AMTH 428 / E&EB 428 / EPS 428/528 / PHYS 428
Assignment #2

Due: 10:30 AM on September 28, 2020

Finite difference methods and 2-D dynamical system exercises.

1. Consider the following initial-value problem:

$$\frac{du}{dt} = au, \tag{1}$$

with $u(0) = 10$ and $a = -5$.

(a) (10 point) Write a MATLAB script to integrate the equation from $t = 0$ to $t = 4$, using the forward Euler method (i.e., you write your own differential equation solver). Let the maximum time step for stability be Δt_m, and use a half of this value for your time step. Compare the numerical solution with the exact solution. Plot the numerical and exact solutions in one panel, and plot the error in another panel. Use a logarithmic scale if appropriate.

(b) (10 point) Repeat the above with smaller time steps, $\Delta t_m/4$, $\Delta t_m/8$, and $\Delta t_m/16$, and see how the error reduces with the decreasing time step. For this comparison, use the following root-mean-square (RMS) relative error:

$$E = \left(\frac{\sum_i (u_i - u(t_i))^2}{\sum_i u(t_i)^2} \right)^{1/2}, \tag{2}$$

where u_i is a numerical solution at $t = t_i$, and the summation covers the period of t from 0 to 4. Plot the RMS relative error against the time step. Your plot should make it clear that the order of this method is 1 (if you know how to fit a line by linear regression, it may help).

(c) (10 points) Repeat the above, but this time with the 4th-order Runge-Kutta method (again, you need to write your own solver with this method). You should demonstrate that the order of this method is indeed 4.

2. (10 points) Suppose you have implemented two methods to solve an ODE. Method A is 1st order, and it takes 5 seconds to integrate up to $t = 100$ with Δt of 0.1 and 2% error. Method B is 4th order and, using the same Δt, it is twice as slow as Method A but with 0.5% error. Now you want to have the accuracy of 0.01% by decreasing Δt. How long does each method take to integrate up to $t = 10^5$ with the accuracy? Use appropriate time units for your answer (e.g., 1 hour instead of 3600 seconds).

3. Linear stability analysis exercise.

(a) (10 points) Find and classify fixed points in the phase space of a 1-D system, $d^2x/dt^2 = -dV(x)/dx$, with the potential $V(x) = x^4/4 - x^2/2$. Note that this is a 1-D system (i.e., having only one physical variable x), but in the general framework of first-order ordinary differential equations, it is a 2-D system (i.e., x_1=x and $x_2 = \dot{x}$).

(b) (5 points) If we define the total energy of the system as $\frac{1}{2}\dot{x}^2 + V(x)$ (i.e., the sum of kinetic and potential energies), is the total energy conserved in this 1-D system?

(c) (10 points) Plot the phase portrait of this system, with the following items: a flow field (show vectors at regularly spaced points), the fixed points and their eigenvectors (showing the directions of the real-number eigenvectors is enough), and a few representative trajectories (e.g., starting at $(x,\dot{x})$ = (0.5, 0), (0, 0.5), (0, 1.5), and (-0.7, 0), for t=0 to 100). You can use MATLAB's ODE solver `ode45` or your own solver to calculate trajectories. Either way, make sure that your trajectories are accurate enough so that they do not contradict with your answer for (b).

(d) (5 points) Interpret the portrait physically. For example, you can regard the given equation as the equation of motion for a marble rolling along a valley with a hump and use this analogy to describe the nature of phase trajectories you draw in the above.