Introduction to Game Theory

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Summer semester 2020

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Exercise Sheet 9 — Solutions

Exercise 9.1 (May's theorem, 1 + 1 + 1 points)

May's theorem: A social choice function $f : L^n \to A$ for a set of two alternatives $A = \{x, y\}$ satisfies anonymity, neutrality and monotonicity iff it is the plurality method (i.e., $f(\prec_1, \dots, \prec_n) = x$ iff $\#\{i \mid y \prec_i x\} \geq \frac{n}{2}$). We assume n is odd to avoid tie-breaking issues that could violate neutrality. Show that each of the three conditions is necessary for May's theorem: For each condition, find a counterexample (a social choice function) that fulfills all other conditions but the one in question and that is not the plurality method.

(a) anonymity, i.e., $f(\prec_1, \dots, \prec_n) = f(\prec_{\pi(1)}, \dots, \prec_{\pi(n)})$ for all permutations π of the voters $\{1, \dots, n\}$.

Solution:

Counterexample:

$$f : (\prec_1, \dots, \prec_n) \mapsto \begin{cases} x, & \text{if } y \prec_1 x \\ y, & \text{otherwise} \end{cases}$$

(b) neutrality, i.e., $f(\prec_1, \dots, \prec_n) = x$ iff $f(\prec'_1, \dots, \prec'_n) = y$, where $x \prec'_i y$ iff $y \prec_i x$ for all $i = 1, \dots, n$.

Solution:

Counterexample:
$f : (\prec_1, \dots, \prec_n) \mapsto x$

(c) monotonicity, i.e., if $f(\prec_1, \dots, \prec_n) = x$, then also $f(\prec'_1, \dots, \prec'_n) = x$, where $\prec'_i = \prec_i$ for $i \neq I$ for some voter I such that $x \prec_I y$ and $y \prec'_I x$.

Solution:

Counterexample:

$$f : (\prec_1, \dots, \prec_n) \mapsto \begin{cases} x, & \text{if } |\{i|y \prec_i x\}| < n/2 \\ y, & \text{otherwise} \end{cases}$$

Exercise 9.2 (Single-Peaked Preferences; 2 + 1 + 2 points)

Alice, Bob, Carol, Dave, and Eve own a *shared* bank account and want to make a small donation of x Euros, $0 \leq x \leq 10$, for a good cause from that account. They all have different preferences of how much to donate, though. In fact, they all have unique preferred amounts $x_i^* \in [0, 10]$, $i \in \{A, B, C, D, E\}$, to donate, and the further they move away from those preferred amounts (either donating too little or too much), the less comfortable they feel with the particular amount. In other words, their preference relations are *single-peaked*. Specifically, they are:

$$v_A(x) = 7 - |x - 4|, \qquad v_B(x) = -\frac{1}{10}(x-3)^2 + 5, \qquad v_C(x) = \frac{1}{2}x + 3,$$
$$v_D(x) = 8 - (x-6)^4, \qquad v_E(x) = -\frac{1}{2}(x+2)^2 + 8 \ .$$

(a) Determine the preferred amounts $x_i^* \in [0, 10]$, $i \in \{A, B, C, D, E\}$.

(*Hint:* You do not necessarily have to look at derivatives of the functions v_i in order to determine their maxima. Sometimes they are easy to read off the function terms above.)

Solution:

$x_A = 4, x_B = 3, x_C = 10, x_D = 6, x_E = 0$

(b) To agree on a fixed amount of money $x^* \in [0, 10]$, Alice, Bob, Carol, Dave, and Eve take a vote in which each of them submits a single-peaked preference relation. On what amount of money will they agree using the *median rule*?

Solution:

Values: $0, 3, 4, 6, 10$, median: 4

(c) When using the median rule, does any of the five participants have an incentive to misrepresent their preference relation? If so, give an example. If not, why?

Solution:

No, because the median rule is incentive compatible for single-peaked preference relations, and all preference relations above are single-peaked.