Modeling, Analysis, and Improvement of Batch-Discrete Manufacturing Systems: A Systems Approach

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***Abstract*— Production systems include both discrete part and batch operations, where an individual part is manufactured in a discrete operation, and a group of parts are processed simultaneously, i.e., in a batch, on one machine for a batch operation. Many manufacturing industries, such as battery, aircraft, and automotive, consist of mixed batch and discrete part operations, referred to as batch-discrete lines. Although such operations are widely encountered, analytical studies of these systems are limited in current literature. In this paper, a systems approach is presented to model and analyze batch-discrete lines. First, a Bernoulli machine reliability model for a two-machine batch-discrete system is introduced. Using a virtual buffer to represent the batch processing feature, performance evaluation formulae are derived and system properties are investigated. Using them, improvement analyses and bottleneck identification are presented. Then, the model is extended to systems with a quality inspection device under different control policies. To illustrate the applicability of the model, a case study in a composite part production process is described. Such a work delivers a quantitative tool for production engineers and managers to design, analyze, and improve batch-discrete manufacturing systems.**

***Note to Practitioners*—Many manufacturing systems in aircraft, automotive, battery, medical device, and defense industries include both batch operation and discrete part processing machines. In a batch operation, multiple parts are manufactured simultaneously on a batch machine, while a single part is made in a discrete part machine. Production lines with mixed batch and discrete part operations are named as batch-discrete systems. Analysis and improvement of such systems are critical to ensure high productivity and quality. However, accurate modeling and analysis of batch-discrete systems are lacking in current literature. To bridge this gap, a novel methodology is presented in this paper. Using a Bernoulli reliability machine model with a virtual buffer concept, performance measures are derived for batch-discrete two-machine lines, as well as the reversed discrete-batch lines. Then system properties, such as monotonicity, interchangeability, and reversibility, are investigated, followed by improvement analysis under constraints and bottleneck analysis. By extending to systems with quality inspections, two quality control policies, to scrap either the current batch or the whole inventory after detecting a degraded part, are studied. In addition, a case study of heating (batch) and trimming (discrete) operations in composite panel production lines is presented, and improvement strategies are investigated, to illustrate how to apply the model and analysis in practice.**

***Index Terms*— Batch-discrete manufacturing, Bernoulli reliability model, production rate, bottleneck analysis, quality inspection.**

I. Introduction

MANUFACTURING systems are moving towards a new paradigm of Industry 4.0 [1]. To improve the efficiency, quality, and sustainability of production operations, numerous research studies have been carried out to model, analyze, and optimize system performance.

In addition to many of manufacturing systems where discrete parts are processed sequentially and independently, batch processing is also prevalent in many production systems. Here batch operations refer to processing of bulk materials simultaneously, i.e., in batches, on a single station, which exist in many discrete material (e.g., heat treatment, washing, and cooling) and continuous flow (such as chemical, liquid, powder mixing or fermenting) manufacturing. Many of such systems have mixed operations, where batch operations are immediately followed by discrete part productions. For example, in Li-ion battery cell manufacturing, barrels of cathode (or anode) material are mixed with solvent and binder in the mixer, then roll-to-roll operations are followed for coating, drying, calendering, and slitting [2]. In automobile powertrain manufacturing, fuel injectors are heated in furnaces and cleaned in washers, both in batches, and then polished and welded individually [3]. Similar examples can be found in composite, medical device, aircraft, and space products manufacturing (e.g., [4]–[6]).

Although such mixed batch and discrete manufacturing systems (referred to as batch-discrete productions in this paper) are widely encountered, the analysis of them is not easy since the connected operations consist of significantly

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different scales in processing time and quantity units. Majority of manufacturing system research considers single or discrete part operation in all stages, and assumes stationary distribution of inventories, which is not applicable to study batch-discrete systems, particularly when machines are unreliable and buffers have finite capacities. Only limited studies on batch operation production lines are available, and many of them assume parallel machines to process parts in batches (in fact, still discrete part manufacturing on each machine). In addition, they often minimize or ignore the time differences and transform batch sizes into operation speed parameters through approximation to discrete part manufacturing, which may not be suitable for accurate analysis. Moreover, such a system also posts significant challenges for quality inspection and control, as any defects due to settings in batch machines (such as heating and pressure parameters) could affect the whole batch, and in some cases, all inventories in the system. To the best of our knowledge, no available study addresses the quality inspection and control issues in current literature for such systems. Thus, there exists a need to develop a novel method to model, analyze, and improve batch-discrete production lines.

In this paper, a systems approach is introduced to evaluate the performance of batch-discrete production systems with Bernoulli machine reliability model. In addition to deriving production rate and other performance measures, system properties, such as monotonicity, interchangeability, and reversibility, are investigated. Improvability analysis and bottleneck identification strategies for system improvement are also studied. Furthermore, two quality control policies in such systems, either scrapping a batch or whole inventory after defects being inspected, are discussed and compared.

The remainder of the paper is structured as follows: The related literature is briefly reviewed in Section II. Section III formulates the problem and Section IV presents the performance evaluation method. System properties are discussed in Section V and improvement analysis is carried out in Section VI. Extensions to systems with quality inspections are described in Section VII. In addition, a case study at a composite part production plant is presented in Section VIII to illustrate the applicability of the method. Finally, Section IX formulates the conclusions and discusses future research directions. All proofs are provided in the Appendix.

II. Literature Review

Manufacturing systems have received substantial research attention for over three decades (e.g., monographs [7]–[11] and reviews [12]–[15]). Most studies focus on modeling, analysis, improvement, and control of system throughput, lead time, and work-in-process, etc., for different types of production systems, such as serial lines, assembly systems, rework and closed loops, or multi-product lines, using aggregation, decomposition, and other analytical methods (see representative papers [16]–[28]). In most cases, single or discrete part processing is assumed.

Batch processing or parallel operation has been studied extensively in production scheduling (see reviews [29]–[31]). Such research typically addresses issues of optimizing makespan or job delivery under various sequencing or routing constraints in job-shop systems, where machine reliability and finite buffer capacity are less considered.

In large volume manufacturing, only limited studies can be found for production systems with batch operations. For instance, paper [32] presents a two-stage model with unreliable batch machines to analyze system performance and compare full- and partial-batch policies. Similarly, reliable two-batch-machine lines are considered in [33] and [34] to investigate the impact of batch control policies on mean cycle time of the system. A Bernoulli serial line model is introduced in [35] and a computation efficient aggregation approach is developed to evaluate throughput for long serial lines, study the impact of parameter changes, and analyze monotonic and reversibility properties. However, such studies are more applicable for production lines with multiple synchronized parallel machines at each stage, while may not be suitable for mixed batch and discrete production lines where significant differences in processing times and non-stationary behavior of material flow exist in these lines.

Both productivity and quality are critical elements in manufacturing systems, particularly in the era of Industry 4.0 [36]. It has been shown that quality and productivity are interacting with each other, i.e., tightly coupled [37], [38]. There have been numerous studies analyzing the impact of quality inspection and control on production performance. For example, paper [39] presents a survey on inspection strategy and sensor distribution in discrete-part manufacturing processes. A review on optimization of quality inspection planning is introduced in [40] for multi-stage manufacturing systems. In addition, integrated quality and operational failures are analyzed in [41] and [42] for two-machine and longer flow lines, respectively. The coupling effects in Bernoulli serial lines are investigated in [43] and applied in automotive paint shops [44]. Furthermore, integrated productivity and quality models are studied in [45] for battery manufacturing systems. Paper [46] introduces a general method to analyze production rate of conforming parts in manufacturing systems with deteriorating machines and preventive maintenance. However, the impact of quality inspection and control policies on batch-discrete production system performance has not been analyzed.

In addition, it is worthy to note that the batch arrival single server queue with finite buffer capacity has been investigated in the literature, for example, papers [47] and [48]. However, in both papers and most of such studies, Poisson arrivals are assumed, and parts or batches can be lost or rejected due to full buffer occupancy, which are not suitable to analyze the batch-discrete lines, since the arrival process to the buffer (or the departure process of the first machine) is impacted by the unreliability of both machines, the capacity of the buffer, and the associated blockage of the first machine, which make it impossible to be described by a Poisson process. Furthermore, blockage-induced scraping is not applicable in most production systems due to high cost of parts and customer demand requirement.

In summary, the existing literature does not provide an effective model and analytical approach to study batch-discrete manufacturing systems. Developing a systems method to

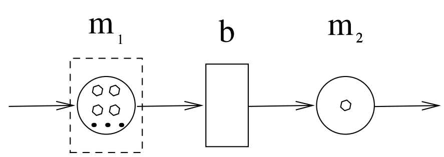


Fig. 1. A batch-discrete production line.

evaluate system performance, investigate system properties, improvement and quality control strategies, and applying the method on the factory floor, are of significant importance, which is the goal of this paper.

III. Problem Formulation

Consider a batch-discrete production line shown in Figure 1. The circles represent the machines. The rectangle with broken lines surrounding the machine indicates batch production, while the rectangle with solid lines is the storage buffer holding intermediate materials. The number of parts in a machine is illustrated by little hexagons inside the circles, where multiple parts are processed simultaneously in batch machine m_1 and only one part is made in discrete machine m_2.

The following assumptions define the machines, the buffer, and interactions between them.

(i) The batch-discrete production line consists of a batch machine, which processes raw materials in batches, and a discrete machine that operates on individual parts, as well as a buffer between the two machines.

(ii) The time axis is slotted by time unit, where one time unit is referred to as cycle time.

(iii) The batch machine, denoted as m_1, loads and processes k units of raw materials in one batch, which takes a total of k cycles to finish the operation. As a batch machine can stop working at any time, in each cycle, m_1 has probabilities p_1 to operate and $1 - p_1$ to be down. The operation will continue after a stoppage during a batch processing period. The whole batch is released to the buffer at the end of the operation.

(iv) The discrete machine, denoted as m_2, loads one unit of intermediate material to process into a finished part per time unit. It has probabilities p_2 to be up and $1 - p_2$ to be down in each cycle.

(v) There is a buffer b of capacity n batches (in boxes, trays, etc.), i.e., a total of kn parts, between the two machines, $0 < n < \infty$.

(vi) The batch machine is never starved but can be blocked if it is up at the beginning of a time unit, and the available buffer space is less than one batch even if the discrete machine takes one unit.

(vii) The discrete machine is never blocked but can be starved if it is up and there is no available intermediate material in the buffer at the beginning of a time unit.

(viii) The machine status changes at the beginning of a time unit, while the buffer contents change at the end of a cycle.

Remark 1: Assumptions (iii) and (iv) formulate a Bernoulli machine reliability model of the line. Such models have been extensively applied in many manufacturing system studies successfully (see case studies in monograph [11] and papers [3], [28], [44], [49]–[55]). The Bernoulli model is mainly suitable for assembly type operations whose downtimes are comparable to their cycle times. However, as shown in [11], for lines with exponential machine reliability models and unequal cycle times, a transformation can be applied to transform into Bernoulli models. Specifically, for machine i with cycle time τ_i, average uptime $T_{up,i}$, average downtime $T_{down,i}$, we obtain

$$p_i = \frac{\min \tau_i}{\tau_i} \cdot e_i, \tag{1}$$

where $e_i = T_{up,i}/(T_{up,i} + T_{down,i})$ represents the efficiency (or availability) of machine i. It has been shown that, using such a transformation, Bernoulli models can still be successfully applied in different types of manufacturing systems (for example, [3], [52], [55]). In future work, we plan to extend the study to other reliability models, such as exponential, phase-type, gamma, or general distributions.

Remark 2: Assumption (vi) implies a block before service policy in production operations. Such a policy is adopted in many batching operations, such as heating, mixing, and fermenting, since the finished materials cannot stay in the machines (e.g., furnaces, mixers, and tanks) for extra time due to quality requirements. In addition, if the discrete machine is up and takes a part to make the available space in the buffer being able to hold exactly one batch, then the batch machine can still load a new batch.

Let PR be the probability to produce a part by machine m_2 per time unit, i.e., the production rate of the system. In addition, introduce WIP, BL_1 and ST_2 as the average buffer occupancy, probabilities of blockage (of machine m_1) and starvation (of machine m_2), respectively. Then, PR, WIP, BL_1, and ST_2 are the key performance indicators (KPIs, [56]) of the system, and are functions of all system parameters p_1, p_2, n, and k, i.e.,

$$\begin{aligned} PR &= f_t(p_1, p_2, n, k), \\ WIP &= f_w(p_1, p_2, n, k), \\ BL_1 &= f_b(p_1, p_2, n, k), \\ ST_2 &= f_s(p_1, p_2, n, k). \end{aligned} \tag{2}$$

The problem to be studied in this paper can be formulated as: *Under assumptions (i)-(viii), develop a method to evaluate system performance measures and investigate system properties and improvement strategies.*

IV. Performance Evaluation

A. Batch-Discrete Line

Assumptions (i)-(viii) define an ergodic stochastic process. However, direct analysis of such a process is difficult due to mixed batch and discrete operations. Considering that m_1 will release the whole batch after processing all k units, it is equivalent to view machine m_1 as processing one part unit per cycle and storing the finished parts in a virtual buffer $\widetilde{b}$. When all k units are available, they will be shifted to buffer b together. Figure 2 illustrates such an equivalent model.

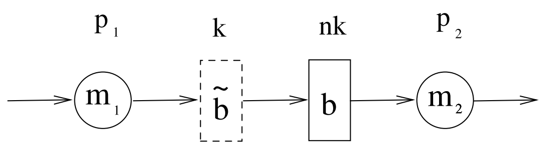


Fig. 2. Bernoulli model of batch-discrete production line.

In this case, the state of the system is denoted as $(\widetilde{h}, h)$, $\widetilde{h} = 0, 1, \ldots, k-1$, $h = 0, 1, \ldots, kn$, where h represents the inventory level in buffer b at the end of the cycle, and $\widetilde{h}$ indicates the occupancy in virtual buffer $\widetilde{b}$ or how many time units have been processed for a batch on machine m_1 (excluding downtimes). Note that due to block before service assumption (vi), states $(i, (n-1)k + j)$, $i > 0$, $j > 0$, $k > 1$, and (i, nk), $i > 0$, $k > 1$, do not exist, because the batch machine m_1 will not load a new batch if the available space in buffer b is less than $k-1$, or the available space is $k-1$ and m_2 is down, at the beginning of a cycle. Thus, the total number of states is

$$K = k[(n-1)k + 2]. \tag{3}$$

Introduce $P_{\widetilde{h},h}$ to define the probability of state $(\widetilde{h}, h)$. The balance equations (A.1) in Subsection A of Appendix A characterize state transitions in the system. As shown in the balance equations, the buffer occupancy which defines the system state is a result of machine status. To illustrate the state transitions, an example of $n = k = 2$ is presented in Figure 3.

Using balance equations (A.1), the transition probabilities can be written in a matrix format, denoted as $\mathcal{Q}$ with dimensions $K \times K$. Let $\mathcal{P}$ be the vector of all state probabilities $P_{i,j}$, $i = 0, 1, \ldots, k-1$, $j = 0, 1, \ldots, nk$, we obtain

$$\mathcal{P} = \mathcal{P} \cdot \mathcal{Q}. \tag{4}$$

In addition,

$$\sum_{i=0}^{k-1} \sum_{j=0}^{nk} P_{i,j} = 1. \tag{5}$$

Solving $P_{i,j}$ from (4) and (5), the system production rate, average inventory level, blockage and starvation probabilities can be evaluated as

$$\begin{aligned} PR &= p_2\left(1 - \sum_{i=0}^{k-1} P_{i,0}\right), \\ WIP &= \sum_{j=1}^{nk} \sum_{i=0}^{k-1} j P_{i,j}, \\ BL_1 &= \sum_{j=(n-1)k+2}^{nk} p_1 P_{0,j} + P_{0,(n-1)k+1}\, p_1(1-p_2), \\ ST_2 &= \sum_{i=0}^{k-1} p_2 P_{i,0}. \end{aligned} \tag{6}$$

From conservation of flow, the production rate can also be evaluated in terms of machine m_1, i.e.,

$$PR = p_1\left(1 - \sum_{j=(n-1)k+2}^{nk} P_{0,j} - P_{0,(n-1)k+1}(1-p_2)\right).$$

Thus, PR can be calculated from blockage and starvation.

$$PR = p_1 - BL_1 = p_2 - ST_2. \tag{7}$$

Remark 3: When $k = 1$, i.e., batch size one, the system becomes a serial production line making a single part each cycle. Then $\widetilde{h}$ can be withdrawn from the state definition, and balance equations can be simplified. Then solutions introduced in [11] can be obtained.

Due to complexity, closed formulae for PR and other performance measures for general cases are all but impossible. Only for a few cases, closed-form expressions can be derived. Particularly, when $n = 1$, i.e., only one batch is available in the buffer, the balance equations can be simplified and the close-form solution is derived as follows:

Proposition 1: Under assumptions (i)-(viii) with $n = 1$, the line production rate can be evaluated as

$$PR = \frac{k p_1 p_2}{k(p_1 + p_2) - p_1 p_2}. \tag{8}$$

Proof: See Appendix B. ∎

When $k = n = 2$, more complex expressions can be derived.

Proposition 2: Under assumptions (i)-(viii) with $n = k = 2$, the line production rate can be evaluated as

$$PR = p_2 - \frac{p_2}{\mathcal{A} - \mathcal{B}}\left(\frac{2 - p_1}{1 - p_1} + \frac{p_1[2(1+\alpha) - p_2]}{p_2(p_1 + \beta^2)}\right), \tag{9}$$

where

$$\begin{aligned} \alpha &= \frac{p_1}{p_2(1-p_1)}, \\ \beta &= (1-p_1)(1+\alpha), \\ \mathcal{A} &= (1+\alpha^2)(1+2\alpha) + \frac{1+\alpha^2(1+p_1)}{1-p_1}, \\ \mathcal{B} &= \frac{1}{p_1+\beta^2}\left[p_1(1+2\alpha) + 2\alpha^2(p_1+\beta)\frac{1+\beta}{1-p_1}\right]. \end{aligned} \tag{10}$$

Proof: See Appendix B. ∎

B. Reversed Line

In many production lines, a batch machine may immediately follows a discrete machine. In other words, individual parts produced by the discrete machine are released to the buffer and accumulated until a batch size k is reached. Then the batch machine loads the whole batch and processes them for k cycles. For instance, in powertrain manufacturing, fuel injectors are polished individually, and then sent to a noble washer in batches [3].

Using a similar approach, such a line can be modeled as a two machine line where the second machine loads a batch and processes each part into a virtual buffer, as illustrated in Figure 5.

Analogous assumptions of the model can be introduced, where the differences lie in blockage and starvation. In a discrete-batch line, machine m_1 is blocked if it is up, the buffer is full, and machine m_2 does not take a batch from the buffer at the beginning of a cycle, while machine m_2 is starved if it is up, and the inventory in the buffer is not enough for a batch, i.e., less than k parts at the beginning of a time unit.

The state of the system is defined by $(h, \widetilde{h})$, where h is the buffer occupancy, and $\widetilde{h}$ represents how many cycles have been

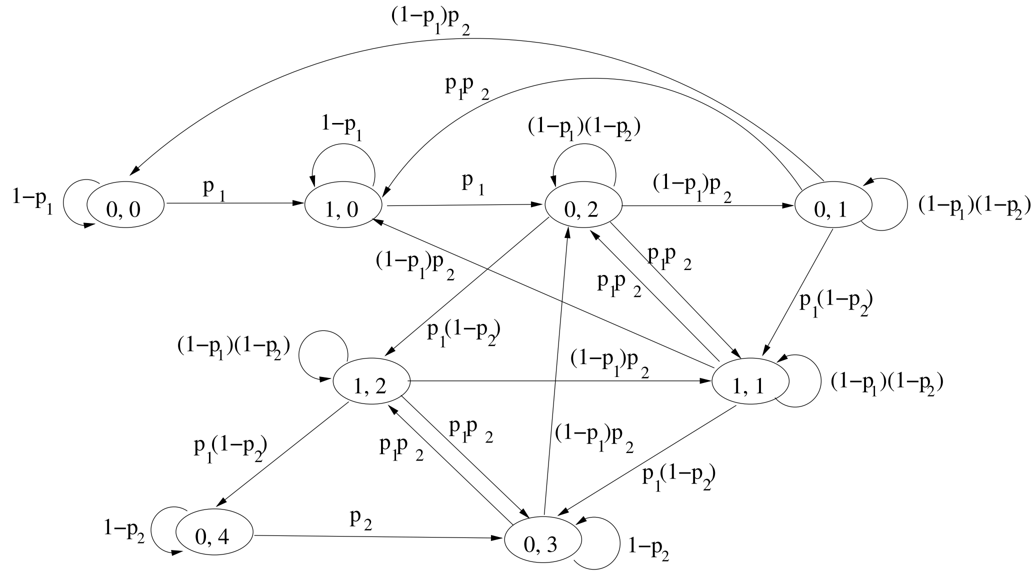


Fig. 3. Example of state transitions for a system with $n = k = 2$.

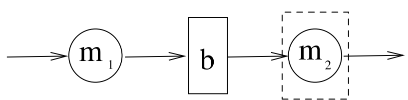


Fig. 4. Reversed line: discrete-batch production line.

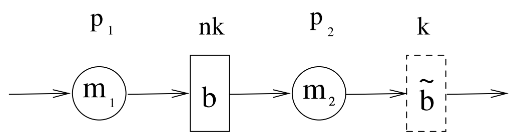


Fig. 5. Bernoulli model of discrete-batch production line.

processed by batch machine m_2 for the batch (or occupancy level in virtual buffer $\widetilde{b}$). Then, a higher number of states is obtained:

$$K = k(nk + 1). \tag{11}$$

Similar analysis can be carried out. The state balance equations (A.3) are derived in Subsection B of Appendix A. Using a similar derivation approach, the state probabilities can be obtained. Then, the system PR, WIP, BL_1, and ST_2 can be evaluated as

$$\begin{aligned}
PR &= p_2\left(1 - \sum_{j=0}^{k-1} P_{j,0}\right),\\
WIP &= \sum_{j=0}^{nk}\sum_{i=0}^{k-1} jP_{j,i},\\
BL_1 &= p_1 \sum_{i=1}^{k-1} P_{nk,i} + p_1(1-p_2)P_{nk,0},\\
ST_2 &= p_2 \sum_{j=0}^{k-1} P_{j,0}.
\end{aligned} \tag{12}$$

Due to the larger number of states in discrete-batch lines, only when $n = 1$ and $k = 2$, a closed formula of production rate can be derived.

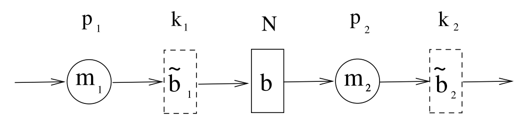


Fig. 6. Bernoulli model of a batch-batch production line.

Proposition 3: Under assumptions (i)-(viii) for reversed line, when $n = 1$ and $k = 2$, the line production rate can be evaluated as

$$PR = \frac{2p_1p_2\mathcal{C}^2}{\mathcal{C}^3 + p_2^2(1-p_1)(p_1+\mathcal{C}) + p_1^2\mathcal{C}} \tag{13}$$

where

$$\mathcal{C} = p_1 + p_2 - p_1p_2. \tag{14}$$

Proof: See Appendix B. ■

Remark 4: Moreover, using a similar modeling and derivation approach, the scenarios of batch-batch operations can be studied. Specifically, for a batch-batch two-machine line, as illustrated in Figure 6 where $\widetilde{b}_i$, $i = 1, 2$, represents the virtual buffer, introduce $k_i > 1$ as the batch size of machine m_i, where buffer capacity N is an integer multiple of the least common multiple of k_1 and k_2. Denote system state as $(\widetilde{h}_1, h, \widetilde{h}_2)$, where $\widetilde{h}_i$ takes values from 0 to $k_i - 1$, representing virtual buffer occupancy, and h indicates actual buffer level from 0 to N, with increment k_d. Here k_d is the greatest common divisor of k_1 and k_2. As the first machine is blocked when buffer occupancy is higher than $N - k_1$, the numbers of system states with the first machine being blocked and not blocked are $1 \cdot \frac{k_1}{k_d} \cdot k_2$ and $k_1 \cdot (\frac{N-k_1}{k_d} + 1) \cdot k_2$, respectively. Thus, the total number of states is

$$\begin{aligned}
K &= \left(\frac{N-k_1}{k_d} + 1\right) \cdot k_1 \cdot k_2 + \frac{k_1}{k_d} \cdot 1 \cdot k_2\\
&= \frac{k_1k_2}{k_d}(N - k_1 + k_d + 1).
\end{aligned} \tag{15}$$

The system performance can be derived as

$$PR = p_2\left(1 - \sum_{i=0}^{k_1-1}\sum_{j=0}^{k_2-k_d} P_{i,j,0}\right),$$

$$WIP = \sum_{i=0}^{k_1-1}\sum_{j=k_d}^{N}\sum_{w=0}^{k_2-1} jP_{i,j,w},$$

$$BL_1 = \begin{cases} \sum_{j=k_d}^{k_1}\sum_{w=1}^{k_2-1} P_{0,N-k_1+j,w}p_1 \\ \quad + \sum_{j=k_d}^{k_2} P_{0,N-k_1+j,0}p_1(1-p_2) \\ \quad + \sum_{j=k_2+k_d}^{k_1} P_{0,N-k_1+j,0}p_1, \quad k_1 > k_2, \\ \sum_{j=k_d}^{k_1}\sum_{w=1}^{k_2-1} P_{0,N-k_1+j,w}p_1 \\ \quad + \sum_{j=k_d}^{k_1} P_{0,N-k_1+j,0}p_1(1-p_2), \quad k_1 \le k_2, \end{cases}$$

$$ST_2 = \sum_{i=0}^{k_1-1}\sum_{j=0}^{k_2-k_d} P_{i,j,0}p_2, \tag{16}$$

where $P_{i,j,w}$'s are solved from the balance equations provided in Subsection C of Appendix A.

V. System Properties

A. Monotonicity

Monotonicity has been observed in many production systems, which can provide a direction for continuous improvement. In batch-discrete lines, as expected, the system production rate is monotonically increasing with respect to p_i, $i = 1, 2$. Specifically, when $n = 1$, we obtain

Proposition 4: Under assumptions (i)-(viii) with $n = 1$,

$$\frac{\partial PR}{\partial p_i} > 0, \quad i = 1, 2. \tag{17}$$

Proof: See Appendix B. ∎

For general cases, although analytically intractable, this property is validated through extensive numerical experiments. Specifically, over 11,000 experiments are conducted by randomly and equiprobably selecting parameters from the following sets:

$$\begin{aligned} p_i &\in (0.5, 1), \\ (k, n) &\in \{(2,2), (2,3), (3,2), (2,4), (4,2), (3,3), \\ &\quad (2,5), (5,2), (2,6), (3,4), (4,3), (6,2)\}. \end{aligned} \tag{18}$$

By setting $\Delta p = 0.001$, the corresponding changes in production rate, ΔPR, are calculated. In all the experiments, we always obtain $\frac{\Delta PR}{\Delta p_i} > 0$. Therefore, the following observation is formulated:

Observation 1: Under assumptions (i)-(viii), the line production rate PR is monotonically increasing with respect to machine efficiency p_i, $i = 1, 2$.

In case of discrete-batch production lines, when $n = 1$ and $k = 2$, we obtain

Proposition 5: In reversed lines with $n = 1$ and $k = 2$,

$$\frac{\partial PR}{\partial p_i} > 0, \quad i = 1, 2. \tag{19}$$

TABLE I

PR as a Function of kn

p_1	p_2	k	n	kn	PR
0.84	0.84	2	3	6	0.8088
0.84	0.84	3	2	6	0.7851
0.84	0.84	2	4	8	0.8187
0.84	0.84	4	2	8	0.7879
0.84	0.84	3	3	9	0.8153

Proof: See Appendix B. ∎

For other general cases in discrete-batch production lines, the same monotonicity property still holds, which is also validated through extensive numerical experiments.

B. Impact of Batch Size

In discrete part manufacturing systems, production rates are also monotonically increasing with respect to buffer capacity. However, such a property may not always hold in batch-discrete lines.

When the total buffer capacity kn is increasing, the production rate may not increase. For example, as shown in Table I, when $p_1 = p_2 = 0.84$, if $k = 2$ and $n = 3$ ($kn = 6$), we obtain $PR = 0.8088$; if the order of k and n are switched, i.e., $k = 3$ and $n = 2$, we have $PR = 0.7851$. When kn is increased to 8, such as $k = 2$ and $n = 4$, PR is increased to 0.8187. But $k = 4$ and $n = 2$ lead to $PR = 0.7879$. If kn is increased to 9, i.e., $k = n = 3$, PR is reduced to 0.8153.

Similar scenarios are observed in other experiments. Such a phenomenon suggests that the total buffer capacity kn is not an independent variable as k and n may lead to opposite directions. To further investigate the impact of batch size, we fix either the number of batches n or the batch size k as constants. Through extensive numerical experiments, we obtain

Observation 2: Under assumptions (i)-(viii), when number of batches $n > 1$ and is a constant, the line production rate PR is monotonically increasing with respect to batch size k. While if batch size k is a constant, PR is monotonically increasing in number of batches n.

When n (respectively k) is fixed, increasing k (respectively n) implies a larger kn, i.e., larger batch size (respectively, more batches) and additional capacity. Thus, the buffer has more possibility to accumulate parts so that the starvation probability will become smaller. Therefore the production rate will increase. As shown in Table I, when $n = 3$, PR approaches 0.8088 and 0.8153 for $k = 2$ and 3, respectively. Similarly, when $k = 3$, PR is increased from 0.7851 to 0.8153 if n changes from 2 to 3.

However, when $n = 1$, PR is monotonically decreasing with respect to k, which is due to that m_1 is always blocked until a batch is completely processed by m_2.

Proposition 6: Under assumptions (i)-(viii) with $n = 1$,

$$\frac{\partial PR}{\partial k} < 0. \tag{20}$$

Proof: See Appendix B. ∎

If we keep the total buffer capacity kn as a constant, then PR may exhibit negative monotonicity with respect to batch size k. When kn is fixed, larger batch size k implies smaller number of batches n, and then higher probability of starvation as batch processing time is longer. From Table I, it is observed that PR decreases from 0.8088 to 0.7851 when k increases from 2 to 3 under condition $kn = 6$, and PR decreases from 0.8187 to 0.7879 when k increases from 2 to 4 under condition $kn = 8$.

Observation 3: Under assumptions (i)-(viii), when kn is a constant, the line production rate PR is monotonically decreasing with respect to batch size k.

In addition, numerical experiments indicate that all above properties still hold in discrete-batch production lines when $n > 1$, i.e., PR is monotonically increasing in k and n when n and k are fixed, respectively, and decreasing in k when kn is fixed. However, when $n = 1$, opposite to batch-discrete lines, PR is increasing in k due to the difference in blockage assumption.

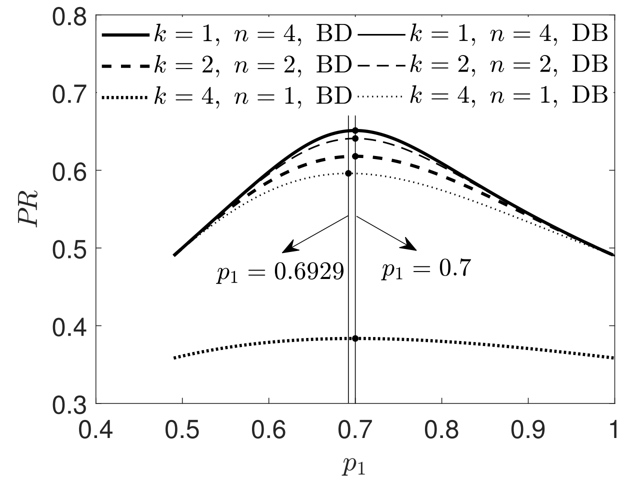


Fig. 7. Production rates comparison.

C. Interchangeability

In batch-discrete lines, if parameters p_1 and p_2 are switched, will the line production rates still be the same? To answer this question, let machine m_1 still be a batch machine but with efficiency p_2, and machine m_2 be a discrete machine with parameter p_1. In this case, exactly the same production rates can be observed. Let PR and PR' be the production rates of lines with parameters $\{p_1, p_2\}$ and $\{p_1', p_2'\}$, where $p_1' = p_2$ and $p_2' = p_1$. Then we obtain

Proposition 7: Under assumptions (i)-(viii), $PR = PR'$ when $n = 1$ and $k = n = 2$.

Proof: See Appendix B. ■

For general cases, numerical experiments are carried out. In all the experiments, we observe

Observation 4: Under assumptions (i)-(viii), the production rates of the original and the interchanged lines, PR and PR', respectively, are the same.

However, in discrete-batch lines, the interchangeability property may not hold anymore. When $n = 1$ and $k = 2$, we obtain

Proposition 8: For discrete-batch lines with $n = 1$ and $k = 2$, if $p_1 > p_2$, then $PR > PR'$, where PR and PR' are production rates of the original line and the interchanged line, respectively.

Proof: See Appendix B. ■

For general cases in discrete-batch lines, the difference between PR and PR' becomes practically diminished when $n > 1$. Within over 52,000 randomly generated numerical experiments, most differences between PR and PR' are in the order of 10^{-6} and the largest one is 0.0002. This may because larger n makes the impact of blockage and starvation due to batching becoming insignificant.

D. Reversibility

Reversibility has been observed in discrete manufacturing lines (see [11], [57]). In batch-discrete lines with machines m_1 and m_2 having parameters p_1 and p_2, reversibility is studied from the perspective of discrete-batch lines with machines m_2 and m_1 having parameters p_2 and p_1. Let PR and PR^r denote the production rates of original and reversed lines, respectively. Then, different with discrete manufacturing lines, reversibility does not hold anymore.

Specifically, when $n = 1$ and $k = 2$, we obtain

Proposition 9: Let PR and PR^r denote the production rates of the original batch-discrete line and the reversed discrete-batch line, respectively. When $n = 1$ and $k = 2$,

$$PR - PR^r = \frac{-2p_1^2 p_2^2[p_2 + \mathcal{C}(1-p_2)]}{(2\mathcal{C} + p_1 p_2)[\mathcal{C}^3 + p_1^2(1-p_2)(p_2+\mathcal{C}) + p_2^2\mathcal{C}]}.$$

Proof: See Appendix B. ■

Therefore, $PR - PR^r < 0$. For other general cases, the following relationship is observed through extensive numerical experiments:

Observation 5: Comparing production rates in a batch-discrete line, PR, and in the reversed discrete-batch line, PR^r, when $k > 1$, we observe

$$PR^r > PR. \tag{21}$$

In addition, when kn is a constant, the difference between PR^r and PR increases with respect to k.

As shown in Figure 7 where $p_1 p_2 = 0.49$, the production rates for batch-discrete (denoted as "BD" and illustrated using thicker lines) and discrete-batch (denoted as "DB" and illustrated using thinner lines) systems are the same when $k = 1$ (solid lines). When $k = 2$ and $k = 4$, the production rates (broken and dot lines, respectively) are higher for discrete-batch systems. In addition, the figure shows that PR increases and then decreases as a function of p_1. Such an observation will be discussed in the next section.

Note that in discrete-batch lines, the discrete machine can keep working until the last one to make buffer full when the batch machine is processing, while in batch-discrete lines, the discrete machine is starved when the buffer is empty, even if the batch machine is working on a batch. This makes a discrete-batch line having a higher production rate than a batch-discrete line.

VI. Improvement Analysis

Continuous improvement has a critical importance to ensure and increase productivity in manufacturing systems. Typically, two types of continuous improvement efforts are available. One is referred to as constrained improvement, i.e., to improve or optimize system performance under given constraints. The other is unconstrained improvement, which is typically referred to as bottleneck analysis [11]. Below, both constrained and unconstrained improvement analyses are introduced.

A. Constrained Improvement

Constraints may exist in many manufacturing systems. For instance, the total workforce may be limited, so that an optimal design of production capacity needs to allocate workforce under the constraint. Similar to improvability analysis introduced in [11], in a batch-discrete line, the workforce constraint can be quantified as

$$p_1 p_2 = p^*, \tag{22}$$

where p^* is a constant.

To investigate when the line production rate can reach its maximum under such a constraint, first, we consider scenarios of $n = 1$. As expected, we obtain

Proposition 10: Under assumptions (i)-(viii) under constraint (22), when $n = 1$, the line production rate PR is maximized if and only if

$$p_1 = p_2 = \sqrt{p^*}. \tag{23}$$

Proof: See Appendix B. ■

Next, when $n > 1$, such a property still holds. Through numerical experiments over 1,700 test cases by selecting parameters from $p^* \in (0.3, 0.9)$ and $kn \in \{2, 4, 6, 8, 10\}$ where $k, n, \in Z^+$, we obtain

Observation 6: Under assumptions (i)-(viii) and constraint (22), when $n > 1$, the line production rate PR is increasing then decreasing with respect to p_1. When the maximum value PR^* is reached, the resulting p_1^* and p_2^* are the same.

From Proposition 7 and Observation 4, when the maximal value PR_{max} is reached at (p_1^*, p_2^*), interchanging p_1^* and p_2^* should also achieve the same PR_{max}. Thus, to ensure the uniqueness of p_1^* and p_2^* to deliver the maximal production rate, $p_1^* = p_2^*$ is needed.

However, such a property may not hold in discrete-batch systems. As shown in Figure 7, the thicker dot curve (batch-discrete line) reaches maximum when $p_1 = 0.7$, while the thinner dot curve (discrete-batch line) achieves maximum when $p_1 > 0.7$, such as $p_1 = 0.7072$ when $k = 4$ and $n = 1$. Similar issues can also be observed from the examples in Table II as well, where p_1^* is slightly larger than 0.7. Such a result is coincident with the conclusion in Proposition 8. As the differences are small, practically, $p_1 = p_2$ can achieve an optimal (or near optimal) design in discrete-batch lines.

TABLE II

DIFFERENCES BETWEEN p_1^* AND p_2^* WHEN $n = 1$

p_1^*	p_2^*	k	PR	$p_1^* - p_2^*$
0.7061	0.6940	2	0.5651	0.0121
0.7072	0.6929	3	0.5832	0.0143
0.7072	0.6929	4	0.5960	0.0143
0.7069	0.6932	5	0.6056	0.0137
0.7065	0.6936	6	0.6130	0.0129
0.7061	0.6940	7	0.6190	0.0121
0.7057	0.6943	8	0.6239	0.0114
0.7054	0.6946	9	0.6280	0.0108
0.7051	0.6949	10	0.6315	0.0102
0.7048	0.6952	11	0.6346	0.0096
0.7046	0.6954	12	0.6373	0.0092

B. Bottleneck Analysis

Bottleneck identification and mitigation have been viewed as the most effective way to improve production system performance (see, for instance, [44], [50], [52], [53], [58]). Here the bottleneck is referred to as the machine whose improvement will lead to the largest improvement in system production rate, the largest $\frac{\partial PR}{\partial p_i}$. It has been shown in [11] that, in discrete two-machine lines, the machine with a lower efficiency, i.e., smaller p_i in Bernoulli lines, is the bottleneck. Such a property is validated in batch-discrete lines with $n = 1$.

Proposition 11: Under assumptions (i)-(viii), machine m_i is the line bottleneck, i.e., $\frac{\partial PR}{\partial p_i} > \frac{\partial PR}{\partial p_j}$, if and only if $p_i < p_j$, $i, j = 1, 2,\ j \neq i$.

Proof: See Appendix B. ■

To investigate the validity of this property in general cases, over 11,000 numerical experiments are carried out with parameters selected randomly and equiprobably from the sets in (18). The results indicate that such a property still holds. In addition, probabilities of blockage and starvation are often used as measurements to identify the bottlenecks without knowing machine parameters. This is also applicable in batch-discrete lines, which can be verified from equation (7) and Observation 7, i.e., when p_1 is small, BL_1 is small; while ST_2 is small when p_2 is small. Thus, the following observation is formulated:

Observation 7: Under assumptions (i)-(viii), machine m_i is the line bottleneck, i.e., $\frac{\Delta PR}{\Delta p_i} > \frac{\Delta PR}{\Delta p_j}$, if and only if $p_i < p_j$, $i, j = 1, 2,\ j \neq i$. In addition, machine m_1 is the line bottleneck if $BL_1 < ST_2$. Otherwise, m_2 is the bottleneck.

For discrete-batch lines, such an identification method is practically valid. By randomly selecting parameters using $p_i \in (0.5, 1)$, $i = 1, 2$, and $kn \in \{2, 4, 6, 8, 9, 10, 12\}$, where $k, n \in Z^+$, the partial derivatives are evaluated using $\frac{\Delta PR}{\Delta p_i}$, where $\Delta p_i = 0.001$, $i = 1, 2$. In all 13,794 test cases with $n > 1$, using smaller p_i can identify the same bottlenecks as calculating $\frac{\Delta PR}{\Delta p_i}$. Among 5082 $n = 1$ test cases, 5026 cases return the same identification by using smaller p_i and larger $\frac{\Delta PR}{\Delta p_i}$. This implies that the identification accuracy is 98.8981%. For the rest 56 test cases where the machine with a larger p_i becomes the bottleneck, the largest differences between p_1 and p_2, and between $\frac{\partial PR}{\partial p_1}$ and $\frac{\partial PR}{\partial p_2}$, are 0.0122 and 0.0229, respectively, which implies that even if the identification is inaccurate, the results are not deviating substantially. This result is also coincident with the conclusion in Proposition 8. Therefore, comparing p_i or BL_1 and ST_2 can be a practical tool to identify line bottlenecks.

Remark 5: The two-machine models presented in this study can be served as building blocks for analysis of longer lines, which will be investigated in future work. Specifically, a single (virtual) machine can be obtained by aggregating two machines. Then such a machine can be aggregated with the next machine into another virtual machine. Such a process can be continued till the last machine in the line. By carrying out such aggregation processes both backward and forward to update virtual machine parameters, upon convergence, the whole production line can be represented by a single machine whose output depicts the performance of the line.

VII. Extensions: Systems With Quality Inspections

A. Inspection Policies

In many batch-discrete manufacturing systems, quality inspection is carried out after the whole process (e.g., the composite part manufacturing described in the case study in Section VIII). Due to the specific feature in batch production, two typical inspection and control strategies are prevailing in such systems.

In Policy 1, the first part in a batch is inspected for defects in batching operation. If the part is in good quality, then the whole batch is assumed with no defects. If the first part quality is degraded, then the whole batch is scraped (or sent to a repair shop). As many quality issues in batch production are due to machine parameter setting, such as pressure, temperature, and processing time, all parts in one batch typically have the same quality issue.

In Policy 2, the same inspection rule is applied but with different scraping (or repairing) strategy, i.e., not only the whole batch, but all intermediate parts will be scraped or repaired, including the parts still in processing (i.e., all parts in buffer b and virtual buffer $\tilde{b}$, and in inspection as well). This is based on the assumption that the same machine parameter setting is adopted and will affect the quality of all parts that have been or are being processed.

To study batch-discrete lines with quality inspections, the state of the system needs to be redefined, to include quality status and batch information, so that the scraping policies can be characterized in the model. Specifically, state (i, j) is extended to state (i, j, q), where $q = 1$ represents there is a part being inspected in the cycle, otherwise $q = 0$. Below, these two policies are modeled and analyzed.

1) Policy 1: Scraping Current Batch: Under this policy, all parts in the current batch will be scraped if the inspected quality is not satisfactory. Then the balance equations can be derived, as equations (A.7) for $k = 1$, (A.8) for $n = 1$ and $k > 1$, and (A.9)-(A.10) for $n > 1$ and $k > 1$, shown in Subsection D in Appendix A.

2) Policy 2: Clearing up All Inventories: To clear up the whole production inventory, state $P_{i,jk-1,1}$, $i = 0, \ldots, k-1$, $j = 1, \ldots, n$ will be transferred to $P_{0,0,0}$ when the product is defective because of machine m_1. Thus, part of balance equations in Policy 1 need to be changed. The balance equations (A.11) for $k = 1$ and (A.12) for $k > 1$ are derived in Subsection E of Appendix A.

B. Production Rate Analysis

1) Production Rate Evaluation: The expressions of PR for the two cases above are same, since a high quality part is

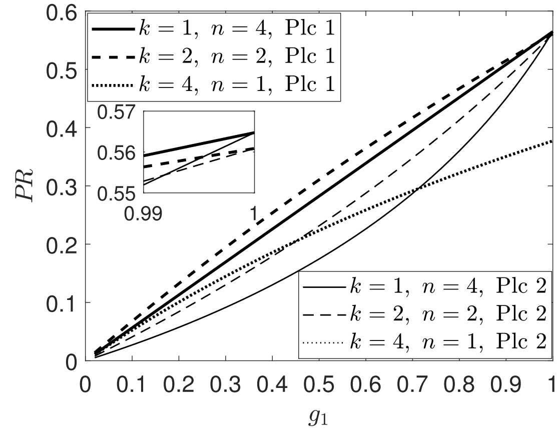


Fig. 8. PR in batch-discrete lines with quality inspection.

produced only when machine m_2 is up and buffer is not empty, and no defects in both operations.

$$PR = \begin{cases} \sum_{j=0}^{n} P_{0,j,1} g_1 g_2, & k = 1, \\ \sum_{i=0}^{k-1} \sum_{j=1}^{n} P_{i,jk-1,1} g_1 g_2 & \\ \quad + \sum_{i=0}^{k-1} \sum_{j=0}^{n-1} \sum_{l=0}^{k-2} P_{i,jk+l,1} g_2, & k > 1, \end{cases} \tag{24}$$

where g_i, $i = 1, 2$, is the yield of machine m_i.

However, as the state transition equations differ between the two policies, the resulting state probabilities $P_{i,j,q}$ will be different, which makes the production rates not the same.

2) Impact of Quality Inspection: As the system production rate is a linear function of g_2, here we ignore g_2 and only focus on investigating the impact of g_1. To do this, the following parameter sets are used for randomly generated experiments:

$$\begin{aligned} p_i &\in (0.5, 1), \\ kn &\in \{4, 6, 8, 10, 12\}, \quad k, n \in Z^+, \\ g_1 &\in \{0.02, 0.04, \ldots, 0.98, 1\}. \end{aligned}$$

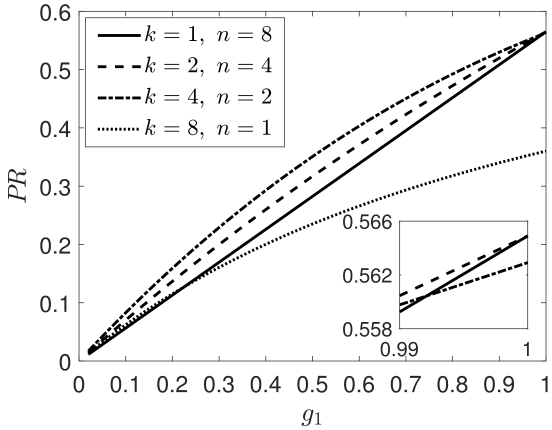
Based on extensive numerical experiments, we observe the following:

Observation 8: The system production rate is monotonically increasing with respect to yield g_1. Moreover,

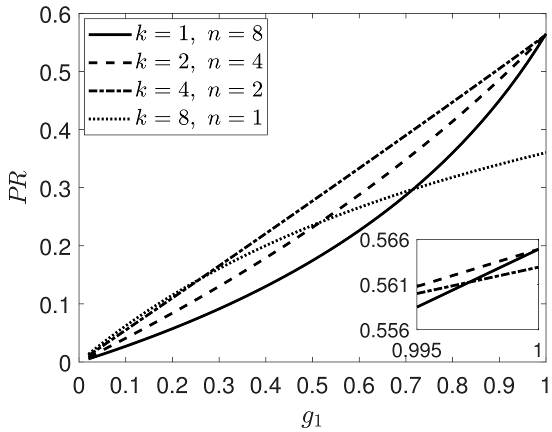
- If Policy 1 (scraping current batch) is adopted, PR is a concave function of g_1.
- If Policy 2 (scraping all inventories) is adopted, PR is a convex function of g_1 when $n > 1$ and a concave function if $n = 1$.

To compare the production rates under these two policies, Figure 8 provides the curves for $kn = 4$, where PR is plotted as a function of g_1 with $p_1 = 0.8842$ and $p_2 = 0.5649$.

As one can see, when $n = 1$, a concave function of g_1 is observed and it is the same under both policies. When $n = 2$ and $n = 4$, thicker lines (Policy 1) are also concave and they are higher than thinner lines (Policy 2) which are convex. When g_1 is close to 1, these curves are very close. To make it clear, the small box in the figure provides a detailed comparison, which indicates that the thicker and thinner lines join when $g_1 = 1$, i.e., the same production rate is obtained when there is no quality issue. Moreover, under $kn = 4$,



(a) Policy 1



(b) Policy 2

Fig. 9. PR as a function of g_1, k, and n.

smaller k results in higher PR when g_1 is close to 1, which coincides with Observation 3.

To further illustrate such observations, the scenarios for $kn = 8$ under Policies 1 and 2 are presented in Figure 9(a) and (b), respectively, where similar results are observed.

VIII. Case Study

A. Process Description

Consider a composite panel production process, where the composite materials are formed into the desired configuration through a plycutter machine, and then stored in a freezer for an extended amount of time to maintain the material's life until the demand is called. Then the parts are pulled, thawed, and layered up, and wait for curing. They will be sent to a composite cure oven to heat up for curing, annealing, drying and hardening synthetic and composite materials to form a solid structure. Next, the parts will be stored in racks, waiting to be cleaned and trimmed into the designed shapes. After trimming, they will be inspected and ready for painting. An illustration of the process is shown in Figure 10.

To model such a process, considering that the plycutting machine operates much faster and the freezer is a shared space with large capacity and long waiting time exists before heating, the oven operation can be decoupled from the upstream, so that the plycutter and freezer can be excluded in the model. Next,

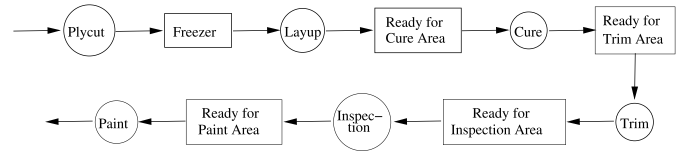


Fig. 10. Illustration of a composite panel production process.

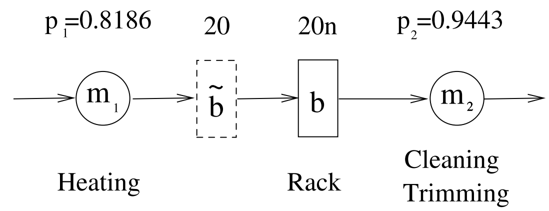


Fig. 11. Simplified composite part production line.

since the cleaning and trimming processes are carried out sequentially and manually by one operator, they can be viewed as one operation. The inspection process is performed by a worker using simple tools, which can be finished quickly with negligible downtimes thus the inspection operation can be ignored in the model. Finally, as a large area is prepared before painting, the downstream painting operation can be decoupled as well. Therefore, we can only focus on heating and trimming operations in the composite panel production process, which can be represented by a two-machine line. Since the cure oven takes 20 parts to heat up each time, it can be modeled as a batch processing machine.

The cycle time of heating operation is 2 hours, and the turnaround time for cleaning and trimming is much shorter, about 5 minutes per piece. Each rack can hold 20 parts. The oven has an average uptime 2500 minutes and average downtime 45 minutes, while the cleaning and trimming operation has an average uptime 1000 minutes and average downtime 59 minutes. Note that although this is a real-world case study, to keep confidentiality of the manufacturing plant, the data have been modified slightly, and are presented for illustration purpose only. However, the nature of the process and data are kept the same.

To study such a system, a Bernoulli batch-discrete model is developed. As it takes 2 hours for the oven to heat 20 parts, it can be viewed as 120 minutes are used to process 20 parts, i.e., $\tau = 6$ minutes per piece. Then the shorter processing time per piece (5 minutes for cleaning and trimming) is used as the cycle time of the line. Using the exponential-Bernoulli transformation introduced in [11] (also see Remark 1 for explanation), we obtain

$$p_1 = \frac{2500}{2500+45} \cdot \frac{5}{6} = 0.8186,$$
$$p_2 = \frac{1000}{1000+59} \cdot \frac{5}{5} = 0.9443.$$

Since the oven loads 20 parts each time, and only two racks after the oven are designed, we have $k = 20$ and $n = 2$. Using these data, we obtain a Bernoulli model of the composite part production line as shown in Figure 11.

B. System Analysis

Using the method introduced in Section IV, the line production rate is calculated as $PR = 0.8175$.

From Observation 7, the heating machine becomes the bottleneck of the system as $p_1 < p_2$. To improve the system, the following potential strategies are proposed:

- *Reducing oven downtime.* Typically the 2 hours cycle time is necessary to heat the parts in the oven. Thus, reducing cycle time is not considered. Now assuming the average downtime of the oven can be reduced from 45 minutes to 30 minutes, we obtain a new $p_1 = 0.8235$. This can lead to $PR = 0.8223$, which improves 0.5838%.
- *Increasing rack size.* If the rack size can be increased to hold more parts, for instance, k is increased from 20 to 22, then the resulting p_1 becomes
$$p_1 = \frac{2500}{2500+45} \cdot \frac{5}{120/22} = 0.9005.$$
This will increase PR from 0.8175 to 0.8942, which is a 9.3752% improvement. Note that in this case, not only p_1 is improved, but also k is increased. Thus more production rate improvement is expected.
- *Increasing number of racks.* If the buffer capacity becomes larger to hold more racks, such as n is increased from 2 to 3, then the line PR is increased to 0.8186, which is a 0.1333% improvement.
- *Combining all strategies.* If all above changes are possible, i.e., average downtime 30 minutes, rack size $k = 22$, number of racks $n = 3$, then $p_1 = 0.9058$, and line PR can be increased to 0.9058, more than 10.7998% improvement, where increasing rack size is the dominate factor for the improvement.

IX. CONCLUSION

In this paper, analytical models of batch-discrete production systems with two Bernoulli machines are studied. In addition to performance evaluation formulae, system properties, such as monotonicity, interchangeability, reversibility, etc., are investigated, and improvement analysis, including constrained improvability and bottleneck identification, are carried out. Moreover, quality inspection and control policies are discussed. Finally, a case study in a composite part production process is presented.

In future work, such studies can be extended in the following directions:

- Extending the study to multi-stage production lines with more than two batch and discrete machines. To reduce computation complexity, decomposition and aggregation approaches can be pursued and the current two-machine models can be used as building blocks.
- Generalizing machine reliability models to other distributions, such as exponential, phase-type, gamma, and arbitrary ones, and approximations will be needed.
- Considering more mixed operation scenarios, such as batch-batch operations, different product types and variable batch sizes.
- Deriving optimal quality inspection and control policies to reduce scrap and increase effective throughput.
- Studying not only the steady state performance, but also transient behavior, and real-time control strategies.
- In addition, time constraints exist for many batch operations, such as washing and mixing, addressing the time dynamics in such lines is of interest.
- As many batch operations may consume substantial energy, such as heating and pressing, designing production lines to optimize system performance under energy constraints becomes important.
- Applying the models on the factory floor in different production settings for more validation and improvement.

The development of this work can provide quantitative tools to study mixed batch and discrete production lines.

APPENDIX A: BALANCE EQUATIONS

A. Batch-Discrete Two-Machine Lines

The balance equations for the batch-discrete two-machine line are presented in expressions (A.1), shown at the bottom of the next page. In addition, as the sum of state probabilities equals to 1, we obtain

$$\sum_{i=0}^{k-1} \sum_{j=0}^{nk} P_{i,j} = 1. \tag{A.2}$$

B. Discrete-Batch Two-Machine Lines

The balance equations for the discrete-batch two-machine lines, i.e., the reversed lines, are presented in expressions (A.3), shown at the bottom of the next page. In addition, we have

$$\sum_{j=0}^{nk} \sum_{i=0}^{k-1} P_{j,i} = 1. \tag{A.4}$$

C. Batch-Batch Two-Machine Lines

Assuming the initial buffer occupancy is zero, the balance equations for a batch-batch two-machine line can be presented in expressions (A.5), shown at the bottom of the page 1579. In addition, the sum of all state probabilities is one.

D. Batch-Discrete Two-Machine Lines With Quality Inspection Policy 1

First, when $k = 1$, the balance equations are presented in expressions (A.7), shown at the bottom of the page 1580. Next, expressions (A.8), shown at the bottom of the page 1580, characterize the balance equations for $n = 1$ and $k > 1$. Finally, when $n > 1$ and $k > 1$, the equations are included in expressions (A.9), shown at the bottom of the page 1581 and (A.10), shown at the bottom of the page 1582.

E. Batch-Discrete Two-Machine Lines With Quality Inspection Policy 2

The state balance equations are shown in expressions (A.11), shown at the bottom of the page 1582 for $k = 1$ and (A.12), shown at the bottom of the page 1582 for $k > 1$.

Batch-discrete line

$$
\begin{aligned}
P_{0,0} &= P_{0,0}(1-p_1) + P_{0,1}(1-p_1)p_2,\\
P_{0,j} &= \begin{cases} P_{0,j}(1-p_2) + P_{0,j+1}p_2, & n=1, \quad j=1,\dots,k-1,\\ P_{0,j}(1-p_1)(1-p_2) + P_{0,j+1}(1-p_1)p_2, & n>1, \quad j=1,\dots,k-1,\end{cases}\\
P_{0,k} &= \begin{cases} P_{0,k}(1-p_2) + P_{k-1,0}p_1, & n=1, \quad k>1,\\ P_{0,k+1}(1-p_1)p_2 + P_{0,k}(1-p_1)(1-p_2) + P_{k-1,0}p_1 + P_{k-1,1}p_1p_2, & n>1,\end{cases}\\
P_{0,j} &= P_{0,j+1}(1-p_1)p_2 + P_{0,j}(1-p_1)(1-p_2) + P_{k-1,j-k}p_1(1-p_2) + P_{k-1,j-k+1}p_1p_2,\\
&\qquad j=k+1,\dots,(n-1)k,\\
P_{0,(n-1)k+j} &= P_{0,(n-1)k+j}(1-p_2) + P_{0,(n-1)k+j+1}p_2 + P_{k-1,(n-2)k+j}p_1(1-p_2)\\
&\quad + P_{k-1,(n-2)k+j+1}p_1p_2, \qquad n>1, \quad j=1,\dots,k-1,\\
P_{0,nk} &= \begin{cases} P_{0,nk}(1-p_2) + P_{0,nk-1}p_1 + P_{0,nk}p_1p_2, & k=n=1,\\ P_{0,nk}(1-p_2) + P_{0,nk-1}p_1(1-p_2) + P_{0,nk}p_1p_2, & k=1, \quad n>1,\\ P_{0,nk}(1-p_2) + P_{k-1,(n-1)k}p_1(1-p_2), & k>1, \quad n>1,\end{cases}\\
P_{i,0} &= P_{i-1,0}p_1 + P_{i,0}(1-p_1) + P_{i,1}(1-p_1)p_2 + P_{i-1,1}p_1p_2, \quad n>1, \quad i=1,\dots,k-1,\\
P_{i,j} &= P_{i-1,j}p_1(1-p_2) + P_{i,j}(1-p_1)(1-p_2) + P_{i-1,j+1}p_1p_2 + P_{i,j+1}(1-p_1)p_2,\\
&\qquad i=1,\dots,k-1, \quad j=1,\dots,(n-1)k-1,\\
P_{1,(n-1)k} &= \begin{cases} P_{1,(n-1)k}(1-p_1) + P_{0,(n-1)k}p_1 + P_{0,(n-1)k+1}p_1p_2, & n=1, \quad k>1,\\ P_{1,(n-1)k}(1-p_1)(1-p_2) + P_{0,(n-1)k}p_1(1-p_2) + P_{0,(n-1)k+1}p_1p_2, & n>1, \quad k>1,\end{cases}\\
P_{i,(n-1)k} &= \begin{cases} P_{i,(n-1)k}(1-p_1) + P_{i-1,(n-1)k}p_1, & n=1, \quad i=2,\dots,k-1,\\ P_{i,(n-1)k}(1-p_1)(1-p_2) + P_{i-1,(n-1)k}p_1(1-p_2), & n>1, \quad i=2,\dots,k-1.\end{cases}
\end{aligned} \tag{A.1}
$$

Discrete-batch line

$$
\begin{aligned}
P_{0,0} &= \begin{cases} P_{0,0}(1-p_1) + P_{1,0}(1-p_1)p_2, & k=1,\\ P_{0,0}(1-p_1) + P_{0,k-1}(1-p_1)p_2, & k>1,\end{cases}\\
P_{j,0} &= P_{j-1,0}p_1 + P_{j,0}(1-p_1) + P_{j,k-1}(1-p_1)p_2 + P_{j-1,k-1}p_1p_2, \quad j=1,\dots,k-1,\\
P_{k,0} &= \begin{cases} P_{k-1,0}p_1 + P_{k,0}(1-p_2) + P_{k,0}p_1p_2, & k=n=1,\\ P_{k-1,0}p_1 + P_{k,0}(1-p_1)(1-p_2) + P_{k+1,0}(1-p_1)p_2 + P_{k,0}p_1p_2, & k=1, \quad n>1,\\ P_{k-1,0}p_1 + P_{k,0}(1-p_2) + P_{k,k-1}p_2 + P_{k-1,k-1}p_1p_2, & k>1, \quad n=1,\\ P_{k-1,0}p_1 + P_{k,0}(1-p_1)(1-p_2) + P_{k,k-1}(1-p_1)p_2 + P_{k-1,k-1}p_1p_2, & k>1, \quad n>1,\end{cases}\\
P_{k+j,0} &= \begin{cases} P_{k+j-1,0}p_1(1-p_2) + P_{k+j,0}(1-p_1)(1-p_2) + P_{k+j,k-1}p_1p_2 + P_{k+j+1,k-1}(1-p_1)p_2, & k=1,\\ P_{k+j-1,0}p_1(1-p_2) + P_{k+j,0}(1-p_1)(1-p_2) + P_{k+j,k-1}(1-p_1)p_2 + P_{k+j-1,k-1}p_1p_2, & k>1,\end{cases}\\
&\qquad j=1,\dots,(n-1)k-1,\\
P_{nk,0} &= \begin{cases} P_{nk,0}(1-p_2) + P_{nk-1,0}p_1(1-p_2) + P_{nk,k-1}p_1p_2, & k=1, \quad n>1,\\ P_{nk,0}(1-p_2) + P_{nk-1,0}p_1(1-p_2) + P_{nk,k-1}p_2 + P_{nk-1,k-1}p_1p_2, & k>1, \quad n>1,\end{cases}\\
P_{0,i} &= \begin{cases} P_{0,1}(1-p_1)(1-p_2) + P_{k,0}(1-p_1)p_2, & i=1, \quad k>1,\\ P_{0,i}(1-p_1)(1-p_2) + P_{0,i-1}(1-p_1)p_2, & i=2,\dots,k-1,\end{cases}\\
P_{j,1} &= \begin{cases} P_{j,1}(1-p_1)(1-p_2) + P_{j-1,1}p_1(1-p_2) + P_{k+j,0}(1-p_1)p_2 & \\ \quad + P_{k+j-1,0}p_1p_2, & k>1, \quad j=1,\dots,(n-1)k-1,\\ P_{(n-1)k,1}(1-p_1)(1-p_2) + P_{(n-1)k-1,1}p_1(1-p_2) & \\ \quad + P_{nk,0}(1-p_1)p_2 + P_{nk-1,0}p_1p_2, & k>1, \quad n>1, \quad j=(n-1)k,\\ P_{(n-1)k+1,1}(1-p_1)(1-p_2) + P_{(n-1)k,1}p_1(1-p_2) + P_{nk,0}p_1p_2, & k>1, \quad j=(n-1)k+1,\\ P_{j,1}(1-p_1)(1-p_2) + P_{j-1,1}p_1(1-p_2), & k>1, \quad j=(n-1)k+2,\dots,nk-1,\\ P_{nk,1}(1-p_2) + P_{nk-1,1}p_1(1-p_2), & j=nk, \quad k>1,\end{cases}\\
P_{j,i} &= \begin{cases} P_{j,i}(1-p_1)(1-p_2) + P_{j-1,i}p_1(1-p_2) + P_{j,i-1}(1-p_1)p_2 + P_{j-1,i-1}p_1p_2, & j=1,\dots,nk-1,\\ P_{nk,i}(1-p_2) + P_{nk-1,i}p_1(1-p_2) + P_{nk,i-1}p_2 + P_{nk-1,i-1}p_1p_2, & j=nk,\end{cases}\\
&\qquad i=2,\dots,k-1.
\end{aligned} \tag{A.3}
$$

Appendix B: Proofs

Proof of Proposition 1: When $n = 1$, the balance equations are simplified to the following:

$$\begin{aligned} P_{0,0} &= P_{0,0}(1-p_1) + P_{0,1}(1-p_1)p_2, \\ P_{0,j} &= P_{0,j}(1-p_2) + P_{0,j+1}p_2, \quad j = 1,\ldots,k-1, \\ P_{0,k} &= P_{0,k}(1-p_2) + P_{k-1,0}p_1, \\ P_{1,0} &= P_{1,0}(1-p_1) + P_{0,0}p_1 + P_{0,1}p_1p_2, \\ P_{i,0} &= P_{i,0}(1-p_1) + P_{i-1,0}p_1, \quad i = 2,\ldots,k-1. \end{aligned}$$

Rewriting these equations, we obtain

$$\begin{aligned} P_{0,1} &= \frac{p_1P_{0,0}}{(1-p_1)p_2}, \\ P_{0,j+1} &= P_{0,j}, \quad j = 1,\ldots,k-1, \\ P_{1,0} &= P_{0,0} + p_2P_{0,1} = \frac{P_{0,0}}{1-p_1}, \\ P_{i,0} &= P_{i-1,0}, \quad i = 2,\ldots,k-1. \end{aligned}$$

Batch-batch line

$$P_{0,j,0} = \begin{cases} P_{0,j,0}(1-p_1) + P_{0,j,k_2-1}(1-p_1)p_2, & j < k_2, \\ P_{0,j,0}(1-p_1)(1-p_2) + P_{0,j,k_2-1}(1-p_1)p_2, & j \ge k_2, \quad \frac{N}{k_1} > 1, \\ P_{0,j,0}(1-p_2) + P_{0,j,k_2-1}p_2, & j \ge k_2, \quad N = k_1, \end{cases}$$
$$j = 0, k_d, \ldots, k_1 - k_d,$$

$$P_{0,j,1} = \begin{cases} P_{0,j,1}(1-p_1)(1-p_2) + P_{0,j+k_2,0}(1-p_1)p_2, & otherwise \\ P_{0,j,1}(1-p_2) + P_{0,j+k_2,0}p_2, & j \ge k_d, \quad N = k_1, \end{cases}$$
$$j = 0, k_d, \ldots, k_1 - k_d,$$

$$P_{0,j,w} = \begin{cases} P_{0,j,w}(1-p_2) + P_{0,j,w-1}p_2, & j \ge k_d, \quad N = k_1, \\ P_{0,j,w}(1-p_1)(1-p_2) + P_{0,j,w-1}(1-p_1)p_2, & otherwise \end{cases}$$
$$j = 0, k_d, \ldots, k_1 - k_d, \quad w = 2, \ldots, k_2 - 1,$$

$$P_{0,j,0} = \begin{cases} P_{0,j,0}(1-p_1) + P_{0,j,k_2-1}(1-p_1)p_2 + P_{k_1-1,j-k_1,0}p_1 + P_{k_1-1,j-k_1,k_2-1}p_1p_2, & j < k_2 \\ P_{0,j,0}(1-p_1)(1-p_2) + P_{0,j,k_2-1}(1-p_1)p_2 + P_{k_1-1,j-k_1,0}p_1 + P_{k_1-1,j-k_1,k_2-1}p_1p_2, & k_2 \le j < k_1 + k_2, \\ P_{0,j,0}(1-p_1)(1-p_2) + P_{0,j,k_2-1}(1-p_1)p_2 + P_{k_1-1,j-k_1,0}p_1(1-p_2) & \\ \quad + P_{k_1-1,j-k_1,k_2-1}p_1p_2, & j \ge k_1 + k_2, \end{cases}$$
$$j = k_1, k_1 + k_d, \ldots, N - k_1,$$

$$P_{0,j,1} = \begin{cases} P_{0,j,1}(1-p_1)(1-p_2) + P_{k_1-1,j-k_1,1}p_1(1-p_2), & j > N - k_2, \\ P_{0,j,1}(1-p_1)(1-p_2) + P_{0,j+k_2,0}(1-p_1)p_2 + P_{k_1-1,j-k_1,1}p_1(1-p_2) + P_{k_1-1,j-k_1+k_2,0}p_1p_2, & otherwise \end{cases}$$
$$j = k_1, k_1 + k_d, \ldots, N - k_1,$$

$$P_{0,j,w} = P_{0,j,w}(1-p_1)(1-p_2) + P_{0,j,w-1}(1-p_1)p_2 + P_{k_1-1,j-k_1,w}p_1(1-p_2) + P_{k_1-1,j-k_1,w-1}p_1p_2, \tag{A.5}$$
$$j = k_1, k_1 + k_d, \ldots, N - k_1, \quad w = 2, \ldots, k_2 - 1,$$

$$P_{0,j,0} = \begin{cases} P_{0,j,0}(1-p_2) + P_{0,j,k_2-1}p_2 + P_{k_1-1,j-k_1,0}p_1 + P_{k_1-1,j-k_1,k_2-1}p_1p_2, & k_1 \le j < k_1 + k_2 \\ P_{0,j,0}(1-p_2) + P_{0,j,k_2-1}p_2 + P_{k_1-1,j-k_1,0}p_1(1-p_2) + P_{k_1-1,j-k_1,k_2-1}p_1p_2, & j \ge k_1 + k_2 \end{cases}$$
$$j = N - k_1 + k_d, \ldots, N,$$

$$P_{0,j,1} = \begin{cases} P_{0,j,1}(1-p_2) + P_{0,j+k_2,0}p_2 + P_{k_1-1,j-k_1,1}p_1(1-p_2) + P_{k_1-1,j-k_1+k_2,0}p_1p_2, & k_1 \le j \le N - k_2 \\ P_{0,j,1}(1-p_2) + P_{k_1-1,j-k_1,1}p_1(1-p_2), & j > N - k_2 \end{cases}$$
$$j = N - k_1 + k_d, \ldots, N,$$

$$P_{0,j,w} = P_{0,j,w}(1-p_2) + P_{0,j,w-1}p_2 + P_{k_1-1,j-k_1,w}p_1(1-p_2) + P_{k_1-1,j-k_1,w-1}p_1p_2,$$
$$j = N - k_1 + k_d, \ldots, N, \quad j \ge k_1, \quad w = 2, \ldots, k_2 - 1,$$

$$P_{i,j,0} = \begin{cases} P_{i,j,0}(1-p_1) + P_{i,j,k_2-1}(1-p_1)p_2 + P_{i-1,j,0}p_1 + P_{i-1,j,k_2-1}p_1p_2, & 0 \le j < k_2 \\ P_{i,j,0}(1-p_1)(1-p_2) + P_{i,j,k_2-1}(1-p_1)p_2 + P_{i-1,j,0}p_1(1-p_2) + P_{i-1,j,k_2-1}p_1p_2, & j \ge k_2 \end{cases}$$
$$i = 1, \ldots, k_1 - 1, \quad j = 0, k_d, \ldots, N - k_1,$$

$$P_{i,j,1} = \begin{cases} P_{i,j,1}(1-p_1)(1-p_2) + P_{i,j+k_2,0}(1-p_1)p_2 + P_{i-1,j,1}p_1(1-p_2) & \\ \quad + P_{i-1,j+k_2,0}p_1p_2, & j \le N - k_1 - k_2 \\ P_{i,j,1}(1-p_1)(1-p_2) + P_{i-1,j,1}p_1(1-p_2) + P_{i-1,j+k_2,0}p_1p_2, & i = 1, \quad N - k_1 < j + k_2 \le N \\ P_{i,j,1}(1-p_1)(1-p_2) + P_{i-1,j,1}p_1(1-p_2), & otherwise \end{cases}$$
$$i = 1, \ldots, k_1 - 1, \quad j = 0, k_d, \ldots, N - k_1,$$

$$P_{i,j,w} = P_{i,j,w}(1-p_1)(1-p_2) + P_{i,j,w-1}(1-p_1)p_2 + P_{i-1,j,w}p_1(1-p_2) + P_{i-1,j,w-1}p_1p_2,$$
$$i = 1, \ldots, k_1 - 1, \quad j = 0, k_d, \ldots, N - k_1, \quad w = 2, \ldots, k_2 - 1.$$

$$\sum_{i=0}^{k_1-1}\sum_{w=0}^{k_2-1}\sum_{j=0}^{N} P_{i,j,w} = 1. \tag{A.6}$$

From equation (5), we have

$$\sum_{i=0}^{k-1}\sum_{j=0}^{k} P_{i,j} = P_{0,0} + kP_{0,1} + (k-1)P_{1,0} = P_{0,0}\frac{k(p_1+p_2)-p_1p_2}{(1-p_1)p_2} = 1.$$

Thus, probability $P_{0,0}$ can be derived.

$$P_{0,0} = \frac{(1-p_1)p_2}{k(p_1+p_2)-p_1p_2}.$$

Then the production rate can be evaluated.

$$PR = p_2\Big(1-\sum_{i=0}^{k-1}P_{i,0}\Big) = \frac{kp_1p_2}{k(p_1+p_2)-p_1p_2}.$$

■

Proof of Proposition 2: Due to space limitation, only the sketch of the proof is presented below. When $k = n = 2$, the balance equations can be simplified as

$$\begin{aligned}
P_{0,0} &= P_{0,0}(1-p_1) + P_{0,1}(1-p_1)p_2,\\
P_{0,1} &= P_{0,1}(1-p_1)(1-p_2) + P_{0,2}(1-p_1)p_2,\\
P_{0,2} &= P_{0,2}(1-p_1)(1-p_2) + P_{0,3}(1-p_1)p_2 + P_{1,0}p_1\\
&\quad + P_{1,1}p_1p_2,\\
P_{0,3} &= P_{0,3}(1-p_2) + P_{0,4}p_2 + P_{1,1}p_1(1-p_2) + P_{1,2}p_1p_2,\\
P_{0,4} &= P_{0,4}(1-p_2) + P_{1,2}p_1(1-p_2),\\
P_{1,0} &= P_{1,0}(1-p_1) + P_{1,1}(1-p_1)p_2 + P_{0,0}p_1 + P_{0,1}p_1p_2,\\
P_{1,1} &= P_{1,1}(1-p_1)(1-p_2) + P_{1,2}(1-p_1)p_2\\
&\quad + P_{0,1}p_1(1-p_2) + P_{0,2}p_1p_2,\\
P_{1,2} &= P_{1,2}(1-p_1)(1-p_2) + P_{0,2}p_1(1-p_2) + P_{0,3}p_1p_2.
\end{aligned}$$

Quality Inspection Policy 1: $k = 1$.

$$\begin{aligned}
P_{0,0,l} &= \begin{cases} P_{0,0,0}(1-p_1) + P_{0,0,1}(1-p_1), & l=0,\\ P_{0,1,0}(1-p_1)p_2 + P_{0,1,1}(1-p_1)p_2, & l=1,\end{cases}\\
P_{0,1,0} &= \begin{cases} P_{0,1,0}(1-p_2) + P_{0,1,1}(1-p_2) + P_{0,0,0}p_1 + P_{0,0,1}p_1, & n=1,\\ P_{0,1,0}(1-p_1)(1-p_2) + P_{0,1,1}(1-p_1)(1-p_2) + P_{0,0,0}p_1 + P_{0,0,1}p_1, & n>1,\end{cases}\\
P_{0,1,1} &= \begin{cases} P_{0,1,0}p_1p_2 + P_{0,1,1}p_1p_2, & n=1,\\ P_{0,2,0}(1-p_1)p_2 + P_{0,2,1}(1-p_1)p_2 + P_{0,1,0}p_1p_2 + P_{0,1,1}p_1p_2, & n>1,\end{cases}\\
P_{0,j,l} &= \begin{cases} P_{0,j,0}(1-p_1)(1-p_2) + P_{0,j,1}(1-p_1)(1-p_2) + P_{0,j-1,0}p_1(1-p_2) + P_{0,j-1,1}p_1(1-p_2), & l=0,\\ P_{0,j+1,0}(1-p_1)p_2 + P_{0,j+1,1}(1-p_1)p_2 + P_{0,j,0}p_1p_2 + P_{0,j,1}p_1p_2, & l=1,\end{cases}\\
&\qquad j = 2,\ldots,n-1,\\
P_{0,n,l} &= \begin{cases} P_{0,n,0}(1-p_2) + P_{0,n,1}(1-p_2) + P_{0,n-1,0}p_1(1-p_2) + P_{0,n-1,1}p_1(1-p_2), & l=0, \quad n>1,\\ P_{0,n,0}p_1p_2 + P_{0,n,1}p_1p_2, & l=1, \quad n>1.\end{cases}
\end{aligned} \tag{A.7}$$

Quality Inspection Policy 1: $n = 1, \quad k > 1$.

$$\begin{aligned}
P_{0,0,0} &= \begin{cases} P_{0,0,0}(1-p_1) + P_{0,0,1}(1-p_1) + P_{0,1,1}[(1-p_1)p_2 + (1-p_2)]\cdot(1-g_1), & k=2,\\ P_{0,0,0}(1-p_1) + P_{0,0,1}(1-p_1) + P_{0,k-1,1}(1-g_1), & k>2,\end{cases}\\
P_{0,0,1} &= \begin{cases} P_{0,1,0}(1-p_1)p_2 + P_{0,1,1}(1-p_1)p_2g_1, & k=2,\\ P_{0,1,0}(1-p_1)p_2 + P_{0,1,1}(1-p_1)p_2, & k>2,\end{cases}\\
P_{0,j,0} &= \begin{cases} P_{0,j,0}(1-p_2) + P_{0,j,1}(1-p_2), & j=1,\ldots,k-2,\\ P_{0,k-1,0}(1-p_2) + P_{0,k-1,1}(1-p_2)g_1, & j=k-1,\end{cases}\\
P_{0,j,1} &= \begin{cases} P_{0,j+1,0}p_2 + P_{0,j+1,1}p_2, & j=1,\ldots,k-3,\\ P_{0,k-1,0}p_2 + P_{0,k-1,1}p_2g_1, & j=k-2,\\ P_{0,k,0}p_2, & j=k-1,\end{cases}\\
P_{0,k,0} &= \begin{cases} P_{0,k,0}(1-p_2) + P_{k-1,0,0}p_1 + P_{1,0,1}p_1, & k=2,\\ P_{0,k,0}(1-p_2) + P_{k-1,0,0}p_1, & k>2,\end{cases}\\
P_{1,0,0} &= \begin{cases} P_{1,0,0}(1-p_1) + P_{1,0,1}(1-p_1) + P_{0,0,0}p_1 + P_{0,0,1}p_1 + P_{0,1,1}p_1p_2(1-g_1), & k=2,\\ P_{1,0,0}(1-p_1) + P_{1,0,1}(1-p_1) + P_{0,0,0}p_1 + P_{0,0,1}p_1, & k>2,\end{cases}\\
P_{1,0,1} &= \begin{cases} P_{0,1,0}p_1p_2 + P_{0,1,1}p_1p_2g_1, & k=2,\\ P_{0,1,0}p_1p_2 + P_{0,1,1}p_1p_2, & k>2,\end{cases}\\
P_{i,0,0} &= \begin{cases} P_{2,0,0}(1-p_1) + P_{1,0,0}p_1 + P_{1,0,1}p_1, & i=2,\\ P_{i,0,0}(1-p_1) + P_{i-1,0,0}p_1, & i=3,\ldots,k-1.\end{cases}
\end{aligned} \tag{A.8}$$

Quality Inspection Policy 1: $n > 1, \quad k > 1,$

$$P_{0,0,l} = \begin{cases} P_{0,0,0}(1-p_1) + P_{0,0,1}(1-p_1) + P_{0,k-1,1}(1-p_1)(1-g_1), & l = 0, \\ P_{0,1,0}(1-p_1)p_2 + P_{0,1,1}(1-p_1)p_2 g_1, & l = 1, \quad k = 2, \\ P_{0,1,0}(1-p_1)p_2 + P_{0,1,1}(1-p_1)p_2, & l = 1, \quad k > 2. \end{cases}$$

$$P_{0,j,0} = \begin{cases} P_{0,j,0}(1-p_1)(1-p_2) + P_{0,j,1}(1-p_1)(1-p_2), & j = 1,\ldots,k-2, \\ P_{0,k-1,0}(1-p_1)(1-p_2) + P_{0,k-1,1}(1-p_1)(1-p_2)g_1, & j = k-1, \end{cases}$$

$$P_{0,j,1} = \begin{cases} P_{0,j+1,0}(1-p_1)p_2 + P_{0,j+1,1}(1-p_1)p_2, & j = 1,\ldots,k-3,k-1, \\ P_{0,k-1,0}(1-p_1)p_2 + P_{0,k-1,1}(1-p_1)p_2 g_1, & j = k-2, \end{cases}$$

$$P_{0,k,0} = \begin{cases} P_{0,k,0}(1-p_1)(1-p_2) + P_{0,k,1}(1-p_1)(1-p_2) + P_{k-1,0,0}p_1 + P_{k-1,0,1}p_1 \\ \quad + P_{k-1,k-1,1}p_1(1-g_1) + P_{0,2k-1,1}[(1-p_1)p_2 + (1-p_2)](1-g_1), & n = k = 2, \\ P_{0,k,0}(1-p_1)(1-p_2) + P_{0,k,1}(1-p_1)(1-p_2) + P_{k-1,0,0}p_1 + P_{k-1,0,1}p_1 \\ \quad + P_{k-1,k-1,1}p_1(1-g_1) + P_{0,2k-1,1}(1-g_1), & n = 2, \quad k > 2, \\ P_{0,k,0}(1-p_1)(1-p_2) + P_{0,k,1}(1-p_1)(1-p_2) + P_{k-1,0,0}p_1 + P_{k-1,0,1}p_1 \\ \quad + P_{k-1,k-1,1}p_1(1-g_1) + P_{0,2k-1,1}(1-p_1)(1-g_1), & n > 2, \end{cases}$$

$$P_{0,k,1} = \begin{cases} P_{0,k+1,0}(1-p_1)p_2 + P_{0,k+1,1}(1-p_1)p_2 g_1 + P_{k-1,1,0}p_1 p_2 + P_{k-1,1,1}p_1 p_2 g_1, & k = 2, \\ P_{0,k+1,0}(1-p_1)p_2 + P_{0,k+1,1}(1-p_1)p_2 + P_{k-1,1,0}p_1 p_2 + P_{k-1,1,1}p_1 p_2, & k > 2, \end{cases}$$

$$P_{0,jk+l,0} = \begin{cases} P_{0,jk+l,0}(1-p_1)(1-p_2) + P_{0,jk+l,1}(1-p_1)(1-p_2) \\ \quad + P_{k-1,(j-1)k+l,0}p_1(1-p_2) + P_{k-1,(j-1)k+l,1}p_1(1-p_2), & j = 1,\ldots,n-2, \quad l = 1,\ldots,k-2, \\ P_{0,(j+1)k-1,0}(1-p_1)(1-p_2) + P_{0,(j+1)k-1,1}(1-p_1)(1-p_2)g_1 \\ \quad + P_{k-1,jk-1,0}p_1(1-p_2) + P_{k-1,jk-1,1}p_1(1-p_2)g_1, & j = 1,\ldots,n-2, \quad l = k-1, \end{cases}$$

$$\begin{aligned} P_{0,jk,0} = {} & P_{0,(j+1)k-1,1}(1-p_1)(1-g_1) + P_{k-1,jk-1,1}p_1(1-g_1) + P_{0,jk,0}(1-p_1)(1-p_2) \\ & + P_{0,jk,1}(1-p_1)(1-p_2) + P_{k-1,(j-1)k,0}p_1(1-p_2) + P_{k-1,(j-1)k,1}p_1(1-p_2), \quad j = 2,\ldots,n-2, \end{aligned}$$

$$P_{0,(n-1)k,0} = \begin{cases} P_{k-1,(n-1)k-1,1}p_1(1-g_1) + P_{0,nk-1,1}[(1-p_1)p_2 + (1-p_2)](1-g_1) + P_{k-1,(n-2)k,0}p_1(1-p_2) \\ \quad + P_{k-1,(n-2)k,1}p_1(1-p_2) + P_{0,(n-1)k,0}(1-p_1)(1-p_2) + P_{0,(n-1)k,1}(1-p_1)(1-p_2), \quad k = 2, \\ P_{0,nk-1,1}(1-g_1) + P_{k-1,(n-1)k-1,1}p_1(1-g_1) + P_{k-1,(n-2)k,0}p_1(1-p_2) + P_{k-1,(n-2)k,1}p_1(1-p_2) \\ \quad + P_{0,(n-1)k,0}(1-p_1)(1-p_2) + P_{0,(n-1)k,1}(1-p_1)(1-p_2), \quad k > 2, \end{cases}$$

$$\begin{aligned} P_{0,jk+l,1} = {} & P_{0,jk+l+1,0}(1-p_1)p_2 + P_{0,jk+l+1,1}(1-p_1)p_2 + P_{k-1,(j-1)k+l+1,0}p_1 p_2 + P_{k-1,(j-1)k+l+1,1}p_1 p_2, \\ & j = 1,\ldots,n-2, \quad l = 1,\ldots,k-3,k-1,k, \end{aligned}$$

$$\begin{aligned} P_{0,jk-2,1} = {} & P_{0,jk-1,0}(1-p_1)p_2 + P_{0,jk-1,1}(1-p_1)p_2 g_1 + P_{k-1,(j-1)k-1,0}p_1 p_2 + P_{k-1,(j-1)k-1,1}p_1 p_2 g_1, \\ & j = 2,\ldots,n-1, \end{aligned}$$

$$P_{0,(n-1)k+j,l} = \begin{cases} P_{0,(n-1)k+j,0}(1-p_2) + P_{0,(n-1)k+j,1}(1-p_2) + P_{k-1,(n-2)k+j,0}p_1(1-p_2) \\ \quad + P_{k-1,(n-2)k+j,1}p_1(1-p_2), & l = 0, \quad j = 1,\ldots,k-2, \\ P_{0,(n-1)k+j+1,0}p_2 + P_{0,(n-1)k+j+1,1}p_2 + P_{k-1,(n-2)k+j+1,0}p_1 p_2 \\ \quad + P_{k-1,(n-2)k+j+1,1}p_1 p_2, & l = 1, \quad j = 1,\ldots,k-3, \end{cases}$$

$$P_{0,nk-1,0} = P_{0,nk-1,0}(1-p_2) + P_{0,nk-1,1}(1-p_2)g_1 + P_{k-1,(n-1)k-1,0}p_1(1-p_2) + P_{k-1,(n-1)k-1,1}p_1(1-p_2)g_1,$$

$$P_{0,nk-2,1} = P_{0,nk-1,0}p_2 + P_{0,nk-1,1}p_2 g_1 + P_{k-1,(n-1)k-1,0}p_1 p_2 + P_{k-1,(n-1)k-1,1}p_1 p_2 g_1,$$

$$P_{0,nk-1,1} = \begin{cases} P_{0,nk,0}p_2 + P_{k-1,(n-1)k,0}p_1 p_2 + P_{k-1,(n-1)k,1}p_1 p_2, & k = 2, \\ P_{0,nk,0}p_2 + P_{k-1,(n-1)k,0}p_1 p_2, & k > 2, \end{cases}$$

$$P_{0,nk,0} = P_{0,nk,0}(1-p_2) + P_{k-1,(n-1)k,0}p_1(1-p_2),$$

$$\begin{aligned} P_{i,0,0} = {} & P_{i,0,0}(1-p_1) + P_{i,0,1}(1-p_1) + P_{i-1,0,0}p_1 + P_{i-1,0,1}p_1 + P_{i,k-1,1}(1-p_1)(1-g_1) \\ & + P_{i-1,k-1,1}p_1(1-g_1), \quad i = 1,\ldots,k-1, \end{aligned}$$

$$P_{i,0,1} = \begin{cases} P_{i,1,0}(1-p_1)p_2 + P_{i-1,1,0}p_1 p_2 + P_{i,1,1}(1-p_1)p_2 g_1 + P_{i-1,1,1}p_1 p_2 g_1, & k = 2, \quad i = 1,\ldots,k-1, \\ P_{i,1,0}(1-p_1)p_2 + P_{i,1,1}(1-p_1)p_2 + P_{i-1,1,0}p_1 p_2 + P_{i-1,1,1}p_1 p_2, & k > 2, \quad i = 1,\ldots,k-1, \end{cases}$$

$$\begin{aligned} P_{i,jk+l,0} = {} & P_{i,jk+l,0}(1-p_1)(1-p_2) + P_{i,jk+l,1}(1-p_1)(1-p_2) + P_{i-1,jk+l,0}p_1(1-p_2) + P_{i-1,jk+l,1}p_1(1-p_2), \\ & i = 1,\ldots,k-1, \quad j = 0,1,\ldots,n-2, \quad l = 1,\ldots,k-2, \end{aligned}$$

$$\begin{aligned} P_{i,jk-1,0} = {} & P_{i,jk-1,0}(1-p_1)(1-p_2) + P_{i,jk-1,1}(1-p_1)(1-p_2)g_1 + P_{i-1,jk-1,0}p_1(1-p_2) \\ & + P_{i-1,jk-1,1}p_1(1-p_2)g_1, \quad i = 1,\ldots,k-1, \; j = 1,\ldots,n-1. \end{aligned} \tag{A.9}$$

From these equations, we can rewrite $P_{0,1}$ and $P_{0,2}$ in terms of $P_{0,0}$, and $P_{1,1}$, $P_{1,2}$, $P_{0,3}$ and $P_{0,4}$ in term of $P_{0,0}$ and $P_{1,0}$. Then the relationship between $P_{0,0}$ and $P_{1,0}$ can be established so that $P_{1,0}$ can also be written in terms of $P_{0,0}$. Thus, denote

Quality Inspection Policy 1: $n > 1, \quad k > 1$ (cont.),

$$\begin{aligned}
P_{i,jk,0} &= P_{i,(j+1)k-1,1}(1-p_1)(1-g_1) + P_{i-1,(j+1)k-1,1}p_1(1-g_1) + P_{i,jk,0}(1-p_1)(1-p_2) + P_{i-1,jk,0}p_1(1-p_2) \\
&\quad + P_{i,jk,1}(1-p_1)(1-p_2) + P_{i-1,jk,1}p_1(1-p_2), \qquad i = 1,\ldots,k-1, \quad j = 1,\ldots,n-2, \\
P_{i,jk-2,1} &= P_{i-1,jk-1,0}p_1p_2 + P_{i-1,jk-1,1}p_1p_2g_1 + P_{i,jk-1,0}(1-p_1)p_2 + P_{i,jk-1,1}(1-p_1)p_2g_1, \\
&\qquad i = 1,\ldots,k-1, \quad j = 1,\ldots,n-1, \\
P_{i,jk+l,1} &= P_{i-1,jk+l+1,0}p_1p_2 + P_{i-1,jk+l+1,1}p_1p_2 + P_{i,jk+l+1,0}(1-p_1)p_2 + P_{i,jk+l+1,1}(1-p_1)p_2, \\
&\qquad i = 1,\ldots,k-1, \quad j = 0,\ldots,n-3, \quad l = 1,\ldots,k-3,k-1,k, \\
&\qquad \text{and} \quad j = n-2, \quad l = 1,\ldots,k-3, \\
&\qquad \text{and} \quad i = 1, \quad j = n-2, \quad l = k-1, \\
P_{i,(n-1)k-1,1} &= \begin{cases} P_{1,(n-1)k,0}p_1p_2 + P_{1,(n-1)k,1}p_1p_2 + P_{2,(n-1)k,0}(1-p_1)p_2, & i = 2, \\ P_{i-1,(n-1)k,0}p_1p_2 + P_{i,(n-1)k,0}(1-p_1)p_2, & i = 3,\ldots,k-1, \end{cases} \\
P_{1,(n-1)k,0} &= \begin{cases} P_{1,(n-1)k,0}(1-p_1)(1-p_2) + P_{1,(n-1)k,1}(1-p_1)(1-p_2) + P_{0,(n-1)k,0}p_1(1-p_2) & \\ \quad + P_{0,(n-1)k,1}p_1(1-p_2) + P_{0,(n-1)k+1,1}p_1p_2(1-g_1), & k = 2, \\ P_{1,(n-1)k,0}(1-p_1)(1-p_2) + P_{1,(n-1)k,1}(1-p_1)(1-p_2) + P_{0,(n-1)k,0}p_1(1-p_2) & \\ \quad + P_{0,(n-1)k,1}p_1(1-p_2), & k > 2 \end{cases} \\
P_{1,(n-1)k,1} &= \begin{cases} P_{0,(n-1)k+1,0}p_1p_2 + P_{0,(n-1)k+1,1}p_1p_2g_1, & k = 2, \\ P_{0,(n-1)k+1,0}p_1p_2 + P_{0,(n-1)k+1,1}p_1p_2, & k > 2, \end{cases} \\
P_{i,(n-1)k,0} &= \begin{cases} P_{2,(n-1)k,0}(1-p_1)(1-p_2) + P_{1,(n-1)k,0}p_1(1-p_2) + P_{1,(n-1)k,1}p_1(1-p_2), & i = 2, \\ P_{i,(n-1)k,0}(1-p_1)(1-p_2) + P_{i-1,(n-1)k,0}p_1(1-p_2), & i = 3,\ldots,k-1. \end{cases}
\end{aligned} \tag{A.10}$$

Quality Inspection Policy 2: $k = 1$.

$$\begin{aligned}
P_{0,0,l} &= \begin{cases} P_{0,0,0}(1-p_1) + P_{0,0,1}(1-p_1)g_1 + \sum_{j=0}^{n} P_{0,j,1}(1-g_1), & l = 0, \\ P_{0,1,0}(1-p_1)p_2 + P_{0,1,1}(1-p_1)p_2g_1, & l = 1, \end{cases} \\
P_{0,1,0} &= \begin{cases} P_{0,1,0}(1-p_2) + P_{0,1,1}(1-p_2)g_1 + P_{0,0,0}p_1 + P_{0,0,1}p_1g_1, & n = 1, \\ P_{0,1,0}(1-p_1)(1-p_2) + P_{0,1,1}(1-p_1)(1-p_2)g_1 + P_{0,0,0}p_1 + P_{0,0,1}p_1g_1, & n > 1, \end{cases} \\
P_{0,1,1} &= \begin{cases} P_{0,1,0}p_1p_2 + P_{0,1,1}p_1p_2g_1, & n = 1, \\ P_{0,2,0}(1-p_1)p_2 + P_{0,2,1}(1-p_1)p_2g_1 + P_{0,1,0}p_1p_2 + P_{0,1,1}p_1p_2g_1, & n > 1, \end{cases} \\
P_{0,j,0} &= P_{0,j,0}(1-p_1)(1-p_2) + P_{0,j,1}(1-p_1)(1-p_2)g_1 + P_{0,j-1,0}p_1(1-p_2) + P_{0,j-1,1}p_1(1-p_2)g_1, \quad j = 2,\ldots,n-1, \\
P_{0,j,1} &= P_{0,j+1,0}(1-p_1)p_2 + P_{0,j+1,1}(1-p_1)p_2g_1 + P_{0,j,0}p_1p_2 + P_{0,j,1}p_1p_2g_1, \quad j = 2,\ldots,n-1, \\
P_{0,n,l} &= \begin{cases} P_{0,n,0}(1-p_2) + P_{0,n,1}(1-p_2)g_1 + P_{0,n-1,0}p_1(1-p_2) + P_{0,n-1,1}p_1(1-p_2)g_1, & l = 0, \quad n > 1, \\ P_{0,n,0}p_1p_2 + P_{0,n,1}p_1p_2g_1, & l = 1, \quad n > 1, \end{cases}
\end{aligned} \tag{A.11}$$

Quality Inspection Policy 2: $k > 1$.

$$\begin{aligned}
P_{0,0,0} &= P_{0,0,0}(1-p_1) + P_{0,0,1}(1-p_1) + \left(\sum_{j=1}^{n} P_{0,jk-1,1} + \sum_{i=1}^{k-1}\sum_{j=1}^{n-1} P_{i,jk-1,1}\right)(1-g_1), \\
P_{0,k,0} &= P_{0,k,0}(1-p_1)(1-p_2) + P_{0,k,1}(1-p_1)(1-p_2) + P_{k-1,0,0}p_1 + P_{k-1,0,1}p_1, \\
P_{0,jk,0} &= P_{0,jk,0}(1-p_1)(1-p_2) + P_{0,jk,1}(1-p_1)(1-p_2) + P_{k-1,(j-1)k,0}p_1(1-p_2) + P_{k-1,(j-1)k,1}p_1(1-p_2), \\
&\qquad j = 2,\ldots,n-2, \\
P_{0,(n-1)k,0} &= P_{k-1,(n-2)k,0}p_1(1-p_2) + P_{k-1,(n-2)k,1}p_1(1-p_2) + P_{0,(n-1)k,0}(1-p_1)(1-p_2) \\
&\quad + P_{0,(n-1)k,1}(1-p_1)(1-p_2), \\
P_{i,0,0} &= P_{i,0,0}(1-p_1) + P_{i,0,1}(1-p_1) + P_{i-1,0,0}p_1 + P_{i-1,0,1}p_1, \quad i = 1,\ldots,k-1, \\
P_{i,jk,0} &= P_{i,jk,0}(1-p_1)(1-p_2) + P_{i,jk,1}(1-p_1)(1-p_2) + P_{i-1,jk,0}p_1(1-p_2) + P_{i-1,jk,1}p_1(1-p_2), \\
&\qquad i = 1,\ldots,k-1, \quad j = 1,\ldots,n-2, \\
P_{1,(n-1)k,0} &= P_{1,(n-1)k,0}(1-p_1)(1-p_2) + P_{1,(n-1)k,1}(1-p_1)(1-p_2) + P_{0,(n-1)k,0}p_1(1-p_2) + P_{0,(n-1)k,1}p_1(1-p_2).
\end{aligned} \tag{A.12}$$

$\alpha_1 = \frac{p_1}{p_2}$ and $\beta_1 = \frac{1}{1-p_1}$, and we obtain

$$\begin{aligned}
P_{0,1} &= \alpha_1\beta_1 P_{0,0},\\
P_{0,2} &= \alpha_1\beta_1(1+\alpha_1\beta_1)P_{0,0},\\
P_{1,0} &= \beta_1\left[1+\frac{2\alpha_1\beta_1(1+\alpha_1\beta_1)-p_1\beta_1}{(1+\alpha_1\beta_1)^2+p_1\beta_1^2}\right]P_{0,0},\\
P_{1,1} &= \alpha_1\beta_1^3\frac{2\alpha_1(1+\alpha_1\beta_1)-p_1}{(1+\alpha_1\beta_1)^2+p_1\beta_1^2}P_{0,0},\\
P_{1,2} &= \alpha_1\beta_1^3\frac{\alpha_1(1+\alpha_1\beta_1)^2-p_1(1+\alpha_1\beta_1+\alpha_1\beta_1^2)}{(1+\alpha_1\beta_1)^2+p_1\beta_1^2}P_{0,0},\\
P_{0,3} &= \alpha_1\beta_1\left[1+\alpha_1\beta_1-\beta_1\frac{1+\alpha_1\beta_1+2\alpha_1\beta_1^2}{(1+\alpha_1\beta_1)^2+p_1\beta_1^2}\right]\\
&\quad\cdot(1+\alpha_1\beta_1)P_{0,0},\\
P_{0,4} &= \alpha_1\beta_1\Bigg[(1+\alpha_1\beta_1)^2+\alpha_1\beta_1^2-\beta_1(1+\alpha_1)(1+\alpha_1\beta_1)\\
&\quad\cdot\frac{1+\alpha_1\beta_1+2\alpha_1\beta_1^2}{(1+\alpha_1\beta_1)^2+p_1\beta_1^2}\Bigg]P_{0,0}.
\end{aligned}$$

From $\sum_{i=0}^{k-1}\sum_{j=0}^{nk} P_{i,j} = 1$, $P_{0,0}$ can be derived using notation α, β, $\mathcal{A}$ and $\mathcal{B}$ in (10).

$$P_{0,0} = \frac{1}{\mathcal{A}-\mathcal{B}}.$$

Then all other $P_{i,j}$ can be obtained. Finally, we have

$$\begin{aligned}
PR &= p_2(1-P_{0,0}-P_{1,0})\\
&= p_2 - \frac{p_2}{\mathcal{A}-\mathcal{B}}\left(\frac{2-p_1}{1-p_1}+\frac{p_1[2(1+\alpha)-p_2]}{p_2(p_1+\beta^2)}\right).
\end{aligned}$$

■

Proof of Proposition 3: When $n = 1$ and $k = 2$, the balance equations are simplified as

$$\begin{aligned}
P_{0,0} &= P_{0,0}(1-p_1) + P_{0,1}(1-p_1)p_2,\\
P_{1,0} &= P_{1,0}(1-p_1) + P_{0,0}p_1 + P_{1,1}(1-p_1)p_2 + P_{0,1}p_1p_2,\\
P_{2,0} &= P_{1,0}p_1 + P_{2,0}(1-p_2) + P_{2,1}p_2 + P_{1,1}p_1p_2,\\
P_{0,1} &= P_{0,1}(1-p_1)(1-p_2) + P_{2,0}(1-p_1)p_2,\\
P_{1,1} &= P_{1,1}(1-p_1)(1-p_2) + P_{0,1}p_1(1-p_2) + P_{2,0}p_1p_2,\\
P_{2,1} &= P_{2,1}(1-p_2) + P_{1,1}p_1(1-p_2).
\end{aligned}$$

Using these equations, we can rewrite $P_{0,1}$, $P_{2,0}$, $P_{1,1}$, $P_{2,1}$ and $P_{1,0}$ in terms of $P_{0,0}$.

$$\begin{aligned}
P_{0,1} &= P_{0,0}\frac{p_1}{p_2(1-p_1)},\\
P_{2,0} &= P_{0,0}\frac{p_1(p_1+p_2-p_1p_2)}{p_2^2(1-p_1)^2},\\
P_{1,1} &= P_{0,0}\frac{p_1^2}{p_2(1-p_1)^2(p_1+p_2-p_1p_2)},\\
P_{2,1} &= P_{0,0}\frac{p_1^3(1-p_2)}{p_2^2(1-p_1)^2(p_1+p_2-p_1p_2)},\\
P_{1,0} &= P_{0,0}\left[\frac{1}{1-p_1}+\frac{p_1}{(1-p_1)(p_1+p_2-p_1p_2)}\right].
\end{aligned}$$

From $\sum P_{i,j} = 1$, let $\mathcal{C} = p_1 + p_2 - p_1p_2$, we derive $P_{0,0}$ as

$$P_{0,0} = \frac{p_2^2(1-p_1)^2\mathcal{C}}{\mathcal{C}^3 + p_2^2(1-p_1)(p_1+\mathcal{C}) + p_1^2\mathcal{C}}.$$

Then PR can be obtained

$$\begin{aligned}
PR &= p_2 - p_2(P_{0,0}+P_{1,0})\\
&= \frac{2p_1p_2\mathcal{C}^2}{\mathcal{C}^3 + p_2^2(1-p_1)(p_1+\mathcal{C}) + p_1^2\mathcal{C}}.
\end{aligned}$$

■

Proof of Proposition 4:

$$\begin{aligned}
\frac{\partial PR}{\partial p_i} &= \frac{kp_j}{k(p_i+p_j)-p_ip_j} - \frac{kp_ip_j(k-p_j)}{[k(p_i+p_j)-p_ip_j]^2}\\
&= \frac{k^2p_j^2}{[k(p_i+p_j)-p_ip_j]^2} > 0, \quad \forall i,j = 1,2, \quad j \neq i.
\end{aligned}$$

■

Proof of Proposition 5:

$$\begin{aligned}
\frac{\partial PR}{\partial p_1} &= \frac{2p_2}{\left[\mathcal{C}^3 + p_2^2(1-p_1)(p_1+\mathcal{C}) + p_1^2\mathcal{C}\right]^2}\Big(\big[2p_1(1-p_2)\mathcal{C}\\
&\quad +\mathcal{C}^2\big]\cdot\left[\mathcal{C}^3 + p_2^2(1-p_1)(p_1+\mathcal{C}) + p_1^2\mathcal{C}\right]\\
&\quad -p_1\mathcal{C}^2\big[3(1-p_2)\mathcal{C}^2 - p_2^2(\mathcal{C}+p_1)\\
&\quad +p_2^2(1-p_1)(2-p_2) + 2p_1\mathcal{C} + p_1^2(1-p_2)\big]\Big)\\
&= \frac{2p_2\mathcal{C}T_1}{\left[\mathcal{C}^3 + p_2^2(1-p_1)(p_1+\mathcal{C}) + p_1^2\mathcal{C}\right]^2}
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= \left[\mathcal{C} + 2p_1(1-p_2)\right]\left[\mathcal{C}^3 + p_2^2(1-p_1)(p_1+\mathcal{C}) + p_1^2\mathcal{C}\right]\\
&\quad -p_1\mathcal{C}\big[3(1-p_2)\mathcal{C}^2 - p_2^2(\mathcal{C}+p_1) + p_2^2(1-p_1)(2-p_2)\\
&\quad +2p_1\mathcal{C} + p_1^2(1-p_2)\big].
\end{aligned}$$

Through algebraic reorganization, T_1 can be simplified to

$$T_1 = 2p_2^2(1-p_1)\mathcal{C}^2 + 2p_1p_2^2\mathcal{C} + 2p_1^2p_2^2(1-\mathcal{C}) > 0.$$

Thus $\frac{\partial PR}{\partial p_1} > 0$. Using similar derivations, we have $\frac{\partial PR}{\partial p_2} > 0$. Therefore, PR is increasing in p_1 and p_2. ■

Proof of Proposition 6:

$$\begin{aligned}
\frac{\partial PR}{\partial k} &= \frac{p_1p_2[k(p_1+p_2)-p_1p_2] - kp_1p_2(p_1+p_2)}{[k(p_1+p_2)-p_1p_2]^2}\\
&= -\frac{p_1^2p_2^2}{[k(p_1+p_2)-p_1p_2]^2} < 0.
\end{aligned}$$

■

Proof of Proposition 7: When $n = 1$, from Proposition 1, it follows directly that

$$PR' = \frac{kp_1'p_2'}{k(p_1'+p_2')-p_1'p_2'} = \frac{kp_2p_1}{k(p_2+p_1)-p_2p_1} = PR.$$

When $n = k = 2$, using Symbolic Toolbox in Matlab, we obtain

$$PR - PR' = 0.$$

■

Proof of Proposition 8: From the formulae of PR and PR',

$$\begin{aligned}
PR &= \frac{2p_1p_2\mathcal{C}^2}{\mathcal{C}^3 + p_2^2(1-p_1)(p_1+\mathcal{C}) + p_1^2\mathcal{C}},\\
PR' &= \frac{2p_1p_2\mathcal{C}^2}{\mathcal{C}^3 + p_1^2(1-p_2)(p_2+\mathcal{C}) + p_2^2\mathcal{C}}.
\end{aligned}$$

Let V and V' denote the denominators in PR and PR', respectively, we obtain

$$\begin{aligned}
V - V' &= \mathcal{C}^3 + p_2^2(1-p_1)(p_1+\mathcal{C}) + p_1^2\mathcal{C} - \mathcal{C}^3\\
&\quad -p_1^2(1-p_2)(p_2+\mathcal{C}) - p_2^2\mathcal{C}\\
&= p_1p_2(p_2-p_1)(1-\mathcal{C}).
\end{aligned}$$

It follows that

$$PR - PR' = \frac{2p_1^2 p_2^2 \mathcal{C}^2 (p_1 - p_2)(1 - \mathcal{C})}{VV'}.$$

Thus, $PR > PR'$ when $p_1 > p_2$. ■

Proof of Proposition 9: Let $p_1^r = p_2$ and $p_2^r = p_1$. When $n = 1$ and $k = 2$, PR and PR^r are derived as follows:

$$PR = \frac{2p_1 p_2}{2(p_1 + p_2) - p_1 p_2} = \frac{2p_1 p_2 \mathcal{C}^2}{2\mathcal{C}^3 + p_1 p_2 \mathcal{C}^2},$$

$$PR^r = \frac{2p_1 p_2 \mathcal{C}^2}{\mathcal{C}^3 + p_1^2(1 - p_2)(p_2 + \mathcal{C}) + p_2^2 \mathcal{C}}.$$

Let U and U^r denote the denominators in PR and PR^r, respectively, then we have

$$\begin{aligned} U - U^r &= 2\mathcal{C}^3 + p_1 p_2 \mathcal{C}^2 - \mathcal{C}^3 - p_1^2(1 - p_2)(p_2 + \mathcal{C}) - p_2^2 \mathcal{C} \\ &= p_1 p_2 \mathcal{C} + p_1 p_2^2 (1 - \mathcal{C}) > 0. \end{aligned}$$

It follows that

$$PR - PR^r = -\frac{2p_1^2 p_2^2 \mathcal{C}^2 [p_2 + \mathcal{C}(1 - p_2)]}{U \cdot U^r}.$$

■

Proof of Proposition 10: Under constraint (22),

$$PR = \frac{kp_1 p_2}{k(p_1 + p_2) - p_1 p_2} = \frac{kp^*}{k(p_1 + \frac{p^*}{p_1}) - p^*}.$$

Thus, maximizing PR is equivalent to minimizing $p_1 + \frac{p^*}{p_1}$. From

$$\frac{\partial \left(p_1 + \frac{p^*}{p_1}\right)}{\partial p_1} = 1 - \frac{p^*}{p_1^2} = 0,$$

we obtain

$$p_1 = \sqrt{p^*} = p_2.$$

■

Proof of Proposition 11: From the proof of Proposition 4, we have

$$\begin{aligned} \frac{\partial PR}{\partial p_i} - \frac{\partial PR}{\partial p_j} &= \frac{k^2 p_j^2}{[k(p_i + p_j) - p_i p_j]^2} - \frac{k^2 p_i^2}{[k(p_i + p_j) - p_i p_j]^2} \\ &= \frac{k^2 (p_j + p_i)(p_j - p_i)}{[k(p_i + p_j) - p_i p_j]^2}. \end{aligned}$$

Thus, $\frac{\partial PR}{\partial p_i} > \frac{\partial PR}{\partial p_j}$ if and only if $p_j > p_i$. ■

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