l.p. with respect to all maps $H(p)$ for p a display map, i.e. if the naturality square

$$\begin{array}{ccc} A(\Gamma) & \xrightarrow{A(p)} & A(\Delta) \\ \downarrow{\scriptstyle f_\Gamma} & & \downarrow{\scriptstyle f_\Delta} \\ B(\Gamma) & \xrightarrow{B(p)} & B(\Delta) \end{array}$$

is a weak pullback for each display map $p : \Gamma \to \Delta$. By the small object argument, the full maps form the right class of a cofibrantly generated WFS

$$(\mathcal{E}, \mathcal{F})$$

on $\mathcal{T}\text{-}\mathsf{Alg}$ whose left maps we call *extensions.*

We call $A \in \mathcal{T}\text{-}\mathsf{Alg}$ a *0-extension*, if $(0 \to A) \in \mathcal{E}$. In particular, all objects $H(\Gamma)$ are 0-extensions, since all terminal projections in $\mathcal{T}$ are display maps and Z sends terminal projections to initial inclusions.

3 The universal property of $\mathcal{T}$-Alg

It is well known that for $\mathbb{C}$ a small category, the presheaf category $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathsf{op}}, \mathsf{Set}]$ is the *cocompletion* of $\mathbb{C}$, in the sense that for every cocomplete category $\mathfrak{X}$, precomposition with the Yoneda embedding $よ : \mathbb{C} \to \widehat{\mathbb{C}}$ induces an equivalence

$$\mathsf{CoCont}(\widehat{\mathbb{C}}, \mathfrak{X}) \xrightarrow{\simeq} [\mathbb{C}, \mathfrak{X}] \tag{3.1}$$

between the category of functors $F : \mathbb{C} \to \mathfrak{X}$ and the category of cocontinuous functors $F_! : \widehat{\mathbb{C}} \to \mathfrak{X}$. Specifically, the cocontinuous functor $F_! : \widehat{\mathbb{C}} \to \mathfrak{X}$ corresponding to a functor $F : \mathbb{C} \to \mathfrak{X}$ is the left Kan extension of F along $よ : \mathbb{C} \to \widehat{\mathbb{C}}$ and can be written as $F_!(A) = \mathsf{colim}(\underleftarrow{\mathsf{elts}}(A) \to \mathbb{C} \xrightarrow{F} \mathfrak{X})$. If $\mathfrak{X}$ is locally small then $F_!$ has a right adjoint $F^* : \mathfrak{X} \to \widehat{\mathbb{C}}$ given by $F^*(X) = \mathfrak{X}(F(-), X)$. We call F^* and $F_!$ the *nerve* and *realization* functors of F, respectively, and $F_! \dashv F^*$ the nerve–realization adjunction.

Theorem 3.1 (The universal property of $\mathcal{T}$-Alg) *Let $\mathcal{T}$ be a small clan.*

1. *The co-restricted Yoneda embedding $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Alg}$ is a coclan morphism.*
2. *For every cocomplete and locally small category $\mathfrak{X}$ and coclan morphism $F : \mathcal{T}^{\mathsf{op}} \to \mathfrak{X}$, the restriction of the realization functor $F_! : [\mathbb{C}, \mathsf{Set}] \to \mathfrak{X}$ to $\mathcal{T}\text{-}\mathsf{Alg}$ — which is also the Kan extension of F along $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Alg}$ — is cocontinuous, and the nerve functor $F^* : \mathfrak{X} \to [\mathbb{C}, \mathsf{Set}]$ factors through $\mathcal{T}\text{-}\mathsf{Alg}$.*

In particular, the equivalence (3.1) *'restricts' to an equivalence*

$$\mathsf{CoCont}(\mathcal{T}\text{-}\mathsf{Alg}, \mathfrak{X}) \xrightarrow{\simeq} \mathsf{CoClan}(\mathcal{T}^{\mathsf{op}}, \mathfrak{X})$$

between coclan morphisms $F : \mathcal{T}^{\mathsf{op}} \to \mathfrak{X}$ and cocontinuous functors $F_! : \mathcal{T}\text{-}\mathsf{Alg} \to \mathfrak{X}$.

Proof. An analogous statement holds more generally for arbitrary small realized[2] limit sketches. As Brandenburg points out on MathOverflow [Bra], the earliest reference for this seems to be [Pul70, Theorem 2.5]. See also [Bra21] which gives a careful account of an even more general statement for non-small sketches. □

3.1 Slicing and co-slicing

Definition 3.2 For $\mathcal{T}$ a clan and $\Gamma \in \mathcal{T}$, we write $\mathcal{T}(\Gamma)$ for the full subcategory of the slice category $\mathcal{T}/\Gamma$ on display maps. Then $\mathcal{T}(\Gamma)$ is a clan where an arrow in $\mathcal{T}(\Gamma)$ is a display map if its underlying map is in $\mathcal{T}_\dagger$. Compare [Joy17, Proposition 1.1.6]. ♢

The following is easy to see.

Lemma 3.3 *For $\mathcal{T}$ a small clan and $\Gamma \in \mathcal{T}$, the functor*

$$H_\Gamma : \mathcal{T}(\Gamma)^{\mathsf{op}} \to H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg} \tag{3.2}$$

which sends $d : \Delta \twoheadrightarrow \Gamma$ to $H(d) : H(\Gamma) \rightarrowtail H(\Delta)$ is a coclan morphism. □

Proposition 3.4 *For any small clan $\mathcal{T}$ and object $\Gamma \in \mathcal{T}$, the categories $\mathcal{T}(\Gamma)$-Alg and $H(\Gamma)/\mathcal{T}$-Alg are equivalent by means of the nerve–realization adjunction*

$$(H_\Gamma)_! \dashv (H_\Gamma)^* \ : \ H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg} \ \to \ \mathcal{T}(\Gamma)\text{-}\mathsf{Alg}$$

induced by the functor (3.2)*, and moreover the equivalence identifies the extension/full WFS on $\mathcal{T}(\Gamma)$-Alg with the coslice WFS on $H(\Gamma)/\mathcal{T}$-Alg.*

[2] A sketch is called 'realized' if all its designated cones are limiting.

Proof. Since arrows $H(\Gamma) \to A$ correspond to elements of $A(\Gamma)$, we can identify the coslice category $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg}$ with the category of 'Γ-pointed $\mathcal{T}$-algebras', i.e. pairs (A, x) of a $\mathcal{T}$-algebra A and an element $x \in A(\Gamma)$, and morphisms preserving chosen elements.

Under this identification, we first verify that the functor $(H_\Gamma)^*$ is given by

$$(H_\Gamma)^*(A,x)(\Delta \overset{d}{\twoheadrightarrow} \Gamma) = \{y \in A(\Delta) \mid d \cdot y = x\},$$

and then that it is an equivalence with inverse $\Phi : \mathcal{T}(\Gamma)\text{-}\mathsf{Alg} \to H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg}$ given by

$$\Phi(B) = (B(- \times \Gamma \overset{\pi_2}{\twoheadrightarrow} \Gamma), \delta \cdot \star)$$

where $\star$ is the unique element of $B(\mathrm{id}_\Gamma)$ and $\delta : \Gamma \to \Gamma \times \Gamma$ is the diagonal map viewed as global element of $\pi_2 : \Gamma \times \Gamma \twoheadrightarrow \Gamma$ in $\mathcal{T}(\Gamma)$. Thus, $(H_\Gamma)_! = \Phi$.

For the second claim we first note that the coslice WFS on $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg}$ is cofibrantly generated by commutative triangles

(3.3)
$$\begin{tikzcd} H(\Gamma) \arrow[d, "H(\pi_2)"'] \arrow[dr, "H(\pi_2)"] & \\ H(\Delta \times \Gamma) \arrow[r, "H(d\times\Gamma)"'] & H(\Theta \times \Gamma) \end{tikzcd}$$

for display maps $d : \Theta \twoheadrightarrow \Gamma$ [Hir21, Theorem 2.7]. On the other hand, since $(H_\Gamma)_! \circ H = H_\Gamma$, the functor $(H_\Gamma)_!$ sends the generators of the extension/full WFS on $\mathcal{T}(\Gamma)\text{-}\mathsf{Alg}$ to triangles

(3.4)
$$\begin{tikzcd} H(\Gamma) \arrow[d, "H(e)"'] \arrow[dr, "H(f)"] & \\ H(\Delta) \arrow[r, "H(d)"'] & H(\Theta) \end{tikzcd}$$

for arbitrary display maps d, e, f in $\mathcal{T}$. Now the triangles of shape (3.4) contain the triangles of shape (3.3), but are contained in their saturation, which is the left class of the coslice WFS. Thus, the two WFSs are equal. $\square$

Proposition 3.5 *Let $\mathcal{T}$ be a small clan, and A a $\mathcal{T}$-algebra. Then the forgetful functor $U : \underrightarrow{\mathsf{elts}}(A) \to \mathcal{T}$ creates a clan structure on $\underrightarrow{\mathsf{elts}}(A)$, i.e. $\underrightarrow{\mathsf{elts}}(A)$ is a clan with display maps those arrows that are mapped to display maps in $\mathcal{T}$ by U. Moreover, the canonical functor*

$$H_A : \underrightarrow{\mathsf{elts}}(A)^{\mathsf{op}} = \underleftarrow{\mathsf{elts}}(A) \simeq \mathcal{T}^{\mathsf{op}}/A \to \mathcal{T}\text{-}\mathsf{Alg}/A$$

is a coclan morphism, and its restricted nerve–realization adjunction

$$(H_A)_! \dashv (H_A)^* : \mathcal{T}\text{-}\mathsf{Alg}/A \to \underrightarrow{\mathsf{elts}}(A)\text{-}\mathsf{Alg}$$

is an equivalence which identifies the extension/full WFS on $\underrightarrow{\mathsf{elts}}(A)\text{-}\mathsf{Alg}$ and the slice WFS on $\mathcal{T}\text{-}\mathsf{Alg}/A$.

Proof. The verification that $\underrightarrow{\mathsf{elts}}(A)$ is a clan and H_A is a coclan morphism is straightforward. The equivalence is a restriction of the well-known equivalence $\widehat{\mathcal{T}^{\mathsf{op}}}/A \simeq \widehat{\mathcal{T}^{\mathsf{op}}/A}$.

The WFSs coincide since the — again by $(H_A)_! \circ H = H_A$ — the functor $(H_A)_!$ sends the generators of the WFS on $\underrightarrow{\mathsf{elts}}(A)$ to commutative triangles

$$\begin{tikzcd} H(\Gamma) \arrow[r, "d", tail] \arrow[dr, "\hat{x}"'] & H(\Delta) \arrow[d, "\hat{y}"] \\ & A \end{tikzcd}$$

in $\mathcal{T}\text{-}\mathsf{Alg}/A$, where $d : \Delta \twoheadrightarrow \Gamma$ is a display map in $\mathcal{T}$ and $x \in A(\Gamma)$ and $y \in A(\Delta)$ are elements with $d \cdot y = x$. By [Hir21, Theorem 1.5], these form a set of generators for the slice WFS on $\mathcal{T}\text{-}\mathsf{Alg}/A$. $\square$

4 $(\mathcal{E}, \mathcal{F})$-categories

Definition 4.1 An $(\mathcal{E}, \mathcal{F})$-category is a locally finitely presentable category $\mathcal{L}$ equipped with a weak factorization system $(\mathcal{E}, \mathcal{F})$ whose maps we call *extensions* and *full maps* respectively.

A *morphism of* $(\mathcal{E}, \mathcal{F})$*-categories* from $\mathcal{L}$ to $\mathcal{M}$ is an adjunction $F_! \dashv F^*$ where

1. the *direct image part* $F_! : \mathcal{L} \to \mathcal{M}$ preserves compact objects and extensions, and

2. the *inverse image part* $F^* : \mathcal{M} \to \mathcal{L}$ preserves filtered colimits and full maps.

A 2-*cell* $\eta : F \to G$ between morphisms of $(\mathcal{E}, \mathcal{F})$-categories is a natural transformation $\eta : F^* \to G^*$ between the *inverse image parts.* We write EFCat for the 2-category of $(\mathcal{E}, \mathcal{F})$-categories. $\diamondsuit$

Remark 4.2 It follows from standard arguments that conditions 1 and 2 in Definition 4.1 are equivalent. Moreover, by the special adjoint functor theorem [Mac98, Section V-8] and the *adjoint functor theorem for presentable categories* [AR94, Theorem 1.66] respectively, the two adjoints can be reconstructed from each other. This means that a morphism from $\mathcal{L}$ to $\mathcal{M}$ of $(\mathcal{E}, \mathcal{F})$-categories is determined equivalently by

- a *cocontinuous* functor $F_! : \mathcal{L} \to \mathcal{M}$ preserving extensions and compact objects, or

- a *continuous* functor $F^* : \mathcal{M} \to \mathcal{L}$ preserving full maps and filtered colimits.

Lemma 4.3 *For any morphism $F : \mathcal{S} \to \mathcal{T}$ between small clans, the precomposition functor*

$$F^* \ : \ \mathcal{T}\text{-}\mathsf{Alg} \to \mathcal{S}\text{-}\mathsf{Alg}$$

is the image inverse part of a morphism of $(\mathcal{E}, \mathcal{F})$-categories.

Proof. The preservation of small limits and filtered colimits is obvious since they are computed pointwise (Remark 2.2-1). To show that F^* preserves full maps, let $f : A \to B$ be full in $\mathcal{T}\text{-}\mathsf{Alg}$. It is sufficient to show that the $(f \circ F)$-naturality squares are weak pullbacks at all display maps p : in $\mathcal{S}\text{-}\mathsf{Alg}$. But the $(f \circ F)$-naturality square at p is the same as the f-naturality square at Fp so the claim follows since f is full and F preserves display maps. $\square$

From Lemma 4.3 it is immediate that the assignment $\mathcal{T} \mapsto \mathcal{T}\text{-}\mathsf{Alg}$ extends to a pseudofunctor

$$(-)\text{-}\mathsf{Alg} \;:\; \mathsf{Clan}_{\mathsf{sm}} \;\to\; \mathsf{EFCat} \tag{4.1}$$

from the 2-category $\mathsf{Clan}_{\mathsf{sm}}$ of small clans to the 2-category of $(\mathcal{E}, \mathcal{F})$-categories.

Proposition 4.4 *The pseudofunctor* (4.1) *has a right biadjoint.*

Proof. Given a small clan $\mathcal{T}$ and an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$, it is easy to see that the natural equivalence

$$\mathsf{CoCont}(\mathcal{T}\text{-}\mathsf{Alg}, |\mathcal{L}|) \;\simeq\; \mathsf{CoClan}_{\mathsf{sm}}(\mathcal{T}^{\mathsf{op}}, |\mathcal{L}|)$$

from Theorem 3.1 (where $|\mathcal{L}|$ is the underlying category of $\mathcal{L}$) restricts to an equivalence

$$\mathsf{EFCat}(\mathcal{T}\text{-}\mathsf{Alg}, \mathcal{L})^{\mathsf{op}} \;\simeq\; \mathsf{CoClan}_{\mathsf{sm}}(\mathcal{T}^{\mathsf{op}}, \mathfrak{C}(\mathcal{L}))$$

where $\mathfrak{C}(\mathcal{L})$ is the coclan whose underlying category is the full subcategory of $\mathcal{L}$ on compact 0-extensions, and whose co-display maps are the extensions. Taking opposite categories on both sides we get

$$\mathsf{EFCat}(\mathcal{T}\text{-}\mathsf{Alg}, \mathcal{L}) \;\simeq\; \mathsf{Clan}_{\mathsf{sm}}(\mathcal{T}, \mathfrak{C}(\mathcal{L})^{\mathsf{op}}),$$

which shows that the 2-presheaf $\mathsf{EFCat}((-)\text{-}\mathsf{Alg}, \mathcal{L})$ is birepresented by $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$. □

In the following we will show that this biadjunction

$$(-)\text{-}\mathsf{Alg} \;\dashv\; \mathfrak{C}(-)^{\mathsf{op}} : \mathsf{EFCat} \to \mathsf{Clan}_{\mathsf{sm}}$$

between small clans and $(\mathcal{E}, \mathcal{F})$-categories is *idempotent* (in the sense that the associated monad and comonad are), and characterize the fixed-points on both sides.

5 Cauchy-complete clans and the fat small object argument

Definition 5.1 A clan $\mathcal{T}$ is called *Cauchy complete* if its underlying category is Cauchy complete (i.e. idempotents split), and retracts of display maps are display maps. ♢

Clearly, every clan of the form $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$ is Cauchy complete, thus Cauchy completeness is a necessary condition for the unit $\eta_{\mathcal{T}} : \mathcal{T} \to \mathfrak{C}(\mathcal{T}\text{-}\mathsf{Alg})^{\mathsf{op}}$ of the biadjunction to be an equivalence. We will show that it is also sufficient, but for this we need the *fat small object argument.*

Proposition 5.2 (Fat small object argument) *For any small clan $\mathcal{T}$, the 0-extensions in $\mathcal{T}$-Alg are* flat*, i.e. filtered colimits of* hom*-algebras.*

Proof. This is a special case of [MRV14, Corollary 5.1], but we give a direct proof in the appendix (Corollary A.9), which simplifies onsiderably in the case of clans. □

Definition 5.3 Let $\mathfrak{X}$ be a cocomplete locally small category.

1. We say that an arrow $f : A \to B$ is *left orthogonal* to a small diagram $D : \mathbb{J} \to \mathfrak{X}$, and write $f \perp D$, if the square

$$\begin{tikzcd}
\mathsf{colim}_{j\in\mathbb{J}}\,\mathfrak{X}(B, D_j) \arrow[r] \arrow[d] & \mathfrak{X}(B, \mathsf{colim}(D)) \arrow[d] \\
\mathsf{colim}_{j\in\mathbb{J}}\,\mathfrak{X}(A, D_j) \arrow[r] & \mathfrak{X}(A, \mathsf{colim}(D))
\end{tikzcd}$$

   is a pullback in Set.

2. We call f *compact*, if it is left orthogonal to all small filtered diagrams. $\diamondsuit$

Lemma 5.4 *Let $\mathfrak{X}$ be a locally small and cocomplete category.*

(i) *An object $A \in \mathfrak{X}$ is compact in the usual sense that $\mathfrak{X}(A, -)$ preserves filtered colimits if and only if the arrow $0 \to A$ is compact in the sense of Definition 5.3.*

(ii) *If the arrow g in a commutative triangle* $\begin{tikzcd}[row sep=small, column sep=small] A \arrow[r, "g"] \arrow[dr, "f"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$ *is compact, then f is compact if and only if h is compact. In other words, compact arrow are closed under composition and have the right cancellation property.*

(iii) *If $f : A \to B$ is compact as an arrow in $\mathfrak{X}$, then it is compact as an object in $A/\mathfrak{X}$.*

(iv) *If $h : B \to C$ is an arrow between compact objects in $\mathfrak{X}$, then h is compact.*

Proof. (i) is obvious, and (ii) follows from the pullback lemma.

For (iii) assume that f is compact as an arrow in $\mathfrak{X}$ and consider a filtered diagram in $A/\mathfrak{X}$, given by a filtered diagram $D : \mathbb{I} \to \mathfrak{X}$ and a cocone $\gamma = (\gamma_i : A \to D_i)_{i\in\mathbb{I}}$. Note that since the forgetful functor $A/\mathfrak{X} \to \mathfrak{X}$ creates connected colimits, we have $\mathsf{colim}(\gamma) : A \to \mathsf{colim}(D)$. Also because $\mathbb{I}$ is connected, all γ_i are in the same equivalence class in $\mathsf{colim}_{i\in\mathbb{I}}\,\mathfrak{X}(A, D_i)$, which we denote by $\overline{\gamma} : 1 \to \mathsf{colim}_{i\in\mathbb{I}}\,\mathfrak{X}(A, D_i)$. We have to show that the canonical map

$$\mathsf{colim}_i (A/\mathfrak{X})(f, \gamma_i) \quad\longrightarrow\quad (A/\mathfrak{X})(f, \mathsf{colim}(\gamma))$$

is a bijection. This follows because this function can be presented as a pullback in Set^2 as in the following diagram.

$$\begin{tikzcd}
& \mathsf{colim}_i\,\mathfrak{X}(B, D_i) \arrow[rr] \arrow[dd] & & \mathfrak{X}(B, \mathsf{colim}(D)) \arrow[dd] \\
\mathsf{colim}_i (A/\mathfrak{X})(f, \gamma_i) \arrow[ur] \arrow[rr] \arrow[dd] & & (A/\mathfrak{X})(f, \mathsf{colim}(\gamma)) \arrow[ur] \arrow[dd] & \\
& \mathsf{colim}_i\,\mathfrak{X}(A, D_i) \arrow[rr] & & \mathfrak{X}(A, \mathsf{colim}(D)) \\
1 \arrow[ur, "\overline{\gamma}"] \arrow[rr] & & 1 \arrow[ur, "\mathsf{colim}(\gamma)"'] &
\end{tikzcd}$$

The front square is a pullback since the back one is by compactness of f as an arrow, and the side ones are pullbacks by construction; thus the gray horizontal arrow is a bijection since $1 \to 1$ is.

Finally, claim (iv) now follows directly from (i), (ii), and (iii). $\square$

Remark 5.5 One can show the implication of Lemma 5.4(iii) is actually an equivalence, i.e. $f : A \to B$ is compact as an arrow if and only if it is so as an object of the coslice category, but the other direction is more awkward to write down and we don't need it.

Corollary 5.6 *If $\mathcal{T}$ is small and Cauchy-complete then*

1. *the functor $H : \mathcal{T}^{\mathsf{op}} \to \mathcal{T}\text{-}\mathsf{Alg}$ co-restricts to an equivalence between $\mathcal{T}^{\mathsf{op}}$ and the coclan $\mathfrak{C}(\mathcal{T}\text{-}\mathsf{Alg}) \subseteq \mathcal{T}\text{-}\mathsf{Alg}$ of compact 0-extensions, and*

2. *an arrow $f : \Delta \to \Gamma$ is a display map in in $\mathcal{T}$ if and only if $H(f) : H(\Gamma) \to H(\Delta)$ is an extension in $\mathcal{T}\text{-}\mathsf{Alg}$.*

Proof. Let $C \in \mathcal{T}\text{-}\mathsf{Alg}$ be a compact 0-extension. By Proposition 5.2 there exists a filtered diagram $D : \mathbb{J} \to \mathcal{T}^{\mathsf{op}}$ and a colimit cocone $(\sigma_j : H(D_j) \to C)_{j \in \mathbb{J}}$. Since C is compact, the identity arrow id_C factors through one of the cocone maps σ_j, i.e. C is a retract of $\hom(D_j, -)$. By Cauchy-completeness, C is itself representable. Thus, we have an equivalence of categories.

For the second claim, we know that $H(f)$ is an extension whenever f is a display map. Conversely, assume that $f : \Delta \to \Gamma$ is an arrow in $\mathcal{T}$ such that $H(f) : H(\Gamma) \to H(\Delta)$ is an extension. Then $H(f)$ is compact in $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg}$ by Lemma 5.4(iv) and $H(\Gamma)/\mathcal{T}\text{-}\mathsf{Alg} \simeq \mathcal{T}(\Gamma)\text{-}\mathsf{Alg}$ by Proposition 3.4. This means that the object corresponding to $H(f)$ in $\mathcal{T}(\Gamma)\text{-}\mathsf{Alg}$ is a compact 0-extension, and thus it is isomorphic to a hom-algebra $\mathcal{T}(\Gamma)(d, -)$ for a display map $d : \Theta \twoheadrightarrow \Gamma$ by the first claim. This means that f is isomorphic to d over Γ, and therefore a display map. $\square$

6 Clan-algebraic categories

Given an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$, the counit $E : \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg} \to \mathcal{L}$ of the biadjunction is given by the *nerve-realization adjunction*

$$\begin{tikzcd}
\mathfrak{C}(\mathcal{L}) \arrow[r, hook, "E"] \arrow[d, "H"'] & \mathcal{L} \arrow[dl, bend left, "E^*"] \\
\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg} \arrow[ur, bend left, "E_!"] \arrow[ur, phantom, "\dashv"] &
\end{tikzcd}$$

where E^* is the nerve of J given by $E^*(L) = \mathcal{L}(J(-), L)$, and its left adjoint $E_!$ is the Kan extension of H along J, given by

$$E_!(A) = \mathsf{colim}(\mathsf{elts}(A) \to \mathfrak{C}(\mathcal{L}) \xrightarrow{J} \mathcal{L}),$$

where $\mathsf{elts}(A)$ is the (contravariant) category of elements. In this section we show that E is an equivalence in EFCat if and only if $\mathcal{L}$ is *clan-algebraic* in the sense of the following definition.

Definition 6.1 An $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ is called *clan-algebraic* if

(D) the subcategory $\mathfrak{C}(\mathcal{L})$ is dense,

(CG) $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by $\mathcal{E} \cap \mathrm{mor}(\mathfrak{C}(\mathcal{L}))$, and

(FQ) quotients of componentwise-full equivalence relations are effective and have full quotient maps. $\diamondsuit$

Theorem 6.2 *For every clan $\mathcal{T}$, the category $\mathcal{T}$-Alg is clan-algebraic.*

Proof. $\mathfrak{C}(\mathcal{L})$ is dense since it contains the representables. $(\mathcal{E}, \mathcal{F})$ is cofibrantly generated by maps between representables, and and therefore a fortiori by maps between compact 0-extensions.

For the third condition, let

$$r = \langle r_1, r_2 \rangle : R \rightarrowtail A \times A$$

be an equivalence relation such that r_1 and r_2 are full maps. This means that we have an equivalence relation $\sim$ on each $A(\Gamma)$, such that

1. for all arrows $s : \Delta \to \Gamma$, the function $A(s) = s \cdot (-) : A(\Delta) \to A(\Gamma)$ preserves this relation, and

2. for every display map $p : \Gamma^+ \twoheadrightarrow \Gamma$ and all $a, b \in A(\Gamma)$ and $c \in A(\Gamma^+)$ such that $a \sim b$ and $p \cdot c = a$, there exists a $d \in A(\Gamma^+)$ with $c \sim d$ and $p \cdot d = b$.

We show first that the pointwise quotient A/R is an algebra. Clearly $(A/R)(1) = 1$, and it remains to show that given a pullback

$$\begin{tikzcd} \Delta^+ \arrow[r, "t"] \arrow[d, two heads, "q"] & \Gamma^+ \arrow[d, two heads, "p"] \\ \Delta \arrow[r, "s"] & \Gamma \end{tikzcd}$$

with p and q display maps, and elements $a \in A(\Delta)$, $b \in A(\Gamma^+)$ with $s \cdot a \sim p \cdot b$, there exists a unique-up-to-$\sim$ $c \in A(\Delta^+)$ with $q \cdot c \sim a$ and $t \cdot c \sim b$. Since p is a display map, there exists a b' with $b \sim b'$ and $p \cdot b' = s \cdot a$, and since A is an algebra there exists therefore a c with $q \cdot c = a$ and $t \cdot c = b'$. For uniqueness assume that $c, c' \in A(\Delta^+)$ with $q \cdot c \sim q \cdot c'$ and $t \cdot c \sim t \cdot c'$. Then $c \sim c'$ follows from the fact that R is an algebra. This shows that A/R is an algebra, and also that the quotient is effective, since the kernel pair is computed pointwise. The fact that $A \to A/R$ is full is similarly easy to see. □

The following lemma is a kind of converse to (FQ) .

Lemma 6.3 *Full maps in clan-algebraic categories are regular epimorphisms.*

Proof. Given a full map in a clan-algebraic category $\mathcal{L}$, the lifting property against (compact) 0-extensions implies that $E^*(f)$ is componentwise surjective in $\mathfrak{C}(\mathcal{L})^{\mathsf{op}}$-$\mathsf{Alg}$, and therefore the coequalizer of its kernel pair. Since left adjoints preserve regular epis, we deduce that $E_!(E^*(f))$ is regular epic in $\mathcal{L}$ and the claim follows since $E_! \circ E^* \cong \mathrm{id}$ by (D). □

Remark 6.4 Observe that we only used property (D) in the proof, no exactness.

Lemma 6.5 *The class $\mathcal{F}$ of full maps in a clan-algebraic category $\mathcal{L}$ has the* right cancellation property, *i.e. we have $g \in \mathcal{F}$ whenever $gf \in \mathcal{F}$ and $f \in \mathcal{F}$ for composable pairs $f : A \to B$, $g : B \to C$.*

Proof. By (CG) it suffices to show that g has the r.l.p. with respect to extensions $e : I \hookrightarrow J$ between compact 0-extensions I, J. Let

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[d, hook, "e"] & B \arrow[d, "g"] \\ J \arrow[r, "k"] & C \end{tikzcd}$$

be a filling problem. Since I is a 0-extension and f is full, there exists a map $h' : I \to A$ with $fh' = h$. We obtain a new filling problem

$$\begin{tikzcd} I \arrow[r, "h'"] \arrow[d, hook, "e"] & A \arrow[d, two heads, "gf"] \\ J \arrow[r, "k"] & C \end{tikzcd}$$

which can be filled by a map $m : J \to A$ since gf is full. Then fm is a filler for the original problem. □

Lemma 6.6 *Let $\mathcal{L}$ be a clan-algebraic category, let $f : A \to B$ be an arrow in $\mathcal{L}$ with componentwise full kernel pair $p, q : R \twoheadrightarrow A$, and let $e : A \twoheadrightarrow C$ be the coequalizer of p and q. Then the unique $m : C \to B$ with $me = f$ is monic.*

Proof. By (D) it is sufficient to test monicity of m on maps out of compact 0-extensions E. Let $h, k : E \to C$ such that $mh = mk$. Since e is full by (FQ) , there exist $h', k' : E \to A$ with $eh' = h$ and $ek' = k$. In particular we have $fh' = fk'$ and therefore there is an $u : E \to R$ with $pu = h'$ and $qu = k'$. Thus we can argue

$$h = eh' = epu = equ = ek' = k$$

which shows that m is monic. □

Lemma 6.7 *If $A \in \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg}$ is flat, then $A \to E^*(E_!(A))$ is an isomorphism, thus $E_!$ restricted to flat algebras is fully faithful.*

Proof. We have

$$\begin{aligned} E^*(E_!(A))(C) &= \mathcal{L}(C, \mathsf{colim}(\mathsf{elts}(A) \to \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L})) \\ &\cong \mathsf{colim}(\mathsf{elts}(A) \to \mathfrak{C}(\mathcal{L}) \xrightarrow{よ(C)} \mathsf{Set}) \qquad \text{since } \mathsf{elts}(A) \text{ is filtered} \\ &\cong A \otimes よ(C) \cong A(C). \end{aligned}$$

The second claim follows since for flat B, the mapping

$$(\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg})(A, B) \to \mathcal{L}(E_!A, E_!B)$$

can be decomposed as

$$(\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg})(A, B) \to (\mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg})(A, E^*E_!B) \to \mathcal{L}(E_!A, E_!B)$$ □

Lemma 6.8 *The following are equivalent for a cone $\phi : \Delta C \to D$ on a diagram $D : \mathbb{J} \to \mathcal{L}$ in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$.*

1. *Given an extension $e : A \to B$, an arrow $h : A \to C$, and a cone $\kappa : \Delta B \to D$ such that $\phi_j \circ h = \kappa_j e$ for all $j \in \mathbb{J}$, there exists $l : B \to C$ such that $le = h$ and $\phi_j l = \kappa_j$ for all $j \in \mathbb{J}$.*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "e"'] & C \arrow[d, "\phi_j"] \\ B \arrow[r, "\kappa_j"] \arrow[ur, dashed, "l"] & D_j \end{tikzcd}$$

2. *The mediating arrow $: C \to \mathsf{lim}(D)$ is full.*

Proof. The data of e, h, κ is equivalent to e, h, and $k : B \to \mathsf{lim}(D)$ such that

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "e"'] & C \arrow[d, "f"] \\ B \arrow[r, "k"] & \mathsf{lim}(D) \end{tikzcd}$$

commutes, and $l : B \to C$ fills the latter square iff it fills all the squares with the D_j. □

Definition 6.9 We call a cone $\phi : \Delta C \to D$ satisfying the conditions of the lemma *jointly full.* ♢

Remark 6.10 The interest of this is that it allows us to talk about full 'covers' of limits without actually computing the limits, which is useful when talking about cones and diagrams in the full subcategory of a clan-algebraic category on 0-extensions, which does not admit limits.

Definition 6.11 A *nice diagram* in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ is a 2-truncated semi-simplicial diagram

$$A_\bullet = \left(\begin{tikzcd} A_2 \arrow[r, shift left=2, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_2"'] & A_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & A_0 \end{tikzcd} \right)$$

where

1. A_0, A_1, and A_2 are 0-extensions,
2. the maps $d_0, d_1 : A_1 \to A_0$ are full,
3. in the commutative square $\begin{tikzcd} A_2 \arrow[r, "d_0"] \arrow[d, "d_2"'] & A_1 \arrow[d, "d_1"] \\ A_1 \arrow[r, "d_0"] & A_0 \end{tikzcd}$ the span constitutes a jointly full cone over the cospan,
4. there exists a 'symmetry' map $\begin{tikzcd} A_1 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, "\sigma"] & A_0 \\ A_0 & A_1 \arrow[l, "d_1"] \arrow[u, "d_0"'] \end{tikzcd}$ making the triangles commute, and
5. there exists a 0-extension $\tilde{A}$ and full maps $f, g : \tilde{A} \twoheadrightarrow A_1$ constituting a jointly full cone over the diagram ♢

$$\begin{tikzcd} A_1 \arrow[d, "d_0"'] \arrow[dr, "d_0"'] & A_1 \arrow[d, "d_1"] \arrow[dl, "d_1"'] \\ A_0 & A_0 \end{tikzcd}.$$

Lemma 6.12 *If $A_\bullet$ is a nice diagram in a clan-algebraic category $\mathcal{L}$, the pairing $\langle d_0, d_1\rangle : A_1 \to A_0 \times A_0$ factors as $A_1 \xrightarrow{f} R \xrightarrow{\langle r_0, r_1\rangle} A_0 \times A_0$ where f is full, and $r = \langle r_0, r_1\rangle$ is monic and a componentwise full equivalence relation.*

Proof. Condition 5 of the preceding definition gives us the following diagram

$$\begin{tikzcd}
\tilde{A} \arrow[dr, two heads, "h"'] \arrow[drr, two heads, bend left, "g"] \arrow[ddr, two heads, bend right, "f"'] & & \\
& S \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, phantom, "\lrcorner", very near start] & A_1 \arrow[d, "\langle d_0,d_1\rangle"] \\
& A_1 \arrow[r, "\langle d_0,d_1\rangle"] & A_0 \times A0
\end{tikzcd},$$

i.e. S is the kernel of $\langle d_0, d_1\rangle$ with projections p, q, $\tilde{A}$ is a 0-extension, and f, g, h are full. By right cancellation we deduce that p and q are full, and the existence of the factorization follows from Lemma 6.6. Fullness of r_0, r_1 follows again from right cancellation because f, d_0, and d_1 are full.

It remains to show that r is an equivalence relation. This is easy: condition 4 gives symmetry, and condition 3 gives transitivity, and reflexivity follows from the fact that r_0 admits a section as a full map into a 0-extension, together with symmetry (we internalize the argument that if in a symmetric and transitive relation everything is related to *something*, then it is reflexive.) □

Definition 6.13 A *fully extended cover of an object A* in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ is a full map $f : E \twoheadrightarrow A$ from a 0-extension E. ♢

Fully extended can be constructed by factoring the initial map $0 \to A$. So they're like cofibrant replacements.

Lemma 6.14 *For every object A in an $(\mathcal{E}, \mathcal{F})$-category $\mathcal{L}$ there exists a nice diagram $A_\bullet$ with colimit A.*

Proof. A_0 is given as fully extended cover $f_0 : A_0 \twoheadrightarrow A$ of A, A_1 is obtained as fully extended cover of $f_1 : A_1 \twoheadrightarrow A_0 \times_A A_0$, and A_2 as fully extended cover $f_2 : A_2 \twoheadrightarrow P$ of the pullback

$$\begin{tikzcd}
P \arrow[r, "p_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_1 \arrow[d, "d_0"] \\
A_1 \arrow[r, "d_1"] & A_0
\end{tikzcd},$$

with $d_0, d_1, d_2 : A_2 \to A_1$ given by $d_0 = p_0 \circ f$, $d_2 = p_1 \circ f$, and d_1 a lifting of $\langle d_0 \circ d_0, d_1 \circ d_2\rangle$ along f_1.

The symmetry map σ is constructed as a lifting of the symmetry of $A_0 \times_A A_0$ along f_1.

$\tilde{A}$ is a fully extended cover of the kernel of f_1. □

Lemma 6.15 *For any clan-algebraic category $\mathcal{L}$, the realization functor $E_!$ preserves jointly full cones in flat algebras, and nice diagrams.*

Proof. The first claim follows since $E_!$ is fully faithful by Lemma 6.7 and in both sides the weak factorization system determined by the same generators. Thus there's a one-to-one correspondence between lifting problems. The second claim follows since $E_!$ preserves 0-extensions and 0-extensions are flat. $\square$

Lemma 6.16 *For any clan-algebraic category $\mathcal{L}$, the nerve functor $E_* : \mathcal{L} \to \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg}$ preserves quotients of nice diagrams.*

Proof. Given a nice diagram $A_\bullet$ in $\mathcal{L}$, its colimit is the coequalizer of $d_0, d_1 : A_1 \to A_0$. By Lemma 6.12, $\langle d_0, d_1 \rangle$ factors as $\langle r_0, r_1 \rangle \circ f$ with f full and $r = \langle r_0, r_1 \rangle$ an equivalence relation. The pairs d_0, d_1 and r_0, r_1 have the same coequalizer (since f is epic), and E_* preserves the coequalizer of r_0, r_1 since it preserves full maps and kernel pairs. Finally, the coequalizer of $E_*(r_0), E_*(r_1)$ is also the coequalizer of $E_*(d_0), E_0(d_1)$ since $E_*(f)$ is full and therefore epic. $\square$

Theorem 6.17 *If $\mathcal{L}$ is clan-algebraic, then $\varepsilon : \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg} \to \mathcal{L}$ is an equivalence in* EFCat.

Proof. By density, ε^* is fully faithful. It remains to show that it is essentially surjective. It remains to show that the unit map $\eta_A : A \to E^*(E_!(A))$ is an isomorphism for all $A \in \mathfrak{C}(\mathcal{L})^{\mathsf{op}}\text{-}\mathsf{Alg}$. Let $A_\bullet$ be a nice diagram with colimit A. We have

$$\begin{aligned} E^*(E_!(A)) &= E^*(E_!(\mathsf{colim}(A_\bullet))) \\ &= \mathsf{colim}(E^*(E_!(A_\bullet))) \\ &= \mathsf{colim}(A_\bullet) \\ &= A \end{aligned}$$

$\square$

A The fat small object argument

A.1 Colimit decomposition formula and pushouts of sieves

In this subsection we discuss two results that we need in our proof of the fat small object argument.

Theorem A.1 (Colimit decomposition formula (CDF)) *Let $\mathbb{C} : \mathbb{J} \to \mathsf{Cat}$ be a small diagram in the 1-category of small categories, let $D : \mathsf{colim}(\mathbb{C}) \to \mathfrak{X}$ be a diagram in a category $\mathfrak{X}$ such that*

1. *for each $j \in \mathbb{J}$, the colimit of $\mathsf{colim}_{c \in \mathbb{C}_j} D_{\sigma_j c}$ exists, and*
2. *$\mathsf{colim}_{j \in \mathbb{J}} \mathsf{colim}_{c \in \mathbb{C}_j} D_{\sigma_j c}$ exists.*

Then $\mathsf{colim}_{j \in \mathbb{J}} \mathsf{colim}_{c \in \mathbb{C}_j} D_{\sigma_j c}$ is a colimit of D.

Proof. Peschke and Tholen [PT20] give three proofs of this under the additional assumption that $\mathfrak{X}$ is cocomplete. The third proof (Section 5.3, 'via Fubini') easily generalizes to the situation where only the necessary colimits are assumed to exist. We sketch a slightly simplified argument here. Let $\int \mathbb{C}$ be the covariant Grothendieck construction of $\mathbb{C}$, whose projection $\int \mathbb{C} \to \mathbb{J}$ is a split opfibration. Then $\mathsf{colim}(\mathbb{C})$ is the 'joint coidentifier' of the splitting, i.e. there is a functor $E : \int \mathbb{C} \to \mathsf{colim}(\mathbb{C})$ such that for every category $\mathfrak{X}$, the precomposition functor

$$(- \circ E) : [\mathsf{colim}(\mathbb{C}), \mathfrak{X}] \to [\textstyle\int \mathbb{C}, \mathfrak{X}]$$

restricts to an isomorphism between the functor category $[\mathsf{colim}(\mathbb{C}), \mathfrak{X}]$ and the full subcategory of $[\int\mathbb{C}, \mathfrak{X}]$ on functors which send the arrows of the splitting to identities. In particular, $(- \circ E)$ is fully faithful and thus it induces an isomorphism

$$(\mathsf{colim}(\mathbb{C}))(D, \Delta-) \overset{\cong}{\rightrightarrows} (\textstyle\int\mathbb{C})(D \circ E, \Delta-) : \mathfrak{X} \to \mathsf{Set}$$

of co-presheaves of cocones for every diagram $D : \mathsf{colim}(\mathbb{C}) \to \mathfrak{X}$. In other words, E is *final*, which is the crucial point of the argument, and for which Peschke and Tholen give a more complicated proof in [PT20, Theorem 5.8].

Finality of E means that D has a colimit iff $D \circ E$ has a colimit, and the existence of the latter follows if successive left Kan extensions along the composite $\int\mathbb{C} \to \mathbb{J} \to 1$ exist. The first of these can be computed as fiberwise colimit since $\int\mathbb{C} \to \mathbb{J}$ is a cofibration [PT20, Theorem 4.6], which yields the inner term in the double colimit in the proposition. $\square$

In the following we use the CDF specifically for pushouts of sieve inclusions of posets. Recall that a *sieve* (a.k.a. *downset*) in a poset P is a subset $U \subseteq P$ satisfying

$$x \in U \wedge y \leq x \implies y \in U$$

for all $x, y \in P$. A monotone map $f : P \to Q$ is called a *sieve inclusion* if it is order-reflecting and its image $\mathsf{im}(f) = f[P]$ is a sieve in Q. The proof of the following lemma is straightforward, but we state it explicitly since it will play a crucial role.

Lemma A.2 *1. If $f : P \to Q$ and $g : P \to R$ are sieve inclusions of posets, a pushout of f and g in the 1-category Cat of small categories is given by*

$$\begin{CD} P @>g>> R \\ @VfVV @VV\sigma_2V \\ Q @>\sigma_1>> Q +_P R \end{CD}$$

where $Q +_P R$ is the set-theoretic pushout, ordered by

$$\begin{array}{ll} \sigma_1(x) \leq \sigma_1(y) \text{ iff } x \leq y & \quad \sigma_1(x) \leq \sigma_2(y) \text{ iff } \exists z \,.\, x = f(z) \wedge g(z) \leq y \\ \sigma_2(x) \leq \sigma_2(y) \text{ iff } x \leq y & \quad \sigma_2(x) \leq \sigma_1(y) \text{ iff } \exists z \,.\, x = g(z) \wedge f(z) \leq y. \end{array}$$

In particular, the maps σ_1 and σ_2 are also sieve inclusions.

2. If U and V are sieves in a poset P then the square

$$\begin{CD} U \cap V @>>> V \\ @VVV @VVV \\ U @>>> U \cup V \end{CD}$$

is a pushout in Cat, where the sieves are equipped with the induced ordering. $\square$

A.2 The fat small object argument

Throughout this section let $\mathcal{C}$ be a *small* coclan.

We start by establishing some notation. Given a poset P and an element $x \in P$, we write $P_{\leq x} = \{y \in P \mid y \leq x\}$ for the principal sieve generated by x, and $P_{<x} = \{y \in P \mid y < x\}$ for its subset on elements that are strictly smaller than x. If x is a maximal element of P, we write $P \backslash x$ for the sub-poset obtained by removing x. Given a diagram $D : P \to \mathcal{C}$, we write $D_{\leq x}$, $D_{<x}$, and $D \backslash x$ for the restrictions of D to $P_{\leq x}$, $P_{<x}$, and $P \backslash x$, respectively. More generally we write D_U for the restriction of D to arbitrary sieves $U \subseteq P$.

Note that we have $P_{\leq x} = P_{<x} \star 1$, where $\star$ is the *join* or *ordinal sum*, thus diagrams $D : P_{\leq x} \to \mathcal{C}$ correspond to cocones on $D_{<x}$ with vertex D_x, and to arrows $\mathsf{colim}(D_{<x}) \to D_x$ whenever the colimit exists.

Definition A.3 A *finite* $\mathcal{C}$*-complex* is a pair (P, D) of a finite poset P and a diagram $D : P \to \mathcal{C}$, such that for $x, y \in P$:

1. The colimit $\mathsf{colim}(D_{<x})$ exists, and the induced map $\alpha_x : \mathsf{colim}(D_{<x}) \to D_x$ is co-display.
2. We have $x = y$ whenever $P_{<x} = P_{<y}$, $D_x = D_y$, and $\alpha_x = \alpha_y : \mathsf{colim}(D_x) \to D_x$.

An *inclusion of finite* $\mathcal{C}$*-complexes* $f : (P, D) \to (Q, E)$ is a sieve inclusion $f : P \to Q$ such that $D = E \circ f$. We write $\mathsf{FC}(\mathcal{C})$ for the category of finite $\mathcal{C}$-complexes and inclusions. $\diamondsuit$

Remark A.4 We view a finite $\mathcal{C}$-complex as a construction of an object by a finite (though not necessarily lineally ordered) number of 'cell attachments', represented by the co-display maps $\alpha_x : \mathsf{colim}(D_{<x}) \to D_x$. Condition 2 should be read as saying that 'every cell can only be attached once at the same stage'. This is needed in Lemma A.7 to show that $\mathsf{FC}(\mathcal{C})$ is a preorder.

Lemma A.5 *1. For every finite* $\mathcal{C}$*-complex* (P, D)*, the colimit* $\mathsf{colim}(D)$ *exists.*

2. The induced functor $\mathsf{colim} : \mathsf{FC}(\mathcal{C}) \to \mathcal{C}$ *sends inclusions of finite* $\mathcal{C}$*-complexes to co-display maps.*

Proof. The first claim is shown by induction on $|P|$. For empty P the statement is true since coclans have initial objects. For $|P| = n + 1$ assume that $x \in P$ is a maximal element. Then the quare

$$\begin{tikzcd} P_{<x} \arrow[r] \arrow[d] & P \backslash x \arrow[d] \\ P_{\leq x} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & P \end{tikzcd}$$

is a pushout in Cat by Lemma A.2, which by the colimit decomposition formula A.1 means that the pushout of the span

$$\begin{tikzcd} \mathsf{colim}(D_{<x}) \arrow[r] \arrow[d, tail] & \mathsf{colim}(D \backslash x) \arrow[d, tail, dashed] \\ D_x \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner", very near end] & \mathsf{colim}(D) \end{tikzcd} \tag{A.1}$$

– which exists since the left arrow is a co-display map by A.3-1 – is a colimit of D in $\mathcal{C}$.

For the second claim let $f : (E, Q) \to (D, P)$ be an inclusion of finite $\mathcal{C}$-complexes. Since co-display maps compose and every inclusion of finite $\mathcal{C}$-complexes can be decomposed into 'atomic' inclusions with $|P \backslash f[Q]| = 1$ by successively removing maximal elements from the codomain, we may assume without loss of generality that $Q = P \backslash x$ for some maximal element $x \in P$. Then the image of f under colim is the right dashed arrow in (A.1), which is a co-display map since co-displays are stable under pushout. □

Remark A.6 Lemma A.5 implies that the assumption 'The colimit $\mathsf{colim}(D_{<x})$ exists' in Definition A.3-1 is redundant, since the colimits in question are colimits of finite subcomplexes.

Lemma A.7 *The category* $\mathsf{FC}(\mathcal{C})$ *is an essentially small preorder with finite suprema.*

Proof. $\mathsf{FC}(\mathcal{C})$ is essentially small as a collection of finite diagrams in a small category. To see that it is a preorder let $f, g : (P, D) \to (Q, E)$ be inclusions of finite $\mathcal{C}$-complexes. We show that $f(x) = g(x)$ by well-founded induction on $x \in P$. Let $x \in P$ and assume that $f|_{P_{<x}} = g|_{P_{<x}}$. Then since f and g are sieve inclusions we have $Q_{<f(x)} = Q_{<g(x)}$ and since $E\, f = D = E\, g$ we have the equalities

$$\big(E_y \to E_{f(x)}\big)_{y<f(x)} = (D_y \to D_x)_{y<x} = \big(E_y \to E_{g(x)}\big)_{y<g(x)}$$

of cocones, whence $f(x) = g(x)$ by A.3-2.

It remains to show that $\mathsf{FC}(\mathcal{C})$ has finite suprema. The empty complex is clearly initial. We show that a supremum of (P, D) and (Q, E) exists by induction on $|P|$. The empty case is trivial, so assume that P is inhabited and let x be a maximal element. Let (R, F) be a supremum of $(P \backslash x, D \backslash x)$ and (Q, E), with inclusion maps $f : (P \backslash x, D \backslash x) \to (R, F)$ and $g : (Q, E) \longrightarrow (R, F)$. If there exists a $y \in R$ such that $R_{<y} = f[P_{<x}]$ and $(D_z \to D_x)_{z<x} = \big(R_{f(z)} \to R_y\big)_{z<x}$ then 'the cell-attachment corresponding to x is already contained in (R, F)', i.e. f extends to an inclusion $f' : (P, D) \to (R, F)$ of finite complexes with $f'(x) = y$, whence (R, F) is a supremum of (P, D) and (Q, E).

If no such y exists then a supremum of (P, D) and (R, F) is given by $(P +_{P \backslash x} R, [D, F])$, as in the pushout diagram

$$\begin{tikzcd}
P \backslash x \arrow[r, "f"'] \arrow[d, hook] & R \arrow[d] \arrow[ddr, bend left, "F"] & \\
P \arrow[r] \arrow[drr, bend right, "D"'] & P +_{P \backslash x} R \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "{[D,F]}"] & \\
 & & \mathcal{C}
\end{tikzcd}$$

constructed as in Lemma A.2. □

Lemma A.8 *The object* $C = \mathsf{colim}_{(P,D) \in \mathsf{FC}(\mathcal{C})} H(\mathsf{colim}(D))$ *is a* 0*-extension in* $\mathcal{C}^{\mathsf{op}}\text{-}\mathsf{Alg}$ *and* $C \to 1$ *is full.*

Proof. To see that $C \to 1$ is full, let $e : I \rightarrowtriangle J$ be a co-display map and let $f : H(I) \to C$. Since $\mathsf{FC}(\mathcal{C})$ is filtered and $H(I)$ is compact, f factors through a colimit inclusion as

$f = (H(I) \xrightarrow{H(g)} H(\mathsf{colim}(D)) \xrightarrow{\sigma_{(P,D)}} C)$ for some finite complex (P, D). We form the pushout

$$\begin{tikzcd} I \arrow[r, "g"] \arrow[d, tail, "e"'] & \mathsf{colim}(D) \arrow[d, tail, "k"] \\ J \arrow[r] & K \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

and extend the finite complex (P, D) to $(P \star 1, D \star K)$ where $P \star 1$ is the join of P and 1, and $D \star K : P \star 1 \to \mathcal{C}$ is the functor corresponding to the D-cocone corresponding to the arrow $k : \mathsf{colim}(D) \rightarrowtail K$. Then $K = \mathsf{colim}(D \star K)$ and $k = \mathsf{colim}((P, D) \to (P \star 1, D \star K))$, thus we obtain an extension of f along $H(e)$ as in the following diagram.

$$\begin{tikzcd} H(I) \arrow[r, "H(g)"'] \arrow[d, hook, "H(e)"'] \arrow[rr, bend left, "f"] & H(\mathsf{colim}(D)) \arrow[r, "\sigma_{(P,D)}"'] \arrow[d, hook, "H(k)"] & C \\ H(J) \arrow[r] & H(K) \arrow[ur, bend right, "\sigma_{(P\star 1, D\star K)}"'] \arrow[ul, phantom, very near start, "\ulcorner"] & \end{tikzcd}$$

To see that C is a 0-extension, consider a full map $f : Y \to X$ in $\mathcal{C}^{\mathsf{op}}\text{-}\mathsf{Alg}$ and an arrow $h : C \to X$. To show that h lifts along f we construct a lift of the cocone

$$\left(H(\mathsf{colim}(D)) \xrightarrow{\sigma_{(P,D)}} C \xrightarrow{h} X\right)_{(P,D)\in\mathsf{FC}(\mathcal{C})}$$

by induction over the preorder $\mathsf{FC}(\mathcal{C})$ which is well-founded since every finite $\mathcal{C}$-complex has only finitely many subcomplexes. Given a finite complex (D, P) it is sufficient to exhibit a lift $\kappa_{(P,D)} : H(\mathsf{colim}(D)) \to Y$ satisfying

(A.2) $\quad f \circ \kappa_{(P,D)} = h \circ \sigma_{(P,D)}$ and

(A.3) $\quad \kappa_{(P,D)} \circ H(\mathsf{colim}\, j) = \kappa_{(Q,E)}$ for all subcomplexes $j : (Q, E) \to (P, D)$,

where we may assume that the $\kappa_{(Q,E)}$ satisfy the analogous equations by induction hypothesis. We distinguish two cases:

1. If P has a greatest element x then we can take $\kappa_{(P,D)}$ to be a lift in the square

$$\begin{tikzcd} H(\mathsf{colim}(D_{<x})) \arrow[rr, "\kappa_{(P_{<x}, D_{<x})}"] \arrow[d] & & Y \arrow[d, "f"] \\ H(D_x) \arrow[r, "\sigma_{(P,D)}"'] \arrow[urr, dashed, "\kappa_{(P,D)}"] & C \arrow[r, "h"'] & X \end{tikzcd}$$

whose left side is an extension by Lemma A.5 and whose right side is full by assumption. Then (A.2) holds by construction, and (A.3) holds for all subcomplexes since it holds for the largest strict subcomplex $(P_{<x}, D_{<x}) \to (P, D)$.

2. If P *doesn't* have a greatest element we can write $P = U \cup V$ as union of two strict sub-sieves, wence we have pushouts

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & P \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{colim}(D_{U\cap V}) \arrow[r] \arrow[d] & \mathsf{colim}(D_V) \arrow[d] \\ \mathsf{colim}(D_U) \arrow[r] & \mathsf{colim}(D) \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

by Lemma A.2 and the CDF. This means that condition (A.3) forces us to define $\kappa_{(P,D)}$ to be the unique arrow fitting into

$$
\text{(A.4)} \qquad
\begin{tikzcd}
H(\mathsf{colim}(D_{U\cap V})) \arrow[r, "\phi_V^{U\cap V}"'] \arrow[d, "\phi_U^{U\cap V}"] & H(\mathsf{colim}(D_V)) \arrow[d, "\phi_P^V"] \arrow[drr, "\kappa_{(V,D_V)}"] & & \\
H(\mathsf{colim}(D_U)) \arrow[r, "\phi_P^U"] \arrow[rrr, bend right=10, "\kappa_{(U,D_U)}"'] & H(\mathsf{colim}(D)) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[rr, dashed, "\kappa_{(P,D)}"] & & Y
\end{tikzcd} ,
$$

where for the remainder of the proof we write $\phi_W^X : H(\mathsf{colim}(D_X)) \to H(\mathsf{colim}(D_W))$ for the canonical arrows induced by successive sieve inclusions $X \subseteq W \subseteq P$. Using the fact that the ϕ_P^U and ϕ_P^V are jointly epic it is easy to see that the $\kappa_{(P,D)}$ defined in this way satisfies condition (A.2), and it remains to show that (A.3) is satisfied for arbitrary sieves $W \subseteq P$, i.e. $\kappa_{(P,D)} \circ \phi_P^W = \kappa_{(W,D_W)} : H(\mathsf{colim}(D_W)) \to Y$. Since

$$
\begin{tikzcd}
H(\mathsf{colim}(D_{U\cap V\cap W})) \arrow[r, "\phi_{V\cap W}^{U\cap V\cap W}"'] \arrow[d, "\phi_{U\cap W}^{U\cap V\cap W}"] & H(\mathsf{colim}(D_{V\cap W})) \arrow[d, "\phi_W^{V\cap W}"] \\
H(\mathsf{colim}(D_{U\cap W})) \arrow[r, "\phi_W^{U\cap W}"] & H(\mathsf{colim}(D_W)) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

is a pushout it is enough to verify this equation after precomposing with $\phi_W^{U\cap W}$ and $\phi_W^{V\cap W}$. We have

$$
\begin{aligned}
\kappa_{(P,D)} \circ \phi_P^W \circ \phi_W^{U\cap W} &= \kappa_{(P,D)} \circ \phi_P^U \circ \phi_U^{U\cap W} && \text{by functoriality} \\
&= \kappa_{(U,D_U)} \circ \phi_U^{U\cap W} && \text{by (A.4)} \\
&= \kappa_{(U\cap W, D_{U\cap W})} && \text{by (A.3)} \\
&= \kappa_{(W,D_W)} \circ \phi_W^{U\cap W} && \text{by (A.3)}
\end{aligned}
$$

and the case with $\phi_W^{V\cap W}$ is analogous. □

Corollary A.9 *For any small clan $\mathcal{T}$, the 0-extensions in $\mathcal{T}$-Alg are flat algebras.*

Proof. Let $E \in \mathcal{T}$-Alg be a 0-extension. By applying Lemma A.8 in $\mathcal{T}$-Alg/E (using Proposition 3.5), we obtain full map $f : F \twoheadrightarrow E$ where F is a 0-extension f is a filtered colimit of arrows $H(\Gamma) \to E$ in $\mathcal{T}$-Alg/A. Since $\mathcal{T}$-$\mathsf{Alg}/A \to \mathcal{T}$-$\mathsf{Alg}$ creates colimits this means that F is a filtered colimit of hom-algebras in $\mathcal{T}$-Alg, and therefore flat (Lemma 2.4). Since f is a full map into a 0-extension it has a section, thus E is a retract of F and therefore flat as well. □

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