Numerical Optimal Control
Lecture 6: Direct Collocation

Sébastien Gros

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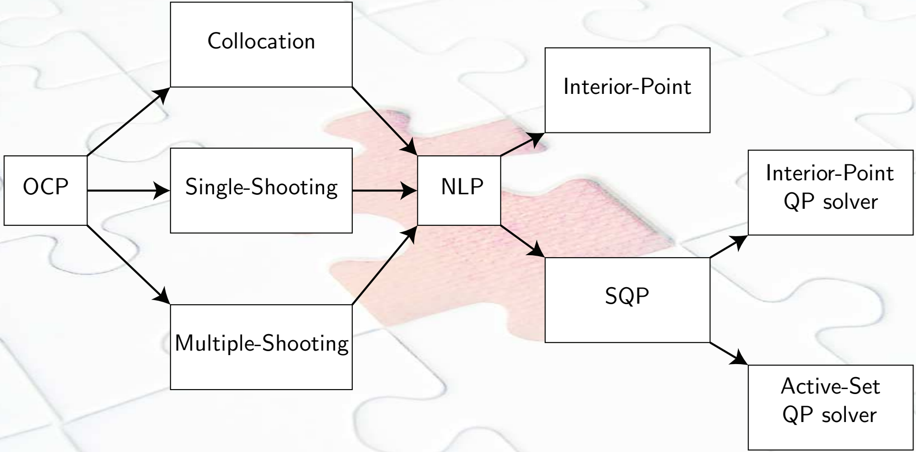
NTNU PhD course

Survival map of Direct Optimal Control



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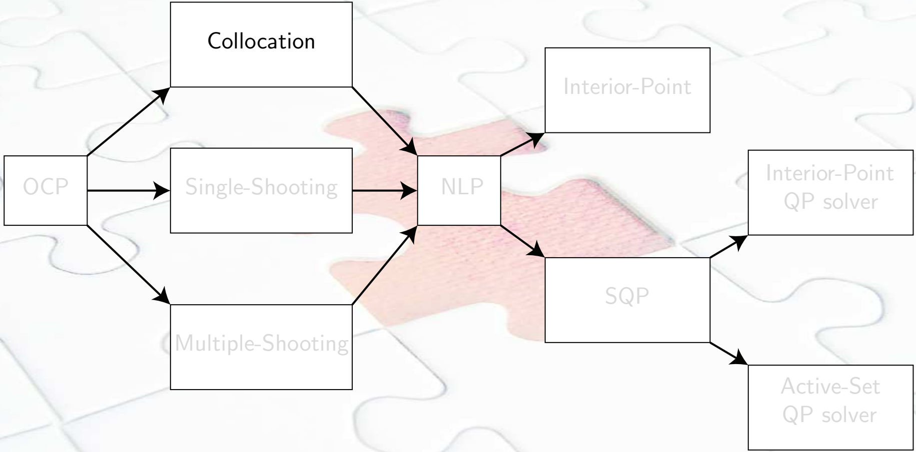

Another way for going from OCP to NLP

Outline

1. Polynomial interpolation
2. Collocation-based integration
3. Collocation in multiple-shooting
4. Direct Collocation
5. NLP from direct collocation

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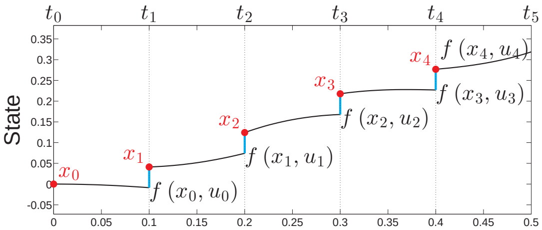
Polynomial interpolation

Consider a time grid:

$$\{t_{k,0}, \ldots, t_{k,K}\} \in [t_k,\ t_{k+1}]$$



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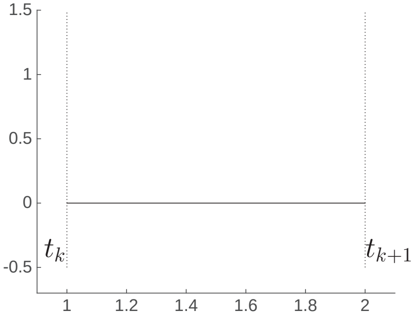

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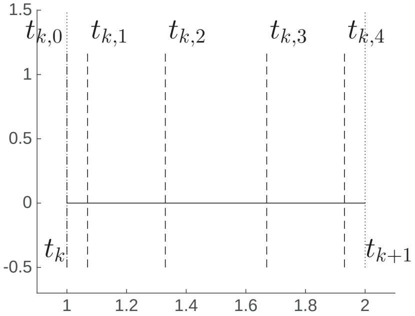

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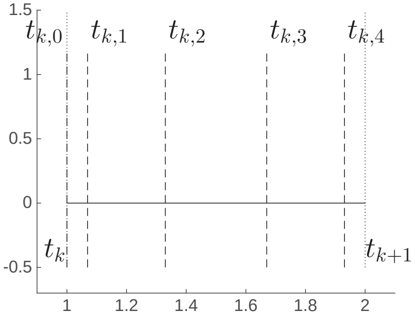

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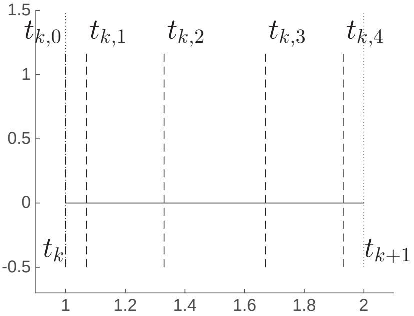

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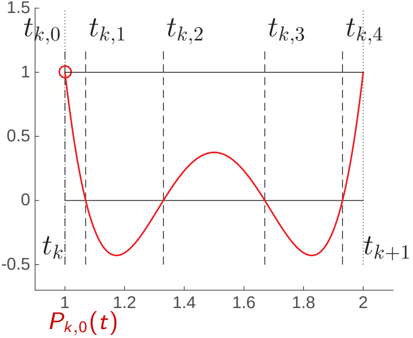

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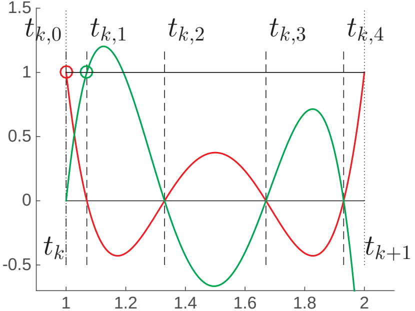

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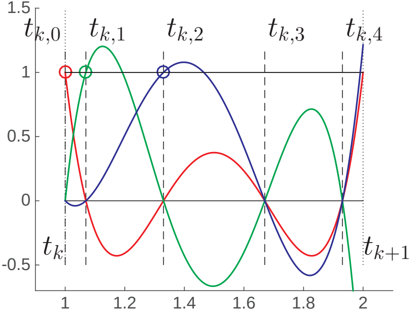

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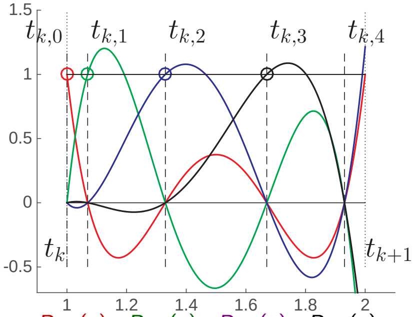

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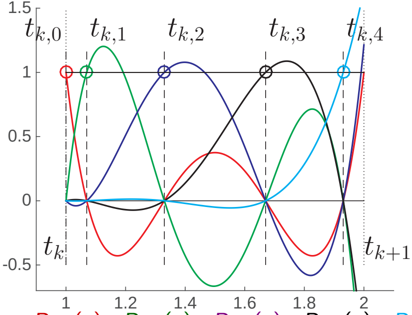

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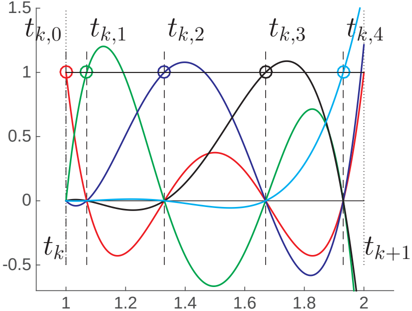

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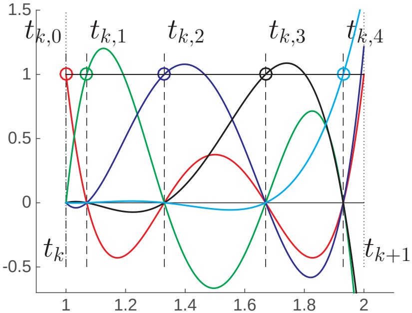

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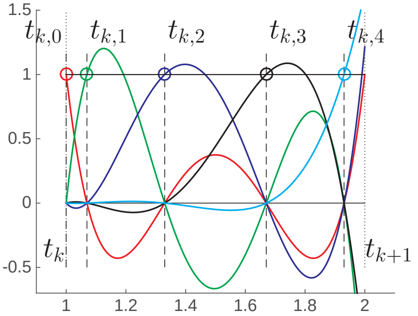

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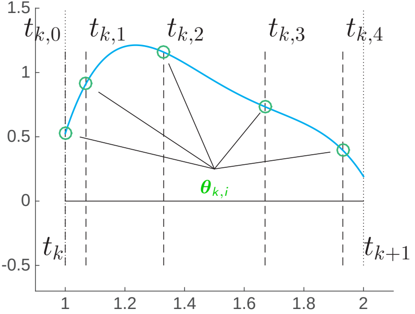

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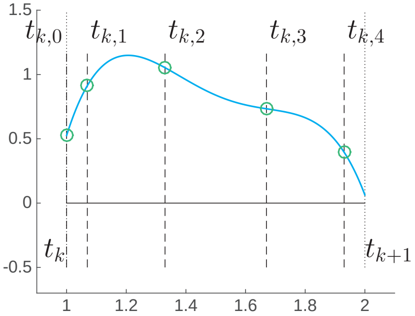

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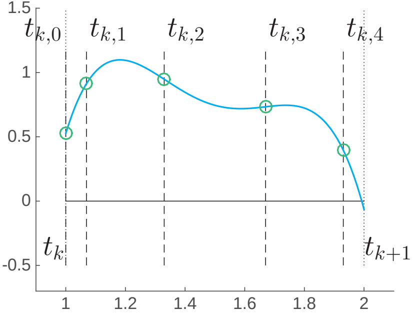

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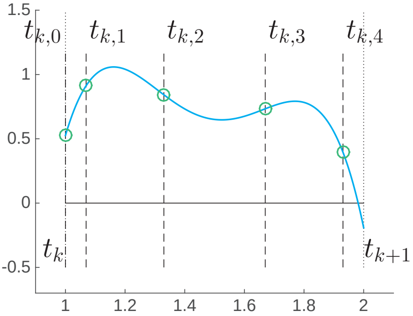

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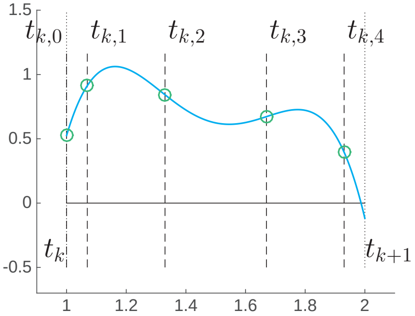

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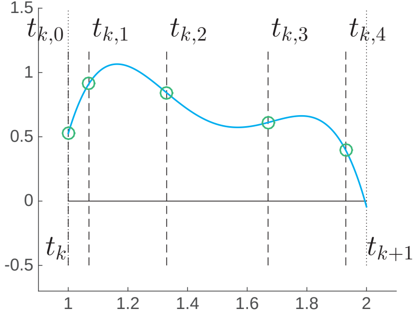

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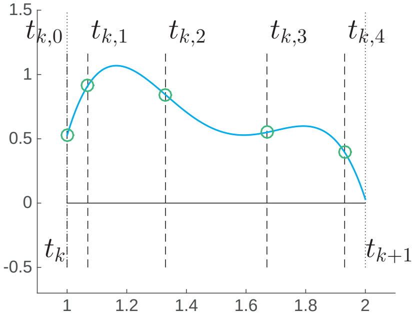

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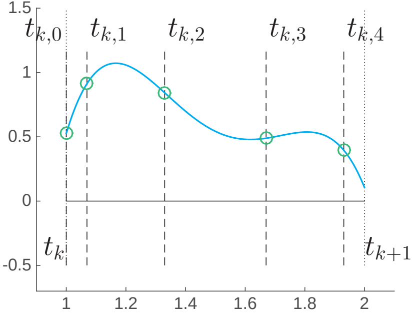

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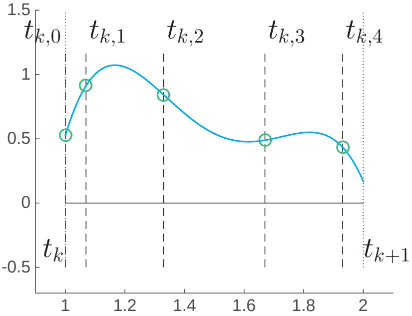

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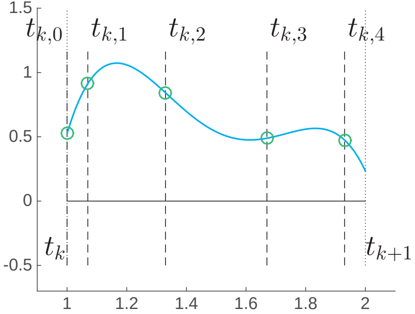

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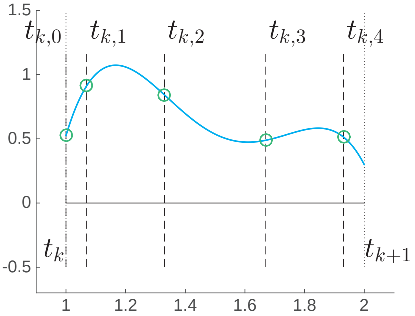

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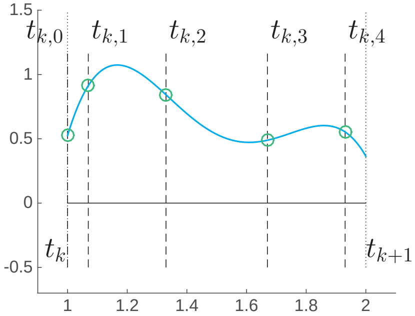

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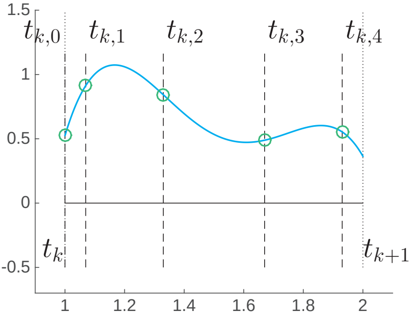


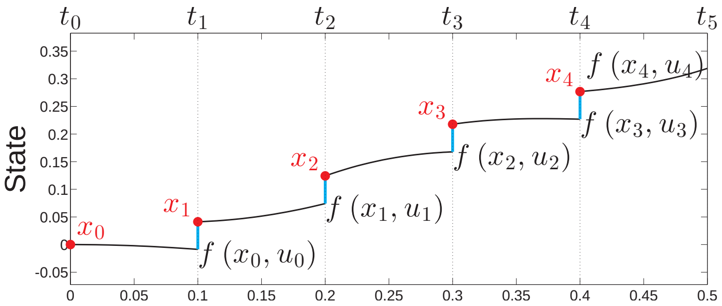
Note: the Lagrange polynomials are orthogonal, i.e.

$$\int_{t_k}^{t_{k+1}} P_{k,i}(t) P_{k,j}(t)\, \mathrm{d}t = 0, \quad \forall i \neq j$$

Outline

Collocation methods - key idea

Approximate state trajectory $\mathbf{s}(t)$ via polynomials (order K)



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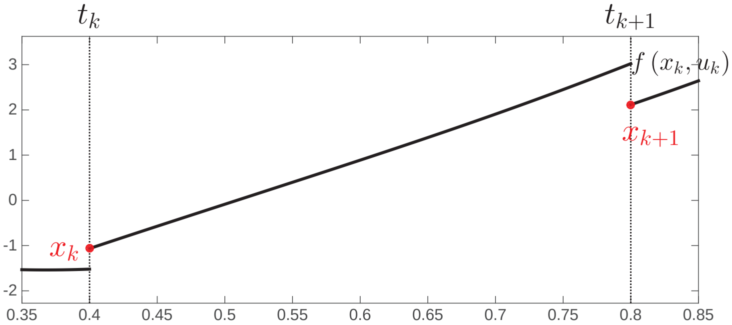

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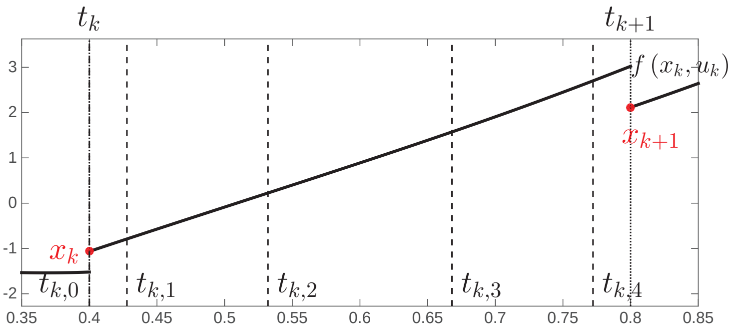

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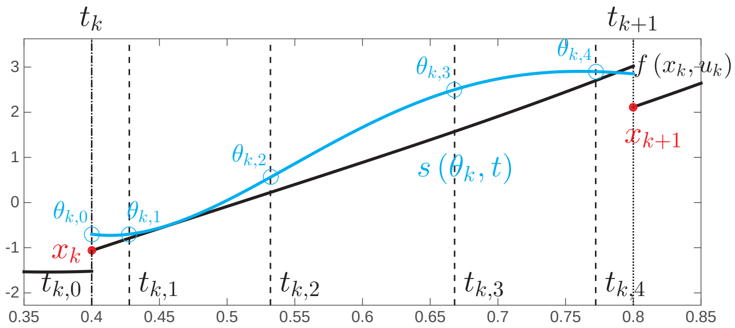

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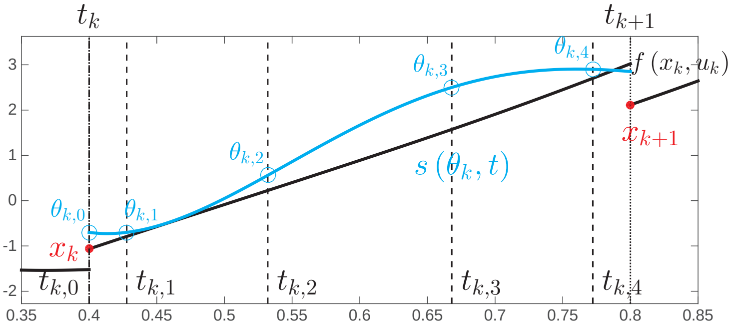

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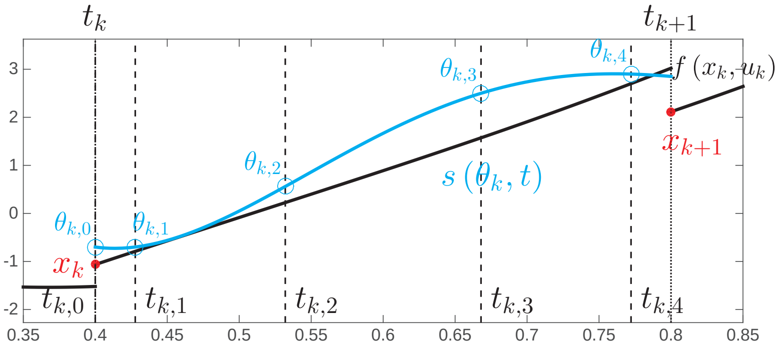

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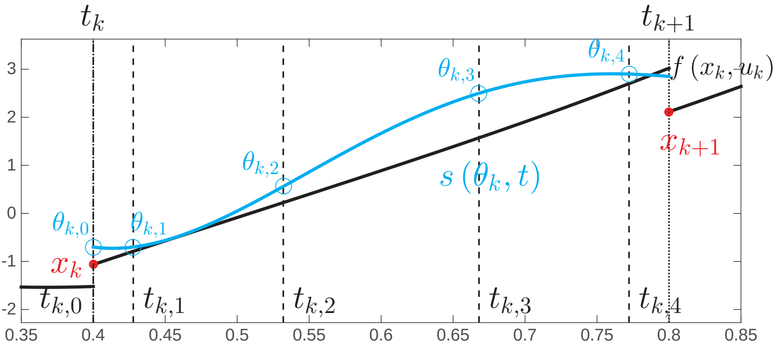

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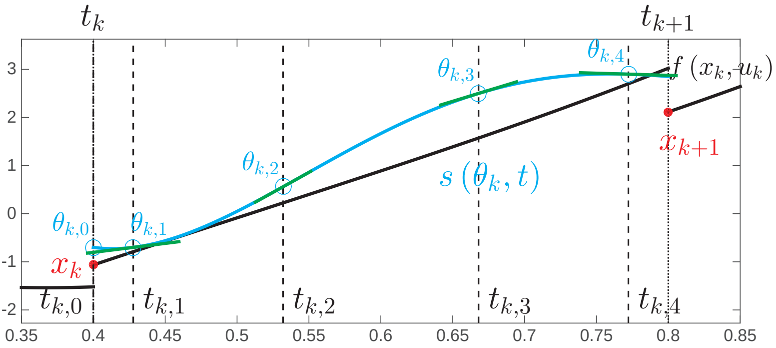

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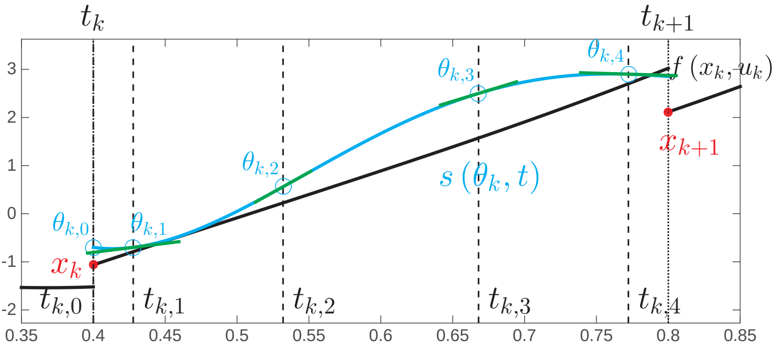

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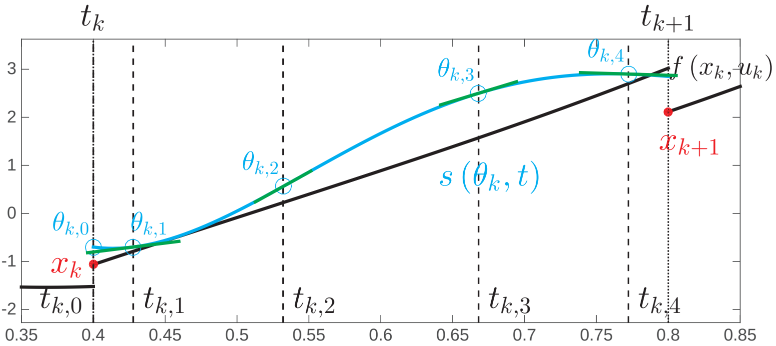

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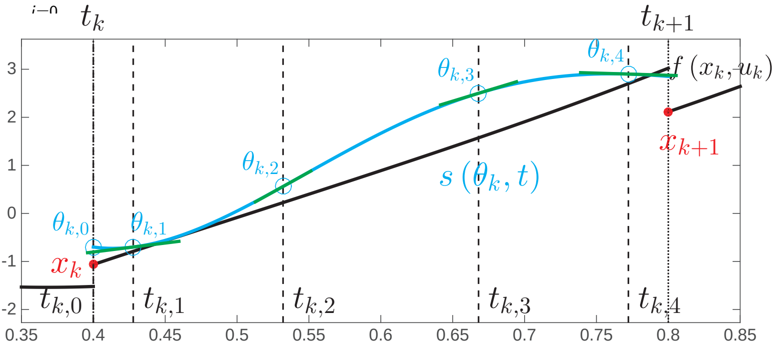

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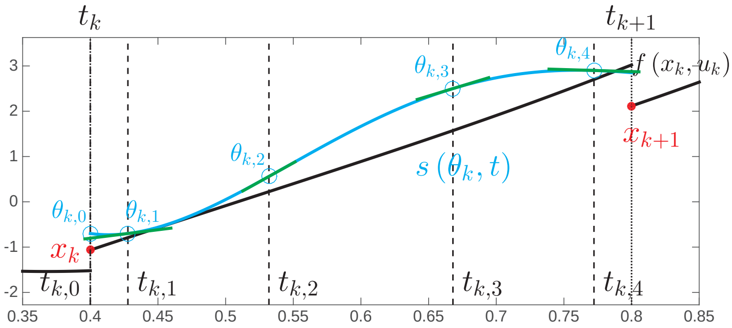

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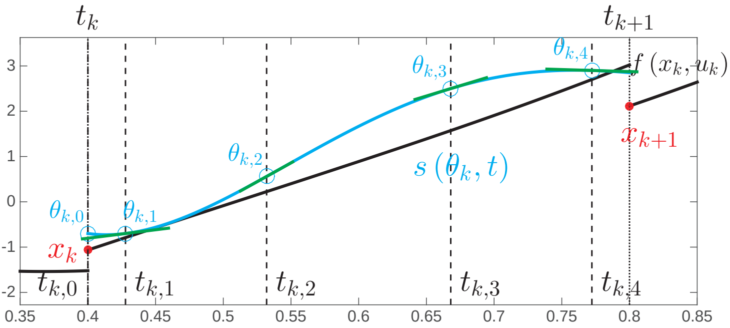

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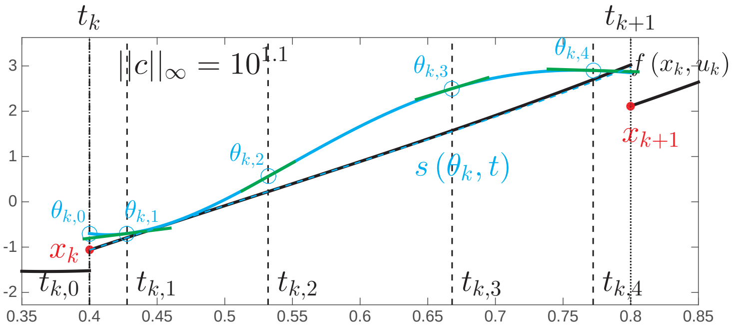

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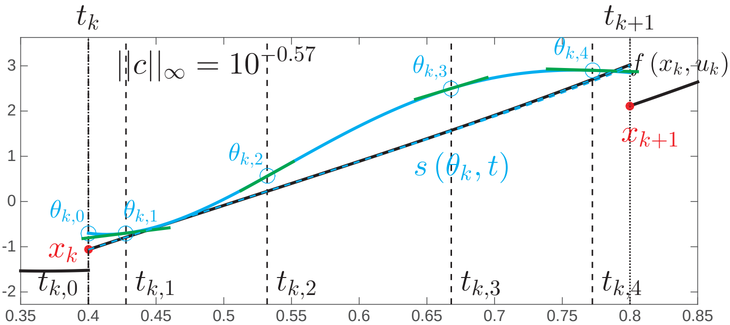

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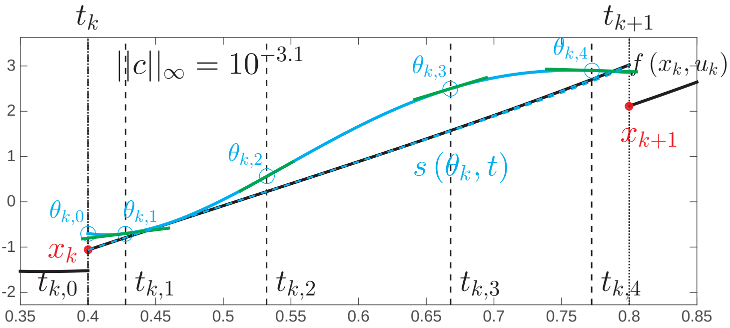

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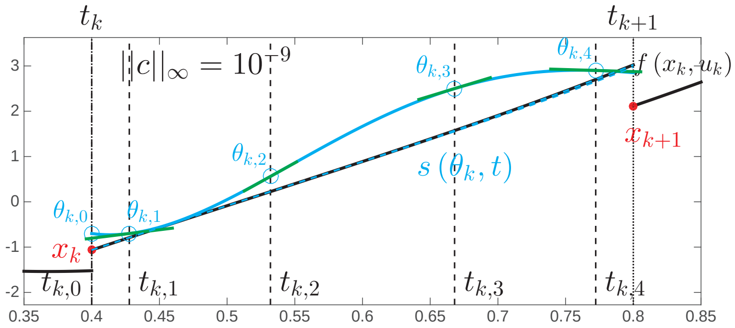

Collocation methods - Implementation

Collocation uses the constraints:

$$\boldsymbol{\theta}_{k,0} = \mathbf{x}_k$$

$$\sum_{i=0}^{K} \boldsymbol{\theta}_{k,i} \cdot \dot{P}_{k,i}(t_{k,j}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,j}, \mathbf{u}_k\right)$$

for $j = 1, ..., K$. End-state:

$$\mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right) = \sum_{i=0}^{K} \boldsymbol{\theta}_{k,i} \cdot P_{k,i}(t_{k+1})$$

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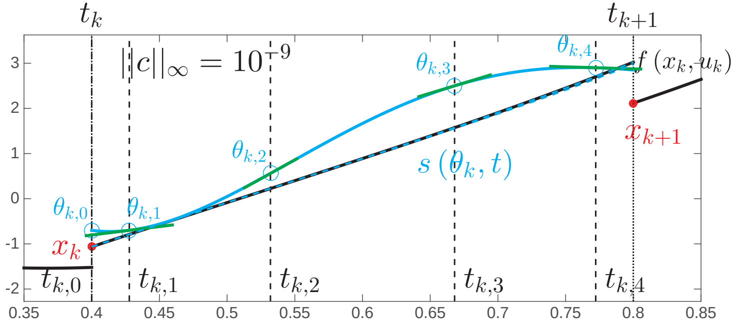

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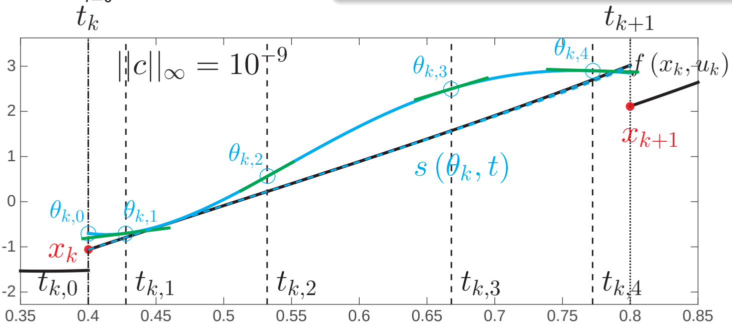
Shooting constraints

End-state reported to the NLP solver:

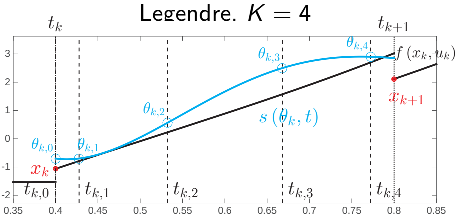
$$\underbrace{\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)}_{=\mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right)} - \mathbf{x}_{k+1} = 0$$

Shooting constraints also reads as:

$$\sum_{i=0}^{K} \boldsymbol{\theta}_{k,i} P_{k,i}(t_{k+1}) - \mathbf{x}_{k+1} = 0$$



Selection of the time grid $t_{k,i}$



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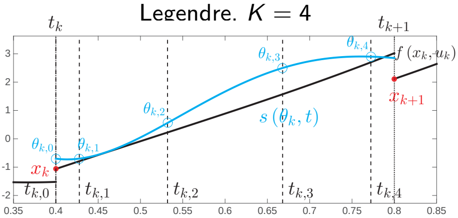

+one point at t_k to enforce continuity !

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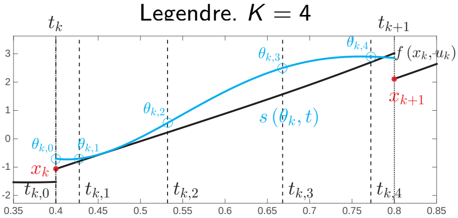

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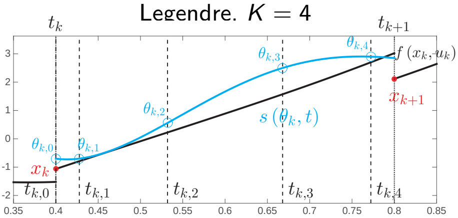

Why these points ?!? They deliver an exact integration for any polynomial $\mathbf{P}$ of order $< 2K$ (Legendre) and $< 2K - 1$ (Radau). I.e. for

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}) = \mathbf{P}(t)$$

the collocation equations deliver an exact solution, namely:

$$\mathbf{s}\left(t_{k+1}, \boldsymbol{\theta}_k\right) = \mathbf{x}_k + \int_{t_k}^{t_{k+1}} \mathbf{P}(\tau)\, \mathrm{d}\tau$$

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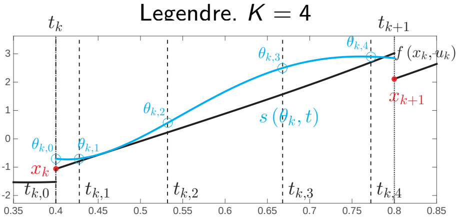

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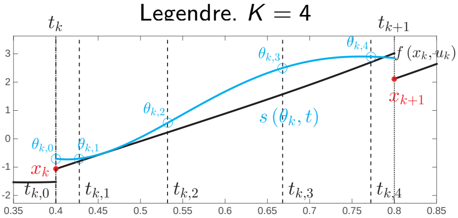

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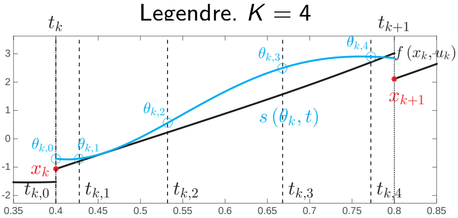

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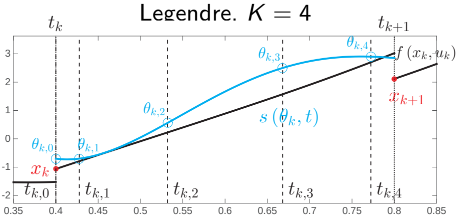

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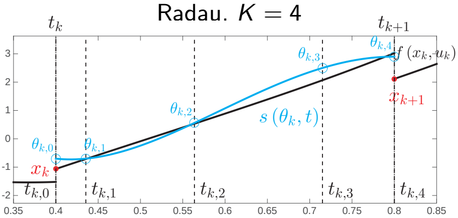

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Order & Stability

- Collocation methods are **A-stable** (i.e. can handle stiff equations). They have no stability limitation on the time intervals $h = t_{k+1} - t_k$ for stiff problems. I.e. even large time steps $h = t_{k+1} - t_k$ allow for capturing steady state and slow dynamics correctly in the presence of very fast dynamics.

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- Collocation-based integration is an **Implicit Runge-Kutta** scheme (e.g. an order 1 collocation scheme is implicit Euler !)

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Collocation constraints...

$$\boldsymbol{\theta}_{k,0} = \mathbf{x}_k$$

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right), \quad i = 1, ..., K$$

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$$\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{x}_k} + \frac{\partial \mathbf{c}}{\partial \mathbf{x}_k} = 0, \qquad \frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{u}_k} + \frac{\partial \mathbf{c}}{\partial \mathbf{u}_k} = 0$$

Hence:

$$\frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{x}_k} = -\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k}^{-1} \frac{\partial \mathbf{c}}{\partial \mathbf{x}_k}, \qquad \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{u}_k} = -\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k}^{-1} \frac{\partial \mathbf{c}}{\partial \mathbf{u}_k}$$

Collocation - Sensitivity

Collocation constraints...

$$\boldsymbol{\theta}_{k,0} = \mathbf{x}_k$$

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right), \quad i = 1, ..., K$$

... can be seen as

$$\mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right) = 0$$

Solved by iterating:

$$\Delta\boldsymbol{\theta}_k = -\frac{\partial \mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right)}{\partial \boldsymbol{\theta}_k}^{-1} \mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right)$$

Integrator function given by:

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right)$$

Get sensitivities using:

$$\frac{\partial \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)}{\partial \mathbf{x}_k} = \frac{\partial \mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right)}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{x}_k}, \qquad \frac{\partial \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)}{\partial \mathbf{u}_k} = \frac{\partial \mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right)}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{u}_k}$$

Implicit function theorem states that

$$\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{x}_k} + \frac{\partial \mathbf{c}}{\partial \mathbf{x}_k} = 0, \qquad \frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k} \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{u}_k} + \frac{\partial \mathbf{c}}{\partial \mathbf{u}_k} = 0$$

Hence:

$$\frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{x}_k} = -\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k}^{-1} \frac{\partial \mathbf{c}}{\partial \mathbf{x}_k}, \qquad \frac{\partial \boldsymbol{\theta}_k}{\partial \mathbf{u}_k} = -\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k}^{-1} \frac{\partial \mathbf{c}}{\partial \mathbf{u}_k}$$

Note that $\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}_k}^{-1}$ is computed in the Newton iteration, i.e. it comes for free !!

Outline

Collocation-based integrators in Multiple-shooting

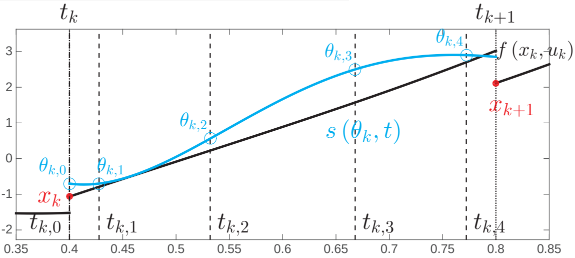
Collocation-based integrator solves:

$$\mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right) = 0$$

on each time interval $[t_k,\ t_{k+1}]$, provides:

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right)$$

with sensitivities.



Collocation-based integrators in Multiple-shooting

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$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right)$$

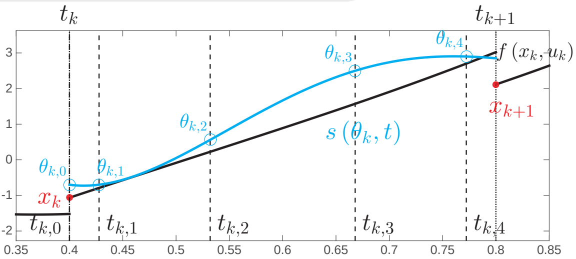
with sensitivities.

NLP with multiple-shooting

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \mathbf{x}_0 - \bar{\mathbf{x}}_0 \\ \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) - \mathbf{x}_N \end{bmatrix}$$

where $\mathbf{w} = \left\{\mathbf{x}_0,\ \mathbf{u}_0, \ldots, \mathbf{x}_{N-1},\ \mathbf{u}_{N-1}, \mathbf{x}_N\right\}$



Collocation-based integrators in Multiple-shooting

Collocation-based integrator solves:

$$\mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right) = 0$$

on each time interval $[t_k,\ t_{k+1}]$, provides:

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right)$$

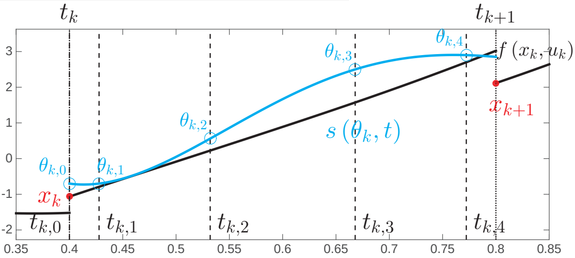
with sensitivities.

NLP with multiple-shooting

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \mathbf{x}_0 - \bar{\mathbf{x}}_0 \\ \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) - \mathbf{x}_N \end{bmatrix}$$

where $\mathbf{w} = \left\{\mathbf{x}_0,\ \mathbf{u}_0, \dots, \mathbf{x}_{N-1},\ \mathbf{u}_{N-1}, \mathbf{x}_N\right\}$



NLP:

$$\nabla_{\mathbf{w}} \mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}\right) = 0$$

$$\mathbf{g}\left(\mathbf{w}\right) = 0$$

Collocation-based integrators in Multiple-shooting

Collocation-based integrator solves:

$$\mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right) = 0$$

on each time interval $[t_k,\ t_{k+1}]$, provides:

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right)$$

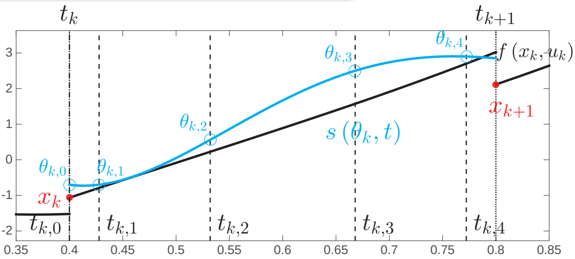
with sensitivities.

NLP with multiple-shooting

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \mathbf{x}_0 - \bar{\mathbf{x}}_0 \\ \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right) - \mathbf{x}_1 \\ \dots \\ \mathbf{f}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) - \mathbf{x}_N \end{bmatrix}$$

where $\mathbf{w} = \left\{\mathbf{x}_0,\ \mathbf{u}_0, \dots, \mathbf{x}_{N-1},\ \mathbf{u}_{N-1}, \mathbf{x}_N\right\}$



NLP:

$$\nabla_{\mathbf{w}}\mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}\right) = 0$$
$$\mathbf{g}\left(\mathbf{w}\right) = 0$$

Each $\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ and $\nabla\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ is provided by the "collocation code"

Collocation-based integrators in Multiple-shooting

Collocation-based integrator solves:

$$\mathbf{c}\left(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k\right) = 0$$

on each time interval $[t_k,\ t_{k+1}]$, provides:

$$\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right) = \mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right)$$

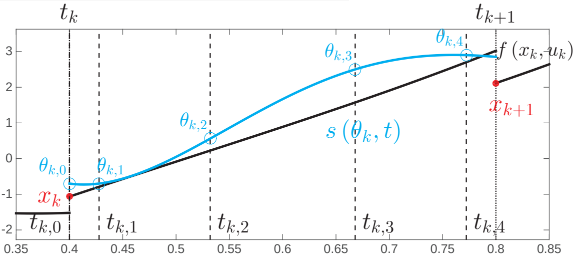
with sensitivities.

NLP with multiple-shooting

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \mathbf{x}_0 - \bar{\mathbf{x}}_0 \\ \mathbf{f}\left(\mathbf{x}_0, \mathbf{u}_0\right) - \mathbf{x}_1 \\ \ldots \\ \mathbf{f}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right) - \mathbf{x}_N \end{bmatrix}$$

where $\mathbf{w} = \left\{\mathbf{x}_0,\ \mathbf{u}_0, \ldots, \mathbf{x}_{N-1},\ \mathbf{u}_{N-1}, \mathbf{x}_N\right\}$



NLP:

$$\nabla_{\mathbf{w}}\mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}\right) = 0$$
$$\mathbf{g}\left(\mathbf{w}\right) = 0$$

Each $\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ and $\nabla\mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k\right)$ is provided by the "collocation code"

Collocation-based integrator inside the NLP becomes a two-level Newton scheme !!

Collocation-based integrators in Multiple-shooting (cont')

NLP solver
$$\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}) = 0$$
$$\mathbf{g}(\mathbf{w}) = 0$$

$$\mathbf{w} = \{\mathbf{x}_0, \mathbf{u}_0, ..., \mathbf{x}_{N-1}, \mathbf{u}_{N-1}, \mathbf{x}_N\}$$

Collocation-based integrators in Multiple-shooting (cont')

NLP solver

$$\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}) = 0$$

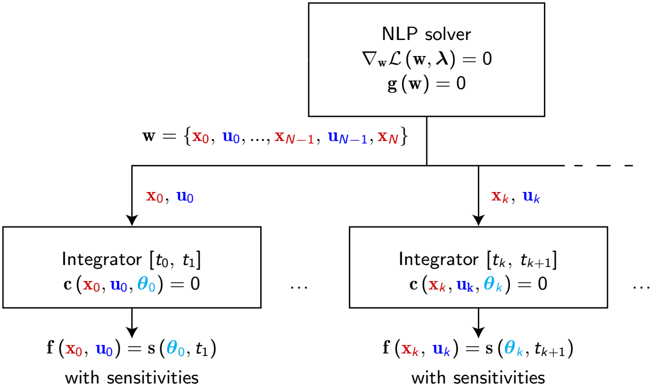
$$\mathbf{g}(\mathbf{w}) = 0$$

$$\mathbf{w} = \{\mathbf{x}_0,\, \mathbf{u}_0, \ldots, \mathbf{x}_{N-1},\, \mathbf{u}_{N-1}, \mathbf{x}_N\}$$

NLP level

- Constraints $\mathbf{g} = 0$
- Newton iterations (SQP/IP)

Collocation-based integrators in Multiple-shooting (cont')



NLP level

- Constraints $\mathbf{g} = 0$
- Newton iterations (SQP/IP)

Collocation-based integrators in Multiple-shooting (cont')

NLP solver
$$\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}) = 0$$
$$\mathbf{g}(\mathbf{w}) = 0$$

$$\mathbf{w} = \{\mathbf{x}_0,\ \mathbf{u}_0, \dots, \mathbf{x}_{N-1},\ \mathbf{u}_{N-1}, \mathbf{x}_N\}$$

$\mathbf{x}_0,\ \mathbf{u}_0$

Integrator $[t_0,\ t_1]$
$$\mathbf{c}(\mathbf{x}_0, \mathbf{u}_0, \boldsymbol{\theta}_0) = 0$$

$$\mathbf{f}(\mathbf{x}_0,\ \mathbf{u}_0) = \mathbf{s}(\boldsymbol{\theta}_0, t_1)$$
with sensitivities

...

$\mathbf{x}_k,\ \mathbf{u}_k$

Integrator $[t_k,\ t_{k+1}]$
$$\mathbf{c}(\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\theta}_k) = 0$$

$$\mathbf{f}(\mathbf{x}_k,\ \mathbf{u}_k) = \mathbf{s}(\boldsymbol{\theta}_k, t_{k+1})$$
with sensitivities

...

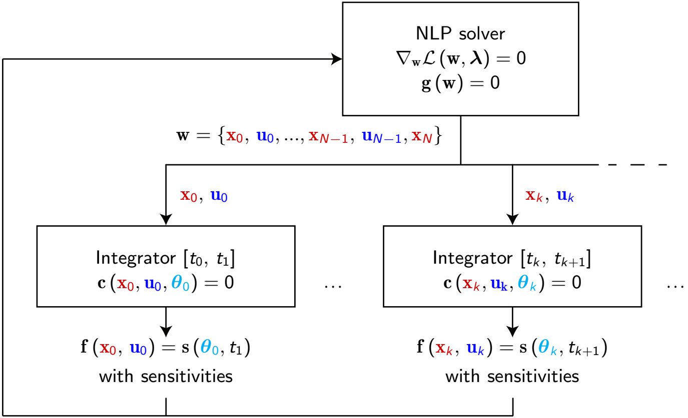
NLP level

- Constraints $\mathbf{g} = 0$
- Newton iterations (SQP/IP)

Integrator level

- Constraints $\mathbf{c} = 0$
- Newton iterations

Collocation-based integrators in Multiple-shooting (cont')



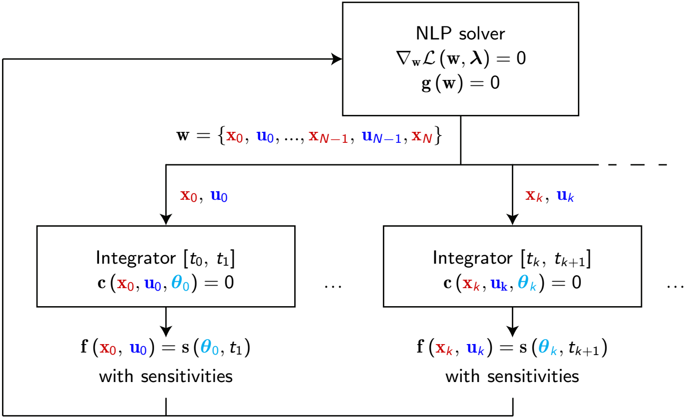
NLP level

- Constraints $\mathbf{g} = 0$
- Newton iterations (SQP/IP)

Integrator level

- Constraints $\mathbf{c} = 0$
- Newton iterations

Collocation-based integrators in Multiple-shooting (cont')



NLP level

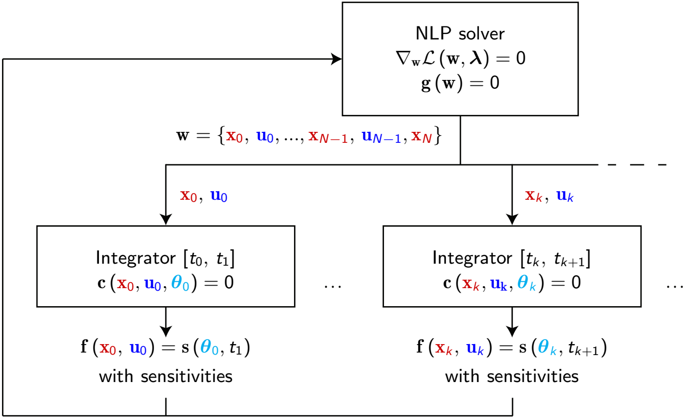
- Constraints $\mathbf{g} = 0$
- Newton iterations (SQP/IP)

Integrator level

- Constraints $\mathbf{c} = 0$
- Newton iterations

Constraints are solved at the NLP and at the integrator level separately !!

Collocation-based integrators in Multiple-shooting (cont')



NLP level

- Constraints $\mathbf{g}=0$
- Newton iterations (SQP/IP)

Integrator level

- Constraints $\mathbf{c}=0$
- Newton iterations

Constraints are solved at the NLP and at the integrator level separately !!

... what about handling them **altogether** in the NLP ?!?

Outline

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

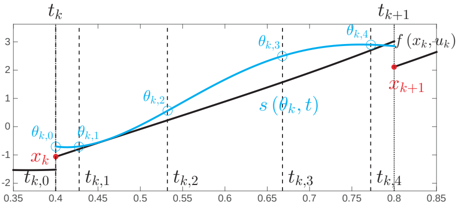
$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}_k\right)$$

is approximated using:

$$\mathrm{s}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathrm{s}\left(\boldsymbol{\theta}_{k,i}, t_{k,i}\right) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

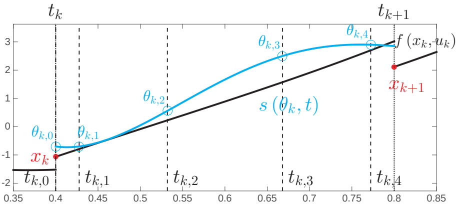
$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}_k\right)$$

is approximated using:

$$\mathbf{s}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{s}\left(\boldsymbol{\theta}_{k,i}, t_{k,i}\right) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



Integration constraints $(i = 1, ..., K)$

$$\frac{\partial}{\partial t}\mathbf{s}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

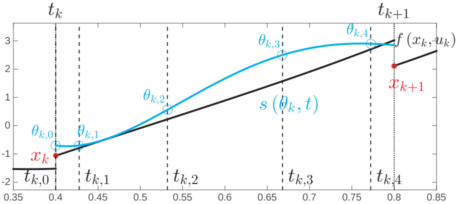
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}_k)$$

is approximated using:

$$\mathbf{s}(\boldsymbol{\theta}_k, t) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{s}(\boldsymbol{\theta}_{k,i}, t_{k,i}) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \left[\quad \right]$$

Integration constraints ($i = 1, ..., K$)

$$\frac{\partial}{\partial t}\mathbf{s}(\boldsymbol{\theta}_k, t_{k,i}) = \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

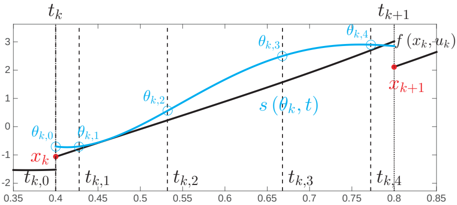
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}_k)$$

is approximated using:

$$\mathrm{s}(\boldsymbol{\theta}_k, t) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathrm{s}(\boldsymbol{\theta}_{k,i}, t_{k,i}) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \\ \\ \\ \end{bmatrix}$$

Initial conditions $\bar{\mathbf{x}}_0$

Integration constraints $(i = 1, \dots, K)$

$$\frac{\partial}{\partial t}\mathrm{s}(\boldsymbol{\theta}_k, t_{k,i}) = \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

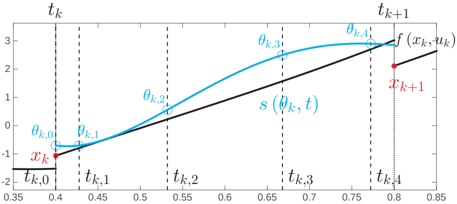
$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}_k\right)$$

is approximated using:

$$\mathbf{s}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{s}\left(\boldsymbol{\theta}_{k,i}, t_{k,i}\right) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{s}\left(\boldsymbol{\theta}_0, t_1\right) - \boldsymbol{\theta}_{1,0} \\ \\ \\ \\ \end{bmatrix}$$

Continuity constraints ($\equiv$ shooting gaps)

Integration constraints ($i = 1, \ldots, K$)

$$\frac{\partial}{\partial t}\mathbf{s}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

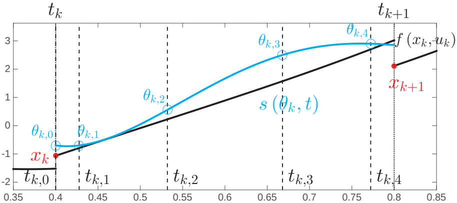
$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}_k\right)$$

is approximated using:

$$\mathbf{s}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{s}\left(\boldsymbol{\theta}_{k,i}, t_{k,i}\right) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{s}\left(\boldsymbol{\theta}_0, t_1\right) - \boldsymbol{\theta}_{1,0} \\ \mathbf{F}\left(\boldsymbol{\theta}_{0,i}, \mathbf{u}_0\right) - \sum_{j=0}^{K} \boldsymbol{\theta}_{0,j} \dot{P}_{0,j}(t_{0,i}) \\ \\ \\ \\ \end{bmatrix}$$

Integration constraints for $k = 0$

Integration constraints $(i = 1, \ldots, K)$

$$\frac{\partial}{\partial t}\mathbf{s}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

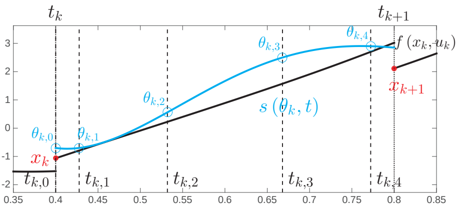
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{u}_k)$$

is approximated using:

$$\mathbf{s}(\boldsymbol{\theta}_k, t) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{s}(\boldsymbol{\theta}_{k,i}, t_{k,i}) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi(\mathbf{w})$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{s}(\boldsymbol{\theta}_0, t_1) - \boldsymbol{\theta}_{1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{0,i}, \mathbf{u}_0) - \sum_{j=0}^{K} \boldsymbol{\theta}_{0,j} \dot{P}_{0,j}(t_{0,i}) \\ \dots \\ \mathbf{s}(\boldsymbol{\theta}_k, t_{k+1}) - \boldsymbol{\theta}_{k+1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) \\ \dots \end{bmatrix}$$

Remaining integration constraints $k = 1, ..., N-1$

Integration constraints $(i = 1, ..., K)$

$$\frac{\partial}{\partial t}\mathbf{s}(\boldsymbol{\theta}_k, t_{k,i}) = \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)$$

Direct collocation - Give all constraints to the NLP solver

On each interval $[t_k, t_{k+1}]$

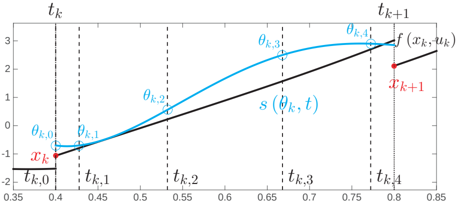
$$\dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}_k\right)$$

is approximated using:

$$\mathbf{s}\left(\boldsymbol{\theta}_k, t\right) = \sum_{i=0}^{K} \underbrace{\boldsymbol{\theta}_{k,i}}_{\text{parameters}} \cdot \underbrace{P_{k,i}(t)}_{\text{polynomials}}$$

Note:

- $\mathbf{s}\left(\boldsymbol{\theta}_{k,i}, t_{k,i}\right) = \boldsymbol{\theta}_{k,i}$
- $K+1$ degrees of freedom per state.



NLP with direct collocation

$$\min_{\mathbf{w}} \quad \Phi\left(\mathbf{w}\right)$$

$$\text{s.t.} \quad \mathbf{g}\left(\mathbf{w}\right) = \left[\begin{array}{c} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{s}\left(\boldsymbol{\theta}_0, t_1\right) - \boldsymbol{\theta}_{1,0} \\ \mathbf{F}\left(\boldsymbol{\theta}_{0,i}, \mathbf{u}_0\right) - \sum_{j=0}^{K} \boldsymbol{\theta}_{0,j} \dot{P}_{0,j}(t_{0,i}) \\ \dots \\ \mathbf{s}\left(\boldsymbol{\theta}_k, t_{k+1}\right) - \boldsymbol{\theta}_{k+1,0} \\ \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) \\ \dots \end{array}\right]$$

Decision variables:

$$\mathbf{w} = \left\{\boldsymbol{\theta}_{0,0}, \dots, \boldsymbol{\theta}_{0,K}, \mathbf{u}_0, \dots, \boldsymbol{\theta}_{N-1,0}, \dots, \boldsymbol{\theta}_{N-1,K}, \mathbf{u}_{N-1}\right\}$$

Integration constraints $(i = 1, \dots, K)$

$$\frac{\partial}{\partial t}\mathbf{s}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

i.e.

$$\sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

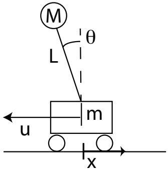
Direct Collocation - Example: swing-up of a pendulum

OCP

$$\min_{u_0,\dots,u_{N-1}} \quad \sum_{k=0}^{N-1} u_k^2$$

$$\text{s.t.} \quad \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, u_k\right), \quad \forall t \in [t_k, t_{k+1}]$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 & \pi & 0 & 0 \end{bmatrix}, \quad \mathbf{x}\left(t_{\mathrm{f}}\right) = 0$$



$$\mathbf{x} = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^\top$$

Direct Collocation - Example: swing-up of a pendulum

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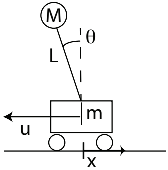

$$\mathbf{x} = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^\top$$

$N = 20$
$K = 4$ with Legendre, order 8 !!
420 variables
404 constraints

Reminder:

$$\mathbf{s}(\boldsymbol{\theta}_k, t) = \sum_{i=0}^{K} \boldsymbol{\theta}_{k,i} \cdot P_{k,i}(t)$$

$$\mathbf{s}(\boldsymbol{\theta}_k, t_{k,i}) = \boldsymbol{\theta}_{k,i}$$

NLP with direct collocation

$$\min_{\mathbf{w}} \quad \sum_{k=0}^{N-1} u_k^2$$

$$\text{s.t.} \quad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{s}(\boldsymbol{\theta}_0, t_1) - \boldsymbol{\theta}_{1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{0,i}, \mathbf{u}_0) - \sum_{j=0}^{K} \boldsymbol{\theta}_{0,j} \dot{P}_{0,j}(t_{0,i}) \\ \dots \\ \mathbf{s}(\boldsymbol{\theta}_k, t_{k+1}) - \boldsymbol{\theta}_{k+1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) \\ \dots \\ \mathbf{s}(\boldsymbol{\theta}_{N-1}, t_N) \end{bmatrix} = 0$$

Direct Collocation - Example: swing-up of a pendulum

OCP

$$\min_{u_0,\dots,u_{N-1}} \quad \sum_{k=0}^{N-1} u_k^2$$

$$\text{s.t.} \quad \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, u_k), \quad \forall t \in [t_k, t_{k+1}]$$

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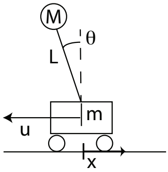

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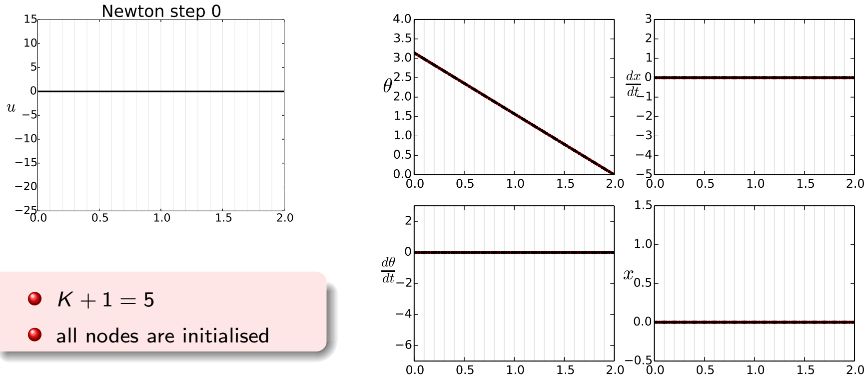

- $K + 1 = 5$
- all nodes are initialised

Direct Collocation - Example: swing-up of a pendulum

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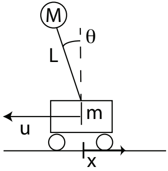

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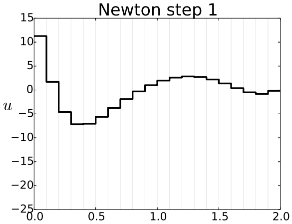

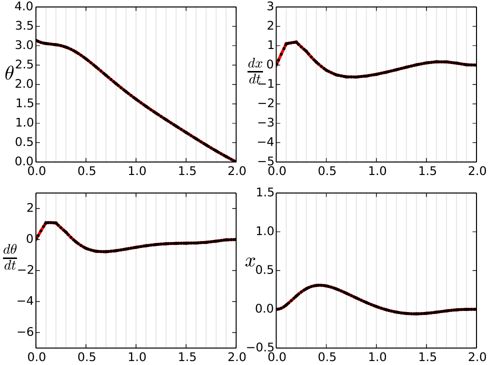

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Direct Collocation - Example: swing-up of a pendulum

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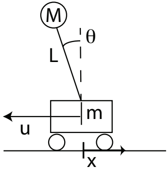

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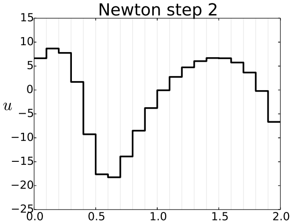

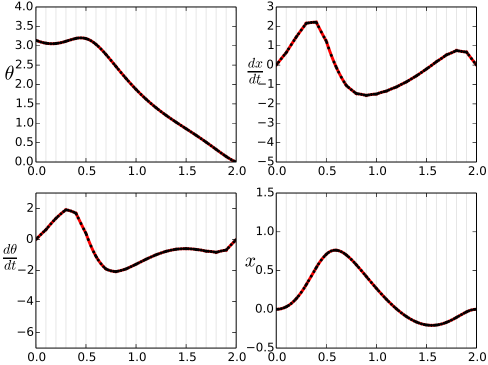

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Direct Collocation - Example: swing-up of a pendulum

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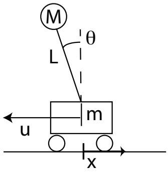

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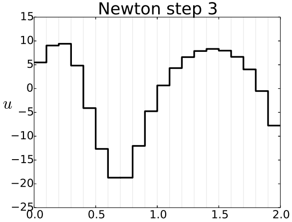

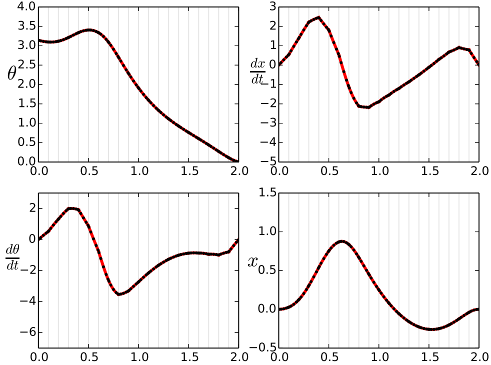

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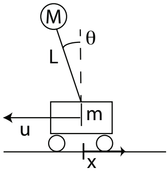

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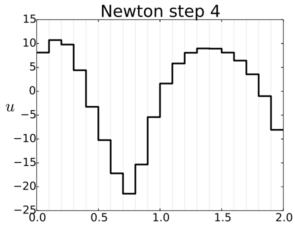

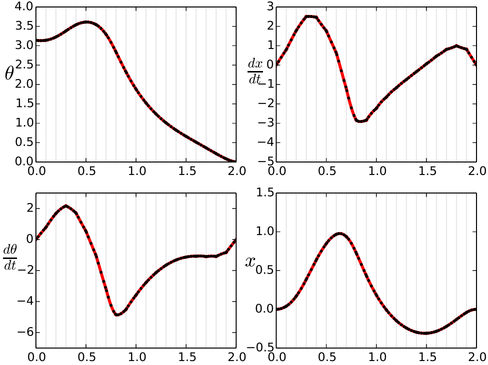

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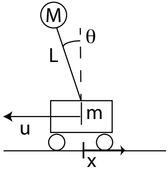

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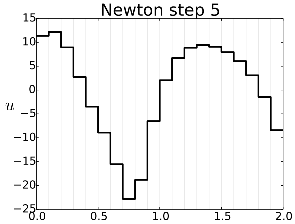

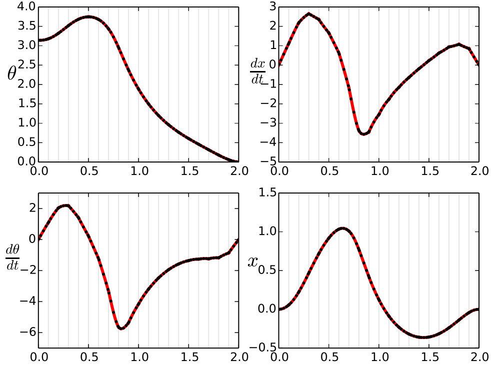

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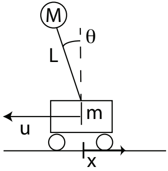

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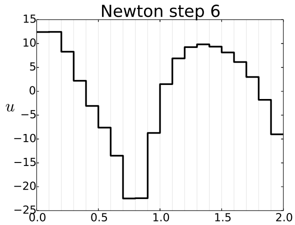

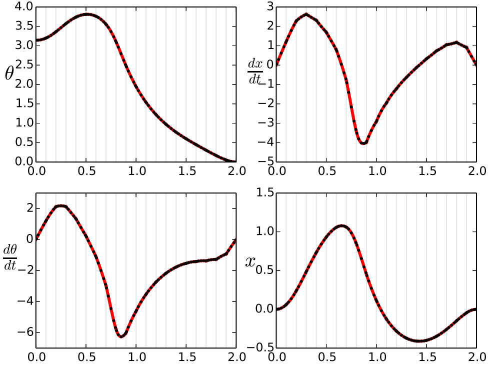

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Direct Collocation - Example: swing-up of a pendulum

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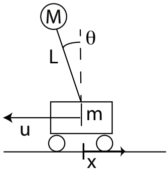

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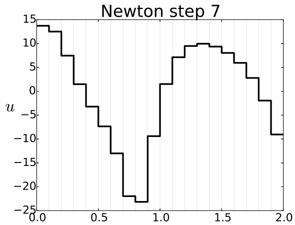

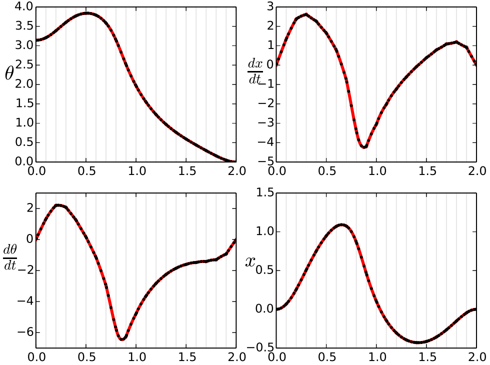

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Direct Collocation - Example: swing-up of a pendulum

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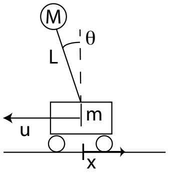

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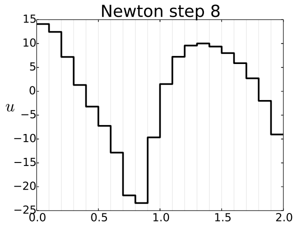

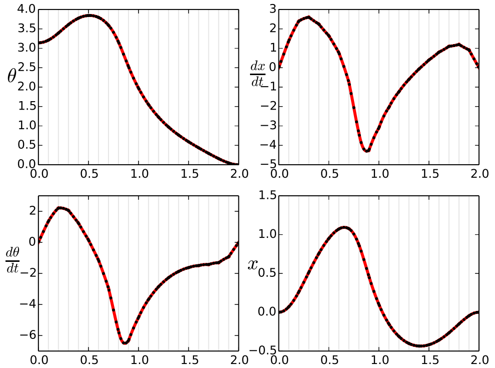

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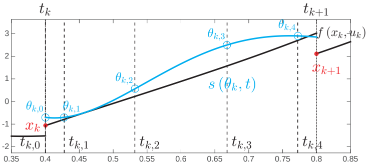
Cost and constraints discretisation in Direct Collocation

OCP:

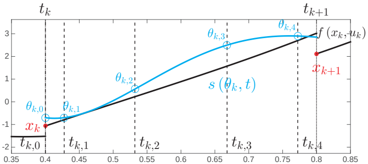
$$\begin{aligned} \min \quad & T\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right)+\int_{0}^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}}=\mathbf{F}\left(\mathbf{x},\mathbf{u}\right) \\ & \mathbf{h}\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) \leq 0 \end{aligned}$$



Cost and constraints discretisation in Direct Collocation

OCP:

$$\begin{aligned} \min \quad & T\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) + \int_0^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) \\ & \mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0 \end{aligned}$$



- Inequality constraints: $\mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0$ can be enforced on all collocation nodes:

$$\mathbf{h}\left(\boldsymbol{\theta}_k, t_{k,i}, \mathbf{u}_k\right) \leq 0, \quad \forall\, k = 0, \dots, N-1, \quad i = 0, \dots, K$$

but often only on the "shooting" nodes $t_{0,0}$, $t_{1,0}$, ..., $t_{N,0}$

Cost and constraints discretisation in Direct Collocation

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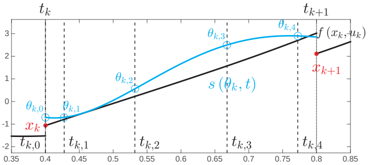

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$$\mathbf{h}\left(\boldsymbol{\theta}_k, t_{k,i}, \mathbf{u}_k\right) \leq 0, \quad \forall\, k = 0, ..., N-1, \quad i = 0, ..., K$$

  but often only on the "shooting" nodes $t_{0,0}$, $t_{1,0}$, ..., $t_{N,0}$

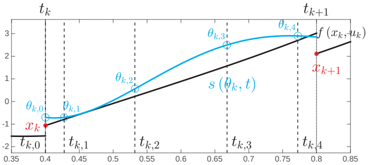
- Cost function often approximated as (rectangular quadrature):

$$T\left(\mathbf{x}\left(\boldsymbol{\theta}_{N-1}, t_{N-1,K}\right)\right) + \sum_{k=0}^{N-1} \left(t_{k+1} - t_k\right) L\left(\boldsymbol{\theta}_{k,0}, \mathbf{u}_k\right)$$

Cost and constraints discretisation in Direct Collocation

OCP:

$$\begin{aligned}\min \quad & T\left(\mathbf{x}\left(t_{\mathrm{f}}\right)\right)+\int_0^{t_{\mathrm{f}}} L\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}}=\mathbf{F}\left(\mathbf{x},\mathbf{u}\right) \\ & \mathbf{h}\left(\mathbf{x}\left(t\right),\mathbf{u}\left(t\right)\right)\leq 0\end{aligned}$$



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$$\mathbf{h}\left(\boldsymbol{\theta}_k, t_{k,i}, \mathbf{u}_k\right)\leq 0, \quad \forall\, k=0,\dots,N-1, \quad i=0,\dots,K$$

  but often only on the "shooting" nodes $t_{0,0}$, $t_{1,0}$, ..., $t_{N,0}$

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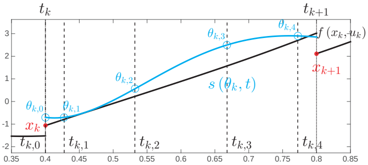
$$T\left(\mathbf{x}\left(\boldsymbol{\theta}_{N-1}, t_{N-1,K}\right)\right)+\sum_{k=0}^{N-1}\left(t_{k+1}-t_k\right) L\left(\boldsymbol{\theta}_{k,0}, \mathbf{u}_k\right)$$

Careful: if you want to use $\boldsymbol{\theta}_{k,i}$ for $i=1,\dots,K$, the time grid is not uniform !!

Cost and constraints discretisation in Direct Collocation

OCP:

$$\begin{aligned} \min \quad & T\left(\mathbf{x}\left(t_\mathrm{f}\right)\right) + \int_0^{t_\mathrm{f}} L\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) dt \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{F}\left(\mathbf{x}, \mathbf{u}\right) \\ & \mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0 \end{aligned}$$



- Inequality constraints: $\mathbf{h}\left(\mathbf{x}\left(t\right), \mathbf{u}\left(t\right)\right) \leq 0$ can be enforced on all collocation nodes:

$$\mathbf{h}\left(\boldsymbol{\theta}_k, t_{k,i}, \mathbf{u}_k\right) \leq 0, \quad \forall\, k = 0, ..., N-1, \quad i = 0, ..., K$$

but often only on the "shooting" nodes $t_{0,0}$, $t_{1,0}$, ..., $t_{N,0}$

- Cost function often approximated as (rectangular quadrature):

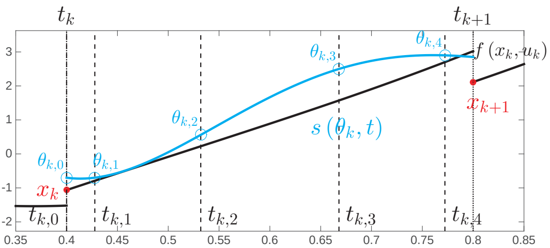
$$T\left(\mathbf{x}\left(\boldsymbol{\theta}_{N-1}, t_{N-1,K}\right)\right) + \sum_{k=0}^{N-1} \left(t_{k+1} - t_k\right) L\left(\boldsymbol{\theta}_{k,0}, \mathbf{u}_k\right)$$

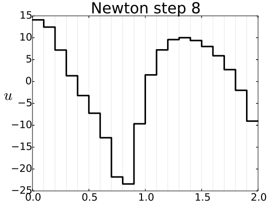
Careful: if you want to use $\boldsymbol{\theta}_{k,i}$ for $i = 1, ..., K$, the time grid is not uniform !!

- Quadratic term in cost function $L\left(\mathbf{x}, \mathbf{u}\right) = \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + ...$ can be implemented using:

$$\int_{t_k}^{t_{k+1}} \frac{1}{2}\mathbf{x}\left(t\right)^\top Q\mathbf{x}\left(t\right)\,\mathrm{d}t = \frac{1}{2}\sum_{l=0}^{K}\sum_{j=0}^{K} \boldsymbol{\theta}_{k,l}{}^\top Q\boldsymbol{\theta}_{k,j} \underbrace{\int_{t_k}^{t_{k+1}} P_{k,l}(t)P_{k,j}(t)dt}_{=\alpha_j\delta_{l,j}\ (P\text{:s are orthogonal})} = \frac{1}{2}\sum_{j=0}^{K} \alpha_j \boldsymbol{\theta}_{k,j}{}^\top Q\boldsymbol{\theta}_{k,j}$$

Some remarks



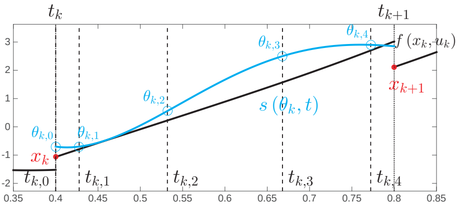


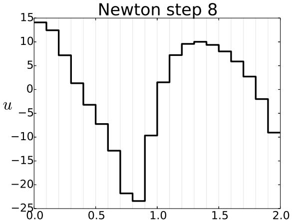
- Direct collocation is a "fully simultaneous" approach, as the **integration and the optimization are performed together** in the NLP solver.
- The decision variables are:

$$\mathbf{w} = \{\boldsymbol{\theta}_{0,0}, ..., \boldsymbol{\theta}_{0,K}, \mathbf{u}_0, ..., \boldsymbol{\theta}_{N-1,0}, ..., \boldsymbol{\theta}_{N-1,K}, \mathbf{u}_{N-1}\}$$

Observe that $\boldsymbol{\theta}_{k,i}$, i.e. the state at the collocation point $t_{k,i}$ of the interval $[t_k, \ t_{k+1}]$ is in $\mathbb{R}^n$ (size of the state). Manipulating these variables properly in a computer code can be tricky.

Refining the input discretization





- Input $\mathbf{u}(t)$ is usually chosen piecewise-constant, i.e. constant in every $[t_k,\ t_{k+1}]$

Collocation constraints:

$$\boldsymbol{\theta}_{k,0} = \mathbf{x}_k$$

$$\frac{\partial}{\partial t}\mathbf{s}\left(\boldsymbol{\theta}_k, t_{k,i}\right) = \mathbf{F}\left(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k\right)$$

for $i = 1, \ldots, K$

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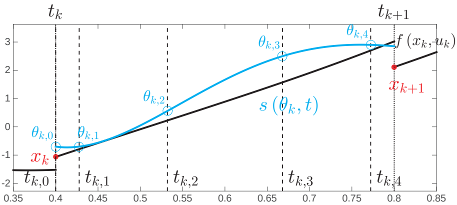

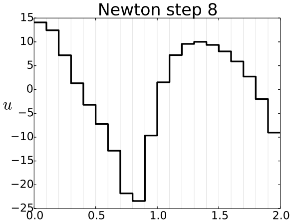

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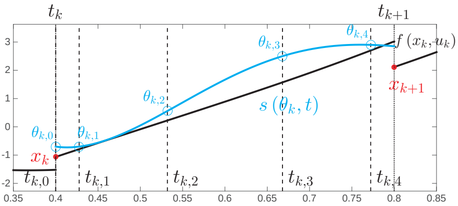

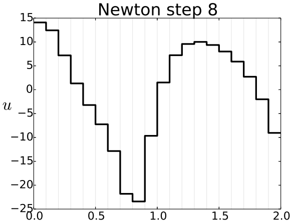

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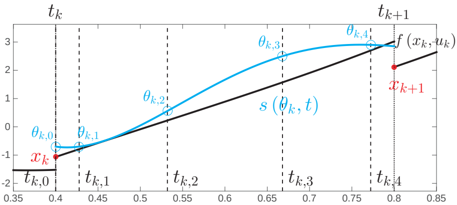

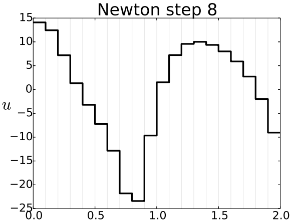

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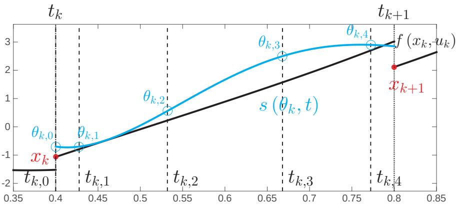

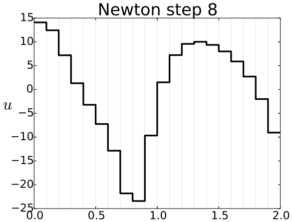

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- Drawbacks: **1.** the input profile can present important "oscillations", **2.** the linear algebra tends to loose conditioning

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Outline

1. Polynomial interpolation
2. Collocation-based integration
3. Collocation in multiple-shooting
4. Direct Collocation
5. NLP from direct collocation

Hessian in Direct Collocation

Lagrange function:

$$\mathcal{L}\left(\mathbf{w}, \boldsymbol{\lambda}\right) = \Phi\left(\mathbf{w}\right) + \boldsymbol{\lambda}^{\top}\mathbf{g}\left(\mathbf{w}\right) + \boldsymbol{\mu}^{\top}\mathbf{h}\left(\mathbf{w}\right)$$

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Reminder: dynamics yield

$$\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \boldsymbol{\theta}_{0,0} - \bar{\mathbf{x}}_0 \\ \mathbf{s}(\boldsymbol{\theta}_0, t_1) - \boldsymbol{\theta}_{1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{0,i}, \mathbf{u}_0) - \sum_{j=0}^{K} \boldsymbol{\theta}_{0,j} \dot{P}_{0,j}(t_{0,i}) \\ \dots \\ \mathbf{s}(\boldsymbol{\theta}_k, t_{k+1}) - \boldsymbol{\theta}_{k+1,0} \\ \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) \\ \dots \end{bmatrix}$$

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Nonlinear contributions of the dynamics:

$$\nabla^2_{\mathbf{w}} \left(\boldsymbol{\lambda}^\top \mathbf{g}\right) = \nabla^2_{\mathbf{w}} \left[\sum_{k=0,\dots,N-1} \sum_{i=1,..,K} \boldsymbol{\lambda}_{k,i}^\top \left(\mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) \right) \right]$$

Hessian in Direct Collocation

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$$= \sum_{k=0,\dots,N-1} \sum_{i=1,..,K} \nabla^2_{\mathbf{w}} \left(\boldsymbol{\lambda}_{k,i}^\top \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)\right)$$

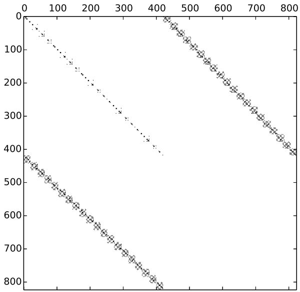
Function $\mathbf{F}$ is simply your ODE. With $\mathbf{w} = \{\boldsymbol{\theta}_{0,0}, \dots, \boldsymbol{\theta}_{0,K}, \mathbf{u}_0, \dots, \boldsymbol{\theta}_{N-1,0}, \dots, \boldsymbol{\theta}_{N-1,K}, \mathbf{u}_{N-1}\}$, the contributions

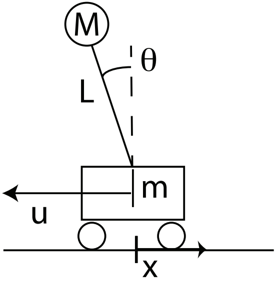
$$\nabla^2_{\mathbf{w}} \left(\boldsymbol{\lambda}_{k,i}^\top \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k)\right)$$

are very sparse and trivial to compute !! (e.g. CasADi)

Sparsity pattern

For the pendulum, KKT matrix M is:



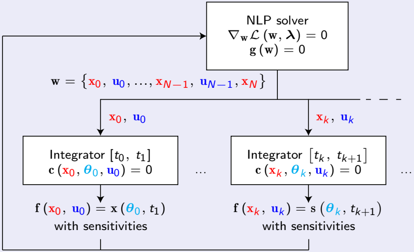


$$M = \begin{bmatrix} H & \nabla \mathbf{g} \\ \nabla \mathbf{g}^\top & 0 \end{bmatrix}$$

- Direct collocation yields **very large but very sparse** NLPs. Typically not a problem for dedicated NLP solvers (e.g. ipopt)
- Exact Hessian is inexpensive to build and compute, unlike in multiple-shooting

Multiple-shooting vs. Direct Collocation

NLP with multiple-shooting & collocation integrators



NLP with direct collocation

$$\begin{aligned}\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w},\boldsymbol{\lambda}) &= 0\\ \mathbf{g}(\mathbf{w}) &= 0\end{aligned}$$

where

$$\mathbf{w} = \{\boldsymbol{\theta}_{0,0}, \dots, \boldsymbol{\theta}_{0,K}, \mathbf{u}_0, \dots, \boldsymbol{\theta}_{N-1,0}, \dots, \boldsymbol{\theta}_{N-1,K}, \mathbf{u}_{N-1}\}$$

and

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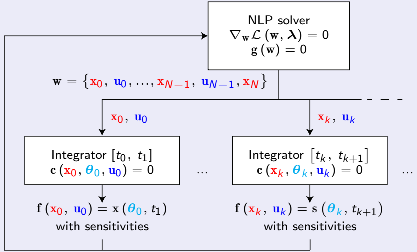

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- NLP has $n_{\mathrm{x}}(N+1) + n_{\mathrm{u}}N$ variables
- N integrators with $n_{\mathrm{x}}(K+1)$ variables, **parallelizable**

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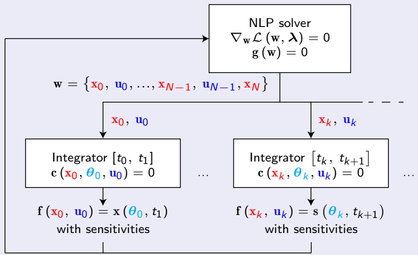

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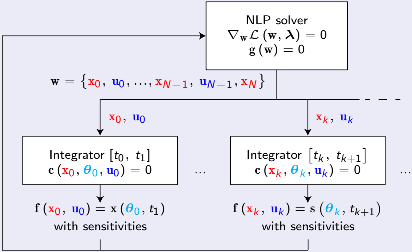

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"Multiple-shooting + collocation integrators" does not converge as efficiently as direct collocation...

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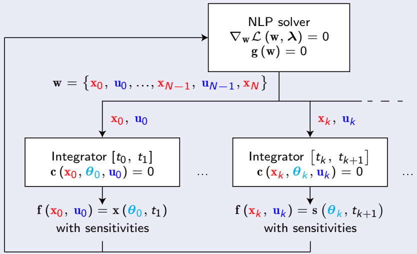

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NLP with direct collocation

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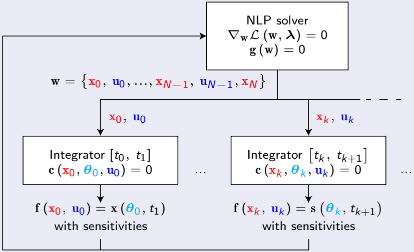
"Multiple-shooting + collocation integrators" does not converge as efficiently as direct collocation... Unless:

Lifted implicit integrators for direct optimal control, R. Quirynen, S. Gros, M. Diehl, NMPC Workshop 2015
Lifted implicit integrators for NMPC based on Multiple Shooting, R. Quirynen, S. Gros, M. Diehl, CDC 2015
Lifted Collocation Integrators for Direct Optimal Control in ACADO Toolkit, R. Quirynen, S. Gros, B. Houska, M. Diehl, Journal of Math. Prog. Comp. 2017

Multiple-shooting vs. Direct Collocation

NLP with multiple-shooting & collocation integrators



- NLP has $n_{\mathrm{x}}(N+1) + n_{\mathrm{u}}N$ variables
- N integrators with $n_{\mathrm{x}}(K+1)$ variables, **parallelizable**

NLP with direct collocation

$$\begin{aligned} \nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}) &= 0 \\ \mathbf{g}(\mathbf{w}) &= 0 \end{aligned}$$

where

$$\mathbf{w} = \{\boldsymbol{\theta}_{0,0}, \dots, \boldsymbol{\theta}_{0,K}, \mathbf{u}_0, \dots, \boldsymbol{\theta}_{N-1,0}, \dots, \boldsymbol{\theta}_{N-1,K}, \mathbf{u}_{N-1}\}$$

and

$$\mathbf{g}(\mathbf{w}) = \begin{bmatrix} \mathbf{s}(\boldsymbol{\theta}_0, t_0) - \bar{\mathbf{x}}_0 \\ \dots \\ \mathbf{s}(\boldsymbol{\theta}_k, t_{k+1}) - \mathbf{s}(\boldsymbol{\theta}_{k+1}, t_{k+1}) \\ \mathbf{F}(\boldsymbol{\theta}_{k,i}, \mathbf{u}_k) - \sum_{j=0}^{K} \boldsymbol{\theta}_{k,j} \dot{P}_{k,j}(t_{k,i}) \\ \dots \end{bmatrix} = 0$$

- NLP has $n_{\mathrm{x}}N(K+1) + n_{\mathrm{u}}N$ variables
- **Integration** performed in the NLP, converges together with optimization

Consequence: there is a systematic parallel linear algebra for Direct Collocation !!