Resonances in Dynamical Systems

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1
Introduction to Dynamical Systems

Roughly speaking, a dynamical system is a system that evolves (changes) in time according to some rules. This includes the position of the planets, moons, and stars, flows of fluids, and the movement of a wave or particle.

The rules that govern these systems might be complex and unknown. We use physical priniciples to derive a set of rules.

Definition 1.1 (Dynamical System)**.** A dynamical system is a 3-tuple of

- a state space X,
- a set of times T,
- a function $\Phi\colon T \times X \to X$.

Usually, one also adds a semigroup structure both to T and Φ.

We will use $X = \mathbb{R}^n$ or $X = M$ a euclidean manifold (or X a function space). For continuous systems, we have $T = \mathbb{R}$ or $T = \mathbb{R}_+ = [0, \infty)$. For discrete systems, we have $T = \mathbb{Z}$ or $T = \mathbb{N}_0$.

$\Phi(t, x_0)$ gives the state of the system at time $t \in T$ if the system was initially (that is say at $t = 0$) in the state $x_0 \in X$.

This definition is incredibly abstract and much too general for the purpose of this course. In practice, we will think of a continuous dynamical system to be the flow of an autonomous ordinary differential equation

$$\dot{x} = f(x). \tag{1.1}$$

Assuming (1.1) has global-in-time unique solutions (for example if f is Lipschitz), we may define the flow via

$$\Phi(t, x_0) = x(t), \quad \text{where } x \text{ is the unique solution to } \begin{cases} \dot{x} = f(x), \\ x(0) = x_0. \end{cases}$$

Note that Φ is a semigroup: it holds for all $x_0 \in X$ and $t, s \in \mathbb{R}$

(1) $\Phi(0, x_0) = x_0$,

(2) $\Phi(t, \Phi(s, x_0)) = \Phi(s, \Phi(t, x_0)) = \Phi(t + s, x_0)$.

Observe in particular that Φ is invertible. In fact, if f is smooth, Φ is a continuous family of diffeomorphisms of the state space X.

A discrete dynamical system might come from sampling a continuous time system. Thereforem, choosing a time-step size t_0, we may define the map

$$F(x_0) = \Phi(t_0, x_0).$$

Then we obtain a discrete dynamical system

$$\widetilde{\Phi}\colon \mathbb{Z} \times X \to X : \tilde{\Phi}(n, x_0) = \Phi(nt_0, x_0) = F^n(x_0).$$

Here and in the future, we denote by F^n the n-th iterate of F[1]. We will only consider discrete dynamical systems to be of this form, i.e. to come from a homeomorphism[2] $F\colon X \to X$.

[1] That is

$$F^n = \underbrace{F \circ \cdots \circ F}_{n \text{ times}}.$$

[2] Or very soon from a diffeomorphism in a specific regularity class.

Example 1.2 (Continuous systems)**.**

(a) Simple harmonic oscillator: consider a mass m hanging of a spring. If x denotes the displacement of a spring from its equilibrium position and using Hooke's law[3], then the dynamical behaviour of the spring is given by

$$m\ddot{x} = -kx.$$

This equation can be transformed into a first-order autonomous ODE of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ mv \end{pmatrix} = \begin{pmatrix} v \\ -kx \end{pmatrix}$$

[3] Hooke's law: the restoring force of a spring is proportional to the displacement

$$F_{\text{Hooke}} = -kx,$$

k the spring constant.

(b) Two-body problem: the motion of two bodies through physical space $\mathbb{R}^3$ with masses m_1, m_2 considering gravitational pull is given by

$$\begin{aligned} m_1\ddot{x}_1 &= F_{12}(x_1 - x_2) = \tfrac{gm_1m_2(x_2-x_1)}{|x_1-x_2|^3} \\ m_2\ddot{x}_2 &= F_{21}(x_1 - x_2) = -F_{12}(x_1 - x_2). \end{aligned}$$

Here $g > 0$ denotes the gravitational constant.

(c) Hamiltonian systems: there is an underlying structure for (a) and (b). If $H = H(q,p)\colon X \times TX \to \mathbb{R}$ is a function, we define the correspoding *Hamiltonian system* via

$$\begin{aligned} \dot{q} &= \tfrac{\partial H}{\partial q} \\ \dot{p} &= -\tfrac{\partial H}{\partial p}. \end{aligned}$$

Here, TX is the abstract tangent space of X. So if $X = \mathbb{R}^n$, then we will identify $TX \cong X$.

We interpret q to be the position and p as the momentum of our moving body.

For (a), we have $X = TX = \mathbb{R}$ and the Hamiltonian is given by

$$H(q,p) = \frac{k^2}{2} - \frac{kq^2}{2m}.$$

For (b), we have $X = TX = \mathbb{R}^3 \times \mathbb{R}^3$ and we have

$$\begin{aligned} H(q_1, q_2, p_1, p_2) &= K - U, \\ K(p_1, p_2) &= \frac{|p_1|^2}{2m_1} + \frac{|p_2|^2}{2m_2}, \\ U(q_1, q_2) &= \frac{gm_1m_2}{|q_1 - q_2|}. \end{aligned}$$

K denotes the *kinetic energy* of the system and U denotes the *potential energy* of the system.

(d) Infinite dimensional dynamical systems: such as the wave equation

$$\partial_t^2 u - \Delta u = 0.$$

Example 1.3 (Discrete dynamical systems). (a) Any diffeomorphism of a euclidean manifold M, $f\colon M \to M$ gives rise to a dynamical system

$$\Phi\colon \mathbb{Z} \times M \to M : \Phi(n, x_0) = f^n(x_0).$$

(b) Rotations of the circle: denote by $S^1 = \{e^{2\pi i\theta} : \theta \in [0,1]\}$ the circle and let $M = S^1$. For $\alpha \in [0,1)$, consider the diffeomorphism $R_\alpha\colon S^1 \to S^1$ given by rotation with angle $2\pi\alpha$, that is

$$R_\alpha(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha)}.$$

Observe that then

$$R_\alpha(e^{2\pi i\theta})^n = e^{2\pi i(\theta+n\alpha)} = R_{n\alpha}(e^{2\pi i\theta}).$$

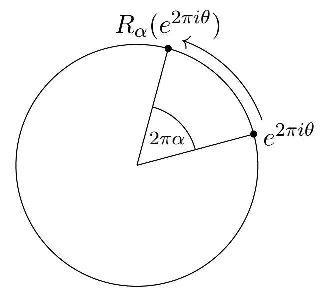

Fig. 1.1: Rotation with angle $2\pi\alpha$

(c) The Chirikov standard map: denote by $\mathbb{T}^n$ the n-dimensional torus[4] and consider the diffeomorphism

$$f\colon \mathbb{T}^2 \to \mathbb{T}^2 : f(e^{2\pi i\theta}, e^{2\pi ip}) = (e^{2\pi i(\theta+p+K\sin\theta)}, e^{2\pi i(p+K\sin\theta)}).$$

[4] $\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$.

We will study the rotations on the circle in more detail in Section 2.

A general dynamical system can have very different kinds of behaviour. There might be order or chaos. We collect different notions of order.

Definition 1.4. Given a dynamical system (X, T, Φ), we define

(i) We define the *orbit* starting at $x_0 \in X$ by

$$\gamma_{x_0} = \{\Phi(t, x_0) : t \in T\}.$$

(ii) We call $x_0 \in X$ a *stationary point*[5] if

$$\gamma_{x_0} = \{x_0\}.$$

[5] Observe that for an autonomous ODE $\dot{x} = f(x)$, a point is a stationary point if and only if $f(x_0) = 0$. For a discrete system given by a homeomorphism $F\colon X \to X$, a point is stationary if and only if it is a fixed point $F(x_0) = x_0$.

(iii) An orbit γ_{x_0} is called *periodic* if there is $t \neq 0$ such that

$$\Phi(t, x_0) = x_0.$$

(iv) An orbit γ_{x_0} is called *quasi-periodic* if $t \mapsto \Phi(t, x_0)$ is a quasi-periodic function[6], that is if

$$\Phi(t, x_0) = f(\omega_1 t, \ldots, \omega_n t),$$

where $f\colon T^n \to X$, for some $n \in \mathbb{N}$, is periodic in each component and $\omega_1, \ldots, \omega_n \in \mathbb{R}$ are rationally independent.

[6] For example the function

$$t \mapsto \sin(at) + \sin(bt)$$

with a and b rationally independent is quasi-periodic.

2

KAM THEORY FOR DIFFEOMORPHISMS OF THE CIRCLE

In this chapter, we will discuss the KAM theory for diffeomorphisms of the circle. We begin by analysing the dynamics of the rotation map. Before we study small analytic perturbations of the rotations, we need to define the rotation number and we will prove Denjoy's theorem.

Before we start, we need to fix some notation: consider the circle $S^1 = \{e^{2\pi i\theta} \in \mathbb{C} : \theta \in [0,1)\} \cong \mathbb{R}/\mathbb{Z}$[1]. Then $\mathbb{R}$ forms a *covering space* of S^1 with cover given by

$$\pi\colon \mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}.$$

In particular, $\pi(x+z) = \pi(x)$ for all $z \in \mathbb{Z}$.

[1] We will relatively freely identify both constructions.

Definition 2.1. Let $f\colon S^1 \to S^1$ a continuous map of the circle. Then $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is called a *lift* of f to R if

$$\pi \circ \tilde{f} = f \circ \pi.$$

Lemma 2.2. *Every continuous map $f\colon S^1 \to S^1$ has a lift $\tilde{f}$ to $\mathbb{R}$.*

Proof. Exercise. □

Remark 2.3.
1) If $f\colon S^1 \to S^1$ is a homeomorphism of the circle, then $\tilde{f}$ is strictly monotone. If f is orientation-preserving, then $\tilde{f}$ is increasing. If f is orientation-reversing, then $\tilde{f}$ decreases.
2) If $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is a lift of $f\colon S^1 \to S^1$, then so is $\tilde{f} + m$ for any $m \in \mathbb{Z}$. Vice versa, all lifts differ only by a translation by $m \in \mathbb{Z}$.
3) For every lift it holds
$$\tilde{f}(x+1) = \tilde{f}(x) + d.$$
If $f\colon S^1 \to S^1$ is a homeomorphism, then $d \in \{\pm 1\}$.[2] Here, $d = 1$ if and only if f is orientation-preserving. d is called the *degree* of f and $x \mapsto F(x) - dx$ is periodic with period 1.
4) The map $\tilde{f}(x) - x$ is periodic with period 1.

[2] The opposite direction is false in general.

Exercise. Let $f\colon S^1 \to S^1$ be an orientation-reversing homeomorphism. Show that f has exactly two fixed points.

Example 2.4. Let $f\colon S^1 \to S^1 : f(e^{2\pi i\theta}) = e^{2\pi i(\theta+\alpha)}$ the rotation by angle $\alpha \in [0,1)$. Then $\tilde{f}(x) = x + \alpha$ is a lift.

1 *The dynamics of the rotation map*

Definition 2.5. For $\alpha \in [0,1)$ we define the map

$$R_\alpha \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} : R_\alpha(x) = (x+\alpha) \bmod \mathbb{Z}$$

as the rotation of the circle $S^1 = \mathbb{R}/\mathbb{Z}$.

Observe that $R_\alpha^n(x_0) = x_0 + n\alpha \bmod \mathbb{Z}$ for any $n \in \mathbb{Z}$. We will now study the dynamics of the rotation map.

Proposition 2.6. *Let* $\alpha \in [0,1)$.

(1) *If* $\alpha \in \mathbb{Q}$ *is rational, then every orbit of* R_α *is periodic. If* $\alpha = \frac{p}{q}$ *with* p, q *coprime, then*q *is the period of each orbit.*

(2) *If* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *is irrational, then every orbit is dense*[3] *in* $\mathbb{R}/\mathbb{Z}$.

[3] Sometimes, dynamical systems with this property are called minimal.

In order to prove the density of orbits, we rely on a result from number theory.

Lemma 2.7 (Dirichlet's approximation theorem)**.** *Let* $\alpha \in \mathbb{R}$ *and* $N \in \mathbb{N}$. *Then there is a pair of integers* $(p,q) \in \mathbb{Z} \times \mathbb{N}$ *with* $1 \le q \le N$ *such that*

$$|\alpha q - p| < \frac{1}{N}.$$

Proof. [4] This can be proved via the pigeonhole principle. Consider the $N+1$ numbers $k\alpha$, $k = 0, \dots, N$. They can be uniquely written as

$$k\alpha = m_k + x_k, \quad m_k \in \mathbb{Z},\ 0 \le x_k < 1.$$

Now the set $\{x_0, \dots, x_N\} \subset [0,1)$ consists of $N+1$ numbers, hence by the pigeonhole principle there must be two numbers x_i and x_j with $i < j$ such that

$$|x_j - x_i| < \frac{1}{N}.$$

But now observe that

$$|(j-i)\alpha - (m_j - m_i)| = |j\alpha - m_j - (i\alpha - m_i)| = |x_j - x_i| < \frac{1}{N}.$$

This proves the theorem with the choice $q = (j-i) \in \mathbb{N}$ and $p = m_j - m_i \in \mathbb{Z}$. □

Using that $1 \le q \le N$, the following corollary is an immediate consequence.

Corollary 2.8. *For any real number* $\alpha \in \mathbb{R}$ *there exist infinitely many pairs of integers* $(p,q) \in \mathbb{Z} \times \mathbb{N}$ *such that*

$$\Big|\alpha - \frac{p}{q}\Big| < \frac{1}{q^2}.$$

Now we are able to prove Proposition 2.6.

[4] There is a deep connection to continued fractions: for a number $\alpha \in \mathbb{R}$, we define

$$\begin{aligned} \alpha &= \lfloor \alpha \rfloor + \tfrac{1}{\alpha_1} = a_0 + \tfrac{1}{\alpha_1} \\ \alpha_1 &= \lfloor \alpha_1 \rfloor + \tfrac{1}{\alpha_2} = a_1 + \tfrac{1}{\alpha_2} \\ &\vdots \end{aligned}$$

then

$$\alpha = a_0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\vdots}{a_{n-1} + \cfrac{1}{\alpha_n}}}}}$$

and we call

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\vdots}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

the n*-th convergent* of α. Then the following results hold true:

$$\Big|\alpha - \frac{p_n}{q_n}\Big| < \frac{1}{q_n^2}.$$

And vice versa, if

$$\Big|\alpha - \frac{p}{q}\Big| < \frac{1}{q^2},$$

then

$$\frac{p}{q} \in \left\{ \frac{p_n}{q_n}, \frac{p_{n+1} + p_n}{q_{n+1} + q_n}, \frac{p_{n+2} - p_{n+1}}{q_{n+2} - q_{n+1}} \right\}$$

Proof. **Step 1: rational angles** If $\alpha = \frac{p}{q}$ and $x_0 \in \mathbb{R}/\mathbb{Z}$ is arbitrary, then

$$R_\alpha^q(x_0) = x_0 + q \cdot \frac{p}{q} \mod \mathbb{Z} = x_0.$$

Hence, γ_{x_0} is a periodic orbit. If p and q are coprime, then q is the smallest number such that $q \cdot \frac{p}{q} \in \mathbb{Z}$.
Step 2: irrational angles Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $x_0 \in [0, 1)$. We first prove:
Claim 1: if $m \neq n$, then $R_\alpha^m(x_0) \neq R_\alpha^n(x_0)$. Indeed, assume that $R_\alpha^m(x_0) = R_\alpha^n(x_0)$, then

$$x + n\alpha \mod \mathbb{Z} = x + m\alpha \mod \mathbb{Z},$$

which implies that

$$(m - n)\alpha \in \mathbb{Z}.$$

But since $\alpha \notin \mathbb{Q}$, this can only be satisfied if $m = n$.
Claim 2: the orbit is dense[5]. Therefore, let $\varepsilon > 0$. By Corollary 2.8 there is a pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with $\frac{1}{q} < \varepsilon$ such that

$$|q\alpha - p| < \frac{1}{q} < \varepsilon.$$

But this readily implies that

$$|(R_\alpha^q(x_0) - x_0) \mod \mathbb{Z}| = |q\alpha - p| < \varepsilon.$$

Consider, for M large enough the set of points

$$\{x_0, R_\alpha^q(x_0), R_\alpha^{2q}(x_0), \ldots, R_\alpha^{Mq}(x_0)\}.$$

Then this set breaks S^1 into M intervals of length smaller than ε. Hence, for every $x \in [0, 1)$ there must exist $k \in \{0, \ldots, M\}$ such that

$$|x - R_\alpha^{kq}(x_0)| < \varepsilon,$$

which proves the density of the orbit. □

[5] One could also use Weyl's equidistribution criterion. This states that a sequence $(x_n)_{n\in\mathbb{N}_0}$ is equidistributed in [0, 1] if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i \ell x_n} = 0$$

holds for every $\ell \neq 0$.
Applying this to the sequence $x_n = R_\alpha^n(x_0)$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ gives equidistribution. Actually, the dynamical system is ergodic: every measurable, invariant set is either a Lebesgue null set or of full Lebesgue measure.

2 *Rotation number and Denjoy's theorem*

In the remaining part of this chapter we will be concerned with the question: when does an arbitrary (orientation-preserving) homeomorphism (or diffeomorphism) behave like a rotation? To understand which rotation to choose, we introduce the rotation number.

Definition 2.9. Let $f\colon S^1 \to S^1$ an orientation-preserving homeomorphism and $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ a lift of f and let $x_0 \in [0, 1)$. Then the *rotation number* of f is given by

$$\rho(f) = \Big[\lim_{|n|\to\infty} \frac{\tilde{f}^n(x_0) - x_0}{n}\Big] \mod \mathbb{Z}. \tag{2.1}$$

Theorem 2.10. *Let $f\colon S^1 \to S^1$ an orientation-preserving homeomorphism. Then the rotation number $\rho(f)$ exists and is independent of the choice of $x_0 \in [0, 1)$.*

Proof. **Step 1:** The rotation number is independent of the choice of x_0. Fix $x_0, y_0 \in [0,1)$. Let w.l.o.g. $x_0 < y_0$. Then it holds by Remark 2.3 that

$$\tilde{f}^n(y_0) - \tilde{f}^n(x_0) < \tilde{f}^n(y_0+1) - \tilde{f}^n(y_0) = 1.$$

Now we can estimate

$$\begin{aligned} \left| \frac{\tilde{f}^n(x_0) - x_0 - (\tilde{f}^n(y_0) - y_0)}{n} \right| &\leq \frac{|\tilde{f}^n(x_0) - \tilde{f}^n(y_0)| + |x_0 - y_0|}{n} \\ &\leq \frac{2}{n} \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

Hence, the limit is independent of the choice of x_0.
Step 2: Existence of the limit. We will distinguish the cases: f has a periodic point and f has no periodic point.

If f has a periodic point with period $m \in \mathbb{N}$, this implies that there must be an element $z \in \mathbb{Z}$ and $x_0 \in [0,1]$ such that $\tilde{f}^m(x_0) = x_0 + z$. Hence, it must hold

$$\tilde{f}^{km}(x_0) = x_0 + kz, \quad k \in \mathbb{N}.$$

But from this, we conclude that

$$\lim_{k\to\infty} \frac{|\tilde{f}^{km}(x_0) - x_0|}{km} = \frac{kz}{km} = \frac{z}{m} \in \mathbb{Q}.$$

We need to show that the full sequence is converging. Therefore, write $n = km + r$, with $0 \leq r < m$. Since $\tilde{f} - \mathrm{Id}$ is periodic and continuous, we find a number $M > 0$ such that

$$|\tilde{f}(x) - x| \leq M$$

for all $x \in \mathbb{R}$ and we may conclude

$$\frac{|\tilde{f}^n(x_0) - x_0 - (\tilde{f}^{km}(x_0) - x_0)}{n} = \frac{|\tilde{f}^r(\tilde{f}^{km}(x_0)) - \tilde{f}^{km}(x_0)|}{n} \leq \frac{M}{n} \xrightarrow[n\to\infty]{} 0.$$

If f has no periodic points, then we know that $\tilde{f}^n(x) - x \notin \mathbb{Z}$ for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. Since $\tilde{f}^n - \mathrm{Id}$ is periodic and continuous, this means that there is $z \in \mathbb{Z}$ such that

$$z < \tilde{f}^n(x) - x < z + 1$$

for all $x \in \mathbb{R}$. Choosing $x = 0$, we find that

$$z < \tilde{f}^n(0) < z + 1$$

and choosing $x = \tilde{f}^n(0)$ und using monotonicity, we find

$$\tilde{f}^n(0) + z = \tilde{f}^n(z) \leq \tilde{f}^{2n}(0) \leq \tilde{f}^n(z+1) = \tilde{f}^n(0) + z + 1.$$

Hence,

$$z < \tilde{f}^{2n}(0) - \tilde{f}^n(0) < z + 1$$

and by induction

$$z < \tilde{f}^{kn}(0) - \tilde{f}^{(k-1)n}(0) < z + 1$$

holds for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. Summing over k, we find

$$kz < \tilde{f}^{kn}(0) < k(z+1)$$

for all $k, n \in \mathbb{N}$, or dividing by kn, we have

$$\frac{z}{n} < \frac{\tilde{f}^{kn}(0) - 0}{kn} < \frac{z+1}{n}$$

for every $k, n \in \mathbb{N}$. In particular, it holds

$$\left| \frac{\tilde{f}^{kn}(0) - 0}{kn} - \frac{\tilde{f}^{n}(0) - 0}{n} \right| < \frac{1}{n}. \tag{2.2}$$

But now, we obtain by (2.2) that

$$\begin{aligned} \left| \frac{\tilde{f}^{n}(0) - 0}{n} - \frac{\tilde{f}^{m}(0) - 0}{m} \right| &\leq \left| \frac{\tilde{f}^{n}(0)}{n} - \frac{\tilde{f}^{mn}(0) - 0}{nm} \right| + \left| \frac{\tilde{f}^{mn}(0) - 0}{mn} - \frac{\tilde{f}^{m}(0)}{m} \right| \\ &\leq \frac{1}{n} + \frac{1}{m}. \end{aligned}$$

This proves that the sequence $\left(\frac{\tilde{f}^{n}(0)}{n}\right)_n$ is Cauchy and so it has a limit. ☐

Example 2.11. For $\alpha \in [0,1)$ it is $\rho(R_\alpha) = \alpha$.

Lemma 2.12. *Let $f\colon S^1 \to S^1$ an orientation-preserving homeomorphism.*

1) If $h\colon S^1 \to S^1$ is an orientation-preserving homeomorphism, then

$$\rho(f) = \rho(h^{-1} \circ f \circ h).$$

2) If $m \in \mathbb{N}$, then

$$\rho(f^m) = m\rho(f) \mod \mathbb{Z}.$$

3) $\rho(f) \in \mathbb{Q}$ is rational if and only if f has a periodic orbit.

Proof. **2)** Note that

$$\frac{(\tilde{f}^m)^n(x_0) - x_0}{n} = \frac{\tilde{f}^{mn}(x_0) - x_0}{n} = m\frac{\tilde{f}^{mn}(x_0) - x_0}{mn}.$$

3) We have already seen that if we have a periodic orbit, then the rotation number is rational. Now assume that the rotation number is rational with $\rho(f) = \frac{p}{q} \in \mathbb{Q}$. We will show that we can construct an orbit of period q.

By 2), we have

$$\rho(f^q) = q\rho(f) \mod \mathbb{Z} = 0.$$

It hence suffices to show that if for a general homeomorphism we have $\rho(f) = 0$, then f has a stationary point. We prove the contraposition: assume f does not have a stationary point, that is $f(x_0) \neq x_0$ for every $x_0 \in S^1$. Consider the lift with $\tilde{f}(0) \in [0,1)$. Then also $\tilde{f}(x) - x \notin \mathbb{Z}$ for every $x \in [0,1)$, and by periodicity we conclude that $\tilde{f}(x_0) - x_0 \notin \mathbb{Z}$

for every $x \in \mathbb{R}$. By continuity, the assumption that $0 < \tilde{f}(0) < 1$ and compactness of the interval $[0, 1]$ this implies that there is a positive $\delta > 0$ such that

$$\delta \leq \tilde{f}(x_0) - x_0 \leq 1 - \delta$$

for every $x_0 \in \mathbb{R}$. Choosing $x_0 = \tilde{f}^n(0)$, $n \in \mathbb{N}$, we obtain

$$\delta \leq \tilde{f}^{n+1}(x_0) - \tilde{f}^n(x_0) \leq 1 - \delta.$$

for $n \in \mathbb{N}_0$. Summing over the first n, we obtain

$$n\delta \leq \tilde{f}^n(0) - 0 \leq n(1 - \delta)$$

for every $n \in \mathbb{N}$, and hence $\rho(f) \in [\delta, 1 - \delta]$. In particular, $\rho(f) \neq 0$. This concludes the proof. □

Later, we will consider diffeomorphisms of the form $f(x) = x + \rho + \eta(x)$, where η is periodic.

Lemma 2.13. *Let η be a continuous, periodic function. If the diffeomorphism $f(x) = x + \rho + \eta(x)$ satisfies $\rho(f) = \rho$, then there exists $x_0 \in S^1$ such that $\eta(x_0) = 0$.*

Proof. Observe by induction that

$$\tilde{f}^n(x_0) = x_0 + n\rho + \sum_{k=0}^{n-1} \eta \circ \tilde{f}^k(x_0).$$

Hence, if $\rho(f) = \rho$, we get

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \eta \circ \tilde{f}^k(x_0) = 0.$$

But this must mean that η has a root since otherwise by continuity it would be strictly bounded away from zero. □

Exercise. Consider the diffeomorphism

$$f(x) = x + \frac{1}{2} + \frac{1}{4\pi} \sin(2\pi x).$$

(a) Compute $\rho(f)$.

(b) Find two periodic orbits.

(c) Find a non-periodic orbit.

2.1 Irrational rotation numbers and Denjoy's theorem

To understand the dynamic even better, we consider the set of all limit points of an orbit, the so-called ω-limit set.

Definition 2.14. Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism and $x_0 \in S^1$. We define the ω*-limit set* of the orbit γ_{x_0} by

$$\omega(x_0) = \{y \in S^1 : \exists n_1 < n_2 < \dots \text{ s.t. } f^{n_k}(x_0) \xrightarrow[k\to\infty]{} y\}.$$

We can rephrase the result from Proposition 2.6.

Example 2.15. If $\alpha \in [0,1) \setminus \mathbb{Q}$ and $f = R_\alpha$ is the rotation, then

$$\omega(x_0) = S^1 \quad \text{for all } x_0 \in S^1.$$

That all orbits have the same limit set is no coincidence as the following theorem demonstrates.

Theorem 2.16. *Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism and $\rho(f) \in [0,1) \setminus \mathbb{Q}$. Then $\omega(x_0) = \omega(y_0)$ for every $x_0, y_0 \in S^1$.*

Proof. Let $x_0 \in S^1$ and $x \in \omega(x_0)$. Let $\varepsilon > 0$ fixed. Then there must be a pair of integers $m > n$ such that

$$|f^m(x_0) - x| < \varepsilon \quad \text{and} \quad |f^n(x_0) - x| < \varepsilon.$$

Consider the closed interval $I = [f^m(x_0), f^n(x_0)] \subset [x - \varepsilon, x + \varepsilon]$ in S^1.
Claim: for every $y_0 \in S^1$, there exists $k \in \mathbb{N}$ such that $f^k(y_0) \in I$. Indeed, once we have demonstrated this claim, the proof of the theorem follows: since ε was chosen arbitrarily, this shows that we may construct a sequence $y_\ell = f^{k_\ell}(y_{\ell-1})$ starting from y_0 and such that $|y_\ell - x| < \frac{1}{\ell}$. From this we conclude that $x \in \omega(y_0)$ and since the assertion is symmetric in x_0 and y_0 the theorem.

It remains to prove the claim. We trace back, from where we can end up in I: define the sets $I_0 = I$ and

$$I_k = f^{-k(m-n)}(I) = [f^{kn-(k-1)m}(x), f^{(k+1)n-km}(x)],$$

hence the right endpoint of I_k is the left endpoint of I_{k+1} and the intervals wrap around the circle. Consider now the union of all of these intervals

$$\bigcup_{k=0}^{n} I_k.$$

This is one large, closed interval. We now want to show that there is N large enough so that $\bigcup_{k=0}^{n} I_k = S^1$. Suppose for the contrary, that this is not the case. Then the right endpoint of the interval must be bounded to the left of $f^m(x_0)$ and so

$$\lim_{k\to\infty} f^{-k(m-n)}(x_0) = p \in S^1$$

must exist. However, this cannot be since

$$\begin{aligned} p &= \lim_{k\to\infty} f^{-k(m-n)}(x_0) \\ &= \lim_{k\to\infty} f^{-(k-1)(m-n)}(x_0) \\ &= f^{(m-n)}\big(\lim_{k\to\infty} f^{-k(m-n)}(x_0)\big) \\ &= f^{(m-n)}(p), \end{aligned}$$

so that$p$ were a periodic point. But since the rotation number ρ was irrational, there are no periodic points. So, we conclude

$$S_1 = \bigcup_{k=0}^{N} I_k$$

for some $N \in \mathbb{N}$. For $y_0 \in S^1$ fixed, there must be $k \in \mathbb{N}$ with $y_0 \in I_k$ and so

$$f^{k(m-n)}(y_0) \in I.$$

This proves the claim and so the theorem. □

Next, we want to understand the structure of $\omega = \omega(x_0)$ for irrational rotation numbers $\rho(f) \notin \mathbb{Q}$.

Recall that a set C is called a *Cantor set* if it is a compact, totally disconnected set without isolated points.

Theorem 2.17. *Given any Cantor set $C \subset [0,1)$ and any $\rho \in [0,1) \setminus \mathbb{Q}$, there is an orientation-preserving homeomorphism $f\colon S^1 \to S^1$ such that*

$$\omega(x_0) = C \quad \textit{and} \quad \rho(f) = \rho.$$

Remark 2.18. One can show that for any orientation-preserving homeomorphism of the circle either $\omega = S^1$ or ω is a Cantor set.

We now introduce an equivalence class of dynamical systems.

Definition 2.19. Let $f, g\colon S^1 \to S^1$ be orientation-preserving diffeomorphisms. Then f and g are called *(topologically) conjugate* if there is a homeomorphism $h\colon S^1 \to S^1$ such that

$$g \circ h = h \circ f.$$

Note that we could have equivalently written that $f = h^{-1} \circ g \circ h$. If f and g are topologically conjugate, then all topological dynamical properties are the same. In particular, if $\omega \neq S^1$ and $\rho(f) \notin \mathbb{Q}$, then f cannot be conjugate to a rotation.

What are sufficient conditions for an orientation-preserving homeomorphism with $\rho(f) \notin \mathbb{Q}$ to be conjugate to the rotation R_ρ

We will cite here two theorems due to Denjoy: the first gives a positive answer under sufficient regularity, the second a negative answer under lack of said regularity.

Theorem 2.20 (Denjoy, 1932)**.** *If $f\colon S^1 \to S^1$ is a C^2-diffeomorphism with $\rho = \rho(f) \notin \mathbb{Q}$ irrational, then f is topologically conjugate to the rotation R_ρ. It suffices to assume that f' has bounded variation.*

Theorem 2.21 (Denjoy, 1946)**.** *Let $\rho \in [0,1) \setminus \mathbb{Q}$ and $\varepsilon > 0$. There exists a $C^{2-\varepsilon}$-diffeomorphism $f\colon S^1 \to S^1$ with $\rho(f) = f$ which is not conjugate to a rotation.*

With these results, we conclude the discussion about general diffeomorphisms on the circle.

3 KAM theory for analytic perturbations of rotations

The previous results show that C^2-diffeormophisms on the circle with irrational rotation number are topologically conjugate to an irrational rotation. But then h is merely a homeomorphism.

Question: can we say more about the regularity of h? For example, can we say that h is as regular as f? This is a very difficult questions for which the methods developed by Poincaré and Denjoy do not work. It will turn out that if we can even say something, then we also encounter a loss of regularity.

We will consider only small analytic perturbations of the rotations, that is we will consider analytic diffeomorphism $f\colon S^1 \to S^1$ of the form

$$f(x) = x + \rho + \eta(x) \bmod \mathbb{Z},$$

with $\rho \notin \mathbb{Q}$ and η periodic and analytic. Note that we can assume that $\rho(f) = \rho$: if $f(x) = x + \alpha + \mu(x)$ satisfies $\rho(f) = \rho$, then we can consider

$$\tilde{f}(x) = x + \rho + (\alpha - \rho + \mu(x)) =: x + \rho + \eta(x)$$

instead. We want to measure the regularity and size of η, as we think of η to be small.

Definition 2.22. For fixed $\sigma > 0$, we define the strip

$$S_\theta = \{z \in \mathbb{C} : |\operatorname{Im} z| < \sigma\}$$

and we define the set of analytic functions

$$B_\sigma = \big\{\eta \in C(\overline{S}_\sigma; \mathbb{C}) : \eta \text{ is analytic in } S_\sigma,\ \eta(z) = \eta(z+1) \text{ for all } z \in S_\sigma \\ \text{and } \|\eta\|_\sigma < \infty\big\}\,.$$

Here, we define

$$\|\eta\|_\sigma = \sup_{z \in S_\sigma} |\eta(z)|.$$

We will assume that for some $\sigma > 0$

$$\eta \in B_\sigma \quad \text{and} \quad \|\eta\|_\sigma < \varepsilon,$$

where $\varepsilon > 0$ will be specified later[6].

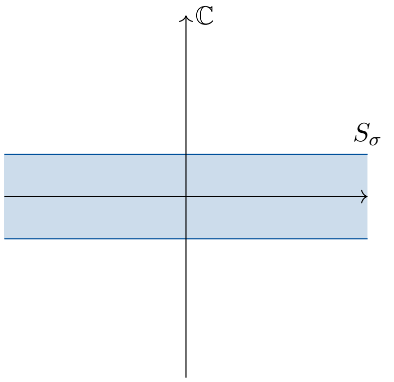

Fig. 2.1: The strip S_σ

[6] We will see that the change of coordinates h will only be analytic in $B_{\sigma-\delta}$ for some $\delta > 0$.

3.1 Heuristics and Diophantine numbers

We start by discussing a heuristical approach. The results of this will be used later to set up an iterative scheme.

Recall that we consider the diffeomorphism

$$f(x) = x + \rho + \eta(x).$$

We look for an analytic function $h\colon S^1 \to S^1$ such that

$$f \circ h = h \circ R_\rho.$$

Making the ansatz that

$$h(x) = x + H(x),$$

where H is an analytic and periodic function, we find that

$$f(x + H(x)) = x + H(x) + \rho + \eta(x + H(x)) = x + \rho + H(x + \rho) = h \circ R_\rho(x).$$

Rearranging this, we obtain

$$H(x + \rho) - H(x) = \eta(x + H(x)) = \eta(x) + \eta'(x)H(x) + \text{h.o.t.} \approx \eta(x).$$

Consequently, as an approximation, we first consider the equation

$$H(x + \rho) - H(x) = \eta(x). \tag{2.3}$$

Then, since H and η are both periodic, we can apply the Fourier transform[7] on both sides to obtain

$$e^{2\pi i n\rho}\hat{H}_n - \hat{H}_n = \hat{\eta}_n, \quad n \in \mathbb{Z}.$$

[7] $\hat{\eta}_n = \int_0^1 \eta(x) e^{-2\pi i n x} \, \mathrm{d}x$

Since $\rho \notin \mathbb{Q}$, it is $e^{2\pi i n\rho} - 1 \neq 0$ for every $n \neq 0$, hence we obtain

$$H(x) = \sum_{n \neq 0} \frac{\hat{\eta}_n}{e^{2\pi i n\rho} - 1} e^{2\pi i n x}. \tag{2.4}$$

Two problems ensue from this definition:

1) Equation (2.3) does not hold since

$$H(x + \rho) - H(x) = \sum_{n \neq 0} \hat{\eta}_n e^{2\pi i n x} = \eta(x) - \hat{\eta}_0 = \eta(x) - \int_0^1 \eta(y) \, \mathrm{d}y. \tag{2.5}$$

We can deal with this problem later in the proof.

2) Does the series in (2.4) even have a chance to converge? The problem is that $e^{2\pi i n\rho} - 1$ might be very small very often.

The second problem is called the *problem of small denominators*. This will appear again when we study the n-body problem of celestial mechanics. For a given rotation number $\rho \notin \mathbb{Q}$, we don't know how well-behaved these denominators are. But, we can resort to number theory again.

Last time, we have seen that any $\rho \in \mathbb{R}$ could be approximated by rational numbers such that

$$\left|\rho - \frac{p}{q}\right| < \frac{1}{q^2}.$$

It turns out that some numbers can be approximate better than others. This leads us to the Diophantine classes of numbers.

Definition 2.23. The irrational number ρ is of *Diophantine type* (K, ν) with $K, \nu > 0$ if

$$\left|\rho - \frac{p}{q}\right| > K|q|^{-\nu}$$

holds true for all $(p, q) \in \mathbb{Z} \times \mathbb{N}$.

We can w.l.o.g. assume that $K \leq 1$. In particular, there is no number of Diophantine type $(1, 2)$, but generically that is the best we can do approximating a number as the following proposition demonstrates.

Proposition 2.24. *For every $\nu > 2$, almost every irrational number ρ is of type (K, ν) for some $K > 0$.*

Proof. Let $\nu > 2$. It is enough to show that every irrational number $\rho \in [0, 1]$ is of the type (K, ν) for some $K > 0$.

First fix $K > 0$ and $\frac{p}{q} \in \mathbb{Q}$. Then define

$$I_{K,p,q} := \{\rho \in [0,1] \setminus \mathbb{Q} : \big|\rho - \frac{p}{q}\big| \leq K|q|^{-\nu}\}.$$

We will show that

$$\left|\bigcap_{n\in\mathbb{N}} \bigcup_{q\in\mathbb{N}} \bigcup_{p=1}^{q} I_{\frac{1}{n},p,q}\right| = 0. \tag{2.6}$$

First note that $I_{K,p,q}$ is essentially an interval up to a set of measure zero, hence we obtain

$$|I_{K,p,q}| \leq 2K|q|^{-\nu}.$$

But this implies, taking the union over all possible $p = 1, \dots, q$ that still for fixed q

$$\left|\bigcup_{p=1}^{q} I_{K,p,q}\right| \leq 2K|q|^{-\nu+1}.$$

Now, taking the union over all possible q, we find

$$\left|\bigcup_{q\in\mathbb{N}} \bigcup_{p=1}^{q} I_{K,p,q}\right| \leq 2K \sum_{q\in\mathbb{N}} q^{-\nu+1} < 2cK$$

for some $c > 0$ since $-\nu + 1 < 1$ and hence we obtain convergence of the series. But this readily implies (2.6). ☐

We can now use the Diophantine type of a number to get explicit control of the small denominator.

Lemma 2.25. *If $\rho \in [0, 1) \setminus \mathbb{Q}$ is of Diophantine type (K, ν), then*

$$|e^{2\pi i n\rho} - 1| \geq 4K|n|^{-(\nu-1)}$$

for all $n \neq 0$.

Proof. Let $m \in \mathbb{N}$. Then using $e^{2\pi i m} = 1$, we obtain

$$\begin{aligned} |e^{2\pi i n\rho} - 1| &= |e^{2\pi i m}(e^{2\pi i(n\rho - m)} - 1)| \\ &= |e^{2\pi i(n\rho - m)} - 1| \\ &= 2|\sin(\pi(\rho n - m))|, \end{aligned}$$

where we used in the last step that $|e^{ix} - 1| = 2\sin(x/2)$. Using the inequality

$$|\sin(\pi x)| \geq 2|x|, \quad |x| \leq \frac{1}{2},$$

we conclude that

$$|e^{2\pi i n\rho} - 1| \geq 4|\rho n - m| \geq 4K|n|^{-(\nu-1)}.$$

This concludes the proof. □

Using Cauchy's theorem, we get exponential decay for the Fourier coefficients of an analytic function.

Lemma 2.26. *Let $\eta \in B_\sigma$. Then*

$$|\hat{\eta}_n| \ \leq \|\eta\|_\sigma e^{-2\pi\sigma|n|}. \tag{2.7}$$

Proof. Recall that

$$\hat{\eta}_n = \int_0^1 \eta(x) e^{-2\pi i n x} \,\mathrm{d}x.$$

Denote by C the contour in $\mathbb{C}$ given by concatenating the path $[0,1]$, $[1, 1 \pm i\sigma]$, $[1 \pm i\sigma, \pm i\sigma]$ and $[\pm i\sigma, 0]$. By Cauchy's integral theorem, it holds

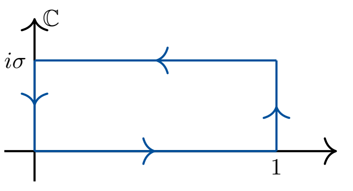

Fig. 2.2: The contour C

$$\int_C \eta(z) e^{-2\pi i n z} \,\mathrm{d}z = 0.$$

Combining this with periodicity, we find that

$$\int_{[0,1]} \eta(z) e^{-2\pi i n z} \,\mathrm{d}z = \int_{[\pm i\sigma, 1 \pm i\sigma]} \eta(z) e^{-2\pi i n z} \,\mathrm{d}z = \int_0^1 \eta(x \pm i\sigma) e^{2\pi i n x} e^{\mp 2\pi n\sigma} \,\mathrm{d}x.$$

Choosing either the pass on the upper or lower halfplane, depending on the sign of n, we obtain

$$|\hat{\eta}_n| \leq \|\eta\|_\sigma e^{2\pi|n|\sigma}$$

which concludes the proof. □

Remark 2.27. Also the inverse statement is true. Assume that (2.7) holds true, then the function

$$\eta(x) = \sum_{n \in \mathbb{Z}} \hat{\eta}_n e^{2\pi i n x}$$

is analytic in the strip S_σ and

$$\|\eta\|_{\sigma-\delta} \leq \frac{C}{\delta}.$$

We may apply these results to the function

$$H(x) = \sum_{n \neq 0} \frac{\hat{\eta}_n}{e^{2\pi i n\rho} - 1} e^{2\pi i n x}.$$

and obtain the following bound.

Proposition 2.28. *If ρ is of type (K,ν) and $\eta \in B_\sigma$ and $\delta < \sigma$, then $H \in B_{\sigma-\delta}$ with*

$$\|H\|_{\sigma-\delta} \leq \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma, \tag{2.8}$$

where Γ denotes the standard Γ-function

$$\Gamma(\nu) = \int_0^1 x^{\nu-1}e^{-x}\,\mathrm{d}x.$$

Proof. It suffices to prove absolute convergence of the series locally uniformly in $S_{\sigma-\delta}$, then analyticity follows. Indeed, using Lemmas 2.25 and 2.26, we obtain for $z \in S_{\sigma-\delta}$

$$\begin{aligned} |H(z)| &\leq \sum_{n\neq 0} \frac{|\hat{\eta}_n|}{|e^{2\pi i n\rho}-1|}|e^{2\pi i n z}| \\ &\leq \sum_{n\neq 0} \frac{|n|^{\nu-1}}{4K}\|\eta\|_\sigma e^{-2\pi\sigma\ |n|}e^{2\pi|n|(\sigma-\delta)} \\ &\leq \frac{\|\eta\|_\sigma}{4K}\sum_{n\neq 0}|n|^{\nu-1}e^{-2\pi\delta|n|} \\ &= \frac{\|\eta\|_\sigma}{2K}\sum_{n=1}^{\infty} n^{\nu-1}e^{-2\pi\delta n} \\ &\leq \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma. \end{aligned}$$

Here, in the last step, we use

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\nu-1}e^{-2\pi\delta n} &\leq \int_0^\infty y^{\nu-1}e^{-2\pi\delta y}\,\mathrm{d}y \\ &= \frac{1}{(2\pi\delta)^\nu}\int_0^\infty x^{\nu-1}e^{-x}\,\mathrm{d}x \\ &= \frac{1}{(2\pi\delta)^\nu}\Gamma(\nu), \end{aligned}$$

which completes the estimate. □

Next, we need to make sure that we have actually constructed an analytic diffeomorphism.

Proposition 2.29. *Assume that $2\delta < \sigma$ and*

$$\frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma < 1.$$

The map $h(z) = z + H(z)$ is analytic with analytic inverse on the domain $S_{\sigma-2\delta}$. Furthermore, h^{-1} is well-defined on $S_{\sigma-3\delta}$.

To prove the proposition, we rely on another lemma from complex analysis.

Lemma 2.30. *Let $\eta \in B_\sigma$. There is a constant C_1 such that for every $0 < \delta < \sigma$ it holds*

$$\|\eta'\|_{\sigma-\delta} \leq \frac{2\pi}{\delta}\|\eta\|_\sigma.$$

Proof. From Cauchy's integral formula, we know that for every $z \in S_{\sigma-\delta}$ and every ball $B_r(z) \subset S_\sigma$, we may write

$$\eta'(z) = \int_{\partial B_r(z)} \frac{\eta(w)}{(w-z)^2}\,\mathrm{d}w.$$

Hence, we obtain the estimate

$$|\eta'(z)| \leq \|\eta\|_\sigma \int_{\partial B_r(z)} \frac{dw}{r^2} = \frac{2\pi}{r}\|\eta\|_\sigma.$$

Sending r to δ proves the claim. □

Proof of Proposition 2.29. Analyticity of H and thus of h on $S_{\sigma-\delta}$ follows by construction. We only need to make sure that h is invertible. Recall that it is enough to find a domain so that $\|H'\| < 1$ holds true as this guarantees injectivity.

Now Lemma 2.30 and Proposition 2.28 imply that

$$\|H'\|_{\sigma-2\delta} \leq \frac{2\pi}{\delta}\|H\|_{\sigma-\delta} \leq \frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma < 1. \tag{2.9}$$

Hence, h restricted to $S_{\sigma-2\delta}$ is a diffeomorphism onto its image.

For the second part of the claim, we realise that by Proposition 2.28 and the assumption

$$\|H\|_{\sigma-\delta} \leq \frac{\Gamma(\nu)}{K(2\pi\delta)^{\nu}}\|\eta\|_\sigma < \delta. \tag{2.10}$$

Thus, if $z \in S_{\sigma-2\delta}$, it must hold $h(z) = z + H(z) \in S_{\sigma-\delta}$. Furthermore, $h(x) \in \mathbb{R}$ for every $x \in \mathbb{R}$ by construction. Finally, it is $|\operatorname{Im} h(z)| > \sigma - 3\delta$, whenever $\operatorname{Im} z = \sigma \pm 2\delta$. With this, we conclude that the image of $S_{\sigma-2\delta}$ contains the set $S_{\sigma-3\delta}$. □

To summarise: we have found an analytic change of coordinates $h(x) = x + H(x)$. Two problems remain though:

1) h is not the correct change of variables, as we have ignored higher order terms. Intuitively though, h is a step into the right direction and we may try to iterate the procedure.

2) If we constantly lose a fixed δ in analyticity during the iteration, we cannot have hope of constructing an analytic diffeomorphism. We need to choose a clever iteration scheme.

3.2 *Main theorem and Newton's method*

Before it is time to discuss how we can iterate the procedure introduced above, it is time to state the main theorem.

Theorem 2.31 (Arnold 1961)**.** *Assume that ρ is of Diophantine type (K, ν) and $\sigma > 0$. Then there exists $\varepsilon = \varepsilon(K, \nu, \sigma) > 0$ such that if*

$$f(x) = x + \rho + \eta(x)$$

has rotation number ρ and $\eta \in B_\sigma$ satisfies $\|\eta\|_\sigma < \varepsilon$, then there exists an analytic and invertible change of variables h which conjugates f to the rotation R_ρ:

$$R_\rho = h^{-1} \circ f \circ h.$$

Recall that so far we have heuristically taken the following approach: we want to find $h = \mathrm{Id} + H$ such that

$$H \circ R_\rho - R_\rho = \eta \circ h \approx \eta,$$

that is, we have linearised the equation to find a solution. Compare this to Newton's method:

Consider a map $F \in C^2(\mathbb{R})$ and we look for a root $\bar{x}$ of F. To do so, consider some point x_0 and define the linearisation around x_0 by

$$L_{x_0}(x) = F(x_0) + F'(x_0)(x - x_0).$$

We can find a root of L_{x_0} (provided $F'(x_0) \neq 0$), that is

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}.$$

Now, we want to iterate this and set

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}.$$

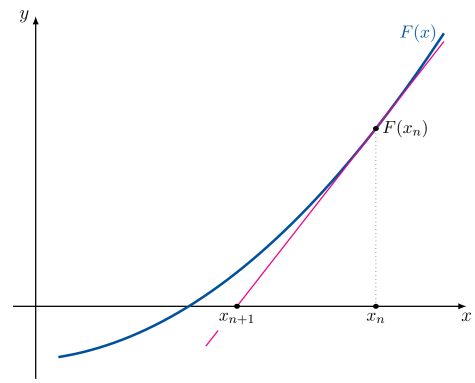

Fig. 2.3: Iteration step for Newton's method

For this to work, we require that $F' \neq 0$ in a neighborhood of the root $\bar{x}$. We can show very fast convergence: applying Taylor's theorem, we get

$$L_y(\bar{x}) - F(\bar{x}) = \frac{1}{2} F''(\xi)(\bar{x} - y)^2$$

for some $\xi \in [\bar{x}, y]$. Now set $\varepsilon_n = |\bar{x} - x_n|$. From the formula, we conclude that there is $\xi_n \in [\bar{x}, x_n]$ such that

$$F(x_n) - (\bar{x} - x_n)F'(x_n) = \frac{1}{2} F''(\xi_n)(x_n - \bar{x})^2.$$

Dividing by $F'(x_n)$ and using the definition of x_{n+1}, this leads to

$$x_{n+1} - \bar{x} = \frac{F''(\xi_n)}{2F'(x_n)}(x_n - \bar{x})^2$$

If $|F'|$ is bounded from below in a neighborhood of $\bar{x}$ and since F'' is bounded from above, there is a constant $C > 0$ such that

$$\varepsilon_{n+1} \leq C\varepsilon_n^2$$

and hence by iteration

$$|x_n - \bar{x}| \leq C\varepsilon_0^{2n}$$

superexponential convergence of x_n to $\bar{x}$.

Now, with Newton's method in mind, we will prove Theorem 2.31 by an iterative argument relying on the heuristics studied in the previous subsection. In order to achieve this, we need to understand $f_1 = h^{-1} \circ f \circ h$, where $h(x) = x + H(x)$ is the analytic change of variables constructed before. This is part of the following two propositions. They make rigorous our intuitive guess that $f_1 = R_\rho + \tilde{\eta}$, where $\|\tilde{\eta}\| \lesssim \|\eta\|_\sigma^2$.

Proposition 2.32. *Assume that* $4\delta < \sigma$ *and*

$$\frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma < 1.$$

Then it holds $h^{-1}(z) = z - H(z) + g(z)$, *where*

$$\|g\|_{\sigma-4\delta} \leq \frac{(2\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}}\|\eta\|_\sigma^2.$$

Proof. Define g by

$$g(z) = h^{-1}(z) - z + H(z).$$

Then using that $h^{-1} = z - H(z) + g(z)$, we obtain for every $z \in S_{\sigma-2\delta}$

$$z = h^{-1} \circ h(z) = h^{-1}(z + H(z)) = z + H(z) - H(z + H(z)) + g(z + H(z)). \tag{2.11}$$

Solving for g, we find

$$g(z + H(z)) = H(z + H(z)) - H(z) = \int_0^1 H'(z + sH(z))H(z)\,\mathrm{d}s.$$

So, we may define g for each $\xi = h(z)$ by

$$g(\xi) = H(h^{-1}(\xi))\int_0^1 H'(h^{-1}(\xi) + sH(h^{-1}(\xi)))\,\mathrm{d}s.)$$

As in the proof of Proposition 2.29, we may argue using $\|h\|_{\sigma-\delta} < \delta$ that since the image of $S_{\sigma-3\delta}$ under h must contain $S_{\sigma-4\delta}$, we have for $\xi \in S_{\sigma-4\delta}$ that $h^{-1}(\xi) \in S_{\sigma-3\delta}$ and furthermore $h^{-1}(\xi) + sH(h^{-1}(\xi)) \in S_{\sigma-2\delta}$ so that we may apply the estimates from equations (2.8) and (2.9) to obtain

$$\|g\|_{\sigma-4\delta} \leq \|H\|_{\sigma-\delta}\|H'\|_{\sigma-2\delta} \leq \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma \frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma.$$

This proves the claim. □

Finally, we need to understand how large is the error that we make by linearisation. Therefore, define $f_1 = h^{-1} \circ f \circ h$. Intuitively, f_1 is closer to a rotation and we expect from Newton's method that the error is also quadratic in terms of $\|\eta\|_\sigma^2$. This is demonstrated in the following proposition.

Proposition 2.33. *Assume that* $4\delta < \sigma$ *and*

$$\frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma < 1.$$

Then $f_1(x) = h^{-1} \circ f \circ h(x) = x + \rho + \eta_1(x)$, *where*

$$\|\eta_1\|_{\sigma-6\delta} \leq \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}}\|\eta\|_\sigma^2.$$

Proof. Using the previous results, we can spell out f_1 explicitly:

$$\begin{aligned} f_1(x) &= h^{-1} \circ f \circ h(x) \\ &= h^{-1}(x + H(x) + \rho + \eta(x + H(x))) \\ &= x + H(x) + \rho + \eta(x + H(x)) - H(x + H(x) + \rho + \eta(x + H(x))) \\ &\quad + g(x + H(x) + \rho + \eta(x + H(x))) \\ &= x + \rho + [H(x) - H(x + \rho) + \eta(x)] + [\eta(x + H(x)) - \eta(x)] \\ &\quad + [H(x + \rho) - H(x + \rho + H(x) + \eta(x + H(x))] \\ &\quad + g(x + H(x) + \rho + \eta(x + H(x))). \end{aligned} \tag{2.12}$$

Now define

$$\eta_1(x) = f_1(x) - x - \rho.$$

We want to obtain an estimate for η_1. The reason for splitting the term in this particular fashion becomes clear immediately: the first bracket was the linearised equation for H, see (2.5), so we find

$$H(x) - H(x + \rho) + \eta(x) = \hat{\eta}_0.$$

The second bracket we may rewrite as

$$\eta(x + H(x)) - \eta(x) = \int_0^1 \eta'(x + sH(x))H(x)\,\mathrm{d}s$$

and as before we obtain a bound for this term using the assumptions

$$\|\eta(z + H(z)) - \eta(z)\|_{S_{\sigma-4\delta}} \leq \|H\|_{\sigma-\delta}\|\eta'\|\sigma - 2\delta \leq \frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma \frac{2\pi}{\delta}\|\eta\|_\sigma. \tag{2.13}$$

The same applies to the third bracket:

$$\begin{aligned} &H(z + \rho + H(z) + \eta(z + H(z)) - H(z + \rho) \\ &= \int_0^1 H'(z + \rho + s(H(z) + \eta(z + H(z))))(H(z) + \eta(z + H(z)))\,\mathrm{d}s. \end{aligned}$$

We can bound the norm of this term for $z \in S_{\sigma-4\delta}$ using the assumed bounds:

$$\|H(z + \rho + H(z) + \eta(z + H(z)) - H(z + \rho)\|_{\sigma-4\delta} \leq (\|H\|_{\sigma-\delta} + \|\eta\|_\sigma)\|H'\|_{\sigma-2\delta}.$$

We obtain the explicit bound

$$\begin{aligned} (\|H\|_{\sigma-\delta} + \|\eta\|_\sigma)\|H'\|_{\sigma-2\delta} &\leq \left(\frac{\Gamma(\nu)}{K(2\pi\delta)^\nu}\|\eta\|_\sigma + \|\eta\|_\sigma\right)\frac{(2\pi)^2\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma \\ &\leq \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2. \end{aligned} \tag{2.14}$$

For the last term, notice that if $z \in S_{\sigma-6\delta}$, then $z + H(z) + \rho + \eta(z + H(z)) \in S_{\sigma-4\delta}$ and we may apply Proposition 2.32 to obtain

$$\|g(z + H(z) + \rho + \eta(z + H(z)))\|_{\sigma-6\delta} \leq \frac{(2\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}}\|\eta\|_\sigma^2. \tag{2.15}$$

We are left with finding a corresponding bound for $\hat{\eta}_0$. Since f_1 and f are conjugate, the rotation numbers must be equal $\rho(f_1) = \rho(f) = \rho$. We may apply Lemma 2.13 to f_1 and find that there must be $x_0 \in [0, 1)$ such that $\eta_1(x_0) = 0$ and hence $f_1(x_0) = x_0 + \rho$. Using this expression in (2.12), we obtain

$$\begin{aligned}\hat{\eta}_0 = -[\eta(x + H(x)) - \eta(x)] \\ &- [H(x + \rho) - H(x + \rho + H(x) + \eta(x + H(x))] \\ &- g(x + H(x) + \rho + \eta(x + H(x))).\end{aligned}$$

We have already bounded each of the objects on the right-hand side in (2.13), (2.14) and (2.15) so that we obtain

$$|\hat{\eta}_0| \leq \frac{(2\pi^2)\Gamma(\nu)}{K(2\pi\delta)^{\nu+1}}\|\eta\|_\sigma^2 + \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{2\nu+1}}\|\eta\|_\sigma^2 + \frac{(2\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}}\|\eta\|_\sigma^2. \tag{2.16}$$

Combining the four estimates (2.13), (2.14), (2.15) and (2.16), the theorem follows. □

Before we come to the technical details of the iteration scheme, this is a good time to summarise our approach.

1. Linearisation. We want to solve $h \circ R_\rho = f \circ h$, that is

$$H \circ R_\rho - H = \eta \circ h,$$

where we write $h = \mathrm{Id} + H$. Then we linearise this equation and obtain the approximate equation

$$H \circ R_\rho - H = \eta.$$

2. Solving the linear equation. We solved the linear equation for analytic right-hand side η and obtained corresponding bounds for H in a smaller strip in Proposition 2.28. In Proposition 2.29 we showed that we obtained a diffeomorphism.
3. Error term in linearised solution. Defining $f_1 = h^{-1} \circ f \circ h$, we miss R_ρ by an error which is of the order $\|\eta\|_\sigma^2$. In the process, we lose regularity.
4. Fast convergence due to Newton scheme. If we define inductively $f_{n+1} = h_n^{-1} \circ f_n \circ h_n$, then the fast convergence of the Newton scheme will allow us to show that the maps

$$h_0 \circ h_1 \circ \cdots \circ h_n$$

actually converge to an analytic diffeomorphism of the circle conjugating f to the rotation R_ρ. Now we are in the position to iterate this scheme. We define

$$f_0(x) = f(x) = R_\rho(x) + \eta(x), \quad \eta_0(x) = \eta(x). \tag{2.17}$$

We define $h_0(x) = h(x) = x + H(x)$ and

$$f_1(x) = h_0^{-1} \circ f_0 \circ h_0 = R_\rho + \eta_1.$$

Then we inductively define

$$f_{n+1} = h_n^{-1} \circ f_n \circ h_n = R_\rho + \eta_{n+1},$$

where $h_n = \mathrm{Id} + H_n$ are constructed as before, i.e. they solve

$$H_n(x+\rho) - H_n(x) = \eta_n(x) - \hat{\eta}_n(0).$$

Define for $\sigma_0 = \sigma$ and $\varepsilon_0 = \|\eta\|_\sigma$ the inductive constants:

$$\begin{aligned} \delta_n &= \frac{\sigma}{36(1+n^2)}, \\ \sigma_{n+1} &= \sigma_n - 6\delta_n, \\ \varepsilon_n &= \varepsilon_0^{(3/2)^n} \end{aligned}$$

if $n \geq 0$. We also define

$$\sigma^* = \lim_{n\to\infty} \sigma_n > \frac{\sigma}{2} > 0.$$

We need to make sure that these constants are chosen so that our inductive scheme works correctly before we may prove the main theorem.

Lemma 2.34. *If*

$$\|\eta\|_\sigma = \varepsilon_0 < \left(\frac{K}{16\pi\Gamma(\nu)} \left(\frac{\sigma}{36}\right)^{\nu+1} \right)^8,$$

then $f_{n+1}(x) = x + \rho + \eta_{n+1}(x)$ *with* $\eta_{n+1} \in B_{\sigma_{n+1}}$ *and*

$$\|\eta_{n+1}\|_{\sigma_{n+1}} \leq \varepsilon_{n+1}.$$

Furthermore, $h_n = \mathrm{Id} + H_n$ *satisfies*

$$\|H_n\|_{\sigma_n - \delta_n} \leq \frac{\Gamma(\nu)\varepsilon_n}{K(2\pi\delta_n)^\nu},$$

and $h_n^{-1} = x - h_n + g_n$, *where*

$$\|g_n\|_{\sigma_n - 4\delta_n} \leq \frac{(2\pi)^2\Gamma(\nu)^2\varepsilon_n^2}{K^2(2\pi\delta_n)^{(2\nu+1)}}.$$

Proof. For $n = 0$, the estimates for H_n and g_n were demonstrated in Propositions 2.28 and 2.32. Proposition 2.33 gives the estimate

$$\begin{aligned} \|\eta_1\|_{\sigma-6\delta} &\leq \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}} \|\eta\|_\sigma^2 < \varepsilon_0^{3/2} \left(\frac{K}{16\pi\Gamma(\nu)} \left(\frac{\sigma}{36}\right)^{\nu+1} \right)^4 \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta)^{(2\nu+1)}} \\ &\leq \varepsilon_0^{3/2}. \end{aligned}$$

Now suppose the induction holds up to step $n-1$ so that we know that $\|\eta_n\|_{\sigma_n} \leq \varepsilon_n$. Then Propositions Propositions 2.28 and 2.32 give the corresponding bounds for H_n and g_n, respectively. Again by Proposition 2.33, we find

$$\|\eta_{n+1}\|_{\sigma_{n+1}} \leq \frac{(4\pi)^2\Gamma(\nu)^2}{K^2(2\pi\delta_n)^{(2\nu+1)}} \varepsilon_n^2 \leq \varepsilon_{n+1}^{3/2}$$

as before. ∎

Now we can finally prove Arnold's theorem.

Proof of Theorem 2.31. Define the change of coordinates via $\psi_0 = h_0$ and

$$\psi_n = h_n \circ \psi_{n-1} = h_n \circ \cdots \circ h_1 \circ h_0.$$

Then it is

$$\begin{aligned}\psi_n(x) &= x + H_n(x) + h_{n-1}(x + H_n(x)) + h_{n-2}(x + H_n(x) + h_{n-1}(x + H_n(x))) \\ &\quad + \cdots + h_0(x + H_1(x + \cdots) + \cdots).\end{aligned}$$

From Lemma 2.34, we know that ψ_n is analytic on $S_{\sigma_n - 2\delta_n}$ and also

$$\|\psi_n - \operatorname{Id}\|_{\sigma_n - 2\delta_n} \leq \sum_{k=0}^{\infty} \frac{\Gamma(\nu)\varepsilon_k}{K(2\pi\delta_n)^\nu} =: \Delta < \infty.$$

We need to show that ψ_n converges to an analytic limit. For this, note that

$$\begin{aligned}\psi_{n+1}(z) - \psi_n(z) = \psi_n \circ h_{n+1}(z) - \psi_n(z) &= \psi_n(z + H_{n+1}(z)) - \psi_n(z) \\ &= \int_0^1 \psi_n'(z + sH_{n+1}(z))H_{n+1}(z)\,\mathrm{d}s.\end{aligned}$$

We may use that $\psi_n - \operatorname{Id}$ is bounded, to obtain the bound $\|\psi_n'\|_{\sigma_n - 4\delta_n} \leq \|(\psi_n - \operatorname{Id})'\|_{\sigma_n - 4\delta_n} + 1 \leq \frac{2\pi\Delta}{\delta_n} + 1$ and hence

$$\|\psi_{n+1} - \psi_n\|_{\sigma_{n+1}} \leq \|H_{n+1}\|_{\sigma_{n+1}} \|\psi_n'\|_{\sigma_{n+1}} \leq \Big(\frac{2\pi\Delta}{\delta_n} + 1\Big) \frac{\Gamma(\nu)\varepsilon_{n+1}}{K(2\pi\delta_{n+1})^\nu}.$$

The series

$$\sum_{n=0}^{\infty} \Big(\frac{2\pi\Delta}{\delta_n} + 1\Big) \frac{\Gamma(\nu)\varepsilon_{n+1}}{K(2\pi\delta_{n+1})^\nu} \tag{2.18}$$

converges since ε_n is converging exponentially fast. Hence, the sequence $(\psi_n)_n$ is Cauchy in B_{σ^*} and converges there uniformly to a limit $h \in B_\sigma$[8]. We may write

[8] Note that uniform limits of analytic functions are analytic.

$$h(z) = z + H(z)$$

and find that

$$\|H'\|_{\sigma^* - \delta^*} \leq \frac{\Delta}{\delta^*} < \delta^* \tag{2.19}$$

provided $\delta^* < \min\{\frac{\sigma^*}{16}, 1\}$. But this implies, since $\delta^* < 1$ that h is invertible on the image of $S_{\sigma^* - \delta^*}$ and that this image contains $S_{\sigma^* - 2\delta^*}$. We conclude the proof by noticing that by induction and using $f_0 = f$ together with $f_n \circ h_{n+1} = h_{n+1} \circ f_{n+1}$ it holds

$$f \circ \psi_n = \psi_n \circ f_n.$$

We may use this to obtain

$$f \circ h(z) = \lim_{n\to\infty} f \circ \psi_n(z) = \lim_{n\to\infty} \psi_n \circ f_n(z) = \lim_{n\to\infty} \psi_n(z + \rho + \eta_n(z)) = h \circ R_\rho(z)$$

due to the uniform convergence of $\psi_n \to h$ and η_n to zero. This proves that we have indeed constructed an analytic diffeomorphism h that conjugates f to the rotation R_ρ. $\square$

Bibliography

[Arn63] V. I. Arnol'd. "Small Denominators and Problems of Stability of Motion in Classical and Celestial Mechanics". In: *Russian Mathematical Surveys* 18.6 (1963). Publisher: IOP Publishing, p. 85.

[Arn78] V. I. Arnold. *Mathematical Methods of Classical Mechanics*. Red. by C. C. Moore. Vol. 60. Graduate Texts in Mathematics. New York, NY: Springer New York, 1978.

[Arn88] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*. Red. by M. Artin et al. Vol. 250. Grundlehren der mathematischen Wissenschaften. New York, NY: Springer New York, 1988.

[CW93] W. Craig and C. E. Wayne. "Newton's method and periodic solutions of nonlinear wave equations". In: *Communications on Pure and Applied Mathematics* 46.11 (1993), pp. 1409–1498.

[Dum14] H. S. Dumas. *The KAM story: a friendly introduction to the content, history, and significance of classical Kolmogorov-Arnold-Moser theory*. [Hackensack], New Jersey: World Scientific, 2014. 361 pp.

[GH83] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Vol. 42. Applied Mathematical Sciences. New York, NY: Springer New York, 1983.

[Her79] M. R. Herman. "Sur la Conjugaison Différentiable des Difféomorphismes du Cercle a des Rotations". In: *Publications mathématiques de l'IHÉS* 49.1 (1979), pp. 5–233.

[Kol54] A. N. Kolmogorov. "On conservation of conditionally periodic motions for a small change in Hamilton's function". In: *Dokl. Akad. Nauk SSSR*. Vol. 98. 1954, pp. 527–530.

[Kuk93] S. B. Kuksin. *Nearly integrable infinite dimensional Hamiltonian systems*. Lecture notes in mathematics 1556. Berlin Heidelberg: Springer, 1993. 101 pp.

[Las13] J. Laskar. "Is the Solar System Stable?" In: *Chaos*. Ed. by B. Duplantier, S. Nonnenmacher, and V. Rivasseau. Vol. 66. Series Title: Progress in Mathematical Physics. Basel: Springer Basel, 2013, pp. 239–270.

[MHO09] K. Meyer, G. Hall, and D. Offin. *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*. Red. by J. Hale et al. Vol. 90. Applied Mathematical Sciences. New York, NY: Springer New York, 2009.

[Mos16] J. Moser. *Stable and Random Motions in Dynamical Systems: With Special Emphasis on Celestial Mechanics (AM-77)*. Publication Title: Stable and Random Motions in Dynamical Systems. Princeton University Press, 2016.

[Mos66a] J. Moser. "A rapidly convergent iteration method and non-linear differential equations = II". In: *Annali della Scuola Normale Superiore di Pisa - Scienze Fisiche e Matematiche* 20.3 (1966), pp. 499–535.

[Mos66b] J. Moser. "A rapidly convergent iteration method and non-linear partial differential equations - I". In: *Annali della Scuola Normale Superiore di Pisa - Scienze Fisiche e Matematiche* 20.2 (1966), pp. 265–315.

[Sch96] W. M. Schmidt. *Diophantine approximations and diophantine equations*. Lecture notes in mathematics 1467. Berlin ; New York: Springer, 1996. 217 pp.

[Way08] C. E. Wayne. "An introduction to KAM theory". In: *http://math.bu.edu/people/cew/preprints/introkam.pdf* (2008).

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