Game Theory

1. Introduction

1.1. Rational Agents, History, Course Outline

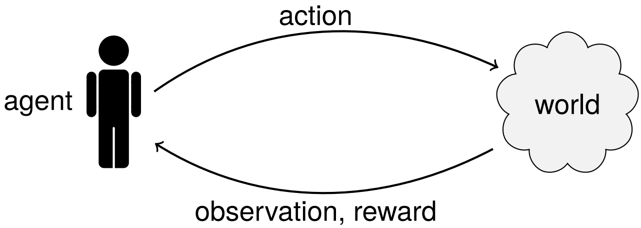
Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Rational Agents

Consider rationally acting agents:

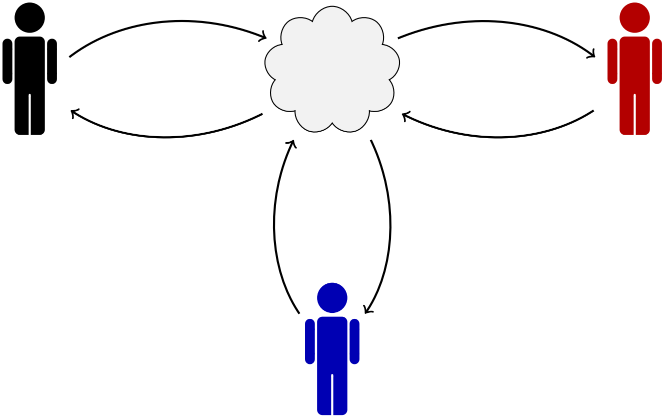


Rational agents maximize their (expected) utility:

- decision theory
- Markov decision processes (MDPs)
- reinforcement learning
- AI planning
- ...

Rational Agents in Game Theory

Situation in game theory:



Rational Agents in Game Theory

Multiple rational agents interacting in strategic decision situations

- resulting utility depends on what other agents do
- all agents know that other agents are rational (this is even common knowledge)

Interesting questions:

- how to model such strategic situations
- how to solve such strategic situations
- how to design games that have desired solutions

Game theory is the study and analysis of such strategic decision situations.

History of Game Theory

- originally part of mathematics and theoretical economics
- today ubiquitous
- here: artificial intelligence and computer science perspective
  - rationality assumptions (“homo economicus”) more warranted for artificial agents than for humans
  - interesting algorithmic questions

Rationality

Rationality:

- General assumption: All players want to maximize their own utility and nothing else.
- Contrasts:
  - Altruistic agents want to maximize utility of other agents
  - Cooperative agents want to maximize group utility
  - Byzantine agents want to minimize utility of other agents

Limitations:

- agents may not foresee all consequences of their decisions (bounded rationality)
- agents may not know all relevant information about the game structure (incomplete information)
- agents may not know all relevant information about the current state of the game (imperfect information)

Course Outline

- strategic games
- extensive games (with perfect and imperfect information)
- repeated games
- social choice theory
- mechanism design

Game Theory

1. Introduction

1.2. Application Examples

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Board and Card Games

Two-player board and card games:

- very special
- whatever is good for one player is bad for the other (strictly competitive games)
- recent visible success in heads-up no-limit hold'em Poker: Libratus (Brown and Sandholm, 2018)

Successful extension to multi-player variant:

- Pluribus (Brown and Sandholm, 2019)

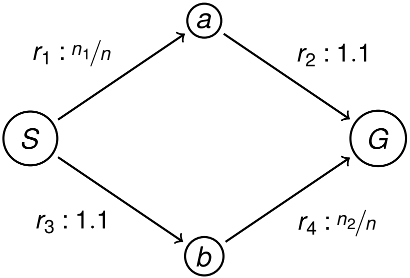
Auctions

Auctions: think of eBay, Google AdWords, …

- setting: one object should be allocated to one out of a number of bidders
- questions:
  - what bidding protocol to use?
  - who is the winner?
  - what does the winning bidder have to pay?

Congestion Games

Congestion games: road network with travel costs dependent on the number of agents choosing a particular road

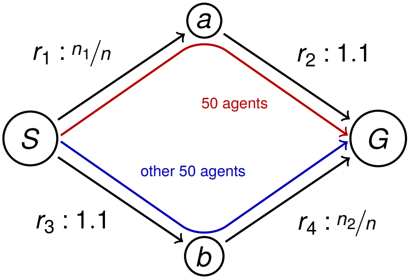


Question: Assume that there are $n = 100$ agents.
Which routes will they choose?

Average travel cost per agent: ?

Congestion Games

Congestion games: road network with travel costs dependent on the number of agents choosing a particular road

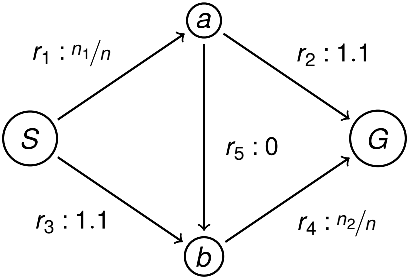


Question: Assume that there are $n = 100$ agents.
Which routes will they choose?

Average travel cost per agent: 1.6

Congestion Games

Congestion games: road network with travel costs dependent on the number of agents choosing a particular road

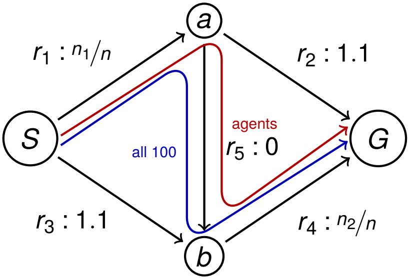


Question: Assume that there are $n = 100$ agents.
Which routes will they choose now (with free new road)?

Average travel cost per agent: ?

Congestion Games

Congestion games: road network with travel costs dependent on the number of agents choosing a particular road



Question: Assume that there are $n = 100$ agents.

Which routes will they choose now (with free new road)?

Average travel cost per agent: $2 > 1.6$

Security Games

Security games:

- setting: a facility (e. g., an airport) has to be guarded to avoid attacks
- possible methods:
  - visit all critical places
  - choose the places probabilistically
  - find a probability distribution for the routing that minimizes expected damage even under the assumption that the attacker can observe the guards

Elections

- setting: a set of alternatives (candidates) and a set of voters, determine winner or ranking
- questions:
  - what questions to ask?
  - how to determine a winner / ranking?
  - what is the computational complexity of determining a winner?
  - can the protocol be made manipulation-safe?

Game Theory

2. Strategic Games

2.1. Preliminaries and Examples

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Informally:

- one-shot games of finitely many players with given action sets and payoff functions
- perfect information

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Definition (Strategic game)

A strategic game is a tuple $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where

- a nonempty finite set N of players,
- for each player $i \in N$, a nonempty set A_i of actions (or strategies), and
- for each player $i \in N$, a payoff function $u_i : A \to \mathbb{R}$, where $A = \prod_{i \in N} A_i$.

A strategic game G is called finite if A is finite.

A strategy profile is a tuple $a = (a_1, \ldots, a_{|N|}) \in A$.

Strategic Games

We can describe finite strategic games using payoff matrices.

Example: Two-player game where player 1 has actions T and B, and player 2 has actions L and R, with payoff matrix

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\text{player 2}} \\
 & & L & R \\
\hline
\text{player 1} & T & w_1, w_2 & x_1, x_2 \\
\hline
 & B & y_1, y_2 & z_1, z_2 \\
\hline
\end{array}
$$

Read: **If** player 1 plays T and player 2 plays L
then player 1 gets payoff w_1 and player 2 gets payoff w_2, etc.

Prisoner's Dilemma

Example (Prisoner's Dilemma (informally))

Two prisoners are interrogated separately, and have the options to either cooperate (C) with their fellow prisoner and stay silent, or defect (D) and accuse the fellow prisoner of the crime.

Possible outcomes:

- Both cooperate: no hard evidence against either of them, only short prison sentences for both.
- One cooperates, the other defects: the defecting prisoner is set free immediately, and the cooperating prisoner gets a very long prison sentence.
- Both confess: both get medium-length prison sentences.

Prisoner's Dilemma

Example (Prisoner's Dilemma (payoff matrix))

Strategies $A_1 = A_2 = \{C, D\}$.

		player 2: C	player 2: D
player 1	C	$3,3$	$0,4$
player 1	D	$4,0$	$1,1$

Hawk and Dove

An anti-coordination game:

Example (Hawk and Dove (informally))

In a fight for resources two players can behave either like a dove (*D*), yielding, or like a hawk (*H*), attacking.

Possible outcomes:

- Both players behave like doves: both players share the benefit.
- A hawk meets a dove: the hawk wins and gets the bigger part.
- Both players behave like hawks: the benefit gets lost completely because they will fight each other.

Example (Hawk and Dove (payoff matrix))

Strategies $A_1 = A_2 = \{D, H\}$.

player 2

		D	H
player 1	D	$3,3$	$1,4$
	H	$4,1$	$0,0$

Matching Pennies

A strictly competitive game:

Example (Matching Pennies (informally))

Two players can choose either heads (H) or tails (T) of a coin.

Possible outcomes:

- Both players make the same choice: player 1 receives one Euro from player 2.
- The players make different choices: player 2 receives one Euro from player 1.

Matching Pennies

Example (Matching Pennies (payoff matrix))

Strategies $A_1 = A_2 = \{H, T\}$.

		player 2: H	player 2: T
player 1	H	$1, -1$	$-1, 1$
player 1	T	$-1, 1$	$1, -1$

Bach or Stravinsky (aka Battle of the Sexes)

A coordination game:

Example (Bach or Stravinsky (informally))

Two persons, one of whom prefers Bach whereas the other prefers Stravinsky want to go to a concert together. For both it is more important to go to the same concert than to go to their favorite one. Let B be the action of going to the Bach concert and S the action of going to the Stravinsky concert.

Possible outcomes:

- Both players make the same choice: the player whose preferred option is chosen gets high payoff, the other player gets medium payoff.
- The players make different choices: they both get zero payoff.

Bach or Stravinsky (aka Battle of the Sexes)

Example (Bach or Stravinsky (payoff matrix))

Strategies $A_1 = A_2 = \{B, S\}$.

		Stravinsky enthusiast	
		B	S
Bach enthusiast	B	$2,1$	$0,0$
	S	$0,0$	$1,2$

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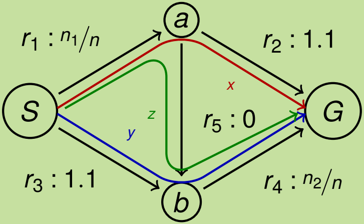
Example (A congestion game)



		player 2		
		x	y	z
player 1	x	$-2.1, -2.1$	$-1.6, -1.6$	$-2.1, -1.5$
	y	$-1.6, -1.6$	$-2.1, -2.1$	$-2.1, -1.5$
	z	$-1.5, -2.1$	$-1.5, -2.1$	$-2, -2$

Notation

We want to write down strategy profiles where one player's strategy is removed or replaced.

Let $a = (a_1, \ldots, a_{|N|}) \in A = \prod_{i \in N} A_i$ be a strategy profile.

We write:

- $A_{-i} := \prod_{j \in N \setminus \{i\}} A_j$,
- $a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{|N|})$, and
- $(a_{-i}, a'_i) := (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_{|N|})$.

Example

Let $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, $A_3 = \{X, Y, Z\}$, and $a := (T, R, Z)$.
Then $a_{-1} = (R, Z)$, $a_{-2} = (T, Z)$, $a_{-3} = (T, R)$.
Moreover, $(a_{-2}, L) = (T, L, Z)$.

Game Theory

2. Strategic Games

2.2. Strict Dominance

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Solution Concepts

Question: What is a “solution” of a strategic game?

Answer:

- a strategy profile where all players play strategies that are rational (i. e., in some sense optimal)
- note: different ways of making the above item precise (different solution concepts)
- solution concept: formal rule for predicting how a game will be played

In the following, we will consider some solution concepts:

- strict and weak dominance
- Nash equilibria
- maximinimizers

Strictly Dominated Strategies

Question: What strategy should an agent avoid?

One answer: obviously irrational strategies (can be eliminated)

A strategy is obviously irrational if there is another strategy that is always better, no matter what the other players do.

Strictly Dominated Strategies

Definition (Strictly dominated strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A strategy $a_i \in A_i$ is called strictly dominated in G if there is a strategy $a_i^+ \in A_i$ such that for all strategy profiles $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that a_i^+ strictly dominates a_i.

If $a_i^+ \in A_i$ strictly dominates every other strategy $a_i' \in A_i \setminus \{a_i^+\}$, we call a_i^+ strictly dominant in G.

Remark: Playing strictly dominated strategies is irrational.

Strictly Dominated Strategies

This suggest a solution concept:
iterative elimination of strictly dominated strategies:

> **while** some strictly dominated strategy is left:
>
> > eliminate some strictly dominated strategy
>
> **if** a unique strategy profile remains:
>
> > this unique profile is the solution

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

player 2

		C	D
player 1	C	$3,3$	$0,4$
	D	$4,0$	$1,1$

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

player 2

		C	D
player 1	C	$3, 3$	$0, 4$
	D	$4, 0$	$1, 1$

- Step 1: eliminate row C (strictly dominated by row D)
- Step 2: eliminate column C (strictly dominated by col. D)

Strictly Dominated Strategies

Example (Iterative elim. of strictly dominated strategies)

player 2

		L	R
player 1	T	$2, 1$	$0, 0$
	M	$1, 2$	$2, 1$
	B	$0, 0$	$1, 1$

Strictly Dominated Strategies

Example (Iterative elim. of strictly dominated strategies)

player 2

		L	R
player 1	T	$2,1$	$0,0$
	M	$1,2$	$2,1$
	B	$0,0$	$1,1$

- Step 1: eliminate row B (strictly dominated by row M)
- Step 2: eliminate column R (strictly dominated by col. L)
- Step 3: eliminate row M (strictly dominated by row T)

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

Stravinsky enthusiast

		B	S
Bach enthusiast	B	$2,1$	$0,0$
	S	$0,0$	$1,2$

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

Stravinsky enthusiast

		B	S
Bach enthusiast	B	$2,1$	$0,0$
	S	$0,0$	$1,2$

- No strictly dominated strategies.
- All strategies survive iterative elimination of strictly dominated strategies.
- All strategies rationalizable.

Strictly Dominated Strategies

Remark

Strict dominance between actions is rather rare.
We should identify more constraints on “solutions”, better solution concepts.

Proposition

The result of iterative elimination of strictly dominated strategies is unique, i. e., independent of the elimination order.

Proof.

Homework. □

Game Theory

2. Strategic Games

2.3. Weak Dominance

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Solution Concepts

Question: What is a “solution” of a strategic game?

Answer:

- a strategy profile where all players play strategies that are rational (i. e., in some sense optimal)
- note: different ways of making the above item precise (different solution concepts)
- solution concept: formal rule for predicting how a game will be played

In the following, we will consider some solution concepts:

- strict and weak dominance
- Nash equilibria
- maximinimizers

Weakly Dominated Strategies

Definition (Weakly dominated strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A strategy $a_i \in A_i$ is called weakly dominated in G if there is a strategy $a_i^+ \in A_i$ such that for all profiles $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) \leq u_i(a_{-i}, a_i^+)$$

and that for at least one profile $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that a_i^+ weakly dominates a_i.

If $a_i^+ \in A_i$ weakly dominates every other strategy $a_i' \in A_i \setminus \{a_i^+\}$, we call a_i^+ weakly dominant in G.

Weakly Dominated Strategies

What about
iterative elimination of weakly dominated strategies
as a solution concept?

Let's see what happens.

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

player 2

		L	R
	T	$2, 1$	$0, 0$
player 1	M	$2, 1$	$1, 1$
	B	$0, 0$	$1, 1$

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

player 2

		L	R
player 1	T	$2,1$	$0,0$
	M	$2,1$	$1,1$
	B	$0,0$	$1,1$

- Step 1: eliminate row B (weakly dominated by row M, $u_1(M,L) = 2 > 0 = u_1(B,L)$ and $u_1(M,R) = 1 = u_1(B,R)$)
- Step 2: eliminate column R (weakly dominated by col. L)

Here, two solution profiles remain.

Weakly Dominated Strategies

Iterative elimination of weakly dominated strategies:

- leads to smaller games,
- can also lead to situations where only a single solution remains,
- but: the result can depend on the elimination order! (see example on next slide)

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

player 2

		L	R
	T	$2, 1$	$0, 0$
player 1	M	$2, 1$	$1, 1$
	B	$0, 0$	$1, 1$

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

player 2

		L	R
	T	$2, 1$	$0, 0$
player 1	M	$2, 1$	$1, 1$
	B	$0, 0$	$1, 1$

- Step 1: eliminate row T (weakly dominated by row M)
- Step 2: eliminate column L (weakly dominated by col. R)

Different elimination order, different result,
even different payoffs ($1, 1$ vs. $2, 1$)!

Weakly Dominated Strategies

Consequence:
Iterative elimination of weakly dominated strategies not such a useful solution concept.

Let's look for something more useful.

Game Theory

2. Strategic Games

2.4. Nash Equilibria

2.4.1. Definitions and Examples

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Solution Concepts

Question: What is a “solution” of a strategic game?

Answer:

- a strategy profile where all players play strategies that are rational (i. e., in some sense optimal)
- note: different ways of making the above item precise (different solution concepts)
- solution concept: formal rule for predicting how a game will be played

In the following, we will consider some solution concepts:

- strict and weak dominance
- Nash equilibria
- maximinimizers

Nash Equilibria

Question: Which strategy profiles are stable?

Possible answer:

- strategy profiles where no player benefits from playing a different strategy
- equivalently: strategy profiles where every player's strategy is a best response to the other players' strategies

Such strategy profiles are called Nash equilibria, one of the most-used solution concepts in game theory.

Remark: In following examples, for non-Nash equilibria, only one possible profitable deviation is shown (even if there are more).

Definition (Nash equilibrium)

A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a strategy profile $a^* \in A$ such that for every player $i \in N$,

$$u_i(a^*) \geq u_i(a^*_{-i}, a_i) \qquad \text{for all } a_i \in A_i.$$

Nash Equilibria

Remark: There is an alternative definition of Nash equilibria (which we consider because it gives us a slightly different perspective on Nash equilibria).

Definition (Best response)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game, $i \in N$ a player, and $a_{-i} \in A_{-i}$ a strategy profile of the players other than i. Then a strategy $a_i \in A_i$ is a best response of player i to a_{-i} if

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a_i') \qquad \text{for all } a_i' \in A_i.$$

We write $B_i(a_{-i})$ for the set of best responses of player i to a_{-i}.

For a strategy profile $a \in A$, we write $B(a) = \prod_{i \in N} B_i(a_{-i})$.

Nash Equilibria

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Definition (Nash equilibrium, alternative 1)

A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that for every player $i \in N$, $a_i^* \in B_i(a_{-i}^*)$.

Definition (Nash equilibrium, alternative 2)

A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that $a^* \in B(a^*)$.

Proposition

The three definitions of Nash equilibria are equivalent.

Proof.

Homework. $\square$

Nash Equilibria

Example (Nash Equilibria in the Prisoner's Dilemma)

		player 2 C	player 2 D
player 1	C	3,3	0,4
player 1	D	4,0	1,1

- (C,C): no Nash equilibrium (player 1: $C \to D$)
- (C,D): no Nash equilibrium (player 1: $C \to D$)
- (D,C): no Nash equilibrium (player 2: $C \to D$)
- (D,D): Nash equilibrium!

Nash Equilibria

Example (Nash Equilibria in Hawk and Dove)

player 2

		D	H
player 1	D	$3,3$	$1,4$
	H	$4,1$	$0,0$

- (D, D): no Nash equilibrium (player 1: $D \to H$)
- (D, H): Nash equilibrium!
- (H, D): Nash equilibrium!
- (H, H): no Nash equilibrium (player 1: $H \to D$)

Nash Equilibria

Example (Nash Equilibria in Matching Pennies)

player 2

		H	T
player 1	H	$1,-1$	$-1,\ \ 1$
	T	$-1,\ \ 1$	$1,-1$

- (H,H): no Nash equilibrium (player 2: $H \to T$)
- (H,T): no Nash equilibrium (player 1: $H \to T$)
- (T,H): no Nash equilibrium (player 1: $T \to H$)
- (T,T): no Nash equilibrium (player 2: $T \to H$)

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Example (Nash Equilibria in Bach or Stravinsky)

Stravinsky enthusiast

Bach enthusiast		B	S
	B	$2,1$	$0,0$
	S	$0,0$	$1,2$

- (B,B): Nash equilibrium!
- (B,S): no Nash equilibrium (player 1: $B \rightarrow S$)
- (S,B): no Nash equilibrium (player 2: $S \rightarrow B$)
- (S,S): Nash equilibrium!

Game Theory

2. Strategic Games

2.4. Nash Equilibria

2.4.2. Example: NEs in Sealed-bid Auctions

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Example: Sealed-Bid Auctions

We consider a slightly larger example: sealed-bid auctions

Setting:

- An object has to be assigned to a winning bidder in exchange for a payment.
- For each player (“bidder”) $i = 1, \dots, n$, let v_i be the private value that bidder i assigns to the object.
  (We assume that $v_1 > v_2 > \cdots > v_n > 0$.)
- The bidders simultaneously give their bids $b_i \geq 0$, $i = 1, \dots, n$.
- The object is given to the bidder i with the highest bid b_i.
  (Ties are broken in favor of bidders with lower index, i.e., if $b_i = b_j$ are the highest bids, then bidder i will win iff $i < j$.)

Example: Sealed-Bid Auctions

Question: What should the winning bidder have to pay?

One possible answer: the highest bid.

Definition (First-price sealed-bid auction)

- $N = \{1, \dots, n\}$ with $v_1 > v_2 > \dots > v_n > 0$,
- $A_i = \mathbb{R}_0^+$ for all $i \in N$,
- Bidder $i \in N$ wins if b_i is maximal among all bids (+ possible tie-breaking by index), and
- $u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - b_i & \text{otherwise} \end{cases}$
  where $b = (b_1, \dots, b_n)$.

Example: Sealed-Bid Auctions

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Example (First-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids

$$v_1 = 100, \qquad v_2 = 80, \qquad v_3 = 53,$$
$$b_1 = 90, \qquad b_2 = 85, \qquad b_3 = 45.$$

Observations:

- Bidder 1 wins, pays 90, gets utility
  $u_1(b) = v_1 - b_1 = 100 - 90 = 10.$
- Bidders 2 and 3 pay nothing, get utility 0.
- (Bidder 2 over-bids.)
- Bidder 1 could still win, but pay less, by bidding $b'_1 = 85$ instead. Then $u_1(b_{-1}, b'_1) = v_1 - b'_1 = 100 - 85 = 15.$

Example: Sealed-Bid Auctions

Question: How to avoid untruthful bidding and incentivize truthful revelation of private valuations?

Different answer to question about payments: Winner pays the second-highest bid.

Definition (Second-price sealed-bid auction)

- $N = \{1, \dots, n\}$ with $v_1 > v_2 > \dots > v_n > 0$,
- $A_i = \mathbb{R}_0^+$ for all $i \in N$,
- Bidder $i \in N$ wins if b_i is maximal among all bids (+ possible tie-breaking by index), and
- $u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - \max b_{-i} & \text{otherwise} \end{cases}$
  where $b = (b_1, \dots, b_n)$.

Example: Sealed-Bid Auctions

Example (Second-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids

$$v_1 = 100, \qquad v_2 = 80, \qquad v_3 = 53,$$
$$b_1 = 90, \qquad b_2 = 85, \qquad b_3 = 45.$$

Observations:

- Bidder 1 wins, pays 85, gets utility
  $u_1(b) = v_1 - b_2 = 100 - 85 = 15.$
- Bidders 2 and 3 pay nothing, get utility 0.
- Bidder 1 has no incentive to bid strategically and guess the other bidders' private valuations.

Example: Sealed-Bid Auctions

Proposition

In a second-price sealed-bid auction, bidding ones own valuation, $b_i^+ = v_i$, is a weakly dominant strategy.

Example: Sealed-Bid Auctions

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Proof.

We have to show that b_i^+ weakly dominates every other strategy b_i of player i.

Example: Sealed-Bid Auctions

Proposition

In a second-price sealed-bid auction, bidding ones own valuation, $b_i^+ = v_i$, is a weakly dominant strategy.

Proof.

We have to show that b_i^+ weakly dominates every other strategy b_i of player i.

For that, it suffices to show that

1. for all $b_i \in A_i$, we have
$$u_i(b_{-i}, b_i^+) \geq u_i(b_{-i}, b_i) \text{ for all } b_{-i} \in A_{-i}, \text{ and that}$$
2. for all $b_i \in A_i \setminus \{b_i^+\}$, we have
$$u_i(b_{-i}, b_i^+) > u_i(b_{-i}, b_i) \text{ for at least one } b_{-i} \in A_{-i}.$$

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (1) [regardless of what the other bidders do,
b_i^+ is always a best response]:

- Case I) bidder i wins:

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Proof (ctd.)

Ad (1) [regardless of what the other bidders do,
b_i^+ is always a best response]:

- Case I) bidder i wins:
  bidder i pays $\max b_{-i} \leq v_i$, gets $u_i(b_{-i}, b_i^+) \geq 0$.
  - Case I.a) bidder i decreases bid:
    this does not help, since he might still win and pay the same as before, or lose and get utility 0.
  - Case I.b) bidder i increases bid:
    bidder i still wins and pays the same as before.

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (1) (ctd.):

- Case II) bidder *i* loses:

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (1) (ctd.):

- Case II) bidder i loses:
  bidder i pays nothing, gets $u_i(b_{-i}, b_i^+) = 0$.
  - Case II.a) bidder i decreases bid:
    bidder i still loses and gets utility 0.
  - Case II.b) bidder i increases bid:
    either bidder i still loses and gets utility 0, or becomes the winner and pays more than the object is worth to him, leading to a negative utility.

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (2) [for each alternative b_i to b_i^+, there is an opponent profile b_{-i} against which b_i^+ is strictly better than b_i]:

Let b_i be some strategy other than b_i^+.

- Case I) $b_i < b_i^+$:

□

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (2) [for each alternative b_i to b_i^+, there is an opponent profile b_{-i} against which b_i^+ is strictly better than b_i]:

Let b_i be some strategy other than b_i^+.

- Case I) $b_i < b_i^+$:
  Consider b_{-i} with $b_i < \max b_{-i} < b_i^+$.
  With b_i, bidder i does not win any more, i. e., we have $u_i(b_{-i}, b_i^+) > 0 = u_i(b_{-i}, b_i)$.

$\square$

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Proof (ctd.)

Ad (2) (ctd.):

Let b_i be some strategy other than b_i^+.

- Case II) $b_i > b_i^+$:

□

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (2) (ctd.):

Let b_i be some strategy other than b_i^+.

- Case II) $b_i > b_i^+$:
  Consider b_{-i} with $b_i > \max b_{-i} > b_i^+$.
  With b_i, bidder i overbids and pays more than the object is worth to him, i. e., we have $u_i(b_{-i}, b_i^+) = 0 > u_i(b_{-i}, b_i)$.

$\square$

Example: Sealed-Bid Auctions

Proposition

Profiles of weakly dominant strategies are Nash equilibria.

Proof.

Homework. □

Proposition

In a second-price sealed-bid auction, if all bidders bid their true valuations, this is a Nash equilibrium.

Proof.

Follows immediately from the previous two propositions. □

Remark: This is not the only Nash equilibrium in second-price sealed-bid auctions, though.

Game Theory

2. Strategic Games

2.5. Strict Dominance vs. Nash Equilibria

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Summer semester 2020

Iterative Elimination and Nash Equilibria

Motivation: We have seen two different solution concepts,

- surviving iterative elimination of strictly dominated strategies
- Nash equilibria

Obvious question: Is there any relationship between the two?

Answer: Yes, Nash equilibria refine the concept of iterative elimination of strictly dominated strategies. We will formalize this on the next slides.

Iterative Elimination and Nash Equilibria

Lemma (preservation of Nash equilibria)

Let G and G' be two strategic games where G' is obtained from G by elimination of one strictly dominated strategy. Then a strategy profile a^* is a Nash equilibrium of G if and only if it is Nash equilibrium of G'.

Proof.

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ and $G' = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$.

Let a'_i be the eliminated strategy.

Then there is a strategy a^+_i such that for all $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a'_i) < u_i(a_{-i}, a^+_i). \tag{1}$$

Proof (ctd.)

"$\Rightarrow$": Let a^* be a Nash equilibrium of G.

- Nash equilibrium strategies are not eliminated: For players $j \neq i$, this is clear, because none of their strategies are eliminated.

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Proof (ctd.)

"$\Rightarrow$": Let a^* be a Nash equilibrium of G.

- Nash equilibrium strategies are not eliminated: For players $j \neq i$, this is clear, because none of their strategies are eliminated.

  For player i, action a_i^* is a best response to a_{-i}^*, and in particular at least as good a response as a_i^+:

$$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+).$$

Iterative Elimination and Nash Equilibria

Proof (ctd.)

"$\Rightarrow$": Let a^* be a Nash equilibrium of G.

- Nash equilibrium strategies are not eliminated: For players $j \neq i$, this is clear, because none of their strategies are eliminated.

  For player i, action a_i^* is a best response to a_{-i}^*, and in particular at least as good a response as a_i^+:

  $$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+).$$

  With (1) $u_i(a_{-i}, a_i^+) > u_i(a_{-i}, a_i')$, we get $u_i(a_{-i}^*, a_i^*) > u_i(a_{-i}^*, a_i')$ and hence $a_i^* \neq a_i'$.

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Proof (ctd.)

"$\Rightarrow$": Let a^* be a Nash equilibrium of G.

- Nash equilibrium strategies are not eliminated: For players $j \neq i$, this is clear, because none of their strategies are eliminated.

  For player i, action a_i^* is a best response to a_{-i}^*, and in particular at least as good a response as a_i^+:

  $$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+).$$

  With (1) $u_i(a_{-i}, a_i^+) > u_i(a_{-i}, a_i')$, we get $u_i(a_{-i}^*, a_i^*) > u_i(a_{-i}^*, a_i')$ and hence $a_i^* \neq a_i'$.

  Thus, the Nash equilibrium strategy a_i^* is not eliminated.

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Proof (ctd.)

"$\Rightarrow$" (ctd.):

- Best responses remain best responses: For all players $j \in N$, a_j^* is a best response to a_{-j}^* in G. Since in G', no potentially better responses are introduced ($A_j' \subseteq A_j$) and the payoffs are unchanged, this also holds in G'.

Hence, a^* is also a Nash equilibrium of G'.

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Proof (ctd.)

"$\Rightarrow$" (ctd.):

- Best responses remain best responses: For all players $j \in N$, a_j^* is a best response to a_{-j}^* in G. Since in G', no potentially better responses are introduced ($A_j' \subseteq A_j$) and the payoffs are unchanged, this also holds in G'.

Hence, a^* is also a Nash equilibrium of G'.

"$\Leftarrow$": Let a^* be a Nash equilibrium of G'.

- For player $j \neq i$: a_j^* is a best response to a_{-j}^* in G as well, since the responses available to player j in G and G' are the same.

Iterative Elimination and Nash Equilibria

Proof (ctd.)

"$\Leftarrow$" (ctd.):

- For player i: Since $A_i = A'_i \cup \{a_i\}$ and a^*_i is a best response to a^*_{-i} among the strategies in A'_i, it suffices to show that a_i is no better response.

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Proof (ctd.)

"$\Leftarrow$" (ctd.):

- For player i: Since $A_i = A'_i \cup \{a_i\}$ and a^*_i is a best response to a^*_{-i} among the strategies in A'_i, it suffices to show that a_i is no better response.

  Because a^* is a Nash equilibrium in G' and a^+_i is a strategy in A'_i, we have $u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a^+_i)$.

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Proof (ctd.)

"$\Leftarrow$" (ctd.):

- For player i: Since $A_i = A'_i \cup \{a_i\}$ and a^*_i is a best response to a^*_{-i} among the strategies in A'_i, it suffices to show that a_i is no better response.

  Because a^* is a Nash equilibrium in G' and a^+_i is a strategy in A'_i, we have $u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a^+_i)$.

  Since a^+_i strictly dominates a_i, we have $u_i(a^*_{-i}, a^+_i) > u_i(a^*_{-i}, a_i)$, and hence $u_i(a^*_{-i}, a^*_i) > u_i(a^*_{-i}, a_i)$.

Iterative Elimination and Nash Equilibria

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Proof (ctd.)

"$\Leftarrow$" (ctd.):

- For player i: Since $A_i = A'_i \cup \{a_i\}$ and a^*_i is a best response to a^*_{-i} among the strategies in A'_i, it suffices to show that a_i is no better response.

  Because a^* is a Nash equilibrium in G' and a^+_i is a strategy in A'_i, we have $u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a^+_i)$.

  Since a^+_i strictly dominates a_i, we have $u_i(a^*_{-i}, a^+_i) > u_i(a^*_{-i}, a_i)$, and hence $u_i(a^*_{-i}, a^*_i) > u_i(a^*_{-i}, a_i)$.

  Therefore, a_i cannot be a better response to a^*_{-i} than a^*_i.

Hence, a^* is also a Nash equilibrium of G. $\square$

Iterative Elimination and Nash Equilibria

Corollary

If iterative elimination of strictly dominated strategies results in a *unique* strategy profile a^*, then a^* is the unique Nash equilibrium of the original game.

Proof.

Assume that a^* is the unique remaining strategy profile. By definition, a^* must be a Nash equilibrium of the remaining game.

Iterative Elimination and Nash Equilibria

Corollary

If iterative elimination of strictly dominated strategies results in a *unique* strategy profile a^*, then a^* is the unique Nash equilibrium of the original game.

Proof.

Assume that a^* is the unique remaining strategy profile. By definition, a^* must be a Nash equilibrium of the remaining game.

We can inductively apply the previous lemma (preservation of Nash equilibria) and see that a^* (and no other strategy profile) must have been a Nash equilibrium before the last elimination step, and before that step, . . . , and in the original game. □

Game Theory

2. Strategic Games

2.6. Zero-Sum Games

2.6.1. Definition, Examples, Maximinimizers

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Summer semester 2020

Playing it Safe (in Two-Player Games)

Motivation: What happens if both players try to “play it safe”?

Question: What does it even mean to “play it safe”?

Answer: Choose a strategy that guarantees the highest worst-case payoff.

Example

player 2

		L	R
	T	$2, 1$	$2, -20$
player 1	M	$3, 0$	$-10, 1$
	B	$-100, 2$	$3, 3$

Playing it Safe (in Two-Player Games)

Example

		player 2	
		L	R
	T	$2, 1$	$2, -20$
player 1	M	$3, 0$	$-10, \quad 1$
	B	$-100, 2$	$3, \quad 3$

Worst-case payoff for player 1:

- if playing T: 2
- if playing M: -10
- if playing B: -100

$\rightsquigarrow$ play T.

Worst-case payoff for player 2:

- if playing L: 0
- if playing R: -20

$\rightsquigarrow$ play L.

Playing it Safe (in Two-Player Games)

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Example

		player 2	
		L	R
player 1	T	$2,1$	$2,-20$
	M	$3,0$	$-10,\quad 1$
	B	$-100,2$	$3,\quad 3$

Worst-case payoff for player 1:

- if playing T: 2
- if playing M: -10
- if playing B: -100

$\rightsquigarrow$ play T.

Worst-case payoff for player 2:

- if playing L: 0
- if playing R: -20

$\rightsquigarrow$ play L.

However: Unlike (B,R), the profile (T,L) is not a Nash equilibrium.

Playing it Safe (in Two-Player Games)

Observation: In general, pairs of maximinimizers, like (T, L) in the example above, are not the same as Nash equilibria.

Claim: However, in zero-sum games, pairs of maximinimizers and Nash equilibria are essentially the same.

(Tiny restriction: This does not hold if the considered game has no Nash equilibrium at all, because unlike Nash equilibria, pairs of maximinimizers always exist.)

Reason (intuitively): In zero-sum games, the worst-case assumption that the other player tries to harm you as much as possible is justified, because harming the other is the same as maximizing ones own payoff. Playing it safe is rational.

Zero-Sum Games

Definition (Zero-sum game)

A zero-sum game is a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and

$$u_1(a) = -u_2(a)$$

for all $a \in A$.

Example (Matching Pennies as a zero-sum game)

		player 2: H	player 2: T
player 1	H	$1, -1$	$-1, 1$
player 1	T	$-1, 1$	$1, -1$

Maximinimizers

Definition (Maximinimizer)

Let $G = \langle \{1,2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a zero-sum game.

An action $x^* \in A_1$ is called maximinimizer for player 1 in G if

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y) \qquad \text{for all } x \in A_1,$$

and $y^* \in A_2$ is called maximinimizer for player 2 in G if

$$\min_{x \in A_1} u_2(x, y^*) \geq \min_{x \in A_1} u_2(x, y) \qquad \text{for all } y \in A_2.$$

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Example (Zero-sum game with three actions each)

player 2

		L	C	R
	T	$8, -8$	$3, -3$	$-6, \ \ 6$
player 1	M	$2, -2$	$-1, \ \ 1$	$3, -3$
	B	$-6, \ \ 6$	$4, -4$	$8, -8$

Maximinimizers

Example (Zero-sum game with three actions each)

player 2

		L	C	R
player 1	T	$8, -8$	$3, -3$	$-6, \quad 6$
	M	$2, -2$	$-1, \quad 1$	$3, -3$
	B	$-6, \quad 6$	$4, -4$	$8, -8$

Guaranteed worst-case payoffs:

- T: -6, M: -1, B: -6 $\rightsquigarrow$ maximinimizer M
- L: -8, C: -4, R: -8 $\rightsquigarrow$ maximinimizer C

$\rightsquigarrow$ pair of maximinimizers (M, C) with payoffs $(-1, 1)$
(not a Nash equilibrium; this game has no Nash equilibrium.)

Maximinimizers

Example (Maximinimization vs. minimaximization)

		player 2 L	player 2 R
player 1	T	$1, -1$	$2, -2$
player 1	B	$-2, \ 2$	$-4, \ 4$

Worst-case payoffs (player 2):

- L: -1, R: -2
- Maximize: -1

Best-case payoffs (player 1):

- L: $+1$, R: $+2$
- Minimize: $+1$

Observation: Results identical up to different sign.

Maximinimizers

Lemma

Let $G = \langle\{1,2\},(A_i)_{i\in N},(u_i)_{i\in N}\rangle$ *be a zero-sum game. Then*

$$\max_{y\in A_2}\min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2}\max_{x\in A_1} u_1(x,y). \tag{1}$$

Proof.

For any real-valued function f, we have

$$\min_z -f(z) = -\max_z f(z). \tag{2}$$

Maximinimizers

Proof (ctd.)

Thus, for all $y \in A_2$,

$$
\begin{aligned}
-\min_{y\in A_2}\max_{x\in A_1} u_1(x,y) &\overset{(2)}{=} \max_{y\in A_2} -\max_{x\in A_1} u_1(x,y) \\
&\overset{(2)}{=} \max_{y\in A_2}\min_{x\in A_1} -u_1(x,y) \\
&\overset{\mathrm{ZS}}{=} \max_{y\in A_2}\min_{x\in A_1} u_2(x,y).
\end{aligned}
$$

□

Game Theory

2. Strategic Games

2.6. Zero-Sum Games

2.6.2. Nash-Equilibria vs. Maximinimizers

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Summer semester 2020

Zero-Sum Games

Recall:

Definition (Zero-sum game)

A zero-sum game is a strategic game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ with $N = \{1,2\}$ and

$$u_1(a) = -u_2(a)$$

for all $a \in A$.

Lemma

Let $G = \langle \{1,2\}, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a zero-sum game. Then

$$\max_{y\in A_2}\min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2}\max_{x\in A_1} u_1(x,y). \tag{1}$$

Nash Equilibria in Zero-Sum Games

Now, we are ready to prove our
main theorem about zero-sum games and Nash equilibria.

In zero-sum games:

1. Every Nash equilibrium is a pair of maximinimizers.
2. All Nash equilibria have the same payoffs.
3. If there is at least one Nash equilibrium, then every pair of maximinimizers is a Nash equilibrium.

Nash Equilibria in Zero-Sum Games

Theorem (Maximinimizer theorem)

Let $G = \langle \{1,2\}, (A_i)_{i\in N}, (u_i)_{i\in N} \rangle$ be a zero-sum game. Then:

1. *If (x^*, y^*) is a Nash equilibrium of G, then x^* and y^* are maximinimizers for player 1 and player 2, respectively.*
2. *If (x^*, y^*) is a Nash equilibrium of G, then*

$$\max_{x\in A_1} \min_{y\in A_2} u_1(x,y) = \min_{y\in A_2} \max_{x\in A_1} u_1(x,y) = u_1(x^*, y^*).$$

3. *If* $\max_{x\in A_1} \min_{y\in A_2} u_1(x,y) = \min_{y\in A_2} \max_{x\in A_1} u_1(x,y)$, *and x^* and y^* are maximinimizers of player 1 and player 2 respectively, then (x^*, y^*) is a Nash equilibrium.*

Nash Equilibria in Zero-Sum Games

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Proof.

1. Let (x^*, y^*) be a Nash equilibrium. Then

$$u_2(x^*, y^*) \geq u_2(x^*, y) \quad \text{for all } y \in A_2.$$

Nash Equilibria in Zero-Sum Games

Proof.

1. Let (x^*, y^*) be a Nash equilibrium. Then

$$u_2(x^*, y^*) \geq u_2(x^*, y) \quad \text{for all } y \in A_2.$$

With $u_1 = -u_2$, this implies

$$u_1(x^*, y^*) \leq u_1(x^*, y) \quad \text{for all } y \in A_2.$$

Nash Equilibria in Zero-Sum Games

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Proof.

1. Let (x^*, y^*) be a Nash equilibrium. Then

$$u_2(x^*, y^*) \geq u_2(x^*, y) \quad \text{for all } y \in A_2.$$

With $u_1 = -u_2$, this implies

$$u_1(x^*, y^*) \leq u_1(x^*, y) \quad \text{for all } y \in A_2.$$

Thus

$$u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y) \leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{2}$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

1. (ctd.)
   Furthermore, since (x^*, y^*) is a Nash equilibrium, also

$$u_1(x^*, y^*) \geq u_1(x, y^*) \quad \text{for all } x \in A_1.$$

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Proof (ctd.)

1. (ctd.)
   Furthermore, since (x^*, y^*) is a Nash equilibrium, also

$$u_1(x^*, y^*) \geq u_1(x, y^*) \quad \text{for all } x \in A_1.$$

Hence

$$u_1(x^*, y^*) \geq \max_{x \in A_1} u_1(x, y^*).$$

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Proof (ctd.)

1. (ctd.)

   Furthermore, since (x^*, y^*) is a Nash equilibrium, also

   $$u_1(x^*, y^*) \geq u_1(x, y^*) \quad \text{for all } x \in A_1.$$

   Hence

   $$u_1(x^*, y^*) \geq \max_{x \in A_1} u_1(x, y^*).$$

   This implies

   $$u_1(x^*, y^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{3}$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

1. (ctd.)
   Inequalities (2) and (3) together imply that

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{4}$$

   Thus, x^* is a maximinimizer for player 1.

Nash Equilibria in Zero-Sum Games

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Proof (ctd.)

1. (ctd.)
   Inequalities (2) and (3) together imply that

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{4}$$

   Thus, x^* is a maximinimizer for player 1.

   Similarly, we can show that y^* is a maximinimizer for player 2:

$$u_2(x^*, y^*) = \max_{y \in A_2} \min_{x \in A_1} u_2(x, y). \tag{5}$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

2. We only need to put things together:

$$
\begin{aligned}
\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) &\overset{(4)}{=} u_1(x^*,y^*) \\
&\overset{\text{ZS}}{=} -u_2(x^*,y^*) \\
&\overset{(5)}{=} -\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) \\
&\overset{(1)}{=} \min_{y \in A_2} \max_{x \in A_1} u_1(x,y).
\end{aligned}
$$

In particular, it follows that all Nash equilibria share the same payoff profile.

Nash Equilibria in Zero-Sum Games

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Proof (ctd.)

3. Let x^* and y^* be maximinimizers for player 1 and 2, respectively, and assume that

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) =: v^*. \tag{6}$$

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Proof (ctd.)

3 Let x^* and y^* be maximinimizers for player 1 and 2, respectively, and assume that

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) =: v^*. \tag{6}$$

With Equation (1) from the previous lemma, we get

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -v^*. \tag{7}$$

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Proof (ctd.)

3 Let x^* and y^* be maximinimizers for player 1 and 2, respectively, and assume that

$$\max_{x\in A_1} \min_{y\in A_2} u_1(x,y) = \min_{y\in A_2} \max_{x\in A_1} u_1(x,y) =: v^*. \tag{6}$$

With Equation (1) from the previous lemma, we get

$$\max_{y\in A_2} \min_{x\in A_1} u_2(x,y) = -v^*. \tag{7}$$

With x^* and y^* being maximinimizers, (6) and (7) imply

$$u_1(x^*,y) \geq v^* \quad \text{for all } y \in A_2, \text{ and} \tag{8}$$

$$u_2(x,y^*) \geq -v^* \quad \text{for all } x \in A_1. \tag{9}$$

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Proof (ctd.)

3. (ctd.)
   Special cases of (8) and (9) for $x = x^*$ and $y = y^*$:

$$u_1(x^*, y^*) \geq v^* \qquad \text{and} \qquad u_2(x^*, y^*) \geq -v^*.$$

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Proof (ctd.)

3 (ctd.)
Special cases of (8) and (9) for $x = x^*$ and $y = y^*$:

$$u_1(x^*, y^*) \geq v^* \qquad \text{and} \qquad u_2(x^*, y^*) \geq -v^*.$$

With $u_1 = -u_2$, the latter is equivalent to $u_1(x^*, y^*) \leq v^*$, which gives us

$$u_1(x^*, y^*) = v^*. \tag{10}$$

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Proof (ctd.)

3. (ctd.)
   Plugging (10) into the right-hand side of (8) gives us

$$u_1(x^*, y) \geq u_1(x^*, y^*) \quad \text{for all } y \in A_2.$$

Nash Equilibria in Zero-Sum Games

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Proof (ctd.)

3. (ctd.)
   Plugging (10) into the right-hand side of (8) gives us

   $$u_1(x^*, y) \geq u_1(x^*, y^*) \quad \text{for all } y \in A_2.$$

   With $u_1 = -u_2$, this is equivalent to

   $$u_2(x^*, y) \leq u_2(x^*, y^*) \quad \text{for all } y \in A_2.$$

   In other words, y^* is a best response to x^*.

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Proof (ctd.)

3. (ctd.)
   Similarly, we can plug (10) into the right-hand side of (9) and obtain

$$u_2(x, y^*) \geq -u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

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Proof (ctd.)

3 (ctd.)
Similarly, we can plug (10) into the right-hand side of (9) and obtain

$$u_2(x, y^*) \geq -u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

Again using $u_1 = -u_2$, this is equivalent to

$$u_1(x, y^*) \leq u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

In words, x^* is also a best response to y^*.

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Proof (ctd.)

3 (ctd.)

Similarly, we can plug (10) into the right-hand side of (9) and obtain

$$u_2(x, y^*) \geq -u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

Again using $u_1 = -u_2$, this is equivalent to

$$u_1(x, y^*) \leq u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

In words, x^* is also a best response to y^*.

Hence, (x^*, y^*) is a Nash equilibrium.

□

Nash Equilibria in Zero-Sum Games

Corollary

Let $G = \langle \{1,2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a zero-sum game, and let (x_1^*, y_1^*) and (x_2^*, y_2^*) be two Nash equilibria of G.

Then (x_1^*, y_2^*) and (x_2^*, y_1^*) are also Nash equilibria of G.

In other words: Nash equilibria of zero-sum games can be arbitrarily recombined.

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Proof.

With part (1) of the maximinimizer theorem, we get that x_1^* and x_2^* are maximinimizers for player 1 and that y_1^* and y_2^* are maximinimizers for player 2.

Nash Equilibria in Zero-Sum Games

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Proof.

With part (1) of the maximinimizer theorem, we get that x_1^* and x_2^* are maximinimizers for player 1 and that y_1^* and y_2^* are maximinimizers for player 2.

With part (2) of the maximinimizer theorem, we get that $\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$.

Nash Equilibria in Zero-Sum Games

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Proof.

With part (1) of the maximinimizer theorem, we get that x_1^* and x_2^* are maximinimizers for player 1 and that y_1^* and y_2^* are maximinimizers for player 2.

With part (2) of the maximinimizer theorem, we get that $\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$.

With this equality, with x_1^*, x_2^*, y_1^*, and y_2^* all being maximinimizers, and with part (3) of the maximinimizer theorem, we get that (x_1^*, y_2^*) and (x_2^*, y_1^*) are also Nash equilibria of G. □

Summary

- In zero-sum games, one player's gain is the other player's loss. Thus, playing it safe is rational. Relevant concept: maximinimizers.
- Relation to Nash equilibria: In zero-sum games, Nash equilibria are pairs of maximinimizers, and, if at least one Nash equilibrium exists, pairs of maximinimizers are also Nash equilibria.
- In zero-sum games, Nash equilibrium strategies can be recombined.

Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.1. Motivation and Definitions

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Mixed Strategies

Observation: Not every strategic game has a pure-strategy Nash equilibrium (e. g. matching pennies).

Question:

- Can we do anything about that?
- Which strategy to play then?

Idea: Consider randomized strategies.

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Notation

Let X be a (finite) set.

Then $\Delta(X)$ denotes the set of probability distributions over X.

That is, each $p \in \Delta(X)$ is a mapping $p : X \to [0, 1]$ with

$$\sum_{x \in X} p(x) = 1.$$

Mixed Strategies

A mixed strategy is a strategy where a player is allowed to randomize his action (throw a dice mentally and then act according to what he has decided to do for each outcome).

Definition (Mixed strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A mixed strategy of player i in G is a probability distribution $\alpha_i \in \Delta(A_i)$ over player i's actions.

For $a_i \in A_i$, $\alpha_i(a_i)$ is the probability for playing a_i.

Terminology: When we talk about strategies in A_i specifically, to distinguish them from mixed strategies, we sometimes also call them pure strategies.

Mixed Strategies

Note: Pure strategies can be seen as a special case of mixed strategies.

Notation

Since each pure strategy $a_i \in A_i$ is equivalent to its induced mixed strategy $\hat{a}_i$

$$\hat{a}_i(a'_i) = \begin{cases} 1 & \text{if } a'_i = a_i \\ 0 & \text{otherwise,} \end{cases}$$

we sometimes abuse notation and write a_i instead of $\hat{a}_i$.

Mixed Strategies

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Definition (Mixed strategy profile)

A profile $\alpha = (\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$ of mixed strategies induces a probability distribution p_α over $A = \prod_{i \in N} A_i$ as follows:

$$p_\alpha(a) = \prod_{i \in N} \alpha_i(a_i).$$

For $A' \subseteq A$, we define

$$p_\alpha(A') = \sum_{a \in A'} p_\alpha(a).$$

Mixed Strategies

Example (Mixed strategies for matching pennies)

	H	T
H	$1, -1$	$-1, \ 1$
T	$-1, \ 1$	$1, -1$

$$\alpha = (\alpha_1, \alpha_2), \ \ \alpha_1(H) = 2/3, \ \ \alpha_1(T) = 1/3, \ \ \alpha_2(H) = 1/3, \ \ \alpha_2(T) = 2/3.$$

This induces a probability distribution over $\{H, T\} \times \{H, T\}$:

$$\begin{aligned} p_\alpha(H,H) &= \alpha_1(H) \cdot \alpha_2(H) = 2/9, & u_1(H,H) &= +1, \\ p_\alpha(H,T) &= \alpha_1(H) \cdot \alpha_2(T) = 4/9, & u_1(H,T) &= -1, \\ p_\alpha(T,H) &= \alpha_1(T) \cdot \alpha_2(H) = 1/9, & u_1(T,H) &= -1, \\ p_\alpha(T,T) &= \alpha_1(T) \cdot \alpha_2(T) = 2/9, & u_1(T,T) &= +1. \end{aligned}$$

Expected Utility

Definition (Expected utility)

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile.

The expected utility of α for player i is

$$U_i(\alpha) = U_i\big((\alpha_j)_{j \in N}\big) := \sum_{a \in A} p_\alpha(a)\, u_i(a) = \sum_{a \in A} \Big(\prod_{j \in N} \alpha_j(a_j)\Big) u_i(a).$$

Example (Mixed strategies for matching pennies (ctd.))

The expected utilities for player 1 and player 2 are

$$U_1(\alpha_1, \alpha_2) = -1/9 \qquad \text{and} \qquad U_2(\alpha_1, \alpha_2) = +1/9.$$

Expected Utility

Remark: The expected utility functions U_i are linear in all mixed strategies.

Proposition

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile, $\beta_i, \gamma_i \in \Delta(A_i)$ mixed strategies, and $\lambda \in [0,1]$. Then

$$U_i(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i) = \lambda U_i(\alpha_{-i}, \beta_i) + (1-\lambda)U_i(\alpha_{-i}, \gamma_i).$$

Moreover,

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) \cdot U_i(\alpha_{-i}, a_i)$$

Proof.

Homework. □

Mixed Extension

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Definition (Mixed extension)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

The mixed extension of G is the game $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$ where

- $\Delta(A_i)$ is the set of probability distributions over A_i and
- $U_i : \prod_{j \in N} \Delta(A_j) \to \mathbb{R}$ assigns to each mixed strategy profile α the expected utility for player i according to the induced probability distribution p_α.

Nash Equilibria in Mixed Strategies

Definition (Nash equilibrium in mixed strategies)

Let G be a strategic game.

A Nash equilibrium in mixed strategies (or mixed-strategy Nash equilibrium, or MSNE) of G is a Nash equilibrium in the mixed extension of G.

Summary

- Not every strategic game has a pure-strategy Nash equilibrium.
- Randomization sometimes seems rational (e. g., matching pennies)
  $\rightsquigarrow$ mixed strategies

- This section: definition of mixed strategies, mixed extension, MSNE
- Next sections: characterization of MSNE, existence proof, computation

Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.2. Support Lemma

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Summer semester 2020

Support

Intuition:

- It does not make sense to assign positive probability to a pure strategy that is not a best response to what the other players do.
- Claim: A profile of mixed strategies α is a Nash equilibrium if and only if everyone only plays best pure responses to what the others play.

Definition (Support)

Let α_i be a mixed strategy.

The support of α_i is the set

$$supp(\alpha_i) = \{a_i \in A_i \mid \alpha_i(a_i) > 0\}$$

of actions played with nonzero probability.

Support Lemma

Lemma (Support lemma)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite strategic game.

Then $\alpha^* \in \prod_{i \in N} \Delta(A_i)$ is a mixed-strategy Nash equilibrium in G if and only if for every player $i \in N$, every pure strategy in the support of α_i^* is a best response to α_{-i}^*.

For a single player—given all other players stick to their mixed strategies—it does not make a difference whether he plays the mixed strategy or whether he plays any single pure strategy from the support of the mixed strategy.

Support Lemma

Example (Support lemma)

Matching pennies, strategy profile $\alpha = (\alpha_1, \alpha_2)$ with

$$\alpha_1(H) = {}^2\!/_3, \quad \alpha_1(T) = {}^1\!/_3, \quad \alpha_2(H) = {}^1\!/_3, \text{ and } \quad \alpha_2(T) = {}^2\!/_3.$$

For α to be a Nash equilibrium, both actions in $supp(\alpha_2) = \{H, T\}$ have to be best responses to α_1. Are they?

$$\begin{aligned} U_2(\alpha_1, H) &= \alpha_1(H) \cdot u_2(H, H) + \alpha_1(T) \cdot u_2(T, H) \\ &= {}^2\!/_3 \cdot (-1) + {}^1\!/_3 \cdot (+1) = -{}^1\!/_3, \\ U_2(\alpha_1, T) &= \alpha_1(H) \cdot u_2(H, T) + \alpha_1(T) \cdot u_2(T, T) \\ &= {}^2\!/_3 \cdot (+1) + {}^1\!/_3 \cdot (-1) = {}^1\!/_3. \end{aligned}$$

$\Rightarrow$ $H \in supp(\alpha_2)$, but $H \notin B_2(\alpha_1)$.

$\xRightarrow{\text{Support lemma}}$ α can not be a Nash equilibrium.

Support Lemma

Proof.

"$\Rightarrow$": Let α^* be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Support Lemma

Proof.

"$\Rightarrow$": Let α^* be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Assume that a_i is not a best response to α_{-i}^*. Because U_i is linear, player i can improve his utility by shifting probability in α_i^* from a_i to a better response.

Support Lemma

Proof.

"$\Rightarrow$": Let α^* be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Assume that a_i is not a best response to α^*_{-i}. Because U_i is linear, player i can improve his utility by shifting probability in α_i^* from a_i to a better response.

This makes the modified α_i^* a better response than the original α_i^*, i. e., the original α_i^* was not a best response, which contradicts the assumption that α^* is a Nash equilibrium.

Support Lemma

Proof.

"$\Rightarrow$": Let α^* be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Assume that a_i is not a best response to α^*_{-i}. Because U_i is linear, player i can improve his utility by shifting probability in α_i^* from a_i to a better response.

This makes the modified α_i^* a better response than the original α_i^*, i. e., the original α_i^* was not a best response, which contradicts the assumption that α^* is a Nash equilibrium. So each pure strategy in the support of α_i must be a best response.

Support Lemma

Proof (ctd.)

"$\Leftarrow$": Assume that α^* is not a Nash equilibrium.

Support Lemma

Proof (ctd.)

"$\Leftarrow$": Assume that α^* is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy α'_i such that $U_i(\alpha^*_{-i}, \alpha'_i) > U_i(\alpha^*_{-i}, \alpha^*_i)$.

Support Lemma

Proof (ctd.)

"$\Leftarrow$": Assume that α^* is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy α'_i such that $U_i(\alpha^*_{-i}, \alpha'_i) > U_i(\alpha^*_{-i}, \alpha^*_i)$.

Because U_i is linear, there must be a pure strategy $a'_i \in supp(\alpha'_i)$ that has higher utility than some pure strategy $a''_i \in supp(\alpha^*_i)$.

Support Lemma

Proof (ctd.)

"$\Leftarrow$": Assume that α^* is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy α'_i such that $U_i(\alpha^*_{-i}, \alpha'_i) > U_i(\alpha^*_{-i}, \alpha^*_i)$.

Because U_i is linear, there must be a pure strategy $a'_i \in supp(\alpha'_i)$ that has higher utility than some pure strategy $a''_i \in supp(\alpha^*_i)$.

Therefore, $supp(\alpha^*_i)$ does not only contain best responses to α^*_{-i}. $\square$

Summary

- Characterization of mixed-strategy Nash equilibria: players only play best responses with positive probability (support lemma).

  $\rightsquigarrow$ only need to look at pure candidate best responses against other players' mixed strategy profile when computing MSNE. (See later sections.)

Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.3. Computing Mixed-Strategy Nash Equilibria

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Computing Mixed-Strategy Nash Equilibria

Example (Mixed-strategy Nash equilibria in BoS)

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

We already know: (B, B) and (S, S) are pure Nash equilibria.

Possible supports (excluding "pure-vs-pure" strategies) are:

$$\{B\} \text{ vs. } \{B,S\}, \quad \{S\} \text{ vs. } \{B,S\}, \quad \{B,S\} \text{ vs. } \{B\},$$
$$\{B,S\} \text{ vs. } \{S\} \quad \text{and} \quad \{B,S\} \text{ vs. } \{B,S\}$$

Observation: In Bach or Stravinsky, pure strategies have unique best responses. Therefore, there can be no Nash equilibria of "pure-vs-strictly-mixed" type.

Computing Mixed-Strategy Nash Equilibria

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Example (Mixed-strategy Nash equilibria in BoS (ctd.))

Consequence: Only need to search for additional Nash equilibria with support sets $\{B, S\}$ vs. $\{B, S\}$.
Assume that (α_1^*, α_2^*) is a Nash equilibrium with $0 < \alpha_1^*(B) < 1$ and $0 < \alpha_2^*(B) < 1$. Then

Computing Mixed-Strategy Nash Equilibria

Example (Mixed-strategy Nash equilibria in BoS (ctd.))

Consequence: Only need to search for additional Nash equilibria with support sets $\{B, S\}$ vs. $\{B, S\}$.
Assume that (α_1^*, α_2^*) is a Nash equilibrium with $0 < \alpha_1^*(B) < 1$ and $0 < \alpha_2^*(B) < 1$. Then

$$
\begin{aligned}
& & U_1(B, \alpha_2^*) &= U_1(S, \alpha_2^*) \\
&\Rightarrow & 2 \cdot \alpha_2^*(B) + 0 \cdot \alpha_2^*(S) &= 0 \cdot \alpha_2^*(B) + 1 \cdot \alpha_2^*(S) \\
&\Rightarrow & 2 \cdot \alpha_2^*(B) &= 1 - \alpha_2^*(B) \\
&\Rightarrow & 3 \cdot \alpha_2^*(B) &= 1 \\
&\Rightarrow & \alpha_2^*(B) &= 1/3 \quad (\text{and } \alpha_2^*(S) = 2/3)
\end{aligned}
$$

Similarly, we get $\alpha_1^*(B) = 2/3$ and $\alpha_1^*(S) = 1/3$.
The payoff profile of this equilibrium is $(2/3, 2/3)$.

Remark

Let $G = \langle\{1,2\},(A_i),(u_i)\rangle$ with $A_1 = \{T,B\}$ and $A_2 = \{L,R\}$ be a two-player game with two actions each, and (T,α_2^*) with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of G.

Then at least one of the profiles (T,L) and (T,R) is also a Nash equilibrium of G.

Remark

Let $G = \langle \{1,2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$ be a two-player game with two actions each, and (T, α_2^*) with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of G.

Then at least one of the profiles (T, L) and (T, R) is also a Nash equilibrium of G.

Reason: Both L and R are best responses to T. Assume that T was neither a best response to L nor to R. Then B would be a better response than T both to L and to R.

Support Lemma

Remark

Let $G = \langle \{1,2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$ be a two-player game with two actions each, and (T, α_2^*) with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of G.

Then at least one of the profiles (T, L) and (T, R) is also a Nash equilibrium of G.

Reason: Both L and R are best responses to T. Assume that T was neither a best response to L nor to R. Then B would be a better response than T both to L and to R.

With the linearity of U_1, B would also be a better response to α_2^* than T is. Contradiction.

Support Lemma

Example

Consider the Nash equilibrium $\alpha^* = (\alpha_1^*, \alpha_2^*)$ with

$$\alpha_1^*(T) = 1, \qquad \alpha_1^*(B) = 0, \qquad \alpha_2^*(L) = 1/10, \qquad \alpha_2^*(R) = 9/10$$

in the following game:

	L	R
T	1, 1	1, 1
B	2, 2	$-5, -5$

Here, (T, R) is also a Nash equilibrium.

Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.4. Nash's Theorem: Introduction

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Nash's Theorem

Motivation: When does a strategic game have a mixed-strategy Nash equilibrium?

We already discussed necessary and sufficient conditions for the existence of Nash equilibria for the special case of zero-sum games. Can we make other claims?

Nash's Theorem

Theorem (Nash's theorem)

Every finite strategic game has a mixed-strategy Nash equilibrium.

Proof sketch.

Consider the best-response function B with

$$B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i}).$$

A mixed strategy profile α is a Nash equilibrium iff it is a fixpoint of B iff $\alpha \in B(\alpha)$.

Under certain conditions that are satisfied by B, B has such a fixpoint (Kakutani's Fixpoint Theorem!). Therefore, the game has a mixed-strategy Nash equilibrium. □

Nash's Theorem

Outline for the formal proof:

1. Review of necessary mathematical definitions
   $\rightsquigarrow$ Subsection "Nash's Theorem: Required Background"
2. Statement of a fixpoint theorem used to prove Nash's theorem (without proof)
   $\rightsquigarrow$ Subsection "Nash's Theorem: Required Background"
3. Proof of Nash's theorem using fixpoint theorem
   $\rightsquigarrow$ Subsection "Nash's Theorem: Proof"

Game Theory

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2.7.5. Nash's Theorem: Required Background

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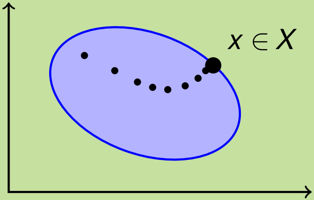
Definition

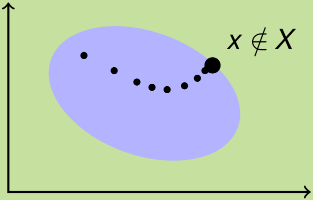
A set $X \subseteq \mathbb{R}^n$ is closed if X contains all its limit points, i. e., if $(x_k)_{k\in\mathbb{N}}$ is a sequence of elements in X and $\lim_{k\to\infty} x_k = x$, then also $x \in X$.

Example

Closed:



Not closed:



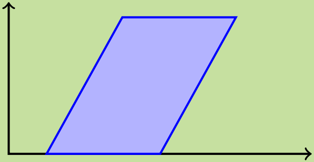
Definition

A set $X \subseteq \mathbb{R}^n$ is bounded if for each $i = 1, \ldots, n$ there are lower and upper bounds $a_i, b_i \in \mathbb{R}$ such that

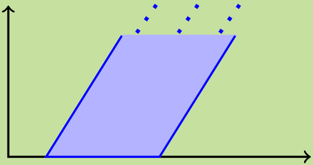
$$X \subseteq \prod_{i=1}^{n} [a_i, b_i].$$

Example

Bounded:



Not bounded:

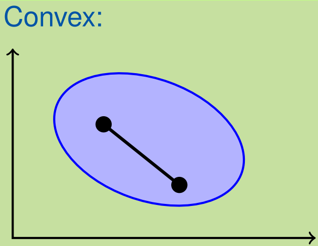


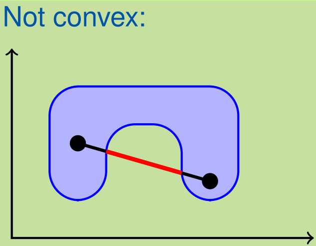
Definition

A set $X \subseteq \mathbb{R}^n$ is convex if for all $x, y \in X$ and all $\lambda \in [0,1]$,

$$\lambda x + (1-\lambda)y \in X.$$

Example





Definition

For a function $f : X \to 2^X$, the graph of f is the set

$$Graph(f) = \{(x,y) \mid x \in X,\ y \in f(x)\}.$$

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Theorem (Kakutani's fixpoint theorem)

Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed, bounded and convex set and let $f : X \to 2^X$ be a function such that

- *for all $x \in X$, the set $f(x) \subseteq X$ is nonempty and convex, and*
- *$Graph(f)$ is closed.*

Then there is an $x \in X$ with $x \in f(x)$, i. e., f has a fixpoint.

Proof.

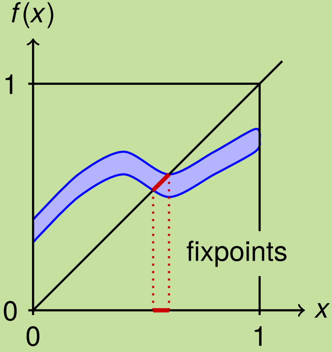
See Shizuo Kakutani, A generalization of Brouwer's fixed point theorem, 1941, or your favorite advanced calculus textbook, or the Internet.

For German speakers: Harro Heuser, Lehrbuch der Analysis, Teil 2, also has a proof (Abschnitt 232). □

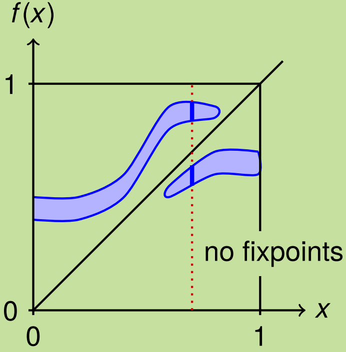
Example

Let $X = [0, 1]$.

Kakutani's theorem applicable:



Kakutani's theorem not applicable:



Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.6. Nash's Theorem: Proof

Albert-Ludwigs-Universität Freiburg

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Nash's Theorem

Recall:

Theorem (Kakutani's fixpoint theorem)

Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed, bounded and convex set and let $f : X \to 2^X$ be a function such that

- *for all $x \in X$, the set $f(x) \subseteq X$ is nonempty and convex, and*
- *$Graph(f)$ is closed.*

Then there is an $x \in X$ with $x \in f(x)$, i. e., f has a fixpoint.

We use this to prove:

Theorem (Nash's theorem)

Every finite strategic game has a mixed-strategy Nash equilibrium.

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Proof.

Apply Kakutani's fixpoint theorem using $X = \mathscr{A} = \prod_{i \in N} \Delta(A_i)$ and $f = B$, where $B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i})$.

We have to show:

1. $\mathscr{A}$ is nonempty,
2. $\mathscr{A}$ is closed,
3. $\mathscr{A}$ is bounded,
4. $\mathscr{A}$ is convex,
5. $B(\alpha)$ is nonempty for all $\alpha \in \mathscr{A}$,
6. $B(\alpha)$ is convex for all $\alpha \in \mathscr{A}$, and
7. $Graph(B)$ is closed.

Proof (ctd.)

Some notation:

- Assume without loss of generality that $N = \{1, \ldots, n\}$.
- A profile of mixed strategies can be written as a vector of $M = \sum_{i \in N} |A_i|$ real numbers in the interval $[0, 1]$ such that numbers for the same player add up to 1.

  For example, $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1(T) = 0.7$, $\alpha_1(M) = 0.0$, $\alpha_1(B) = 0.3$, $\alpha_2(L) = 0.4$, $\alpha_2(R) = 0.6$ can be seen as the vector

$$(\underbrace{0.7,\ 0.0,\ 0.3}_{\alpha_1},\ \underbrace{0.4,\ 0.6}_{\alpha_2})$$

- This allows us to interpret the set $\mathscr{A}$ of mixed strategy profiles as a subset of $\mathbb{R}^M$.

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Proof (ctd.)

1. $\mathscr{A}$ nonempty: Trivial. $\mathscr{A}$ contains the tuple

$$(1, \underbrace{0,\dots,0}_{|A_1|-1 \text{ times}}, \dots, 1, \underbrace{0,\dots,0}_{|A_n|-1 \text{ times}}).$$

Proof (ctd.)

1. $\mathscr{A}$ nonempty: Trivial. $\mathscr{A}$ contains the tuple

$$(1, \underbrace{0, \ldots, 0}_{|A_1|-1 \text{ times}}, \ldots, 1, \underbrace{0, \ldots, 0}_{|A_n|-1 \text{ times}}).$$

2. $\mathscr{A}$ closed: Let $\alpha_1, \alpha_2, \ldots$ be a sequence in $\mathscr{A}$ that converges to $\lim_{k \to \infty} \alpha_k = \alpha$. Suppose $\alpha \notin \mathscr{A}$. Then either there is some component of α that is less than zero or greater than one, or the components for some player i add up to a value other than one.

Proof (ctd.)

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   Since α is a limit point, the same must hold for some α_k in the sequence. But then, $\alpha_k \notin \mathscr{A}$, a contradiction. Hence $\mathscr{A}$ is closed.

Proof (ctd.)

3. $\mathscr{A}$ bounded: Trivial. All entries are between 0 and 1, i. e., $\mathscr{A}$ is bounded by $[0, 1]^M$.

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Proof (ctd.)

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$$\begin{aligned}\min(\gamma) &= \min(\lambda\alpha + (1-\lambda)\beta) \\ &\geq \lambda \cdot \min(\alpha) + (1-\lambda) \cdot \min(\beta) \\ &\geq \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0,\end{aligned}$$

and similarly, $\max(\gamma) \leq 1$.

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Proof (ctd.)

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   and similarly, $\max(\gamma) \leq 1$.

   Hence, all entries in γ are still in $[0,1]$.

Proof (ctd.)

4 $\mathscr{A}$ convex (ctd.): Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be the sections of α, β and γ, respectively, that determine the probability distribution for player *i*. Then

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Proof (ctd.)

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$$\begin{aligned}\sum \tilde{\gamma} &= \sum (\lambda \tilde{\alpha} + (1-\lambda)\tilde{\beta}) \\ &= \lambda \cdot \sum \tilde{\alpha} + (1-\lambda) \cdot \sum \tilde{\beta} \\ &= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1.\end{aligned}$$

Proof (ctd.)

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Hence, all probabilities for player *i* in γ still sum up to 1.

Altogether, $\gamma \in \mathscr{A}$, and therefore, $\mathscr{A}$ is convex.

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Proof (ctd.)

5 $B(\alpha)$ nonempty: For a fixed α_{-i}, U_i is linear in the mixed strategies of player i, i. e., for $\beta_i, \gamma_i \in \Delta(A_i)$,

$$U_i\left(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i\right) = \lambda U_i\left(\alpha_{-i}, \beta_i\right) + (1-\lambda)U_i\left(\alpha_{-i}, \gamma_i\right) \tag{1}$$

for all $\lambda \in [0, 1]$.

Proof (ctd.)

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Hence, U_i is continous on $\Delta(A_i)$.

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Continous functions on closed and bounded sets take their maximum in that set.

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Proof (ctd.)

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for all $\lambda \in [0,1]$.

Hence, U_i is continous on $\Delta(A_i)$.

Continous functions on closed and bounded sets take their maximum in that set.

Therefore, $B_i(\alpha_{-i}) \neq \emptyset$ for all $i \in N$, and thus $B(\alpha) \neq \emptyset$.

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Proof (ctd.)

6. $B(\alpha)$ convex: This follows, since each $B_i(\alpha_{-i})$ is convex. To see this, let $\alpha_i', \alpha_i'' \in B_i(\alpha_{-i})$.

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   Then $U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'')$.

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Proof (ctd.)

6 $B(\alpha)$ convex: This follows, since each $B_i(\alpha_{-i})$ is convex. To see this, let $\alpha_i', \alpha_i'' \in B_i(\alpha_{-i})$.

Then $U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'')$.

With Equation (1), this implies

$$\lambda \alpha_i' + (1 - \lambda)\alpha_i'' \in B_i(\alpha_{-i}).$$

Hence, $B_i(\alpha_{-i})$ is convex.

Proof (ctd.)

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7. *$Graph(B)$ closed:* Let (α^k, β^k) be a convergent sequence in $Graph(B)$ with $\lim_{k\to\infty}(\alpha^k, \beta^k) = (\alpha, \beta)$.

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Proof (ctd.)

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7 $Graph(B)$ closed: Let (α^k, β^k) be a convergent sequence in $Graph(B)$ with $\lim_{k\to\infty}(\alpha^k, \beta^k) = (\alpha, \beta)$.

So, $\alpha^k, \beta^k, \alpha, \beta \in \prod_{i\in N} \Delta(A_i)$ and $\beta^k \in B(\alpha^k)$.

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Proof (ctd.)

6. $B(\alpha)$ convex: This follows, since each $B_i(\alpha_{-i})$ is convex. To see this, let $\alpha_i', \alpha_i'' \in B_i(\alpha_{-i})$.
   Then $U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'')$.
   With Equation (1), this implies
   $$\lambda \alpha_i' + (1-\lambda)\alpha_i'' \in B_i(\alpha_{-i}).$$
   Hence, $B_i(\alpha_{-i})$ is convex.
7. *Graph*(B) closed: Let (α^k, β^k) be a convergent sequence in *Graph*(B) with $\lim_{k\to\infty}(\alpha^k, \beta^k) = (\alpha, \beta)$.
   So, $\alpha^k, \beta^k, \alpha, \beta \in \prod_{i\in N} \Delta(A_i)$ and $\beta^k \in B(\alpha^k)$.
   We need to show that $(\alpha, \beta) \in \mathit{Graph}(B)$, i. e., that $\beta \in B(\alpha)$.

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Proof (ctd.)

7 *Graph*(*B*) closed (ctd.): It holds for all $i \in N$:

$$
\begin{aligned}
U_i(\alpha_{-i}, \beta_i) &\overset{\text{(D)}}{=} U_i\big(\lim_{k\to\infty}(\alpha_{-i}^k, \beta_i^k)\big) \\
&\overset{\text{(C)}}{=} \lim_{k\to\infty} U_i(\alpha_{-i}^k, \beta_i^k) \\
&\overset{\text{(B)}}{\geq} \lim_{k\to\infty} U_i(\alpha_{-i}^k, \beta_i') \qquad \text{for all } \beta_i' \in \Delta(A_i) \\
&\overset{\text{(C)}}{=} U_i\big(\lim_{k\to\infty}\alpha_{-i}^k, \beta_i'\big) \qquad \text{for all } \beta_i' \in \Delta(A_i) \\
&\overset{\text{(D)}}{=} U_i(\alpha_{-i}, \beta_i') \qquad \text{for all } \beta_i' \in \Delta(A_i).
\end{aligned}
$$

(D): def. α_i, β_i; (C) continuity; (B) β_i^k best response to α_{-i}^k.

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Proof (ctd.)

7. *Graph*(B) closed (ctd.): It follows that β_i is a best response to α_{-i} for all $i \in N$.

   Thus, $\beta \in B(\alpha)$ and finally $(\alpha, \beta) \in \mathit{Graph}(B)$.

Proof (ctd.)

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Therefore, all requirements of Kakutani's fixpoint theorem are satisfied.

Proof (ctd.)

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   Thus, $\beta \in B(\alpha)$ and finally $(\alpha, \beta) \in Graph(B)$.

Therefore, all requirements of Kakutani's fixpoint theorem are satisfied.

Applying Kakutani's theorem establishes the existence of a fixpoint of B, which is, by definition/construction, the same as a mixed-strategy Nash equilibrium. □

Summary

Take-home message:

- Nash's theorem: Every finite strategic game has a mixed-strategy Nash equilibrium.
- Proof idea: Apply Kakutani's fixpoint theorem to the best-response function.
- Encode mixed strategy profiles as real-valued vectors, apply standard techniques from real analysis.

Game Theory

2. Strategic Games

2.8. Correlated Equilibria

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Summer semester 2020

Recall: There are three Nash equilibria in Bach or Stravinsky

- (B, B) with payoff profile $(2, 1)$
- (S, S) with payoff profile $(1, 2)$
- (α_1^*, α_2^*) with payoff profile $(2/3, 2/3)$ where
  - $\alpha_1^*(B) = 2/3$, $\alpha_1^*(S) = 1/3$,
  - $\alpha_2^*(B) = 1/3$, $\alpha_2^*(S) = 2/3$.

All of them are somewhat unsatisfactory:

- (B, B) and (S, S) because of unclear coordination and uneven payoffs.
- (α_1^*, α_2^*) because of low payoffs.

Correlated Equilibria

Question: Can the players somehow do better?

Yes! Idea: Use a publicly visible coin toss to decide which action from a mixed strategy is played. This can lead to higher payoffs.

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Example (Correlated equilibrium in BoS)

With a fair coin that both players can observe, the players can agree to play as follows:

- If the coin shows heads, both play B.
- If the coin shows tails, both play S.

This is stable in the sense that no player has an incentive to deviate from this agreed-upon rule, as long as the other player keeps playing his/her strategy (cf. definition of Nash equilibria).

Expected payoffs: $(3/2, 3/2)$ instead of $(2/3, 2/3)$.

Observations and Information Partitions

We assume that observations are made based on a finite probability space (Ω, π), where Ω is a set of states and π is a probability measure on Ω.

Agents might not be able to distingush all states from each other. In order to model this, we assume, for each player i, an information partition $\mathscr{P}_i = \{P_{i1}, P_{i2}, \dots, P_{ik_i}\}$. This means that $\bigcup \mathscr{P}_i = \Omega$ for all i, and for all $P_j, P_k \in \mathscr{P}_i$ with $P_j \neq P_k$, we have $P_j \cap P_k = \emptyset$.

Observations and Information Partitions

Example

$\Omega = \{x, y, z\}$, $\mathscr{P}_1 = \{\{x\}, \{y, z\}\}$, $\mathscr{P}_2 = \{\{x, y\}, \{z\}\}$.

A function $f : \Omega \to X$ respects an information partition for player i if $f(\omega) = f(\omega')$ whenever $\omega, \omega' \in P_i$ for some $P_i \in \mathscr{P}_i$.

Example

f respects $\mathscr{P}_1$ if $f(y) = f(z)$.

Correlated Equilibria – Formally

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Definition

A correlated equilibrium of a strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ consists of

- a finite probability space (Ω, π),
- for each player $i \in N$, an information partition $\mathscr{P}_i$ of Ω,
- for each player $i \in N$, a function $\sigma_i : \Omega \to A_i$ that respects $\mathscr{P}_i$ (σ_i is player i's strategy)

such that for every $i \in N$ and every function $\tau_i : \Omega \to A_i$ that respects $\mathscr{P}_i$ (i.e. for every possible strategy of player i), we have

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)).$$

Correlated Equilibria

Example

	L	R
T	$6,6$	$2,7$
B	$7,2$	$0,0$

Nash equilibria: (T,R) with payoffs $(2,7)$, (B,L) with payoffs $(7,2)$, and $((\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}))$ with payoffs $(4+\frac{2}{3},4+\frac{2}{3})$.

Correlated Equilibria

Example

	L	R
T	$6,6$	$2,7$
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Nash equilibria: (T,R) with payoffs $(2,7)$, (B,L) with payoffs $(7,2)$, and $((\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}))$ with payoffs $(4+\frac{2}{3},4+\frac{2}{3})$.

Better correlated equilibrium: Assume $\Omega = \{x,y,z\}$, $\pi(x) = \pi(y) = \pi(z) = \frac{1}{3}$, $\mathscr{P}_1 = \{\{x\},\{y,z\}\}$, $\mathscr{P}_2 = \{\{x,y\},\{z\}\}$.
Set $\sigma_1(x) = B, \sigma_1(y) = \sigma_1(z) = T$ and $\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R$.

This is a correlated equilibrium with payoffs $(5,5)$.

Note: This example only works with uncertainty about states.

Connection to Nash Equilibria

Proposition

For every mixed strategy Nash equilibrium α of a finite strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, there is a correlated equilibrium $\langle (\Omega, \pi), (\mathscr{P}_i), (\sigma_i) \rangle$ in which, for each player i, the distribution on A_i induced by σ_i is α_i.

This means that correlated equilibria are a generalization of Nash equilibria.

Connection to Nash Equilibria

Proof.

Let $\Omega = A$ and define $\pi(a) = \prod_{j \in N} \alpha_j(a_j)$. For each player i, let a, b be in the same $P \in \mathscr{P}_i$ iff $a_i = b_i$. Define $\sigma_i(a) = a_i$ for each $a \in A$.

Then $\langle (\Omega, \pi), (\mathscr{P}_i), (\sigma_i) \rangle$ is a correlated equilibrium since the left hand side of the "best-response inequality"

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega))$$

is the Nash equilibrium payoff, and for each player i at least as good any other strategy τ_i respecting the information partition. Furthermore, the distribution induced by σ_i is α_i. $\square$

Summary

- In correlated equilibria, players can make their actions dependent on a signal received before the game.
- Players may be unable to distinguish some signals.
- In a correlated equilibrium, each player's state-to-action mapping is a best response to the others' state-to-action mappings in the context of the possible states and their probabilities (which are part of the correlated equilibrium).
- Equivalently: for every possible state, each player's action for that state is optimal given the other players' strategies and its knowledge about the state.
- Correlated equilibria generalize MSNE.
- They can lead to higher payoffs than MSNE.

Game Theory

3. Nash Equilibrium Computation Algorithms

3.1. Finite Zero-Sum Games

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Summer semester 2020

Motivation

- We know: In finite strategic games, mixed-strategy Nash equilibria are guaranteed to exist.
- We don't know: How to systematically find them?
- Challenge: There are infinitely many mixed strategy profiles to consider. How to do this in finite time?

This section:

- Computation of mixed-strategy Nash equilibria for finite zero-sum games.

Next section:

- Computation of mixed-strategy Nash equilibria for general finite two player games.

Mixed-Strategy Nash Equilibria in Finite Zero-Sum Games

We start with finite zero-sum games for two reasons:

- They are easier to solve than general finite two-player games.
- Understanding how to solve finite zero-sum games facilitates understanding how to solve general finite two-player games.

Mixed-Strategy Nash Equilibria in Finite Zero-Sum Games

In the following, we will exploit the zero-sum property of a game G when searching for mixed-strategy Nash equilibria. For that, we need the following result.

Proposition

Let G be a finite zero-sum game. Then the mixed extension of G is also a zero-sum game.

Proof.

Homework. □

Mixed-Strategy Nash Equilibria in Finite Zero-Sum Games

Let G be a finite zero-sum game with mixed extension G'.

Then we know the following:

1. Previous proposition implies: G' is also a zero-sum game.
2. Nash's theorem implies: G' has a Nash equilibrium.
3. Maximinimizer theorem + 1 + 2 implies: Nash equilibria and pairs of maximinimizers in G' are the same.

Mixed-Strategy Nash Equilibria in Finite Zero-Sum Games

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Consequence:
When looking for mixed-strategy Nash equilibria in G, it is sufficient to look for pairs of maximinimizers in G'.

Method: Linear Programming

Linear Program Encoding

Approach:

- Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite zero-sum game:
  - $N = \{1, 2\}$.
  - A_1 and A_2 are finite.
  - $U_1(\alpha, \beta) = -U_2(\alpha, \beta)$ for all $\alpha \in \Delta(A_1), \beta \in \Delta(A_2)$.
- Player 1 looks for a maximinimizer mixed strategy α.
- For each possible α of player 1:
  - Determine expected utility against best response of pl. 2. (Only need to consider finitely many pure candidates for best responses because of Support Lemma).
  - Maximize expected utility over all possible α.

Linear Program Encoding

- Result: maximinimizer α for player 1 in G'
  (= Nash equilibrium strategy for player 1)
- Analogously: obtain maximinimizer β for player 2 in G'
  (= Nash equilibrium strategy for player 2)
- With maximinimizer theorem: we can combine α and β into a Nash equilibrium.

Linear Program Encoding

"For each possible α of player 1, determine expected utility against best response of player 2, and maximize."

translates to the following LP:

$$
\begin{aligned}
\text{Maximize } u \qquad & \text{subject to} \\
\alpha(a) \geq 0 \qquad & \text{for all } a \in A_1 \\
\sum_{a \in A_1} \alpha(a) = 1 \qquad & \\
\underbrace{\sum_{a \in A_1} \alpha(a) \cdot u_1(a,b)}_{=U_1(\alpha,b)} \geq u \qquad & \text{for all } b \in A_2
\end{aligned}
$$

Note: Each $\alpha(a)$ is a single LP variable, and so is u.
The values $u_1(a,b)$ are constant coefficients.

Linear Program Encoding

Example (Matching pennies)

	H	T
H	$1, -1$	$-1, \quad 1$
T	$-1, \quad 1$	$1, -1$

Linear program for player 1:

Maximize u subject to the constraints

$$\alpha(H) \geq 0,\ \alpha(T) \geq 0,\ \alpha(H) + \alpha(T) = 1,$$
$$\alpha(H) \cdot u_1(H,H) + \alpha(T) \cdot u_1(T,H) = \alpha(H) - \alpha(T) \geq u,$$
$$\alpha(H) \cdot u_1(H,T) + \alpha(T) \cdot u_1(T,T) = -\alpha(H) + \alpha(T) \geq u.$$

Solution: $\alpha(H) = \alpha(T) = 1/2$, $u = 0$.

Linear Program Encoding

Theorem

A mixed strategy α is a maximinimizer with payoff u if and only if it is a solution to the LP encoding over α and u.

Proof.

By construction. □

Similarly with β and v for the opposite player.

Linear Program Encoding

Resulting LPs can be solved using off-the-shelf LP solver, e. g.:

- lp_solve
- CLP
- GLPK
- CPLEX
- gurobi

Linear Program Encoding

- Remark: There is an alternative encoding based on the observation that in zero-sum games that have a Nash equilibrium, maximinimization and minimaximization yield the same result.
- Idea: Formulate linear program with inequalities

$$U_1(a,\beta) \leq u \qquad \text{for all } a \in A_1$$

and minimize u. Analogously for β.

Summary

Summary:

- Computing mixed-strategy Nash equilibria in finite zero-sum games can be reduced to solving certain linear programs.
- Some theory is required to justify the reduction: Nash's theorem, maximinimizer theorem, support lemma.
- Resulting LPs are of linear size.
  $\rightsquigarrow$ polynomial-time Nash equilibrium computation

Software:

- Gambit (`http://www.gambit-project.org`) can be used to compute Nash equilibria.
- It also has LP solving built-in as one of the solution methods.

Game Theory

3. Nash Equilibrium Computation Algorithms

3.2. General Finite Two-Player Games

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Motivation

- We know: In finite strategic games, mixed-strategy Nash equilibria are guaranteed to exist.
- We don't know: How to systematically find them?
- Challenge: There are infinitely many mixed strategy profiles to consider. How to do this in finite time?

Previous section:

- Computation of mixed-strategy Nash equilibria for finite zero-sum games.

This section:

- Computation of mixed-strategy Nash equilibria for general finite two player games.

General Finite Two-Player Games

- For general finite two-player games, the LP approach does not work.
- Instead, use instances of the linear complementarity problem (LCP):
  - Linear (in-)equalities as with LPs.
  - Additional constraints of the form $x_i \cdot y_i = 0$ (or equivalently $x_i = 0 \vee y_i = 0$) for variables $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$, and $i \in \{1, \dots, k\}$.
  - no objective function.
- With LCPs, we can compute Nash equilibria for arbitrary finite two-player games.

General Finite Two-Player Games

Let A_1 and A_2 be finite and let (α, β) be a Nash equilibrium with payoff profile (u, v). Then consider this LCP encoding:

$$
\begin{aligned}
u - U_1(a, \beta) &\geq 0 && \text{for all } a \in A_1 && (1)\\
v - U_2(\alpha, b) &\geq 0 && \text{for all } b \in A_2 && (2)\\
\alpha(a) \cdot (u - U_1(a, \beta)) &= 0 && \text{for all } a \in A_1 && (3)\\
\beta(b) \cdot (v - U_2(\alpha, b)) &= 0 && \text{for all } b \in A_2 && (4)\\
\alpha(a) &\geq 0 && \text{for all } a \in A_1 && (5)\\
\sum_{a \in A_1} \alpha(a) &= 1 && && (6)\\
\beta(b) &\geq 0 && \text{for all } b \in A_2 && (7)\\
\sum_{b \in A_2} \beta(b) &= 1 && && (8)
\end{aligned}
$$

General Finite Two-Player Games

Remarks about the encoding:

- In (3) and (4): for instance,

$$\alpha(a) \cdot (u - U_1(a, \beta)) = 0$$

  if and only if

$$\alpha(a) = 0 \qquad \text{or} \qquad u - U_1(a, \beta) = 0.$$

  This holds in every Nash equilibrium, because:
  - if $a \notin \mathit{supp}(\alpha)$, then $\alpha(a) = 0$, and
  - if $a \in \mathit{supp}(\alpha)$, then $a \in B_1(\beta)$, thus $U_1(a, \beta) = u$.
- With additional variables, the above LCP formulation can be transformed into LCP normal form.

General Finite Two-Player Games

Theorem

A mixed strategy profile (α, β) with payoff profile (u, v) is a Nash equilibrium if and only if it is a solution to the LCP encoding over (α, β) and (u, v).

Proof.

- Nash equilibria are solutions to the LCP: Obvious because of the support lemma.
- Solutions to the LCP are Nash equilibria: Let (α, β, u, v) be a solution to the LCP. Because of (5)–(8), α and β are mixed strategies.

Proof (ctd.)

- Solutions to the LCP are Nash equilibria (ctd.): Because of (1), u is at least the maximal payoff over all possible pure responses, and because of (3), u is exactly the maximal payoff.

  If $\alpha(a) > 0$, then, because of (3), the payoff for player 1 against β is u.

  The linearity of the expected utility implies that α is a best response to β.

  Analogously, we can show that β is a best response to α and hence (α, β) is a Nash equilibrium with payoff profile (u, v). □

Solution Algorithm for LCPs

Naïve algorithm:

Enumerate all $(2^n - 1) \cdot (2^m - 1)$ possible pairs of support sets.

For each such pair $(supp(\alpha), supp(\beta))$:

- Convert the LCP into an LP:
  - Linear (in-)equalities are preserved.
  - Constraints of the form $\alpha(a) \cdot (u - U_1(a, \beta)) = 0$ are replaced by a new linear equality:
    - $u - U_1(a, \beta) = 0$, if $a \in supp(\alpha)$, and
    - $\alpha(a) = 0$, otherwise,

    Analogously for $\beta(b) \cdot (v - U_2(\alpha, b)) = 0$.
  - Objective function: maximize constant zero function.
- Apply solution algorithm for LPs to the transformed program.

Solution Algorithm for LCPs

- Runtime of the naïve algorithm: $O(p(n+m) \cdot 2^{n+m})$, where p is some polynomial.
- Better in practice: Lemke-Howson algorithm.
- Complexity:
  - unknown whether $\textsc{LcpSolve} \in \mathbf{P}$.
  - $\textsc{LcpSolve} \in \mathbf{NP}$ is clear (naïve algorithm can be seen as a nondeterministic polynomial-time algorithm).

Summary

- Previous section: Computation of mixed-strategy Nash equilibria for finite zero-sum games using linear programs.
  $\rightsquigarrow$ polynomial-time computation
- This section: Computation of mixed-strategy Nash equilibria for general finite two player games using linear complementarity problem.
  $\rightsquigarrow$ computation in **NP**

Game Theory

3. Nash Equilibrium Computation Algorithms
Appendix. Linear Programming

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Linear Programming

Appendix:

In this appendix, we briefly discuss linear programming.
(We need it to find Nash equilibria.)

Goal of linear programming:

Solving a system of linear inequalities over n real-valued variables while optimizing some linear objective function.

Linear Programming

Example

Production of two sorts of items with time requirements and profit per item. Objective: Maximize profit.

	Cutting	Assembly	Postproc.	Profit per item
(x) sort 1	25	60	68	30
(y) sort 2	75	60	34	40
per day	≤ 450	≤ 480	≤ 476	maximize!

Goal: Find numbers of pieces x of sort 1 and y of sort 2 to be produced per day such that the resource constraints are met and the objective function is maximized.

Example (ctd., formalization)

$$x \geq 0,\ y \geq 0 \tag{1}$$

$$25x + 75y \leq 450 \quad (\text{or } y \leq 6 - {}^1\!/_3\, x) \tag{2}$$

$$60x + 60y \leq 480 \quad (\text{or } y \leq 8 - x) \tag{3}$$

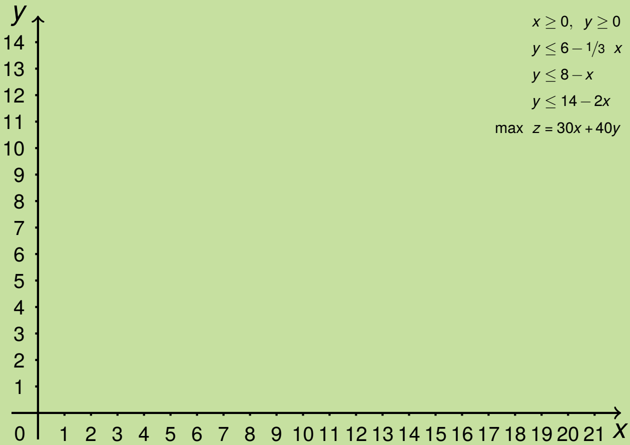
$$68x + 34y \leq 476 \quad (\text{or } y \leq 14 - 2x) \tag{4}$$

$$\text{maximize } z = 30x + 40y \tag{5}$$

- Inequalities (1)–(4): Admissible solutions
  (They form a convex set in $\mathbb{R}^2$.)
- Line (5): Objective function

Linear Programming

Example (ctd., visualization)



$x \geq 0, \quad y \geq 0$

$y \leq 6 - 1/3 \ x$

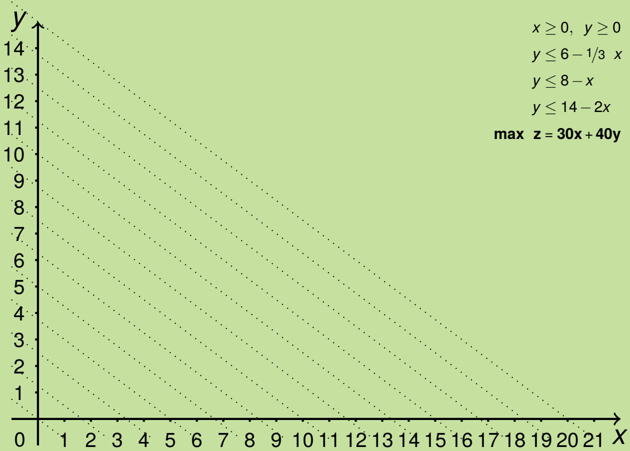
$y \leq 8 - x$

$y \leq 14 - 2x$

max $z = 30x + 40y$

Linear Programming

Example (ctd., visualization)



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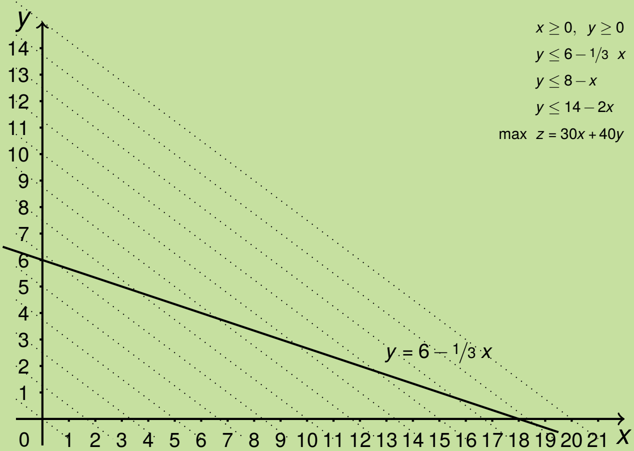
max $\mathbf{z = 30x + 40y}$

Example (ctd., visualization)



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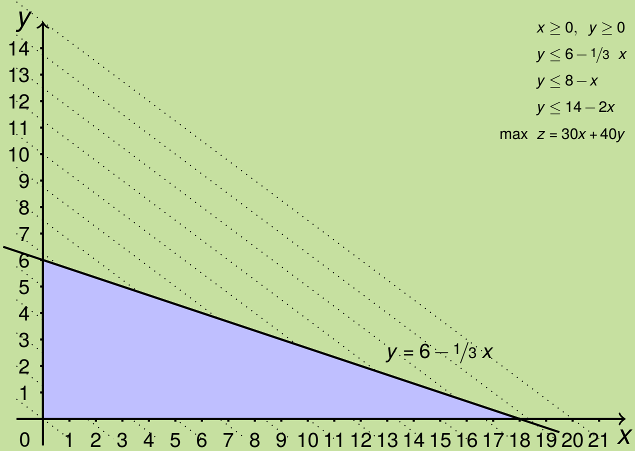
Example (ctd., visualization)



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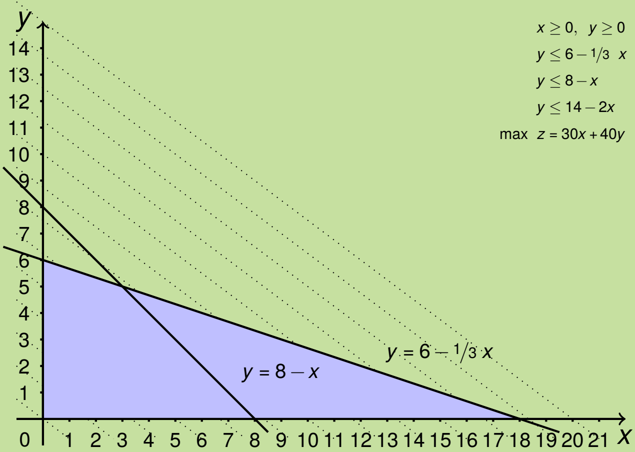
Example (ctd., visualization)



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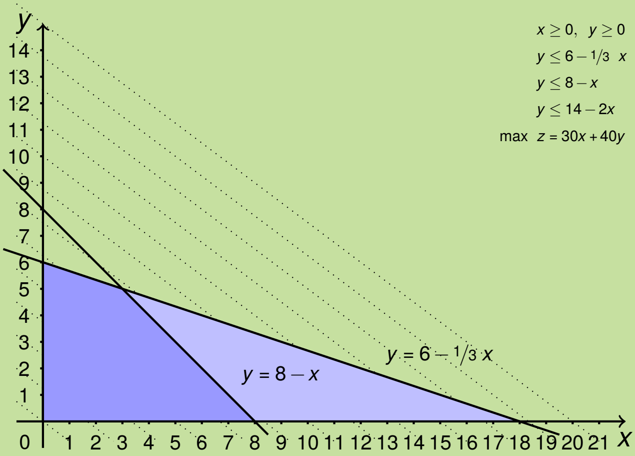
Example (ctd., visualization)



$x \geq 0, \quad y \geq 0$

$y \leq 6 - 1/3 \; x$

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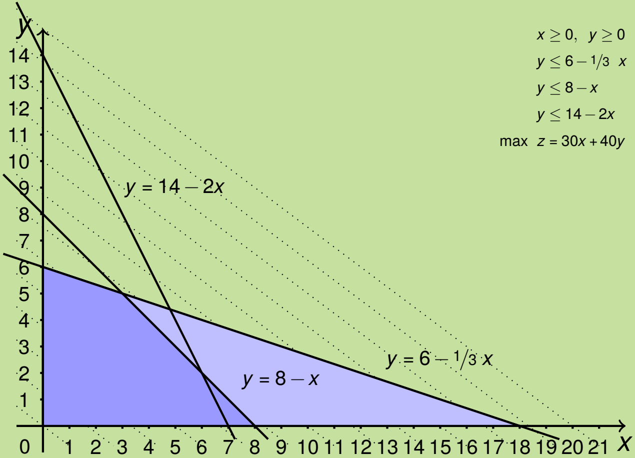
Linear Programming

Example (ctd., visualization)



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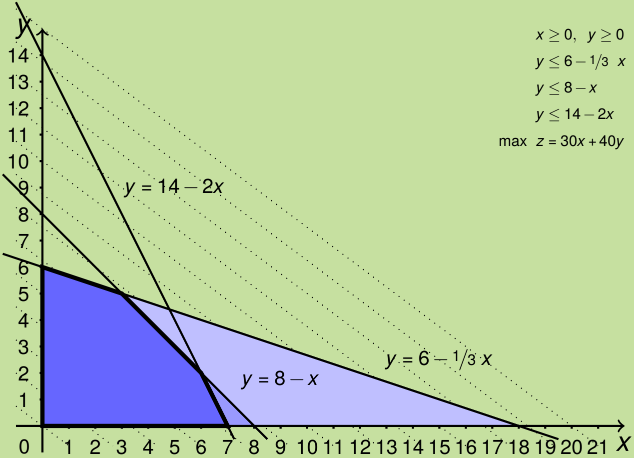
Example (ctd., visualization)



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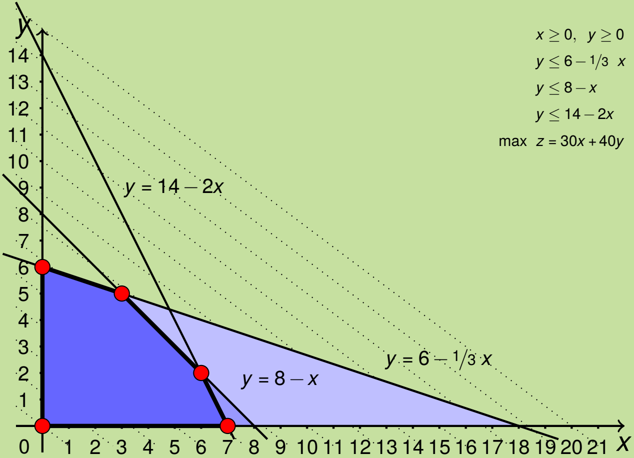
Example (ctd., visualization)



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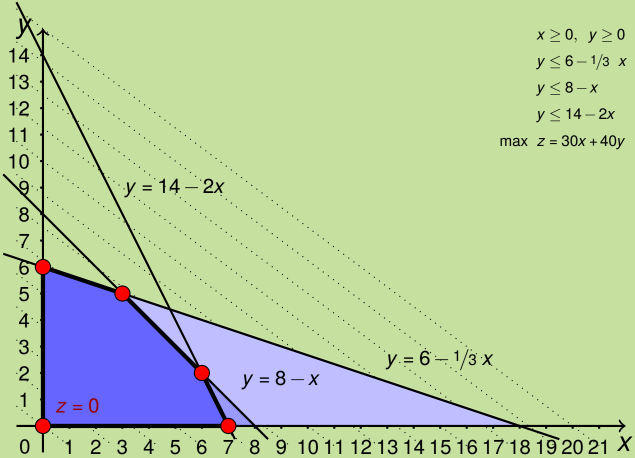

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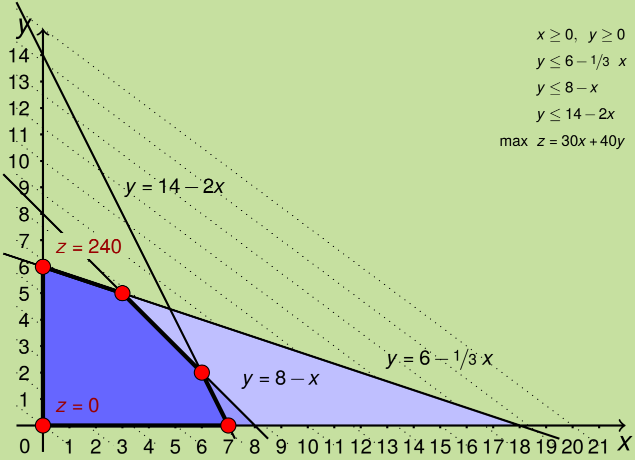
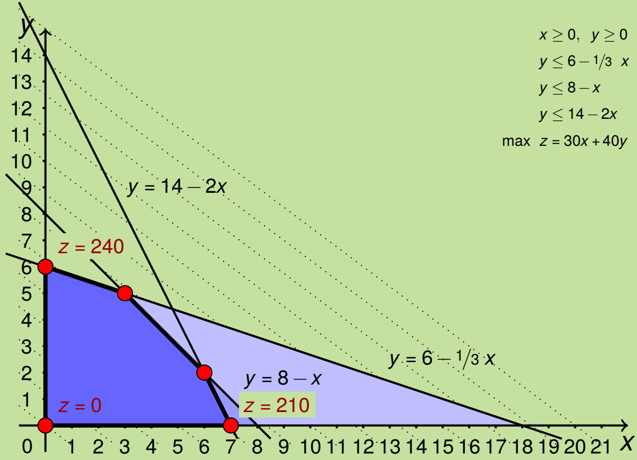


Example (ctd., visualization)



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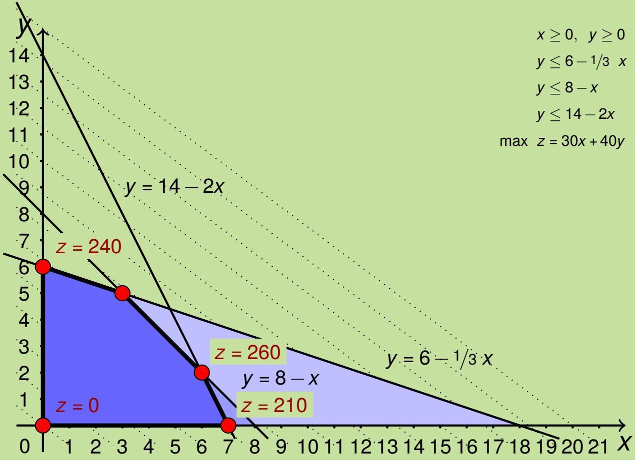

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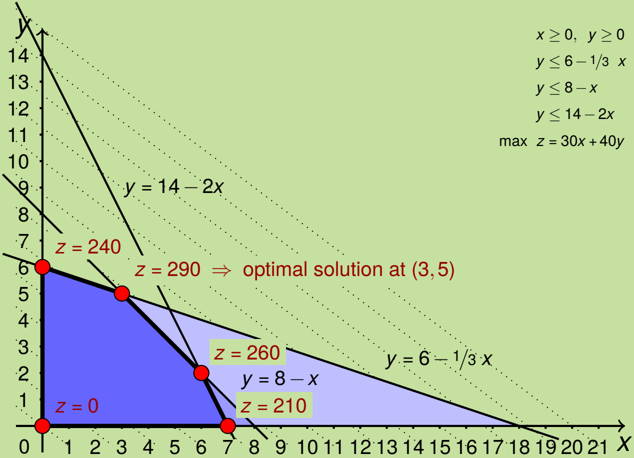

Linear Programming

Definition (Linear program)

A linear program (LP) in standard from consists of

- n real-valued variables x_i; n coefficients b_i;
- m constants c_j; $n \cdot m$ coefficients a_{ij};
- m constraints of the form

$$c_j \leq \sum_{i=1}^{n} a_{ij} x_i,$$

- and an objective function to be minimized ($x_i \geq 0$)

$$\sum_{i=1}^{n} b_i x_i.$$

Solution of an LP:
assignment of values to the x_i satisfying the constraints and minimizing the objective function.

Remarks:

- Maximization instead of minimization: easy, just change the signs of all the b_i's, $i = 1, \ldots, n$.
- Equalities instead of inequalities: $x + y \leq c$ if and only if there is a $z \geq 0$ such that $x + y + z = c$ (z is called a slack variable).

Linear Programming

Solution algorithms:

- Usually, one uses the simplex algorithm (which is worst-case exponential!).
- There are also polynomial-time algorithms such as interior-point or ellipsoid algorithms.

Tools and libraries:

- lp_solve
- CLP
- GLPK
- CPLEX
- gurobi

Game Theory

4. Computational Complexity

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Summer semester 2020

Motivation

Motivation: We already know some algorithms for finding Nash equilibria in restricted settings from the previous chapter, and upper bounds on their complexity.

- For finite zero-sum games: polynomial-time computation.
- For general finite two-player games: computation in **NP**.

Question: What about lower bounds for those cases and in general?

Approach to an answer: In this chapter, we study the computational complexity of finding Nash equilibria.

Finding Nash Equilibria as a Search Problem

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Definition (The problem of computing a Nash equilibrium)

NASH

Given: A finite two-player strategic game G.

Find: A mixed-strategy Nash equilibrium (α, β) of G.

Remarks:

- No need to add restriction "…if one exists, else 'fail'", because existence is guaranteed by Nash's theorem.
- The corresponding decision problem can be trivially solved in constant time (always return "true").
  Hence, we really need to consider the search problem version instead.

Finding Nash Equilibria as a Search Problem

In this form, NASH looks similar to other search problems, e. g.:

SAT

Given: A propositional formula φ in CNF.

Find: A truth assignment that makes φ true, if one exists, else ‘fail’.

Note: This is the search version of the usual decision problem.

Search Problems

A search problem is given by a binary relation $R(x, y)$.

Definition (Search problem)

A search problem is a problem that can be stated in the following form, for a given binary relation $R(x, y)$ over strings:

SEARCH-R

Given: x.

Find: Some y such that $R(x, y)$ holds, if such a y exists, else 'fail'.

Complexity Classes for Search Problems

Some complexity classes for search problems:

- **FP**: class of search problems that can be solved by a deterministic Turing machine in polynomial time.
- **FNP**: class of search problems that can be solved by a nondeterministic Turing machine in polynomial time.
- **TFNP**: class of search problems in **FNP** where the relation R is total, i. e., $\forall x \exists y . R(x, y)$.
- **PPAD**: class of search problems that can be polynomially reduced to END-OF-LINE.

  (PPAD: Polynomial Parity Argument in Directed Graphs)

To understand **PPAD**, we need to understand what the END-OF-LINE problem is.

The END-OF-LINE Problem

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Definition (END-OF-LINE instance)

Consider a directed graph $\mathcal{G}$ with node set $\{0,1\}^n$ such that each node has in-degree and out-degree at most one and there are no isolated vertices. The graph $\mathcal{G}$ is specified by two polynomial-time computable functions π and σ:

- $\pi(v)$: returns the predecessor of v, or $\perp$ if v has no predecessor.
- $\sigma(v)$: returns the successor of v, or $\perp$ if v has no successor.

In $\mathcal{G}$, there is an arc from v to v' if and only if $\sigma(v) = v'$ and $\pi(v') = v$.

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Definition (END-OF-LINE instance (ctd.))

We call a triple (π, σ, v) consisting of such functions π and σ and a node v in $\mathscr{G}$ with in-degree zero (a “source”) an END-OF-LINE instance.

With this, we can define the END-OF-LINE problem:

Definition (END-OF-LINE problem)

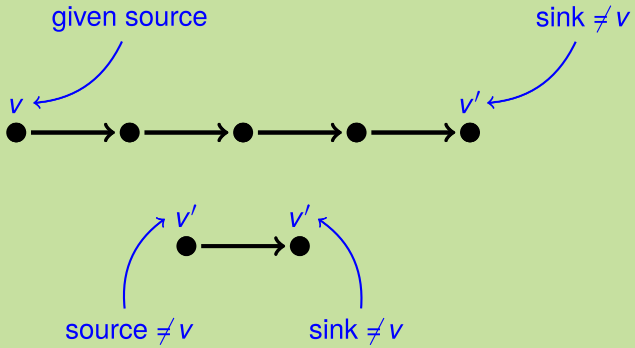
END-OF-LINE

Given: An END-OF-LINE instance (π, σ, v).

Find: Some node $v' \neq v$ such that v' has out-degree zero (a “sink”) or in-degree zero (another “source”).

The END-OF-LINE Problem

Example (END-OF-LINE)



Comparison of Search Complexity Classes

Relationship of different search complexity classes:

$$FP \subseteq PPAD \subseteq TFNP \subseteq FNP$$

Compare to upper runtime bound that we already know:
Lemke-Howson algorithm has exponential time complexity in the worst case.

PPAD-Completeness of NASH

Theorem (Daskalakis et al., 2006)

NASH *is* ***PPAD****-complete.*

The same holds for k-player instead of just two-player NASH. □

Thus, NASH is presumably "simpler" than the SAT search problem, but presumably "harder" than any polynomial search problem.

FNP-Completeness of 2ND-NASH

Another search problem related to Nash equilibria is the problem of finding a second Nash equilibrium (given a first one has already been found). As it turns out, this is at least as hard as finding a first Nash equilibrium.

Definition (2ND-NASH problem)

2ND-NASH

Given: A finite two-player game G and a mixed-strategy Nash equilibrium of G.

Find: A second different mixed-strategy Nash equilibrium of G, if one exists, else 'fail'.

Theorem (Conitzer and Sandholm, 2003)

2ND-NASH *is **FNP**-complete.* □

Some Further Hardness Results

Theorem (Conitzer and Sandholm, 2003)

*For each of the following properties P^ℓ, $\ell = 1,2,3,4$, given a finite two-player game G, it is **NP**-hard to decide whether there exists a mixed-strategy Nash equilibrium (α,β) in G that has property P^ℓ.*

P^1 : *player 1 (or 2) receives a payoff $\geq k$ for some given k.*
("Guaranteed payoff problem")

P^2 : $U_1(\alpha,\beta) + U_2(\alpha,\beta) \geq k$ *for some given k.*
("Guaranteed social welfare problem")

P^3 : *player 1 (or 2) plays some given action a with prob. > 0.*

P^4 : (α,β) *is Pareto-optimal, i. e., there is no strategy profile (α',β') such that*

- $U_i(\alpha',\beta') \geq U_i(\alpha,\beta)$ *for both $i \in \{1,2\}$, and*
- $U_i(\alpha',\beta') > U_i(\alpha,\beta)$ *for at least one $i \in \{1,2\}$.* □

Summary

- **PPAD** is the complexity class for which the END-OF-LINE problem is complete.
- Finding a mixed-strategy Nash equilibrium in a finite two-player strategic game is **PPAD**-complete.
- **FNP** is the search-problem equivalent of the class **NP**.
- Finding a second mixed-strategy Nash equilibrium in a finite two-player strategic game is **FNP**-complete.
- Several decision problems related to Nash equilibria are **NP**-complete:
  - guaranteed payoff
  - guaranteed social welfare
  - inclusion in support
  - Pareto-optimality of Nash equilibria

Game Theory

5. Extensive Games

5.1. Extensive Games with Perfect Information

5.1.1. Motivation and Definitions

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Motivation

- So far: All players move simultaneously, and then the outcome is determined.
- Often in practice: Several moves in sequence (e. g. in chess).
  $\rightsquigarrow$ cannot be directly reflected by strategic games.
- Extensive games (with perfect information) reflect such situations by modeling games as game trees.
- Idea: Players have several decision points where they can decide how to play.
- Strategies: Mappings from decision points in the game tree to actions to be played.

Outline

- Section 5.1: extensive games with perfect information (players know the state of the game and the actions chosen by other players, e. g., chess)
- Section 5.2: extensive games with imperfect information (players may not know the state of the game or the actions chosen by other players, e. g., poker)

Extensive Games

Definition (Extensive game with perfect information)

An extensive game with perfect information is a tuple $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ that consists of:

- a finite non-empty set N of players,
- a set H of (finite or infinite) sequences, called histories, such that
  - it contains the empty sequence $\langle\rangle \in H$,
  - H is closed under prefixes: if $\langle a^1, \dots, a^k \rangle \in H$ for some $k \in \mathbb{N} \cup \{\infty\}$, and $l < k$, then also $\langle a^1, \dots, a^l \rangle \in H$, and
  - H is closed under limits: if for some infinite sequence $\langle a^i \rangle_{i=1}^{\infty}$, we have $\langle a^i \rangle_{i=1}^{k} \in H$ for all $k \in \mathbb{N}$, then $\langle a^i \rangle_{i=1}^{\infty} \in H$.

  All infinite histories and all histories $\langle a^i \rangle_{i=1}^{k} \in H$, for which there is no a^{k+1} such that $\langle a^i \rangle_{i=1}^{k+1} \in H$ are called terminal histories Z. Components of a history are called actions.

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Definition (Extensive game with perfect information, ctd.)

- a player function $P : H \setminus Z \to N$ that determines which player's turn it is to move after a given nonterminal history, and
- for each player $i \in N$, a utility function (or payoff function) $u_i : Z \to \mathbb{R}$ defined on the set of terminal histories.

The game is called finite, if H is finite. It has a finite horizon, if the lenght of histories is bounded from above.

Assumption: All ingredients of Γ are common knowledge amongst the players of the game.

Terminology: In the rest of Section 5.1, we will write extensive games instead of extensive games with perfect information.

Example (Division game)

- Two identical objects should be divided among two players.
- Player 1 proposes an allocation.
- Player 2 agrees or rejects.
  - on agreement: allocation as proposed
  - on rejection: nobody gets anything

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Example (Division game)

- Two identical objects should be divided among two players.
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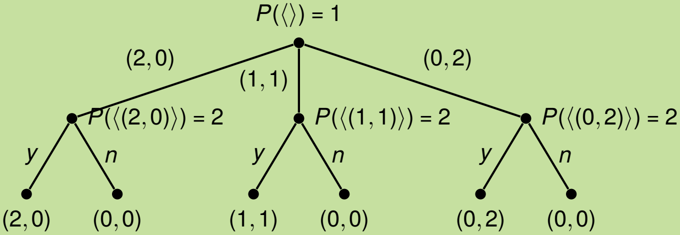
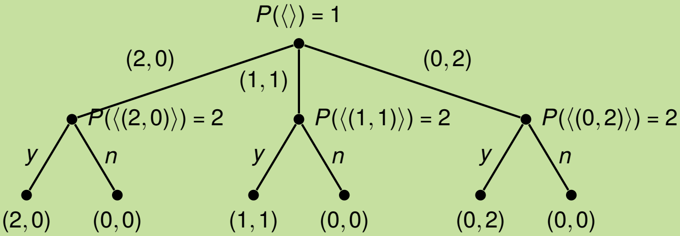


Example (Division game, formally)



- $N = \{1,2\}$
- $H = \{\langle\rangle, \langle(2,0)\rangle, \langle(1,1)\rangle, \langle(0,2)\rangle, \langle(2,0),y\rangle, \langle(2,0),n\rangle, \ldots\}$
- $P(\langle\rangle) = 1$, $P(h) = 2$ for all $h \in H \setminus Z$ with $h \neq \langle\rangle$
- $u_1(\langle(2,0),y\rangle) = 2$, $u_2(\langle(2,0),y\rangle) = 0$, etc.

Notation:

Let $h = \langle a^1, \dots, a^k \rangle$ be a history, and a an action.

- Then (h, a) is the history $\langle a^1, \dots, a^k, a \rangle$.
- If $h' = \langle b^1, \dots, b^\ell \rangle$, then (h, h') is the history $\langle a^1, \dots, a^k, b^1, \dots, b^\ell \rangle$.
- The set of actions from which player $P(h)$ can choose after a history $h \in H \setminus Z$ is written as

$$A(h) = \{a \mid (h, a) \in H\}.$$

Question: What is $A(\langle (2, 0) \rangle)$ in the division game?

Strategies

Definition (Strategy in an extensive game)

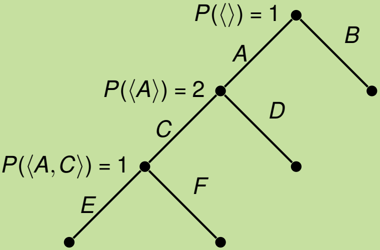
A strategy of a player i in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a function s_i that assigns to each nonterminal history $h \in H \setminus Z$ with $P(h) = i$ an action $a \in A(h)$. The set of strategies of player i is denoted as S_i.

Remark: Strategies require us to assign actions to histories h, even if it is clear that they will never be played (e. g., because h will never be reached because of some earlier action).

Notation (for finite games): A strategy for a player is written as a string of actions at decision nodes as visited in a breadth-first order.

Strategies

Example (Strategies in an extensive game)



- Strategies for player 1: AE, AF, BE and BF
- Strategies for player 2: C and D.

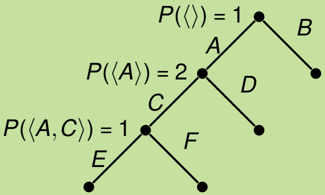
Outcome

Definition (Outcome)

The outcome $O(s)$ of a strategy profile $s = (s_i)_{i \in N}$ is the (possibly infinite) terminal history $h = \langle a^i \rangle_{i=1}^{k}$, with $k \in \mathbb{N} \cup \{\infty\}$, such that for all $\ell \in \mathbb{N}$ with $0 \leq \ell < k$,

$$s_{P(\langle a^1,\ldots,a^\ell \rangle)}(\langle a^1, \ldots, a^\ell \rangle) = a^{\ell+1}.$$

Example (Outcome)



$O(AF, C) = \langle A, C, F \rangle$

$O(AE, D) = \langle A, D \rangle.$

Summary

- Extensive games allow several moves in sequence (cf. game trees).
- Formalized using players, histories, player function, payoff functions for terminal histories.
- Strategies: mappings from decision points to actions
- Outcome: terminal history resulting from strategy profile

Game Theory

5. Extensive Games

5.1. Extensive Games with Perfect Information

5.1.2. Solution Concepts

Albert-Ludwigs-Universität Freiburg

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Motivation

- So far: definition of extensive games
- Now: solution concepts for extensive games
  - transfer the idea of Nash equilibria
  - identify problems with Nash equilibria
    $\rightsquigarrow$ subgame-perfect equilibria

Nash Equilibria

Definition (Nash equilibrium in an extensive game)

A Nash equilibrium in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a strategy profile s^* such that for every player $i \in N$ and for all strategies $s_i \in S_i$,

$$u_i(O(s^*_{-i}, s^*_i)) \geq u_i(O(s^*_{-i}, s_i)).$$

Induced Strategic Game

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Definition (Induced strategic game)

The strategic game G induced by an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is defined by $G = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$, where

- $A'_i = S_i$ for all $i \in N$, and
- $u'_i(a) = u_i(O(a))$ for all $i \in N$.

Proposition

The Nash equilibria of an extensive game Γ are exactly the Nash equilibria of the induced strategic game G of Γ. $\square$

Induced Strategic Game

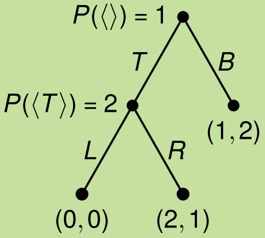
Remarks:

- Each extensive game can be transformed into a strategic game, but the resulting game can be exponentially larger.
- The other direction does not work, because in extensive games, we do not have simultaneous actions.

Empty Threats

Example (Empty threat)

Extensive game:



Induced strategic game:

	L	R
T		
B		

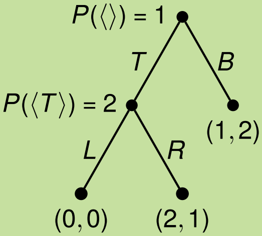
Strategies:

- Player 1: T and B
- Player 2: L and R

Empty Threats

Example (Empty threat)

Extensive game:



Induced strategic game:

	L	R
T	$0, 0$	$2, 1$
B	$1, 2$	$1, 2$

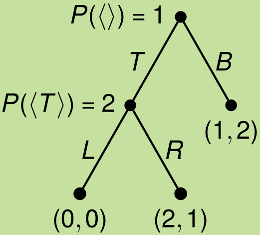
Strategies:

- Player 1: T and B
- Player 2: L and R

Empty Threats

Example (Empty threat)

Extensive game:



Strategies:

- Player 1: T and B
- Player 2: L and R

Induced strategic game:

	L	R
T	$0,0$	$2,1$
B	$1,2$	$1,2$

Nash equilibria: (B,L) and (T,R).

However, (B,L) is not realistic:

- Player 1 plays B, "fearing" response L to T.
- But player 2 would never play L against T in the extensive game. $\rightsquigarrow$ (B,L) involves "empty threat".

Subgames

Idea: Exclude empty threats.

How? Demand that a strategy profile is not only a Nash equilibrium in the entire game, but also in every subgame.

Definition (Subgame)

A subgame of an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$, starting after history h, is the game $\Gamma(h) = \langle N, H|_h, P|_h, (u_i|_h)_{i \in N} \rangle$, where

- $H|_h = \{h' \mid (h, h') \in H\}$,
- $P|_h(h') = P(h, h')$ for all $h' \in H|_h$, and
- $u_i|_h(h') = u_i(h, h')$ for all $h' \in H|_h$.

Subgames

Definition (Strategy in a subgame)

Let Γ be an extensive game and $\Gamma(h)$ a subgame of Γ starting after some history h.

For each strategy s_i of Γ, let $s_i|_h$ be the strategy induced by s_i for $\Gamma(h)$. Formally, for all $h' \in H|_h$,

$$s_i|_h(h') = s_i(h, h').$$

The outcome function of $\Gamma(h)$ is denoted by O_h.

Subgame-Perfect Equilibria

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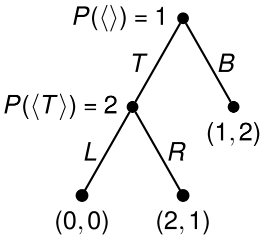
Definition (Subgame-perfect equilibrium)

A strategy profile s^* in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a subgame-perfect equilibrium (SPE) if and only if for every player $i \in N$ and every nonterminal history $h \in H \setminus Z$ with $P(h) = i$,

$$u_i|_h(O_h(s^*_{-i}|_h, s^*_i|_h)) \geq u_i|_h(O_h(s^*_{-i}|_h, s_i))$$

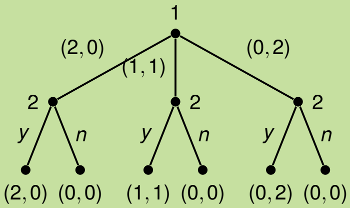
for every strategy $s_i \in S_i$ in subgame $\Gamma(h)$.



Two Nash equilibria:

- (T,R): subgame-perfect, because:
  - In history $h = \langle T\rangle$: subgame-perfect.
  - In history $h = \langle\rangle$: player 1 obtains utility 1 when choosing B and utility of 2 when choosing T.
- (B,L): not subgame-perfect, since L does not maximize the utility of player 2 in history $h = \langle T\rangle$.

Example (Subgame-perfect equilibria in division game)



Equilibria in subgames:

- in $\Gamma(\langle(2,0)\rangle)$: y and n
- in $\Gamma(\langle(1,1)\rangle)$: only y
- in $\Gamma(\langle(0,2)\rangle)$: only y
- in $\Gamma(\langle\rangle)$: $((2,0),yyy)$ and $((1,1),nyy)$

Nash equilibria (red: no SPE):

- $((2,0),yyy)$, $((2,0),yyn)$, $((2,0),yny)$, $((2,0),ynn)$, $((2,0),nny)$, $((2,0),nnn)$,
- $((1,1),nyy)$, $((1,1),nyn)$,
- $((0,2),nny)$.

Summary

- Nash equilibria in extensive game with perfect information
  - defined directly or
  - defined via induced strategic game
- Problems with Nash equilibria:
  - exponentially many strategies to consider
  - empty threats
- Alternative/better solution concept without empty threats: subgame-perfect equilibria
  - have to be an equilibrium in every subgame

Game Theory

5. Extensive Games

5.1. Extensive Games with Perfect Information

5.1.3. One-Deviation Property

Albert-Ludwigs-Universität Freiburg

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Motivation

- So far:
  - definition of extensive games
  - solution concept: subgame-perfect equilibria (SPE)
- Now:
  - Existence:
    - Does every extensive game have a subgame-perfect equilibrium?
    - If not, which extensive games do have a subgame-perfect equilibrium?
  - Computation:
    - If a subgame-perfect equilibrium exists, how to compute it?
    - How complex is that computation?

Motivation

Positive case (a subgame-perfect equilibrium exists):

- Step 1: Show that is suffices to consider local deviations from strategies (for finite-horizon games).
  (Section 5.1.3)
- Step 2: Show how to systematically explore such local deviations to find a subgame-perfect equilibrium (for finite games).
  (Section 5.1.4)

Step 1: One-Deviation Property

Definition

Let Γ be a finite-horizon extensive game. Then $\ell(\Gamma)$ denotes the length of the longest history of Γ.

Step 1: One-Deviation Property

Definition (One-deviation property)

A strategy profile s^* in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ satisfies the one-deviation property if and only if for every player $i \in N$ and every nonterminal history $h \in H \setminus Z$ with $P(h) = i$,

$$u_i|_h(O_h(s^*_{-i}|_h, s^*_i|_h)) \geq u_i|_h(O_h(s^*_{-i}|_h, s_i))$$

for every strategy $s_i \in S_i$ in subgame $\Gamma(h)$ that differs from $s^*_i|_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Note: Without the highlighted part in the end, this is just the definition of subgame-perfect equilibria!

Step 1: One-Deviation Property

Lemma

Let $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ be a finite-horizon extensive game. Then a strategy profile s^* is a subgame-perfect equilibrium of Γ if and only if it satisfies the one-deviation property.

Proof

- ($\Rightarrow$) Clear.
- ($\Leftarrow$) By contradiction:

  Suppose that s^* is not a subgame-perfect equilibrium.

  Then there is a history h and a player i such that s_i is a profitable deviation for player i in subgame $\Gamma(h)$.

  …

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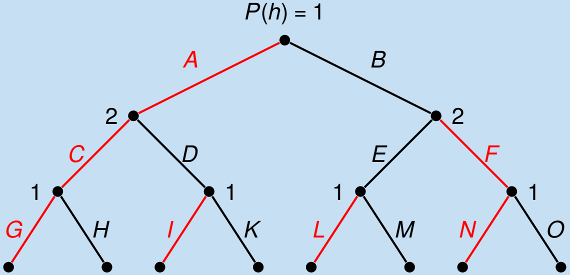
Proof (ctd.)

- ($\Leftarrow$) …WLOG, the number of histories h' with $s_i(h') \neq s_i^*|_h(h')$ is at most $\ell(\Gamma(h))$ and hence finite (finite horizon assumption!), since deviations not on resulting outcome path are irrelevant.

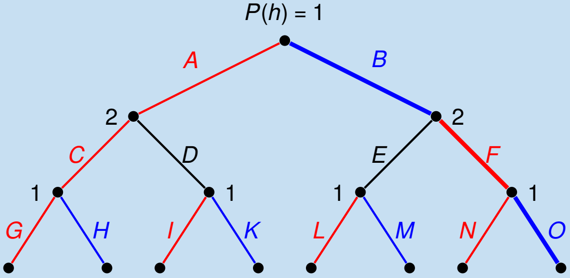
  Illustration: strategies $s_1^*|_h = AGILN$ and $s_2^*|_h = CF$:



Step 1: One-Deviation Property

Proof (ctd.)

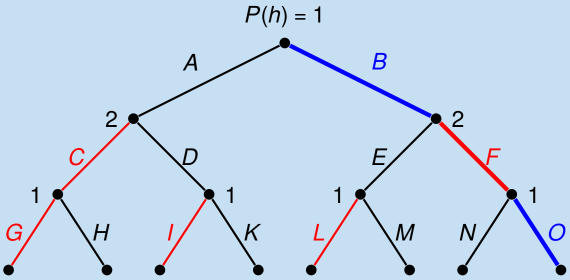
- (⇐) …Illustration for WLOG assumption: Assume $s_1 = BHKMO$ (blue) profitable deviation:



Then only B and O really matter.

Proof (ctd.)

- (⇐) ... Illustration for WLOG assumption: And hence $\tilde{s}_1 = BGILO$ (blue) also profitable deviation:



Step 1: One-Deviation Property

Proof (ctd.)

- ($\Leftarrow$) ...

  Choose profitable deviation s_i in $\Gamma(h)$ with minimal number of deviation points (such s_i must exist).

  Let h^* be the longest history in $\Gamma(h)$ with $s_i(h^*) \neq s_i^*|_h(h^*)$, i.e., "deepest" deviation point for s_i.

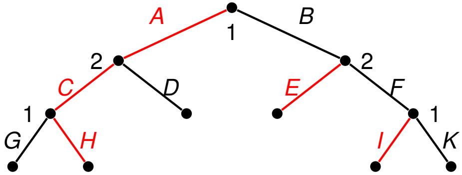
  Then in $\Gamma(h, h^*)$, $s_i|_{h^*}$ differs from $s_i^*|_{(h,h^*)}$ only in the initial history.

  Moreover, $s_i|_{h^*}$ is a profitable deviation in $\Gamma(h, h^*)$, since otherwise fewer deviation points would suffice.

  So, $\Gamma(h, h^*)$ is the desired subgame where a one-step deviation is sufficient to improve utility. □

Step 1: One-Deviation Property

Example



To show that (AHI, CE) is a subgame-perfect equilibrium, it suffices to check these deviating strategies:

Player 1:

- G in subgame $\Gamma(\langle A, C \rangle)$
- K in subgame $\Gamma(\langle B, F \rangle)$
- BHI in Γ

Player 2:

- D in subgame $\Gamma(\langle A \rangle)$
- F in subgame $\Gamma(\langle B \rangle)$

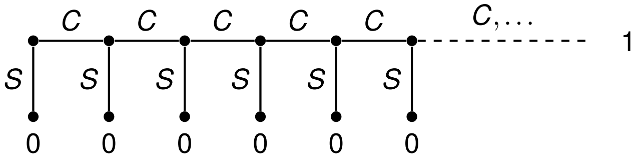
In particular, e.g., no need to check if strategy BGK of player 1 is profitable in Γ.

Step 1: One-Deviation Property

Remark on Infinite-Horizon Games

The corresponding proposition for infinite-horizon games does not hold.

Counterexample (one-player case):



Strategy s_i with $s_i(h) = S$ for all $h \in H \setminus Z$

- satisfies one deviation property, but
- is not a subgame-perfect equilibrium, since it is dominated by s_i^* with $s_i^*(h) = C$ for all $h \in H \setminus Z$.

Summary

- For finite-horizon extensive games, it suffices to consider local deviations when looking for better strategies.
- This simplifies verifying whether a strategy profile is an SPE (or finding one).
- For infinite-horizon games, this is not true in general.

Game Theory

5. Extensive Games

5.1. Extensive Games with Perfect Information

5.1.4. Kuhn’s Theorem

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Motivation

- So far:
  - definition of extensive games
  - solution concept: subgame-perfect equilibria (SPE)
  - one deviation property
- Now:
  - Existence:
    - Does every extensive game have a subgame-perfect equilibrium?
    - If not, which extensive games do have a subgame-perfect equilibrium?
  - Computation:
    - If a subgame-perfect equilibrium exists, how to compute it?
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Motivation

Positive case (a subgame-perfect equilibrium exists):

- Step 1: Show that is suffices to consider local deviations from strategies (for finite-horizon games).
  (Section 5.1.3)
- Step 2: Show how to systematically explore such local deviations to find a subgame-perfect equilibrium (for finite games).
  (Section 5.1.4)

Step 2: Kuhn's Theorem

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Theorem (Kuhn)

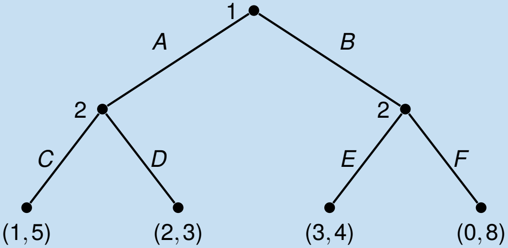
Every finite extensive game has a subgame-perfect equilibrium.

Proof idea:

- Proof is constructive and builds a subgame-perfect equilibrium bottom-up (aka backward induction).
- For those familiar with the Foundations of AI lecture: generalization of Minimax algorithm to general-sum games with possibly more than two players.

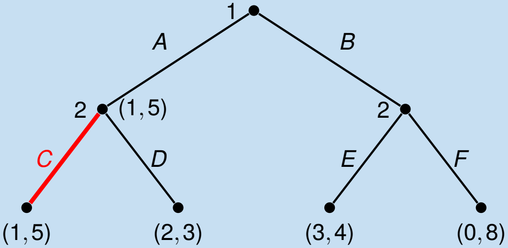
Step 2: Kuhn's Theorem

Example



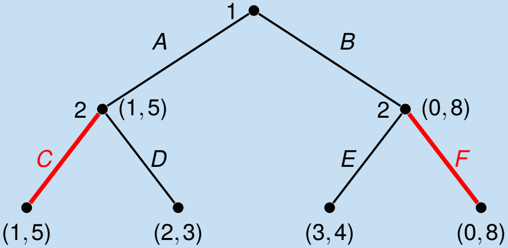
Step 2: Kuhn's Theorem

Example



$s_2(\langle A \rangle) = C$ $\quad$ $t_1(\langle A \rangle) = 1$ $\quad$ $t_2(\langle A \rangle) = 5$

Example

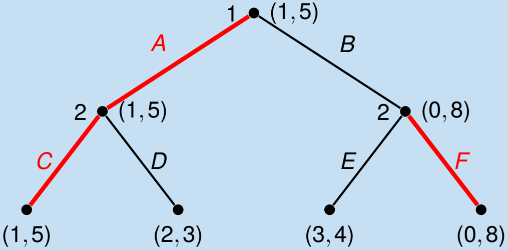


$s_2(\langle A\rangle) = C$ $\quad t_1(\langle A\rangle) = 1$ $\quad t_2(\langle A\rangle) = 5$

$s_2(\langle B\rangle) = F$ $\quad t_1(\langle B\rangle) = 0$ $\quad t_2(\langle B\rangle) = 8$

Step 2: Kuhn's Theorem

Example



$$s_2(\langle A\rangle) = C \qquad t_1(\langle A\rangle) = 1 \qquad t_2(\langle A\rangle) = 5$$

$$s_2(\langle B\rangle) = F \qquad t_1(\langle B\rangle) = 0 \qquad t_2(\langle B\rangle) = 8$$

$$s_1(\langle\rangle) = A \qquad t_1(\langle\rangle) = 1 \qquad t_2(\langle\rangle) = 5$$

Step 2: Kuhn's Theorem

A bit more formally:

Proof

Let $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ be a finite extensive game.

Construct a subgame-perfect equilibrium by induction on $\ell(\Gamma(h))$ for all subgames $\Gamma(h)$. In parallel, construct functions $t_i : H \to \mathbb{R}$ for all players $i \in N$ s. t. $t_i(h)$ is the payoff for player i in a subgame-perfect equilibrium in subgame $\Gamma(h)$.

Base case: If $\ell(\Gamma(h)) = 0$, then $t_i(h) = u_i(h)$ for all $i \in N$.

...

Step 2: Kuhn's Theorem

Proof (ctd.)

Inductive case: If $t_i(h)$ already defined for all $h \in H$ with $\ell(\Gamma(h)) \leq k$, consider $h^* \in H$ with $\ell(\Gamma(h^*)) = k + 1$ and $P(h^*) = i$.

For all $a \in A(h^*)$, $\ell(\Gamma(h^*, a)) \leq k$, let

$$s_i(h^*) := \operatorname*{argmax}_{a \in A(h^*)} t_i(h^*, a) \quad \text{and}$$

$$t_j(h^*) := t_j(h^*, s_i(h^*)) \quad \text{for all players } j \in N.$$

Inductively, we obtain a strategy profile s that satisfies the one-deviation property.

With the one-deviation property lemma it follows that s is a subgame-perfect equilibrium. □

Step 2: Kuhn's Theorem

- In principle: sample subgame-perfect equilibrium effectively computable using the technique from the above proof.
- In practice: often game trees not enumerated in advance, hence unavailable for backward induction.
- E.g., for branching factor b and depth m, procedure needs time $O(b^m)$.

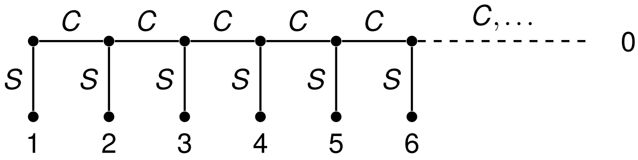
Corresponding proposition for infinite games does not hold.

Counterexamples (both for one-player case):

A) finite horizon, infinite branching factor:

Infinitely many actions $a \in A = [0, 1)$ with payoffs $u_1(\langle a \rangle) = a$ for all $a \in A$.
There exists no subgame-perfect equilibrium in this game.

B) infinite horizon, finite branching factor:



$u_1(CCC\ldots) = 0$ and $u_1(\underbrace{CC\ldots C}_{n}S) = n + 1$.

No subgame-perfect equilibrium.

Step 2: Kuhn's Theorem

Uniqueness:

Kuhn's theorem tells us nothing about uniqueness of subgame-perfect equilibria. However, if no two histories get the same evaluation by any player, then the subgame-perfect equilibrium is unique.

Extended Example: Pirate Game

1. There are five rational pirates, A, B, C, D and E. They find 100 gold coins. They must decide how to distribute them.
2. The pirates have a strict order of seniority: A is senior to B, who is senior to C, who is senior to D, who is senior to E.
3. The pirate world's rules of distribution say that the most senior pirate first proposes a distribution of coins. The pirates, including the proposer, then vote on whether to accept this distribution (in order from most junior to senior). In case of a tie vote, the proposer has the casting vote. If the distribution is accepted, the coins are disbursed and the game ends. If not, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to apply the method again.

Pirates: General Setting & Utility

4. The pirates do not trust each other, and will neither make nor honor any promises between pirates apart from a proposed distribution plan that gives a whole number of gold coins to each pirate.
5. Pirates base their decisions on three factors. First of all, each pirate wants to survive. Second, everything being equal, each pirate wants to maximize the number of gold coins each receives. Third, each pirate would prefer to throw another overboard, if all other results would otherwise be equal.

Pirates: Formalization

- Players $N = \{A, B, C, D, E\}$;
- actions are:
  - proposals by a pirate: $\langle A : x_A, B : x_B, C : x_C, D : x_D, E : x_E \rangle$, with $\sum_{i \in \{A,B,C,D,E\}} x_i = 100$;
  - votings: y for accepting, n for rejecting;
- histories are sequences of a proposal, followed by votings of the alive pirates;
- utilities:
  - for pirates who are alive: utilities are according to the accepted proposal plus $x/100$, x being the number of dead pirates;
  - for dead pirates: -100.

Remark: Very large game tree!

Pirates: Analysis by Backward Induction

1. Assume only D and E are still alive. D can propose $\langle A:0, B:0, C:0, D:100, E:0\rangle$, because D has the casting vote!

Pirates: Analysis by Backward Induction

1. Assume only D and E are still alive. D can propose $\langle A:0, B:0, C:0, D:100, E:0\rangle$, because D has the casting vote!
2. Assume C, D, and E are alive. For C it is enough to offer 1 coin to E to get his vote: $\langle A:0, B:0, C:99, D:0, E:1\rangle$.

Pirates: Analysis by Backward Induction

1. Assume only D and E are still alive. D can propose $\langle A:0, B:0, C:0, D:100, E:0\rangle$, because D has the casting vote!
2. Assume C, D, and E are alive. For C it is enough to offer 1 coin to E to get his vote: $\langle A:0, B:0, C:99, D:0, E:1\rangle$.
3. Assume B, C, D, and E are alive. B offering D one coin is enough because of the casting vote: $\langle A:0, B:99, C:0, D:1, E:0\rangle$.

Pirates: Analysis by Backward Induction

1. Assume only D and E are still alive. D can propose $\langle A:0, B:0, C:0, D:100, E:0\rangle$, because D has the casting vote!
2. Assume C, D, and E are alive. For C it is enough to offer 1 coin to E to get his vote: $\langle A:0, B:0, C:99, D:0, E:1\rangle$.
3. Assume B, C, D, and E are alive. B offering D one coin is enough because of the casting vote: $\langle A:0, B:99, C:0, D:1, E:0\rangle$.
4. Assume A, B, C, D, and E are alive. A offering C and E each one coin is enough: $\langle A:98, B:0, C:1, D:0, E:1\rangle$ (note that giving 1 to D instaed to E does not help).

Summary

- Every finite extensive game has a subgame-perfect equilibrium.
- This does not generally hold for infinite games, no matter if the game is infinite due to infinite branching factor or infinitely long histories (or both).
- Subgame-perfect equilibria in finite extensive game can be identified using backward induction.

Game Theory

5. Extensive Games

5.1. Extensive Games with Perfect Information

5.1.5 Simultaneous Moves and Chance Moves

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Summer semester 2020

Motivation

- So far:
  - Finite-horizon extensive games with perfect information: one-deviation property
  - Finite extensive games with perfect information: Kuhn's theorem
- Now: what about those results if we allow
  - simultaneous moves or
  - chance moves?

Definition

An extensive game with simultaneous moves is a tuple $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$, where

- N, H, P and (u_i) are defined as before, and
- $P : H \setminus Z \to 2^N$ assigns to each nonterminal history a set of players to move; for all $h \in H \setminus Z$, there exists a family $(A_i(h))_{i \in P(h)}$ such that

$$A(h) = \{a \,|\, (h, a) \in H\} = \prod_{i \in P(h)} A_i(h).$$

Simultaneous Moves

- Intended meaning of simultaneous moves: all players from $P(h)$ move simultaneously
- Strategies: functions $s_i : h \mapsto a_i$ with $a_i \in A_i(h)$
- Histories: sequences of vectors of actions
- Outcome: terminal history reached when tracing strategy profile
- Payoffs: utilities at outcome history

Observations:

- The one-deviation property still holds for extensive games with perfect information and simultaneous moves.
- Kuhn's theorem does not hold for extensive game with simultaneous moves.

  Example: Matching Pennies can be viewed as extensive game with simultaneous moves. No Nash equilibrium/subgame-perfect equilibrium.

  player 2

		H	T
player 1	H	$1, -1$	$-1, \ \ 1$
	T	$-1, \ \ 1$	$1, -1$

  $\rightsquigarrow$ Need more sophisticated solution concepts (cf. mixed strategies). Not covered in this course.

Setting:

- Three players have to split a cake fairly.
- Player 1 suggest split: shares $x_1, x_2, x_3 \in [0,1]$ s.t. $x_1 + x_2 + x_3 = 1$.
- Then players 2 and 3 simultaneously and independently decide whether to accept ("y") or reject ("n") the suggested splitting.
- If both accept, each player i gets their allotted share (utility x_i). Otherwise, no player gets anything (utility 0).

Formally:

$$
\begin{aligned}
N &= \{1, 2, 3\} \\
X &= \{(x_1, x_2, x_3) \in [0,1]^3 \mid x_1 + x_2 + x_3 = 1\} \\
H &= \{\langle\rangle\} \cup \{\langle x\rangle \mid x \in X\} \cup \{\langle x, z\rangle \mid x \in X, z \in \{\mathrm{y},\mathrm{n}\} \times \{\mathrm{y},\mathrm{n}\}\} \\
P(\langle\rangle) &= \{1\} \\
P(\langle x\rangle) &= \{2, 3\} \text{ for all } x \in X \\
u_i(\langle x, z\rangle) &= \begin{cases} 0 & \text{if } z \in \{(\mathrm{y},\mathrm{n}), (\mathrm{n},\mathrm{y}), (\mathrm{n},\mathrm{n})\} \\ x_i & \text{if } z = (\mathrm{y},\mathrm{y}). \end{cases} \text{ for all } i \in N
\end{aligned}
$$

Subgame-perfect equilibria:

- Subgames after legal split (x_1, x_2, x_3) by player 1:
  - NE (y, y) (both accept)
  - NE (n, n) (neither accepts)
  - If $x_2 = 0$, NE (n, y) (only player 3 accepts)
  - If $x_3 = 0$, NE (y, n) (only player 2 accepts)

Subgame-perfect equilibria (ctd.):

- Entire game:
  Let s_2 and s_3 be any two strategies of players 2 and 3 such that for all splits $x \in X$ the profile $(s_2(\langle x \rangle), s_3(\langle x \rangle))$ is one of the NEs from above.
  Let $X_y = \{x \in X \mid s_2(\langle x \rangle) = s_3(\langle x \rangle) = \mathrm{y}\}$ be the set of splits accepted under s_2 and s_3. Distinguish three cases:
  - $X_y = \emptyset$ or $x_1 = 0$ for all $x \in X_y$. Then (s_1, s_2, s_3) is a subgame-perfect equilibrium for any possible s_1.
  - $X_y \neq \emptyset$ and there are splits $x_{max} = (x_1, x_2, x_3) \in X_y$ that maximize $x_1 > 0$. Then (s_1, s_2, s_3) is a subgame-perfect equilibrium if and only if $s_1(\langle\rangle)$ is such a split x_{max}.
  - $X_y \neq \emptyset$ and there are no splits $(x_1, x_2, x_3) \in X_y$ that maximize x_1. Then there is no subgame-perfect equilibrium, in which player 2 follows strategy s_2 and player 3 follows strategy s_3.

Chance Moves

Definition

An extensive game with chance moves is a tuple
$\Gamma = \langle N, H, P, f_c, (u_i)_{i \in N} \rangle$, where

- N, A, H and u_i are defined as before,
- the player function $P : H \setminus Z \to N \cup \{c\}$ can also take the value c for a chance node, and
- for each $h \in H \setminus Z$ with $P(h) = c$, the function $f_c(\cdot|h)$ is a probability distribution on $A(h)$ such that the probability distributions for all $h \in H \setminus Z$ with $P(h) = c$ are independent of each other.

Chance Moves

- Intended meaning of chance moves: in chance node, an applicable action is chosen randomly with probability according to f_c
- Strategies: as before
- Outcome: for a given strategy profile, the outcome is a probability distribution on the set of terminal histories
- Payoffs: for player i, U_i is the expected payoff (with weights according to outcome probabilities)

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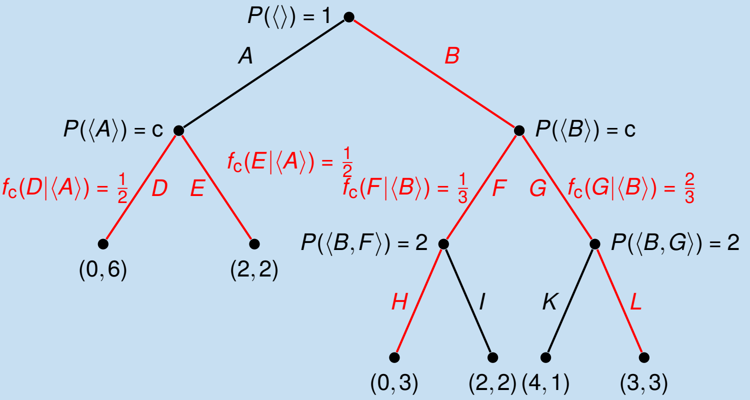
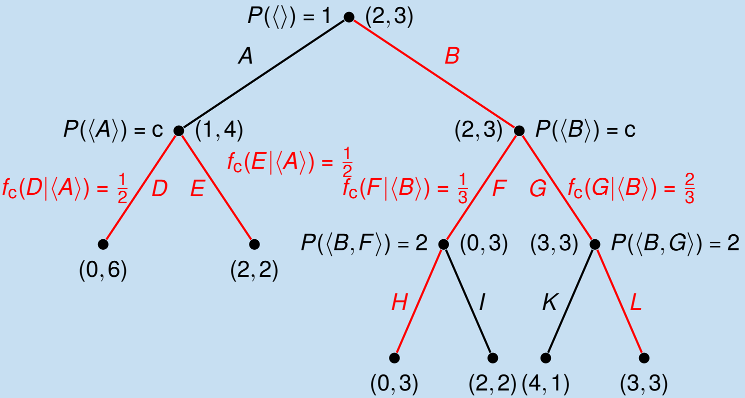
Example



Example


$P(\langle\rangle) = 1$
$(2,3)$
A
B
$P(\langle A\rangle) = c$
$(1,4)$
$(2,3)$
$P(\langle B\rangle) = c$
$f_c(E|\langle A\rangle) = \frac{1}{2}$
$f_c(D|\langle A\rangle) = \frac{1}{2}$
D
E
$f_c(F|\langle B\rangle) = \frac{1}{3}$
F
G
$f_c(G|\langle B\rangle) = \frac{2}{3}$
$(0,6)$
$(2,2)$
$P(\langle B,F\rangle) = 2$
$(0,3)$
$(3,3)$
$P(\langle B,G\rangle) = 2$
H
I
K
L
$(0,3)$
$(2,2)$
$(4,1)$
$(3,3)$

Remark:

The one-deviation property and Kuhn's theorem still hold in the presence of chance moves. When proving Kuhn's theorem, expected utilities have to be used.

Summary

- With simultaneous moves:
  - ✔ One-deviation property still holds.
  - ✗ Kuhn's theorem no longer holds.
- With chance moves:
  - ✔ One-deviation property still holds.
  - ✔ Kuhn's theorem still holds.

Game Theory

5. Extensive Games

5.2. Extensive Games with Imperfect Information

5.2.1. Motivation and Definitions

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Summer semester 2020

Motivation

- So far: All state information is common knowledge among all players
- Often in practice: Only partial knowledge (e. g. in card games)
- Extensive games with imperfect information model such situations using information sets of indistinguishable histories.

Motivation

- Decision points are now such information sets.
- Strategies:
  - Pure: InfoSets $\rightarrow$ Actions
  - Mixed: (InfoSets $\rightarrow$ Actions) $\rightarrow$ Probabilities
    (randomization over pure strategies)
  - Behavioral: InfoSets $\rightarrow$ (Actions $\rightarrow$ Probabilities)
    (collections of independent randomized decisions for each information set)
- Different from incomplete information games, in which there is uncertainty about the utility functions of the other players.

Extensive Games with Imperfect Information

Definition (Extensive game)

An extensive game is a tuple $\Gamma = \langle N, H, P, f_c, (\mathscr{I}_i)_{i\in N}, (u_i)_{i\in N}\rangle$ that consists of:

- a finite non-empty set N of players,
- a set H of (finite or infinite) sequences, called histories, such that
  - it contains the empty sequence $\langle\rangle \in H$,
  - H is closed under prefixes: if $\langle a^1, \dots, a^k\rangle \in H$ for some $k \in \mathbb{N} \cup \{\infty\}$, and $l < k$, then also $\langle a^1, \dots, a^l\rangle \in H$, and
  - H is closed under limits: if for some infinite sequence $\langle a^i\rangle_{i=1}^{\infty}$, we have $\langle a^i\rangle_{i=1}^{k} \in H$ for all $k \in \mathbb{N}$, then $\langle a^i\rangle_{i=1}^{\infty} \in H$.

  All infinite histories and all histories $\langle a^i\rangle_{i=1}^{k} \in H$, for which there is no a^{k+1} such that $\langle a^i\rangle_{i=1}^{k+1} \in H$ are called terminal histories Z. Components of a history are called actions.

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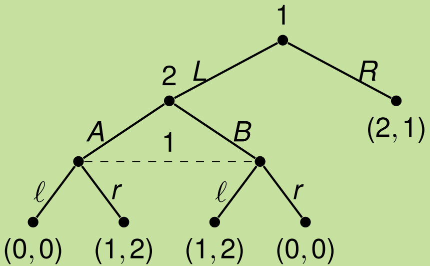
Definition (Extensive game, ctd.)

- a player function $P : H \setminus Z \to N \cup \{c\}$ that determines which player's turn it is to move after a given nonterminal history, c signifying a chance move,
- a probability distribution $f_c(\cdot|h)$ over $A(h)$,
- an information partition $\mathscr{I}_i$ for player i of $\{h \in H | P(h) = i\}$ with the property that $A(h) = A(h')$ whenever h and h' are in the same member $I_i \in \mathscr{I}_i$ of the partition (notation: $A(I_i)$, $P(I_i)$; members I_i of the partition are called information sets), and
- for each player $i \in N$, a utility function (or payoff function) $u_i : Z \to \mathbb{R}$ defined on the set of terminal histories.

Γ is finite, if H is finite; finite horizon, if histories are bounded.

Example



After player 1 chooses L, player 2 makes a move (A or B) that player 1 cannot observe. Formally:

$$\mathscr{I}_1 = \{I_{11}, I_{12}\} \quad \text{with} \quad I_{11} = \{\langle\rangle\} \text{ and } I_{12} = \{\langle L, A\rangle, \langle L, B\rangle\}$$

$$\mathscr{I}_2 = \{I_{21}\} \quad \text{with} \quad I_{21} = \{\langle L\rangle\}$$

Simultaneous Moves

- Question: We already have chance moves, but could/should we extend the model with simultaneous moves as well?
- Answer: We could, but we don't need to.
  Actually, we can already model them somehow.
- In the example game after the history $\langle L \rangle$, we have essentially a simultaneous move of players 1 and 2:
  - When player 2 moves, he does not know what player 1 will do.
  - After player 2 has made his move, player 1 does not know whether A or B was chosen.
  - Only after both players have acted, they are presented with the outcome.

Simultaneous Moves

- Consequence: We will need randomized strategies as part of a reasonable solution concept, since:
  - already for strategic games, we need randomized strategies to guarantee equilibrium existence (in the finite case),
  - strategic games are a special case of extensive games with perfect information and simultaneous moves, and
  - extensive games with perfect information and simultaneous moves are a special case of extensive games with imperfect information.

Summary

- Extensive games with imperfect information can model situations in which the players know only part of the world.
- Modeled by information sets, which are the histories an agent cannot distinguish.
- In the model, we allow chance moves (explicitly) and simultaneous moves (implicitly).

Game Theory

5. Extensive Games

5.2. Extensive Games with Imperfect Information

5.2.2. Perfect Recall

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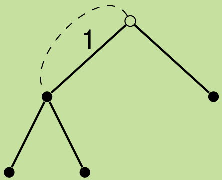
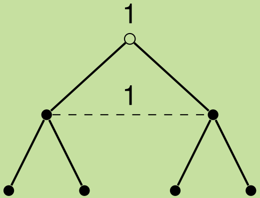
Summer semester 2020

Recall

Information sets can be arbitrary. However, often we want to assume that agents always remember what they have learned before and which actions they have performed: perfect recall.

Example (Imperfect recall)





- Left: player 1 forgets that he made a move!
- Right: player 1 cannot remember what his last move was.

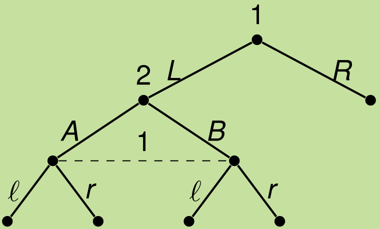
Experience Record

Definition (Experience record)

Given a history h of an extensive game, $X_i(h)$ is the sequence consisting of information sets that player i encounters in h and the actions that player i takes at them. X_i is called the experience record of player i in h.

Experience Record

Example



Player 1 encounters two information sets in the history $h = \langle L, A \rangle$, namely $I_{11} = \{\langle\rangle\}$ and $I_{12} = \{\langle L, A \rangle, \langle L, B \rangle\}$. In the first information set, he chooses L.

Hence, $X_1(h) = \langle I_{11}, L, I_{12} \rangle$.

Perfect Recall

Definition (Perfect recall)

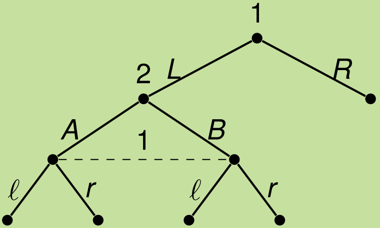
An extensive game has perfect recall if for each player i, we have $X_i(h) = X_i(h')$ whenever the histories h and h' are in the same information set of player i.

Conversely, whenever an agent has made different experiences (own actions, observations) when arriving at h and h', he can distinguish between them.

In most cases, our games will have perfect recall.

Perfect Recall

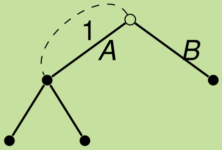
Example

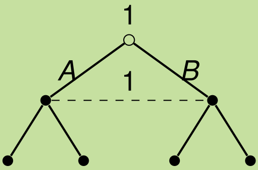


This game has perfect recall: the only $h \neq h'$ in the same information set of some player are $h = \langle L, A \rangle$ and $h' = \langle L, B \rangle$ in information set $I_{12} = \{h, h'\}$. They satisfy the condition, since $X_1(h) = X_1(h') = \langle I_{11}, L, I_{12} \rangle$.

Perfect Recall

Example





No perfect recall:

- Left: player 1 cannot distinguish between $h = \langle\rangle$ and $h' = \langle A\rangle$, although $X_1(h) = \langle\{h, h'\}\rangle \neq \langle\{h, h'\}, A, \{h, h'\}\rangle = X_1(h')$.
- Right: player 1 cannot distinguish between $h = \langle A\rangle$ and $h' = \langle B\rangle$, although $X_1(h) = \langle\{\langle\rangle\}, A, \{h, h'\}\rangle \neq \langle\{\langle\rangle\}, B, \{h, h'\}\rangle = X_1(h')$.

Summary

- Perfect recall requires that agents remember what they have done and learned.
- Formalized using experience records.
- For perfect recall, different experience records must be sufficient for a player to be able to distinguish between two histories.

Game Theory

5. Extensive Games

5.2. Extensive Games with Imperfect Information

5.2.3. Strategies and Outcomes

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Summer semester 2020

Strategies

- Decision points are information sets.
- Types of strategies:
  - Pure: InfoSets $\rightarrow$ Actions
  - Mixed: (InfoSets $\rightarrow$ Actions) $\rightarrow$ Probabilities
    (randomization over pure strategies)
  - Behavioral: InfoSets $\rightarrow$ (Actions $\rightarrow$ Probabilities)
    (collections of independent randomized decisions for each information set)

Pure Strategies

Definition (Pure strategy in an extensive game)

A pure strategy of a player i in an extensive game $\Gamma = \langle N, H, P, f_c, (\mathscr{I}_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a function s_i that assigns an action from $A(I_i)$ to each information set I_i.

Remark: Note that the outcome of a strategy profile s is now a probability distribution (because of the chance moves).

Remark: Because of the chance moves and because of the imperfect information, it probably makes more sense to consider randomized strategies.

Mixed and Behavioral Strategies

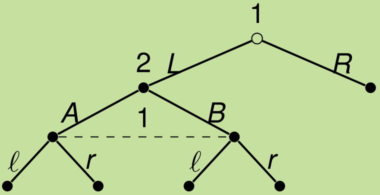
Definition (Mixed and behavioral strategies)

A mixed strategy σ_i of a player i in an extensive game $\Gamma = \langle N, H, P, f_c, (\mathscr{I}_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a probability distribution over the set of player i's pure strategies.

A behavioral strategy of player i is a collection $(\beta_i(I_i))_{I_i \in \mathscr{I}_i}$ of independent probability distributions, where $\beta_i(I_i)$ is a probability distribution over $A(I_i)$.
For any history $h \in I_i \in \mathscr{I}_i$ and action $a \in A(h)$, we denote by $\beta_i(h)(a)$ the probability $\beta_i(I_i)(a)$ assigned by $\beta_i(I_i)$ to action a.

Mixed and Behavioral Strategies

Example



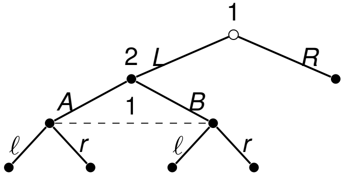
- Player 1 has four pure strategies (two information sets, two actions at each): $L\ell$, Lr, $R\ell$, Rr.
- A mixed strategy is a probability distribution over those.
- A behavioral strategy is a pair of probability distributions, one over $\{L,R\}$ for $\{\langle\rangle\}$, and one over $\{\ell,r\}$ for $\{\langle L,A\rangle,\langle L,B\rangle\}$.

Outcomes

The outcome of a (mixed or behavioral) strategy profile σ is a probability distribution over histories $O(\sigma)$, resulting from following the individual strategies:

- For any history $h = \langle a^1, \dots, a^k \rangle$, define a pure strategy s_i of i to be consistent with h if for any prefix $h' = \langle a^1, \dots, a^\ell \rangle$ of h with $P(h') = i$, we have $s_i(h') = a^{\ell+1}$.
- For any history h, let $\pi_i(h)$ be the sum of probabilities of pure strategies s_i from σ_i that are consistent with h.
- Then for any mixed profile σ, the probability that $O(\sigma)$ assigns to a terminal history h is: $\prod_{i \in N \cup \{c\}} \pi_i(h)$ (where $\pi_c(h)$ is the product of the $f_c(\cdot|\cdot)$ values along h).
- For any behavioral profile β, the probability that $O(\beta)$ assigns to $h = \langle a^1, \dots, a^K \rangle$ is: $\prod_{k=0}^{K-1} \beta_{P(\langle a^1,\dots,a^k \rangle)}(\langle a^1, \dots, a^k \rangle)(a^{k+1})$.

Outcomes: Mixed Strategies – Example



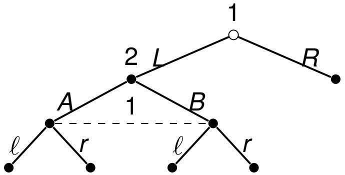
Assume $\sigma_1 = \{L\ell \mapsto {}^{28}/_{70}, Lr \mapsto {}^{21}/_{70}, R\ell \mapsto {}^{12}/_{70}, Rr \mapsto {}^{9}/_{70}\}$,
$\sigma_2 = \{A \mapsto {}^{1}/_{2}, B \mapsto {}^{1}/_{2}\}$, and $\sigma = (\sigma_1, \sigma_2)$.

Then, e. g., $s_{13} = R\ell$, $s_{14} = Rr$, $s_{21} = A$, and $s_{22} = B$ all consistent with $h = \langle R \rangle$, but $s_{11} = L\ell$ and $s_{12} = Lr$ not.

$\pi_1(\langle R \rangle) = \sigma_1(R\ell) + \sigma_1(Rr) = {}^{12}/_{70} + {}^{9}/_{70} = {}^{3}/_{10}$,
$\pi_2(\langle R \rangle) = \sigma_2(A) + \sigma_2(B) = {}^{1}/_{2} + {}^{1}/_{2} = 1$, and
$O(\sigma)(\langle R \rangle) = \pi_1(\langle R \rangle) \cdot \pi_2(\langle R \rangle) = {}^{3}/_{10} \cdot 1 = {}^{3}/_{10}$.

Similarly,
$O(\sigma)(\langle L, A, \ell \rangle) = \pi_1(\langle L, A, \ell \rangle) \cdot \pi_2(\langle L, A, \ell \rangle) = {}^{28}/_{70} \cdot {}^{1}/_{2} = {}^{2}/_{10}$.

Outcomes: Behavioral Strategies – Example



Assume $\beta_1(\{\langle\rangle\}) = \{L \mapsto 7/10, R \mapsto 3/10\}$,
$\beta_1(\{\langle L, A\rangle, \langle L, B\rangle\}) = \{\ell \mapsto 4/7, r \mapsto 3/7\}$,
$\beta_2(\{\langle L\rangle\}) = \{A \mapsto 1/2, B \mapsto 1/2\}$, and $\beta = (\beta_1, \beta_2)$.

Then, e. g., $O(\beta)(\langle R\rangle) = \beta_1(\{\langle\rangle\})(R) = 3/10$.

Similarly, $O(\beta)(\langle L, A, \ell\rangle) = \beta_1(\{\langle\rangle\})(L) \cdot \beta_2(\{\langle L\rangle\})(A) \cdot \beta_1(\{\langle L, A\rangle, \langle L, B\rangle\})(\ell) = 7/10 \cdot 1/2 \cdot 4/7 = 2/10$.

Outcome Equivalence

Definition

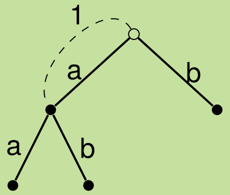
Two (mixed or behavioral) strategies of a player i are called outcome-equivalent if for every partial profile of pure strategies of the other players, the two strategies induce the same outcome.

Question: Can we find outcome-equivalent mixed strategies for behavioral strategies and vice versa?

Partial answer: Sometimes.

Counterexample (1)

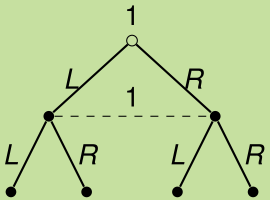
Example (Behavioral strategy without a mixed strategy)



- A behavioral strategy assigning non-zero probability to *a* and *b* generates outcomes $\langle a, a \rangle$, $\langle a, b \rangle$, and $\langle b \rangle$ with non-zero probabilities.
- Since there are only the pure strategies of playing *a* or *b*, no mixed strategy can produce $\langle a, b \rangle$.

Counterexample (2)

Example (Mixed strategy without a behavioral strategy)



- Mix the two pure strategies *LL* and *RR* equally, resulting in the distribution $(1/2, 0, 0, 1/2)$ over the terminal histories.
- No behavioral strategy can accomplish this.

Equivalence of Behavioral and Mixed Strategies

If we restrict ourselves to games with perfect recall, however, everything works.

Theorem (Equivalence of mixed and behavioral strategies (Kuhn))

In a game of perfect recall, any mixed strategy of a given agent can be replaced by an outcome-equivalent behavioral strategy, and any behavioral strategy can be replaced by an outcome-equivalent mixed strategy.

Summary

- Types of strategies:
  - Pure: InfoSets $\rightarrow$ Actions
  - Mixed: (InfoSets $\rightarrow$ Actions) $\rightarrow$ Probabilities
    (randomization over pure strategies)
  - Behavioral: InfoSets $\rightarrow$ (Actions $\rightarrow$ Probabilities)
    (collections of independent randomized decisions for each information set)
- Mixed and behavioral are equivalent (induce same outcome probabilities) in the case of perfect recall.
- Otherwise not.

Game Theory

5. Extensive Games

5.2. Extensive Games with Imperfect Information

5.2.4. Solution Concepts

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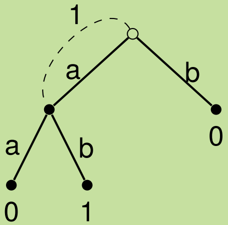
Expected Utility

Similar to the case of mixed strategies for strategic games, we define the utility for mixed and behavioral strategies as expected utility, summing over all terminal histories:

$$U_i(\sigma) = \sum_{h \in Z} u_i(h) \cdot O(\sigma)(h)$$

Example



- Mixed strategy (mixing a and b) σ: $U_1(\sigma) = 0$.
- Behavioral strategy β with $p = 1/2$ for a: $U_1(\beta) = 1/4$.

Nash Equilibria

Definition (Nash equilibrium in mixed strategies)

A Nash equilibrium in mixed strategies is a profile σ^* of mixed strategies with the property that for every player i:

$$U_i(\sigma^*_{-i}, \sigma^*_i) \geq U_i(\sigma^*_{-i}, \sigma_i) \text{ for every mixed strategy } \sigma_i \text{ of } i.$$

Note: Support lemma applies here as well.

Definition (Nash equilibrium in behavioral strategies)

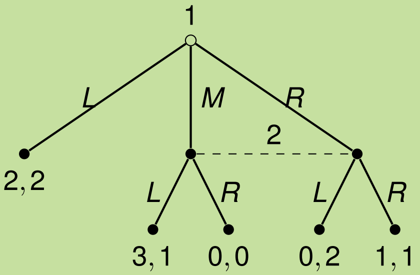
A Nash equilibrium in behavioral strategies is a profile β^* of behavioral strategies with the property that for every player i:

$$U_i(\beta^*_{-i}, \beta^*_i) \geq U_i(\beta^*_{-i}, \beta_i) \text{ for every behavioral strategy } \beta_i \text{ of } i.$$

Remark: Equivalent, provided we have perfect recall.

Eliminating Imperfect Equilibria

Example

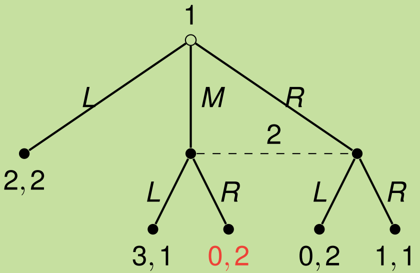


Nash equilibria: (M, L) and (L, R).

Unreasonable ones: (L, R), because in the information set of player 2, L dominates R (cf. empty threats).

How have we got here?

Example



Nash equilibria: (L, R).

What should player 2 do in his information set?

This depends on his belief: if the probability that M has been played is $\geq 1/2$, then R is optimal, otherwise L.

Assessments

Let us take the beliefs about what has been played into account when defining an equilibrium.

Definition (Assessment)

An assessment in an extensive game is a pair (β, μ), where β is a profile of behavioral strategies and μ is a function that assigns to every information set a probability distribution on the set of histories in the information set.

$\mu(I)(h)$ is the probability that player $P(I)$ assigns to the history $h \in I$, given that I is reached.

Outcome

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We have to modify the outcome function. Let $h^* = \langle a^1, \dots, a^K \rangle$ be a terminal history. Then:

- $O(\beta, \mu \,|\, I)(h^*) = 0$, if there is no subhistory of h^* in I (i. e., h^* is unreachable from I), and
- $O(\beta, \mu \,|\, I)(h^*) =$ $\mu(I)(h) \cdot \prod_{k=L}^{K-1} \beta_{P(\langle a^1,\dots,a^k \rangle)}(\langle a^1, \dots, a^k \rangle)(a^{k+1})$, if a subhistory $h = \langle a^1, \dots, a^L \rangle$ of h^* with $L \leq K$ is in I.

Remark 1: This is well-defined, since the subhistory $h = \langle a^1, \dots, a^L \rangle$ in the second case is unique if the game has perfect recall.

Remark 2: For the inital history, we have $O(\beta, \mu \,|\, \langle\rangle)(h^*) = O(h^*)$.

Sequential Rationality

Similar to the outcome function, we generalize the expected utility functions:

$$U_i(\beta, \mu \mid I_i) = \sum_{h \in Z} u_i(h) \cdot O(\beta, \mu \mid I_i)(h)$$

Definition (Sequential rationality)

Let Γ be an extensive game with perfect recall. An assessment (β, μ) is sequentially rational if for every player i and every information set $I_i \in \mathscr{I}_i$, we have

$$U_i(\beta, \mu \mid I_i) \geq U_i((\beta_{-i}, \beta_i'), \mu \mid I_i) \quad \text{for every strategy } \beta_i' \text{ of player } i.$$

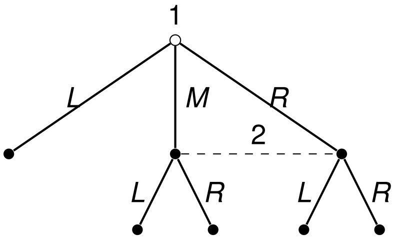
Note: restrictions on μ still missing!

Consistency with Strategies

We would at least require that the beliefs μ are consistent with the strategies, meaning they should be derived from the strategies using Bayes' rule.

In our earlier example, player 2's belief should be derived from the behavioral strategy of player 1.

Consistency with Strategies



E.g., the probability that M has been played should be:

$$\mu(\{\langle M\rangle, \langle R\rangle\})(\langle M\rangle) = \frac{\beta_1(\langle\rangle)(M)}{\beta_1(\langle\rangle)(M) + \beta_1(\langle\rangle)(R)}.$$

However, what to do when the denominator is 0? (I. e., in this example, if player 1 only plays L, i. e., if $\beta_1(\langle\rangle)(L) = 1$.)

Consistency

By viewing an assessment as a limit of a sequence of completely mixed strategy profiles (all strategies are in the support), one can enforce the Bayes condition also on information set that are not reached by an equilibrium profile.

Definition (Consistency)

Let Γ by a finite extensive game with perfect recall. An assessment (β, μ) is consistent if there is a sequence $((\beta^n, \mu^n))_{n=1}^{\infty}$ of assessments that converges to (β, μ) in Euclidian space and has the properties that each strategy profile β^n is completely mixed and that each belief system μ^n is derived from β^n using Bayes' rule.

Note: Kreps (1990) wrote: "a lot of bodies are buried in this definition."

Sequential Equilibria

Definition (Sequential equilibrium)

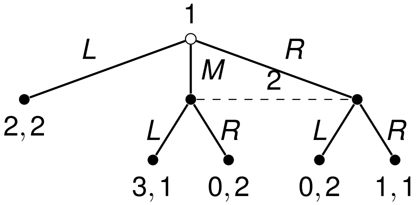
An assessment is a sequential equilibrium of a finite extensive game with perfect recall if it is sequentially rational and consistent.

Theorem (Kreps and Wilson, 1982)

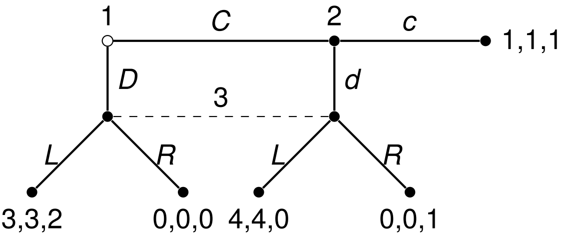
Every finite extensive game with perfect recall has a sequential equilibrium. □

Theorem (Kreps and Wilson, 1982)

Sequential equilibria generalize subgame-perfect equilibria. □



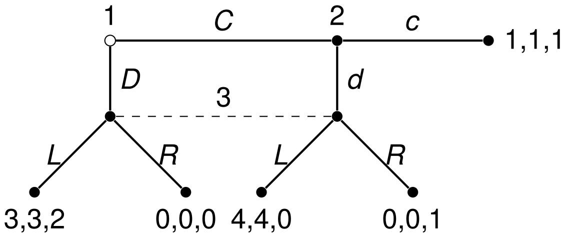
- Let (β, μ) be as follows: $\beta_1(L) = 1$, $\beta_2(R) = 1$, $\mu(\{\langle M\rangle, \langle R\rangle\})(\langle M\rangle) = \alpha$ with $0 \leq \alpha \leq 1$.
- Then (β, μ) is consistent since $\beta_1^n = (1-\varepsilon, \alpha\varepsilon, (1-\alpha)\varepsilon)$, $\beta_2^n = (\varepsilon, 1-\varepsilon)$, for $\varepsilon = {}^1\!/_n$, and $\mu^n(\{\langle M\rangle, \langle R\rangle\})(\langle M\rangle) = \alpha$ converges to (β, μ) for $n \to \infty$.
- For $\alpha \geq 1/2$, (β, μ) is sequentially rational.



Two types of NE (for $I = \{\langle D\rangle, \langle C, d\rangle\}$):

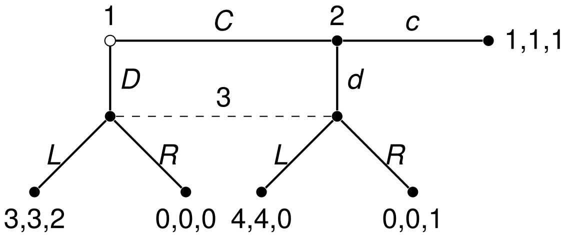
1. $\beta_1(\langle\rangle)(D) = 1$, $1/3 \leq \beta_2(\langle C\rangle)(c) \leq 1$, $\beta_3(I)(L) = 1$
2. $\beta_1(\langle\rangle)(C) = 1$, $\beta_2(\langle C\rangle)(c) = 1$, $3/4 \leq \beta_3(I)(R) \leq 1$

Are these also sequential equilibria?



1. $\beta_1(\langle\rangle)(D) = 1$, $1/3 \leq \beta_2(\langle C\rangle)(c) \leq 1$, $\beta_3(I)(L) = 1$:

violates sequential rationality for player 2!



2 $\beta_1(\langle\rangle)(C) = 1$, $\beta_2(\langle C\rangle)(c) = 1$, ${}^3\!/\!{}_4 \leq \beta_3(I)(R) \leq 1$:

for each NE of this form, there exists a sequential equilibrium (β, μ) with $\mu(I)(D) = {}^1\!/\!{}_3$.

For consistency consider: $\beta_1^n(\langle\rangle)(D) = \varepsilon$, $\beta_2^n(\langle C\rangle)(d) = {}^{2\varepsilon}\!/\!{}_{1-\varepsilon}$, $\beta_3^n(I)(R) = \beta_3(I)(R) - \varepsilon \quad (\varepsilon = {}^1\!/\!{}_n)$.

Note: $\beta_1^n(\langle\rangle)(D) + (\beta_1^n(\langle\rangle)(C) \cdot \beta_2^n(\langle C\rangle)(d)) = 3\varepsilon$.

Summary

- Nash equilibria can be defined for extensive games, however, similiar to perfect information games, are not always reasonable.
- Sequential equilibria are the refinement, which is based an assessments (behavioral strategies + beliefs).
- Beliefs should be consistent with strategies.
- Strategies should be best responses in each information set, given beliefs.

Game Theory

6. Repeated Games

6.1. Example: Repeated Prisoners' Dilemma

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Bernhard Nebel and Robert Mattmüller

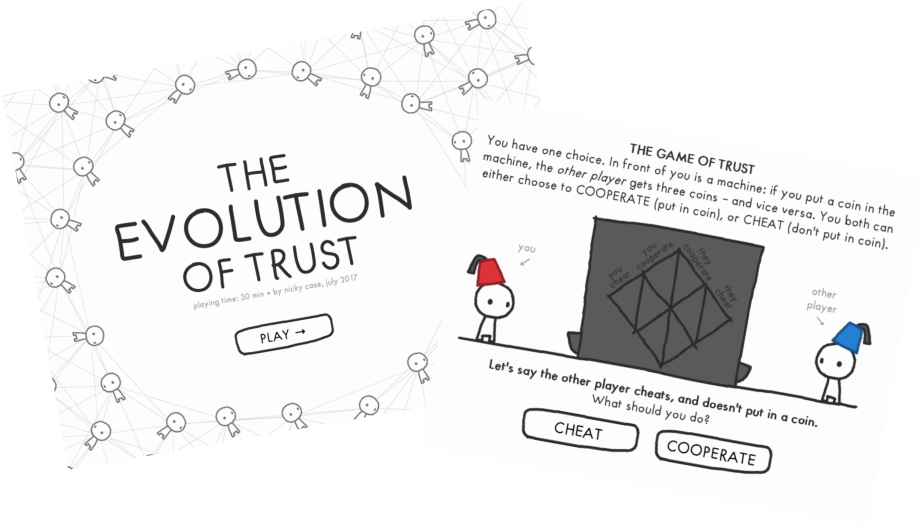
Summer semester 2020

Motivation and Intuition

- Remember: the Prisoner's Dilemma leads to the unsatisfying result (that both defect) because there is neither experience nor future encounters.
- What if the game is played repeatedly?
- Model this as an extensive game where in each turn, we repeat a given base game.
- Will social norms evolve?
- Will punishments, which can lead to short-term costs, nevertheless be played (are these potential punishments credible threats?)

See (and play!): The Evolution of Trust

(http://ncase.me/trust/)



Reminder: Prisoners' Dilemma

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

(D,D) (i. e., both players defect) is the unique Nash equilibrium, the pair of maximinimizers and the pair of strictly dominant strategies.

So, in a single encounter, there is no argument for rationally playing C!

Repeated Prisoners' Dilemma

Terminology: The game that is played repeatedly is called the stage game.

Question: How often to repeat the stage game?

Possible answers:

- Finitely: repeat pre-specified number of times k.
- Infinitely: repeat infinitely often.
- Indefinitely: after each step, terminate with probability $0 < p < 1$.

Finitely Repeated Prisoners' Dilemma

Assume we play the prisoners' dilemma a pre-specified number of times k.

$\rightsquigarrow$ extensive game with perfect information and simultaneous moves

Finitely Repeated Prisoners' Dilemma

What will be a subgame perfect equilibirium?

Use backward induction:

- (D, D) is the NE in the last subgame, since this would be the only NE in the one-shot game.
- So, (D, D) will also be played in period $k - 1$.
- ...
- So, the (only) subgame-perfect equilibrium and the only NE of this repeated game is $(D, D), (D, D), \ldots, (D, D)$.

$\rightsquigarrow$ players still defect all the time

$\rightsquigarrow$ allowing finitely many repetitions not really helpful!

Infinitely Repeated Prisoners' Dilemma

If we play the prisoners' dilemma infinitely often, we need to solve two problems:

1. How to define a strategy?
2. How to define the payoffs or preferences?

How to specify a strategy using only finite resources?

- In general: one could use an algorithm.
- Usually done in game theory: use Moore automata, i.e., finite state automata, where the inputs are actions of the other players, and in each state, a response action to the previous actions is generated.
- A Nash equilibrium is a profile of automata (strategies) such that no deviation is profitable.

Infinitely Repeated Prisoners' Dilemma

Payoffs/Preferences

How to specify players' preferences using only finite resources?

- Use preferences from the stage game.
- Derive preferences over infinite repetitions using:
  - discounting future payoffs, or
  - limit of means criterion, or
  - overtaking criterion

Indefinitely Repeated Prisoners' Dilemma

Terminating with probability $0 < p < 1$ is equivalent to discounting future payoffs in infinitely repeated game with discount factor $1 - p$.

$\leadsto$ no need to study indefinitely repeated games separately

$\leadsto$ focus on infinitely repeated games

Summary

- Repeated games are extensive games with perfect information and simultaneous moves, in which a base strategic game (the stage game) is played in each round.
- Finitely, infinitely, or indefinitely many repetitions possible.
- Often, finitely many repetitions not helpful.

Game Theory

6. Repeated Games

6.2. Strategies and Preferences in Infinite Games

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

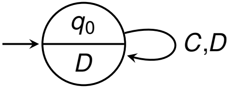
Some Possible Strategies

Using Moore automata, we can specify what to do in response to the new input (action played by others) and the state we are in. Since the automata are finite, this requires only finite memory!

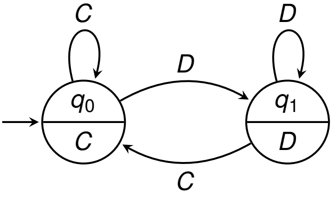
- Unconditionally cooperative: Always play C.
- Unconditonally uncooperative: Always play D.
- Tit-for-Tat: Start with C and then reply with C to each C and with D to each D.
- Grim: Start with C. After any play of D, play D in the future forever.
- Bipolar: Start with D and then always alternate between C and D.

Strategies as Automata

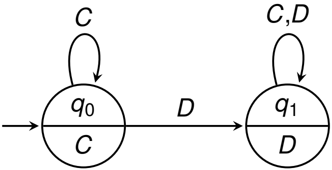
Uncooperative:



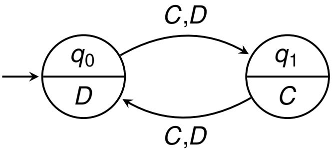
Tit-for-tat:



Grim:

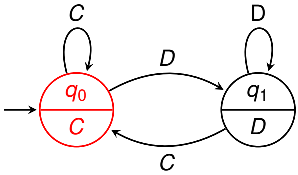


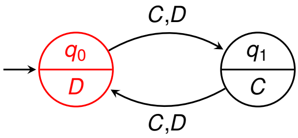
Bipolar:



Game Traces

Player 1 plays tit-for-tat, player 2 plays bipolar; 4 rounds.

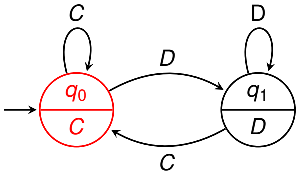


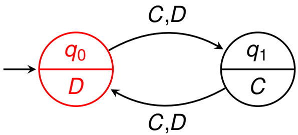


Round	Action	Utility	Accumulated payoff

Game Traces

Player 1 plays tit-for-tat, player 2 plays bipolar; 4 rounds.

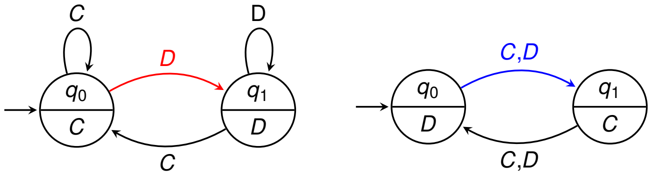




Round	Action	Utility	Accumulated payoff
1	(C,D)	(0,4)	(0,4)

Game Traces

Player 1 plays tit-for-tat, player 2 plays bipolar; 4 rounds.



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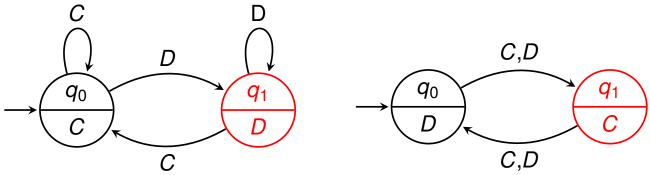

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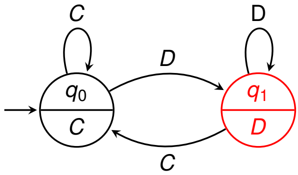
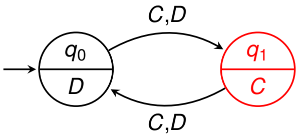
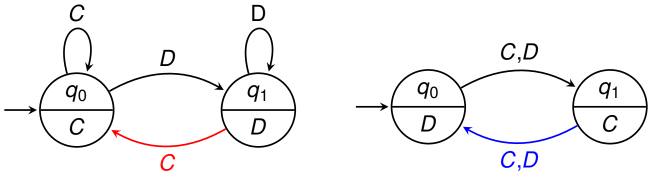




Round	Action	Utility	Accumulated payoff
1	(C,D)	(0,4)	(0,4)
2	(D,C)	(4,0)	(4,4)

Game Traces

Player 1 plays tit-for-tat, player 2 plays bipolar; 4 rounds.



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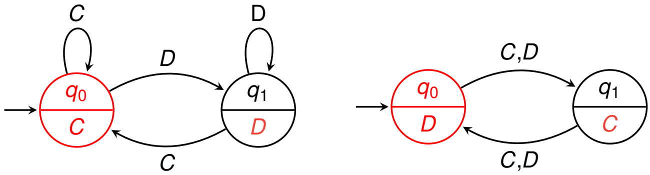

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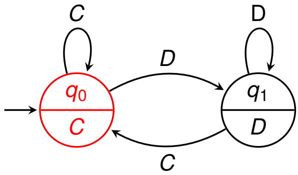
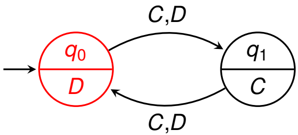
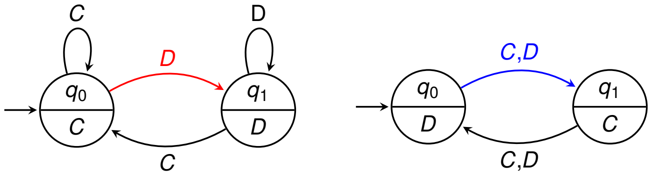




Round	Action	Utility	Accumulated payoff
1	(C,D)	(0,4)	(0,4)
2	(D,C)	(4,0)	(4,4)
3	(C,D)	(0,4)	(4,8)

Game Traces

Player 1 plays tit-for-tat, player 2 plays bipolar; 4 rounds.



Round	Action	Utility	Accumulated payoff
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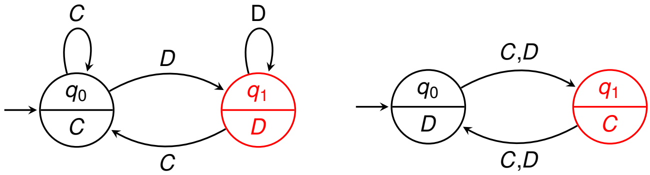
3	(C,D)	(0,4)	(4,8)

Game Traces

Player 1 plays tit-for-tat, player 2 plays bipolar; 4 rounds.



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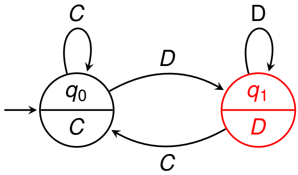

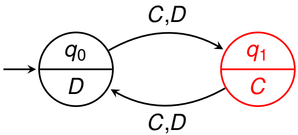

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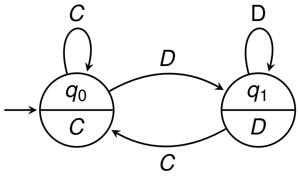
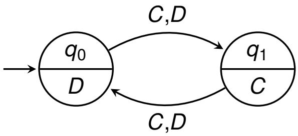
Player 1 plays tit-for-tat, player 2 plays bipolar; 4 rounds.





Round	Action	Utility	Accumulated payoff
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2	(D,C)	(4,0)	(4,4)
3	(C,D)	(0,4)	(4,8)
4	(D,C)	(4,0)	(8,8)

Preferences over Payoff Traces

How to define the payoff of an infinite game or whether to prefer one outcome over another one?

Given two infinite sequences $(v_i^t)_{t=1}^{\infty}$ and $(w_i^t)_{t=1}^{\infty}$ of payoffs, we will define when the first is preferred over the second by player i: $(v_i^t)_{t=1}^{\infty} \succsim_i (w_i^t)_{t=1}^{\infty}$.

(Strict preference $\succ_i$ and indifference $\approx_i$ defined similarly.)

Preferences over Payoff Traces

Option 1: Discounting

Discount future payoffs by a discount factor $\delta \in (0,1)$:

$$(v_i^t)_{t=1}^{\infty} \succsim_i^{\delta} (w_i^t)_{t=1}^{\infty} \qquad \text{iff} \qquad \sum_{t=1}^{\infty} \delta^{t-1}(v_i^t - w_i^t) \geq 0$$

$\sum_{t=1}^{\infty} \delta^{t-1} v_i^t$ is considered the payoff in the repeated game.

Example

- $(1,-1,0,0,\ldots) \succ_i^{\delta} (0,0,0,0,\ldots)$ for any $\delta \in (0,1)$
- $(-1,2,0,0,\ldots) \succ_i^{\delta} (0,0,0,0,\ldots)$ iff $\delta > 1/2$
- $(1,1,1,1,\ldots) \approx_i^{\delta} (1/(1-\delta),0,0,0,\ldots)$

Note: $\sum_{t=1}^{\infty} \delta^{t-1} = 1/(1-\delta)$

Preferences over Payoff Traces

Option 2: Limit of Means

Compare average payoffs in the limit (limit of means criterion):

$$(v_i^t)_{t=1}^{\infty} \succsim_i^{\text{lom}} (w_i^t)_{t=1}^{\infty} \qquad \text{iff} \qquad \liminf_{T\to\infty} \sum_{t=1}^{T} (v_i^t - w_i^t)/T \geq 0$$

If $\lim_{T\to\infty} \sum_{t=1}^{T} v_i^t/T$ exists, this is considered the payoff in the repeated game.

Example

- $(1,-1,0,0,\ldots) \approx_i^{\text{lom}} (0,0,0,0,\ldots)$
- $(-1,2,0,0,\ldots) \approx_i^{\text{lom}} (0,0,0,0,\ldots)$
- $(\underbrace{0,\ldots,0}_{m},1,1,1,\ldots) \succ_i^{\text{lom}} (1,0,0,0,\ldots)$ for all $m \in \mathbb{N}$

  (Note: For every δ there exists an m^* such that for all $m > m^*$ the preference under discounting is reversed.)

Overtaking criterion:

$$(v_i^t)_{t=1}^\infty \succsim_i^{\mathrm{ot}} (w_i^t)_{t=1}^\infty \qquad \text{iff} \qquad \liminf_{T\to\infty} \sum_{t=1}^{T} (v_i^t - w_i^t) \geq 0$$

Example

- $(1, -1, 0, 0, \ldots) \approx_i^{\mathrm{ot}} (0, 0, 0, 0, \ldots)$
- $(-1, 2, 0, 0, \ldots) \succ_i^{\mathrm{ot}} (0, 0, 0, 0, \ldots)$

Infinitly Repeated Games Formally

Definition (Infinitely repeated game of G)

Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be a strategic game with $A = \prod_{i\in N} A_i$. An infinitely repeated game of G is an extensive game with perfect information and simultaneous moves $\langle N, H, P, (\succsim_i)_{i\in N}\rangle$ in which

- $H = \{\langle\rangle\} \cup (\bigcup_{t=1}^{\infty} A^t) \cup A^\infty$,
- $P(h) = N$ for all nonterminal histories $h \in H$, and
- $\succsim_i$ is a preference relation on A^∞ that is based on discounting, limit of means or overtaking.

Summary

- Strategies in infinitely repeated games are described using finite Moore automata.
- For preferences over the outcomes of infinitely repeated games, different preference criteria are possible:
  - discounting
  - limit of means
  - overtaking

Game Theory

6. Repeated Games

6.3. Analysis of the Infinitely Repeated Prisoners' Dilemma

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Let us consider the Grim strategy. Is (Grim, Grim) a Nash equilibrium under all the preference criteria?

Recall: (Grim, Grim) leads to outcome $(C,C),(C,C),(C,C),\ldots$.

Limit of means and overtaking criterion: (Grim, Grim) is a Nash equilibrium! Any deviation will result in getting ≤ 1 instead of 3 infinitely often.

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Discounting: This is a bit more complicated.

- If a player gets v for every round, he will accumulate the following payoff:

$$v + \delta v + \delta^2 v + \ldots = \sum_{i=1}^{\infty} \delta^{i-1} v$$

- Since we know that $\sum_{i=0}^{\infty} \delta^i = \frac{1}{1-\delta}$ (for $0 < \delta < 1$), we have:

$$\sum_{i=1}^{\infty} \delta^{i-1} v = v \sum_{i=0}^{\infty} \delta^i = \frac{v}{1-\delta}$$

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Assume that both players play Grim for the first $k-1$ rounds. Then player 1 deviates and plays D once. In the remainder he must play D in order to get at least 1 in each round.

- Starting in round k, he receives:

$$4+\delta+\delta^2+\ldots = 3+\sum_{i=0}^{\infty}\delta^i = 3+\frac{1}{1-\delta}$$

- If he had not deviated, the accumulated payoff starting at round k would have been:

$$3+3\delta+3\delta^2+\ldots = 3\sum_{i=0}^{\infty}\delta^i = \frac{3}{1-\delta}$$

- $\rightsquigarrow$ deviation is profitable iff $3+1/(1-\delta) > 3/(1-\delta)$ iff $\delta < 1/3$.
- $\rightsquigarrow$ Grim is a NE strategy for $\delta \geq 1/3$.

Under which preference criteria is (Tit-for-tat,Tit-for-tat) an equilibrium?

Recall: (Tit-for-tat, Tit-for-tat) leads to outcome $(C,C),(C,C),(C,C),\dots$.

Limit of means: Finitely many deviations do not change the payoff profile in the limit. Infinitely many deviations lead to lower payoff. So Tit-for-tat is an NE strategy under this preference criterion.

Overtaking: Even only one deviation leads to a payoff of 5 over two rounds instead of 6. So, in no case, a deviation can lead to a better payoff.

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Discounting: Deviating only in one move in round k and then returning to being cooperative leads to $4 + 0 + \ldots$ instead of $3 + \delta 3 + \ldots$ in round k.

- $\rightsquigarrow$ deviation is profitable iff $4 > 3 + \delta 3$ iff $\delta < 1/3$ (This is the best case for a deviation!)
- $\rightsquigarrow$ Tit-for-tat is a NE strategy for $\delta \geq 1/3$.

- With outcome $(C,C),(C,C),\ldots$:
  - (Grim, Grim)
  - (Tit-for-tat, Tit-for-tat)
  - (Grim, Tit-for-tat)
- With outcome $(D,D),(D,D),\ldots$:
  - (Always-defect, Always-defect)

Summary

- In the repeated Prisoners' Dilemma, it is possible to play Nash Equilibrium strategies that result in infinite (C, C) sequences, i. e., infinite cooperation.
- Outlook: can also be studied from an evolutionary perspective – which strategies survive if whole populations of players are considered?

Game Theory

6. Repeated Games

6.4. Punishments and Enforceable Outcomes

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Punishments

Observe that the Nash equilibrium strategies are based on being able to punish a deviating player.

Definition (Minmax payoff)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ a strategic game. Player i's minimax payoff in G, also written as $v_i(G)$, is the lowest payoff that the other players can force upon player i:

$$v_i(G) = \min_{a_{-i} \in A_{-i}} \max_{a_i \in A_i} u_i(a_{-i}, a_i).$$

The idea is that the other players all punish a deviating player in the next round(s) and allow him only to get $v_i(G)$.

Enforceable Payoffs

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Definition (Feasible payoff profile)

Given a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a vector $v \in \mathbb{R}^N$ is called payoff profile of G if there exists $a \in A$ such that $v = u(a)$. $v \in \mathbb{R}^N$ is called feasible payoff profile if there exists a vector $(\alpha_a)_{a \in A} \in \mathbb{Q}^A$ with $\sum \alpha_a = 1$ and $v = \sum \alpha_a u(a)$.

Note: Such payoffs can be generated in a repeated game by playing a for β_a rounds in a set of γ games with $\gamma = \sum_{a \in A} \beta_a$ and $\alpha_a = \beta_a / \gamma$.

Definition (Enforceable payoff)

Given a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a payoff profile w with $w_i \geq v_i(G)$ for all $i \in N$ is called enforceable. If $w_i > v_i(G)$ for all $i \in N$, it is said to be strictly enforceable.

Many Different Equilibria …

- Using the concept of enforceable payoffs, one can construct many different Nash equilibria and payoff profiles for the repeated prisoners' dilemma!
- E. g., $(^{10}/_{3}, 2)$ is a feasible payoff profile, because $4 \times (C, C)$ and $2 \times (D, C)$ leads to $(^{(4\times3+2\times4)}/_{6}, ^{(4\times3+2\times0)}/_{6})$.
- Construct two automata that implement this repeated sequence and in case of deviation revert to playing D.
- These two automata implement Nash equilibrium strategies, since deviating leads to a payoff of 1 instead of $^{10}/_{3}$ or 2!
- Folk theorems stating that all enforceable outcomes are reachable have been proven for the general case.

Summary

- In the repeated prisoners' dilemma, it is possible to achieve any feasible payoff profile under the limit of means criterion.

Game Theory

7. Social Choice Theory

7.1. Introduction and Examples

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Summer semester 2020

Social Choice Theory

Motivation: Aggregation of individual preferences

Examples:

- political elections
- council decisions
- Eurovision Song Contest

Question: If voters' preferences are private, then how to implement aggregation rules such that voters vote truthfully (no "strategic voting")?

Definition (Social welfare and social choice function)

Let A be a set of alternatives (candidates) and L be the set of all linear orders on A. For n voters, a function

$$F : L^n \to L$$

is called a social welfare function. A function

$$f : L^n \to A$$

is called a social choice function.

Notation: Linear orders $\prec \in L$ express preference relations.
$a \prec_i b$: voter i prefers candidate b over candidate a.
$a \prec b$: candidate b socially preferred over candidate a.

- Plurality voting (aka first-past-the-post or winner-takes-all):
  - only top preferences taken into account
  - candidate with most top preferences wins

  Drawback: wasted votes, compromising, spoiler effect, winner only preferred by minority

- Plurality voting with runoff:
  - first round: two candidates with most top votes proceed to second round (unless absolute majority)
  - second round: runoff

  Drawback: still, tactical voting and strategic nomination possible

- Instant runoff voting:
  - each voter submits his preference order
  - iteratively candidates with fewest top preferences are eliminated until one candidate has absolute majority

  Drawback: tactical voting still possible

- Borda count:
  - each voter submits his preference order over the m candidates
  - if a candidate is in position j of a voter's list, he gets $m-j$ points from that voter
  - points from all voters are added
  - candidate with most points wins

  Drawback: tactical voting still possible ("voting opponent down")

- Condorcet winner:
  - each voter submits his preference order
  - perform pairwise comparisons between candidates
  - if one candidate wins all his pairwise comparisons, he is the Condorcet winner

  Drawback: Condorcet winner does not always exist.

$\rightsquigarrow$ Is there any voting system without such problems? Or is there some deeper underlying reason for all those problems?

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

Plurality voting:

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

Plurality voting: candidate e wins (8 votes)

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

Plurality voting with runoff:

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

Plurality voting with runoff:

- first round: candidates e (8 votes) and a (6 votes) proceed
- second round: candidate a $(6+4+3+1=14$ votes) beats candidate e $(8+1=9$ votes)

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

Instant runoff voting:

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

Instant runoff voting:

- first elimination: d
- second elimination: b
- third elimination: a
- now c has absolute majority and wins.

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1	
1st	e	a	b	c	d	d	4 points
2nd	d	b	c	b	c	c	3 points
3rd	b	c	d	d	a	b	2 points
4th	c	e	a	a	b	e	1 point
5th	a	d	e	e	e	a	0 points

Borda count:

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1	
1st	e	a	b	c	d	d	4 points
2nd	d	b	c	b	c	c	3 points
3rd	b	c	d	d	a	b	2 points
4th	c	e	a	a	b	e	1 point
5th	a	d	e	e	e	a	0 points

Borda count:

- Cand. a: $8 \cdot 0 + 6 \cdot 4 + 4 \cdot 1 + 3 \cdot 1 + 1 \cdot 2 + 1 \cdot 0 = 33$ pts
- Cand. b: $8 \cdot 2 + 6 \cdot 3 + 4 \cdot 4 + 3 \cdot 3 + 1 \cdot 1 + 1 \cdot 2 = 62$ pts
- Cand. c: $8 \cdot 1 + 6 \cdot 2 + 4 \cdot 3 + 3 \cdot 4 + 1 \cdot 3 + 1 \cdot 3 = 50$ pts
- Cand. d: $8 \cdot 3 + 6 \cdot 0 + 4 \cdot 2 + 3 \cdot 2 + 1 \cdot 4 + 1 \cdot 4 = 46$ pts
- Cand. e: $8 \cdot 4 + 6 \cdot 1 + 4 \cdot 0 + 3 \cdot 0 + 1 \cdot 0 + 1 \cdot 1 = 39$ pts

$\leadsto$ Candidate b wins.

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

Condorcet winner:

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

Condorcet winner: Ex.: a $\prec_i$ b 16 times, b $\prec_i$ a 7 times

	a	b	c	d	e
a	–	0	0	0	1
b	1	–	1	1	1
c	1	0	–	1	1
d	1	0	0	–	0
e	0	0	0	1	–

$\longleftarrow$ candidate b wins.

Social Choice Functions

Examples: Different Winners

23 voters, candidates a, b, c, d, e.

# voters	8	6	4	3	1	1
1st	e	a	b	c	d	d
2nd	d	b	c	b	c	c
3rd	b	c	d	d	a	b
4th	c	e	a	a	b	e
5th	a	d	e	e	e	a

- Plurality voting: candidate e wins.
- Plurality voting with runoff: candidate a wins.
- Instant runoff voting: candidate c wins.
- Borda count / Condorcet winner: candidate b wins.

- Different winners for different voting systems.
- Which voting system to prefer? Can even strategically choose voting system!

Summary

- Multitude of possible social welfare functions (plurality voting with or without runoff, instant runoff voting, Borda count, ...).
- Tactical voting seems to be possible in all of them.
- May lead to different winners.
- Strategic choice of voting system.

Game Theory

7. Social Choice Theory

7.2. Condorcet Methods

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Summer semester 2020

Example: voters 1, 2, 3; candidates a, b, c.

$$a \prec_1 b \prec_1 c$$
$$b \prec_2 c \prec_2 a$$
$$c \prec_3 a \prec_3 b$$

Then we have cyclical preferences.

	a	b	c
a	–	0	1
b	1	–	0
c	0	1	–

$a \prec b$, $b \prec c$, $c \prec a$: violates transitivity of linear order consistent with these preferences.

Condorcet Methods

Definition

A Condorcet method returns a Condorcet winner, if one exists.

One particular Condorcet method: the Schulze method.

Relatively new: proposed in 1997

Already many users: Debian, Ubuntu, Pirate Parties, Associated Student Government at Uni Freiburg (Studierendenrat, StuRa), ...

Schulze Method

Notation: $d(X, Y)$ = number of pairwise comparisons won by X against Y

Definition

For candidates X and Y, there exists a path $C_1, \ldots, C_n$ between X and Y of strength z if

- $C_1 = X$,
- $C_n = Y$,
- $d(C_i, C_{i+1}) > d(C_{i+1}, C_i)$ for all $i = 1, \ldots, n-1$, and
- $d(C_i, C_{i+1}) \geq z$ for all $i = 1, \ldots, n-1$ and there exists $j = 1, \ldots, n-1$ such that $d(C_j, C_{j+1}) = z$

Example: path between a and d of strength 5:

$$a \xrightarrow{8} b \xrightarrow{5} c \xrightarrow{6} d$$

Schulze Method

Definition

Let $p(X, Y)$ be the maximal value z such that there exists a path of strength z from X to Y, and $p(X, Y) = 0$ if no such path exists.

Then, the Schulze winner is the Condorcet winner, if it exists. Otherwise, a potential winner is a candidate a such that $p(a, X) \geq p(X, a)$ for all $X \neq a$.

Tie-breaking is used between potential winners.

Schulze Method

Example

# voters	3	2	2	2
1st	a	d	d	c
2nd	b	a	b	b
3rd	c	b	c	d
4th	d	c	a	a

Is there a Condorcet winner?

	a	b	c	d
a	–	1	1	0
b	0	–	1	1
c	0	0	–	1
d	1	0	0	–

$\rightsquigarrow$ No!

Schulze Method

Example

# voters	3	2	2	2
1st	*a*	*d*	*d*	*c*
2nd	*b*	*a*	*b*	*b*
3rd	*c*	*b*	*c*	*d*
4th	*d*	*c*	*a*	*a*

Weights $d(X, Y)$:

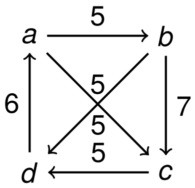
	a	*b*	*c*	*d*
a	–	5	5	3
b	4	–	7	5
c	4	2	–	5
d	6	4	4	–

# voters	3	2	2	2
1st	*a*	*d*	*d*	*c*
2nd	*b*	*a*	*b*	*b*
3rd	*c*	*b*	*c*	*d*
4th	*d*	*c*	*a*	*a*

Weights $d(X, Y)$:

	a	*b*	*c*	*d*
a	–	5	5	3
b	4	–	7	5
c	4	2	–	5
d	6	4	4	–

As a graph:



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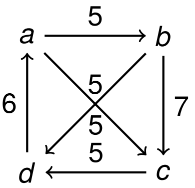
# voters	3	2	2	2
1st	a	d	d	c
2nd	b	a	b	b
3rd	c	b	c	d
4th	d	c	a	a

Weights $d(X, Y)$:

	a	b	c	d
a	–	5	5	3
b	4	–	7	5
c	4	2	–	5
d	6	4	4	–

As a graph:



Path strengths $p(X, Y)$:

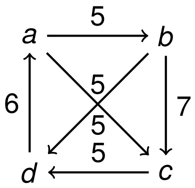
	a	b	c	d
a	–	5	5	5
b	5	–	7	5
c	5	5	–	5
d	6	5	5	–

# voters	3	2	2	2
1st	*a*	*d*	*d*	*c*
2nd	*b*	*a*	*b*	*b*
3rd	*c*	*b*	*c*	*d*
4th	*d*	*c*	*a*	*a*

Weights $d(X, Y)$:

	a	*b*	*c*	*d*
a	–	5	5	3
b	4	–	7	5
c	4	2	–	5
d	6	4	4	–

As a graph:



Path strengths $p(X, Y)$:

	a	*b*	*c*	*d*
a	–	5	5	5
b	5	–	7	5
c	5	5	–	5
d	6	5	5	–

Potential winners: *b* and *d*.

Schulze Method

Theorem

There is always at least one potential winner.

Proof.

Homework. □

Summary

- Condorcet paradox: cyclical social preferences
  $\rightsquigarrow$ Condorcet winner may not exist
- Condorcet methods produce Condorcet winner if it exists
- Example: Schulze method
  (satisfies many desirable criteria, see
  `https://en.wikipedia.org/wiki/Schulze_method#Satisfied_criteria`)

Game Theory

7. Social Choice Theory

7.3. Arrow's Impossibility Theorem

7.3.1. Properties of Social Welfare Functions

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Motivation: It appears as if all considered voting systems encourage strategic voting.

Question: Can this be avoided or is it a fundamental problem?

Answer (simplified): It is a fundamental problem!

Desirable properties of social welfare functions:

Definition (unanimity)

A social welfare function satisfies

- total unanimity if for all $\prec \in L$, $F(\prec, \dots, \prec) = \prec$.
- partial unanimity if for all $\prec_1, \prec_2, \dots, \prec_n \in L$, $a, b \in A$,

$$a \prec_i b \text{ for each } i = 1, \dots, n \implies a \prec b$$

where $\prec := F(\prec_1, \dots, \prec_n)$.

Remark

Partial unanimity implies total unanimity, but not vice versa.

Properties of Social Welfare Functions

Non-Dictatorship and Independence of Irrelevant Alternatives

Desirable properties of social welfare functions:

Definition (non-dictatorship)

A voter i is called a dictator for F, if $F(\prec_1, \dots, \prec_i, \dots, \prec_n) = \prec_i$ for all orders $\prec_1, \dots, \prec_n \in L$.
F is called non-dictatorial if there is no dictator for F.

Definition (independence of irrelevant alternatives (IIA))

F satisfies independence of irrelevant alternatives (IIA) if for all alternatives a, b, the social preference between a and b depends only on the preferences of the voters between a and b.
Formally, for all $(\prec_1, \dots, \prec_n), (\prec'_1, \dots, \prec'_n) \in L^n$,
$\prec := F(\prec_1, \dots, \prec_n)$, and $\prec' := F(\prec'_1, \dots, \prec'_n)$,

$$a \prec_i b \text{ iff } a \prec'_i b, \text{ for each } i = 1, \dots, n \implies a \prec b \text{ iff } a \prec' b.$$

Properties of Social Welfare Functions

Total vs. Partial Unanimity

Lemma

Total unanimity and independence of irrelevant alternatives together imply partial unanimity.

Proof.

Consider any $\prec_1, \dots, \prec_n \in L$ with $a \prec_i b$ for all voters i.

To show: $a \prec b$, where $\prec := F(\prec_1, \dots, \prec_n)$.

Define $\prec'_1, \dots, \prec'_n$ with $\prec'_i := \prec_1$ for each voter i.

By total unanimity, $\prec' := F(\prec'_1, \dots, \prec'_n) = F(\prec_1, \dots, \prec_1) = \prec_1$.

Hence, we have $a \prec' b$.

Moreover, $a \prec_i b$ iff $a \prec'_i b$, for all voters i.

By IIA, it follows $a \prec b$ iff $a \prec' b$.

From $a \prec' b$ we conclude that $a \prec b$ must hold. $\square$

Pairwise Neutrality

Neutrality $\approx$ candidates are treated symmetrically
(i. e., no bias, “names” of the candidates do not matter)

Definition (pairwise neutrality)

A social welfare function F satisfies pairwise neutrality if, for any two preference profiles $(\prec_1, \dots, \prec_n)$ and $(\prec'_1, \dots, \prec'_n)$,

$$a \prec_i b \text{ iff } c \prec'_i d \text{ for each } i = 1, \dots, n \implies a \prec b \text{ iff } c \prec' d$$

where $\prec := F(\prec_1, \dots, \prec_n)$ and $\prec' := F(\prec'_1, \dots, \prec'_n)$.

Pairwise Neutrality

Lemma

(Total or partial) unanimity and independence of irrelevant alternatives together imply pairwise neutrality.

Proof sketch.

Assume that a, b, c, d are pairwise different. WLOG, $a \prec b$.

Construct a new preference profile $(\prec''_1, \dots, \prec''_n)$, where $c \prec''_i a$ and $b \prec''_i d$ for all $i = 1, \dots, n$, the order of the pairs (a, b) is taken from $\prec_i$, and the order of the pairs (c, d) is taken from $\prec'_i$.

By unanimity, we get $c \prec'' a$ and $b \prec'' d$ ($\prec'' := F(\prec''_1, \dots, \prec''_n)$). Because of IIA, we have $a \prec'' b$. By transitivity, we obtain $c \prec'' d$. With IIA, it follows that $c \prec' d$.

The proof for the opposite direction is similar.

[Technical details if a, b, c, d not pairwise different omitted.] $\square$

Summary

- Relevant properties of social welfare functions:
  - unanimity (total or partial)
  - non-dictatorship
  - independence of irrelevant alternatives (IIA)
  - pairwise neutrality
- Given IIA, total and partial unanimity are the same.
- Unanimity and IIA imply pairwise neutrality.

Game Theory

7. Social Choice Theory

7.3. Arrow's Impossibility Theorem

7.3.2. Proof

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

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Arrow's Impossibility Theorem

Every social welfare function over more than two alternatives that satisfies unanimity and independence of irrelevant alternatives is necessarily dictatorial.

Proof

We assume unanimity and independence of irrelevant alternatives.

Consider two elements $a, b \in A$ with $a \neq b$ and construct a sequence $(\pi^i)_{i=0,\dots,n}$ of preference profiles such that in π^i exactly the first i voters prefer b to a, i.e., $a \prec_j b$ iff $j \leq i$:

...

Arrow's Impossibility Theorem

Proof (ctd.)

	π^0	$\dots$	π^{i^*-1}	π^{i^*}	$\dots$	π^n
1:	$b \prec_1 a$	$\dots$	$a \prec_1 b$	$a \prec_1 b$	$\dots$	$a \prec_1 b$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
i^*-1:	$b \prec_{i^*-1} a$	$\dots$	$a \prec_{i^*-1} b$	$a \prec_{i^*-1} b$	$\dots$	$a \prec_{i^*-1} b$
i^*:	$b \prec_{i^*} a$	$\dots$	$b \prec_{i^*} a$	$a \prec_{i^*} b$	$\dots$	$a \prec_{i^*} b$
i^*+1:	$b \prec_{i^*+1} a$	$\dots$	$b \prec_{i^*+1} a$	$b \prec_{i^*+1} a$	$\dots$	$a \prec_{i^*+1} b$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n:	$b \prec_n a$	$\dots$	$b \prec_n a$	$b \prec_n a$	$\dots$	$a \prec_n b$
F:	$b \prec^0 a$	$\dots$	$b \prec^{i^*-1} a$	$a \prec^{i^*} b$	$\dots$	$a \prec^n b$

Unanimity $\Rightarrow b \prec^0 a$ for $\prec^0 = F(\pi^0)$, $a \prec^n b$ for $\prec^n := F(\pi^n)$.

Thus, there must exist a minimal index i^* such that $b \prec^{i^*-1} a$ and $a \prec^{i^*} b$ for $\prec^{i^*-1} := F(\pi^{i^*-1})$ and $\prec^{i^*} = F(\pi^{i^*})$.

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Proof (ctd.)

Show that i^* is a dictator.

Consider two alternatives $c, d \in A$ with $c \neq d$ and show that for all $(\prec_1, \dots, \prec_n) \in L^n$, $c \prec_{i^*} d$ implies $c \prec d$, where $\prec = F(\prec_1, \dots, \prec_{i^*}, \dots, \prec_n)$.

Consider $e \notin \{c, d\}$ and construct preference profile $(\prec'_1, \dots, \prec'_n)$, where:

$$
\begin{array}{llcl}
\text{for } j < i^*: & e \prec'_j c \prec'_j d & \text{or} & e \prec'_j d \prec'_j c \\
\text{for } j = i^*: & c \prec'_j e \prec'_j d & \text{or} & d \prec'_j e \prec'_j c \\
\text{for } j > i^*: & c \prec'_j d \prec'_j e & \text{or} & d \prec'_j c \prec'_j e
\end{array}
$$

depending on whether $c \prec_j d$ or $d \prec_j c$.

...

Arrow's Impossibility Theorem

Proof (ctd.)

Let $\prec' = F(\prec'_1, \dots, \prec'_n)$.

Independence of irrelevant alternatives implies $c \prec' d$ iff $c \prec d$.

	π^{i^*-1}	$(\prec'_i)_{i=1,\dots,n}$	π^{i^*}	$(\prec'_i)_{i=1,\dots,n}$
1:	$a \prec_1 b$	$e \prec'_1 c$	$a \prec_1 b$	$e \prec'_1 d$
i^*-1:	$a \prec_{i^*-1} b$	$e \prec'_{i^*-1} c$	$a \prec_{i^*-1} b$	$e \prec'_{i^*-1} d$
i^*:	$b \prec_{i^*} a$	$c \prec'_{i^*} e$	$a \prec_{i^*} b$	$e \prec'_{i^*} d$
n:	$b \prec_n a$	$c \prec'_n e$	$b \prec_n a$	$d \prec'_n e$
F:	$b \prec^{i^*-1} a$	$c \prec' e$	$a \prec^{i^*} b$	$e \prec' d$

For (e, c) we have the same preferences in $\prec'_1, \dots, \prec'_n$ as for (a, b) in π^{i^*-1}. Pairwise neutrality implies $c \prec' e$.

For (e, d) we have the same preferences in $\prec'_1, \dots, \prec'_n$ as for (a, b) in π^{i^*}. Pairwise neutrality implies $e \prec' d$.

...

Arrow's Impossibility Theorem

Proof (ctd.)

With transitivity, we get $c \prec' d$.

By construction of $\prec'$ and independence of irrelevant alternatives, we get $c \prec d$.

Opposite direction: similar. □

Arrow's Impossibility Theorem

Remark:

Unanimity and non-dictatorship often satisfied in social welfare functions. Problem usually lies with independence of irrelevant alternatives.

Closely related to possibility of strategic voting: insert "irrelevant" candidate between favorite candidate and main competitor to help favorite candidate (only possible if independence of irrelevant alternatives is violated).

Summary

- All social welfare functions for more than two alternatives suffer from Arrow's Impossibility Theorem:

  Every social welfare function over more than two alternatives that satisfies unanimity and independence of irrelevant alternatives is necessarily dictatorial.

- Typical handling of this issue: use unanimous, non-dictatorial social welfare functions – violate independence of irrelevant alternatives

  $\rightsquigarrow$ strategic voting inevitable

Game Theory

7. Social Choice Theory

7.4. Gibbard-Satterthwaite Theorem

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Summer semester 2020

Gibbard-Satterthwaite Theorem

Motivation:

- Arrow's Impossibility Theorem only applies to social welfare functions.
- Can this be transferred to social choice functions?
- Yes! Intuitive result: Every "reasonable" social choice function is susceptible to manipulation (strategic voting).

Strategic Manipulation and Incentive Compatibility

Definition (strategic manipulation, incentive compatibility)

A social choice function f can be strategically manipulated by voter i if there are preferences $\prec_1, \dots, \prec_i, \dots, \prec_n, \prec_i' \in L$ such that $a \prec_i b$ for $a = f(\prec_1, \dots, \prec_i, \dots, \prec_n)$ and $b = f(\prec_1, \dots, \prec_i', \dots, \prec_n)$.

The function f is called incentive compatible if f cannot be strategically manipulated.

Definition (monotonicity)

A social choice function is monotone if $f(\prec_1, \dots, \prec_i, \dots, \prec_n) = a$, $f(\prec_1, \dots, \prec_i', \dots, \prec_n) = b$ and $a \neq b$ implies $b \prec_i a$ and $a \prec_i' b$.

Incentive Compatibility and Monotonicity

Proposition

A social choice function is monotone iff it is incentive compatible.

Proof.

Let f be monotone. If $f(\prec_1, \dots, \prec_i, \dots, \prec_n) = a$, $f(\prec_1, \dots, \prec_i', \dots, \prec_n) = b$ and $a \neq b$, then also $b \prec_i a$ and $a \prec_i' b$.

Then there cannot be any $\prec_1, \dots, \prec_n, \prec_i' \in L$ such that $f(\prec_1, \dots, \prec_i, \dots, \prec_n) = a$, $f(\prec_1, \dots, \prec_i', \dots, \prec_n) = b$ and $a \prec_i b$.

Conversely, violated monotonicity implies that there is a possibility for strategic manipulation. □

Dictatorship in Social Choice Functions

Definition (dictatorship)

Voter i is a dictator in a social choice function f if for all $\prec_1, \dots, \prec_i, \dots, \prec_n \in L$, $f(\prec_1, \dots, \prec_i, \dots, \prec_n) = a$, where a is the unique candidate with $b \prec_i a$ for all $b \in A$ with $b \neq a$.

The function f is a dictatorship if there is a dictator in f.

We are going to prove the theorem of Gibbard and Satterthwaite:

Every incentive compatible and surjective social choice function with three or more alternatives is necessarily a dictatorship.

Approach:

- We prove the result using Arrow's Theorem.
- To that end, construct social welfare function from social choice function.

Notation:

Let $S \subseteq A$ and $\prec \in L$. By $\prec^S$ we denote the order obtained by moving all elements from S "to the top" in $\prec$, while preserving the relative orderings of the elements in S and of those in $A \setminus S$.

More formally:

- for $a, b \in S$: $a \prec^S b$ iff $a \prec b$,
- for $a, b \notin S$: $a \prec^S b$ iff $a \prec b$,
- for $a \notin S$, $b \in S$: $a \prec^S b$.

These conditions uniquely define $\prec^S$.

Example

Let $d \prec a \prec c \prec b \prec e$, and $S = \{a, b\}$.
Then $d \prec^S c \prec^S e \prec^S a \prec^S b$.

Lemma (top preference)

Let f be an incentive compatible and surjective social choice function. Then for all $\prec_1, \dots, \prec_n \in L$ and all $\emptyset \neq S \subseteq A$, we have $f(\prec_1^S, \dots, \prec_n^S) \in S$.

Proof.

Let $a \in S$.

Since f is surjective, there are $\prec_1', \dots, \prec_n' \in L$ such that $f(\prec_1', \dots, \prec_n') = a$.

Now, sequentially, for $i = 1, \dots, n$, change the relation $\prec_i'$ to $\prec_i^S$. At no point during this sequence of changes will f output any candidate $b \notin S$, because f is monotone. $\square$

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Definition (extension of a social choice function)

The function $F : L^n \to L$ that extends the social choice function f is defined as $F(\prec_1, \ldots, \prec_n) = \prec$, where $a \prec b$ iff $f(\prec_1^{\{a,b\}}, \ldots, \prec_n^{\{a,b\}}) = b$ for all $a, b \in A, a \neq b$.

Lemma

If f is an incentive compatible and surjective social choice function, then its extension F is a social welfare function.

Proof.

We show that $\prec$ is a strict linear order, i.e., asymmetric, total and transitive.

…

Gibbard-Satterthwaite Theorem

Extension of a Social Choice Function

Proof (ctd.)

- Asymmetry and totality: Because of the top-preference lemma, $f(\prec_1^{\{a,b\}}, \dots, \prec_n^{\{a,b\}})$ is either a or b, i.e., $a \prec b$ or $b \prec a$, but not both (asymmetry) and not neither (totality).
- Transitivity: We may already assume totality. Suppose that $\prec$ is not transitive, i. e., $a \prec b$ and $b \prec c$, but not $a \prec c$, for some a, b and c. Because of totality, $c \prec a$. Consider $S = \{a,b,c\}$ and WLOG, $f(\prec_1^{\{a,b,c\}}, \dots, \prec_n^{\{a,b,c\}}) = a$. Due to monotonicity of f, we get $f(\prec_1^{\{a,b\}}, \dots, \prec_n^{\{a,b\}}) = a$ by successively changing $\prec_i^{\{a,b,c\}}$ to $\prec_i^{\{a,b\}}$. Thus, we get $b \prec a$ in contradiction to our assumption. □

Lemma (extension lemma)

If f is an incentive compatible, surjective, and non-dictatorial social choice function, then its extension F is a social welfare function that satisfies unanimity, independence of irrelevant alternatives, and non-dictatorship.

Proof.

We already know that F is a social welfare function and still have to show unanimity, independence of irrelevant alternatives, and non-dictatorship.

- Unanimity: Let $a \prec_i b$ for all i. Then $(\prec_i^{\{a,b\}})^{\{b\}} = \prec_i^{\{a,b\}}$. Because of the top-preference lemma, $f(\prec_1^{\{a,b\}}, \ldots, \prec_n^{\{a,b\}}) = b$, hence $a \prec b$.
- ...

Proof (ctd.)

- Independence of irrelevant alternatives: If for all i, $a \prec_i b$ iff $a \prec'_i b$, then $f(\prec_1^{\{a,b\}}, \dots, \prec_n^{\{a,b\}}) = f(\prec_1'^{\{a,b\}}, \dots, \prec_n'^{\{a,b\}})$ must hold, since due to monotonicity the result does not change when $\prec_i^{\{a,b\}}$ is successively replaced by $\prec_i'^{\{a,b\}}$.
- Non-dictatorship: Obvious. $\square$

Gibbard-Satterthwaite Theorem

Theorem (Gibbard-Satterthwaite)

If f is an incentive compatible and surjective social choice function with three or more alternatives, then f is a dictatorship. $\square$

The purpose of mechanism design is to alleviate the negative results of Arrow and Gibbard and Satterthwaite by changing the underlying model. The two usually investigated modifications are:

- Introduction of money (Sections 8.1–8.3)
- Restriction of admissible preference relations (Sections 7.5.2, 8.4)

Summary

- Result corresponding to Arrow's theorem for social choice functions (Gibbard-Satterthwaite):

  Every incentive compatible and surjective social choice function with three or more alternatives is necessarily a dictatorship.

- Proof: reduction to Arrow's theorem
- Outlook (not further discussed here): score vs. ranked voting systems?

Game Theory

7. Social Choice Theory

7.5. Some Positive Results

7.5.1 May's Theorem

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May's Theorem

We had some negative results on social choice and welfare functions so far: Arrow, Gibbard-Satterthwaite.

Question: any positive results for special cases?

First special case: only two alternatives

Intuition: with only two alternatives, no point in misrepresenting preferences

May’s Theorem

Axioms for voting systems:

- Neutrality: “Names” of candidates/alternatives should not be relevant.
- Anonymity: “Names” of voters should not be relevant.
- Monotonicity: If a candidate wins, he should still win if one voter ranks him higher.

May's Theorem

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Theorem (May, 1958)

A voting method for two alternatives satisfies anonymity, neutrality, and monotonicity if and only if it is the plurality method.

Proof.

$\Leftarrow$: Obvious.

$\Rightarrow$: For simplicity, we assume that the number of voters is odd.

Anonymity and neutrality imply that only the numbers of votes for the candidates matter.

Let A be the set of voters that prefer candidate a, and let B be the set of voters that prefer candidate b. Consider a vote with $|A| = |B| + 1$.

May's Theorem

Proof (ctd.)

- Case 1: Candidate a wins. Then by monotonicity, a still wins whenever $|A| > |B|$. With neutrality, we also get that b wins whenever $|B| > |A|$. This uniquely characterizes the plurality method.
- Case 2: Candidate b wins. Assume that one voter for a changes his preference to b. Then $|A'| + 1 = |B'|$. By monotonicity, b must still win. This is completely symmetric to the original vote. Hence, by neutrality, a should win. This is a contradiction, implying that case 2 cannot occur. $\square$

Remark: For three or more alternatives, there are no voting methods that satisfy such a small set of desirable criteria.

Summary

- With only two alternatives, there is a positive result.
- May's theorem:
  A voting method for two alternatives satisfies anonymity, neutrality, and monotonicity if and only if it is the plurality method.
- Note:
  $|A| = 2 \Rightarrow$ plurality = plurality+runoff = IRV = Borda = ...

Game Theory

7. Social Choice Theory

7.5. Some Positive Results

7.5.2 Single-Peaked Preferences

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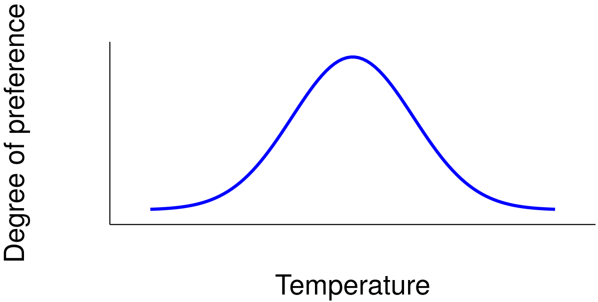
Single-Peaked Preferences

The results by Arrow and Gibbard-Satterthwaite only apply is there are no restrictions on the preference orders.

Second special case: restrictions on preference orders



Single-Peaked Preferences

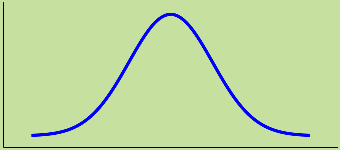
Definition (single-peaked preference)

A preference relation $\prec_i$ over the interval $[0,1]$ is called a single-peaked preference relation if there exists a value $p_i \in [0,1]$ such that for all $x \in [0,1] \setminus p_i$ and for all $\lambda \in [0,1)$,

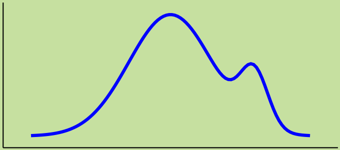
$$x \prec_i \lambda x + (1-\lambda) p_i.$$

Example

Single-peaked:



Not single-peaked:



Single-Peaked Preferences

First idea: Use arithmetic mean of all peak values.

Example

Preferred room temperatures:

- Voter 1: 10 °C
- Voter 2: 20 °C
- Voter 3: 21 °C

Arithmetic mean: 17 °C. Is this incentive compatible?

Single-Peaked Preferences

First idea: Use arithmetic mean of all peak values.

Example

Preferred room temperatures:

- Voter 1: $10\,^\circ\mathrm{C}$
- Voter 2: $20\,^\circ\mathrm{C}$
- Voter 3: $21\,^\circ\mathrm{C}$

Arithmetic mean: $17\,^\circ\mathrm{C}$. Is this incentive compatible?

No! Voter 1 can misrepresent his peak value as, e.g., $-11\,^\circ\mathrm{C}$.

Then the mean is $10\,^\circ\mathrm{C}$, his favorite value!

Question: What is a good way to design incentive compatible social choice functions for this setting?

Median Rule

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Definition (median rule)

Let $p_1, \dots, p_n$ be the peaks for the preferences $\prec_1, \dots, \prec_n$ ordered such that we have $p_1 \le p_2 \le \dots \le p_n$. Then the median rule is the social choice function f with

$$f(\prec_1, \dots, \prec_n) = p_{\lceil n/2 \rceil}.$$

Example

Preferred room temperatures:

- Voter 1: 10 °C
- Voter 2: 20 °C
- Voter 3: 21 °C

Median: 20 °C.

Is this incentive compatible?

Median Rule

Theorem

The median rule is surjective, incentive compatible, anonymous, and non-dictatorial.

Proof.

- Surjective: Obvious, because the median rule satisfies unanimity.
- Incentive compatible: Assume that p_i is below the median. Then reporting a lower value does not change the median ($\rightsquigarrow$ does not help), and reporting a higher value can only increase the median ($\rightsquigarrow$ does not help, either). Similarly, if p_i is above the median.
- Anonymous: Is implicit in the rule.
- Non-dictatorial: Follows from anonymity. □

Summary

- With restricted type of preferences, there is a positive result.
- The median rule returns the median value among the reported peaks (of single-peaked preferences).
- The median rule is surjective, incentive compatible, anonymous, and non-dictatorial.

Game Theory

8. Mechanism Design

8.1. Introduction and Example

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Motivation

Objective: Design the rules of the game such that desirable behavior is dominant behavior.

Here: desirable $\approx$

- truthful about one's own preferences +
- contributing to maximizing social welfare

Motivation

Model:

- Strict linear orders $\prec$ contain no information about "by how much" one alternative is preferred.
- Idea: Instead, use money to measure this.
- Use money also for transfers between players "for compensation".

Setting

Given:

- set of alternatives A
- set of n players N
- valuation functions $v_i : A \to \mathbb{R}$ such that $v_i(a)$ denotes the value player i assigns to alternative a

Find:

- a chosen alternative $a \in A$.
- payments $p_i \in \mathbb{R}$ to be paid by player i

Utility of player i: $u_i(a) = v_i(a) - p_i$.

Example: Vickrey Auctions

Second price auctions (aka Vickrey auctions):

- There are n players bidding for a single item.
- Player i's private valuations of item: w_i.
- Desired outcome: Player with highest private valuation wins bid.
- Players should reveal their valuations truthfully.
- Winner i pays price p^* and has utility $w_i - p^*$.
- Non-winners pay nothing and have utility 0.

Example: Vickrey Auctions

Formally:

- $A = N$
- $v_i(a) = \begin{cases} w_i & \text{if } a = i \\ 0 & \text{else} \end{cases}$
- What about payments? Say player i wins:
  - $p^* = 0$ (winner pays nothing): bad idea, players would manipulate and publicly declare values $w_i' \gg w_i$.
  - $p^* = w_i$ (winner pays his valuation): bad idea, players would manipulate and publicly declare values $w_i' = w_i - \varepsilon$.
  - better: $p^* = \max_{j \neq i} w_j$ (winner pays second highest bid).

Vickrey Auction

Definition (Vickrey Auction)

The winner of the Vickrey Auction (aka second price auction) is the player i with the highest declared value w_i. He has to pay the second highest declared bid $p^* = \max_{j \neq i} w_j$.

Proposition (Vickrey)

Let i be one of the players and w_i his valuation for the item, u_i his utility if he truthfully declares w_i as his valuation of the item, and u_i' his utility if he falsely declares w_i' as his valuation of the item. Then $u_i \geq u_i'$.

Proof

See
`http://en.wikipedia.org/wiki/Vickrey_auction`. □

Summary

- New preference model: with money.
- To ensure truthful revelation of preferences, we need the right payment functions.
- Example: Vickrey auctions.

Game Theory

8. Mechanism Design

8.2. Incentive Compatible Mechanisms

8.2.1. VCG Mechanisms

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Incentive Compatible Mechanisms

- Idea: Generalization of Vickrey auctions.
- Preferences modeled as functions $v_i : A \to \mathbb{R}$.
- Let V_i be the space of all such functions for player i.
- Unlike for social choice functions: Not only decide about chosen alternative, but also about payments.

Incentive Compatible Mechanisms

Definition (mechanism)

A mechanism $\langle f, p_1, \dots, p_n \rangle$ consists of

- a social choice function $f : V_1 \times \cdots \times V_n \to A$ and
- for each player i, a payment function $p_i : V_1 \times \cdots \times V_n \to \mathbb{R}$.

Definition (incentive compatibility)

A mechanism $\langle f, p_1, \dots, p_n \rangle$ is called incentive compatible if for each player $i = 1, \dots, n$, for all preferences $v_1 \in V_1, \dots, v_n \in V_n$ and for each preference $v_i' \in V_i$,

$$v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i}).$$

VCG Mechanisms

- If $\langle f, p_1, \ldots, p_n \rangle$ is incentive compatible, truthfully declaring ones preference is dominant strategy.
- The Vickrey-Clarke-Groves mechanism is an incentive compatible mechanism that maximizes “social welfare”, i.e., the sum over all individual utilities $\sum_{i=1}^{n} v_i(a)$.
- Idea: Reflect other players’ utilities in payment functions, align all players’ incentives with goal of maximizing social welfare.

VCG Mechanisms

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Definition (Vickrey-Clarke-Groves mechanism)

A mechanism $\langle f, p_1, \dots, p_n \rangle$ is called a Vickrey-Clarke-Groves mechanism (VCG mechanism) if

1. $f(v_1, \dots, v_n) \in \operatorname{argmax}_{a \in A} \sum_{i=1}^{n} v_i(a)$ for all $v_1, \dots, v_n$ and
2. there are functions $h_1, \dots, h_n$ with $h_i : V_{-i} \to \mathbb{R}$ such that $p_i(v_1, \dots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \dots, v_n))$ for all $i = 1, \dots, n$ and $v_1, \dots, v_n$.

Note: $h_i(v_{-i})$ independent of player i's declared preference $\Rightarrow$ $h_i(v_{-i}) = c$ constant from player i's perspective.

Utility of player i $= v_i(f(v_1, \dots, v_n)) + \sum_{j \neq i} v_j(f(v_1, \dots, v_n)) - c = \sum_{j=1}^{n} v_j(f(v_1, \dots, v_n)) - c =$ social welfare $- c$.

VCG Mechanisms

Theorem (Vickrey-Clarke-Groves)

Every VCG mechanism is incentive compatible.

Proof.

Let i, v_{-i}, v_i and v_i' be given. Show: Declaring true preference v_i dominates declaring false preference v_i'.

Let $a = f(v_i, v_{-i})$ and $a' = f(v_i', v_{-i})$.

$$\text{Utility player } i = \begin{cases} v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) & \text{if declaring } v_i \\ v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \text{if declaring } v_i' \end{cases}$$

Alternative $a = f(v_i, v_{-i})$ maximizes social welfare

$\Rightarrow v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a')$.

$\Rightarrow v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i})$. $\square$

Summary

- New preference model: with money.
- VCG mechanisms generalize Vickrey auctions.
- VCG mechanisms are incentive compatible mechanisms maximizing social welfare.

Game Theory

8. Mechanism Design

8.2. Incentive Compatible Mechanisms

8.2.2. Clarke Pivot Functions

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Clarke Pivot Function

- So far: functions h_i unspecified
  $\rightsquigarrow$ payment functions p_i only partially specified
- One possibility: $h_i(v_{-i}) = 0$ for all h_i and v_{-i}
  Drawback: too much money distributed among players (more that necessary)
- Further requirements:
  - Players should pay at most as much as they value the outcome.
  - Players should only pay, never receive money.

Individual Rationality, Positive Transfers

Definition (individual rationality)

A mechanism is individually rational if all players always get a nonnegative utility, i.e., if for all $i = 1, \ldots, n$ and all $v_1, \ldots, v_n$,

$$v_i(f(v_1, \ldots, v_n)) - p_i(v_1, \ldots, v_n) \geq 0.$$

Definition (positive transfers)

A mechanism has no positive transfers if no player is ever paid money, i.e., if for all $i = 1, \ldots, n$ and all $v_1, \ldots, v_n$,

$$p_i(v_1, \ldots, v_n) \geq 0.$$

Clarke Pivot Function

Definition (Clarke pivot function)

The Clarke pivot function is the function

$$h_i(v_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b).$$

- This leads to payment functions

$$p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$

  for $a = f(v_1, \ldots, v_n)$.
- Player i pays the difference between what the other players could achieve without him and what they achieve with him.
- Each player internalizes the externalities he causes.

Clarke Pivot Function

Example

- Players $N = \{1,2\}$, alternatives $A = \{a,b\}$.
- Values: $v_1(a) = 10$, $v_1(b) = 2$, $v_2(a) = 9$ and $v_2(b) = 15$.
- Without player 1: b best, since $v_2(b) = 15 > 9 = v_2(a)$.
- With player 1: a best, since $v_1(a) + v_2(a) = 10 + 9 = 19 > 17 = 2 + 15 = v_1(b) + v_2(b)$.
- With player 1, other players (i.e., player 2) lose $v_2(b) - v_2(a) = 6$ units of utility.

$\Rightarrow$ Clarke pivot function $h_1(v_2) = 15$

$\Rightarrow$ payment function

$$p_1(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq 1} v_j(b) - \sum_{j \neq 1} v_j(a) = 15 - 9 = 6.$$

Clarke Pivot Rule

Lemma (Clarke pivot rule)

A VCG mechanism with Clarke pivot functions has no positive transfers. If $v_i(a) \geq 0$ for all $i = 1, \dots, n$, $v_i \in V_i$ and $a \in A$, then the mechanism is also individually rational.

Proof.

Let $a = f(v_1, \dots, v_n)$ be the alternative maximizing $\sum_{j=1}^{n} v_j(a)$, and b the alternative maximizing $\sum_{j \neq i} v_j(b)$.

Payment function for i: $p_i(v_1, \dots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$.

Since b maximizes $\sum_{j \neq i} v_j(b)$: $p_i(v_1, \dots, v_n) \geq 0$ ($\rightsquigarrow$ no positive transfers).

Utility of player i: $u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b)$.

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Proof (ctd.)

Individual rationality: Since $v_i(b) \geq 0$,

$$u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j=1}^{n} v_j(a) - \sum_{j=1}^{n} v_j(b).$$

Since a maximizes $\sum_{j=1}^{n} v_j(a)$,

$$\sum_{j=1}^{n} v_j(a) \geq \sum_{j=1}^{n} v_j(b)$$

and hence $u_i \geq 0$.

Therefore, the mechanism is also individually rational. $\square$

Summary

- Recall: VCG mechanisms are incentive compatible mechanisms maximizing social welfare.
- With Clarke pivot functions:
  - no positive transfers and
  - individual rationality (if nonnegative valuations).

Game Theory

8. Mechanism Design

8.2. Incentive Compatible Mechanisms

8.2.3. Examples

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Vickrey Auction as a VCG Mechanism

- $A = N$. Valuations: w_i. $v_a(a) = w_a$, $v_i(a) = 0$ $(i \neq a)$.
- a maximizes social welfare $\sum_{i=1}^{n} v_i(a)$ iff a maximizes w_a.
- Let $a = f(v_1, \ldots, v_n) = \operatorname{argmax}_{j \in A} w_j$ be the highest bidder.
- Payments: $p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$.
- But $\max_{b \in A} \sum_{j \neq i} v_j(b) = \max_{b \in A \setminus \{i\}} w_b$.
- Winner pays value of second highest bid:

$$p_a(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq a} v_j(b) - \sum_{j \neq a} v_j(a)$$

$$= \max_{b \in A \setminus \{a\}} w_b - 0 = \max_{b \in A \setminus \{a\}} w_b.$$

- Non-winners pay nothing: For $i \neq a$,

$$p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$

$$= \max_{b \in A \setminus \{i\}} w_b - w_a = w_a - w_a = 0.$$

Example: Bilateral Trade

- Seller s offers item he values with $0 \leq w_s \leq 1$.
- Potential buyer b values item with $0 \leq w_b \leq 1$.
- Alternatives $A = \{\textit{no-trade}, \textit{trade}\}$.
- Valuations:

$$\begin{aligned} v_s(\textit{no-trade}) &= 0, & v_s(\textit{trade}) &= -w_s, \\ v_b(\textit{no-trade}) &= 0, & v_b(\textit{trade}) &= w_b. \end{aligned}$$

- VCG mechanism maximizes $v_s(a) + v_b(a)$.
- We have

$$v_s(\textit{no-trade}) + v_b(\textit{no-trade}) = 0,$$
$$v_s(\textit{trade}) + v_b(\textit{trade}) = w_b - w_s$$

i.e., *trade* maximizes social welfare iff $w_b \geq w_s$.

Example: Bilateral Trade (ctd.)

- Requirement: if *no-trade* is chosen, neither player pays anything:

$$p_s(v_s, v_b) = p_b(v_s, v_b) = 0.$$

- To that end, choose Clarke pivot function for buyer:

$$h_b(v_s) = \max_{a \in A} v_s(a).$$

- For seller: Modify Clarke pivot function by an additive constant and set

$$h_s(v_b) = \max_{a \in A} v_b(a) - w_b.$$

Example: Bilateral Trade (ctd.)

- For alternative *no-trade*,

$$\begin{aligned}
p_s(v_s, v_b) &= \max_{a \in A} v_b(a) - w_b - v_b(\textit{no-trade}) \\
&= w_b - w_b - 0 = 0 \quad \text{and} \\
p_b(v_s, v_b) &= \max_{a \in A} v_s(a) - v_s(\textit{no-trade}) \\
&= 0 - 0 = 0.
\end{aligned}$$

- For alternative *trade*,

$$\begin{aligned}
p_s(v_s, v_b) &= \max_{a \in A} v_b(a) - w_b - v_b(\textit{trade}) \\
&= w_b - w_b - w_b = -w_b \quad \text{and} \\
p_b(v_s, v_b) &= \max_{a \in A} v_s(a) - v_s(\textit{trade}) \\
&= 0 + w_s = w_s.
\end{aligned}$$

Example: Bilateral Trade (ctd.)

- Because $w_b \geq w_s$, the seller gets at least as much as the buyer pays, i.e., the mechanism subsidizes the trade.
- Without subsidies, no incentive compatible bilateral trade possible.
- Note: Buyer and seller can exploit the system by colluding.

Example: Public Project

- Project costs C units.
- Each citizen i privately values the project at w_i units.
- Government will undertake project if $\sum_i w_i > C$.
- Alternatives: $A = \{\textit{no-project}, \textit{project}\}$.
- Valuations:

$$v_G(\textit{no-project}) = 0, \qquad v_G(\textit{project}) = -C,$$
$$v_i(\textit{no-project}) = 0, \qquad v_i(\textit{project}) = w_i.$$

- VCG mechanism with Clarke pivot rule: for each citizen i,

$$h_i(v_{-i}) = \max_{a \in A} \left(\sum_{j \neq i} v_j(a) + v_G(a) \right)$$
$$= \begin{cases} \sum_{j \neq i} w_j - C, & \text{if } \sum_{j \neq i} w_j > C \\ 0, & \text{otherwise.} \end{cases}$$

Example: Public Project (ctd.)

- Citizen i pivotal if $\sum_j w_j > C$ and $\sum_{j \neq i} w_j \leq C$.
- Payment function for citizen i:

$$p_i(v_{1..n}, v_G) = h_i(v_{-i}) - \left(\sum_{j \neq i} v_j(f(v_{1..n}, v_G)) + v_G(f(v_{1..n}, v_G)) \right)$$

  - Case 1: Project undertaken, i pivotal:

$$p_i(v_{1..n}, v_G) = 0 - \left(\sum_{j \neq i} w_j - C \right) = C - \sum_{j \neq i} w_j$$

  - Case 2: Project undertaken, i not pivotal:

$$p_i(v_{1..n}, v_G) = \left(\sum_{j \neq i} w_j - C \right) - \left(\sum_{j \neq i} w_j - C \right) = 0$$

  - Case 3: Project not undertaken:

$$p_i(v_{1..n}, v_G) = 0$$

Example: Public Project (ctd.)

- I.e., citizen i pays nonzero amount

$$C - \sum_{j \neq i} w_j$$

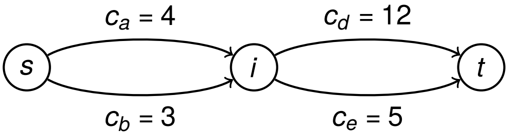
  only if he is pivotal.
- He pays difference between value of project to fellow citizens and cost C, in general less than w_i.
- Generally,

$$\sum_i p_i(\mathit{project}) \leq C$$

  i.e., project has to be subsidized.

Example: Buying a Path in a Network

- Communication network modeled as $G = (V, E)$.
- Each link $e \in E$ owned by different player e.
- Each link $e \in E$ has cost c_e if used.
- Objective: procure communication path from s to t.
- Alternatives: $A = \{\pi \mid \pi \text{ path from } s \text{ to } t\}$.
- Valuations: $v_e(\pi) = -c_e$, if $e \in \pi$, and $v_e(\pi) = 0$, if $e \notin \pi$.
- Maximizing social welfare:
  minimize $\sum_{e \in \pi} c_e$ over all paths π from s to t.
- Example:



Example: Buying a Path in a Network (ctd.)

- For $G = (V, E)$ and $e \in E$ let $G \setminus e = (V, E \setminus \{e\})$.
- VCG mechanism:

$$h_e(v_{-e}) = \max_{\pi' \in G \setminus e} \sum_{e' \in \pi'} -c_{e'}$$

i.e., the cost of the cheapest path from s to t in $G \setminus e$.
(Assume that G is 2-connected, s.t. such π' exists.)

- Payment functions: for chosen path $\pi = f(v_1, \dots, v_n)$,

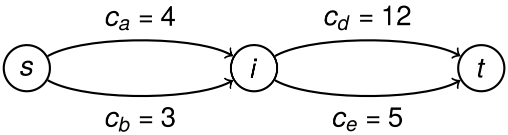
$$p_e(v_1, \dots, v_n) = h_e(v_{-e}) - \sum_{e \neq e' \in \pi} -c_{e'}.$$

  - Case 1: $e \notin \pi$. Then $p_e(v_1, \dots, v_n) = 0$.
  - Case 2: $e \in \pi$. Then

$$p_e(v_1, \dots, v_n) = \max_{\pi' \in G \setminus e} \sum_{e' \in \pi'} -c_{e'} - \sum_{e \neq e' \in \pi} -c_{e'}.$$

Example: Buying a Path in a Network (ctd.)

- Example:



  - Cost along b and e: 8
  - Cost without e: 3
  - Cost of cheapest path without e: 15 (along b and d)
  - Difference is payment: $-15 - (-3) = -12$
    I.e., owner of arc e gets payed 12 for using his arc.

- Note: Alternative path after deletion of e does not necessarily differ from original path at only one position. Could be totally different.

Summary

We saw some examples of applications of VCG mechanisms:

- Vickrey Auctions
- Bilateral Trade
- Public Projects
- Buying a Path in a Network

Game Theory

8. Mechanism Design

8.3. Mechanisms without Money

8.3.1. House Allocation

Albert-Ludwigs-Universität Freiburg

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Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Mechanisms without Money

Motivation 1:

- According to Gibbard-Satterthwaite:
  In general, nontrivial social choice functions manipulable.
- One way out: introduction of money
  (cf. VCG mechanisms)
- Other way out: restriction of preferences
  (cf. single-peaked preferences; this chapter)

Motivation 2:

- Introduction of central concept from cooperative game theory: the core

Examples:

- House allocation problem (Sec. 8.3.1)
- Stable matchings (Sec. 8.3.2)

House Allocation Problem

- Players $N = \{1, \ldots, n\}$.
- Each player i owns house i.
- Each player i has strict linear preference order $\lhd_i$ over the set of houses.
  Example: $j \lhd_i k$ means player i prefers house k to house j.
- Alternatives A: allocations of houses to players (permutations $\pi \in S_n$ of N).
  Example: $\pi(i) = j$ means player i gets house j.
- Objective: reallocate the houses among the agents "appropriately".

House Allocation Problem

- Note on preference relations:
  - arbitrary (strict linear) preference orders $\lhd_i$ over houses,
  - but no arbitrary preference orders $\preceq_i$ over A.
- Rather: player i indifferent between different allocations π_1 and π_2 as long as $\pi_1(i) = \pi_2(i)$.
  Indifference denoted as $\pi_1 \approx_i \pi_2$.
- If player i is not indifferent: $\pi_1 \prec_i \pi_2$ iff $\pi_1(i) \lhd_i \pi_2(i)$.
- Notation: $\pi_1 \preceq_i \pi_2$ iff $\pi_1 \prec_i \pi_2$ or $\pi_1 \approx_i \pi_2$.
- This makes Gibbard-Satterthwaite inapplicable.

House Allocation Problem

- Important new aspect of house allocation problem: players control resources to be allocated.
- Allocation can be subverted by subset of agents breaking away and trading among themselves.
- How to avoid such allocations?
- How to make allocation mechanism non-manipulable?

House Allocation Problem

Notation: For $M \subseteq N$, let

$$A(M) = \{\pi \in A \,|\, \forall i \in M : \pi(i) \in M\}$$

be the set of allocations that can be achieved by the agents in M trading among themselves.

Definition (blocking coalition)

Let $\pi \in A$ be an allocation. A set $M \subseteq N$ is called a blocking coalition for π if there exists a $\pi' \in A(M)$ such that

- $\pi \preceq_i \pi'$ for all $i \in M$ and
- $\pi \prec_i \pi'$ for at least one $i \in M$.

House Allocation Problem

Intuition:

A blocking coalition can receive houses everyone from the coalition likes at least as much as under allocation π, with at least one player being strictly better off, by trading among themselves.

Definition (core)

The set of allocations that is not blocked by any subset of agents is called the core.

Question: Is the core nonempty?

Top Trading Cycle Algorithm (TTCA)

- Algorithm to construct allocation
- Let $G = \langle V, A, c \rangle$ be an arc-colored directed graph where:
  - $V = N$ (i.e., one vertex for each player),
  - $A = V \times V$, and
  - $c : A \to N$ such that $c(i, j) = k$ if house j is player i's kth ranked choice according to $\lhd_i$.
- Note: Loops (i, i) are allowed. We treat them as cycles of length 0.

Top Trading Cycle Algorithm (TTCA)

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Pseudocode:

let $\pi(i) = i$ for all $i \in N$.
while players unaccounted for **do**
  consider subgraph G' of G where each vertex has
    only one outgoing arc: the least-colored one from G.
  identify cycles in G'.
  add corresponding cyclic permutations to π.
  delete players accounted for and incident edges from G.
end while
output π.

Notation:

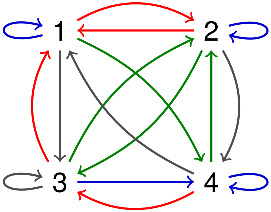
Let N_i be the set of vertices on cycles identified in iteration i.

Top Trading Cycle Algorithm (TTCA)

Example:

- Player 1: $3 \lhd_1 1 \lhd_1 4 \lhd_1 2$
- Player 2: $4 \lhd_2 2 \lhd_2 3 \lhd_2 1$
- Player 3: $3 \lhd_3 4 \lhd_3 2 \lhd_3 1$
- Player 4: $1 \lhd_4 4 \lhd_4 2 \lhd_4 3$

Corresponding graph:



Top Trading Cycle Algorithm (TTCA)

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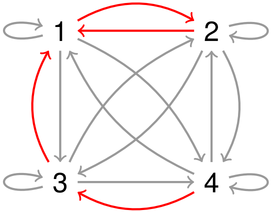

Example:

- Player 1: $3 \lhd_1 1 \lhd_1 4 \lhd_1 2$
- Player 2: $4 \lhd_2 2 \lhd_2 3 \lhd_2 1$
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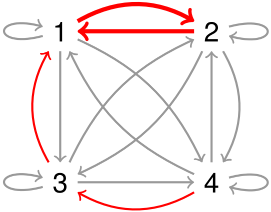


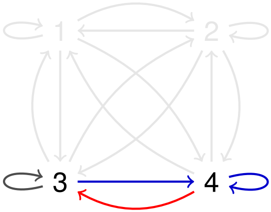
- Iteration 1: $\pi(1) = 2$, $\pi(2) = 1$.

Top Trading Cycle Algorithm (TTCA)

Example:

- Player 1: $3 \lhd_1 1 \lhd_1 4 \lhd_1 2$
- Player 2: $4 \lhd_2 2 \lhd_2 3 \lhd_2 1$
- Player 3: $3 \lhd_3 4 \lhd_3 2 \lhd_3 1$
- Player 4: $1 \lhd_4 4 \lhd_4 2 \lhd_4 3$

Corresponding graph:



- Iteration 1: $\pi(1) = 2$, $\pi(2) = 1$.

Top Trading Cycle Algorithm (TTCA)

Example:

- Player 1: $3 \lhd_1 1 \lhd_1 4 \lhd_1 2$
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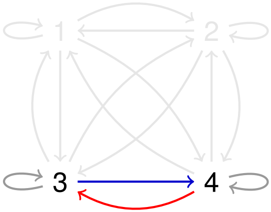
Corresponding graph:



- Iteration 1: $\pi(1) = 2$, $\pi(2) = 1$.

Top Trading Cycle Algorithm (TTCA)

Example:

- Player 1: $3 \lhd_1 1 \lhd_1 4 \lhd_1 2$
- Player 2: $4 \lhd_2 2 \lhd_2 3 \lhd_2 1$
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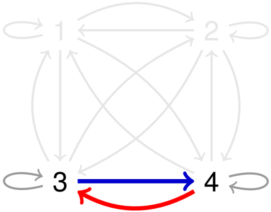
Corresponding graph:



- Iteration 1: $\pi(1) = 2$, $\pi(2) = 1$.
- Iteration 2: $\pi(3) = 4$, $\pi(4) = 3$.

Top Trading Cycle Algorithm (TTCA)

Example:

- Player 1: $3 \lhd_1 1 \lhd_1 4 \lhd_1 2$
- Player 2: $4 \lhd_2 2 \lhd_2 3 \lhd_2 1$
- Player 3: $3 \lhd_3 4 \lhd_3 2 \lhd_3 1$
- Player 4: $1 \lhd_4 4 \lhd_4 2 \lhd_4 3$

Corresponding graph:

- Iteration 1: $\pi(1) = 2$, $\pi(2) = 1$.
- Iteration 2: $\pi(3) = 4$, $\pi(4) = 3$.
- Done: $\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 4$, $\pi(4) = 3$.

Top Trading Cycle Algorithm (TTCA)

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Theorem

The core of the house allocation problem consists of exactly one matching.

Proof sketch

At most one matching: Show that if a matching is in the core, it must be the one returned by the TTCA.

In TTCA, each player in N_1 receives his favorite house.

Therefore, N_1 would form a blocking coalition to any allocation that does not assign to all of those players the houses they would receive in TTCA.

…

Top Trading Cycle Algorithm (TTCA)

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Proof sketch (ctd.)

That is, any core allocation must assign N_1 to houses as TTCA assigns them.

Argument can be extended inductively to N_k, $2 \leq k \leq n$.

At least one matching: Show that TTCA allocation is in the core, i.e., that there is no other blocking coalition $M \subseteq N$. Homework. □

Top Trading Cycle Mechanism (TTCM)

Question: What about manipulability?

Definition (top trading cycle mechanism)

The top trading cycle mechanism (TTCM) is the function that, for each profile of preferences, returns the allocation computed by the TTCA.

Theorem

The TTCM cannot be manipulated.

Proof

Homework. □

Summary

- Avoid Gibbard-Satterthwaite by restricting domain of preferences.
- House allocation problem:
  - Solved using top trading cycle algorithm.
  - Algorithm finds unique solution in the core, where no blocking coalition of players has an incentive to break away.
  - The top trading cycle mechanism cannot be manipulated.

Game Theory

8. Mechanism Design

8.3. Mechanisms without Money

8.3.2. Stable Matchings

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Summer semester 2020

Mechanisms without Money

Motivation 1:

- According to Gibbard-Satterthwaite:
  In general, nontrivial social choice functions manipulable.
- One way out: introduction of money
  (cf. VCG mechanisms)
- Other way out: restriction of preferences
  (cf. single-peaked preferences; this chapter)

Motivation 2:

- Introduction of central concept from cooperative game theory: the core

Examples:

- House allocation problem (Sec. 8.3.1)
- Stable matchings (Sec. 8.3.2)

Stable Matchings

Problem statement:

- Given disjoint finite sets M of men and W of women.
- Assume WLOG that $|M| = |W|$ (introduce dummy-men/dummy-women).
- Each $m \in M$ has strict preference ordering $\prec_m$ over W.
- Each $w \in W$ has strict preference ordering $\prec_w$ over M.
- Matching: "appropriate" assignment of men to women such that each man is assigned to at most one woman and vice versa.

Stable Matchings

Note: A group of players can subvert a matching by opting out.

Definition (stability, blocking pair)

A matching is called unstable if there are two men m, m' and two women w, w' such that

- m is matched to w,
- m' is matched to w', and
- $w \prec_m w'$ and $m' \prec_{w'} m$.

The pair $\langle m, w' \rangle$ is called a blocking pair.
A matching that has no blocking pairs is called stable.

Definition (core)

The core of the matching game is the set of all stable matchings.

Stable Matchings

Example:

- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Two matchings:

Stable Matchings

Example:

- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Two matchings:

- Matching $\{\langle m_1, w_1\rangle, \langle m_2, w_2\rangle, \langle m_3, w_3\rangle\}$

Stable Matchings

Example:

- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Two matchings:

- Matching $\{\langle m_1, w_1\rangle, \langle m_2, w_2\rangle, \langle m_3, w_3\rangle\}$
  - unstable ($\langle m_1, w_2\rangle$ is a blocking pair)

Stable Matchings

Example:

- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Two matchings:

- Matching $\{\langle m_1, w_1\rangle, \langle m_2, w_2\rangle, \langle m_3, w_3\rangle\}$
  - unstable ($\langle m_1, w_2\rangle$ is a blocking pair)
- Matching $\{\langle m_1, w_1\rangle, \langle m_3, w_2\rangle, \langle m_2, w_3\rangle\}$
  - stable

Stable Matchings

Question: Is there always a stable matching?

Answer: Yes! And it can even be efficiently constructed.

How? Deferred acceptance algorithm!

Deferred Acceptance Algorithm

Definition (deferred acceptance algorithm, male proposals)

1. Each man proposes to his top-ranked choice.
2. Each woman who has received at least one proposal (including tentatively kept one from earlier rounds) tentatively keeps top-ranked proposal and rejects rest.
3. If no man is left rejected, stop.
4. Otherwise, each man who has been rejected proposes to his top-ranked choice among the women who have not rejected him. Then, goto 2.

Deferred Acceptance Algorithm

Note:

- Algorithm has polynomial runtime.
- No man is assigned to more than one woman.
- No woman is assigned to more than one man.
- $\rightsquigarrow$ matching

Deferred Acceptance Algorithm

Example:

- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Deferred acceptance algorithm:

Deferred Acceptance Algorithm

Example:

- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Deferred acceptance algorithm:

1. m_1 proposes to w_2, m_2 to w_1, and m_3 to w_1.

Deferred Acceptance Algorithm

Example:

- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Deferred acceptance algorithm:

1. m_1 proposes to w_2, m_2 to w_1, and m_3 to w_1.
2. w_1 keeps m_3 and rejects m_2, w_2 keeps m_1.

Deferred Acceptance Algorithm

Example:

- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Deferred acceptance algorithm:

1. m_1 proposes to w_2, m_2 to w_1, and m_3 to w_1.
2. w_1 keeps m_3 and rejects m_2, w_2 keeps m_1.
3. m_2 now proposes to w_3.

Deferred Acceptance Algorithm

Example:

- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Deferred acceptance algorithm:

1. m_1 proposes to w_2, m_2 to w_1, and m_3 to w_1.
2. w_1 keeps m_3 and rejects m_2, w_2 keeps m_1.
3. m_2 now proposes to w_3.
4. w_3 keeps m_2.

Deferred Acceptance Algorithm

Example:

- Man 1: $w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- Man 2: $w_2 \prec_{m_2} w_3 \prec_{m_2} w_1$
- Man 3: $w_3 \prec_{m_3} w_2 \prec_{m_3} w_1$
- Woman 1: $m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- Woman 2: $m_2 \prec_{w_2} m_1 \prec_{w_2} m_3$
- Woman 3: $m_2 \prec_{w_3} m_3 \prec_{w_3} m_1$

Deferred acceptance algorithm:

1. m_1 proposes to w_2, m_2 to w_1, and m_3 to w_1.
2. w_1 keeps m_3 and rejects m_2, w_2 keeps m_1.
3. m_2 now proposes to w_3.
4. w_3 keeps m_2.

Resulting matching: $\{\langle m_1, w_2\rangle, \langle m_2, w_3\rangle, \langle m_3, w_1\rangle\}$.

Deferred Acceptance Algorithm

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Theorem

The deferred acceptance algorithm with male proposals terminates in a stable matching.

Proof.

Suppose not.

Then there exists a blocking pair $\langle m_1, w_1 \rangle$ with m_1 matched to some w_2 and w_1 matched to some m_2.

Since $\langle m_1, w_1 \rangle$ is blocking and $w_2 \prec_{m_1} w_1$, in the proposal algorithm, m_1 would have proposed to w_1 before w_2.

Since m_1 was not matched with w_1 by the algorithm, it must be because w_1 received a proposal from a man she ranked higher than m_1. $\ldots$

Deferred Acceptance Algorithm

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Proof (ctd.)

Since the algorithm matches her to m_2 it follows that $m_1 \prec_{w_1} m_2$.

This contradicts the fact that $\langle m_1, w_1 \rangle$ is a blocking pair. $\square$

Analogous version where the women propose: outcome would also be a stable matching.

Deferred Acceptance Algorithm

Denote a matching by μ. The woman assigned to man m in μ is $\mu(m)$, and the man assigned to woman w is $\mu(w)$.

Definition (optimality)

A matching μ is male-optimal if there is no stable matching ν such that $\mu(m) \prec_m \nu(m)$ or $\mu(m) = \nu(m)$ for all $m \in M$ and $\mu(m) \prec_m \nu(m)$ for at least one $m \in M$. Female-optimal: similar.

Theorem

- *The stable matching produced by the (fe)male-proposal deferred acceptance algorithm is (fe)male-optimal.*
- *In general, there is no stable matching that is male-optimal and female-optimal.* □

Deferred Acceptance Algorithm

Theorem

The mechanism associated with the (fe)male-proposal algorithm cannot be manipulated by the (fe)males. □

Note: The mechanism associated with the male-proposal algorithm can be manipulated by the females and vice versa.

(Idea: strategically reject a proposal who then binds your main competitor for your favorite partner in the next round, freeing up that partner for you $\rightsquigarrow$ try this out with our running example!)

Summary

- Avoid Gibbard-Satterthwaite by restricting domain of preferences.
- Stable matchings:
  - Solved using deferred acceptance algorithm.
  - Algorithm finds a stable matching in the core, where no blocking pair of players has an incentive to break away.
  - The mechanism associated with the (fe)male-proposal algorithm cannot be manipulated by the (fe)males.

Game Theory

8. Mechanism Design

8.4. Combinatorial Auctions

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Combinatorial Auctions

Motivation:

- Multiple items are auctioned concurrently.
- Bidders have preferences for combinations (bundles) of items.
- Items can complement or substitute one another.
  - complement: left and right shoe together.
  - substitute: two right shoes.
- Aim: socially optimal allocation of items to bidders.

Combinatorial Auctions

Applications:

- Spectrum auctions (with combinations of spectrum bands and geographical areas)
- Procurement of transportation services for multiple routes
- ...

Notation:

- Items: $G = \{1, \ldots, m\}$
- Bidders: $N = \{1, \ldots, n\}$

Definition (valuation)

A valuation is a function $v : 2^G \to \mathbb{R}^+$ with $v(\emptyset) = 0$ and $v(S) \leq v(T)$ for $S \subseteq T \subseteq G$.

- Requirement $v(\emptyset) = 0$ to “normalize” valuations.
- Requirement $v(S) \leq v(T)$ for $S \subseteq T \subseteq G$: monotonicity (or “free disposal”).

Let $S, T \subseteq G$ be disjoint.

- S and T are complements to each other if $v(S \cup T) > v(S) + v(T)$.
- S and T are substitutes if $v(S \cup T) < v(S) + v(T)$.

Allocations

Definition (allocation)

An allocation of the items to the bidders is a tuple $\langle S_1, \dots, S_n \rangle$ with $S_i \subseteq G$ for $i = 1, \dots, n$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.

The social welfare obtained by an allocation is $\sum_{i=1}^{n} v_i(S_i)$ if $v_1, \dots, v_n$ are the valuations of the bidders.

An allocation is called socially efficient if it maximizes social welfare among all allocations.

Let A be the set of all allocations.

Winner Determination Problem

Definition (winner determination problem)

Let $v_i : 2^G \to \mathbb{R}^+$, $i = 1, \ldots, n$, be the declared valuations of the bidders. The winner determination problem (WDP) is the problem of finding a socially efficient allocation $a \in A$ for these valuations.

Aim: Develop mechanism for WDP.

Challenges:

- Incentive compatibility
- Complexity of representation and communication of preferences (exponentially many subsets of items!)
- Computational complexity

Special Case: Single-Minded Bidders

Motivation:

- Focus on single-minded bidders: cuts complexity of representation down to polynomial space.
- Idea: single-minded bidder focuses on one bundle, has fixed valuation v^* for that bundle (and its supersets), valuation 0 for all other bundles.

Single-Minded Bidders

Definition (single-minded bidder)

A valuation v is called single-minded if there is a bundle $S^* \subseteq G$ and a value $v^* \in \mathbb{R}^+$ such that

$$v(S) = \begin{cases} v^* & \text{if } S^* \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

A single-minded bid is a pair $\langle S^*, v^* \rangle$.

- Representational complexity: solved.
- Computational complexity: not solved.

Allocation Problem for Single-Minded Bidders

Definition (allocation problem for single-minded bidders)

The allocation problem for single-minded bidders (APSMB) is defined by the following input and output.

- INPUT: Bids $\langle S_i^*, v_i^* \rangle$ for $i = 1, \dots, n$
- OUTPUT: $W \subseteq \{1, \dots, n\}$ with $S_i^* \cap S_j^* = \emptyset$ for $i, j \in W$, $i \neq j$ such that $\sum_{i \in W} v_i^*$ is maximized.

Claim: APSMB is NP-complete.

Allocation Problem for Single-Minded Bidders

Since APSMB is an optimization problem, consider the corresponding decision problem:

Definition (allocation problem for single-minded bidders, decision problem)

The decision problem version of APSMB (APSMB-D) is defined by the following input and output.

- INPUT: Bids $\langle S_i^*, v_i^* \rangle$ for $i = 1, \ldots, n$ and $k \in \mathbb{N}$
- OUTPUT: Is there a $W \subseteq \{1, \ldots, n\}$ with $S_i^* \cap S_j^* = \emptyset$ for $i, j \in W$, $i \neq j$ such that $\sum_{i \in W} v_i^* \geq k$?

Theorem

APSMB-D is NP-complete.

APSMB-D is NP-complete

Proof

NP-hardness: reduction from INDEPENDENT-SET.

INDEPENDENT-SET instance:

- undirected graph $\langle V, E \rangle$ and $k_{IS} \in \mathbb{N}$.
- Question: Is there an independent set of size k_{IS} in $\langle V, E \rangle$?

Corresponding APSMB-D instance:

- $k = k_{IS}$, items $G = E$, bidders $N = V$, and
- for each bidder $i \in V$ the bid $\langle S_i^*, v_i^* \rangle$ with $S_i^* = \{e \in E \mid i \in e\}$ and $v_i^* = 1$.
- Question: Is there an allocation with social welfare $\geq k$?
- (Intuitively: Vertices bid for their incident edges.)

APSMB-D is NP-complete

Proof (ctd.)

Since $S_i^* \cap S_j^* = \emptyset$ for $i, j \in W$, $i \neq j$, the set of winners W represents an independent set of cardinality

$$|W| = \sum_{i \in W} v_i^*.$$

Therefore, there is an independent set of cardinality at least k_{IS} iff there is a set of winners W with $\sum_{i \in W} v_i^* \geq k$.
This proves NP-hardness.

APSMB-D $\in$ NP: obvious (guess and verify set of winners).

APSMB-D is NP-complete

Consequences:

- Solving APSMB optimally: too costly.
- Alternatives:
  - approximation algorithm
  - heuristic approach
  - special cases
- Here: approximation algorithm

Definition (approximation factor)

Let $c \geq 1$. An allocation $\langle S_1, \ldots, S_n \rangle$ is a *c-approximation* of an optimal allocation if

$$\sum_{i=1}^{n} v_i(T_i) \leq c \cdot \sum_{i=1}^{n} v_i(S_i)$$

for an optimal allocation $\langle T_1, \ldots, T_n \rangle$.

Proposition

Approximating APSMB within a factor of $c \leq m^{1/2-\varepsilon}$ for any $\varepsilon > 0$ is NP-hard. □

Approximation Algorithms

Best we can still hope for in case of single-minded bidders:

- incentive compatible
- $m^{1/2}$-approximation algorithm
- with polynomial runtime.

Good news:

- Such an algorithm exists!

Mechanism for Single-Minded Bidders

Definition (mechanism for single-minded bidders)

Let V_{sm} be the set of all single-minded bids and A the set of all allocations.

A mechanism for single-minded bidders is a tuple $\langle f, p_1, \dots, p_n \rangle$ consisting of

- a social choice function $f : V_{sm}^n \to A$ and
- payment functions $p_i : V_{sm}^n \to \mathbb{R}$ for all $i = 1, \dots, n$.

Mechanism for Single-Minded Bidders

Definition (efficient computability)

A mechanism for single-minded bidders is efficiently computable if f and all p_i can be computed in polynomial time.

Definition (incentive compatibility)

A mechanism for single-minded bidders is incentive compatible if

$$v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i})$$

for all $i = 1, \dots, n$ and all $v_1, \dots, v_n, v_i' \in V_{sm}$, where $v_i(a) = v_i^*$ if i wins in a (gets the desired bundle), and $v_i(a) = 0$, otherwise.

Mechanism for Single-Minded Bidders

How to build such a mechanism?

- In principle: could use a VCG mechanism.
- Problem with VCG: incentive compatible, but not efficiently computable
  (need to compute social welfare, which is NP-hard)
- Alternative idea: VCG-like mechanism that approximates social welfare
- Problem with alternative: efficiently computable, but not incentive compatible
- Solution: forget VCG, use specific mechanism for single-minded bidders.

Greedy Mechanism for Single-Minded Bidders

Definition (greedy mechanism for single-minded bidders)

The greedy mechanism for single-minded bidders (GMSMB) is defined as follows.

Let the bidders $1, \ldots, n$ be ordered such that

$$\frac{v_1^*}{\sqrt{|S_1^*|}} \geq \frac{v_2^*}{\sqrt{|S_2^*|}} \geq \cdots \geq \frac{v_n^*}{\sqrt{|S_n^*|}}.$$

...

Greedy Mechanism for Single-Minded Bidders

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Definition (greedy mechanism for single-minded bidders, ctd.)

Let the set $W \subseteq \{1, \dots, n\}$ be procedurally defined by the following pseudocode:

$W \leftarrow \emptyset$
for $i = 1, \dots, n$ **do**
  if $S_i^* \cap \left(\bigcup_{j \in W} S_j^* \right) = \emptyset$ **then**
    $W \leftarrow W \cup \{i\}$
  end if
end for

...

Greedy Mechanism for Single-Minded Bidders

Definition (greedy mechanism for single-minded bidders, ctd.)

Result: allocation a where exactly the bidders in W win.

Payments:

- Case 1: If $i \in W$ and there is a smallest index j such that $S_i^* \cap S_j^* \neq \emptyset$ and for all $k < j$, $k \neq i$, $S_k^* \cap S_j^* = \emptyset$, then

$$p_i(v_1, \dots, v_n) = \frac{v_j^*}{\sqrt{|S_j^*|/|S_i^*|}},$$

- Case 2: Otherwise,

$$p_i(v_1, \dots, v_n) = 0.$$

Greedy Mechanism for Single-Minded Bidders

Example

Let $N = \{1,2,3,4\}$ and $G = \{1,\ldots,13\}$.

i	**Package** S_i^*	**Val.** v_i^*	$v_i^* / \sqrt{|S_i^*|}$	**Assignm. order**
1	$\{1,2,3,4,5,6,7,8,9\}$	15		
2	$\{3,4,5,6,7,8,9,12,13\}$	3		
3	$\{1,2,10,11\}$	12		
4	$\{10,11,12,13\}$	8		

Positions in assignment order? Winner set? Assignment? Social welfare of winner set?

Example (ctd.)

Assignments:

1. Bidder 3 gets $\{1, 2, 10, 11\}$.
2. Bidder 1 gets nothing (obj. 1 and 2 already assigned).
3. Bidder 4 gets nothing (obj. 10 and 11 already assigned).
4. Bidder 2 gets the remainder, i.e., $\{3, 4, 5, 6, 7, 8, 9, 12, 13\}$.

Payments:

1. Bidder 3 pays

$$\frac{v_1^*}{\sqrt{|S_1^*|/|S_3^*|}} = \frac{15}{\sqrt{9/4}} = \frac{15}{3/2} = 10.$$

2. Bidders 1, 4 and 2 pay 0.

Greedy Mechanism for Single-Minded Bidders

Example (ctd.)

Therefore:

- Winner set: $W = \{2,3\}$.
  Social welfare: $U = 12+3 = 15$.
- Optimal winner set: $W^* = \{1,4\}$.
  Optimal social welfare: $U^* = 15+8 = 23$.
- Approximation ratio: $23/15 < 2 < 3 < \sqrt{13} = \sqrt{m}$

Greedy Mechanism for Single-Minded Bidders: Efficient Computability

Theorem

GMSMB is efficiently computable. □

Open questions:

- What about incentive compatibility?
- What about approximation factor of $\sqrt{m}$?

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

To prove incentive compatibility:

- Step 1: Show that GMSMB is monotone.
- Step 2: Show that GMSMB uses critical payments.
- Step 3: Show that in GMSMB losers pay nothing.
- Step 4: Show that every mechanism for single-minded bidders that is monotone, that uses critical payments, and where losers pay nothing is incentive compatible.

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Definition (monotonicity)

A mechanism for single-minded bidders is monotone if a bidder who wins with bid $\langle S^*, v^* \rangle$ would also win with any bid $\langle S', v' \rangle$ where $S' \subseteq S^*$ and $v' \geq v^*$ (for fixed bids of the other bidders).

Definition (critical payments)

A mechanism for single-minded bidders uses critical payments if a bidder who wins pays the minimal amount necessary for winning, i.e., the infimum of all v' such that $\langle S^*, v' \rangle$ still wins.

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

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Lemma

GMSBM is monotone, uses critical payments, and losers pay nothing.

Proof

Monotonicity: Increasing v_i^* or decreasing S_i^* can only move bidder i up in the greedy order, making it easier to win.

Critical payments: Bidder i wins as long as he is before bidder j in the greedy order (if such a j exists). This holds iff

$$\frac{v_i^*}{\sqrt{|S_i^*|}} \geq \frac{v_j^*}{\sqrt{|S_j^*|}} \quad \text{iff} \quad v_i^* \geq \frac{v_j^* \sqrt{|S_i^*|}}{\sqrt{|S_j^*|}} = \frac{v_j^*}{\sqrt{|S_j^*|/|S_i^*|}} = p_i.$$

Losers pay nothing: Obvious. □

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Lemma

A mechanism for single-minded bidders that is monotone, that uses critical payments, and where losers pay nothing is incentive compatible.

Proof

(A) Truthful bids never lead to negative utility.

- If the declared bid loses, bidder has utility 0.
- If the declared bid wins, he has utility $v^* - p^* \geq 0$, since $v^* \geq p^*$, because p^* is the critical payment, and if the bid wins, the bidder must have (truthfully) bid a value v^* of at least p^*.

...

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Proof (ctd.)

(B) Truthful bids never lead to lower utility than untruthful bids.

Suppose declaration of untruthful bid $\langle S', v' \rangle$ deviating from truthful bid $\langle S^*, v^* \rangle$.

(B.1) Case 1: untruthful bid is losing or not useful for bidder.

Suppose $\langle S', v' \rangle$ is losing or $S^* \not\subseteq S'$ (bidder does not get the bundle he wants). Then utility ≤ 0 in $\langle S', v' \rangle$, i.e., no improvement over utility when declaring $\langle S^*, v^* \rangle$ (cf. (A)).

(B.2) Case 2: untruthful bid is winning and useful for bidder.

Assume $\langle S', v' \rangle$ is winning and $S^* \subseteq S'$. To show that $\langle S^*, v^* \rangle$ is at least as good a bid as $\langle S', v' \rangle$, show that $\langle S^*, v' \rangle$ is at least as good as $\langle S', v' \rangle$ and that $\langle S^*, v^* \rangle$ is at least as good as $\langle S^*, v' \rangle$.

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Proof (ctd.)

- (B.2.a) Lying about desired bundle does not help.
  Show that $\langle S^*, v' \rangle$ is at least as good as $\langle S', v' \rangle$.

  Let p' be the payment for bid $\langle S', v' \rangle$ and p the payment for bid $\langle S^*, v' \rangle$.

  For all $x < p$, $\langle S^*, x \rangle$ is losing, since p is the critical payment for S^*.

  Due to monotonicity, also $\langle S', x \rangle$ is losing for all $x < p$. Hence, the critical payment p' for S' is at least p.

  Thus, $\langle S^*, v' \rangle$ is still winning, if $\langle S', v' \rangle$ was, and leads to the same or even lower payment.

...

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Proof (ctd.)

- (B.2.b) Lying about valuation does not help.
  Show that $\langle S^*, v^* \rangle$ is at least as good as $\langle S^*, v' \rangle$.
  - (B.2.b.i) Case 1: $\langle S^*, v^* \rangle$ is winning with payment p^*.
    If $v' > p^*$, then $\langle S^*, v' \rangle$ is still winning with the same payment, so there is no incentive to deviate to $\langle S^*, v' \rangle$.

    If $v' \leq p^*$, then $\langle S^*, v' \rangle$ is losing, so there is also no incentive to deviate to $\langle S^*, v' \rangle$.
  - (B.2.b.ii) Case 2: $\langle S^*, v^* \rangle$ is losing.
    Then v^* is less than the critical payment, i.e., the payment p' for a winning bid $\langle S^*, v' \rangle$ would be greater than v^*, making a deviation to $\langle S^*, v' \rangle$ unprofitable. $\square$

Greedy Mechanism for Single-Minded Bidders: Incentive Compatibility

Corollary

The greedy mechanism for single-minded bidders is incentive compatible. □

Open question:

- What about approximation factor of $\sqrt{m}$?

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

In the next proof, we will need the Cauchy-Schwarz inequality:

Theorem (Cauchy-Schwarz inequality)

Let $x_j, y_j \in \mathbb{R}$. Then

$$\sum_j x_j y_j \leq \sqrt{\sum_j x_j^2} \cdot \sqrt{\sum_j y_j^2}.$$

□

Lemma

GMSBM produces a winner set W that induces a social welfare that is at most a factor $\sqrt{m}$ worse than the optimal social welfare.

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

Proof

- Let W^* be a set of winning bidders such that $\sum_{i \in W^*} v_i^*$ is maximal and $S_i^* \cap S_j^* = \emptyset$ for $i, j \in W^*$, $i \neq j$.
- Let W be the result of GMSMB.

Show:

$$\sum_{i \in W^*} v_i^* \leq \sqrt{m} \sum_{i \in W} v_i^*.$$

For $i \in W$ let

$$W_i^* = \{j \in W^* \mid j \geq i \text{ and } S_i^* \cap S_j^* \neq \emptyset\}$$

be the winners in W^* identical with i or not contained in W because of bidder i. ...

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

Proof (ctd.)

Since no $j \in W_i^*$ is before i in the greedy ordering, for such j,

$$v_j^* \leq \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|S_j^*|} \qquad \text{and, summing over } j \in W_i^*$$

$$\sum_{j \in W_i^*} v_j^* \leq \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in W_i^*} \sqrt{|S_j^*|}. \tag{1}$$

With Cauchy-Schwarz for $x_j = 1$ and $y_j = \sqrt{|S_j^*|}$:

$$\sum_{j \in W_i^*} \sqrt{|S_j^*|} \leq \sqrt{\sum_{j \in W_i^*} 1^2} \sqrt{\sum_{j \in W_i^*} |S_j^*|} = \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|}. \tag{2}$$

...

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

Proof (ctd.)

For all $j \in W_i^*$, $S_i^* \cap S_j^* \neq \emptyset$, i.e., there is a $g(j) \in S_i^* \cap S_j^*$.

Since W^* induces an allocation, for all $j_1, j_2 \in W_i^*$, $j_1 \neq j_2$,

$$S_{j_1}^* \cap S_{j_2}^* = \emptyset$$

Hence,

$$(S_i^* \cap S_{j_1}^*) \cap (S_i^* \cap S_{j_2}^*) = \emptyset$$

i.e., $g(j_1) \neq g(j_2)$ for $j_1, j_2 \in W_i^*$ with $j_1 \neq j_2$, making g an injective function from W_i^* to S_i^*.
Thus,

$$|W_i^*| \leq |S_i^*|. \tag{3}$$

...

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

Proof (ctd.)

Since W^* induces an allocation and $W_i^* \subseteq W^*$,

$$\sum_{j \in W_i^*} |S_j^*| \leq m. \tag{4}$$

...

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

Proof (ctd.)

Recall inequalities (1), (2), (3), and (4):

$$\sum_{j \in W_i^*} v_j^* \overset{(1)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in W_i^*} \sqrt{|S_j^*|}, \qquad |W_i^*| \overset{(3)}{\leq} |S_i^*|,$$

$$\sum_{j \in W_i^*} \sqrt{|S_j^*|} \overset{(2)}{\leq} \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|}, \qquad \sum_{j \in W_i^*} |S_j^*| \overset{(4)}{\leq} m.$$

With these, we get (5):

$$\sum_{j \in W_i^*} v_j^* \overset{(1)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sum_{j \in W_i^*} \sqrt{|S_j^*|} \overset{(2)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|W_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|}$$

$$\overset{(3)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|S_i^*|} \sqrt{\sum_{j \in W_i^*} |S_j^*|} \overset{(4)}{\leq} \frac{v_i^*}{\sqrt{|S_i^*|}} \sqrt{|S_i^*|} \sqrt{m} = \sqrt{m} v_i^*. \qquad \ldots$$

Proof (ctd.)

Recall that for $i \in W$,

$$W_i^* = \{j \in W^* \mid j \geq i \text{ and } S_i^* \cap S_j^* \neq \emptyset\}.$$

Let $j \in W^*$.

- If $j \in W$: then by definition, $j \in W_j^*$ (assuming, WLOG, $S_j^* \neq \emptyset$).
- If $j \notin W$: then there must be some $i \in W$ such that $j \geq i$ and $S_i^* \cap S_j^* \neq \emptyset$, i.e., $j \in W_i^*$.

Therefore, for each $j \in W^*$, there is an $i \in W$ such that $j \in W_i^*$:

$$W^* \subseteq \bigcup_{i \in W} W_i^*. \qquad \ldots \tag{6}$$

Greedy Mechanism for Single-Minded Bidders: Approximation Factor

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Proof (ctd.)

Recall (5) and (6):

$$\sum_{j \in W_i^*} v_j^* \overset{(5)}{\leq} \sqrt{m} v_i^*, \qquad W^* \overset{(6)}{\subseteq} \bigcup_{i \in W} W_i^*.$$

With these, we finally obtain the desired estimation

$$\sum_{i \in W^*} v_i^* \overset{(6)}{\leq} \sum_{i \in W} \sum_{j \in W_i^*} v_j^* \overset{(5)}{\leq} \sum_{i \in W} \sqrt{m} v_i^* = \sqrt{m} \sum_{i \in W} v_i^*.$$

Thus, the social welfare of W differs from the optimal social welfare by a factor of at most $\sqrt{m}$. □

Greedy Mechanism for Single-Minded Bidders

The following theorem summarizes the results in this chapter:

Theorem

The greedy mechanism for single-minded bidders is efficiently computable, incentive compatible, and leads to an allocation whose social welfare is a $\sqrt{m}$-approximation of the optimal social welfare. □

Summary

- In combinatorial auctions, bidders bid for bundles of items.
- Exponential space needed just to represent and communicate valuations.
- Therefore: Focus on special case of single-minded bidders (compact representation of valuations).
- Unfortunately, still, optimal allocation NP-hard.
- Solution: approximate optimal allocation.
- Polynomial-time approximation possible for approximation factor no better than $\sqrt{m}$.
- Greedy mechanism for single-minded bidders:
  - achieves $\sqrt{m}$-approximation of social welfare,
  - is efficiently computable, and
  - is incentive compatible.