Foundations of Artificial Intelligence

Prof. Dr. J. Boedecker, Prof. Dr. W. Burgard, Prof. Dr. F. Hutter, Prof. Dr. B. Nebel,
Dr. rer. nat. M. Tangermann
T. Schulte, M. Krawez, R. Rajan, S. Adriaensen, K. Sirohi
Summer Term 2020

University of Freiburg
Department of Computer Science

Exercise Sheet 6 — Solutions

Exercise 6.1 (Semantics of Predicate Logic)

Consider the Interpretation $\mathcal{I} =< \mathcal{D}, \cdot^{\mathcal{I}} >$ with

- $\mathcal{D} = \{0, 1, 2, 3\}$
- $even^{\mathcal{I}} = \{0, 2\}$
- $odd^{\mathcal{I}} = \{1, 3\}$
- $lessThan^{\mathcal{I}} = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$
- $two^{\mathcal{I}} = 2$
- $plus^{\mathcal{I}} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}, plus^{\mathcal{I}}(a, b) = (a + b) \mod 4$

and the variable assignment $\alpha = \{(x, 0), (y, 1)\}$.
Decide for the following formulae θ_i if $\mathcal{I}$ is a model for θ_i under α, i.e. if $\mathcal{I}, \alpha \models \theta_i$.
Explain your answer by formally applying the semantics.

(a) $\theta_1 = odd(y) \wedge even(two)$

(b) $\theta_2 = \forall x\ (even(x) \vee odd(x))$

(c) $\theta_3 = \forall x \exists y\ lessThan(x, y)$

(d) $\theta_4 = \forall x\ (even(x) \Rightarrow \exists y\ lessThan(x, y))$

(e) $\theta_5 = \forall x\ (odd(x) \Rightarrow even(plus(x, y)))$

Solution:

(a) $\theta_1^{\mathcal{I}} =$
$(odd(y) \wedge even(two))^{\mathcal{I}} =$
$odd^{\mathcal{I}}(y^{\mathcal{I}}) \wedge even^{\mathcal{I}}(two^{\mathcal{I}}) =$
$odd^{\mathcal{I}}(1) \wedge even^{\mathcal{I}}(2) =$
$\top \wedge \top = \top$, since $1 \in odd^{\mathcal{I}}$ and $2 \in even^{\mathcal{I}}$.
Thus, $\mathcal{I}, \alpha \models \theta_1$.

(b) Let $\theta_2 = \forall x \phi_2$ with $\phi_2 = even(x) \vee odd(x)$.
Then $\mathcal{I}, \alpha \models \theta_2$ iff $\mathcal{I}, \alpha[x/d](\phi_2) = \top$ for all $d \in \mathcal{D}$.
Notice that here x is bound by a quantifier and therefore the variable assignment in the definition of α is overridden by $\alpha[x/d]$.

Since our universe $\mathcal{D}$ is finite (and small), we can evaluate the above for all $d \in \mathcal{D}$:

$\mathcal{I}, \alpha[x/0](\phi_2) = even^{\mathcal{I}}(0) \vee odd^{\mathcal{I}}(0) = \top \vee \bot = \top$
$\mathcal{I}, \alpha[x/1](\phi_2) = even^{\mathcal{I}}(1) \vee odd^{\mathcal{I}}(1) = \bot \vee \top = \top$
$\mathcal{I}, \alpha[x/2](\phi_2) = even^{\mathcal{I}}(2) \vee odd^{\mathcal{I}}(2) = \top \vee \bot = \top$
$\mathcal{I}, \alpha[x/3](\phi_2) = even^{\mathcal{I}}(3) \vee odd^{\mathcal{I}}(3) = \bot \vee \top = \top$

Thus, $\mathcal{I}, \alpha \models \theta_2$.

(c) Let $\theta_3 = \forall x \exists y \phi_3$ with $\phi_3 = lessThan(x, y)$.
Then $\mathcal{I}, \alpha \models \theta_3$ iff for all $d_1 \in \mathcal{D}$ there exists a $d_2 \in \mathcal{D}$ so that $\mathcal{I}, \alpha[x/d_1, y/d_2](\phi_3) = \top$.
In this case it is clear that for $d_1 = 3$ no such d_2 exists, more formally, $lessThan^{\mathcal{I}}(3, d_2) = \bot$ for all $d_2 \in \mathcal{D}$ and therefore $I, \alpha \not\models \theta_3$.

(d) Let $\theta_4 = \forall x \phi_4$ with $\phi_4 = even(x) \Rightarrow \exists y \; lessThan(x, y)$.
Then $\mathcal{I}, \alpha \models \theta_4$ iff $\mathcal{I}, \alpha[x/d_1](\phi_4) = \top$ for all $d_1 \in \mathcal{D}$
We again consider all cases, starting odd numbers, i.e., $d_1 \in \{1, 3\}$: $(even(x) \Rightarrow \exists y \; lessThan(x, y))^{\mathcal{I}} = \neg even^{\mathcal{I}}(d_1) \vee ... =$
$\neg \bot \vee ... = \top \vee ... = \top$.
For $d_1 \in \{0, 2\}$ we have $\neg even^{\mathcal{I}}(d_1) = \bot$, thus we have to find a corresponding $d_2 \in \mathcal{D}$ with $\mathcal{I}, \alpha[x/d_1, y/d_2](lessThan(x, y)) = \top$. Here, we have $(0, 1) \in lessThan^{\mathcal{I}}$ and $(2, 3) \in lessThan^{\mathcal{I}}$.
Summing up all of the above, we showed that $\mathcal{I}, \alpha \models \theta_4$.

(e) Let $\theta_5 = \forall x \phi_5$ with $\phi_5 = odd(x) \Rightarrow even(plus(x, y))$.
Then $\mathcal{I}, \alpha \models \theta_5$ iff $\mathcal{I}, \alpha[x/d](\phi_5) = \top$ for all $d \in \mathcal{D}$
$\mathcal{I}, \alpha[x/d](\phi_5) = \neg odd^{\mathcal{I}}(d) \vee even^{\mathcal{I}}(plus^{\mathcal{I}}(d, 1))$
Notice that here y is a free variable and therefore the assignment $[y/1]$ is applied.
Now we consider all cases for d:
$\mathcal{I}, \alpha[x/0](\phi_5) = \neg odd^{\mathcal{I}}(0) \vee even^{\mathcal{I}}(plus^{\mathcal{I}}(0, 1)) = \neg odd^{\mathcal{I}}(0) \vee even^{\mathcal{I}}(1) = \top \vee \bot = \top$
$\mathcal{I}, \alpha[x/1](\phi_5) = \neg odd^{\mathcal{I}}(1) \vee even^{\mathcal{I}}(plus^{\mathcal{I}}(1, 1)) = \neg odd^{\mathcal{I}}(1) \vee even^{\mathcal{I}}(2) = \bot \vee \top = \top$
$\mathcal{I}, \alpha[x/2](\phi_5) = \neg odd^{\mathcal{I}}(2) \vee even^{\mathcal{I}}(plus^{\mathcal{I}}(2, 1)) = \neg odd^{\mathcal{I}}(2) \vee even^{\mathcal{I}}(3) = \top \vee \bot = \top$
$\mathcal{I}, \alpha[x/3](\phi_5) = \neg odd^{\mathcal{I}}(3) \vee even^{\mathcal{I}}(plus^{\mathcal{I}}(3, 1)) = \neg odd^{\mathcal{I}}(3) \vee even^{\mathcal{I}}(0) = \bot \vee \top = \top$
Thus, $\mathcal{I}, \alpha \models \theta_5$.

Exercise 6.2 (Semantics of Predicate Logic)
Consider the Interpretation $\mathcal{I} =< \mathcal{D}, \cdot^{\mathcal{I}} >$ and the variable assignment α:

- $\mathcal{D} = \{a, b, c\}$
- $P^{\mathcal{I}} = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, a)\}$
- $Q^{\mathcal{I}} = \{a, b\}$
- $R^{\mathcal{I}} = \{(a, a), (a, b), (a, c), (b, c), (c, b)\}$

- $\alpha = \{(v, a), (w, b)\}$

Decide for the following formulae θ_i if $\mathcal{I}$ is a model for θ_i under α, i.e. if $\mathcal{I}, \alpha \models \theta_i$. Explain your answer by formally applying the semantics.

(a) $\theta_1 = \forall x\big(P(x, w) \Rightarrow Q(x)\big)$

(b) $\theta_2 = \exists x\big(R(v, x) \Rightarrow P(x, x)\big)$

(c) $\theta_3 = \forall x \forall y\big(R(x, y) \iff Q(y)\big)$

(d) $\theta_4 = \Big[\neg \forall x \forall y\big(Q(y) \vee P(x, y)\big)\Big] \wedge \Big[\exists z\big(Q(z) \vee P(w, z)\big)\Big]$

Solution:

(a) Let $\theta_1 = \forall x \phi_1$ with $\phi_1 = \big(P(x, w) \Rightarrow Q(x)\big)$.
Then $\mathcal{I}, \alpha \models \theta_1$ iff $\mathcal{I}, \alpha[x/d](\phi_1) = \top$ for all $d \in \mathcal{D}$.
Notice that here x is bound by a quantifier and therefore the variable assignment in the definition of α is overridden by $\alpha[x/d]$.
Also w is a free variable and therefore the assignment $[w/b]$ is applied.
Since our universe $\mathcal{D}$ is finite (and small), we can evaluate the above for all $d \in \mathcal{D}$:
$\mathcal{I}, \alpha[x/a](\phi_1) = \neg P^{\mathcal{I}}(a, b) \vee Q^{\mathcal{I}}(a) = \bot \vee \top = \top$
$\mathcal{I}, \alpha[x/b](\phi_1) = \neg P^{\mathcal{I}}(b, b) \vee Q^{\mathcal{I}}(b) = \bot \vee \top = \top$
$\mathcal{I}, \alpha[x/c](\phi_1) = \neg P^{\mathcal{I}}(c, b) \vee Q^{\mathcal{I}}(c) = \top \vee \bot = \top$
Thus, $\mathcal{I}, \alpha \models \theta_1$.

(b) Let $\theta_2 = \exists x \phi_2$ with $\phi_2 = \big(R(v, x) \Rightarrow P(x, x)\big)$.
Then $\mathcal{I}, \alpha \models \theta_3$ if there exists a $d \in \mathcal{D}$
so that $\mathcal{I}, \alpha[x/d](\phi_2) = \top$.
Also v is a free variable and therefore the assignment $[w/a]$ is applied.
One case which can be applied is $d = b$.
$\mathcal{I}, \alpha[x/b](\phi_2) = \neg R^{\mathcal{I}}(a, b) \vee P^{\mathcal{I}}(b, b) = \bot \vee \top = \top$
Thus $\mathcal{I}, \alpha \models \theta_2$.

(c) Let $\theta_3 = \forall x \forall y \phi_3$ with $\phi_3 = \big(R(x, y) \iff Q(y)\big)$.
Then $\mathcal{I}, \alpha \models \theta_3$ iff $\mathcal{I}, \alpha[x/d1][y/d2](\phi_1) = \top$ for all $d1$ and $d2 \in \mathcal{D}$.
However, it can does not satisfy by assigning x=a and y = c,
$\mathcal{I}, \alpha[x/a][y/c](\phi_3) = \big(R^{\mathcal{I}}(a, c) \iff Q^{\mathcal{I}}(c)\big) = \bot \iff \top = \bot$
Hence $\mathcal{I}, \alpha \not\models \theta_3$.

(d) $\mathcal{I}, \alpha \models \theta_4$
Let $\theta_4 = \neg \theta_4^a \wedge \theta_4^b$ with
$\theta_4^a = \forall x \forall y\big(Q(y) \vee P(x, y)\big)$ and
$\theta_4^b = \exists z\big(Q(z) \vee P(w, z)\big)$
Thus, $\mathcal{I}, \alpha$ must model θ_4^b but not θ_4^a.
$\mathcal{I}, \alpha \not\models \theta_4^a$, since for $y = c$, $c \notin Q^{\mathcal{I}}$ and $(c, c) \notin P^{\mathcal{I}}$
$\mathcal{I}, \alpha \models \theta_4^b$, since, for instance, $a \in Q^{\mathcal{I}}$.