Homotopy in regular hyperdoctrines

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Abstract

We construct a category of fibrant objects $\mathbb{C}\langle\mathcal{P}\rangle$ in the sense of Brown from any regular hyperdoctrine $\mathcal{P} : \mathbb{C} \xrightarrow{\mathsf{op}} \mathbf{Pos}$, and show that its homotopy category is isomorphic to the category $\mathbb{C}[\mathcal{P}]$ of partial equivalence relations and compatible functional relations.

1 Introduction

This work is related in spirit to work by van den Berg [Ber18] Rosolini [Ros15]. It was first presented at topos at IHES in 2015.

2 Regular hyperdoctrines

Definition 2.1 A *regular hyperdoctrine* is a contravariant functor

$$\mathcal{P} : \mathbb{C} \xrightarrow{\mathsf{op}} \mathbf{Pos}$$

from a finite-limit category $\mathbb{C}$ into the locally ordered category **Pos** of posets and monotone maps, subject to the following conditions.

(Inf) For every object A, the poset $\mathcal{P}(A)$ has finite meets.

(Ex) For every morphism $f : A \to B$, the *reindexing map*

$$f^* \;=\; \mathcal{P}(f) \;:\; \mathcal{P}(B) \to \mathcal{P}(A)$$

has a left adjoint $\exists_f : \mathcal{P}(A) \to \mathcal{P}(B)$.

(Fr) We have $(\exists_f \varphi) \wedge \psi = \exists_f(\varphi \wedge f^*\psi)$ for all $f : A \to B$, $\varphi \in \mathcal{P}(A)$, $\psi \in \mathcal{P}(B)$.

(BC) We have $\exists_k \circ h^* = g^* \circ \exists_f$ for all pullback squares $\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$ in $\mathbb{C}$. $\diamondsuit$

Observe that (Ex) implies that the reindexing maps preserve finite meets. (Fr) is known as the *Frobenius condition*, and (BC) as the *Beck-Chevalley* condition.

Hyperdoctrines were originally introduced by Lawvere [Law69; Law70] to give a categorical account of the connectives of predicate logic in terms of adjoint functors. The semantics of first-order logic in hyperdoctrines was developed more systematically in [See83].

The definition of *regular* hyperdoctrine above postulates the structure required to interpret 'regular logic', which is the fragment of first order logic allowing only the connectives of equality, existential quantification and conjunction. Compared to Lawvere's original definition the hyperdoctrines considered here are degenerate in that their fibers are posets rather than general categories.

Notions similar to regular hyperdoctrines appear under various names in the literature – compare e.g. to *regular fibrations* in [Jac01, Definition 4.2.1] and to *elementary existential doctrines* in [MR12], which both only require finite products in the base category and use a weaker version of the Beck-Chevalley condition. Stekelenburg's *fibered locales* [Ste13] require the Beck-Chevalley condition for all existing pullback squares, but the definition is stated for arbitrary base categories.

Definition 2.2 A *tripos* is a regular hyperdoctrine $\mathcal{P} : \mathbb{C} \xrightarrow{\mathsf{op}} \mathbf{Pos}$ such that

1. $\mathcal{P}(A)$ is a Heyting algebra for all $A \in \mathbb{C}$,
2. besides the left adjoints $\exists_f$, the monotone maps $f^* : \mathcal{P}(A) \to \mathcal{P}(B)$ have right adjoints $\forall_f$ for all $f : B \to A$ in $\mathbb{C}$, and
3. every object $A \in \mathbb{C}$ has a *power object*, i.e. an object $\mathfrak{P}(A)$ together with a predicate $\varepsilon_A \in \mathcal{P}(A \times \mathfrak{P}(A))$ such that for all objects B and predicates $\varphi \in \mathcal{P}(A \times B)$ there exists a (not necessarily unique) morphism $\ulcorner\varphi\urcorner : B \to \mathfrak{P}(A)$ satisfying
$$(A \times \ulcorner\varphi\urcorner)(\varepsilon_A) = \varphi. \qquad \diamondsuit$$

Triposes were introduced by Hyland, Johnstone and Pitts [HJP80] over **Set**, and subsequently generalized by Pitts [Pit81] to base categories having only finite limits.

2.1 The internal language of a regular hyperdoctrine

The *internal language* of a regular hyperdoctrine $\mathcal{P} : \mathbb{C} \xrightarrow{\mathsf{op}} \mathbf{Pos}$ is a many-sorted first-order language in the sense of [Joh02, Section D1.1]. It is generated from a signature whose sorts are the objects of $\mathbb{C}$, whose function symbols of arity $A_1 \dots A_n \to B$ are the morphisms of type $A_1 \times \dots \times A_n \to B$ in $\mathbb{C}$, and whose relation symbols of arity $A_1 \dots A_n$ are the elements of $\mathcal{P}(A_1 \times \dots \times A_n)$.

Over this signature, we consider *terms* – which are built up from sorted variables and function symbols, subject to matching arities and sorts – and *regular formulas*, which are generated from *atomic formulas* $\varphi(\vec{t})$ (where φ a relation symbol and $\vec{t}$ is a list of terms matching its arity) and $s = t$ (where s and t are terms of the same sort), using the connectives of conjunction $\wedge$, truth $\top$, and existential quantification $\exists$.

We write $\mathfrak{s}(x)$ and $\mathfrak{s}(t)$ for the sort of a variable and a term, respectively, and we use the shorthand $\mathfrak{s}(x_1, \dots, x_n) = \mathfrak{s}(x_1) \times \dots \times \mathfrak{s}(x_n)$ for lists of variables.

A *term-in-context* is an expressions of the form $(\vec{x} \mid t)$, where $\vec{x}$ is a list of sorted variables and t is a term containing only variables from $\vec{x}$. Similarly, a *formula-in-context* is a pair $(\vec{x} \mid \varphi)$ of a list $\vec{x}$ of sorted variables, and a formula whose *free* variables are contained in $\vec{x}$.

The *interpretation* of terms-in-context and formlas-in-context is defined by structural induction by the clauses in Table 1. In general, the interpretation

$$
\begin{aligned}
[\![x_i]\!]_{\vec{x}} &= \pi_i \\
[\![f(t_1, \dots, t_n)]\!]_{\vec{x}} &= f \circ \langle [\![t_1]\!]_{\vec{x}}, \dots, [\![t_n]\!]_{\vec{x}} \rangle \\
[\![\varphi(t_1, \dots, t_n)]\!]_{\vec{x}} &= \langle [\![t_1]\!]_{\vec{x}}, \dots, [\![t_n]\!]_{\vec{x}} \rangle^*(\varphi) \\
[\![t = u]\!]_{\vec{x}} &= \langle [\![t]\!]_{\vec{x}}, [\![u]\!]_{\vec{x}} \rangle^*(\exists_\delta \top) \\
[\![\varphi \wedge \psi]\!]_{\vec{x}} &= [\![\varphi]\!]_{\vec{x}} \wedge [\![\psi]\!]_{\vec{x}} \\
[\![\top]\!]_{\vec{x}} &= \top \\
[\![\exists y \,.\, \varphi]\!]_{\vec{x}} &= \exists_\pi([\![\varphi]\!]_{\vec{x},y})
\end{aligned}
$$

In the fourth clause δ is the diagonal map $\mathfrak{s}(t) \to \mathfrak{s}(t) \times \mathfrak{s}(t)$, and in the last clause π is the projection $\mathfrak{s}(\vec{x}) \times \mathfrak{s}(y) \to \mathfrak{s}(\vec{x})$.

Table 1: Interpretation of the internal language

of a term-in-context $(\vec{x} \mid t)$ is a morphism $[\![t]\!]_{\vec{x}} : \mathfrak{s}(\vec{x}) \to \mathfrak{s}(t)$ in $\mathbb{C}$, and the interpretation of a formula-in-context $(\vec{x} \mid \varphi)$ is a predicate $[\![\varphi]\!]_{\vec{x}} \in \mathcal{P}(\mathfrak{s}(\vec{x}))$. Observe that by (BC) applied to the pullback

$$
\begin{tikzcd}
U \arrow[r] \arrow[d, "m"'] & \mathfrak{s}(t) \arrow[d] \\
\mathfrak{s}(\vec{x}) \arrow[r, "\langle [\![t]\!]_{\vec{x}}, [\![u]\!]_{\vec{x}} \rangle"] & \mathfrak{s}(t) \times \mathfrak{s}(t)
\end{tikzcd}
$$

we have $[\![t = u]\!]_{\vec{x}} = \exists_m \top$, where m is the equalizer of $[\![t]\!]_{\vec{x}}$ and $[\![u]\!]_{\vec{x}}$. The following standard lemmas are verified by structural induction.

Lemma 2.3 (Weakening) *We have*

- $[\![t]\!]_{\vec{x},y,\vec{z}} = [\![t]\!]_{\vec{x},\vec{z}} \circ \pi$
- $[\![\varphi]\!]_{\vec{x},y,\vec{z}} = \pi^*([\![\varphi]\!]_{\vec{x},\vec{z}})$

for all terms-in-context $(\vec{x}, \vec{z} \mid t)$ and formulas-in-context $(\vec{x}, \vec{z} \mid \varphi)$, where $\pi : \mathfrak{s}(\vec{x}, y, \vec{z}) \to \mathfrak{s}(\vec{x}, \vec{z})$ is the obvious projection. ■

Lemma 2.4 (Substitution) *We hvae*

- $[\![t[u/y]]\!]_{\vec{x}} = [\![t]\!]_{\vec{x},y} \circ \langle \mathrm{id}_{\mathfrak{s}(\vec{x})}, [\![u]\!]_{\vec{x}} \rangle$
- $[\![\varphi[u/y]]\!]_{\vec{x}} = \langle \mathrm{id}_{\mathfrak{s}(\vec{x})}, [\![u]\!]_{\vec{x}} \rangle^*([\![t]\!]_{\vec{x},y})$

for all formulas-in-context $(\vec{x}, y \mid \varphi)$ and terms-in-context $(\vec{x}, y \mid t)$, $(\vec{x} \mid u)$ such that $\mathfrak{s}(y) = \mathfrak{s}(u)$. ■

We call terms-in-context $(\vec{x} \mid t)$ and $(\vec{x} \mid u)$ (or formulas-in-context $(\vec{x} \mid \varphi)$ and $(\vec{x} \mid \psi)$) *semantically equal*, if $[\![t]\!]_{\vec{x}} = [\![u]\!]_{\vec{x}}$ (or $[\![\varphi]\!]_{\vec{x}} = [\![\psi]\!]_{\vec{x}}$).

Lemma 2.5 (Congruence) *Semantic equality of terms and formulas in-context is a* congruence, *in the sense that it is preserved by the formation of bigger terms/formulas from smaller ones.*

$\dfrac{}{\varphi_1, \ldots, \varphi_n \vdash_{\vec{x}} \varphi_i}$	$\dfrac{\Gamma \vdash_{\vec{x},y} \theta}{\Gamma \vdash_{\vec{x}} \exists y \,.\, \theta}$	$\dfrac{\Gamma \vdash_{\vec{x}} \exists y \,.\, \theta \qquad \Gamma, \theta \vdash_{\vec{x},y} \varphi}{\Gamma \vdash_{\vec{x}} \varphi}$
$\dfrac{}{\Gamma \vdash_{\vec{x}} \top}$	$\dfrac{}{\Gamma \vdash_{\vec{x}} t = t}$	$\dfrac{\Gamma \vdash_{\vec{x}} \theta[s/y] \qquad \Gamma \vdash_{\vec{x}} s = t}{\Gamma \vdash_{\vec{x}} \theta[t/y]}$
$\dfrac{\Gamma \vdash_{\vec{x}} \varphi \wedge \psi}{\Gamma \vdash_{\vec{x}} \varphi}$	$\dfrac{\Gamma \vdash_{\vec{x}} \varphi \wedge \psi}{\Gamma \vdash_{\vec{x}} \psi}$	$\dfrac{\Gamma \vdash_{\vec{x}} \varphi \qquad \Gamma \vdash_{\vec{x}} \psi}{\Gamma \vdash_{\vec{x}} \varphi \wedge \psi}$

Table 2: The rules of regular logic

The preceding lemma justifies *local rewriting*, i.e. replacing subterms-in-context (or subformulas-in-context) of a formula-in-context $(\vec{x} \mid \varphi)$ by semantically equal ones without changing the interpretation.

A *judgment* in the internal language is an expression of the form

$$\varphi_1, \ldots, \varphi_n \vdash_{\vec{x}} \psi,$$

where $\varphi_1, \ldots, \varphi_n$ and ψ are formulas in context $\vec{x}$. We say that the judgment is *valid* (or *holds*), if

$$[\![\varphi_1]\!]_{\vec{x}} \wedge \cdots \wedge [\![\varphi_n]\!]_{\vec{x}} \leq [\![\psi]\!]_{\vec{x}} \quad \text{in} \quad \mathcal{P}(\mathfrak{s}(\vec{x})).$$

Theorem 2.6 (Soundness) *The set of valid judgments is closed under the rules of regular logic in Table 2.*

The following lemma gives an equality rule relative to a pullback square, which we use in the proof of Theorem 4.5.

Lemma 2.7 *Given a pullback square*

$$\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd},$$

in $\mathbb{C}$, the set of valid judgments is closed under the following rule.

$$\frac{\Gamma[hp, kp] \vdash_{\vec{x},p} \varphi[hp, kp]}{\Gamma[x, u], f(x) = g(u) \vdash_{\vec{x},x,u} \varphi[x, u]}$$

Proof. The claim follows from the Beck-Chevalley condition since

$$\begin{tikzcd} D \arrow[r] \arrow[d] & C \arrow[d] \\ A \times B \arrow[r] & C \times C \end{tikzcd}$$

is a pullback. ∎

3 The category $\mathbb{C}\langle\mathcal{P}\rangle$

Definition 3.1 Let $\mathcal{P}\mathbb{C}^{\mathsf{op}} \to \mathbf{Ord}$ be a regular hyperdoctrine. The category $\mathbb{C}\langle\mathcal{P}\rangle$ is defined as follows.

- Objects are pairs $(A \in \mathbb{C}, \rho \in \mathcal{P}(A \times A))$ such that the judgments

  (sym) $\rho(x,y) \vdash_{x,y} \rho(y,x)$ and

  (trans) $\rho(x,y), \rho(y,z) \vdash_{x,y,z} \rho(x,z)$

  hold.

- Morphisms from (A,ρ) to (B,σ) are functions $f : A \to B$ such that

  (compat) $\rho(x,y) \vdash_{x,y} \sigma(fx, fy)$

  holds.

- Composition and identities are inherited from $\mathbb{C}$. $\diamondsuit$

Thus, objects of $\mathbb{C}\langle\mathcal{P}\rangle$ are partial equivalence relations in $\mathcal{P}$, and morphisms are compatible functions. Adopting common practice we normally write ρx instead of $\rho(x,x)$ for the diagonal ('support') of a partial equivalence relation $\rho \in \mathcal{P}(A \times A)$. When reasoning variable-freely (i.e. not in the internal language) we use the notation

$$\rho_0 := \delta_A^* \rho$$

for the 'restriction' of a partial equivalence relation along the diagonal.

The definition of $\mathbb{C}\langle\mathcal{P}\rangle$ is similar to the definition of $\boldsymbol{Q}\mathcal{P}$ given in [Fre15] for the special case of triposes, the difference being that whereas in $\mathbb{C}\langle\mathcal{P}\rangle$ morphisms are compatible functions, in $\boldsymbol{Q}\mathcal{P}$, morphisms are *equivalence classes* of compatible functions, where $f, g : (A,\rho) \to (B,\sigma)$ are identified whenever

(equiv) $\rho(x) \vdash_x \sigma(fx, gx)$

holds. Thus there is a full identity on objects functor $\mathbb{C}\langle\mathcal{P}\rangle \to \boldsymbol{Q}\mathcal{P}$. In the next section, we will describe a structure of category of fibrant objects on $\mathbb{C}\langle\mathcal{P}\rangle$, and recover this equivalence relation as the homotopy relation induced by the path objects. However, our aim is not to recover the category $\boldsymbol{Q}\mathbb{C}$ but rather the *topos* $\mathbb{C}[\mathcal{P}]$. This will turn out just right, since the homotopy category of a category of fibrant objects is not simply the morphisms quotiented by homotopy, but something a bit more complicated (because contrary to model structures we don't have cofibrant replacements).

Before coming to homotopy, we establish some basic properties of $\mathbb{C}\langle\mathcal{P}\rangle$.

Lemma 3.2 *Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Ord}$ be a regular hyperdoctrine.*

1. *The forgetful functor $U : \mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{C}$ has a right adjoint.*
2. *$\mathbb{C}\langle\mathcal{P}\rangle$ has finite limits.*
3. *$f : (A,\rho) \to (B,\sigma)$ is iso in $\mathbb{C}\langle\mathcal{P}\rangle$ iff f is iso in $\mathbb{C}$ and $(f \times f)^*\sigma = \rho$.*

Proof. The right adjoint is given by $R(A) = (A, \top)$. The terminal object is $(1, \top)$. A pullback of $(A, \rho) \xrightarrow{f} (C, \tau) \xleftarrow{g} (B, \sigma)$ is given by

$$\begin{tikzcd}
(D, \rho \bowtie_C \sigma) \arrow[r, "k"'] \arrow[d, "h"'] & (B, \sigma) \arrow[d, "g"] \\
(A, \rho) \arrow[r, "f"] & (C, \tau)
\end{tikzcd},$$

where $\begin{tikzcd} D \arrow[r, "k"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$ is a pullback in $\mathbb{C}$ and $(\rho \bowtie_C \sigma)(p, q) \equiv \rho(hp, hq) \wedge \sigma(kp, kq)$.

For the third claim, the necessity of the conditions becomes obvious by considering an inverse to $f : (A, \rho) \to (B, \sigma)$. Conversely, the conditions also allow to construct this inverse. ∎

3.1 Functoriality

Let $\mathcal{P}, \mathcal{Q} : \mathbb{C}^{\mathsf{op}} \to \mathbf{Ord}$ be regular hyperdoctrines, and let $\Phi : \mathcal{P} \to \mathcal{Q}$ be a natural transformation all of whose components

$$\Phi_A : \mathcal{P}(A) \to \mathcal{Q}(A) \qquad (\text{for } A \in \mathbb{C})$$

preserve finite meets. The functor

$$\mathbb{C}\langle\Phi\rangle \;:\; \mathbb{C}\langle\mathcal{P}\rangle \;\to\; \mathbb{C}\langle\mathcal{Q}\rangle$$

sends objects $(A, \rho) \in \mathbb{C}\langle\mathcal{P}\rangle$ to objects $(A, \Phi_{A\times A}(\rho)) \in \mathbb{C}\langle\mathcal{Q}\rangle$, and morphisms in $\mathbb{C}\langle\mathcal{P}\rangle$ to morphisms in $\mathbb{C}\langle\mathcal{Q}\rangle$ having the same underlying map in $\mathbb{C}$. It is easy to see that $\mathbb{C}\langle\Phi\rangle$ preserves finite limits.

4 $\mathbb{C}\langle\mathcal{P}\rangle$ as a category of fibrant objects

We recall the following definition from [Bro73].

Definition 4.1 A *category of fibrant objects* is a category $\mathbb{C}$ with finite products, together with two distinguished classes $\mathcal{F}, \mathcal{W} \subseteq \mathrm{mor}(\mathbb{C})$ of morphisms whose elements are called *fibrations* and *weak equivalences*, respectively. Morphisms in $\mathcal{F} \cap \mathcal{W}$ are called *trivial fibrations.* These classes are subject to the following axioms.

(A) $\mathcal{W}$ contains all isomorphisms, and for any composable pair $A \xrightarrow{f} B \xrightarrow{g} C$, if either two of the three morphisms f, g, and gf are in $\mathcal{W}$, then so is the third.

(B) $\mathcal{F}$ contains all isomorphisms and is closed under composition.

(C) Pullbacks of fibrations along arbitrary maps exist and are fibrations. Pullbacks of trivial fibrations are trivial fibrations.

(D) For any $X \in \mathbb{C}$ there exists a *path object*, i.e. a factorization

$$X \xrightarrow{s} X^I \xrightarrow{d=\langle d_0, d_1\rangle} X \times X$$

of the diagonal, where $s \in \mathcal{W}$ and $d \in \mathcal{F}$.

(E) For any $X \in \mathbb{C}$, the map $X \to 1$ is a fibration. $\diamondsuit$

To endow $\mathbb{C}\langle\mathcal{P}\rangle$ with the structure of a category of fibrant objects, we define fibrations and weak equivalences.

Definition 4.2 A morphism $f : (A, \rho) \to (B, \sigma)$ in $\mathbb{C}\langle\mathcal{P}\rangle$ is a *fibration* if

(fib) $\rho x, \sigma(fx, u) \vdash_{x,u} \exists y \,.\, \rho(x, y) \wedge fy = u$

holds. It is a *weak equivalence* if

(inj) $\rho x, \sigma(fx, fy), \rho y \vdash_{x,y} \rho(x, y)$ and

(esurj) $\sigma u \vdash_u \exists x \,.\, \rho x \wedge \sigma(fx, u)$

hold. $\diamondsuit$

Lemma 4.3 *$f : (A, \rho) \to (B, \sigma)$ is a trivial fibration if and only if (inj) and*

(surj) $\sigma u \vdash_u \exists x \,.\, \rho x \wedge fx = u$

hold.

Proof. Easy, see Appendix A. ∎

Remark 4.4 Stated variable-freely, the condition (surj) reduces to the inequality $\sigma_0 \leq \exists_f \rho_0$, and since the reverse inequality follows from (compat), it is equivalent to the equality $\sigma_0 = \exists_f \rho_0$. $\diamondsuit$

Theorem 4.5 *$\mathbb{C}\langle\mathcal{P}\rangle$ with the classes of fibrations and weak equivalences from Definition 4.2 is a category of fibrant objects.*

Proof. It is easy to see that the properties (fib), (inj), and (esurj) hold for isomorphisms (using Lemma 3.2-3) and are stable under composition. Given a composable pair $(A, \rho) \xrightarrow{f} (B, \sigma) \xrightarrow{g} (C, \tau)$, if (inj) holds for gf, then it holds for f, and if (esurj) holds for gf, then it holds for g; for the same reason that initial segments of injective functions are injective, and end segments of surjective functions are surjective. Furthermore it is easy to show that (esurj) for gf and (inj) for g implies (esurj) for f, and that (inj) for gf and (esurj) for f implies (inj) for g, again formalizing set theoretic arguments. This shows conditions (A) and (B).

For condition (C) consider a pullback square

$$\begin{tikzcd}
(D, \rho \bowtie_C \sigma) \arrow[r, "k"] \arrow[d, "h"'] & (B, \sigma) \arrow[d, "g"] \\
(A, \rho) \arrow[r, "f"'] & (C, \tau)
\end{tikzcd},$$

and assume that g is a fibration. The validity of (fib) for h (abbreviated (fib)(h)) is shown as follows (the step from 5 to 6 we uses the rule from Lemma 2.7).

1. (compat) $\Rightarrow$ $\rho(hp, x) \vdash_{p,x} \tau(g(kp), fx)$
2. (fib)(g) $\Rightarrow$ $\sigma(kp), \tau(g(kp), fx) \vdash_{p,x} \exists v \,.\, \sigma(kp, v) \wedge gv = fx$

3. 1, 2 $\Rightarrow$ $\sigma(kp), \rho(hp, x) \vdash_{p,x} \exists v \,.\, \sigma(kp, v) \wedge gv = fx$

4. $\Rightarrow$ $\rho(hp, hq^*), \sigma(kp, kq^*) \vdash_{p,q^*} \rho(hp, hq^*) \wedge \sigma(kp, kq^*) \wedge hq^* = hq^*$

5. 4 $\Rightarrow$ $\rho(hp, hq^*), \sigma(kp, kq^*) \vdash_{p,q^*} \exists q \,.\, \rho(hp, hq) \wedge \sigma(kp, kq) \wedge hq = hq^*$

6. 5 $\Rightarrow$ $\rho(hp, x), \sigma(kp, v), gv = fx \vdash_{p,x,v} \exists q \,.\, \rho(hp, hq), \sigma(kp, kq), hq = x$

7. 3, 6 $\Rightarrow$ $\sigma(kp), \rho(hp, x) \vdash_{p,x} \exists q \,.\, \rho(hp, hq) \wedge \sigma(kp, kq), hq = x$

8. 7 $\Rightarrow$ (fib)(h)

This shows that fibrations are stable under pullback. To show that *trivial* fibrations are stable under pullback, we show pullback stability of conditions (inj) and (surj) separately.

Pullback stability of (surj) is shown as follows.

1. (surj)(g), (compat)(f) $\Rightarrow$ $\rho x \vdash_x \exists u \,.\, \sigma u \wedge gu = fx$

2. $\Rightarrow$ $\rho(hp^*), \sigma(kp^*) \vdash_{p^*} \rho(hp^*) \wedge \sigma(kp^*) \wedge hp^* = hp^*$

3. 2 $\Rightarrow$ $\rho(hp^*), \sigma(kp^*) \vdash_{p^*} \exists p \,.\, \rho(hp) \wedge \sigma(kp) \wedge hp = hp^*$

4. 3 $\Rightarrow$ $\rho x, \sigma u, gu = fx \vdash_{x,u} \exists p \,.\, \rho(hp) \wedge \sigma(kp) \wedge hp = x$

5. 1, 3 $\Rightarrow$ $\rho x \vdash_x \exists p \,.\, \rho(hp) \wedge \sigma(kp) \wedge hp = x$

Pullback stability of (inj) is shown as follows.

1. (inj)(g) $\Rightarrow$ $\sigma(kp), \sigma(kq), \tau(gkp, gkq) \vdash_{p,q} \sigma(kp, kq)$

2. (compat)(f), $fh = gk$ $\Rightarrow$ $\rho(hp, hq) \vdash_{p,q} \tau(gkp, gkq)$

3. 1, 2 $\Rightarrow$ $\sigma(kp), \rho(hp, hq), \sigma(kq) \vdash_{p,q} \sigma(kp, kq)$

4. 3 $\Rightarrow$ $\rho(hp), \sigma(kp), \rho(hp, hq), \rho(hq), \sigma(hq) \vdash_{p,q} \rho(hp, hq) \wedge \sigma(kp, kq)$

A path object for (A, ρ) is given by

$$(A, \rho) \xrightarrow{s} (A \times A, \tilde{\rho}) \xrightarrow{d} (A, \rho) \times (A, \rho) = (A \times A, \rho \bowtie \rho) \tag{4.1}$$

with

$$\tilde{\rho}((x, y), (x', y')) \;\equiv\; \rho(x, x') \wedge \rho(y, y') \wedge \rho(x, y),$$

and where the underlying maps of s and d are δ and id, respectively. It is easy to see that this is well defined, and that s is a weak equivalence and d is a fibration, as required.

Finally, it is easy to check that terminal projections $(A, \rho) \to 1$ are fibrations, and this finishes the proof. ∎

Remark 4.6 It can easily be seen that the fibration part of all path object factorizations (4.1) is monic (since the underlying map is iso, and the forgetful functor reflects monomorphisms). This implies that the ∞-*localization of* $\mathbb{C}\langle\mathcal{P}\rangle$ – i.e. the ∞-category obtained by weakly inverting weak equivalences – is degenerate in the sense that all of its objects are 0-truncated. Indeed, if the second factor of a path object factorization $X \to PX \to X \times X$ is monic, then $\begin{array}{ccc} PX & \longrightarrow & PX \\ \downarrow & & \downarrow \\ PX & \longrightarrow & X \times X \end{array}$ is a homotopy-pullback (since it is a pullback of a span of fibrations) and thus the diagonal of X is a homotopy embedding. $\diamondsuit$

Remark 4.7 (Fibrations from restrictions) Given a regular hyperdoctrine $\mathbb{C} \xrightarrow{\mathsf{op}} \mathbf{Pos}$, an object $(A, \rho) \in \mathbb{C}\langle \mathcal{P} \rangle$ and a predicate $\varphi \in \mathcal{P}(A)$, we say that φ is *compatible* with ρ if the judgments

$$\varphi x \vdash_x \rho x \qquad \text{and} \qquad \varphi x, \rho(x, y) \vdash_{x,y} \varphi y$$

hold in $\mathcal{P}$. If this is the case, we define the *restriction* $\rho|_\varphi$ of ρ to φ by

$$(\rho|_\varphi)(x, y) \;\equiv\; \rho(x, y) \wedge \varphi(x)$$

Then $\rho|_\varphi$ is a partial equivalence relation, and the identity $\mathrm{id} : A \to A$ in $\mathbb{C}$ induces a monomorphism

$$(A, \rho|_\varphi) \to (A, \rho)$$

in $\mathbb{C}\langle \mathcal{P} \rangle$ which is easily seen to be a fibration. $\diamondsuit$

5 The homotopy category

In this section we show that $\mathbb{C}[\mathcal{P}]$ is the homotopy category of $\mathbb{C}\langle \mathcal{P} \rangle$. Rather than making making use of the description of the homotopy category in [Bro73], we directly establish the universal property: $\mathbb{C}[\mathcal{P}]$ is obtained from $\mathbb{C}\langle \mathcal{P} \rangle$ by freely inverting weak equivalences. This insight can be viewed as the main result of the paper, and since we prove it directly, it turns out that in the end we do not really need the machinery of 'categories of fibrant objects'.

Recall from [Pit02, Def. 3.1] that the category $\mathbb{C}[\mathcal{P}]$ has the same objects as $\mathbb{C}\langle \mathcal{P} \rangle$, its morphisms from (A, ρ) to (B, σ) are predicates $\phi \in \mathcal{P}(A \times B)$ satisfying the judgments

(strict) $\phi(x, u) \vdash_{x,u} \rho x \wedge \sigma u$

(cong) $\rho(y, x), \phi(x, u), \sigma(u, v) \vdash_{x,y,u,v} \phi(y, v)$

(singval) $\phi(x, u), \phi(x, v) \vdash_{x,u,v} \sigma(u, v)$

(tot) $\rho x \vdash_x \exists u \,.\, \phi(x, u)$,

and composition of morphisms $(A, \rho) \xrightarrow{\phi} (B, \sigma) \xrightarrow{\gamma} (C, \tau)$ is given by

$$(\gamma \circ \phi)(x, r) \;\equiv\; \exists u \,.\, \phi(x, u) \wedge \gamma(u, r).$$

The identity morphism on (A, ρ) is given by the predicate ρ itself.

We define a functor $E : \mathbb{C}\langle \mathcal{P} \rangle \to \mathbb{C}[\mathcal{P}]$ by $E(A, \rho) = (A, \rho)$ on objects, and $E(f) = \phi^f$ with

$$\phi^f(x, u) \;\equiv\; \rho(x) \wedge \sigma(fx, u)$$

on morphisms $f : (A, \rho) \to (B, \sigma)$.

Lemma 5.1 *A map $\phi : (A, \rho) \to (B, \sigma)$ in $\mathbb{C}[\mathcal{P}]$ is an isomorphism if and only if the judgments*

(inj*) $\phi(x, u), \phi(y, u) \vdash_{x,y,u} \rho(x, y)$

(esurj*) $\sigma u \vdash_u \exists x \,.\, \phi(x, u)$

hold in $\mathcal{P}$.

Proof. Observe that (inj*) and (esurj*) are dual to (singval) and (tot), respectively, by interchanging ρ and σ, and the first and second argument of ϕ. Since (strict) and (cong) are self-dual under this operation, the predicate $\phi^\circ \in \mathcal{P}(B \times A)$ given by

$$\phi^\circ(u,x) \;\equiv\; \phi(x,u)$$

(i.e. the *reciprocal relation* in the sense of [FS90]) represents a morphism $[\phi^\circ] : (B,\sigma) \to (A,\rho)$ if and only if (inj*) and (esurj*) hold. If this is the case, then $[\phi^\circ]$ is easily seen to be inverse to $[\phi]$. Conversely, if $[\phi]$ has an inverse $[\gamma]$ then one can show $\gamma \cong \phi^\circ$ which implies that (singval) and (tot) hold for ϕ° and hence (inj*) and (esurj*) hold for ϕ. ∎

Theorem 5.2 1. *A morphism $f : (A,\rho) \to (B,\sigma)$ in $\mathbb{C}\langle\mathcal{P}\rangle$ is a weak equivalence if and only if $E(f)$ is an isomorphism in $\mathbb{C}[\mathcal{P}]$.*

2. *For any category $\mathbb{D}$ and any functor $F : \mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{D}$ sending weak equivalences to isomorphisms there exists a unique $\widetilde{F} : \mathbb{C}[\mathcal{P}] \to \mathbb{D}$ satisfying $\widetilde{F} \circ E = F$.*

Proof. The first claim follows from Lemma 5.1 and the facts that (inj) holds for f if and only if (inj*) holds for ϕ_f, and that (esurj) holds for f if and only if (esurj*) holds for ϕ_f, as is easily verified.

For the second claim assume that $F : \mathbb{C}\langle\mathcal{P}\rangle \to \mathbb{D}$ inverts weak equivalences. Since E is identity on objects, we only have to define $\widetilde{F}$ on morphisms. Let $[\phi] : (A,\rho) \to (B,\sigma)$ in $\mathbf{Set}[\mathcal{P}]$. We construct the span

$$(A,\rho) \xleftarrow{\phi_l} (A \times B, (\rho \bowtie \sigma)|_\phi) \xrightarrow{\phi_r} (B,\sigma)$$

in $\mathbb{C}\langle\mathcal{P}\rangle$, where the underlying functions of ϕ_l and ϕ_r are the projections, and $(\rho \bowtie \sigma)|_\phi)$ is defined as in Remark 4.7. Then ϕ_l is a weak equivalence (even a trivial fibration), and furthermore we have $\phi \circ E(\phi_l) = E(\phi_r)$ in $\mathbf{Set}[\mathcal{P}]$.

$$\begin{tikzcd}
(A \times B, (\rho \bowtie \sigma)|_\phi) \arrow[d, "E(\phi_l)"'] \arrow[dr, "E(\phi_r)"] & \\
(A,\rho) \arrow[r, "\phi"'] & (B,\sigma)
\end{tikzcd}$$

Since F inverts weak equivalences we can deduce $\phi = E(\phi_r) \circ E(\phi_l)^{-1}$, and thus we necessarily have

$$\begin{aligned}
\widetilde{F}(\phi) &= \widetilde{F}(E(\phi_r)) \circ \widetilde{F}(E(\phi_l)^{-1}) \\
&= \widetilde{F}(E(\phi_r)) \circ \widetilde{F}(E(\phi_l))^{-1} \\
&= F(\phi_r) \circ F(\phi_l)^{-1}.
\end{aligned}$$

It remains to show that the equation

$$\widetilde{F}(\phi) = F(\phi_r) \circ F(\phi_l)^{-1}$$

gives rise to a functor $\mathbf{Set}[\mathcal{P}] \to \mathbb{D}$ satisfying $\widetilde{F} \circ E = F$.

To start we show that the construction is independent of representatives, and assume that γ is another representative of $[\phi]$, i.e. $\phi \cong \gamma \in \mathcal{P}(A \times B)$. Then the identity on $A \times B$ gives rise to an isomorphism $j : (A \times B, (\rho \bowtie \sigma)|_\phi) \to (A \times B, (\rho \bowtie \sigma)|_\gamma)$ which makes the two triangles in

$$\begin{tikzcd}
 & (A \times B, (\rho \bowtie \sigma)|_\phi) \arrow[dl, "\phi_l"'] \arrow[dr, "\phi_r"] \arrow[d, "j"] & \\
(A, \rho) & (A \times B, (\rho \bowtie \sigma)|_\gamma) \arrow[l, "\gamma_l"] \arrow[r, "\gamma_r"'] & (B, \sigma)
\end{tikzcd}$$

and we can argue

$$F(\gamma_r) \circ F(\gamma_l)^{-1} = F(\gamma_r) \circ F(j) \circ F(\phi_l)^{-1} = F(\phi_r) \circ F(\phi_l)^{-1}.$$

Applying the construction to identity morphisms, we obtain precisely the projections

$$(A, \rho) \xleftarrow{d_0} (A\times, \tilde{\rho}) \xrightarrow{d_1} (A, \rho)$$

of the path object, and we have

$$F(d_1) \circ F(d_0)^{-1} = \mathrm{id}_{F(A,\rho)}$$

since d_0 and d_1 are weak equivalences with a common section s, which is transformed to a common inverse by F.

To see that the construction preserves composition, let

$$(A, \rho) \xrightarrow{[\phi]} (B, \sigma) \xrightarrow{[\gamma]} (C, \tau)$$

be a composable pair in $\mathbf{Set}[\mathcal{P}]$, define $\xi \in \mathcal{P}(A \times B \times C)$ and $\theta \in \mathcal{P}(A \times C)$ by

$$\begin{aligned} \xi(x, u, r) &\equiv \phi(x, u) \wedge \gamma(u, r) \quad \text{and} \\ \theta(x, r) &\equiv \exists u \,.\, \xi(x, u, r), \end{aligned}$$

such that $[\theta] = [\gamma] \circ [\phi]$. Consider the following diagram.

$$\begin{tikzcd}
 & & (A\times C, (\rho\bowtie\tau)|_\theta) \arrow[dll, "\theta_l", "\sim"'] \arrow[drr, "\theta_r"] & & \\
(A, \rho) & & & & (C, \tau) \\
 & & (A\times B\times C, (\rho\bowtie\sigma\bowtie\tau)|_\xi) \arrow[uu, "\partial_1"', "\sim"] \arrow[dll, "\sim", "\partial_2"'] \arrow[drr, "\partial_0"] & & \\
(A\times B, (\rho\bowtie\sigma)|_\phi) \arrow[uu, "\phi_l", "\sim"'] \arrow[drr, "\phi_r"'] & & & & (B\times C, (\sigma\bowtie\tau)|_\gamma) \arrow[uu, "\gamma_r"'] \arrow[dll, "\sim"', "\gamma_l"] \\
 & & (B, \sigma) & &
\end{tikzcd}$$

The three squares commute since the underlying maps are simply projections, ϕ_l, γ_l, and θ_l are weak equivalences as we already remarked earlier, ∂_2 is a weak equivalence since it is the pullback of γ_l (or just do a direct verification), and

∂_1 is a weak equivalence by the 3/2 property (or a simple direct verification as well). Applying F we can argue

$$\begin{aligned}\widetilde{F}([\gamma]) \circ \widetilde{F}([\phi]) &= F(\gamma_r) \circ F(\gamma_l)^{-1} \circ F(\phi_r) \circ F(\phi_l)^{-1} \\ &= F(\gamma_r) \circ F(\partial_0) \circ F(\partial_2)^{-1} \circ F(\phi_l)^{-1} \\ &= F(\theta_r) \circ F(\partial_1) \circ F(\partial_2)^{-1} \circ F(\phi_l)^{-1} \\ &= F(\theta_r) \circ F(\theta_l)^{-1} = \widetilde{F}([\theta]) = \widetilde{F}([\gamma] \circ [\phi])\end{aligned}$$

which shows that $\widetilde{F}$ is compatible with composition and thus a functor.

To see that $\widetilde{F} \circ E = F$, let $f : (A, \rho) \to (B, \sigma)$ in $\mathbb{C}\langle \mathcal{P} \rangle$, and consider the diagram

$$\begin{tikzcd} (A \times B, (\rho \bowtie \sigma)|_{\phi^f}) \arrow[d, bend right, "\phi_l^f"'] \arrow[dr, "\phi_r^f"] & \\ (A, \rho) \arrow[u, bend right, "s"'] \arrow[r, "f"'] & (B, \sigma) \end{tikzcd}$$

where s has underlying map $\langle \mathrm{id}_A, f \rangle$. Then $\phi_r^f \circ s = f$, furthermore s is a section of the weak equivalence ϕ_l^f, which means that $F(s)$ is an inverse of $F(\phi_l^f)$ and we can argue

$$\widetilde{F}(E(f)) = F(\phi_r^f) \circ F(\phi_l^f)^{-1} = F(\phi_r^f) \circ F(s) = F(f)$$

as required. ∎

6 Cofibrant objects

Following Baues [Bau89, Section I.1], we call an object C of a category of fibrant objects $\mathbb{C}$ *cofibrant*, if every trivial fibration $f : B \to C$ admits a section. $\mathbb{C}$ is said to have *enough* cofibrant objects, if every $A \in \mathbb{C}$ admits a *cofibrant replacement*, i.e. for every object A there exists a cofibrant object C and a trivial fibration $f : C \to A$. We shall give a sufficient condition on a regular hyperdoctrine $\mathcal{P}$ for $\mathbb{C}\langle \mathcal{P} \rangle$ to have a enough cofibrant objects. For this we require the following definition.

Definition 6.1 Let $\mathcal{P} : \mathbb{C} \xrightarrow{\mathsf{op}} \mathbf{Pos}$ be a regular hyperdoctrine. A predicate $\varpi \in \mathcal{P}(I)$ is called *$\exists$-prime*, if for every composable pair $I \xleftarrow{u} J \xleftarrow{v} K$ of maps and every predicate $\psi \in \mathcal{P}(K)$ satisfying $u^*\varpi \leq \exists_v \psi$, there exists a section s of v such that $u^*\varpi \leq s^*\psi$.

We say that $\mathcal{P}$ has *enough* $\exists$-prime predicates, if for every predicate $\varphi \in \mathcal{P}(I)$ there exists a $\exists$-prime predicate $\varpi \in \mathcal{P}(J)$ and a map $e : J \to I$ such that $\varphi = \exists_e \varpi$. ♢

Proposition 6.2 *Let $\mathcal{P} : \mathbb{C} \xrightarrow{\mathsf{op}} \mathbf{Pos}$ be a regular hyperdoctrine. If $\mathcal{P}$ has enough $\exists$-prime predicates, then $\mathbb{C}\langle \mathcal{P} \rangle$ has enough cofibrant objects.*

Proof. Let $(A, \rho) \in \mathbb{C}\langle \mathcal{P} \rangle$. By assumption there exists an object $C \in \mathbb{C}$, an $\exists$-prime predicate $\varpi \in \mathcal{P}(C)$, and a morphism $e : C \to A$ such that $\exists_e \varpi = \rho_0$. We claim that a cofibrant replacement of (A, ρ) is given by (C, τ), where

$$\tau(c, c') \;\equiv\; \varpi(c) \wedge \rho(ec, ec').$$

It is easy to see that $\tau_0 = \varpi$, and using Lemma 4.3 and Remark 4.4 that e constitutes a trivial fibration from (C, τ) to (A, ρ).

To see that (C, τ) is cofibrant, let $f : (B, \sigma) \to (C, \tau)$ be a trivial fibration. Again using Lemma 4.3 and Remark 4.4 we deduce $\varpi = \tau_0 \leq \exists_f \sigma_0$, and since ϖ is $\exists$-prime this implies that f has a section $s : C \to B$ such that $\varpi \leq s^* \sigma_0$, i.e. the judgment $\tau(c) \vdash_c \sigma(sc)$ holds. The judgment (compat) for s then follows from this and (inj) for f. Thus, s constitutes a morphism of type $(C, \tau) \to (B, \sigma)$ in $\mathbb{C}\langle\mathcal{P}\rangle$, which gives the required section. ∎

7 Derived functors

Recall from [Dwy+04, Section I.2.3] that a *category with weak equivalences* (or *we-category*) is a category $\mathcal{C}$ equipped with a class $\mathcal{W}$ of arrows – called *weak equivalences* – which satisfies the 3-for-2 property.

Definition 7.1 Let $\mathcal{C}$ be a we-category with localization functor $E : \mathcal{C} \to \mathrm{ho}(\mathcal{C})$.

1. We call $X \in \mathcal{C}$ *quasi-fibrant*, if
$$E_{A,X} \ : \ \mathcal{C}(A, X) \to \mathrm{ho}(\mathcal{C})(EA, EX)$$
is surjective for all $A \in \mathcal{C}$.

2. A *path object* for $A \in \mathcal{C}$ consists of an object P and weak equivalences $p, q : P \to A$ admitting a common section.

3. A *right homotopy* between parallel arrows $f, g : A \to B$ consists of a path object $p, q : P \to A$ and an arrow $h : A \to P$ such that $ph = f$ and $qh = g$.

4. We call $\mathcal{C}$ *right-derivable*, if the following conditions hold.
   (a) Every object $A \in \mathcal{C}$ admits a weak equivalence $\iota_A : A \to \overline{A}$ into a strongly fibrant object.
   (b) If X is strongly fibrant and $f, g : A \to X$ are such that $Ef = Eg$, then f and g are right homotopic by means of a strongly fibrant path object. ◊

Remark 7.2 By dualizing, we obtain the notions of *strongly cofibrant object*, *cylinder object*, *left homotopy*, and *left-derivable we-category*. ◊

The term ‘right-derivable’ is justified by the following result.

Theorem 7.3 *Assume that $\mathcal{C}$ is a right-derivable we-category with localization functor $E : \mathcal{C} \to \mathrm{ho}(\mathcal{C})$, $\mathbb{D}$ is a category, and $F : \mathcal{C} \to \mathbb{D}$ is a functor inverting weak equivalences between strongly fibrant objects. Then F admits a left Kan extension along E.*

Proof. Define $\widetilde{F} : \mathrm{ho}(\mathcal{C}) \to \mathcal{X}$ as follows. On objects, we set $\widetilde{F}(EA) = F(\overline{A})$ (this covers all objects since E is bijective on objects). Given $f : EA \to EB$, by strong fibrantness of $\overline{B}$, and since $E(\iota_A)$ is iso, there exists a $f^\dagger : \overline{A} \to \overline{B}$ such that

$$\begin{array}{ccc} EA & \xrightarrow{f} & EB \\ \downarrow{\scriptstyle E\iota_A} & & \downarrow{\scriptstyle E\iota_B} \\ E\overline{A} & \xrightarrow{Ef^\dagger} & E\overline{B} \end{array}$$

commutes. We set $\widetilde{F}(f) = F(f^\dagger)$. This is independent of the choice of $f^\dagger$, since F identifies parallel maps which are related by a strongly fibrant right homotopy, and with this it follows easily that $\widetilde{F}$ is well defined and functorial. We define a natural transformation $\eta : F \to \widetilde{F} \circ E$ by $\eta_A = F\iota_A$ and claim that this exhibits $\widetilde{F}$ as a right Kan extension of F along E.

To verify this, we have to show that for each $G : \mathrm{ho}(\mathcal{C}) \to \mathcal{X}$ and $\theta : F \to G \circ E$ there is a unique $\xi : \widetilde{F} \to G$ such that

$$(\xi \cdot E) \circ \eta = \theta. \tag{$*$}$$

For uniqueness, assume that we had such a ξ for given G and θ. For $A \in \mathcal{C}$ we can instantiate ($*$) at $\overline{A}$ and argue:

$$\begin{aligned} \theta_{\overline{A}} &= \xi_{E\overline{A}} \circ \eta_{\overline{A}} && \text{by } (*) \\ &= \xi_{E\overline{A}} \circ F\iota_{\overline{A}} && \text{by definition of } \eta \\ &= \xi_{E\overline{A}} \circ \widetilde{F}E\iota_A && \text{by definition of } \widetilde{F} \text{ and since } \iota_{\overline{A}} \text{ is a choice of } (E\iota_A)^\dagger \\ &= GE\iota_A \circ \xi_{EA} && \text{by naturality ot } \xi \end{aligned}$$

Since $GE\iota_A$ is invertible, ξ is uniquely determined by the equation

$$\xi_{EA} = (GE\iota_A)^{-1} \circ \theta_{\overline{A}}.$$

It is easy to see that this equation does indeed define a natural transformation satisfying ($*$). ∎

Theorem 7.4 *Let $\mathcal{P} : \mathbb{C}^{\mathsf{op}} \to$ **Ord** be a regular hyperdoctrine.*

1. *If $\mathcal{P}$ is a tripos, then $\mathbb{C}\langle\mathcal{P}\rangle$ is right derivable.*
2. *If $\mathcal{P}$ has enough $\exists$-prime predicates, then $\mathbb{C}\langle\mathcal{P}\rangle$ is left-derivable.*

Proof. Assume first that $\mathcal{P}$ is a tripos. In this case, the property of being strongly fibrant is precisely what is called *weakly complete* in [HJP80, Definition 3.2] – every object is equivalent to a weakly complete one, and ∎

Theorem 7.5 *Let $\mathcal{P}, \mathcal{Q} : \mathbb{C}^{\mathsf{op}} \to$ **Ord** be regular hyperdoctrines, and let $\Phi : \mathcal{P} \to \mathcal{Q}$ be an indexed monotone map whose components preserve finite meets.*

1. *If $\mathcal{P}$ has enough $\exists$-prime predicates, then $\mathbb{C}\langle\Phi\rangle$ has a left derived functor.*
2. *If $\mathcal{P}$ is a tripos, then $\mathbb{C}\langle\Phi\rangle$ has a right derived functor*

Let $F : \mathbb{C} \to \mathbb{D}$ be a functor between categories of fibrant objects. In this section, we want to study conditions under which F has a right derived functor, i.e. there exists a functor $\tilde{F}$ and a natural transformation η

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\eta \Downarrow"] & \mathbb{D} \arrow[d, "\gamma"] \\
\mathrm{ho}\mathbb{C} \arrow[r, "\tilde{F}"'] & \mathrm{ho}\mathbb{C}
\end{tikzcd}
$$

exhibiting $\tilde{F}$ as a left Kan extension of EF along E.

A Proofs

Proof of Lemma 4.3. Implication (inj), (surj) $\Rightarrow$ (fib):

1. (sym), (trans) $\Rightarrow$ $\sigma(fx, u) \vdash_{x,u} \sigma u$
2. 1, (surj) $\Rightarrow$ $\sigma(fx, u) \vdash_{x,u} \exists y \,.\, \rho y \wedge fy = u$
3. (inj) $\Rightarrow$ $\rho x, \sigma(fx, u), \rho y, fy = u \vdash_{x,y,u} \rho(x, y)$
4. 3 $\Rightarrow$ $\rho x, \sigma(fx, u), \rho y, fy = u \vdash_{x,y,u} \rho(x, y) \wedge fy = u$
5. 4 $\Rightarrow$ $\rho x, \sigma(fx, u), \rho y, fy = u \vdash_{x,y,u} \exists y \,.\, \rho(x, y) \wedge fy = u$
6. 2, 5 $\Rightarrow$ $\rho x, \sigma(fx, u) \vdash_{x,u} \exists y \,.\, \rho(x, y) \wedge fy = u$

Implication (surj) $\Rightarrow$ (esurj):

1. (compat) $\Rightarrow$ $\rho x \vdash_x \sigma(fx)$
2. 1 $\Rightarrow$ $\rho x, fx = u \vdash_{x,u} \sigma(fx, u)$
3. 2 $\Rightarrow$ $\rho x, fx = u \vdash_u \rho x \wedge \sigma(fx, u)$
4. 3 $\Rightarrow$ $\exists x \,.\, \rho x \wedge fx = u \vdash_u \exists x \,.\, \rho x \wedge \sigma(fx, u)$
5. 4, (surj) $\Rightarrow$ $\sigma u \vdash_u \exists x \,.\, \rho x \wedge \sigma(fx, u)$

Implication (esurj), (fib) $\Rightarrow$ (surj):

1. (sym), (trans) $\Rightarrow$ $\rho(x, y) \vdash_{x,y,u} \rho y$
2. 1 $\Rightarrow$ $\rho(x, y), fy = u \vdash_{x,y,u} \rho y \wedge fy = u$
3. 2 $\Rightarrow$ $\exists y \,.\, \rho(x, y) \wedge fy = u \vdash_{x,u} \exists y \,.\, \rho y \wedge fy = u$
4. 3, (fib) $\Rightarrow$ $\rho x, \sigma(fx, u) \vdash_{x,u} \exists y \,.\, \rho y \wedge fy = u$
5. 4, (esurj) $\Rightarrow$ $\sigma u \vdash_u \exists y \,.\, \rho y \wedge fy = u$ ∎

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