1.

den 16 september 2018 22:00

Uppgift 1. Låt $Ax = b$ vara ett linjärt ekvationssystem, där A är en $m \times n$-matris (d.v.s. m ekvationer i n variabler) och $Ax = 0$ det tillhörande homogena ekvationssystemet. Motivera varför var ett av följande påståenden är sant eller fel:

- Om $Ax = 0$ har en nollskild lösning, så har $Ax = b$ också en lösning, oavsett m och n.
- Om $Ax = b$ har en lösning, så har $Ax = 0$ en nollskild lösning, oavsett m och n.
- Om $m < n$ så kan $Ax = b$ inte ha en unik lösning.
- Om $m > n$ så kan $Ax = b$ inte ha en unik lösning.
- Om $Ax = b$ har en entydig lösning så är $x = 0$ den enda lösningen till $Ax = 0$.

Ett enkelt motexempel är den bästa motiveringen till varför ett påstående är fel!

a) Låt $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, och låt $\bar{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Då har Falskt!

$A\bar{x} = \bar{b}$ ingen lösning.

b) Om lösningen är unik så har $A\bar{x} = \bar{0}$ bara den triviala lsg av satsen om nästan allt!

c) sant, systemet är antingen inkonsistent eller har oändligt många lösningar.

d) Falskt, t.ex.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} a \\ b \\ c \\ b \end{bmatrix}.$$

e) Sant, ty då är A:s kolonnvektorer linjärt oberoende.

2.

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Uppgift 2. Matrisen

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

är ett speciallfall av en typ av matriser som ofta förekommer i olika tillämpningar, exempelvis i samband med diskretisering av differentialekvationer för numerisk lösning. Använd rad- eller kolonnoperationer för att beräkna determinanten av matrisen A.

2. Med hjälp av *elementära* radoperationer transformerar vi matrisen till en övertriangulär matris vars determinant sedan enkelt kan beräknas, eftersom determinanten av en triangulär matris är lika med produkten av dess diagonalelement.

   I själva verket räcker det i detta exempel med upprepad användning av operationen "addera en multipel av en rad till en annan rad", vilket som bekant inte ändrar determinantens värde.

$$\begin{vmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} r_1 \\ r_2 + \frac{1}{2} r_1 \\ r_3 \\ r_4 \\ r_5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} r_1 \\ r_2 \\ r_3 + \frac{2}{3} r_2 \\ r_4 \\ r_5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 + \frac{3}{4} r_3 \\ r_5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & -1 \end{vmatrix} = \begin{vmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{vmatrix}$$

$$
= \begin{vmatrix} 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & -1 \\ 0 & 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} r_3 \\ r_4 \\ r_5 + \frac{4}{5}\, r_4 \end{vmatrix}
$$

$$
= \begin{vmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & -1 \\ 0 & 0 & 0 & 0 & \frac{6}{5} \end{vmatrix} = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \frac{6}{5} = 6.
$$

3.

den 16 september 2018 22:12

Uppgift 3. De fyra vektorerna

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{och} \quad \vec{u}_4 = \begin{bmatrix} 3 \\ 5 \\ 3 \\ 0 \end{bmatrix}$$

i $\mathbb{R}^4$ är linjärt beroende. Skriv en av dem som en linjärkombination av de andra.

Ekvivalent med

$$\bar{u}_4 = t \cdot \bar{u}_1 + s \cdot \bar{u}_2 + v \cdot \bar{u}_3$$

$$\begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} t \\ s \\ v \end{bmatrix} = \bar{u}_4$$

Ger systemet

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ -1 & 2 & 2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} t \\ s \\ v \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \\ 0 \end{bmatrix} \sim \left\{ \begin{matrix} r_2 - r_1 \\ r_3 + r_1 \\ r_4 - 2r_1 \end{matrix} \right\} \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 4 & 6 \\ \cancel{0} & \cancel{-3} & \cancel{-4} & -6 \end{array}\right] \sim$$

$$\sim \{ r_3 - 3r_2 \} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow \quad v = 0$

$s = 2$

$t = 1$

Kontroll:

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}}_{\bar{u}_1} + 2\underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}}_{\bar{u}_2} = \begin{bmatrix} 3 \\ 5 \\ 3 \\ 0 \end{bmatrix} = \bar{u}_4$$

4.

den 16 september 2018 22:31

Uppgift 4. En *tetraeder* är en tredimensionell kropp med fyra hörn och där sidorna är trianglar. De fyra punkterna $O = (0,0,0)$, $A = (1,2,-3)$, $B = (3,1,0)$ och $C = (0,2,1)$ utgör hörnen i en tetraeder. Volymen av tetraedern kan beräknas som en sjättedel av absolutbeloppet av trippelprodukten, $(\vec{u} \times \vec{v}) \cdot \vec{w}$, av de tre vektorer som går från ett av hörnen till de tre andra.

(a) Beräkna vektorprodukten av de två vektorerna $\vec{u} = \overrightarrow{OA}$ och $\vec{v} = \overrightarrow{OB}$

(b) Beräkna volymen av den tetraeder som har hörn i punkterna O, A, B och C.

a) $\overline{OA} \times \overline{OB} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & -3 \\ 3 & 1 & 0 \end{vmatrix} = \hat{x} \cdot 3 - \hat{y} \cdot 9 + \hat{z} \cdot (-4) = \begin{bmatrix} 3 \\ -9 \\ -4 \end{bmatrix}$.

b) $V = \dfrac{|(\overline{OA} \times \overline{OB}) \cdot \overline{OC}|}{6} = \dfrac{\left| \begin{bmatrix} 3 \\ -9 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right|}{6} = |-18-4| = \dfrac{|-22| = 22.}{6}$

Diskussion

den 16 september 2018 22:35

ANNAT

Här är några andra moment som är viktiga och intressanta att diskutera.

- Vad är definitionen of linjärt oberoende? Vad är ekvivalenta sätt att uttrycka det?
- Vilken metod är bäst för att beräkna en determinant?
- Kan man definiera kryssprodukten för vektorer av annan längd än 3 på ett vettigt sätt?

- Se boken.
- reducera till 0-or på någon rad eller kolumn och kofaktorexpandera! Typ alltid enklast.
-

Generalizations [edit]

There are several ways to generalize the cross product to the higher dimensions.

Lie algebra [edit]

Main article: Lie algebra

The cross product can be seen as one of the simplest Lie products, and is thus generalized by Lie algebras, which are axiomatized as binary products satisfying the axioms of multilinearity, skew-symmetry, and the Jacobi identity. Many Lie algebras exist, and their study is a major field of mathematics, called Lie theory.

For example, the Heisenberg algebra gives another Lie algebra structure on $\mathbf{R}^3$. In the basis $\{x, y, z\}$, the product is $[x, y] = z, [x, z] = [y, z] = 0.$

Quaternions [edit]

Further information: quaternions and spatial rotation

The cross product can also be described in terms of quaternions, and this is why the letters **i**, **j**, **k** are a convention for the standard basis on $\mathbf{R}^3$. The unit vectors **i**, **j**, **k** correspond to "binary" (180 deg) rotations about their respective axes (Altmann, S. L., 1986, Ch. 12), said rotations being represented by "pure" quaternions (zero real part) with unit norms.

For instance, the above given cross product relations among **i**, **j**, and **k** agree with the multiplicative relations among the quaternions *i*, *j*, and *k*. In general, if a vector $[a_1, a_2, a_3]$ is represented as the quaternion $a_1i + a_2j + a_3k$, the cross product of two vectors can be obtained by taking their product as quaternions and deleting the real part of the result. The real part will be the negative of the dot product of the two vectors.

Alternatively, using the above identification of the 'purely imaginary' quaternions with $\mathbf{R}^3$, the cross product may be thought of as half of the commutator of two quaternions.

Octonions [edit]

See also: Seven-dimensional cross product and Octonion

A cross product for 7-dimensional vectors can be obtained in the same way by using the octonions instead of the quaternions. The nonexistence of nontrivial vector-valued cross products of two vectors in other dimensions is related to the result from Hurwitz's theorem that the only normed division algebras are the ones with dimension 1, 2, 4, and 8.

Exterior product [edit]

Main article: Exterior algebra

In general dimension, there is no direct analogue of the binary cross product that yields specifically a vector. There is however the exterior product, which has similar properties, except that the exterior product of two vectors is now a 2-vector instead of an ordinary vector. As mentioned above, the cross product can be interpreted as the exterior product in three dimensions by using the Hodge star operator to map 2-vectors to vectors. The Hodge dual of the exterior product yields an $(n-2)$-vector, which is a natural generalization of the cross product in any number of dimensions.

The exterior product and dot product can be combined (through summation) to form the geometric product.

External product [edit]

As mentioned above, the cross product can be interpreted in three dimensions as the Hodge dual of the exterior product. In any finite *n* dimensions, the Hodge dual of the exterior product of *n*-1 vectors is a vector. So, instead of a binary operation, in arbitrary finite dimensions, the cross product is generalized as the Hodge dual of the exterior product of some given *n*-1 vectors. This generalization is called **external product**[15]

Multilinear algebra [edit]

In the context of multilinear algebra, the cross product can be seen as the (1,2)-tensor (a mixed tensor, specifically a bilinear map) obtained from the 3-dimensional volume form,[note 2] a (0,3)-tensor, by raising an index.

In detail, the 3-dimensional volume form defines a product $V \times V \times V \to \mathbf{R}$, by taking the determinant of the matrix given by these 3 vectors. By duality, this is equivalent to a function $V \times V \to V^*$, (fixing any two inputs gives a function $V \to \mathbf{R}$ by evaluating on the third input) and in the presence of an inner product (such as the dot product; more generally, a non-degenerate bilinear form), we have an isomorphism $V \to V^*$, and thus this yields a map $V \times V \to V$, which is the cross product: a (0,3)-tensor (3 vector inputs, scalar output) has been transformed into a (1,2)-tensor (2 vector inputs, 1 vector output) by "raising an index".

Translating the above algebra into geometry, the function "volume of the parallelepiped defined by $(a, b, -)$" (where the first two vectors are fixed and the last is an input), which defines a function $V \to \mathbf{R}$, can be *represented* uniquely as the dot product with a vector: this vector is the cross product $a \times b$. From this perspective, the cross product is *defined* by the scalar triple product, $\mathrm{Vol}(a, b, c) = (a \times b) \cdot c$.

In the same way, in higher dimensions one may define generalized cross products by raising indices of the *n*-dimensional volume form, which is a $(0, n)$-tensor. The most direct generalizations of the cross product are to define either:

- a $(1, n-1)$-tensor, which takes as input $n-1$ vectors, and gives as output 1 vector – an $(n-1)$-ary vector-valued product, or
- a $(n-2, 2)$-tensor, which takes as input 2 vectors and gives as output skew-symmetric tensor of rank *n* − 2 – a binary product with rank *n* − 2 tensor values. One can also define $(k, n-k)$-tensors for other *k*.

These products are all multilinear and skew-symmetric, and can be defined in terms of the determinant and parity.

The $(n-1)$-ary product can be described as follows: given $n-1$ vectors $v_1, \ldots, v_{n-1}$ in $\mathbf{R}^n$, define their generalized cross product $v_n = v_1 \times \cdots \times v_{n-1}$ as:

- perpendicular to the hyperplane defined by the v_i,
- magnitude is the volume of the parallelotope defined by the v_i, which can be computed as the Gram determinant of the v_i,
- oriented so that $v_1, \ldots, v_n$ is positively oriented.

This is the unique multilinear, alternating product which evaluates to $e_1 \times \cdots \times e_{n-1} = e_n$, $e_2 \times \cdots \times e_n = e_1$, and so forth for cyclic permutations of indices.

In coordinates, one can give a formula for this $(n-1)$-ary analogue of the cross product in $\mathbf{R}^n$ by:

$$\bigwedge(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}) = \begin{vmatrix} {v_1}^1 & \cdots & {v_1}^n \\ \vdots & \ddots & \vdots \\ {v_{n-1}}^1 & \cdots & {v_{n-1}}^n \\ \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{vmatrix}.$$

This formula is identical in structure to the determinant formula for the normal cross product in $\mathbf{R}^3$ except that the row of basis vectors is the last row in the determinant rather than the first. The reason for this is to ensure that the ordered vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \Lambda(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}))$ have a positive orientation with respect to $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$. If *n* is odd, this modification leaves the value unchanged, so this convention agrees with the normal definition of the binary product. In the case that *n* is even, however, the distinction must be kept. This $(n-1)$-ary form enjoys many of the same properties as the vector cross product: it is alternating and linear in its arguments, it is perpendicular to each argument, and its magnitude gives the hypervolume of the region bounded by the arguments. And just like the vector cross product, it can be defined in a coordinate independent way as the Hodge dual of the wedge product of the arguments.

Skew-symmetric matrix [edit]

If the cross product is defined as a binary operation, it takes as *input* exactly two vectors. If its *output* is not required to be a vector or a pseudovector but instead a matrix, then it can be generalized in an arbitrary number of dimensions.[16][17][18]

In mechanics, for example, the angular velocity can be interpreted either as a pseudovector ω or as a anti-symmetric matrix or skew-symmetric tensor Ω. In the latter case, the velocity law for a rigid body looks:

$$\mathbf{v}_P - \mathbf{v}_Q = \Omega \cdot (\mathbf{r}_P - \mathbf{r}_Q)$$

where Ω is formally defined from the rotation matrix $R^{N \times N}$ associated to body's frame: $\Omega \triangleq \frac{dR}{dt} R^{\mathrm{T}}$. In three-dimensions holds:

$$\Omega = [\omega]_\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

In quantum mechanics the angular momentum L is often represented as an anti-symmetric matrix or tensor. More precisely, it is the result of cross product involving position $\mathbf{x}$ and linear momentum $\mathbf{p}$:

$$L_{ij} = x_i p_j - p_i x_j$$

Since both $\mathbf{x}$ and $\mathbf{p}$ can have an arbitrary number N of components, that kind of cross product can be extended to any dimension, holding the "physical" interpretation of the operation.

See § Alternative ways to compute the cross product for numerical details.

History [edit]

Ja! Definitivt. men det blir väldigt abstrakt väldigt snabbt..