Problem set 1

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1 Independent t trials with probability at least p

1.1 (a)

The actual probability of at least one success is:

$$Pr(B(t,p') > 0) = 1 - Pr(B(t,p) = 0)$$

$$Pr(B(t,p) = 0) = \binom{t}{0} \cdot (1-p')^t = (1-p')^t$$

Where q' is the actual probability and since $p' >= p$

$$Pr(B(t,p') > 0) >= 1 - (1-p)^t$$

Making the lower bound:

$$Pr(B(t,p') > 0) = 1 - (1-p)^t$$

1.2 (b)

$$1 - 0.99 = 0.01 = Pr(B(c,p) = 0) >= (1-p)^c$$

Using that $log_b(x^v) = v \cdot log_b(x)$:

$$lg((1-p)^c) = c * lg((1-p)) = lg(0.01) = -2$$

$$c = ceil(-2/lg((1-p)))$$

A graph of the trend can be seen in figure 1.

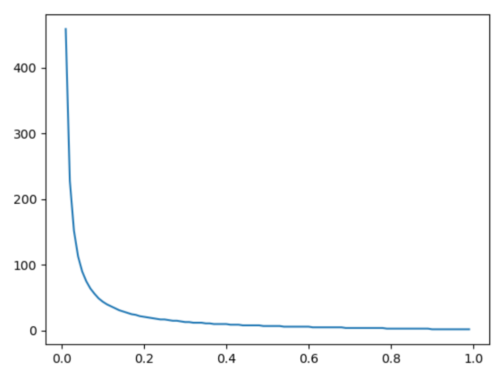

Figure 1: Plotting (p,c) points, x-axis=p, y-axis=c

1.3 (c)

Let F be the event that no minimum cut is selected in any of the t trials. The probability of one trial finding a min-cut is at least $\frac{2}{n(n-1)}$ and therefore:

$$Pr(F) = 1 - 0.99 <= e^{0.99}$$

$$Pr(F) = (1 - \frac{2}{n(n-1)})^t$$

$$<= \left(e^{-\frac{2}{n(n-1)}}\right)^t \qquad \text{using: } 1 - x <= e^{-x}$$

$$= e^{-\frac{2t}{n(n-1)}}$$

Using that $log_b(x^v) = v \cdot log_b(x)$:

$$e^{0.99} <= e^{-\frac{2t}{n(n-1)}}$$

$$ln(e^{0.99}) = 0.99 <= ln\left(e^{-\frac{2t}{n(n-1)}}\right)$$

$$= -\frac{2t}{n(n-1)} \cdot ln(e) = -\frac{2t}{n(n-1)}$$

Therefor it holds that:

$$0.99 \cdot n(n-1) <= -2t$$

$$t >= \frac{0.99 \cdot n(n-1)}{-2}$$

$$t >= 0.495 \cdot n(n-1)$$

2 No two consecutive 1s in n length bit string.

Let B_i be the event that bit string i and i+1 are both 1. Let C_i be the event that all $B_j = 0$ for all j=0..i.

$$Pr(B_i = 0) = 3/4 \qquad \text{as both bits have probability } 1/2 \text{ of being } 1$$

$$Pr(p_n = 0) = Pr(C_n = 1) = \prod_{i=1}^{n} Pr(B_i = 0) \qquad \text{since each event } B_i \text{are mutually indpendent}$$

$$= \prod_{i=1}^{n} 3/4$$

$$= (3/4)^n$$

Therefore, $Pr(p_n) \to 0$ as $n \to$ inf, since $(3/4)^{inf} = 0$.

3 Business card passing

If it is from person k then must have passed from n-k people. Therefore, n-k people must have passed giving probability:

$$Pr(X = 1) = (1/k) \cdot \prod_{i=k+1}^{n} (1 - 1/i)$$

$$Pr(X = 1) = (1/k) \cdot \prod_{i=k+1}^{n} ((i - 1)/i)$$

$$Pr(X = 1) = (1/k) \cdot \prod_{i=k+1}^{n} (i - 1) \cdot \prod_{i=k+1}^{n} 1/i$$

The First series of multiplications f_1 become:

$$(k + 1 - 1)(k + 2 - 1)(k + 3 - 1)...(n - 1)$$

$$(k)(k + 1)(k + 2)...(n - 1)$$

The second series of multiplications f_2 become:

$$(1/(k + 1))(1/(k + 2))...(1/(n))$$

Therefore the second term of f_1 is canceled out by the first term of f_2. Following that it becomes:

$$Pr(X = 1) = (1/k) \cdot (k) \cdot (1/n)$$

$$Pr(X = 1) = 1/n$$

4 Closed walk on graph G

4.1 (a)

- Convert G to G'=(V',E') by converting each node v ∈ V 2 to nodes in V', an in node and an out node. Each edge (v,u) ∈ E are becoming 2 edges (v-out,u-in) and (u-out,v-in) ∈ E'.
- Make a total of k augmented matrices A_i, one for each edge color and an extra A_0 for the edges connecting v-in and u-out nodes for v != u.
  $O(n^2)$
- For each k! permutation of 1,..,k do 2k matrix multiplications of the augmented matrices in the same order as the permutations with a multiplication of A_0 between each permutation and at the end.
  $O(k! \cdot k \cdot n^\omega)$
- For each permutation, test if any of the resulting matrix contains a non-zero at the diagonal, answer yes.
  If at the end no such matrix is computed, return no.
  $O(k! \cdot k \cdot n^\omega)$

The total running time is $O(k! \cdot k \cdot n^\omega)$.

A slightly faster solution would be for each k! color combination make a bfs with all nodes of first color being put into the frontier and only proceeding to the next vertex in order of the color combination. Total here would be $O(k! \cdot (n + m))$

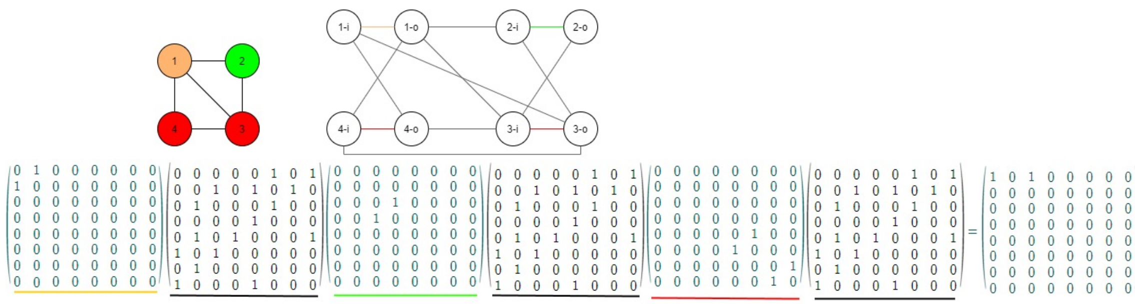

Figure 2: Example

4.2 (b)

4.2.1 (i)

The total number of colorings of a k-cycle is: k^k.
The total number of good colorings are:

$$
\begin{aligned}
k \cdot (k-1) \cdot (k-2) \cdot \ldots \cdot 1 & \qquad \text{k choices for first color, k-1 for second,...} \\
= k! &
\end{aligned}
$$

Because of that, the probability of a correct ordering is:

$$Pr(correct) = \frac{k!}{k^k}$$

4.2.2 (ii)

It should hold that:

$$\frac{k!}{k^k} >= \frac{1}{e^k}$$

$$\iff \frac{k^k}{k!} <= e^k = \sum_{n=0}^{\infty} \frac{k^n}{n!}$$

When n = k.

$$\frac{k^k}{k!} <= \frac{k^k}{k!} + \sum_{n=0, n!=k}^{\infty} \frac{k^n}{n!}$$

$$0 <= \sum_{n=0, n!=k}^{\infty} \frac{k^n}{n!}$$

Therefore, the probability is not larger than $1/e^k$

4.2.3 (iii)

Since it is not a cycle, at least 2 nodes u,v in the walk are identical and thus have the same color, meaning it can't be colorful making the probability 0.

4.3 (c)

Algorithm
Pick a starting node n_0 and iteratively pick an edge from the current node as the next edge in the k-path, until a path of suitable length has been found.
If all vertices in the pack are distinct and if the last edge in the path is connected to n_0 then answer yes else answer no.

Analysis
The probability p for a random k-path is a cycle is $\frac{k!}{k^k}$.
So the amount of trials t needed to answer yes with probability of at least 0.99 is therefore:

$$1 - 0.99 = 0.01 = Pr(B(t, p) = 0) >= (1 - p)^t$$

Using that $log_b(x^v) = v \cdot log_b(x)$:

$$lg(0.01) = -2 == lg((1 - p)^t) = t \cdot lg(1 - p)$$

$$t = ceil(-2/lg(1 - p))$$

Inserting value for p therefore give:

$$t >= ceil\left(\frac{-2}{lg\left(1 - \frac{k!}{k^k}\right)}\right)$$