Introduction to Game Theory

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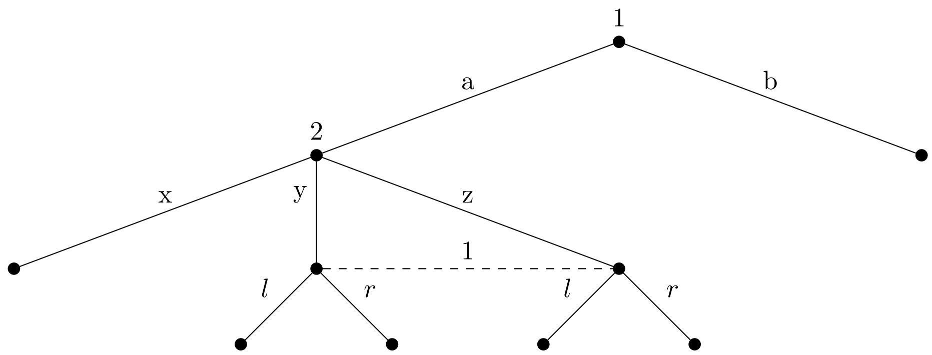
Exercise Sheet 6 — Solutions

Exercise 6.1 (Imperfect Information Games, 1+1+1+2 points)

Consider the following extensive form game with imperfect information:



(a) Is this a game of perfect or imperfect recall? Justify your answer.

Solution:

Yes, it is perfect recall. The players neither forget which moves they made, nor that they have made them. Their experience records are the same for all histories that are in the same information set. The only non-singleton information set $\{\langle a, y\rangle, \langle a, z\rangle\}$ satisfies this condition: $X_1(\langle a, y\rangle) = X_1(\langle a, z\rangle) = \langle\langle\rangle, a, \{\langle a, y\rangle, \langle a, z\rangle\}\rangle$.

(b) Specify the information partition $\mathcal{I}_i$ for player 1 and 2.

Solution:

$$\mathcal{I}_1 = \{\langle\rangle, \{\langle a, y\rangle, \langle a, z\rangle\}\}$$
$$\mathcal{I}_2 = \{\langle a\rangle\}$$

(c) Specify player 1's experience record of the following histories: $\langle a, x\rangle, \langle a, y, l\rangle, \langle a, z, r\rangle$

Solution:

An experience record $X_i(h)$ of player i for some history h is the sequence consisting of information sets that player i encounters in h and the actions that player i takes at them.

$$X_1(\langle a, x\rangle) = \langle\langle\rangle, a\rangle$$
$$X_1(\langle a, y, l\rangle) = \langle\langle\rangle, a, \{\langle a, y\rangle, \langle a, z\rangle\}, l\rangle$$
$$X_1(\langle a, z, r\rangle) = \langle\langle\rangle, a, \{\langle a, y\rangle, \langle a, z\rangle\}, r\rangle$$

(d) Find the behavioral strategy of player 1 that is outcome-equivalent to her mixed strategy in which she plays (b, r) with probability 0.4, (b, l) with probability 0.1, and (a, l) with probability 0.5.

Solution:

The following behavioral strategy β_1 of player 1 is outcome-equivalent to the mixed strategy σ_1 described above:

$$\beta_1(\{\langle\rangle\})(a) = 0.5$$
$$\beta_1(\{\langle\rangle\})(b) = 0.5$$
$$\beta_1(\{\langle a, y\rangle, \langle a, z\rangle\})(l) = 1$$
$$\beta_1(\{\langle a, y\rangle, \langle a, z\rangle\})(r) = 0$$

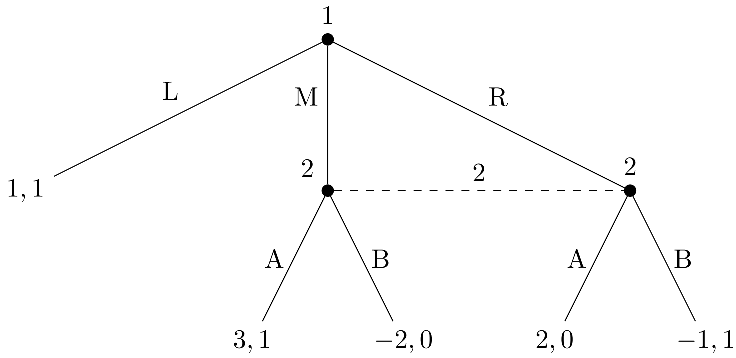
Explanation: The probability of each outcome is the same when player 1 plays his behavioral strategy β_1 or his mixed strategy σ_1 and player 2 plays the same strategy in both cases. We can verify this claim by checking whether, for each terminal history $h = \langle a^1, \ldots, a^K \rangle$, it holds that:

$$\pi_1(h) = \prod_{\substack{0 \leq k < K \\ P(\langle a^1, \ldots, a^k \rangle) = 1}} \beta_1(\langle a^1, \ldots, a^k \rangle)(a^{k+1})$$

Example:

$$\begin{aligned}
\pi_1(\langle b \rangle) = \sigma_1((b,l)) + \sigma_1((b,r)) & = 0.5 = \beta_1(\{\langle\rangle\})(b) \\
\pi_1(\langle a, y, l \rangle) = \sigma_1((a,l)) & = 0.5 = \beta_1(\{\langle\rangle\})(a) \cdot \beta_1(\{\langle a, y \rangle, \langle a, z \rangle\})(l) \\
\pi_1(\langle a, y, r \rangle) = \sigma_1((a,r)) & = 0 = \beta_1(\{\langle\rangle\})(a) \cdot \beta_1(\{\langle a, y \rangle, \langle a, z \rangle\})(r) \\
& \vdots
\end{aligned}$$

Exercise 6.2 (Sequential equilibria, 3 points)
Consider the following imperfect information game:



Find the set of sequential equilibria of this game. (Hint: There are three types of sequential equilibria:

(a) $\beta_1(\langle\rangle)(L) = 1,\ x \leq \beta_2(\{\langle M \rangle, \langle R \rangle\})(B) \leq y,\ \mu(\langle M \rangle) = \mu(\langle R \rangle) = \frac{1}{2}$

(b) $\beta_1(\langle\rangle)(L) = 1,\ \beta_2(\{\langle M \rangle, \langle R \rangle\})(B) = 1,\ x' \leq \mu(\langle R \rangle) \leq y'$

(c) $\beta_1(\langle\rangle)(M) = 1,\ \beta_2(\{\langle M \rangle, \langle R \rangle\})(A) = z,\ \mu(\langle M \rangle) = 1$

Find values of x, y, x', y', and z such that this list covers all sequential equilibria and only those.)

Solution:

The values are: $x = \frac{3}{8}, y = 1, x' = \frac{1}{2}, y' = 1, z = 1$, yielding the following sequential equilibria:

(a) $\beta_1(\langle\rangle)(L) = 1,\ \frac{3}{8} \leq \beta_2(\{\langle M \rangle, \langle R \rangle\})(B) \leq 1,\ \mu(\langle M \rangle) = \mu(\langle R \rangle) = \frac{1}{2}$

(b) $\beta_1(\langle\rangle)(L) = 1,\ \beta_2(\{\langle M \rangle, \langle R \rangle\})(B) = 1,\ \frac{1}{2} \leq \mu(\langle R \rangle) \leq 1$

(c) $\beta_1(\langle\rangle)(M) = 1,\ \beta_2(\{\langle M \rangle, \langle R \rangle\})(A) = 1,\ \mu(\langle M \rangle) = 1$