Introduction to Game Theory

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Exercise Sheet 3 — Solutions

Exercise 3.1 (Support lemma, 3 Punkte)

Let α be a mixed strategy profile, $a_i \in \mathit{supp}(\alpha_i)$, $a_i \notin B_i(\alpha_{-i})$, $a'_i \in B_i(\alpha_{-i})$ and α'_i defined by $\alpha'_i(a_i) = 0$, $\alpha'_i(a'_i) = \alpha_i(a'_i) + \alpha_i(a_i)$ and $\alpha'_i(a''_i) = \alpha_i(a''_i)$ for all $a''_i \in A_i \setminus \{a_i, a'_i\}$. Show formally that $U_i(\alpha'_i, \alpha_{-i}) > U_i(\alpha_i, \alpha_{-i})$. Use the definition of the expected reward.

Solution:

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$$\begin{aligned}
U_i(\alpha'_i, \alpha_{-i}) &= \sum_{\tilde{a}_i \in A_i} \alpha'_i(\tilde{a}_i) \cdot U_i(\tilde{a}_i, \alpha_{-i}) \\
&= \alpha'_i(a'_i) \cdot U_i(a'_i, \alpha_{-i}) + \alpha'_i(a_i) \cdot U_i(a_i, \alpha_{-i}) + \sum_{\substack{\tilde{a}_i \in A_i \\ \tilde{a}_i \notin \{a_i, a'_i\}}} \alpha'_i(\tilde{a}_i) \cdot U_i(\tilde{a}_i, \alpha_{-i}) \\
&= (\alpha_i(a'_i) + \alpha_i(a_i)) \cdot U_i(a'_i, \alpha_{-i}) + 0 \cdot U_i(a_i, \alpha_{-i}) + \sum_{\substack{\tilde{a}_i \in A_i \\ \tilde{a}_i \notin \{a_i, a'_i\}}} \alpha_i(\tilde{a}_i) \cdot U_i(\tilde{a}_i, \alpha_{-i}) \\
&= \alpha_i(a'_i) \cdot U_i(a'_i, \alpha_{-i}) + \alpha_i(a_i) \cdot U_i(a'_i, \alpha_{-i}) + \sum_{\substack{\tilde{a}_i \in A_i \\ \tilde{a}_i \notin \{a_i, a'_i\}}} \alpha_i(\tilde{a}_i) \cdot U_i(\tilde{a}_i, \alpha_{-i}) \\
&> \alpha_i(a'_i) \cdot U_i(a'_i, \alpha_{-i}) + \alpha_i(a_i) \cdot U_i(a_i, \alpha_{-i}) + \sum_{\substack{\tilde{a}_i \in A_i \\ \tilde{a}_i \notin \{a_i, a'_i\}}} \alpha_i(\tilde{a}_i) \cdot U_i(\tilde{a}_i, \alpha_{-i}) \\
&= \sum_{\tilde{a}_i \in A_i} \alpha_i(\tilde{a}_i) \cdot U_i(\tilde{a}_i, \alpha_{-i}) \\
&= U_i(\alpha_i, \alpha_{-i}).
\end{aligned}$$

Inequality holds because $U_i(a'_i, \alpha_{-i}) > U_i(a_i, \alpha_{-i})$ which follows directly from $a_i \notin B_i(\alpha_{-i})$, $a'_i \in B_i(\alpha_{-i})$, as specified.

Exercise 3.2 (Mixed strategy Nash equilibria, 2 points)

Consider the following strategic game:

		Player 2: X	Player 2: Y
Player 1	A	$1, 1$	$3, 2$
	B	$2, 2$	$1, 1$

Determine and write down all mixed strategy Nash equilibria.

Solution:

There are two pure strategy NE: (A, Y) and (B, X). Thus, the only possible MSNE[1] α exists for the support sets $supp(\alpha_1) = \{A, B\}$ and $supp(\alpha_2) = \{X, Y\}$. According to the support lemma, the following must hold:

$$\begin{aligned}
U_1(A, \alpha_2) &= U_1(B, \alpha_2) \\
\Rightarrow 1 \cdot \alpha_2(X) + 3 \cdot \alpha_2(Y) &= 2 \cdot \alpha_2(X) + 1 \cdot \alpha_2(Y) \\
\Rightarrow -\alpha_2(X) + 2 \cdot \alpha_2(Y) &= 0 \\
\Rightarrow -\alpha_2(X) + 2 \cdot (1 - \alpha_2(X)) &= 0 \\
\Rightarrow -3 \cdot \alpha_2(X) &= -2 \\
\Rightarrow \alpha_2(X) &= \frac{2}{3} \quad \text{and } \alpha_2(Y) = \frac{1}{3}
\end{aligned}$$

Similarly, we get $\alpha_1(A) = \alpha_1(B) = \frac{1}{2}$. Thus, the MSNE is: $(\alpha_1, \alpha_2) = ((\frac{1}{2}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{3}))$

Exercise 3.3 (Linear Complementarity Problem, 2 + 1 points)

Consider the strategic game given by the following payoff matrix:

		Player 2		
		x	y	z
Player 1	a	$0, 0$	$3, 1$	$3, 3$
	b	$1, 1$	$0, 0$	$1, 3$
	c	$1, 1$	$1, 1$	$0, 0$

(a) For the following pair of support sets formulate the corresponding linear program: $(supp(\alpha), supp(\beta)) = (\{a, b, c\}, \{x, y, z\})$.

Solution:

For simplicity we denote $\alpha(a)$ with α_a, $\alpha(b)$ with α_b, and so on.

$$\begin{aligned}
u - (0\beta_x + 3\beta_y + 3\beta_z) &\geq 0 & \qquad v - (0\alpha_a + 1\alpha_b + 1\alpha_c) &\geq 0 \\
u - (1\beta_x + 0\beta_y + 1\beta_z) &\geq 0 & v - (1\alpha_a + 0\alpha_b + 1\alpha_c) &\geq 0 \\
u - (1\beta_x + 1\beta_y + 0\beta_z) &\geq 0 & v - (3\alpha_a + 3\alpha_b + 0\alpha_c) &\geq 0
\end{aligned}$$

$$\begin{aligned}
\beta_x \cdot (v - (0\alpha_a + 1\alpha_b + 1\alpha_c)) &= 0 & \qquad \alpha_a \cdot (u - (0\beta_x + 3\beta_y + 3\beta_z)) &= 0 \\
\beta_y \cdot (v - (1\alpha_a + 0\alpha_b + 1\alpha_c)) &= 0 & \alpha_b \cdot (u - (1\beta_x + 0\beta_y + 1\beta_z)) &= 0 \\
\beta_z \cdot (v - (3\alpha_a + 3\alpha_b + 0\alpha_c)) &= 0 & \alpha_c \cdot (u - (1\beta_x + 1\beta_y + 0\beta_z)) &= 0
\end{aligned}$$

$$\begin{aligned}
\beta_x &\geq 0 & \qquad \alpha_a &\geq 0 \\
\beta_y &\geq 0 & \alpha_b &\geq 0 \\
\beta_z &\geq 0 & \alpha_c &\geq 0 \\
\beta_x + \beta_y + \beta_z &= 1 & \alpha_a + \alpha_b + \alpha_c &= 1
\end{aligned}$$

Solution:

[1] Mixed strategy Nash equilibrium.

The corresponding linear program:

$$\begin{aligned} u - (0\beta_x + 3\beta_y + 3\beta_z) &= 0 & v - (0\alpha_a + 1\alpha_b + 1\alpha_c) &= 0 \\ u - (1\beta_x + 0\beta_y + 1\beta_z) &= 0 & v - (1\alpha_a + 0\alpha_b + 1\alpha_c) &= 0 \\ u - (1\beta_x + 1\beta_y + 0\beta_z) &= 0 & v - (3\alpha_a + 3\alpha_b + 0\alpha_c) &= 0 \\ \beta_x + \beta_y + \beta_z &= 1 & \alpha_a + \alpha_b + \alpha_c &= 1 \end{aligned}$$

$$\begin{aligned} \beta_x &\geq 0 & \alpha_a &\geq 0 \\ \beta_y &\geq 0 & \alpha_b &\geq 0 \\ \beta_z &\geq 0 & \alpha_c &\geq 0 \end{aligned}$$

(b) Solve the linear program and provide values for each $\alpha(a_1)$ and $\beta(a_2)$, $a_1 \in \{a, b, c\}, a_2 \in \{x, y, z\}$. What is the expected payoff (u, v) of the NE computed above?

Solution:

The solution obtained by solving the linear program above is the MSNE (α, β), with:

$$\begin{aligned} \alpha(a) &= 1/7 & \beta(x) &= 5/7 \\ \alpha(b) &= 1/7 & \beta(y) &= 1/7 \\ \alpha(c) &= 5/7 & \beta(z) &= 1/7 \end{aligned}$$

Expected payoffs (u, v) are $u = v = 6/7 \approx 0.8571$