Game Theory

4. Computational Complexity

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Motivation

Motivation: We already know some algorithms for finding Nash equilibria in restricted settings from the previous chapter, and upper bounds on their complexity.

- For finite zero-sum games: polynomial-time computation.
- For general finite two-player games: computation in **NP**.

Question: What about lower bounds for those cases and in general?

Approach to an answer: In this chapter, we study the computational complexity of finding Nash equilibria.

Finding Nash Equilibria as a Search Problem

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Definition (The problem of computing a Nash equilibrium)

NASH

Given: A finite two-player strategic game G.

Find: A mixed-strategy Nash equilibrium (α, β) of G.

Remarks:

- No need to add restriction "…if one exists, else 'fail'", because existence is guaranteed by Nash's theorem.
- The corresponding decision problem can be trivially solved in constant time (always return "true").
  Hence, we really need to consider the search problem version instead.

Finding Nash Equilibria as a Search Problem

In this form, NASH looks similar to other search problems, e. g.:

SAT

Given: A propositional formula φ in CNF.

Find: A truth assignment that makes φ true, if one exists, else 'fail'.

Note: This is the search version of the usual decision problem.

Search Problems

A search problem is given by a binary relation $R(x,y)$.

Definition (Search problem)

A search problem is a problem that can be stated in the following form, for a given binary relation $R(x,y)$ over strings:

SEARCH-R

Given: x.

Find: Some y such that $R(x,y)$ holds, if such a y exists, else ‘fail’.

Complexity Classes for Search Problems

Some complexity classes for search problems:

- **FP**: class of search problems that can be solved by a deterministic Turing machine in polynomial time.
- **FNP**: class of search problems that can be solved by a nondeterministic Turing machine in polynomial time.
- **TFNP**: class of search problems in **FNP** where the relation *R* is total, i. e., $\forall x \exists y . R(x, y)$.
- **PPAD**: class of search problems that can be polynomially reduced to END-OF-LINE.

  (PPAD: Polynomial Parity Argument in Directed Graphs)

To understand **PPAD**, we need to understand what the END-OF-LINE problem is.

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Definition (END-OF-LINE instance)

Consider a directed graph $\mathscr{G}$ with node set $\{0,1\}^n$ such that each node has in-degree and out-degree at most one and there are no isolated vertices. The graph $\mathscr{G}$ is specified by two polynomial-time computable functions π and σ:

- $\pi(v)$: returns the predecessor of v, or $\perp$ if v has no predecessor.
- $\sigma(v)$: returns the successor of v, or $\perp$ if v has no successor.

In $\mathscr{G}$, there is an arc from v to v' if and only if $\sigma(v) = v'$ and $\pi(v') = v$.

The End-of-Line Problem

Definition (End-of-Line instance (ctd.))

We call a triple (π, σ, v) consisting of such functions π and σ and a node v in $\mathscr{G}$ with in-degree zero (a "source") an End-of-Line instance.

With this, we can define the End-of-Line problem:

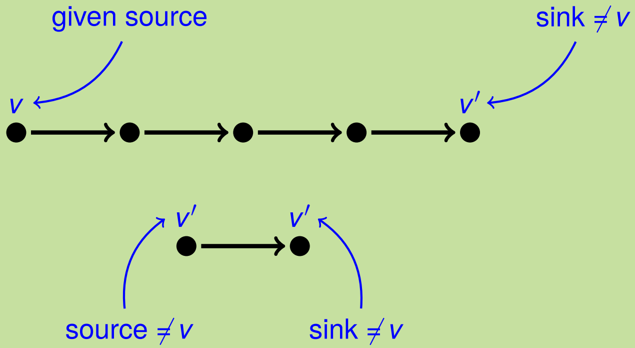
Definition (End-of-Line problem)

End-of-Line

Given: An End-of-Line instance (π, σ, v).

Find: Some node $v' \neq v$ such that v' has out-degree zero (a "sink") or in-degree zero (another "source").

Example (END-OF-LINE)



Comparison of Search Complexity Classes

Relationship of different search complexity classes:

$$FP \subseteq PPAD \subseteq TFNP \subseteq FNP$$

Compare to upper runtime bound that we already know:

Lemke-Howson algorithm has exponential time complexity in the worst case.

PPAD-Completeness of NASH

Theorem (Daskalakis et al., 2006)

NASH *is **PPAD**-complete.*

The same holds for k-player instead of just two-player NASH. □

Thus, NASH is presumably "simpler" than the SAT search problem, but presumably "harder" than any polynomial search problem.

FNP-Completeness of 2ND-NASH

Another search problem related to Nash equilibria is the problem of finding a second Nash equilibrium (given a first one has already been found). As it turns out, this is at least as hard as finding a first Nash equilibrium.

Definition (2ND-NASH problem)

2ND-NASH

Given: A finite two-player game G and a mixed-strategy Nash equilibrium of G.

Find: A second different mixed-strategy Nash equilibrium of G, if one exists, else 'fail'.

Theorem (Conitzer and Sandholm, 2003)

2ND-NASH *is **FNP**-complete.* □

Some Further Hardness Results

Theorem (Conitzer and Sandholm, 2003)

*For each of the following properties P^ℓ, $\ell = 1, 2, 3, 4$, given a finite two-player game G, it is **NP**-hard to decide whether there exists a mixed-strategy Nash equilibrium (α, β) in G that has property P^ℓ.*

P^1 : *player 1 (or 2) receives a payoff $\geq k$ for some given k.*
("Guaranteed payoff problem")

P^2 : $U_1(\alpha, \beta) + U_2(\alpha, \beta) \geq k$ *for some given k.*
("Guaranteed social welfare problem")

P^3 : *player 1 (or 2) plays some given action a with prob. > 0.*

P^4 : (α, β) *is Pareto-optimal, i. e., there is no strategy profile* (α', β') *such that*

- $U_i(\alpha', \beta') \geq U_i(\alpha, \beta)$ *for both $i \in \{1, 2\}$, and*
- $U_i(\alpha', \beta') > U_i(\alpha, \beta)$ *for at least one $i \in \{1, 2\}$.* □

Summary

- **PPAD** is the complexity class for which the End-of-Line problem is complete.
- Finding a mixed-strategy Nash equilibrium in a finite two-player strategic game is **PPAD**-complete.
- **FNP** is the search-problem equivalent of the class **NP**.
- Finding a second mixed-strategy Nash equilibrium in a finite two-player strategic game is **FNP**-complete.
- Several decision problems related to Nash equilibria are **NP**-complete:
  - guaranteed payoff
  - guaranteed social welfare
  - inclusion in support
  - Pareto-optimality of Nash equilibria