Lower Bounds for CSP Hierarchies Through Ideal Reduction

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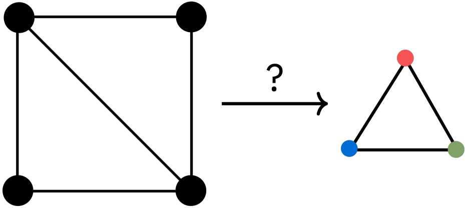
Graph Coloring

Can the vertices of a given graph G be colored with c colors so that no edge is monochromatic?

Equivalently: is there a homomorphism $G \to K_c$?



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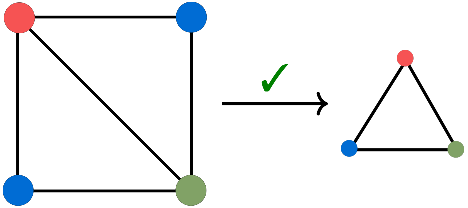

Homomorphisms and CSPs

Special case of constraint satisfaction problem $\mathrm{CSP}(\mathscr{A}, \mathscr{T})$:

Given relational structures $\mathbf{A} \in \mathscr{A}$ *and* $\mathbf{T} \in \mathscr{T}$*, decide whether* $\mathbf{A} \to \mathbf{T}$.

Target $\mathbf{T}$ often fixed; then called the *template*

Graph c-coloring is $\mathrm{CSP}(\mathscr{G}, K_c)$

Promise CSPs [AGH17; BG21]

Decide if instance satisfiable in strong sense, or not even in weak sense

Approximate graph coloring [GJ76]: decide whether input graph is c-colorable or not even ℓ-colorable, for $\ell \geq c$

Conjectured NP-hard for all $c, \ell \geq 3$; unknown even for $c = 3,\ \ell \geq 6$...

In general, for $\mathbf{A}$ and $(\mathbf{S}, \mathbf{T})$ such that $\mathbf{S} \to \mathbf{T}$, decide whether $\mathbf{A} \to \mathbf{S}$ or $\mathbf{A} \not\to \mathbf{T}$

Algorithms for (P)CSPs

Conjecture [FV98]: Every fixed-template CSP either in P or NP-complete

Resolved by Bulatov [Bul17] and Zhuk [Zhu20]

Proof: algorithms for all remaining conjectured tractable cases

One per template; is there a *uniform* algorithm?

Also useful for PCSPs...

Algorithms for (P)CSPs

(P)CSP algorithms in general combine two building blocks:

1. strong local consistency: check if small subproblems satisfiable; accept iff local solutions consistent with each other
2. weak global consistency: e.g., relax domain of 0-1 ILP formulation to field $\mathbb{F}$ or to $\mathbb{Z}$

Examples of (P)CSP Algorithms

Recent work proposes and studies simpler (P)CSP algorithms:

- integer linear equations [BG17; BG19; BBKO21]
- LP/SDP + integer linear equations [BGWŽ20; CŽ23; CŽ24; CNP24]
- k-consistency + integer linear equations: $\mathbb{Z}$-*affine* and *cohomological* variants [DO24; ÓCo22]

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Lichter and Pago [LP25]: most algorithms above fail on tractable CSP

Moreover, cohomological k-consistency correctly solves CSP hard for other algorithms!

Hierarchies and Fooling Instances

All mentioned algorithms are *hierarchies*, parametrized by *level* k

- tighter relaxations for larger k
- k-th level runs in time $m^{O(k)}$ on instance of size m
- $\omega(1)$ level lower bound $\implies$ not guaranteed polynomial running time

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Level lower bound for $\mathrm{CSP}(\mathbf{T})$ through *fooling instance*:

- structure $\mathbf{A}$ such that $\mathbf{A} \not\to \mathbf{T}$, but is accepted by $\leq k$-th level
- for $\mathrm{PCSP}(\mathbf{S}, \mathbf{T})$: $\mathbf{A} \not\to \mathbf{T}$, but hierarchy for $\mathrm{CSP}(\mathbf{S})$ accepts $\mathbf{A}$

Main Results

Optimal hardness of approximate graph coloring for cohomological k-consistency:

Theorem

Cohomological k-consistency does not solve c vs ℓ-coloring on n-vertex graphs, for any $\ell \geq c \geq 3$ and $k \leq \Omega(n)$.

Best previous result [CN25]: c vs $\left(\binom{c}{\lfloor c/2 \rfloor} - 1\right)$-coloring for $c \geq 4$ and $k \leq \Omega(n)$

(Reduction to best known NP-hardness result [KOWŽ23])

Main Results

Show connection to *polynomial calculus*

Informally: *Alekhnovich–Razborov method* [AR03] for degree lower bounds carries over to cohomological k-consistency level

Can then use degree lower bounds from [CdRN+25]

Polynomial Calculus [CEI96]

To prove polynomials in set $\mathscr{P} = \{p_1, \ldots, p_m\}$ have no common root over field $\mathbb{F}$, derive polynomials in ideal $\langle \mathscr{P} \rangle$ by

$$\text{Linear combination: } \frac{p \quad q}{\alpha p + \beta q} \qquad \alpha, \beta \in \mathbb{F}$$

$$\text{Multiplication: } \frac{p}{x \cdot p} \qquad x \text{ any variable}$$

Refutation of $\mathscr{P}$ is derivation of 1

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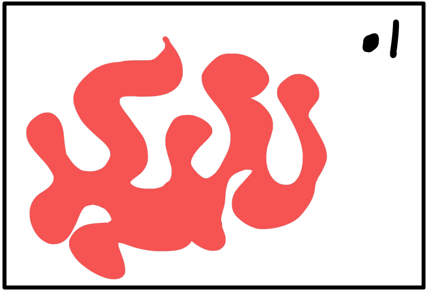
Complexity measures:

- **Size:** Total # of monomials in proof lines (with multiplicities)
- **Degree:** Largest degree among monomials in proof lines

Degree Lower Bounds Through Operators



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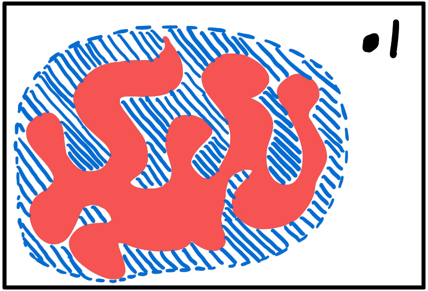

- Derivable in degree $\leq D$
- Overapproximation

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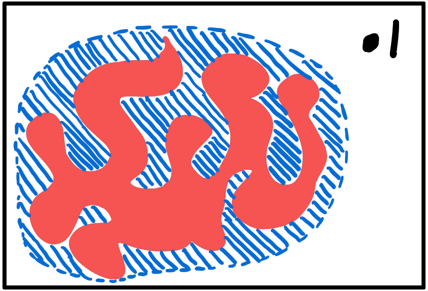

Derivable in degree $\leq D$

Overapproximation

Define so-called R**-operator** [Raz98] on polynomials such that

- $R(p) = 0$ for each input polynomial p
- $R(p) + R(q) = R(p+q)$
- If $R(p) = 0$ then $R(x \cdot p) = 0$, for all p of degree $\leq D-1$
- $R(1) = 1$

Overapproximation is kernel of R

R as in Reduction

Put (admissible) order $\prec$ on monomials in $\mathbb{F}[x_1, \ldots, x_n]$, where 1 smallest

Leading term LT of p is term with largest monomial w.r.t. $\prec$

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For ideal $\langle \mathscr{P} \rangle$, define *reduction operator* $R_{\langle \mathscr{P} \rangle} : p \mapsto r$

Algorithm Reduction of p modulo $\langle \mathscr{P} \rangle$

1: $r \leftarrow p$
2: **while** $\exists q \in \langle \mathscr{P} \rangle$ such that $\mathrm{LT}(q) \in r$ **do**
3: $\quad r \leftarrow r - q$
4: **end while**
5: **return** r

"generalized row-echelon form + remainder modulo division"

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Then, R-operator can be reduction modulo $\langle \mathscr{P} \rangle$:

- $R(p) = 0$ for each polynomial $p \in \mathscr{P}$
- $R(p) + R(q) = R(p+q)$ by definition
- If $R(p) = 0$, then $R(x \cdot p) = 0$ for *all* p
- $R(1) = 1$ since 1 smallest and not in ideal

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For unsatisfiable input, *pseudo-reduction* operator R should behave as $R_{\langle \mathscr{P} \rangle}$ would on low-degree polynomials if $\mathscr{P}$ satisfiable.

Alekhnovich–Razborov Operators

Based on this heuristic, Alekhnovich and Razborov [AR03] define *local pseudo-reduction* operator for $\mathscr{P}$:

- to each low-degree m, associate satisfiable $S(m) \subseteq \mathscr{P}$
- define local reduction $R(m) = R_{\langle S(m) \rangle}(m)$
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(Technical challenge: show R agrees with multiplication)

Cohomological k-Consistency

Maintain set $\mathscr{H}$ of partial homomorphisms $\mathbf{A} \to \mathbf{T}$ with domain size $\leq k$

Initialize $\mathscr{H} = \{\mathrm{Hom}(\mathbf{A}[X], \mathbf{T}) : X \in \binom{A}{\leq k}\}$; repeat steps 1-2 until stabilizes:

1. find maximal *k-consistent* subset $\mathscr{C}$ in $\mathscr{H}$; update $\mathscr{H} \leftarrow \mathscr{C}$
2. for each $\varphi \in \mathscr{C}$, solve relaxation over $\mathbb{Z}$ of 0-1 ILP $L_k(\mathbf{A}, \mathbf{T})$ encoding $\mathbf{A} \to \mathbf{T}$, subject to:
   - solution supported on $\mathscr{C}$
   - require $x_\varphi = 1$ and $x_\psi = 0$ for all $\psi \neq \varphi$ with same domain as φ

   Remove φ from $\mathscr{H}$ next loop if no solution exists

Reject if $\mathscr{H}$ empty on some domain; otherwise accept

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Proof Overview

Define set of polynomials $\mathscr{P}_{\mathbf{A}\to\mathbf{T}}$ over variables $\mathbf{x}_{A,T} = \{x_{a,t} : a \in A, t \in T\}$

Common roots are homomorphisms $\mathbf{A} \to \mathbf{T}$; partial homomorphism φ encoded as monomial $m_\varphi = \prod x_{a,\varphi(a)}$

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1. support $\{\varphi : R(m_\varphi) \neq 0\}$ is d-consistent
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The Lex Game

Polynomials $R(m_\varphi)$ all have integer coefficients if $\mathrm{char}(\mathbb{F}) = 0$

Lex Game [FRR06] characterizes reducibility of m modulo ideal I for lexicographic $\prec$ (generalizes *pigeon dance* [Raz98])

Game between Lea and Stan; winning strategy for Lea iff m reducible mod I

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Proof idea: encode Lea's strategy as polynomial p with leading term m and integer coefficients

$\implies$ all reduction steps can be done with integer polynomials!

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Thank you!

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