Problem 1. Let the symmetric group S_n act on the vector space k^n in the usual way: S_n acts on the standard coordinate basis $(e_1, \dots, e_n)$ by $\sigma e_i = e_{\sigma(i)}$ for $\sigma \in S_n$ and we extend this action to k^n by requiring it to be linear : $\sigma\left(\sum c_i e_i\right) := \sum c_i \sigma(e_i)$.

(a) Let $a := (a_1, \dots, a_n) \in k^n$. Show that $\sigma(a) = (a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)})$. [So care must be taken when we say that S_n acts by permuting the coordinates.]

(b) Show that the resulting representation $\rho : S_n \to \mathsf{GL}(n,k)$ contains a subrepresentation isomorphic to the trivial representation $\mathbb{1}$.

(c) If k has characteristic zero, show that $\rho = \mathbb{1} \oplus \rho'$ is the direct sum of the trivial representation and a representation ρ' of dimension $n-1$.

(d) Assume k has characteristic 2 and $n = 2$. Show that ρ is not a direct sum of two nontrivial subspaces. In other words, ρ is reducible but not completely reducible. What goes wrong as opposed to **(c)**?

(e) Let p be a prime and assume k has characteristic p. Show that ρ is a direct sum as in **(c)** if and only if p does not divide n.

Solution.

(a) We have

$$\sigma(a) = \sigma\left(\sum_i a_i e_i\right) = \sum_i a_i e_{\sigma(i)}.$$

Since a permutation is a bijection, if we change variable to $j = \sigma^{-1}(i)$, we are simply changing the order of summation. This yields the same summand in a finite sum. Therefore

$$\sigma(a) = \sum_i a_i e_{\sigma(i)} = \sum_j a_j e_{\sigma(j)} = \sum_{\sigma^{-1}(i)} a_{\sigma^{-1}(i)} e_{\sigma(\sigma^{-1}(i))} = \sum_{\sigma^{-1}(i)} a_{\sigma^{-1}(i)} e_i = (a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)}),$$

as desired.

(b) ρ maps the permutations to the appropriate permutation matrices, which are $n \times n$-matrices of rank n whose rows consist of zeros except for a single non-zero element, which is 1. The vector $(1, \dots, 1) \in k^n$ is an eigenvector of any permutation matrix with eigenvalue 1. Therefore, the subspace spanned by $(1, \dots, 1)$ constitutes a S_n-stable subspace E of k^n on which S_n acts trivially. Therefore $\rho|_E$, the restriction of ρ onto E, is a subrepresentation isomorphic to the trivial representation.

(c) Let W be the orthogonal complement of E, so that $E + W = k^n$. Define $\rho' := \rho|_W$. Denote $(1, 1, \dots, 1) := 1_n$. W contains precisely the vectors v such that $v \cdot 1_n = 0$, that is, the column sum of all $v \in W$ is 0. The action of S_n on W simply exchanges coordinates, which does not affect the row sum. Therefore W is also S_n-stable, so $\rho|_W$ is a subrepresentation. Moreover, $E \oplus W = k^n$ and E has dimension 1 (it has a basis 1_n), so W has dimension $n-1$. Hence $\rho = \rho|_E \oplus \rho|_W \cong \mathbb{1} \oplus \rho|_W$, where $\rho|_W$ has dimension $n-1$, as desired.

(d) Suppose we can indeed write ρ as the product of two non-trivial subspaces. By Theorem 1.10. in Isaacs, this is equivalent to every S_n-stable subspace having a S_n-stable complement. Consider the subspace E as in **(c)**. E is the span of $(1,1)$, so any candidate W must be a space which is the span of a basis vector $w \in W$ which is linearly independent from $(1,1)$. That is, $W = \mathsf{Span}((a,b), a \neq b)$. The only choice of (a,b) that makes W S_n-stable is putting $b = -a$ (since S_n switches the coordinates of (a,b)), but this is not possible since $a = -a$ in characteristic 2, so $(a,-a) \in E$. Hence E has no S_n-stable complement, which in turn proves that ρ cannot be written as a direct sum of two nontrivial subspaces. The problem is precisely that $a \neq -a$ in characteristic 0, but not necessarily in other characteristics.

(e) Suppose $\rho = \mathbb{1} \oplus \rho'$, and suppose p does not divide n. By Maschke's theorem, it follows that every G-stable subspace has a G-stable complement. Again, we have the G-stable subspace E spanned by 1_n on which the restriction $\rho|_E$ is isomorphic to the trivial representation. Maschke's theorem thus yields that E has a G-stable complement W, which must then be of dimension $n-1$ since $E \oplus W = V$. Hence ρ decomposes as $\mathbb{1} \oplus \rho'$ where ρ' has dimension $n-1$.

For the other direction, we prove the contrapositive, i.e. if p divides n then ρ does not decompose as in **(c)**. Define $\phi : k^n \to k; v \overset{\phi}{\mapsto} \sum_i v_i$. Clearly, $\mathsf{Ker}(\phi)$ forms a subspace of k^n, and $E \subseteq \mathsf{Ker}(\phi)$ since E is spanned by 1_n which evaluates to $n = 0$ in k if $\mathsf{Char}(k) = p$ divides n. Suppose $\mathsf{Ker}(\phi)$ has a G-stable complement W. Let $w \in W$. Since E is assumed to not be in W, there is a $\rho(g)$ such that $\rho(g)w \neq w$.

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Problem 2. Some representations of cyclic and dihedral groups.

(a) Let $n \geq 2$. Describe n distinct, one-dimensional representations of $\mathbb{Z}/n$ over $\mathbb{C}$ and explain why they are pairwise non-isomorphic.

(b) Let $D_{2n} := \langle x, y \mid x^n = y^2 = 1, yxy^{-1} = x^{-1}\rangle$ be the dihedral group of order $2n$. Show that setting

$$\rho(x) := \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad \rho(y) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

gives a unique, well-defined representation of D_{2n} over $\mathbb{R}$.

(c) Show that ρ is irreducible over $\mathbb{C}$. [A priori, this is at least as strong as showing that ρ is irreducible over $\mathbb{R}$.]

(d) Show that the restriction $\mathrm{Res}^{D_{2n}}_{\langle x\rangle}$ of ρ to $\langle x\rangle$ is irreducible over $\mathbb{R}$ if and only if $n \geq 3$.

(e) By contrast, show that the same restriction $\mathrm{Res}^{D_{2n}}_{\langle x\rangle}$ is always reducible over $\mathbb{C}$.

Solution.

(a) Since $\mathbb{Z}/n$ is cyclic, any representation is completely determined by its mapping of a generator of $\mathbb{Z}/n$, say 1. The proposed representations are

$$\begin{aligned} \rho_k &: \mathbb{Z}/n \to \mathbb{C}, \\ & 1 \mapsto \exp(2\pi i k/n), k = 1, \dots, n, \\ \rho_k(l) &= p_k(1)^l. \end{aligned}$$

Indeed, this is a representation because

$$\rho_k(a+b) = \exp(2\pi i k/n)^{a+b} = \exp(2\pi i k/n)^a \exp(2\pi i k/n)^b = \rho_k(a)\rho_k(b).$$

These representations are also non-isomorphic because their characters are not equal. This can be seen by considering the character of a generator: $\rho_k(1) = \exp(2\pi i k/n) \neq \exp(2\pi i l/n)$ for $l \neq k, 1 \leq, k, l \leq n$.

(b) We have to verify that $\rho(x), \rho(y)$ fulfil the same relations as x, y. We have

$$\rho(x)^n = \left(\begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}\right)^n = \begin{bmatrix} \cos(2\pi n/n) & -\sin(2\pi n/n) \\ \sin(2\pi n/n) & \cos(2\pi n/n) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \rho(0),$$

$$\rho(y)^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \rho(0),$$

$$\begin{aligned} \rho(y)\rho(x)\rho(y)^{-1} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix} = \rho(x)^{-1}, \text{ since} \\ \begin{bmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}\rho(x) &= \begin{bmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}\begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix} \\ &= \begin{bmatrix} \cos^2(2\pi/n) + \sin^2(2\pi/n) & 0 \\ 0 & \cos^2(2\pi/n) + \sin^2(2\pi/n) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

The first relation $\rho(x)^n = 1$ is indeed in one-to-one correspondence with the relation $x^n = 1$ since n is the smallest non-negative integer k such that $\cos(2\pi k/n), \sin(2\pi k/n) = 1, 0$ respectively which implies that $\rho(x)^k \neq 1$ for $k < n$.

(c) Let χ_ρ be the trace of ρ. We know that irreducible representations are characterized by $[\chi_\rho, \chi_\rho] = 1$ over $\mathbb{C}$. We have

$$\begin{aligned}
[\chi_\rho, \chi_\rho] &= \frac{1}{2n} \sum_{g \in D_{2n}} \chi_\rho(g)\overline{\chi_\rho(g)} = \frac{1}{2n} \sum_{j=1}^{n} \chi_\rho(x^j)\overline{\chi_\rho(x^j)} + \frac{1}{2n} \sum_{j=1}^{n} \chi_\rho(yx^j)\overline{\chi_\rho(yx^j)} \\
&= \frac{1}{2n} \sum_{j=1}^{n} \mathsf{Tr}\begin{bmatrix} \cos(2\pi j/n) & -\sin(2\pi j/n) \\ \sin(2\pi j/n) & \cos(2\pi j/n) \end{bmatrix} \overline{\mathsf{Tr}\begin{bmatrix} \cos(2\pi j/n) & -\sin(2\pi j/n) \\ \sin(2\pi j/n) & \cos(2\pi j/n) \end{bmatrix}} \\
&+ \frac{1}{2n} \sum_{j=1}^{n} \mathsf{Tr}\begin{bmatrix} \sin(2\pi j/n) & \cos(2\pi j/n) \\ \sin(2\pi j/n) & -\sin(2\pi j/n) \end{bmatrix} \overline{\mathsf{Tr}\begin{bmatrix} \sin(2\pi j/n) & \cos(2\pi j/n) \\ \sin(2\pi j/n) & -\sin(2\pi j/n) \end{bmatrix}} \\
&= \frac{1}{2n} \sum_{j=1}^{n} 4\cos^2(2\pi j/n).
\end{aligned}$$

We use a result hailing from the famous mathematician Wolf Ramalpha which states that, for $n \geq 2$,

$$\sum_{j=1}^{n} \cos^2(2\pi j/n) = n/2.$$

It follows that $[\chi_\rho, \chi_\rho] = 1$, so ρ is irreducible over $\mathbb{C}$ as desired.

(d) We have that

$$\begin{aligned}
(\chi_{\mathsf{Res}^{D_{2n}}_{\langle x \rangle}}, \chi_{\mathsf{Res}^{D_{2n}}_{\langle x \rangle}}) &= \frac{1}{n} \sum_{g \in G} \chi_\rho(g)\chi_\rho(g^{-1}) \\
&= \frac{1}{n} \sum_{j=1}^{n} \chi_\rho(x^j)\chi_\rho(x^{n-j}).
\end{aligned}$$

For $n = 2$, this is

$$\begin{aligned}
&\frac{1}{2} \sum_{j=1}^{2} \chi_\rho(x^j)\chi_\rho(x^{n-j}) \\
&= \frac{1}{2}(\chi_\rho(x)\chi_\rho(x) + \chi(\mathrm{Id})\chi(\mathrm{Id})) \\
&= \frac{1}{2}(4 + 4),
\end{aligned}$$

which is 4, not 1. For $n \geq 3$, this is

$$\begin{aligned}
&\frac{1}{n}\sum_{j=1}^{n}\chi_\rho(x^j)\chi_\rho(x^{n-j}) \\
&= \frac{1}{n}\sum_{j=1}^{n}\chi_\rho(x^j)\chi_\rho(x^{n-j}) \\
&= \frac{1}{n}\sum_{j=1}^{n}4\cos(2\pi j/n)\cos(2\pi(n-j)/n).
\end{aligned}$$

This evaluates to 2, not 1, which is unfortunate. There must be something I'm missing.

(e) This follows from realizing that

$$\begin{aligned}
[\chi_{\mathrm{Res}_{\langle x\rangle}^{D_{2n}}}, \chi_{\mathrm{Res}_{\langle x\rangle}^{D_{2n}}}] &= \frac{1}{n}\sum_{g\in\langle x\rangle}\chi_\rho(g)\overline{\chi_\rho(g)} \\
&= \frac{1}{n}\sum_{j=1}^{n}\chi_\rho(x^j)\overline{\chi_\rho(x^j)} \\
&= 2\frac{1}{2n}\sum_{j=1}^{n}\chi_\rho(x^j)\overline{\chi_\rho(x^j)} \\
&= 2[\chi_\rho, \chi_\rho] \\
&= 2.
\end{aligned}$$

Since $2 \neq 1, \mathrm{Res}_{\langle x\rangle}^{D_{2n}}$ must be reducible. We could also have noted that all irreducible representations of abelian groups have dimension 1.

■

Problem 3. Let ρ be as in in Problem 1 with $k = \mathbb{C}$.

(a) Describe explicitly the character χ_ρ of ρ: Given a permutation σ, what is the value $\chi_\rho(\sigma)$ in terms of its cycle type?

(b) Assume a finite group G acts on a finite set X. Let $\mathsf{Fix}(g) := \{x \in X \mid gx = x\}$; the fixed points of g. What is $\sum_{g\in G} |\mathsf{Fix}(g)|$?

(c) Deduce that $[\chi_\rho, \mathbb{1}] = 1$.

(d) Recall that G acts doubly-transitively on X if the action of G on $X \times X$ has precisely two orbits (necessarily the diagonal and the rest). If G acts doubly-transitively on X and r is the corresponding linear representation of G, show that $[\chi_r, \mathbb{1}] = 1$ and $[\chi_r, \chi_r] = 2$. [Hint: Consider also the character of the linear representation gotten from the action of G on $X \times X$.]

(e) Deduce that $[\chi_{\rho'}, \chi_{\rho'}] = 1$.

Solution.

(a) A permutation matrix σ has exactly one nonzero element per row, and it is contained in the i:th diagonal if and only if it keeps the i:th coordinate fixed. Therefore, a permutation on n coordinates that keeps k coordinates fixed has character k.

(b) By Burnside's lemma, the number of orbits of G in X (denoted by $|X/G|$) is

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |\mathsf{Fix}(g)|,$$

so $\sum_{g\in G} |\mathsf{Fix}(g)|$ is $|G| \cdot |X/G|$.

(c) A group action $G \curvearrowright X$ induces a homomorphism $\phi : G \to S_X$, where S_X is the group of permutations of X under function composition, by $g \mapsto \psi_g, \psi_g(x) = g \cdot x$. For the finite case, $S_X = S_n$, where $n = |X|$ and S_n is the group of permutations on n letters. Therefore, we can think of $g \in G$ as permutation matrices acting on the n-element vector of elements $x_1, \dots x_n \in X$. Then, our reasoning in **(a)** shows that $|\mathsf{Fix}(g)|$ is the trace of its corresponding permutation matrix M_g, since the number of fixed points is the number of coordinates of the vector $(x_1, \dots, x_n)$ left fixed by the permutation M_g. That is,

$$\sum_{g\in G} |\mathsf{Fix}(g)| = \sum_{g\in G} \chi(g).$$

Over $\mathbb{C}$, this expression is equal to $|G| \cdot [\chi_\rho, \mathbb{1}]$. By **(b)** (Burnside's lemma does not require that X is finite), we have

$$|G|[\chi_\rho, \mathbb{1}] = \sum_{g\in G} |\mathsf{Fix}(g)| = |G||X/G|,$$

where $|X/G|$ is the number of orbits of G in X. But the symmetric group acts transitively on $\mathbb{C}^n$ with the given action, so by the above we get $[\chi, \mathbb{1}] = 1$.

■

Problem 4. Let $k = \mathbb{C}$. Let ρ be the representation of the dihedral group from Problem 2.

(a) Compute the character of ρ.

(b) Let χ, ψ be one-dimensional characters of a finite group G. Show that

$$(\chi, \psi) = \delta_{\rho,\psi} := \begin{cases} 1, \chi = \psi, \\ 0, \text{ otherwise} \end{cases}$$

[This is the first orthogonality relation in the special case of one-dimensional characters. You may not cite the general orthogonality relation or use its proof here.]

(c) Let $n \geq 2$. Show that $\sum_{m=1}^{n} e^{2\pi i m/n} = 0$ in two ways: Use **(b)** and use the polynomial $x^n - 1$.

(d) Show that $[\chi_\rho, \chi_\rho] = 1$. [Hint: Use **(c)**. I found it helpful to distinguish between the cases n even and n odd.]

Solution.

(a) The details of this calculation are covered in problem 2. The trace of x^j is $2\cos(2\pi j/n)$ and the trace of yx^j is 0.

(b) We have that

$$(\chi, \psi) =$$

(c) Note that $\sum_{m=1}^{n} e^{2\pi i m/n} = [\rho, \mathbb{1}]$, where ρ is the restricted representation of $\langle x \rangle$ from the dihedral group from problem 2, but over the complex numbers ($x^m \mapsto e^{2\pi i m/n}$), and 1 is the trivial representation of the restricted group $\langle x \rangle$ from the same problem. By **(b)**, their sum must be 0.

One can also view this as a geometric series $\sum_{m=1}^{n} \rho(1)^m$ which by standard methods is equal to $\rho(1)(\rho(1)^n - 1)/(\rho(1) - 1)$. Since ρ is a one-dimensional representation and $x^n = 1$ in $\langle x \rangle$, we have that $\rho(1)^n = 1$ in $\mathbb{C}*$, so the sum evaluates to 0.

(d) I did this in Problem 2.

■

Problem 5. Assume k is algebraically closed and $\mathsf{Char}(k) \neq 2$.

(a) Let (V, ρ) be a self-dual irreducible representation of G over k. Show that ρ is either symplectic or orthogonal.

(b) Assume (V, ρ) is a representation which is both symplectic and orthogonal. Show that (V, ρ) is reducible.

Solution.

(a) Suppose (V, ρ) is self-dual. Then, for all $g \in G$, we have that $\rho(g^{-1})^T = \rho(g)$.

■

Problem 6. Consider the two permutation representations

$$\rho_1, \rho_2 : \mathbb{Z}/2 \times \mathbb{Z}/2 \to S_6$$

given by

$$\rho_1((1,0)) = \rho_2((1,0)) = (12)(34),$$

while

$$\rho_1((0,1)) = (13)(24), \rho_2((0,1)) = (12)(56).$$

(a) Show that ρ_1 and ρ_2 are element-conjugate.
(b) Show that ρ_1 and ρ_2 are not globally conjugate.

Solution.

(a) Of the given cases, we need only examine how $\rho_1((0,1))$ relates to $\rho_2((0,1))$. What we have to show is that there is some $\tau \in S_6$ such that $\tau\rho_2((0,1))\tau^{-1} = \rho_1((0,1))$. Take $\tau = (235)(64)$. Then

$$\tau\rho_2((0,1))\tau^{-1} = (235)(64)(12)(56)(64)(532) = (13)(24).$$

By representations being homomorphisms, we have that

$$\rho_1((1,1)) = \rho_1((1,0)+(0,1)) = \rho_1((1,0))\rho_1((0,1)) = (12)(34)(13)(24) = (14)(23)$$

and

$$\rho_2((1,1)) = \rho_2((1,0)+(0,1)) = \rho_2((1,0))\rho_2((0,1)) = (12)(34)(12)(56) = (34)(56).$$

Take $\sigma = (31526)$. Then

$$\sigma\rho_1((1,1))\sigma^{-1} = (31526)(34)(56)(62513) = (14)(23) = \rho_2((1,1)).$$

Hence we have proved that ρ_1 and ρ_2 are element-conjugate.

(b) Suppose they are globally conjugate by τ. Then we must have that

$$\tau\rho_1((0,1))\tau^{-1} = \rho_2((0,1)) \iff \tau(13)(24)\tau^{-1} = (12)(56),$$
$$\tau\rho_1((1,1))\tau^{-1} = \rho_2((1,1)) \iff \tau(14)(23)\tau^{-1} = (34)(56).$$

This would mean

$$\tau(13)(24)\tau^{-1}\tau(14)(23)\tau^{-1} = (12)(56)(34)(56).$$

By decomposition into disjoint cycles, this is equivalent to

$$\tau(1342)\tau^{-1} = (12)(34)$$

which is impossible since two permutations are conjugate iff they have the same cycle type. Hence ρ_1, ρ_2 are not globally conjugate.

■