Game Theory

2. Strategic Games

2.1. Preliminaries and Examples

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Strategic Games

Informally:

- one-shot games of finitely many players with given action sets and payoff functions
- perfect information

Strategic Games

Definition (Strategic game)

A strategic game is a tuple $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where

- a nonempty finite set N of players,
- for each player $i \in N$, a nonempty set A_i of actions (or strategies), and
- for each player $i \in N$, a payoff function $u_i : A \to \mathbb{R}$, where $A = \prod_{i \in N} A_i$.

A strategic game G is called finite if A is finite.

A strategy profile is a tuple $a = (a_1, \dots, a_{|N|}) \in A$.

Strategic Games

We can describe finite strategic games using payoff matrices.

Example: Two-player game where player 1 has actions T and B, and player 2 has actions L and R, with payoff matrix

player 2

		L	R
player 1	T	w_1, w_2	x_1, x_2
	B	y_1, y_2	z_1, z_2

Read: **If** player 1 plays T and player 2 plays L
then player 1 gets payoff w_1 and player 2 gets payoff w_2, etc.

Prisoner’s Dilemma

Example (Prisoner’s Dilemma (informally))

Two prisoners are interrogated separately, and have the options to either cooperate (C) with their fellow prisoner and stay silent, or defect (D) and accuse the fellow prisoner of the crime.

Possible outcomes:

- Both cooperate: no hard evidence against either of them, only short prison sentences for both.
- One cooperates, the other defects: the defecting prisoner is set free immediately, and the cooperating prisoner gets a very long prison sentence.
- Both confess: both get medium-length prison sentences.

Prisoner's Dilemma

Example (Prisoner's Dilemma (payoff matrix))

Strategies $A_1 = A_2 = \{C, D\}$.

		player 2	
		C	D
player 1	C	$3, 3$	$0, 4$
	D	$4, 0$	$1, 1$

Hawk and Dove

An anti-coordination game:

Example (Hawk and Dove (informally))

In a fight for resources two players can behave either like a dove (*D*), yielding, or like a hawk (*H*), attacking.

Possible outcomes:

- Both players behave like doves: both players share the benefit.
- A hawk meets a dove: the hawk wins and gets the bigger part.
- Both players behave like hawks: the benefit gets lost completely because they will fight each other.

Hawk and Dove

Example (Hawk and Dove (payoff matrix))

Strategies $A_1 = A_2 = \{D, H\}$.

		player 2	
		D	*H*
player 1	*D*	$3,3$	$1,4$
	H	$4,1$	$0,0$

Matching Pennies

A strictly competitive game:

Example (Matching Pennies (informally))

Two players can choose either heads (H) or tails (T) of a coin.

Possible outcomes:

- Both players make the same choice: player 1 receives one Euro from player 2.
- The players make different choices: player 2 receives one Euro from player 1.

Matching Pennies

Example (Matching Pennies (payoff matrix))

Strategies $A_1 = A_2 = \{H, T\}$.

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\text{player 2}} \\
 & & H & T \\
\hline
\text{player 1} & H & 1, -1 & -1, \ 1 \\
\hline
 & T & -1, \ 1 & 1, -1 \\
\hline
\end{array}$$

Bach or Stravinsky (aka Battle of the Sexes)

A coordination game:

Example (Bach or Stravinsky (informally))

Two persons, one of whom prefers Bach whereas the other prefers Stravinsky want to go to a concert together. For both it is more important to go to the same concert than to go to their favorite one. Let *B* be the action of going to the Bach concert and *S* the action of going to the Stravinsky concert.

Possible outcomes:

- Both players make the same choice: the player whose preferred option is chosen gets high payoff, the other player gets medium payoff.
- The players make different choices: they both get zero payoff.

Bach or Stravinsky (aka Battle of the Sexes)

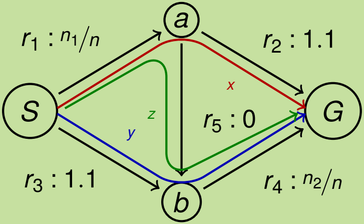
Example (Bach or Stravinsky (payoff matrix))

Strategies $A_1 = A_2 = \{B, S\}$.

		Stravinsky enthusiast	
		B	*S*
Bach enthusiast	*B*	2, 1	0, 0
	S	0, 0	1, 2

Congestion Game

Example (A congestion game)



player 2

		x	y	z
	x	−2.1, −2.1	−1.6, −1.6	−2.1, −1.5
player 1	y	−1.6, −1.6	−2.1, −2.1	−2.1, −1.5
	z	−1.5, −2.1	−1.5, −2.1	−2, −2

Notation

We want to write down strategy profiles where one player's strategy is removed or replaced.

Let $a = (a_1, \ldots, a_{|N|}) \in A = \prod_{i \in N} A_i$ be a strategy profile.

We write:

- $A_{-i} := \prod_{j \in N \setminus \{i\}} A_j$,
- $a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{|N|})$, and
- $(a_{-i}, a'_i) := (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots, a_{|N|})$.

Example

Let $A_1 = \{T, B\}$, $A_2 = \{L, R\}$, $A_3 = \{X, Y, Z\}$, and $a := (T, R, Z)$.
Then $a_{-1} = (R, Z)$, $a_{-2} = (T, Z)$, $a_{-3} = (T, R)$.
Moreover, $(a_{-2}, L) = (T, L, Z)$.

Game Theory

2. Strategic Games

2.2. Strict Dominance

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Solution Concepts

Question: What is a “solution” of a strategic game?

Answer:

- a strategy profile where all players play strategies that are rational (i. e., in some sense optimal)
- note: different ways of making the above item precise (different solution concepts)
- solution concept: formal rule for predicting how a game will be played

In the following, we will consider some solution concepts:

- strict and weak dominance
- Nash equilibria
- maximinimizers

Strictly Dominated Strategies

Question: What strategy should an agent avoid?

One answer: obviously irrational strategies (can be eliminated)

A strategy is obviously irrational if there is another strategy that is always better, no matter what the other players do.

Strictly Dominated Strategies

Definition (Strictly dominated strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A strategy $a_i \in A_i$ is called strictly dominated in G if there is a strategy $a_i^+ \in A_i$ such that for all strategy profiles $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that a_i^+ strictly dominates a_i.

If $a_i^+ \in A_i$ strictly dominates every other strategy $a_i' \in A_i \setminus \{a_i^+\}$, we call a_i^+ strictly dominant in G.

Remark: Playing strictly dominated strategies is irrational.

Strictly Dominated Strategies

This suggest a solution concept:
iterative elimination of strictly dominated strategies:

```
while some strictly dominated strategy is left:
    eliminate some strictly dominated strategy
if a unique strategy profile remains:
    this unique profile is the solution
```

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		C	*D*
player 1	*C*	3, 3	0, 4
	D	4, 0	1, 1

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for the prisoner's dilemma)

		player 2	
		C	*D*
player 1	*C*	$3,3$	$0,4$
	D	$4,0$	$1,1$

- Step 1: eliminate row *C* (strictly dominated by row *D*)
- Step 2: eliminate column *C* (strictly dominated by col. *D*)

Strictly Dominated Strategies

Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	R
player 1	T	$2, 1$	$0, 0$
	M	$1, 2$	$2, 1$
	B	$0, 0$	$1, 1$

Strictly Dominated Strategies

Example (Iterative elim. of strictly dominated strategies)

		player 2	
		L	*R*
	T	2, 1	0, 0
player 1	*M*	1, 2	2, 1
	B	0, 0	1, 1

- Step 1: eliminate row *B* (strictly dominated by row *M*)
- Step 2: eliminate column *R* (strictly dominated by col. *L*)
- Step 3: eliminate row *M* (strictly dominated by row *T*)

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

Stravinsky enthusiast

Bach enthusiast

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

Strictly Dominated Strategies

Example (Iterative elimination of strictly dominated strategies for Bach or Stravinsky)

Stravinsky enthusiast

		B	S
Bach enthusiast	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

- No strictly dominated strategies.
- All strategies survive iterative elimination of strictly dominated strategies.
- All strategies rationalizable.

Strictly Dominated Strategies

Remark

Strict dominance between actions is rather rare.
We should identify more constraints on “solutions”, better solution concepts.

Proposition

The result of iterative elimination of strictly dominated strategies is unique, i. e., independent of the elimination order.

Proof.

Homework. □

Game Theory

2. Strategic Games

2.3. Weak Dominance

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Solution Concepts

Question: What is a “solution” of a strategic game?

Answer:

- a strategy profile where all players play strategies that are rational (i. e., in some sense optimal)
- note: different ways of making the above item precise (different solution concepts)
- solution concept: formal rule for predicting how a game will be played

In the following, we will consider some solution concepts:

- strict and weak dominance
- Nash equilibria
- maximinimizers

Weakly Dominated Strategies

Definition (Weakly dominated strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A strategy $a_i \in A_i$ is called weakly dominated in G if there is a strategy $a_i^+ \in A_i$ such that for all profiles $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) \leq u_i(a_{-i}, a_i^+)$$

and that for at least one profile $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i) < u_i(a_{-i}, a_i^+).$$

We say that a_i^+ weakly dominates a_i.

If $a_i^+ \in A_i$ weakly dominates every other strategy $a_i' \in A_i \setminus \{a_i^+\}$, we call a_i^+ weakly dominant in G.

Weakly Dominated Strategies

What about
iterative elimination of weakly dominated strategies
as a solution concept?

Let's see what happens.

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	*R*
player 1	*T*	2,1	0,0
	M	2,1	1,1
	B	0,0	1,1

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

player 2

		L	R
	T	$2, 1$	$0, 0$
player 1	M	$2, 1$	$1, 1$
	B	$0, 0$	$1, 1$

- Step 1: eliminate row B (weakly dominated by row M, $u_1(M, L) = 2 > 0 = u_1(B, L)$ and $u_1(M, R) = 1 = u_1(B, R)$)
- Step 2: eliminate column R (weakly dominated by col. L)

Here, two solution profiles remain.

Weakly Dominated Strategies

Iterative elimination of weakly dominated strategies:

- leads to smaller games,
- can also lead to situations where only a single solution remains,
- but: the result can depend on the elimination order! (see example on next slide)

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

player 2

		L	R
	T	$2,1$	$0,0$
player 1	M	$2,1$	$1,1$
	B	$0,0$	$1,1$

Weakly Dominated Strategies

Example (Iterative elim. of weakly dominated strategies)

		player 2	
		L	R
	T	$2, 1$	$0, 0$
player 1	M	$2, 1$	$1, 1$
	B	$0, 0$	$1, 1$

- Step 1: eliminate row T (weakly dominated by row M)
- Step 2: eliminate column L (weakly dominated by col. R)

Different elimination order, different result,
even different payoffs ($1, 1$ vs. $2, 1$)!

Weakly Dominated Strategies

Consequence:

Iterative elimination of weakly dominated strategies not such a useful solution concept.

Let's look for something more useful.

Game Theory

2. Strategic Games

2.4. Nash Equilibria

2.4.1. Definitions and Examples

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Solution Concepts

Question: What is a “solution” of a strategic game?

Answer:

- a strategy profile where all players play strategies that are rational (i. e., in some sense optimal)
- note: different ways of making the above item precise (different solution concepts)
- solution concept: formal rule for predicting how a game will be played

In the following, we will consider some solution concepts:

- strict and weak dominance
- Nash equilibria
- maximinimizers

Nash Equilibria

Question: Which strategy profiles are stable?

Possible answer:

- strategy profiles where no player benefits from playing a different strategy
- equivalently: strategy profiles where every player's strategy is a best response to the other players' strategies

Such strategy profiles are called Nash equilibria, one of the most-used solution concepts in game theory.

Remark: In following examples, for non-Nash equilibria, only one possible profitable deviation is shown (even if there are more).

Nash Equilibria

Definition (Nash equilibrium)

A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is a strategy profile $a^* \in A$ such that for every player $i \in N$,

$$u_i(a^*) \geq u_i(a^*_{-i}, a_i) \qquad \text{for all } a_i \in A_i.$$

Nash Equilibria

Remark: There is an alternative definition of Nash equilibria (which we consider because it gives us a slightly different perspective on Nash equilibria).

Definition (Best response)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game, $i \in N$ a player, and $a_{-i} \in A_{-i}$ a strategy profile of the players other than i. Then a strategy $a_i \in A_i$ is a best response of player i to a_{-i} if

$$u_i(a_{-i}, a_i) \geq u_i(a_{-i}, a_i') \qquad \text{for all } a_i' \in A_i.$$

We write $B_i(a_{-i})$ for the set of best responses of player i to a_{-i}.

For a strategy profile $a \in A$, we write $B(a) = \prod_{i \in N} B_i(a_{-i})$.

Nash Equilibria

Definition (Nash equilibrium, alternative 1)

A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a strategy profile $a^* \in A$ such that for every player $i \in N$, $a_i^* \in B_i(a_{-i}^*)$.

Definition (Nash equilibrium, alternative 2)

A Nash equilibrium of a strategic game $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a strategy profile $a^* \in A$ such that $a^* \in B(a^*)$.

Proposition

The three definitions of Nash equilibria are equivalent.

Proof.

Homework. □

Nash Equilibria

Example (Nash Equilibria in the Prisoner's Dilemma)

		player 2	
		C	D
player 1	C	$3,3$	$0,4$
	D	$4,0$	$1,1$

- (C,C): no Nash equilibrium (player 1: $C \to D$)
- (C,D): no Nash equilibrium (player 1: $C \to D$)
- (D,C): no Nash equilibrium (player 2: $C \to D$)
- (D,D): Nash equilibrium!

Nash Equilibria

Example (Nash Equilibria in Hawk and Dove)

player 2

		D	H
player 1	D	$3,3$	$1,4$
	H	$4,1$	$0,0$

- (D,D): no Nash equilibrium (player 1: $D \rightarrow H$)
- (D,H): Nash equilibrium!
- (H,D): Nash equilibrium!
- (H,H): no Nash equilibrium (player 1: $H \rightarrow D$)

Nash Equilibria

Example (Nash Equilibria in Matching Pennies)

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\text{player 2}} \\
 & & H & T \\
\hline
\text{player 1} & H & 1,-1 & -1,\ 1 \\
\hline
 & T & -1,\ 1 & 1,-1 \\
\hline
\end{array}$$

- (H,H): no Nash equilibrium (player 2: $H \rightarrow T$)
- (H,T): no Nash equilibrium (player 1: $H \rightarrow T$)
- (T,H): no Nash equilibrium (player 1: $T \rightarrow H$)
- (T,T): no Nash equilibrium (player 2: $T \rightarrow H$)

Nash Equilibria

Example (Nash Equilibria in Bach or Stravinsky)

		Stravinsky enthusiast	
		B	S
Bach enthusiast	B	$2,1$	$0,0$
	S	$0,0$	$1,2$

- (B,B): Nash equilibrium!
- (B,S): no Nash equilibrium (player 1: $B \to S$)
- (S,B): no Nash equilibrium (player 2: $S \to B$)
- (S,S): Nash equilibrium!

Game Theory

2. Strategic Games

2.4. Nash Equilibria

2.4.2. Example: NEs in Sealed-bid Auctions

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Example: Sealed-Bid Auctions

We consider a slightly larger example: sealed-bid auctions

Setting:

- An object has to be assigned to a winning bidder in exchange for a payment.
- For each player ("bidder") $i = 1, \ldots, n$, let v_i be the private value that bidder i assigns to the object.
  (We assume that $v_1 > v_2 > \cdots > v_n > 0$.)
- The bidders simultaneously give their bids $b_i \geq 0$, $i = 1, \ldots, n$.
- The object is given to the bidder i with the highest bid b_i.
  (Ties are broken in favor of bidders with lower index, i.e., if $b_i = b_j$ are the highest bids, then bidder i will win iff $i < j$.)

Example: Sealed-Bid Auctions

Question: What should the winning bidder have to pay?

One possible answer: the highest bid.

Definition (First-price sealed-bid auction)

- $N = \{1, \ldots, n\}$ with $v_1 > v_2 > \cdots > v_n > 0$,
- $A_i = \mathbb{R}_0^+$ for all $i \in N$,
- Bidder $i \in N$ wins if b_i is maximal among all bids (+ possible tie-breaking by index), and
- $u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - b_i & \text{otherwise} \end{cases}$
  where $b = (b_1, \ldots, b_n)$.

Example: Sealed-Bid Auctions

Example (First-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids

$$\begin{array}{lll} v_1 = 100, & v_2 = 80, & v_3 = 53, \\ b_1 = 90, & b_2 = 85, & b_3 = 45. \end{array}$$

Observations:

- Bidder 1 wins, pays 90, gets utility
  $u_1(b) = v_1 - b_1 = 100 - 90 = 10$.
- Bidders 2 and 3 pay nothing, get utility 0.
- (Bidder 2 over-bids.)
- Bidder 1 could still win, but pay less, by bidding $b'_1 = 85$ instead. Then $u_1(b_{-1}, b'_1) = v_1 - b'_1 = 100 - 85 = 15$.

Example: Sealed-Bid Auctions

Question: How to avoid untruthful bidding and incentivize truthful revelation of private valuations?

Different answer to question about payments: Winner pays the second-highest bid.

Definition (Second-price sealed-bid auction)

- $N = \{1, \dots, n\}$ with $v_1 > v_2 > \dots > v_n > 0$,
- $A_i = \mathbb{R}_0^+$ for all $i \in N$,
- Bidder $i \in N$ wins if b_i is maximal among all bids (+ possible tie-breaking by index), and
- $u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ v_i - \max b_{-i} & \text{otherwise} \end{cases}$
  where $b = (b_1, \dots, b_n)$.

Example: Sealed-Bid Auctions

Example (Second-price sealed-bid auction)

Assume three bidders 1, 2, and 3, with valuations and bids

$$v_1 = 100, \qquad v_2 = 80, \qquad v_3 = 53,$$
$$b_1 = 90, \qquad b_2 = 85, \qquad b_3 = 45.$$

Observations:

- Bidder 1 wins, pays 85, gets utility $u_1(b) = v_1 - b_2 = 100 - 85 = 15$.
- Bidders 2 and 3 pay nothing, get utility 0.
- Bidder 1 has no incentive to bid strategically and guess the other bidders' private valuations.

Example: Sealed-Bid Auctions

Proposition

In a second-price sealed-bid auction, bidding ones own valuation, $b_i^+ = v_i$, is a weakly dominant strategy.

Example: Sealed-Bid Auctions

Proposition

In a second-price sealed-bid auction, bidding ones own valuation, $b_i^+ = v_i$, is a weakly dominant strategy.

Proof.

We have to show that b_i^+ weakly dominates every other strategy b_i of player i.

Example: Sealed-Bid Auctions

Proposition

In a second-price sealed-bid auction, bidding ones own valuation, $b_i^+ = v_i$, is a weakly dominant strategy.

Proof.

We have to show that b_i^+ weakly dominates every other strategy b_i of player i.

For that, it suffices to show that

1. for all $b_i \in A_i$, we have
   $$u_i(b_{-i}, b_i^+) \geq u_i(b_{-i}, b_i) \text{ for all } b_{-i} \in A_{-i}, \text{ and that}$$
2. for all $b_i \in A_i \setminus \{b_i^+\}$, we have
   $$u_i(b_{-i}, b_i^+) > u_i(b_{-i}, b_i) \text{ for at least one } b_{-i} \in A_{-i}.$$

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (1) [regardless of what the other bidders do, b_i^+ is always a best response]:

- Case I) bidder *i* wins:

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (1) [regardless of what the other bidders do,
b_i^+ is always a best response]:

- Case I) bidder *i* wins:
  bidder *i* pays $\max b_{-i} \leq v_i$, gets $u_i(b_{-i}, b_i^+) \geq 0$.
  - Case I.a) bidder *i* decreases bid:
    this does not help, since he might still win and pay the same as before, or lose and get utility 0.
  - Case I.b) bidder *i* increases bid:
    bidder *i* still wins and pays the same as before.

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (1) (ctd.):

- Case II) bidder *i* loses:

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (1) (ctd.):

- Case II) bidder *i* loses:
  bidder *i* pays nothing, gets $u_i(b_{-i}, b_i^+) = 0$.
  - Case II.a) bidder *i* decreases bid:
    bidder *i* still loses and gets utility 0.
  - Case II.b) bidder *i* increases bid:
    either bidder *i* still loses and gets utility 0, or becomes the winner and pays more than the object is worth to him, leading to a negative utility.

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (2) [for each alternative b_i to b_i^+, there is an opponent profile b_{-i} against which b_i^+ is strictly better than b_i]:

Let b_i be some strategy other than b_i^+.

- Case I) $b_i < b_i^+$:

☐

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (2) [for each alternative b_i to b_i^+, there is an opponent profile b_{-i} against which b_i^+ is strictly better than b_i]:

Let b_i be some strategy other than b_i^+.

- Case I) $b_i < b_i^+$:
  Consider b_{-i} with $b_i < \max b_{-i} < b_i^+$.
  With b_i, bidder i does not win any more, i. e., we have
  $u_i(b_{-i}, b_i^+) > 0 = u_i(b_{-i}, b_i)$.

☐

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (2) (ctd.):

Let b_i be some strategy other than b_i^+.

- Case II) $b_i > b_i^+$:

☐

Example: Sealed-Bid Auctions

Proof (ctd.)

Ad (2) (ctd.):

Let b_i be some strategy other than b_i^+.

- Case II) $b_i > b_i^+$:
  Consider b_{-i} with $b_i > \max b_{-i} > b_i^+$.
  With b_i, bidder i overbids and pays more than the object is worth to him, i. e., we have $u_i(b_{-i}, b_i^+) = 0 > u_i(b_{-i}, b_i)$.

$\square$

Example: Sealed-Bid Auctions

Proposition

Profiles of weakly dominant strategies are Nash equilibria.

Proof.

Homework. ☐

Proposition

In a second-price sealed-bid auction, if all bidders bid their true valuations, this is a Nash equilibrium.

Proof.

Follows immediately from the previous two propositions. ☐

Remark: This is not the only Nash equilibrium in second-price sealed-bid auctions, though.

Game Theory

2. Strategic Games

2.5. Strict Dominance vs. Nash Equilibria

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Iterative Elimination and Nash Equilibria

Motivation: We have seen two different solution concepts,

- surviving iterative elimination of strictly dominated strategies
- Nash equilibria

Obvious question: Is there any relationship between the two?

Answer: Yes, Nash equilibria refine the concept of iterative elimination of strictly dominated strategies. We will formalize this on the next slides.

Iterative Elimination and Nash Equilibria

Lemma (preservation of Nash equilibria)

Let G and G' be two strategic games where G' is obtained from G by elimination of one strictly dominated strategy. Then a strategy profile a^* is a Nash equilibrium of G if and only if it is Nash equilibrium of G'.

Proof.

Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ and $G' = \langle N, (A_i')_{i\in N}, (u_i')_{i\in N}\rangle$.

Let a_i' be the eliminated strategy.
Then there is a strategy a_i^+ such that for all $a_{-i} \in A_{-i}$,

$$u_i(a_{-i}, a_i') < u_i(a_{-i}, a_i^+). \tag{1}$$

Iterative Elimination and Nash Equilibria

Proof (ctd.)

"$\Rightarrow$": Let a^* be a Nash equilibrium of G.

- Nash equilibrium strategies are not eliminated: For players $j \neq i$, this is clear, because none of their strategies are eliminated.

Iterative Elimination and Nash Equilibria

Proof (ctd.)

"$\Rightarrow$": Let a^* be a Nash equilibrium of G.

- Nash equilibrium strategies are not eliminated: For players $j \neq i$, this is clear, because none of their strategies are eliminated.

  For player i, action a_i^* is a best response to a_{-i}^*, and in particular at least as good a response as a_i^+:

  $$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+).$$

Iterative Elimination and Nash Equilibria

Proof (ctd.)

"$\Rightarrow$": Let a^* be a Nash equilibrium of G.

- Nash equilibrium strategies are not eliminated: For players $j \neq i$, this is clear, because none of their strategies are eliminated.

  For player i, action a_i^* is a best response to a_{-i}^*, and in particular at least as good a response as a_i^+:

  $$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+).$$

  With (1) $u_i(a_{-i}, a_i^+) > u_i(a_{-i}, a_i')$, we get $u_i(a_{-i}^*, a_i^*) > u_i(a_{-i}^*, a_i')$ and hence $a_i^* \neq a_i'$.

Iterative Elimination and Nash Equilibria

Proof (ctd.)

"$\Rightarrow$": Let a^* be a Nash equilibrium of G.

- Nash equilibrium strategies are not eliminated: For players $j \neq i$, this is clear, because none of their strategies are eliminated.

  For player i, action a_i^* is a best response to a_{-i}^*, and in particular at least as good a response as a_i^+:

  $$u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+).$$

  With (1) $u_i(a_{-i}, a_i^+) > u_i(a_{-i}, a_i')$, we get $u_i(a_{-i}^*, a_i^*) > u_i(a_{-i}^*, a_i')$ and hence $a_i^* \neq a_i'$.

  Thus, the Nash equilibrium strategy a_i^* is not eliminated.

Iterative Elimination and Nash Equilibria

Proof (ctd.)

“$\Rightarrow$” (ctd.):

- Best responses remain best responses: For all players $j \in N$, a_j^* is a best response to a_{-j}^* in G. Since in G', no potentially better responses are introduced ($A_j' \subseteq A_j$) and the payoffs are unchanged, this also holds in G'.

Hence, a^* is also a Nash equilibrium of G'.

Iterative Elimination and Nash Equilibria

Proof (ctd.)

"$\Rightarrow$" (ctd.):

- Best responses remain best responses: For all players $j \in N$, a_j^* is a best response to a_{-j}^* in G. Since in G', no potentially better responses are introduced ($A_j' \subseteq A_j$) and the payoffs are unchanged, this also holds in G'.

Hence, a^* is also a Nash equilibrium of G'.

"$\Leftarrow$": Let a^* be a Nash equilibrium of G'.

- For player $j \neq i$: a_j^* is a best response to a_{-j}^* in G as well, since the responses available to player j in G and G' are the same.

Iterative Elimination and Nash Equilibria

Proof (ctd.)

“$\Leftarrow$” (ctd.):

- For player i: Since $A_i = A'_i \cup \{a_i\}$ and a^*_i is a best response to a^*_{-i} among the strategies in A'_i, it suffices to show that a_i is no better response.

Iterative Elimination and Nash Equilibria

Proof (ctd.)

“$\Leftarrow$” (ctd.):

- For player i: Since $A_i = A'_i \cup \{a_i\}$ and a^*_i is a best response to a^*_{-i} among the strategies in A'_i, it suffices to show that a_i is no better response.

  Because a^* is a Nash equilibrium in G' and a^+_i is a strategy in A'_i, we have $u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a^+_i)$.

Iterative Elimination and Nash Equilibria

Proof (ctd.)

“$\Leftarrow$” (ctd.):

- For player i: Since $A_i = A'_i \cup \{a_i\}$ and a_i^* is a best response to a_{-i}^* among the strategies in A'_i, it suffices to show that a_i is no better response.

  Because a^* is a Nash equilibrium in G' and a_i^+ is a strategy in A'_i, we have $u_i(a_{-i}^*, a_i^*) \geq u_i(a_{-i}^*, a_i^+)$.

  Since a_i^+ strictly dominates a_i, we have $u_i(a_{-i}^*, a_i^+) > u_i(a_{-i}^*, a_i)$, and hence $u_i(a_{-i}^*, a_i^*) > u_i(a_{-i}^*, a_i)$.

Iterative Elimination and Nash Equilibria

Proof (ctd.)

“$\Leftarrow$” (ctd.):

- For player i: Since $A_i = A'_i \cup \{a_i\}$ and a^*_i is a best response to a^*_{-i} among the strategies in A'_i, it suffices to show that a_i is no better response.

  Because a^* is a Nash equilibrium in G' and a^+_i is a strategy in A'_i, we have $u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a^+_i)$.

  Since a^+_i strictly dominates a_i, we have $u_i(a^*_{-i}, a^+_i) > u_i(a^*_{-i}, a_i)$, and hence $u_i(a^*_{-i}, a^*_i) > u_i(a^*_{-i}, a_i)$.

  Therefore, a_i cannot be a better response to a^*_{-i} than a^*_i.

Hence, a^* is also a Nash equilibrium of G. ☐

Iterative Elimination and Nash Equilibria

Corollary

If iterative elimination of strictly dominated strategies results in a *unique* strategy profile a^*, then a^* is the unique Nash equilibrium of the original game.

Proof.

Assume that a^* is the unique remaining strategy profile. By definition, a^* must be a Nash equilibrium of the remaining game.

Iterative Elimination and Nash Equilibria

Corollary

If iterative elimination of strictly dominated strategies results in a *unique* strategy profile a^*, then a^* is the unique Nash equilibrium of the original game.

Proof.

Assume that a^* is the unique remaining strategy profile. By definition, a^* must be a Nash equilibrium of the remaining game.

We can inductively apply the previous lemma (preservation of Nash equilibria) and see that a^* (and no other strategy profile) must have been a Nash equilibrium before the last elimination step, and before that step, ..., and in the original game. □

Game Theory

2. Strategic Games

2.6. Zero-Sum Games

2.6.1. Definition, Examples, Maximinimizers

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Playing it Safe (in Two-Player Games)

Motivation: What happens if both players try to “play it safe”?

Question: What does it even mean to “play it safe”?

Answer: Choose a strategy that guarantees the highest worst-case payoff.

Playing it Safe (in Two-Player Games)

Example

player 2

		L	*R*
	T	2, 1	2, −20
player 1	*M*	3, 0	−10, 1
	B	−100, 2	3, 3

Playing it Safe (in Two-Player Games)

Example

player 2

		L	R
	T	$2, 1$	$2, -20$
player 1	M	$3, 0$	$-10, 1$
	B	$-100, 2$	$3, 3$

Worst-case payoff for player 1:

- if playing T: 2
- if playing M: -10
- if playing B: -100

$\rightsquigarrow$ play T.

Worst-case payoff for player 2:

- if playing L: 0
- if playing R: -20

$\rightsquigarrow$ play L.

Playing it Safe (in Two-Player Games)

Example

player 2

		L	R
	T	$2, 1$	$2, -20$
player 1	M	$3, 0$	$-10, 1$
	B	$-100, 2$	$3, 3$

Worst-case payoff for player 1:

- if playing T: 2
- if playing M: -10
- if playing B: -100

$\rightsquigarrow$ play T.

Worst-case payoff for player 2:

- if playing L: 0
- if playing R: -20

$\rightsquigarrow$ play L.

However: Unlike (B, R), the profile (T, L) is not a Nash equilibrium.

Playing it Safe (in Two-Player Games)

Observation: In general, pairs of maximinimizers, like (T, L) in the example above, are not the same as Nash equilibria.

Claim: However, in zero-sum games, pairs of maximinimizers and Nash equilibria are essentially the same.

(Tiny restriction: This does not hold if the considered game has no Nash equilibrium at all, because unlike Nash equilibria, pairs of maximinimizers always exist.)

Reason (intuitively): In zero-sum games, the worst-case assumption that the other player tries to harm you as much as possible is justified, because harming the other is the same as maximizing ones own payoff. Playing it safe is rational.

Zero-Sum Games

Definition (Zero-sum game)

A zero-sum game is a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and

$$u_1(a) = -u_2(a)$$

for all $a \in A$.

Example (Matching Pennies as a zero-sum game)

		player 2	
		H	T
player 1	H	$1, -1$	$-1, \ 1$
	T	$-1, \ 1$	$1, -1$

Maximinimizers

Definition (Maximinimizer)

Let $G = \langle \{1,2\}, (A_i)_{i\in N}, (u_i)_{i\in N} \rangle$ be a zero-sum game.

An action $x^* \in A_1$ is called maximinimizer for player 1 in G if

$$\min_{y\in A_2} u_1(x^*,y) \geq \min_{y\in A_2} u_1(x,y) \qquad \text{for all } x \in A_1,$$

and $y^* \in A_2$ is called maximinimizer for player 2 in G if

$$\min_{x\in A_1} u_2(x,y^*) \geq \min_{x\in A_1} u_2(x,y) \qquad \text{for all } y \in A_2.$$

Maximinimizers

Example (Zero-sum game with three actions each)

player 2

		L	C	R
	T	$8, -8$	$3, -3$	$-6, \ 6$
player 1	M	$2, -2$	$-1, \ 1$	$3, -3$
	B	$-6, \ 6$	$4, -4$	$8, -8$

Maximinimizers

Example (Zero-sum game with three actions each)

			player 2	
		L	*C*	*R*
	T	8, −8	3, −3	−6, 6
player 1	*M*	2, −2	−1, 1	3, −3
	B	−6, 6	4, −4	8, −8

Guaranteed worst-case payoffs:

- T: -6, M: -1, B: -6 ⇝ maximinimizer M
- L: -8, C: -4, R: -8 ⇝ maximinimizer C

⇝ pair of maximinimizers (M, C) with payoffs $(-1, 1)$
(not a Nash equilibrium; this game has no Nash equilibrium.)

Maximinimizers

Example (Maximinimization vs. minimaximization)

player 2

		L	*R*
player 1	*T*	$1, -1$	$2, -2$
	B	$-2, 2$	$-4, 4$

Worst-case payoffs (player 2):

- *L*: -1, *R*: -2
- Maximize: -1

Best-case payoffs (player 1):

- *L*: $+1$, *R*: $+2$
- Minimize: $+1$

Observation: Results identical up to different sign.

Maximinimizers

Lemma

Let $G = \langle \{1,2\}, (A_i)_{i\in N}, (u_i)_{i\in N} \rangle$ *be a zero-sum game. Then*

$$\max_{y\in A_2} \min_{x\in A_1} u_2(x,y) = -\min_{y\in A_2} \max_{x\in A_1} u_1(x,y). \qquad (1)$$

Proof.

For any real-valued function f, we have

$$\min_z -f(z) = -\max_z f(z). \qquad (2)$$

Maximinimizers

Proof (ctd.)

Thus, for all $y \in A_2$,

$$
\begin{aligned}
-\min_{y \in A_2} \max_{x \in A_1} u_1(x,y) &\overset{(2)}{=} \max_{y \in A_2} -\max_{x \in A_1} u_1(x,y) \\
&\overset{(2)}{=} \max_{y \in A_2} \min_{x \in A_1} -u_1(x,y) \\
&\overset{\text{ZS}}{=} \max_{y \in A_2} \min_{x \in A_1} u_2(x,y).
\end{aligned}
$$

☐

Game Theory

2. Strategic Games

2.6. Zero-Sum Games

2.6.2. Nash-Equilibria vs. Maximinimizers

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Zero-Sum Games

Recall:

Definition (Zero-sum game)

A zero-sum game is a strategic game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1,2\}$ and

$$u_1(a) = -u_2(a)$$

for all $a \in A$.

Lemma

Let $G = \langle \{1,2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a zero-sum game. Then

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -\min_{y \in A_2} \max_{x \in A_1} u_1(x,y). \tag{1}$$

Nash Equilibria in Zero-Sum Games

Now, we are ready to prove our
main theorem about zero-sum games and Nash equilibria.

In zero-sum games:

1. Every Nash equilibrium is a pair of maximinimizers.
2. All Nash equilibria have the same payoffs.
3. If there is at least one Nash equilibrium, then every pair of maximinimizers is a Nash equilibrium.

Nash Equilibria in Zero-Sum Games

Theorem (Maximinimizer theorem)

Let $G = \langle \{1,2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ *be a zero-sum game. Then:*

1. *If* (x^*, y^*) *is a Nash equilibrium of G, then* x^* *and* y^* *are maximinimizers for player 1 and player 2, respectively.*
2. *If* (x^*, y^*) *is a Nash equilibrium of G, then*

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) = u_1(x^*, y^*).$$

3. *If* $\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$, *and* x^* *and* y^* *are maximinimizers of player 1 and player 2 respectively, then* (x^*, y^*) *is a Nash equilibrium.*

Nash Equilibria in Zero-Sum Games

Proof.

1. Let (x^*, y^*) be a Nash equilibrium. Then

$$u_2(x^*, y^*) \geq u_2(x^*, y) \quad \text{for all } y \in A_2.$$

Nash Equilibria in Zero-Sum Games

Proof.

1. Let (x^*, y^*) be a Nash equilibrium. Then

$$u_2(x^*, y^*) \geq u_2(x^*, y) \quad \text{for all } y \in A_2.$$

With $u_1 = -u_2$, this implies

$$u_1(x^*, y^*) \leq u_1(x^*, y) \quad \text{for all } y \in A_2.$$

Nash Equilibria in Zero-Sum Games

Proof.

1. Let (x^*, y^*) be a Nash equilibrium. Then

$$u_2(x^*, y^*) \geq u_2(x^*, y) \quad \text{for all } y \in A_2.$$

With $u_1 = -u_2$, this implies

$$u_1(x^*, y^*) \leq u_1(x^*, y) \quad \text{for all } y \in A_2.$$

Thus

$$u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y) \leq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \qquad (2)$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

1. (ctd.)
   Furthermore, since (x^*, y^*) is a Nash equilibrium, also

$$u_1(x^*, y^*) \geq u_1(x, y^*) \quad \text{for all } x \in A_1.$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

1. (ctd.)
   Furthermore, since (x^*, y^*) is a Nash equilibrium, also

   $$u_1(x^*, y^*) \geq u_1(x, y^*) \quad \text{for all } x \in A_1.$$

   Hence

   $$u_1(x^*, y^*) \geq \max_{x \in A_1} u_1(x, y^*).$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

1. (ctd.)

   Furthermore, since (x^*, y^*) is a Nash equilibrium, also

   $$u_1(x^*, y^*) \geq u_1(x, y^*) \quad \text{for all } x \in A_1.$$

   Hence

   $$u_1(x^*, y^*) \geq \max_{x \in A_1} u_1(x, y^*).$$

   This implies

   $$u_1(x^*, y^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{3}$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

1. (ctd.)

   Inequalities (2) and (3) together imply that

   $$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{4}$$

   Thus, x^* is a maximinimizer for player 1.

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

1 (ctd.)

Inequalities (2) and (3) together imply that

$$u_1(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \tag{4}$$

Thus, x^* is a maximinimizer for player 1.

Similarly, we can show that y^* is a maximinimizer for player 2:

$$u_2(x^*, y^*) = \max_{y \in A_2} \min_{x \in A_1} u_2(x, y). \tag{5}$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

2 We only need to put things together:

$$\begin{aligned}\max_{x\in A_1}\min_{y\in A_2} u_1(x,y) &\overset{(4)}{=} u_1(x^*,y^*)\\ &\overset{\text{ZS}}{=} -u_2(x^*,y^*)\\ &\overset{(5)}{=} -\max_{y\in A_2}\min_{x\in A_1} u_2(x,y)\\ &\overset{(1)}{=} \min_{y\in A_2}\max_{x\in A_1} u_1(x,y).\end{aligned}$$

In particular, it follows that all Nash equilibria share the same payoff profile.

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3 Let x^* and y^* be maximinimizers for player 1 and 2, respectively, and assume that

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) =: v^*. \qquad (6)$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3. Let x^* and y^* be maximinimizers for player 1 and 2, respectively, and assume that

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) =: v^*. \tag{6}$$

With Equation (1) from the previous lemma, we get

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -v^*. \tag{7}$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3. Let x^* and y^* be maximinimizers for player 1 and 2, respectively, and assume that

$$\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y) =: v^*. \tag{6}$$

With Equation (1) from the previous lemma, we get

$$\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -v^*. \tag{7}$$

With x^* and y^* being maximinimizers, (6) and (7) imply

$$u_1(x^*,y) \geq v^* \quad \text{for all } y \in A_2\text{, and} \tag{8}$$

$$u_2(x,y^*) \geq -v^* \quad \text{for all } x \in A_1. \tag{9}$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3 (ctd.)

Special cases of (8) and (9) for $x = x^*$ and $y = y^*$:

$$u_1(x^*, y^*) \geq v^* \qquad \text{and} \qquad u_2(x^*, y^*) \geq -v^*.$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3 (ctd.)

Special cases of (8) and (9) for $x = x^*$ and $y = y^*$:

$$u_1(x^*, y^*) \geq v^* \qquad \text{and} \qquad u_2(x^*, y^*) \geq -v^*.$$

With $u_1 = -u_2$, the latter is equivalent to $u_1(x^*, y^*) \leq v^*$, which gives us

$$u_1(x^*, y^*) = v^*. \tag{10}$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3. (ctd.)

   Plugging (10) into the right-hand side of (8) gives us

$$u_1(x^*, y) \geq u_1(x^*, y^*) \quad \text{for all } y \in A_2.$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3 (ctd.)

Plugging (10) into the right-hand side of (8) gives us

$$u_1(x^*,y) \geq u_1(x^*,y^*) \quad \text{for all } y \in A_2.$$

With $u_1 = -u_2$, this is equivalent to

$$u_2(x^*,y) \leq u_2(x^*,y^*) \quad \text{for all } y \in A_2.$$

In other words, y^* is a best response to x^*.

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3. (ctd.)

   Similarly, we can plug (10) into the right-hand side of (9) and obtain

$$u_2(x,y^*) \geq -u_1(x^*,y^*) \quad \text{for all } x \in A_1.$$

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3 (ctd.)

Similarly, we can plug (10) into the right-hand side of (9) and obtain

$$u_2(x, y^*) \geq -u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

Again using $u_1 = -u_2$, this is equivalent to

$$u_1(x, y^*) \leq u_1(x^*, y^*) \quad \text{for all } x \in A_1.$$

In words, x^* is also a best response to y^*.

Nash Equilibria in Zero-Sum Games

Proof (ctd.)

3 (ctd.)

Similarly, we can plug (10) into the right-hand side of (9) and obtain

$$u_2(x,y^*) \geq -u_1(x^*,y^*) \quad \text{for all } x \in A_1.$$

Again using $u_1 = -u_2$, this is equivalent to

$$u_1(x,y^*) \leq u_1(x^*,y^*) \quad \text{for all } x \in A_1.$$

In words, x^* is also a best response to y^*.

Hence, (x^*,y^*) is a Nash equilibrium.

□

Nash Equilibria in Zero-Sum Games

Corollary

Let $G = \langle \{1,2\}, (A_i)_{i\in N}, (u_i)_{i\in N} \rangle$ be a zero-sum game, and let (x_1^*, y_1^*) and (x_2^*, y_2^*) be two Nash equilibria of G.

Then (x_1^*, y_2^*) and (x_2^*, y_1^*) are also Nash equilibria of G.

In other words: Nash equilibria of zero-sum games can be arbitrarily recombined.

Nash Equilibria in Zero-Sum Games

Proof.

With part (1) of the maximinimizer theorem, we get that x_1^* and x_2^* are maximinimizers for player 1 and that y_1^* and y_2^* are maximinimizers for player 2.

Nash Equilibria in Zero-Sum Games

Proof.

With part (1) of the maximinimizer theorem, we get that x_1^* and x_2^* are maximinimizers for player 1 and that y_1^* and y_2^* are maximinimizers for player 2.

With part (2) of the maximinimizer theorem, we get that $\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)$.

Nash Equilibria in Zero-Sum Games

Proof.

With part (1) of the maximinimizer theorem, we get that x_1^* and x_2^* are maximinimizers for player 1 and that y_1^* and y_2^* are maximinimizers for player 2.

With part (2) of the maximinimizer theorem, we get that $\max_{x \in A_1} \min_{y \in A_2} u_1(x,y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x,y)$.

With this equality, with x_1^*, x_2^*, y_1^*, and y_2^* all being maximinimizers, and with part (3) of the maximinimizer theorem, we get that (x_1^*, y_2^*) and (x_2^*, y_1^*) are also Nash equilibria of G. □

Summary

- In zero-sum games, one player’s gain is the other player’s loss. Thus, playing it safe is rational. Relevant concept: maximinimizers.
- Relation to Nash equilibria: In zero-sum games, Nash equilibria are pairs of maximinimizers, and, if at least one Nash equilibrium exists, pairs of maximinimizers are also Nash equilibria.
- In zero-sum games, Nash equilibrium strategies can be recombined.

Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.1. Motivation and Definitions

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Mixed Strategies

Observation: Not every strategic game has a pure-strategy Nash equilibrium (e. g. matching pennies).

Question:

- Can we do anything about that?
- Which strategy to play then?

Idea: Consider randomized strategies.

Mixed Strategies

Notation

Let X be a (finite) set.

Then $\Delta(X)$ denotes the set of probability distributions over X.

That is, each $p \in \Delta(X)$ is a mapping $p : X \to [0, 1]$ with

$$\sum_{x \in X} p(x) = 1.$$

Mixed Strategies

A mixed strategy is a strategy where a player is allowed to randomize his action (throw a dice mentally and then act according to what he has decided to do for each outcome).

Definition (Mixed strategy)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

A mixed strategy of player i in G is a probability distribution $\alpha_i \in \Delta(A_i)$ over player i's actions.

For $a_i \in A_i$, $\alpha_i(a_i)$ is the probability for playing a_i.

Terminology: When we talk about strategies in A_i specifically, to distinguish them from mixed strategies, we sometimes also call them pure strategies.

Mixed Strategies

Note: Pure strategies can be seen as a special case of mixed strategies.

Notation

Since each pure strategy $a_i \in A_i$ is equivalent to its induced mixed strategy $\hat{a}_i$

$$\hat{a}_i(a'_i) = \begin{cases} 1 & \text{if } a'_i = a_i \\ 0 & \text{otherwise,} \end{cases}$$

we sometimes abuse notation and write a_i instead of $\hat{a}_i$.

Mixed Strategies

Definition (Mixed strategy profile)

A profile $\alpha = (\alpha_i)_{i \in N} \in \prod_{i \in N} \Delta(A_i)$ of mixed strategies induces a probability distribution p_α over $A = \prod_{i \in N} A_i$ as follows:

$$p_\alpha(a) = \prod_{i \in N} \alpha_i(a_i).$$

For $A' \subseteq A$, we define

$$p_\alpha(A') = \sum_{a \in A'} p_\alpha(a).$$

Mixed Strategies

Example (Mixed strategies for matching pennies)

	H	T
H	$1,-1$	$-1,\ \ 1$
T	$-1,\ \ 1$	$1,-1$

$$\alpha = (\alpha_1, \alpha_2),\ \ \alpha_1(H) = {}^2\!/_3,\ \ \alpha_1(T) = {}^1\!/_3,\ \ \alpha_2(H) = {}^1\!/_3,\ \ \alpha_2(T) = {}^2\!/_3.$$

This induces a probability distribution over $\{H, T\} \times \{H, T\}$:

$$\begin{aligned}
p_\alpha(H,H) &= \alpha_1(H) \cdot \alpha_2(H) = {}^2\!/_9, & u_1(H,H) &= +1,\\
p_\alpha(H,T) &= \alpha_1(H) \cdot \alpha_2(T) = {}^4\!/_9, & u_1(H,T) &= -1,\\
p_\alpha(T,H) &= \alpha_1(T) \cdot \alpha_2(H) = {}^1\!/_9, & u_1(T,H) &= -1,\\
p_\alpha(T,T) &= \alpha_1(T) \cdot \alpha_2(T) = {}^2\!/_9, & u_1(T,T) &= +1.
\end{aligned}$$

Expected Utility

Definition (Expected utility)

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile.

The expected utility of α for player i is

$$U_i(\alpha) = U_i\big((\alpha_j)_{j \in N}\big) := \sum_{a \in A} p_\alpha(a)\, u_i(a) = \sum_{a \in A} \Big(\prod_{j \in N} \alpha_j(a_j)\Big) u_i(a).$$

Example (Mixed strategies for matching pennies (ctd.))

The expected utilities for player 1 and player 2 are

$$U_1(\alpha_1, \alpha_2) = -{}^1\!/_9 \qquad \text{and} \qquad U_2(\alpha_1, \alpha_2) = +{}^1\!/_9.$$

Expected Utility

Remark: The expected utility functions U_i are linear in all mixed strategies.

Proposition

Let $\alpha \in \prod_{i \in N} \Delta(A_i)$ be a mixed strategy profile, $\beta_i, \gamma_i \in \Delta(A_i)$ mixed strategies, and $\lambda \in [0, 1]$. Then

$$U_i(\alpha_{-i}, \lambda\beta_i + (1 - \lambda)\gamma_i) = \lambda U_i(\alpha_{-i}, \beta_i) + (1 - \lambda)U_i(\alpha_{-i}, \gamma_i).$$

Moreover,

$$U_i(\alpha) = \sum_{a_i \in A_i} \alpha_i(a_i) \cdot U_i(\alpha_{-i}, a_i)$$

Proof.

Homework. ☐

Mixed Extension

Definition (Mixed extension)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game.

The mixed extension of G is the game $\langle N, (\Delta(A_i))_{i \in N}, (U_i)_{i \in N} \rangle$ where

- $\Delta(A_i)$ is the set of probability distributions over A_i and
- $U_i : \prod_{j \in N} \Delta(A_j) \to \mathbb{R}$ assigns to each mixed strategy profile α the expected utility for player i according to the induced probability distribution p_α.

Nash Equilibria in Mixed Strategies

Definition (Nash equilibrium in mixed strategies)

Let G be a strategic game.

A Nash equilibrium in mixed strategies (or mixed-strategy Nash equilibrium, or MSNE) of G is a Nash equilibrium in the mixed extension of G.

Summary

- Not every strategic game has a pure-strategy Nash equilibrium.
- Randomization sometimes seems rational (e. g., matching pennies)
  $\rightsquigarrow$ mixed strategies

- This section: definition of mixed strategies, mixed extension, MSNE
- Next sections: characterization of MSNE, existence proof, computation

Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.2. Support Lemma

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Support

Intuition:

- It does not make sense to assign positive probability to a pure strategy that is not a best response to what the other players do.
- Claim: A profile of mixed strategies α is a Nash equilibrium if and only if everyone only plays best pure responses to what the others play.

Definition (Support)

Let α_i be a mixed strategy.

The support of α_i is the set

$$supp(\alpha_i) = \{a_i \in A_i \,|\, \alpha_i(a_i) > 0\}$$

of actions played with nonzero probability.

Support Lemma

Lemma (Support lemma)

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ *be a finite strategic game.*

Then $\alpha^* \in \prod_{i \in N} \Delta(A_i)$ *is a mixed-strategy Nash equilibrium in* G *if and only if for every player* $i \in N$, *every pure strategy in the support of* α_i^* *is a best response to* α_{-i}^*.

For a single player—given all other players stick to their mixed strategies—it does not make a difference whether he plays the mixed strategy or whether he plays any single pure strategy from the support of the mixed strategy.

Support Lemma

Example (Support lemma)

Matching pennies, strategy profile $\alpha = (\alpha_1, \alpha_2)$ with

$$\alpha_1(H) = {}^2\!/_3, \quad \alpha_1(T) = {}^1\!/_3, \quad \alpha_2(H) = {}^1\!/_3, \text{ and } \quad \alpha_2(T) = {}^2\!/_3.$$

For α to be a Nash equilibrium, both actions in $supp(\alpha_2) = \{H, T\}$ have to be best responses to α_1. Are they?

$$\begin{aligned} U_2(\alpha_1, H) &= \alpha_1(H) \cdot u_2(H, H) + \alpha_1(T) \cdot u_2(T, H) \\ &= {}^2\!/_3 \cdot (-1) + {}^1\!/_3 \cdot (+1) = -{}^1\!/_3, \\ U_2(\alpha_1, T) &= \alpha_1(H) \cdot u_2(H, T) + \alpha_1(T) \cdot u_2(T, T) \\ &= {}^2\!/_3 \cdot (+1) + {}^1\!/_3 \cdot (-1) = {}^1\!/_3. \end{aligned}$$

$\Rightarrow$ $H \in supp(\alpha_2)$, but $H \notin B_2(\alpha_1)$.

$\overset{\text{Support lemma}}{\Rightarrow}$ α can not be a Nash equilibrium.

Support Lemma

Proof.

"$\Rightarrow$": Let α^* be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Support Lemma

Proof.

"$\Rightarrow$": Let α^* be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Assume that a_i is not a best response to α_{-i}^*. Because U_i is linear, player i can improve his utility by shifting probability in α_i^* from a_i to a better response.

Support Lemma

Proof.

"$\Rightarrow$": Let α^* be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Assume that a_i is not a best response to α^*_{-i}. Because U_i is linear, player i can improve his utility by shifting probability in α_i^* from a_i to a better response.

This makes the modified α_i^* a better response than the original α_i^*, i. e., the original α_i^* was not a best response, which contradicts the assumption that α^* is a Nash equilibrium.

Support Lemma

Proof.

"$\Rightarrow$": Let α^* be a Nash equilibrium with $a_i \in supp(\alpha_i^*)$.

Assume that a_i is not a best response to α^*_{-i}. Because U_i is linear, player i can improve his utility by shifting probability in α_i^* from a_i to a better response.

This makes the modified α_i^* a better response than the original α_i^*, i. e., the original α_i^* was not a best response, which contradicts the assumption that α^* is a Nash equilibrium. So each pure strategy in the support of α_i must be a best response.

Support Lemma

Proof (ctd.)

“$\Leftarrow$”: Assume that α^* is not a Nash equilibrium.

Support Lemma

Proof (ctd.)

"$\Leftarrow$": Assume that α^* is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy α_i' such that $U_i(\alpha_{-i}^*, \alpha_i') > U_i(\alpha_{-i}^*, \alpha_i^*)$.

Support Lemma

Proof (ctd.)

“$\Leftarrow$”: Assume that α^* is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy α_i' such that $U_i(\alpha^*_{-i}, \alpha_i') > U_i(\alpha^*_{-i}, \alpha_i^*)$.

Because U_i is linear, there must be a pure strategy $a_i' \in supp(\alpha_i')$ that has higher utility than some pure strategy $a_i'' \in supp(\alpha_i^*)$.

Support Lemma

Proof (ctd.)

"$\Leftarrow$": Assume that α^* is not a Nash equilibrium.

Then there must be a player $i \in N$ and a strategy α_i' such that $U_i(\alpha_{-i}^*, \alpha_i') > U_i(\alpha_{-i}^*, \alpha_i^*)$.

Because U_i is linear, there must be a pure strategy $a_i' \in supp(\alpha_i')$ that has higher utility than some pure strategy $a_i'' \in supp(\alpha_i^*)$.

Therefore, $supp(\alpha_i^*)$ does not only contain best responses to α_{-i}^*. □

Summary

- Characterization of mixed-strategy Nash equilibria: players only play best responses with positive probability (support lemma).

  $\rightsquigarrow$ only need to look at pure candidate best responses against other players' mixed strategy profile when computing MSNE. (See later sections.)

Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.3. Computing Mixed-Strategy Nash Equilibria

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Computing Mixed-Strategy Nash Equilibria

Example (Mixed-strategy Nash equilibria in BoS)

	B	S
B	$2,1$	$0,0$
S	$0,0$	$1,2$

We already know: (B,B) and (S,S) are pure Nash equilibria.

Possible supports (excluding "pure-vs-pure" strategies) are:

$$\{B\} \text{ vs. } \{B,S\}, \quad \{S\} \text{ vs. } \{B,S\}, \quad \{B,S\} \text{ vs. } \{B\},$$
$$\{B,S\} \text{ vs. } \{S\} \quad \text{and} \quad \{B,S\} \text{ vs. } \{B,S\}$$

Observation: In Bach or Stravinsky, pure strategies have unique best responses. Therefore, there can be no Nash equilibria of "pure-vs-strictly-mixed" type.

Computing Mixed-Strategy Nash Equilibria

Example (Mixed-strategy Nash equilibria in BoS (ctd.))

Consequence: Only need to search for additional Nash equilibria with support sets $\{B, S\}$ vs. $\{B, S\}$.

Assume that (α_1^*, α_2^*) is a Nash equilibrium with $0 < \alpha_1^*(B) < 1$ and $0 < \alpha_2^*(B) < 1$. Then

Computing Mixed-Strategy Nash Equilibria

Example (Mixed-strategy Nash equilibria in BoS (ctd.))

Consequence: Only need to search for additional Nash equilibria with support sets $\{B, S\}$ vs. $\{B, S\}$.

Assume that (α_1^*, α_2^*) is a Nash equilibrium with $0 < \alpha_1^*(B) < 1$ and $0 < \alpha_2^*(B) < 1$. Then

$$\begin{aligned}
& & U_1(B, \alpha_2^*) &= U_1(S, \alpha_2^*) \\
&\Rightarrow & 2 \cdot \alpha_2^*(B) + 0 \cdot \alpha_2^*(S) &= 0 \cdot \alpha_2^*(B) + 1 \cdot \alpha_2^*(S) \\
&\Rightarrow & 2 \cdot \alpha_2^*(B) &= 1 - \alpha_2^*(B) \\
&\Rightarrow & 3 \cdot \alpha_2^*(B) &= 1 \\
&\Rightarrow & \alpha_2^*(B) &= 1/3 \quad (\text{and } \alpha_2^*(S) = 2/3)
\end{aligned}$$

Similarly, we get $\alpha_1^*(B) = 2/3$ and $\alpha_1^*(S) = 1/3$.
The payoff profile of this equilibrium is $(2/3, 2/3)$.

Support Lemma

Remark

Let $G = \langle \{1,2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$ be a two-player game with two actions each, and (T, α_2^*) with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of G.

Then at least one of the profiles (T, L) and (T, R) is also a Nash equilibrium of G.

Support Lemma

Remark

Let $G = \langle \{1,2\}, (A_i), (u_i) \rangle$ with $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$ be a two-player game with two actions each, and (T, α_2^*) with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of G.

Then at least one of the profiles (T, L) and (T, R) is also a Nash equilibrium of G.

Reason: Both L and R are best responses to T. Assume that T was neither a best response to L nor to R. Then B would be a better response than T both to L and to R.

Support Lemma

Remark

Let $G = \langle\{1,2\},(A_i),(u_i)\rangle$ with $A_1 = \{T,B\}$ and $A_2 = \{L,R\}$ be a two-player game with two actions each, and (T,α_2^*) with $0 < \alpha_2^*(L) < 1$ be a Nash equilibrium of G.

Then at least one of the profiles (T,L) and (T,R) is also a Nash equilibrium of G.

Reason: Both L and R are best responses to T. Assume that T was neither a best response to L nor to R. Then B would be a better response than T both to L and to R.

With the linearity of U_1, B would also be a better response to α_2^* than T is. Contradiction.

Support Lemma

Example

Consider the Nash equilibrium $\alpha^* = (\alpha_1^*, \alpha_2^*)$ with

$$\alpha_1^*(T) = 1, \qquad \alpha_1^*(B) = 0, \qquad \alpha_2^*(L) = {}^1\!/_{10}, \qquad \alpha_2^*(R) = {}^9\!/_{10}$$

in the following game:

	L	R
T	1, 1	1, 1
B	2, 2	−5, −5

Here, (T, R) is also a Nash equilibrium.

Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.4. Nash's Theorem: Introduction

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Nash's Theorem

Motivation: When does a strategic game have a mixed-strategy Nash equilibrium?

We already discussed necessary and sufficient conditions for the existence of Nash equilibria for the special case of zero-sum games. Can we make other claims?

Nash's Theorem

Theorem (Nash's theorem)

Every finite strategic game has a mixed-strategy Nash equilibrium.

Proof sketch.

Consider the best-response function B with

$$B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i}).$$

A mixed strategy profile α is a Nash equilibrium iff it is a fixpoint of B iff $\alpha \in B(\alpha)$.

Under certain conditions that are satisfied by B, B has such a fixpoint (Kakutani's Fixpoint Theorem!). Therefore, the game has a mixed-strategy Nash equilibrium. □

Nash's Theorem

Outline for the formal proof:

1. Review of necessary mathematical definitions
   $\rightsquigarrow$ Subsection "Nash's Theorem: Required Background"
2. Statement of a fixpoint theorem used to prove Nash's theorem (without proof)
   $\rightsquigarrow$ Subsection "Nash's Theorem: Required Background"
3. Proof of Nash's theorem using fixpoint theorem
   $\rightsquigarrow$ Subsection "Nash's Theorem: Proof"

Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.5. Nash's Theorem: Required Background

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

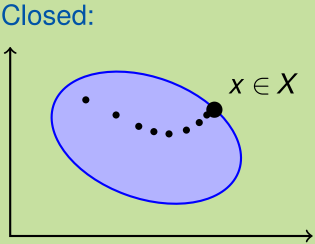
Nash's Theorem

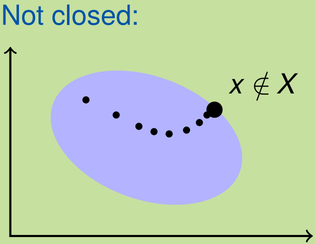
Definitions

Definition

A set $X \subseteq \mathbb{R}^n$ is closed if X contains all its limit points, i. e., if $(x_k)_{k \in \mathbb{N}}$ is a sequence of elements in X and $\lim_{k \to \infty} x_k = x$, then also $x \in X$.

Example





Nash's Theorem

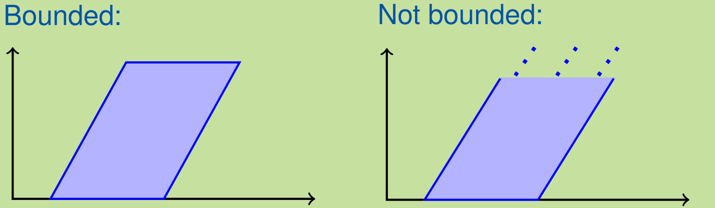
Definitions

Definition

A set $X \subseteq \mathbb{R}^n$ is bounded if for each $i = 1, \ldots, n$ there are lower and upper bounds $a_i, b_i \in \mathbb{R}$ such that

$$X \subseteq \prod_{i=1}^{n} [a_i, b_i].$$

Example



Nash's Theorem

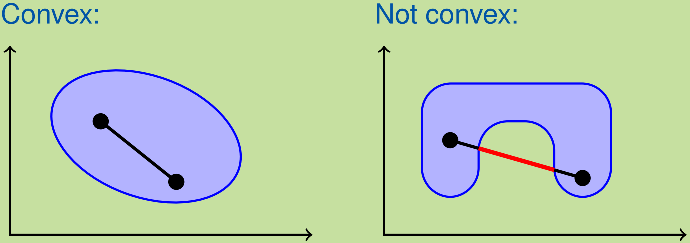
Definitions

Definition

A set $X \subseteq \mathbb{R}^n$ is convex if for all $x, y \in X$ and all $\lambda \in [0, 1]$,

$$\lambda x + (1 - \lambda)y \in X.$$

Example


Convex:

Not convex:

Nash's Theorem

Definitions

Definition

For a function $f : X \to 2^X$, the graph of f is the set

$$Graph(f) = \{(x, y) \mid x \in X,\ y \in f(x)\}.$$

Nash's Theorem

Kakutani's Fixpoint Theorem

Theorem (Kakutani's fixpoint theorem)

Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed, bounded and convex set and let $f : X \to 2^X$ be a function such that

- *for all $x \in X$, the set $f(x) \subseteq X$ is nonempty and convex, and*
- *$Graph(f)$ is closed.*

Then there is an $x \in X$ with $x \in f(x)$, i. e., f has a fixpoint.

Proof.

See Shizuo Kakutani, A generalization of Brouwer's fixed point theorem, 1941, or your favorite advanced calculus textbook, or the Internet.

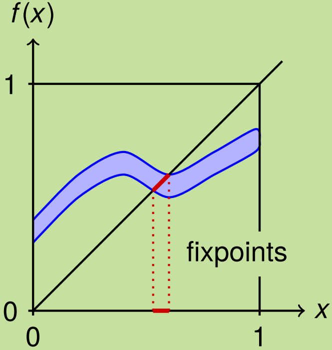
For German speakers: Harro Heuser, Lehrbuch der Analysis, Teil 2, also has a proof (Abschnitt 232). □

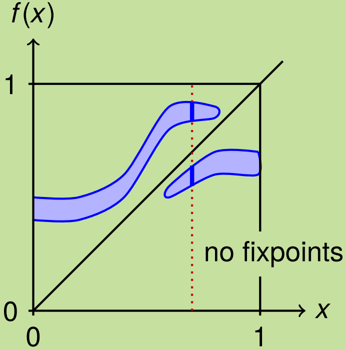
Example

Let $X = [0, 1]$.

Kakutani's theorem applicable:

Kakutani's theorem not applicable:





Game Theory

2. Strategic Games

2.7. Mixed Strategies

2.7.6. Nash's Theorem: Proof

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Nash's Theorem

Recall:

Theorem (Kakutani's fixpoint theorem)

Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed, bounded and convex set and let $f : X \to 2^X$ be a function such that

- *for all $x \in X$, the set $f(x) \subseteq X$ is nonempty and convex, and*
- *$Graph(f)$ is closed.*

Then there is an $x \in X$ with $x \in f(x)$, i. e., f has a fixpoint.

We use this to prove:

Theorem (Nash's theorem)

Every finite strategic game has a mixed-strategy Nash equilibrium.

Nash's Theorem

Proof

Proof.

Apply Kakutani's fixpoint theorem using $X = \mathscr{A} = \prod_{i \in N} \Delta(A_i)$ and $f = B$, where $B(\alpha) = \prod_{i \in N} B_i(\alpha_{-i})$.

We have to show:

1. $\mathscr{A}$ is nonempty,
2. $\mathscr{A}$ is closed,
3. $\mathscr{A}$ is bounded,
4. $\mathscr{A}$ is convex,
5. $B(\alpha)$ is nonempty for all $\alpha \in \mathscr{A}$,
6. $B(\alpha)$ is convex for all $\alpha \in \mathscr{A}$, and
7. $\mathit{Graph}(B)$ is closed.

Nash's Theorem

Proof

Proof (ctd.)

Some notation:

- Assume without loss of generality that $N = \{1, \dots, n\}$.
- A profile of mixed strategies can be written as a vector of $M = \sum_{i \in N} |A_i|$ real numbers in the interval $[0, 1]$ such that numbers for the same player add up to 1.

  For example, $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1(T) = 0.7$, $\alpha_1(M) = 0.0$, $\alpha_1(B) = 0.3$, $\alpha_2(L) = 0.4$, $\alpha_2(R) = 0.6$ can be seen as the vector

$$(\underbrace{0.7,\ 0.0,\ 0.3}_{\alpha_1},\ \underbrace{0.4,\ 0.6}_{\alpha_2})$$

- This allows us to interpret the set $\mathscr{A}$ of mixed strategy profiles as a subset of $\mathbb{R}^M$.

Nash's Theorem

Proof

Proof (ctd.)

1. $\mathscr{A}$ nonempty: Trivial. $\mathscr{A}$ contains the tuple

$$(1, \underbrace{0, \dots, 0}_{|A_1|-1 \text{ times}}, \dots, 1, \underbrace{0, \dots, 0}_{|A_n|-1 \text{ times}}).$$

Nash's Theorem

Proof

Proof (ctd.)

1. $\mathscr{A}$ nonempty: Trivial. $\mathscr{A}$ contains the tuple

$$(1, \underbrace{0,\ldots,0}_{|A_1|-1 \text{ times}}, \ldots, 1, \underbrace{0,\ldots,0}_{|A_n|-1 \text{ times}}).$$

2. $\mathscr{A}$ closed: Let $\alpha_1, \alpha_2, \ldots$ be a sequence in $\mathscr{A}$ that converges to $\lim_{k\to\infty} \alpha_k = \alpha$. Suppose $\alpha \notin \mathscr{A}$. Then either there is some component of α that is less than zero or greater than one, or the components for some player i add up to a value other than one.

Nash's Theorem

Proof

Proof (ctd.)

1. $\mathscr{A}$ nonempty: Trivial. $\mathscr{A}$ contains the tuple

$$(1, \underbrace{0, \dots, 0}_{|A_1|-1 \text{ times}}, \dots, 1, \underbrace{0, \dots, 0}_{|A_n|-1 \text{ times}}).$$

2. $\mathscr{A}$ closed: Let $\alpha_1, \alpha_2, \dots$ be a sequence in $\mathscr{A}$ that converges to $\lim_{k \to \infty} \alpha_k = \alpha$. Suppose $\alpha \notin \mathscr{A}$. Then either there is some component of α that is less than zero or greater than one, or the components for some player i add up to a value other than one.

   Since α is a limit point, the same must hold for some α_k in the sequence. But then, $\alpha_k \notin \mathscr{A}$, a contradiction. Hence $\mathscr{A}$ is closed.

Nash's Theorem

Proof

Proof (ctd.)

3. $\mathscr{A}$ bounded: Trivial. All entries are between 0 and 1, i. e., $\mathscr{A}$ is bounded by $[0,1]^M$.

Nash's Theorem

Proof

Proof (ctd.)

3. $\mathscr{A}$ bounded: Trivial. All entries are between 0 and 1, i. e., $\mathscr{A}$ is bounded by $[0,1]^M$.
4. $\mathscr{A}$ convex: Let $\alpha, \beta \in \mathscr{A}$ and $\lambda \in [0,1]$, and consider $\gamma = \lambda\alpha + (1-\lambda)\beta$. Then

Nash's Theorem

Proof

Proof (ctd.)

3. $\mathscr{A}$ bounded: Trivial. All entries are between 0 and 1, i. e., $\mathscr{A}$ is bounded by $[0,1]^M$.
4. $\mathscr{A}$ convex: Let $\alpha, \beta \in \mathscr{A}$ and $\lambda \in [0,1]$, and consider $\gamma = \lambda\alpha + (1-\lambda)\beta$. Then

$$\begin{aligned}\min(\gamma) &= \min(\lambda\alpha + (1-\lambda)\beta) \\ &\geq \lambda \cdot \min(\alpha) + (1-\lambda) \cdot \min(\beta) \\ &\geq \lambda \cdot 0 + (1-\lambda) \cdot 0 = 0,\end{aligned}$$

and similarly, $\max(\gamma) \leq 1$.

Nash's Theorem

Proof

Proof (ctd.)

3. $\mathscr{A}$ bounded: Trivial. All entries are between 0 and 1, i. e., $\mathscr{A}$ is bounded by $[0,1]^M$.

4. $\mathscr{A}$ convex: Let $\alpha, \beta \in \mathscr{A}$ and $\lambda \in [0,1]$, and consider $\gamma = \lambda\alpha + (1-\lambda)\beta$. Then

$$\begin{aligned}\min(\gamma) &= \min(\lambda\alpha + (1-\lambda)\beta)\\ &\geq \lambda \cdot \min(\alpha) + (1-\lambda)\cdot\min(\beta)\\ &\geq \lambda \cdot 0 + (1-\lambda)\cdot 0 = 0,\end{aligned}$$

   and similarly, $\max(\gamma) \leq 1$.

   Hence, all entries in γ are still in $[0,1]$.

Nash's Theorem

Proof

Proof (ctd.)

4 $\mathscr{A}$ convex (ctd.): Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be the sections of α, β and γ, respectively, that determine the probability distribution for player i. Then

Nash's Theorem

Proof

Proof (ctd.)

4 $\mathscr{A}$ convex (ctd.): Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be the sections of α, β and γ, respectively, that determine the probability distribution for player *i*. Then

$$\begin{aligned}\sum \tilde{\gamma} &= \sum (\lambda \tilde{\alpha} + (1 - \lambda)\tilde{\beta}) \\ &= \lambda \cdot \sum \tilde{\alpha} + (1 - \lambda) \cdot \sum \tilde{\beta} \\ &= \lambda \cdot 1 + (1 - \lambda) \cdot 1 = 1.\end{aligned}$$

Nash's Theorem

Proof

Proof (ctd.)

4. $\mathscr{A}$ convex (ctd.): Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be the sections of α, β and γ, respectively, that determine the probability distribution for player *i*. Then

$$\begin{aligned}\sum \tilde{\gamma} &= \sum (\lambda \tilde{\alpha} + (1-\lambda)\tilde{\beta}) \\ &= \lambda \cdot \sum \tilde{\alpha} + (1-\lambda) \cdot \sum \tilde{\beta} \\ &= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1.\end{aligned}$$

Hence, all probabilities for player *i* in γ still sum up to 1.

Nash's Theorem

Proof

Proof (ctd.)

4. $\mathscr{A}$ convex (ctd.): Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be the sections of α, β and γ, respectively, that determine the probability distribution for player *i*. Then

$$\begin{aligned}\sum \tilde{\gamma} &= \sum(\lambda\tilde{\alpha} + (1-\lambda)\tilde{\beta}) \\ &= \lambda \cdot \sum \tilde{\alpha} + (1-\lambda) \cdot \sum \tilde{\beta} \\ &= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1.\end{aligned}$$

Hence, all probabilities for player *i* in γ still sum up to 1.

Altogether, $\gamma \in \mathscr{A}$, and therefore, $\mathscr{A}$ is convex.

Nash's Theorem

Proof

Proof (ctd.)

5. *$B(\alpha)$ nonempty:* For a fixed α_{-i}, U_i is linear in the mixed strategies of player i, i. e., for $\beta_i, \gamma_i \in \Delta(A_i)$,

$$U_i\big(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i\big) = \lambda U_i\big(\alpha_{-i}, \beta_i\big) + (1-\lambda)U_i\big(\alpha_{-i}, \gamma_i\big) \tag{1}$$

for all $\lambda \in [0,1]$.

Nash's Theorem

Proof

Proof (ctd.)

5. $B(\alpha)$ nonempty: For a fixed α_{-i}, U_i is linear in the mixed strategies of player i, i. e., for $\beta_i, \gamma_i \in \Delta(A_i)$,

$$U_i\big(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i\big) = \lambda U_i\big(\alpha_{-i}, \beta_i\big) + (1-\lambda)U_i\big(\alpha_{-i}, \gamma_i\big) \tag{1}$$

for all $\lambda \in [0, 1]$.

Hence, U_i is continous on $\Delta(A_i)$.

Nash's Theorem

Proof

Proof (ctd.)

5 $B(\alpha)$ nonempty: For a fixed α_{-i}, U_i is linear in the mixed strategies of player i, i. e., for $\beta_i, \gamma_i \in \Delta(A_i)$,

$$U_i\big(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i\big) = \lambda U_i\big(\alpha_{-i}, \beta_i\big) + (1-\lambda)U_i\big(\alpha_{-i}, \gamma_i\big) \tag{1}$$

for all $\lambda \in [0,1]$.

Hence, U_i is continous on $\Delta(A_i)$.

Continous functions on closed and bounded sets take their maximum in that set.

Nash's Theorem

Proof

Proof (ctd.)

5. $B(\alpha)$ nonempty: For a fixed α_{-i}, U_i is linear in the mixed strategies of player i, i. e., for $\beta_i, \gamma_i \in \Delta(A_i)$,

$$U_i\big(\alpha_{-i}, \lambda\beta_i + (1-\lambda)\gamma_i\big) = \lambda U_i\big(\alpha_{-i}, \beta_i\big) + (1-\lambda)U_i\big(\alpha_{-i}, \gamma_i\big) \tag{1}$$

for all $\lambda \in [0, 1]$.

Hence, U_i is continous on $\Delta(A_i)$.

Continous functions on closed and bounded sets take their maximum in that set.

Therefore, $B_i(\alpha_{-i}) \neq \emptyset$ for all $i \in N$, and thus $B(\alpha) \neq \emptyset$.

Nash's Theorem

Proof

Proof (ctd.)

6. *$B(\alpha)$ convex:* This follows, since each $B_i(\alpha_{-i})$ is convex. To see this, let $\alpha'_i, \alpha''_i \in B_i(\alpha_{-i})$.

Nash's Theorem

Proof

Proof (ctd.)

6 $B(\alpha)$ convex: This follows, since each $B_i(\alpha_{-i})$ is convex. To see this, let $\alpha_i', \alpha_i'' \in B_i(\alpha_{-i})$.

Then $U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'')$.

Nash's Theorem

Proof

Proof (ctd.)

6. **$B(\alpha)$ convex:** This follows, since each $B_i(\alpha_{-i})$ is convex. To see this, let $\alpha_i', \alpha_i'' \in B_i(\alpha_{-i})$.

   Then $U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'')$.

   With Equation (1), this implies

   $$\lambda \alpha_i' + (1 - \lambda)\alpha_i'' \in B_i(\alpha_{-i}).$$

   Hence, $B_i(\alpha_{-i})$ is convex.

Proof (ctd.)

6. $B(\alpha)$ convex: This follows, since each $B_i(\alpha_{-i})$ is convex. To see this, let $\alpha_i', \alpha_i'' \in B_i(\alpha_{-i})$.

   Then $U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'')$.

   With Equation (1), this implies

   $$\lambda \alpha_i' + (1 - \lambda)\alpha_i'' \in B_i(\alpha_{-i}).$$

   Hence, $B_i(\alpha_{-i})$ is convex.

7. *Graph*(B) closed: Let (α^k, β^k) be a convergent sequence in *Graph*(B) with $\lim_{k\to\infty}(\alpha^k, \beta^k) = (\alpha, \beta)$.

Nash's Theorem

Proof

Proof (ctd.)

6. $B(\alpha)$ convex: This follows, since each $B_i(\alpha_{-i})$ is convex. To see this, let $\alpha_i', \alpha_i'' \in B_i(\alpha_{-i})$.

   Then $U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'')$.

   With Equation (1), this implies

   $$\lambda \alpha_i' + (1 - \lambda)\alpha_i'' \in B_i(\alpha_{-i}).$$

   Hence, $B_i(\alpha_{-i})$ is convex.

7. $\mathit{Graph}(B)$ closed: Let (α^k, β^k) be a convergent sequence in $\mathit{Graph}(B)$ with $\lim_{k\to\infty}(\alpha^k, \beta^k) = (\alpha, \beta)$.

   So, $\alpha^k, \beta^k, \alpha, \beta \in \prod_{i \in N} \Delta(A_i)$ and $\beta^k \in B(\alpha^k)$.

Nash's Theorem

Proof

Proof (ctd.)

6 $B(\alpha)$ convex: This follows, since each $B_i(\alpha_{-i})$ is convex. To see this, let $\alpha_i', \alpha_i'' \in B_i(\alpha_{-i})$.

Then $U_i(\alpha_{-i}, \alpha_i') = U_i(\alpha_{-i}, \alpha_i'')$.

With Equation (1), this implies

$$\lambda \alpha_i' + (1 - \lambda)\alpha_i'' \in B_i(\alpha_{-i}).$$

Hence, $B_i(\alpha_{-i})$ is convex.

7 *Graph*(B) closed: Let (α^k, β^k) be a convergent sequence in *Graph*(B) with $\lim_{k\to\infty}(\alpha^k, \beta^k) = (\alpha, \beta)$.

So, $\alpha^k, \beta^k, \alpha, \beta \in \prod_{i\in N} \Delta(A_i)$ and $\beta^k \in B(\alpha^k)$.

We need to show that $(\alpha, \beta) \in$ *Graph*(B), i. e., that $\beta \in B(\alpha)$.

Nash's Theorem

Proof

Proof (ctd.)

7 *Graph*(*B*) closed (ctd.): It holds for all $i \in N$:

$$
\begin{aligned}
U_i(\alpha_{-i}, \beta_i) &\overset{\text{(D)}}{=} U_i\big(\lim_{k\to\infty} (\alpha^k_{-i}, \beta^k_i)\big) \\
&\overset{\text{(C)}}{=} \lim_{k\to\infty} U_i(\alpha^k_{-i}, \beta^k_i) \\
&\overset{\text{(B)}}{\geq} \lim_{k\to\infty} U_i(\alpha^k_{-i}, \beta'_i) \qquad \text{for all } \beta'_i \in \Delta(A_i) \\
&\overset{\text{(C)}}{=} U_i\big(\lim_{k\to\infty} \alpha^k_{-i}, \beta'_i\big) \qquad \text{for all } \beta'_i \in \Delta(A_i) \\
&\overset{\text{(D)}}{=} U_i(\alpha_{-i}, \beta'_i) \qquad \text{for all } \beta'_i \in \Delta(A_i).
\end{aligned}
$$

(D): def. α_i, β_i; (C) continuity; (B) β^k_i best response to α^k_{-i}.

Nash's Theorem

Proof

Proof (ctd.)

7. *Graph*(*B*) closed (ctd.): It follows that β_i is a best response to α_{-i} for all $i \in N$.

   Thus, $\beta \in B(\alpha)$ and finally $(\alpha, \beta) \in \mathit{Graph}(B)$.

Nash's Theorem

Proof

Proof (ctd.)

7. *Graph*(B) closed (ctd.): It follows that β_i is a best response to α_{-i} for all $i \in N$.

   Thus, $\beta \in B(\alpha)$ and finally $(\alpha, \beta) \in \mathit{Graph}(B)$.

Therefore, all requirements of Kakutani's fixpoint theorem are satisfied.

Nash's Theorem

Proof

Proof (ctd.)

7. *Graph*(B) closed (ctd.): It follows that β_i is a best response to α_{-i} for all $i \in N$.

   Thus, $\beta \in B(\alpha)$ and finally $(\alpha, \beta) \in \mathit{Graph}(B)$.

Therefore, all requirements of Kakutani's fixpoint theorem are satisfied.

Applying Kakutani's theorem establishes the existence of a fixpoint of B, which is, by definition/construction, the same as a mixed-strategy Nash equilibrium. □

Summary

Take-home message:

- Nash's theorem: Every finite strategic game has a mixed-strategy Nash equilibrium.
- Proof idea: Apply Kakutani's fixpoint theorem to the best-response function.
- Encode mixed strategy profiles as real-valued vectors, apply standard techniques from real analysis.

Game Theory

2. Strategic Games

2.8. Correlated Equilibria

Albert-Ludwigs-Universität Freiburg

Bernhard Nebel and Robert Mattmüller

Summer semester 2020

Correlated Equilibria

Recall: There are three Nash equilibria in Bach or Stravinsky

- (B, B) with payoff profile $(2, 1)$
- (S, S) with payoff profile $(1, 2)$
- (α_1^*, α_2^*) with payoff profile $(2/3, 2/3)$ where
  - $\alpha_1^*(B) = 2/3$, $\alpha_1^*(S) = 1/3$,
  - $\alpha_2^*(B) = 1/3$, $\alpha_2^*(S) = 2/3$.

All of them are somewhat unsatisfactory:

- (B, B) and (S, S) because of unclear coordination and uneven payoffs.
- (α_1^*, α_2^*) because of low payoffs.

Correlated Equilibria

Question: Can the players somehow do better?

Yes! Idea: Use a publicly visible coin toss to decide which action from a mixed strategy is played. This can lead to higher payoffs.

Correlated Equilibria

Example (Correlated equilibrium in BoS)

With a fair coin that both players can observe, the players can agree to play as follows:

- If the coin shows heads, both play B.
- If the coin shows tails, both play S.

This is stable in the sense that no player has an incentive to deviate from this agreed-upon rule, as long as the other player keeps playing his/her strategy (cf. definition of Nash equilibria).

Expected payoffs: $(3/2, 3/2)$ instead of $(2/3, 2/3)$.

Observations and Information Partitions

We assume that observations are made based on a finite probability space (Ω, π), where Ω is a set of states and π is a probability measure on Ω.

Agents might not be able to distingush all states from each other. In order to model this, we assume, for each player i, an information partition $\mathscr{P}_i = \{P_{i1}, P_{i2}, \ldots, P_{ik_i}\}$. This means that $\bigcup \mathscr{P}_i = \Omega$ for all i, and for all $P_j, P_k \in \mathscr{P}_i$ with $P_j \neq P_k$, we have $P_j \cap P_k = \emptyset$.

Observations and Information Partitions

Example

$\Omega = \{x, y, z\}$, $\mathscr{P}_1 = \{\{x\}, \{y, z\}\}$, $\mathscr{P}_2 = \{\{x, y\}, \{z\}\}$.

A function $f : \Omega \to X$ respects an information partition for player i if $f(\omega) = f(\omega')$ whenever $\omega, \omega' \in P_i$ for some $P_i \in \mathscr{P}_i$.

Example

f respects $\mathscr{P}_1$ if $f(y) = f(z)$.

Correlated Equilibria – Formally

Definition

A correlated equilibrium of a strategic game $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ consists of

- a finite probability space (Ω, π),
- for each player $i \in N$, an information partition $\mathscr{P}_i$ of Ω,
- for each player $i \in N$, a function $\sigma_i : \Omega \to A_i$ that respects $\mathscr{P}_i$ (σ_i is player i's strategy)

such that for every $i \in N$ and every function $\tau_i : \Omega \to A_i$ that respects $\mathscr{P}_i$ (i.e. for every possible strategy of player i), we have

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega)).$$

Correlated Equilibria

Example

	L	R
T	6, 6	2, 7
B	7, 2	0, 0

Nash equilibria: (T, R) with payoffs $(2, 7)$, (B, L) with payoffs $(7, 2)$, and $((\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}))$ with payoffs $(4 + \frac{2}{3}, 4 + \frac{2}{3})$.

Correlated Equilibria

Example

$$\begin{array}{c|c|c|}
 & L & R \\
\hline
T & 6,6 & 2,7 \\
\hline
B & 7,2 & 0,0 \\
\hline
\end{array}$$

Nash equilibria: (T,R) with payoffs $(2,7)$, (B,L) with payoffs $(7,2)$, and $((\frac{2}{3},\frac{1}{3}),(\frac{2}{3},\frac{1}{3}))$ with payoffs $(4+\frac{2}{3},4+\frac{2}{3})$.

Better correlated equilibrium: Assume $\Omega = \{x,y,z\}$, $\pi(x) = \pi(y) = \pi(z) = \frac{1}{3}$, $\mathscr{P}_1 = \{\{x\},\{y,z\}\}$, $\mathscr{P}_2 = \{\{x,y\},\{z\}\}$. Set $\sigma_1(x) = B, \sigma_1(y) = \sigma_1(z) = T$ and $\sigma_2(x) = \sigma_2(y) = L, \sigma_2(z) = R$.

This is a correlated equilibrium with payoffs $(5,5)$.

Note: This example only works with uncertainty about states.

Connection to Nash Equilibria

Proposition

For every mixed strategy Nash equilibrium α of a finite strategic game $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$, there is a correlated equilibrium $\langle(\Omega, \pi), (\mathscr{P}_i), (\sigma_i)\rangle$ in which, for each player i, the distribution on A_i induced by σ_i is α_i.

This means that correlated equilibria are a generalization of Nash equilibria.

Connection to Nash Equilibria

Proof.

Let $\Omega = A$ and define $\pi(a) = \prod_{j \in N} \alpha_j(a_j)$. For each player i, let a, b be in the same $P \in \mathscr{P}_i$ iff $a_i = b_i$. Define $\sigma_i(a) = a_i$ for each $a \in A$.

Then $\langle(\Omega, \pi), (\mathscr{P}_i), (\sigma_i)\rangle$ is a correlated equilibrium since the left hand side of the "best-response inequality"

$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \sigma_i(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\sigma_{-i}(\omega), \tau_i(\omega))$$

is the Nash equilibrium payoff, and for each player i at least as good any other strategy τ_i respecting the information partition. Furthermore, the distribution induced by σ_i is α_i. □

Summary

- In correlated equilibria, players can make their actions dependent on a signal received before the game.
- Players may be unable to distinguish some signals.
- In a correlated equilibrium, each player's state-to-action mapping is a best response to the others' state-to-action mappings in the context of the possible states and their probabilities (which are part of the correlated equilibrium).
- Equivalently: for every possible state, each player's action for that state is optimal given the other players' strategies and its knowledge about the state.
- Correlated equilibria generalize MSNE.
- They can lead to higher payoffs than MSNE.