Introduction to Game Theory

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Exercise Sheet 8 — Solutions

Exercise 8.1 (Voting procedures, 2 points)

In the following, we assume that ties are broken in favor of the candidate with the lower index. Consider the following preference relations:

20 voters have the preference $b \prec_i c \prec_i e \prec_i d \prec_i a$
10 voters have the preference $d \prec_i e \prec_i c \prec_i b \prec_i a$
15 voters have the preference $b \prec_i d \prec_i a \prec_i e \prec_i c$
12 voters have the preference $a \prec_i b \prec_i c \prec_i e \prec_i d$
13 voters have the preference $a \prec_i e \prec_i c \prec_i d \prec_i b$

(a) Determine the winner according to the Borda count method.

Solution:

The winner is candidate d with 162 counts (a: 150, b: 94, c: 150, d: 162, e: 144).

(b) Determine the winner according to the instant-runoff voting method.

Solution:

The winner is candidate c. In round one, candidate d is eliminated with 12 votes. Then candidate e is eliminated (again 12 votes), followed by candidate b (13 votes) and finally candidate a (with 30 vs 40 votes).

(c) Determine the set of possible winners according to the Schulze-method[1].

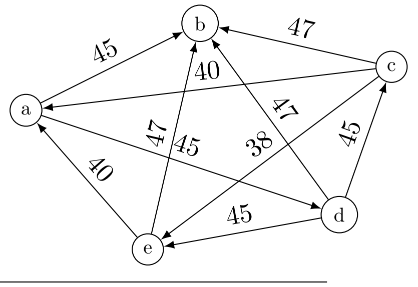
Solution:

In the first step we determine whether a Condorcet winner exists. To check for a Condorcet winner we build the pairwise comparison table.

$d(\cdot,\cdot)$	a	b	c	d	e
a	-	45	30	45	30
b	25	-	23	23	23
c	40	47	-	25	38
d	25	47	45	-	45
e	40	47	32	25	-

Because no candidate is dominating all other candidates (i.e. having the pairwise majority over all other candidates) there exists no Condorcet winner.

Results of the pairwise comparison between the candidates can be represented as a graph over the set of candidates, where between each two candidates X and Y exactly one edge directed from the winner to the looser exists. Each edge is depicted with the number of voters that prefer the winner over the looser.



$p(\cdot,\cdot)$	a	b	c	d	e
a	-	45	45	45	45
b	0	-	0	0	0
c	40	47	-	40	40
d	40	47	45	-	45
e	40	47	40	40	-

[1] `http://en.wikipedia.org/wiki/Schulze_method`

We determine the values for $p(\cdot,\cdot)$ by considering strongest paths in the graph.

To get the potential winner, we have to find a candidate X for which $p(X,Y) \geq p(Y,X)$ for all other candidates $Y \neq X$. This is only the case for candidate a which therefore is the Schulze winner.

Exercise 8.2 (Properties of voting procedures, 3 points)
Consider the following properties of voting procedures:

(a) **Majority criterion:**

If for more than half of the voters $i, b \prec_i a$ for all $b \in A \setminus \{a\}$, then $f(\prec_1, \dots, \prec_n) = a$.

(b) **Reversal symmetry:**

If $f(\prec_1, \dots, \prec_n) = a$ and $a \prec'_i b$ iff $b \prec_i a$ for all $i = 1, \dots, n$ and $a, b \in A$, then $f(\prec'_1, \dots, \prec'_n) \neq a$.

(c) **Incentive compatibility:**

$f(\prec_1, \dots, \prec'_i, \dots, \prec_n) \preceq_i f(\prec_1, \dots, \prec_i, \dots, \prec_n)$ for all $\prec_1, \dots, \prec_n, \prec'_i \in L$.

For each of the above properties, show that the Borda count method satisfies the property or give a counterexample. For simplicity, we assume that ties are broken in favor of the candidate with the lower index. Moreover, $|A| \geq 3$.

Solution:

In the following list of counterexamples, we abbreviate expressions of the form $c \prec_i b \prec_i a$ as abc for simplification.

(a) **Majority criterion:** Assume there are four candidates a, b, c and d and three voters with preference relations $abcd$, $abcd$ and $bcda$. Candidate a has the most top preferences but still does not win the vote as candidate b has a higher Borda count of 7 (compared to the 6 of a). Consequently, the majority criterion does not hold.

(b) **Reversal symmetry:** Assume there are three candidates a, b and c and two voters with the preference relations abc, cba. Here, all candidates have the same border count of 2. Due to tiebreaking, candidate a wins. If the preference relations are changed so that the positions of candidates a and c are reversed, the Borda count is still 2 for each candidate and, again, candidate a wins. Thus, reversal symmetry does not hold.

(c) **Incentive compatibility:** Assume there are three candidates a, b and c, as well as three voters with preferences abc, bca and cab. Due to tiebreaking, candidate a wins. If voter 2 changes his preferences from bca to cba, then candidate c wins instead of a. As voter 2 prefers candidate c over a, incentive compatibility does not hold.

Exercise 8.3 (Social welfare functions: unanimity, 2 + 1 points)
A social welfare function $F : L^n \to L$ satisfies

- **total unanimity** if for all $\prec \in L, F(\prec, \dots, \prec) = \prec$.
- **partial unanimity** if for all $\prec_1, \prec_2, \dots, \prec_n \in L, a, b \in A$,

$$a \prec_i b \text{ for all } i = 1, \dots, n \implies a \prec b, \text{ where } \prec := F(\prec_1, \dots, \prec_n).$$

(a) Proof that *partial unanimity* implies *total unanimity*.

Solution:

Let F be a social welfare function which satisfies partial unanimity. Then, for all $\prec_1, \prec_2, \dots, \prec_n \in L, a, b \in A$, it holds that

$$a \prec_i b \text{ for all } i = 1, \dots, n \implies a \prec b, \text{ where } \prec := F(\prec_1, \dots, \prec_n).$$

Let $\prec' = \prec_1 = \prec_2 = \dots = \prec_n$, then for all $a, b \in A$, it holds that

$$a \prec' b \implies a \prec b, \text{ where } \prec := F(\prec', \dots, \prec').$$

Since $a \prec' b \implies a \prec b$ for all $a, b \in A$, it holds that $a \prec b \implies a \prec' b$ for all $a, b \in A$. Thus $\prec = \prec'$. Consequently it holds that for all $\prec \in L, F(\prec, \dots, \prec) = \prec$.

(b) Proof by counter-example that *total unanimity* does not imply *partial unanimity*.
Hint: specify a social welfare function F that satisfies *total unanimity* but does not satisfy *partial unanimity*.

Solution:

Let $\{a_1, \dots a_k\} = A,\ \prec_1, \dots, \prec_n \in L$

$$F(\prec_1, \dots, \prec_n) = \begin{cases} \prec_1, & \text{if } \prec_1 = \prec_2 = \dots = \prec_n \\ a_1 \prec a_2 \prec \dots \prec a_k, & \text{otherwise} \end{cases}$$