$$
\begin{aligned}
\mathcal{H}_N(t) &= \sum_{v=1}^{\infty} \frac{(-i)^v}{v!} \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \cdots \int_0^t \mathrm{d}t_v \left\langle \mathbb{T}\, \hat{\mathrm{V}}^{\mathrm{I}}(t_1)\hat{\mathrm{V}}^{\mathrm{I}}(t_2)\cdots \hat{\mathrm{V}}^{\mathrm{I}}(t_v) \right\rangle_{\mathrm{c}(N)} \\
&= -i \int_0^t \mathrm{d}t_1 \frac{\left\langle N \middle| \hat{\mathrm{V}}^{\mathrm{I}}(t_1) \middle| \Psi^{\mathrm{S}} \right\rangle}{\langle N|\Psi^{\mathrm{S}}\rangle} \\
&\quad - \frac{1}{2} \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \left(\frac{\left\langle N \middle| \mathbb{T}\hat{\mathrm{V}}^{\mathrm{I}}(t_1)\hat{\mathrm{V}}^{\mathrm{I}}(t_2) \middle| \Psi^{\mathrm{S}} \right\rangle}{\langle N|\Psi^{\mathrm{S}}\rangle} - \frac{\left\langle N \middle| \hat{\mathrm{V}}^{\mathrm{I}}(t_1) \middle| \Psi^{\mathrm{S}} \right\rangle \cdot \left\langle N \middle| \hat{\mathrm{V}}^{\mathrm{I}}(t_2) \middle| \Psi^{\mathrm{S}} \right\rangle}{\langle N|\Psi^{\mathrm{S}}\rangle^2} \right) \\
&\quad + \cdots
\end{aligned}
$$