$\mathsf{Sh}(\mathsf{Cart})$ as a classifying topos

$\mathsf{Cart} \subseteq \mathsf{Top}$ is the full subcategory of the category of topological spaces on $\{\mathbb{R}^n \mid n \in \mathbb{N}\}$, with the open cover topology. This is equivalent to taking all separable topological manifolds.

1 Real-continuous algebras

Cart is a Lawvere theory, let's call its models *real-continuous algebras*, and write

$$\mathsf{RCAlg} \; = \; \mathsf{FP}(\mathsf{Cart}, \mathsf{Set})$$

for the category of models. Real-continuous algebras are in particular real algebras, i.e. $\mathbb{R}$-vector spaces with a commutative monoid structure. As usual, we obtain an embedding

$$\mathsf{Cart}^{\mathsf{op}} \to \mathsf{RCAlg}, \qquad \mathbb{R}^n \mapsto C(\mathbb{R}^n, \mathbb{R}).$$

More generally, we obtain a nerve functor $C(-,\mathbb{R}) : \mathsf{Top}^{\mathsf{op}} \to \mathsf{RCAlg}$ which maps spaces to their algebras of continuous functions, and maps colimits of spaces to limits of algebras. This functor is fully faithful on *realcompact* spaces [Joh86], i.e. closed subspaces of powers of $\mathbb{R}$. If no measurable cardinals exist, then every metrizable space, in particular every discrete space is realcompact in this case.

The category $\mathsf{RCAlg}_{\mathsf{fp}}$ is the full subcategory of RCAlg on *finitely presentable objects*, i.e. coequalizers of parallel maps $f, g : C(\mathbb{R}^n) \to C(\mathbb{R}^k)$. These correspond to maps in the opposite direction between the spaces, thus we can the coequalizers as representing formal equalizers of continuous maps. Do the formal equalizers correspond to actual subspaces? Probably not?

The category $\mathsf{RCAlg}_{\mathsf{fg}}$ is the category of *finitely generated* real continuous algebras, which are representable by congruence relations on $C(\mathbb{R}^n)$, which in turn come from ring theoretic ideals. Every such ideal gives rise to a filter of closed sets, but it's unclear whether this assignment is injective.

1.1 Real-continuous spaces and loci

Let's write $\mathsf{RCS} = \mathsf{RCAlg}^{\mathsf{op}}$ and $\mathsf{RCLoc} = \mathsf{RCAlg}_{\mathsf{fg}}^{\mathsf{op}}$ for the opposite categories of (finitely generated) real-continuous algebras, and call them *real-continuous spaces* and *real-continuous loci*, respectively. Write $\mathsf{RCLoc}_{\mathsf{fp}}$ for the category of *finitely presented real-continuous loci*, i.e. the opposite of the category $\mathsf{RCAlg}_{\mathsf{fp}}$. The term 'locus' is adapted from the theory of C^∞-rings treated in [MR91].

We get a conerve/corealization adjunction $K : \mathsf{Top} \leftrightarrow \mathsf{RCS} : |\cdot|$. Which limits does the conerve functor K preserve? The question that we're actually interested in is whether the formal equalizers that constitute finitely-presented real-continuous loci can be computed as equalizers of spaces. Since $\mathbb{R}^n$ is perfectly normal, this would mean that real-continuous loci correspond to closed sets in $\mathbb{R}^n$. We know that the spectrum preserves equalizers, which means that the spectrum of every locus is the corresponding equalizer, but this is not enough.

In the smooth case, the formal 0-locus of x^2 is different from the actual one, and this is a feature. But what about the topological case?

To understand this better we have to think about congruence relations and ideals.

1.2 Congruence relations and ideals

A congruence relation on a real-continuous algebra A is a set $R \subseteq A \times A$ such that for all operations f we have $f(\vec{a}) \sim f(\vec{b})$ whenever $a_i \sim b_i$ for all i.

We have $a \sim b$ iff $a - b \sim 0$, thus the congruence relation is determined by a ring-theoretic ideal. However, not every ring ideal is a real-continuous ideal. A real-continuous ideal is an ideal such that $f(\vec{x}) - f(\vec{y}) \in I$ whenever all $x_i - y_i$ are.

1.2.1 Congruence relations from subsets

Proposition 1 *If $A \subseteq X$ is a subset of a topological space, then*

$$I(A) = \{f \in C(X, \mathbb{R})^2 \mid f{\restriction_A} = 0\}$$

is a real-continuous ideal. ∎

Not every real-continuous ideal is of this form. A counterexample is

$$L(0) := \bigcup_{n \in \mathbb{N}} I([-\tfrac{1}{n}, \tfrac{1}{n}]) \subseteq C(\mathbb{R}).$$

However, we have the following.

Lemma 2 *Let $f : C(\mathbb{R}^n)$, and let $Z(f) = \{(\vec{x}) \in \mathbb{R}^n \mid f(\vec{x}) = 0\}$. Then we have $\langle f \rangle = I(Z(f))$.*

Proof. Clearly, $\langle f \rangle \subseteq I(Z(f))$. Conversely, let $g \in I(Z(f))$. Define $b : \mathbb{R} \to \mathbb{R}$ by

$$b(x) = \begin{cases} -1 & \text{if } x <= -1 \\ x & \text{if } x \in [-1, 1] \\ 1 & \text{if } x \geq 1. \end{cases}$$

and define $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$h(\vec{x}, y) = \begin{cases} b(\frac{y}{f(\vec{x})})g(\vec{x}) & \text{if } f(\vec{x}) \neq 0 \\ 0 & \text{else.} \end{cases}$$

The function b is clearly continuous, and we claim that h is as well. This only has to be verified at points $(\vec{x}, y)$ such that $f(\vec{x}) = 0$, so assume that $(\vec{x}_n, y_n) \xrightarrow{n \to \infty} (\vec{x}, y)$ with $\vec{x} \in Z(f)$. We have

$$|h(\vec{x}_n, y_n)| \leq |g(\vec{x}_n)| \xrightarrow{n \to \infty} g(\vec{x}) = 0 = h(\vec{x}, y)$$

since $g \in I(Z(f))$, showing continuity of h. Now since $f - 0 \in \langle f \rangle$ we have $h \circ \langle \mathrm{id}, f \rangle - h \circ \langle \mathrm{id}, 0 \rangle \in \langle f \rangle$. Evaluate:

$$\begin{aligned} (h \circ \langle \mathrm{id}, f \rangle - h \circ \langle \mathrm{id}, 0 \rangle)(\vec{x}) &= h(\vec{x}, f(\vec{x})) - h(\vec{x}, 0) \\ &= g(\vec{x}), \end{aligned}$$

thus $g \in \langle f \rangle$. ∎

Corollary 3 *Every finitely generated real-continuous ideal in $C(\mathbb{R}^n)$ is given by a single generator.*

Proof. Given $f_1, \dots, f_n \in C(\mathbb{R}^n)$ we claim that

$$\langle f_1, \dots, f_n \rangle = \langle \sum_i f_i^2 \rangle$$

It is clear that the RHS is contained in the LHS. In the other direction, the lemma implies that it's enough to show that $f_1, \dots, f_n \in I(Z(\sum_i f_i^2))$, i.e. that the f_i vanish on $Z(\sum_i f_i^2)$, which is clear. ∎

Together we get the following 'Nullstellensatz'.

Corollary 4 *We have $I(Z(\mathfrak{a})) = \mathfrak{a}$ for all finitely generated $\mathfrak{a} \subseteq C(\mathbb{R}^n)$.*

Proof. Since every finitely generated $\mathfrak{a}$ is in fact given by a single generator. ∎

What does this say about the conerve?

Lemma 5 $K : \mathsf{Top} \to \mathsf{RCS}$

1.3 The classifying topos of real continuous algebras

The classifying topos of real continuous algebras is $[\mathsf{RCAlg}_{\mathsf{fp}}, \mathsf{Set}]$, the generic algebra being the forgetful functor. The subtopos $[\mathsf{Cart}^{\mathsf{op}}, \mathsf{Set}]$ classifies *flat* algebras, i.e. filtered colimits of algebras $C(\mathbb{R}^n)$.

A concrete criterion for flatness is the following (written in the geometric direction): A is flat, if for every fork $A \to \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ factors through a fork $\mathbb{R}^m \to \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. This excludes e.g. spheres, so I'd like to view it as a kind of contractibility.

We have the following sequence of geometric inclusions of toposes:

$$\mathsf{Sh}(\mathsf{Cart}) \hookrightarrow [\mathsf{Cart}^{\mathsf{op}}, \mathsf{Set}] \hookrightarrow [\mathsf{RCAlg}_{\mathsf{fp}}, \mathsf{Set}]$$

We have described what the second and third describe, it remains to address the first. It's a geometric sub-theory of real-continuous algebras, obtained by adding geometric axioms saying that the covering families are jointly surjective. It seems to be sufficient to consider covers of $\mathbb{R}$.

Proposition 6 *(i) The sheaf topos for the finite cover topology on* Cart *classifies* local *flat real-continuous algebras, i.e. flat real-continuous algebras in which $x + y = 1 \Rightarrow (\exists z \,.\, xz = 1) \vee (\exists z \,.\, yz = 1)$ holds.*

(ii) The sheaf topos for the small cover topology on Cart *classifies* local Archimedean flat real-continuous algebras*, where 'Archimedean' means that $x : A \mid \cdot \vdash \bigvee_n \exists y : A \,.\, x = b_n(y)$, where b_n is a homeomorphism from $\mathbb{R}$ to $(-n, n)$.*

Proof. We claim that adding a topology to Cart amounts to adding axioms to the theory saying that the covering families are jointly epic. Furthermore, we claim that it's sufficient to consider coverings on $\mathbb{R}$. Thus, for each open cover $(u_i : \mathbb{R} \to \mathbb{R})_{i \in I}$ we postulate

$$x \mid \cdot \vdash \bigvee_i \exists y \,.\, u_i(y) = x.$$

Geometrically this means that $\mathsf{spec}(A)$ is so 'small' that every function on it factors through a part of an arbitrary (finite) cover. For example, no function

$f : \mathsf{spec}(A) \to \mathbb{R}$ can cover an interval, since I can always choose a cover none of whose parts entirely contains the interval.

It is relatively easy to see that local and Archimedean axioms are instances. Conversely, let's show that all finite cover axioms follow from the local axiom.

Consider a cover $(u_i)_i$ and an element $x : A$ (thinking about an actual set model). We can choose a partition of unity $(\alpha_i)(i)$ corresponding to the cover. Then $x = \sum_i \alpha_i(x)$ and because A is local, one of the $\alpha_i(x)$ has to be invertible. This means that the range of x is entirely within the support of α_i and therefore within u_i.

If A is furthermore archimedean, then given x we can first find a finite interval containing the range of x. Then every cover of $\mathbb{R}$ contains a finite subcover of the closure of the interval.

More things to do:

- think $\mathsf{Sh}(\mathsf{Cart})$ as a colimit of spaces $\mathbb{R}^n$.

Counterexamples:

- bounded functions on $\mathbb{N}$ mod ultrafilter seems to be local and Archimedean but not flat?
- local neighborhood of closed interval is flat and archimedean but not local?
- does archimedean follow from local and flat? idea of proof: to show: $x \in A$ factors through an element of an arbitrary covering. idea: A has a point, giving rise to an element of x this one is in a neighborhood. task: show that every flat local algebra has a point! or do we need archimedanness for that?

  is the topos for the finite cover topology well defined? does it have different set points?

2 Flat local real-continuous algebras are archimedean?

A real-continuous algebra A is flat if it satisfies

$$\vec{f}(\vec{x}) = \vec{g}(\vec{x}) \vdash_{\vec{x}} \bigvee_m \bigvee_{\substack{\vec{h} \in C(\mathbb{R}^m, \mathbb{R})^k \\ \vec{f} \circ \vec{h} = \vec{g} \circ \vec{h}}} \exists \vec{y} \,.\, \vec{h}(\vec{y}) = \vec{x} \cdot$$

for all $\vec{f}, \vec{g} \in C(\mathbb{R}^k, \mathbb{R})^n$. Since every conjunction of equations is equivalent to the sum of the differences being 0, this simplifies to the condition that

$$f(\vec{x}) = 0 \vdash_{\vec{x}} \bigvee_m \bigvee_{\substack{\vec{h} \in C(\mathbb{R}^m, \mathbb{R})^k \\ f \circ \vec{h} = 0}} \exists \vec{y} \,.\, \vec{h}(\vec{y}) = \vec{x} \,.$$

holds for all $f \in C(\mathbb{R}^k, \mathbb{R})$.

Thinking about elements of A as functions on $\mathsf{spec}(A)$, we can draw the following picture.

$$\begin{tikzcd}
& \mathsf{spec}(A) \arrow[dl, dashed, "\vec{y}"'] \arrow[d, "\vec{x}"] \arrow[dr, "0"] & \\
\mathbb{R}^m \arrow[r, dashed, "\vec{h}"'] & \mathbb{R}^k \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

This means that whenever $\vec{x}$ maps into the 0-locus of a function f, it factors through a mapping of a ball into the same 0-locus. This can be seen as a kind of contractibility since it excludes for example $\mathbb{R}^2$ (whose spectrum is 2), where no such factorization exists for $k = 1$, $x = (-1, 1)$ and $f(x) = x^2$. Spheres can be excluded by a similar argument.

References

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