Geometric Bounds for Steklov Eigenvalues on Graphs

Jonas Conneryd

2021

Overview of presentation

Introduction

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$$\frac{\partial u}{\partial n} = \sigma u \text{ in } \partial M.$$

Under reasonable niceness assumptions, discrete spectrum with finite eigenvalue multiplicities.

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Spectrum of Steklov problem coincides with the spectrum of the *Dirichlet-to-Neumann operator* Λ; for function f on ∂M,

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Eigenvalues are called *Steklov eigenvalues*.

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Analogue of Δ is a *matrix* L, called the *Laplacian*.

The *harmonic extension* $u_f \in \mathbb{R}^n$ of $f : B \to \mathbb{R}$ defined on B:

$$\begin{aligned} Lu_f &= 0 \text{ in } V \setminus B \\ u_f &= f \text{ in } B. \end{aligned}$$

The *Dirichlet-to-Neumann* operator Λ_L is then

$$\Lambda_L : f \mapsto Lu_f \mid_B .$$

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Eigenvalues of DtN map Λ_L are called *Steklov eigenvalues.*

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Will present two such results; will only have time to talk in detail about one of the proofs.

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Will be concerned with finite, edge weighted graphs $G = (V, E, w)$. Unless otherwise stated, n denotes the number of vertices.

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Will often consider distinguished $B \subset V$, called the *boundary* of G. Number of boundary vertices usually denoted b.

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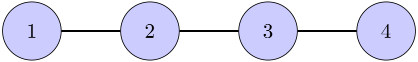

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

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In particular, if $Lf(i) = 0$ then $f(i)$ is weighted average of neighboring function values:

$$0 = (Lf)(i) = \sum_{j:(i,j)\in E} w_{ij}(f(i) - f(j)) \implies \sum_{j:(i,j)\in E} w_{ij} f(i) = \sum_{j:(i,j)\in E} w_{ij} f(j),$$

which in turn yields

$$f(i) = \frac{\sum_{j:(i,j)\in E} w_{ij} f(j)}{\sum_{j:(i,j)\in E} w_{ij}}.$$

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Especially the smallest non-zero eigenvalue - *spectral gap* - has been shown to be related to many graph invariants.

Unfortunately, there is not enough time for a thorough overview.

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If one uses a 'normalized' Laplacian, there is a corresponding 'normalized' DtN map. Will not talk more about this version here. See thesis!

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Borrow some imagery and terminology from electrical networks theory:

$$\begin{aligned} \text{Edge weight} &\iff \text{Conductance } (= 1/\text{Resistance}) \\ \text{Function on (subset of) vertices} &\iff \text{Potential} \end{aligned}$$

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Ohm's law for the current I flowing through an edge (i, j):

$$U_{ij} = R_{ij} I_{ij} \quad \Longrightarrow \quad I_{ij} = \frac{U_{ij}}{R_{ij}},$$

where U is the potential difference $f(i) - f(j)$ and $1/R_{ij} = w_{ij}$.

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Hence, the function $u_g : V \longrightarrow \mathbb{R}$ s.t. $u_g = g$ on B and Kirchhoff's law is satisfied in $V \setminus B$ is harmonic extension of g to V, since $Lu_g = 0$ on $V \setminus B$.

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Therefore, $\Lambda_L : g \mapsto (Lu_g)\,|_B$ maps *potential* $g(i)$ at i in B to *net current* $(Lu_g)(i)$, in network with potential g on B and in which Kirchhoff's law holds in $V \setminus B$.

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Therefore, if one wants to study B more closely, one can use the DtN map to study a smaller network on B, which affects B in a similar way as the whole network.

Interesting and deep topic, but no time to get into details. The interested listener can consult Dörfler and Bullo [DB13].

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Interpretation of the eigenvalues of an $n \times n$ symmetric matrix M as extremal values of the *Rayleigh quotient*

$$R_M(f) = \frac{(f, Mf)}{(f, f)},$$

for suitable restrictions on $f \in \mathbb{R}^n$.

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Note: Rayleigh quotient of eigenvector v with eigenvalue λ_v is

$$R_M(v) = \frac{(v, Mv)}{(v, v)} = \frac{(v, \lambda_v v)}{(v, v)} = \frac{\lambda_v (v, v)}{(v, v)} = \lambda_v.$$

The Courant-Fischer Theorem

Courant-Fischer Theorem gives rather explicit expressions for the smallest non-zero eigenvalues of the (combinatorial) Laplacian and DtN map as follows:

Corollary (Variational characterization of the spectral gap of the Laplacian)

Let $G = (V, E, w)$ be a connected graph with n vertices enumerated as $1, 2, \ldots, n$ and combinatorial Laplacian L. Let $\lambda_1(L)$ denote the spectral gap of the Laplacian. Then

$$\lambda_1(L) = \min_{\substack{f \in \mathbb{R}^n \\ f \neq \mathbf{0}}} \left\{ \frac{\sum_{(i,j) \in E} w_{ij}(f(i) - f(j))^2}{\sum_{i=1}^{n} f(i)^2} \mid \sum_{i=1}^{n} f(i) = 0 \right\}.$$

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This expression is very commonly used in proofs of bounds on Laplacian eigenvalues.

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Let $G = (V, E, w)$ be a connected graph with n vertices, enumerated as $1, 2, \ldots, n$. Let B be the boundary of G, and let L denote the combinatorial Laplacian of G. Denote the combinatorial DtN map of G by Λ_L. Let σ_1 denote the spectral gap of the DtN map. Then

$$\sigma_1(\Lambda_L) = \min_{\substack{f \in \mathbb{R}^n \\ f|_B \neq \mathbf{0}}} \left\{ \frac{\sum_{(i,j)\in E} w_{ij}(f(i) - f(j))^2}{\sum_{i \in B} f(i)^2} \mid \sum_{i \in B} f(i) = 0 \right\},$$

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There is a similar expression for the normalized DtN map. The interested listener is referred to the thesis.

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The expression for the DtN map is quite similar to that of the Laplacian:

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This is the idea behind both novel results in this thesis.

The DtN map on planar graphs

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(Reminder: Measure $m(i)$ is sum of edge weights of edges adjacent to vertex i.)

Theorem

Let $G = (V, E, w)$ be a planar weighted graph with boundary B. Suppose the number of boundary vertices in G, denoted b, is at least 5. Then the spectral gap $\sigma_1(\Lambda_L)$ of the combinatorial DtN map on G w.r.t. B satisfies

$$\sigma_1(\Lambda_L) \leq \frac{8 \max_{i \in V} m(i)}{b}.$$

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The proof is based on a famous paper by Spielman and Teng [ST07], generalized by Plümer [Plü20] to the weighted case.

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So, in the above we are essentially placing n numbers on the real line so that the sum of numbers corresponding to the boundary is 0.

The DtN map on planar graphs

The proof is based on a geometric interpretation of the expression for $\sigma_1(\Lambda_L)$ in the Courant-Fischer Theorem:

$$\sigma_1(\Lambda_L) = \min_{\substack{f \in \mathbb{R}^n \\ f|_B \neq \mathbf{0}}} \left\{ \frac{\sum_{(i,j)\in E} w_{ij}(f(i) - f(j))^2}{\sum_{i\in B} f(i)^2} \mid \sum_{i\in B} f(i) = 0 \right\}.$$

Functions $f : V \to \mathbb{R} \iff$ vectors v in $\mathbb{R}^n$: $f(i) \mapsto v_i$.

So, in the above we are essentially placing n numbers on the real line so that the sum of numbers corresponding to the boundary is 0.

$\sigma_1(\Lambda_L)$ is the minimum of the Rayleigh quotient

$$R_{\Lambda_L}(f) = \frac{\sum_{(i,j)\in E} w_{ij}(f(i) - f(j))^2}{\sum_{i\in B} f(i)^2}$$

among such collections of n real numbers.

The DtN map on planar graphs

Do we get the same minimum if we instead choose n vectors $\mathbf{v}_i$, $i = 1, \dots, n$ in $\mathbb{R}^l$, l arbitrary integer, so that their vector sum is $\mathbf{0}$ and input them in the analogous Rayleigh quotient

$$R'_{\Lambda_L}(\mathbf{v}_1, \dots, \mathbf{v}_n) = \frac{\sum_{(i,j)\in E} w_{ij}\|\mathbf{v}_i - \mathbf{v}_j\|^2}{\sum_{i\in B}\|\mathbf{v}_i\|^2}?$$

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I.e., can we write

$$\sigma_1(\Lambda_L) = \min_{\mathbf{v}_1,\ldots,\mathbf{v}_n\in\mathbb{R}^l} \left\{ \frac{\sum_{(i,j)\in E} w_{ij}\|\mathbf{v}_i - \mathbf{v}_j\|^2}{\sum_{i\in B}\|\mathbf{v}_i\|^2} \mid \sum_{i\in B} \mathbf{v}_i = \mathbf{0}, \{\mathbf{v}_i\}_{i\in B} \text{ not all } \mathbf{0} \right\}?$$

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The answer is yes!

The DtN map on planar graphs

The expression

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is a *minimum* - if we just take a collection of $\mathbf{v}_i$ s.t. $\sum_{i\in B} \mathbf{v}_i = \mathbf{0}$ and plug it into the Rayleigh quotient

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we will get an upper bound on $\sigma_1(\Lambda_L)$.

We will choose the $\mathbf{v}_i$ in a way which lets us bound both the numerator and denominator of the Rayleigh quotient, using that G is planar.

The DtN map on planar graphs

The classical *Koebe-Andreev-Thurston Theorem* or *Circle Packing Theorem*:

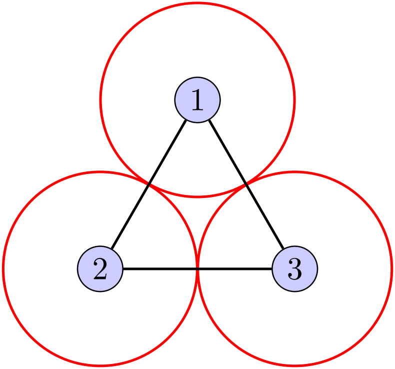
Theorem (Circle Packing Theorem)

Let $G = (V, E)$ be a planar graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. Then there is a set of disks $\mathcal{D} = \{D_1, \ldots, D_n\}$ in the plane with disjoint interiors such that D_i and D_j have a single point in common if and only if (i, j) is in E. In fact, a graph is planar if and only if there is such a set of disks.

Call $\mathcal{D} = \{D_1, \ldots, D_n\}$ a *kissing disk embedding* of G.

The DtN map on planar graphs

An example: The graph K_3.



The DtN map on planar graphs

Take a kissing disk embedding $\mathcal{D} = \{D_1, \dots, D_n\}$ of G, and map it to the unit sphere S^2 in $\mathbb{R}^3$ using (inverse) stereographic projection.

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The kissing disks D_i become *caps* C_i, i.e. intersections of S^2 with half-spaces. C_i and C_j share point iff (i, j) in E.

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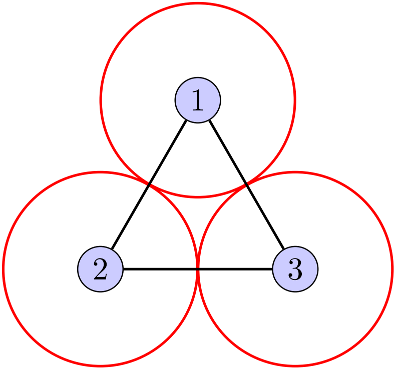
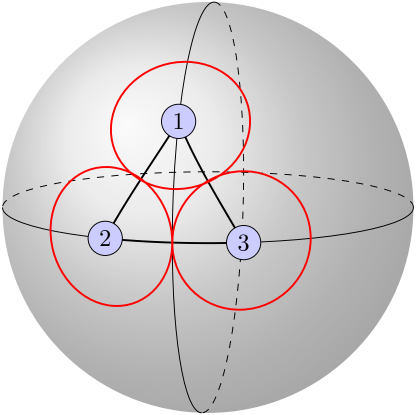
Boundaries of caps are circles. Call the point on C_i equidistant to whole boundary the *center* $p(C_i)$ of C_i. (Euclidean) distance between the boundary of C_i and $p(C_i)$ is the *radius* r_i of C_i.

The DtN map on planar graphs



(a) A kissing disk embedding of the graph K_3 in the plane.



(b) A kissing cap embedding of K_3 on the unit sphere.

Figure: Examples of kissing disk and kissing cap embeddings of K_3.

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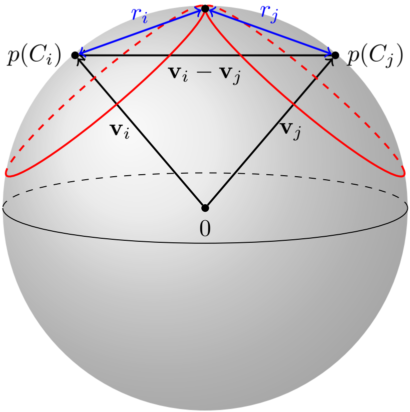
Place $\mathbf{v}_i$ at the center $p(C_i)$ of C_i.

The DtN map on planar graphs

With this embedding, $\|\mathbf{v}_i - \mathbf{v}_j\|^2 \leq (r_i + r_j)^2$:



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Young's inequality implies that $(r_i + r_j)^2 \leq 2(r_i^2 + r_j^2)$.

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Then we can write

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Moreover, since the area of the cap C_i is πr_i^2 and the area of all of S^2 is 4π,

$$\sum_{i=1}^{n} \pi r_i^2 \leq \{\text{combined area of caps}\} \leq 4\pi.$$

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\sigma_1(\Lambda_L) &\leq \frac{\sum_{(i,j)\in E} w_{ij} \|\mathbf{v}_i - \mathbf{v}_j\|^2}{\sum_{i\in B} \|\mathbf{v}_i\|^2} \\
&\leq \frac{2 \max_{i\in V} m(i) \sum_{i=1}^{n} r_i^2}{\sum_{i\in B} \|\mathbf{v}_i\|^2} \\
&= \frac{2 \max_{i\in V} m(i) \sum_{i=1}^{n} r_i^2}{b} \\
&\leq \frac{8 \max_{i\in V} m(i)}{b},
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and we are done - except for one detail.

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Main idea: Take arbitrary kissing cap embedding, stereographically project it to the tangent plane H_β of suitable β in S^2, dilate the plane H_β, and then project back onto S^2.

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The image under these transformations is another kissing cap embedding $\widetilde{C} = \{\widetilde{C}_i\}_{i=1}^n$ s.t. $\sum_{i \in B} \widetilde{C}_i = \mathbf{0}$.

The DtN map and edge connectivity

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There is a link between λ_1 and edge connectivity as well, shown by Fiedler [Fie73] in 1973:

Theorem

Let $G = (V, E)$ be a finite, connected combinatorial graph with n vertices, edge connectivity η, and combinatorial Laplacian L. Then the spectral gap $\lambda_1(L)$ of L satisfies

$$\eta + 1 \geq \lambda_1(L) \geq 2\eta \left[1 - \cos\left(\frac{\pi}{n}\right)\right].$$

The DtN map and edge connectivity

New proof, using the variational characterization

$$\lambda_1(L) = \min_{\substack{f \in \mathbb{R}^n \\ f \neq \mathbf{0}}} \left\{ \frac{\sum_{(i,j)\in E} w_{ij}(f(i) - f(j))^2}{\sum_{i=1}^n f(i)^2} \mid \sum_{i=1}^n f(i) = 0 \right\},$$

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We use the method in that paper as part of the proof of a generalization of the lower bound to the DtN map:

Theorem

Let $G = (V, E, w)$, be a connected weighted graph with boundary B. Denote the number of vertices in G by n and the number of boundary vertices in G by b. Suppose $b > 1$, and suppose G has weighted edge connectivity ν. Then the spectral gap $\sigma_1(\Lambda_L)$ of the combinatorial DtN map Λ_L of G w.r.t. B satisfies

$$\sigma_1(\Lambda_L) \geq \frac{2\nu}{n - b + 1} \left[1 - \cos\left(\frac{\pi}{b}\right) \right].$$

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Proof method (if there is time): Manipulate edges of G while maintaining control of the Rayleigh quotient in the variational characterization

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Unlike in the previous result, the same method cannot be applied to the normalized DtN map, for various reasons.

The DtN map and edge connectivity

Will not talk in detail about the whole proof.

Idea in [Ber+17]: Take f^0 to be a minimizer in the variational characterization. Label vertices of G as $\{v_i\}_{i=1}^n$, via

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Replace each edge (v_i, v_j) with a sequence of edges between (v_i, v_{i+1}), $(v_{i+1}, v_{i+2}), \ldots, (v_{j-1}, v_j)$.

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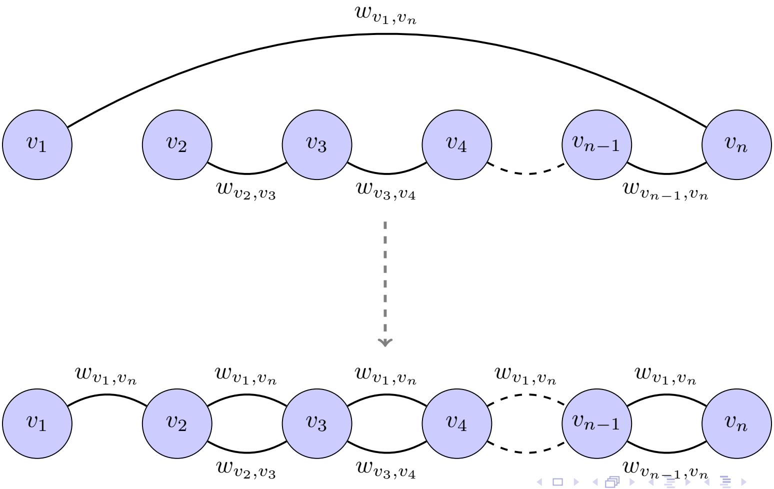
For example, if there is an edge (v_1, v_n) with weight w_{v_1,v_n} (inspiration taken from Figure 1 in [Ber+17]):



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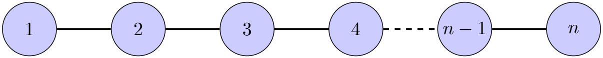
As suggested by the figure, if one does this procedure for all edges in the graph, the result is a *path graph* $\widetilde{G}$ on n vertices.



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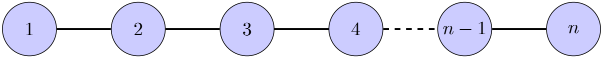

Easy to see that Rayleigh quotient

$$R_G(f^0) = \frac{\sum_{(i,j)\in E} w_{ij} \left(f^0(i) - f^0(j)\right)^2}{\sum_{i\in B} f^0(i)^2}$$

decreases, while edge connectivity does not decrease when going from G to $\widetilde{G}$.

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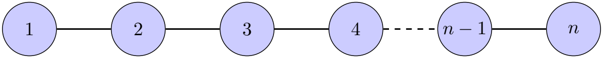

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In a path graph, edge connectivity $\iff$ minimal edge weight, so minimal edge weight in $\widetilde{G} \geq$ edge connectivity of G.

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With some more work, one can show that $R_G(f^0)$ is bounded from below by the spectral gap of the Laplacian of a path graph, whose vertices are the boundary vertices in G and whose edge weights are all equal to $\frac{\nu}{n-b+1}$.

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Therefore, we get

$$\sigma_1(\Lambda_L) = R_G(f^0) \geq \frac{2\nu}{n-b+1}\left[1-\cos\left(\frac{\pi}{b}\right)\right],$$

as we wanted.

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At least, the proved bound is better than the 'trivial' bound $\sigma_1(\Lambda_L) \leq 2 \max_{i \in V} m(i)$ (bounds all eigenvalues of L and Λ_L) when $b > 4$.

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$$\sigma_1(\Lambda_L) \leq \frac{8 \max_{i \in V} m(i)}{b},$$

the author could not come up with such a case.

At least, the proved bound is better than the 'trivial' bound $\sigma_1(\Lambda_L) \leq 2 \max_{i \in V} m(i)$ (bounds all eigenvalues of L and Λ_L) when $b > 4$.

The author has not seen any comparable upper bounds on $\sigma_1(\Lambda_L)$; lower bounds seem more widely studied.

How good are these bounds?

For the lower bound

$$\sigma_1(\Lambda_L) \geq \frac{2\nu}{n-b+1}\left[1-\cos\left(\frac{\pi}{b}\right)\right],$$

the situation is a bit better.

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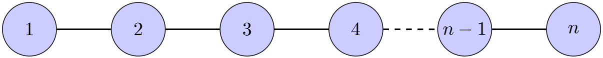
In the case of a path graph on n vertices with 2 end vertices as boundary and unit edge weights, it is known that

$$\sigma_1(\Lambda_L) = \frac{2}{n-1}.$$



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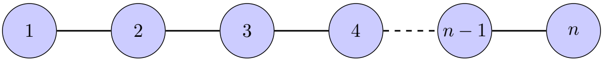

Edge connectivity is 1, and there are 2 boundary vertices. Our bound then gives

$$\sigma_1(\Lambda_L) \geq \frac{2}{n-1},$$

which is tight.

The end!

Thank you for listening!

The end!

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Questions?

[Ber+17] Gregory Berkolaiko, James B. Kennedy, Pavel Kurasov, and Delio Mugnolo. "Edge Connectivity and the Spectral Gap of Combinatorial and Quantum Graphs". In: *Journal of Physics A: Mathematical and Theoretical* 50.36 (Sept. 8, 2017), p. 365201. ISSN: 1751-8113, 1751-8121. DOI: 10.1088/1751-8121/aa8125.

[DB13] Florian Dörfler and Francesco Bullo. "Kron Reduction of Graphs with Applications to Electrical Networks". In: *IEEE Transactions on Circuits and Systems. I. Regular Papers* 60.1 (2013), pp. 150–163.

[Fie73] Miroslav Fiedler. "Algebraic Connectivity of Graphs". In: *Czechoslovak Mathematical Journal* 23.2 (1973), pp. 298–305. ISSN: 0011-4642, 1572-9141. DOI: 10.21136/CMJ.1973.101168.

[Plü20] Marvin Plümer. *Upper Eigenvalue Bounds for the Kirchhoff Laplacian on Embbeded Metric Graphs.* Apr. 9, 2020. URL: http://arxiv.org/abs/2004.03230 (visited on 02/11/2021).

[ST07] Daniel A Spielman and Shang-Hua Teng. "Spectral Partitioning Works: Planar Graphs and Finite Element Meshes". In: *Linear Algebra and its Applications* 421 (2007), p. 22.