Jonas Conneryd
conneryd@kth.se

Homework 2
MM8042 Algebraic Topology

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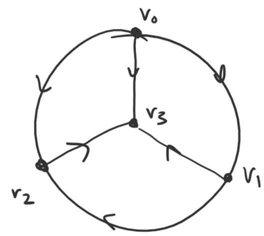
Problem 1. (W) Let D be the unit disk in the complex plane, and let X be the quotient space of D obtained from the relation $z \sim z \cdot \exp(2\pi i/3)$ for every z on the boundary of D. Find a Δ-structure on X (there exists one with 2 zero-simplices, 4 one-simplices, and 3 two-simplices) and use it to calculate the Δ-homology of X with this structure.

Solution. Certainly all homology groups $H_n^\Delta(X)$ with $n \geq 3$ are trivial since there are no simplices of degree 3 or higher in the given Δ-structure. The calculations for the other homology groups are provided below. Please excuse my handwriting.



$$[v_0] \sim [v_1] \sim [v_2]$$
$$[v_0v_1] \sim [v_1v_2] \sim -[v_0v_2]$$

$$\partial_2[v_0v_1v_3] = [v_1v_3] - [v_0v_3] + [v_0v_1]$$
$$\partial_2[v_1v_2v_3] = [v_2v_3] - [v_1v_3] + [v_1v_2] \sim [v_0v_1]$$
$$\partial_2[v_0v_2v_3] = [v_2v_3] - [v_0v_3] + [v_0v_2] \quad -[v_0v_1]$$

$$\partial_1[v_0v_1] = \partial_2[v_1v_2] = \partial_2[v_0v_2] = [v_0] - [v_0] = 0$$
$$\partial_1[v_0v_3] = \partial_2[v_1v_3] = \partial_2[v_2v_3] = [v_3] - [v_0]$$
$$\partial_0[v_0] = \partial_0[v_1] = \partial_0[v_2] = \partial_0[v_3] = 0$$

$$H_1^\Delta(X) = \ker\partial_1/\operatorname{im}\partial_2 = \langle \underset{}{\overset{a}{[v_0v_1]}}, \overset{b}{[v_2v_3] - [v_0v_3]}, \underset{c}{[v_2v_3] - [v_1v_3]} \mid \overset{c}{[v_2v_3] - [v_0v_3]} + \overset{a}{[v_0v_1]}, \overset{b-c}{[v_1v_3] - [v_0v_3]} + \overset{a}{[v_0v_1]}, \underset{b}{[v_2v_3] - [v_0v_3]} - \underset{a}{[v_0v_1]} \rangle$$

$$\Rightarrow a - b = a + c = a + b - c$$
$$\Rightarrow a = b = -c,\ 3a = 0$$

$$\Rightarrow H_1^\Delta(X) = \langle a \mid 3a \rangle \cong \mathbb{Z}/3\mathbb{Z}.$$

$$H_0^\Delta(X) = \ker\partial_0/\operatorname{im}\partial_1$$
$$= \{[v_0] \sim [v_1] \sim [v_2]\} = \langle [v_0], [v_3] \mid [v_3] - [v_0] \rangle \cong \mathbb{Z}.$$

Noting that ∂_2 is injective which means $H_2(X)$ is trivial, we summarize:

- $H_n^\Delta(X) = 0$ for $n \geq 3$.
- $H_2^\Delta(X) = 0$.
- $H_1^\Delta(X) = \mathbb{Z}/3\mathbb{Z}$.
- $H_0^\Delta(X) = \mathbb{Z}$.

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Problem 2. **(W)** For a finitely generated abelian group A, the *rank* of A is the maximal rank of a free abelian subgroup of A.

(a) Suppose we have a short exact sequence of abelian groups

$$0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$$

(this means that i is injective, q is surjective, and $\mathsf{Im}(i) = \mathsf{Ker}(q)$). Prove that $\mathsf{Rank}(B) = \mathsf{Rank}(A) + \mathsf{Rank}(C)$.

(b) Prove that the Euler characteristic of a finite Δ-complex X equals

$$\sum_{i=0}^{\infty} (-1)^i \mathsf{Rank}(H_i^{\Delta}(X)).$$

Hint: Observe that there exist short exact sequences

$$0 \to \mathsf{Im}(\partial_{n+1}) \xrightarrow{i} \mathsf{Ker}(\partial_n) \xrightarrow{q} H_n(X) \to 0$$

and

$$0 \to \mathsf{Ker}(\partial_n) \xrightarrow{i} C_n^{\Delta}(X) \xrightarrow{q} \mathsf{Im}(\partial_n) \to 0.$$

Solution.

(a) Let F_A be a maximal free subgroup of A, generated by $\{f_\alpha \mid \alpha \in \mathscr{A}\}$, for some index set $\mathscr{A}$. By injectivity of i, each of these basis elements f_α are mapped to distinct elements $i(f_\alpha)$. Since A, B, C are abelian, each word $\sum_{\alpha \in \mathscr{A}} f_\alpha^{k_\alpha} \in F_A$ is mapped to

$$i\left(\sum_{\alpha \in \mathscr{A}} f_\alpha^{k_\alpha}\right) = \sum_{\alpha \in \mathscr{A}} i(f_\alpha)^{k_\alpha},$$

so $\mathsf{Im}_i(F_A)$ is a free abelian subgroup of B generated by $\{i(f_\alpha) \mid \alpha \in \mathscr{A}\}$ with rank equal to F_A. By exactness, $\mathsf{Im}(i) = \mathsf{Ker}(q)$ so $\mathsf{Im}(F_A) < \mathsf{Ker}(q)$. The First Isomorphism Theorem implies that $B/\mathsf{Ker}(q) \cong \mathsf{Im}(q) \iff B/\mathsf{Im}(i) \cong \mathsf{Im}(q) = C$, since q is surjective. Take a maximal free subgroup $F_C < C$. Then this is also a maximal free subgroup F_C' of $B/\mathsf{Im}(i)$, which extends to a free subgroup of B by forming the subgroup consisting of words formed by one representative per basis element of F_C' from the preimage of the quotient homomorphism. By abuse of notation, let this group also be denoted by F_C'.

Now, the group F_B generated by the generators of $\mathsf{Im}(F_A)$ together with the generators of F_C' is a free subgroup of B of rank $R = \mathsf{Rank}(A) + \mathsf{Rank}(C)$, since $\mathsf{Rank}(\mathsf{Im}(F_A)) = \mathsf{Rank}(F_A) = \mathsf{Rank}(A)$, $\mathsf{Rank}(F_C') = \mathsf{Rank}(F_C) = \mathsf{Rank}(C)$ and F_C' and $\mathsf{Im}(F_A)$ are mutually disjoint. The restriction of F_B to $\mathsf{Ker}(q) = \mathsf{Im}(i)$ is isomorphic to F_A and the quotient of F_B by $\mathsf{Ker}(q)$ is isomorphic to F_C. The claim is that this group is maximally free in B; suppose on the contrary that there is some free group $F_B' < B$ of higher rank than F_B. Since $B/\mathsf{Ker}(q) \cong C$ and F_C is maximally free in C, we cannot have that the image of F_B' under the quotient homomorphism by $\mathsf{Ker}(q)$ has higher rank than F_C. Hence, the only possibility is that the restriction of F_B' to $\mathsf{Ker}(q)$ has higher rank than F_A, but this is also impossible since F_A is maximally free in A. Hence F_B is maximally free in B with rank $R = \mathsf{Rank}(A) + \mathsf{Rank}(C)$, and so $\mathsf{Rank}(B) = \mathsf{Rank}(A) + \mathsf{Rank}(C)$, as desired.

(b) Recall that the Euler characteristic is

$$\chi(X) = \sum_n (-1)^n c_n,$$

where c_n is the number of n-cells (which in our case will be n-simplices) of X. By definition, c_n is equal to $\mathsf{Rank}(C_n^{\Delta}(X))$.

The existence of a short exact sequence

$$0 \to \mathsf{Im}(\partial_{n+1}) \xrightarrow{i} \mathsf{Ker}(\partial_n) \xrightarrow{q} H_n(X) \to 0.$$

follows from the fact that $\mathsf{Ker}(\partial_n)/\mathsf{Im}(\partial_{n+1}) \cong H_n(X)$ by definition. Moreover, certainly $C_n^{\Delta}(X)/\mathsf{Ker}(\partial_n) \cong \mathsf{Im}(\partial_n)$ by the first isomorphism theorem since ∂_n is a homomorphism, which implies the existence of a short exact sequence

$$0 \to \mathsf{Ker}(\partial_n) \xrightarrow{i} C_n^{\Delta}(X) \xrightarrow{q} \mathsf{Im}(\partial_n) \to 0.$$

By part **(a)**, we have that

$$\mathsf{Rank}(C_n^{\Delta}(X)) = \mathsf{Rank}(\mathsf{Ker}(\partial_n)) + \mathsf{Rank}(\mathsf{Im}(\partial_n))$$

while

$$\mathsf{Rank}(\mathsf{Ker}(\partial_n)) = \mathsf{Rank}(\mathsf{Im}(\partial_{n+1})) + \mathsf{Rank}(H_n(X)),$$

so

$$\mathsf{Rank}(C_n^{\Delta}(X)) = \mathsf{Rank}(\mathsf{Im}(\partial_{n+1})) + \mathsf{Rank}(H_n(X)) + \mathsf{Rank}(\mathsf{Im}(\partial_n)).$$

Plugging into the formula for the Euler characteristic we find

$$\begin{aligned}
\chi(X) = \sum_n (-1)^n c_n &= \sum_{n=0}^{\infty} (-1)^n \mathsf{Rank}(C_n^{\Delta}(X)) \\
&= \sum_{n=0}^{\infty} (-1)^n \left[\mathsf{Rank}(\mathsf{Im}(\partial_{n+1})) + \mathsf{Rank}(H_n(X)) + \mathsf{Rank}(\mathsf{Im}(\partial_n))\right] \\
&= \sum_{n=0}^{\infty} (-1)^n \mathsf{Rank}(H_n(X)) + \sum_{n=0}^{\infty} (-1)^n \mathsf{Rank}(\mathsf{Im}(\partial_{n+1})) + \sum_{n=0}^{\infty} (-1)^n \mathsf{Rank}(\mathsf{Im}(\partial_n)) \\
&= \mathsf{Rank}(\mathsf{Im}(\partial_0)) + \sum_{n=0}^{\infty} (-1)^n \mathsf{Rank}(H_n(X)) + \sum_{n=1}^{\infty} (-1)^n \left[-\mathsf{Rank}(\mathsf{Im}(\partial_n)) + \mathsf{Rank}(\mathsf{Im}(\partial_n))\right] \\
&= \{\text{By finiteness, the second sum is } 0\} \\
&= \mathsf{Rank}(\mathsf{Im}(\partial_0)) + \sum_{n=0}^{\infty} (-1)^n \mathsf{Rank}(H_n(X)) \\
&= \sum_{n=0}^{\infty} (-1)^n \mathsf{Rank}(H_n(X)),
\end{aligned}$$

since the image of ∂_0 is the trivial group.

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