Introduction to Game Theory

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Summer semester 2020

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Exercise Sheet 10 — Solutions

Exercise 10.1 (Vickrey-Clarke-Groves Mechanism; 2+2 points)

In a *k-item auction*, k identical items are to be sold. Each bidder $i = 1, \dots, n$ can get at most one of the items and has a privately known valuation w_i for the item. For simplicity, assume that $w_1 > w_2 > \cdots > w_n$. The set of alternatives $A = N_k$ consists of all k-ary subsets of players. Each alternative represents the players who will receive an item.

(a) Formalize the k-item auction as a VCG mechanism $\mathcal{M} = \langle f, (p_i)_{i\in N}\rangle$ that uses *Clarke pivot functions.*

Solution:

The players valuations over the alternatives $a \in A$ are

$$v_i(a) = \begin{cases} w_i, & \text{if } i \in a. \\ 0, & \text{otherwise.} \end{cases}$$

$$f(v_1, \dots, v_n) = \{i \in N | 1 \leq i \leq k\}$$

$$p_i(a) = \begin{cases} w_{k+1}, & \text{if } i \in a. \\ 0, & \text{otherwise.} \end{cases}$$

(b) Consider the mechanism $\mathcal{M}' = \langle f', (p'_i)_{i\in N}\rangle$ implementing a k-item auction, with

- social choice function $f'(v_1, \dots, v_n) = \{i \in N \mid 1 \leq i \leq k\}$, and
- payment functions $p'_i(a) = \begin{cases} w_{i+1}, & \text{if } i \in a, \\ 0, & \text{otherwise,} \end{cases}$ for all $a \in A$.

Here, the i-th highest bidding winner has to pay the $(i + 1)$-st highest bid, i.e., the highest bidding player pays the second highest bid, the second highest bidder pays the third highest bid, and so on. Non-winning players pay nothing. Construct a counterexample with only three bidders that proves that $\mathcal{M}'$ is *not* incentive compatible.

Solution:

Let $k = 2$. Consider the following players' privately known valuations: $w_1 = 10, w_2 = 8, w_3 = 0$ with $p'_1(v_1, v_2, v_3) = 8$. Now imagine player 1 deviates from her true valuation in the following way: $v'_1 = 7$ with $p'_1(v'_1, v_2, v_3) = 0$. Therefore: $u_1(f'(v_1, v_2, v_3)) = 2 < 10 = u_1(f'(v'_1, v_2, v_3))$.

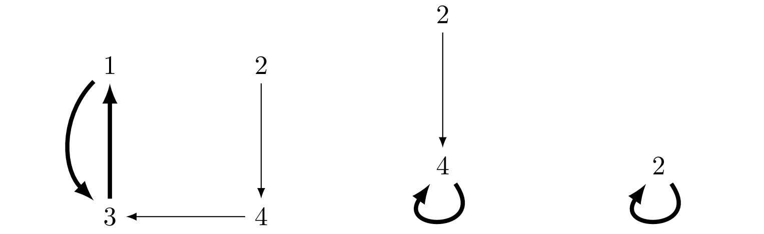
Exercise 10.2 (Top trading cycle method, 2 points)

(a) Apply the top trading cycle algorithm to the following problem and state what happens in the iterations:

- *Player 1:* $1 \lhd_1 4 \lhd_1 2 \lhd_1 3$
- *Player 2:* $3 \lhd_2 2 \lhd_2 1 \lhd_2 4$
- *Player 3:* $2 \lhd_3 3 \lhd_3 4 \lhd_3 1$
- *Player 4:* $2 \lhd_4 1 \lhd_4 4 \lhd_4 3$

Preferences are given from lowest (left) to highest (right).

Solution:

Round 1	Round 2	Round 3
		
Assign $\pi(1) = 3, \pi(3) = 1$.	Assign $\pi(4) = 4$	Assign $\pi(2) = 2$

Assignment: $\pi = (3, 2, 1, 4)$

Exercise 10.3 (Stable matchings, 2 points)

Apply the deferred acceptance algorithm with male proposals to the following problem and state what happens in the iterations:

- *Man 1:* $w_4 \prec_{m_1} w_3 \prec_{m_1} w_1 \prec_{m_1} w_2$
- *Man 2:* $w_3 \prec_{m_2} w_2 \prec_{m_2} w_1 \prec_{m_2} w_4$
- *Man 3:* $w_4 \prec_{m_3} w_2 \prec_{m_3} w_3 \prec_{m_3} w_1$
- *Man 4:* $w_4 \prec_{m_4} w_1 \prec_{m_4} w_3 \prec_{m_4} w_2$
- *Woman 1:* $m_4 \prec_{w_1} m_2 \prec_{w_1} m_3 \prec_{w_1} m_1$
- *Woman 2:* $m_2 \prec_{w_2} m_1 \prec_{w_2} m_4 \prec_{w_2} m_3$
- *Woman 3:* $m_1 \prec_{w_3} m_3 \prec_{w_3} m_2 \prec_{w_3} m_4$
- *Woman 4:* $m_4 \prec_{w_4} m_1 \prec_{w_4} m_2 \prec_{w_4} m_3$

Preferences are given from lowest (left) to highest (right).

Solution:

i	w_1	w_2	w_3	w_4
1	m_3	m_1, m_4		m_2
2	m_1, m_3	m_4		m_2
3	m_1	m_4	m_3	m_2

Matching: $\{\langle m_1, w_1\rangle, \langle m_2, w_4\rangle, \langle m_3, w_3\rangle, \langle m_4, w_2\rangle\}$