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Homework 1

MM8021 Representation Theory for Finite Groups

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Problem 1. Let the symmetric group S_n act on the vector space k^n in the usual way: S_n acts on the standard coordinate basis $(e_1, \dots, e_n)$ by $\sigma e_i = e_{\sigma(i)}$ for $\sigma \in S_n$ and we extend this action to k^n by requiring it to be linear : $\sigma\left(\sum c_i e_i\right) := \sum c_i \sigma(e_i)$.

- **(a)** Let $a := (a_1, \dots, a_n) \in k^n$. Show that $\sigma(a) = (a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)})$. [So care must be taken when we say that S_n acts by permuting the coordinates.]
- **(b)** Show that the resulting representation $\rho : S_n \to \mathsf{GL}(n,k)$ contains a subrepresentation isomorphic to the trivial representation $\mathbb{1}$.
- **(c)** If k has characteristic zero, show that $\rho = \mathbb{1} \oplus \rho'$ is the direct sum of the trivial representation and a representation ρ' of dimension $n-1$.
- **(d)** Assume k has characteristic 2 and $n = 2$. Show that ρ is not a direct sum of two nontrivial subspaces. In other words, ρ is reducible but not completely reducible. What goes wrong as opposed to **(c)**?
- **(e)** Let p be a prime and assume k has characteristic p. Show that ρ is a direct sum as in **(c)** if and only if p does not divide n.

Solution.

(a) We have

$$\sigma(a) = \sigma\left(\sum_i a_i e_i\right) = \sum_i a_i e_{\sigma(i)}.$$

Since a permutation is a bijection, if we change variable to $j = \sigma^{-1}(i)$, we are simply changing the order of summation. This yields the same summand in a finite sum. Therefore

$$\sigma(a) = \sum_i a_i e_{\sigma(i)} = \sum_j a_j e_{\sigma(j)} = \sum_{\sigma^{-1}(i)} a_{\sigma^{-1}(i)} e_{\sigma(\sigma^{-1}(i))} = \sum_{\sigma^{-1}(i)} a_{\sigma^{-1}(i)} e_i = (a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(n)}),$$

as desired.

(b) ρ maps the permutations to the appropriate permutation matrices, which are $n \times n$-matrices of rank n whose rows consist of zeros except for a single non-zero element, which is 1. The vector $(1, \dots, 1) \in k^n$ is an eigenvector of any permutation matrix with eigenvalue 1. Therefore, the subspace spanned by $(1, \dots, 1)$ constitutes a S_n-stable subspace E of k^n on which S_n acts trivially. Therefore $\rho|_E$, the restriction of ρ onto E, is a subrepresentation isomorphic to the trivial representation.

(c) Let W be the orthogonal complement of E, so that $E + W = k^n$. Define $\rho' := \rho|_W$. Denote $(1, 1, \dots, 1) := 1_n$. W contains precisely the vectors v such that $v \cdot 1_n = 0$, that is, the column sum of all $v \in W$ is 0. The action of S_n on W simply exchanges coordinates, which does not affect the row sum. Therefore W is also S_n-stable, so $\rho|_W$ is a subrepresentation. Moreover, $E \oplus W = k^n$ and E has dimension 1 (it has a basis 1_n), so W has dimension $n-1$. Hence $\rho = \rho|_E \oplus \rho|_W \cong \mathbb{1} \oplus \rho|_W$, where $\rho|_W$ has dimension $n-1$, as desired.

(d) Suppose we can indeed write ρ as the product of two non-trivial subspaces. By Theorem 1.10. in Isaacs, this is equivalent to every S_n-stable subspace having a S_n-stable complement. Consider the subspace E as in **(c)**. E is the span of $(1,1)$, so any candidate W must be a space which is the span of a basis vector $w \in W$ which is linearly independent from $(1,1)$. That is, $W = \mathsf{Span}((a,b), a \neq b)$. The only choice of (a,b) that makes W S_n-stable is putting $b = -a$ (since S_n switches the coordinates of (a,b)), but this is not possible since $a = -a$ in characteristic 2, so $(a,-a) \in E$. Hence E has no S_n-stable complement, which in turn proves that ρ cannot be written as a direct sum of two nontrivial subspaces. The problem is precisely that $a \neq -a$ in characteristic 0, but not necessarily in other characteristics.

(e) Suppose $\rho = \mathbb{1} \oplus \rho'$, and suppose p does not divide n. Again, we have the subspace E spanned by 1_n. The question is whether we can find $n-1$ linearly independent vectors, all of which are linearly independent to 1_n, whose span is G-stable.

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Problem 2. Some representations of cyclic and dihedral groups.

(a) Let $n \geq 2$. Describe n distinct, one-dimensional representations of $\mathbb{Z}/n$ over $\mathbb{C}$ and explain why they are pairwise non-isomorphic.

(b) Let $D_{2n} := \langle x, y \mid x^n = y^2 = 1, yxy^{-1} = x^n \rangle$ be the dihedral group of order $2n$. Show that setting

$$\rho(x) := \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \quad \rho(y) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

gives a unique, well-defined representation of D_{2n} over $\mathbb{R}$.

(c) Show that ρ is irreducible over $\mathbb{C}$. [A priori, this is at least as strong as showing that ρ is irreducible over $\mathbb{R}$.]

(d) Show that the restriction $\mathrm{Res}^{D_{2n}}_{\langle x \rangle}$ of ρ to $\langle x \rangle$ is irreducible over $\mathbb{R}$ if and only if $n \geq 3$.

(e) By contrast, show that the same restriction $\mathrm{Res}^{D_{2n}}_{\langle x \rangle}$ is always reducible over $\mathbb{C}$.

Solution.

(a) Since $\mathbb{Z}/n$ is cyclic, any representation is completely determined by its mapping of a generator of $\mathbb{Z}/n$, say 1. The proposed representations are

$$\rho_k : \mathbb{Z}/n \to \mathbb{C},$$
$$1 \mapsto \exp(2\pi i k/n), k = 1, \dots, n,$$
$$\rho_k(l) = p_k(1)^l.$$

Indeed, this is a representation because

$$\rho_k(a+b) = \exp(2\pi i k/n)^{a+b} = \exp(2\pi i k/n)^a \exp(2\pi i k/n)^b = \rho_k(a)\rho_k(b).$$

These representations are also non-isomorphic because their characters are not equal. This can be seen by considering the character of a generator: $\rho_k(1) = \exp(2\pi i k/n) \neq \exp(2\pi i l/n)$ for $l \neq k, 1 \leq, k, l \leq n$.

(b) We have to verify that x, y and x^n,

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Problem 3. Let ρ be as in in Problem 1 with $k = \mathbb{C}$.

(a) Describe explicitly the characted χ_ρ of ρ: Given a permutation σ, what is the value $\chi_\rho(\sigma)$ in terms of its cycle type?

(b) Assume a finite group G acts on a finite set X. Let $\mathsf{Fix}(g) := \{x \in X \mid gx = x\}$; the fixed points of g. What is $\sum_{g\in G} |\mathsf{Fix} g|$?

(c) Deduce that $[\chi_\rho, \mathbb{1}] = 1$.

(d) Recall that G acts doubly-transitively on X if the action of G on $X \times X$ has precisely two orbits (necessarily the diagonal and the rest). If G acts doubly-transitively on X and r is the corresponding linear representation of G, show that $[\chi_r, \mathbb{1}] = 1$ and $[\chi_r, \chi_r] = 2$. [Hint: Consider also the character of the linear representation gotten from the action of G on $X \times X$.]

(e) Deduce that $[\chi_{\rho'}, \chi_{\rho'}] = 1$.

Solution.

(a)

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Problem 4. Let $k = \mathbb{C}$. Let ρ be the representation of the dihedral group from Problem 2.

(a) Compute the character of ρ.

(b) Let χ, ψ be one-dimensional characters of a finite group G. Show that

$$(\chi, \psi) = \delta_{\rho,\psi} := \{1, \chi = \psi, 0, \text{ otherwise}$$

[This is the first orthogonality relation in the special case of one-dimensional characters. You may not cite the general orthogonality relation or use its proof here.]

(c) Let $n \geq 2$. Show that $\sum_{m=1}^{n} e^{2\pi i m/n} = 0$ in two ways: Use **(b)** and use the polynomial $x^n - 1$.

(d) Show that $[\chi_\rho, \chi_\rho] = 1$. [Hint: Use **(c)**. I found it helpful to distinguish between the cases n even and n odd.]

Solution.

(a)

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Problem 5. Assume k is algebraically closed and $\mathsf{Char}(k) \neq 2$.

(a) Let (V, ρ) be a self-dual irreducible representation of G over k. Show that ρ is either symplectic or orthogonal.

(b) Assume (V, ρ) is a representation which is both symplectic and orthogonal. Show that (V, ρ) is reducible.

Solution.

(a)

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Problem 6. Consider the two permutation representations

$$\rho_1, \rho_2 : \mathbb{Z}/2 \times \mathbb{Z}/2 \to S_6$$

given by

$$\rho_1((1,0)) = \rho_2((1,0)) = (12)(34),$$

while

$$\rho_1((0,1)) = (13)(24), \rho_2((0,1)) = (12)(56).$$

(a) Show that ρ_1 and ρ_2 are element-conjugate.
(b) Show that ρ_1 and ρ_2 are not globally conjugate.

Solution.

(a) Of the given cases, we need only examine how $\rho_1((0,1))$ relates to $\rho_2((0,1))$. What we have to show is that there is some $\tau \in S_6$ such that $\tau\rho_2((0,1))\tau^{-1} = \rho_1((0,1))$. Take $\tau = (235)(64)$. Then

$$\tau\rho_2((0,1))\tau^{-1} = (235)(64)(12)(56)(64)(532) = (13)(24).$$

By representations being homomorphisms, we have that $\rho_1((1,1)) = \rho_1((1,0)+(0,1)) = \rho_1((1,0))\rho_1((0,1)) = (12)(34)(13)(24) = (14)(23)$ and $\rho_2((1,1)) = \rho_2((1,0)+(0,1)) = \rho_2((1,0))\rho_2((0,1)) = (12)(34)(12)(56) = (34)(56)$. take $\sigma = (31526)$. Then

$$\sigma\rho_1((1,1))\sigma^{-1} = (31526)(34)(56)(62513) = (14)(23) = \rho_2((1,1)).$$

Hence we have proved that ρ_1 and ρ_2 are element-conjugate.

(b) Suppose they are globally conjugate by τ. Then we must have that

$$\tau\rho_1((0,1))\tau^{-1} = \rho_2((0,1)) \iff \tau(13)(24)\tau^{-1} = (12)(56),$$
$$\tau\rho_1((1,1))\tau^{-1} = \rho_2((1,1)) \iff \tau(14)(23)\tau^{-1} = (34)(56).$$

Now, $(12)(56) = [(13)(24)](34)(56)[(13)(24)]$, so substituting in the above we get

$$\tau(13)(24)\tau^{-1} = (13)(24)\tau(14)(23)\tau^{-1}(13)(24)$$
$$\iff \tau(13)(24)\tau^{-1} = (13)(24)\tau(14)(23)\tau^{-1}(13)(24)\tau\tau^{-1}$$

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