Numerical Optimal Control
Lecture 12: Optimal Control with DAEs

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Outline

Optimal Control with ODEs / DAEs

ODE-constrained

$$\begin{aligned} \min_{\mathbf{x}(.),\mathbf{u}(.)} \quad & \phi(\mathbf{x}(.),\mathbf{u}(.)) \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\mathbf{u},t) \\ & \mathbf{h}(\mathbf{u}(t),\mathbf{x}(t),t) \leq 0 \\ & \mathbf{b}(\mathbf{x}(t_0),\mathbf{x}(t_f)) = 0 \end{aligned}$$

DAE-constrained

$$\begin{aligned} \min_{\mathbf{x},\mathbf{z},\mathbf{u}} \quad & \phi(\mathbf{x}(.),\mathbf{z}(.),\mathbf{u}(.)) \\ \text{s.t.} \quad & \mathbf{F}(\dot{\mathbf{x}},\mathbf{x},\mathbf{z},\mathbf{u}) = 0, \\ & \mathbf{G}(\mathbf{x},\mathbf{z},\mathbf{u}) = 0 \\ & \mathbf{h}(\mathbf{u}(t),\mathbf{x}(t)),\mathbf{z}(t)) \leq 0 \\ & \mathbf{b}(\mathbf{x}(t_0),\mathbf{x}(t_f)) = 0 \end{aligned}$$

- Intuitive definition: some variables in the dynamics are not time-differentiated
- Size of $\mathbf{F}$ + size of $\mathbf{G}$ = size of $\mathbf{x}$ + size of $\mathbf{z}$

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Large systems made of many subsystems:

- ODEs describe each subsystems independently
- Algebraic relationships describe e.g. balance equations, flow, etc...
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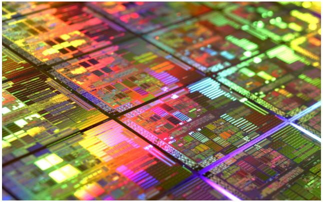

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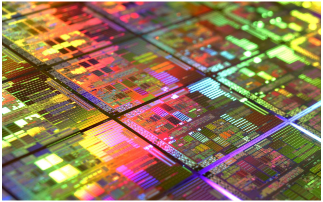

Mechanical/Robotic systems:

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- Modelling procedure is often easier using DAEs
- Often yields more "natural" representations of the system

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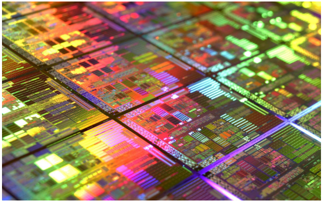

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Delta robot

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Fully-implicit DAE:

$$\mathbf{F}\left(\mathbf{x}, \dot{\mathbf{x}}, \mathbf{u}\right) = 0$$

with $\frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}}$ rank-deficient.

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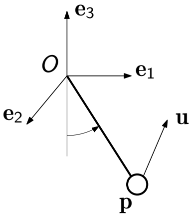
An example of a DAE - 3D pendulum

Position given by $\mathbf{p} \in \mathbb{R}^3$, dynamics:

$$m\ddot{\mathbf{p}} = \mathbf{u} - mg\mathbf{e}_3$$



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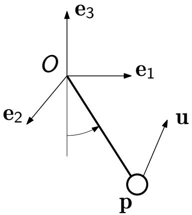

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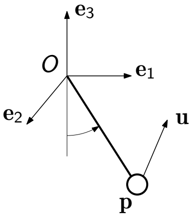

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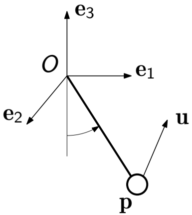

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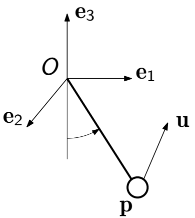

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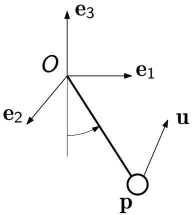

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$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{F}(\mathbf{x}, \mathbf{u}, z) \\ 0 &= \mathbf{G}(\mathbf{x}) \end{aligned}$$

Semi-explicit with $\mathbf{G}$ independent of z...

Outline

Easy & Hard DAEs

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Implicit Function Theorem:

$$\left[\begin{array}{cc} \frac{\partial \mathbf{F}}{\partial \dot{\mathbf{x}}} & \frac{\partial \mathbf{F}}{\partial \mathbf{z}} \end{array} \right]$$

full-rank at $\mathbf{x}, \mathbf{u}$ guarantees that the DAE is "solvable" at $\dot{\mathbf{x}}, \mathbf{z}$.

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For semi-explicit DAEs

$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$
$$0 = \mathbf{G}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}\right)$$

$\frac{\partial \mathbf{G}}{\partial \mathbf{z}}$ full rank makes the DAE "solvable".

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Semi-explicit DAEs - Differential Index

Definition

For a **semi-explicit DAE** the differential index is the minimum i such that:

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Remark: for an index-1 semi-explicit DAE:

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is well-defined, i.e. $\frac{\partial \mathbf{G}}{\partial \mathbf{z}}$ is full rank !!

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$\equiv$

“easy” DAEs

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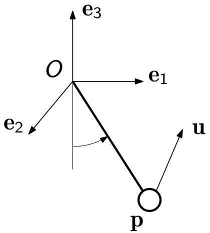

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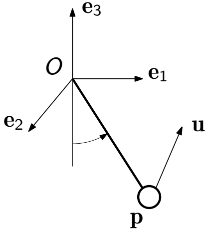

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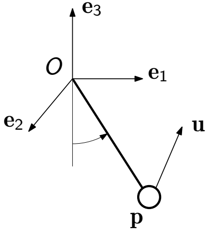

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which is solvable for z:

$$z = \frac{1}{\mathbf{p}^\top \mathbf{p}}\left(\mathbf{p}^\top \mathbf{u} - mg\mathbf{p}^\top \mathbf{e}_3 + m\dot{\mathbf{p}}^\top \dot{\mathbf{p}}\right)$$

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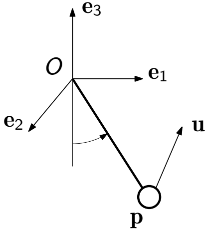

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$$\mathbf{p}^\top\left(\frac{1}{m}\mathbf{u} - g\mathbf{e}_3 - \frac{1}{m}z\mathbf{p}\right) + \dot{\mathbf{p}}^\top\dot{\mathbf{p}} = 0$$

which is solvable for z:

$$z = \frac{1}{\mathbf{p}^\top\mathbf{p}}\left(\mathbf{p}^\top\mathbf{u} - mg\mathbf{p}^\top\mathbf{e}_3 + m\dot{\mathbf{p}}^\top\dot{\mathbf{p}}\right)$$

one extra time differentiation yields an ODE ($\dot{z}$ appears)

For a **semi-explicit DAE** the differential index is the minimum i such that:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
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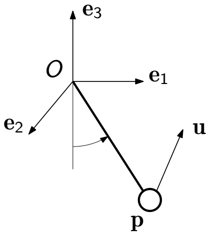

Differential Index - 3D pendulum

Example: 3D pendulum

$$m\ddot{\mathbf{p}} = \mathbf{u} - mg\mathbf{e}_3 - z\mathbf{p}$$
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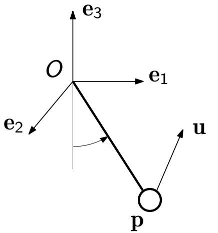

The 3D pendulum is an index-3 DAE !! Note: this is a general observation for mechanical systems modelled via DAEs

Differential Index - 3D pendulum

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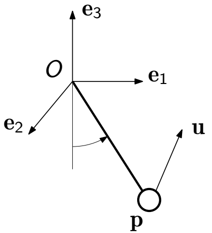

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Differential Index - 3D pendulum

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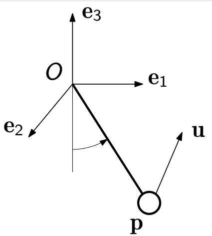

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Differential Index - 3D pendulum

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This is an index-1 (i.e. "easy") DAE (implicit) !!

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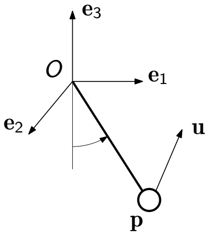

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Differential Index - 3D pendulum

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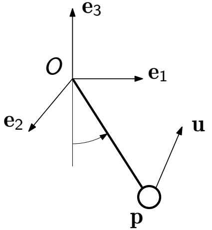

We have converted the index-3 DAE into an index-1 DAE !!

Differential Index - 3D pendulum

Example: 3D pendulum

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Perform **two** time differentiations on $\mathbf{G}$ yields:

$$\ddot{\mathbf{G}} = \mathbf{p}^\top\ddot{\mathbf{p}} + \dot{\mathbf{p}}^\top\dot{\mathbf{p}} = 0$$

Transforming a high-index DAE into an equivalent lower-index one is labelled **index reduction**. This is the most popular approach to tackle high-order DAEs numerically.

For a **semi-explicit DAE** the differential index is the minimum i such that:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{z}, \mathbf{u})$$
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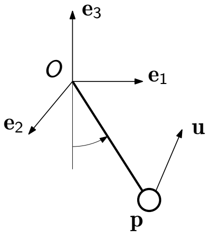

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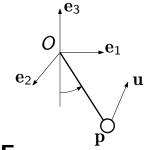
Outline

DAE Consistency - 3D pendulum



Does the index reduction really yield equivalent models ?

Index-3 DAE

$$m\ddot{\mathbf{p}} = \mathbf{u} - mg\mathbf{e}_3 - z\mathbf{p}$$

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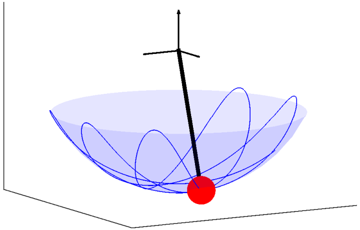
Index-1 DAE

$$\begin{bmatrix} mI & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix}$$



DAE Consistency - 3D pendulum

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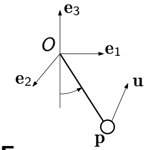

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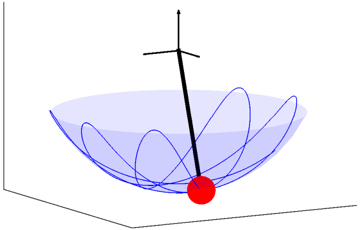
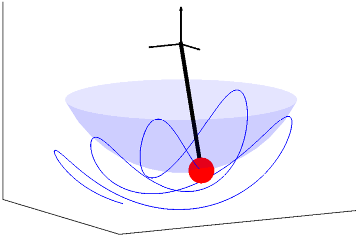
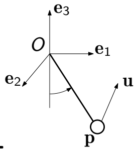




DAE Consistency - 3D pendulum



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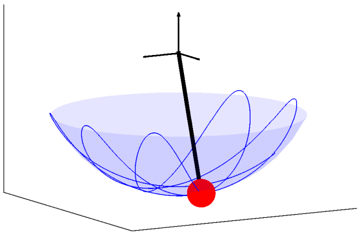

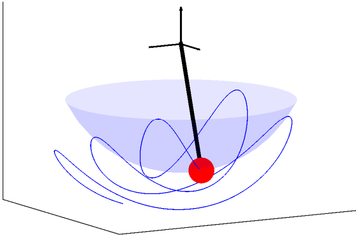

What is going on ??

DAE Consistency - 3D pendulum

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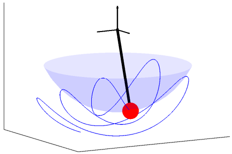

DAE Consistency - 3D pendulum

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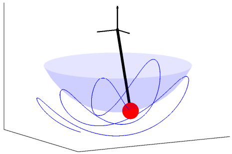

Index reduction

$$\mathbf{G} = \frac{1}{2}\left(\mathbf{p}^\top \mathbf{p} - L^2\right)$$
$$\dot{\mathbf{G}} = \mathbf{p}^\top \dot{\mathbf{p}}$$
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DAE Consistency - 3D pendulum

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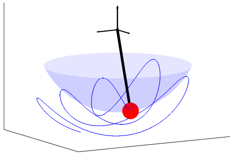

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DAE Consistency - 3D pendulum

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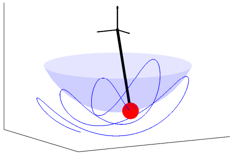

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DAE Consistency - 3D pendulum

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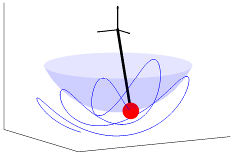

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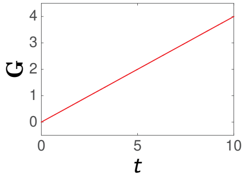

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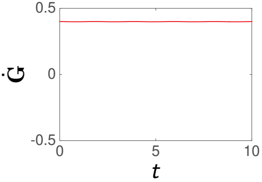

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DAE Consistency - 3D pendulum

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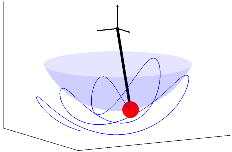

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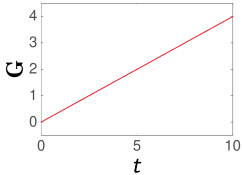

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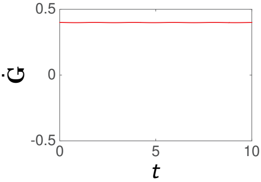

How can we address that ??

DAE Consistency - 3D pendulum

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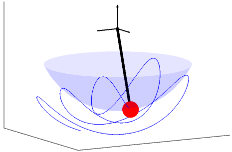

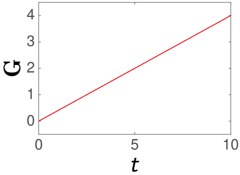

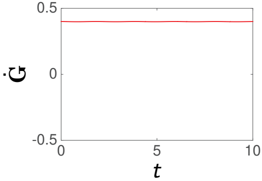

DAE Consistency - 3D pendulum

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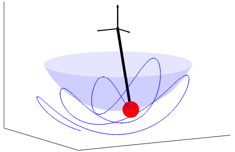
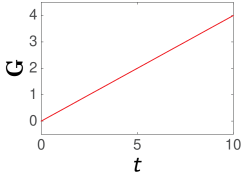
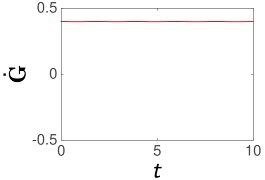
Then if $\mathbf{G} = 0$ and $\dot{\mathbf{G}} = 0$ are satisfied at **any** time on the trajectory, then they are satisfied at **all** time.







DAE Consistency - 3D pendulum

Index-1 DAE

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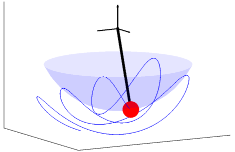

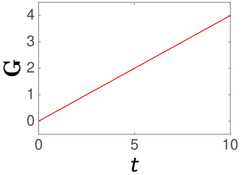

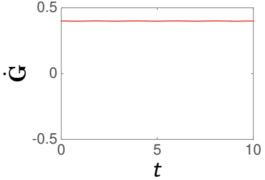

An index-reduced DAE must come with **consistency conditions**. E.g. for the 3D pendulum, the index-1 DAE should be given as:

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DAE Consistency - 3D pendulum

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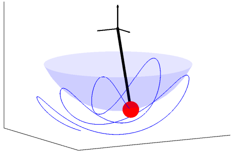

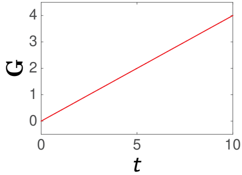

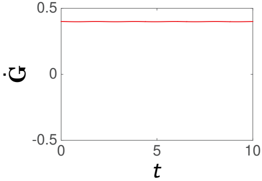

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with the consistency conditions:

$$\mathbf{G} = \frac{1}{2}\left(\mathbf{p}^\top \mathbf{p} - L^2\right) = 0, \qquad \dot{\mathbf{G}} = \mathbf{p}^\top \dot{\mathbf{p}} = 0$$

... to be satisfied e.g. at t_0.

DAE Consistency - 3D pendulum

Index-1 DAE

$$\left[\begin{array}{cc} ml & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{array}\right]\left[\begin{array}{c} \ddot{\mathbf{p}} \\ z \end{array}\right] = \left[\begin{array}{c} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top\dot{\mathbf{p}} \end{array}\right]$$

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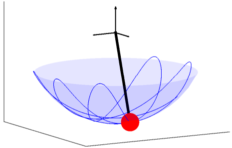

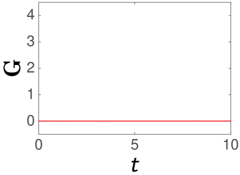

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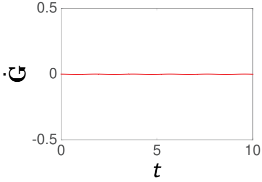

Outline

1 Introduction

2 Differential Index

3 Index Reduction & Consistency

4 Multiple Shooting & Direct Collocation with DAEs

Formulating an OCP

Formulating an OCP

Semi-explicit DAE-constrained OCP

$$
\begin{aligned}
\min_{\mathbf{x}(.),\mathbf{z}(.),\mathbf{u}(.)} \quad & \phi\left(\mathbf{x}\left(.\right),\mathbf{z}\left(.\right),\mathbf{u}\left(.\right)\right) \\
\text{s.t.} \quad & \dot{\mathbf{x}}\left(t\right) = \mathbf{F}\left(\mathbf{z}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) \\
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Formulating an OCP

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Fully implicit DAE-constrained OCP

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- OCPs based on **index-1 DAEs** are the most common

Formulating an OCP

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- OCPs based on **index-1 DAEs** are the most common
- The selected initial condition $\bar{\mathbf{x}}_0$ has to be **consistent**

Formulating an OCP

Semi-explicit DAE-constrained OCP

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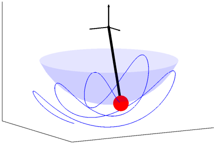
- OCPs based on **index-1 DAEs** are the most common
- The selected initial condition $\bar{\mathbf{x}}_0$ has to be **consistent**
- If "exotic" boundary conditions are needed (e.g. point-to-point or periodic motion), then they ought to be imposed very carefully† !!



† *Numerical Periodic Optimal Control in the Presence of Invariants*, Trans. on Automatic Control, S. Gros, M. Zanon

Multiple-Shooting for DAE-constrained OCPs

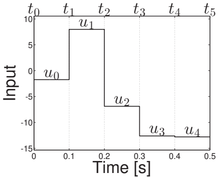
Integrator for index-1 DAE:

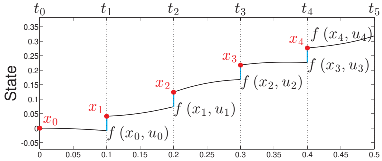
$$\mathbf{F}\left(\dot{\mathbf{x}}(t), \mathbf{z}(t), \mathbf{x}(t), \mathbf{u}(t)\right) = 0$$

Provides the function:

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$$

delivering the integration of the DAE over a time interval $[t_k,\ t_{k+1}]$.





Integrations on the time intervals $[t_k,\ t_{k+1}]$

Multiple-Shooting for DAE-constrained OCPs

Integrator for index-1 DAE:

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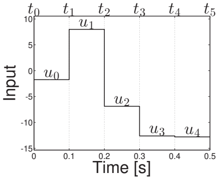
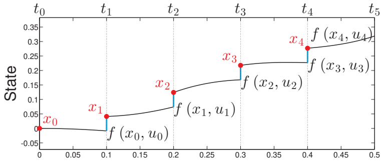
E.g. semi-explicit DAE:

$$\begin{aligned} \dot{\mathbf{x}}\left(t\right) &= \mathbf{F}\left(\mathbf{z}\left(t\right), \mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \\ 0 &= \mathbf{G}\left(\mathbf{z}\left(t\right), \mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \end{aligned}$$





Integrations on the time intervals $[t_k,\ t_{k+1}]$

Multiple-Shooting for DAE-constrained OCPs

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with one-step implicit Euler:

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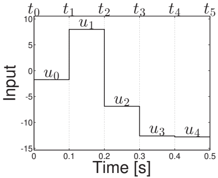

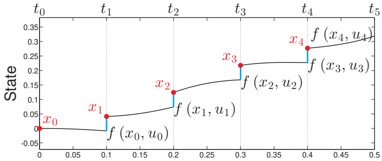

Integrations on the time intervals $[t_k,\ t_{k+1}]$

Multiple-Shooting for DAE-constrained OCPs

Integrator for index-1 DAE:

$$\mathbf{F}\left(\dot{\mathbf{x}}\left(t\right), \mathbf{z}\left(t\right), \mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) = 0$$

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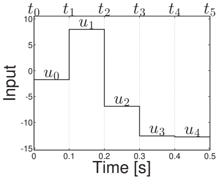
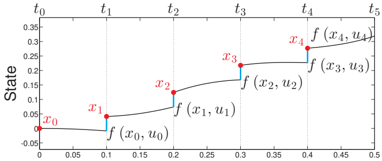
- Return $\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) \equiv \mathbf{x}_+$





Integrations on the time intervals $[t_k,\ t_{k+1}]$

Multiple-Shooting for DAE-constrained OCPs

Integrator for index-1 DAE:

$$\mathbf{F}\left(\dot{\mathbf{x}}\left(t\right),\mathbf{z}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right)=0$$

Provides the function:

$$\mathbf{f}\left(\mathbf{x}_k,\mathbf{u}_k\right)$$

delivering the integration of the DAE over a time interval $[t_k,\ t_{k+1}]$.

> Note that the integrator "eliminates" the algebraic variables $\mathbf{z}\left(.\right)$ by treating them "internally". We have "hidden" the complexity !

E.g. semi-explicit DAE:

$$\begin{aligned}\dot{\mathbf{x}}\left(t\right)&=\mathbf{F}\left(\mathbf{z}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right)\\0&=\mathbf{G}\left(\mathbf{z}\left(t\right),\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right)\end{aligned}$$

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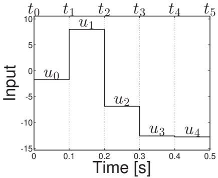
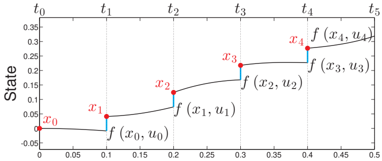
- Return $\mathbf{f}\left(\mathbf{x}_k,\mathbf{u}_k\right)\equiv\mathbf{x}_+$





Integrations on the time intervals $[t_k,\ t_{k+1}]$

Multiple-Shooting for DAE-constrained OCPs

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Is that all ??

E.g. semi-explicit DAE:

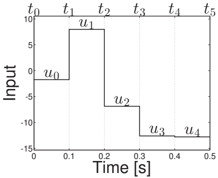
$$\begin{aligned} \dot{\mathbf{x}}\left(t\right) &= \mathbf{F}\left(\mathbf{z}\left(t\right), \mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \\ 0 &= \mathbf{G}\left(\mathbf{z}\left(t\right), \mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \end{aligned}$$

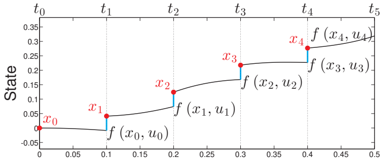
with one-step implicit Euler:

- Solve for $\mathbf{x}_{k+1}, \mathbf{z}_{k+1}$:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{x}_k + h\mathbf{F}\left(\mathbf{x}_{k+1}, \mathbf{z}_{k+1}, \mathbf{u}_k\right) \\ 0 &= \mathbf{G}\left(\mathbf{x}_{k+1}, \mathbf{z}_{k+1}, \mathbf{u}_k\right) \end{aligned}$$

- Return $\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) \equiv \mathbf{x}_+$

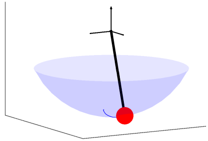


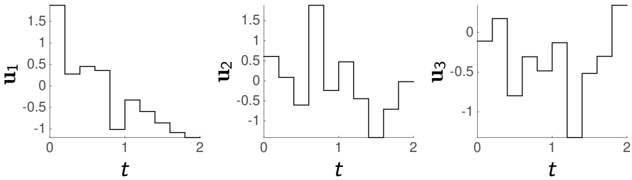


Integrations on the time intervals $[t_k,\ t_{k+1}]$

Algebraic variables & discrete inputs

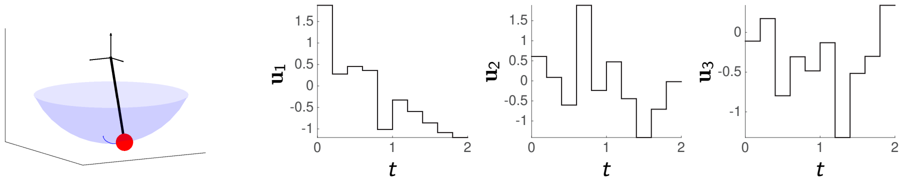
3D pendulum with discretized inputs: (force on the mass)





Algebraic variables & discrete inputs

3D pendulum with discretized inputs: (force on the mass)

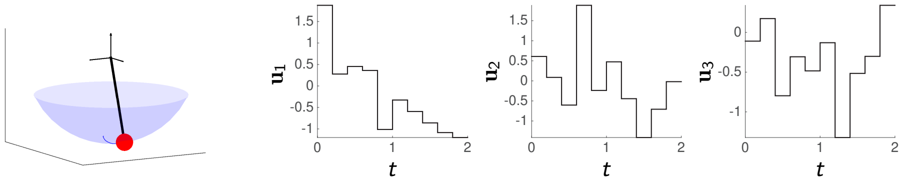


Index-1 DAE:

$$\begin{bmatrix} ml & \mathbf{p} \\ \mathbf{p}^\top & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{u} - mg\mathbf{e}_3 \\ -\dot{\mathbf{p}}^\top \dot{\mathbf{p}} \end{bmatrix}$$

Algebraic variables & discrete inputs

3D pendulum with discretized inputs: (force on the mass)



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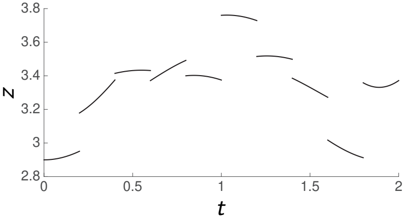
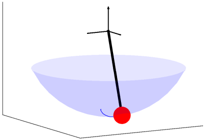
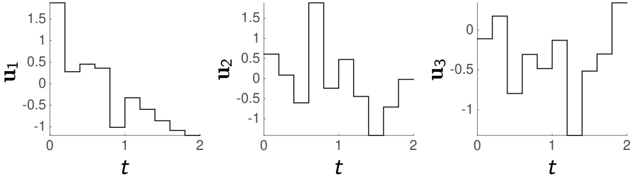


Algebraic variables & discrete inputs

3D pendulum with discretized inputs: (force on the mass)





Index-1 DAE:

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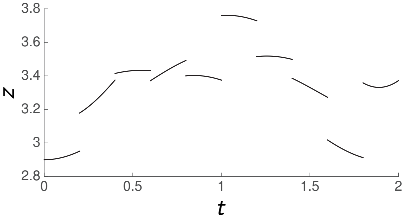
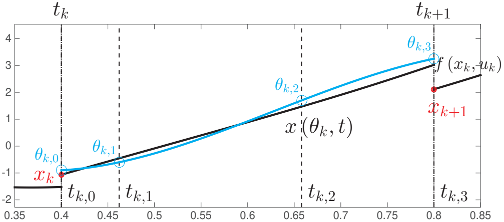
When using a discontinuous input parametrization, the algebraic variables **can** also be discontinuous !!



Direct Collocation for DAE-constrained problems

On each interval $[t_k, t_{k+1}]$ with:

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}_k\right) = 0$$



Direct Collocation for DAE-constrained problems

On each interval $[t_k, t_{k+1}]$ with:

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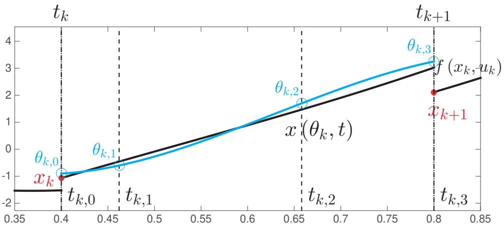
Integration is approximated using:

$$\mathbf{x}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \overbrace{\boldsymbol{\theta}_{k,i}}^{\text{parameters}} \cdot \overbrace{P_{k,i}(t)}^{\text{polynomials}}$$

$$\mathbf{z}\left(\mathbf{z}_k, t\right) = \sum_{i=1}^{K} \underbrace{\mathbf{z}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$



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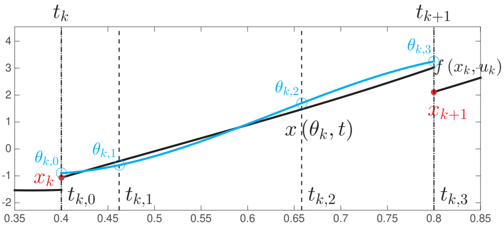

Note:

- $\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \boldsymbol{\theta}_{k,i}$
- $\mathbf{z}\left(\mathbf{z}_k, t_{k,i}\right) = \mathbf{z}_{k,i}$
- $K + 1$ d.o.f. per differential state
- K d.o.f. per algebraic state

Direct Collocation for DAE-constrained problems

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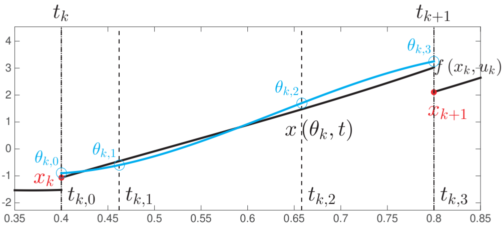

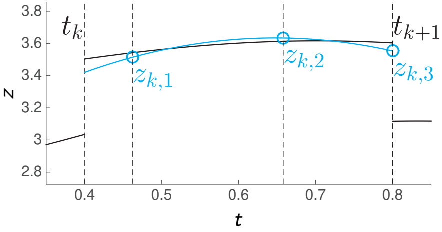

Direct Collocation for DAE-constrained problems

On each interval $[t_k, t_{k+1}]$ with:

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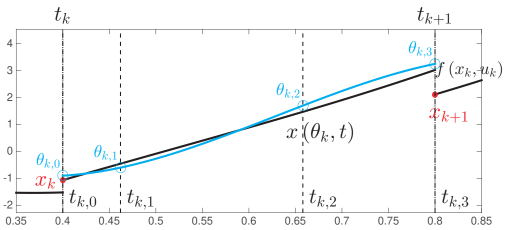
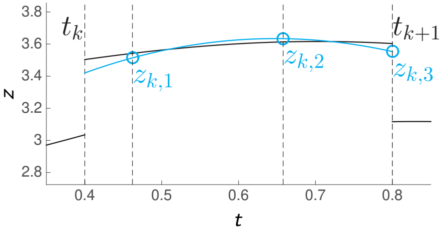


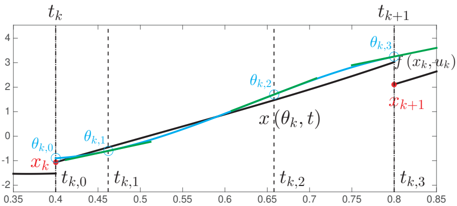


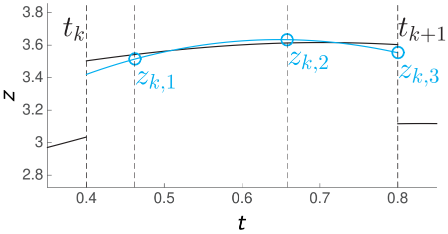
Why different d.o.f ? The differential states need an extra degree of freedom (hence $K+1$) for continuity (i.e. to close the shooting gaps). Algebraic states can be discontinuous and therefore need only K degrees of freedom !

Direct Collocation for DAE-constrained problems

Fully implicit DAE:

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}_k\right) = 0$$





Direct Collocation for DAE-constrained problems

Fully implicit DAE:

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}_k\right) = 0$$

Interpolation:

$$\mathbf{x}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \boldsymbol{\theta}_{k,i} P_{k,i}(t)$$

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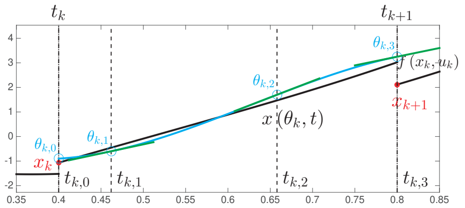

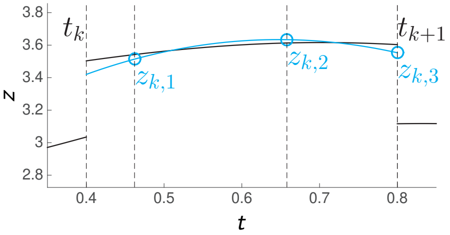

Direct Collocation for DAE-constrained problems

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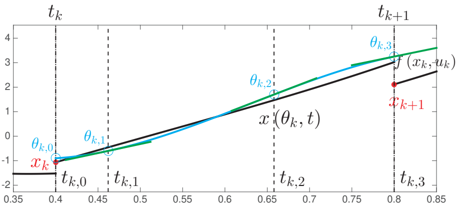
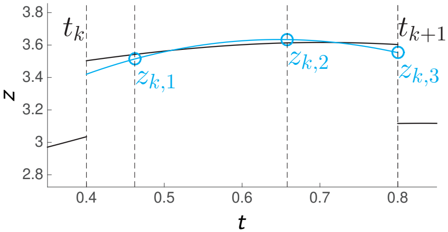
Collocation uses the constraints:

$$\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right) - \mathbf{x}\left(\boldsymbol{\theta}_{k+1}, t_k\right) = 0 \quad \textbf{continuity}$$

$$\mathbf{F}\left(\frac{\partial}{\partial t}\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right), \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right), \mathbf{z}_{k,i}, \mathbf{u}_k\right) = 0 \quad \textbf{dynamics}$$

with $k = 0, ..., N-1$, and $i = 1, ..., K$.





Direct Collocation for DAE-constrained problems

Fully implicit DAE:

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Interpolation:

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Collocation uses the constraints:

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$$\mathbf{F}\left(\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}), \boldsymbol{\theta}_{k,i}, \mathbf{z}_{k,i}, \mathbf{u}_k\right) = 0 \quad \textbf{dynamics}$$

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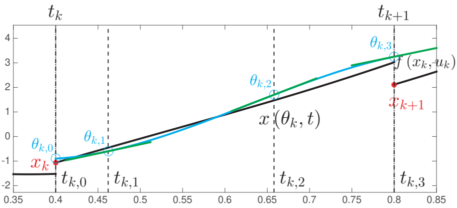

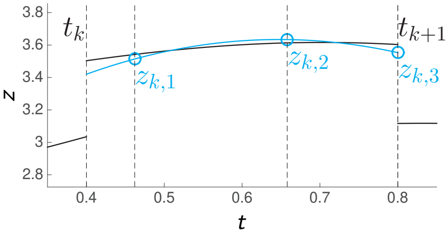

Direct Collocation for DAE-constrained problems

Fully implicit DAE:

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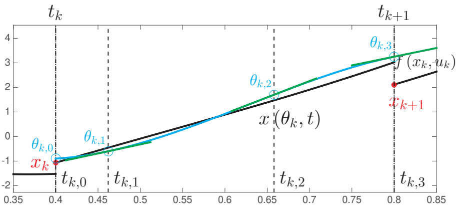
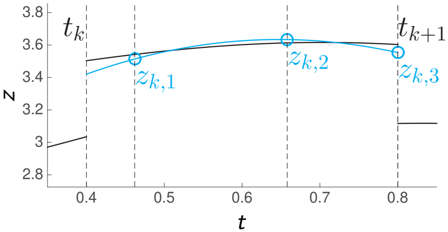
with $k = 0, ..., N-1$, and $i = 1, ..., K$.

Note: algebraic states appear only in the **dynamics** ($i = 1, ..., K$ hence K equations !!), hence only K are needed.

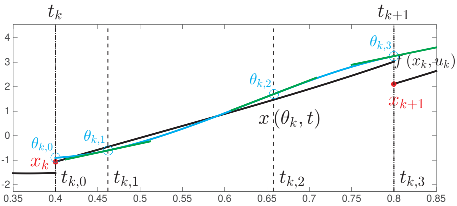


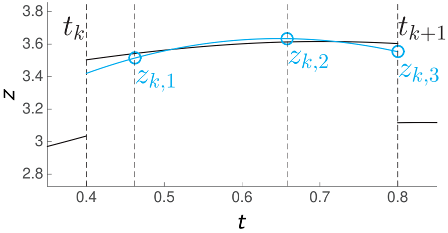


Direct Collocation for DAE-constrained problems

Semi-explicit DAE

$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}_k\right)$$
$$0 = \mathbf{G}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}_k\right)$$





Direct Collocation for DAE-constrained problems

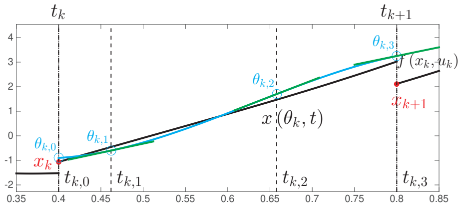
Semi-explicit DAE

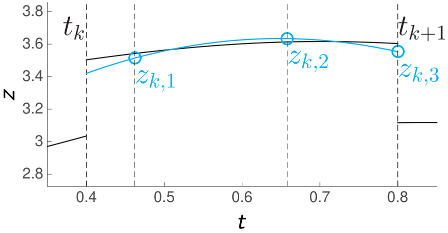
$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}_k\right)$$
$$0 = \mathbf{G}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}_k\right)$$

Collocation uses the constraints:

$$0 = \mathbf{x}\left(\boldsymbol{\theta}_k, t_{k+1}\right) - \mathbf{x}\left(\boldsymbol{\theta}_{k+1}, t_k\right) \quad \textbf{continuity}$$
$$\frac{\partial}{\partial t}\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \mathbf{F}\left(\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right), \mathbf{z}_{k,i}, \mathbf{u}_k\right) \quad \textbf{dynamics}$$
$$0 = \mathbf{G}\left(\mathbf{x}\left(\boldsymbol{\theta}_k, t_{k,i}\right), \mathbf{z}_{k,i}, \mathbf{u}_k\right) \quad \textbf{algebraic}$$

with $k = 0, ..., N-1$, and $i = 1, ..., K$.





Direct Collocation for DAE-constrained problems

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$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{z}, \mathbf{u}_k\right)$$

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Collocation uses the constraints:

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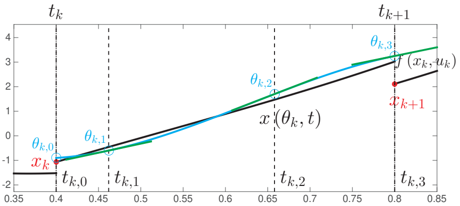
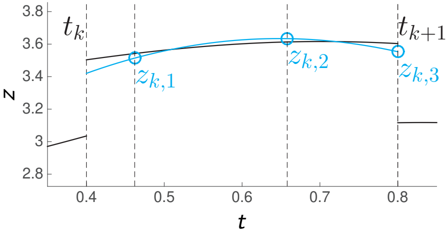




Selection of the collocation points for DAEs

What collocation scheme to use for DAEs ?!?

Selection of the collocation points for DAEs

What collocation scheme to use for DAEs ?!?

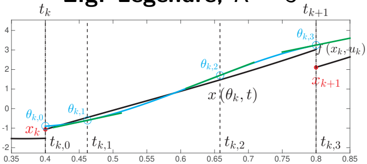
K	Legendre	Radau
1	0.5	1.0
2	0.211325 0.788675	0.333333 1.000000
3	0.112702 0.500000 0.887298	0.155051 0.644949 1.000000
4	0.069432 0.330009 0.669991 0.930568	0.088588 0.409467 0.787659 1.000000
5	0.046910 0.230765 0.500000 0.769235 0.953090	0.057104 0.276843 0.583590 0.860240 1.000000

c.f. Lecture "Direct Collocation"

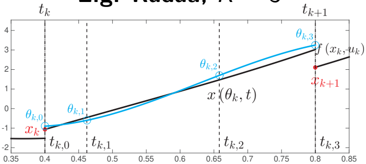
Selection of the collocation points for DAEs

What collocation scheme to use for DAEs ?!?

E.g. Legendre, $K = 3$



E.g. Radau, $K = 3$



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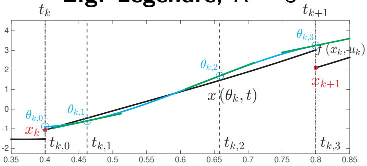
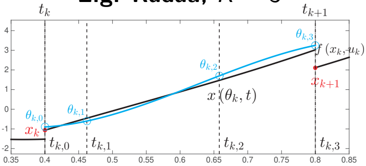


- has a collocation point at t_k all others inside $[t_k,\ t_{k+1}]$

E.g. Radau, $K = 3$

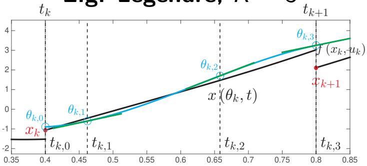


- has collocation points at t_k and t_{k+1}

Selection of the collocation points for DAEs

What collocation scheme to use for DAEs ?!?

E.g. Legendre, $K = 3$



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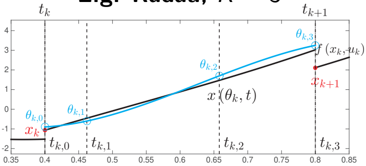

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Selection of the collocation points for DAEs

What collocation scheme to use for DAEs ?!?

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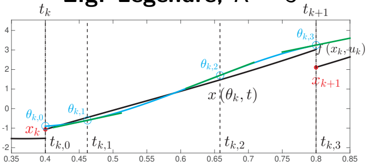

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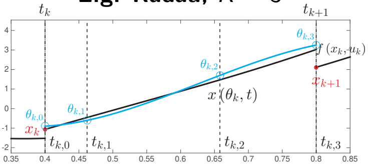

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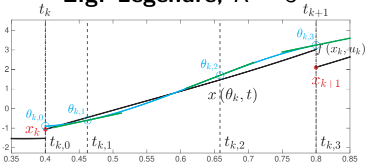

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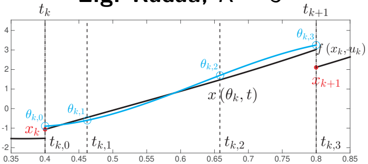

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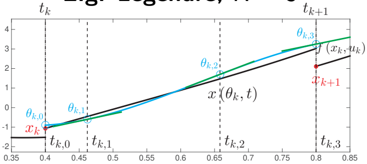

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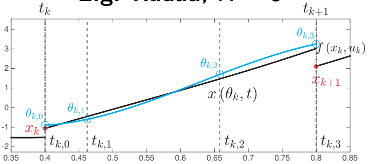

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Careful: using a very high-order collocation setup *can* deteriorate the conditioning of your KKT matrices and hinder the linear algebra underlying the NLP solver !!

NLP from Direct Collocation for DAE-constrained OCPs

Fully implicit DAE:

$$\mathbf{F}\left(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{z}, \mathbf{u}_k\right) = 0$$

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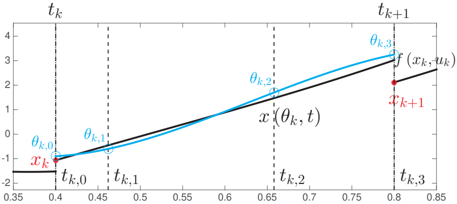

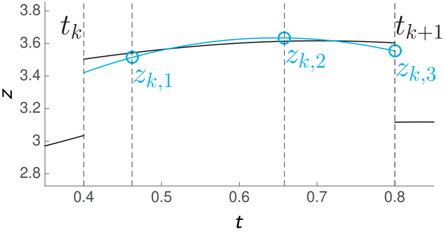

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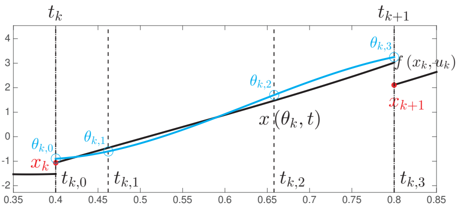
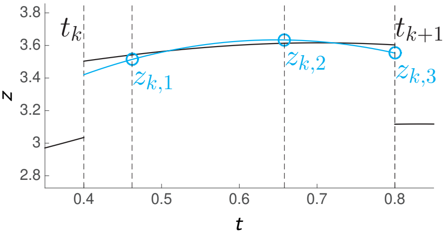
NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \quad \end{bmatrix}$$





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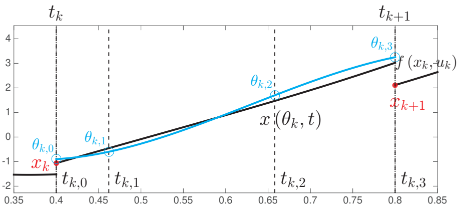
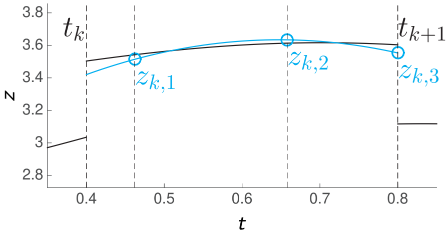
Initial conditions $\bar{\mathbf{x}}_0$





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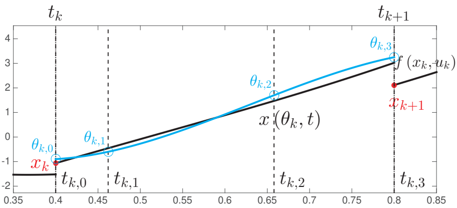
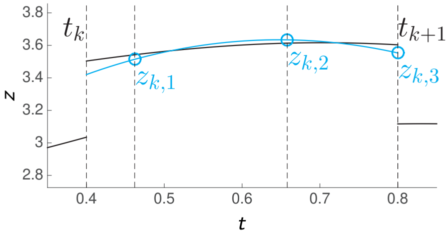
Continuity constraints ($\equiv$ shooting gaps)





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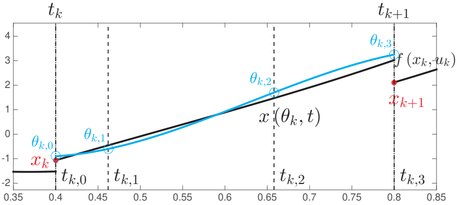
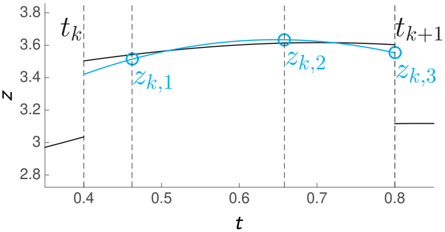
Integration constraints for $k = 0$





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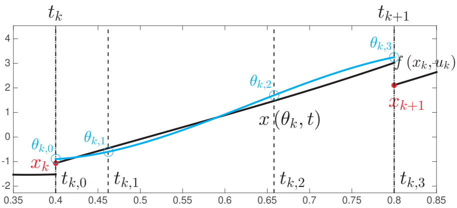
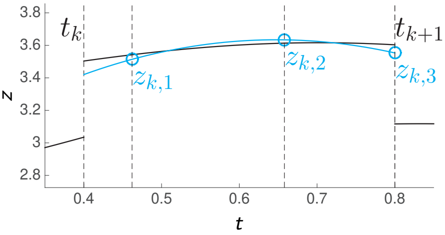
Remaining integration constraints $k = 1, \dots, N-1$





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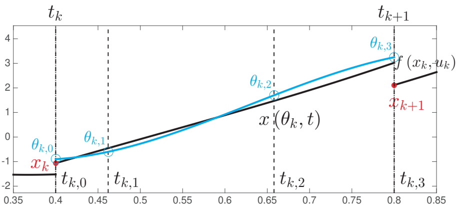
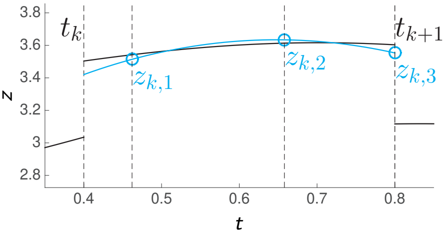
Decision variables ($k = 0, ..., N-1$)

$$\mathbf{w} = \left\{..., \boldsymbol{\theta}_{k,0}, \boldsymbol{\theta}_{k,1}, \mathbf{z}_{k,1}, ..., \boldsymbol{\theta}_{k,K}, \mathbf{z}_{k,K}, \mathbf{u}_k, ...\right\}$$





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Note: for $\mathbf{z}$, the interpolation plays no role in the collocation equations !

NLP with direct collocation

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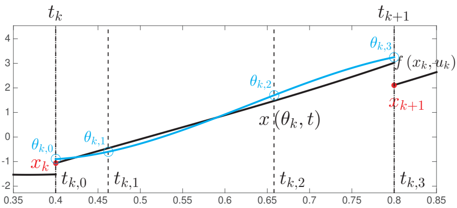
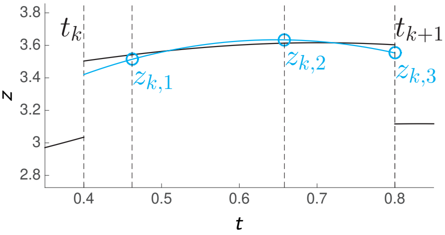
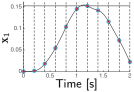


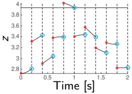


Direct Methods for DAE-based OCPs - Wrap up

Multiple-shooting

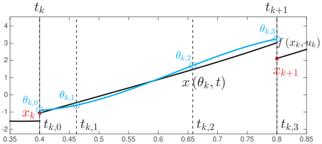
- Hides the algebraic variables $\mathbf{z}$ in the integrator
- If they are needed in the constraints/cost, the integrator needs to report them back to the NLP solver, with sensitivities.

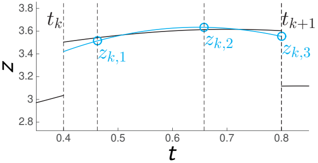




Direct Collocation:

- Collocation equations are *almost* the same as for ODEs
- A discrete instance of the algebraic variables exists at every collocation time <u>but the first one</u> (associated to the continuity conditions)
- Use the Radau collocation times
- Carefule about very high orders in the collocation polynomial !





What about an "Optimal control discussion group" at NTNU?

- Regular meetings (e.g. monthly) where people can share questions / results related to optimal control in their research
- Act as a support group to
  - Unlock difficulties in research
  - Promote "good practices" in optimal control
  - Make sure that NTNU use state-of-the-art approaches in its research
- Act as a platform to communicate ideas