

CAN ALGORITHMS BE CONSTRUCTED NON-CONSTRUCTIVELY?

JONAS CONNERYD

CONNERYD@KTH.SE

MM7042 MATHEMATICAL COMMUNICATION

2020

ABSTRACT. The attitudes of mathematicians toward non-constructive results throughout the 20th and 21st centuries are exemplified and explored. L.E.J. Brouwer's school of intuitionism and its relation to the controversy regarding the first proof of the Hilbert Basis Theorem are briefly surveyed. The algorithmic implications of the Robertson-Seymour Theorem in Graph Theory are informally presented, and in particular that it implies a non-constructive result about algorithms, i.e. a proof that an effective algorithm exists for a certain class of computational problems that does not provide the algorithm itself. The discussions surrounding this fact are contrasted to those surrounding those regarding the Hilbert Basis Theorem.

1. INTRODUCTION

In modern mathematics, the notion of what constitutes a mathematical proof seems to be clear enough that results can be published in journals, prizes can be awarded, and incorrect proofs can be refuted. (However, it seems that when you ask mathematicians what exactly a proof is, you're likely to get an answer similar to what Louis Armstrong is supposed to have replied when asked to define jazz: "If you have to ask, you'll never know"). But this wasn't always as universally agreed upon; in the start of the 20:th century, Hilbert produced a powerful but non-constructive existence proof that sparked a debate among mathematicians about the validity of such proofs. While this debate was eventually essentially settled in favor of non-constructive existence proofs, constructive mathematics seems to have gotten a spiritual successor in the form of algorithms research. Surely, to prove a result of the form "There exists an algorithm that solves a certain computational problem with a certain efficiency", what else could one do than provide an algorithm that solves the problem at hand, and prove that it does so with the claimed efficiency? This essay will survey a result which goes about proving such a result in a non-constructive way, and compare the subsequent discussions regarding this result with those surrounding Hilbert's result.

2. THE RESULT THAT STARTED IT ALL

The proof of Hilbert's Basis Theorem, originally from 1888 but coming into prominence some time later, sparked an intense debate regarding the nature of proof [8, p. 1]. The result itself requires more prerequisite theory than is necessary to introduce here, so it is omitted¹. What matters in this essay is however *how* he proved it. To prove this result, one needs to show that a finite collection of mathematical objects with a set of certain specifications exists. Hilbert did so without actually *constructing* them, instead showing that their non-existence would be contradictory [10, p. 33].

3. THE OBJECTIONS

A cohort of mathematicians that were sympathetic towards the constructivist view of mathematics, in which a proof of existence is one where the object of interest is actually constructed, were not convinced by this proof. Paul Gordan is supposed to have said about this result (albeit with unclear intention, cf. [8]) that

"This is not mathematics. This is theology!"[8, p. 1].

¹The interested reader can consult any textbook on Algebraic Geometry. Note however that the proof contained in the chosen textbook is probably not Hilbert's original proof.

The fiercest opponent to such non-constructive methods as was used in Hilbert's proof was however probably L. E. J. Brouwer (of Brouwer's fixed point theorem fame) and the followers of his *intuitionist* view of mathematics. There are other schools of constructive mathematics² but in consideration of space, only intuitionism is (briefly) surveyed in this essay.

4. WHAT IS INTUITIONISM?

Hilbert was a proponent of *formalism* [14], which maintained the view that mathematics is essentially a game about symbol manipulation in which the rules are clear and well defined, and results obtained through these rules are, in some sense, objectively true within the bounds of the game³. Brouwer, on the other hand, thought that mathematics is inherently a subjective activity - the result of constructive mental activity by humans [6]. This implies in particular that mathematical truth is subjective, with the following consequence: A mathematical object is considered a product of a construction produced by a mind, and therefore existence of an object is *equivalent* to its possibility of being constructed. This stands in sharp contrast to the usual approach, in which it is valid to prove existence of an object through refuting its non-existence. Put succinctly, yet possibly a bit crudely, intuitionists in the sense of Brouwer refute the validity of the *law of excluded middle*: That for any proposition, either that proposition or its negation are true [6]. A follower of intuitionism would then not be convinced of a proof which essentially boils down to a statement of the form " $\neg A$ ", where A in the case of Hilbert's Basis Theorem would be the statement establishing non-existence of the sought mathematical objects.

5. ALGORITHMIC RESEARCH AND NON-CONSTRUCTIVISM

The mathematics of the 21st century seems to not be especially concerned with issues similar to those discussed in the Hilbert-Brouwer controversy; indeed, the debate over the law of excluded middle seems to have been decisively settled in favor of it, as most any practitioner of mathematics would be ready to confirm. But it seems that branches of mathematics whose only real concern is actual construction of mathematical objects has made somewhat of a comeback with the advent of the mathematical theory of computation, more precisely the mathematical study of algorithms.

Complexity Theory, that part of mathematics that deals with classifying computational problems after how efficiently they can be solved by algorithms. In Complexity Theory, the notion of an "effective" algorithm is defined to be an algorithm whose running time scales polynomially with the size of the input to the algorithm⁴. Complexity Theory is a proper subfield of mathematics, and results in Complexity Theory are in general answers to questions of the form

Does there exist an algorithm that solves $\langle \text{computational problem} \rangle$ in $\langle \text{conjectured efficiency} \rangle$?

How would one prove an affirmative result in Complexity Theory? The answer seems to invariably be to construct an algorithm that indeed solves the computational problem at hand with sufficient efficiency⁵. Nonetheless, it turns out that there are non-constructive existence results in Complexity Theory, and the remainder of this essay aims to introduce one such result (or rather, an infinite collection of such results).

6. THE ROBERTSON-SEYMOUR THEOREM

The results in question are the algorithmic implications of the *Robertson-Seymour Theorem*, which follow from the combination of this theorem with another theorem, also by Robertson and Seymour. Both results are introduced in this essay. The results concern *graphs*, which are collections of *vertices* with *edges* drawn between them. Some definitions are needed to state both theorems by Robertson and Seymour. It should be noted that the definitions and theorem statements in this essay are mostly of the more informal kind, since

²For an overview, see Section 3 in [2].

³This is obviously, and in this medium probably unavoidably, a gross simplification of the ideas Hilbert subscribed to in the philosophy of mathematics. For a more detailed account and its connection to the project to axiomatize mathematics initiated by Hilbert, see [14].

⁴For a discussion on this definition of "effectiveness" in Complexity Theory, see Section 1.5 in [1].

⁵Negative results are another story. A general way to prove negative results in Complexity Theory is worth at least a million dollars in prize money, cf. [9].

a more rigorous treatment of the subject would demand a great deal more space to convey essentially the same content.

Definition 1 (Informal). A graph $G = (V, E)$ is a collection of vertices (points) and edges (lines between the points).

Graphs model pairwise relations between objects and are of immense importance in algorithms research and mathematical modelling, as well as very fascinating mathematical objects in their own right⁶. Two very simple examples of graphs can be found in Figure 1.

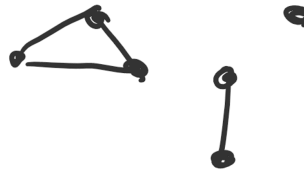


Figure 1. Two graphs with three vertices. The graph to the left has three edges and the the graph to the right has two edges.

It should be noted that the visual representation of the graph is immaterial; the two graphs in Figure 2 are the same graph.

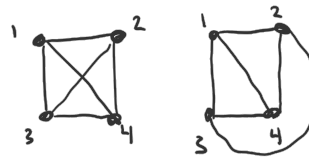


Figure 2. Two equivalent visual representations of a graph with four vertices and six edges.

The goal of Graph Theory is, naturally, to study properties defined on graphs and to study how different graphs relate to each other. One example of a graph property is whether it is possible to draw the graph on the plane without any of its edges intersection, in which case we say the graph is *planar*. An example of a graph relation, which is the relation of interest in this essay, is when a graph H is a *minor* of another graph G .

Definition 2. Let H and G be graphs. If we can form H from G by deleting vertices, deleting edges and contracting two edge-connected vertices into one, we say that H is a minor of G . The process of deleting vertices, deleting edges and contracting two edge-connected vertices into one is denoted by taking minors.

A visual representation of each of the operations allowed under taking minors can be seen in Figure 3.

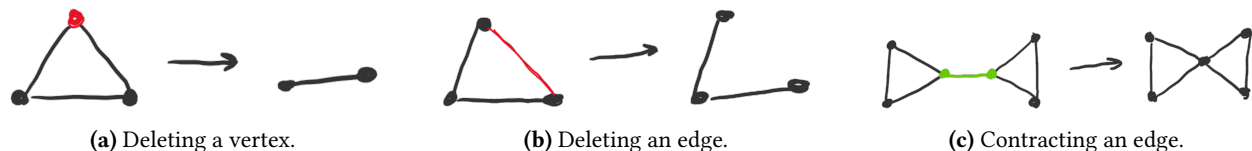


Figure 3. Operations allowed when taking minors.

Planarity is one of many properties a graph can possess that are retained under taking minors. Such graph properties are called *minor-closed*. The Robertson-Seymour Theorem relates to such properties of graphs, and the concepts defined so far are sufficient to state it.

⁶A rigorous introduction to the subject covering most of its major developments, including a more detailed exposition of the results covered in this essay, is [3].

Theorem 1 (Robertson-Seymour). *Suppose a graph property \mathcal{P} is retained when taking minors. Then there is a finite list of graphs such that a graph G has property \mathcal{P} if and only if none of the graphs in the list can be formed from G by taking minors^a.*

^aFor a detailed account of this theorem (albeit not its full proof, for good reason as will become clear), see Chapter 12 of [3].

Remark 1. *The graphs in the finite list described in the Robertson-Seymour Theorem belonging to a graph property \mathcal{P} are called the forbidden minors of \mathcal{P} .*

At first glance, the (extremely deep) ramifications of the Robertson-Seymour Theorem might not be apparent. It should also be noted that despite the seemingly simple statement of the theorem, its proof took 20 years and comparably many long journal articles totalling over 500 pages, to complete [11]. Its power in the field of Complexity Theory will become apparent when combined with the following result by the same authors.

Theorem 2. *Let H be a fixed graph, and let $G = (V, E)$ be an arbitrary graph. Then there is an algorithm that decides if H is a minor of G whose running time scales polynomially with the number of vertices in G .*

For a proof of Theorem 3, see [12]. The proof is constructive in that it is a description of an algorithm that checks whether H is a minor of G in the allotted time. It is notable that just the analysis of the running time of the algorithm takes around 50 pages.

Together, Theorem 2 and Theorem 3 will yield a very strong result about algorithms regarding the computational problem of determining whether a graph G has some property \mathcal{P} which is minor-closed. We know by Theorem 2 that belonging to \mathcal{P} is some finite list of graphs such that a graph G has property \mathcal{P} if and only if none of the graphs in the list is a minor of G . This list is the same regardless of the input graph, so we have to check a constant, finite number of cases to determine whether G has property \mathcal{P} . By Theorem 3 we know that we can check each case in time polynomially related to the number of vertices of G , so the total number of cases can in fact also be checked in time polynomially related to the number of vertices of G , since a constant times a polynomial is still a polynomial. Therefore, we have the following algorithmic result:

Theorem 3. *Determining whether a given graph $G = (V, E)$ has a property \mathcal{P} that is closed under taking minors can be done in time polynomially related to the number of vertices in G .*

One example of a minor-closed property was mentioned earlier, namely planarity. More generally, one family (out of many) of minor-closed properties of a graph is being able to be drawn on a surface, such as a sphere or donut surface, without edges crossing [5].

This result is indeed a non-constructive proof of the existence of an efficient algorithm for the computational problem of deciding whether a graph G has a given minor-closed property \mathcal{P} , since the Robertson-Seymour Theorem does not explicitly say what the forbidden minors of \mathcal{P} are, only that they exist and that there are finitely many of them. It is in fact in general highly non-trivial to find these forbidden minors, and only some cases are known. For example, the list of forbidden minors that characterize planarity of a graph contains exactly two graphs; this result is known as *Wagner's Theorem* (for a proof of this, see Section 4.4 of [3]). However, the list of forbidden minors that characterize whether a graph can be drawn on a donut surface without intersecting edges is known to contain at least 17 000 graphs, and it is not known whether this list is exhaustive [13].

Another caveat of this result is that the term "polynomially related", and hence the characterization of "efficient" used in Complexity Theory, does not take into account constant factors in the running time. For the algorithm in Theorem 3 it turns out that this constant factor is so large that it makes the algorithm unusable in even the most trivial cases; it is at least on the order of 2^{500} [4, p. 151].

7. SUBSEQUENT CONTROVERSY

While no new schools of Complexity Theory were started after this result appeared (at least to the knowledge of the author), this result did call into question whether the categorizations chosen for when to

call an algorithm "efficient", and what constitutes a proof in Complexity Theory, are really the right ones⁷. For one, the case for constructive results is much stronger when it comes to algorithms - if the purpose of Complexity Theory is to find efficient algorithms for important problems, what good is it, really, to know that an algorithm exists if we have no way of constructing, let alone use it? Also, can an algorithm really be called "efficient" if running it will take many times the lifetime of the universe for the most trivial of cases? Owing to these considerations, one famous computer scientist is supposed to have said about this result:

"This is not computer science. This is a mathematical curiosity!"[4, p. 152],

much in the spirit of Gordan's saying.

8. CONCLUDING REMARKS

There are some interesting similarities between the proofs and subsequent controversies regarding the Hilbert Basis Theorem and Robertson-Seymour Theorem and related results. In both cases, the *usefulness* of the results were questioned, and forced mathematicians to ponder whether the conceptual frames in which they worked were the right ones. Both results were also very influential and impactful and propelled their respective areas forward. However, there is also an important difference between the two situations: At no point was the *validity* of the proof of the Robertson-Seymour Theorem and related results questioned, while this question was central in the discussions surrounding the Hilbert Basis Theorem. If there is any conclusion to be drawn from both situations, it is that mathematical results that push preconceived boundaries both in proof method and implications are a net positive that helps mathematicians better understand what mathematics is and can be.

⁷For some perspectives on both of these questions, see [7] as well as [4, p. 151-152].

REFERENCES

- [1] Sanjeev Arora and Boaz Barak. *Computational Complexity: A Modern Approach*. Cambridge: Cambridge University Press, 2009. ISBN: 978-0-511-80409-0. DOI: [10.1017/CB09780511804090](https://doi.org/10.1017/CB09780511804090). URL: <http://ebooks.cambridge.org/ref/id/CB09780511804090> (visited on 10/06/2020).
- [2] Douglas Bridges and Erik Palmgren. “Constructive Mathematics”. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Summer 2018. Metaphysics Research Lab, Stanford University, 2018. URL: <https://plato.stanford.edu/archives/sum2018/entries/mathematics-constructive/> (visited on 10/06/2020).
- [3] Reinhard Diestel. *Graph Theory*. Vol. 173. Graduate Texts in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2017. ISBN: 978-3-662-53621-6 978-3-662-53622-3. DOI: [10.1007/978-3-662-53622-3](https://doi.org/10.1007/978-3-662-53622-3). URL: <http://link.springer.com/10.1007/978-3-662-53622-3> (visited on 10/06/2020).
- [4] R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Red. by David Gries and Fred B. Schneider. Monographs in Computer Science. New York, NY: Springer New York, 1999. ISBN: 978-1-4612-6798-0 978-1-4612-0515-9. DOI: [10.1007/978-1-4612-0515-9](https://doi.org/10.1007/978-1-4612-0515-9). URL: <http://link.springer.com/10.1007/978-1-4612-0515-9> (visited on 10/06/2020).
- [5] *Graph Minor*. In: *Wikipedia*. Sept. 12, 2020. URL: https://en.wikipedia.org/w/index.php?title=Graph_minor&oldid=978000162 (visited on 10/06/2020).
- [6] Rosalie Iemhoff. “Intuitionism in the Philosophy of Mathematics”. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2020. Metaphysics Research Lab, Stanford University, 2020. URL: <https://plato.stanford.edu/archives/fall2020/entries/intuitionism/> (visited on 10/06/2020).
- [7] David S Johnson. “The NP-Completeness Column: An Ongoing Guide”. In: (), p. 18.
- [8] Colin Mclarty. “Theology and Its Discontents: The Origin Myth of Modern Mathematics”. In: (Apr. 15, 2008), p. 19.
- [9] *P vs NP Problem* | *Clay Mathematics Institute*. URL: <https://claymath.org/millennium-problems/p-vs-np-problem> (visited on 10/06/2020).
- [10] Constance Reid. *Hilbert*. New York, NY: Springer New York, 1996. ISBN: 978-0-387-94674-0 978-1-4612-0739-9. DOI: [10.1007/978-1-4612-0739-9](https://doi.org/10.1007/978-1-4612-0739-9). URL: <http://link.springer.com/10.1007/978-1-4612-0739-9> (visited on 10/06/2020).
- [11] *Robertson–Seymour Theorem*. In: *Wikipedia*. Aug. 29, 2020. URL: https://en.wikipedia.org/w/index.php?title=Robertson%E2%80%93Seymour_theorem&oldid=975608138 (visited on 10/06/2020).
- [12] P.D. Seymour and Neil Robertson. “Graph Minors XIII: The Disjoint Paths Problem”. In: *Journal of Combinatorial Theory, Series B* 63 (1995), pp. 65–110.
- [13] *Toroidal Graph*. In: *Wikipedia*. Sept. 29, 2019. URL: https://en.wikipedia.org/w/index.php?title=Toroidal_graph&oldid=918703839 (visited on 10/06/2020).
- [14] Richard Zach. “Hilbert’s Program”. In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2019. Metaphysics Research Lab, Stanford University, 2019. URL: <https://plato.stanford.edu/archives/fall2019/entries/hilbert-program/> (visited on 10/06/2020).