A Universal Characterisation of the Tripos-to-Topos Construction

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1. Introduction

1.1. Motivation

The starting point for the ideas presented in this thesis was that I wanted to get a better understanding of the various functors and geometric morphisms that arise in the context of different realisability constructions.

Let $\mathcal{P}$ be a tripos over $\mathbf{Set}$. Then we may construct the topos $\mathbf{Set}[\mathcal{P}]$. There are two functors that we may construct for each such topos, namely the global sections functor

$$\Gamma : \mathbf{Set}[\mathcal{P}] \longrightarrow \mathbf{Set}$$

and the 'constant objects' functor

$$\Delta : \mathbf{Set} \longrightarrow \mathbf{Set}[\mathcal{P}].$$

For localic toposes, we have $\Delta \dashv \Gamma$, but for realisability toposes Δ and Γ mysteriously exchange places and we have $\Gamma \dashv \Delta$. In relative realisability, Γ has a right adjoint and Δ has a left adjoint, but they do not pair any more. In modified realisability, Γ has a right adjoint, but Δ has no adjoint at all. Moreover, the different realisability constructions are related through geometric morphisms.

It turns out that all these functors are induced by mappings between the underlying triposes and these are much easier to analyse. But in order to systematically reduce questions about functors between tripos-induced toposes to questions about mappings between triposes it is desirable to have an abstract characterisation of the construction which maps triposes to toposes and morphisms between triposes to functors between toposes. The present work intends to give such a characterisation.

The most important problem is to find the right framework for this characterisation. We want the tripos-to-topos construction to be functorial, but between which (2-)categories? The traditional approach to consider geometric morphisms as the 'right' class of maps between toposes as well as triposes seems not satisfactory, because

1. the constant objects functor, which seems to be the only construction that works for every tripos, does not come as a geometric morphism in a canonical way, and

2. already in [4], it is remarked that the construction that transforms maps of triposes to functors already works if the tripos morphism (which is really a natural transformation between presheaves of Heyting algebras) preserves only finite meets.

In order to capture the construction in its greatest possible generality, we will therefore consider finite meet preserving tripos-morphisms and finite limit preserving functors as arrows between triposes and toposes respectively[1]. However, we have to pay a price for this generality: On cartesian tripos-morphisms, the tripos-to-topos construction does

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[1]Conforming with contemporary literature, we will use the adjective *cartesian* to say that something has or preserves finite limits or meets

not act pseudofunctorial, but merely *oplax functorial*. To see this we do not even have to use complicated realisability-constructions, finite locales suffice.

Let $\mathbb{B} = \{\mathsf{true}, \mathsf{false}\}$ be the locale of booleans, with $\mathsf{false} \leq \mathsf{true}$. Then $\mathbf{Set}(-, \mathbb{B})$ and $\mathbf{Set}(-, \mathbb{B} \times \mathbb{B})$ are triposes, and the induced toposes are equivalent to $\mathbf{Set}$ and $\mathbf{Set} \times \mathbf{Set}$, respectively. Between the locales we consider the meet-preserving maps

$$\delta = \langle \mathrm{id}, \mathrm{id} \rangle : \mathbb{B} \longrightarrow \mathbb{B} \times \mathbb{B} \qquad \text{and} \qquad \wedge : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$$

These maps give rise to cartesian tripos-morphisms, which in turn give rise to functors and the reader may believe me for the moment that these are just the familiar

$$\Delta = \langle \mathrm{id}, \mathrm{id} \rangle : \mathbf{Set} \longrightarrow \mathbf{Set} \times \mathbf{Set} \qquad \text{and} \qquad (- \times -) : \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}.$$

Forming the composition of the maps, we get $\wedge \circ \delta = \mathrm{id}_{\mathbb{B}}$ and this gives rise to the identity functor. Therefore we obtain a *noninvertible* constraint cell

where $\eta_I = \delta_I : I \longrightarrow I \times I$ is the unit of the adjunction $\Delta \dashv (- \times -)$.

Hence, the tripos-to-topos construction is generally only oplax. However, we are dealing with a pretty well-behaved form of oplaxness. In particular the constraint cells for the identities are always invertible and the constraint cells for the composition are invertible whenever the second arrow is *regular*, by which we mean that a tripos-morphism commutes with existential quantification.

So far, we have given an outline of the behaviour of the tripos-to-topos construction, but our original goal was to give an abstract characterisation. This characterisation comes in the form of a (generalised) biadjunction. More precisely, we want to justify the tripos-to-topos construction by showing that it is adjoint to some 'forgetful' functor. This functor has to forget from toposes to triposes and the obvious way to do so is to assign to each topos its subobject fibration. This deliberation also suggests an answer to another design question that we have not even posed yet: The forgetful functor is only definable if we do not consider triposes over a fixed base category, but put the triposes over all bases in one big 2-category. The right notion of arrow between triposes over different bases is then the concept of fibred functor *over* a functor between the bases. This is presented e.g. in [11].

Having sketched what the involved 2-categories look like, we can now give them names. The 2-category of triposes is called $\mathfrak{Trip}_{\mathsf{c}}$ and the 2-category of toposes (with finite-limit preserving functors) is called $\mathfrak{Top}_{\mathsf{c}}$. We have already described the object part of the forgetful functor $\boldsymbol{S} : \mathfrak{Top}_{\mathsf{c}} \longrightarrow \mathfrak{Trip}_{\mathsf{c}}$, the morphism part is straightforward. The tripos-to-topos construction will be denoted by $\boldsymbol{T} : \mathfrak{Trip}_{\mathsf{c}} \longrightarrow \mathfrak{Top}_{\mathsf{c}}$.

We also have something that looks like a unit for the adjunction: for a given fibration $\mathcal{P}$ over $\mathcal{C}$, this is a fibred functor over the constant objects functor, which assigns to

every predicate over I the canonical subobject of ΔI. Naturality constraint cells exist, but they are only invertible if the involved tripos morphism is regular.

$$
\begin{array}{ccc}
\mathcal{P} & \longrightarrow & \mathcal{Q} \\
\downarrow & \swarrow\!\!\!\swarrow & \downarrow \\
\boldsymbol{ST}\mathcal{P} & \longrightarrow & \boldsymbol{ST}\mathcal{Q}
\end{array}
$$

For any topos $\mathcal{E}$, we have $\mathcal{E} \simeq \boldsymbol{TS}\mathcal{E}$ and this settles the question about the counit.

Hence, we have all the data that are necessary for an adjunction and it turns out that the triangle equalities also hold up to equivalence. But unfortunately, this is not a biadjunction. The concept of biadjunction is only defined for pseudofunctors, and in general, if we try to generalise bicategorical concepts and try to find lax/oplax versions, we get huge problems. For example, the horizontal composition $F \circ \eta$ of a natural transformation with a lax functor is not even definable in general. In our case, all composites in the triangle equalities are well defined, but we can not infer the uniqueness of the adjoint anymore in the usual way.

We provide two ways to get a 'good' biadjunction after all.

1. If we restrict our attention to regular tripos morphisms and regular functors between toposes, then all constraint cells become invertible, we get ordinary pseudofunctors and pseudo-natural transformations and all works out. At first, this is surprising[2], because the constant objects functor seems not to be regular in general (although I do not know a counterexample). But this is not a problem, because in the construction, the constant objects functor only appears as base part of $\mathcal{P} \longrightarrow \boldsymbol{ST}\mathcal{P}$ and the regularity requirement just concerns the fibred part of the tripos morphism.

   In this way, we obtain a universal characterisation of the topos $\boldsymbol{T}\mathcal{P}$ for a given tripos $\mathcal{P}$, but the approach is not entirely satisfactory, as it does not capture the 'full spirit' of the tripos-to-topos construction. This approach is presented in section 3.

2. As mentioned above, if we include *cartesian* tripos morphisms and functors, we obtain something that looks like an adjunction but is not really one, because the occurring functors and natural transformations are partly oplax. Just as I wanted to give up investigating this any further, Thomas Streicher gave me the important hint that Peter Johnstone used an adjunction-like concept with similar features in his paper [6] to give a definition of fibration that works in arbitrary 2-categories: the *semi-lax* adjunction. Most importantly, Johnstone describes a way to sustain the desired uniqueness of adjoints (up to equivalence) by postulating additional data. The basic idea is that we introduce a class of 'good' arrows, which is a subclass of the class of all 1-cells and add axioms that ensure that all lax data is actually pseudo on the good arrows.

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[2]and I did not notice it myself, Thomas Streicher had to point it out to me

I had to generalise Johnstone's concept to adapt it to the situation of the tripos-to-topos construction. These ideas are developed in section 6, culminating in an 'adjointability style' characterisation of those functors that allow a semi-lax left adjoint. The tripos theoretical main result of this thesis can then be phrased as 'the forgetful functor from toposes to triposes is left semi-lax adjointable'. We decided to present the concrete case first, therefore this result can be found already in section 5, of course not in abstract form but spelled out elementary.

1.2. Conventions and Notation

In the previous section I stated that triposes are presheaves of pre-Heyting algebras, but from now on, we shall view them as fibrations and not as indexed categories. Apart from philosophical/foundational considerations, this has a practical advantage: from the fibrational point of view, a tripos morphism is a single functor and not a family of functors, and this helps us to save lots of indizes.

Also concerning fibrations, we use the traditional names *cartesian arrow* and *co-cartesian arrow*, but we use *fibred functor* and *fibred natural transformation* instead of *cartesian functor* and *cartesian natural transformation* to avoid naming conflicts with finite limit related concepts.

In diagrams, we draw cartesian arrows in the form $\rightsquigarrow$.

A *cartesian category* is a category with finite limits and a *cartesian functor* is a functor between cartesian categories that preserves finite limits.

When composing functors and natural transformations in various ways, or in general 2-categorical reasoning we use the convention from [7] to denote composition of 1-cells and the vertical composition of 2-cells by juxtaposition and horizontal composition of 2-cells by ∘.

Instead of the 2-category of toposes and geometric morphisms, which is the primary 2-category of toposes e.g. in [7], we consider *two* 2-categories of toposes:

- $\mathfrak{Top}_c$ is the 2-category of toposes, cartesian functors and natural transformations.

- $\mathfrak{Top}_r$ is the 2-category of toposes, regular (i.e. cartesian and epi-preserving) functors and natural transformations.

Finally, we have to depart from traditional tripos theoretical notation in two points. Firstly, although we used Δ for the constant objects functor in the introduction, we will denote it by D from now on, because we want to reserve greek capital letters for the 'fibred functor'-part of tripos morphisms. Secondly, we will write $\boldsymbol{T}\mathcal{P}$ instead of the traditional $\mathcal{C}[\mathcal{P}]$ to emphasise the functorial nature of the tripos-to-topos construction, and we will generally use capital boldface letters for functors between 2-categories.

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I want to thank my supervisor Thomas Streicher for introducing me to a fascinating field of study, and for his support and great patience while I was working on this thesis.

2. Triposes

Triposes were introduced in [4]. They give a class of models for constructive higher order logic that is even more general than toposes. Using fibrational language[3], we can define them concisely as follows.

Definition 2.1   *1. Let $\mathcal{C}$ be a cartesian category, and $\mathcal{P}$ a fibration over $\mathcal{C}$. We say that $\mathcal{P}$ has* power objects[4] *if for each object $I \in \mathsf{Obj}(\mathcal{C})$ there exists an object $\mathfrak{P}(I) \in \mathsf{Obj}(\mathcal{C})$ and an object $(\ni_I) \in \mathcal{P}_{\mathfrak{P}(I) \times I}$ such that for all $J \in \mathsf{Obj}(\mathcal{C})$ and all $\varphi \in \mathcal{P}_{J \times I}$, there exists an arrow $\{\varphi\} : J \longrightarrow \mathfrak{P}(I)$ such that $\varphi \cong (\{\varphi\} \times I)^*(\ni_I)$.*

$$\varphi \rightsquigarrow \ni_I$$

$$J \times I \xrightarrow{\{\varphi\} \times I} \mathfrak{P}(I) \times I$$

   2. A posetal hyperdoctrine *is a fibration of pre-Heyting algebras (pHa's) over a cartesian category that has internal sums and products.*

   3. A tripos *is a posetal hyperdoctrine that has power objects.*

Most triposes that we will encounter are in fact based on toposes and the following lemma shows how the requirement to have power objects can be simplified in this case.

Lemma 2.2 *A fibration over a properly[5] cartesian closed category $\mathcal{C}$ has power objects iff it has a generic predicate, i.e. there exists an object Prop in $\mathcal{C}$ and a predicate tr over Prop such that for all objects I of $\mathcal{C}$ and all predicates φ over I there exists $\ulcorner \varphi \urcorner : I \longrightarrow \mathsf{Prop}$ with*

$$\varphi \rightsquigarrow \mathsf{tr}$$

$$I \xrightarrow{\ulcorner \varphi \urcorner} \mathsf{Prop}$$

Proof. The necessity of the condition is clear because for each tripos, $\ni_{\mathbf{1}}$ is a generic predicate.

For the converse direction, choose $\mathfrak{P}(J) = \mathsf{Prop}^J$ and $\ni_J = \mathsf{eval}^*_{\mathsf{Prop}}\mathsf{tr}$, where $\mathsf{eval}_{\mathsf{Prop}} : \mathsf{Prop}^J \times J \longrightarrow \mathsf{Prop}$ is the counit of the adjunction $(-) \times J \dashv (-)^J : \mathcal{C} \longrightarrow \mathcal{C}$ at Prop.∎

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[3]The author's primary reference on fibrations is [11], everything that is not defined here can be looked up there.

[4]These should be more precisely called *weak power objects*, because the arrow $\{\varphi\}$ is not required to be unique.

[5]in the sense of Johnstone, i.e. cartesian closed with all finite limits

2.1. The internal language of a tripos

Triposes are models for higher order logic. More precisely, for any tripos there is a canonical way to interpret in it full predicate logic with equality, supplemented by product- and power types. The interpretation of predicate logic is fairly standard, therefore we will only explain equality and power types.

Let $\mathcal{P}$ be a tripos over $\mathcal{C}$, $A \in \mathsf{Obj}(\mathcal{C})$. Then the equality predicate on A is given by $\exists_\delta(\top)$, where $\delta = \langle \mathrm{id}, \mathrm{id} \rangle : A \longrightarrow A \times A$ and $\top \in \mathcal{P}_A$ is a greatest element of $\mathcal{P}_A$. This interpretation is not new, it goes back to Lawvere and it validates the usual equality axioms $(a \mid \vdash a = a)$ and $(a, b \mid \varphi(a), a = b \vdash \varphi(b))$. We only mention it here, since in the standard references [4, 10], triposes are presented as models for higher order logic *without* equality.

The power types come syntactically with the following rules.

$$\frac{\Gamma, x{:}A \mid \phi(x)}{\Gamma \mid \{x{:}A \mid \phi(x)\} : \mathfrak{P}A} \qquad \frac{\Gamma \mid t : A \qquad \Gamma \mid M : \mathfrak{P}A}{\Gamma \mid t \in M}$$

They are interpreted by the power objects and the interpretation validates

$$\Gamma \mid \vdash t \in \{x{:}A \mid \varphi(x)\} \leftrightarrow \varphi(t)$$

However, the schema

$$m, n : \mathfrak{P}A \mid \forall x{:}A \,.\, x \in m \leftrightarrow x \in n \vdash m = n$$

is only validated in rare cases (for example if the tripos is in fact the subobject fibration of a topos or the fibration of j-stable predicates for some local operator j on a topos).

The *internal language* of a tripos $\mathcal{P} : \mathrm{dom}(\mathcal{P}) \longrightarrow \mathcal{C}$ is a language of higher order logic in the above sense where the symbols for base types are the objects of $\mathcal{C}$, all morphisms $f : A \longrightarrow B$ of $\mathcal{C}$ serve as function symbols of the corresponding signature, and the predicate symbols are the objects of $\mathrm{dom}(\mathcal{P})$ where each $\varphi \in \mathsf{Obj}(\mathcal{C})$ expects a single argument of type $\mathcal{P}(\varphi)$.

The semantics that we always have in mind when using the internal language is given as usual by interpreting all type, function, and predicate symbols by themselves.

2.2. Examples of triposes

1. Let $\mathcal{C}$ be a category, and let $\boldsymbol{M}\mathcal{C}$ be the full subcategory of $\mathcal{C}{\downarrow}\mathcal{C}$ on the monic arrows. The codomain projection $\partial_1 : \boldsymbol{M}\mathcal{C} \longrightarrow \mathcal{C}$ is a fibration iff in $\mathcal{C}$, pullbacks of monos along arbitrary morphisms exist.

    We call this fibration the *subobject fibration* of $\mathcal{C}$ and denote it by $\boldsymbol{S}\mathcal{C}$. A sufficient condition for $\boldsymbol{S}\mathcal{C}$ to be a tripos is that $\mathcal{C}$ is a topos, and this claim is most easily verified in the internal logic of $\mathcal{C}$ (hint: id_Ω is the generic predicate). However, there are also other categories aside from toposes whose subobject fibrations are triposes, for example the category $\mathbf{Asm}(\mathcal{A})$ for a partial combinatory algebra $\mathcal{A}$ (See [13]).

2. Let $\mathcal{E}$ be a topos and $(A, (\leq) \rightarrowtail A \times A)$ an internal locale (complete Heyting algebra). We view A as an internal category and construct its externalisation $[A]$[6]. Spelled out, this looks as follows.

   The objects of $\text{dom}([A])$ are maps $\varphi : I \longrightarrow A$ and morphisms from $\varphi : I \longrightarrow A$ to $\psi : J \longrightarrow A$ are maps $u : I \longrightarrow J$ with the property that $\langle \varphi, \psi u \rangle$ factors through $(\leq)$, or in other words the judgement $(i \mid\vdash \varphi(i) \leq \psi(ui))$ holds in $\mathcal{E}$. The functor part is given by the obvious projection.

   The ensuing fibration is a tripos with generic predicate id_A.

   Of course, when we take $(\Omega, \Rightarrow)$ as our locale, we obtain a fibration that is equivalent to the subobject fibration as in Example 1.

3. A special case of 2 that is a bit more general than 1 is the following. Let $\mathcal{E}$ be a topos, and let $j : \Omega \longrightarrow \Omega$ be a *local operator* (see e.g. [7]; local operators are traditionally called *Lawvere-Tierney topologies*). We denote by Ω_j the image of j as a subobject of Ω. It is an internal locale with respect to the ordering induced by Ω. The induced tripos consists of the j-stable predicates.

4. Let $\mathcal{E}$ be a topos and $(\mathcal{A}, (- \cdot -) : \mathcal{A} \times \mathcal{A} \rightharpoonup \mathcal{A})$ an internal partial combinatory algebra (pca) in the sense of [1].

   The total category of the *realisability tripos* $\mathbf{rt}(\mathcal{A})$ has arrows $\varphi : I \longrightarrow \Omega^{\mathcal{A}}$ as objects and given two such objects $\varphi : I \longrightarrow \Omega^{\mathcal{A}}$ and $\psi : J \longrightarrow \Omega^{\mathcal{A}}$ a morphism from φ to ψ is an arrow $u : I \longrightarrow J$ such that the topos validates

   $$\vdash \exists e{:}\mathcal{A}\ \forall i\ \forall r{:}\mathcal{A} \,.\, r \in \varphi(i) \Rightarrow e{\cdot}r \in \psi(ui) \tag{2.1}$$

   and again the functor is the obvious projection.

   To see that this is a fibration of pHa's, we have to find corresponding constructions for all connectives of predicate calculus. This is presented e.g. in [13]. A generic predicate is given by $\text{id}_{\Omega^{\mathcal{A}}}$

5. Let $\mathcal{E}$ be a topos and $\mathcal{A}_{\#} \rightarrowtail \mathcal{A}$ an *inclusion* of pca's in $\mathcal{E}$; that is a pca $\mathcal{A}$ together with a subobject $\mathcal{A}_{\#}$ that is closed under application and contains two elements that serve as k and s for $\mathcal{A}$.

   The total category of the *relative realisability tripos* $\mathbf{rt}_{\mathsf{r}}(\mathcal{A}_{\#}, \mathcal{A})$ is the lluf subcategory of $\text{dom}(\mathbf{rt}(\mathcal{A}))$ on the arrows for which the strengthening

   $$\vdash \exists e{:}\mathcal{A}_{\#}\ \forall i\ \forall r{:}\mathcal{A} \,.\, r \in \varphi(i) \Rightarrow e{\cdot}r \in \psi(ui)$$

   of (2.1) holds in $\mathcal{E}$, the functor part is again given by projection.

   The first order structure and the generic predicate are given in the same way as for $\mathbf{rt}(\mathcal{A})$, so the only thing that distinguishes $\mathbf{rt}(\mathcal{A})$ and $\mathbf{rt}(\mathcal{A}_{\#}, \mathcal{A})$ is that for the latter one the entailment relation in the fibres is sparser.

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[6]The notation is from [5] with the difference that they use $[A]$ not for the functor, but for the total category.

6. Let $\mathcal{A}$ be a pca in $\mathcal{E}$. The *modified realisability tripos* $\mathbf{rt_m}(\mathcal{A})$ is given as follows. Objects of the total category are pairs $\varphi = (\varphi_a, \varphi_p)$ of maps with $\varphi_a, \varphi_p : I \longrightarrow \Omega^{\mathcal{A}}$ such that

   a) $\forall i \,.\, \varphi_a(i) \subset \varphi_p(i)$, and

   b) $\bigcap_{i \in I} \varphi_p(i)$ is inhabited.

   A morphism from φ to ψ is a morphism $u : I \longrightarrow J$ such that

   $$\vdash \exists e{:}\mathcal{A} \; \forall i \; \forall r{:}\mathcal{A} \,.\, (r \in \varphi_a(i) \Rightarrow e{\cdot}r \in \psi_a(ui)) \wedge (r \in \varphi_p(i) \Rightarrow e{\cdot}r \in \psi_p(ui))$$

   The functor part is as above. A generic predicate can be defined as follows.

   $$\mathsf{Prop} = \{(M, N) \in \Omega^{\mathcal{A}} \times \Omega^{\mathcal{A}} \mid M \subset N, \mathsf{k} \in N\}$$
   $$\mathsf{tr}_a(M, N) = M$$
   $$\mathsf{tr}_p(M, N) = N$$

   This tripos and the corresponding topos were first examined by van Oosten in [14].

2.3. Tripos morphisms

Fibrations form 2-categories in a natural way, the 1– and 2–cells being the fibred functors and fibred natural transformations. For triposes, we only consider fibred functors that are compatible with a part of the logical structure.

Definition 2.3 *Let $\mathcal{P} : \mathrm{dom}(\mathcal{P}) \longrightarrow \mathcal{C}$ and $\mathcal{Q} : \mathrm{dom}(\mathcal{Q}) \longrightarrow \mathcal{D}$ be two triposes.*

- *A (cartesian) tripos morphism is a pair of functors (F, Φ) with $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $\Phi : \mathrm{dom}(\mathcal{P}) \longrightarrow \mathrm{dom}(\mathcal{Q})$ such that*

   1. *F preserves finite limits,*

   2. *the square*

$$\begin{array}{ccc} \mathrm{dom}(\mathcal{P}) & \xrightarrow{\ \Phi\ } & \mathrm{dom}(\mathcal{Q}) \\ {\scriptstyle \mathcal{P}}\downarrow & & \downarrow{\scriptstyle \mathcal{Q}} \\ \mathcal{C} & \xrightarrow[\ F\ ]{} & \mathcal{D} \end{array}$$

   commutes (on the nose),

   3. *Φ maps cartesian arrows to cartesian arrows, and*

   4. *For each $A \in \mathsf{Obj}(\mathcal{C})$, the restricted functor $\Phi_A : \mathcal{P}_A \longrightarrow \mathcal{Q}_{FA}$ preserves finite meets.*

- *A tripos morphism (F, Φ) is called* regular *if Φ maps cocartesian arrows to cocartesian arrows.*

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Conditions 2 and 3 in the definition of cartesian tripos morphism say that Φ is a fibred functor over F. The others are compatibility postulates. Their effect is best understood in terms of the internal logic.

If (F, Φ) is an arbitrary fibred functor, then we can infer

$$\frac{x{:}A \mid \varphi(x) \vdash \psi(x)}{x{:}FA \mid \Phi\varphi(x) \vdash \Phi\psi(x)}$$

If F is cartesian, then we can use several variables:

$$\frac{x_1{:}A_1 \ldots x_n{:}A_n \mid \varphi(x_1 \ldots x_n) \vdash \psi(x_1 \ldots x_n)}{x_1{:}FA_1 \ldots x_n{:}FA_n \mid \Phi\varphi(x_1 \ldots x_n) \vdash \Phi\psi(x_1 \ldots x_n)}$$

Moreover, cartesianness makes the translation commute with the interpretation of terms.

If Φ satisfies 4., the translation along (F, Φ) is compatible with interpretation of Horn clauses.

$$\frac{\Gamma \mid \varphi_1 \ldots \varphi_n \vdash \psi}{\Phi\Gamma \mid \Phi\varphi_1 \ldots \Phi\varphi_n \vdash \Phi\psi}$$

Clearly, this scheme can be extended, e.g. interpretation of formulas of regular logic commutes with translation along regular tripos morphisms. However, it will turn out later that plain Horn logic takes us surprisingly far.

Examples of tripos morphisms

It turns out that the most convenient and natural way to define tripos morphisms is to pretend that the triposes were not fibred, but indexed posets; i.e. instead of defining a fibred functor between the triposes we define a pseudo-natural transformation between the associated presheaves of **pHa**'s. Thus, if we want to define a tripos morphism between two triposes $\mathcal{P} : \mathrm{dom}(\mathcal{P}) \longrightarrow \mathcal{C}, \mathcal{Q} : \mathrm{dom}(\mathcal{Q}) \longrightarrow \mathcal{D}$, we have to provide a cartesian functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ and for each $A \in \mathsf{Obj}(\mathcal{C})$ a monotone map $\Phi_A : \mathcal{P}_A \longrightarrow \mathcal{Q}_{FA}$ such that

- **Naturality:** For all $f : A \longrightarrow B$ in $\mathcal{C}$ and all $\psi \in \mathsf{Obj}(\mathcal{P}_B)$ we have

$$(Ff)^*(\Phi_B\psi) \dashv\vdash \Phi_A(f^*\psi)$$

- **Meet-stability:** All Φ_A preserve finite meets.

Existential quantification appears in the indexed setting in the form of left adjoints to the reindexing maps, and a tripos morphism is regular iff we have

- **Regularity:** For all $f : A \longrightarrow B$ in $\mathcal{C}$ and all $\psi \in \mathsf{Obj}(\mathcal{P}_A)$ we have

$$\exists_{Ff}(\Phi_A\psi) \dashv\vdash \Phi_B(\exists_f\psi)$$

Consider a tripos $\mathcal{P}$ over a topos $\mathcal{E}$. There is a cartesian tripos morphism $(\mathrm{id}_{\mathcal{E}}, \Delta)$: $\boldsymbol{S}\mathcal{E} \longrightarrow \mathcal{P}$ with constant base, defined by

$$\Delta_A(m : U \rightarrowtail A) = (a \mid \exists u . \, m(u) = a),$$

where the right hand formula has to be interpreted in $\mathcal{P}$.

Lemma 2.4 *Let $\mathcal{P}$ be a tripos over a topos $\mathcal{E}$. The following are equivalent.*

1. *$\mathcal{P}$ recognises* epis; *i.e. for each epimorphism $e : A \twoheadrightarrow Q$ in $\mathcal{E}$,*

$$q{:}Q \mid\vdash \exists a . \, e(a) = q$$

   holds in $\mathcal{P}$.

2. *For each epimorphism $e : A \twoheadrightarrow Q$ in $\mathcal{E}$, the predicate $\exists_e \top$ is maximal in $\mathcal{P}_Q$.*

3. *The tripos morphism $(\mathrm{id}_{\mathcal{E}}, \Delta)$ defined above is regular.*

4. *$\mathcal{P}$ has fibrewise quantification in the sense of Pitts [10].*

Proof. The equivalence of 1–3 is straightforward, and the equivalence of 2. and 4. is proved in [10]. ∎

Just as **Set** has a special status among categories, **Set**-based triposes have special status among triposes. This can be seen from the following construction that only works for **Set**-based triposes.

Let $\mathcal{P}$ be a tripos over **Set**. The tripos morphism $(\mathrm{id}_{\mathbf{Set}}, \Gamma) : \mathcal{P} \longrightarrow \boldsymbol{S}(\mathbf{Set})$ is given by

$$\Gamma_I(\varphi) = \{i \mid \top_1 \vdash_1 \varphi(i)\}$$

Γ is sometimes regular (e.g. for realisability triposes) and sometimes not (e.g. for locale-induced triposes for locales where $\top$ is a non-trivial join). Now, the natural question to ask is: 'Is this Γ the same as comprehension[7] in the sense of Lawvere?' — It is, whenever $\mathcal{P}$ *has* comprehension, and this is the case if and only if $\mathcal{P}$ comes from a locale (without proof).

Another natural source for tripos morphisms are meet preserving maps between locales. Indeed, given two locales A, B in the same topos $\mathcal{E}$ and a map $f : A \longrightarrow B$ that preserves finite meets, we can define a cartesian tripos morphism $(\mathrm{id}_{\mathcal{E}}, [f]) : [A] \longrightarrow [B]$ by letting

$$[f]_I(\varphi : I \longrightarrow A) = f \circ \varphi.$$

This tripos morphism is regular iff f preserves all joins.

Finally, we give a somehow peculiar example from realisability that we will need later. Consider the modified realisability tripos $\mathbf{rt_m}(\mathcal{A})$ for a pca $\mathcal{A}$ over **Set**. Although this tripos is 2-valued in the sense that the only predicates in the fibre over the terminal

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[7]If you do not know comprehension, just ignore this paragraph

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object are true and false, the tripos morphism $(\mathrm{id}, \Delta) : \boldsymbol{S}(\mathbf{Set}) \longrightarrow \mathbf{rt_m}(\mathcal{A})$ is not the only nontrivial embedding of the classical predicates into the modified realisability tripos. The reason for this is that the predicates of the form ΔU in $\mathbf{rt_m}(\mathcal{A})$ are in general not $\neg\neg$-stable. If we postcompose (id, Δ) with double negation, we obtain the tripos transformation (id, ∇) with

$$(\nabla_A U)_a(x) = \{r \in \mathcal{A} \mid x \in U\} \qquad \text{and}$$
$$(\nabla_A U)_p(x) = \mathcal{A}$$

for $U \subset A$ and $x \in A$. This tripos morphism commutes with existential quantification only along epimorphisms. In particular, it does not commute with equality.

2.4. Tripos transformations and 2-categories of triposes

We will consider two 2-categories of triposes. The first has the cartesian tripos morphisms as 1-cells and the second contains just the regular tripos morphisms. The 2-cells will be the *fibred natural transformations* (we will simply call them transformations), that we will now define.

Definition 2.5 *Let* $\mathcal{P} : \mathrm{dom}(\mathcal{P}) \longrightarrow \mathcal{C}$ *and* $\mathcal{Q} : \mathrm{dom}(\mathcal{Q}) \longrightarrow \mathcal{D}$ *be two triposes and consider two cartesian tripos morphisms* $(F, \Phi), (G, \Gamma) : \mathcal{P} \longrightarrow \mathcal{Q}$. *A* transformation *from* (F, Φ) *to* (G, Γ) *is a natural transformation* $\eta : F \Longrightarrow G$ *with the property that for all* $A \in \mathsf{Obj}(\mathcal{C})$ *and all* $\psi \in \mathsf{Obj}(\mathcal{P}_A)$, *we have*

$$a{:}FA \mid \Phi\psi(a) \vdash \Gamma\psi(\eta_A(a)),$$

or diagrammatically

$$\psi \qquad\qquad \Phi\psi \longrightarrow \Gamma\psi$$

$$A \qquad\qquad FA \xrightarrow{\ \eta_A\ } GA$$

The 2-categories of triposes are then defined as follows.

Definition 2.6   • $\mathfrak{Trip}_\mathsf{c}$ *is the 2-category of triposes, cartesian tripos morphisms and transformations.*

     • $\mathfrak{Trip}_\mathsf{r}$ *is the 2-category of triposes, regular tripos morphisms and transformations.*

Observe how these 2-categories correspond to the 2-categories $\mathfrak{Top}_\mathsf{c}$ and $\mathfrak{Top}_\mathsf{r}$ defined in section 1.2.

To conclude this section, we define the forgetful 2-functor from $\mathfrak{Top}_\mathsf{c}$ to $\mathfrak{Trip}_\mathsf{c}$ that we want to left adjoin later.

Definition 2.7 *The 2-functor* $\boldsymbol{S} : \mathfrak{Top}_\mathsf{c} \longrightarrow \mathfrak{Trip}_\mathsf{c}$ *is defined by*

$$\begin{array}{ccc}
\mathcal{E} & \longmapsto & \boldsymbol{S}\mathcal{E} \\
(F : \mathcal{E} \longrightarrow \mathcal{F}) & \longmapsto & (F, \boldsymbol{M}(F)) \\
(\eta : F \longrightarrow G) & \longmapsto & \eta
\end{array}$$

where $\boldsymbol{M}(F) : \boldsymbol{M}(\mathcal{E}) \longrightarrow \boldsymbol{M}(\mathcal{F})$ maps $m : U \rightarrowtail A$ to $Fm : FU \rightarrowtail FA$ which is a mono again, because F preserves finite limits.

It is straightforward to see that $\boldsymbol{S}$ restricts to a pseudofunctor of type $\mathfrak{Top}_r \longrightarrow \mathfrak{Trip}_r$ which we also denote by $\boldsymbol{S}$. In the next section, we will show that this restricted $\boldsymbol{S}$ is left biadjointable.

3. The tripos-to-topos construction

In this section, we want to show that the 2-functor $\boldsymbol{S} : \mathfrak{Top}_r \longrightarrow \mathfrak{Trip}_r$ has a left biadjoint $\boldsymbol{T} : \mathfrak{Trip}_r \longrightarrow \mathfrak{Top}_r$. To achieve this, we have to show that for a given tripos, the presheaf $\mathfrak{Trip}_r(\mathcal{P}, \boldsymbol{S}-)$ of categories is representable, what means that we have to find a topos $\boldsymbol{T}\mathcal{P}$ and an object (D, Ξ) of $\mathfrak{Trip}_r(\mathcal{P}, \boldsymbol{S}\boldsymbol{T}\mathcal{P})$ such that the induced pseudo-natural transformation $\mathfrak{Top}_r(\boldsymbol{T}\mathcal{P}, -) \longrightarrow \mathfrak{Trip}_r(\mathcal{P}, \boldsymbol{S}-)$ is an equivalence.

Fortunately, to show that a pseudo-natural transformation is an equivalence it suffices to show that it is a fibrewise equivalence, whence it remains to show that for a given topos $\mathcal{E}$, the functor

$$
\begin{array}{rcl}
\mathfrak{Top}_r(\boldsymbol{T}\mathcal{P}, \mathcal{E}) & \longrightarrow & \mathfrak{Trip}_r(\mathcal{P}, \boldsymbol{S}\mathcal{E}) \\
F & \longmapsto & (FD, \boldsymbol{M}(F)\Xi) \\
\eta & \longmapsto & \eta \circ D
\end{array}
$$

is full, faithful and essentially surjective.

Having sketched our general strategy, we start by defining $\boldsymbol{T}\mathcal{P}$ for a tripos $\mathcal{P}$ based on some cartesian category $\mathcal{C}$. $\boldsymbol{T}\mathcal{P}$ is of course the same as the familiar $\mathcal{C}[\mathcal{P}]$ known from [4, 10].

Definition 3.1 *Let $\mathcal{P}$ be a tripos over a cartesian category $\mathcal{C}$. The category $\boldsymbol{T}\mathcal{P}$ is given by the following data:*

- **Objects:**
  Objects of $\boldsymbol{T}\mathcal{P}$ are objects of $\mathcal{C}$ together with partial equivalence relations in $\mathcal{P}$; more precisely, objects of $\boldsymbol{T}\mathcal{P}$ are pairs (A, ρ) with $A \in \mathsf{Obj}(\mathcal{C})$ and $\rho \in \mathcal{P}_{A \times A}$ such that the following judgements hold in $\mathcal{P}$.

  (trans) $\rho(a, b) \wedge \rho(b, c) \vdash \rho(a, c)$
  (symm) $\rho(a, b) \vdash \rho(b, a)$

- **Morphisms:**
  Arrows are given as functional relations (again in $\mathcal{P}$): A morphism from (A, ρ) to (B, σ) is a $(\dashv\vdash)$-equivalence class of predicates over $A \times B$ such that for some (or equivalently any) representative γ the following judgements hold.

  (strict) $\gamma(a, b) \vdash \rho(a) \wedge \sigma(b)$
  (cong) $\gamma(a, b) \wedge \rho(a, a') \wedge \sigma(b, b') \vdash \gamma(a', b')$
  (singval) $\gamma(a, b) \wedge \gamma(a, b') \vdash \sigma(b, b')$
  (tot) $\rho(a) \vdash \exists b \,.\, \gamma(a, b)$

  Here, we use $\rho(a)$ as abbreviation for $\rho(a, a)$ for partial equivalence relations ρ.

  Given a functional relation ϕ, we write $[\phi]$ for the equivalence class, i.e. the corresponding morphism in $\boldsymbol{T}\mathcal{P}$. We use the same notation also for externalisations of internal categories (see example 2 in subsection 2.2), but this will not lead to any confusion.

- **Composition:**
  Given a pair $A \xrightarrow{[\phi]} B \xrightarrow{[\gamma]} C$, we define $[\gamma][\phi] = [\gamma\phi]$, where

  $$\gamma\phi = (a, c \mid \exists b \,.\, \phi(a, b) \wedge \gamma(b, c))$$

- **Identity:**
  The identity morphism of (A, ρ) is $[\rho]$.

It is not difficult to see that $\boldsymbol{T}\mathcal{P}$ is actually a category.

As a side note, the reader may have noticed that we do not need all the structure of the tripos for the construction, but only finite meets and existential quantification. So, can we do the same construction for arbitrary posetal fibrations with finite meets and existential quantification? There is one obstacle: Natural deduction is not conservative over its $(\wedge, \exists)$-fragment, and in particular we can not prove the composition to be associative in the weaker fragment. However, it all works out if we add as an additional axiom the *Frobenius law.*

$$\varphi(x) \wedge \exists y \,.\, \psi(x, y) \vdash \exists y \,.\, \varphi(x) \wedge \psi(x, y) \qquad \text{Frobenius law}$$

This axiom can be appropriately reformulated for fibrations and indeed for posetal fibrations that have finite limits, existential quantification *and* validate Frobenius, the above construction works.

What we gain when starting with a tripos and not with some fibration with less structure, is that the ensuing category is a *topos.*

Theorem 3.2 *For any tripos* $\mathcal{P}$, $\boldsymbol{T}\mathcal{P}$ *is a topos.*

Proof. A detailed proof is given in [10]. We recall the basic steps for reference.

We start with the observation that a morphism $[\phi] : (A, \rho) \longrightarrow (B, \sigma)$ in $\boldsymbol{T}\mathcal{P}$ is a mono iff the tripos validates $(a, a', b \mid \phi(a, b), \phi(a', b) \vdash \rho(a, a'))$. From this we deduce that the subobjects of a given object (A, ρ) can be represented by *strict predicates* in the following sense: A predicate φ on A is called *strict* (with respect to ρ) if $(a \mid \varphi(a) \vdash \rho(a))$ and $(a, a' \mid \varphi(a), \rho(a, a') \vdash \varphi(a'))$ hold. Such a φ induces a mono $[\rho|_\varphi] : (A, \rho|_\varphi) \rightarrowtail (A, \rho)$, where $\rho|_\varphi = (a, a' \mid \varphi(a) \wedge \rho(a, a'))$.

Now we show that $\boldsymbol{T}\mathcal{P}$ has finite limits and power objects.

- **Terminal object:** Given by $(\mathbf{1}, \top_{\mathbf{1}})$.

- **Binary products:** A product of (A, ρ) and (B, σ) is given by $(A \times B, (a, b, a', b' \mid \rho(a, a') \wedge \sigma(b, b')))$. $(a, b, a' \mid \rho(a, a') \wedge \sigma(b))$ gives a representative of the first projection, the second projection is given analogously.

- **Equalisers:** An equaliser of $[\phi], [\gamma] : (A, \rho) \longrightarrow (B, \sigma)$ is induced by the strict predicate $(a \mid \exists b \,.\, \phi(a, b) \wedge \gamma(a, b))$.

- **Power objects:** A power object of (A, ρ) is given by $(\mathfrak{P}A, \mathfrak{P}\rho)$, where

$$\mathfrak{P}\rho = (m, n \mid (\forall a \,.\, a \in m \to \rho(a))$$
$$\wedge (\forall a, a' \,.\, a \in m \wedge \rho(a, a') \to a' \in m)$$
$$\wedge (\forall a \,.\, a \in m \leftrightarrow a \in n))$$

  and the subobject of $A \times \mathfrak{P}A$ giving the element relation is induced by $(a, m \mid \mathfrak{P}\rho(m) \wedge a \in m)$. ∎

Next, we have to define the tentative unit $(D, \Xi) : \mathcal{P} \longrightarrow \boldsymbol{ST}\mathcal{P}$.

Definition 3.3 (The constant objects functor) $D : \mathcal{C} \longrightarrow \boldsymbol{T}\mathcal{P}$ *is defined by*

$$\begin{array}{rcl} A & \longmapsto & (A, (a, a' \mid a = a')) \\ (f : A \to B) & \longmapsto & (a, b \mid f(a) = b) \end{array}$$

Lemma 3.4   *1. D preserves finite limits.*

2. *Every object of $\boldsymbol{T}\mathcal{P}$ is a subquotient of some DA.*

Proof. *Ad 1.* We show that D preserves the terminal object and pullbacks.

The image of the terminal object is $(\mathbf{1}, \top)$ and the unique arrow from (A, ρ) to $(\mathbf{1}, \top)$ is given by $[(a, * \mid \rho(a))]$.

For the pullbacks, first of all observe that a square

$$
\begin{array}{ccc}
(P, \upsilon) & \xrightarrow{\;[o]\;} & (A, \rho) \\
{\scriptstyle [\pi]}\big\downarrow & & \big\downarrow{\scriptstyle [\phi]} \\
(B, \sigma) & \xrightarrow[{[\gamma]}]{} & (C, \gamma)
\end{array}
$$

is a pullback iff the judgements

$$a, b, c \mid \phi(a, c), \gamma(b, c) \vdash \exists p \,.\, o(p, a) \wedge \pi(p, b)$$
$$p, p', a, b \mid o(p, a), o(p', a), \pi(p, b), \pi(p', b) \vdash \upsilon(p, p')$$

hold in $\mathcal{P}$ and then verify that this is the case for the image of a pullback in $\mathcal{C}$ under D.

Ad 2. For an object (A, ρ) of $\boldsymbol{T}\mathcal{P}$, the canonical mono-epi pair is given by

$$DA \xleftarrow{\;[\mathsf{supp}(\rho)]\;} (A, \mathsf{supp}(\rho)) \xrightarrow{\;[\rho]\;} (A, \rho),$$

where

$$\mathsf{supp}(\rho) = (a, a' \mid \rho(a) \wedge a = a')$$

∎

The next lemma anticipates the counit of the biadjunction.

Lemma 3.5 *If $\mathcal{P}$ is the subobject fibration of a topos, then D is an equivalence. Hence $\mathcal{E} \simeq \boldsymbol{TS}\mathcal{E}$ for any topos $\mathcal{E}$.* ∎

In the proof of 3.2, we remarked that the subobjects of a given object (A, ρ) of $\boldsymbol{T}\mathcal{P}$ correspond to the predicates on A that are strict with respect to ρ. As all predicates on A are strict with respect to the equality, the subobjects of DA correspond precisely to the predicates on A. These subobjects occur so often, that we introduce a special notation: We write $\|\psi\|$ for the canonical subobject of DA that is induced by $\psi \in \mathcal{P}_A$.

Lemma 3.6 *The assignment*

$$
\begin{array}{ccc}
\psi & \longmapsto & \|\psi\| \\
\vdots & & \vee \\
\big\downarrow & & \big\downarrow \\
A & & DA
\end{array}
$$

defines a fibred functor $\Xi : \mathrm{dom}(\mathcal{P}) \longrightarrow \boldsymbol{M}(\boldsymbol{T}\mathcal{P})$ over D that is a fibrewise equivalence in the sense that all Ξ_A's are equivalences.

Because Ξ is a fibrewise equivalence, (D, Ξ) is a tripos morphism that preserves all first order logical structure and thus is in particular regular. ∎

Having defined the candidate for the unit of the adjunction, we are now ready to prove the main theorem.

Theorem 3.7 *For any topos $\mathcal{E}$, the functor*

$$
\begin{array}{ccc}
(\check{-}) \ : \ \mathfrak{Top}_{\mathsf{r}}(\boldsymbol{T}\mathcal{P}, \mathcal{E}) & \longrightarrow & \mathfrak{Trip}_{\mathsf{r}}(\mathcal{P}, \boldsymbol{S}\mathcal{E}) \\
\check{F} & = & \boldsymbol{S}F \circ (D, \Xi) \\
\check{\eta} & = & \eta \circ D
\end{array}
$$

is full, faithful and essentially surjective.

Proof. We first show that $(\check{-})$ is faithful. Consider two natural transformations

$$
\eta, \theta : F \longrightarrow G : \boldsymbol{T}\mathcal{P} \longrightarrow \mathcal{E}
$$

such that $\eta \circ D = \theta \circ D$. Let $X \in \mathsf{Obj}(\boldsymbol{T}\mathcal{P})$ and represent X as as a subquotient

$$
DA \longleftarrow\!\!\!< Q \longrightarrow\!\!\!\!\!\rightarrow X
$$

The diagram

$$
\begin{array}{ccccc}
FDA & \longleftarrow\!\!\!< & FQ & \longrightarrow\!\!\!\!\!\rightarrow & FX \\
\big\downarrow{\scriptstyle \eta_{DA}=\theta_{DA}} & & \big\downarrow & & \big\downarrow \\
GDA & \longleftarrow\!\!\!< & GQ & \longrightarrow\!\!\!\!\!\rightarrow & GX
\end{array}
$$

has at most one pair of mediating arrows, but because of naturality, η_Q, η_X as well as θ_Q, θ_X mediate. Therefore $\eta_X = \theta_X$.

The proof that $(\tilde{-})$ is full is more interesting. Consider a transformation

$$\zeta : (FD, \boldsymbol{M}F \ \Xi) \longrightarrow (GD, \boldsymbol{M}G \ \Xi) : \mathcal{P} \longrightarrow \boldsymbol{S}\mathcal{E},$$

i.e., a natural transformation from FD to GD such that for all $A \in \mathsf{Obj}(\mathcal{C})$ and (because Ξ is fibrewise essentially surjective) for all $\psi \in (\boldsymbol{S}\boldsymbol{T}\mathcal{P})_{DA}$ we have

$$a \mid F\psi(a) \vdash G\psi(\zeta_A(a)).$$

We have to construct a natural transformation $\tilde{\zeta} : F \longrightarrow G$ such that $\tilde{\zeta} \circ D = \zeta$. To construct $\tilde{\zeta}_X$ for some $X \in \mathsf{Obj}(\boldsymbol{T}\mathcal{P})$, we first choose a subquotient representation $DA \overset{m}{\longleftarrow} Q \overset{e}{\longrightarrow} X$ with corresponding partial equivalence relation $\rho \rightarrowtail DA \times DA \cong D(A \times A)$. Since we have

$$F\rho(a, a') \vdash G\rho(\zeta_A(a), \zeta_A(a')),$$

there is a unique pair of mediating arrows in

$$
\begin{array}{ccc}
FDA \longleftarrow FQ \longrightarrow FX \\
\zeta_A \downarrow \qquad \vdots \qquad \vdots \\
GDA \longleftarrow GQ \longrightarrow GX
\end{array}
$$

(Lemma A.1), and the right hand vertical arrow gives us $\tilde{\zeta}_X$.

To see that this really gives a natural transformation, we still have to check the naturality condition. Let $X' \in \mathsf{Obj}(\boldsymbol{T}\mathcal{P})$ be a second object, with subquotient representation $DA' \overset{m'}{\longleftarrow} Q' \overset{e'}{\longrightarrow} X'$ and corresponding partial equivalence relation $\psi' \rightarrowtail D(A' \times A')$, and consider a morphism $f : X \longrightarrow X'$. We define a predicate on $DA \times DA' \cong D(A \times A')$ by

$$\theta = (a, a' \mid \exists q, q' \,.\, m(q) = a \wedge m'(q') = a' \wedge f(e(q)) = e'(q')).$$

Again, as ζ is a tripos transformation, we have

$$a, a' \mid F\theta(a, a') \vdash G\theta(\zeta_A(a), \zeta_{A'}(a')),$$

but because θ is built up completely from regular logic which is stable under F and G, this can be rephrased as

$$
\exists q, q' \,.\, Fm(q) = a \wedge Fm'(q') = a' \wedge F(fe)(q) = Fe'(q')
$$
$$
\vdash \exists q, q' \,.\, Gm(q) = \zeta_A(a) \wedge Gm'(q') = \zeta_{A'}(a') \wedge G(fe)(q) = Ge'(q').
$$

From this, we can deduce

$$x{:}FX \mid \vdash \zeta_{X'}(Ff(x)) = Gf(\zeta_X(x)).$$

Hint: Look at the diagram

$$DA \xmapsfrom{m} U \xrightarrow{e} X \xrightarrow{f} X' \xmapsfrom{e'} U' \xrightarrow{m'} DA'$$

$$
\begin{array}{ccccccccccc}
FDA & \xleftarrow{} & FU & \xrightarrow{} & FX & \xrightarrow{} & FX' & \xleftarrow{} & FU' & \xrightarrow{} & FDA' \\
\downarrow{\scriptstyle\zeta_A} & & \downarrow & & \downarrow{\scriptstyle\tilde{\zeta}_X} & {\scriptstyle ?} & \downarrow{\scriptstyle\tilde{\zeta}_{X'}} & & \downarrow & & \downarrow{\scriptstyle\zeta_{A'}} \\
GDA & \xleftarrow{} & GU & \xrightarrow{} & GX & \xrightarrow{} & GX' & \xleftarrow{} & GU' & \xrightarrow{} & GDA'
\end{array}
$$

and think in terms of relations. θ, for example, is given as relational composition of the top row. Monomorphisms and epimorphisms can be neatly characterised in terms of relational composition. A morphism f is monic iff $f^\circ \circ f = \mathrm{id}$ and epic iff $f \circ f^\circ = \mathrm{id}$ $((-)^\circ$ denotes the opposite of a relation). Using these principles, the verification of the claimed equality is straightforward.

It remains to show that $(\breve{})$ is essentially surjective. Consider a regular tripos transformation

$$(F, \Phi) : \mathcal{P} \longrightarrow S\mathcal{E}.$$

We have to construct a regular functor $G = \widehat{(F, \Phi)} : T\mathcal{P} \longrightarrow \mathcal{E}$ such that

$$(F, \Phi) \cong (GD, (\boldsymbol{M}G)\Xi). \tag{3.1}$$

Because $\mathcal{E} \simeq TS\mathcal{E}$ by Lemma 3.5, we may construct a functor of type $T\mathcal{P} \longrightarrow TS\mathcal{E}$ instead, and this can be done simply by letting

$$
\begin{array}{rcl}
(A, \rho) & \longmapsto & (FA, \Phi\rho) \\
[\phi] & \longmapsto & [\Phi\phi].
\end{array}
$$

It is easy to deduce from the regularity of (F, Φ) that this is well defined and functorial. Limiting cones can be characterised completely in regular logic (as exemplified for pullbacks in the proof of Lemma 3.4) and are therefore also stable under the construction.

Finally, a morphism $(A, \rho) \xrightarrow{[\varepsilon]} (B, \sigma)$ in $T\mathcal{P}$ is epic iff $(\sigma(b) \vdash \exists a \,.\, \varepsilon(a, b))$ and this is stable under (F, Φ) as well. The verification of (3.1) is then straightforward. ∎

4. Weakly complete objects

When one first encounters the category $\boldsymbol{T}\mathcal{P}$, e.g. for the effective topos, one will feel that its structure is somehow difficult to access and to comprehend. One reason for this is its notion of morphism that differs from what we are used to from "normal" mathematics. One effect of this is that the question whether two objects are isomorphic becomes nontrivial. Consider for example the tripos $[\mathbb{B} \times \mathbb{B}]$ over **Set**. We define a predicate φ over **2** by

$$
\begin{array}{rccl}
\varphi : & \mathbf{2} & \longrightarrow & \mathbb{B} \times \mathbb{B} \\
& 0 & \longmapsto & (\mathsf{true}, \mathsf{false}) \\
& 1 & \longmapsto & (\mathsf{false}, \mathsf{true})
\end{array}
$$

Then the object $\|\varphi\|$ is terminal, a fact that becomes only clear by constructing a proof (or by sheaf theoretic intuition).

The situation becomes a bit more intuitive if we restrict our attention to certain well-behaved representatives of (isomorphism classes of) objects, the so-called weakly complete ones.

Definition 4.1 *Let $\mathcal{P}$ be a tripos over a cartesian category $\mathcal{C}$.*

- *Let $[\phi] : (A, \rho) \longrightarrow (B, \sigma)$ in $\boldsymbol{T}\mathcal{P}$. We say that $f : A \longrightarrow B$ is a tracking morphism for $[\phi]$ (alternatively '$[\phi]$ is tracked by f') if one of the following equivalent judgements*

  1. $(a, b \mid \phi(a, b) \dashv\vdash \rho(a) \wedge \sigma(fa, b))$

  2. $(a \mid \rho(a) \vdash \phi(a, fa))$

  holds in $\mathcal{P}$.

- *An object $(A, \sigma) \in \mathsf{Obj}(\boldsymbol{T}\mathcal{P})$ is called weakly complete if all morphisms in $\boldsymbol{T}\mathcal{P}$ with codomain (A, σ) are tracked by some arrow in the base.*

The relevance of tracking arrows and weakly complete objects is that we get something like *skolem functions* (or better skolem morphisms) witnessing in the base certain existence statements (especially totality-statements for functional relations) in the fibration. In this 'witnessed' form, the judgements are stable under tripos morphisms even if these are merely cartesian, for the simple reason that we do not use the $\exists$-symbol. Concretely, we will make use of the following lemma.

Lemma 4.2 *Let $[\phi] : (A, \rho) \longrightarrow (B, \sigma)$ and $[\gamma] : (B, \sigma) \longrightarrow (C, \tau)$ be morphisms in $\boldsymbol{T}\mathcal{P}$, with tracking morphisms f and g, and let $(F, \Phi) : \mathcal{P} \longrightarrow \mathcal{Q}$ be a tripos morphism.*

1. *gf is a tracking morphism for $[\gamma][\phi] = [\gamma\phi]$.*

2. *$\Phi\phi$ is a functional relation with respect to $(FA, \Phi\rho)$ and $(FB, \Phi\sigma)$ and Ff is a tracking morphism for $[\Phi\phi]$*

3. *If $[\phi]$ is a monomorphism then $[\Phi\phi]$ is also a monomorphism.*

4. $[\Phi(\gamma\phi)] = [\Phi\gamma\Phi\phi]$, *i.e. application of* Φ *commutes with relational composition.*

Proof. 1, 2 and 3 are straightforward, and 4 follows from 1 and 2 because $F(gf)$ is a tracking morphism of $[\Phi(\gamma\phi)]$ as well as $\Phi\gamma\Phi\phi$. $\blacksquare$

It is important to notice that being weakly complete should not be seen as a property of objects, but as a property of *representations* of objects. In particular, the class of weakly complete objects is not closed under isomorphisms. Rather, every object is isomorphic to a weakly complete one (and only this property makes them interesting for us).

Lemma 4.3 *Every* $(A, \rho) \in \mathsf{Obj}(\boldsymbol{T}\mathscr{P})$ *is isomorphic to a weakly complete object.*

Proof. The object is $(\mathfrak{P}A, S_\rho)$ with

$$S_\rho = (m, n \mid \exists a \,.\, \rho(a) \wedge \forall a' \,.\, a' \in m \leftrightarrow a' \in n \leftrightarrow \rho(a, a')).$$

'S' is for *singleton*, because $S_\rho(m)$ has to be read as 'm is a singleton with respect to ρ'. It is straightforward to verify that the predicate $\xi_{(A,\rho)} = (a, m \mid S_\rho(m) \wedge a \in m)$ represents an isomorphism from (A, ρ) to $(\mathfrak{P}A, S_\rho)$. We have to show that $(\mathfrak{P}A, S_\rho)$ is weakly complete. Let $[\phi] : (B, \sigma) \longrightarrow (\mathfrak{P}A, S_\rho)$. By the definition of tripos, there is a morphism $f : B \longrightarrow \mathfrak{P}A$ with

$$(b, a \mid \exists m \,.\, \phi(b, m) \wedge a \in m) \rightsquigarrow \ni_A$$

$$B \times A \xrightarrow{\quad f \times \mathrm{id}_A \quad} \mathfrak{P}A \times A$$

that is

$$b, a \mid \exists m \,.\, \phi(b, m) \wedge a \in m \dashv\vdash a \in fb$$

and it is again routine to check that this f tracks $[\phi]$. $\blacksquare$

Of course, because its codomain is weakly complete, $\gamma_{(A,\rho)}$ itself is tracked by some arrow and this arrow is in fact $\{\rho\}$.

We denote the full subcategory of weakly complete objects of $\boldsymbol{T}\mathscr{P}$ by $\mathsf{wc}(\boldsymbol{T}\mathscr{P})$. The previous result tells us that the embedding $\mathsf{wc}(\boldsymbol{T}\mathscr{P}) \hookrightarrow \boldsymbol{T}\mathscr{P}$ is an equivalence.

In the sequel, we will need the singleton predicate mostly for constant objects $(A, =_A)$. There, we will write S_A instead of $S_{=_A}$ to avoid double indizes.

Next, we present a technical lemma that helps to decide wether a given object is weakly complete.

Lemma 4.4 *1. Let* $\sigma \in \mathcal{P}(A \times A)$ *be an equivalence relation on* (A, ρ), *i.e.* σ *transitive and symmetric such that*

$$\rho(a, b) \vdash \sigma(a, b) \quad and \quad \sigma(a) \vdash \rho(a).$$

Then $[\phi] : (A, \sigma) \longrightarrow (C, \tau)$ *has a tracking arrow iff the composition* $[\phi][\sigma]$ *with the quotient mapping has a tracking arrow.*

2. (C, τ) *is weakly complete iff all morphisms of the form* $\|\psi\| \longrightarrow (C, \tau)$ *(i.e. with canonically subconstant domain) have tracking arrows.*

Proof. 1 is straightforward and 2 follows from 1 and the second proposition of 3.4. ∎

$\boldsymbol{T}\mathcal{P}((A, \rho), (B, \sigma))$ **as a subquotient of** $\mathcal{C}(A, B)$

By definition, every morphism with weakly complete codomain has a tracking arrow. Conversely, for a given arrow $f : A \longrightarrow B$ in the base, the predicate

$$a, b \mid \rho(a) \wedge \sigma(fa, b)$$

represents a morphism from (A, ρ) to (B, σ) *iff* we have

$$a, a' \mid \rho(a, a') \vdash \sigma(fa, fa').$$

Abusing notation, we denote this morphism also by $[f]$. Given two such arrows $f, g : A \longrightarrow B$, we have

$$[f] = [g] \quad \text{iff} \quad (a \mid \rho(a) \vdash \sigma(fa, ga)).$$

Thus, for weakly complete (B, σ), we obtain a description of $\boldsymbol{T}\mathcal{P}((A, \rho), (B, \sigma))$ as a subquotient of $\mathcal{C}(A, B)$.

4.1. Weakly complete objects and 'canonical' constructions

Weakly complete objects enable us to work with maps instead of relations, therefore we would like to use them whenever possible. Could we maybe restrict our attention exclusively to the category $\mathsf{wc}(\boldsymbol{T}\mathcal{P})$? — In principle this is possible, but the 'completion' process described in 4.3 increases the complexity of the representations. In particular we have to check for all categorical operations like constant objects, limits and colimits whether their underlying canonical constructions yield weakly complete objects/transfer weakly complete objects to weakly complete objects, or if we need additional completion.

Constant objects

Constant objects are not weakly complete in general. To see this, consider $D\boldsymbol{2}$ in $\boldsymbol{T}[\mathbb{B} \times \mathbb{B}]$. This object has four global elements (because it corresponds to $(\boldsymbol{2}, \boldsymbol{2})$ in $\mathbf{Set} \times \mathbf{Set}$) and these can not possibly all be tracked by functions in the base.

In realisability toposes, however, non-empty constant objects *are* weakly complete. This follows from Lemma 4.4 and the well-known fact that morphisms between assemblies can be represented by functions. The same argument works for relative realisability.

In modified realisability this argument fails, because there the constant objects lack the assembly property [8]. Consequently, constant objects in modified realisability are not in general weakly complete.

---

[8] By 'assembly property', I mean here just the fact that morphisms between assemblies are representable by arrows in the base.

Finite limits

Lemma 4.5 *Canonical subobjects and products of weakly complete objects are weakly complete* ∎

Arbitrary finite limits can be constructed canonically as subobjects of products. From the preceding lemma follows that this construction returns a weakly complete result on weakly complete inputs.

When forming such a limit, the predicate on the product that gives us the subobject is given as a formula involving data from the underlying diagram. *This predicate can be expressed using only Horn logic!* This is easy to see and will be important later.

Another consequence of the lemma is that non-empty assemblies in (relative) realisability toposes are weakly complete.

Finite colimits

If canonical quotients of weakly complete objects were always weakly complete, then the last sentence of the preceding section together with Lemma 3.4 would imply that all non-initial objects of (relative) realisability toposes are weakly complete. This is too good to be true, and here is a counterexample in the effective topos:

$\{0\}$	$\{2\}$	$\varnothing$	$\varnothing$	0	$\{1\}$	$\varnothing$	$\varnothing$
$\{1\}$	$\{0\}$	$\varnothing$	$\varnothing$	1	$\{2\}$	$\varnothing$	$\varnothing$
$\varnothing$	$\varnothing$	$\{0\}$	$\varnothing$	2	$\varnothing$	$\{1\}$	$\varnothing$
$\varnothing$	$\varnothing$	$\varnothing$	$\{0\}$	3	$\varnothing$	$\varnothing$	$\{2\}$
0	1	2	3		a	b	c
				a	$\{1,2\}$	$\varnothing$	$\varnothing$
				b	$\varnothing$	$\{1\}$	$\varnothing$
				c	$\varnothing$	$\varnothing$	$\{2\}$

The left and the lower part of the table are partial equivalence relations, the upper right region is the functional relation. The direction of the mapping is from the x-axis to the y-axis, like in calculus. The functional relation does not have a tracking function, and this implies that the object given by the left per — although it is a canonical quotient of $D4$ — is not weakly complete.

Canonical sums of weakly complete objects need not be weakly complete either, this can again be seen in $\boldsymbol{T}[\mathbb{B} \times \mathbb{B}]$, because $D\boldsymbol{2} = D\boldsymbol{1} + D\boldsymbol{1}$.

The colimits are presented here just for completeness, they are not relevant in the sequel.

5. The lifting for non-regular tripos morphisms

In this section, we show how the lifting $(F, \Phi) \mapsto \widehat{(F, \Phi)}$ can be extended to possibly non-regular tripos morphisms, and how this gives rise to an *oplax* functor of type $\mathfrak{Trip}_c \longrightarrow \mathfrak{Top}_c$. It will be explained in section 6 how this can be understood as a kind of generalised adjunction, but I was advised to present the concrete case first, to avoid loosing the reader in pages full of higher-dimensional generalities.

Therefore, here comes the theorem that in my opinion gives a satisfying characterisation of the 'full' tripos-to-topos construction.

Theorem 5.1 *Let $\mathcal{P}$ be a tripos over $\mathcal{C}$, let $\mathcal{E}$ be a topos and let $(F, \Phi) : \mathcal{P} \longrightarrow \boldsymbol{S}\mathcal{E}$ be a tripos morphism.*

1. *The category $(\mathcal{P} \diagup \boldsymbol{S})((D, \Xi), (F, \Phi))$ has an initial object $(\widehat{(F, \Phi)}, \alpha)$.*

$$
\begin{array}{ccc}
 & \mathcal{P} & \\
\scriptstyle(D,\Xi)\downarrow & & \searrow\scriptstyle(F,\Phi) \\
 & \alpha\Downarrow & \\
\boldsymbol{ST}\mathcal{P} & \longrightarrow & \boldsymbol{S}\mathcal{E} \\
\boldsymbol{T}\mathcal{P} & \underset{\widehat{(F,\Phi)}}{\longrightarrow} & \mathcal{E}
\end{array}
$$

2. *If (F, Φ) is regular, then $\widehat{(F, \Phi)}$ is also regular and α is invertible.*

3. *$(\mathrm{id}_{\boldsymbol{T}\mathcal{P}}, \mathrm{id}_{(D,\Xi)})$ is initial in $(\mathcal{P} \diagup \boldsymbol{S})((D, \Xi), (D, \Xi))$.*

4. *For all toposes $\mathcal{F}$ and all regular functors $K : \mathcal{E} \longrightarrow \mathcal{F}$, $(K\widehat{(F, \Phi)}, K \circ \alpha)$ is initial in $(\mathcal{P} \diagup \boldsymbol{S})((D, \Xi), \boldsymbol{S}K(F, \Phi))$.*

$$
\begin{array}{cccc}
 & \mathcal{P} & & \\
\scriptstyle(D,\Xi)\downarrow & & \searrow\scriptstyle(F,\Phi) & \\
 & \alpha\Downarrow & & \\
\boldsymbol{ST}\mathcal{P} & \longrightarrow & \boldsymbol{S}\mathcal{E} & \longrightarrow \boldsymbol{S}\mathcal{F} \\
\boldsymbol{T}\mathcal{P} & \underset{\widehat{(F,\Phi)}}{\longrightarrow} & \mathcal{E} & \underset{K}{\longrightarrow} \mathcal{F}
\end{array}
$$

Remark on notation

Strictly speaking, D and Ξ have to be indexed by $\mathcal{P}$ and α has to be indexed by (F, Φ). We omit these indices for conciseness. If there are several (D, Ξ)'s belonging to different triposes or several α's induced by different tripos morphisms in the same diagram, we will distinguish them by equipping them with number subscripts or primes ($\alpha_0, \alpha_1, \alpha_2 \ldots$ or $\alpha, \alpha', \alpha'' \ldots$).

5.1. Proof of Theorem 5.1

The proof of Theorem 5.1 makes use of the facts that $\boldsymbol{T}\mathcal{P} \simeq \mathsf{wc}(\boldsymbol{T}\mathcal{P})$ and $\mathcal{E} \simeq \boldsymbol{TS}\mathcal{E}$. Because the proof is rather involved, we have to make these equivalences explicit as adjoint equivalences

$$(I^\bullet \dashv I : \mathsf{wc}(\boldsymbol{T}\mathcal{P}) \longrightarrow \boldsymbol{T}\mathcal{P}, \eta, \varepsilon) \quad \text{and} \quad (J^\bullet \dashv J : \mathcal{E} \longrightarrow \boldsymbol{TS}\mathcal{E}, \eta, \varepsilon)^9.$$

I is the identity embedding and J is the constant objects functor for the tripos $\boldsymbol{S}\mathcal{E}$. We specify the adjunctions in 'adjointability form', i.e. by giving the object part of the left adjoint and the unit, where the appropriate lifting condition has to hold (see e.g. [12]). For the second adjunction, we assume that we have a generic construction that assigns subquotients of the form $A \xleftarrow{\ m\ } \bullet \xrightarrow{\ e\ } A/\rho$ to partial equivalence relations $\rho \in \boldsymbol{S}\mathcal{E}(A \times A)$. Then the adjointability diagrams look as follows.

$$
\begin{array}{ccc}
(A, \rho) & & \\
{\scriptstyle [\{\rho\}]}\Big\downarrow\cong & \searrow^{[f]} & \\
(\mathfrak{P}A, S_\rho) & -\,-\!\!\succ & (B, \sigma)
\end{array}
\qquad\qquad
\begin{array}{ccc}
(A, \rho) & & \\
{\scriptstyle [\mathsf{gr}(e)\mathsf{gr}(m)^\circ]}\Big\downarrow\cong & \searrow^{[\phi]} & \\
(A/\rho, =) & -\,-\!\!\succ & (B, =)
\end{array}
$$

$$A/\rho - - - - \succ B$$

The right hand diagram requires some exemplification. e and m come from the subquotient cone, for any $f : A \longrightarrow B$ in $\mathcal{E}$, $\mathsf{gr}(f)$ denotes its graph as a subobject of $A \times B$, that is a relation between A and B, and by $(-)^\circ$ we mean the opposite of a relation. Juxtaposition of relations denotes relational composition, which coincides with the composition of functional relations that we defined in definition 3.1, but is more general because it does not require the relations to be functional.

The relational composition $\phi\mathsf{gr}(m)\mathsf{gr}(e)^\circ$ *is* functional (with respect to equality) and by unique choice give us the mediating arrow.

We will not write down the explicit definition of $I^\bullet, J^\bullet$ and the four natural isomorphisms, as they can be more or less directly read off the diagrams.

Now we start proving the theorem.

Construction of $\widehat{(F, \Phi)}$

Instead of $\widehat{(F, \Phi)} : \boldsymbol{T}\mathcal{P} \longrightarrow \mathcal{E}$ we first define a functor G between the equivalent categories by

$$
\begin{array}{ccccc}
G \;:\; \mathsf{wc}(\boldsymbol{T}\mathcal{P}) & \longrightarrow & & \boldsymbol{TS}\mathcal{E} & \\
(A, \rho) & & & (FA, \Phi\rho) & \\
{\scriptstyle [f]}\Big\downarrow & \longmapsto & & \Big\downarrow{\scriptstyle [Ff]} & \quad . \\
(B, \sigma) & & & (FB, \Phi\sigma) &
\end{array}
$$

---

^{9}The notation is adopted from Gurski

By Lemma 4.2, this is well defined and functorial, and a careful rereading of the paragraph about finite limits in section 4.1 shows that these are preserved as well. $\widehat{(F, \Phi)}$ can now be defined as

$$\widehat{(F, \Phi)} = J^\bullet G I^\bullet.$$

In fact, the construction of G is well defined and functorial not only on $\mathsf{wc}(\boldsymbol{T}\mathcal{P})$, but even on the *lluf* subcategory of $\boldsymbol{T}\mathcal{P}$ on tracked morphisms, and we will make use of this fact later.

Construction of α

Instead of $\alpha : F \longrightarrow \widehat{(F, \Phi)}D$, we define $\tilde{\alpha} : JF \longrightarrow GI^\bullet D$. $\tilde{\alpha}_A$ is given by

$$\cfrac{(FA, =) \xrightarrow{\text{[id]}} (FA, \Phi=) \qquad \cfrac{(A, =) \xrightarrow{\eta_{(A,=)}} (\mathfrak{P}A, S_A)}{(FA, \Phi=) \rightarrowtail (F\mathfrak{P}A, \Phi S_A)}}{(FA, =) \longrightarrow (F\mathfrak{P}A, \Phi S_A)} \;(*)$$

The inference $(*)$ makes use of the fact that the construction of G works for *all* tracked arrows, as remarked above.

Naturality of $\tilde{\alpha}$ is established by the following inference.

$$\cfrac{\cfrac{(FA, =) \longrightarrow (FA, \Phi=)}{(FB, =) \longrightarrow (FB, \Phi=)} \quad \begin{array}{c} (A, =) \longrightarrow (\mathfrak{P}A, S_A) \\ \downarrow \qquad\qquad \downarrow \\ (B, =) \longrightarrow (\mathfrak{P}B, S_B) \\ \hline (FA, \Phi=) \rightarrowtail (F\mathfrak{P}A, \Phi S_A) \\ \downarrow \qquad\qquad \downarrow \\ (FB, \Phi=) \rightarrowtail (F\mathfrak{P}B, \Phi S_B) \end{array}}{\begin{array}{c} (FA, =) \longrightarrow (F\mathfrak{P}A, \Phi S_A) \\ \downarrow \qquad\qquad \downarrow \\ (FB, =) \longrightarrow (F\mathfrak{P}B, \Phi S_B) \end{array}}$$

The left square commutes, since the tracking arrows compose to the same arrow and the right upper square commutes because of the naturality of η.

Similarly, we can establish the fact that $\tilde{\alpha}$ is a tripos transformation from $\boldsymbol{S}J(F, \Phi)$ to $\boldsymbol{S}(GI^\bullet)(D, \Xi)$. Let $\psi \in \mathcal{P}_A$. For appropriately defined θ (it is the strict predicate on

$(\mathfrak{P}A, S_A)$ that corresponds to ψ via the isomorphism), we can infer

$$
\begin{array}{ccc}
(A, =|_\psi) & \longrightarrow & (\mathfrak{P}A, S_A|_\theta) \\
\downarrow & & \downarrow \\
(A, =) & \longrightarrow & (\mathfrak{P}A, S_A)
\end{array}
$$

$$
\begin{array}{ccc}
(FA, =|_{\Phi\psi}) \longrightarrow (FA, \Phi=|_{\Phi\psi}) & & (FA, \Phi=|_{\Phi\psi}) \rightarrowtail (F\mathfrak{P}A, \Phi S_A|_{\Phi\theta}) \\
\downarrow \qquad\qquad \downarrow & & \downarrow \qquad\qquad \downarrow \\
(FA, =) \longrightarrow (FA, \Phi=) & & (FA, \Phi=) \longrightarrow (F\mathfrak{P}A, \Phi S_A)
\end{array}
$$

$$
\begin{array}{ccc}
(FA, =|_{\Phi\psi}) & \rightarrowtail & (F\mathfrak{P}A, \Phi S_A|_{\Phi\theta}) \\
\downarrow & & \downarrow \\
(FA, =) & \longrightarrow & (F\mathfrak{P}A, \Phi S_A)
\end{array}
$$

and the bottom square is a diagrammatic representation of the desired judgement

$$
(J(\Phi\psi))(x) \vdash (G(I^\bullet(\Xi\psi)))(\tilde\alpha(x))
$$

in $\boldsymbol{STS}\mathcal{E}$.

α is constructed from $\tilde\alpha$ via the equivalences, spelled out beneath.

$$
\alpha = (\varepsilon^{-1} \circ F)(J^\bullet \circ \alpha)
$$

Next, we have to show that $(\widehat{(F, \Phi)}, \alpha)$ is initial in $(\mathfrak{P} \diagdown \boldsymbol{S})((D, \Xi), (F, \Phi))$. Let (H, β) be as second object in $(\mathfrak{P} \diagdown \boldsymbol{S})((D, \Xi), (F, \Phi))$. We have to show that there is a unique mediator ι in the following diagram.

In particular, we have to carry out the following steps:

- construct ι

- verify that ι mediates

- show uniqueness

Construction of ι

The construction becomes most transparent when expressed relative to $\mathsf{wc}(\boldsymbol{T}\mathcal{P})$ and $\mathcal{E}$, therefore we define

$$\tilde{\iota} : J^\bullet G \longrightarrow HI : \mathsf{wc}(\boldsymbol{T}\mathcal{P}) \longrightarrow \mathcal{E}.$$

The construction of $\tilde{\iota}_{(A,\rho)}$ is displayed in Figure 1. Basically, we chase the per ρ through the diagram

$$(5.1)$$

To understand why $\tilde{\iota}$ is natural take a morphism $(A, \rho) \xrightarrow{[f]} (B, \sigma)$ in $\mathsf{wc}(\boldsymbol{T}\mathcal{P})$ and imagine a copy of the diagram in figure 1 with A and ρ substituted by B and σ on a plane parallel to the paper. Then connect the two copies of the diagram by arrows constructed from f and convince yourself that everything commutes and that this gives really an instance of the naturality condition.

Finally, ι is obtained from $\tilde{\iota}$ via the following equation.

$$\iota = (\tilde{\iota} \circ I^\bullet)(H \circ \eta^{-1})$$

Proof that ι mediates

We have to show that $(\iota \circ D)\alpha = \beta$, and this rewrites as

If we evaluate the natural transformation represented by the left diagram at A, we obtain the composite

$$FA \longrightarrow FA/= \longrightarrow F\mathfrak{P}A/\Phi S_A \longrightarrow H(\mathfrak{P}A, S_A) \longrightarrow H(A, =) \qquad (5.2)$$

in $\mathcal{E}$ and we have to prove this to be equal to β_A. The idea of the proof is similar to the proof of naturality for $\tilde{\iota}$. The idea of the naturality proof was to chase the left hand

Figure 1: Construction of $\tilde{\imath}$. The component $\tilde{\imath}_{(A,\rho)} : FA/\Phi\rho \to H(A,\rho)$ of $\tilde{\imath}$ is given by composing the two dashed arrows on the lower right hand side of the diagram. It is explained in A.2 why there is a pair of mediators for the left transition of spans.
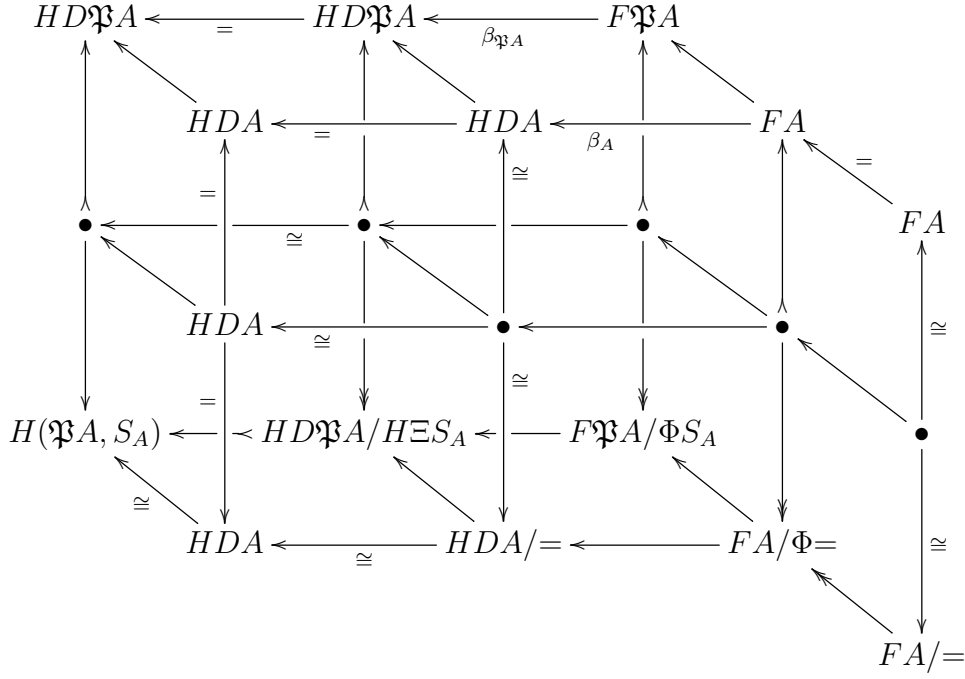
Figure 2: Proof that $(\iota \circ D)\alpha = \beta$.

pair of arrows beneath through the diagram (5.1) and then switch to subquotients as depicted in Figure 1. Now we do the same for the right hand pair.

$$
\begin{array}{ccc}
\rho \longrightarrow \sigma & \qquad = \longrightarrow S_A \\
\downarrow \qquad \downarrow & \qquad \downarrow \qquad \downarrow \\
A \times A \longrightarrow B \times B & \quad A \times A \longrightarrow \mathfrak{P}A \times \mathfrak{P}A
\end{array}
$$

The resulting diagram, with two squares appended for the transition from $=$ to $\Phi=$, is displayed in Figure 2. In this diagram, we recover the composite (5.2), and careful inspection shows that the asserted equality holds.

Uniqueness of ι

We have to show that the mapping

$$
(\xi : \widehat{(F, \Phi)} \longrightarrow H) \mapsto ((\xi \circ D)\alpha : F \longrightarrow HD)
$$

is injective. By general principles about equivalences, it is injective if and only if

$$
(\xi : GI^{\bullet} \longrightarrow JH) \mapsto ((\xi \circ D)\tilde{\alpha} : JF \longrightarrow JHD)
$$

is injective. Therefore, let $\xi, \upsilon : GI^{\bullet} \longrightarrow JH$ and assume that $(\xi \circ D)\tilde{\alpha} = (\upsilon \circ D)\tilde{\alpha}$. To show $\xi = \upsilon$ it suffices to verify $\xi \circ I = \upsilon \circ I$; that means that we only have to prove $\xi_{(A,\rho)} = \upsilon_{(A,\rho)}$ for weakly complete (A, ρ).

Let (A, ρ) be weakly complete. Consider the construction

$$
\begin{array}{ccccc}
(A, \rho) & \longleftarrow\!\!\!\!\longleftarrow (A, \mathsf{supp}(\rho)) \rightarrowtail & (A, =) \\
\cong \downarrow & \cong \downarrow & \cong \downarrow \\
(\mathfrak{P}A, S_\rho) & \longleftarrow (\mathfrak{P}A, S_{\mathsf{supp}(\rho)}) \rightarrowtail & (\mathfrak{P}A, S_A) \\
\hline
(FA, \Phi\rho) & \longleftarrow (FA, \Phi\mathsf{supp}(\rho)) \rightarrowtail & (FA, \Phi{=}) \\
\cong \downarrow & \downarrow & \downarrow \\
(F\mathfrak{P}A, \Phi S_\rho) & \longleftarrow\!\!\!\!\longleftarrow (F\mathfrak{P}A, \Phi S_{\mathsf{supp}(\rho)}) \rightarrowtail & (F\mathfrak{P}A, \Phi S_A)
\end{array}
$$

Here, we again use the trick of applying the construction of G to not necessarily weakly complete input. In this case, the construction even preserves the two epimorphisms, because they have identity tracking arrows, what allows us to express their 'being epimorphic' without using '$\exists$'. The left hand arrow in the lower diagram is an isomorphism, because (A, ρ) is already weakly complete. Furthermore, the lower row of the lower pair of squares is the $GI^\bullet$-image of the top row of the upper pair of squares, and from $GI^\bullet$ we may pass over to JH via ξ and υ. This looks as follows.

$$
\begin{array}{ccc}
& (FA, =) & \\
& \downarrow & \\
(FA, \Phi\rho) \longleftarrow (FA, \Phi\mathsf{supp}(\rho)) \rightarrowtail (FA, \Phi{=}) & \Big\} \tilde{\alpha}_A \\
\cong \downarrow \quad\quad \downarrow \quad\quad \downarrow & \\
(F\mathfrak{P}A, \Phi S_\rho) \longleftarrow (F\mathfrak{P}A, \Phi S_{\mathsf{supp}(\rho)}) \rightarrowtail (F\mathfrak{P}A, \Phi S_A) & \\
\xi_{(A,\rho)} \Downarrow \Downarrow \upsilon_{(A,\rho)} \quad \Downarrow \quad\quad \Downarrow & \\
JH(A, \rho) \longleftarrow JH(A, \mathsf{supp}(\rho)) \rightarrowtail JH(A, =) &
\end{array}
$$

By hypothesis, the two composites along the right hand side of the diagram are equal, and by a simple diagram chasing argument we get $\xi_{(A,\rho)} = \upsilon_{(A,\rho)}$.

This concludes the proof of the first claim of Theorem 5.1. Fortunately, claims 2–4 are much easier. 2 follows directly from the definitions of $\widehat{(F, \Phi)}$ and α, and the initiality claims of 3 and 4 can be verified by comparing with the initial objects that are given by the construction and observing that the mediators are isomorphisms.

5.2. Theorem 5.1 as an adjointability condition

Proposition 1 of Theorem 5.1 was deliberately stated in a way that reminds us of the concept of left adjointability for a(n ordinary) functor. The same pattern that leads us from left adjointability to the left adjoint can also be applied here, only the resulting functor $\boldsymbol{T} : \mathfrak{Trip}_\mathsf{c} \longrightarrow \mathfrak{Top}_\mathsf{c}$ is merely *oplax*.
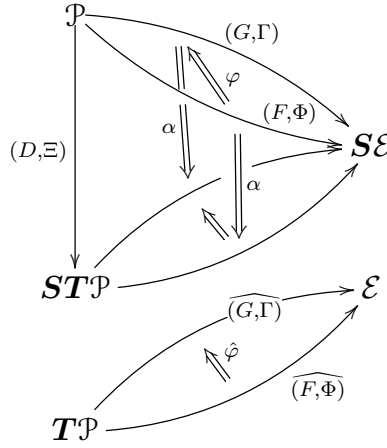
The object part of $\boldsymbol{T}$ is given by $\mathcal{P} \mapsto \boldsymbol{T}\mathcal{P}$. Before defining the morphism part, we observe that the lifting

$$((F, \Phi) : \mathcal{P} \longrightarrow \boldsymbol{S}\mathcal{E}) \longmapsto (\widehat{(F, \Phi)} : \boldsymbol{T}\mathcal{P} \longrightarrow \mathcal{E})$$

can be extended to a functor

$$(\hat{-}) : \mathfrak{Trip}_{\mathsf{c}}(\mathcal{P}, \boldsymbol{S}\mathcal{E}) \longrightarrow \mathfrak{Top}_{\mathsf{c}}(\boldsymbol{T}\mathcal{P}, \mathcal{E}),$$

mapping a transformation $\varphi : (F, \Phi) \longrightarrow (G, \Gamma)$ to the mediator in the diagram beneath which exists and is unique because of the initiality of $(\widehat{(F, \Phi)}, \alpha)$.

The morphism part of $\boldsymbol{T}$ is now given by

$$\widehat{(D, \Xi) \circ} (-) = \boldsymbol{T}_{\mathcal{P}, \mathcal{Q}} : \mathfrak{Trip}_{\mathsf{c}}(\mathcal{P}, \mathcal{Q}) \longrightarrow \mathfrak{Top}_{\mathsf{c}}(\boldsymbol{T}\mathcal{P}, \boldsymbol{T}\mathcal{Q}).$$

The construction of the constraints for identity $(\boldsymbol{T}(\mathrm{id}_{\mathcal{P}}) \longrightarrow \mathrm{id}_{\boldsymbol{T}\mathcal{P}})$ and composition $(\boldsymbol{T}((G, \Gamma) \circ (F, \Phi)) \longrightarrow \boldsymbol{T}(G, \Gamma) \circ \boldsymbol{T}(F, \Phi))$ is suggested by the following diagrams (again making use of initiality)
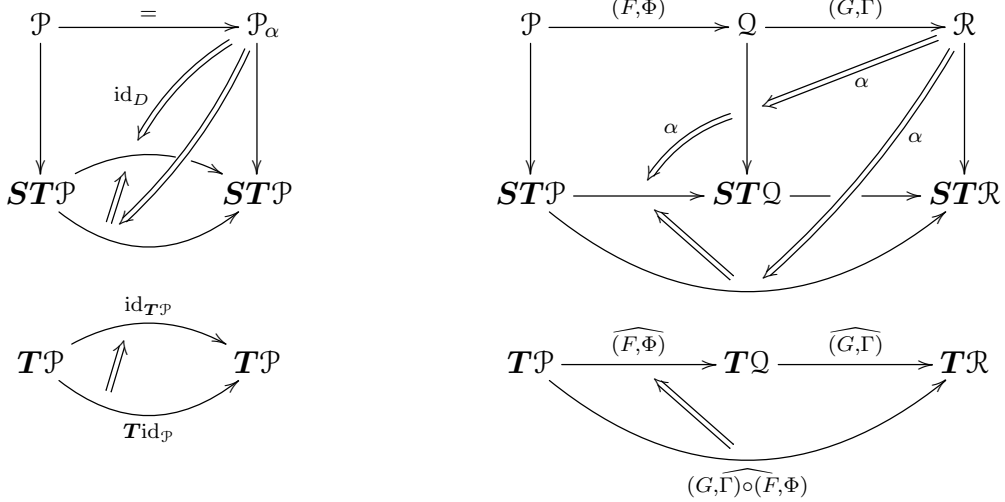
The three axioms for the constraints follow directly from the uniqueness of the mediators.

Exploiting the metaphor of adjointability further, we can use Theorem 5.1 to construct (oplax) natural transformations.

$$\eta : \mathrm{id}_{\mathfrak{Trip}_c} \longrightarrow \boldsymbol{ST} \quad \text{and} \quad \varepsilon : \boldsymbol{TS} \longrightarrow \mathrm{id}_{\mathfrak{Top}_c}.$$

η at $\mathcal{P}$ is just given by (D', Ξ'), and the constraint $\eta_{(F,\Phi)}$ is given by the appropriate α.

$$
\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\ (F,\Phi)\ } & \mathcal{Q} \\
\downarrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow \\
\boldsymbol{ST}\mathcal{P} & \longrightarrow & \boldsymbol{ST}\mathcal{Q}
\end{array}
$$

$$\boldsymbol{T}\mathcal{P} \longrightarrow \boldsymbol{T}\mathcal{Q}$$

The axioms are straightforward.

ε is a bit more complicated.

$\varepsilon_{\mathcal{E}}$ is given by $\widehat{\mathrm{id}_{\boldsymbol{S}\mathcal{E}}}$, as usual. It is easy to see that $\widehat{\mathrm{id}_{\boldsymbol{S}\mathcal{E}}}$ is isomorphic to $J^{\bullet} : \boldsymbol{TS}\mathcal{E} \longrightarrow \mathcal{E}$ and hence an equivalence, but we will ignore this fact for the moment and concentrate on the general pattern.

The construction of constraints for ε is the most involved. Up to now, we only made use of statement 1 of Theorem 5.1, but now we also need statements 2 and 4. Consider the diagram

$$(5.3)$$

By Theorem 5.1 2, $\widehat{\mathrm{id}_{\boldsymbol{S}\mathcal{F}}}$ is regular and α_2 is invertible. By 5.1 4,

$$(\widehat{\mathrm{id}_{\boldsymbol{S}\mathcal{F}}} \circ \boldsymbol{TS}F, \boldsymbol{S}\widehat{\mathrm{id}_{\boldsymbol{S}\mathcal{F}}} \circ \alpha_1) \quad \textit{is initial in} \quad (\mathcal{P} \diagup \boldsymbol{S})((D, \Xi), \boldsymbol{S}\widehat{\mathrm{id}_{\boldsymbol{S}\mathcal{F}}} \circ (D', \Xi') \circ \boldsymbol{S}F),$$

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and because α_2 is an isomorphism, it is easy to see that appending it does not destroy initiality, i.e.

$$(\widehat{\mathrm{id}_{\boldsymbol{S}\mathcal{F}}} \circ \boldsymbol{T}\boldsymbol{S}F, (\boldsymbol{S}\widehat{\mathrm{id}_{\boldsymbol{S}\mathcal{F}}} \circ \alpha_1)(\alpha_2 \circ \boldsymbol{S}F)) \quad \textit{is initial in} \quad (\mathcal{P}\!\diagdown\!\boldsymbol{S})((D,\Xi),\boldsymbol{S}F).$$

$(\widehat{F\mathrm{id}_{\boldsymbol{S}\mathcal{E}}}, \boldsymbol{S}F \circ \alpha_0)$ is a second object in $(\mathcal{P}\!\diagdown\!\boldsymbol{S})((D,\Xi),\boldsymbol{S}F)$, and the unique mediator (dashed in the diagram) gives us the desired constraint. The axioms follow again from initiality considerations.

In our concrete case, the fact that $\boldsymbol{T}\boldsymbol{S}\mathcal{E} \simeq \mathcal{E}$ for all toposes $\mathcal{E}$ lets us conjecture that ε is an equivalence, and this is in fact the case (in particular, the constraint cells are all invertible). However, the verification of this claim involves a lot of fiddling with equivalences and isomorphisms and we will not carry it out here. If you want to prove it yourself, notice that it is easier if you replace the pair $(\widehat{\mathrm{id}_{\boldsymbol{S}\mathcal{E}}}, \alpha)$ as it is constructed in the proof by a hand-tailored initial object in the respective category, and similarly for the lifting of $(D',\Xi') \circ \boldsymbol{S}F$.

Up to now, the analogy between one-dimensional adjointability and the conditions of Theorem 5.1 was quite successful, because we were able to more or less straightforwardly translate the constructions of the left adjoint and the two transformations.

It is easy to see that the triangle equalities hold up to isomorphism, although strictly speaking, we have to introduce new data in form of isomorphic modifications, subject to new axioms. More on this in greater generality in the next section.

Finally, it is interesting to ask if the conditions of Theorem 5.1 are sufficient to characterise the pair $(\boldsymbol{T}\mathcal{P}, (D,\Xi))$ up to equivalence. In the one-dimensional case, $\eta_A : A \longrightarrow UFA$ is given as an initial object in $A{\downarrow}U$ and therefore uniquely determined up to isomorphism. 5.1 1 replaces initiality in a category with the condition that for fixed I and varying A, both objects of the same 2-category $\mathfrak{A}$, the hom-categories $\mathfrak{A}(I,A)$ all have initial objects. Unfortunately, this condition does not suffice to characterise I up to equivalence[10]. However, 5.1 1–4 together suffice to characterise $(\boldsymbol{T}\mathcal{P}, (D,\Xi))$ up to isomorphism among pairs with *regular* second component, what can be seen as follows.

Assume $(\boldsymbol{T}'\mathcal{P}, (D',\Xi'))$ also satisfies 5.1 1–4 and (D',Ξ') is regular. Then we have initial mediators

where α, β are isomorphisms and F, G are regular because of 5.1 2.

5.1 4 tells us that $(GF, \boldsymbol{S}G \circ \alpha)$ is initial in $(\mathcal{P}\!\diagdown\!\boldsymbol{S})((D,\Xi), \boldsymbol{S}G \circ (D',\Xi'))$ and composing

---

[10]In the 2-category of posets with least elements, for example, all objects have this property

an isomorphism, we see that $(GF, (\boldsymbol{S}G \circ \alpha)\beta)$ is initial in $(\mathcal{P} \diagup \boldsymbol{S})((D,\Xi),(D,\Xi))$.

$$
\begin{array}{c}
\mathcal{P} \\
\swarrow_{(D,\Xi)} \quad \downarrow_{(D',\Xi')} \quad \searrow^{(D,\Xi)} \\
\boldsymbol{ST}\mathcal{P} \xrightarrow{\quad} \boldsymbol{ST'}\mathcal{P} \xrightarrow{\quad} \boldsymbol{ST}\mathcal{P} \\
\ \\
\boldsymbol{T}\mathcal{P} \xrightarrow{\quad} \boldsymbol{T'}\mathcal{P} \xrightarrow{\quad} \boldsymbol{T}\mathcal{P}
\end{array}
$$

By 5.1 3, we have a second initial object $(\mathrm{id}_{\boldsymbol{T}\mathcal{P}}, \mathrm{id}_{(D,\Xi)})$, consequently we have $GF \cong \mathrm{id}_{\boldsymbol{T}\mathcal{P}}$. Similarly, we get $FG \cong \mathrm{id}_{\boldsymbol{T'}\mathcal{P}}$.

Thus, 5.1 can be seen as a universal characterisation of $\boldsymbol{T}\mathcal{P}$ and (D,Ξ).

We had already given such a characterisation in section 3, making use only of regular functors and morphisms, and in the present section, we also had to appeal to regularity to show uniqueness. Hence, one could say that the tripos-to-topos construction gets its stability and universality from the regular arrows, and the non-regular arrows are only further decoration.

5.3. Examples

We give two examples of how α fails to be invertible if the trips morphism (F,Φ) in 5.1 is not regular. In both cases we diverge from the 'canonical' description of (D,Ξ) and $\widehat{(F,\Phi)}$ given in the previous section. In concrete cases, especially the construction of $\widehat{(F,\Phi)}$ can be quite complicated to spell out because of the involved equivalences, therefore it is often useful to just propose a pair of functor and natural transformation and then prove that this pair is initial in the respective hom-category.

We start by reconsidering the example sketched in the introduction.

Consider the tripos-morphism

$$
(\mathrm{id}_{\mathbf{Set}}, [\wedge]) : [\mathbb{B} \times \mathbb{B}] \longrightarrow [\mathbb{B}] \simeq \boldsymbol{S}(\mathbf{Set})
$$

that is induced by the meet-preserving map $\wedge : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$ as sketched at the end of 2.3. We know that $\boldsymbol{T}[\mathbb{B} \times \mathbb{B}]$ is equivalent to $\mathbf{Sh}(\mathbb{B} \times \mathbb{B})$, the category of sheaves on $\mathbb{B} \times \mathbb{B}$[11], and it is easy to see that this in turn is equivalent to $\mathbf{Set} \times \mathbf{Set}$. Via this pair of equivalences, $(D,\Xi) : [\mathbb{B} \times \mathbb{B}] \longrightarrow \boldsymbol{ST}[\mathbb{B} \times \mathbb{B}]$ corresponds (up to isomorphism) to the tripos morphism

$$
\begin{array}{ccc}
\bullet & \xrightarrow{\tilde{\Xi}:(U,V)\mapsto U \times V} & \bullet \\
\scriptstyle{[\mathbb{B}\times\mathbb{B}]}\Big\downarrow & & \Big\downarrow\scriptstyle{\boldsymbol{S}(\mathbf{Set}\times\mathbf{Set})} \\
\mathbf{Set} & \xrightarrow[\langle \mathrm{id},\mathrm{id}\rangle]{} & \mathbf{Set} \times \mathbf{Set}
\end{array}
$$

where we implicitly identify predicates in $[\mathbb{B} \times \mathbb{B}]$ over I with pairs of subsets of I.

---

[11]This is mentioned in [4] and a proof can be found in [8], C1.3.

If we want to verify our claim from the introduction, we have to show that the pair

$$(P : \mathbf{Set} \times \mathbf{Set} \longrightarrow \mathbf{Set}, \delta : \mathrm{id} \longrightarrow P \circ \langle \mathrm{id}, \mathrm{id} \rangle)$$
$$P(A, B) = A \times B$$
$$\delta_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle : A \longrightarrow A \times A$$

is initial in $([\mathbb{B} \times \mathbb{B}] \swarrow \boldsymbol{S})((\langle \mathrm{id}, \mathrm{id} \rangle, \tilde{\Xi}), (\mathrm{id}, [\wedge]))$.

Let (F, θ) be a second object in $([\mathbb{B} \times \mathbb{B}] \swarrow \boldsymbol{S})((\langle \mathrm{id}, \mathrm{id} \rangle, \tilde{\Xi}), (\mathrm{id}, [\wedge]))$. Then we can construct a mediator $\iota : P \longrightarrow F$ by defining its component $\iota_{(A,B)}$ as

$$A \times B \xrightarrow{\theta_{A \times B}} F(A \times B, A \times B) \xrightarrow{F(\pi_1, \pi_2)} F(A, B).$$

It is easy to see that this is natural and does really mediate. For uniqueness, assume that we have two natural transformations $\xi, \upsilon : P \longrightarrow F$ such that for all A we have $\xi_{(A,A)} \eta_A = \upsilon_{(A,A)} \eta_A$. To see that $\xi_{(A,B)} = \upsilon_{(A,B)}$ consider the following diagram.

Because the left hand triangle commutes and the two upper composites are equal, the assertion follows.

We observe that in this example, the constraint transformtation (which is called α in the generic construction and δ here) is a pointwise monomorphism. Next, we consider an example where this transformation is a pointwise epimorphism.

Assume that $\mathcal{A}$ is a partial combinatory algebra over $\mathbf{Set}$ and consider the composition

$$\boldsymbol{S}(\mathbf{Set}) \xrightarrow{(\mathrm{id}, \nabla)} \mathbf{rt}_{\mathsf{m}}(\mathcal{A}) \xrightarrow{(D, \Xi)} \boldsymbol{ST}\mathbf{rt}_{\mathsf{m}}(\mathcal{A})$$

of tripos morphisms.

Then $(D, \Xi) : \boldsymbol{SSet} \longrightarrow \boldsymbol{STSSet}$ is equivalent to id $: \boldsymbol{SSet} \longrightarrow \boldsymbol{SSet}$ in $(\boldsymbol{SSet}\diagdown \boldsymbol{S})$, and we claim that an initial object in $(\boldsymbol{SSet}\diagdown \boldsymbol{S})(\mathrm{id}, (D, \Xi\nabla))$ is given by (F, α) with

$$F : \mathbf{Set} \longrightarrow \boldsymbol{T}(\mathbf{rt}_{\mathrm{m}}(\mathcal{A})) \qquad\qquad \alpha : D \longrightarrow F$$
$$FA = (A, \nabla{=}) \qquad\qquad\qquad \alpha_A = [\mathrm{id}] : (A, =) \longrightarrow (A, \nabla{=})$$
$$Ff = [f]$$

It is straightforward to see that F is well defined as a cartesian functor, and α as a tripos transformation. Moreover, we observe that α is a pointwise epimorphism. Now, let (G, β) be a second object in $(\boldsymbol{SSet}\diagdown \boldsymbol{S})(\mathrm{id}, (D, \Xi\nabla))$. We have to show that there exists one and only one $\iota : F \longrightarrow G$ with $\iota\alpha = \beta$. ι_A has to be a mediator in the diagram

$$
\begin{array}{ccc}
 & DA & \\
{\scriptstyle \alpha_A}\downarrow & & \searrow{\scriptstyle \beta_A} \\
FA & \dashrightarrow & GA
\end{array}
$$

Because α_A is an epimorphism, this mediator is necessarily unique; it exists iff β_A equalises the kernel pair of α_A. To check this, we make use of the fact that β is not only a natural transformation, but moreover a tripos transformation. If we take the equality predicate on A in $\boldsymbol{SSet}$ and map it through $(D, \Xi\nabla)$ and $\boldsymbol{S}G$, we get therefore a mediator

$$
\begin{array}{ccc}
\|\nabla{=}\| & \dashrightarrow & FA \\
\downarrow & & \downarrow \\
(A \times A, =) & \xrightarrow{\ \beta_{A\times A}\ } & F(A \times A) \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \cong} \\
(A, =) \times (A, =) & \xrightarrow[\ \beta_A \times \beta_A\ ]{} & FA \times FA
\end{array}
$$

and the left hand side of the diagram is the kernel pair.

6. dc-categories

In this section, we introduce a new class of 2-categories.

The starting point for the development of the ideas that are presented here was the attempt to generalise the concept of *semi-lax* right adjoint introduced by Peter Johnstone in [6] to include the tripos-to-topos construction. Johnstone's trick was to sustain the desirable uniqueness up to equivalence of the adjoint (normally lost when we go lax) by introducing additional data in form of lluf 1-subcategories of the involved 2-categories. This is the idea that we are now going to exploit systematically.

Before starting to read this section, the reader may want to have a look at Appendix B which contains basic 2-categorical definitions.

Definition 6.1   *1. A dc-category is a 2-category $\mathfrak{A}$ together with a designated subclass $\mathfrak{A}_r$ of the class of all 1-cells such that*

- *$\mathfrak{A}_r$ contains all equivalences,*

- *$\mathfrak{A}_r$ is closed under composition, and*

- *$\mathfrak{A}_r$ is closed under vertical isomorphisms; i.e if $f \in \mathfrak{A}_r$ and $f \cong g$, then $g \in \mathfrak{A}_r$.*

We think of the 1-cells in $\mathfrak{A}_r$ as of arrows with especially good properties and we call them regular *arrows. We use the arrow symbol '$\longmapsto$' for regular arrows to highlight them visually.*

2. *A* semi-lax functor *between dc-categories $\mathfrak{A}$ and $\mathfrak{B}$ is a lax functor $(F, \phi) : \mathfrak{A} \longrightarrow \mathfrak{B}$ such that*

- *F maps regular arrows in $\mathfrak{A}$ to regular arrows in $\mathfrak{B}$,*

- *for all objects A of $\mathfrak{A}$ the constraint cell $\mathrm{id}_{FA} \xrightarrow{\phi_A} F(\mathrm{id}_A)$ is invertible, and*

- *for all composable pairs f, g of 1-cells in $\mathfrak{A}$ with regular g, the constraint cell $Fg\,Ff \xrightarrow{\phi_{(f,g)}} F(gf)$ is invertible.*

3. *A semi-lax transformation between semi-lax functors F, G is a lax natural transformation $\eta : F \longrightarrow G$ such that*

- *For each object A, η_A is regular, and*

- *For each regular $f : A \longrightarrow B$ the constraint cell*

$$
\begin{array}{ccc}
FA & \xrightarrow{\ Ff\ } & FB \\
\eta_A \downarrow & \nearrow\!\!\!\!\nearrow & \downarrow \eta_B \\
GA & \xrightarrow[\ Gf\ ]{} & GB
\end{array}
$$

is invertible.

Given two dc-categories $\mathfrak{A}, \mathfrak{B}$, the semi-lax functors between them together with semi-lax natural transformations and modifications form a sub-2-category of $\mathbf{Lax}(\mathfrak{A}, \mathfrak{B})$ that we denote by $\mathbf{SLax}(\mathfrak{A}, \mathfrak{B})$

Comparison with Johnstone's definition and naming

Johnstone [6] uses the term 'semi-lax' only for his *semi-lax adjunctions*, the intended meaning there is 'adjunction, where part of the data (one of the transformations)' is lax. He gives the definition of what we call a semi-lax transformation, but he does not name this concept and he considers these transformations only between pseudofunctors.

We use the name *semi-lax functor/transformation* in the sense of *partially lax* functor/transformation[12]. Consequently, we use the term 'semi-lax adjunction' in a more general sense than Johnstone, namely in the sense 'adjunction where all of the data is semi-lax'. It will turn out that the right adjoint is a pseudofunctor anyway, but this is a theorem and not a definition.

The definition of semi-lax functor is new; it is suggested by the behaviour of the tripos-to-topos construction[13].

6.1. Composition of semi-lax functors and transformations

Pseudo-natural transformations are composable horizontally and vertically. However, the horizontal composition is a derived concept. The primitive operations are the composition ϑF of a natural transformation after a functor and the composition $G\eta$ of a natural transformation before a functor. Given

$$\mathfrak{A} \underset{F'}{\overset{F}{\Rightarrow\eta}} \mathfrak{B} \underset{G'}{\overset{G}{\Rightarrow\theta}} \mathfrak{C},$$

to define $\theta \circ \eta$ one has to choose between one of the paths around the square

$$
\begin{array}{ccc}
G \circ F & \xrightarrow{G\eta} & G \circ F' \\
{\scriptstyle \theta F}\downarrow & {\overset{\mathsf{c}_{(\eta,\theta)}}{\Longrightarrow}} & \downarrow{\scriptstyle \theta F'} \\
G' \circ F & \xrightarrow[G\eta]{} & G' \circ F'
\end{array}
\qquad (6.1)
$$

The constraints of θ give rise to an isomorphic modification $\mathsf{c}_{(\eta,\theta)}$ (which we call *exchange modification*) between the two choices.

However, if θ is merely lax, then we lose the invertibility of $\mathsf{c}_{(\eta,\theta)}$. If we, by convention, choose one of the possibilities to define $\theta \circ \eta$, we get the problem that horizontal composition is not pseudofunctorial any more.

---

[12] Actually this is not optimal, as it could be misinterpreted as meaning something even weaker than lax. Maybe 'semi-strong' would be better, but this can not be dualised.

[13] Although in the case of the tripos-to-topos construction, everything is oplax, we treat everything here in the $(-)^{\mathsf{co}}$-dual setting to save prefixes.

If we consider lax functors instead of pseudofunctors, the situation becomes even worse. If G is lax, the composition $G\eta$ is not even definable, because if we try to construct $(G\eta)_f$ for $f : A \longrightarrow B$, we end up with

$$
\begin{array}{ccc}
GFA & \xrightarrow{\;GFf\;} & GFB \\
\end{array}
$$

$$\tag{6.2}$$

and this does not compose.

Luckily, for semi-lax functors and semi-lax transformations, it all works out. Convince yourself — the disturbing 2-cell in (6.2) becomes invertible because η_B is regular by definition, and for similar reasons $\mathsf{c}_{(\eta,\theta)}$ in (6.1) is componentwise isomorphic.

Thus the really great thing about about semi-lax functors and transformations is that they are lax data that nevertheless share all the good properties of pseudofunctors and pseudo-natural transformations.

Because semi-lax functors and transformations behave so similar to the pseudo ones, I conjectured that they give rise to a *tricategory* in the sense of [3]. Unfortunately, my attempts to prove this failed because of the overwhelming complexity of the definition. Nevertheless, the intuition behind our treatment of semi-lax adjunctions is that we see them as ordinary biadjunctions — just not in **2-ℭat** or 𝔅𝔦𝔠𝔞𝔱, but in some other three-dimensional structure.

6.2. Semi-lax adjunctions

Definition 6.2 *Let* $\mathfrak{A}, \mathfrak{B}$ *be* dc-*categories. A semi-lax adjunction between* $\mathfrak{A}$ *and* $\mathfrak{B}$ *is given by two semi-lax functors*

$$(F, \psi) : \mathfrak{A} \longrightarrow \mathfrak{B} \qquad and \qquad (U, \phi) : \mathfrak{B} \longrightarrow \mathfrak{A},$$

two semi-lax natural transformations,

$$\eta : \mathrm{id}_{\mathfrak{A}} \longrightarrow UF \qquad and \qquad \varepsilon : FU \longrightarrow \mathrm{id}_{\mathfrak{B}}$$

and two invertible *modifications*

$$\mu : \mathrm{id}_U \xRightarrow{\;\cong\;} (U \circ \varepsilon)(\eta \circ U) \qquad and \qquad \nu : \mathrm{id}_F \xRightarrow{\;\cong\;} (\varepsilon \circ F)(F \circ \eta).$$

$$
\begin{array}{ccc}
\eta_A & & \varepsilon_B \\
\downarrow{\scriptstyle\psi\circ\mathrm{id}_{\eta_A}} & & \downarrow{\scriptstyle\mathrm{id}_{\varepsilon_B}\circ\phi} \\
U\mathrm{id}_{FA}\circ\eta_A & & \varepsilon_B\circ F\mathrm{id}_{UB} \\
\downarrow{\scriptstyle U\nu_A\circ\mathrm{id}_{\eta_A}} & & \downarrow{\scriptstyle\mathrm{id}_{\varepsilon_B}\circ F\mu_B} \\
U(\varepsilon_{FA}\circ F\eta_A)\circ\eta_A & & \varepsilon_B\circ F(U\varepsilon_B\circ\eta_{UB}) \\
\downarrow{\scriptstyle\psi\circ\mathrm{id}_{\eta_A}} \quad =\mathrm{id}_{\eta_A} & & \downarrow{\scriptstyle\mathrm{id}_{\varepsilon_B}\circ\phi} \quad =\mathrm{id}_{\varepsilon_B} \\
U\varepsilon_{FA}\circ UF\eta_A\circ\eta_A & & \varepsilon_B\circ FU\varepsilon_B\circ F\eta_{UB} \\
\downarrow{\scriptstyle\mathrm{id}_{U\varepsilon_{FA}}\circ\eta_A} & & \downarrow{\scriptstyle\varepsilon_B^{-1}\circ\mathrm{id}_{F\eta_{UB}}} \\
U\varepsilon_{FA}\circ\eta_{UFA}\circ\eta_A & & \varepsilon_B\circ\varepsilon_{FUB}\circ F\eta_{UB} \\
\downarrow{\scriptstyle\mu_{FA}^{-1}\circ\mathrm{id}_{\eta_A}} & & \downarrow{\scriptstyle\mathrm{id}_{\varepsilon_B}\circ\nu_{UB}^{-1}} \\
\eta_A & & \varepsilon_B
\end{array}
$$

Figure 3: The axioms for semi-lax adjunctions in one-dimensional form. The constraints for F are denoted by ϕ and the constraints for U are denoted by ψ.

such that the canonical isomorphic 2-cell fill-ins in the following diagrams compose to identities for $A \in \mathsf{Obj}(\mathfrak{A})$ and $B \in \mathsf{Obj}(\mathfrak{B})$ respectively.

An alternative, one-dimensional presentation of the axioms is given in Figure 3.

At this point, we have to make some remarks on dualisation. Adjunctions between ordinary categories are self-dual, therefore left adjoints are symmetric to right adjoints. 2-categories (and dc-categories) can be dualised in two ways: $\mathfrak{A} \mapsto \mathfrak{A}^{\mathsf{op}}$ reverses the 1-cells and $\mathfrak{A} \mapsto \mathfrak{A}^{\mathsf{co}}$ reverses the 2-cells. However, none of the dualisations transfers semi-lax adjunctions to semi-lax adjunctions. Rather, they both lead out of the framework of semi-lax functors and semi-lax natural transformations into dual frameworks:

- If we reverse all 1-cells, lax natural transformations are turned into oplax natural

transformations (lax functors, however, remain lax[14]).

- If we reverse all 2-cells, lax natural transformations become oplax natural transformations and lax functors become oplax functors.

- If we combine both dualisations, we get lax natural transformations and oplax functors.

Therefore, semi-lax adjunctions are not self-symmetric in any sense, in particular the left-right symmetry is missing. This becomes visible especially in Lemma 6.5.

Our first lemma says that semi-lax adjunctions are unique up to equivalence.

Lemma 6.3 *Let $\mathfrak{A}, \mathfrak{B}$ be two* **dc**-*categories.*

1. *Assume $U : \mathfrak{B} \longrightarrow \mathfrak{A}$ is a semi-lax functor, part of two semi-lax adjunctions $((F, \psi) \dashv (U, \phi) : \mathfrak{B} \longrightarrow \mathfrak{A}, \eta, \varepsilon, \mu, \nu)$ and $((F', \psi') \dashv (U, \phi) : \mathfrak{B} \longrightarrow \mathfrak{A}, \eta', \varepsilon', \mu', \nu')$. Then $F \simeq F'$.*

2. *Assume $F : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a semi-lax functor, part of two semi-lax adjunctions $((F, \psi) \dashv (U, \phi) : \mathfrak{B} \longrightarrow \mathfrak{A}, \eta, \varepsilon, \mu, \nu)$ and $((F, \psi) \dashv (U', \phi') : \mathfrak{B} \longrightarrow \mathfrak{A}, \eta', \varepsilon', \mu', \nu')$. Then $U \simeq U'$.*

Proof. *Ad 1.* We have semi-lax transformations

$$F \xrightarrow{F\eta'} FUF' \xrightarrow{\varepsilon F'} F' \quad \text{and} \quad F' \xrightarrow{F'\eta} F'UF \xrightarrow{\varepsilon' F} F.$$

We have to show that the compositions both ways round are isomorphic to id_F respectively $\mathrm{id}_{F'}$. The following pasting diagram shows the construction of one of the two required modifications.
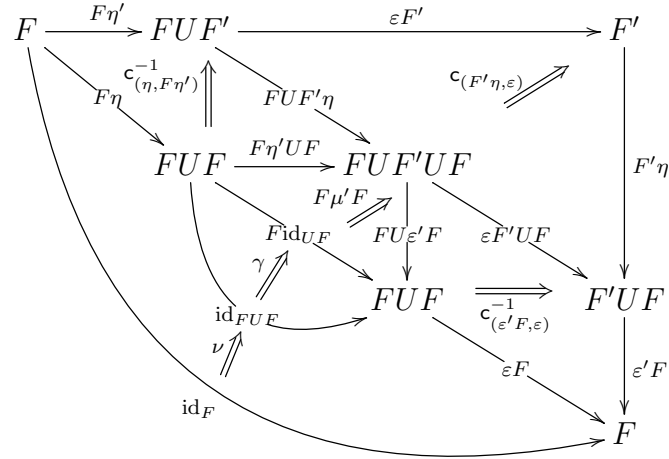
The $c_{(\alpha,\beta)}$ are the exchange modifications mentioned after (6.1). γ has as components the constraints $\mathrm{id}_{FUFA} \xrightarrow{\psi} F\mathrm{id}_{UFA}$. It is easily seen that this is natural (or rather 'modificational'), and γ is invertible by definition of semi-lax functor. Because all faces are isomorphisms, the diagram composes to an isomorphic modification.

Following the same pattern, one can construct an isomorphic modification of type $\mathrm{id}_U \longrightarrow \varepsilon F'\ F\eta'\ \varepsilon'F\ F'\eta$. Thus, $\varepsilon F'\ F\eta'$ and $\varepsilon'F\ F'\eta$ together give an equivalence between F and F' in $\mathbf{SLax}(\mathfrak{A},\mathfrak{B})$.

Remark. By general principles, every equivalence in a bicategory can be made into an adjoint equivalence by modifying the 2-cells. However, it is natural to ask if the equivalence as given by the above construction already *is* an adjoint equivalence. I have not checked that yet, but I would be very surprised if this was not the case.

Ad 2. The proof of the second assertion follows the same pattern again. ∎

The next lemma is a categorification of the well-known correspondence

$$
\begin{array}{ccc}
\begin{array}{ccc}
FA & \xrightarrow{Ff} & FA' \\
\downarrow & & \downarrow \\
B & \xrightarrow{g} & B'
\end{array}
&
\hat{=}
&
\begin{array}{ccc}
A & \xrightarrow{f} & A' \\
\downarrow & & \downarrow \\
UB & \xrightarrow{Ug} & UB'
\end{array}
\end{array}
$$

between commutative squares (f, g fixed) for one-dimensional adjunctions. Instead of commutativity, we allow arbitrary 2-cells. However, for this to work it is necessary that the vertical arrows at the sides of the squares are regular.

Lemma 6.4 (Conjugate squares)   *1. Given a semi-lax adjunction*

$$((F,\psi) \dashv (U,\phi) : \mathfrak{B} \longrightarrow \mathfrak{A}, \eta, \varepsilon, \mu, \nu)$$

and morphisms $A \xrightarrow{f} A'$ in $\mathfrak{A}$ and $B \xrightarrow{g} B'$ in $\mathfrak{B}$, the mappings

$$
\begin{array}{ccc}
\begin{array}{ccc}
FA & \xrightarrow{Ff} & FA' \\
r\downarrow & \rho \nearrow\!\!\!\nearrow & \downarrow r' \\
B & \xrightarrow{g} & B'
\end{array}
&
\mapsto
&
\begin{array}{c}
\end{array}
\end{array}
$$

and

$$
\begin{array}{ccc}
A \xrightarrow{\ f\ } A' & & \\
\Big\downarrow{\scriptstyle s} \quad {\scriptstyle \sigma}\nearrow\!\!\!\nearrow \quad \Big\downarrow{\scriptstyle s'} & \mapsto & \\
UB \xrightarrow[\ Ug\ ]{} UB' & &
\end{array}
$$

(left square with $A \xrightarrow{f} A'$, s, s' vertical, σ 2-cell, bottom $UB \xrightarrow{Ug} UB'$)

maps to the diagram with $FA \xrightarrow{Ff} FA'$, $F(s'f)$, ψ, $U\sigma$, Fs, Fs', ψ, $F(Ugs)$, $FUB \xrightarrow{FUg} FUB'$, ε_B, ε_f, ε'_B, $B \xrightarrow{g} B'$

— *both called* conjugation — *are inverse to each other in the sense that if we conjugate a square twice, then the vertical sides of the original square and the vertical sides of the resulting square are canonically isomorphic and if we compose the 2-cell in the resulting square on both sides with these canonical isomorphisms, then we get the 2-cell in the original square.*

2. *Conjugation commutes with the composition 2-cells on the top and bottom on the square. More precisely, for given*

$$
\begin{array}{ccc}
FA \xrightarrow{Ff} FA' & & \\
{\scriptstyle r}\Big\uparrow \ {\scriptstyle \rho}\nearrow\!\!\!\nearrow \ \Big\downarrow{\scriptstyle r'} & , \quad A \overset{f'}{\underset{f}{\rightrightarrows}}{\scriptstyle \Uparrow\alpha} A' \quad and \quad B \overset{g}{\underset{g'}{\rightrightarrows}}{\scriptstyle \Uparrow\beta} B' & \\
B \xrightarrow[g]{} B' & &
\end{array}
$$

it does not make a difference if we first compose $F\alpha$ and β to the top and the bottom of the square and then conjugate, or first conjugate and then compose α and $U\beta$.

The same is true for the other direction of conjugation.

Proof. *Ad 1.* If we conjugate the square

$$
\begin{array}{ccc}
FA \xrightarrow{Ff} FA' & & \\
{\scriptstyle r}\Big\downarrow \ {\scriptstyle \rho}\nearrow\!\!\!\nearrow \ \Big\downarrow{\scriptstyle r'} & & \\
B \xrightarrow[g]{} B' & &
\end{array}
$$

twice, we obtain a square with left hand side $\varepsilon_B \circ F(Ur \circ \eta_A)$ and the canonical isomorphism to r is given by

$$
\varepsilon_B \circ F(Ur \circ \eta_A) \xrightarrow{\ \mathrm{id}\circ\psi\ } \varepsilon_B \circ FUr \circ F\eta_A \xrightarrow{\ \varepsilon_r^{-1}\circ\mathrm{id}\ } r \circ \varepsilon_{FA} \circ \eta_A \xrightarrow{\ \mathrm{id}\circ\nu_A\ } r,
$$

similarly for the right hand side.

The 2-cell in the result square is built up as follows (From now on, we leave it to the reader to insert the names of the 2-cells).

$$g \circ \varepsilon_B \circ F(Ur \circ \eta_A) \qquad \varepsilon_{B'} \circ F(Ur' \circ UFf \circ \eta_A) \longrightarrow \varepsilon_{B'} \circ F(Ur' \circ \eta_{A'} \circ f)$$

$$\downarrow \qquad\qquad\qquad\qquad \cong\uparrow \qquad\qquad\qquad\qquad\qquad \cong\downarrow$$

$$\varepsilon_{B'} \circ FUg \circ F(Ur \circ \eta_A) \qquad \varepsilon_{B'} \circ F(U(r' \circ Ff) \circ \eta_A) \qquad \varepsilon_{B'} \circ F(Ur' \circ \eta_{A'}) \circ Ff$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad \uparrow$$

$$\varepsilon_{B'} \circ F(Ug \circ Ur \circ \eta_A) \longrightarrow \varepsilon_{B'} \circ F(U(g \circ r) \circ \eta_A)$$

The proof that this, composed with the two canonical isomorphisms, is equal to ρ is displayed in Figure 4. In this diagram, the path along the upper side is the mentioned composition, the path along the lower side has ρ at the beginning and the fact that the remainder is the identity is an instance of the modification axiom for ν, composed with r'. Beneath the diagram, there are some comments on why the different faces commute.

This was the first half of the proof. When we start with a square of the form

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & A' \\
s\downarrow & \overset{\sigma}{\Longrightarrow} & \downarrow s' \\
UB & \xrightarrow[Ug]{} & UB'
\end{array}
$$

and conjugate twice, the proof is similar[15], and we do not carry it out here.

Ad 2. We carry out the proof only for one direction of conjugation. The assertion rewrites as

$$
\begin{array}{ccc}
\begin{array}{c}
Ug'\,Ur\,\eta_A \\
\downarrow \\
U(g'\,r)\,\eta_A \\
\downarrow \\
U(g\,r)\,\eta_A \\
\downarrow \\
U(r'\,Ff)\,\eta_A \\
\downarrow \\
U(r'\,Ff')\,\eta_A \\
\downarrow \\
Ur'\,UFf'\,\eta_A \\
\downarrow \\
Ur'\,\eta'_A\,f'
\end{array}
& = &
\begin{array}{c}
Ug'\,Ur\,\eta_A \\
\downarrow \\
Ug\,Ur\,\eta_A \\
\downarrow \\
U(g\,r)\,\eta_A \\
\downarrow \\
U(r'\,Ff)\,\eta_A \\
\downarrow \\
Ur'\,UFf\,\eta_A \\
\downarrow \\
Ur'\,\eta'_A\,f \\
\downarrow \\
Ur'\,\eta'_A\,f'
\end{array}
\end{array}
$$

---

[15]However, the other case is not symmetric. Remember that semi-lax adjunction is not a self-symmetric concept.

and the proof is given by the following diagram.

$$U g'\, U r\, \eta_A$$

$$U(g'\, r)\, \eta_A \qquad\qquad U g\, U r\, \eta_A$$

$$U(g\, r)\, \eta_A$$

$$U(r'\, Ff)\, \eta_A$$

$$U(r'\, Ff')\, \eta_A \qquad\qquad U r'\, UFf\, \eta_A$$

$$U r'\, UFf'\, \eta_A \qquad\qquad U r'\, \eta'_A\, f$$

$$U r'\, \eta'_A\, f'$$

$\blacksquare$

Lemma 6.5 *Assume that we have a semi-lax adjunction*

$$((F,\psi) \dashv (U,\phi) : \mathfrak{B} \longrightarrow \mathfrak{A}, \eta, \varepsilon, \mu, \nu).$$

Then all constraint cells of U are invertible and thus U is a pseudofunctor.

Proof. $\mathrm{id}_{(UB)} \cong U(\mathrm{id}_B)$ holds for any semi-lax functor, therefore it remains to show that for all composable pairs of arrows $A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$ in $\mathfrak{B}$, the constraint cell

$$(Ug)(Uf) \xrightarrow{\ \phi_{(f,g)}\ } U(gf) \tag{6.3}$$

is invertible.

Using the technique of conjugation, we can obtain a candidate for the inverse 2-cell by conjugating the square

$$\begin{array}{c}
FUA \xrightarrow{\ F(Ug\,Uf)\ } FUC \\
\varepsilon_A \Big\downarrow \quad \psi_{(Uf,Ug)} \Uparrow \quad FUg \quad \Big\downarrow \varepsilon_C \\
FUB \\
\varepsilon_f \quad \varepsilon_B \quad \varepsilon_g \\
A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C
\end{array} \tag{6.4}$$

and pre- and postcomposing the result with μ_A and μ_C. It remains to check wether the

resulting 2-cell $\begin{array}{c} UA \xrightarrow{UgUf} UC \\ \mathrm{id}\Big\downarrow \quad \gamma\!\!\nearrow \quad \Big\downarrow\mathrm{id} \\ UA \xrightarrow[U(gf)]{} UC \end{array}$ is in fact inverse to $\phi_{(f,g)}$.

Figure 4: (1) and (2) commute because of the second axiom for lax transformations for the pairs (g,r) and (Ff,r'), respectively. (3), (4), (5) are instances of the second axiom for lax functors. The commutativity of the other faces follows either from plain 2-category axioms or from the naturality of the constraints for composition.

$$
\begin{array}{c}
g\ r \longrightarrow g\ r\ \varepsilon_{FA}\ F\eta_A \\
r'\ Ff \longrightarrow r'\ Ff\ \varepsilon_{FA}\ F\eta_A \\[2mm]
g\ \varepsilon_B\ FUr\ F\eta_A \longrightarrow g\ \varepsilon_B\ F(Ur\ \eta_A) \\
\varepsilon_{B'}\ FUg\ FUr\ F\eta_A \longrightarrow \varepsilon_{B'}\ FUg\ F(Ur\ F\eta_A) \quad (3) \\
(1)\quad \varepsilon_{B'}\ F(Ug\ Ur)\ F\eta_A \longrightarrow \varepsilon_{B'}\ F(Ug\ Ur\ F\eta_A) \\
\varepsilon_{B'}\ FU(g\ r)\ F\eta_A \longrightarrow \varepsilon_{B'}\ F(U(g\ r)\ \eta_A) \\
\varepsilon_{B'}\ FU(r'\ Ff)\ F\eta_A \longrightarrow \varepsilon_{B'}\ F(U(r'\ Ff)\ \eta_A) \\
\varepsilon_{B'}\ F(Ur'\ UFf)\ F\eta_A \longrightarrow \varepsilon_{B'}\ F(Ur'\ UFf\ \eta_A) \\
(4)\quad \varepsilon_{B'}\ FUr'\ FUFf\ F\eta_A \longrightarrow \varepsilon_{B'}\ FUr'\ F(UFf\ \eta_A) \longrightarrow \varepsilon_{B'}\ F(Ur'\ \eta'_A\ f) \longrightarrow \varepsilon_{B'}\ F(Ur'\ \eta'_A)\ Ff \\
\varepsilon_{B'}\ FUr'\ FUFf\ F\eta_A \longrightarrow \varepsilon_{B'}\ FUr'\ F(UFf\ \eta_A) \longrightarrow \varepsilon_{B'}\ FUr'\ F(\eta'_A\ f) \longrightarrow \varepsilon_{B'}\ FUr'\ F\eta'_A\ Ff \quad (5) \\
(2)\quad r'\ \varepsilon_{FA'}\ FUFf\ F\eta_A \longrightarrow r'\ \varepsilon_{FA'}\ F(UFf\ \eta_A) \longrightarrow r'\ \varepsilon_{FA'}\ F(\eta_{A'}\ f) \longrightarrow r'\ \varepsilon_{FA'}\ F\eta_{A'}\ Ff \\
r'\ Ff
\end{array}
$$

To show that it is right inverse to $\phi_{(f,g)}$ we can make use of the compatibility of conjugation with composition on the upper side of the square. Via conjugation, we have the correspondence (modulo canonical equivalences on the vertical sides)

$$
\begin{array}{ccc}
UA \xrightarrow{U(gf)} UC & & FUA \xrightarrow{FU(gf)} FUC \\
\text{id}\downarrow\ \ \overset{\text{id}}{\nearrow}\ \ \downarrow\text{id} & \hat{=} & \varepsilon_A\downarrow\ \ \overset{\varepsilon_{gf}}{\nearrow}\ \ \downarrow\varepsilon_C \\
UA \xrightarrow{U(gf)} UC & & A \xrightarrow{gf} C
\end{array}
$$

Therefore, the assertion

$$
\begin{array}{ccc}
UA \xrightarrow{Ugf} UC & & UA \longrightarrow UC \\
\text{id}\uparrow\ \overset{\phi_{(f,g)}}{\underset{Uf}{\searrow}}\ UB\ \overset{Ug}{\nearrow}\ \downarrow\text{id} & = & \downarrow\ \overset{\text{id}}{\nearrow}\ \downarrow \\
UA \underset{\gamma}{\nearrow} \xrightarrow{Ugf} UC & & UA \longrightarrow UC
\end{array}
$$

is equivalent to

$$
\begin{array}{ccc}
\overset{FU(gf)}{\overset{\frown}{}} & & \\
FUA \xrightarrow{F(UgUf)} FUC & & FUA \xrightarrow{FU(gf)} FUC \\
FUf\downarrow\ \ \psi_{(Uf,Ug)}\ FUB\ FUg & = & \varepsilon_A\downarrow\ \ \overset{\varepsilon_{gf}}{\nearrow}\ \ \downarrow\varepsilon_C \\
\varepsilon_A\downarrow\ \overset{\varepsilon_f}{\nearrow}\ \varepsilon_B\ \overset{\varepsilon_g}{\nearrow}\ \downarrow\varepsilon_C & & A \xrightarrow{gf} C \\
A \xrightarrow{f} B \xrightarrow{g} C & &
\end{array}
$$

and this is an instance of the second axiom for lax transformations.

It remains to show that composing the other way round also gives the identity. If we unfold the conjugation operation, we observe that the 2-cell that we want to prove identic is built up as follows

$$
\begin{array}{c}
UA \longrightarrow UB \longrightarrow UC \\
\nearrow \\
UFUA \xrightarrow{UF(UgUf)} UFUC \\
\text{id}_{UA}\left(\ \cong\ \right)\qquad\cong\qquad\left(\ \cong\ \right)\text{id}_{UC} \\
\nearrow\ \nearrow\ \nearrow \\
UA \longrightarrow UC \\
\searrow\ \Uparrow\ \nearrow \\
UB
\end{array}
\tag{6.5}
$$

where the curved arrows in the square are $U(gf\varepsilon_A), U(g\varepsilon_B(FUf)), U(\varepsilon_C(FUg)(FUf))$ and $U(\varepsilon_C(F((Ug)(Uf))))$ from left to right. We first concentrate on the subdiagram consisting of this square and the lower triangle. This subdiagram corresponds to the path along the left side in the following diagram (living in $\mathfrak{A}(UFUA, UC)$).

$$
\begin{array}{ccccc}
U\varepsilon_C UF(UgUf) & \longleftarrow & U\varepsilon_C U(FUgFUf) & \longleftarrow & U\varepsilon_C UFUgUFUf \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
U(\varepsilon_C F(UgUf)) & \longleftarrow & U(\varepsilon_C FUgFUf) & \longleftarrow & U(\varepsilon_C FUg)UFUf \\
& & \uparrow & & \uparrow \\
U(gf)U\varepsilon_A \longrightarrow U(gf\varepsilon_A) & \longrightarrow & U(g\varepsilon_B FUf) & \longleftarrow & U(g\varepsilon_B)UFUf \\
\uparrow & \uparrow & \uparrow & & \uparrow \\
UgUfU\varepsilon_A \longrightarrow UgU(f\varepsilon_A) & \longrightarrow & UgU(\varepsilon_B FUf) & \overset{\cong}{\longleftarrow} & UgU(\varepsilon_B)UFUf
\end{array}
$$

Because all squares commute, we can replace the subdigram in (6.5) by a diagram that visualizes the path counterclockwise along the lower, right and upper side of the above diagram. This looks as follows.

The lax naturality of η allows us to rewrite the upper rectangle and we arrive at

Now if we view the small squares grouped columnwise, we see that the left column is $((U \circ \varepsilon)(\eta \circ U))_f$ and the right one is $((U \circ \varepsilon)(\eta \circ U))_g$, and the proposition follows from the modification axiom for μ. ∎

In Lemma 6.5, we have observed that a necessary condition for a semi-lax functor to have a left semi-lax adjoint is to be pseudo. Next, we will present a necessary and sufficient condition which is inspired by the characterisation by representability of one-dimensional adjunctions.

Theorem 6.6 *Let $\mathfrak{A}, \mathfrak{B}$ be dc-categories and let $(U, \phi) : \mathfrak{B} \longrightarrow \mathfrak{A}$ be a semi-lax pseudofunctor (i.e. a pseudofunctor that maps regular arrows to regular arrows). Then U has a left semi-lax adjoint* iff

1. *For each $A \in \mathsf{Obj}(\mathfrak{A})$ there is an object $FA \in \mathsf{Obj}(\mathfrak{B})$ and a regular arrow $\eta_A : A \longmapsto UFA$ such that for all $B \in \mathsf{Obj}(\mathfrak{B})$ and all $f : A \longrightarrow UB$, the category $(A \nearrow U)(\eta_A, f)$ has a terminal object $(\hat{f}, \alpha_f)$*

$$
\begin{array}{ccc}
& A & \\
{\scriptstyle \eta_A}\downarrow & {\scriptstyle \alpha_f}\Uparrow & \searrow{\scriptstyle f} \\
UFA & \longrightarrow & UB
\end{array}
$$

$$FA \xrightarrow{\quad \hat{f} \quad} B$$

2. *If $f : A \longrightarrow UB$ is regular then $\hat{f}$ is also regular and α_f is invertible.*

3. *$(\mathrm{id}_{FA}, \phi_{FA}^{-1} \circ \eta_A)$ is terminal in $(A \nearrow U)(\eta_A, \eta_A)$.*

$$
\begin{array}{c}
A \\
{\scriptstyle \eta_A}\downarrow \qquad \overset{\eta_A}{\searrow} \\
UFA \xrightarrow{\;\mathrm{id}_{UFA}\;} UFA \\
{\scriptstyle \Uparrow \phi_{FA}^{-1}} \\
FA \xrightarrow[\mathrm{id}_{FA}]{U\mathrm{id}_{FA}} FA
\end{array}
$$

4. *For all $f : A \longrightarrow UB$ and all regular $g : B \longmapsto C$, $(g\hat{f}, (\phi_{(\hat{f},g)}^{-1} \circ \eta_A)(Ug \circ \alpha_f))$ is terminal in $(A \nearrow U)(\eta_A, Ugf)$.*

$$
\begin{array}{c}
A \\
{\scriptstyle \eta_A}\downarrow \quad {\scriptstyle \alpha_f}\Uparrow \quad \searrow{\scriptstyle f} \\
UFA \longrightarrow UB \longmapsto UC \\
{\scriptstyle \Uparrow \phi_{(\hat{f},g)}^{-1}} \\
U(g\hat{f}) \\
FA \xrightarrow{\hat{f}} B \xrightarrow{g} C
\end{array}
$$

Proof. Assume we have a semi-lax adjunction $((F, \psi) \dashv (U, \phi) : \mathfrak{B} \longrightarrow \mathfrak{A}, \eta, \varepsilon, \mu, \nu)$.

Ad 1. The terminal object in $(A \nearrow U)(\eta_A, f)$ is given by

$$(\hat{f}, \alpha_f) = (\varepsilon_B \, Ff, (\mu_U^{-1} \circ f)(U\varepsilon_B \circ \eta_f)(\phi_{(Ff, \varepsilon_B)}^{-1} \circ \eta_A)),$$

as in the following diagram.

$$
\begin{array}{c}
\text{(6.6)}
\end{array}
$$

Given a second object (G, β) in $(A \nearrow U)(\eta_A, f)$, a mediator is given by

$$
\begin{array}{c}
g \\
\downarrow \\
g \; \varepsilon_{FA} \; F\eta_A \\
\downarrow \\
\varepsilon_B \; FUg \; F\eta_A \\
\downarrow \\
\varepsilon_B \; F(Ug \; \eta_A) \\
\downarrow \\
\varepsilon_B \; Ff
\end{array}
$$

The proof that this really mediates involves drawing a big diagram in the style of Figure 4. We do not carry that out here, we only mention that it is necessary to use the first axiom for semi-lax adjunctions (Figure 3).

To show that the mediator is unique, one has to show that any $\iota : g \longrightarrow \varepsilon_B Ff$ can be reconstructed from $\alpha_f(U\iota \circ \eta_A)$ via the construction for the mediator. Again, the proof involves a big diagram, and this time we need the second axiom for semi-lax adjunctions.

Ad 2. This becomes obvious when looking at (6.6).

Ad 3. The construction from 1 gives us the following terminal object in $(A \nearrow U)(\eta_A, \eta_A)$.

It is straightforward to check that ν_A gives an isomorphism to the proposed alternative terminal object.

Ad 4. This is proved similarly to 3.

For the converse direction, assume that we are given a pseudofunctor $(U, \phi) : \mathfrak{B} \longrightarrow \mathfrak{A}$ such that conditions 1–4 hold.

The construction of the left adjoint and the transformations η and ε have already been described by the example of the tripos-to-topos construction in section 5.2.

The modification μ at $B \in \mathsf{Obj}(\mathfrak{B})$ is given by α_{UB}^{-1}

and if we have a look at (5.3), we observe that the constraints for ε were defined to mediate a diagram that is precisely the modification axiom for μ.

ν_A is defined as the unique mediator between

These diagrams are both terminal: the left one because of properties 2 and 4, and the right one because of property 3. To check the modification axiom, one again has to make use of terminality in the appropriate hom-category of $(A \nearrow U)$.

Finally, we have to check the two axioms for semi-lax adjunctions.

If we compare the diagrams in Definition 6.2 with the definition of ν_A, then we see that the defining condition for ν is precisely the first axiom.

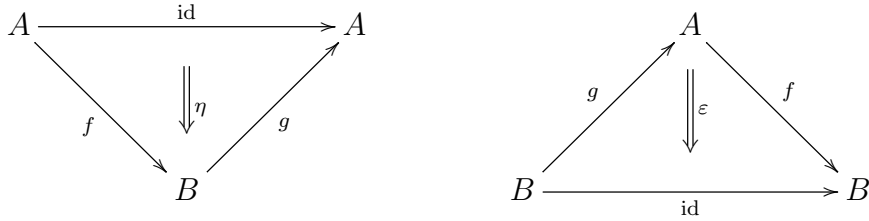For the second axiom have a look at the diagram

$$
\begin{array}{c}
\text{diagram with } UB, UFUB, UFUFUB, \text{ and their arrows and 2-cells}
\end{array}
$$
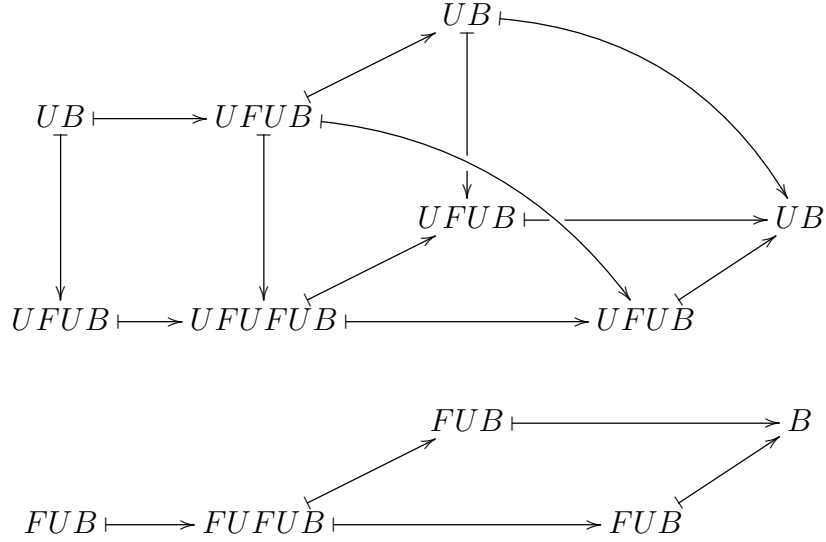
Insert some arrows and 2-cells and observe that the composition that we want to prove identic is an endo-map of (ε_B, μ_B) in $(\eta_{UB}\nearrow U)$. This proves the assertion because (ε_B, μ_B) is terminal. ∎

6.3. dc-categories and (1-)adjunctions

It is well known that 2-functors and pseudofunctors map adjunctions to adjunctions.

However, if we are given an adjunction

$$
\begin{array}{ccc}
A \xrightarrow{\ \ \text{id}\ \ } A & \qquad & A \\
f \searrow \ \Downarrow\eta\ \nearrow g & & g \nearrow\ \Downarrow\varepsilon\ \searrow f \\
B & & B \xrightarrow{\ \ \text{id}\ \ } B
\end{array}
$$

in $\mathfrak{A}$ and apply a *semi-lax* functor (F, ϕ) to the data, we get

$$
\begin{array}{ccc}
FA \ \ FA & \qquad & FA \\
Ff\quad Fg & & Fg\quad Ff \\
FB & & FB\quad FB
\end{array}
$$
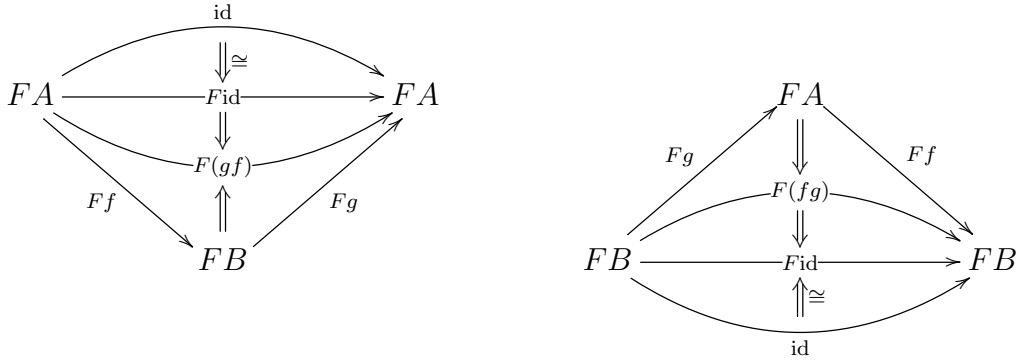
and we can not define a new unit, because one of the 2-cells points in the wrong direction.

If the dc-category $\mathfrak{A}$ has the property that *all right adjoints* in $\mathfrak{A}$ are regular, then the disturbing 2-cell becomes invertible and we can define 2-cells id $\longrightarrow FgFf$ and $FfFg \longrightarrow$ id. It is then easy to see that the triangle equalities hold.

A related concept are monads. Monads in $\mathfrak{A}$ can be defined as lax functors from the terminal 2-category into $\mathfrak{A}$. Because we can compose lax functors, it follows that semi-lax functors map monads to monads.

For comonads, this does not work, because we can not compose lax and oplax functors. However, if a comonad splits into an adjunction then we can map this adjunction and then form the associated comonad in the image-2-category. This tactic has the nice property that it maps idempotent comonads to idempotent comonads, as can be seen in the above diagram (if η is isomorphic, then the induced transformation of type id $\longrightarrow FgFf$ is also isomorphic).

Now, let us try to apply this to the dc-categories $\mathfrak{Trip}_\mathsf{c}$ and $\mathfrak{Top}_\mathsf{c}$. Because we consider semi-*oplax* functors and transformations between them, we have to dualise and therefore exchange left and right adjoints as well as monads and comonads.

Indeed, right adjointable functors and tripos morphisms are necessarily regular, which is just what we need. Idempotent monads (also known as cartesian reflections or topologies) split in $\mathfrak{Trip}_\mathsf{c}$ as well as $\mathfrak{Top}_\mathsf{c}$.

Thus, we have finally arrived at a conceptual explanation why the tripos-to-topos construction behaves well with respect to these two concepts. In particular, we recover the geometric morphisms, that are so important for topos theory, as adjunctions in $\mathfrak{Top}_\mathsf{c}$.

Note: The dc-category that was considered by Johnstone has the property that all right adjoints are regular.

A. Partial equivalence relations in toposes

Let $\mathcal{E}$ be a topos. A partial equivalence relation (per) on $A \in \mathsf{Obj}(\mathcal{E})$ is a subobject $R \rightarrowtail A \times A$ that is symmetric and transitive as a predicate in the internal logic. Its *support* A_0 is the denotation of the predicate $(a \mid R(a,a))$; given by the pullback

$$
\begin{array}{ccc}
A_0 & \rightarrowtail & R \\
\downarrow & & \downarrow \\
A & \underset{\delta}{\rightarrowtail} & A \times A
\end{array}
$$

We have the following decomposition:

$$
A_0 \;\rightarrowtail\; R \;\overset{\langle r_0, r_1 \rangle}{\rightarrowtail}\; A_0 \times A_0 \;\rightarrowtail\; A \times A \tag{A.1}
$$

Partial equivalence relations are used to form *subquotients*, intuitively the 'set of equivalence classes'. Category-theoretically, the subquotient defined by R is the coequaliser

$$
R \underset{p^1}{\overset{p_0}{\rightrightarrows}} A_0 \twoheadrightarrow Q(R)
$$

of r_0 and r_1; r_0, r_1 form a kernel pair for this coequaliser because any topos is effective regular.

It is easy to see that up to isomorphism, every subquotient

$$
A \overset{m}{\leftarrowtail} U \overset{e}{\twoheadrightarrow} Q
$$

comes from a partial equivalence relation. The predicate that classifies this relation is

$$
a, a' \mid \exists u, u' .\, m(u) = a \wedge m(u') = a' \wedge e(u) = e(u').
$$

A.1. Comparison of partial equivalence relations

Lemma A.1 *Consider two per's* $R \rightarrowtail A \times A$ *and* $S \rightarrowtail B \times B$, *inducing the subquotients* $Q(R) \twoheadleftarrow A_0 \rightarrowtail A$ *and* $Q(S) \twoheadleftarrow B_0 \rightarrowtail B$; *and a morphism* $f : A \to B$.

Then there exists a mediating arrow for the left diagram iff there exists a pair of mediating arrows for the right diagram.

$$
\begin{array}{ccc}
R & \dashrightarrow & S \\
\downarrow & & \downarrow \\
A \times A & \longrightarrow & B \times B
\end{array}
\qquad\qquad
\begin{array}{ccc}
Q(R) & \dashrightarrow & Q(S) \\
\uparrow & & \uparrow \\
A_0 & \dashrightarrow & B_0 \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
$$
$\blacksquare$

A.2. Partial equivalence relations and cartesian functors

Now consider a second topos $\mathcal{F}$ and a cartesian functor $F : \mathcal{E} \to \mathcal{F}$. The image of a per $R \rightarrowtail A \times A$ under a cartesian functor is again a per, and the decomposition (A.1) is (up to canonical isos) also stable under F. The only thing that is not stable without regularity of F is the coequaliser, here a non-invertible mono $m_{R,F}$ creeps in, illustrated in the following diagram.

$$FR \underset{Fr_1}{\overset{Fr_0}{\rightrightarrows}} FA_0 \twoheadrightarrow Q(FR)$$

$$\downarrow m_{R,F}$$

$$F(Q(R))$$

B. 2-categorical basics

A 2-category is a $\mathfrak{Cat}$-enriched category. The $\mathfrak{Cat}$-enriched functors and natural transformations are called 2-functors and 2-natural transformations. However, there is a bigger class of functors and transformations between 2-categories that can not be captured by the enrichment idea: pseudofunctors and pseudo-natural transformations, and more generally lax functors and lax transformations. We state the relevant definitions here for reference.

Definition B.1 (Lax functor) *Let $\mathfrak{A}, \mathfrak{B}$ be 2-categories. A* lax functor

$$(F, \phi) : \mathfrak{A} \longrightarrow \mathfrak{B}$$

is given by the following data

- *a mapping*
$$F_0 : \mathsf{Obj}(\mathfrak{A}) \longrightarrow \mathsf{Obj}(\mathfrak{B})$$

- *for each pair of objects $X, Y \in \mathsf{Obj}(\mathfrak{A})$, a functor*
$$F_{XY} : \mathfrak{A}(X, Y) \longrightarrow \mathfrak{B}(FX, FY)$$

- *for each object $X \in \mathsf{Obj}(\mathfrak{A})$, an arrow*
$$\phi_X : \mathrm{id}_{FX} \longrightarrow F\mathrm{id}_X$$

- *for each triple $X, Y, Z \in \mathsf{Obj}(\mathfrak{A})$, a natural transformation ϕ_{XYZ}*

subject to the following axioms

- *For all $f : X \longrightarrow Y$ in $\mathfrak{A}$, we have*

- *For every sequence $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ of 1-cells in $\mathfrak{A}$, we have*

$$
\begin{array}{ccc}
FX \xrightarrow{Fg} FY & & FX \xrightarrow{Fg} FY \\
Ff \uparrow \quad F(hg) \quad \downarrow Fh & = & Ff \uparrow \quad F(gf) \quad \downarrow Fh \\
FW \xrightarrow[F(hgf)]{} FZ & & FW \xrightarrow[F(hgf)]{} FZ
\end{array}
$$

The arrows ϕ_X and the components of the transformations ϕ_{XYZ} are called *constraint cells*. A lax functor with all constraint cells invertible is called a *pseudofunctor*. The 2-functors that we mentioned at the beginning can be recovered as the lax functors whose constraints are all identities. We can dualise the definition of lax functor by reversing all constraints. The resulting concept is called *oplax functor*.

Definition B.2 (Lax transformation) *Let $\mathfrak{A}, \mathfrak{B}$ be 2-categories and $F, G : \mathfrak{A} \longrightarrow \mathfrak{B}$ lax functors. A* lax (natural) transformation $\eta : F \longrightarrow G$ *is given by the following data.*

- *for each object $X \in \mathsf{Obj}(\mathfrak{A})$ a 1-cell*

$$
\eta_X : FX \longrightarrow GX
$$

- *for each pair of objects $X, Y \in \mathsf{Obj}(\mathfrak{A})$ a natural transformation*

$$
\begin{array}{ccc}
\mathfrak{A}(X,Y) & \xrightarrow{F_{XY}} & \mathfrak{B}(FX, FY) \\
G_{XY} \downarrow & \quad \eta_{XY} \quad & \downarrow \eta_Y \circ (-) \\
\mathfrak{B}(GX, GY) & \xrightarrow[(-) \circ \eta_X]{} & \mathfrak{B}(FX, GY)
\end{array}
$$

  (Remark: We denote the component of η_{XY} at $f : X \longrightarrow Y$ by η_f instead of $(\eta_{XY})_f$.)

subject to the following axioms

- *For all objects $X \in \mathsf{Obj}(\mathfrak{A})$ we have*

$$
\begin{array}{ccc}
FX \xrightarrow{F\mathrm{id}} FX & & FX \xrightarrow{F\mathrm{id}} FX \\
\eta_X \downarrow \quad \eta_{\mathrm{id}} \quad \downarrow \eta_X & = & \quad \mathrm{id} \quad \downarrow \eta_X \\
GX \xrightarrow[G\mathrm{id}]{} GX & & GX \\
\quad \mathrm{id} & &
\end{array}
$$

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- *For all composable pairs $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathfrak{A}$ we have*

$$
\begin{array}{ccc}
FX \xrightarrow{F(gf)} FZ & & FX \xrightarrow{F(gf)} FZ \\
\eta_X \Big\downarrow \quad \eta_{gf} \quad \Big\downarrow \eta_Z & = & \eta_X \Big\downarrow \quad Ff \nearrow \ FY \ \nwarrow Fg \quad \Big\downarrow \eta_Z \\
GX \xrightarrow{G(gf)} GZ & & GX \qquad \eta_Y \qquad GZ \\
Gf \searrow \ GY \ \nearrow Gg & & Gf \searrow \ GY \ \nearrow Gg
\end{array}
$$

The constraint cells of a lax transformation are the components of the natural transformations η_{XY}. A lax transformation with all constraints invertible is called *pseudo-natural transformation*. If we require all constraints to be identities, we recover 2-natural transformations. If we reverse the constraints, we get what is called an *oplax* transformation[16].

Definition B.3 (Modification) *Let $\mathfrak{A}, \mathfrak{B}$ be 2-categories, $F, G : \mathfrak{A} \longrightarrow \mathfrak{B}$ lax functors and $\eta, \theta : F \longrightarrow G$ lax natural transformations. A modification $\alpha : \eta \longrightarrow \theta$ is a mapping that assigns to each $X \in \mathsf{Obj}(\mathfrak{A})$ a 2-cell*

$$
\alpha_X : \eta_X \longrightarrow \theta_X
$$

such that for all $f : X \longrightarrow Y$ in $\mathfrak{A}$ we have

$$
\begin{array}{ccc}
FX \xrightarrow{Ff} FY & & FX \xrightarrow{Ff} FY \\
\eta_X \Big(\xRightarrow{\alpha_X} \Big) \theta_X \nearrow \theta_f \ \Big) \theta_Y & = & \eta_X \Big(\ \eta_f \nearrow \ \eta_Y \ \Big(\xRightarrow{\alpha_Y} \Big) \theta_Y \\
GX \xrightarrow{Gf} GY & & GX \xrightarrow{Gf} GY
\end{array}
$$

Given 2-categories $\mathfrak{A}, \mathfrak{B}$, the collection of all lax functors between them, lax transformations between the functors, and modifications between the transformations, forms a 2-category in a natural way (with the obvious compositions and identities), denoted by **Lax**$(\mathfrak{A}, \mathfrak{B})$.

Here is a basic lemma about adjoint equivalences in **Lax**$(\mathfrak{A}, \mathfrak{B})$.

---

[16]In the literature, there is no consensus about which direction of constraints should be called lax and which oplax for transformations. While [9, 6, 2] use the same convention as we do, in [7, 3] lax and oplax transformations are exchanged.

Lemma B.4 *Let* $F, G : \mathfrak{A} \longrightarrow \mathfrak{B}$ *such that* $\phi \dashv \gamma : G \longrightarrow F$ *constitute an adjoint equivalence (The isomorphic modifications for unit and counit remain nameless). Then* ϕ *and* γ *are* pseudo-*natural transformations.*

Proof. We only carry out the proof for ϕ; thus, given some $f : A \longrightarrow B$ in $\mathfrak{A}$, we have to show that ϕ_f is an isomorphic 2-cell. To establish this fact, it suffices to show that ϕ_f has a left and a right inverse. Again, we only construct the right inverse, the left one is symmetrical.

First of all, although unit and counit are nameless, they are nevertheless important, and most important about them are the triangle equalities. They can be presented as

and the message is that in both cases, the composition of the isomorphisms in the faces yields the identity.

Now, because $\gamma\phi \cong \mathrm{id}_F$, the 2-cells in the following diagram compose to id_{Ff}.

whence

has a right inverse. Composing another invertible 2-cell, we see that

$$
\begin{array}{ccc}
FA & \longrightarrow & FB \\
\downarrow & \nearrow\!\!\!\nearrow & \downarrow \cong \searrow \\
GA & \longrightarrow GB & \longrightarrow FB \\
& & \searrow \cong \downarrow \\
& & GB
\end{array}
$$

is also right-invertible, but this is just ϕ_f, since the two triangles cancel by the triangle equality. ∎

Finally, here is a 2-dimensional version of the comma category construction.

Definition B.5 (Lax comma category) *Assume we are given three 2-categories $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$, a lax functor $F : \mathfrak{A} \longrightarrow \mathfrak{C}$ and an oplax functor $G : \mathfrak{B} \longrightarrow \mathfrak{C}$. The 2-category $(F \nearrow G)$ is given by the follwing data.*

- *Objects are triples (A, B, f) with $A \in \mathsf{Obj}(\mathfrak{A}), B \in \mathsf{Obj}(\mathfrak{B})$ and $f : FA \longrightarrow GB$. Normally we suppress A and B in the notation because they can be deduced from the context.*

- *A morphism from (A, B, f) to (C, D, g) is a triple (h, k, α) with $h : A \longrightarrow C$, $k : B \longrightarrow D$ and $\alpha : Gkf \longrightarrow gFh$.*

$$
\begin{array}{ccc}
FA & \xrightarrow{Fh} & FC \\
f \downarrow & \alpha\nearrow\!\!\!\nearrow & \downarrow g \\
GB & \xrightarrow[Gk]{} & GD
\end{array}
$$

- *A 2-cell from (h, k, α) to (l, m, β) is given by a pair (φ, ψ) with $\varphi : h \longrightarrow l$ and $\psi : k \longrightarrow m$ such that $(g \circ \varphi)\alpha = \beta(\psi \circ f)$.*

$$
\begin{array}{ccc}
Gkf & \xrightarrow{\alpha} & gFh \\
\psi \circ f \downarrow & & \downarrow g \circ \varphi \\
Gmf & \xrightarrow[\beta]{} & gFl
\end{array}
$$

Identities and compositions are defined in the obvious way. It is important that F is lax and G is oplax because otherwise the composition of 1-cells is not definable.

We are especially interested in the situation where $\mathfrak{A}$ is the terminal 2-category and F is a strict functor, in which case we use the notation $(A \nearrow G)$ where A is the only value of F. In this case, the first component of the tuples defining 0-, 1-, and 2-cells does not carry any information and we can omit it.

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