Chapter 1

Time to Level of Brownian motion

1.1 Arrivals of batches

Assuming that the in-flow from the previous section in the production obeys the following SDE

$$dS_t = rdt + \sigma dB_t$$

I.e. Brownian motion with drift. And assuming that every time the accumulated mass hits a level l, the batch is ready to be processed by the next step, we wish to first find the distribution for these times. Note that the above model allows for negative flow and thus also negative accumulated mass. However, for $\sigma << r$ this becomes very unlikely as

$$\mathbb{P}\left(S_t \leq 0\right) = \Phi\left(\frac{-r\sqrt{t}}{\sigma}\right)$$

and thus only for small t this is probable, as otherwise it is dominated by $\frac{r}{\sigma}$

which is large and thus the probability very low.

Furthermore, if one allows periods without inflow, the running maximum could be a good model. Either way, the probability distribution for between batch times is the same.

To derive the distribution for the between batch times, T, we shall use the Girsanov Theorem as well as the joint distribution of the maximum of a standard Brownian motion and is running maximum. Thus, let B_t be a standard Brownian motion, and M_t the running maximum defined as

$$M_t := \sup_{s\in[0,t]} \{B_s\}$$

- Udledning af joint fordeling mellem M og B
- Change of measure for at opnå med drift og sigma
- Marginal fordeling for M

1.1.1 Joint distribution of Brownian and its running maximum

To derive the joint density of a standard Brownian motion and its running maximum, consider the following probability

$$\mathbb{P}\left(M_t \geq m, B_t \leq w\right)$$

Let T_m be defined as the first time B_t hits the level m, i.e. $T_m := \inf_t \left(B_t = m\right)$. Then $M_t \geq m \iff T_m \leq t$. Thus, the above probability is reexpressed as

$$\mathbb{P}\left(M_t \geq m, B_t \leq w\right) = \mathbb{P}\left(T_m \leq t, B_t \leq w\right)$$

To proceed, we use the principle of reflection which is admissible due to B_t being a martingale. In particular, we define $\tilde{B}_t$ as follows

$$\tilde{B}_t := \begin{cases} B_t & t \leq T_m \\ 2m - B_t & t > T_m \end{cases}$$

It follows that $\tilde{B}_t$ is also a standard Brownian motion. By the definition of $\tilde{B}_t$, we then have that

$$\mathbb{P}\left(T_m \leq t, B_t \leq w\right) = \mathbb{P}\left(T_m \leq t, 2m - w \leq \tilde{B}_t\right)$$

Notice that the original expression is only sensible for $m \geq w$ as $w > m$ is a contradiction to the definition of M_t. Thus, $2m - w \geq m$ hence $\tilde{B}_t \geq 2m - w$ implies that the original Brownian motion B_t has hit the level m and thus $T_m \leq t$. This means that

$$\mathbb{P}\left(T_m \leq t, 2m - w \leq \tilde{B}_t\right) = \mathbb{P}\left(2m - w \leq \tilde{B}_t\right) = 1 - \Phi\left(\frac{2m - w}{\sqrt{t}}\right)$$

Thus, in total we have found that

$$\mathbb{P}\left(M_t \geq m, B_t \leq w\right) = 1 - \Phi\left(\frac{2m - w}{\sqrt{t}}\right)$$

And thus, the joint distribution is obtained by differentiation

$$\begin{aligned} f_{M_t,B_t}(m, w) &= \frac{\partial^2}{\partial m\, \partial w} \mathbb{P}\left(M_t \leq m, B_t \leq w\right) \\ &= \frac{\partial^2}{\partial m\, \partial w} \left(\mathbb{P}\left(B_t \leq w\right) - \mathbb{P}\left(M_t \geq m, B_t \leq w\right)\right) \\ &= \frac{\partial^2}{\partial m\, \partial w} \Phi\left(\frac{2m - w}{\sqrt{t}}\right) \\ &= \frac{2(2m - w)}{t^{3/2}} \phi\left(\frac{2m - w}{\sqrt{t}}\right) \qquad , m \leq w,\ m \geq 0 \end{aligned}$$

Note:

Now, define instead $\tilde{B}_t = \sigma B_t$. We then find a similar expression for the joint density of ... and its running maximum. Namely, as

$$\mathbb{P}\left(\tilde{M}_t \geq m, \tilde{B}_t \leq w\right) = \mathbb{P}\left(\sigma M_t \geq m, \sigma B_t \leq w\right)$$

Same formula, but with m and w divided by σ

1.1.2 Joint distribution with drift and arbitrary variance

Let B_t be a standard Brownian motion defined on the probability space, $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, define $\tilde{B}_t$ to be a Brownian motion with drift as follows

$$\tilde{B}_t := \tilde{\mu} t + B_t$$

To derive the joint density $f_{\tilde{M}_t, \tilde{B}_t}(m, w)$ on measure $\mathbb{P}$, we use a corollary of the Girsanov theorem. Namely, suppose B_t is Brownian motion under measure $\mathbb{P}$, then there exists a measure $\mathbb{Q}$ such that $\tilde{B}_t = B_t - \langle B, X \rangle_t$ is a Brownian motion (without drift) under this new measure given that X_t is an adapted process. Furthermore, as $\tilde{B}_t$ is a martingale, the Radon-Nikodym derivative is equal to the stochastic exponential $Z_t = \exp\left(X_t - \frac{1}{2}\langle X \rangle_t\right)$.

Now, if X_t is of the form $\int_0^t Y_s \, dB_s$ where $\mathbb{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T Y_s^2 \, ds\right)\right] < \infty$, a special case, the Cameron-Martin-Girsanov implies that $\tilde{B}_t = B_t - \int_0^t Y_s \, ds$ is then a $\mathbb{Q}$ Brownian motion. This can easily be shown when Y_s fulfills Noviko's condition, then Z_t is a martingale and the Girsanov theorem applies as clearly X_t is also adapted to B_t. Then, from the above corollary,

$$\begin{aligned}
\tilde{B}_t &= B_t - \langle B, X \rangle_t \\
&= B_t - \lim_{||P|| \to 0} \sum_i \left(B_{t_{i+1}} - B_{t_i}\right) \left(\int_{t_i}^{t_{i+1}} Y_s \, dB_s\right) \\
&= B_t - \lim_{||P|| \to 0} \sum_i \left(B_{t_{i+1}} - B_{t_i}\right)^2 Y_{t_i^*} \\
&= B_t - \int_0^t Y_s \, ds
\end{aligned}$$

As it has now been shown that there exists as measure $\mathbb{Q}$ under which $\tilde{B}_t$ is a Brownian motion as choosing $Y_s = -\tilde{mu}$ we reproduce the initial definition of $\tilde{B}_t$. To then derive the joint distribution of $\tilde{B}_t$ and its running maximum $\tilde{M}_t$, we compute the Radon-Nikodym derivative, Z_t, hence given by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t = \exp\left(\int_0^t Y_s\, dB_s - \frac{1}{2}\int_0^t Y_s^2\, ds\right)$$
$$= \exp\left(-\tilde{\mu}\int_0^t dB_s - \frac{1}{2}\tilde{\mu}^2\int_0^t ds\right)$$
$$= \exp\left(-\tilde{\mu}B_t - \frac{1}{2}\tilde{\mu}^2 t\right)$$
$$= \exp\left(-\tilde{\mu}\tilde{B}_t + \frac{1}{2}\tilde{\mu}^2 t\right)$$

With the above derivative, we have that

$$\mathbb{Q}\,(A) = \int_A Z_t\, d\mathbb{P}$$

And thus also

$$\mathbb{P}\,(A) = \int_A Z_t^{-1}\, d\mathbb{Q}$$

as $Z_t : X \to (0,\infty)$. It then simply follows that

$$f_{\tilde{M}_t,\tilde{B}_t}(m,w) = \tilde{f}_{\tilde{M}_t,\tilde{B}_t}(m,w)e^{\tilde{\mu}w - \frac{1}{2}\tilde{\mu}^2 t}$$

where $\tilde{f}$ is the probability distribution under measure $\mathbb{Q}$. Hence,

$$f_{\tilde{M}_t,\tilde{B}_t}(m,w) = \frac{2(2m-w)}{t^{3/2}}e^{\tilde{\mu}w - \frac{1}{2}\tilde{\mu}^2 t}\phi\left(\frac{2m-w}{\sqrt{t}}\right)$$

To introduce the standard deviation σ, first define $\tilde{\mu} = \mu/\sigma$ and $\hat{B}_t = \sigma\tilde{B}_t$. Then, $\hat{B}_t$ is also a Brownian with drift, μ, but with variance $\sigma^2 t$. Furthermore, the joint distribution is

$$f_{\hat{M}_t,\hat{B}_t}(m,w) = \frac{2(2m-w)}{\sigma^3 t^{3/2}}e^{\frac{1}{\sigma^2}\left(\mu w - \frac{1}{2}\mu^2 t\right)}\phi\left(\frac{2m-w}{\sigma\sqrt{t}}\right)$$

1.1.3 Distribution of maximum of Brownian motion with drift

The distribution of the running maximum $\hat{M}_t$ is given by the marginal of the above, namely

$$f_{\hat{M}_t}(m) = \int_{-\infty}^{m} f_{\hat{M}_t,\hat{B}_t}(m,w)\,dw$$

Integration by parts admits

$$f_{\hat{M}_t}(m) = \frac{2}{\sigma\sqrt{t}}\phi\left(\frac{m-\mu t}{\sigma\sqrt{t}}\right) - \frac{2\mu}{\sigma^2}e^{\frac{2m\mu}{\sigma^2}}\Phi\left(-\frac{m+\mu t}{\sigma\sqrt{t}}\right)$$

1.1.4 Cumulative distribution of maximum

As we shall later need the survival function of $\hat{M}_t$, we first compute the cumulative distribution. Namely

$$\mathbb{P}\left(\hat{M}_t \leq m\right) = \int_0^m \int_{-\infty}^{\eta} f_{\hat{M}_t,\hat{B}_t}(\eta,w)\,dw\,d\eta$$

To compute the above, we split the inner integral over the line $w=0$ in the η, w plane and reformulate

$$\mathbb{P}\left(\hat{M}_t \leq m\right) = \underbrace{\int_0^m \int_w^m f_{\hat{M}_t,\hat{B}_t}(\eta,w)\,d\eta\,dw}_{I_1} + \underbrace{\int_{-\infty}^0 \int_0^m f_{\hat{M}_t,\hat{B}_t}(\eta,w)\,d\eta\,dw}_{I_2}$$

The antiderivative of $f_{\hat{M}_t,\hat{B}_t}(m,w)$ w.r.t. m is simple and calculated to be

$$\int f_{\hat{M}_t,\hat{B}_t}(m,w)\,dm = -\frac{1}{\sigma\sqrt{2\pi t}}e^{\frac{1}{\sigma^2}\left(\mu w-\frac{1}{2}\mu^2 t\right)}e^{-\frac{1}{2}\left(\frac{2m-w}{\sigma\sqrt{t}}\right)^2}$$

The first of the above integrals, I_1, is then

$$I_1 = -\frac{1}{\sigma\sqrt{2\pi t}}e^{-\frac{1}{2\sigma^2}\mu^2 t}\int_0^m e^{\frac{\mu w}{\sigma^2}-\frac{1}{2}\left(\frac{2m-w}{\sigma\sqrt{t}}\right)^2} - e^{\frac{\mu w}{\sigma^2}-\frac{1}{2}\left(\frac{w}{\sigma\sqrt{t}}\right)^2}\,dw$$

And similar for the second integral I_2

$$I_2 = -\frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{1}{2\sigma^2}\mu^2 t} \int_{-\infty}^{0} e^{\frac{\mu w}{\sigma^2} - \frac{1}{2}\left(\frac{2m-w}{\sigma\sqrt{t}}\right)^2} - e^{\frac{\mu w}{\sigma^2} - \frac{1}{2}\left(\frac{w}{\sigma\sqrt{t}}\right)^2} dw$$

It is observed that the integrands are the same, thus

$$\mathbb{P}\left(\hat{M}_t \le m\right) = -\frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{1}{2\sigma^2}\mu^2 t} \int_{-\infty}^{m} e^{\frac{\mu w}{\sigma^2} - \frac{1}{2}\left(\frac{2m-w}{\sigma\sqrt{t}}\right)^2} - e^{\frac{\mu w}{\sigma^2} - \frac{1}{2}\left(\frac{w}{\sigma\sqrt{t}}\right)^2} dw$$

From simple substitution, and a few calculations, on gets that

$$\mathbb{P}\left(\hat{M}_t \le m\right) = \Phi\left(\frac{m-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2m\mu}{\sigma^2}} \Phi\left(-\frac{m+\mu t}{\sigma\sqrt{t}}\right)$$

1.1.5 Distribution of time to level

As $\mathbb{P}(M_t \ge l) = \mathbb{P}(T_l \le t)$. It thus follows that $f_{T_l}(t) = \frac{d}{dt}\mathbb{P}(M_t \ge l)$ which is easily calculated from the above. Namely

$$\begin{aligned} f_{T_l}(t) &= \frac{d}{dt}\left(1 - \mathbb{P}(M_t \le l)\right) \\ &= -\frac{d}{dt}\left(\Phi\left(\frac{l-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2l\mu}{\sigma^2}} \Phi\left(-\frac{l+\mu t}{\sigma\sqrt{t}}\right)\right) \\ &= \frac{\mu t + l}{2\sigma t^{3/2}} \phi\left(\frac{l-\mu t}{\sigma\sqrt{t}}\right) + \frac{l-\mu t}{2\sigma t^{3/2}} e^{\frac{2\mu l}{\sigma^2}} \phi\left(-\frac{\mu t + l}{\sigma\sqrt{t}}\right) \end{aligned}$$

Note that although the distribution above is parameterized by 3 parameters, it can be completely specified by $\tilde{\mu} = \mu/\sigma$ and $\tilde{l} = l/\sigma$ which is clear also from the following

Let $Z_t = \mu t + \sigma B_t$ and similarly $\tilde{Z}_t = Z_t/\sigma = \tilde{\mu} + B_t$. Then $\mathbb{P}(T_l \le t) = \mathbb{P}(M_t \ge l) = \mathbb{P}\left(\tilde{M}_t \ge \tilde{l}\right) = \mathbb{P}\left(\tilde{T}_{\tilde{l}} \le t\right)$ where $\tilde{M}_t$ and $\tilde{T}_{\tilde{l}}$ are the running maximum and time to level of $\tilde{Z}_t$. Thus, equivalent to a probability of non-scaled Brownian motion with drift.

To verify the above probability distribution, a Monte-Carlo simulation is carried out for 100.000 simulations with parameters $l = 10$, $\mu = 0.1$, $\sigma = 0.5$. As the shape resembles a gamma distribution, a simple fit, matching the mean and variance is also plotted. Although the gamma family of probability distributions is also a two-parameter family, they do not quite overlap as can be seen in the following plot.

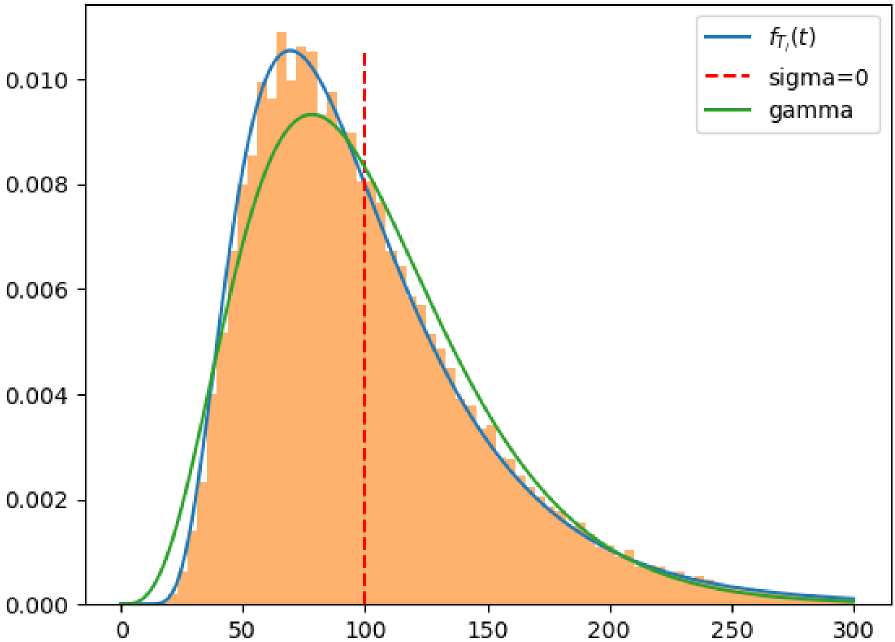

Figure 1.1: Example of simulation and actual distribution. The marked $\sigma = 0$ shows the limit as $\sigma \to 0$ corresponding to no noise on the input flow

1.1.6 MGF

$$
\begin{aligned}
\mathbb{E}\left[e^{\theta T_l}\right] &= \int_0^\infty e^{\theta t} f_{T_l}(t)\, dt \\
&= \underbrace{\int_0^\infty e^{\theta t} \frac{\mu t + l}{2\sigma t^{3/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{l-\mu t}{\sigma\sqrt{t}}\right)^2} dt}_{I_1} + e^{\frac{2\mu l}{\sigma^2}} \underbrace{\int_0^\infty e^{\theta t} \frac{l - \mu t}{2\sigma t^{3/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(-\frac{\mu t + l}{\sigma\sqrt{t}}\right)^2} dt}_{I_2}
\end{aligned}
$$

We shall only consider the first integral I_1 as the second follows directly from the result of the first by substituting μ with $-\mu$

$$\begin{aligned}
I_1 &= \int_0^\infty \frac{\mu t + l}{2\sigma t^{3/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(l-\mu t)^2 - 2\theta\sigma^2 t^2}{\sigma^2 t}}\, dt \\
&= \int_0^\infty \frac{\mu t + l}{2\sigma t^{3/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(\mu^2 - 2\theta\sigma^2)t^2 - 2l\mu t + l^2}{\sigma^2 t}}\, dt \\
&= e^{\frac{l\mu - l\sqrt{\mu^2 - 2\theta\sigma^2}}{\sigma^2}} \int_0^\infty \frac{\mu t + l}{2\sigma t^{3/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\sqrt{\mu^2 - 2\theta\sigma^2}t - l}{\sigma\sqrt{t}}\right)^2}\, dt \\
&= e^{\frac{l\mu - l\sqrt{\mu^2 - 2\theta\sigma^2}}{\sigma^2}} \int_0^\infty \left(\frac{\sqrt{\mu^2 - 2\theta\sigma^2}t + l}{2\sigma t^{3/2}} + \frac{\mu - \sqrt{\mu^2 - 2\theta\sigma^2}}{2\sigma\sqrt{t}} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\sqrt{\mu^2 - 2\theta\sigma^2}t - l}{\sigma\sqrt{t}}\right)^2}\, dt
\end{aligned}$$

Once again, we split the integral, now as follows

$$\begin{aligned}
I_1 = e^{\frac{l\mu - l\sqrt{\mu^2 - 2\theta\sigma^2}}{\sigma^2}} \Bigg(& \underbrace{\int_0^\infty \frac{\sqrt{\mu^2 - 2\theta\sigma^2}t + l}{2\sigma t^{3/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\sqrt{\mu^2 - 2\theta\sigma^2}t - l}{\sigma\sqrt{t}}\right)^2}\, dt}_{I_{11}} \\
& + \frac{\mu - \sqrt{\mu^2 - 2\theta\sigma^2}}{\sigma} \underbrace{\int_0^\infty \frac{1}{2\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\sqrt{\mu^2 - 2\theta\sigma^2}t - l}{\sigma\sqrt{t}}\right)^2}\, dt}_{I_{12}} \Bigg)
\end{aligned}$$

For the first integral, the substitution $u = \frac{\sqrt{\mu^2 - 2\theta\sigma^2}t - l}{\sigma\sqrt{t}}$ reveals that

$$I_{11} = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du = 1$$

As for the second integral I_{12} it can be rewritten as

$$I_{12} = e^{\frac{l\sqrt{\mu^2 - 2\theta\sigma^2}}{\sigma^2}} \int_0^\infty \frac{1}{2\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{\mu^2}{2\sigma^2} - \theta\right)t - \frac{l^2}{2\sigma^2}t^{-1}}\, dt$$

Then, substituting $u = \sqrt{\frac{\mu^2}{2\sigma^2} - \theta}\sqrt{t}$

$$I_{12} = e^{\frac{l\sqrt{\mu^2 - 2\theta\sigma^2}}{\sigma^2}} \frac{1}{\sqrt{2\pi}\sqrt{\frac{\mu^2}{2\sigma^2} - \theta}} \int_0^\infty e^{-u^2 - \frac{l^2}{2\sigma^2}\left(\frac{\mu^2}{2\sigma^2} - \theta\right)u^{-2}}\, du$$

To solve the above integral, we consider the following family of integrals, parameterized by s

$$I(s) = \int_0^\infty e^{-u^2 - s^2 u^{-2}}\, du$$

It follows that

$$I'(s) = -2 \int_0^\infty e^{-u^2 - s^2 u^{-2}} s u^{-2}\, du$$

letting $z = s/u$, it follows that $dz = -s/u^2 du$ and (assuming $s > 0$)

$$I'(s) = -2 \int_0^\infty e^{-s^2 z^{-2} - z^2}\, dz = -2I(s)$$

Also,

$$I(0) = \int_0^\infty e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$$

Hence

$$I(s) = \frac{\sqrt{\pi}}{2} e^{-2s} \quad \text{, for } s \geq 0$$

Note that due to symmetry, $I(s) = I(-s)$, and hence

$$I(s) = \frac{\sqrt{\pi}}{2} e^{-2|s|}$$

Thus, letting $s = \frac{l}{2\sigma}\sqrt{\frac{\mu^2}{\sigma^2} - 2\theta}$ i.e. resulting in the integral from I_{12}, the integral I_{12} is simply

$$\begin{aligned} I_{12} &= e^{\frac{l\sqrt{\mu^2 - 2\theta\sigma^2}}{\sigma^2}} \frac{1}{\sqrt{2\pi}\sqrt{\frac{\mu^2}{2\sigma^2} - \theta}} \frac{\sqrt{\pi}}{2} e^{-2\frac{l}{2\sigma}\sqrt{\frac{\mu^2}{\sigma^2} - 2\theta}} \\ &= \frac{\sigma}{2\sqrt{\mu^2 - 2\theta\sigma^2}} \end{aligned}$$

Combining the above results, we finally have I_1

$$
\begin{aligned}
I_1 &= e^{\frac{l\mu - l\sqrt{\mu^2-2\theta\sigma^2}}{\sigma^2}} \left(1 + \frac{\mu - \sqrt{\mu^2-2\theta\sigma^2}}{\sigma}\frac{\sigma}{2\sqrt{\mu^2-2\theta\sigma^2}}\right) \\
&= e^{\frac{l\mu - l\sqrt{\mu^2-2\theta\sigma^2}}{\sigma^2}} \left(\frac{1}{2} + \frac{\mu}{2\sqrt{\mu^2-2\theta\sigma^2}}\right)
\end{aligned}
$$

Similar calculations results in the integral I_2 or simply by letting $\mu = -\mu$ as discussed before. In total, we find the moment generating function to be

$$
\begin{aligned}
\mathbb{E}\left[e^{\theta T_l}\right] &= e^{\frac{l\mu - l\sqrt{\mu^2-2\theta\sigma^2}}{\sigma^2}} \left(\frac{1}{2} + \frac{\mu}{2\sqrt{\mu^2-2\theta\sigma^2}}\right) \\
&\quad + e^{\frac{2\mu l}{\sigma^2}} e^{\frac{-l\mu - l\sqrt{\mu^2-2\theta\sigma^2}}{\sigma^2}} \left(\frac{1}{2} - \frac{\mu}{2\sqrt{\mu^2-2\theta\sigma^2}}\right) \\
&= e^{\frac{l}{\sigma^2}\left(\mu - \sqrt{\mu^2-2\theta\sigma^2}\right)}
\end{aligned}
$$

From the above calculations, this is clearly defined for θ in some neighborhood of 0, thus the above is indeed a proper moment generating function. Furthermore, all derivatives exists at $\theta = 0$.

This also shows that T_l does not belong to neither the Gamma family nor Phase-Type

The first 3 moments are given by

$$
\mathbb{E}\left[T_l\right] = \frac{l}{\mu}, \quad \mathbb{E}\left[T_l^2\right] = \frac{l\sigma^2}{\mu^3} + \frac{l^2}{\mu^2}, \quad \mathbb{E}\left[T_l^3\right] = \frac{3l\sigma^4}{\mu^5} + \frac{3l^2\sigma^2}{\mu^4} + \frac{l^3}{\mu^3}
$$

Interestingly, the average is exactly what one would expect if no stochasticity was present. Furthermore, the variance has a simple nice form, namely $\frac{l\sigma^2}{\mu^3}$.

Continuing the simulation from above, the theoretical mean evaluates to 100 whereas the simulated mean evaluated to 100.343. The theoretical variance is 2500 whereas the variance from simulation was 2468.224.