Introduction to Game Theory

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Summer semester 2020

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Exercise Sheet 5 — Solutions

Exercise 5.1 (Uniqueness of SPE, 2 + 1 points)

Let Γ be an extensive two-player game with s^* and r^* being subgame perfect equilibria of Γ. Show (for $i \in N$):

(a) If Γ is a ZSG, then $u_i(O(s^*)) = u_i(O(r^*))$.

Solution:

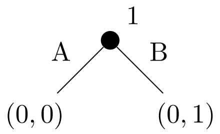
Let $G = \langle N, (A_i), (v_i) \rangle$ be the induced strategic game of Γ, s.t. $N = \{1, 2\}$, $A_i = \{s_i : H_i \to \bigcup_{h \in H_i} A(h) \mid \forall h \in H_i : s_i(h) \in A(h)\}$, where $H_i = \{h \in H \setminus Z \mid P(h) = i\}$, and $v_i(s_1, s_2) = u_i(O(s_1, s_2))$.

Let s^* and r^* be subgame-perfect equilibria of Γ. Then s^* and r^* are also Nash equilibria in G. According to the maximinimizer theorem, all Nash equilibria of a ZSG have the same payoffs. Thus, $v_1(s^*) = v_1(r^*)$ and $v_2(s^*) = v_2(r^*)$, and, accordingly, $u_1(O(s^*)) = u_1(O(r^*))$ and $u_2(O(s^*)) = u_2(O(r^*))$.

(b) For general extensive games, $u_i(O(s^*)) = u_i(O(r^*))$ is not necessarily true.

Solution:

Consider the extensive game $\Gamma = \langle N, H, P, (u_i) \rangle$ with $N = \{1, 2\}$, $H = \{\langle\rangle, \langle A \rangle, \langle B \rangle\}$, $P = \{\langle\rangle \mapsto 1\}$, $u_1(\langle A \rangle) = u_1(\langle B \rangle) = u_2(\langle A \rangle) = 0$ and $u_2(\langle B \rangle) = 1$, i.e.,

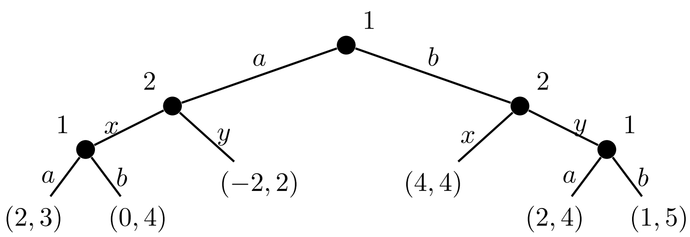


Both $s^* = (s_1^*, s_2^*) = (\{\langle\rangle \mapsto A\}, \emptyset)$ and $r^* = (r_1^*, r_2^*) = (\{\langle\rangle \mapsto B\}, \emptyset)$ are SPE, but their payoffs differ:

$$u(O(s^*)) = u(\langle A \rangle) = (0, 0) \neq (0, 1) = u(\langle B \rangle) = u(O(r^*)).$$

Exercise 5.2 (Subgame perfect equilibria, 2 points)

Determine all subgame perfect equilibria of the extensive form game defined by the following game tree.



Solution:

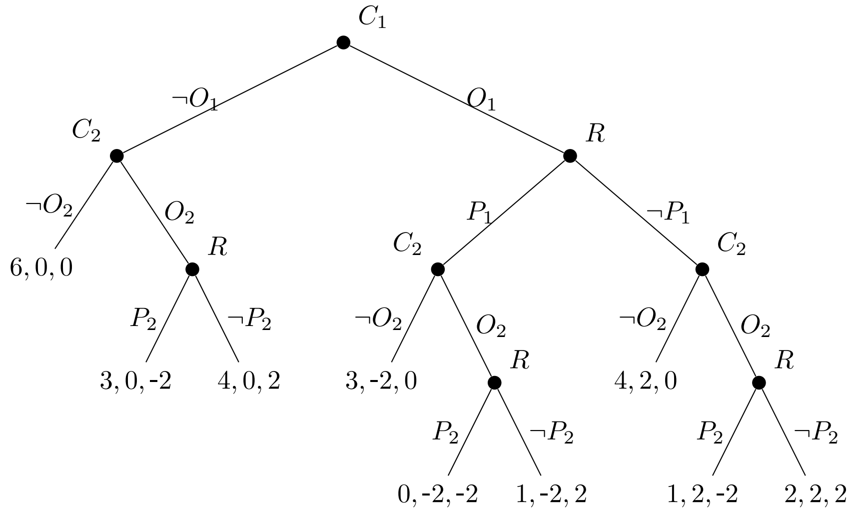
SPEs are: $(aaa, xy), (baa, xx), (baa, xy)$

Exercise 5.3 (Extensive Games, 1 + 1 + 1 points)

The owner of a retail chain R operates stores in K cities. In each city k, $1 \leq k \leq K$, there is a potential competitor C_k who can decide to open up a store (O_k) or to stay out of business ($\neg O_k$). If competitor C_k opens a store, R can either start a price war (P_k) or ignore the competitor ($\neg P_k$). The competitors make their decisions sequentially, i.e. when C_k makes its decision, $C_1, \ldots, C_{k-1}$ have already made their decisions and C_k is aware of their choice and the reactions of R. In every city k competitor C_k gets payoff 0 if he chooses to stay out of business, payoff 2 if he opens a store and R is not starting a price war, and payoff -2 if he opens a store and R starts a price war. The retail chain owner R gets a payoff of $3K$ if no competitor opens a store. For every competitor opening a store R's payoff is reduced by 2. For every price war R decides to start the payoff is additionally reduced by 1. Regard the special case of $K = 2$.

(a) Model this situation as an extensive game with perfect information and specify the game tree.

Solution:



Note: payoffs are given in the order (R, C_1, C_2).

(b) Specify each players set of strategies.

Solution:

$S_R = \{P_1P_2P_2P_2, P_1P_2P_2\neg P_2, \ldots, \neg P_1\neg P_2\neg P_2\neg P_2\}$
$S_{C_1} = \{O_1, \neg O_1\}$
$S_{C_2} = \{O_2O_2O_2, O_2O_2\neg O_2, \ldots, \neg O_2\neg O_2\neg O_2\}$

Note: Each strategy is written as a string of actions at decision nodes of the respective player as visited in a breadth-first order. For example, $O_2\neg O_2O_2$ denotes the following strategy of player C_2: $\{\langle\neg O_1\rangle \to O_2, \langle O_1, P_1\rangle \to \neg O_2, \langle O_1, \neg P_1\rangle \to O_2\}$.

(c) Determine a subgame perfect equilibrium.

Solution:

$(s_R, s_{C_1}, s_{C_2}) = (\neg P_1\neg P_2\neg P_2\neg P_2, O_1, O_2O_2O_2)$