PROBLEM 1. For points $y_0, \dots, y_k$ in a convex set Z, let $\overline{y_0 \dots y_k} = \frac{y_0 + \dots + y_k}{k+1}$ denote their barycenter.

(A) Let

$$\Delta^n = [y_0, \dots, y_n] = \left\{ t_0 y_0 + \dots + t_n y_n | t_i \geq 0, \sum_i t_i = 1 \right\}$$

be the n-dimensional simplex with our usual notation. Let $(i_0, \dots, i_n)$ be a permutation of $(0, \dots, n)$. Prove that

$$[\overline{y_{i_0}}, \overline{y_{i_0} y_{i_1}}, \dots, \overline{y_{i_0} \dots y_{i_n}}] = \{ t_0 y_0 + \dots + t_n y_n | t_{i_0} \geq t_{i_1} \geq \dots t_{i_n} \}.$$

(B) Prove that the simplices $[\overline{y_{i_0}}, \overline{y_{i_0} y_{i_1}}, \dots, \overline{y_{i_0} \dots y_{i_n}}]$, ranging over all permutations $(i_0, \dots, i_n)$ of $[n] = (0, \dots, n)$, triangulate Δ^n (that is, their union is Δ^n and the intersection of any two is a face of both).

Solution.

(A) By definition, we have

$$\begin{aligned}
[\overline{y_{i_0}}, \overline{y_{i_0} y_{i_1}}, \dots, \overline{y_{i_0} \dots y_{i_n}}] &= \left\{ t_0 y_{i_0} + t_1 \frac{y_{i_0} + y_{i_1}}{2} + \dots t_n \frac{\sum_{i=0}^n t_i}{n+1} \right\} \\
&= \left\{ y_{i_0} \sum_{i=0}^n \frac{t_i}{i+1} + y_{i_1} \sum_{i=1}^n \frac{t_i}{i+1} + y_{i_2} \sum_{i=2}^n \frac{t_i}{i+1} + \dots + y_{i_n} \frac{t_n}{n+1} \right\} \\
&= \left\{ y_{i_0} \sum_{i=0}^n \frac{t_i}{i+1} + y_{i_1} \left(\sum_{i=0}^n \frac{t_i}{i+1} - t_0 \right) + \dots + y_{i_n} \left(\sum_{i=0}^n \frac{t_i}{i+1} - \sum_{i=0}^{n-1} \frac{t_i}{i+1} \right) \right\} \\
&= \left\{ s_{i_0} y_{i_0} + s_{i_1} y_{i_1} + \dots + s_{i_n} y_{i_n} \right\}
\end{aligned}$$

Since $t_i \geq 0$ for all i, we get by the above that $s_{i_0} \geq s_{i_1} \geq \dots \geq s_{i_n}$. Since the terms in all the sums have strictly increasing denominators and the sum of the t_i is 1, the maximal value for each s_{i_k} is its value when the first t_i appearing in the corresponding sum (see the second row in the above equation) is set to 1 and all other t_k to 0. Therefore all s_{i_k} are bounded by 1, with the bound being sharp for s_{i_0}. By convexity of Z, the set described in this paragraph is a subset of Δ^n. Finally, the sum $\sum_k s_{i_k}$ is 1 since

$$\begin{aligned}
\sum_{k=0}^n s_{i_k} &= \sum_{i=0}^n \frac{t_i}{i+1} + \left(\sum_{i=0}^n \frac{t_i}{i+1} - t_0 \right) + \dots + \left(\sum_{i=0}^n \frac{t_i}{i+1} - \sum_{i=0}^{n-2} \frac{t_i}{i+1} \right) + \left(\sum_{i=0}^n \frac{t_i}{i+1} - \sum_{i=0}^{n-1} \frac{t_i}{i+1} \right) \\
&= (n+1) \frac{t_n}{n+1} + n \frac{t_{n-1}}{n} + \dots + 2 \frac{t_1}{2} + t_0 \\
&= \sum_{i=0}^n t_i,
\end{aligned}$$

which by definition is 1. In summary, $[\overline{y_{i_0}}, \overline{y_{i_0} y_{i_1}}, \dots, \overline{y_{i_0} \dots y_{i_n}}] = \{ s_0 y_0 + \dots + s_n y_n | s_{i_0} \geq s_{i_1} \geq \dots \geq s_{i_n} \}$, which was the sought result.

(B) Let i be a permutation. Let $[\overline{y_{i_0}}, \overline{y_{i_0} y_{i_1}}, \dots, \overline{y_{i_0} \dots y_{i_n}}] = \left\{ t_0 y_{i_0} + t_1 \frac{y_{i_0} + y_{i_1}}{2} + \dots t_n \frac{\sum_{i=0}^n t_i}{n+1} \right\} = \{ s_0 y_0 + \dots + s_n y_n | s_{i_0} \geq s_{i_1} \geq \dots s_{i_n} \} := A_i$. A face of A_i obeys the same conditions as above, with the added constraint that one or more of the t_i are zero. How does this look in terms of the s_i? As in **(A)**, we have

$$\left\{ t_0 y_{i_0} + t_1 \frac{y_{i_0} + y_{i_1}}{2} + \dots t_n \frac{\sum_{i=0}^n t_i}{n+1} \right\} = \left\{ y_{i_0} \sum_{i=0}^n \frac{t_i}{i+1} + y_{i_1} \sum_{i=1}^n \frac{t_i}{i+1} + y_{i_2} \sum_{i=2}^n \frac{t_i}{i+1} + \dots + y_{i_n} \frac{t_n}{n+1} \right\}.$$

From this point of view, we see that setting $t_i = 0$ gives equality between s_i and s_{i+1}. Hence, we can identify faces of A_i with equalities in the sequence of relations $s_{i_0} \geq \dots \geq s_{i_n}$.

Let i,k be permutations and consider the intersection $A_{ik} = A_i \cap A_k$. By **(A)**, we can think of this set as

$$A_{ik} = \{s_0 y_0 + \ldots + s_n y_n | s_{i_0} \geq s_{i_1} \geq \ldots s_{i_n} \text{ and } s_{k_0} \geq s_{k_1} \geq \ldots s_{k_n}\}.$$

We consider the restriction on A_{ik} more closely:

$$s_{i_0} \geq s_{i_i} \geq \ldots \geq s_{i_r} \geq \ldots \geq s_{i_n}$$
$$s_{k_0} \geq s_{k_i} \geq \ldots \geq s_{k_r} \geq \ldots \geq s_{i_n}.$$

Suppose $i_l = k_l$ for $l = 0, \ldots, r$, and $i_{l+1} \neq l_{l+1}$. Then k_{l+1} is i_s for some $s > r$ since all the lower indices fulfil $i_l = k_l$, (and therefore $s_{k_{r+1}}$ is to the right of $s_{i_{r+1}}$ in the above picture. Therefore we have the relation $s_{i_{r+1}} \geq s_{k_{r+1}}$. At the same time, by a symmetric argument $s_{k_{r+1}} \geq s_{ir+1}$, so we get equality for all s_{i_j} with indices j between $r+1$ and s (which is guaranteed to be at least two elements). We can repeat this process for higher indices, which leads to constraints on $A_i k$ which will be of the form $s_{i_0} R_0 s_{i_1} R_1 \ldots R_{n-1} s_{i_n}$, where the R_l are either $\geq$ or $=$. By the above discussion, this defines a face of A_i, since at least one R_l will be $=$ for two distinct permutations i,k as we saw above. Entirely symmetrically, this process leads to constraints on A_{ik} of the form $s_{k_0} R'_0 s_{k_1} R'_1 \ldots R'_{n-1} s_{k_n}$, which by the above discussion defines a face of A_k. Hence the intersection of A_i and A_k is a face of both of them.

Next, we show that the union of all A_i is Δ^n. By **(A)** we have

$$\begin{aligned} \bigcup_{\text{permutations } i} [\overline{y_{i_0}}, \overline{y_{i_0} y_{i_1}}, \ldots, \overline{y_{i_0} \ldots y_{i_n}}] &= \bigcup_{\text{permutations } i} \{t_0 y_0 + \ldots + t_n y_n | t_{i_0} \geq t_{i_1} \geq \ldots t_{i_n}\} \\ &= \{t_0 y_0 + \ldots + t_n y_n | t_{i_0} \geq t_{i_1} \geq \ldots t_{i_n} \text{ or } t_{i'_0} \geq t_{i'_1} \geq \ldots t_{i'_n} \text{ or } \ldots\}, \end{aligned}$$

that is, $\bigcup_i A_i$ is the set of all sums$\sum_k t_k y_k$ where $t_k \geq 0, \sum_k t_k = 1$ and the t_k are ordered according to *some* linear ordering out of the set of all linear orderings of $[n]$ (which is in one-to-one correspondence with the set of all permutations on $n+1$ indices). Since any configuration of the t_i will obey some linear ordering out of the set of all linear orderings, we get that $\sum_i A_i$ is just the set of all sums $\sum_k t_k y_k$ where $t_k \geq 0, \forall k$ and $\sum_k t_k = 1$. This is precisely Δ^n.

In conclusion, the simplices $[\overline{y_{i_0}}, \overline{y_{i_0} y_{i_1}}, \ldots, \overline{y_{i_0} \ldots y_{i_n}}]$, ranging over all permutations of $[n]$, triangulate Δ^n. The proof is complete.

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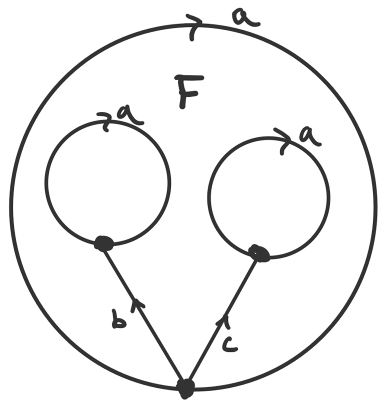
Problem 2. Let X be the space obtained from the closed unit disk D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying the three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.

Describe a CW structure on X, and use it to calculate the cellular (and therefore also the singluar) homology of X.

Solution. We equip X with the following CW-structure, consisting of one 2-cell, three 1-cells and one 0-cell:



Starting at the bottom vertex, a bit to the right of a, we can follow along the 1-cells to see that the attaching map for the 2-cell amounts to attaching along the word $aca^{-1}c^{-1}ba^{-1}b^{-1}$. Since the sum of the exponents of a in the word is -1 and the sum of the exponents of b and c in the word is 0, we can use the Cellular Boundary Formula to deduce that $d_2(F) = -a$. Furthermore, since X is connected and there is only one 0.cell (all points are identified with each other in our CW structure), we know that d_1 is the zero map. Moreover d_0 is the zero map. All maps d_n for $n \geq 3$ are also the zero maps since all homology groups $H_n(X)$ are zero for $n \geq 3$, which follows from the fact that the highest dimensional n-skeleton is X^2 in X. We have a sequence

$$0 \xrightarrow{d_3} H_2(X^2, X^1) \xrightarrow{d_2} H_1(X^1, X^0) \xrightarrow{d_1} H_0(X^0, X^{-1}) \xrightarrow{d_0} 0$$

which is

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{\oplus 3} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0.$$

$H_1(X^1, X^0)$ is generated by a, b, c. We have

$$\begin{aligned} H_2(X) &= \mathsf{Ker}(d_2)/\mathsf{Im}(d_3) = 0; \\ H_1(X) &= \mathsf{Ker}(d_1)/\mathsf{Im}(d_2) = \langle a, b, c\rangle/\langle -a\rangle \cong \mathbb{Z}^{\oplus 2}; \\ H_0(X) &= \mathsf{Ker}(d_0)/\mathsf{Im}(d_1) \cong \mathbb{Z}. \end{aligned}$$

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