I wish I had more time to spend on this assignment. As a result of this lack of time, some solutions are incomplete, and others are nonsensical. I apologize in advance for this.

PROBLEM 1 (2.3)**.** Let χ be a character of G. Define $\det\chi : G \to \mathbb{C}$ as follows: Choose ρ affording χ and set

$$(\det\chi)(g) = \det\rho(g).$$

Show that $\det\chi$ is a uniquely defined linear character of G.

SOLUTION. To prove uniqueness, we must prove that $\det\chi$ does not depend on the choice of ρ. Let ρ and ψ be representations affording χ. Since ρ and ψ both afford χ, we know that they are similar by Corollary 2.9: $\rho(g) = P\psi(g)P^{-1}$ for some matrix P. Then by the multiplicativity of det we have $\det(\rho(g)) = \det(P\psi(g)P^{-1}) = \det(P)\det(\psi(g))\det(P^{-1}) = \det(P)\det(P^{-1})\det(\psi(g)) = \det(PP^{-1})\det(\psi(g)) = \det(\psi(g))$.

Now for the degree of $\det(\chi)$. Any representation ρ has $\rho(1) = \mathbb{1}$, so $\det\chi(1) = \det(\mathbb{1}) = 1$, so $\det\chi$ is linear. Therefore it is a homomorphism from G to $\mathbb{C}$ and thus a character of degree 1.

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Problem 2 (2.4).

(A) Let G be a nonabelian group of order 8. Show that G has a unique nonlinear irreducible character χ. Show that $\chi(1) = 2, \chi(z) = -2, \chi(x) = 0$, where $z \in G' - \{1\}$ and $x \in G - G'$.
(B) If $G \cong D_8$, show that $\det \chi \neq 1_G$.
(C) If $G \cong Q_8$, show that $\det \chi = 1_G$.

HINT: Show that $\mathrm{Ker}(\det \chi)$ **contains all elements of order 4. Use Lemma 2.15.** DIHEDRAL HAR 2 element av order 4, Sym har 6

Solution.

(A) Suppose G is nonabelian of order 8. We know by Corollary 2.6. that a group is abelian iff all its irreducible characters are linear, so we know that there is at least one nonlinear irreducible character. Let k be the number of conjugacy classes of G, which we know is not 8. Then

$$8 = \sum_{i=1}^{k} \chi_i(1)^2,$$

where we know that the $\chi_i(1)$ are integers. There are two possibilities: either there are two conjugacy classes $1, 2$ and both of their characters $\chi_1(1)$ and $\chi_2(1)$ fulfill $\chi_i(1) = 2$, or there is one conjugacy class i with $\chi_i(1) = 2$ and four conjugacy classes who have linear characters. But certainly the trivial character is linear, so the first option is not possible. We conclude that G has a unique nonlinear irreducible character χ and that $\chi(1) = 2$.

By Corollary 2.23 we know that $|G : G'| =$ the number of linear characters in G, and by the above paragraph we know that this number is 4. Therefore Lagrange's theorem implies that there are two elements in G', one of which is the identity and one of which we denote by z. Certainly no element except for the identity is conjugate to the identity. Also, by Corollary 2.23, $\chi(z) = 1$ on linear characters. By the second orthogonality relation

$$\begin{aligned} 0 &= \sum_{\chi' \in \mathrm{Irr}(G)} \chi'(1)\overline{\chi'(z)} \\ &= \sum_{\chi' \text{ linear } \in \mathrm{Irr}(G)} \chi'(1)\overline{\chi'(z)} + \chi(1)\overline{\chi(z)} \\ &= 4 + 2\overline{\chi(z)} \end{aligned}$$

so $\chi(z) = \overline{\chi(z)} = -2$. Finally, by the first orthogonality relation

$$\begin{aligned} 8 = |G| &= \sum_{g \in G} \chi(g)\chi(g^{-1}) \\ &= \sum_{g \in G} \chi(g)\overline{\chi(g)} \\ &= \sum_{g \in G'} \chi(g)\overline{\chi(g)} + \sum_{g \in G - G'} \chi(g)\overline{\chi(g)} \\ &= 2^2 + (-2)^2 + \sum_{g \in G - G'} \chi(g)\overline{\chi(g)} \\ &= 8 + \sum_{g \in G - G'} \chi(g)\overline{\chi(g)}, \end{aligned}$$

which implies that $\chi(g) = 0$ for all $g \in G - G'$. This completes the proof.

(B) Note that $D_8 = \langle s, r \mid s^2, r^8, srsr \rangle$ has two elements of order 4, namely r and sr. Note that $\det \chi$ is a linear character. Therefore, for an element d of order 4 we have $1 = \det \chi(1) = \det \chi(d^4) = \det \chi(d)^4$. At the same time, we can write $\chi(d) = \sum_i \varepsilon_i = 0$ and $\det \chi(d) = \Pi_i \varepsilon_i$. Therefore we can conclude that

$\Pi_i \varepsilon_i$ has modulus 1, which by Lemma 2.15. means that each of the ε_i have modulus 1. At the same time, their sum is 0. Finally, the ε_i are all of order 4, so $\chi()$

The elements of order 4 are all in $G - G'$.

(C) Recall that Q_8 has presentation $\langle \bar{e}, i, j, k \mid \bar{e}^2 = e, i^2 = j^2 = k^2 = ijk = \bar{e} \rangle$ where $\bar{e}$ commutes with the other elements of the group. Suppose we know that all elements of order 4 are in the kernel of $\det \chi$. From the relation $i^2 = j^2 = k^2 = ijk = \bar{e}$ we see that we can generate all elements of Q_8 from two elements of order four (these are i, j, k). Therefore, by linearity of the determinant, it follows that if all elements of order 4 are in the kernel of the determinant $\det \chi$, then $\det \chi$ is the identity map.

Note that $\det \chi(ijk) = \det \chi(i^2) = \det \chi(j^2) = \det \chi(k^2) = \det \chi(\bar{e})$. By linearity all of these must square to 1. But then

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PROBLEM 3 (2.5).

(A) Find a real representation of D_8 which affords the character χ of problem 2.4 **(A)**.
(B) Show that this cannot be done for the group Q_8.

SOLUTION.

(A) Recall that the dihedral group D_8 has presentation $\langle r, s | s^2, r^4 \rangle$ Consider the representation

$$r \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

It is not hard to see that $G' = \{1, s\}$ and $G - G' = \{r^i, sr^i\}, i < 4$. Seeing that the required trace formulae are satisfied is now a matter of computation, which we omit from this document.

(B) Recall that the det(χ) of such a representation is the identity representation.

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PROBLEM 4 (2.8)**.** Let χ be a faithful character of G. Show that $H \subseteq G$ is abelian iff every irreducible constituent of χ_H is linear.

SOLUTION. Recall that a group is abelian iff all its irreducible characters are linear, by Corollary 2.6. Suppose $H \subseteq G$ is abelian. Then every irreducible constituent of χ_H is linear in light of the fact stated above, since they are characters of H.

Conversely, suppose every irreducible constituent of χ_H is linear. Since χ is faithful over G, we get that χ_H is faithful over H. Consider the commutator subgroup H' of H. We know that $G' = \cap\{\mathsf{Ker}(\lambda) : \lambda \in \mathsf{Irr}(H), \lambda(1) = 1\}$. Therefore, the commutator subgroup of H is contained in the intersection of the kernels of the (linear) irreducible constituents of χ_H. But the intersection of the kernels of the irreducible constituents of χ_H is in the kernel of χ_H, and since χ_H is faithful, $\mathsf{Ker}(\chi_H) = 1$. Therefore the commutator subgroup of H is trivial, so H is abelian. ■

PROBLEM 5 (2.9).

(A) Let χ be a character of an abelian group A. Show that
$$\sum_{a\in A}|\chi(x)|^2 \geq |A|\chi(1).$$

(B) Let $A \subseteq G$ with A abelian and $|G:A| = n$. Show that $\chi(1) \leq n$ for all $\chi \in \mathrm{Irr}(G)$.

SOLUTION.

(A) By Corollary 2.17, we have that $[\chi,\chi] \geq 1$. Therefore
$$[\chi,\chi] = \frac{1}{|A|}\sum_{a\in A}\chi(a)\overline{\chi(a)} = \frac{1}{|A|}\sum_{a\in A}|\chi(a)|^2 \geq 1$$
$$\iff \sum_{a\in A}|\chi(a)|^2 \geq |A|.$$
In an abelian group all characters are linear, so $\chi(1) = 1$. Therefore
$$\sum_{a\in A}|\chi(a)|^2 \geq |A| = |A|\chi(1).$$
The proof is complete.

(B) Let $\chi \in \mathrm{Irr}(G)$. By part **(A)** and the first orthogonality relation we have
$$|G| = \sum_{g\in G}|\chi(g)|^2 \geq \sum_{a\in A}|\chi(a)|^2 \geq |A| = |A|\chi_H(1).$$
EJ KLAR Since A is abelian, $A = Z(A)$. By Corollary 2.28,

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PROBLEM 6 (2.10)**.** Suppose $G = \bigcup_{i=1}^{n} A_i$, where the A_i are abelian subgroups of G and $A_i \cap A_j = 1$ if $i \neq j$.

(A) Let $\chi \in \mathrm{Irr}(G)$. Show that if $\chi(1) > 1$, then $\chi(1) \geq |G|/(n-1)$.

(B) If G is nonabelian, then $|A_i| \leq n-1$ for each i and $n-1 \geq |G|^{1/2}$.

SOLUTION.

(A)

(B) If G is nonabelian, there is at least one irreducible nonlinear character χ. By part **(A)**, we have that $\chi(1) \geq |G|/(n-1)$. Moreover, by Problem 2.9. we know that $\chi(1) \leq |G : A_i| = |G|/|A_i|$. Therefore $|G|/|A_i| \geq \chi(1) \geq |G|/(n-1)$, so $|A_i| \leq n-1$. It follows immediately that $n-1 \geq |G|/\chi(1)$. By Corollary 2.30 we also have $\chi(1)^2 \leq |G : Z(\chi)| \leq |G|$ for $\chi(g) \in \mathrm{Irr}(G)$, from which it follows that $\chi(1) \leq |G|^{1/2}$. Hence $n-1 \geq |G|/\chi(1) \geq |G|/|G|^{1/2} = |G|^{1/2}$, which is our desired result.

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Problem 7 (2.11). Let $g \in G$. Show that g is conjugate to g^{-1} in G iff $\chi(g)$ is real for all characters χ of G.

Solution. Suppose $\chi(g)$ is real for all characters χ of G. In any case, for $\chi = \mathbb{1}$ we have $\mathbb{1}(g)\mathbb{1}(g^{-1}) = \mathbb{1}(g)\overline{\mathbb{1}(g)} = 1$. Then we have

$$\sum_{\chi \in \mathrm{Irr}(g)} \chi(g)\overline{\chi(g^{-1})} = \sum_{\chi \in \mathrm{Irr}(g)} \chi(g)^2 > 0,$$

since $\chi(g)$ are real and not all zero because the trivial character is nonzero. By the second orthogonality relation, g and g^{-1} are conjugate in G.

Conversely, suppose g and g^{-1} are conjugate in G. Then by Theorem 2.2.1. in the course notes, $\chi(g) = \chi(g^{-1})$ for all irreducible characters (and therefore all characters) χ. On the other hand $\chi(g^{-1}) = \overline{\chi(g)}$ for all characters χ, so $\overline{\chi(g)} = \chi(g)$ for all χ. We conclude that $\chi(g)$ is real for all characters. The proof is complete. ■