PROBLEM 1. Suppose we have homomorphisms of abelian groups $A \xrightarrow{i} B \xrightarrow{q} C$. Prove that the following are equivalent:

(A) The sequence $0 \to A \xrightarrow{i} B \xrightarrow{q} C$ is exact and the homomorphism q has a section (i.e. there is a homomorphism $s : C \to B$ such that $q \circ s$ is the identity map on C).

(B) The sequence $A \xrightarrow{i} B \xrightarrow{q} C \to 0$ is exact, and the homomorphism i has a retraction (i.e. there is a homomorphism $r : B \to A$ such that $r \circ i$ is the identity on A).

(C) The sequence $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$ is exact, and moreover it is isomorphic to the sequence $0 \xrightarrow{\iota} A \to A \oplus C \xrightarrow{p} C \to 0$, where the homomorphisms in the latter sequence are the canonical inclusion of the direct summand A and the canonical proection onto the direct summand C.

REMARK 1. A sequence satisfying these conditions is called a *split* exact sequence.

A question for you to ponder in your free time: Which of these implications hold if the groups are *not* assumed to be abelian?

Solution. We begin by proving that **(C)** implies **(A)** and **(B)**. Assume **(C)**. Then we have short exact sequences $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$ and $0 \to A \xrightarrow{\iota} A \oplus C \xrightarrow{p} C \to 0$ where we have an isomorphism $f : B \to A \oplus C$ such that $f \circ i = \iota$ and $q \circ f^{-1} = p$.

Certainly the sequences $0 \to A \xrightarrow{i} B \xrightarrow{q} C$ and $A \xrightarrow{i} B \xrightarrow{q} C \to 0$ are exact if the above sequences are exact. Let s' be the inclusion of C into $A \oplus C$. Then $p \circ s' = \mathsf{Id}_C$, since p was defined to be the projection of $A \oplus C$ onto C. But $p = q \circ f^{-1}$ so $\mathsf{Id}_C = p \circ s' = q \circ f^{-1} \circ s'$, so we have a homomorphism $s := f^{-1} \circ s'$ such that $q \circ s = \mathsf{Id}_C$. Similarly, let r' be the projection of $A \oplus C$ onto A. Then $r' \circ \iota = \mathsf{Id}_A$. But $\iota = f \circ i$ so with $r = r' \circ f$ we have a homomorphism $r : B \to A$ such that $r \circ i = \mathsf{Id}_A$.

Next, we show that **(A)** implies **(C)**, and **(B)** implies **(C)**. Assume **(A)**. We show q is surjective, which will imply exactness of $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$. Let $c \in C$. Then $c = \mathsf{Id}_C(c) = q(s(c))$, so $c \in \mathsf{Im}(q)$. Hence q is surjective which gives the desired exact sequence. Now, instead assume **(B)**. We show i is injective which implies exactness of $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$. Let $a_1, a_2 \in A$ and suppose $i(a_1) = i(a_2)$. Then $r(i(a_1)) = r(i(a_2)) = \mathsf{Id}_C(a_1) = \mathsf{Id}_C(a_2)$, so $a_1 = a_2$. Hence i is injective as desired.

We will invoke the famous five-lemma to complete the proofs. Assume **(A)**. Then there is a homomorphism $s : C \to B$ such that $q \circ s = \mathsf{Id}_C$. The map $f : A \oplus C \to B; (a, c) \mapsto i(a) + s(c)$ is then clearly well-defined, and a homomorphism since both i and s are homomorphisms: $f(a + a', c + c') = i(a + a') + s(c + c') = i(a) + i(a') + s(c) + s(c') = f(a, c) + f(a', c')$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\mathsf{Id}_0"] & A \arrow[r, "\iota"] \arrow[d, "\mathsf{Id}_A"] & A \oplus C \arrow[r, "p"] \arrow[d, "f"] & C \arrow[r] \arrow[d, "\mathsf{Id}_C"] & 0 \arrow[d, "\mathsf{Id}_0"] \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "q"] & C \arrow[r] & 0
\end{tikzcd}$$

Certainly the leftmost and rightmost squares commute; since $\mathsf{Id}_0, \mathsf{Id}_A, \mathsf{Id}_C$ are isomorphisms, it remains to show that the middle left and middle right squares commute for the five-lemma to guarantee that f is an isomorphism. We have that $f(\iota(a)) = f(a, 0) = i(a) = i(\mathsf{Id}_A(a))$ and $q(f(a, c)) = q(i(a) + s(c)) = q(i(a)) + \mathsf{Id}_C(c) = c$, since $i(a) \in \mathsf{Ker}(q)$ by exactness. Hence the diagram commutes, so f is an isomorphism by the five-lemma. This also implies that f^{-1} is defined and is an isomorphism. Going the path $A \xrightarrow{i} B \xrightarrow{f^{-1}} A \oplus C$ in the diagram, we see that $f^{-1} \circ i = \iota$, and going the path $A \oplus C \xrightarrow{f} B \xrightarrow{q} C$, we get that $q \circ f = p$. Therefore we also have an isomorphism of sequences.

Now, assume **(B)**. Then there is a homomorphism $r : B \to A$ such that $r \circ i$ is the identity of A. Then $g : B \to A \oplus C; b \mapsto (r(b), q(b))$ is well-defined, and a homomorphism since both r and q are well-defined homomorphisms: $g(b + b') = (r(b + b'), q(b + b')) = (r(b), q(b)) + (r(b'), q((b')) = g(b) + g(b')$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\mathsf{Id}_0"] & A \arrow[r, "i"] \arrow[d, "\mathsf{Id}_A"] & B \arrow[r, "q"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "\mathsf{Id}_C"] & 0 \arrow[d, "\mathsf{Id}_0"] \\
0 \arrow[r] & A \arrow[r, "\iota"] & A \oplus C \arrow[r, "p"] & C \arrow[r] & 0
\end{tikzcd}$$

As above, we have to show that it commutes for the five-lemma to imply that g is an isomorphism, and as above we need only check the two middle squares. We have that $g(i(a)) = (r(i(a)), q(i(a))) = (a, 0) = \iota(a) = \iota(\mathsf{Id}_A(a))$ where we have used exactness at B. Moreover $p(g(b)) = p(r(b), q(b)) = q(b) = \mathsf{Id}_C(q(b))$. Hence the diagram commutes, and by the five-lemma g is an isomorphism. Going the path $A \xrightarrow{i} B \xrightarrow{g} A \oplus C$ we see that $g \circ i = \iota$ and going the path $A \oplus C \xrightarrow{g^{-1}} B \xrightarrow{q} C$ we see that $q \circ g^{-1} = p$ and so we have an isomorphism of sequences in this case as well. This proves that **(A)** $\implies$ **(C)** and **(B)** $\implies$ **(C)**, which concludes the proof. ■

PROBLEM 2. Prove the following variation on the theme of excision. Suppose $X = X_1 \cup X_2$, $X_0 = X_1 \cap X_2$, where X_1, X_2 are *closed* subsets of X. Suppose that for $i = 1, 2$ there exists an open subset in the subspace topology $U_i \subset X_i$ containing X_0, U_i strong deformation retract onto X_0. Prove that the homology groups of X_0, X_1, X_2 and X fit in a Mayer-Vietoris sequence. You may assume the homotopy invariance of singular homology, and the Mayer-Vietoris sequence for decompositions into unions of open sets.

Solution. Since X_1, X_2 are closed, their complements are open. We have that $X_1^C = X \setminus X_1 = X_2 \setminus X_0$ and $X_2^C = X \setminus X_2 = X_1 \setminus X_0$. Since U_i is open in the subspace topology on X_i, there is an open set $U_i^* \subseteq X$ such that $U_i^* \cap X_i = U_i$. Let $V_1 := X_1^C \cup U_1^*$ and $V_2 := X_2^C \cup U_2^*$. Then V_1 and V_2 are open in X since they are unions of open sets. Moreover $V_1 = X_1^C \cup U_1^* = (X_2 \setminus X_0) \cup U_1^* = X_2 \cup U_1$ and $V_2 = X_2^C \cup U_2^* = (X_1 \setminus X_0) \cup U_2^* = X_1 \cup U_2$, because $U_1^* \setminus U_1 \subseteq X_2$ and $U_2^* \setminus U_2 \subseteq X_1$. Now $X = V_1 \cup V_2$ and $V_0 := V_1 \cap V_2 = U_1 \cup U_2$, and V_1, V_2, V_0 fit into a Meyer-Vietoris sequence since V_1, V_2 are open. More explicitly, we have the exact sequence

$$\ldots \to H_n(V_1 \cap V_2) \xrightarrow{\phi} H_n(V_1) \oplus H_n(V_2) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(V_1 \cap V_2) \to \ldots$$

Our next step is to associate the homology groups present in this sequence with those of X_1, X_2, and X_0. Since U_1, U_2 strong deformation retract onto X_0 via homotopies H_1, H_2, the unique map obtained from the gluing lemma applied to H_1, H_2 (which we can apply since $H_1 = H_2 = \mathrm{Id}$ on $U_1 \cap U_2 = X_0$ and U_1, U_2 is a finite closed cover of $U_1 \cup U_2$) will make $U_1 \cup U_2$ strong deformation retract onto X_0 as well. By homotopy invariance of singular homology, we must then have that $H_n(V_0 = U_1 \cup U_2) \cong H_n(X_0)$. We have that $V_1 = X_1 \cup U_2$ and U_2 strong deformation retracts onto $X_0 \subseteq X_1$ and $X_1 \cap U_2 = X_0$. Since the strong deformation retraction H of U_2 onto X_0 is the identity on $U_2 \cap X_1 = X_0$, the map obtained from the gluing lemma on H together with the identity map on X_1 (which can be used since U_2, X_1 is a finite closed cover of $X_1 \cup U_2$ in the subspace topology on $X_1 \cup U_2$) will be a homotopy $X_1 \cup U_2 \simeq X_1$. Entirely analogously, we get a homotopy $X_2 \cup U_1 \simeq X_2$. Again by the homotopy invariance of singular homology, we get that $H_n(V_1 = X_1 \cup U_2) \cong H_n(X_1)$ and $H_n(V_2 = X_2 \cup U_1) \cong H_n(X_2)$. Therefore we have a one-to one correspondence[1] between the Mayer-Vietoris sequence of V_1, V_2, V_0

$$\ldots \to H_n(V_0) \xrightarrow{\phi} H_n(V_1) \oplus H_n(V_2) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(V_0) \to \ldots$$

and the following sequence

$$\ldots \to H_n(X_0) \xrightarrow{\phi'} H_n(X_1) \oplus H_n(X_2) \xrightarrow{\psi'} H_n(X) \xrightarrow{\partial} H_{n-1}(X_0) \to \ldots$$

where we have replaced all homology groups from the first sequence with their isomorphic counterparts. This second sequence is a Meyer-Vietoris sequence for X_1, X_2, X_0, as desired. ■

[1] If this term has a hyper-specific category-theoretical meaning, I'm probably not using it correctly. I simply mean that all the homology groups in the first sequence are isomorphic to their corresponding groups in the second sequence.