

# Introduction to Game Theory

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## Exercise Sheet 10

Due: Friday, July 24, 2020

**Exercise 10.1** (Vickrey-Clarke-Groves Mechanism; 2+2 points)

In a  $k$ -item auction,  $k$  identical items are to be sold. Each bidder  $i = 1, \dots, n$  can get at most one of the items and has a privately known valuation  $w_i$  for the item. For simplicity, assume that  $w_1 > w_2 > \dots > w_n$ . The set of alternatives  $A = N_k$  consists of all  $k$ -ary subsets of players. Each alternative represents the players who will receive an item.

- (a) Formalize the  $k$ -item auction as a VCG mechanism  $\mathcal{M} = \langle f, (p_i)_{i \in N} \rangle$  that uses *Clarke pivot functions*.
- (b) Consider the mechanism  $\mathcal{M}' = \langle f', (p'_i)_{i \in N} \rangle$  implementing a  $k$ -item auction, with
  - social choice function  $f'(v_1, \dots, v_n) = \{i \in N \mid 1 \leq i \leq k\}$ , and
  - payment functions  $p'_i(a) = \begin{cases} w_{i+1}, & \text{if } i \in a, \\ 0, & \text{otherwise,} \end{cases}$  for all  $a \in A$ .

Here, the  $i$ -th highest bidding winner has to pay the  $(i + 1)$ -st highest bid, i.e., the highest bidding player pays the second highest bid, the second highest bidder pays the third highest bid, and so on. Non-winning players pay nothing. Construct a counterexample with only three bidders that proves that  $\mathcal{M}'$  is *not* incentive compatible.

**Exercise 10.2** (Top trading cycle method, 2 points)

- (a) Apply the top trading cycle algorithm to the following problem and state what happens in the iterations:
  - *Player 1*:  $1 \triangleleft_1 4 \triangleleft_1 2 \triangleleft_1 3$
  - *Player 2*:  $3 \triangleleft_2 2 \triangleleft_2 1 \triangleleft_2 4$
  - *Player 3*:  $2 \triangleleft_3 3 \triangleleft_3 4 \triangleleft_3 1$
  - *Player 4*:  $2 \triangleleft_4 1 \triangleleft_4 4 \triangleleft_4 3$

Preferences are given from lowest (left) to highest (right).

**Exercise 10.3** (Stable matchings, 2 points)

Apply the deferred acceptance algorithm with male proposals to the following problem and state what happens in the iterations:

- *Man 1*:  $w_4 \prec_{m_1} w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- *Man 2*:  $w_3 \prec_{m_2} w_2 \prec_{m_2} w_1 \prec_{m_2} w_4$
- *Man 3*:  $w_4 \prec_{m_3} w_2 \prec_{m_3} w_3 \prec_{m_3} w_1$
- *Man 4*:  $w_4 \prec_{m_4} w_1 \prec_{m_4} w_3 \prec_{m_4} w_2$
- *Woman 1*:  $m_4 \prec_{w_1} m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- *Woman 2*:  $m_2 \prec_{w_2} m_1 \prec_{w_2} m_4 \prec_{w_2} m_3$
- *Woman 3*:  $m_1 \prec_{w_3} m_3 \prec_{w_3} m_2 \prec_{w_3} m_4$
- *Woman 4*:  $m_4 \prec_{w_4} m_1 \prec_{w_4} m_2 \prec_{w_4} m_3$

Preferences are given from lowest (left) to highest (right).