Frey, Zwanziger — (1,2)-cosmoses and tagged linear logic

Recall: An **(elementary) topos** is a category $\mathcal{E}$ with finite limits such that all presheaves $\mathbf{Rel}_{\mathcal{E}}(-, A) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ are representable.

> *Definition*
>
> A $(1, 2)$**-cosmos** is a **Pos**-enriched category $\mathcal{E}$ with finite limits, such that all
>
> $$\mathbf{Prof}_{\mathcal{E}}(-, A) : \mathcal{E}^{\mathrm{coop}} \to \mathbf{Pos} \quad \text{and} \quad \mathbf{Prof}_{\mathcal{E}}(A, -) : \mathcal{E}^{\mathrm{op}} \to \mathbf{Pos}$$
>
> are representable.

- representing objects are *lower* and *upper power objects* $P_{\downarrow}A$, $P_{\uparrow}A$
- monadicity fails – $(P_{\downarrow} \circ P_{\uparrow})$-algebras on **Pos** are *completely dist. lattices*

virtual double category $\mathbf{Prof}(\mathcal{E})$ — *tagged linear logic with substitutions*

$$\begin{tikzcd}
B_0 \arrow[rrr, "\psi"', "|" marking] & & & B_1 \\
A_0 \arrow[u, "f"] \arrow[r, "\phi_1", "|" marking] & A_1 \;\cdots\; A_{n-1} \arrow[rr, "\phi_n", "|" marking] & & A_n \arrow[u, "g"']
\end{tikzcd} \quad — \quad \phi_1, \ldots, \phi_n \vdash \psi[f, g]$$

Thm: $\mathbf{Prof}(\mathcal{E})$ is *closed* and has *compositions*; $\mathbf{Prof}_{\mathcal{E}}(A, 1)$ is Heyting alg.