PROBLEM 1 (2.3)**.** Let χ be a character of G. Define $\det\chi : G \to \mathbb{C}$ as follows: Choose ρ affording χ and set

$$(\det\chi)(g) = \det\rho(g).$$

Show that $\det\chi$ is a uniquely defined linear character of G.

Solution. ■

PROBLEM 2 (2.4).

- **(A)** Let G be a nonabelian group of order 8. Show that G has a unique nonlinear irreducible character χ. Show that $\chi(1) = 2$, $\chi(z) = -2$, $\chi(x) = 0$, where $z = G - \{1\}$ and $x \in G - G'$.
- **(B)** If $G \cong D_8$, show that $\det \chi \neq 1_G$.
- **(C)** If $G \cong Q_8$, show that $\det \chi = 1_G$.

HINT: Show that $\mathsf{Ker}(\det \chi)$ **contains all elements of order 4. Use Lemma 2.15.**

Solution. ■

Problem 3 (2.5).

(A) Find a real representation of D_8 which affords the character χ of problem 2.4 **(A)**.
(B) Show that this cannot be done for the group Q_8.

Solution. ■

PROBLEM 4 (2.8)**.** Let χ be a faithful character of G. Show that $H \subseteq G$ is abelian iff every irreducible constituent of χ_H is linear.

Solution. ■

PROBLEM 5 (2.9). **(A)** Let χ be a character of an abelin group A. Show that

$$\sum_{a \in A} |\chi(x)|^2 \geq |A|\chi(1).$$

(B) Let $A \subseteq G$ with A abelian and $|G : A| = n$. Show that $\chi(1) \leq n$ for all $\chi \in \mathrm{Irr}(G)$.

Solution. ∎

Problem 6 (2.10)**.** Suppose $G = \bigcup_{i=1}^{n} A_i$, where the A_i are abelian subgroups of G and $A_i \cap A_j = 1$ if $i \neq j$.

(A) Let $\chi \in \mathrm{Irr}(G)$. Show that if $\chi(1) > 1$, then $\chi(1) \geq |G|/(n-1)$.

(B) If G is nonabelian, then $|A_i| \leq n-1$ for each i and $n-1 \geq |G|^{1/2}$.

Solution. ■

PROBLEM 7 (2.11)**.** Let $g \in G$. SHow that g is conjugate to g^{-1} in G iff $\chi(g)$ is real for all characters χ of G.

Solution. ■