Numerical Optimal Control
Lecture 7: QP solvers for Direct Optimal Control

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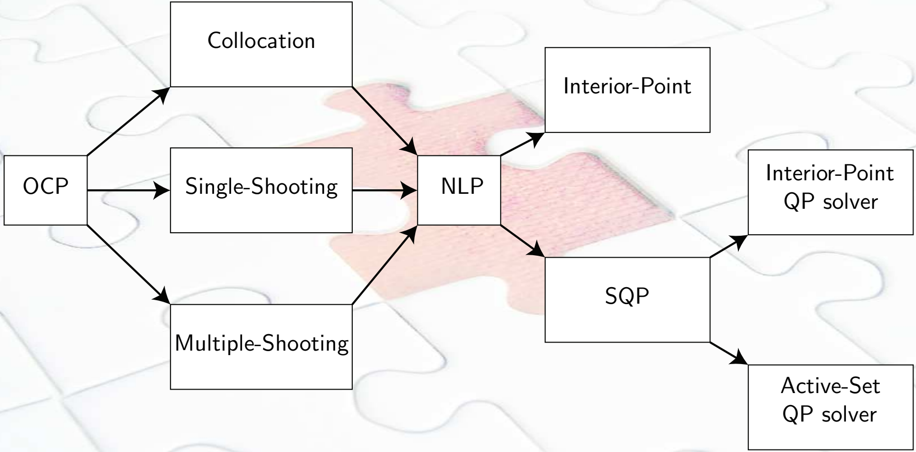
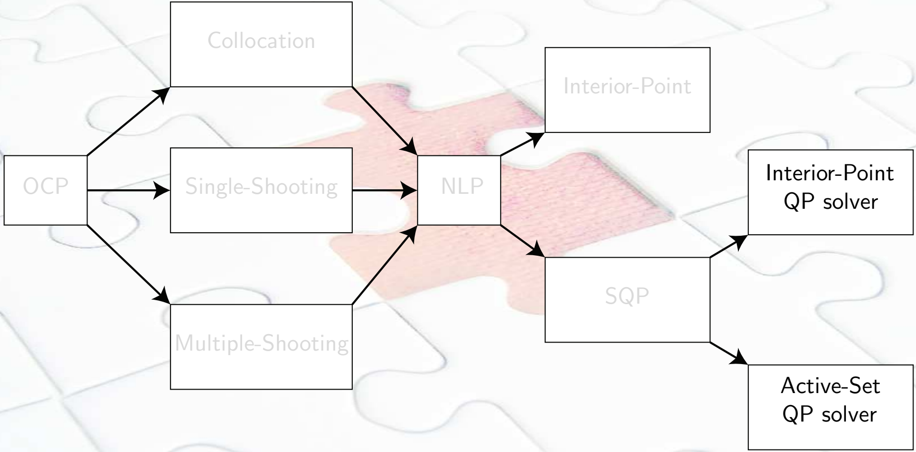
NTNU PhD course

Survival map of Direct Optimal Control



Survival map of Direct Optimal Control



Outline

1. Quadratic Programming
2. Active Set method
3. Interior-Point method

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SQP - Quadratic Programming

OCP

$$\begin{aligned} \min_{\mathbf{u},\mathbf{x}} \quad & E\left(\mathbf{x}_N\right) + \sum_{k=0}^{N-1} L\left(\mathbf{x}_k, \mathbf{u}_k\right) \\ \text{s.t.} \quad & \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) - \mathbf{x}_{k+1} = 0 \\ & \mathbf{h}_k\left(\mathbf{x}_k, \mathbf{u}_k\right) \leq 0 \end{aligned}$$

Iterate on the QP

Using:

$$\Delta\mathbf{w} = \begin{bmatrix} \Delta\mathbf{x}_0 \\ \Delta\mathbf{u}_0 \\ \dots \\ \Delta\mathbf{x}_{N-1} \\ \Delta\mathbf{u}_{N-1} \\ \Delta\mathbf{x}_N \end{bmatrix}$$

QP:

$$\begin{aligned} \min_{\Delta\mathbf{w}} \quad & \frac{1}{2}\Delta\mathbf{w}\top H\Delta\mathbf{w} + \nabla\Phi\top\Delta\mathbf{w} \\ \text{s.t.} \quad & \nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0 \\ & \nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} \leq 0 \end{aligned}$$

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$$\min_{\mathbf{u},\mathbf{x}} \quad E\left(\mathbf{x}_N\right) + \sum_{k=0}^{N-1} L\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

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$$\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} \leq 0$$

Different problems have different features:

- Problem: small, medium, large-scale
- Horizon N: short / long
- Inequality constraints : simple bounds / elaborate polytopes
- Ratio #inputs / #states
- Dynamics: stable / unstable

Different features call for different QP solvers

Outline

1. Quadratic Programming
2. **Active Set method**
3. Interior-Point method

Active Set Method - Property of the Active Set

OCP

$$\begin{aligned}\min_{\mathbf{u},\mathbf{x}} \quad & E\left(\mathbf{x}_N\right) + \sum_{k=0}^{N-1} L\left(\mathbf{x}_k, \mathbf{u}_k\right) \\ \text{s.t.} \quad & \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) - \mathbf{x}_{k+1} = 0 \\ & \mathbf{h}_k\left(\mathbf{x}_k, \mathbf{u}_k\right) \leq 0\end{aligned}$$

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$$\nabla\mathbf{h}^\top \Delta\mathbf{w} + \mathbf{h} \leq 0$$

KKT conditions *for the QP*

Find $\Delta\mathbf{w}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ s.t.

$$H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} = 0$$

$$\nabla\mathbf{g}^\top \Delta\mathbf{w} + \mathbf{g} = 0$$

$$\nabla\mathbf{h}^\top \Delta\mathbf{w} + \mathbf{h} \leq 0$$

$$\boldsymbol{\mu} \geq 0$$

$$\boldsymbol{\mu}_i{}^\top \left(\nabla\mathbf{h}^\top \Delta\mathbf{w} + \mathbf{h}\right)_i = 0$$

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The Active Set $\mathbb{A}$

Let $\mathbb{A}$ be the **set of (strictly) active constraints** at the solution, then the solution fulfils:

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$$\nabla\mathbf{g}^\top \Delta\mathbf{w} + \mathbf{g} = 0$$
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with $\boldsymbol{\mu}_{\mathbb{A}} > 0$ **and** $\nabla\mathbf{h}_{\bar{\mathbb{A}}}^\top \Delta\mathbf{w} + \mathbf{h}_{\bar{\mathbb{A}}} < 0$

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with $\boldsymbol{\mu}_{\mathbb{A}} > 0$ **and** $\nabla\mathbf{h}_{\bar{\mathbb{A}}}^\top \Delta\mathbf{w} + \mathbf{h}_{\bar{\mathbb{A}}} < 0$

Active Set Method - the key idea

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Iterate QP

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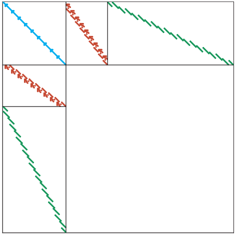

Active Set Method - the key idea

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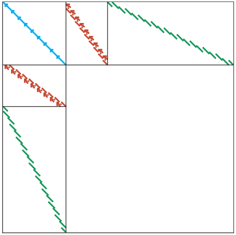
Checking a candidate set $\mathbb{A}$ takes "only" a sparse matrix factorization



Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

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Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
2. Solve :

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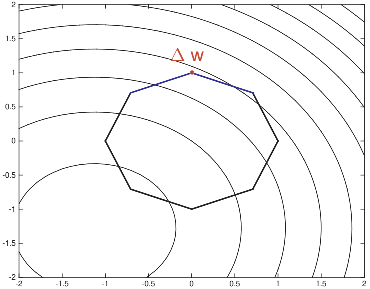

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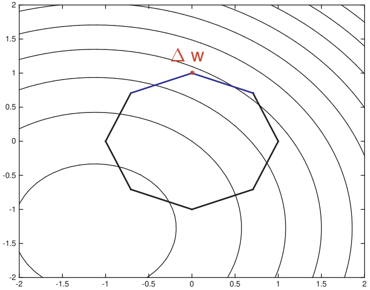
3. Move $\Delta w \rightarrow \Delta w^+$ until hitting a constraint



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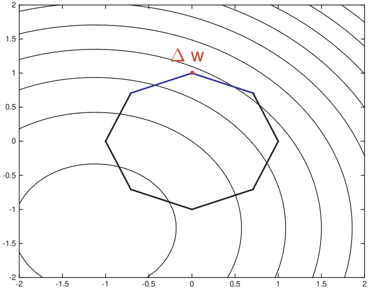

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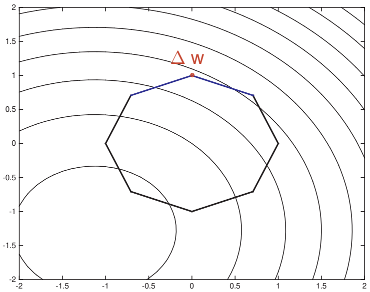

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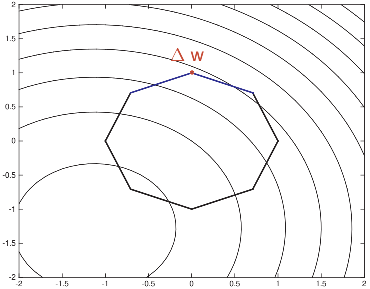
6. If $\boldsymbol{\mu}^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**



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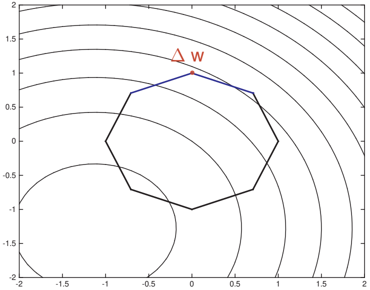

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$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

3. Move $\Delta w \rightarrow \Delta w^+$ until hitting a constraint
4. Add the new constraint to $\mathbb{A}$
5. If $\boldsymbol{\mu}_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\boldsymbol{\mu}^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**

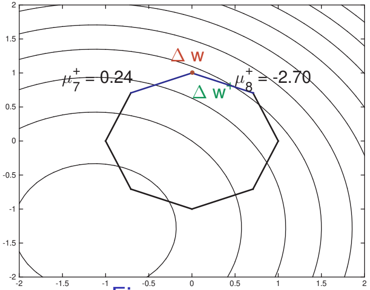

Figure: Step 2

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
2. Solve :

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

3. Move $\Delta w \rightarrow \Delta w^+$ until hitting a constraint
4. Add the new constraint to $\mathbb{A}$
5. If $\mu_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\mu^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**

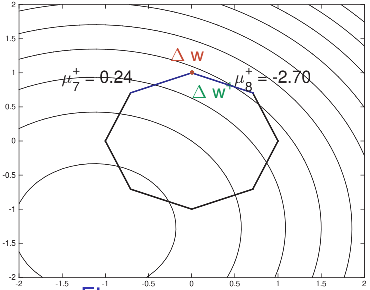

Figure: Step 3 and 4

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
2. Solve :

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

3. Move $\Delta w \rightarrow \Delta w^+$ until hitting a constraint
4. Add the new constraint to $\mathbb{A}$
5. If $\boldsymbol{\mu}_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\boldsymbol{\mu}^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**

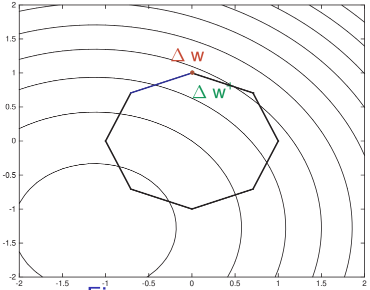

Figure: Step 5 → 1

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
2. Solve :

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

3. Move $\Delta w \rightarrow \Delta w^+$ until hitting a constraint
4. Add the new constraint to $\mathbb{A}$
5. If $\boldsymbol{\mu}_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\boldsymbol{\mu}^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**

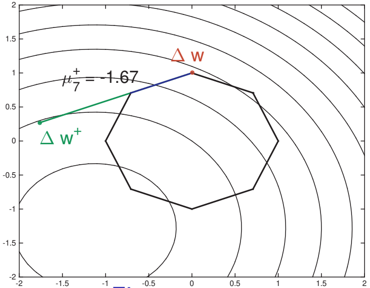

Figure: Step 2

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7,8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
2. Solve :

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

3. Move $\Delta w \to \Delta w^+$ until hitting a constraint
4. Add the new constraint to $\mathbb{A}$
5. If $\mu_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\mu^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**

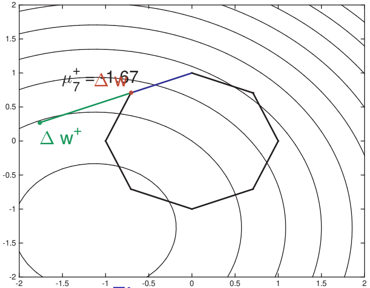

Figure: Step 3

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
2. Solve :

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

3. Move $\Delta w \to \Delta w^+$ until hitting a constraint
4. Add the new constraint to $\mathbb{A}$
5. If $\boldsymbol{\mu}_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\boldsymbol{\mu}^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**

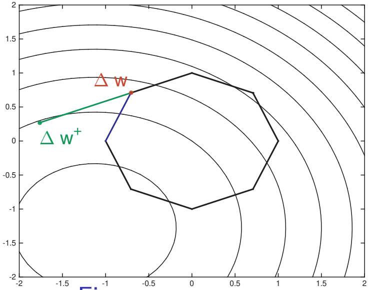

Figure: Step 4,5 → 1

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^{\top}$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
2. Solve :
$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^{\top} & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^{\top} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^{+} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^{+} \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$
3. Move $\Delta w \rightarrow \Delta w^{+}$ until hitting a constraint
4. Add the new constraint to $\mathbb{A}$
5. If $\boldsymbol{\mu}_k^{+} \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\boldsymbol{\mu}^{+} > 0$ and $\Delta w^{+} = \Delta w$ reached then **exit**
7. Else **goto 1**

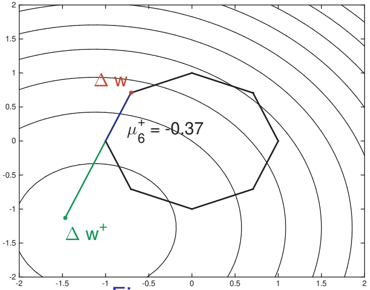

Figure: Step 2

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
2. Solve :

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

3. Move $\Delta w \rightarrow \Delta w^+$ until hitting a constraint
4. Add the new constraint to $\mathbb{A}$
5. If $\boldsymbol{\mu}_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\boldsymbol{\mu}^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**

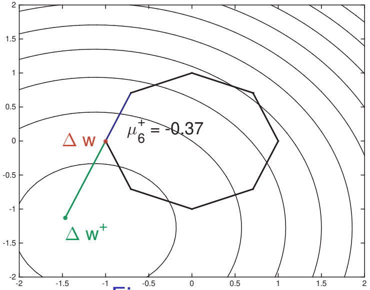

Figure: Step 3

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
2. Solve :
$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$
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5. If $\mu_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\mu^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**

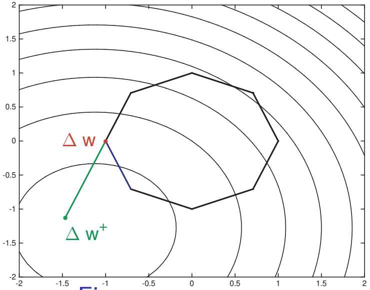

Figure: Step 4,5 → 1

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
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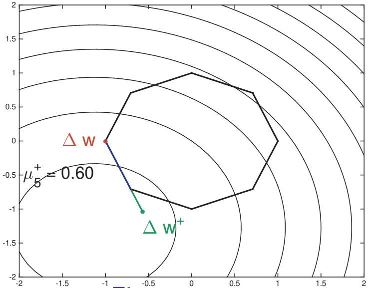

Figure: Step 2

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^{\top}$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
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$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^{\top} & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^{\top} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^{+} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^{+} \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

3. Move $\Delta w \rightarrow \Delta w^{+}$ until hitting a constraint
4. Add the new constraint to $\mathbb{A}$
5. If $\boldsymbol{\mu}_k^{+} \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\boldsymbol{\mu}^{+} > 0$ and $\Delta w^{+} = \Delta w$ reached then **exit**
7. Else **goto 1**

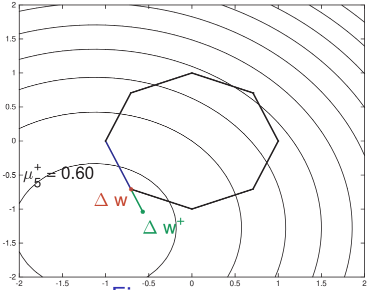

Figure: Step 3

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

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1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
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3. Move $\Delta w \rightarrow \Delta w^+$ until hitting a constraint
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5. If $\boldsymbol{\mu}_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\boldsymbol{\mu}^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**

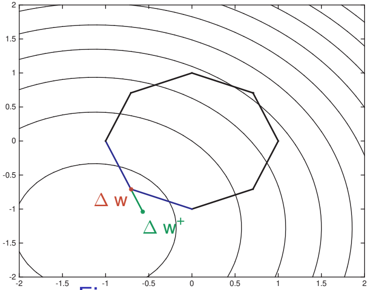

Figure: Step 4,5 → 1

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
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3. Move $\Delta w \rightarrow \Delta w^+$ until hitting a constraint
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5. If $\boldsymbol{\mu}_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
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7. Else **goto 1**

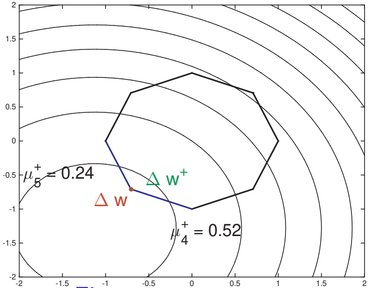


Figure: Step 2 $\rightarrow$ **exit**

Active Set Method - Algorithm

Catch the right Active Set $\mathbb{A}$ as fast as possible !!

Pseudo-algorithm

Guess: $\mathbb{A} = \{7, 8\}$, Δw feasible ("phase-I")

1. Form $\nabla \mathbf{h}_{\mathbb{A}}^\top$ and $\mathbf{h}_{\mathbb{A}}$ (active constraints)
2. Solve :

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

3. Move $\Delta w \rightarrow \Delta w^+$ until hitting a constraint
4. Add the new constraint to $\mathbb{A}$
5. If $\mu_k^+ \leq 0$, then **remove** k from $\mathbb{A}$
6. If $\mu^+ > 0$ and $\Delta w^+ = \Delta w$ reached then **exit**
7. Else **goto 1**

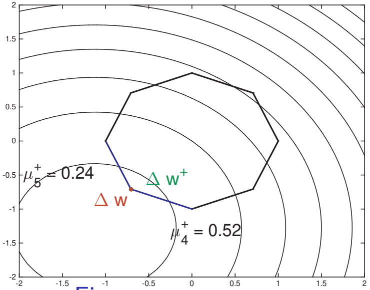


Figure: Step 2 $\rightarrow$ **exit**

1. Very fast for a few changes of the active set
2. No tight complexity bound

Structure exploitation - Condensing

Iterate QP

$$\begin{aligned} \min_{\Delta\mathbf{w}} \quad & \frac{1}{2}\Delta\mathbf{w}^\top H \Delta\mathbf{w} + \nabla\Phi^\top\Delta\mathbf{w} \\ \text{s.t.} \quad & \nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0 \\ & \nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} \leq 0 \end{aligned}$$

Factorization of the linear system

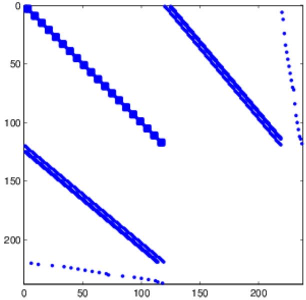
$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h}_{\mathbb{A}} \\ \nabla\mathbf{g}^\top & 0 & 0 \\ \nabla\mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = -\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

is expensive !!

Structure exploitation - Condensing

Iterate QP

$$
\begin{aligned}
\min_{\Delta \mathbf{w}} \quad & \frac{1}{2}\Delta \mathbf{w}^\top H \Delta \mathbf{w} + \nabla \Phi^\top \Delta \mathbf{w} \\
\text{s.t.} \quad & \nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0 \\
& \nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} \leq 0
\end{aligned}
$$



Example: 5 integrators, 1 input, $N = 20$ with input & state bounds.

Factorization of the linear system

$$
\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix}
\begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix}
= -\begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}
$$

is expensive !!

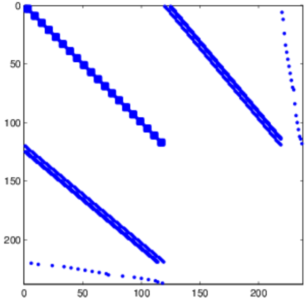
Structure exploitation - Condensing

Iterate QP

$$\min_{\Delta \mathbf{w}} \quad \frac{1}{2}\Delta \mathbf{w}^\top H \Delta \mathbf{w} + \nabla \Phi^\top \Delta \mathbf{w}$$

$$\text{s.t.} \quad \nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0$$

$$\nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} \leq 0$$



Example: 5 integrators, 1 input, $N = 20$ with input & state bounds.

Factorization of the linear system

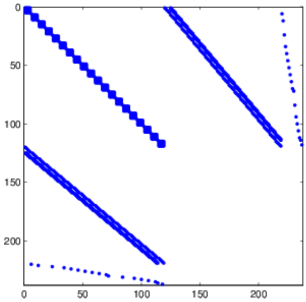
$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h}_{\mathbb{A}} \\ \nabla \mathbf{g}^\top & 0 & 0 \\ \nabla \mathbf{h}_{\mathbb{A}}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu}^+ \end{bmatrix} = - \begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h}_{\mathbb{A}} \end{bmatrix}$$

is expensive !!

Structure exploitation - Condensing

Iterate QP

$$\begin{aligned} \min_{\Delta \mathbf{w}} \quad & \frac{1}{2}\Delta \mathbf{w}\top H \Delta \mathbf{w} + \nabla \Phi \top \Delta \mathbf{w} \\ \text{s.t.} \quad & \nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0 \\ & \nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} \leq 0 \end{aligned}$$



Example: 5 integrators, 1 input, $N = 20$ with input & state bounds.

Condensed QP

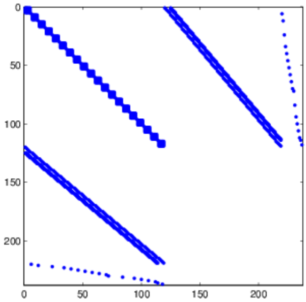
Eliminate the states $\Delta \mathbf{x}_k$ using $\nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0$, i.e.

$$\Delta \mathbf{x}_{k+1} = \nabla_{\mathbf{x}} \mathbf{f}_k{}^\top \Delta \mathbf{x}_k + \nabla_{\mathbf{u}} \mathbf{f}_k{}^\top \Delta \mathbf{u}_k$$

Structure exploitation - Condensing

Iterate QP

$$\begin{aligned} \min_{\Delta \mathbf{w}} \quad & \frac{1}{2}\Delta \mathbf{w}^\top H \Delta \mathbf{w} + \nabla\Phi^\top \Delta \mathbf{w} \\ \text{s.t.} \quad & \nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0 \\ & \nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} \leq 0 \end{aligned}$$



Example: 5 integrators, 1 input, $N = 20$ with input & state bounds.

Condensed QP

Eliminate the states $\Delta \mathbf{x}_k$ using $\nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0$, i.e.

$$\Delta \mathbf{x}_{k+1} = \nabla_{\mathbf{x}} \mathbf{f}_k^\top \Delta \mathbf{x}_k + \nabla_{\mathbf{u}} \mathbf{f}_k^\top \Delta \mathbf{u}_k$$

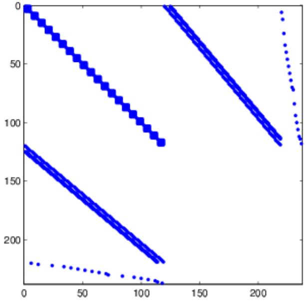
yields by "simulation":

$$\Delta \mathbf{x}_k = \prod_{i=0}^{k-1} \nabla_{\mathbf{x}} \mathbf{f}_i^\top \Delta \mathbf{x}_0 + \sum_{j=0}^{k-1} \prod_{i=j+1}^{k-1} \nabla_{\mathbf{x}} \mathbf{f}_i^\top \nabla_{\mathbf{u}} \mathbf{f}_j^\top \Delta \mathbf{u}_j$$

Structure exploitation - Condensing

Iterate QP

$$\begin{aligned} \min_{\Delta\mathbf{w}} \quad & \frac{1}{2}\Delta\mathbf{w}\top H\Delta\mathbf{w} + \nabla\Phi\top\Delta\mathbf{w} \\ \text{s.t.} \quad & \nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0 \\ & \nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} \leq 0 \end{aligned}$$



Example: 5 integrators, 1 input, $N = 20$ with input & state bounds.

Condensed QP

Eliminate the states $\Delta\mathbf{x}_k$ using $\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0$, i.e.

$$\Delta\mathbf{x}_{k+1} = \nabla_{\mathbf{x}}\mathbf{f}_k{}^\top\Delta\mathbf{x}_k + \nabla_{\mathbf{u}}\mathbf{f}_k{}^\top\Delta\mathbf{u}_k$$

yields by "simulation":

$$\Delta\mathbf{x}_k = \prod_{i=0}^{k-1}\nabla_{\mathbf{x}}\mathbf{f}_i{}^\top\Delta\mathbf{x}_0 + \sum_{j=0}^{k-1}\prod_{i=j+1}^{k-1}\nabla_{\mathbf{x}}\mathbf{f}_i{}^\top\nabla_{\mathbf{u}}\mathbf{f}_j{}^\top\Delta\mathbf{u}_j$$

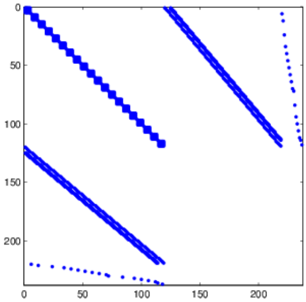
an we can write:

$$\Delta\mathbf{w} = A + M\begin{bmatrix}\Delta\mathbf{u}_0 \\ \dots \\ \Delta\mathbf{u}_{N-1}\end{bmatrix} \equiv A + M\Delta\mathbf{u}$$

Structure exploitation - Condensing

Iterate QP

$$\min_{\Delta \mathbf{w}} \quad \frac{1}{2}\Delta \mathbf{w} \top H \Delta \mathbf{w} + \nabla \Phi \top \Delta \mathbf{w}$$
$$\text{s.t.} \quad \nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0$$
$$\nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} \leq 0$$



Example: 5 integrators, 1 input, $N = 20$ with input & state bounds.

Condensed QP

Eliminate the states $\Delta \mathbf{x}_k$ using $\nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0$, i.e.

$$\Delta \mathbf{x}_{k+1} = \nabla_\mathbf{x} \mathbf{f}_k^\top \Delta \mathbf{x}_k + \nabla_\mathbf{u} \mathbf{f}_k^\top \Delta \mathbf{u}_k$$

yields by "simulation":

$$\Delta \mathbf{x}_k = \prod_{i=0}^{k-1} \nabla_\mathbf{x} \mathbf{f}_i^\top \Delta \mathbf{x}_0 + \sum_{j=0}^{k-1} \prod_{i=j+1}^{k-1} \nabla_\mathbf{x} \mathbf{f}_i^\top \nabla_\mathbf{u} \mathbf{f}_j^\top \Delta \mathbf{u}_j$$

an we can write:

$$\Delta \mathbf{w} = A + M \begin{bmatrix} \Delta \mathbf{u}_0 \\ \ldots \\ \Delta \mathbf{u}_{N-1} \end{bmatrix} \equiv A + M \Delta \mathbf{u}$$

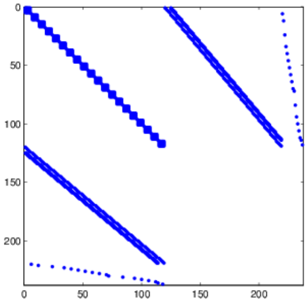
The condensed QP then reads as:

$$\min_{\Delta \mathbf{u}} \quad \frac{1}{2} \Delta \mathbf{u}^\top M^\top H M \Delta \mathbf{u} + \left(\frac{1}{2} A^\top H M + \nabla \Phi^\top M \right) \Delta \mathbf{u}$$
$$\text{s.t.} \quad \nabla \mathbf{h}^\top M \Delta \mathbf{u} + \nabla \mathbf{h}^\top A + \mathbf{g} \leq 0$$

Structure exploitation - Condensing

Iterate QP

$$\begin{aligned} \min_{\Delta \mathbf{w}} \quad & \frac{1}{2}\Delta \mathbf{w}\top {\color{cyan}H}\Delta \mathbf{w} + \nabla \Phi \top \Delta \mathbf{w} \\ \text{s.t.} \quad & {\color{red}\nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0} \\ & {\color{green}\nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} \leq 0} \end{aligned}$$



Example: 5 integrators, 1 input, $N = 20$ with input & state bounds.

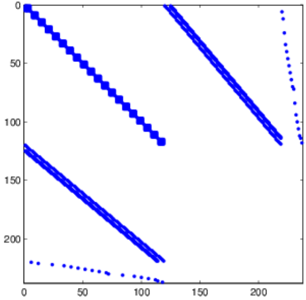
Condensed QP

$$\begin{aligned} \min_{\Delta \mathbf{u}} \quad & \frac{1}{2}\Delta \mathbf{u}^\top M^\top H M \Delta \mathbf{u} + \left(\frac{1}{2}A^\top H M + \nabla \Phi^\top M\right)\Delta \mathbf{u} \\ \text{s.t.} \quad & {\color{green}\nabla \mathbf{h}^\top M \Delta \mathbf{u} + \nabla \mathbf{h}^\top A + \mathbf{g} \leq 0} \end{aligned}$$

Structure exploitation - Condensing

Iterate QP

$$
\begin{aligned}
\min_{\Delta\mathbf{w}} \quad & \frac{1}{2}\Delta\mathbf{w}\top H\Delta\mathbf{w} + \nabla\Phi\top\Delta\mathbf{w} \\
\text{s.t.} \quad & \nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0 \\
& \nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} \leq 0
\end{aligned}
$$



Example: 5 integrators, 1 input, $N = 20$ with input & state bounds.

Condensed QP

$$
\begin{aligned}
\min_{\Delta\mathbf{u}} \quad & \frac{1}{2}\Delta\mathbf{u}^\top M^\top HM\Delta\mathbf{u} + \left(\frac{1}{2}A^\top HM + \nabla\Phi^\top M\right)\Delta\mathbf{u} \\
\text{s.t.} \quad & \nabla\mathbf{h}^\top M\Delta\mathbf{u} + \nabla\mathbf{h}^\top A + \mathbf{g} \leq 0
\end{aligned}
$$

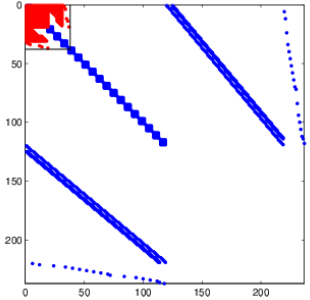
requires the factorisation of the matrix:

$$
\begin{bmatrix} M^\top HM & \left(M^\top\nabla\mathbf{h}\right)_{\mathbb{A}} \\ \left(\nabla\mathbf{h}^\top M\right)_{\mathbb{A}} & 0 \end{bmatrix}
\begin{bmatrix} \Delta\mathbf{u} \\ \tilde{\boldsymbol{\mu}} \end{bmatrix} = \dots
$$

Structure exploitation - Condensing

Iterate QP

$$
\begin{aligned}
\min_{\Delta \mathbf{w}} \quad & \frac{1}{2}\Delta \mathbf{w}^\top H \Delta \mathbf{w} + \nabla \Phi^\top \Delta \mathbf{w} \\
\text{s.t.} \quad & \nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0 \\
& \nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} \leq 0
\end{aligned}
$$



Example: 5 integrators, 1 input, $N = 20$ with input & state bounds, condensed

Condensed QP

$$
\begin{aligned}
\min_{\Delta \mathbf{u}} \quad & \frac{1}{2}\Delta \mathbf{u}^\top M^\top H M \Delta \mathbf{u} + \left(\frac{1}{2} A^\top H M + \nabla \Phi^\top M\right) \Delta \mathbf{u} \\
\text{s.t.} \quad & \nabla \mathbf{h}^\top M \Delta \mathbf{u} + \nabla \mathbf{h}^\top A + \mathbf{g} \leq 0
\end{aligned}
$$

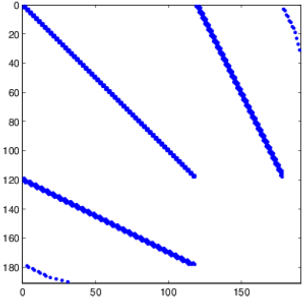
requires the factorisation of the matrix:

$$
\begin{bmatrix} M^\top H M & \left(M^\top \nabla \mathbf{h}\right)_{\mathbb{A}} \\ \left(\nabla \mathbf{h}^\top M\right)_{\mathbb{A}} & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{u} \\ \tilde{\boldsymbol{\mu}} \end{bmatrix} = \ldots
$$

Structure exploitation - Condensing

Iterate QP

$$\begin{aligned} \min_{\Delta \mathbf{w}} \quad & \frac{1}{2}\Delta\mathbf{w}\top H \Delta\mathbf{w} + \nabla\Phi\top\Delta\mathbf{w} \\ \text{s.t.} \quad & \nabla\mathbf{g}^\top \Delta\mathbf{w} + \mathbf{g} = 0 \\ & \nabla\mathbf{h}^\top \Delta\mathbf{w} + \mathbf{h} \leq 0 \end{aligned}$$



Example: 2 integrators, 2 inputs, $N = 60$ with input & state bounds.

Condensed QP

$$\begin{aligned} \min_{\Delta \mathbf{u}} \quad & \frac{1}{2}\Delta\mathbf{u}^\top M^\top H M \Delta\mathbf{u} + \left(\frac{1}{2}A^\top H M + \nabla\Phi^\top M\right)\Delta\mathbf{u} \\ \text{s.t.} \quad & \nabla\mathbf{h}^\top M \Delta\mathbf{u} + \nabla\mathbf{h}^\top A + \mathbf{g} \leq 0 \end{aligned}$$

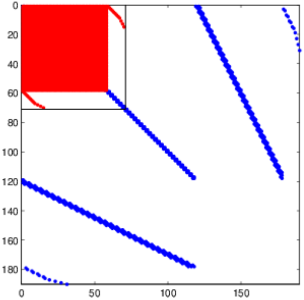
requires the factorisation of the matrix:

$$\begin{bmatrix} M^\top H M & \left(M^\top \nabla\mathbf{h}\right)_{\mathbb{A}} \\ \left(\nabla\mathbf{h}^\top M\right)_{\mathbb{A}} & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathbf{u} \\ \tilde{\boldsymbol{\mu}} \end{bmatrix} = \dots$$

Structure exploitation - Condensing

Iterate QP

$$\begin{aligned}
\min_{\Delta \mathbf{w}} \quad & \frac{1}{2}\Delta \mathbf{w}^\top H \Delta \mathbf{w} + \nabla \Phi^\top \Delta \mathbf{w} \\
\text{s.t.} \quad & \nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0 \\
& \nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} \leq 0
\end{aligned}$$



Example: 2 integrators, 2 inputs, $N = 60$ with input & state bounds, condensed

Condensed QP

$$\begin{aligned}
\min_{\Delta \mathbf{u}} \quad & \frac{1}{2}\Delta \mathbf{u}^\top M^\top H M \Delta \mathbf{u} + \left(\frac{1}{2}A^\top H M + \nabla \Phi^\top M\right) \Delta \mathbf{u} \\
\text{s.t.} \quad & \nabla \mathbf{h}^\top M \Delta \mathbf{u} + \nabla \mathbf{h}^\top A + \mathbf{g} \leq 0
\end{aligned}$$

requires the factorisation of the matrix:

$$\begin{bmatrix} M^\top H M & \left(M^\top \nabla \mathbf{h}\right)_{\mathbb{A}} \\ \left(\nabla \mathbf{h}^\top M\right)_{\mathbb{A}} & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{u} \\ \tilde{\boldsymbol{\mu}} \end{bmatrix} = \ldots$$

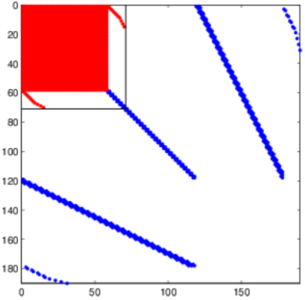
Structure exploitation - Condensing

Iterate QP

$$\min_{\Delta \mathbf{w}} \quad \frac{1}{2}\Delta \mathbf{w}^\top H \Delta \mathbf{w} + \nabla \Phi^\top \Delta \mathbf{w}$$

$$\text{s.t.} \quad \nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0$$

$$\nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} \leq 0$$



Example: 2 integrators, 2 inputs, $N = 60$ with input & state bounds, condensed

Condensed QP

$$\min_{\Delta \mathbf{u}} \quad \frac{1}{2}\Delta \mathbf{u}^\top M^\top H M \Delta \mathbf{u} + \left(\frac{1}{2}A^\top H M + \nabla \Phi^\top M\right)\Delta \mathbf{u}$$

$$\text{s.t.} \quad \nabla \mathbf{h}^\top M \Delta \mathbf{u} + \nabla \mathbf{h}^\top A + \mathbf{g} \leq 0$$

requires the factorisation of the matrix:

$$\begin{bmatrix} M^\top H M & \left(M^\top \nabla \mathbf{h}\right)_{\mathbb{A}} \\ \left(\nabla \mathbf{h}^\top M\right)_{\mathbb{A}} & 0 \end{bmatrix} \begin{bmatrix} \Delta \mathbf{u} \\ \tilde{\boldsymbol{\mu}} \end{bmatrix} = \ldots$$

Large/sparse QP $\rightarrow$ **small/dense** QP, but...

- Condensing is "unstable" for locally unstable systems
- Dense factorization has cubic complexity

Unfavorable for

- unstable systems
- many inputs
- long horizon

Outline

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$
\begin{aligned}
H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} &= 0 \\
\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} &= 0 \\
\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} &\leq 0 \\
\boldsymbol{\mu}_i\left(\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h}\right)_i &= 0 \\
\boldsymbol{\mu} &\geq 0
\end{aligned}
$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$
\begin{aligned}
H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} &= 0 \\
\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} &= 0 \\
\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} &= 0 \\
\boldsymbol{\mu}_i\mathbf{s}_i &= 0 \\
\mathbf{s} \geq 0, \quad \boldsymbol{\mu} &\geq 0
\end{aligned}
$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$
\begin{aligned}
H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} &= 0 \\
\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} &= 0 \\
\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} &= 0 \\
\boldsymbol{\mu}_i\mathbf{s}_i &= \tau \\
\mathbf{s} \geq 0, \quad \boldsymbol{\mu} &\geq 0
\end{aligned}
$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

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\begin{aligned}
H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} &= 0 \\
\nabla\mathbf{g}^\top \Delta\mathbf{w} + \mathbf{g} &= 0 \\
\nabla\mathbf{h}^\top \Delta\mathbf{w} + \mathbf{h} + \mathbf{s} &= 0 \\
\boldsymbol{\mu}_i \mathbf{s}_i &= \tau \\
\mathbf{s} \geq 0, \quad \boldsymbol{\mu} &\geq 0
\end{aligned}
$$

Newton step

$$
\begin{bmatrix}
H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\
\nabla\mathbf{g}^\top & 0 & 0 & 0 \\
\nabla\mathbf{h}^\top & 0 & 0 & I \\
0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu})
\end{bmatrix}
\begin{bmatrix}
\Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+
\end{bmatrix}
= - \underbrace{\begin{bmatrix}
\nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu}
\end{bmatrix}}_{\mathbf{r}}
$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} = 0$$
$$\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0$$
$$\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} = 0$$
$$\boldsymbol{\mu}_i\mathbf{s}_i = \tau$$
$$\mathbf{s} \geq 0, \quad \boldsymbol{\mu} \geq 0$$

Algorithm: PD-IP for QP

Input: Guess $\Delta\mathbf{w}, \boldsymbol{\lambda}$, and $\mathbf{s}, \boldsymbol{\mu} > 0$
while $\|\mathbf{r}\|, \tau > \mathrm{Tol}$ **do**
  Compute $\Delta\mathbf{w}^+, ..., \mathbf{s}^+$
  Chose step-size $t \leq 1$ s.t.
  - $(1-t)\,\mathbf{s} + t\mathbf{s}^+ > 0$
  - $(1-t)\,\boldsymbol{\mu} + t\boldsymbol{\mu}^+ > 0$
  - $\|\mathbf{r}\|$ decreases

  Decrease τ with $\tau \leftarrow \gamma\tau$
return $\Delta\mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}$

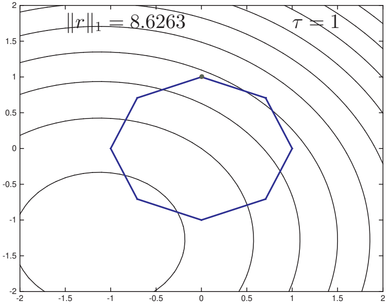
Newton step

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} = 0$$
$$\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0$$
$$\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} = 0$$
$$\boldsymbol{\mu}_i\mathbf{s}_i = \tau$$
$$\mathbf{s} \geq 0, \quad \boldsymbol{\mu} \geq 0$$



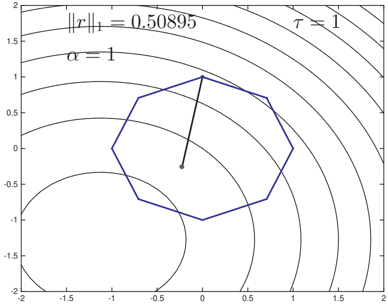
Newton step

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$
\begin{aligned}
H\Delta \mathbf{w} + \nabla \Phi + \nabla \mathbf{g} \boldsymbol{\lambda} + \nabla \mathbf{h} \boldsymbol{\mu} &= 0 \\
\nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} &= 0 \\
\nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} + \mathbf{s} &= 0 \\
\boldsymbol{\mu}_i \mathbf{s}_i &= \tau \\
\mathbf{s} \geq 0, \quad \boldsymbol{\mu} &\geq 0
\end{aligned}
$$



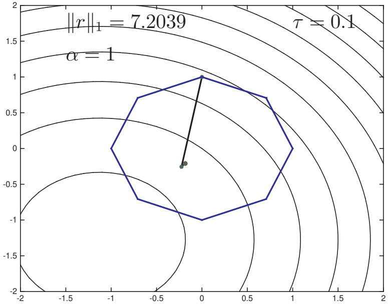
Newton step

$$
\begin{bmatrix}
H & \nabla \mathbf{g} & \nabla \mathbf{h} & 0 \\
\nabla \mathbf{g}^\top & 0 & 0 & 0 \\
\nabla \mathbf{h}^\top & 0 & 0 & I \\
0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu})
\end{bmatrix}
\begin{bmatrix}
\Delta \mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+
\end{bmatrix}
= -\underbrace{\begin{bmatrix}
\nabla \Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu}
\end{bmatrix}}_{\mathbf{r}}
$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$H\Delta \mathbf{w} + \nabla\Phi + \nabla \mathbf{g}\boldsymbol{\lambda} + \nabla \mathbf{h}\boldsymbol{\mu} = 0$$
$$\nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0$$
$$\nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} + \mathbf{s} = 0$$
$$\boldsymbol{\mu}_i \mathbf{s}_i = \tau$$
$$\mathbf{s} \geq 0, \quad \boldsymbol{\mu} \geq 0$$



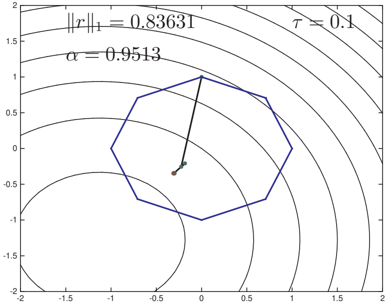
Newton step

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h} & 0 \\ \nabla \mathbf{g}^\top & 0 & 0 & 0 \\ \nabla \mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$\begin{aligned} H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} &= 0 \\ \nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} &= 0 \\ \nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} &= 0 \\ \boldsymbol{\mu}_i\mathbf{s}_i &= \tau \\ \mathbf{s} \geq 0, \quad \boldsymbol{\mu} &\geq 0 \end{aligned}$$



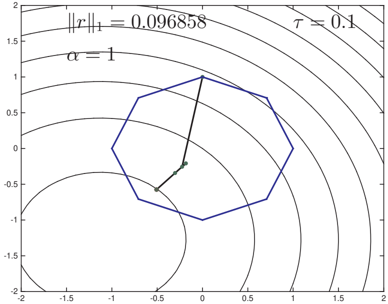
Newton step

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$
\begin{aligned}
H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} &= 0 \\
\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} &= 0 \\
\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} &= 0 \\
\boldsymbol{\mu}_i\mathbf{s}_i &= \tau \\
\mathbf{s} \geq 0, \quad \boldsymbol{\mu} &\geq 0
\end{aligned}
$$



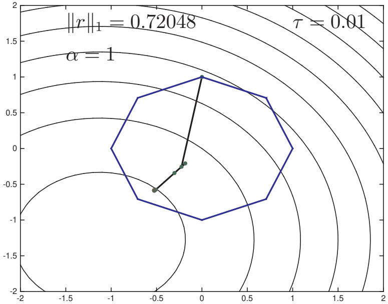
Newton step

$$
\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix}
\begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix}
= -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}
$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} = 0$$
$$\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0$$
$$\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} = 0$$
$$\boldsymbol{\mu}_i\mathbf{s}_i = \tau$$
$$\mathbf{s} \geq 0, \quad \boldsymbol{\mu} \geq 0$$



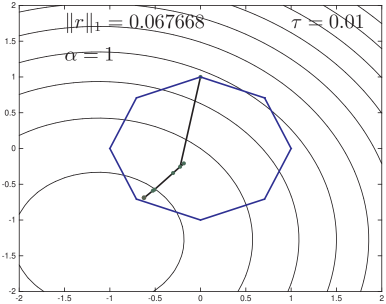
Newton step

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} = 0$$
$$\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0$$
$$\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} = 0$$
$$\boldsymbol{\mu}_i\mathbf{s}_i = \tau$$
$$\mathbf{s} \geq 0, \quad \boldsymbol{\mu} \geq 0$$



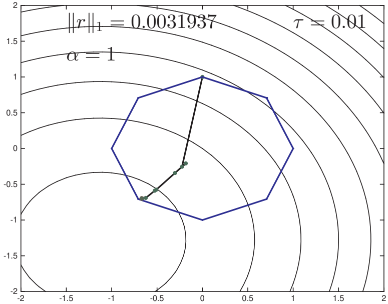
Newton step

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$
\begin{aligned}
H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} &= 0 \\
\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} &= 0 \\
\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} &= 0 \\
\boldsymbol{\mu}_i\mathbf{s}_i &= \tau \\
\mathbf{s} \geq 0, \quad \boldsymbol{\mu} &\geq 0
\end{aligned}
$$



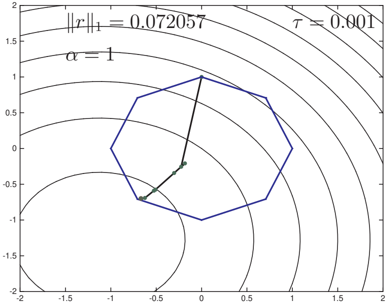
Newton step

$$
\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix}
\begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix}
= -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}
$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$H\Delta \mathbf{w} + \nabla \Phi + \nabla \mathbf{g} \boldsymbol{\lambda} + \nabla \mathbf{h} \boldsymbol{\mu} = 0$$
$$\nabla \mathbf{g}^\top \Delta \mathbf{w} + \mathbf{g} = 0$$
$$\nabla \mathbf{h}^\top \Delta \mathbf{w} + \mathbf{h} + \mathbf{s} = 0$$
$$\boldsymbol{\mu}_i \mathbf{s}_i = \tau$$
$$\mathbf{s} \geq 0, \quad \boldsymbol{\mu} \geq 0$$



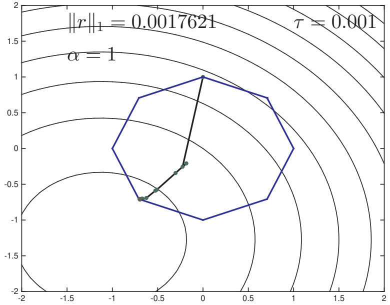
Newton step

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h} & 0 \\ \nabla \mathbf{g}^\top & 0 & 0 & 0 \\ \nabla \mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix} = - \underbrace{\begin{bmatrix} \nabla \Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} = 0$$
$$\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0$$
$$\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} = 0$$
$$\boldsymbol{\mu}_i\mathbf{s}_i = \tau$$
$$\mathbf{s} \geq 0, \quad \boldsymbol{\mu} \geq 0$$



Newton step

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

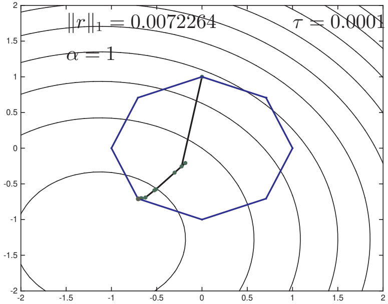
$$H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} = 0$$

$$\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0$$

$$\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} = 0$$

$$\boldsymbol{\mu}_i \mathbf{s}_i = \tau$$

$$\mathbf{s} \geq 0, \quad \boldsymbol{\mu} \geq 0$$



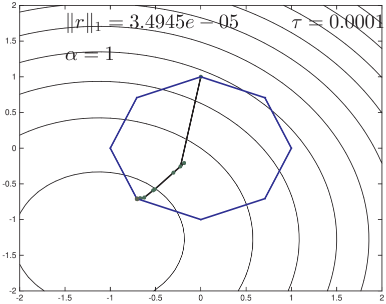
Newton step

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}$$

Primal-dual Interior-Point methods for QPs

KKT cond. with slack

$$H\Delta\mathbf{w} + \nabla\Phi + \nabla\mathbf{g}\boldsymbol{\lambda} + \nabla\mathbf{h}\boldsymbol{\mu} = 0$$
$$\nabla\mathbf{g}^\top\Delta\mathbf{w} + \mathbf{g} = 0$$
$$\nabla\mathbf{h}^\top\Delta\mathbf{w} + \mathbf{h} + \mathbf{s} = 0$$
$$\boldsymbol{\mu}_i\mathbf{s}_i = \tau$$
$$\mathbf{s} \geq 0, \quad \boldsymbol{\mu} \geq 0$$



Newton step

$$\begin{bmatrix} H & \nabla\mathbf{g} & \nabla\mathbf{h} & 0 \\ \nabla\mathbf{g}^\top & 0 & 0 & 0 \\ \nabla\mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta\mathbf{w}^+ \\ \boldsymbol{\lambda}^+ \\ \boldsymbol{\mu}^+ \\ \mathbf{s}^+ \end{bmatrix} = -\underbrace{\begin{bmatrix} \nabla\Phi \\ \mathbf{g} \\ \mathbf{h} \\ \tau + \mathrm{diag}(\mathbf{s})\boldsymbol{\mu} \end{bmatrix}}_{\mathbf{r}}$$

Interior-Point methods - Factorization

Newton step - Factorization

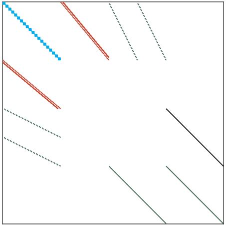
$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h} & 0 \\ \nabla \mathbf{g}^\top & 0 & 0 & 0 \\ \nabla \mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \\ \Delta \mathbf{s} \end{bmatrix} = \mathbf{r}$$

Interior-Point methods - Factorization

Newton step - Factorization

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h} & 0 \\ \nabla \mathbf{g}^\top & 0 & 0 & 0 \\ \nabla \mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \\ \Delta \mathbf{s} \end{bmatrix} = \mathbf{r}$$

KKT matrix:

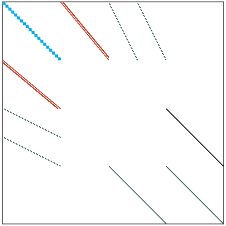


Interior-Point methods - Factorization

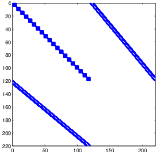
Newton step - Factorization

$$\begin{bmatrix} {\color{cyan}H} & {\color{red}\nabla \mathbf{g}} & {\color{green}\nabla \mathbf{h}} & 0 \\ {\color{red}\nabla \mathbf{g}}^\top & 0 & 0 & 0 \\ {\color{green}\nabla \mathbf{h}}^\top & 0 & 0 & I \\ 0 & 0 & {\color{green}\mathrm{diag}(\mathbf{s})} & {\color{green}\mathrm{diag}(\boldsymbol{\mu})} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \\ \Delta \mathbf{s} \end{bmatrix} = \mathbf{r}$$

KKT matrix:



Augmented



Factorize

Augmented form: eliminate Δs, $\Delta \boldsymbol{\mu}$

$$\begin{bmatrix} \Phi & \nabla \mathbf{g} \\ \nabla \mathbf{g}^\top & 0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta \boldsymbol{\lambda} \end{bmatrix} = \bar{\mathbf{r}}$$

where

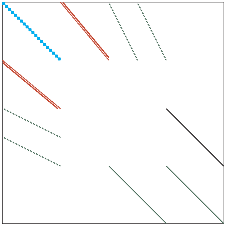
$$\Phi = {\color{cyan}H} + {\color{green}\nabla \mathbf{h} \cdot \mathrm{diag}(\mathbf{s})}^{-1} \cdot \mathrm{diag}(\boldsymbol{\mu}) \cdot {\color{green}\nabla \mathbf{h}}^\top$$

Interior-Point methods - Factorization

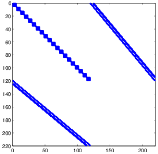
Newton step - Factorization

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h} & 0 \\ \nabla \mathbf{g}^\top & 0 & 0 & 0 \\ \nabla \mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \\ \Delta \mathbf{s} \end{bmatrix} = \mathbf{r}$$

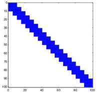
KKT matrix:



Augmented



Normal form:



Factorize

Augmented form: eliminate $\Delta \mathbf{s}$, $\Delta \boldsymbol{\mu}$

$$\begin{bmatrix} \Phi & \nabla \mathbf{g} \\ \nabla \mathbf{g}^\top & 0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta \boldsymbol{\lambda} \end{bmatrix} = \bar{\mathbf{r}}$$

where

$$\Phi = H + \nabla \mathbf{h} \cdot \mathrm{diag}\left(\mathbf{s}\right)^{-1} \cdot \mathrm{diag}\left(\boldsymbol{\mu}\right) \cdot \nabla \mathbf{h}^\top$$

Normal form: eliminate $\Delta \mathbf{w}$

$$W \Delta \boldsymbol{\lambda} = \beta$$

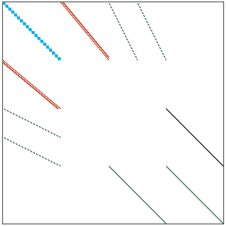
where $W = \nabla \mathbf{g}^\top \Phi^{-1} \nabla \mathbf{g}$ is the Schur complement of Φ (Φ block diagonal)

Interior-Point methods - Factorization

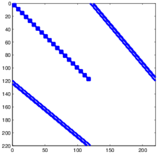
Newton step - Factorization

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h} & 0 \\ \nabla \mathbf{g}^\top & 0 & 0 & 0 \\ \nabla \mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \\ \Delta \mathbf{s} \end{bmatrix} = \mathbf{r}$$

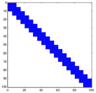
KKT matrix:



Augmented



Normal form:



Factorize

Augmented form: eliminate $\Delta \mathrm{s}$, $\Delta \boldsymbol{\mu}$

$$\begin{bmatrix} \Phi & \nabla \mathbf{g} \\ \nabla \mathbf{g}^\top & 0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta \boldsymbol{\lambda} \end{bmatrix} = \bar{\mathbf{r}}$$

where

$$\Phi = H + \nabla \mathbf{h} \cdot \mathrm{diag}\,(\mathbf{s})^{-1} \cdot \mathrm{diag}\,(\boldsymbol{\mu}) \cdot \nabla \mathbf{h}^\top$$

Normal form: eliminate $\Delta \mathrm{w}$

$$W \Delta \boldsymbol{\lambda} = \beta$$

where $W = \nabla \mathbf{g}^\top \Phi^{-1} \nabla \mathbf{g}$ is the Schur complement of Φ (Φ block diagonal)

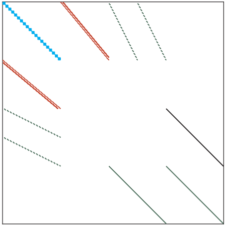
Cholesky/Riccati factorization of the normal form W (banded)

Interior-Point methods - Factorization

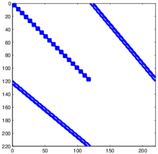
Newton step - Factorization

$$\begin{bmatrix} H & \nabla \mathbf{g} & \nabla \mathbf{h} & 0 \\ \nabla \mathbf{g}^\top & 0 & 0 & 0 \\ \nabla \mathbf{h}^\top & 0 & 0 & I \\ 0 & 0 & \mathrm{diag}(\mathbf{s}) & \mathrm{diag}(\boldsymbol{\mu}) \end{bmatrix} \begin{bmatrix} \Delta \mathbf{w} \\ \Delta \boldsymbol{\lambda} \\ \Delta \boldsymbol{\mu} \\ \Delta \mathbf{s} \end{bmatrix} = \mathbf{r}$$

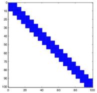
KKT matrix:



Augmented



Normal form:



Factorize

Augmented form: eliminate $\Delta \mathbf{s}$, $\Delta \boldsymbol{\mu}$

$$\begin{bmatrix} \Phi & \nabla \mathbf{g} \\ \nabla \mathbf{g}^\top & 0 \end{bmatrix} \begin{bmatrix} \Delta w \\ \Delta \boldsymbol{\lambda} \end{bmatrix} = \bar{\mathbf{r}}$$

where

$$\Phi = H + \nabla \mathbf{h} \cdot \mathrm{diag}\left(\mathbf{s}\right)^{-1} \cdot \mathrm{diag}\left(\boldsymbol{\mu}\right) \cdot \nabla \mathbf{h}^\top$$

Normal form: eliminate $\Delta \mathbf{w}$

$$W \Delta \boldsymbol{\lambda} = \beta$$

where $W = \nabla \mathbf{g}^\top \Phi^{-1} \nabla \mathbf{g}$ is the Schur complement of Φ (Φ block diagonal)

Cholesky/Riccati factorization of the normal form W (banded)

Complexity of factorization is $N\left(N_{\mathrm{x}} + N_{\mathbf{u}}\right)$, i.e. linear in horizon N

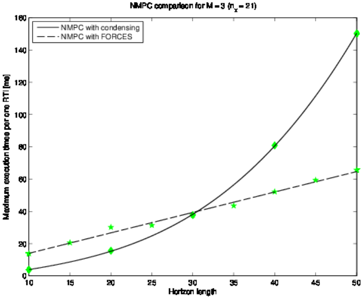
Complexity of Active set method vs. IP

vs.

Active set: **qpOASES**

IP: **Forces**, **HPMPC**

- Very fast for few AS changes !!
- # inputs << # states
- Beware of unstable dynamics !!
- Limited to QP
- *Quadratic* in N



- Speed is consistent
- # inputs irrelevant
- Unstable dynamics ok
- Extension to convex programming
- Linear in N

QP solvers for MPC

QP solvers:

- qpOASES: active-set solver with homotopy strategy for AS change
- FORCES: primal-dual interior point QP solver
- HPMPC: primal-dual interior point QP solver, with efficient implementation, Riccatti factorizations.
- Fiordos: dynamics dualized, stage problems and dual problem solved via fast gradient / proximal fast gradient
- qpDUNES: dynamics dualized, stage problems solved via "prox method" (via qpOASES is non-diagonal Hessian or complex constraints). Dual problem solved via Newton.