Foundations of Artificial Intelligence

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Summer Term 2020

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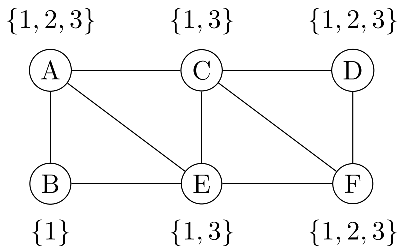
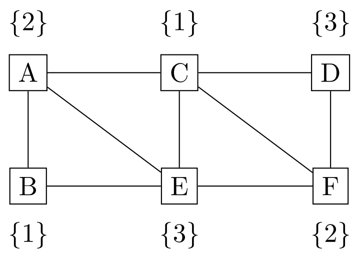
Exercise Sheet 4 — Solutions

Exercise 4.1 (Arc consistency)

Consider the constraint satisfaction problem given by the constraint graph below. The constraints are such that no two adjacent nodes have the same value. Establish arc-consistency.



Solution:



Exercise 4.2 (Satisfiability, Models)

(a) Decide for each of the following propositions whether they are valid, satisfiable or neither valid nor satisfiable.

(a) $\mathit{Smoke} \Rightarrow \mathit{Smoke}$

(b) $\mathit{Smoke} \Rightarrow \mathit{Fire}$

(c) $(\mathit{Smoke} \Rightarrow \mathit{Fire}) \Rightarrow (\neg \mathit{Fire} \Rightarrow \neg \mathit{Smoke})$

(d) $(\mathit{Smoke} \Rightarrow \mathit{Fire}) \Rightarrow ((\mathit{Smoke} \wedge \mathit{Heat}) \Rightarrow \mathit{Fire})$

(e) $\mathit{Spring} \Leftrightarrow \mathit{SunnyWeather}$

Solution:

In all cases it is possible to create a truth table in order to demonstrate validity or to disprove satisfiability. To show satisfiability, it is enough to find a single satisfiable variable assignment. Accordingly, validity is disproven by a single non-satisfiable variable assignment.

(a) $\mathit{Smoke} \Rightarrow \mathit{Smoke}$: This expression simplifies to: $\neg S \vee S$.

S	$\neg S$	$\neg S \vee S$
0	1	1
1	0	1

Hence, valid (truth table) and thus also satisfiable.

(b) $\mathit{Smoke} \Rightarrow \mathit{Fire}$: Satisfiable ($\{S \mapsto 1, F \mapsto 1\}$), but not valid ($\{S \mapsto 1, F \mapsto 0\}$)

(c) $(\mathit{Smoke} \Rightarrow \mathit{Fire}) \Rightarrow (\neg \mathit{Fire} \Rightarrow \neg \mathit{Smoke})$:This expression simplifies to: $\neg(\neg S \vee F) \vee (F \vee \neg S)$.

S	F	$\neg(\neg S \vee F)$	$(F \vee \neg S)$	$\neg(\neg S \vee F) \vee (F \vee \neg S)$
0	0	0	1	1
0	1	0	1	1
1	0	1	0	1
1	1	0	1	1

Hence, valid (truth table) and thus also satisfiable.

(d) $(\mathit{Smoke} \Rightarrow \mathit{Fire}) \Rightarrow ((\mathit{Smoke} \wedge \mathit{Heat}) \Rightarrow \mathit{Fire})$: This expression simplifies to: $\neg(\neg S \vee F) \vee (\neg(S \wedge H) \vee F)$.

H	S	F	$\neg(\neg S \vee F)$	$\neg(S \wedge H) \vee F$	$\neg(\neg S \vee F) \vee (\neg(S \wedge H) \vee F)$
0	0	0	0	1	1
0	0	1	0	1	1
0	1	0	1	1	1
0	1	1	0	1	1
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	0	1	1

Valid (truth table) and thus also satisfiable.

(e) $\mathit{Spring} \Leftrightarrow \mathit{SunnyWeather}$: Satisfiable ($\{Sp \mapsto 1, SW \mapsto 1\}$), but not valid ($\{Sp \mapsto 0, SW \mapsto 1\}$).

(b) Consider a vocabulary with only four propositions, A, B, C, and D. How many models are there for the following formulae? Explain.

(a) $(A \wedge B) \vee (B \wedge C)$

Solution:

Notation: $abcd$ with $a, b, c, d \in \{0, 1, X\}$ as a short form for $\{A \mapsto a, B \mapsto b, C \mapsto c, D \mapsto d\}$. X as notation for: either 0 or 1 is possible.

Models for $A \wedge B$ are all assignments $11XX$ (i.e., four cases: 1100, 1101, 1110, 1111), Models for $B \wedge C$ are all assignments $X11X$ (also four cases). An assignment is a model of $(A \wedge B) \vee (B \wedge C)$ if and only if it is a model of $A \wedge B$ or a model of $B \wedge C$. Thus, there are six models in total, since 1110 and 1111 should not be counted twice.

(b) $A \vee B$

Solution:

The only assignments which are *not* a model of $A \vee B$ are models of $\neg A \wedge \neg B$, i.e., four assignments of the form $00XX$. The remaining 12 out of 16 assignments are models of $A \vee B$.

(c) $(A \leftrightarrow B) \wedge (B \leftrightarrow C)$

Solution:

Models have the form $000X$ or $111X$, i.e., four models in total.

Exercise 4.3 (CNF Transformation, Resolution Method)

The following transformation rules hold, whereby propositional formulae can be transformed into equivalent formulae. Here, φ, ψ, and χ are arbitrary propositional formulae:

$$\begin{aligned}
\neg\neg\varphi &\equiv \varphi && (1)\\
\neg(\varphi \vee \psi) &\equiv \neg\varphi \wedge \neg\psi && (2)\\
\varphi \vee (\psi \wedge \chi) &\equiv (\varphi \vee \psi) \wedge (\varphi \vee \chi) && (3)\\
\neg(\varphi \wedge \psi) &\equiv \neg\varphi \vee \neg\psi && (4)\\
\varphi \wedge (\psi \vee \chi) &\equiv (\varphi \wedge \psi) \vee (\varphi \wedge \chi) && (5)
\end{aligned}$$

Additionally, the operators $\vee$ and $\wedge$ are associative and commutative.

Consider the formula $((C \wedge \neg B) \leftrightarrow A) \wedge (\neg C \to A)$.

(a) Transform the formula into a clause set K using the CNF transformation rules. Write down the steps.

Solution:

Transformation to CNF:

$$\begin{aligned}
((C \wedge \neg B) \leftrightarrow A) \wedge (\neg C \to A) &\equiv ((C \wedge \neg B) \to A) \wedge (A \to (C \wedge \neg B)) \wedge (\neg C \to A)\\
&\equiv (\neg(C \wedge \neg B) \vee A) \wedge (\neg A \vee (C \wedge \neg B)) \wedge (C \vee A)\\
&\equiv (\neg C \vee B \vee A) \wedge (\neg A \vee C) \wedge (\neg A \vee \neg B) \wedge (C \vee A)
\end{aligned}$$

In Clause Normal Form we thus have

$$K = \{\{A, B, \neg C\}, \{\neg A, C\}, \{\neg A, \neg B\}, \{A, C\}\}.$$

(b) Afterwards, using the resolution method, show whether $K \models (\neg B \rightarrow (A \wedge C))$ holds.

Solution:

In order to show that $K \models \varphi$, it is sufficient to show that $K \cup \{\neg\varphi\} \models \bot$. Therefore, we first have to extend K by the clauses corresponding to $\neg(\neg B \rightarrow (A \wedge C))$, and then derive a contradiction (empty set) using resolution. Transformation of $\neg(\neg B \rightarrow (A \wedge C))$ to CNF:

$$\begin{aligned}\neg(\neg B \rightarrow (A \wedge C)) &\equiv \neg B \wedge \neg(A \wedge C)\\ &\equiv \neg B \wedge (\neg A \vee \neg C)\end{aligned}$$

i.e., $\{\{\neg B\}, \{\neg A, \neg C\}\}$.

Resolution (one of many possibilities, recommended notation):

$$\begin{aligned}
&\Big\{A, B, \neg C\Big\} && (1)\\
&\Big\{\neg A, C\Big\} && (2)\\
&\Big\{\neg A, \neg B\Big\} && (3)\\
&\Big\{A, C\Big\} && (4)\\
&\Big\{\neg B\Big\} && (5)\\
&\Big\{\neg A, \neg C\Big\} && (6)\\
(1) + (5):\quad &\Big\{A, \neg C\Big\} && (7)\\
(2) + (6):\quad &\Big\{\neg A\Big\} && (8)\\
(4) + (7):\quad &\Big\{A\Big\} && (9)\\
(8) + (9):\quad &\emptyset && (10)
\end{aligned}$$