Introduction to Game Theory

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Exercise Sheet 2 — Solutions

Exercise 2.1 (Nash equilibria, 2 points)

Consider the following strategic game:

		Player 2		
		X	Y	Z
	A	$1,1$	$2,1$	$0,2$
Player 1	B	$1,1$	$0,1$	$2,2$
	C	$2,2$	$1,0$	$1,1$

Determine and write down all Nash equilibria.

Solution:

Strategies A and Y can be eliminated since they are strictly dominated.

- pure strategy NE: (B, Z) (1p)
- pure strategy NE: (C, X) (1p)

Exercise 2.2 (Minimax strategy profiles, 1.5+1.5 points)

Let G be a two-player zero-sum game that has a Nash equilibrium.

(a) Show that if some of player 1's payoffs are increased in such a way that the resulting game G' is also a zero-sum game then G' has no Nash equilibrium in which player 1 gets a lower payoff than he got in the Nash equilibria of G.

Solution:

Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ and $G' = \langle N, (A_i)_{i\in N}, (u'_i)_{i\in N}\rangle$ be zero sum games with $N = \{1, 2\}$ and $u'_1(a_1, a_2) \geq u_1(a_1, a_2)$ for all $a_1 \in A_1$ and $a_2 \in A_2$.

If there is no Nash equilibrium in G', there is nothing to show.

Let's assume (a'_1, a'_2) is a Nash equilibrium in G'. Furthermore, let (a^*_1, a^*_2) be a Nash equilibrium in G (which exists according to specification). With the maximinimizer theorem the following holds

$$\begin{aligned} u'_1(a'_1, a'_2) &= \max_{a_1\in A_1} \min_{a_2\in A_2} u'_1(a_1, a_2) \\ &= \min_{a_2\in A_2} \max_{a_1\in A_1} u'_1(a_1, a_2) \\ &\geq \min_{a_2\in A_2} \max_{a_1\in A_1} u_1(a_1, a_2) = \max_{a_1\in A_1} \min_{a_2\in A_2} u_1(a_1, a_2) = u_1(a^*_1, a^*_2) \end{aligned}$$

Thus, in every Nash equilibrium (a'_1, a'_2) of G' player 1 gets the same or a higher utility as in the Nash equilibria of G. This does not necessarily hold true for non-ZSGs.

(b) Show that the game G' that results from G by elimination of one of player 1's strategies does not have a Nash equilibrium in which player 1's payoff is higher than it is in the Nash equilibria of G.

Solution:

Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ and $G' = \langle N, (A'_i)_{i\in N}, (u'_i)_{i\in N}\rangle$ be ZSGs with

$$\begin{aligned} N &= \{1,2\}, \\ A'_1 &= A_1 \setminus \{a_1\} \text{ for some } a_1 \in A_1, \\ A'_2 &= A_2, \\ u'_1(a) &= u_1(a) \text{ for all } a \in A'_1 \times A'_2 \end{aligned}$$

Let (a_1^*, a_2^*) be a Nash equilibrium in G, which exists according to specification, and let (a'_1, a'_2) be a NE in G', which we assume exists because otherwise there is nothing to show. As G and G' are ZSGs it holds

$$\begin{aligned} u'_1(a'_1, a'_2) = u_1(a'_1, a'_2) &= \max_{a_1 \in A'_1} \min_{a_2 \in A'_2} u_1(a_1, a_2) \\ &= \max_{a_1 \in A'_1} \min_{a_2 \in A_2} u_1(a_1, a_2) \\ &\leq \max_{a_1 \in A_1} \min_{a_2 \in A_2} u_1(a_1, a_2) = u_1(a_1^*, a_2^*), \end{aligned}$$

Thus, in every NE of G', the utility of player 1 is at best as high as in any NE of G. This (again) does not necessarily hold true for non-ZSGs.

Exercise 2.3 (Best response function, 3 points)
Let $G = \langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ with $N = \{1,2\}$, $A_1 = A_2 = \mathbb{R}^{\geq 0}$, $u_1(a_1, a_2) = a_1(a_2 - a_1)$ and $u_2(a_1, a_2) = a_2(1 - \frac{1}{2}a_1 - a_2)$ for all $(a_1, a_2) \in A$.
Define all Nash equilibria of this game by constructing and analyzing the best response function of both players.

Solution:

A strategy profile (a_1, a_2) is a NE iff a_1 is a best response to a_2 and a_2 is a best response to a_1. Let's first look for best responses of player 1 to actions a_2 of player 2. The utility of player 1 is

$$u_1(a_1, a_2) = a_1 a_2 - a_1^2.$$

Action a_1 is a best response to a_2 iff it maximizes $u_1(a_1, a_2)$ given a_2. We have a maximum that does not lie on the boundaries of the function's domain, iff

$$\frac{\partial}{\partial a_1} u_1(a_1, a_2) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial a_1^2} u_1(a_1, a_2) < 0.$$

We have

$$\frac{\partial}{\partial a_1} u_1(a_1, a_2) = a_2 - 2a_1 = 0 \quad \text{iff} \quad a_1 = \frac{1}{2} a_2 \tag{1}$$

$$\frac{\partial^2}{\partial a_1^2} u_1(a_1, a_2) = -2 < 0 \tag{2}$$

Therefore, the extremum is a maximum. The utility of player 2 is

$$u_2(a_1, a_2) = a_2 - \frac{1}{2}a_1 a_2 - a_2^2.$$

We have

$$\frac{\partial}{\partial a_2} u_2(a_1, a_2) = 1 - \frac{1}{2}a_1 - 2a_2 = 0 \quad \text{iff} \quad a_2 = \frac{1}{2}(1 - \frac{1}{2}a_1) \tag{3}$$

$$\frac{\partial^2}{\partial a_2^2} u_2(a_1, a_2) = -2 \tag{4}$$

Again, the extremum is a maximum. Consequently, we have a NE if $a_1 = \frac{1}{2}a_2$ and $a_2 = \frac{1}{2}(1 - \frac{1}{2}a_1)$. Solving this system of equations yields the following solution: $(a_1, a_2) = (\frac{2}{9}, \frac{4}{9})$. The strategic game G has a unique Nash equilibrium $(a_1, a_2) = (\frac{2}{9}, \frac{4}{9})$ with payoff profile $(\frac{4}{81}, \frac{16}{81})$.