Game Theory

5. Extensive Games

5.1. Extensive Games with Perfect Information

5.1.1. Motivation and Definitions

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Summer semester 2020

Motivation

- So far: All players move simultaneously, and then the outcome is determined.
- Often in practice: Several moves in sequence (e. g. in chess).
  $\rightsquigarrow$ cannot be directly reflected by strategic games.
- Extensive games (with perfect information) reflect such situations by modeling games as game trees.
- Idea: Players have several decision points where they can decide how to play.
- Strategies: Mappings from decision points in the game tree to actions to be played.

Outline

- Section 5.1: extensive games with perfect information (players know the state of the game and the actions chosen by other players, e. g., chess)
- Section 5.2: extensive games with imperfect information (players may not know the state of the game or the actions chosen by other players, e. g., poker)

Extensive Games

Definition (Extensive game with perfect information)

An extensive game with perfect information is a tuple $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ that consists of:

- a finite non-empty set N of players,
- a set H of (finite or infinite) sequences, called histories, such that
  - it contains the empty sequence $\langle\rangle \in H$,
  - H is closed under prefixes: if $\langle a^1, \dots, a^k \rangle \in H$ for some $k \in \mathbb{N} \cup \{\infty\}$, and $l < k$, then also $\langle a^1, \dots, a^l \rangle \in H$, and
  - H is closed under limits: if for some infinite sequence $\langle a^i \rangle_{i=1}^{\infty}$, we have $\langle a^i \rangle_{i=1}^{k} \in H$ for all $k \in \mathbb{N}$, then $\langle a^i \rangle_{i=1}^{\infty} \in H$.

  All infinite histories and all histories $\langle a^i \rangle_{i=1}^{k} \in H$, for which there is no a^{k+1} such that $\langle a^i \rangle_{i=1}^{k+1} \in H$ are called terminal histories Z. Components of a history are called actions.

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Definition (Extensive game with perfect information, ctd.)

- a player function $P : H \setminus Z \to N$ that determines which player's turn it is to move after a given nonterminal history, and
- for each player $i \in N$, a utility function (or payoff function) $u_i : Z \to \mathbb{R}$ defined on the set of terminal histories.

The game is called finite, if H is finite. It has a finite horizon, if the lenght of histories is bounded from above.

Assumption: All ingredients of Γ are common knowledge amongst the players of the game.

Terminology: In the rest of Section 5.1, we will write extensive games instead of extensive games with perfect information.

Example (Division game)

- Two identical objects should be divided among two players.
- Player 1 proposes an allocation.
- Player 2 agrees or rejects.
  - on agreement: allocation as proposed
  - on rejection: nobody gets anything

Extensive Games

Example (Division game)

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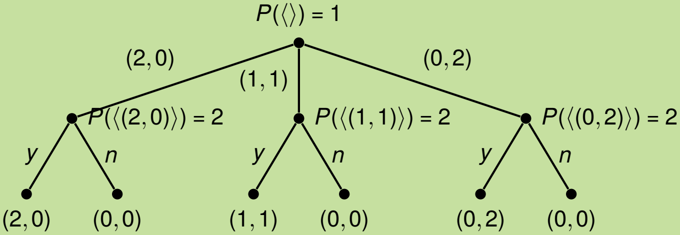
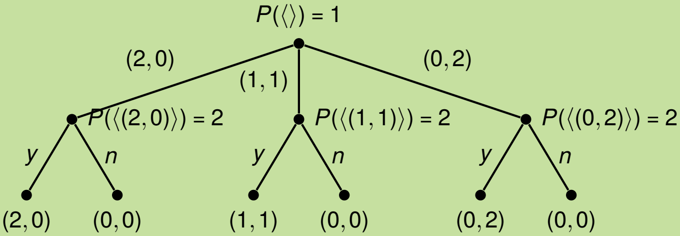


Extensive Games

Example (Division game, formally)



- $N = \{1,2\}$
- $H = \{\langle\rangle, \langle(2,0)\rangle, \langle(1,1)\rangle, \langle(0,2)\rangle, \langle(2,0),y\rangle, \langle(2,0),n\rangle, \dots\}$
- $P(\langle\rangle) = 1$, $P(h) = 2$ for all $h \in H \setminus Z$ with $h \neq \langle\rangle$
- $u_1(\langle(2,0),y\rangle) = 2$, $u_2(\langle(2,0),y\rangle) = 0$, etc.

Notation:

Let $h = \langle a^1, \dots, a^k \rangle$ be a history, and a an action.

- Then (h, a) is the history $\langle a^1, \dots, a^k, a \rangle$.
- If $h' = \langle b^1, \dots, b^\ell \rangle$, then (h, h') is the history $\langle a^1, \dots, a^k, b^1, \dots, b^\ell \rangle$.
- The set of actions from which player $P(h)$ can choose after a history $h \in H \setminus Z$ is written as

$$A(h) = \{a \mid (h, a) \in H\}.$$

Question: What is $A(\langle (2, 0) \rangle)$ in the division game?

Strategies

Definition (Strategy in an extensive game)

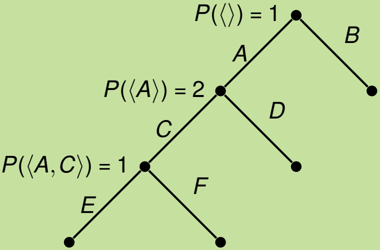
A strategy of a player i in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a function s_i that assigns to each nonterminal history $h \in H \setminus Z$ with $P(h) = i$ an action $a \in A(h)$. The set of strategies of player i is denoted as S_i.

Remark: Strategies require us to assign actions to histories h, even if it is clear that they will never be played (e. g., because h will never be reached because of some earlier action).

Notation (for finite games): A strategy for a player is written as a string of actions at decision nodes as visited in a breadth-first order.

Strategies

Example (Strategies in an extensive game)



- Strategies for player 1: AE, AF, BE and BF
- Strategies for player 2: C and D.

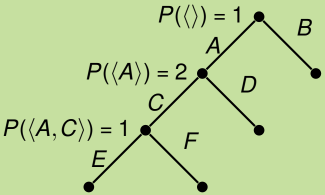
Outcome

Definition (Outcome)

The outcome $O(s)$ of a strategy profile $s = (s_i)_{i \in N}$ is the (possibly infinite) terminal history $h = \langle a^i \rangle_{i=1}^k$, with $k \in \mathbb{N} \cup \{\infty\}$, such that for all $\ell \in \mathbb{N}$ with $0 \leq \ell < k$,

$$s_{P(\langle a^1, \ldots, a^\ell \rangle)}(\langle a^1, \ldots, a^\ell \rangle) = a^{\ell+1}.$$

Example (Outcome)



$$O(AF, C) = \langle A, C, F \rangle$$
$$O(AE, D) = \langle A, D \rangle.$$

Summary

- Extensive games allow several moves in sequence (cf. game trees).
- Formalized using players, histories, player function, payoff functions for terminal histories.
- Strategies: mappings from decision points to actions
- Outcome: terminal history resulting from strategy profile

Game Theory

5. Extensive Games

5.1. Extensive Games with Perfect Information

5.1.2. Solution Concepts

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Motivation

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- So far: definition of extensive games
- Now: solution concepts for extensive games
  - transfer the idea of Nash equilibria
  - identify problems with Nash equilibria
    $\rightsquigarrow$ subgame-perfect equilibria

Nash Equilibria

Definition (Nash equilibrium in an extensive game)

A Nash equilibrium in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a strategy profile s^* such that for every player $i \in N$ and for all strategies $s_i \in S_i$,

$$u_i(O(s^*_{-i}, s^*_i)) \geq u_i(O(s^*_{-i}, s_i)).$$

Induced Strategic Game

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Definition (Induced strategic game)

The strategic game G induced by an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is defined by $G = \langle N, (A'_i)_{i \in N}, (u'_i)_{i \in N} \rangle$, where

- $A'_i = S_i$ for all $i \in N$, and
- $u'_i(a) = u_i(O(a))$ for all $i \in N$.

Proposition

The Nash equilibria of an extensive game Γ are exactly the Nash equilibria of the induced strategic game G of Γ. $\square$

Induced Strategic Game

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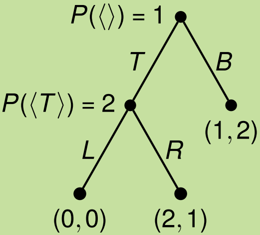
Remarks:

- Each extensive game can be transformed into a strategic game, but the resulting game can be exponentially larger.
- The other direction does not work, because in extensive games, we do not have simultaneous actions.

Empty Threats

Example (Empty threat)

Extensive game:



Induced strategic game:

	L	R
T		
B		

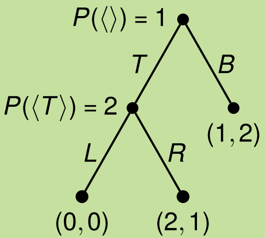
Strategies:

- Player 1: T and B
- Player 2: L and R

Empty Threats

Example (Empty threat)

Extensive game:



Induced strategic game:

	L	R
T	0, 0	2, 1
B	1, 2	1, 2

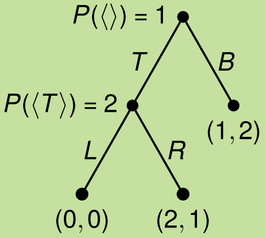
Strategies:

- Player 1: T and B
- Player 2: L and R

Empty Threats

Example (Empty threat)

Extensive game:



Strategies:

- Player 1: T and B
- Player 2: L and R

Induced strategic game:

	L	R
T	$0,0$	$2,1$
B	$1,2$	$1,2$

Nash equilibria: (B,L) and (T,R).

However, (B,L) is not realistic:

- Player 1 plays B, "fearing" response L to T.
- But player 2 would never play L against T in the extensive game.
  $\rightsquigarrow$ (B,L) involves "empty threat".

Subgames

Idea: Exclude empty threats.

How? Demand that a strategy profile is not only a Nash equilibrium in the entire game, but also in every subgame.

Definition (Subgame)

A subgame of an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$, starting after history h, is the game $\Gamma(h) = \langle N, H|_h, P|_h, (u_i|_h)_{i \in N} \rangle$, where

- $H|_h = \{h' \mid (h, h') \in H\}$,
- $P|_h(h') = P(h, h')$ for all $h' \in H|_h$, and
- $u_i|_h(h') = u_i(h, h')$ for all $h' \in H|_h$.

Subgames

Definition (Strategy in a subgame)

Let Γ be an extensive game and $\Gamma(h)$ a subgame of Γ starting after some history h.

For each strategy s_i of Γ, let $s_i|_h$ be the strategy induced by s_i for $\Gamma(h)$. Formally, for all $h' \in H|_h$,

$$s_i|_h(h') = s_i(h, h').$$

The outcome function of $\Gamma(h)$ is denoted by O_h.

Subgame-Perfect Equilibria

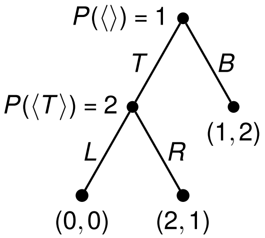
Definition (Subgame-perfect equilibrium)

A strategy profile s^* in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ is a subgame-perfect equilibrium (SPE) if and only if for every player $i \in N$ and every nonterminal history $h \in H \setminus Z$ with $P(h) = i$,

$$u_i|_h(O_h(s^*_{-i}|_h, s^*_i|_h)) \geq u_i|_h(O_h(s^*_{-i}|_h, s_i))$$

for every strategy $s_i \in S_i$ in subgame $\Gamma(h)$.

Subgame-Perfect Equilibria

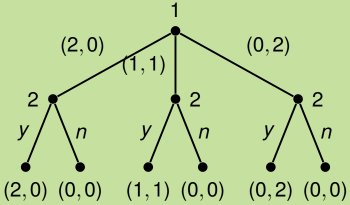


Two Nash equilibria:

- (T, R): subgame-perfect, because:
  - In history $h = \langle T\rangle$: subgame-perfect.
  - In history $h = \langle\rangle$: player 1 obtains utility 1 when choosing B and utility of 2 when choosing T.
- (B, L): not subgame-perfect, since L does not maximize the utility of player 2 in history $h = \langle T\rangle$.

Subgame-Perfect Equilibria

Example (Subgame-perfect equilibria in division game)



Equilibria in subgames:

- in $\Gamma(\langle(2,0)\rangle)$: y and n
- in $\Gamma(\langle(1,1)\rangle)$: only y
- in $\Gamma(\langle(0,2)\rangle)$: only y
- in $\Gamma(\langle\rangle)$: $((2,0), yyy)$ and $((1,1), nyy)$

Nash equilibria (red: no SPE):

- $((2,0), yyy)$, $((2,0), yyn)$, $((2,0), yny)$, $((2,0), ynn)$, $((2,0), nny)$, $((2,0), nnn)$,
- $((1,1), nyy)$, $((1,1), nyn)$,
- $((0,2), nny)$.

Summary

- Nash equilibria in extensive game with perfect information
  - defined directly or
  - defined via induced strategic game
- Problems with Nash equilibria:
  - exponentially many strategies to consider
  - empty threats
- Alternative/better solution concept without empty threats: subgame-perfect equilibria
  - have to be an equilibrium in every subgame

Game Theory

5. Extensive Games

5.1. Extensive Games with Perfect Information

5.1.3. One-Deviation Property

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Summer semester 2020

Motivation

- So far:
  - definition of extensive games
  - solution concept: subgame-perfect equilibria (SPE)
- Now:
  - Existence:
    - Does every extensive game have a subgame-perfect equilibrium?
    - If not, which extensive games do have a subgame-perfect equilibrium?
  - Computation:
    - If a subgame-perfect equilibrium exists, how to compute it?
    - How complex is that computation?

Motivation

Positive case (a subgame-perfect equilibrium exists):

- Step 1: Show that is suffices to consider local deviations from strategies (for finite-horizon games).
  (Section 5.1.3)
- Step 2: Show how to systematically explore such local deviations to find a subgame-perfect equilibrium (for finite games).
  (Section 5.1.4)

Step 1: One-Deviation Property

Definition

Let Γ be a finite-horizon extensive game. Then $\ell(\Gamma)$ denotes the length of the longest history of Γ.

Step 1: One-Deviation Property

Definition (One-deviation property)

A strategy profile s^* in an extensive game $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ satisfies the one-deviation property if and only if for every player $i \in N$ and every nonterminal history $h \in H \setminus Z$ with $P(h) = i$,

$$u_i|_h(O_h(s^*_{-i}|_h, s^*_i|_h)) \geq u_i|_h(O_h(s^*_{-i}|_h, s_i))$$

for every strategy $s_i \in S_i$ in subgame $\Gamma(h)$ that differs from $s^*_i|_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Note: Without the highlighted part in the end, this is just the definition of subgame-perfect equilibria!

Step 1: One-Deviation Property

Lemma

Let $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ be a finite-horizon extensive game. Then a strategy profile s^* is a subgame-perfect equilibrium of Γ if and only if it satisfies the one-deviation property.

Proof

- ($\Rightarrow$) Clear.
- ($\Leftarrow$) By contradiction:

  Suppose that s^* is not a subgame-perfect equilibrium.

  Then there is a history h and a player i such that s_i is a profitable deviation for player i in subgame $\Gamma(h)$.

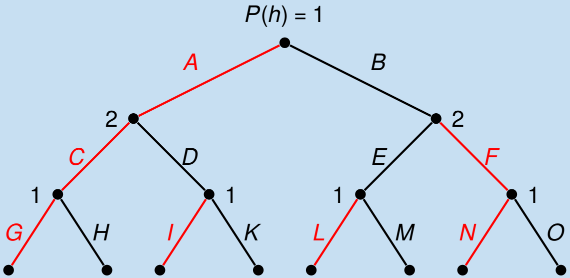
  …

Step 1: One-Deviation Property

Proof (ctd.)

- ($\Leftarrow$) …WLOG, the number of histories h' with $s_i(h') \neq s_i^*|_h(h')$ is at most $\ell(\Gamma(h))$ and hence finite (finite horizon assumption!), since deviations not on resulting outcome path are irrelevant.

  Illustration: strategies $s_1^*|_h = AGILN$ and $s_2^*|_h = CF$:



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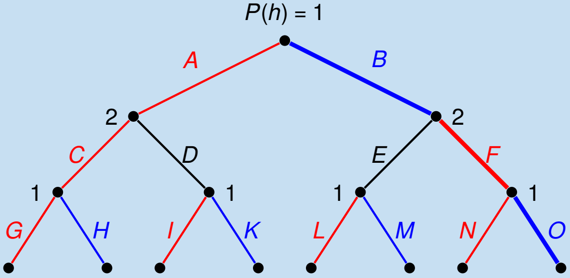
Proof (ctd.)

- (⇐) …Illustration for WLOG assumption: Assume $s_1 = BHKMO$ (blue) profitable deviation:

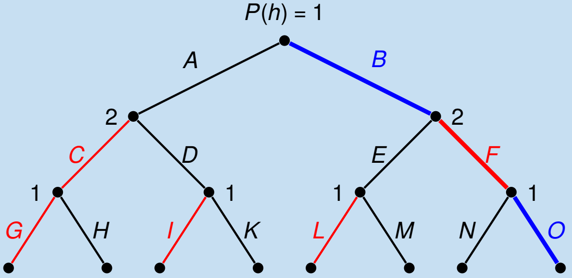


Then only B and O really matter.

Step 1: One-Deviation Property

Proof (ctd.)

- ($\Leftarrow$) …Illustration for WLOG assumption: And hence $\tilde{s}_1 = BGILO$ (blue) also profitable deviation:



Step 1: One-Deviation Property

Proof (ctd.)

- ($\Leftarrow$) …

  Choose profitable deviation s_i in $\Gamma(h)$ with minimal number of deviation points (such s_i must exist).

  Let h^* be the longest history in $\Gamma(h)$ with $s_i(h^*) \neq s_i^*|_h(h^*)$, i.e., "deepest" deviation point for s_i.

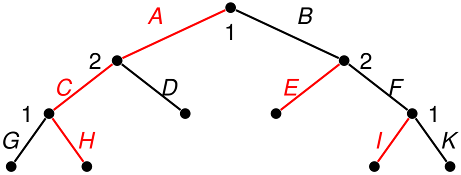
  Then in $\Gamma(h, h^*)$, $s_i|_{h^*}$ differs from $s_i^*|_{(h,h^*)}$ only in the initial history.

  Moreover, $s_i|_{h^*}$ is a profitable deviation in $\Gamma(h, h^*)$, since otherwise fewer deviation points would suffice.

  So, $\Gamma(h, h^*)$ is the desired subgame where a one-step deviation is sufficient to improve utility. $\square$

Step 1: One-Deviation Property

Example



To show that (AHI, CE) is a subgame-perfect equilibrium, it suffices to check these deviating strategies:

Player 1:

- G in subgame $\Gamma(\langle A, C\rangle)$
- K in subgame $\Gamma(\langle B, F\rangle)$
- BHI in Γ

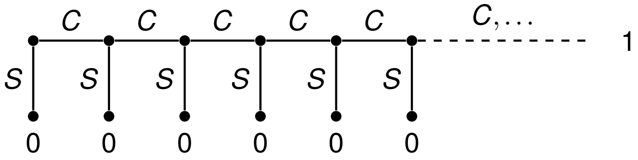
Player 2:

- D in subgame $\Gamma(\langle A\rangle)$
- F in subgame $\Gamma(\langle B\rangle)$

In particular, e.g., no need to check if strategy BGK of player 1 is profitable in Γ.

The corresponding proposition for infinite-horizon games does not hold.

Counterexample (one-player case):



Strategy s_i with $s_i(h) = S$ for all $h \in H \setminus Z$

- satisfies one deviation property, but
- is not a subgame-perfect equilibrium, since it is dominated by s_i^* with $s_i^*(h) = C$ for all $h \in H \setminus Z$.

Summary

- For finite-horizon extensive games, it suffices to consider local deviations when looking for better strategies.
- This simplifies verifying whether a strategy profile is an SPE (or finding one).
- For infinite-horizon games, this is not true in general.

Game Theory

5. Extensive Games

5.1. Extensive Games with Perfect Information

5.1.4. Kuhn's Theorem

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Motivation

- So far:
  - definition of extensive games
  - solution concept: subgame-perfect equilibria (SPE)
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- Now:
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- Step 1: Show that is suffices to consider local deviations from strategies (for finite-horizon games).
  (Section 5.1.3)
- Step 2: Show how to systematically explore such local deviations to find a subgame-perfect equilibrium (for finite games).
  (Section 5.1.4)

Step 2: Kuhn's Theorem

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Theorem (Kuhn)

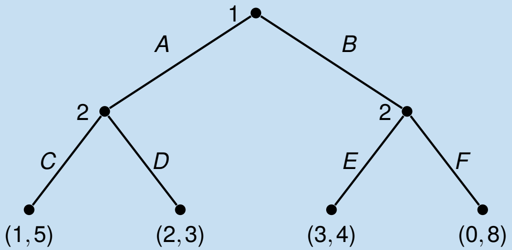
Every finite extensive game has a subgame-perfect equilibrium.

Proof idea:

- Proof is constructive and builds a subgame-perfect equilibrium bottom-up (aka backward induction).
- For those familiar with the Foundations of AI lecture: generalization of Minimax algorithm to general-sum games with possibly more than two players.

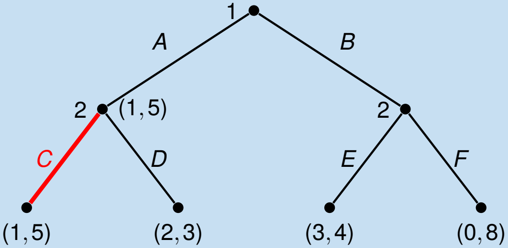
Step 2: Kuhn's Theorem

Example



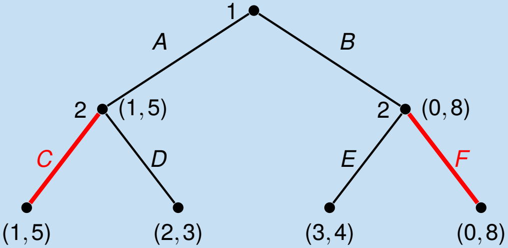
Step 2: Kuhn's Theorem

Example



$s_2(\langle A \rangle) = C$ $\quad t_1(\langle A \rangle) = 1$ $\quad t_2(\langle A \rangle) = 5$

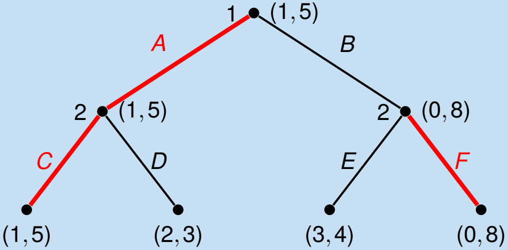
Example



$s_2(\langle A\rangle) = C$ $\quad t_1(\langle A\rangle) = 1$ $\quad t_2(\langle A\rangle) = 5$

$s_2(\langle B\rangle) = F$ $\quad t_1(\langle B\rangle) = 0$ $\quad t_2(\langle B\rangle) = 8$

Example



$s_2(\langle A \rangle) = C$	$t_1(\langle A \rangle) = 1$	$t_2(\langle A \rangle) = 5$
$s_2(\langle B \rangle) = F$	$t_1(\langle B \rangle) = 0$	$t_2(\langle B \rangle) = 8$
$s_1(\langle \rangle) = A$	$t_1(\langle \rangle) = 1$	$t_2(\langle \rangle) = 5$

Step 2: Kuhn's Theorem

A bit more formally:

Proof

Let $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$ be a finite extensive game.

Construct a subgame-perfect equilibrium by induction on $\ell(\Gamma(h))$ for all subgames $\Gamma(h)$. In parallel, construct functions $t_i : H \to \mathbb{R}$ for all players $i \in N$ s. t. $t_i(h)$ is the payoff for player i in a subgame-perfect equilibrium in subgame $\Gamma(h)$.

Base case: If $\ell(\Gamma(h)) = 0$, then $t_i(h) = u_i(h)$ for all $i \in N$.

...

Step 2: Kuhn's Theorem

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Proof (ctd.)

Inductive case: If $t_i(h)$ already defined for all $h \in H$ with $\ell(\Gamma(h)) \leq k$, consider $h^* \in H$ with $\ell(\Gamma(h^*)) = k + 1$ and $P(h^*) = i$.

For all $a \in A(h^*)$, $\ell(\Gamma(h^*, a)) \leq k$, let

$$s_i(h^*) := \operatorname*{argmax}_{a \in A(h^*)} t_i(h^*, a) \quad \text{and}$$

$$t_j(h^*) := t_j(h^*, s_i(h^*)) \quad \text{for all players } j \in N.$$

Inductively, we obtain a strategy profile s that satisfies the one-deviation property.

With the one-deviation property lemma it follows that s is a subgame-perfect equilibrium. □

Step 2: Kuhn's Theorem

- In principle: sample subgame-perfect equilibrium effectively computable using the technique from the above proof.
- In practice: often game trees not enumerated in advance, hence unavailable for backward induction.
- E.g., for branching factor b and depth m, procedure needs time $O(b^m)$.

Step 2: Kuhn's Theorem

Remark on Infinite Games

Corresponding proposition for infinite games does not hold.

Counterexamples (both for one-player case):

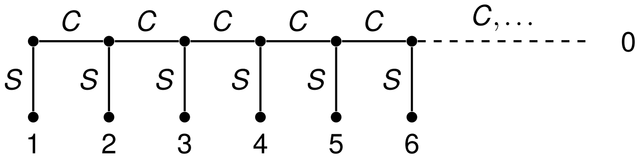
A) finite horizon, infinite branching factor:

Infinitely many actions $a \in A = [0, 1)$ with payoffs $u_1(\langle a \rangle) = a$ for all $a \in A$.
There exists no subgame-perfect equilibrium in this game.

Step 2: Kuhn’s Theorem

Remark on Infinite Games

B) infinite horizon, finite branching factor:



$u_1(CCC\dots) = 0$ and $u_1(\underbrace{CC\dots C}_{n}S) = n + 1$.

No subgame-perfect equilibrium.

Step 2: Kuhn's Theorem

Uniqueness:

Kuhn's theorem tells us nothing about uniqueness of subgame-perfect equilibria. However, if no two histories get the same evaluation by any player, then the subgame-perfect equilibrium is unique.

Extended Example: Pirate Game

1. There are five rational pirates, A, B, C, D and E. They find 100 gold coins. They must decide how to distribute them.
2. The pirates have a strict order of seniority: A is senior to B, who is senior to C, who is senior to D, who is senior to E.
3. The pirate world's rules of distribution say that the most senior pirate first proposes a distribution of coins. The pirates, including the proposer, then vote on whether to accept this distribution (in order from most junior to senior). In case of a tie vote, the proposer has the casting vote. If the distribution is accepted, the coins are disbursed and the game ends. If not, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to apply the method again.

Pirates: General Setting & Utility

4. The pirates do not trust each other, and will neither make nor honor any promises between pirates apart from a proposed distribution plan that gives a whole number of gold coins to each pirate.
5. Pirates base their decisions on three factors. First of all, each pirate wants to survive. Second, everything being equal, each pirate wants to maximize the number of gold coins each receives. Third, each pirate would prefer to throw another overboard, if all other results would otherwise be equal.

Pirates: Formalization

- Players $N = \{A, B, C, D, E\}$;
- actions are:
  - proposals by a pirate: $\langle A : x_A, B : x_B, C : x_C, D : x_D, E : x_E \rangle$, with $\sum_{i \in \{A,B,C,D,E\}} x_i = 100$;
  - votings: y for accepting, n for rejecting;
- histories are sequences of a proposal, followed by votings of the alive pirates;
- utilities:
  - for pirates who are alive: utilities are according to the accepted proposal plus $x/100$, x being the number of dead pirates;
  - for dead pirates: -100.

Remark: Very large game tree!

Pirates: Analysis by Backward Induction

1. Assume only D and E are still alive. D can propose $\langle A:0, B:0, C:0, D:100, E:0\rangle$, because D has the casting vote!

Pirates: Analysis by Backward Induction

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2. Assume C, D, and E are alive. For C it is enough to offer 1 coin to E to get his vote: $\langle A:0, B:0, C:99, D:0, E:1\rangle$.

Pirates: Analysis by Backward Induction

1. Assume only D and E are still alive. D can propose $\langle A:0, B:0, C:0, D:100, E:0\rangle$, because D has the casting vote!
2. Assume C, D, and E are alive. For C it is enough to offer 1 coin to E to get his vote: $\langle A:0, B:0, C:99, D:0, E:1\rangle$.
3. Assume B, C, D, and E are alive. B offering D one coin is enough because of the casting vote: $\langle A:0, B:99, C:0, D:1, E:0\rangle$.

Pirates: Analysis by Backward Induction

1. Assume only D and E are still alive. D can propose $\langle A:0, B:0, C:0, D:100, E:0\rangle$, because D has the casting vote!
2. Assume C, D, and E are alive. For C it is enough to offer 1 coin to E to get his vote: $\langle A:0, B:0, C:99, D:0, E:1\rangle$.
3. Assume B, C, D, and E are alive. B offering D one coin is enough because of the casting vote: $\langle A:0, B:99, C:0, D:1, E:0\rangle$.
4. Assume A, B, C, D, and E are alive. A offering C and E each one coin is enough: $\langle A:98, B:0, C:1, D:0, E:1\rangle$ (note that giving 1 to D instaed to E does not help).

Summary

- Every finite extensive game has a subgame-perfect equilibrium.
- This does not generally hold for infinite games, no matter if the game is infinite due to infinite branching factor or infinitely long histories (or both).
- Subgame-perfect equilibria in finite extensive game can be identified using backward induction.

Game Theory

5. Extensive Games

5.1. Extensive Games with Perfect Information

5.1.5 Simultaneous Moves and Chance Moves

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Summer semester 2020

Motivation

- So far:
  - Finite-horizon extensive games with perfect information: one-deviation property
  - Finite extensive games with perfect information: Kuhn's theorem
- Now: what about those results if we allow
  - simultaneous moves or
  - chance moves?

Definition

An extensive game with simultaneous moves is a tuple $\Gamma = \langle N, H, P, (u_i)_{i \in N} \rangle$, where

- N, H, P and (u_i) are defined as before, and
- $P : H \setminus Z \to 2^N$ assigns to each nonterminal history a set of players to move; for all $h \in H \setminus Z$, there exists a family $(A_i(h))_{i \in P(h)}$ such that

$$A(h) = \{a \mid (h,a) \in H\} = \prod_{i \in P(h)} A_i(h).$$

Simultaneous Moves

- Intended meaning of simultaneous moves: all players from $P(h)$ move simultaneously
- Strategies: functions $s_i : h \mapsto a_i$ with $a_i \in A_i(h)$
- Histories: sequences of vectors of actions
- Outcome: terminal history reached when tracing strategy profile
- Payoffs: utilities at outcome history

Observations:

- The one-deviation property still holds for extensive games with perfect information and simultaneous moves.
- Kuhn's theorem does not hold for extensive game with simultaneous moves.

  Example: MATCHING PENNIES can be viewed as extensive game with simultaneous moves. No Nash equilibrium/subgame-perfect equilibrium.

  player 2

		H	T
player 1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

  $\rightsquigarrow$ Need more sophisticated solution concepts (cf. mixed strategies). Not covered in this course.

Setting:

- Three players have to split a cake fairly.
- Player 1 suggest split: shares $x_1, x_2, x_3 \in [0, 1]$ s.t. $x_1 + x_2 + x_3 = 1$.
- Then players 2 and 3 simultaneously and independently decide whether to accept ("y") or reject ("n") the suggested splitting.
- If both accept, each player i gets their allotted share (utility x_i). Otherwise, no player gets anything (utility 0).

Formally:

$$
\begin{aligned}
N &= \{1,2,3\} \\
X &= \{(x_1,x_2,x_3) \in [0,1]^3 \mid x_1+x_2+x_3 = 1\} \\
H &= \{\langle\rangle\} \cup \{\langle x\rangle \mid x \in X\} \cup \{\langle x,z\rangle \mid x \in X, z \in \{\mathrm{y},\mathrm{n}\} \times \{\mathrm{y},\mathrm{n}\}\} \\
P(\langle\rangle) &= \{1\} \\
P(\langle x\rangle) &= \{2,3\} \text{ for all } x \in X \\
u_i(\langle x,z\rangle) &= \begin{cases} 0 & \text{if } z \in \{(\mathrm{y},\mathrm{n}),(\mathrm{n},\mathrm{y}),(\mathrm{n},\mathrm{n})\} \\ x_i & \text{if } z = (\mathrm{y},\mathrm{y}). \end{cases} \text{ for all } i \in N
\end{aligned}
$$

Subgame-perfect equilibria:

- Subgames after legal split (x_1, x_2, x_3) by player 1:
  - NE (y, y) (both accept)
  - NE (n, n) (neither accepts)
  - If $x_2 = 0$, NE (n, y) (only player 3 accepts)
  - If $x_3 = 0$, NE (y, n) (only player 2 accepts)

Subgame-perfect equilibria (ctd.):

- Entire game:
  Let s_2 and s_3 be any two strategies of players 2 and 3 such that for all splits $x \in X$ the profile $(s_2(\langle x \rangle), s_3(\langle x \rangle))$ is one of the NEs from above.
  Let $X_y = \{x \in X \mid s_2(\langle x \rangle) = s_3(\langle x \rangle) = y\}$ be the set of splits accepted under s_2 and s_3. Distinguish three cases:
  - $X_y = \emptyset$ or $x_1 = 0$ for all $x \in X_y$. Then (s_1, s_2, s_3) is a subgame-perfect equilibrium for any possible s_1.
  - $X_y \neq \emptyset$ and there are splits $x_{max} = (x_1, x_2, x_3) \in X_y$ that maximize $x_1 > 0$. Then (s_1, s_2, s_3) is a subgame-perfect equilibrium if and only if $s_1(\langle\rangle)$ is such a split x_{max}.
  - $X_y \neq \emptyset$ and there are no splits $(x_1, x_2, x_3) \in X_y$ that maximize x_1. Then there is no subgame-perfect equilibrium, in which player 2 follows strategy s_2 and player 3 follows strategy s_3.

Chance Moves

Definition

An extensive game with chance moves is a tuple $\Gamma = \langle N, H, P, f_c, (u_i)_{i \in N} \rangle$, where

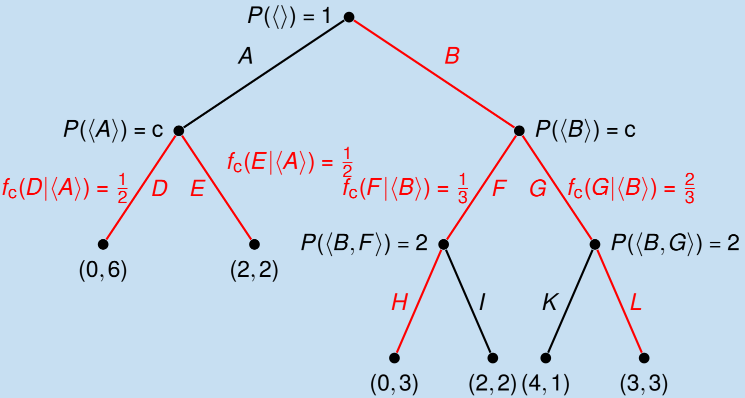
- N, A, H and u_i are defined as before,
- the player function $P : H \setminus Z \to N \cup \{c\}$ can also take the value c for a chance node, and
- for each $h \in H \setminus Z$ with $P(h) = c$, the function $f_c(\cdot|h)$ is a probability distribution on $A(h)$ such that the probability distributions for all $h \in H \setminus Z$ with $P(h) = c$ are independent of each other.

Chance Moves

- Intended meaning of chance moves: in chance node, an applicable action is chosen randomly with probability according to f_c
- Strategies: as before
- Outcome: for a given strategy profile, the outcome is a probability distribution on the set of terminal histories
- Payoffs: for player i, U_i is the expected payoff (with weights according to outcome probabilities)

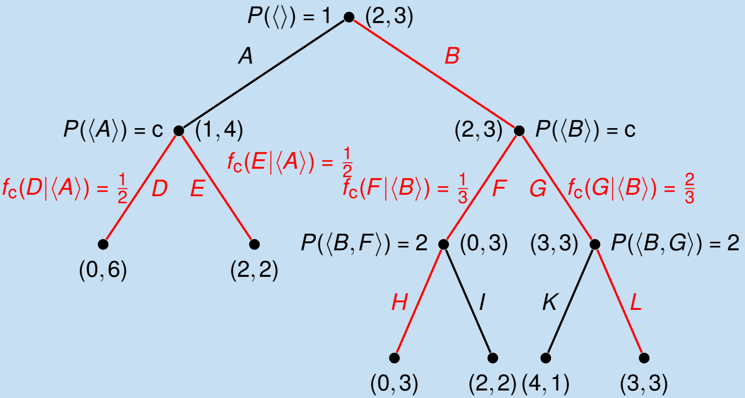
Chance Moves

Example



Chance Moves

Example



Remark:
The one-deviation property and Kuhn's theorem still hold in the presence of chance moves. When proving Kuhn's theorem, expected utilities have to be used.

Summary

- With simultaneous moves:
  - ✔ One-deviation property still holds.
  - ✗ Kuhn's theorem no longer holds.
- With chance moves:
  - ✔ One-deviation property still holds.
  - ✔ Kuhn's theorem still holds.

Game Theory

5. Extensive Games

5.2. Extensive Games with Imperfect Information

5.2.1. Motivation and Definitions

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Motivation

- So far: All state information is common knowledge among all players
- Often in practice: Only partial knowledge (e. g. in card games)
- Extensive games with imperfect information model such situations using information sets of indistinguishable histories.

Motivation

- Decision points are now such information sets.
- Strategies:
  - Pure: InfoSets $\rightarrow$ Actions
  - Mixed: (InfoSets $\rightarrow$ Actions) $\rightarrow$ Probabilities (randomization over pure strategies)
  - Behavioral: InfoSets $\rightarrow$ (Actions $\rightarrow$ Probabilities) (collections of independent randomized decisions for each information set)
- Different from incomplete information games, in which there is uncertainty about the utility functions of the other players.

Extensive Games with Imperfect Information

Definition (Extensive game)

An extensive game is a tuple $\Gamma = \langle N, H, P, f_c, (\mathscr{I}_i)_{i \in N}, (u_i)_{i \in N} \rangle$ that consists of:

- a finite non-empty set N of players,
- a set H of (finite or infinite) sequences, called histories, such that
  - it contains the empty sequence $\langle\rangle \in H$,
  - H is closed under prefixes: if $\langle a^1, \dots, a^k \rangle \in H$ for some $k \in \mathbb{N} \cup \{\infty\}$, and $l < k$, then also $\langle a^1, \dots, a^l \rangle \in H$, and
  - H is closed under limits: if for some infinite sequence $\langle a^i \rangle_{i=1}^{\infty}$, we have $\langle a^i \rangle_{i=1}^{k} \in H$ for all $k \in \mathbb{N}$, then $\langle a^i \rangle_{i=1}^{\infty} \in H$.

  All infinite histories and all histories $\langle a^i \rangle_{i=1}^{k} \in H$, for which there is no a^{k+1} such that $\langle a^i \rangle_{i=1}^{k+1} \in H$ are called terminal histories Z. Components of a history are called actions.

Extensive Games with Imperfect Information

Definition (Extensive game, ctd.)

- a player function $P : H \setminus Z \to N \cup \{c\}$ that determines which player's turn it is to move after a given nonterminal history, c signifying a chance move,
- a probability distribution $f_c(\cdot|h)$ over $A(h)$,
- an information partition $\mathscr{I}_i$ for player i of $\{h \in H | P(h) = i\}$ with the property that $A(h) = A(h')$ whenever h and h' are in the same member $I_i \in \mathscr{I}_i$ of the partition (notation: $A(I_i)$, $P(I_i)$; members I_i of the partition are called information sets), and
- for each player $i \in N$, a utility function (or payoff function) $u_i : Z \to \mathbb{R}$ defined on the set of terminal histories.

Γ is finite, if H is finite; finite horizon, if histories are bounded.

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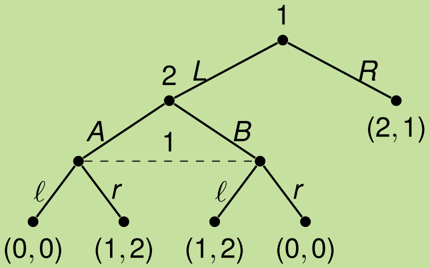
Example



After player 1 chooses L, player 2 makes a move (A or B) that player 1 cannot observe. Formally:

$\mathscr{I}_1 = \{I_{11}, I_{12}\}$ with $I_{11} = \{\langle\rangle\}$ and $I_{12} = \{\langle L, A\rangle, \langle L, B\rangle\}$

$\mathscr{I}_2 = \{I_{21}\}$ with $I_{21} = \{\langle L\rangle\}$

Simultaneous Moves

- Question: We already have chance moves, but could/should we extend the model with simultaneous moves as well?
- Answer: We could, but we don't need to.
  Actually, we can already model them somehow.
- In the example game after the history $\langle L \rangle$, we have essentially a simultaneous move of players 1 and 2:
  - When player 2 moves, he does not know what player 1 will do.
  - After player 2 has made his move, player 1 does not know whether A or B was chosen.
  - Only after both players have acted, they are presented with the outcome.

Simultaneous Moves

- Consequence: We will need randomized strategies as part of a reasonable solution concept, since:
  - already for strategic games, we need randomized strategies to guarantee equilibrium existence (in the finite case),
  - strategic games are a special case of extensive games with perfect information and simultaneous moves, and
  - extensive games with perfect information and simultaneous moves are a special case of extensive games with imperfect information.

Summary

- Extensive games with imperfect information can model situations in which the players know only part of the world.
- Modeled by information sets, which are the histories an agent cannot distinguish.
- In the model, we allow chance moves (explicitly) and simultaneous moves (implicitly).

Game Theory

5. Extensive Games

5.2. Extensive Games with Imperfect Information

5.2.2. Perfect Recall

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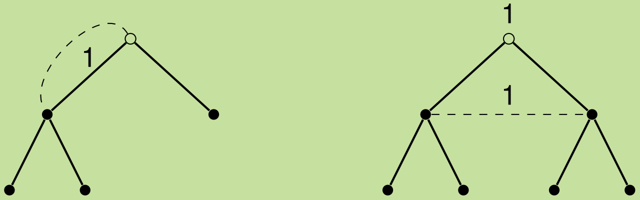
Summer semester 2020

Recall

Information sets can be arbitrary. However, often we want to assume that agents always remember what they have learned before and which actions they have performed: perfect recall.

Example (Imperfect recall)



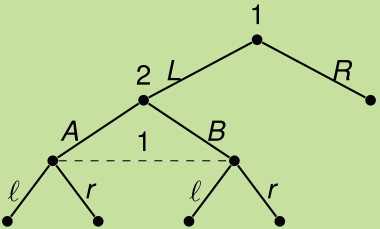
- Left: player 1 forgets that he made a move!
- Right: player 1 cannot remember what his last move was.

Experience Record

Definition (Experience record)

Given a history h of an extensive game, $X_i(h)$ is the sequence consisting of information sets that player i encounters in h and the actions that player i takes at them. X_i is called the experience record of player i in h.

Example



Player 1 encounters two information sets in the history $h = \langle L, A\rangle$, namely $I_{11} = \{\langle\rangle\}$ and $I_{12} = \{\langle L, A\rangle, \langle L, B\rangle\}$. In the first information set, he chooses L.

Hence, $X_1(h) = \langle I_{11}, L, I_{12}\rangle$.

Perfect Recall

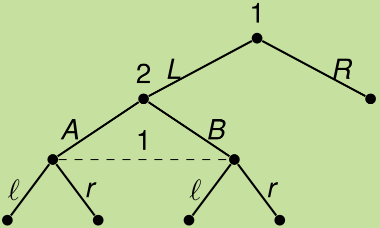
Definition (Perfect recall)

An extensive game has perfect recall if for each player i, we have $X_i(h) = X_i(h')$ whenever the histories h and h' are in the same information set of player i.

Conversely, whenever an agent has made different experiences (own actions, observations) when arriving at h and h', he can distinguish between them.

In most cases, our games will have perfect recall.

Example



This game has perfect recall: the only $h \neq h'$ in the same information set of some player are $h = \langle L, A \rangle$ and $h' = \langle L, B \rangle$ in information set $I_{12} = \{h, h'\}$. They satisfy the condition, since $X_1(h) = X_1(h') = \langle I_{11}, L, I_{12} \rangle$.

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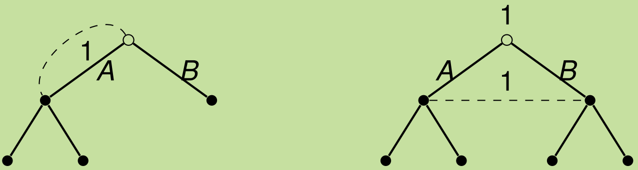
Example



No perfect recall:

- Left: player 1 cannot distinguish between $h = \langle\rangle$ and $h' = \langle A\rangle$, although $X_1(h) = \langle\{h, h'\}\rangle \neq \langle\{h, h'\}, A, \{h, h'\}\rangle = X_1(h')$.
- Right: player 1 cannot distinguish between $h = \langle A\rangle$ and $h' = \langle B\rangle$, although $X_1(h) = \langle\{\langle\rangle\}, A, \{h, h'\}\rangle \neq \langle\{\langle\rangle\}, B, \{h, h'\}\rangle = X_1(h')$.

Summary

- Perfect recall requires that agents remember what they have done and learned.
- Formalized using experience records.
- For perfect recall, different experience records must be sufficient for a player to be able to distinguish between two histories.

Game Theory

5. Extensive Games

5.2. Extensive Games with Imperfect Information

5.2.3. Strategies and Outcomes

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Summer semester 2020

Strategies

- Decision points are information sets.
- Types of strategies:
  - Pure: InfoSets $\rightarrow$ Actions
  - Mixed: (InfoSets $\rightarrow$ Actions) $\rightarrow$ Probabilities
    (randomization over pure strategies)
  - Behavioral: InfoSets $\rightarrow$ (Actions $\rightarrow$ Probabilities)
    (collections of independent randomized decisions for each information set)

Pure Strategies

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Definition (Pure strategy in an extensive game)

A pure strategy of a player i in an extensive game $\Gamma = \langle N, H, P, f_c, (\mathcal{I}_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a function s_i that assigns an action from $A(I_i)$ to each information set I_i.

Remark: Note that the outcome of a strategy profile s is now a probability distribution (because of the chance moves).

Remark: Because of the chance moves and because of the imperfect information, it probably makes more sense to consider randomized strategies.

Mixed and Behavioral Strategies

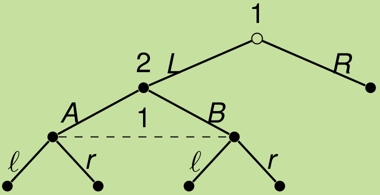
Definition (Mixed and behavioral strategies)

A mixed strategy σ_i of a player i in an extensive game $\Gamma = \langle N, H, P, f_c, (\mathscr{I}_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a probability distribution over the set of player i's pure strategies.

A behavioral strategy of player i is a collection $(\beta_i(I_i))_{I_i \in \mathscr{I}_i}$ of independent probability distributions, where $\beta_i(I_i)$ is a probability distribution over $A(I_i)$.
For any history $h \in I_i \in \mathscr{I}_i$ and action $a \in A(h)$, we denote by $\beta_i(h)(a)$ the probability $\beta_i(I_i)(a)$ assigned by $\beta_i(I_i)$ to action a.

Mixed and Behavioral Strategies

Example



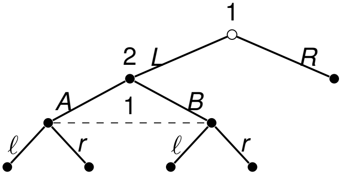
- Player 1 has four pure strategies (two information sets, two actions at each): $L\ell$, Lr, $R\ell$, Rr.
- A mixed strategy is a probability distribution over those.
- A behavioral strategy is a pair of probability distributions, one over $\{L, R\}$ for $\{\langle\rangle\}$, and one over $\{\ell, r\}$ for $\{\langle L, A\rangle, \langle L, B\rangle\}$.

Outcomes

The outcome of a (mixed or behavioral) strategy profile σ is a probability distribution over histories $O(\sigma)$, resulting from following the individual strategies:

- For any history $h = \langle a^1, \dots, a^k \rangle$, define a pure strategy s_i of i to be consistent with h if for any prefix $h' = \langle a^1, \dots, a^\ell \rangle$ of h with $P(h') = i$, we have $s_i(h') = a^{\ell+1}$.
- For any history h, let $\pi_i(h)$ be the sum of probabilities of pure strategies s_i from σ_i that are consistent with h.
- Then for any mixed profile σ, the probability that $O(\sigma)$ assigns to a terminal history h is: $\prod_{i \in N \cup \{c\}} \pi_i(h)$ (where $\pi_c(h)$ is the product of the $f_c(\cdot|\cdot)$ values along h).
- For any behavioral profile β, the probability that $O(\beta)$ assigns to $h = \langle a^1, \dots, a^K \rangle$ is: $\prod_{k=0}^{K-1} \beta_{P(\langle a^1, \dots, a^k \rangle)}(\langle a^1, \dots, a^k \rangle)(a^{k+1})$.

Outcomes: Mixed Strategies – Example



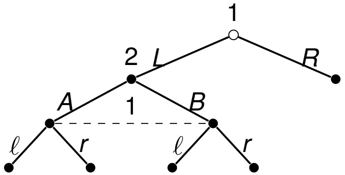
Assume $\sigma_1 = \{L\ell \mapsto {}^{28}/_{70}, Lr \mapsto {}^{21}/_{70}, R\ell \mapsto {}^{12}/_{70}, Rr \mapsto {}^{9}/_{70}\}$, $\sigma_2 = \{A \mapsto {}^{1}/_{2}, B \mapsto {}^{1}/_{2}\}$, and $\sigma = (\sigma_1, \sigma_2)$.

Then, e. g., $s_{13} = R\ell$, $s_{14} = Rr$, $s_{21} = A$, and $s_{22} = B$ all consistent with $h = \langle R \rangle$, but $s_{11} = L\ell$ and $s_{12} = Lr$ not.

$\pi_1(\langle R \rangle) = \sigma_1(R\ell) + \sigma_1(Rr) = {}^{12}/_{70} + {}^{9}/_{70} = {}^{3}/_{10}$,
$\pi_2(\langle R \rangle) = \sigma_2(A) + \sigma_2(B) = {}^{1}/_{2} + {}^{1}/_{2} = 1$, and
$O(\sigma)(\langle R \rangle) = \pi_1(\langle R \rangle) \cdot \pi_2(\langle R \rangle) = {}^{3}/_{10} \cdot 1 = {}^{3}/_{10}$.

Similarly,
$O(\sigma)(\langle L, A, \ell \rangle) = \pi_1(\langle L, A, \ell \rangle) \cdot \pi_2(\langle L, A, \ell \rangle) = {}^{28}/_{70} \cdot {}^{1}/_{2} = {}^{2}/_{10}$.

Outcomes: Behavioral Strategies – Example



Assume $\beta_1(\{\langle\rangle\}) = \{L \mapsto 7/10, R \mapsto 3/10\}$,
$\beta_1(\{\langle L,A\rangle,\langle L,B\rangle\}) = \{\ell \mapsto 4/7, r \mapsto 3/7\}$,
$\beta_2(\{\langle L\rangle\}) = \{A \mapsto 1/2, B \mapsto 1/2\}$, and $\beta = (\beta_1, \beta_2)$.

Then, e. g., $O(\beta)(\langle R\rangle) = \beta_1(\{\langle\rangle\})(R) = 3/10$.

Similarly, $O(\beta)(\langle L,A,\ell\rangle) = \beta_1(\{\langle\rangle\})(L) \cdot \beta_2(\{\langle L\rangle\})(A) \cdot \beta_1(\{\langle L,A\rangle,\langle L,B\rangle\})(\ell) = 7/10 \cdot 1/2 \cdot 4/7 = 2/10$.

Outcome Equivalence

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Definition

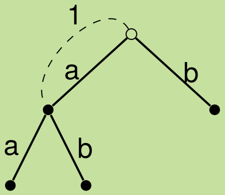
Two (mixed or behavioral) strategies of a player i are called outcome-equivalent if for every partial profile of pure strategies of the other players, the two strategies induce the same outcome.

Question: Can we find outcome-equivalent mixed strategies for behavioral strategies and vice versa?

Partial answer: Sometimes.

Counterexample (1)

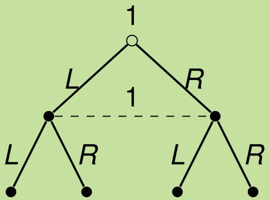
Example (Behavioral strategy without a mixed strategy)



- A behavioral strategy assigning non-zero probability to *a* and *b* generates outcomes $\langle a, a\rangle$, $\langle a, b\rangle$, and $\langle b\rangle$ with non-zero probabilities.
- Since there are only the pure strategies of playing *a* or *b*, no mixed strategy can produce $\langle a, b\rangle$.

Counterexample (2)

Example (Mixed strategy without a behavioral strategy)



- Mix the two pure strategies *LL* and *RR* equally, resulting in the distribution $(1/2, 0, 0, 1/2)$ over the terminal histories.
- No behavioral strategy can accomplish this.

Equivalence of Behavioral and Mixed Strategies

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If we restrict ourselves to games with perfect recall, however, everything works.

Theorem (Equivalence of mixed and behavioral strategies (Kuhn))

In a game of perfect recall, any mixed strategy of a given agent can be replaced by an outcome-equivalent behavioral strategy, and any behavioral strategy can be replaced by an outcome-equivalent mixed strategy.

Summary

- Types of strategies:
  - Pure: InfoSets $\rightarrow$ Actions
  - Mixed: (InfoSets $\rightarrow$ Actions) $\rightarrow$ Probabilities (randomization over pure strategies)
  - Behavioral: InfoSets $\rightarrow$ (Actions $\rightarrow$ Probabilities) (collections of independent randomized decisions for each information set)
- Mixed and behavioral are equivalent (induce same outcome probabilities) in the case of perfect recall.
- Otherwise not.

Game Theory

5. Extensive Games

5.2. Extensive Games with Imperfect Information

5.2.4. Solution Concepts

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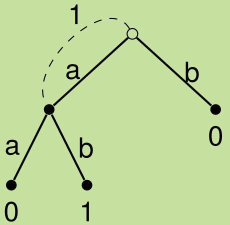
Expected Utility

Similar to the case of mixed strategies for strategic games, we define the utility for mixed and behavioral strategies as expected utility, summing over all terminal histories:

$$U_i(\sigma) = \sum_{h \in Z} u_i(h) \cdot O(\sigma)(h)$$

Example



- Mixed strategy (mixing a and b) σ: $U_1(\sigma) = 0$.
- Behavioral strategy β with $p = {}^1\!/_2$ for a: $U_1(\beta) = {}^1\!/_4$.

Nash Equilibria

Definition (Nash equilibrium in mixed strategies)

A Nash equilibrium in mixed strategies is a profile σ^* of mixed strategies with the property that for every player i:

$$U_i(\sigma^*_{-i}, \sigma^*_i) \geq U_i(\sigma^*_{-i}, \sigma_i) \text{ for every mixed strategy } \sigma_i \text{ of } i.$$

Note: Support lemma applies here as well.

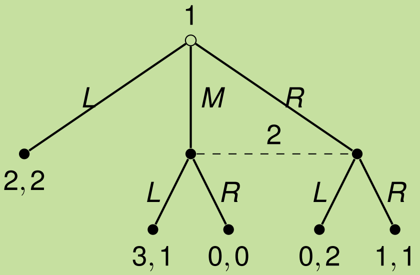
Definition (Nash equilibrium in behavioral strategies)

A Nash equilibrium in behavioral strategies is a profile β^* of behavioral strategies with the property that for every player i:

$$U_i(\beta^*_{-i}, \beta^*_i) \geq U_i(\beta^*_{-i}, \beta_i) \text{ for every behavioral strategy } \beta_i \text{ of } i.$$

Remark: Equivalent, provided we have perfect recall.

Example

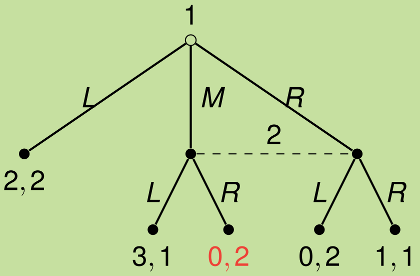


Nash equilibria: (M, L) and (L, R).

Unreasonable ones: (L, R), because in the information set of player 2, L dominates R (cf. empty threats).

How have we got here?

Example



Nash equilibria: (L, R).

What should player 2 do in his information set?

This depends on his belief: if the probability that M has been played is $\geq 1/2$, then R is optimal, otherwise L.

Assessments

Let us take the beliefs about what has been played into account when defining an equilibrium.

Definition (Assessment)

An assessment in an extensive game is a pair (β, μ), where β is a profile of behavioral strategies and μ is a function that assigns to every information set a probability distribution on the set of histories in the information set.

$\mu(I)(h)$ is the probability that player $P(I)$ assigns to the history $h \in I$, given that I is reached.

Outcome

We have to modify the outcome function. Let $h^* = \langle a^1, \ldots, a^K \rangle$ be a terminal history. Then:

- $O(\beta, \mu \mid I)(h^*) = 0$, if there is no subhistory of h^* in I (i. e., h^* is unreachable from I), and
- $O(\beta, \mu \mid I)(h^*) =$ $\mu(I)(h) \cdot \prod_{k=L}^{K-1} \beta_{P(\langle a^1, \ldots, a^k \rangle)}(\langle a^1, \ldots, a^k \rangle)(a^{k+1})$, if a subhistory $h = \langle a^1, \ldots, a^L \rangle$ of h^* with $L \leq K$ is in I.

Remark 1: This is well-defined, since the subhistory $h = \langle a^1, \ldots, a^L \rangle$ in the second case is unique if the game has perfect recall.

Remark 2: For the inital history, we have $O(\beta, \mu \mid \langle\rangle)(h^*) = O(h^*)$.

Sequential Rationality

Similar to the outcome function, we generalize the expected utility functions:

$$U_i(\beta,\mu \mid I_i) = \sum_{h \in Z} u_i(h) \cdot O(\beta,\mu \mid I_i)(h)$$

Definition (Sequential rationality)

Let Γ be an extensive game with perfect recall. An assessment (β,μ) is sequentially rational if for every player i and every information set $I_i \in \mathscr{I}_i$, we have

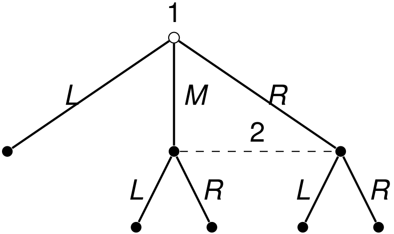
$$U_i(\beta,\mu \mid I_i) \geq U_i((\beta_{-i},\beta_i'),\mu \mid I_i) \quad \text{for every strategy } \beta_i' \text{ of player } i.$$

Note: restrictions on μ still missing!

Consistency with Strategies

We would at least require that the beliefs μ are consistent with the strategies, meaning they should be derived from the strategies using Bayes' rule.

In our earlier example, player 2's belief should be derived from the behavioral strategy of player 1.



E.g., the probability that M has been played should be:

$$\mu(\{\langle M\rangle,\langle R\rangle\})(\langle M\rangle) = \frac{\beta_1(\langle\rangle)(M)}{\beta_1(\langle\rangle)(M) + \beta_1(\langle\rangle)(R)}.$$

However, what to do when the denominator is 0? (I. e., in this example, if player 1 only plays L, i. e., if $\beta_1(\langle\rangle)(L) = 1$.)

Consistency

By viewing an assessment as a limit of a sequence of completely mixed strategy profiles (all strategies are in the support), one can enforce the Bayes condition also on information set that are not reached by an equilibrium profile.

Definition (Consistency)

Let Γ by a finite extensive game with perfect recall. An assessment (β, μ) is consistent if there is a sequence $((\beta^n, \mu^n))_{n=1}^{\infty}$ of assessments that converges to (β, μ) in Euclidian space and has the properties that each strategy profile β^n is completely mixed and that each belief system μ^n is derived from β^n using Bayes' rule.

Note: Kreps (1990) wrote: "a lot of bodies are buried in this definition."

Sequential Equilibria

Definition (Sequential equilibrium)

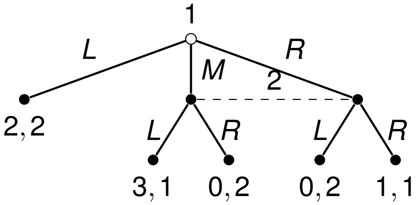
An assessment is a sequential equilibrium of a finite extensive game with perfect recall if it is sequentially rational and consistent.

Theorem (Kreps and Wilson, 1982)

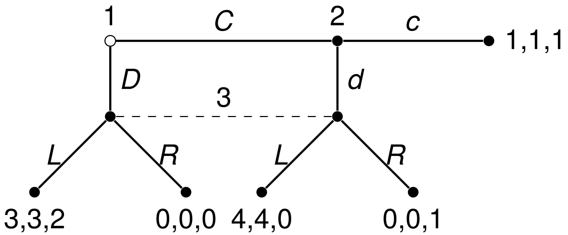
Every finite extensive game with perfect recall has a sequential equilibrium. □

Theorem (Kreps and Wilson, 1982)

Sequential equilibria generalize subgame-perfect equilibria. □



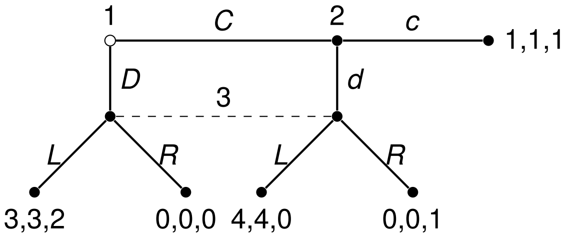
- Let (β,μ) be as follows: $\beta_1(L) = 1$, $\beta_2(R) = 1$, $\mu(\{\langle M\rangle,\langle R\rangle\})(\langle M\rangle) = \alpha$ with $0 \leq \alpha \leq 1$.
- Then (β,μ) is consistent since $\beta_1^n = (1-\varepsilon, \alpha\varepsilon, (1-\alpha)\varepsilon)$, $\beta_2^n = (\varepsilon, 1-\varepsilon)$, for $\varepsilon = {}^1\!/_n$, and $\mu^n(\{\langle M\rangle,\langle R\rangle\})(\langle M\rangle) = \alpha$ converges to (β,μ) for $n \to \infty$.
- For $\alpha \geq 1/2$, (β,μ) is sequentially rational.



Two types of NE (for $I = \{\langle D\rangle, \langle C, d\rangle\}$):

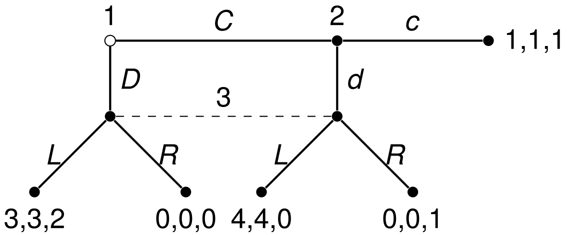
1. $\beta_1(\langle\rangle)(D) = 1$, $1/3 \leq \beta_2(\langle C\rangle)(c) \leq 1$, $\beta_3(I)(L) = 1$
2. $\beta_1(\langle\rangle)(C) = 1$, $\beta_2(\langle C\rangle)(c) = 1$, $3/4 \leq \beta_3(I)(R) \leq 1$

Are these also sequential equilibria?



1. $\beta_1(\langle\rangle)(D) = 1$, $1/3 \leq \beta_2(\langle C \rangle)(c) \leq 1$, $\beta_3(I)(L) = 1$:

violates sequential rationality for player 2!



2 $\beta_1(\langle\rangle)(C) = 1$, $\beta_2(\langle C\rangle)(c) = 1$, ${}^3\!/_4 \leq \beta_3(I)(R) \leq 1$:

for each NE of this form, there exists a sequential equilibrium (β, μ) with $\mu(I)(D) = {}^1\!/_3$.

For consistency consider: $\beta_1^n(\langle\rangle)(D) = \varepsilon$, $\beta_2^n(\langle C\rangle)(d) = {}^{2\varepsilon}\!/_{1-\varepsilon}$, $\beta_3^n(I)(R) = \beta_3(I)(R) - \varepsilon \quad (\varepsilon = {}^1\!/_n)$.

Note: $\beta_1^n(\langle\rangle)(D) + (\beta_1^n(\langle\rangle)(C) \cdot \beta_2^n(\langle C\rangle)(d)) = 3\varepsilon$.

Summary

- Nash equilibria can be defined for extensive games, however, similiar to perfect information games, are not always reasonable.
- Sequential equilibria are the refinement, which is based an assessments (behavioral strategies + beliefs).
- Beliefs should be consistent with strategies.
- Strategies should be best responses in each information set, given beliefs.