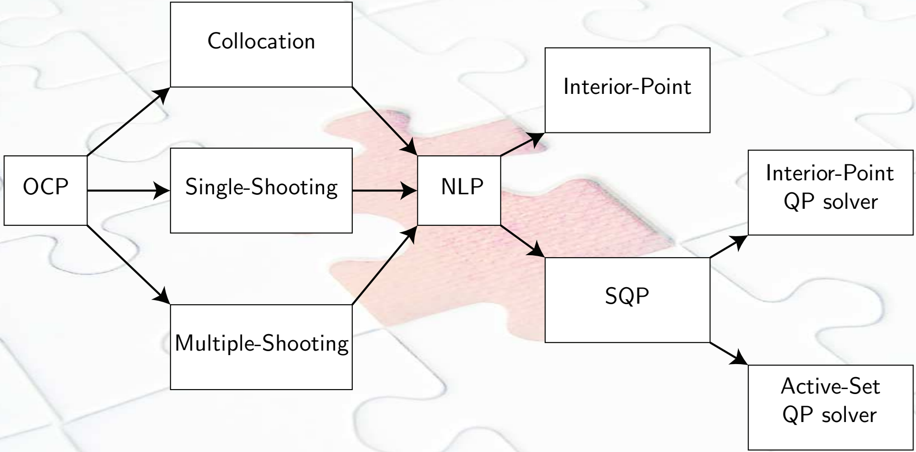
Numerical Optimal Control
Lecture 4: Interior-Point Methods

Sébastien Gros

ITK, NTNU

NTNU PhD course

Survival map of Direct Optimal Control



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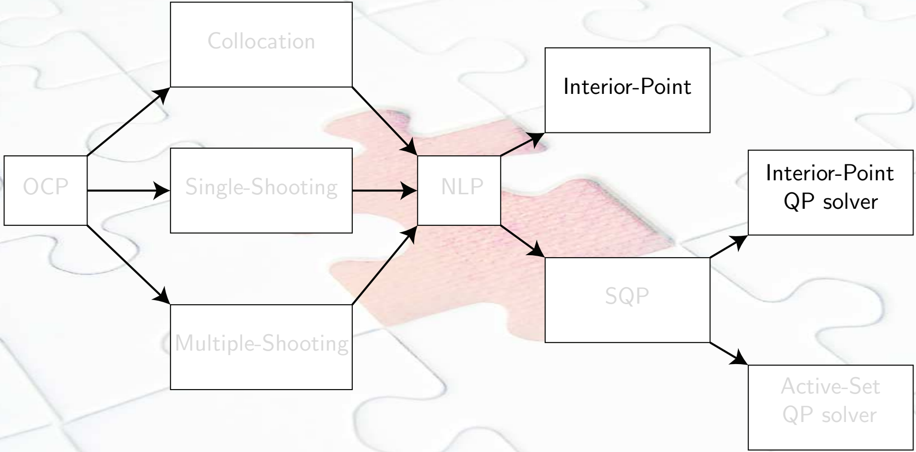

Let's approach again the problem of solving the KKT conditions

Outline

KKT conditions - Reminder

Consider the NLP problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions with $\mathcal{L} = \Phi(\mathbf{w}) + \boldsymbol{\lambda}^\top \mathbf{g}(\mathbf{w}) + \boldsymbol{\mu}^\top \mathbf{h}(\mathbf{w})$

$$\boxed{\begin{array}{rl} \text{Primal Feasibility:} & \mathbf{g}(\mathbf{w}) = 0, \quad \mathbf{h}(\mathbf{w}) \leq 0, \\ \text{Dual Feasibility:} & \nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = 0, \quad \boldsymbol{\mu} \geq 0, \\ \text{Complementarity Slackness:} & \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0, \quad \forall\, i \end{array}}$$

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The difficulty of the KKT conditions is the non-smooth **Complementarity Slackness** conditions resulting from the inequality constraints. Remember: "constraint $\mathbf{h}_i$ can push ($\boldsymbol{\mu}_i > 0$) only when $\mathbf{w}$ touches it (i.e. when $\mathbf{h}_i = 0$)"

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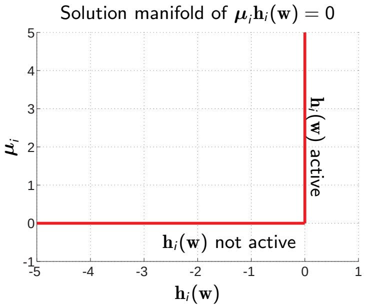

KKT conditions

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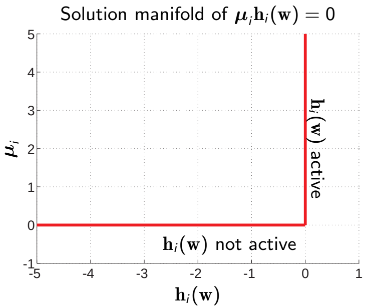

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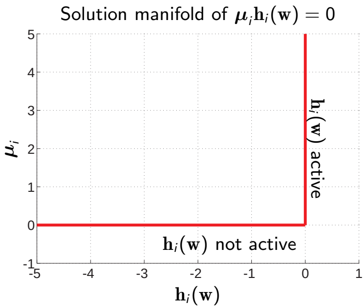

KKT conditions with $\mathcal{L} = \frac{1}{2}w^2 - w + \mu w$

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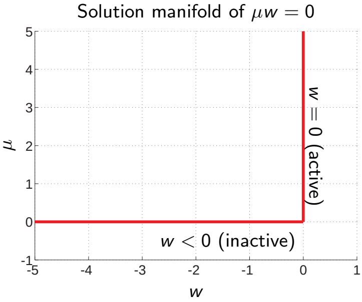

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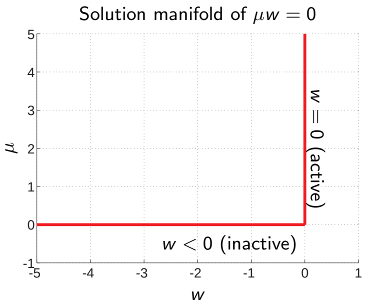

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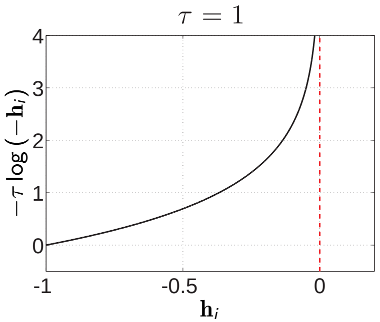
Original idea of the IP method: introduce the inequality constraints in the cost !!
Very large (possibly ∞) penalty for violating feasibility $\equiv$ "barrier"...

Primal Interior-Point Methods

Log-barrier method: introduce the inequality constraints in the cost function

$$\begin{array}{ll}\min\limits_{\mathbf{w}} & \Phi(\mathbf{w}) \\ \text{s.t.} & \mathbf{h}(\mathbf{w}) \leq 0\end{array} \qquad \textit{becomes} \qquad \min_{\mathbf{w}_\tau} \Phi_\tau\left(\mathbf{w}_\tau\right) = \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$



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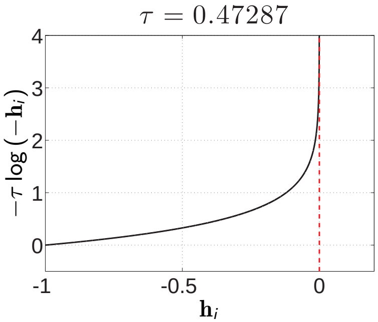

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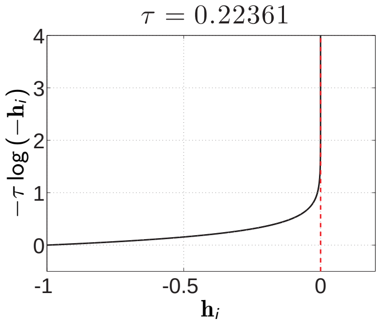

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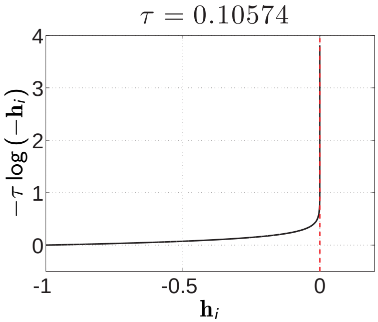

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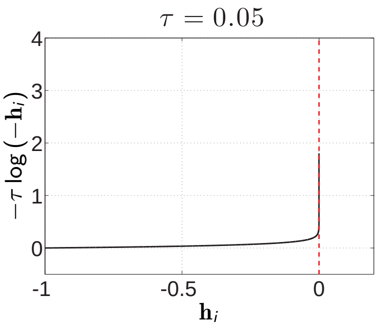
Log-barrier approximates the *characteristic function*

$$\chi\left(\mathbf{h}_i\right) = \begin{cases} 0 & \text{if} \quad \mathbf{h}_i \leq 0 \\ \infty & \text{if} \quad \mathbf{h}_i > 0 \end{cases}$$



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$$\Phi\left(w\right) = \frac{1}{2}w^2 - 2w$$

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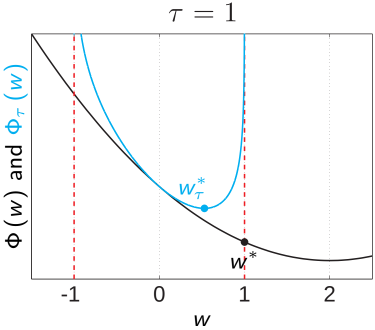

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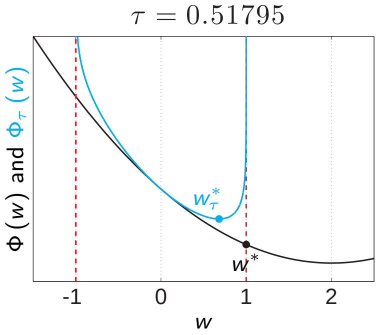

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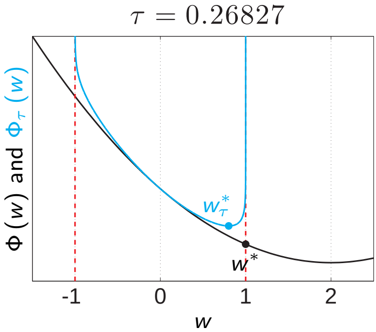

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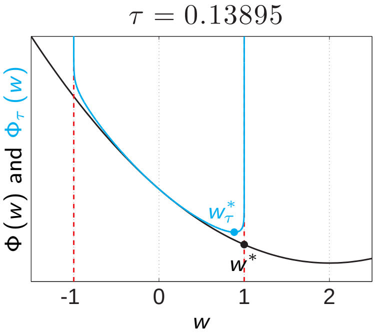

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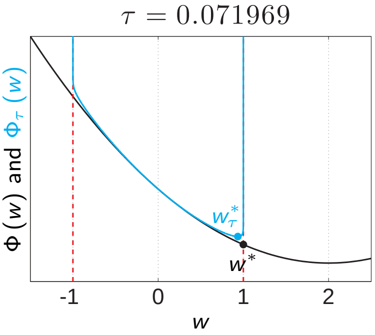

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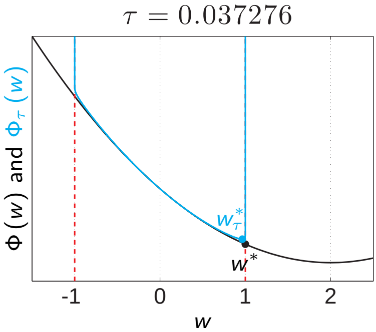

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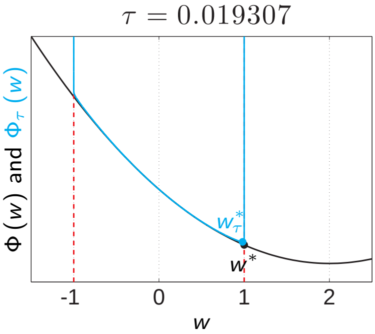

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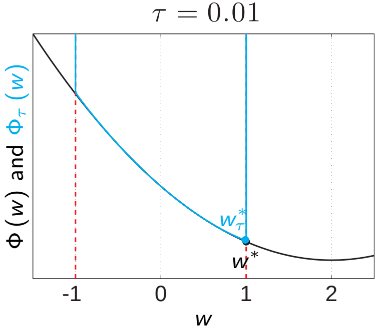

How accurate is the solution w_τ^* ?

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$$\begin{aligned}\min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0\end{aligned} \qquad \textit{becomes} \qquad \min_{\mathbf{w}_\tau} \Phi_\tau(\mathbf{w}_\tau) = \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

Example:

$$\begin{aligned}\min_{w} \quad & \frac{1}{2}w^2 - 2w \\ \text{s.t.} \quad & -1 \leq w \leq 1\end{aligned}$$

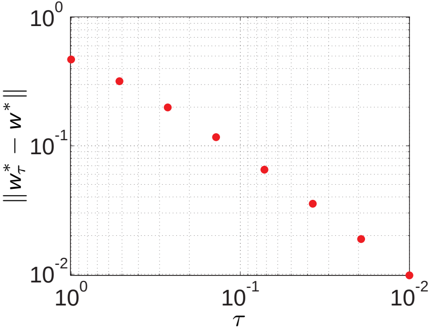
I.e.

$$\Phi(w) = \frac{1}{2}w^2 - 2w$$

$$\mathbf{h}(w) = \begin{bmatrix} -w-1 \\ w-1 \end{bmatrix} \leq 0$$

$$\Phi_\tau(w) = \tfrac{1}{2}w^2 - 2w - \tau\log(w+1) - \tau\log(1-w)$$

How accurate is the solution w_τ^* ?



Primal Interior-Point Methods

Log-barrier method: introduce the inequality constraints in the cost function

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned} \qquad \textit{becomes} \qquad \min_{\mathbf{w}_\tau} \Phi_\tau(\mathbf{w}_\tau) = \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

Example:

$$\begin{aligned} \min_{w} \quad & \frac{1}{2} w^2 - 2w \\ \text{s.t.} \quad & -1 \leq w \leq 1 \end{aligned}$$

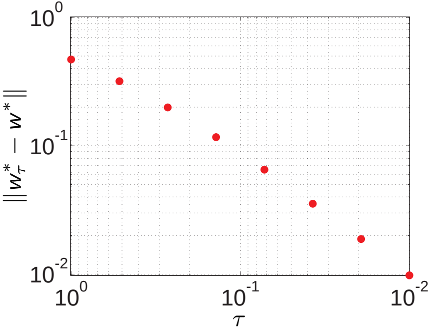
I.e.

$$\Phi(w) = \frac{1}{2} w^2 - 2w$$

$$\mathbf{h}(w) = \begin{bmatrix} -w - 1 \\ w - 1 \end{bmatrix} \leq 0$$

$$\Phi_\tau(w) = \tfrac{1}{2} w^2 - 2w - \tau \log(w+1) - \tau \log(1-w)$$

How accurate is the solution w_τ^* ?



If $\mathbf{w}^*$ has LICQ & SOSC, then

$$\|\mathbf{w}_\tau^* - \mathbf{w}^*\| = O(\tau)$$

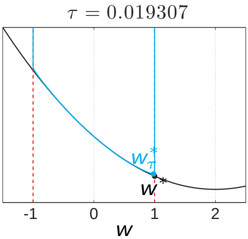
Newton on the Primal Interior-Point method

Problem:

$$\begin{aligned}\min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0\end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}} \; \Phi_\tau(\mathbf{w}) = \min_{\mathbf{w}} \; \Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}))$$



Newton on the Primal Interior-Point method

Problem:

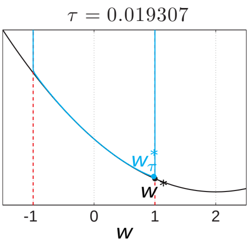
$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}} \ \Phi_\tau(\mathbf{w}) = \min_{\mathbf{w}} \ \Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}))$$



Newton on the Primal Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

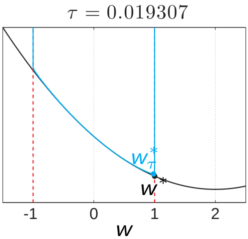
Barrier formulation:

$$\min_{\mathbf{w}} \Phi_\tau(\mathbf{w}) = \min_{\mathbf{w}} \Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}))$$

KKT conditions*:

$$\nabla\Phi_\tau(\mathbf{w}) = \nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-1}\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$



Newton on the Primal Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}} \Phi_\tau(\mathbf{w}) = \min_{\mathbf{w}} \Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}))$$

KKT conditions*:

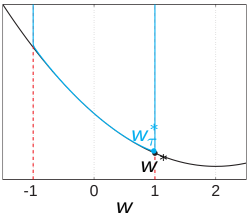
$$\nabla\Phi_\tau(\mathbf{w}) = \nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-1} \nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Newton direction for the Primal Interior-Point KKTs:

$$\underbrace{\left(\nabla^2\Phi(\mathbf{w}) + \tau \sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-2} \nabla\mathbf{h}_i \nabla\mathbf{h}_i^\top\right)}_{=\nabla^2\Phi_\tau(\mathbf{w})} \Delta\mathbf{w} + \nabla\Phi_\tau(\mathbf{w}) = 0$$

for $\mathbf{h}$ affine



Newton on the Primal Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}} \Phi_\tau(\mathbf{w}) = \min_{\mathbf{w}} \Phi(\mathbf{w}) - \tau\sum_{i=1}^{m_i}\log(-\mathbf{h}_i(\mathbf{w}))$$

KKT conditions*:

$$\nabla\Phi_\tau(\mathbf{w}) = \nabla\Phi(\mathbf{w}) - \tau\sum_{i=1}^{m_i}\mathbf{h}_i(\mathbf{w})^{-1}\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

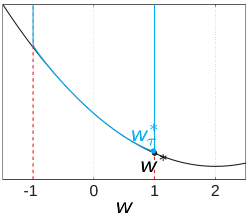
*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Newton direction for the Primal Interior-Point KKTs:

$$\left(\nabla^2\Phi(\mathbf{w}) + \tau\sum_{i=1}^{m_i}\mathbf{h}_i(\mathbf{w})^{-2}\nabla\mathbf{h}_i\nabla\mathbf{h}_i^\top\right)\Delta\mathbf{w} + \Phi_\tau(\mathbf{w}) = 0$$

for $\mathbf{h}$ affine

$\tau = 0.019307$



Newton on the Primal Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}} \Phi_\tau(\mathbf{w}) = \min_{\mathbf{w}} \Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}))$$

KKT conditions*:

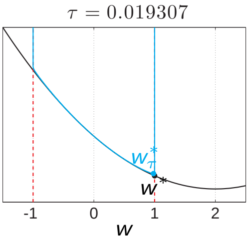
$$\nabla\Phi_\tau(\mathbf{w}) = \nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-1} \nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Newton direction for the Primal Interior-Point KKTs:

$$\left(\nabla^2\Phi(\mathbf{w}) + \tau \sum_{i=1}^{m_i} \mathbf{h}_i(\mathbf{w})^{-2} \nabla\mathbf{h}_i \nabla\mathbf{h}_i^\top\right) \Delta\mathbf{w} + \Phi_\tau(\mathbf{w}) = 0$$

for $\mathbf{h}$ affine



As $\tau \to 0$, the term $\mathbf{h}_i^{-2}(\mathbf{w})$ becomes very large when $\mathbf{h}_i \to 0$, which hinders the convergence (very strong curvature at an active constraint)

Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w})\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Primal-Dual Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w}) \nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w}) \nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Introduce variable $\boldsymbol{\nu}_i = -\tau\mathbf{h}_i^{-1}(\mathbf{w})$

Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w}) \nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Introduce variable $\boldsymbol{\nu}_i = -\tau\mathbf{h}_i^{-1}(\mathbf{w})$, then the Primal-Dual KKT conditions† read as:

$$\nabla\Phi(\mathbf{w}) + \sum_{i=1}^{m_i} \boldsymbol{\nu}_i \nabla\mathbf{h}_i(\mathbf{w}) = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) = -\tau$$

†valid for $\mathbf{h}_i(\mathbf{w}) < 0$, $\boldsymbol{\nu}_i > 0$

Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau\sum_{i=1}^{m_i}\log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau\sum_{i=1}^{m_i}\mathbf{h}_i^{-1}(\mathbf{w})\nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Introduce $\boldsymbol{\nu}_i = -\tau\mathbf{h}_i^{-1}$, then the Primal-Dual Interior-Point KKT conditions read as:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$

Primal-Dual Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w}) \nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Introduce $\boldsymbol{\nu}_i = -\tau\mathbf{h}_i^{-1}$, then the Primal-Dual Interior-Point KKT conditions read as:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} &= 0 \\ \boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau &= 0 \\ \mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} &> 0 \end{aligned}$$

- Primal-Dual IP conditions yield the same solution as the Barrier problem
- Newton steps don't yield anything "nasty"

Primal-Dual Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Barrier formulation:

$$\min_{\mathbf{w}_\tau} \Phi(\mathbf{w}_\tau) - \tau \sum_{i=1}^{m_i} \log(-\mathbf{h}_i(\mathbf{w}_\tau))$$

KKT conditions*:

$$\nabla\Phi(\mathbf{w}) - \tau \sum_{i=1}^{m_i} \mathbf{h}_i^{-1}(\mathbf{w}) \nabla\mathbf{h}_i(\mathbf{w}) = 0$$

*valid for $\mathbf{h}_i(\mathbf{w}) < 0$

Introduce $\boldsymbol{\nu}_i = -\tau\mathbf{h}_i^{-1}$, then the Primal-Dual Interior-Point KKT conditions read as:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} &= 0 \\ \boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau &= 0 \\ \mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} &> 0 \end{aligned}$$

- Primal-Dual IP conditions yield the same solution as the Barrier problem
- Newton steps don't yield anything "nasty"
- Observe the similitude with the original KKT conditions !!

Interpretation of the Primal-Dual Interior-Point method

Problem:

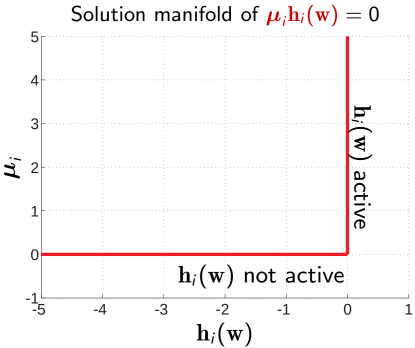
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

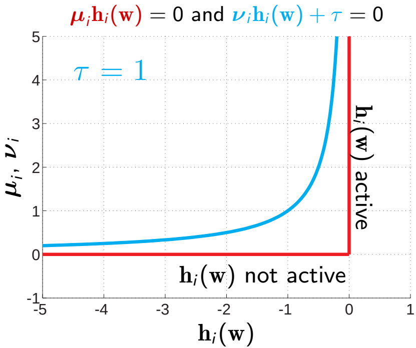
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

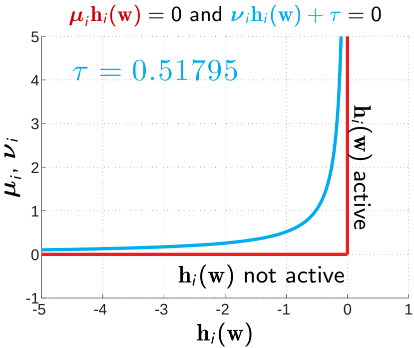
$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

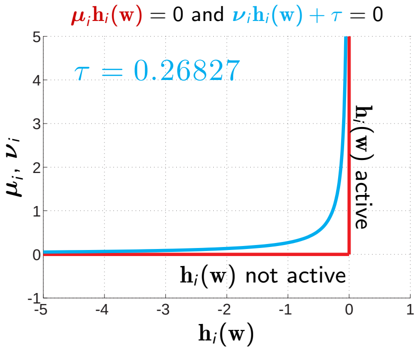
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

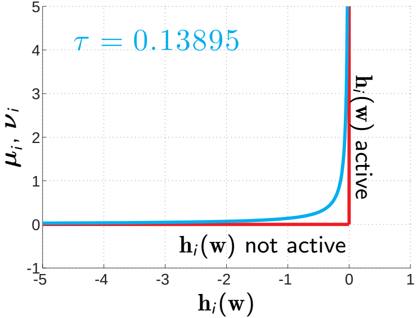
KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$

$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) = 0$ and $\boldsymbol{\nu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0$



Interpretation of the Primal-Dual Interior-Point method

Problem:

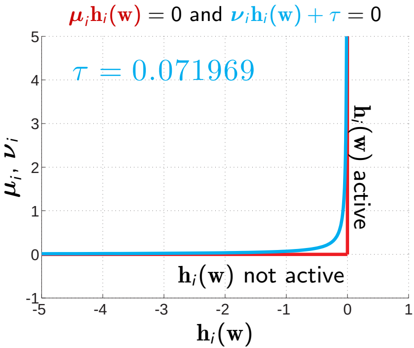
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

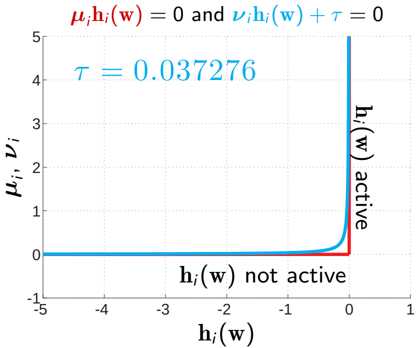
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
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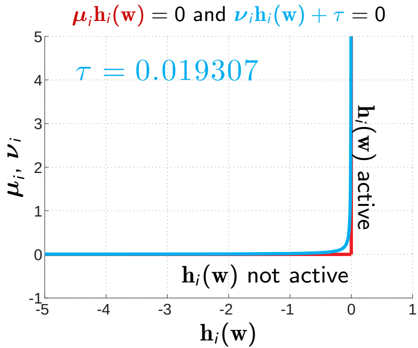
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

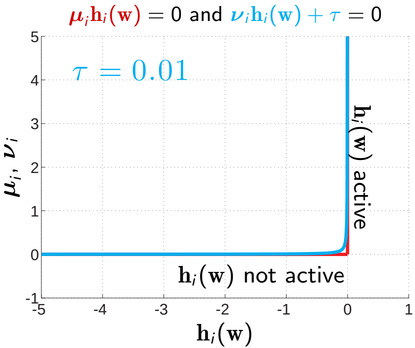
$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$



Interpretation of the Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

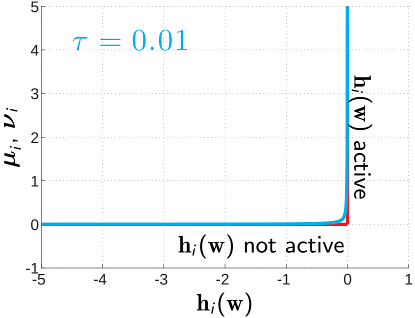
KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$

$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$ and $\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$



- Primal-Dual IP method solves KKT conditions with smoothed Complementarity slackness

Interpretation of the Primal-Dual Interior-Point method

Problem:

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{w}) \leq 0$$

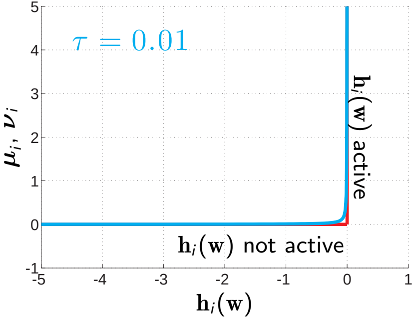
KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$
$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} \geq 0$$

Primal-Dual IP KKT conditions

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} = 0$$
$$\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} > 0$$

$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) = 0$ and $\boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0$



- Primal-Dual IP method solves KKT conditions with smoothed Complementarity slackness
- IP approximation

$$\|\boldsymbol{\mu}^* - \boldsymbol{\nu}^*\| = \mathcal{O}(\tau)$$
$$\|\mathbf{w}^* - \mathbf{w}^*_\tau\| = \mathcal{O}(\tau)$$

$\mathbf{w}^*_\tau$, $\boldsymbol{\nu}^*$ and $\mathbf{w}^*$, $\boldsymbol{\mu}^*$ are not distinguished

Interpretation of the Primal-Dual Interior-Point method

Problem:

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

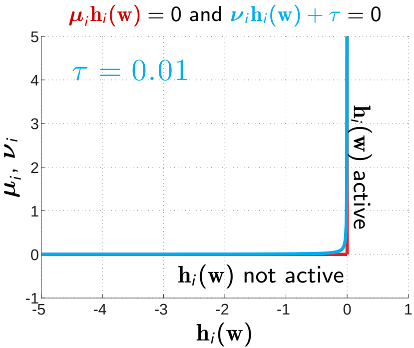
KKT conditions:

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Primal-Dual IP KKT conditions

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\nu} &= 0 \\ \boldsymbol{\nu}_i \mathbf{h}_i(\mathbf{w}) + \tau &= 0 \\ \mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\nu} &> 0 \end{aligned}$$

Note: the PD-IP KKT conditions require that $\mathbf{w}$ is **inside** the feasible domain



- Primal-Dual IP method solves KKT conditions with smoothed Complementarity slackness
- IP approximation

$$\begin{aligned} \|\boldsymbol{\mu}^* - \boldsymbol{\nu}^*\| &= \mathcal{O}(\tau) \\ \|\mathbf{w}^* - \mathbf{w}^*_\tau\| &= \mathcal{O}(\tau) \end{aligned}$$

$\mathbf{w}^*_\tau$, $\boldsymbol{\nu}^*$ and $\mathbf{w}^*$, $\boldsymbol{\mu}^*$ are not distinguished

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

KKT conditions

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{g}(\mathbf{w})\boldsymbol{\lambda} + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \mathbf{g}(\mathbf{w}) &= 0 \\ \boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) \qquad &= 0 \\ \mathbf{h}(\mathbf{w}) \leq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi\left(\mathbf{w}\right) \\ \text{s.t.} \quad & \mathbf{g}\left(\mathbf{w}\right) = 0 \\ & \mathbf{h}\left(\mathbf{w}\right) \leq 0 \end{aligned}$$

PD-IP KKT conditions

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{g}(\mathbf{w})\boldsymbol{\lambda} + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \mathbf{g}\left(\mathbf{w}\right) &= 0 \\ \boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) + \tau &= 0 \\ \mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} &> 0 \end{aligned}$$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

PD-IP KKT conditions

$$\begin{aligned} \nabla \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= 0 \\ \mathbf{g}(\mathbf{w}) &= 0 \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) + \tau &= 0 \\ \mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} &> 0 \end{aligned}$$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

Newton on the conditions (parametrized by τ)

$$\begin{bmatrix} \nabla \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) + \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$$

with $\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$
$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = 0$$
$$\mathbf{h}(\mathbf{w}) \leq 0$$

Newton on the conditions (parametrized by τ)

$$\begin{bmatrix} \nabla \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) + \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$$

with $\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0$

Newton direction $\mathbf{d}$ given by

$$\nabla \mathbf{r}_\tau^\top(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \end{bmatrix} + \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$$

Newton step: updates

$$\begin{bmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} \leftarrow \begin{bmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} + t \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \end{bmatrix}$$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

Newton on the conditions (parametrized by τ)

$$\begin{bmatrix} \nabla \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) + \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$$

with $\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0$

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Newton step: updates

$$\begin{bmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} \leftarrow \begin{bmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} + t \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \end{bmatrix}$$

Step-size: $t \in]0, 1]$ must ensure:

$$\mathbf{h}(\mathbf{w} + t\Delta\mathbf{w}) < 0, \quad \boldsymbol{\mu} + t\Delta\boldsymbol{\mu} > 0$$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

Newton on the conditions (parametrized by τ)

$$\begin{bmatrix} \nabla\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) + \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$$

with $\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0$

Newton direction $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau^\top(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \end{bmatrix} + \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$$

Newton step: updates

$$\begin{bmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} \leftarrow \begin{bmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} + t \begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \end{bmatrix}$$

Step-size: $t \in]0, 1]$ must ensure:

$$\mathbf{h}(\mathbf{w} + t\Delta\mathbf{w}) < 0, \quad \boldsymbol{\mu} + t\Delta\boldsymbol{\mu} > 0$$

Difficulties:

- Selecting t to get

  $$\mathbf{h}(\mathbf{w} + t\Delta\mathbf{w}) < 0$$

  cannot be done simply. Requires evaluating $\mathbf{h}$ for decreasing values of t until the condition is met. Can be expensive !!

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

Newton on the conditions (parametrized by τ)

$$\begin{bmatrix} \nabla \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) + \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$$

with $\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0$

Newton direction $\mathbf{d}$ given by

$$\nabla \mathbf{r}_\tau^\top(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \end{bmatrix} + \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = 0$$

Newton step: updates

$$\begin{bmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} \leftarrow \begin{bmatrix} \mathbf{w} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} + t \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \end{bmatrix}$$

Step-size: $t \in]0, 1]$ must ensure:

$$\mathbf{h}(\mathbf{w} + t\Delta\mathbf{w}) < 0, \quad \boldsymbol{\mu} + t\Delta\boldsymbol{\mu} > 0$$

Difficulties:

- Selecting t to get

  $$\mathbf{h}(\mathbf{w} + t\Delta\mathbf{w}) < 0$$

  cannot be done simply. Requires evaluating $\mathbf{h}$ for decreasing values of t until the condition is met. Can be expensive !!
- We need the initial guess to be feasible for $\mathbf{h}$!!

Slack formulation of the Primal-Dual Interior-Point conditions

Primal-Dual IP KKT conditions:

$$\nabla \Phi(\mathbf{w}) + \nabla \mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0 \qquad (1a)$$

$$\boldsymbol{\mu}_i \mathbf{h}_i(\mathbf{w}) + \tau = 0 \qquad (1b)$$

$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0 \qquad (1c)$$

- Newton steps on (1a)-(1b)
- Backtrack to ensure (1c)

Slack formulation of the Primal-Dual Interior-Point conditions

Primal-Dual IP KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0 \tag{1a}$$

$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0 \tag{1b}$$

$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0 \tag{1c}$$

- Newton steps on (1a)-(1b)
- Backtrack to ensure (1c)

Difficulty: one must ensure that $\mathbf{h}(\mathbf{w})$ starts and remains negative throughout the Newton iterations

Slack formulation of the Primal-Dual Interior-Point conditions

Primal-Dual IP KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0 \tag{1a}$$

$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0 \tag{1b}$$

$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0 \tag{1c}$$

- Newton steps on (1a)-(1b)
- Backtrack to ensure (1c)

Difficulty: one must ensure that $\mathbf{h}(\mathbf{w})$ starts and remains negative throughout the Newton iterations

- need a feasible initial guess

Slack formulation of the Primal-Dual Interior-Point conditions

Primal-Dual IP KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0 \tag{1a}$$
$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0 \tag{1b}$$
$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0 \tag{1c}$$

- Newton steps on (1a)-(1b)
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Difficulty: one must ensure that $\mathbf{h}(\mathbf{w})$ starts and remains negative throughout the Newton iterations

- need a feasible initial guess
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Slack formulation of the Primal-Dual Interior-Point conditions

Primal-Dual IP KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0 \qquad (1a)$$

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$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0 \qquad (1c)$$

- Newton steps on (1a)-(1b)
- Backtrack to ensure (1c)

Difficulty: one must ensure that $\mathbf{h}(\mathbf{w})$ starts and remains negative throughout the Newton iterations

- need a feasible initial guess
- backtracking can be expensive

Slack reformulation: new variable $-\mathbf{s} = \mathbf{h}(\mathbf{w})$

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\ -\boldsymbol{\mu}_i s_i + \tau &= 0 \\ -\mathbf{s} < 0, \quad \boldsymbol{\mu} &> 0 \end{aligned}$$

Slack formulation of the Primal-Dual Interior-Point conditions

Primal-Dual IP KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0 \qquad (1a)$$

$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0 \qquad (1b)$$

$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0 \qquad (1c)$$

- Newton steps on (1a)-(1b)
- Backtrack to ensure (1c)

Difficulty: one must ensure that $\mathbf{h}(\mathbf{w})$ starts and remains negative throughout the Newton iterations

- need a feasible initial guess
- backtracking can be expensive

Slack reformulation: new variable $-\mathbf{s} = \mathbf{h}(\mathbf{w})$

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0$$

$$\mathbf{h}(\mathbf{w}) + \mathbf{s} = 0$$

$$\boldsymbol{\mu}_i s_i - \tau = 0$$

$$\mathbf{s} > 0, \quad \boldsymbol{\mu} > 0$$

Slack formulation of the Primal-Dual Interior-Point conditions

Primal-Dual IP KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0 \qquad (1a)$$

$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0 \qquad (1b)$$

$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0 \qquad (1c)$$

- Newton steps on (1a)-(1b)
- Backtrack to ensure (1c)

Difficulty: one must ensure that $\mathbf{h}(\mathbf{w})$ starts and remains negative throughout the Newton iterations

- need a feasible initial guess
- backtracking can be expensive

Slack reformulation: new variable $-\mathbf{s} = \mathbf{h}(\mathbf{w})$

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\ \boldsymbol{\mu}_i s_i - \tau &= 0 \\ \mathbf{s} > 0, \quad \boldsymbol{\mu} &> 0 \end{aligned}$$

Newton on the slack formulation

- initialize with $\mathbf{s}$, $\boldsymbol{\mu} > 0$ and $\boldsymbol{\mu}_i s_i = \tau$

Slack formulation of the Primal-Dual Interior-Point conditions

Primal-Dual IP KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0 \qquad (1a)$$

$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0 \qquad (1b)$$

$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0 \qquad (1c)$$

- Newton steps on (1a)-(1b)
- Backtrack to ensure (1c)

Difficulty: one must ensure that $\mathbf{h}(\mathbf{w})$ starts and remains negative throughout the Newton iterations

- need a feasible initial guess
- backtracking can be expensive

Slack reformulation: new variable $-\mathbf{s} = \mathbf{h}(\mathbf{w})$

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\ \boldsymbol{\mu}_i s_i - \tau &= 0 \\ \mathbf{s} > 0, \quad \boldsymbol{\mu} &> 0 \end{aligned}$$

Newton on the slack formulation

- initialize with $\mathbf{s}$, $\boldsymbol{\mu} > 0$ and $\boldsymbol{\mu}_i s_i = \tau$
- $\mathbf{h}(\mathbf{w}) > 0$ does not matter at the initial guess or during the iterations. Satisfied in the end.

Slack formulation of the Primal-Dual Interior-Point conditions

Primal-Dual IP KKT conditions:

$$\nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} = 0 \qquad (1a)$$

$$\boldsymbol{\mu}_i\mathbf{h}_i(\mathbf{w}) + \tau = 0 \qquad (1b)$$

$$\mathbf{h}(\mathbf{w}) < 0, \quad \boldsymbol{\mu} > 0 \qquad (1c)$$

- Newton steps on (1a)-(1b)
- Backtrack to ensure (1c)

Difficulty: one must ensure that $\mathbf{h}(\mathbf{w})$ starts and remains negative throughout the Newton iterations

- need a feasible initial guess
- backtracking can be expensive

Slack reformulation: new variable $-\mathbf{s} = \mathbf{h}(\mathbf{w})$

$$\begin{aligned} \nabla\Phi(\mathbf{w}) + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\ \boldsymbol{\mu}_i s_i - \tau &= 0 \\ \mathbf{s} > 0, \quad \boldsymbol{\mu} &> 0 \end{aligned}$$

Newton on the slack formulation

- initialize with $\mathbf{s}$, $\boldsymbol{\mu} > 0$ and $\boldsymbol{\mu}_i s_i = \tau$
- $\mathbf{h}(\mathbf{w}) > 0$ does not matter at the initial guess or during the iterations. Satisfied in the end.
- finding $t \in \,]0, 1]$ to enforce:

$$\begin{aligned} \mathbf{s} + t\Delta\mathbf{s} &> 0 \\ \boldsymbol{\mu} + t\Delta\boldsymbol{\mu} &> 0 \end{aligned}$$

is trivial.

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$
\begin{aligned}
\min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\
\text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\
& \mathbf{h}(\mathbf{w}) \leq 0
\end{aligned}
$$

KKT conditions with slack

$$
\begin{aligned}
\nabla\Phi(\mathbf{w}) + \nabla\mathbf{g}(\mathbf{w})\boldsymbol{\lambda} + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\
\mathbf{g}(\mathbf{w}) &= 0 \\
\mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\
\boldsymbol{\mu}_i s_i &= 0 \\
\mathbf{s} > 0, \quad \boldsymbol{\mu} &> 0
\end{aligned}
$$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$
\begin{aligned}
\min_{\mathbf{w}} \quad & \Phi\left(\mathbf{w}\right) \\
\text{s.t.} \quad & \mathbf{g}\left(\mathbf{w}\right) = 0 \\
& \mathbf{h}\left(\mathbf{w}\right) \leq 0
\end{aligned}
$$

PD-IP KKT conditions with slack

$$
\begin{aligned}
\nabla\Phi(\mathbf{w}) + \nabla\mathbf{g}(\mathbf{w})\boldsymbol{\lambda} + \nabla\mathbf{h}(\mathbf{w})\boldsymbol{\mu} &= 0 \\
\mathbf{g}\left(\mathbf{w}\right) &= 0 \\
\mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\
\boldsymbol{\mu}_i s_i - \tau &= 0 \\
\mathbf{s} \geq 0, \quad \boldsymbol{\mu} &\geq 0
\end{aligned}
$$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

PD-IP KKT conditions

$$\begin{aligned} \nabla \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= 0 \\ \mathbf{g}(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\ \boldsymbol{\mu}_i s_i - \tau &= 0 \\ \mathbf{s} > 0, \quad \boldsymbol{\mu} &> 0 \end{aligned}$$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \le 0 \end{aligned}$$

Newton on the conditions

$$\begin{bmatrix} \nabla \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

with $\mathbf{s} > 0, \quad \boldsymbol{\mu} > 0$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

Newton on the conditions

$$\begin{bmatrix} \nabla\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

with $\mathbf{s} > 0, \quad \boldsymbol{\mu} > 0$

Newton direction $\mathbf{d}$ given by $\nabla\mathbf{r}_\tau^\top(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})\,\mathbf{d} + \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$

Newton on the Primal-Dual Interior-Point KKT conditions

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$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

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$$\begin{bmatrix} \nabla\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

with $\mathbf{s} > 0, \quad \boldsymbol{\mu} > 0$

Newton direction $\mathbf{d}$ given by $\nabla\mathbf{r}_\tau^\top(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})\,\mathbf{d} + \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$

$$\underbrace{\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix}}_{=\nabla\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})} \underbrace{\begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \\ \Delta\mathbf{s} \end{bmatrix}}_{=\mathbf{d}} = -\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})$$

with $H = \nabla^2\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu})$

Newton on the Primal-Dual Interior-Point KKT conditions

NLP

$$\begin{aligned} \min_{\mathbf{w}} \quad & \Phi(\mathbf{w}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{w}) = 0 \\ & \mathbf{h}(\mathbf{w}) \leq 0 \end{aligned}$$

Newton on the conditions

$$\begin{bmatrix} \nabla \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

with $\mathbf{s} > 0, \quad \boldsymbol{\mu} > 0$

Newton direction $\mathbf{d}$ given by $\nabla \mathbf{r}_\tau^\top(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})\,\mathbf{d} + \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$

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with $H = \nabla^2 \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu})$

Observe the specific structure of the matrix $\nabla \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})$!!

Solving an NLP using the Primal-Dual Interior-Point method

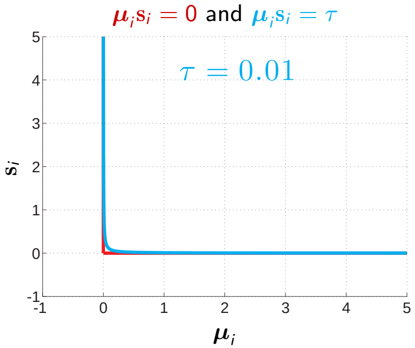
Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$



Solving an NLP using the Primal-Dual Interior-Point method

Solve:

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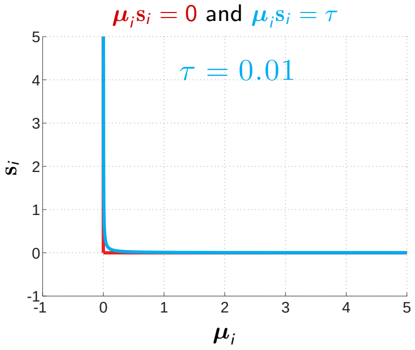
Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$



We want to solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$ *for a very small* τ.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

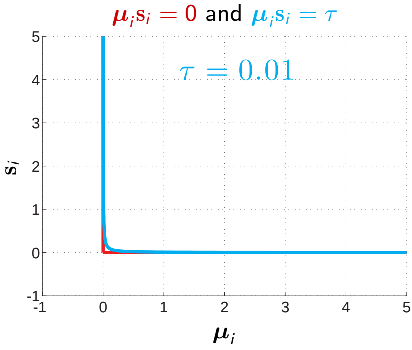
$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top\mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$

Reminder: Newton convergence depends on the Lipschitz constant of $\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)$, i.e. Newton does not "like" strong nonlinearities



We want to solve $\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$ *for a very small* τ.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$

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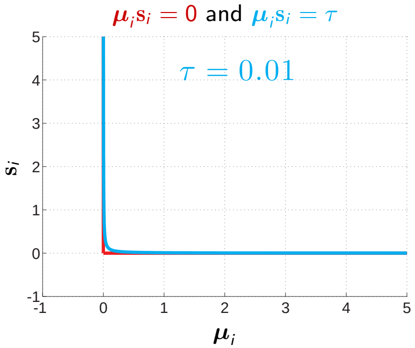


We want to solve $\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$ for a very small τ. But we do not want to get through the "corner" in $\boldsymbol{\mu}_i\mathbf{s}_i = \tau$ when τ is small.

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

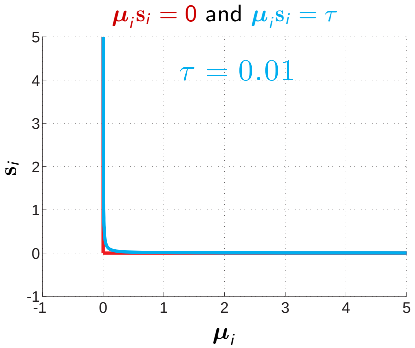
$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=\begin{bmatrix}\nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right)\\ \mathbf{g}\left(\mathbf{w}\right)\\ \mathbf{h}(\mathbf{w})+\mathbf{s}\\ \boldsymbol{\mu}_i s_i-\tau\end{bmatrix}=0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top\mathbf{d}+\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=0$$

Reminder: Newton convergence depends on the Lipschitz constant of $\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)$, i.e. Newton does not "like" strong nonlinearities



We want to solve $\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=0$ *for a very small* τ*. But we do not want to get through the "corner" in* $\boldsymbol{\mu}_i\mathbf{s}_i=\tau$ *when* τ *is small.*

Key idea: solve at large τ, then reduce it while solving again...

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=\left[\begin{array}{c}\nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right)\\ \mathbf{g}\left(\mathbf{w}\right)\\ \mathbf{h}(\mathbf{w})+\mathbf{s}\\ \boldsymbol{\mu}_i s_i-\tau\end{array}\right]=0$$

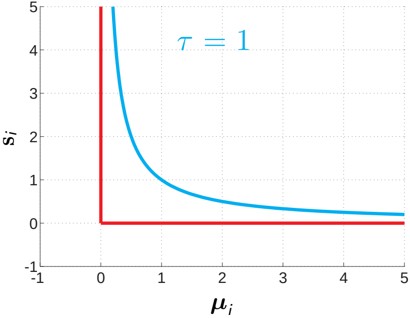
Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top\mathbf{d}+\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=0$$

Reminder: Newton convergence depends on the Lipschitz constant of $\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)$, i.e. Newton does not "like" strong nonlinearities

$\boldsymbol{\mu}_i\mathbf{s}_i=0$ and $\boldsymbol{\mu}_i\mathbf{s}_i=\tau$



We want to solve $\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)=0$ for a very small τ. But we do not want to get through the "corner" in $\boldsymbol{\mu}_i\mathbf{s}_i=\tau$ when τ is small.

Key idea: solve at large τ, then reduce it while solving again...

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

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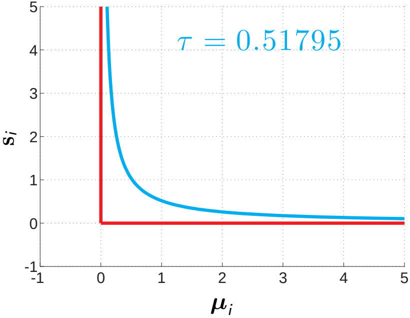


We want to solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$ for a very small τ. But we do not want to get through the "corner" in $\boldsymbol{\mu}_i \mathbf{s}_i = \tau$ when τ is small.

Key idea: solve at large τ, then reduce it while solving again...

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = \begin{bmatrix} \nabla\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

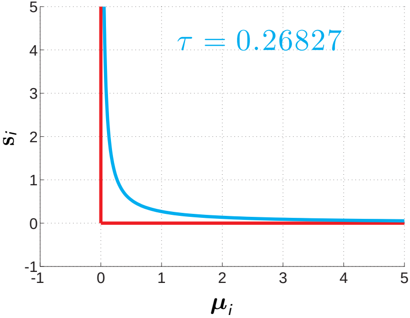
Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})^\top \mathbf{d} + \mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

Reminder: Newton convergence depends on the Lipschitz constant of $\nabla\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})$, i.e. Newton does not "like" strong nonlinearities

$\boldsymbol{\mu}_i \mathbf{s}_i = 0$ and $\boldsymbol{\mu}_i \mathbf{s}_i = \tau$



We want to solve $\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$ *for a very small* τ. *But we do not want to get through the "corner" in* $\boldsymbol{\mu}_i \mathbf{s}_i = \tau$ *when* τ *is small.*

Key idea: solve at large τ, then reduce it while solving again...

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = \begin{bmatrix} \nabla\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w})+\mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})^\top \mathbf{d} + \mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = 0$$

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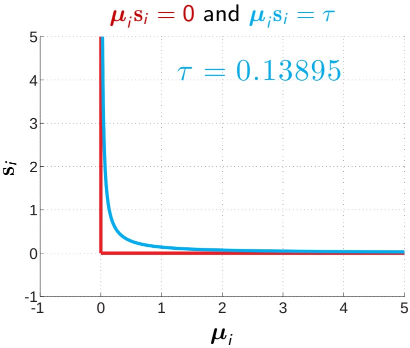

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Key idea: solve at large τ, then reduce it while solving again...

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = \begin{bmatrix} \nabla\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w})+\mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})^\top \mathbf{d} + \mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = 0$$

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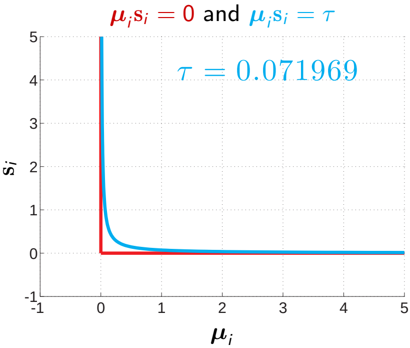

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Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$

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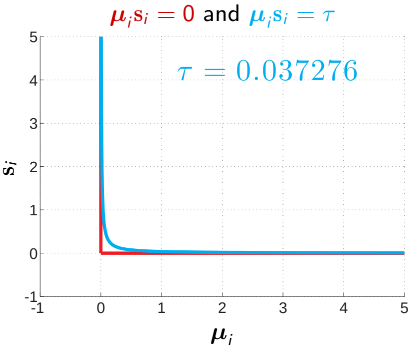

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Key idea: solve at large τ, then reduce it while solving again...

Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = \begin{bmatrix} \nabla\mathcal{L}(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \mathbf{g}(\mathbf{w}) \\ \mathbf{h}(\mathbf{w})+\mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s})^\top \mathbf{d} + \mathbf{r}_\tau(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}) = 0$$

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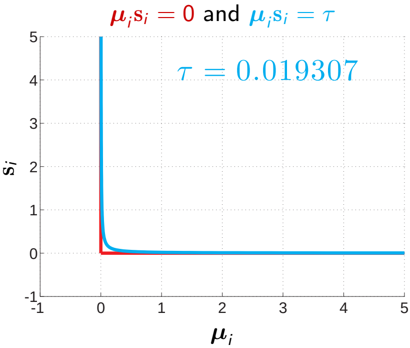

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Solving an NLP using the Primal-Dual Interior-Point method

Solve:

$$\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = \begin{bmatrix} \nabla\mathcal{L}\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu}\right) \\ \mathbf{g}\left(\mathbf{w}\right) \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} \\ \boldsymbol{\mu}_i s_i - \tau \end{bmatrix} = 0$$

Taking steps along the...

Newton direction: $\mathbf{d}$ given by

$$\nabla\mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right)^\top \mathbf{d} + \mathbf{r}_\tau\left(\mathbf{w},\boldsymbol{\lambda},\boldsymbol{\mu},\mathbf{s}\right) = 0$$

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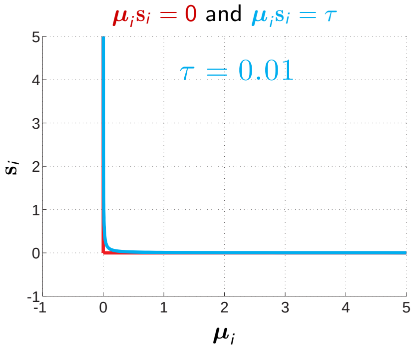

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Key idea: solve at large τ, then reduce it while solving again...

Solving an NLP using the Primal-Dual Interior-Point method

Key idea:

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$, guess $\mathbf{w}, \boldsymbol{\lambda}$
while $\tau > \mathrm{tol}$ **do**
    Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
    Reduce τ
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Solving an NLP using the Primal-Dual Interior-Point method

Key idea:

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathrm{s} \leftarrow 1$, guess $\mathbf{w}, \boldsymbol{\lambda}$
while $\tau > \mathrm{tol}$ **do**
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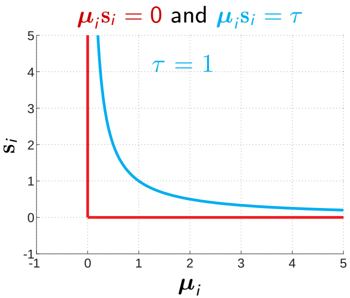


Solving an NLP using the Primal-Dual Interior-Point method

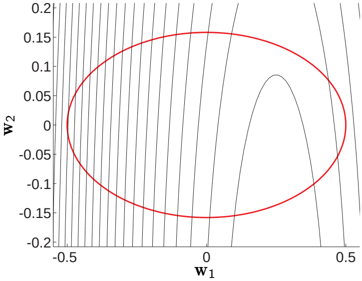
Key idea:

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathrm{s} \leftarrow 1$, guess $\mathbf{w}, \boldsymbol{\lambda}$
while $\tau > \mathrm{tol}$ **do**
 Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}\right) = 0$
 Reduce τ
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}$

Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w} - \mathbf{w}_0)^\top Q\,(\mathbf{w} - \mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S \mathbf{w} \leq 1$$



Solving an NLP using the Primal-Dual Interior-Point method

Key idea:

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$, guess $\mathbf{w}, \boldsymbol{\lambda}$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Reduce τ
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

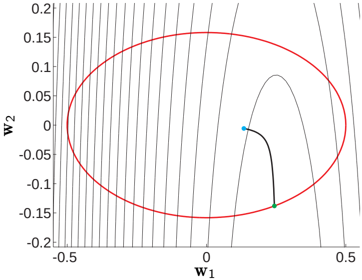
Example

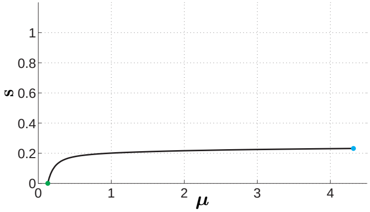
$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\left(\mathbf{w}-\mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$



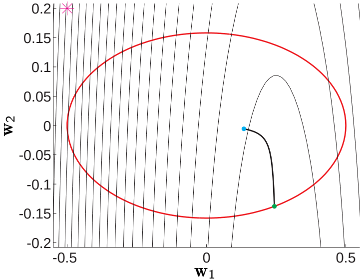


Solving an NLP using the Primal-Dual Interior-Point method

Key idea:

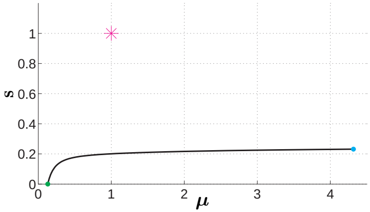
Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$, guess $\mathbf{w}, \boldsymbol{\lambda}$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$
  Reduce τ
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$



Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q(\mathbf{w}-\mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$



Central path: solution manifold of

$$\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

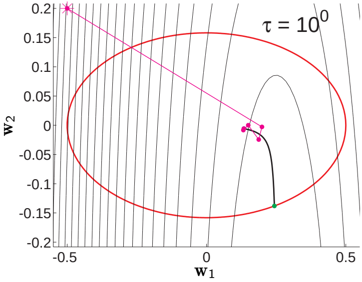
for $\tau \in [1, 0[$

Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

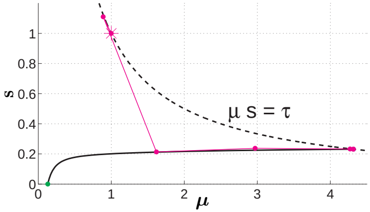
Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \text{tol}$ **do**
  Solve $\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$



Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w} - \mathbf{w}_0)^\top Q\,(\mathbf{w} - \mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$



Central path: solution manifold of

$$\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

for $\tau \in [1, 0[$

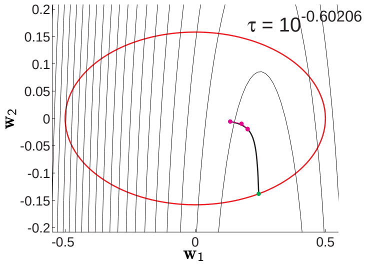
Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

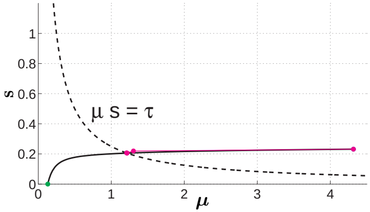
Set $\tau,\ \boldsymbol{\mu},\ \mathrm{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}$

$\gamma = 0.25$



Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}\left(\mathbf{w} - \mathbf{w}_0\right)^\top Q\left(\mathbf{w} - \mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$



Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}\right) = 0$$

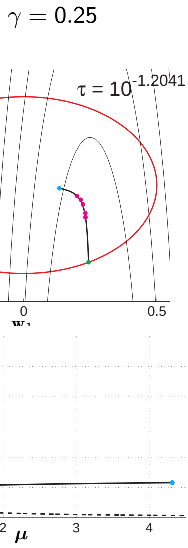
for $\tau \in [1,\ 0[$

Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

Set $\tau,\ \boldsymbol{\mu},\ \mathrm{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$



Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0)$$
$$\mathrm{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$

Solving an NLP using the Primal-Dual Interior-Point method

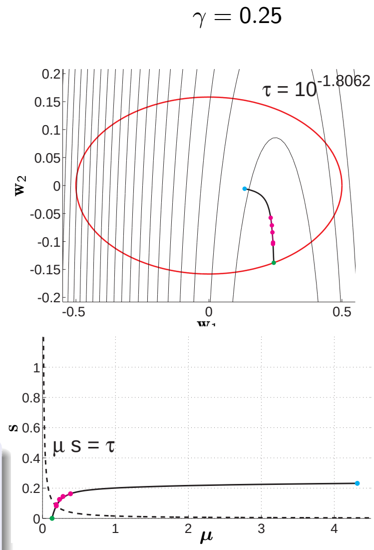
Key idea: homotopy on τ

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau > \text{tol}$ **do**
  Solve $\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q(\mathbf{w}-\mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$



Central path: solution manifold of

$$\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

for $\tau \in [1, 0[$

Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

Set $\tau,\ \boldsymbol{\mu},\ \mathrm{s} \leftarrow 1$
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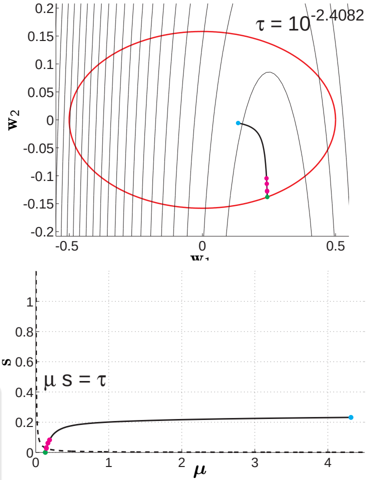
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}$

$\gamma = 0.25$



Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}\left(\mathbf{w}-\mathbf{w}_0\right)^\top Q\left(\mathbf{w}-\mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}\right) = 0$$

for $\tau \in [1,\ 0[$

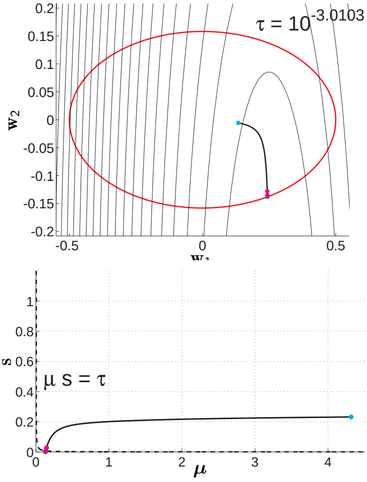
Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

Set $\tau,\ \boldsymbol{\mu},\ \mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

$\gamma = 0.25$



Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}\left(\mathbf{w} - \mathbf{w}_0\right)^\top Q\left(\mathbf{w} - \mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

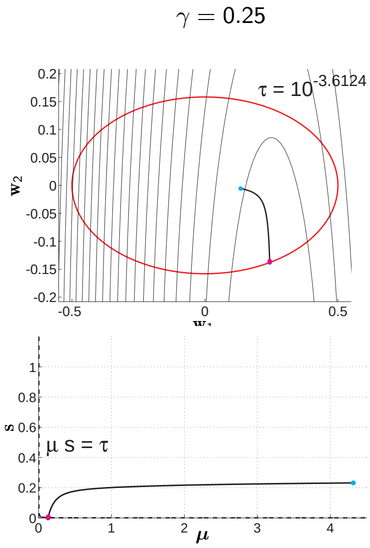
for $\tau \in [1,\ 0[$

Solving an NLP using the Primal-Dual Interior-Point method

Key idea: homotopy on τ

Algorithm: PD-IP solver

Set $\tau,\ \boldsymbol{\mu},\ \mathbf{s} \leftarrow 1$
while $\tau > \mathrm{tol}$ **do**
  Solve $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$
  Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$



Example

$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2}\left(\mathbf{w}-\mathbf{w}_0\right)^\top Q\left(\mathbf{w}-\mathbf{w}_0\right) \\ \text{s.t.} \quad & \mathbf{w}^\top S\mathbf{w} \leq 1 \end{aligned}$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$

Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set $\tau,\ \boldsymbol{\mu},\ \mathbf{s} \leftarrow 1$

while $\tau > \mathrm{tol}$ or $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**

  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$

  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_\mathrm{X} \leq 1$ **then**

    Update $\tau \leftarrow \gamma\tau$

return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

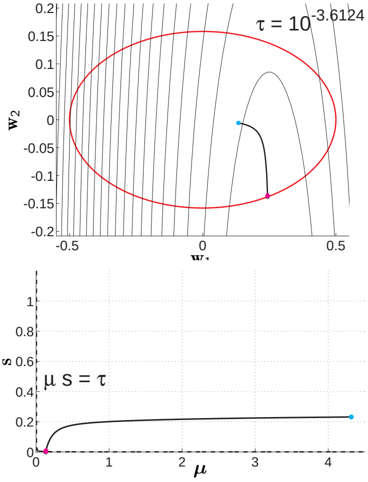
$$\begin{aligned} \min_{\mathbf{w}} \quad & \frac{1}{2}\left(\mathbf{w} - \mathbf{w}_0\right)^\top Q\left(\mathbf{w} - \mathbf{w}_0\right) \\ \mathrm{s.t.} \quad & \mathbf{w}^\top S\mathbf{w} \leq 1 \end{aligned}$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$

$\gamma = 0.25$



Solving an NLP using the Primal-Dual Interior-Point method

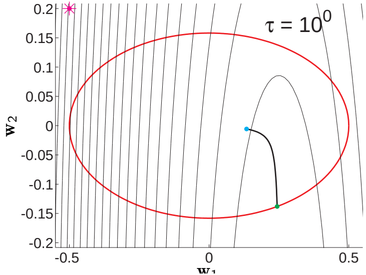
Key idea: path-following

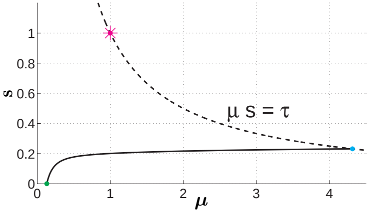
Algorithm: PD-IP solver

Set $\tau,\ \boldsymbol{\mu},\ \mathbf{s} \leftarrow 1$
while $\tau >$ tol or $\|\mathbf{r}_\tau\|_\infty >$ tol **do**
  Newton step on $\mathbf{r}_\tau\,(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})$
  if $\|\mathbf{r}_\tau\,(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})\|_\mathrm{X} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$

return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

$\gamma = 0.1$





Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\,(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

for $\tau \in [1,\ 0[$

Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathrm{s} \leftarrow 1$
while $\tau >$ tol or $\|\mathbf{r}_\tau\|_\infty >$ tol **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}\right)\|_{\mathrm{X}} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}$

Example

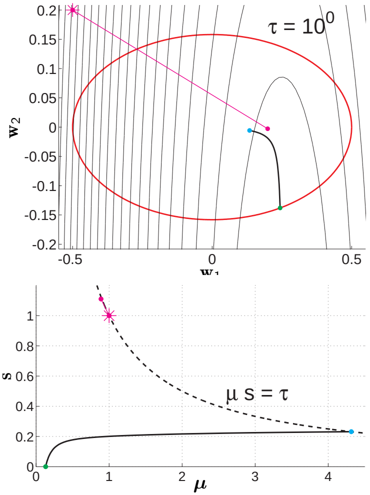
$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q\,(\mathbf{w}-\mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s}\right) = 0$$

for $\tau \in [1, 0[$

$\gamma = 0.1$



Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set $\tau,\ \boldsymbol{\mu},\ \mathbf{s} \leftarrow 1$
while $\tau >$ tol or $\|\mathbf{r}_\tau\|_\infty >$ tol **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_\mathrm{X} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

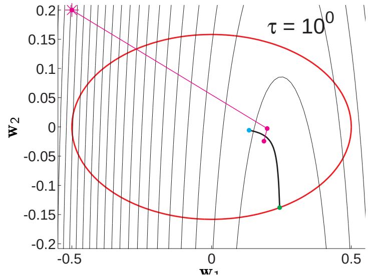
$$\min_{\mathbf{w}} \quad \frac{1}{2}\left(\mathbf{w} - \mathbf{w}_0\right)^\top Q\left(\mathbf{w} - \mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

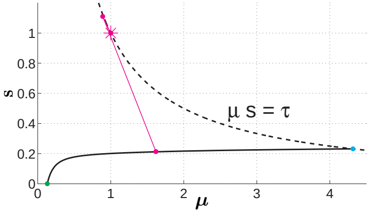
Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$

$\gamma = 0.1$





Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau >$ tol or $\|\mathbf{r}_\tau\|_\infty >$ tol **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_\mathrm{X} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

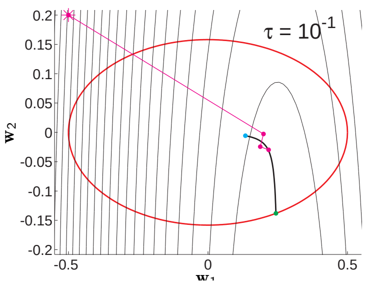
$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w} - \mathbf{w}_0)^\top Q\,(\mathbf{w} - \mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

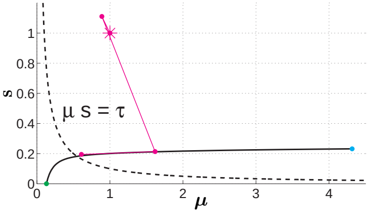
Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\ 0[$

$\gamma = 0.1$





Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

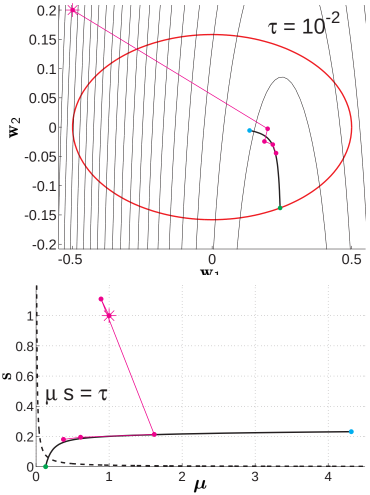
Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau >$ tol or $\|\mathbf{r}_\tau\|_\infty >$ tol **do**
  Newton step on $\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})$
  if $\|\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})\|_{\mathrm{X}} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$
return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}(\mathbf{w}-\mathbf{w}_0)^\top Q(\mathbf{w}-\mathbf{w}_0)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

$\gamma = 0.1$



Central path: solution manifold of

$$\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}) = 0$$

for $\tau \in [1, 0[$

Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau >$ tol or $\|\mathbf{r}_\tau\|_\infty >$ tol **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_{\mathrm{X}} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$

return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

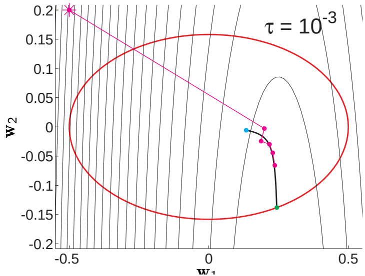
$$\min_{\mathbf{w}} \quad \frac{1}{2}\left(\mathbf{w} - \mathbf{w}_0\right)^\top Q\left(\mathbf{w} - \mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

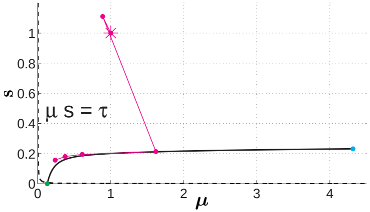
Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1, 0[$

$\gamma = 0.1$





Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$

while $\tau >$ tol or $\|\mathbf{r}_\tau\|_\infty >$ tol **do**

  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$

  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_{\mathrm{X}} \leq 1$ **then**

    Update $\tau \leftarrow \gamma\tau$

return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

$$\min_{\mathbf{w}} \quad \frac{1}{2}\left(\mathbf{w}-\mathbf{w}_0\right)^\top Q\left(\mathbf{w}-\mathbf{w}_0\right)$$

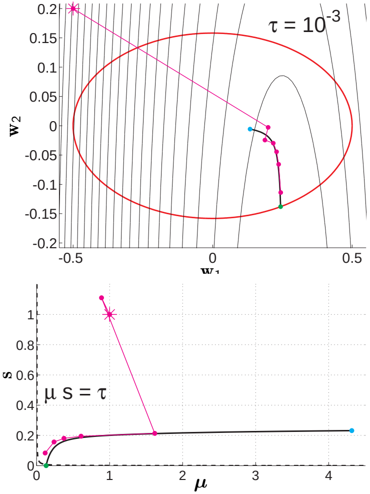
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1,\, 0[$

$\gamma = 0.1$



Solving an NLP using the Primal-Dual Interior-Point method

Key idea: path-following

Algorithm: PD-IP solver

Set τ, $\boldsymbol{\mu}$, $\mathbf{s} \leftarrow 1$
while $\tau >$ tol or $\|\mathbf{r}_\tau\|_\infty >$ tol **do**
  Newton step on $\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)$
  if $\|\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right)\|_\mathrm{X} \leq 1$ **then**
    Update $\tau \leftarrow \gamma\tau$

return $\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}$

Example

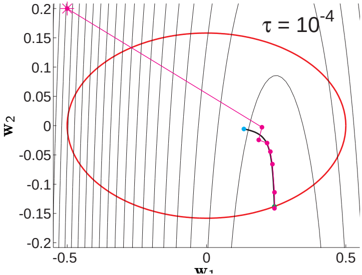
$$\min_{\mathbf{w}} \quad \frac{1}{2}\left(\mathbf{w} - \mathbf{w}_0\right)^\top Q\left(\mathbf{w} - \mathbf{w}_0\right)$$
$$\text{s.t.} \quad \mathbf{w}^\top S\mathbf{w} \leq 1$$

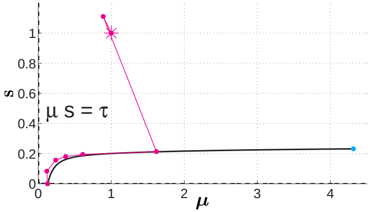
Central path: solution manifold of

$$\mathbf{r}_\tau\left(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s}\right) = 0$$

for $\tau \in [1, 0[$

$\gamma = 0.1$





The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

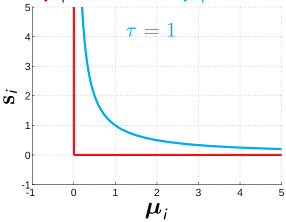
Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau > \text{tol}$ or $\|\mathbf{r}_\tau\|_\infty > \text{tol}$ **do**

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{s}$

$\boldsymbol{\mu}_i \mathbf{s}_i = 0$ and $\boldsymbol{\mu}_i \mathbf{s}_i = \tau$



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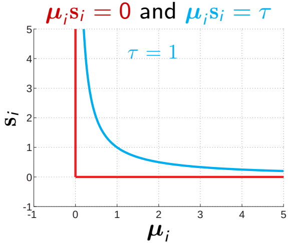
Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathrm{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau >$ tol or $\|\mathbf{r}_\tau\|_\infty >$ tol **do**

  Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, s



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Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

Compute the Newton direction given by

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \\ \Delta\mathbf{s} \end{bmatrix} = -\mathbf{r}_\tau$$

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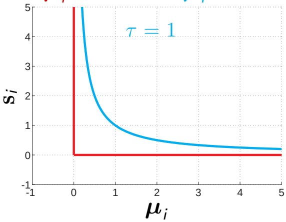

The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau > \mathrm{tol}$ or $\|\mathbf{r}_\tau\|_\infty > \mathrm{tol}$ **do**

Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

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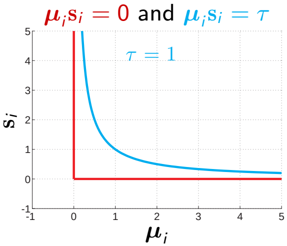
Compute a step-size $t_{\max} \leq 1$ ensuring:

$$\mathbf{s} + t_{\max}\Delta\mathbf{s} \geq \epsilon\mathbf{s}, \quad \boldsymbol{\mu} + t_{\max}\Delta\boldsymbol{\mu} \geq \epsilon\boldsymbol{\mu}$$

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{s}$



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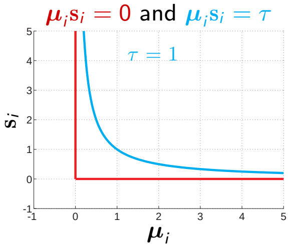
Backtrack $t \in \left]0,\ t_{\max}\right]$ to ensure progress

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, s



The Primal-Dual Interior-Point algorithm

Algorithm: a Primal-dual Interior-Point solver

Input: $\mathbf{w}$

Set $\tau = 1$, $\boldsymbol{\mu} = \mathbf{1}$, $\mathbf{s} = \mathbf{1}$, $\boldsymbol{\lambda} = 0$

while $\tau >$ tol or $\|\mathbf{r}_\tau\|_\infty >$ tol **do**

Evaluate H, $\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{g}$, $\nabla\mathbf{h}$, $\nabla\Phi$

Compute the Newton direction given by

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w} \\ \Delta\boldsymbol{\lambda} \\ \Delta\boldsymbol{\mu} \\ \Delta\mathbf{s} \end{bmatrix} = -\mathbf{r}_\tau$$

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Take Newton step: $\mathbf{w} \leftarrow \mathbf{w} + t\Delta\mathbf{w}$, ...

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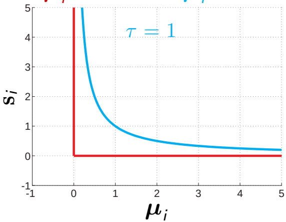

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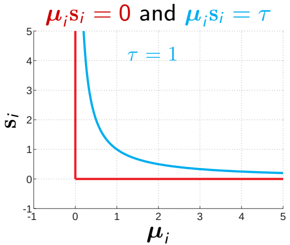
Backtrack $t \in \left]0,\ t_{\max}\right]$ to ensure progress

Take Newton step: $\mathbf{w} \leftarrow \mathbf{w} + t\Delta\mathbf{w}$, ...

if $\|\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathrm{s})\|_{\mathrm{X}} \leq 1$ **then**

Update $\tau \leftarrow \gamma\tau$

return $\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, s



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Backtrack $t \in \,]0,\, t_{\max}]$ to ensure progress

Take Newton step: $\mathbf{w} \leftarrow \mathbf{w} + t\Delta\mathbf{w}, \ldots$

if $\|\mathbf{r}_\tau(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \mathbf{s})\|_{\mathrm{X}} \leq 1$ **then**

Update $\tau \leftarrow \gamma\tau$

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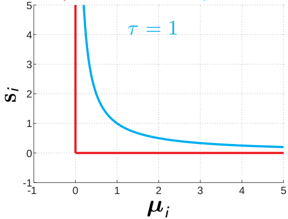

Some subtleties:

- Measuring progress
- Choice of $\|.\|_{\mathrm{X}}$
- Mehrotra predictor
- "Adaptive" γ

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\begin{aligned}
\min_{\mathbf{x},\,\mathbf{u}} \quad & \sum_{k=0}^{N} \frac{1}{2} \left\| \mathbf{x}_k - x_{\mathrm{ref}} \right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\| \mathbf{u}_k - u_{\mathrm{ref}} \right\|_R^2 \\
\mathrm{s.t} \quad & \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right) \\
& \mathbf{x}_0 = \hat{\mathbf{x}}, \qquad x^2 + y^2 \geq r^2 \\
& -\mathbf{u}_{\mathrm{max}} \leq \mathbf{u} \leq \mathbf{u}_{\mathrm{max}}, \quad -v_{\mathrm{min}} \leq v \leq v_{\mathrm{max}}
\end{aligned}$$

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$
\begin{aligned}
\min_{\mathbf{x},\,\mathbf{u}} \quad & \sum_{k=0}^{N} \frac{1}{2} \left\| \mathbf{x}_k - x_{\text{ref}} \right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\| \mathbf{u}_k - u_{\text{ref}} \right\|_R^2 \\
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& -\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}, \quad -v_{\min} \leq v \leq v_{\max}
\end{aligned}
$$

Simple plane dynamics

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \mathbf{F} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ gv^{-1}\tan(\phi) \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}
$$

x, y: position
v: forward velocity
θ: heading
ϕ: bank angle
$\mathbf{u}_1$: roll rate
$\mathbf{u}_2$: forward acceleration

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\text{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\text{ref}}\right\|_R^2$$

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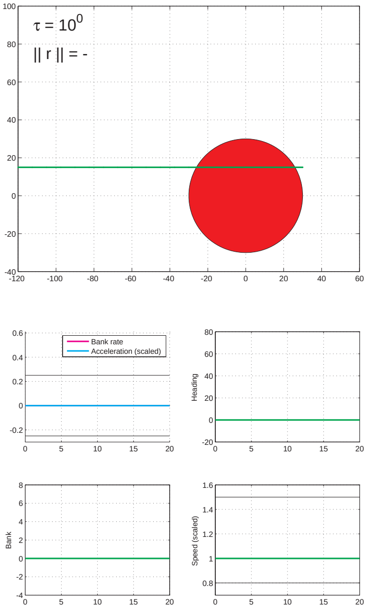

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\text{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\text{ref}}\right\|_R^2$$

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$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \mathbf{F} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ gv^{-1}\tan(\phi) \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

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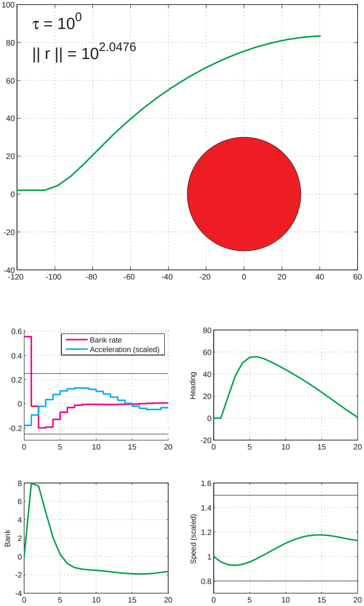

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\text{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\text{ref}}\right\|_R^2$$

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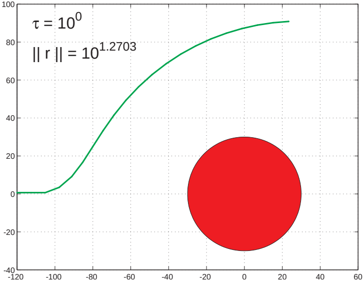

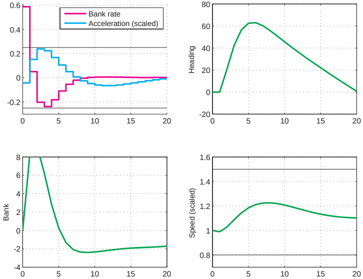

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

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Simple plane dynamics

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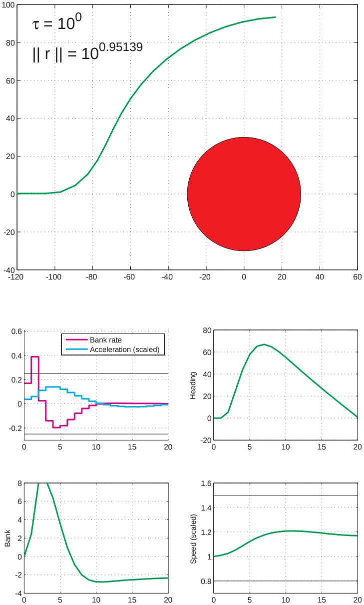

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

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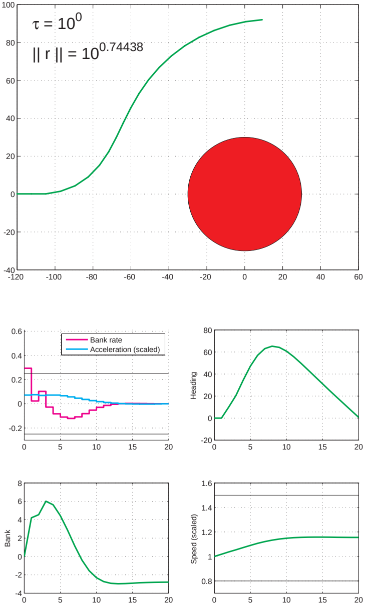

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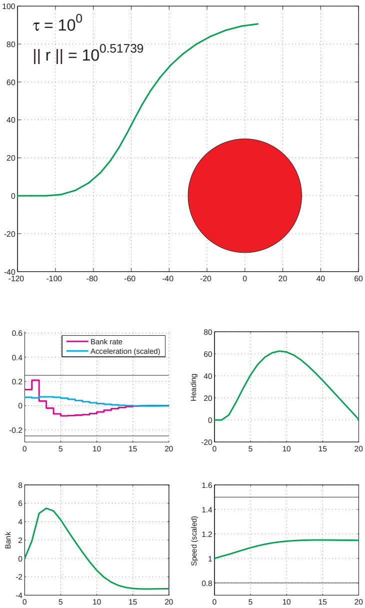

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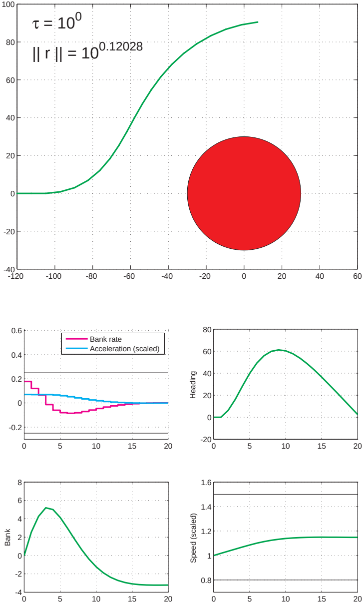
$$\begin{aligned} \text{s.t} \quad & \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right) \\ & \mathbf{x}_0 = \hat{\mathbf{x}}, \qquad x^2 + y^2 \geq r^2 \\ & -\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}, \quad -v_{\min} \leq v \leq v_{\max} \end{aligned}$$

Simple plane dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \mathbf{F} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ gv^{-1}\tan(\phi) \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

x, y: position
v: forward velocity
θ: heading
ϕ: bank angle
$\mathbf{u}_1$: roll rate
$\mathbf{u}_2$: forward acceleration



Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2} \left\| \mathbf{x}_k - x_{\text{ref}} \right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\| \mathbf{u}_k - u_{\text{ref}} \right\|_R^2$$

$$\text{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

$$\mathbf{x}_0 = \hat{\mathbf{x}}, \qquad x^2 + y^2 \geq r^2$$

$$-\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}, \qquad -v_{\min} \leq v \leq v_{\max}$$

Simple plane dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \mathbf{F} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ gv^{-1}\tan(\phi) \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

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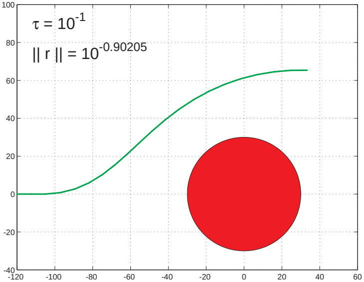
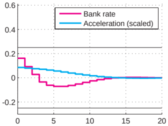
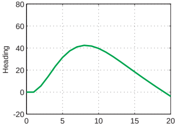
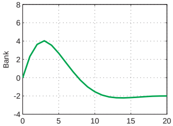
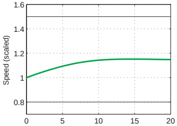










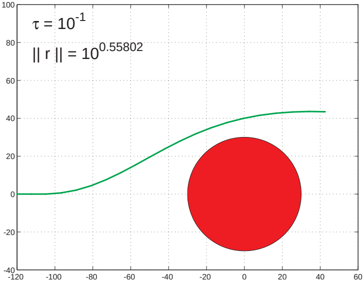
Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2} \left\| \mathbf{x}_k - x_{\text{ref}} \right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\| \mathbf{u}_k - u_{\text{ref}} \right\|_R^2$$

$$\text{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

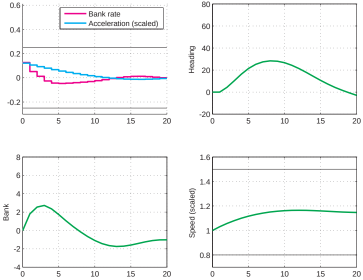
$$\mathbf{x}_0 = \hat{\mathbf{x}}, \qquad x^2 + y^2 \geq r^2$$

$$-\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}, \qquad -v_{\min} \leq v \leq v_{\max}$$



Simple plane dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \mathbf{F} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ g v^{-1} \tan(\phi) \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$



x, y: position
v: forward velocity
θ: heading
ϕ: bank angle
$\mathbf{u}_1$: roll rate
$\mathbf{u}_2$: forward acceleration

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2} \left\| \mathbf{x}_k - x_{\text{ref}} \right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\| \mathbf{u}_k - u_{\text{ref}} \right\|_R^2$$

$$\text{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

$$\mathbf{x}_0 = \hat{\mathbf{x}}, \qquad x^2 + y^2 \geq r^2$$

$$-\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}, \qquad -v_{\min} \leq v \leq v_{\max}$$

Simple plane dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \mathbf{F} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ gv^{-1}\tan(\phi) \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

x, y: position
v: forward velocity
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$\mathbf{u}_1$: roll rate
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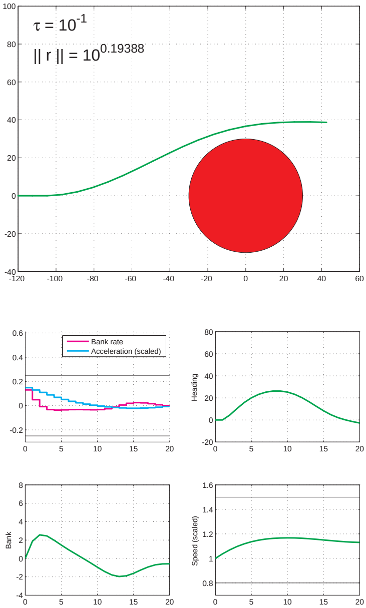

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2} \left\|\mathbf{x}_k - x_{\text{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\|\mathbf{u}_k - u_{\text{ref}}\right\|_R^2$$

$$\text{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

$$\mathbf{x}_0 = \hat{\mathbf{x}}, \qquad x^2 + y^2 \geq r^2$$

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Simple plane dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \mathbf{F} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ gv^{-1}\tan(\phi) \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

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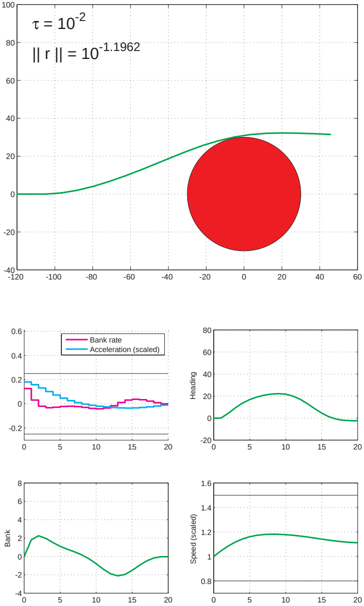

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\text{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\text{ref}}\right\|_R^2$$

$$\text{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t\mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

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Simple plane dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \mathbf{F} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ gv^{-1}\tan(\phi) \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

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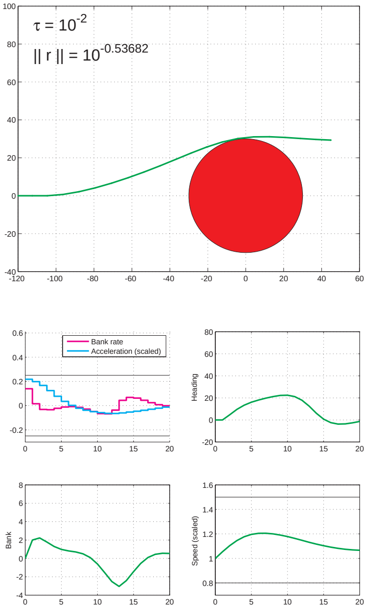

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

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Simple plane dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \mathbf{F} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ gv^{-1}\tan(\phi) \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

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$\mathbf{u}_1$: roll rate
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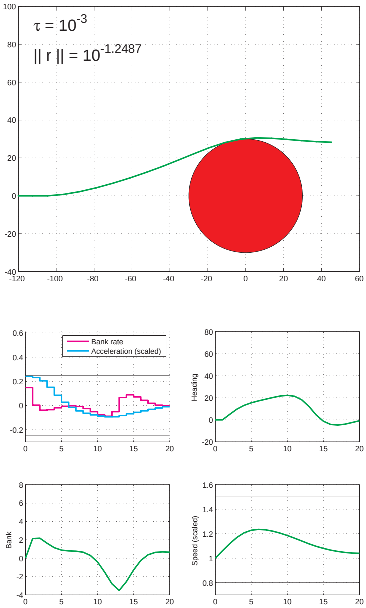

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\text{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\text{ref}}\right\|_R^2$$

$$\text{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

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Simple plane dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{v} \end{bmatrix} = \mathbf{F} = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ gv^{-1}\tan(\phi) \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

x, y: position
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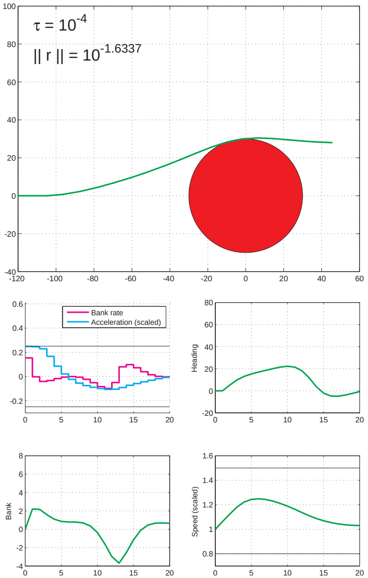
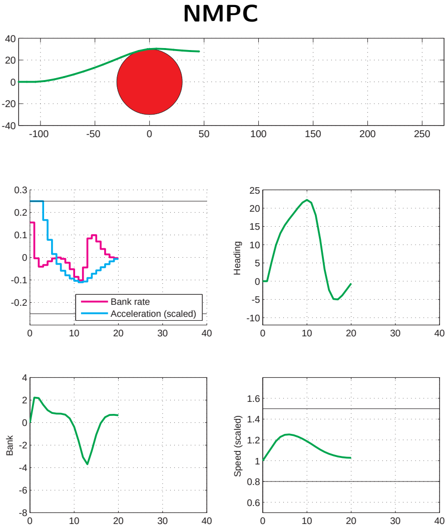


Primal-Dual Interior-Point Algorithm - An Optimal Control Example



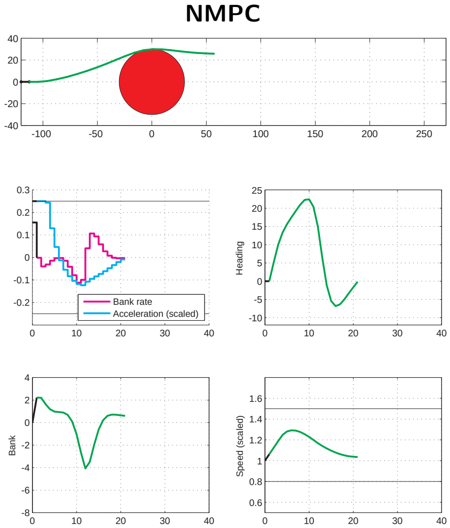
Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2} \left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

$$\begin{aligned} \text{s.t} \quad & \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right) \\ & \mathbf{x}_0 = \hat{\mathbf{x}}, \quad x^2 + y^2 \geq r^2 \\ & -\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max} \end{aligned}$$

solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example



Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

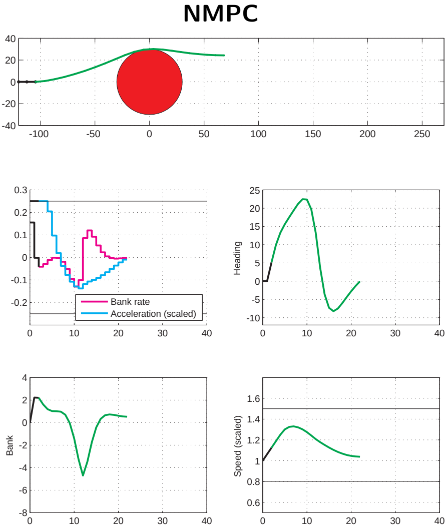
$$\text{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

$$\mathbf{x}_0 = \hat{\mathbf{x}}, \quad x^2 + y^2 \geq r^2$$

$$-\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}$$

solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example



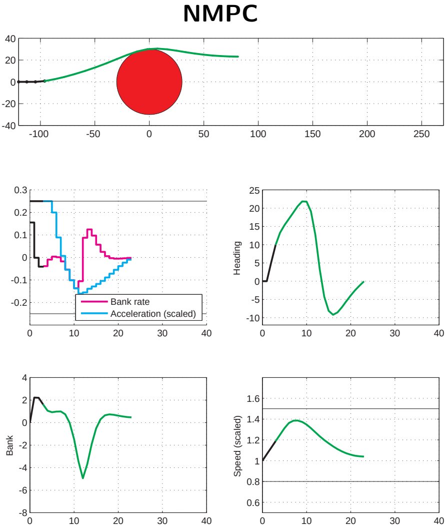
Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2} \left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

$$\begin{aligned} \text{s.t} \quad & \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right) \\ & \mathbf{x}_0 = \hat{\mathbf{x}}, \quad x^2 + y^2 \geq r^2 \\ & -\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max} \end{aligned}$$

solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example



Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2} \left\| \mathbf{x}_k - x_{\mathrm{ref}} \right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\| \mathbf{u}_k - u_{\mathrm{ref}} \right\|_R^2$$

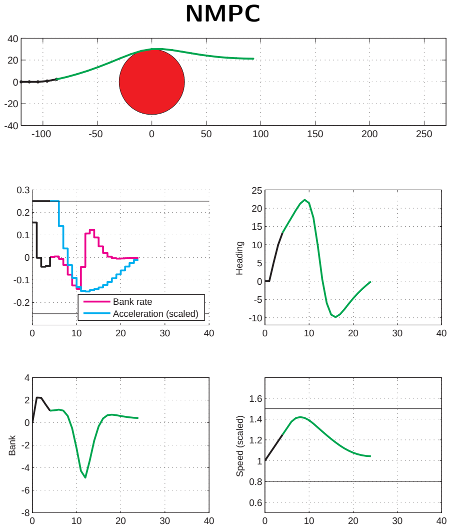
$$\mathrm{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

$$\mathbf{x}_0 = \hat{\mathbf{x}}, \quad x^2 + y^2 \geq r^2$$

$$-\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}$$

solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example



Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

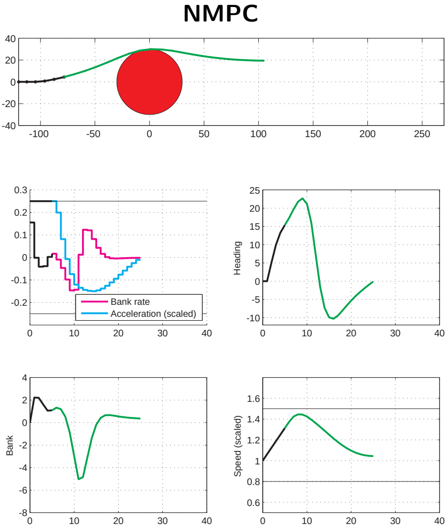
$$\text{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

$$\mathbf{x}_0 = \hat{\mathbf{x}}, \quad x^2 + y^2 \geq r^2$$

$$-\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}$$

solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

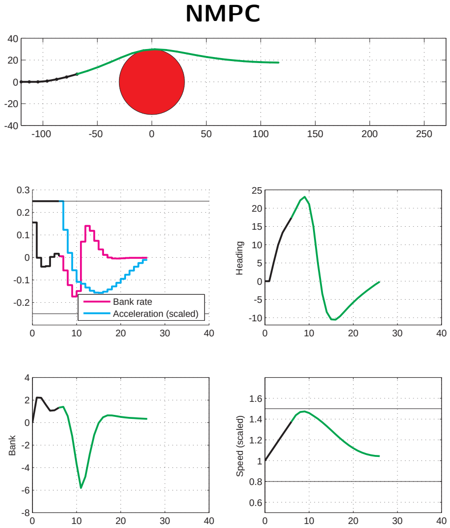


Problem

$$
\begin{aligned}
\min_{\mathbf{x},\,\mathbf{u}} \quad & \sum_{k=0}^{N} \frac{1}{2} \left\| \mathbf{x}_k - x_{\mathrm{ref}} \right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\| \mathbf{u}_k - u_{\mathrm{ref}} \right\|_R^2 \\
\mathrm{s.t} \quad & \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right) \\
& \mathbf{x}_0 = \hat{\mathbf{x}}, \quad x^2 + y^2 \geq r^2 \\
& -\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}
\end{aligned}
$$

solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example



Problem

$$
\begin{aligned}
\min_{\mathbf{x},\,\mathbf{u}} \quad & \sum_{k=0}^{N} \frac{1}{2} \left\| \mathbf{x}_k - x_{\mathrm{ref}} \right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\| \mathbf{u}_k - u_{\mathrm{ref}} \right\|_R^2 \\
\mathrm{s.t} \quad & \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right) \\
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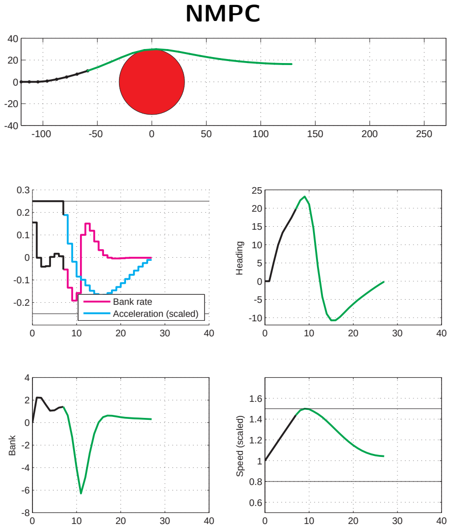
& -\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}
\end{aligned}
$$

solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example



Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

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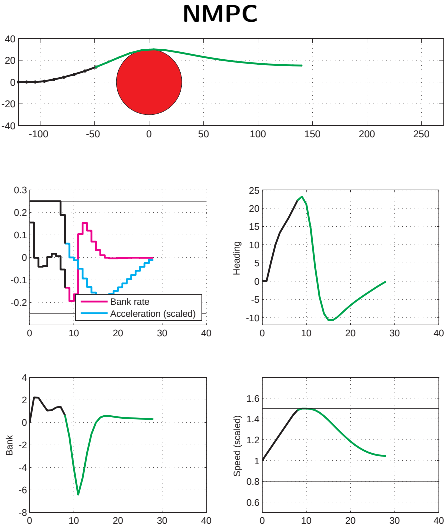
solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example



Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

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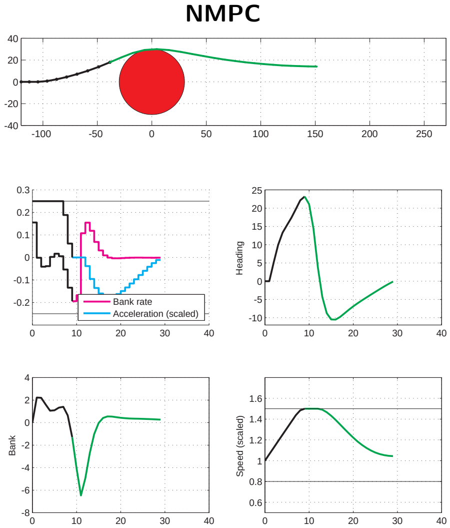
solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example



Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2} \left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

$$\text{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

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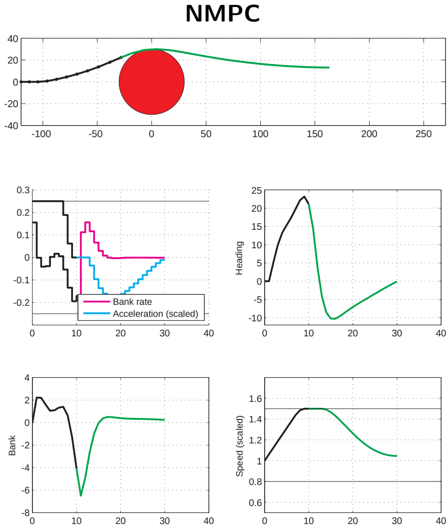
$$-\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}$$

solved repeatedly for $\hat{\mathbf{x}}$ evolving.



Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

$$\begin{aligned}
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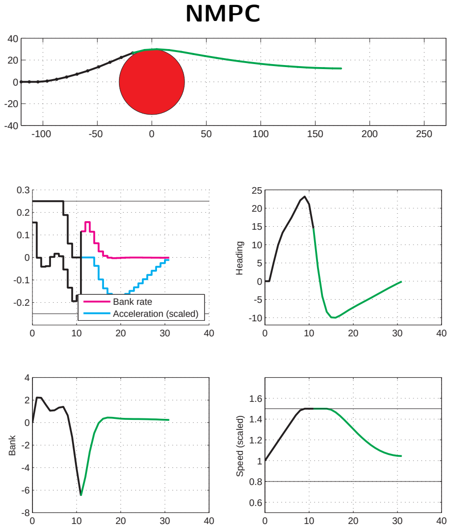
& -\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}
\end{aligned}$$

solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example



Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2}\left\|\mathbf{x}_k - x_{\mathrm{ref}}\right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2}\left\|\mathbf{u}_k - u_{\mathrm{ref}}\right\|_R^2$$

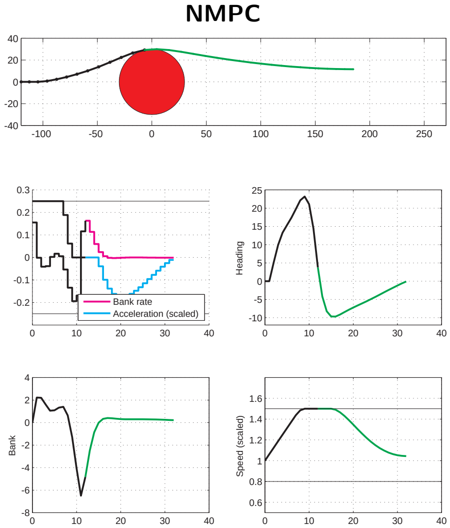
$$\text{s.t} \quad \mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{F}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

$$\mathbf{x}_0 = \hat{\mathbf{x}}, \quad x^2 + y^2 \geq r^2$$

$$-\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}$$

solved repeatedly for $\hat{\mathbf{x}}$ evolving.

Primal-Dual Interior-Point Algorithm - An Optimal Control Example



Problem

$$\min_{\mathbf{x},\,\mathbf{u}} \quad \sum_{k=0}^{N} \frac{1}{2} \left\| \mathbf{x}_k - x_{\mathrm{ref}} \right\|_Q^2 + \sum_{k=0}^{N-1} \frac{1}{2} \left\| \mathbf{u}_k - u_{\mathrm{ref}} \right\|_R^2$$

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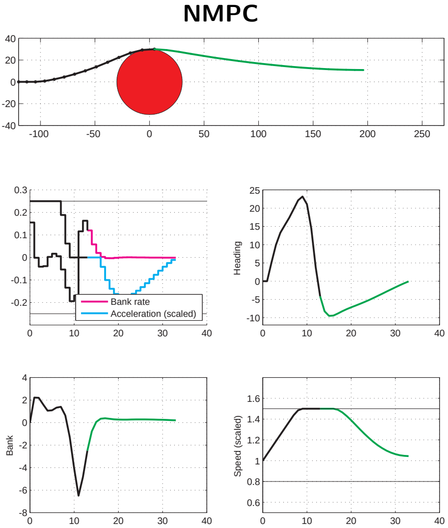

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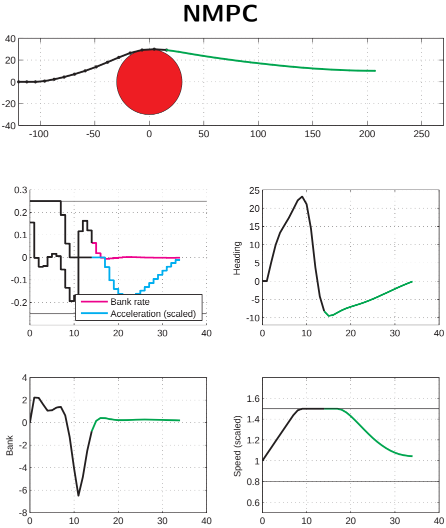

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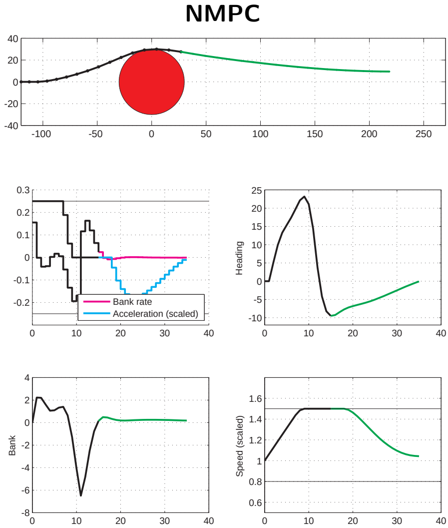
Primal-Dual Interior-Point Algorithm - An Optimal Control Example



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Primal-Dual Interior-Point Algorithm - An Optimal Control Example

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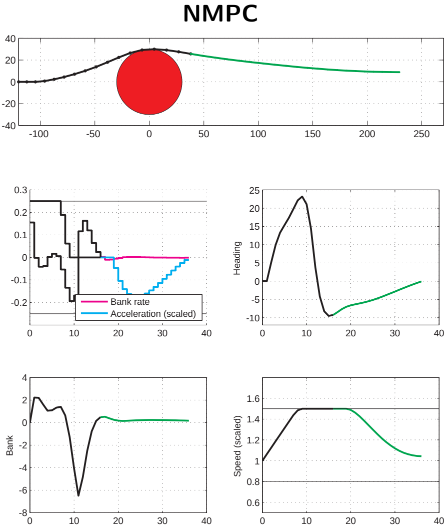

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

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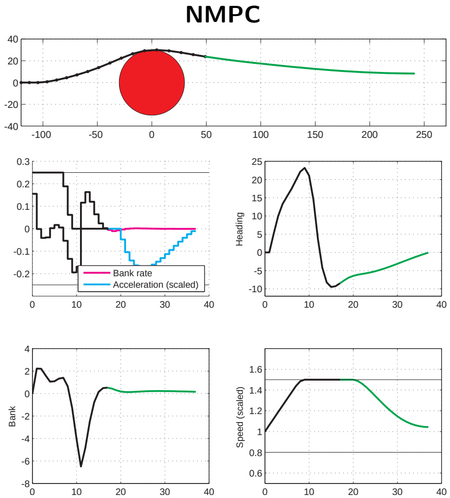

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

Sparsity of the Primal-Dual Interior-Point KKT matrix:

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h} & 0 \\ \nabla \mathbf{g}^\top & 0 & 0 & 0 \\ \nabla \mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}\,(\mathbf{s}) & \mathrm{diag}\,(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \\ \Delta \mathbf{s} \end{bmatrix} = -\mathbf{r}_\tau$$

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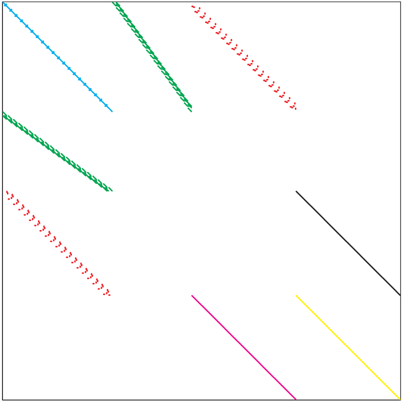

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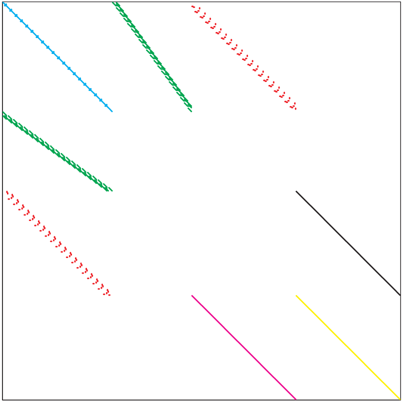

Required ordering:

Primal-Dual Interior-Point Algorithm - An Optimal Control Example

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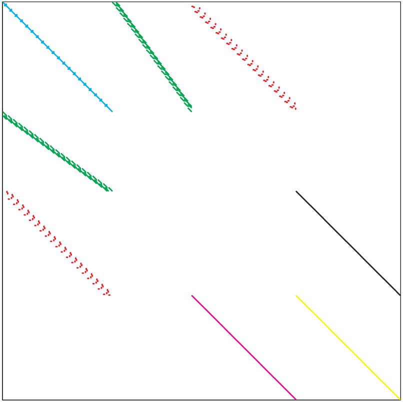

Required ordering:

$$\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \mathbf{x}_0 - \hat{\mathbf{x}} \\ \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}) - \mathbf{x}_N \end{bmatrix},$$

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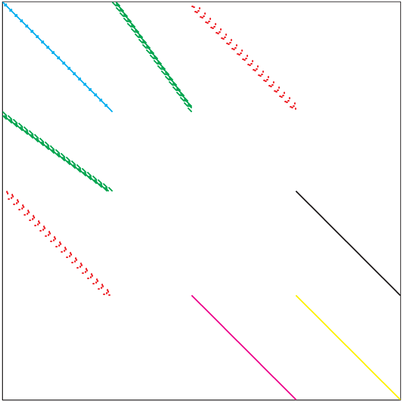

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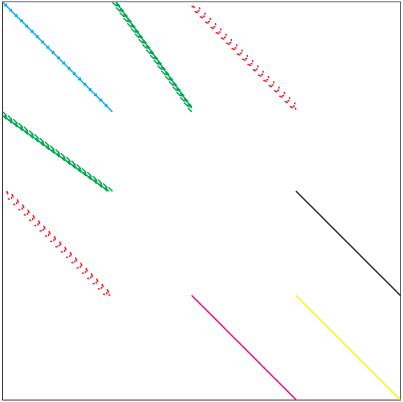

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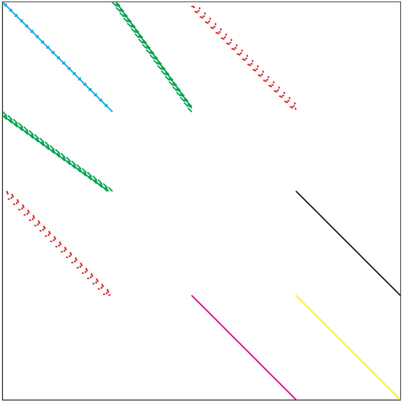

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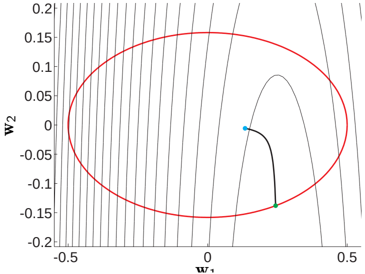
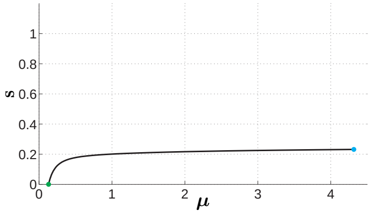
... and attribute dual variables accordingly.

Warm-starting Primal-Dual Interior-Point Algorithms

... what happens if we have a very good guess to warm-start our algorithm ?





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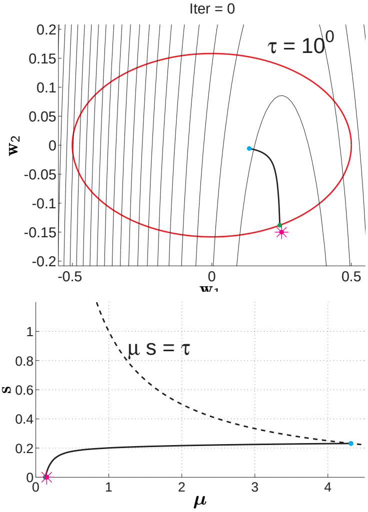

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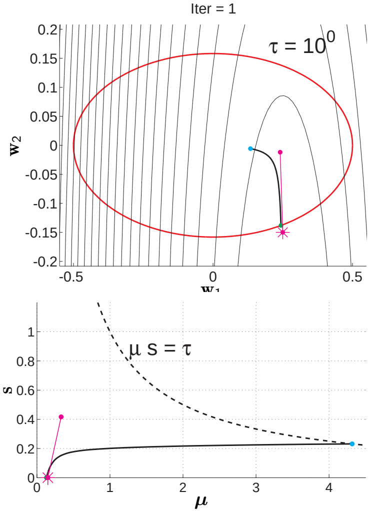

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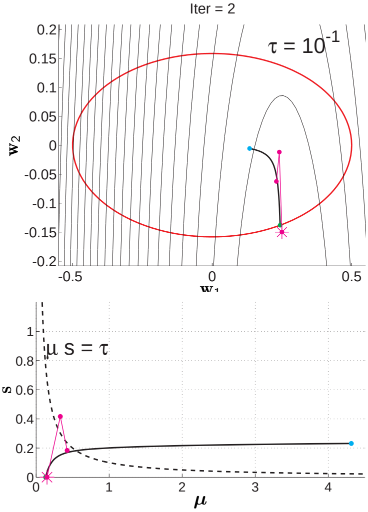

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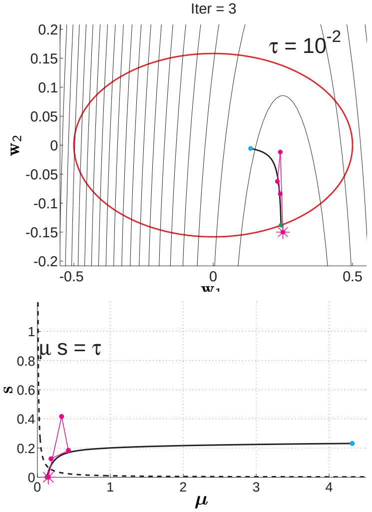

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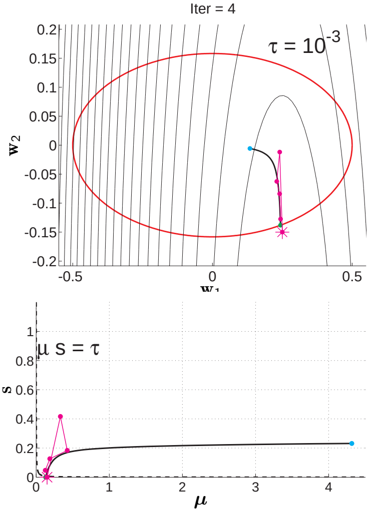

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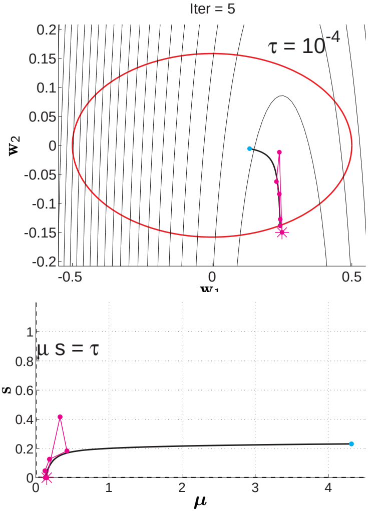

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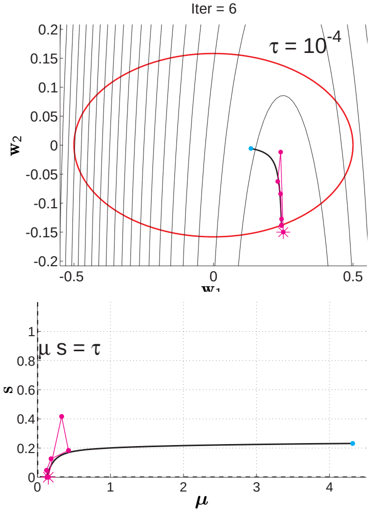

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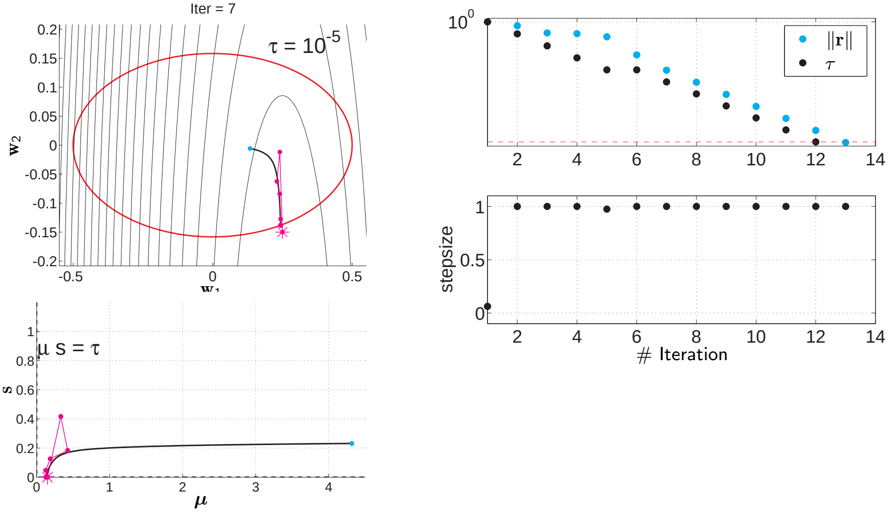

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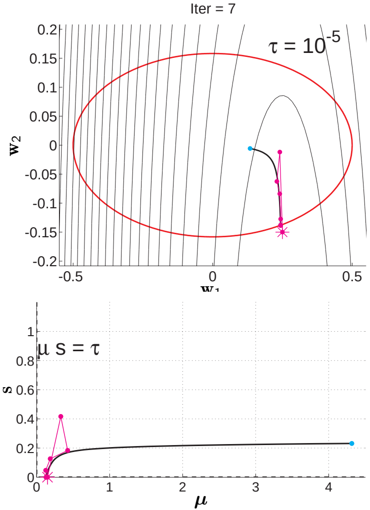

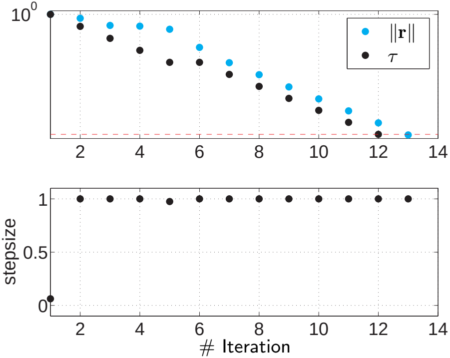

Even with an excellent initial guess interior point methods will **retreat to the central path** before homing onto the solution... what about keeping τ low ?

Warm-starting Primal-Dual Interior-Point Algorithms

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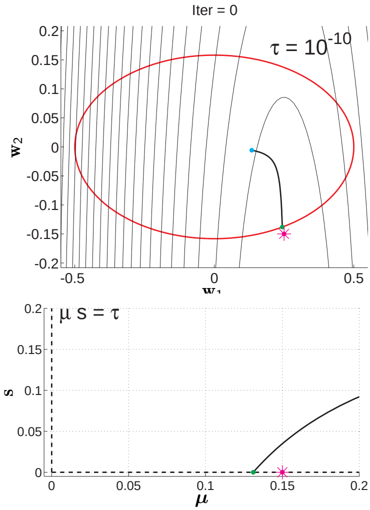

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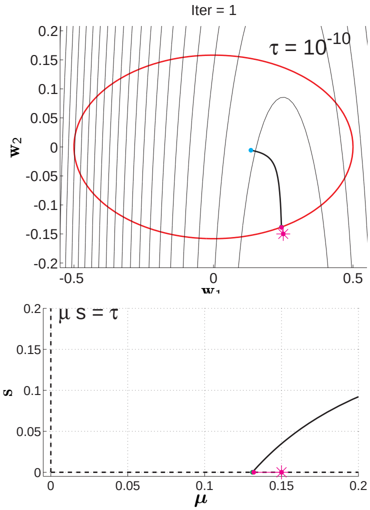

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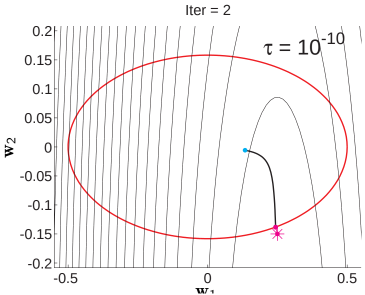

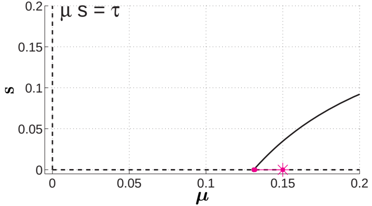

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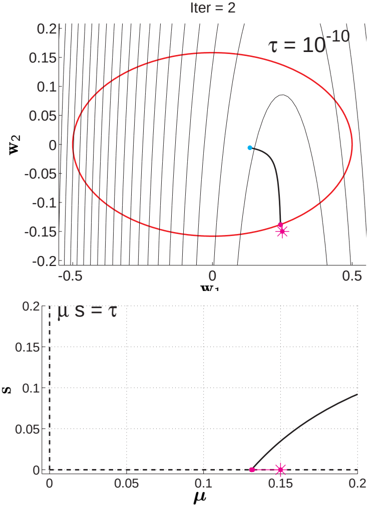

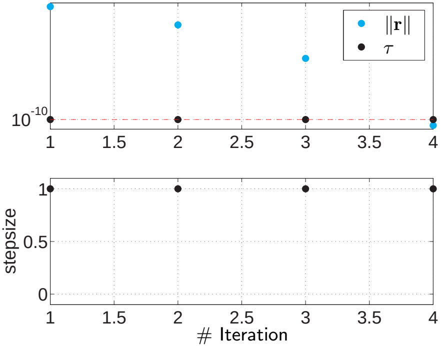

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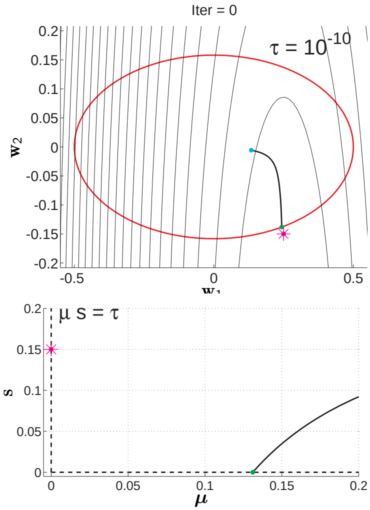

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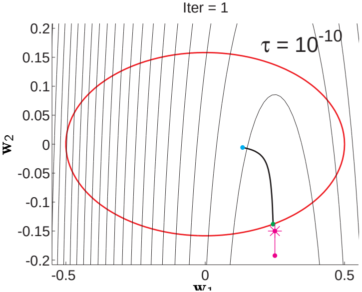

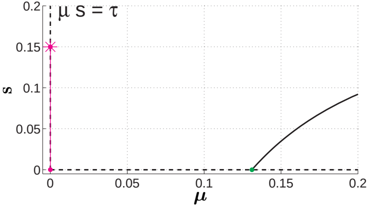

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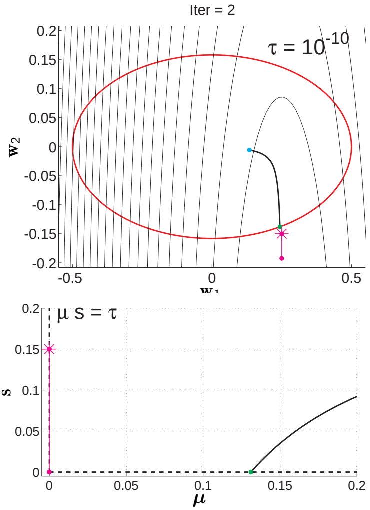

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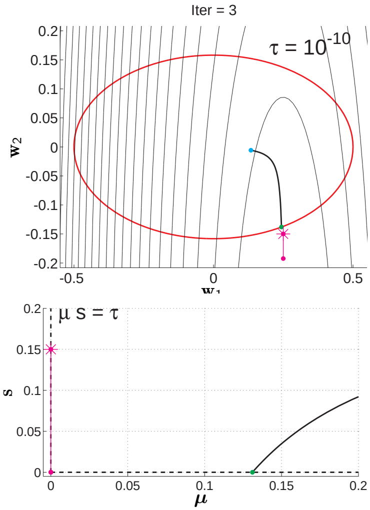
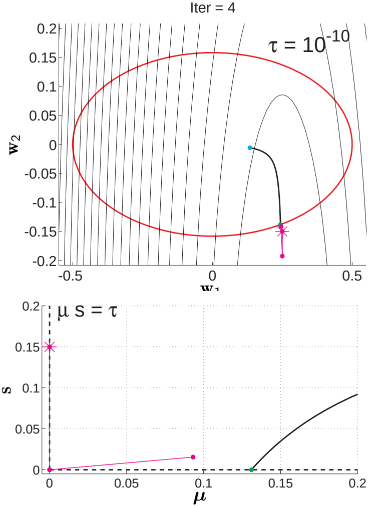


Warm-starting Primal-Dual Interior-Point Algorithms

... what happens if we have a very good guess to warm-start our algorithm ?



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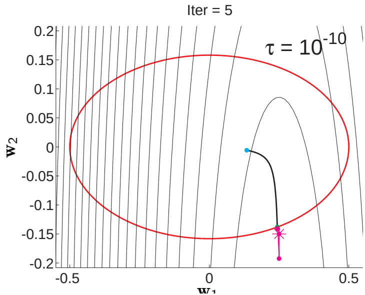

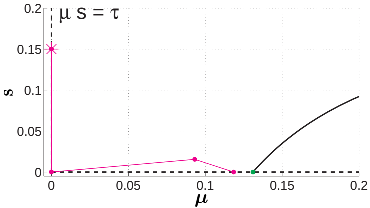

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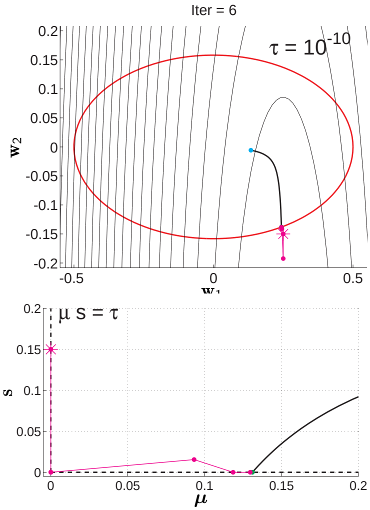

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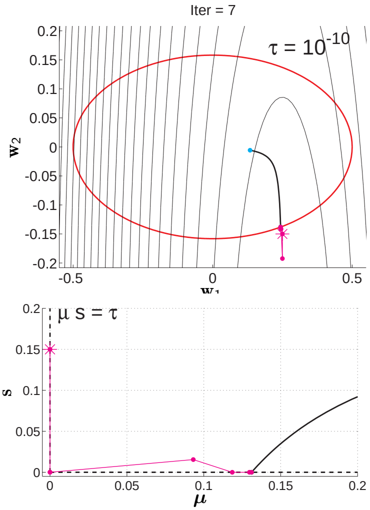

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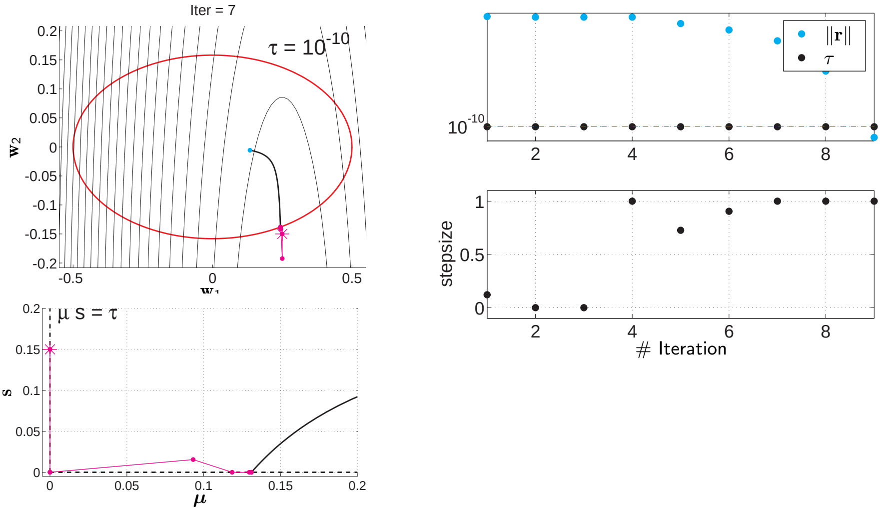

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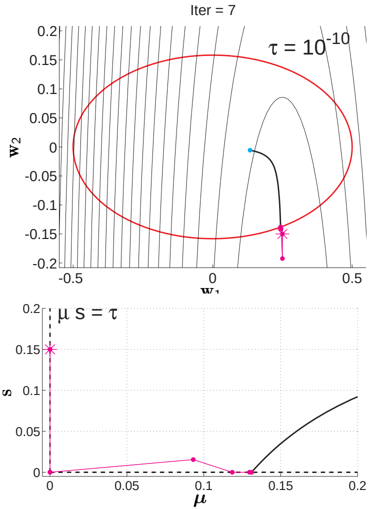

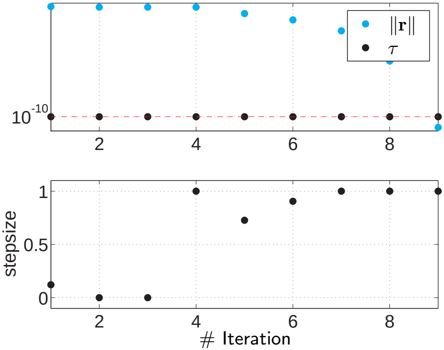

At very low τ, changes of active set are difficult: Newton struggles to get through the sharp turn in $\mu_i s_i = \tau$

Warm-starting Primal-Dual Interior-Point Algorithms

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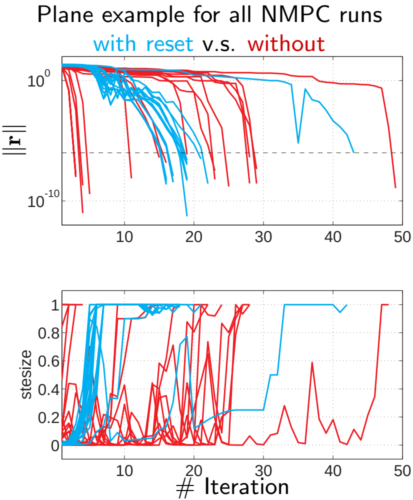
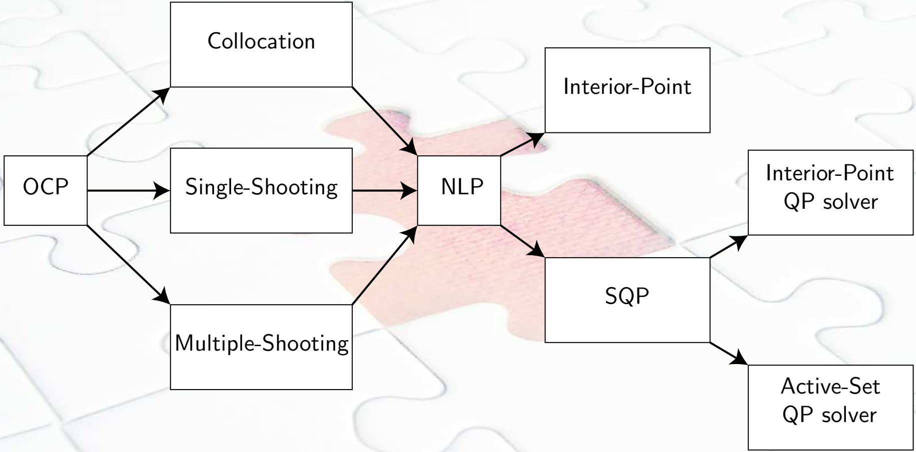


Survival map of Direct Optimal Control



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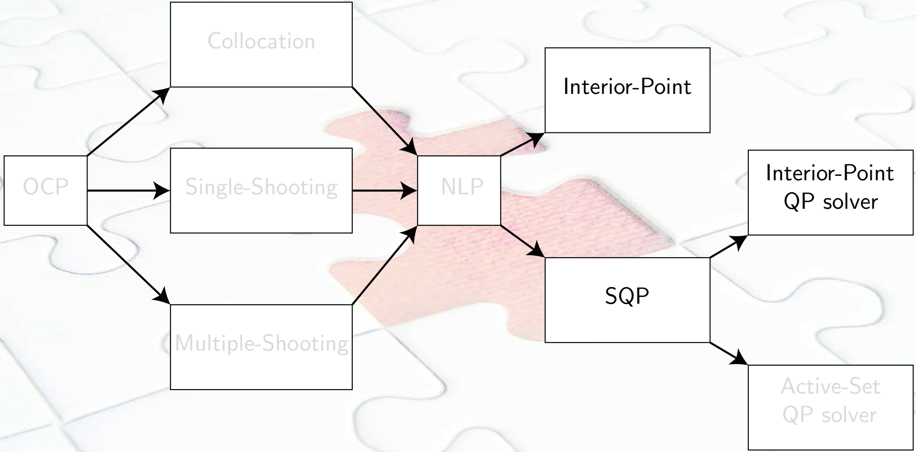

Two approaches for solving NLPs...

Algorithm: SQP (prototype)

while *Not converged* **do**

Form $\nabla^2_{\mathbf{w}}\mathcal{L}$, $\nabla_{\mathbf{w}}\mathcal{L}$, $\mathbf{g}$, $\nabla\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{h}$

Solve QP:

$$\begin{aligned} \min_{\Delta\mathbf{w}} \quad & \frac{1}{2}\Delta\mathbf{w}^\top\nabla^2_{\mathbf{w}}\mathcal{L}\Delta\mathbf{w} + \nabla\Phi\left(\mathbf{w}\right)^\top\Delta\mathbf{w} \\ \text{s.t.} \quad & \mathbf{g}\left(\mathbf{w}\right) + \nabla\mathbf{g}\left(\mathbf{w}\right)^\top\Delta\mathbf{w} = 0 \\ & \mathbf{h}\left(\mathbf{w}\right) + \nabla\mathbf{h}\left(\mathbf{w}\right)^\top\Delta\mathbf{w} \leq 0 \end{aligned}$$

Update

$$\left\{\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\right\} \leftarrow \left\{\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\right\} + \Delta\left\{\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\right\}$$

end

Interior-Point vs. SQP ??

Algorithm: SQP (prototype)

while *Not converged* **do**

  Form $\nabla^2_{\mathbf{w}}\mathcal{L}$, $\nabla_{\mathbf{w}}\mathcal{L}$, $\mathbf{g}$, $\nabla\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{h}$

  while *IPQP not converged* **do**

    Newton step on:

$$\begin{aligned} H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda}^{\mathrm{QP}} + \nabla\mathbf{h}\boldsymbol{\mu}^{\mathrm{QP}} &= 0 \\ \nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} &= 0 \\ \nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s}^{\mathrm{QP}} &= 0 \\ \boldsymbol{\mu}_i^{\mathrm{QP}}\mathbf{s}_i^{\mathrm{QP}} &= \tau \end{aligned}$$

    reduce $\tau \to \epsilon$

  end

  Update

$$\{\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\} \leftarrow \{\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\} + \Delta\{\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\}$$

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end

Algorithm: IP (prototype)

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Form $\nabla^2_{\mathbf{w}}\mathcal{L}$, $\nabla_{\mathbf{w}}\mathcal{L}$, $\mathbf{g}$, $\nabla\mathbf{g}$, $\mathbf{h}$, $\nabla\mathbf{h}$

Newton step on:

$$\begin{aligned} \nabla\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= 0 \\ \mathbf{g}(\mathbf{w}) &= 0 \\ \mathbf{h}(\mathbf{w}) + \mathbf{s} &= 0 \\ \boldsymbol{\mu}_i s_i &= \tau \end{aligned}$$

gives $\Delta\mathbf{w}, \Delta\boldsymbol{\lambda}, \Delta\boldsymbol{\mu}$. Update

$$\{\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\} \leftarrow \{\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\} + \Delta\{\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}\}$$

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