PROBLEM 1. Prove that for all spaces X there is an isomorphism

$$H_n(X \times S^d) \cong H_n(X) \oplus H_{n-d}(X)$$

with the convention that if $n-d<0$ then $H_{n-d}(X)$ is the trivial group.
Suggestion: Prove the following isomorphisms:

$$H_n(X \times S^d) \cong H_n(X) \oplus H_n(X \times S^d, X \times *)$$

and

$$H_n(X \times S^d, X \times *) \cong H_{n-1}(X \times S^{d-1}, X \times *).$$

Solution. We begin by proving the first isomorphism outlined above. Let i_* be the injection induced by the inclusion $X \times * \hookrightarrow X \times S^d$, and let q_* be the map that is the identity on the cycle classes in $H_n(X \times S^d)$ that are also relative cycle classes in $H_n(X \times S^d, X \times *)$, and sends all other cycles to zero. The following sequence is then exact:

$$0 \to H_n(X \times *) \overset{i_*}{\hookrightarrow} H_n(X \times S^d) \xrightarrow{q_*} H_n(X \times S^d, X \times *)$$

Moreover, q has a section via the map that sends a relative cycle class in $H_n(X \times S^d, X \times *)$ to its corresponding cycle class in $H_n(X \times S^d)$. Therefore, the above sequence splits and so we have an isomorphism of sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\mathrm{Id}"] & H_n(X\times *) \arrow[r, "i"] \arrow[d, "\mathrm{Id}"] & H_n(X\times S^d) \arrow[r, "q"] \arrow[d, "g"] & H_n(X\times S^d, X\times *) \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \arrow[d, "\mathrm{Id}"] \\
0 \arrow[r] & H_n(X\times *) \arrow[r, "\iota"] & H_n(X\times *)\oplus H_n(X\times S^d, X\times *) \arrow[r, "p"] & H_n(X\times S^d, X\times *) \arrow[r] & 0
\end{tikzcd}$$

which has the immediate consequence of providing an isomorphism $g : H_n(X \times S^d) \xrightarrow{\sim} H_n(X \times *) \oplus H_n(X \times S^d, X \times *)$. Since $H_n(X) \cong H_n(X \times *)$ because $X \simeq X \times *$, we get an isomorphism $H_n(X \times S^d) \cong H_n(X) \oplus H_n(X \times S^d, X \times *)$.

Now onto the second isomorphism above. We can decompose S^d as a union of two hemispheres (homeomorphic to D^d) intersection S^{d-1}. Let the point $*$ that we have specified lie on this intersection. Recall that if one has a pair of spaces $(X,Y) = (A \cup B, C \cup D)$ where $C \subset A$ and $D \subset B$ and X is the union of the interiors of A and B and Y is the union of the interiors of C and D, we have a relative Mayer-Vietoris sequence

$$\ldots \to H_n(A \cap B, C \cap D) \to H_n(A,C) \oplus H_n(B,D) \to H_n(X,Y) \to H_{n-1}(A \cap B, C \cap D) \ldots$$

Let the first hemisphere be denoted H_1 and let the second hemisphere be denoted H_2. Let their intersection be denoted I. In our case, $A = X \times H_1, B = X \times H_2, C = D = X \times *$. Since the hemispheres are homeomorphic to D^d, we have $H_n(A \cap B, C \cap D) = H_n(X \times S^{d-1}, X \times *)$ (where the point in the second relative homology group is assumed to be in S^{d-1}). Similarly, $H_n(A,C) = H_n(B,D) = H_n(X \times D^d, X \times *)$ (where the point lies in D^d). Then we get the relative Mayer-Vietoris sequence

$$\ldots \to H_n(X \times S^{d-1}, X \times *) \to H_n(X \times D^d, X \times *) \oplus H_n(X \times D^d, X \times *) \to$$
$$\to H_n(X \times S^d, X \times *) \to H_{n-1}(X \times S^{d-1}, X \times *) \to H_{n-1}(X \times D^d, X \times *) \oplus H_{n-1}(X \times D^d, X \times *) \to \ldots$$

Proposition 2.19. in Hatcher states that if two maps $f,g : (X,A) \to (Y,B)$ are homotopic through maps of pairs $(X,A) \to (Y,B)$, then $f_* = g_* : H_n(X,A) \to H_n(Y,B)$. Since $X \times H_i$ is homotopy equivalent to $X \times *$ by contracting the hemisphere to a point via a map f, we get that the induced map is an isomorphism $H_n(X \times D^n, X \times *) \cong H_n(X \times *, X \times *) = (0)$. Therefore our sequence reduces to

$$\ldots \to H_n(X \times S^{d-1}, X \times *) \to 0 \to H_n(X \times S^d, X \times *) \quad \to H_{n-1}(X \times S^{d-1}, X \times *) \to 0 \to \ldots$$

which by exactness immediately implies that

$$H_n(X \times S^d, X \times *) \cong H_{n-1}(X \times S^{d-1}, X \times *).$$

The case of $H_0(X\times S^d, X\times *)$ also follows since the Mayer-Vietoris sequence terminates at 0. We therefore have our desired isomorphisms. Now it remains to show the original isomorphism.

We start with the case $d > n$. Then, consider $\sigma : (0) \to X \times S^j, \sigma \in C_0(X \times S^j)$ and suppose $\sigma(0) = (x, p)$. Define $\eta : \Delta^1 = [0,1] \to X \times S^j$. Let $\eta(0) = (x, p), \eta(1) = (x, *)$. This map is defined for $j \geq 1$ since S^j is then path-connected. Then $\eta \in C_1(X \times S^j)$. We have a commutative diagram

$$\begin{tikzcd}
C_1(X \times S^j) \arrow[r, "\partial"] \arrow[d] & C_0(X \times S^j) \arrow[d] \\
C_1(X \times S^j)/C_1(X \times *) \arrow[r] & C_0(X \times S^j)/C_0(X \times *)
\end{tikzcd}$$

Then $\partial(\eta) = \eta|_{\{0\}} - \eta|_1 = \sigma - \eta|_1$. But in the quotient by $C_0(X \times *)$, $\eta|_1$ is killed and so by surjectivity of the rightmost downward facing map and commutativity of the diagram we must have that $[\eta] \mapsto [\sigma]$ in the relative case. Hence σ is a boundary in the zeroth relative chain complex, and since sigma was arbitrary, the zeroth relative homology group $H_0(X \times S^{d-n}, X \times *)$ must then be trivial. By our second isomorphism we get in the case $n-d < 0$ that $H_n(X \times S^d, X \times *)$ is trivial, so by our first homeomorphism $H_n(X \times S^d) \cong H_n(X)$.

Now onto the case $n - d \geq 0$. By repeated application of the second isomorphism, we get

$$H_n(X \times S^{n-d}, X \times *) \cong H_{n-1}(X \times S^{d-1}, X \times *) \cong \ldots \cong H_{n-d}(X \times S^0, X \times *).$$

But S^0 is simply two points, so the relative homology $H_{n-d}(X \times S^0, X \times *) = H_{n-d}(X \times \{*_1, *_2\}, X \times *)$, which by WLOG excisioning out $X \times *_1$ is equal to $H_{n-d}((X \times *_2) \sqcup *, *) \cong H_{n-d}(X \times *) \cong H_{n-d}(X)$ (we can excision $X \times *$ since it is closed in $X \times \{*_1, *_2\}$). ■

PROBLEM 2. Let $f(z) = a_n z^n + \ldots + a_1 z + a_0$ be a polynomial with complex coefficients. View it as a map $f : \mathbb{C} \to \mathbb{C}$. one can associate with f a map $\hat{f} : S^2 \to S^2$ as follows:

- Let D Be the closed unit disk in $\mathbb{C}$. The preimage $f^{-1}(D)$ of the unit disk is a bounded subset of $\mathbb{C}$. (why?)
- Choose a disk D_r such that the interior of D_r contains $f^{-1}(D)$. Then f restricts to a map $D_r \to \mathbb{C}$ that takes the boundary of D_r to the exterior of D.
- It follows that f induces a map between quotient spaces $D_r/\partial D_r \to \mathbb{C}/(\mathbb{C} \setminus \dot{D})$. Since both of these spaces are canonically homeomorphic to S^2, f induces a map $\hat{f} : S^2 \to S^2$.

Prove that the degree of $\hat{f}$ as a map is the same as the degree of f as a polynomial. Furthermore, prove that for every root of f, the local degree of $\hat{f}$ at the root is the multiplicity of the root.
Note: The problem is based on Hatcher's problem 8 on page 155. I was not sure that everyone knew about one-point compactification, so I used a workaround. If you know what a one-point compactification is, you can also answer Hatcher's version of the problem. Reading carefully the proof of Proposition 2.30 and example 2.32 should help you with this problem.

Solution.

Note: *Most people coming from the Engineering Physics programme at KTH into this master programme have not read complex analysis since it ceased to be a prerequisite for the programme the year we started. This problem seems to lend itself naturally to the methods of complex analysis; I don't know these methods, which is why they are absent in this solution.*

Note after finishing up writing this: *I feel like this solution is a bit iffy, and probably quite unpleasant to read. It was not my intent for that to happen, and if the reader finishes reading this, they have my kudos and respect.*

We want to prove that for every root of f, the local degree of $\hat{f}$ at the root is the multiplicity of the root. It will follow from this that the degree of f as a polynomial is the same as the degree of $\hat{f}$ if we use the fundamental theorem of algebra, i.e. that any polynomial in $\mathbb{C}$ of degree n has n (not necessaily distinct) roots in $\mathbb{C}$, along with the formula $\deg(\hat{f}) = \sum_i \deg \hat{f}|x_i$ where x_i are the roots of f.

Let the one-point compactification of $\mathbb{C}$ be denoted $\mathbb{C}^*$. Then $\mathbb{C}^*$ is homeomorphic to S^2 by stereographic projection, mapping the projection point to infinity. We extend f to $\mathbb{C}^*$ in the obvious way, mapping $\infty \in \mathbb{C}^*$ onto itself. Moving on, our discussion will mostly be on the level of $\mathbb{C}^*$ and not S^2. Viewed as a map $\mathbb{C}^* \to \mathbb{C}^*$, the polynomial f will have $f^{-1}(0) = \{z_1, \ldots, z_k\}$ i.e. the roots of the polynomial, having multiplicities $\{m_1, \ldots, m_k\}$. Let z_i be a root of f with multiplicity m_i. Since we are dealing with a polynomial in $\mathbb{C}$ and $(z - z_i)^{m_i}$ factors into f, we can Taylor expand f in a neighborhood U_i around z_i as $f(z) = (z - z_i)^{m_i}(a_i + (z - z_i)p(z))$, where $|p(z)|$ is bounded in U_i by some A. Take $\epsilon < |a_i|/(1000000A)$ as the radius of our neighborhood around z_i. Then $f(z)$ will certainly be nonzero in $U_i \setminus z_i$. Now, consider the homotopy $H(z,t) = (z - z_i)^{m_i}(a_i + (z - z_i)p(z)(1 - t))$. Then $H(z,0) = f(z)$ and $H(z,1) = a_i(z - z_i)^m_i$. By our choice of ϵ, this homotopy will not hit 0 for any value of t or z, so it is a homotopy of polynomials whose codomain is $V \setminus 0$ for a neighborhood V, chosen suitably large for the homotopy not to change it.

Now, we shift our attention to the polynomial $y(z) = a_i(z - z_i)^{m_i}$. To do this, we have to see how $y_i(z) : U_i \setminus z_i \to V \setminus 0$ fits into the degree of $\hat{f}$. Let s be the inverse stereographic projection that takes $\mathbb{C}^*$ to S^2. We have a diagram

$$
\begin{tikzcd}
 & H_2(U_i, U_i \setminus z_i) \arrow[dl] \arrow[d] \arrow[r, "\hat{f}_*"] & H_2(V, V \setminus 0) \arrow[d, "\sim"] \\
H_2(S^2, S^2 \setminus \hat{z}_i) & H_2(S^2, S^2 \setminus \hat{f}^{-1}(\hat{0})) \arrow[l] \arrow[r, "\hat{f}_*"] & H_2(S^2, S^2 \setminus \hat{0}) \\
 & H_2(S^2) \arrow[ul] \arrow[u] \arrow[r, "\hat{f}_*"] & H_2(S^2) \arrow[u, "\sim"']
\end{tikzcd}
$$

where the top row is abuse of notation since it really should be the images of the sets in S^2, but because those and the relative pairs are homeomorphic to those in the diagram via the inverse projection, it does not matter. We have a diagram

$$
\begin{tikzcd}
\dots \arrow[r] & H_2(U_i) \arrow[r] \arrow[d, "\hat{f}"] & H_2(U_i, U_i \setminus z_i) \arrow[r] \arrow[d, "\hat{f}"] & H_1(U_i \setminus z_i) \arrow[r] \arrow[d, "\hat{f}"] & H_1(U_i) \arrow[r] \arrow[d, "\hat{f}"] & \dots \\
\dots \arrow[r] & H_2(U_i) \arrow[r] & H_2(V, V \setminus 0) \arrow[r] & H_1(V \setminus 0) \arrow[r] & H_1(V) \arrow[r] & \dots
\end{tikzcd}
$$

But U_i and V are both contractible, so we $H_2(U_i) = H_1(U_i) = H_2(V) = H_1(V) = 0$. Therefore we get isomorphisms $H_2(U_i, U_i \setminus z_i) \cong H_1(U_i)$ and $H_2(V, V \setminus 0) \cong H_1(V \setminus 0)$. Now, we can instead look at the induced map from $H_1(U_i \setminus z_i)$ to $H_2(v \setminus 0)$. The first homology group of $U_i \setminus z_i$ will be that of a circle. The cycle $\gamma(\theta) : [0,1] \to U_i \setminus z_i : \theta \mapsto z_0 + \epsilon e^{2\pi i\theta}$ for a suitably small ϵ generates $H_1(U_i \setminus z_i)$. Our polynomial takes γ to $\gamma'(\theta) : [0,1] \to V \setminus \{0\}, \theta \mapsto a_i \epsilon^{m_i} e^{2\pi m_i i\theta}$ which is m_i times a generator of $H_1(V \setminus 0)$ (which is also $\mathbb{Z}$). Therefore the induced map on homology, which is the same as $\hat{f}_*$ since f and y are homotopic and f and $\hat{f}$ differ by homeomorphisms (maybe up to a sign), must take 1 to m_i.

This proves that the local degree of $\hat{f}$ around a root z_i is the multiplicity of the root of f. It follows immediately by the fundamental theorem of algebra and proposition 2.30 in Hatcher that the degree of $\hat{f}$ is the same as the degree of f as a polynomial.

■