

# Duality for generalized algebraic theories

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## Abstract

We exhibit an idempotent biadjunction between a 2-category of small clans and a 2-category of locally finitely presentable categories equipped with a weak factorization system, and characterize the stable subcategories.

## 1 Clans

**Definition 1.1** A *clan* is a category  $\mathcal{T}$  with a distinguished class of morphisms called *display maps*, such that:

1. Arbitrary pullbacks of display maps exist and are again display maps.
2. Isomorphisms and compositions of display maps are display maps.
3.  $\mathcal{T}$  has a terminal object, and terminal projections are display maps.

A *clan morphism* is a functor between clans which preserves display maps, pullbacks of display maps, and the terminal object. We write  $\mathbf{Clan}$  for the 2-category of clans, clan-morphisms, and natural transformations.  $\diamond$

**Remarks 1.2** 1. Clans can be viewed as ‘non-strict’ version of Cartmell’s *contextual categories* [Car78, Car86].

2. The above definition and the term ‘display map’ are due to Taylor [Tay87, §4.3.2], the name ‘clan’ was suggested by Joyal [Joy17, Definition 1.1.1].

**Examples 1.3** 1. Finite-product categories can be viewed as clans where the display maps are the morphisms that are (isomorphic to) product projections. We call such clans *finite-product clans*.

2. Finite-limit categories can be viewed as clans where *all* morphisms are display maps. We call such clans *finite-limit clans*.

3. The base category of every *category with attributes* in the original sense of Cartmell [Car78, Section 3.2] is a clan with display maps the arrows that are isomorphic to projection maps  $\rho_B : \Sigma B \rightarrow A$ .

4. **Kan** is the clan whose underlying category is the full subcategory of the category  $\mathbf{SSet}$  of simplicial sets on *Kan complexes*, and whose display maps are the *Kan fibrations*.

Since it seems to lead to a more readable exposition, we introduce explicit notation and terminology for the dual notion.

**Definition 1.4** A *coclan* is a category  $\mathcal{C}$  with a distinguished class of morphisms called *codisplay maps* satisfying the dual axioms of clans.

The 2-category  $\mathbf{CoClan}$  of coclans is defined dually to that of clans, i.e.  $\mathbf{CoClan}(\mathcal{C}, \mathcal{D}) = \mathbf{Clan}(\mathcal{C}^{\text{op}}, \mathcal{D}^{\text{op}})^{\text{op}}$ .  $\diamond$

## 2 Algebras

**Definition 2.1** Given a small clan  $\mathcal{T}$  and a complete category  $\mathfrak{A}$ , a  $\mathcal{T}$ -*algebra* in  $\mathfrak{A}$  is a clan morphism  $A : \mathcal{T} \rightarrow \mathfrak{A}$  from where  $\mathfrak{A}$  is equipped with the finite-limit clan structure. We write  $\mathcal{T}\text{-Alg}(\mathfrak{A})$  for the category of algebras and natural transformations, i.e.  $\mathcal{T}\text{-Alg}(\mathfrak{A}) = \mathbf{Clan}(\mathcal{T}, \mathfrak{A})$ .

If  $\mathfrak{A} = \mathbf{Set}$  we simply speak of  $\mathcal{T}$ -algebras and write  $\mathcal{T}\text{-Alg}$  for  $\mathcal{T}\text{-Alg}(\mathbf{Set})$ . This is the case we will mainly be concerned with.

**Remarks 2.2** 1. As category of models of a finite-limit sketch,  $\mathcal{T}\text{-Alg}$  is reflective (and therefore closed under arbitrary limits) in  $[\mathcal{T}, \mathbf{Set}]$ , and moreover it is closed under filtered colimits [AR94, Section 1.C]. In particular,  $\mathcal{T}\text{-Alg}$  is locally finitely presentable.

2. The hom-functors  $\mathcal{T}(\Gamma, -) : \mathcal{T} \rightarrow \mathbf{Set}$  are  $\mathcal{T}$ -algebras for all  $\Gamma \in \mathcal{T}$  (we'll refer to them as *hom-algebras*), i.e. the Yoneda embedding  $\mathfrak{y} : \mathcal{T}^{\text{op}} \rightarrow [\mathcal{T}, \mathbf{Set}]$  lifts along the inclusion  $\mathcal{T}\text{-Alg} \hookrightarrow [\mathcal{T}, \mathbf{Set}]$  to a fully faithful functor  $H : \mathcal{T}^{\text{op}} \rightarrow \mathcal{T}\text{-Alg}$ .

$$\begin{array}{ccc} & & \mathcal{T}\text{-Alg} \\ & \nearrow H & \downarrow \\ \mathcal{T}^{\text{op}} & \xrightarrow{\mathfrak{y}} & [\mathcal{T}, \mathbf{Set}] \end{array}$$

3. For  $\Gamma \in \mathcal{T}$ , the hom-functor

$$\mathcal{T}\text{-Alg}(H(\Gamma), -) : \mathcal{T}\text{-Alg} \rightarrow \mathbf{Set}$$

is isomorphic to the evaluation functor  $A \mapsto A(\Gamma)$ , hence it preserves filtered colimits as those are computed in  $[\mathcal{T}, \mathbf{Set}]$  and therefore pointwise. This means that  $H(\Gamma)$  is *compact*<sup>1</sup> in  $\mathcal{T}\text{-Alg}$ .

We call a map  $f : A \rightarrow B$  of algebras *full* if it has the r.l.p. with respect to all maps  $H(p)$  for  $p \in \mathcal{D}$ , i.e. if the naturality square

$$\begin{array}{ccc} A(\Gamma) & \xrightarrow{A(p)} & A(\Delta) \\ \downarrow f_\Gamma & & \downarrow f_\Delta \\ B(\Gamma) & \xrightarrow{B(p)} & B(\Delta) \end{array}$$

<sup>1</sup>Following Lurie [Lur09] we use the shorter term ‘compact’ instead of the more traditional ‘finitely presented’.

is a weak pullback for each display map  $p : \Gamma \rightarrow \Delta$ . By the small object argument, the full maps form the right class of a cofibrantly generated w.f.s.

$$(\mathcal{E}, \mathcal{F})$$

on  $\mathcal{T}\text{-Alg}$  whose left maps we call *extensions*.

We call  $A \in \mathcal{T}\text{-Alg}$  a *0-extension*, if  $(0 \rightarrow A) \in \mathcal{E}$ . In particular, all objects  $H(\Gamma)$  are 0-extensions, since all terminal projections in  $\mathcal{T}$  are display maps and  $Z$  sends terminal projections to initial inclusions.

### 3 Coalgebras and the universal property of $\mathcal{T}\text{-Alg}$

Dually to algebras of a clan, we have *coalgebras of a coclan*. These allow us to formulate the universal property of  $\mathcal{T}\text{-Alg}$ .

**Definition 3.1** Given a small coclan  $\mathcal{C}$  and a cocomplete category  $\mathfrak{X}$ , a  $\mathcal{C}$ -coalgebra in  $\mathfrak{X}$  is a coclan morphism  $C : \mathcal{C} \rightarrow \mathfrak{X}$  from  $\mathcal{C}$  into  $\mathfrak{X}$  equipped with the finite-colimit coclan structure. We write  $\mathcal{C}\text{-CoAlg}(\mathfrak{X})$  for the category of coalgebras and natural transformations.  $\diamond$

**Theorem 3.2 (The universal property of  $\mathcal{T}\text{-Alg}$ )** *For a small clan  $\mathcal{T}$ , the functor  $H : \mathcal{T}^{\text{op}} \rightarrow \mathcal{T}\text{-Alg}$  is the universal  $\mathcal{T}^{\text{op}}$ -coalgebra, in the sense that for every cocomplete category, precomposition with  $H$  induces an equivalence*

$$\text{CoCont}(\mathcal{T}\text{-Alg}, \mathfrak{X}) \xrightarrow{\cong} \text{CoAlg}(\mathcal{T}^{\text{op}}, \mathfrak{X})$$

*between the category of cocontinuous functors from  $\mathcal{T}\text{-Alg}$  to  $\mathfrak{X}$  and the category of  $\mathcal{T}^{\text{op}}$ -coalgebras in  $\mathfrak{X}$ .*

*Proof.* See <https://mathoverflow.net/questions/349409/universal-property-of-the-cocomplete->

TODO: discuss Brandenburg's article.  $\square$

### 4 $(\mathcal{E}, \mathcal{F})$ -categories

**Definition 4.1** An  $(\mathcal{E}, \mathcal{F})$ -category is a locally finitely presentable category  $\mathcal{L}$  equipped with a weak factorization system  $(\mathcal{E}, \mathcal{F})$  whose maps we call *extensions* and *full maps* respectively.

A *morphism of  $(\mathcal{E}, \mathcal{F})$ -categories* from  $\mathcal{L}$  to  $\mathcal{M}$  is an adjunction  $F_! \dashv F^*$  where

1. the *direct image part*  $F_! : \mathcal{L} \rightarrow \mathcal{M}$  preserves compact objects and extensions, and
2. the *inverse image part*  $F^* : \mathcal{M} \rightarrow \mathcal{L}$  preserves filtered colimits and full maps.

A *2-cell*  $\eta : F \rightarrow G$  between morphisms of  $(\mathcal{E}, \mathcal{F})$ -categories is a natural transformation  $\eta : F^* \rightarrow G^*$  between the *inverse image parts*. We write  $\text{EFCat}$  for the 2-category of  $(\mathcal{E}, \mathcal{F})$ -categories.  $\diamond$

**Remark 4.2** It follows from standard arguments that conditions 1 and 2 in Definition 4.1 are equivalent. Moreover, by the special adjoint functor theorem [Mac98, Section V-8] and the *adjoint functor theorem for presentable categories* [AR94, Theorem 1.66] respectively, the two adjoints can be reconstructed from each other. This means that a morphism from  $\mathcal{L}$  to  $\mathcal{M}$  of  $(\mathcal{E}, \mathcal{F})$ -categories is determined equivalently by

- a *cocontinuous* functor  $F_! : \mathcal{L} \rightarrow \mathcal{M}$  preserving extensions and compact objects, and
- a *continuous* functor  $F^* : \mathcal{M} \rightarrow \mathcal{L}$  preserving full maps and filtered colimits.

**Lemma 4.3** *For any morphism  $F : \mathcal{S} \rightarrow \mathcal{T}$  between small clans, the precomposition functor*

$$F^* : \mathcal{T}\text{-Alg} \rightarrow \mathcal{S}\text{-Alg}$$

*is the image inverse part of a morphism of  $(\mathcal{E}, \mathcal{F})$ -categories.*

*Proof.* The preservation of small limits and filtered colimits is obvious since they are computed pointwise (Remark 2.2-1). To show that  $F^*$  preserves full maps, let  $f : A \rightarrow B$  be full in  $\mathcal{T}\text{-Alg}$ . It is sufficient to show that the  $(f \circ F)$ -naturality squares are weak pullbacks at all display maps  $p : \text{in } \mathcal{S}\text{-Alg}$ . But the  $(f \circ F)$ -naturality square at  $p$  is the same as the  $f$ -naturality square at  $Fp$  so the claim follows since  $f$  is full and  $F$  preserves display maps.  $\square$

From Lemma 4.3 it is immediate that the assignment  $\mathcal{T} \mapsto \mathcal{T}\text{-Alg}$  extends to a pseudofunctor

$$(-)\text{-Alg} : \text{Clan}_{\text{sm}} \rightarrow \text{EFCat} \quad (4.1)$$

from the 2-category  $\text{Clan}_{\text{sm}}$  of small clans to the 2-category of  $(\mathcal{E}, \mathcal{F})$ -categories.

**Proposition 4.4** *The pseudofunctor (4.1) has a right biadjoint.*

*Proof.* Given a small clan  $\mathcal{T}$  and an  $(\mathcal{E}, \mathcal{F})$ -category  $\mathcal{L}$ , it is easy to see that the natural equivalence

$$\text{CoCont}(\mathcal{T}\text{-Alg}, |\mathcal{L}|) \simeq \text{CoAlg}(\mathcal{T}^{\text{op}}, |\mathcal{L}|)$$

from Theorem 3.2 (where  $|\mathcal{L}|$  is the underlying category of  $\mathcal{L}$ ) restricts to an equivalence

$$\text{EFCat}(\mathcal{T}\text{-Alg}, \mathcal{L})^{\text{op}} \simeq \text{CoClan}(\mathcal{T}^{\text{op}}, \mathfrak{C}(\mathcal{L}))$$

where  $\mathfrak{C}(\mathcal{L})$  is the coclan whose underlying category is the full subcategory of  $\mathcal{L}$  on compact 0-extensions, and whose co-display maps are the extensions. Taking opposite categories on both sides we get

$$\text{EFCat}(\mathcal{T}\text{-Alg}, \mathcal{L}) \simeq \text{Clan}(\mathcal{T}, \mathfrak{C}(\mathcal{L})^{\text{op}}),$$

which shows that the presheaf  $\text{EFCat}((-)\text{-Alg}, \mathcal{L})$  of categories is birepresented by  $\mathfrak{C}(\mathcal{L})^{\text{op}}$ .  $\square$

In the following we will show that this biadjunction between small clans and  $(\mathcal{E}, \mathcal{F})$ -categories is idempotent, and characterize the fixed-points on both sides.

## 5 Cauchy-complete clans and the fat small object argument

**Definition 5.1** A clan  $\mathcal{T}$  is called *Cauchy-complete*, if its underlying category is Cauchy-complete, and retracts of display maps are display maps.  $\diamond$

Clearly, every clan of the form  $\mathfrak{C}(\mathcal{L})^{\text{op}}$  is Cauchy-complete, thus Cauchy-completeness is a necessary condition for the unit  $\eta_{\mathcal{T}} : \mathcal{T} \rightarrow \mathfrak{C}(\mathcal{T}\text{-Alg})^{\text{op}}$  of the biadjunction to be an equivalence. We will show that it is also sufficient, but for this we need the *fat small object argument*.

**Proposition 5.2 (Fat small object argument)** *For any clan  $\mathcal{T}$ , the 0-extensions in  $\mathcal{T}\text{-Alg}$  are flat, i.e. filtered colimits of hom-algebras.*

*Proof.* This is a special case of [MRV14, Corollary 5.1], but we give a direct proof in the appendix which simplifies considerably in the finitary case.  $\square$

**Corollary 5.3** *If  $\mathcal{T}$  is small and Cauchy-complete then*

- *the functor  $H : \mathcal{T}^{\text{op}} \rightarrow \mathcal{T}\text{-Alg}$  co-restricts to an equivalence between  $\mathcal{T}\text{-Alg}$  and the full subcategory  $\mathfrak{C}(\mathcal{T}\text{-Alg})$  of  $\mathcal{T}\text{-Alg}$  on compact 0-extensions, and*
- *$f : \Delta \rightarrow \Gamma$  in  $\mathcal{T}$  is a display map if and only if  $H(f)$  is an extension.*

*Proof.* Let  $C \in \mathcal{T}\text{-Alg}$  be a compact 0-extension. By Proposition 5.2 there exists a filtered diagram  $D : \mathbb{J} \rightarrow \mathcal{T}^{\text{op}}$  and a limiting cocone  $\sigma : \mathcal{Y} \circ D \rightarrow \Delta(C)$ . Since  $C$  is compact, the identity arrow  $\text{id}_C$  factors through one of the cocone maps  $\sigma_j$ , i.e.  $C$  is a retract of  $\text{hom}(D_j, -)$ . By Cauchy-completeness,  $C$  is itself corepresentable. Thus, we have an equivalence of categories.

TODO: part about display maps.  $\square$

## 6 Clan-algebraic categories

Given an  $(\mathcal{E}, \mathcal{F})$ -category  $\mathcal{L}$ , the counit  $E : \mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Alg} \rightarrow \mathcal{L}$  of the biadjunction is given by the *nerve-realization adjunction*

$$\begin{array}{ccc} \mathfrak{C}(\mathcal{L}) & \xleftarrow{J} & \mathcal{L} \\ H \downarrow & \begin{array}{c} \nearrow E_! \\ \nwarrow E^* \end{array} & \\ \mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Alg} & & \end{array}$$

where  $E^*$  is the nerve of  $J$  given by  $E^*(L) = \mathcal{L}(J(-), L)$ , and its left adjoint  $E_!$  is the Kan extension of  $H$  along  $J$ , given by

$$E_!(A) = \text{colim}(\text{elts}(A) \rightarrow \mathfrak{C}(\mathcal{L}) \xrightarrow{J} \mathcal{L}),$$

where  $\text{elts}(A)$  is the (contravariant) category of elements. In this section we show that  $E$  is an equivalence in  $\text{EFCat}$  if and only if  $\mathcal{L}$  is *clan-algebraic* in the sense of the following definition.

**Definition 6.1** An  $(\mathcal{E}, \mathcal{F})$ -category  $\mathcal{L}$  is called *clan-algebraic* if

- (D) the subcategory  $\mathfrak{C}(\mathcal{L})$  is dense,
- (CG)  $(\mathcal{E}, \mathcal{F})$  is cofibrantly generated by  $\mathcal{E} \cap \text{mor}(\mathfrak{C}(\mathcal{L}))$ , and
- (FQ) quotients of componentwise-full equivalence relations are effective and have full quotient maps.  $\diamond$

**Theorem 6.2** *For every clan  $\mathcal{T}$ ,  $\mathcal{T}\text{-Alg}$  is clan-algebraic.*

*Proof.*  $\mathfrak{C}(\mathcal{L})$  is dense since it contains the corepresentables.  $(\mathcal{E}, \mathcal{F})$  is cofibrantly generated by maps between corepresentables, and therefore a fortiori by maps between compact 0-extensions.

For the third condition, let

$$r = \langle r_1, r_2 \rangle : R \hookrightarrow A \times A$$

be an equivalence relation such that  $r_1$  and  $r_2$  are full maps. This means that we have an equivalence relation  $\sim$  on each  $A(\Gamma)$ , such that

1. for each  $s : \Delta \rightarrow \Gamma$ , the function  $A(s) = s \cdot (-) : A(\Delta) \rightarrow A(\Gamma)$  preserves this relation, and
2. for every display map  $p : \Gamma^+ \rightarrow \Gamma$  and all  $a, b \in A(\Gamma)$  and  $c \in A(\Gamma^+)$  such that  $a \sim b$  and  $p \cdot c = a$ , there exists a  $d \in A(\Gamma^+)$  with  $c \sim d$  and  $p \cdot d = b$ .

We show first that the pointwise quotient  $A/R$  is a model. Clearly  $(A/R)(1) = 1$ , and it remains to show that given a pullback

$$\begin{array}{ccc} \Delta^+ & \xrightarrow{t} & \Gamma^+ \\ \downarrow q & & \downarrow p \\ \Delta & \xrightarrow{s} & \Gamma \end{array}$$

with  $p$  and  $q$  display maps, and elements  $a \in A(\Delta)$ ,  $b \in A(\Gamma^+)$  with  $s \cdot a \sim p \cdot b$ , there exists a unique-up-to- $\sim$   $c \in A(\Delta^+)$  with  $q \cdot c \sim a$  and  $t \cdot c \sim b$ . Since  $p$  is a display map, there exists a  $b'$  with  $b \sim b'$  and  $p \cdot b' = s \cdot a$ , and since  $A$  is a model there exists therefore a  $c$  with  $q \cdot c = a$  and  $t \cdot c = b'$ . For uniqueness assume that  $c, c' \in A(\Delta^+)$  with  $q \cdot c \sim q \cdot c'$  and  $t \cdot c \sim t \cdot c'$ . Then  $c \sim c'$  follows from the fact that  $R$  is a model. This shows that  $A/R$  is an algebra, and also that the quotient is effective, since the kernel pair is computed pointwise. The fact that  $A \rightarrow A/R$  is full is similarly easy to see.  $\square$

The following lemma is a kind of converse to (FQ).

**Lemma 6.3** *Full maps in clan-algebraic categories are regular epimorphisms.*

*Proof.* Given a full map in a clan-algebraic category  $\mathcal{L}$ , the lifting property against (compact) 0-extensions implies that  $E^*(f)$  is componentwise surjective in  $\mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Alg}$ , and therefore the coequalizer of its kernel pair. Since left adjoints preserve regular epis, we deduce that  $E_!(E^*(f))$  is regular epic in  $\mathcal{L}$  and the claim follows since  $E_! \circ E^* \cong \text{id}$  by (D).  $\square$

**Remark 6.4** Observe that we only used property (D) in the proof, no exactness.

**Lemma 6.5** *The class  $\mathcal{F}$  of full maps in a clan-algebraic category  $\mathcal{L}$  has the right cancellation property, i.e. we have  $g \in \mathcal{F}$  whenever  $gf \in \mathcal{F}$  and  $f \in \mathcal{F}$  for composable pairs  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ .*

*Proof.* By (CG) it suffices to show that  $g$  has the r.l.p. with respect to extensions  $e : I \hookrightarrow J$  between compact 0-extensions  $I, J$ . Let

$$\begin{array}{ccc} I & \xrightarrow{h} & B \\ \downarrow e & & \downarrow g \\ J & \xrightarrow{k} & C \end{array}$$

be a filling problem. Since  $I$  is a 0-extension and  $f$  is full, there exists a map  $h' : I \rightarrow A$  with  $fh' = h$ . We obtain a new filling problem

$$\begin{array}{ccc} I & \xrightarrow{h'} & A \\ \downarrow e & & \downarrow gf \\ J & \xrightarrow{k} & C \end{array}$$

which can be filled by a map  $m : J \rightarrow A$  since  $gf$  is full. Then  $fm$  is a filler for the original problem.  $\square$

**Lemma 6.6** *Let  $\mathcal{L}$  be a clan-algebraic category, let  $f : A \rightarrow B$  be an arrow in  $\mathcal{L}$  with componentwise full kernel pair  $p, q : R \rightrightarrows A$ , and let  $e : A \rightrightarrows C$  be the coequalizer of  $p$  and  $q$ . Then the unique  $m : C \rightarrow B$  with  $me = f$  is monic.*

*Proof.* By (D) it is sufficient to test monicity of  $m$  on maps out of compact 0-extensions  $E$ . Let  $h, k : E \rightarrow C$  such that  $mh = mk$ . Since  $e$  is full by (FQ), there exist  $h', k' : E \rightarrow A$  with  $eh' = h$  and  $ek' = k$ . In particular we have  $fh' = fk'$  and therefore there is an  $u : E \rightarrow R$  with  $pu = h'$  and  $qu = k'$ . Thus we can argue

$$h = eh' = epu = equ = ek' = k$$

which shows that  $m$  is monic.  $\square$

**Lemma 6.7** *If  $A \in \mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Alg}$  is flat, then  $A \rightarrow E^*(E_!A)$  is an isomorphism, thus  $E_!$  restricted to flat algebras is fully faithful.*

*Proof.* We have

$$\begin{aligned} E^*(E_!A)(C) &= \mathcal{L}(C, \text{colim}(\text{elts}(A) \rightarrow \mathfrak{C}(\mathcal{L}) \hookrightarrow \mathcal{L})) \\ &\cong \text{colim}(\text{elts}(A) \rightarrow \mathfrak{C}(\mathcal{L}) \xrightarrow{\mathfrak{J}(C)} \text{Set}) \quad \text{since } \text{elts}(A) \text{ is filtered} \\ &\cong A \otimes \mathfrak{J}(C) \cong A(C). \end{aligned}$$

The second claim follows since for flat  $B$ , the mapping

$$(\mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Alg})(A, B) \rightarrow \mathcal{L}(E_!A, E_!B)$$

can be decomposed as

$$(\mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Alg})(A, B) \rightarrow (\mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Alg})(A, E^*E_!B) \rightarrow \mathcal{L}(E_!A, E_!B) \quad \square$$

**Lemma 6.8** *The following are equivalent for a cone  $\phi : \Delta C \rightarrow D$  on a diagram  $D : \mathbb{J} \rightarrow \mathcal{L}$  in an  $(\mathcal{E}, \mathcal{F})$ -category  $\mathcal{L}$ .*

1. *Given an extension  $e : A \rightarrow B$ , an arrow  $h : A \rightarrow C$ , and a cone  $\kappa : \Delta B \rightarrow D$  such that  $\phi_j \circ h = \kappa_j e$  for all  $j \in \mathbb{J}$ , there exists  $l : B \rightarrow C$  such that  $le = h$  and  $\phi_j l = \kappa_j$  for all  $j \in \mathbb{J}$ .*

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ e \downarrow & \nearrow l & \downarrow \phi_j \\ B & \xrightarrow{\kappa_j} & D_j \end{array}$$

2. *The mediating arrow  $l : C \rightarrow \lim(D)$  is full.*

*Proof.* The data of  $e, h, \kappa$  is equivalent to  $e, h$ , and  $k : B \rightarrow \lim(D)$  such that

$$\begin{array}{ccc} A & \xrightarrow{h} & C \\ e \downarrow & & \downarrow f \\ B & \xrightarrow{k} & \lim(D) \end{array}$$

commutes, and  $l : B \rightarrow C$  fills the latter square iff it fills all the squares with the  $D_j$ .  $\square$

**Definition 6.9** We call a cone  $\phi : \Delta C \rightarrow D$  satisfying the conditions of the lemma *jointly full*.  $\diamond$

**Remark 6.10** The interest of this is that it allows us to talk about full ‘covers’ of limits without actually computing the limits, which is useful when talking about cones and diagrams in the full subcategory of a clan-algebraic category on 0-extensions, which does not admit limits.

**Definition 6.11** A *nice diagram* in an  $(\mathcal{E}, \mathcal{F})$ -category  $\mathcal{L}$  is a 2-truncated simplicial diagram

$$\begin{array}{ccccc} & \xrightarrow{d_0} & & \xrightarrow{d_0} & \\ A_2 & \xrightarrow{d_1} & A_1 & \xrightarrow{d_1} & A_0 \\ & \xrightarrow{d_2} & & & \end{array}$$

where

1.  $A_0, A_1$ , and  $A_2$  are 0-extensions,
2. the maps  $d_0, d_1 : A_1 \rightarrow A_0$  are full,

3. in the commutative square  $\begin{array}{ccc} A_2 & \xrightarrow{d_0} & A_1 \\ d_2 \downarrow & & \downarrow d_1 \\ A_1 & \xrightarrow{d_0} & A_0 \end{array}$  the span constitutes a jointly full cone over the cospan,

4. there exists a ‘symmetry’ map  $\begin{array}{ccc} A_1 & \xrightarrow{d_1} & A_0 \\ d_0 \downarrow & \searrow \sigma & \uparrow d_0 \\ A_0 & \xleftarrow{d_1} & A_1 \end{array}$  making the triangles commute, and



5. there exists a 0-extension  $\tilde{A}$  and full maps  $f, g : \tilde{A} \twoheadrightarrow A_1$  constituting a jointly full cone over the diagram

$$\begin{array}{ccc} A_1 & & A_1 \\ d_0 \downarrow & \searrow^{d_1} & \downarrow d_1 \\ A_0 & \xleftarrow{d_0} & A_0 \end{array} \quad \diamond$$

**Lemma 6.12** *If  $A_\bullet$  is a nice diagram in a clan-algebraic category  $\mathcal{L}$ , the pairing  $\langle d_0, d_1 \rangle : A_1 \rightarrow A_0 \times A_0$  factors as  $A_1 \xrightarrow{f} R \xrightarrow{\langle r_0, r_1 \rangle} A_0 \times A_0$  where  $f$  is full, and  $r = \langle r_0, r_1 \rangle$  is monic and a componentwise full equivalence relation.*

*Proof.* Condition 5 of the preceding definition gives us the following diagram

$$\begin{array}{ccccc} \tilde{A} & & & & \\ & \searrow^g & & & \\ & S & \xrightarrow{p} & A_1 & \\ & \downarrow q & \lrcorner & \downarrow \langle d_0, d_1 \rangle & \\ & A_1 & \xrightarrow{\langle d_0, d_1 \rangle} & A_0 \times A_0 & \end{array},$$

$f$  (from  $\tilde{A}$  to  $A_1$ ),  $h$  (from  $\tilde{A}$  to  $S$ )

i.e.  $S$  is the kernel of  $\langle d_0, d_1 \rangle$  with projections  $p, q$ ,  $\tilde{A}$  is a 0-extension, and  $f, g, h$  are full. By right cancellation we deduce that  $p$  and  $q$  are full, and the existence of the factorization follows from Lemma 6.6. Fullness of  $r_0, r_1$  follows again from right cancellation because  $f, d_0$ , and  $d_1$  are full.

It remains to show that  $r$  is an equivalence relation. This is easy: condition 4 gives symmetry, and condition 3 gives transitivity, and reflexivity follows from the fact that  $r_0$  admits a section as a full map into a 0-extension, together with symmetry (we internalize the argument that if in a symmetric and transitive relation everything is related to *something*, then it is reflexive.)  $\square$

**Definition 6.13** A *fully extended cover* of an object  $A$  in an  $(\mathcal{E}, \mathcal{F})$ -category  $\mathcal{L}$  is a full map  $f : E \twoheadrightarrow A$  from a 0-extension  $E$ .  $\diamond$

Fully extended can be constructed by factoring the initial map  $0 \rightarrow A$ . So they're like cofibrant replacements.

**Lemma 6.14** *For every object  $A$  in an  $(\mathcal{E}, \mathcal{F})$ -category  $\mathcal{L}$  there exists a nice diagram  $A_\bullet$  with colimit  $A$ .*

*Proof.*  $A_0$  is given as fully extended cover  $f_0 : A_0 \twoheadrightarrow A$  of  $A$ ,  $A_1$  is obtained as fully extended cover of  $f_1 : A_1 \twoheadrightarrow A_0 \times_A A_0$ , and  $A_2$  as fully extended cover  $f_2 : A_2 \twoheadrightarrow P$  of the pullback

$$\begin{array}{ccc} P & \xrightarrow{p_0} & A_1 \\ p_1 \downarrow & \lrcorner & \downarrow d_0 \\ A_1 & \xrightarrow{d_1} & A_0 \end{array},$$

with  $d_0, d_1, d_2 : A_2 \rightarrow A_1$  given by  $d_0 = p_0 \circ f$ ,  $d_2 = p_1 \circ f$ , and  $d_1$  a lifting of  $\langle d_0 \circ d_0, d_1 \circ d_2 \rangle$  along  $f_1$ .

The symmetry map  $\sigma$  is constructed as a lifting of the symmetry of  $A_0 \times_A A_0$  along  $f_1$ .

$\tilde{A}$  is a fully extended cover of the kernel of  $f_1$ .  $\square$

**Lemma 6.15** *For any clan-algebraic category  $\mathcal{L}$ , the realization functor  $E_!$  preserves jointly full cones in flat algebras, and nice diagrams.*

*Proof.* The first claim follows since  $E_!$  is fully faithful by Lemma 6.7 and in both sides the weak factorization system determined by the same generators. Thus there's a one-to-one correspondence between lifting problems. The second claim follows since  $E_!$  preserves 0-extensions and 0-extensions are flat.  $\square$

**Lemma 6.16** *For any clan-algebraic category  $\mathcal{L}$ , The nerve functor  $E_* : \mathcal{L} \rightarrow \mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Alg}$  preserves quotients of nice diagrams.*

*Proof.* Given a nice diagram  $A_\bullet$  in  $\mathcal{L}$ , its colimit is the coequalizer of  $d_0, d_1 : A_1 \rightarrow A_0$ . By Lemma 6.12,  $\langle d_0, d_1 \rangle$  factors as  $\langle r_0, r_1 \rangle \circ f$  with  $f$  full and  $r = \langle r_0, r_1 \rangle$  an equivalence relation. The pairs  $d_0, d_1$  and  $r_0, r_1$  have the same coequalizer (since  $f$  is epic), and  $E_*$  preserves the coequalizer of  $r_0, r_1$  since it preserves full maps and kernel pairs. Finally, the coequalizer of  $E_*(r_0), E_*(r_1)$  is also the coequalizer of  $E_*(d_0), E_*(d_1)$  since  $E_*(f)$  is full and therefore epic.  $\square$

**Theorem 6.17** *If  $\mathcal{L}$  is clan-algebraic, then  $\varepsilon : \mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Alg} \rightarrow \mathcal{L}$  is an equivalence in EFCat.*

*Proof.* By density,  $\varepsilon^*$  is fully faithful. It remains to show that it is essentially surjective. It remains to show that the unit map  $\eta_A : A \rightarrow E^*(E_!(A))$  is an isomorphism for all  $A \in \mathfrak{C}(\mathcal{L})^{\text{op}}\text{-Alg}$ . Let  $A_\bullet$  be a nice diagram with colimit  $A$ . We have

$$\begin{aligned} E^*(E_!(A)) &= E^*(E_!(\text{colim}(A_\bullet))) \\ &= \text{colim}(E^*(E_!(A_\bullet))) \\ &= \text{colim}(A_\bullet) \\ &= A \end{aligned} \quad \square$$

## References

- [AR94] J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189. Cambridge University Press, 1994.
- [Car78] J. Cartmell. *Generalised algebraic theories and contextual categories*. PhD thesis, Oxford University, 1978.
- [Car86] J. Cartmell. Generalised algebraic theories and contextual categories. *Annals of Pure and Applied Logic*, 32:209–243, 1986.
- [Joy17] A. Joyal. Notes on clans and tribes. *arXiv preprint arXiv:1710.10238*, 2017.
- [Lur09] J. Lurie. *Higher topos theory*. Number 170. Princeton University Press, 2009.
- [Mac98] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

- [MRV14] M. Makkai, J. Rosicky, and L. Vokrinek. On a fat small object argument. *Advances in Mathematics*, 254:49–68, 2014.
- [Tay87] P. Taylor. *Recursive domains, indexed category theory and polymorphism*. PhD thesis, University of Cambridge, 1987.