Advanced Algorithms — Problem set 2

Whenever you give an algorithm, also argue for its running time and correctness. Always feel free to ask in the forum or send a mail if you're stuck, need a clarification, or suspect a typo.

1. Consider a random experiment that succeeds with probability at least p. The experiment is repeated in independent trials.
   (a) Give a lower bound on the probability of succeeding at least once in t trials.
   (b) Depending on p, determine a number $c \in \mathbb{N}$ such that, with probability at least 0.99, the experiment succeeds at least once in c trials.
   (c) In the lecture, we analyzed Karger's algorithm and found that, in a graph with n vertices, it succeeds in finding any fixed minimum cut with probability at least $\frac{2}{n(n-1)}$. How many repetitions of Karger's algorithm are needed to make sure that it succeeds at least once with probability ≥ 0.99? You can use that $1 - x \leq e^{-x}$.

2. Let p_n be the probability that a string $x \in \{0,1\}^n$, chosen uniformly at random, does not contain two consecutive 1s. Show that $p_n \to 0$ as $n \to \infty$. (Hint: You can split $x \in \{0,1\}^n$ into small pieces, determine the probability that each piece satisfies the desired property, and then use this to derive an upper bound of $O(c^n)$ for $c < 2$ on the number of $x \in \{0,1\}^n$ satisfying the property.)

3. A set of n employees align in a queue in front of you. The first employee in the queue (counting from the back) passes his business card to the second employee. The second employee either passes the card he received to the third employee (with probability $\frac{1}{2}$) or discards the card and passes his own card to the third employee (with probability $\frac{1}{2}$). In general, the k-th employee either passes on the card he has received (with probability $1 - \frac{1}{k}$) or discards the card he received and passes on his own card (with probability $\frac{1}{k}$). You receive the card passed on by the n-th employee. What is the probability that this is the card of employee k for $k \in \{1, \ldots, n\}$? Show how you derived the solution.

4. Let $G = (V, E)$ be a graph with n vertices. For $k \in \mathbb{N}$, a *closed walk* of length k in G is a sequence of vertices $W = (v_1, \ldots, v_k)$ such that $\{v_i, v_{i+1}\} \in E$ for all $1 \leq i \leq k$, where we view $v_{k+1} = v_1$. The vertices are *not* required to be distinct. We say that W is a *cycle* if the vertices are all distinct. In the following, we consider $k = O(1)$.
   (a) Let $c : V \to \{1, \ldots, k\}$ be a function, which we view as an assignment of a color (from the palette $1, \ldots, k$) to each vertex of G. A closed walk W of length k is colorful under c if it contains a vertex of each color. Give a polynomial-time algorithm for determining whether a graph $G = (V, E)$, given together with $c : V \to \{1, \ldots, k\}$, has a colorful closed walk of length k. (Hint: Recall here that $k = O(1)$, so you may spend, say, a factor of $k!$ in the running time. This means you can iterate over all possible sequences of colors in the cycle.)
   (b) Let W be any fixed closed walk of length k in $G = (V, E)$. We draw a coloring $c : V \to \{1, \ldots, k\}$ uniformly at random.
      i. If W is a cycle, then what is the probability that W is colorful? (Hint: W has k vertices. Count the good colorings on these vertices and divide by the number of possible colorings.)
      ii. Using the fact that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for any $x \in \mathbb{R}$, show that the above probability is $\geq \frac{1}{e^k}$.
      iii. If W is not a cycle, then what is the probability that W is colorful? (Hint: Trick question.)
   (c) Using the above parts, give a randomized algorithm for testing whether G contains a cycle of length k. Your algorithm should run in polynomial time for $k = O(1)$, even though the running time may depend exponentially (or worse) on k. Your algorithm should always give the right answer when G has no cycle, and it should give the right answer with probability at least 0.99 when G does contain a cycle.