EPS 659a — Problem Set 9

due Friday, December 3, 2021

Before the problems, we need to review a whole lot of definitions and Fourier-transform concepts. We start with an N-point time series $\{x_n\}_{n=0}^{N-1} = x_0, x_1, x_2, \ldots x_{N-1}$, which we interpret as a discretized record of a continuous process $x(t)$, so that $x_n = x(n\Delta t)$ with a uniform sampling interval Δt. The discrete Fourier transform (DFT) translates the time series into the frequency domain, which means it expresses the time series as a sum of oscillatory components, or data representers, each with constant frequency. We can express these oscillatory components in two different ways. As real-valued functions, we can use

$$\cos(2\pi f n\Delta t) \quad \text{and} \quad \sin(2\pi f n\Delta t)$$

with suitable coefficients to express a signal at frequency f. Using complex-valued sinusoidal functions $\exp(-i2\pi f n\Delta t) = \cos(2\pi f n\Delta t) - i\sin(2\pi f n\Delta t)$, we can express a linear combination of sine and cosine functions compactly with

$$(A+iB)\exp(-i2\pi f n\Delta t) = (A\cos(2\pi f n\Delta t)+B\sin(2\pi f n\Delta t))+i(B\cos(2\pi f n\Delta t)-A\sin(2\pi f n\Delta t))$$

By convention, we take the real part of the complex-valued sinusoid to represent a real-valued time series $\{x_n\}_{n=0}^{N-1}$, and discard the imaginary terms. If we define a complex amplitude coefficient $Z = A + iB$, this means that the real-valued signal is

$$\Re\left((A+iB)\exp(-i2\pi f n\Delta t)\right) = A\cos(2\pi f n\Delta t) + B\sin(2\pi f n\Delta t)$$

where the function $\Re$ takes the real part of its argument. Another way to enforce that the sum of the complex Fourier components leads to a real-valued sum is to incorporate Fourier components at negative frequencies $-f$, and specify that their coefficients are complex conjugates of the coefficients at positive f, so that

$$\begin{aligned} Z\exp(-i2\pi f n\Delta t) + Z^*\exp(i2\pi f n\Delta t) &= \\ (A+iB)\exp(-i2\pi f n\Delta t) + (A-iB)\exp(i2\pi f n\Delta t) &= 2(A\cos(2\pi f n\Delta t) + B\sin(2\pi f n\Delta t)) \end{aligned}$$

Remember that Z^* is the complex conjugate of Z, and notice the factor of two! Different DFT algorithms in different computer packages will define the coefficients A and B differently, **and** different packages will define the cyclic components with either $\exp(-i2\pi f n\Delta t)$ or $\exp(+i2\pi f n\Delta t)$ as their phase convention. Each choice will work, as long as you keep the conventions consistent throughout your computation. In this course, we use the convention above, so that

$$x_n = \int_{-f_N}^{f_N} X(f)\exp(-i2\pi f n\Delta t)df$$

for the infinite discrete time series, and we incorporate the negative frequency interval $[-f_N, 0)$ into the definition of a real-valued time series, with the conjugation relation $X(-f) = (X(f))^*$.

The discretization of time restricts the range of resolvable frequencies f in the time series to $-f_N \le f \le f_N$, where f_N is the Nyquist frequency. (Note that this frequency is named after an electrical engineer named Harry Nyquist, see https://en.wikipedia.org/wiki/Nyquist_frequency.)

$$f_N = \frac{1}{2\Delta t}$$

The signal at the Nyquist has a cycle-period $T = 2\Delta t$. At the Nyquist, the cyclic signal has only the cosine component, because the cosine function $\cos(2\pi f_N n\Delta t) = \cos(n\pi) = (-1)^n$ and $\sin(2\pi f_N n\Delta t) = \sin(n\pi) = 0$. Using the complex-valued sinusoidal function, $\exp(-i2\pi f_N n\Delta t) = \exp(-in\pi) = (-1)^n$.

We distinguish between the infinite-time discrete Fourier Transform of the time series $\{x_n\}_{n=-\infty}^{\infty}$ defined as a continuous function of frequency

$$X(f) = \sum_{m=-\infty}^{\infty} x_n \exp(i2\pi f n\Delta t)$$

and the finite-time DFT of the same time series, which is restricted to an N-point time window, denoted with a different letter $Y(f)$ to distinguish it from its infinite-time counterpart:

$$Y(f) = \sum_{n=0}^{N-1} x_n \exp(i2\pi f n\Delta t)$$

The two spectrum definitions are related via the Dirichlet kernel, as seen below. The finite-time DFT can be discretized into an N-point vector $\mathbf{Y}$ with components $Y_m = Y(mf_R)$ for Rayleigh frequency f_R and $m = -N/2+1, -N/2+2, \ldots -2, -1, 0, 1, 2, \ldots N/2-1, N/2$.

A cyclic signal with frequency $f^{(+)}$ "above" the Nyquist frequency f_N will "alias" into a frequency $f^{(-)}$ that is lower than the Nyquist. In particular, consider the case of a frequency $f^{(+)}$ that satisfies $f_N < f^{(+)} < 3f_N$. We can write $f^{(+)} = f_N + \tilde{f}$. Define the aliasing frequency $f^{(-)} = -f_N + \tilde{f}$, which lies within $-f_N < f^{(-)} < f_N$. In this case

$$\begin{aligned}\exp(-i2\pi f^{(+)} n\Delta t) &= \exp(-i2\pi(f^{(-)} + 2f_N)n\Delta t) = \exp(-i2\pi f^{(-)} n\Delta t)\exp(-i2\pi(2f_N n\Delta t)) \\ &= \exp(-i2\pi f^{(-)} n\Delta t)\exp(-i2\pi n) = \exp(-i2\pi f^{(-)} n\Delta t)\end{aligned}$$

because $\exp(-i2\pi n) = 1$ for all integer n. This means that a signal with frequency $f^{(+)} = f_N + \tilde{f}$ and amplitude Z will "alias" to a signal at frequency $f^{(-)} = -f_N + \tilde{f}$ with the same amplitude Z. This result holds whether the time series $\{x_n\}$ is real-valued or complex-valued. You can show by induction that aliasing governs the DFT of signals so that they are periodic in frequency with period $2f_N = (\Delta t)^{-1}$.

Problem 1: Computations Associated With the Discrete Fourier Transform.

(a) The forward DFT transforms a finite time series $\{x_n\}_{n=0}^{N-1}$ into a set of N Fourier coefficients $Y_m = Y(mf_R)$ for Rayleigh frequency $f_R = 1/(N\Delta t)$ and

$$Y(mf_R) = Y_m = \frac{1}{N}\sum_{n=0}^{N-1} x_n \exp(i2\pi m f_R n\Delta t) = \frac{1}{N}\sum_{n=0}^{N-1} x_n \exp(i2\pi mn/N)$$

The DFT expresses a time series $\{x_n\}_{n=0}^{N-1}$ as a linear combination of data representers that express an integer number of cycles within N data points. It is conventional for N to be an even number, so that the Nyquist cycle $T = 2\Delta t$ has an integer number of cycles in the data series.[1]

[1] In practice, if you have an odd number $\tilde{N}$ of data points, you can add a zero at the end of the time series to force $N = \tilde{N} + 1$ to be an even number. In the mid-20th century, the fastest computer algorithms for the DFT required that $N = 2^P$, that is, that N was a power of 2, such as 1024, 4096, 8192, etc. Your instructor's old computer codes all add zeroes to time series, in order to zero-pad the series to the nearest power of 2.

We take $f_m = mf_R = m/(N\Delta t)$, for $m = -N/2+1, -N/2+2, \ldots -2, -1, 0, 1, 2, 3, \ldots N/2$, so that there are N data representers that are N-point vectors $\mathbf{g}_m$ defined by

$$\mathbf{g}_m = (1, \exp(-i2\pi f_m \Delta t), \exp(-i4\pi f_m \Delta t), \exp(-i6\pi f_m \Delta t), \ldots \exp(-i2(N-1)\pi f_m \Delta t)) \quad (0)$$

The time series is a linear combination of these $\mathbf{g}_m$ vectors. If the data series is real-valued, $Y_{-m} = (Y_m)^*$, leading to a total of N free parameters in the Fourier coefficients. For a complex-valued data series $\{x_n\}_{n=0}^{N-1}$ this conjugation relation does not apply, and there are $2N$ free parameters in the Fourier coefficients, equal to the free parameters in a time series of N complex numbers. The coefficients of this linear combination comprise the "frequency-domain" representation of the time series. Confirm that these vectors are mutually orthogonal according to the formula

$$(\mathbf{g}_p)^* \cdot \mathbf{g}_q = \sum_{n=0}^{N-1} \exp(+i2\pi f_p n\Delta t) \exp(-i2\pi f_q n\Delta t) = N\delta_{pq}$$

where δ_{pq} is the Kronecker delta: $\delta_{pq} = 1$ if $p = q$, and $\delta_{pq} = 0$ if $p \neq q$. You will find that the algebra formula

$$\sum_{n=0}^{N-1} r^n = 1 + r + r^2 + \ldots + r^{N-1} = \frac{1-r^N}{1-r} \quad (1)$$

is useful. Formula (1) is undetermined for $p = q$, but you must sum that special case explicitly. (It won't be difficult!)

(b) Derive the convolution relation relevant to tapering a time series before the discrete Fourier transform. Define a data taper $\{w_n\}_{n=0}^{N-1}$ and the tapered DFT of a finite time series $\{x_n\}_{n=0}^{N-1}$ for any frequency $f \in [-f_N, f_N]$ to be

$$Y(f) = \sum_{n=0}^{N-1} w_n x_n \exp(i2\pi f n\Delta t)$$

The tapered DFT is a convolution of the DFTs of the taper and of the untapered data series, defined by

$$W(f) = \sum_{n=0}^{N-1} w_n \exp(i2\pi f n\Delta t) \quad (2)$$

and

$$X(f) = \sum_{m=-\infty}^{\infty} x_m \exp(i2\pi f m\Delta t) \quad (3)$$

The convolution is

$$Y(f) = \Delta t \int_{-f_N}^{f_N} W(f-\tilde{f}) X(\tilde{f}) d\tilde{f} \quad (4)$$

where the Δt prefactor is necessary to preserve units. Demonstrate the veracity of

$$Y(f) = \sum_{n=0}^{N-1} w_n x_n \exp(i2\pi f n\Delta t) = \Delta t \int_{-f_N}^{f_N} W(f-\tilde{f}) X(\tilde{f}) d\tilde{f}$$

by plugging (2) and (3) into the integrand of equation (4), using the specified frequency values. Rearrange the sums and integral, and then evaluate the integral over $\tilde{f}$ first.

Problem 2: The Dirichlet Kernel. The finite-series DFT $Y(mf_R) = Y_m$ of an N-point time series $x_0, x_1, x_2, \ldots, x_{N-1}$ can be expressed with an equation familiar from the standard least-squares problem. Let $\mathbf{G}$ be a matrix with N rows and N columns. Each column is one of the data representers $\mathbf{g}_m$, $m = -N/2+1, -N/2+2, \ldots, -2, -1, 0, 1, 2, \ldots N/2$ from Problem 1(b). The matrix $\mathbf{G}$ can be expressed

$$\mathbf{G} = \left[\mathbf{g}_{-N/2+1}, \ldots \mathbf{g}_{-2}, \mathbf{g}_{-1}, \mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \ldots \mathbf{g}_{N/2}\right]$$

The coefficients of the time series in the frequency domain are Y_m, so that the time-series vector $\mathbf{x}$ relates to the DFT vector $\mathbf{Y}$ via the relation

$$\mathbf{G} \cdot \mathbf{Y} = \mathbf{x}$$

This expression represents the inverse Fourier transform of the $\{Y_m\}$. It will be exact because there are N data and N frequencies, and $\mathbf{G}$ is a nonsingular matrix. Recall that the least-squares solution is ordinarily $\mathbf{X} = \left(\mathbf{G}^T \cdot \mathbf{G}\right)^{-1} \cdot \mathbf{G}^T \cdot \mathbf{x}$ with $\mathbf{G}^T$ the matrix-transpose of $\mathbf{G}$. In the case where $\mathbf{G}$ is a complex-valued matrix, we take its complex conjugate as well as its transpose. This is called the "Hermitian transpose" $\mathbf{G}^H$. With this modification, the least-squares solution becomes

$$\mathbf{Y} = \left(\mathbf{G}^H \cdot \mathbf{G}\right)^{-1} \cdot \mathbf{G}^H \cdot \mathbf{x}$$

From the formula you confirmed in Problem 1(b), you have established that $\mathbf{G}^H \cdot \mathbf{G} = N\mathbf{I}$, where $\mathbf{I}$ is the identity matrix. This reduces the least-squares solution to $\mathbf{Y} = N^{-1} \cdot \mathbf{G}^H \cdot \mathbf{x}$. This matrix formula can be written out as a sum, for each $m = -N/2+1, \ldots, N/2$, that corresponds to the familiar formula for the discrete Fourier transform (DFT)

$$Y_m = \frac{1}{N} \sum_{n=0}^{N-1} x_n \exp(i2\pi f_m n \Delta t) = \frac{1}{N} \sum_{n=0}^{N-1} x_n \exp(i2\pi mn/N)$$

These relations establish that the DFT has many features in common with least-squares parameter inversions and inverse theory.

The Dirichlet kernel is the DFT of a simple phase-coherent sinusoid $x_n = \exp(-i2\pi f n \Delta t)$. If $f = f_p$ for the pth DFT cycle frequency, then the results of Problem 1(a) tell us that $Y_m = \delta_{mp}$, that is, $Y_p = 1$ and all other Y_m vanish. If the time series has a frequency that does not match any of the $N/2$ cycle frequencies or the constant term, we showed in class that

$$\begin{aligned} Y_m \quad = \quad & \frac{1}{2N} \left[\exp(i\pi(f_m - f)(N-1)\Delta t) \left(\frac{\sin(\pi(f_m - f)N\Delta t)}{\sin(\pi(f_m - f)\Delta t)} \right) \right. \\ & \left. + \exp(i\pi(f_m + f)(N-1)\Delta t) \left(\frac{\sin(\pi(f_m + f)N\Delta t)}{\sin(\pi(f_m + f)\Delta t)} \right) \right] \end{aligned} \tag{1}$$

This function is called the Dirichlet kernel, because it represents the way that the DFT integrates the infinite-time spectrum $X(f)$ as it computes Y_m. It consists of a phase factor (the

$\exp(i\pi\ldots)$ factor) and an averaging function that has maximum amplitude at $f = \pm f_m$. At these maxima the phase factor is unity and the amplitude factor obeys the limit

$$\lim_{x\to 0}\left(\frac{\sin Nx}{\sin x}\right) = N$$

so that $Y_m = 1$ if $f_m = \pm f$. (You can verify that the value of each term in (1) is zero at the maximum value of the other term.) In the problems to follow, you first compute with R the Y_m for a simple case where the signal frequency f equals a DFT frequency, and later compute with R the Y_m for a case where f does not match a DFT frequency.

(a) Consider a 100-point time series in R with annual sampling and a signal with a cycle period of 20 years. You can create such a series with the commands

```
time <- 0:99
xdata <- sin(2*pi*time/20)
```

The DFT command in R is called `fft` after the "Fast Fourier Transform" algorithm that it uses internally

```
spec <- fft(xdata,inverse=TRUE)/length(xdata)
```

The parameter `inverse=TRUE` is necessary because R uses the "wrong" DFT phase convention for the forward DFT, that is, it uses the convention that your instructor doesnt like. R also does not normalize its DFT by the number of data, so you must divide-by-N yourself. Plot the absolute values of the DFT against frequency. The DFT in R returns N complex numbers for an N-point time series, computed for the frequencies $f_m = m/T$, for $m = 0, \ldots N-1$, that is, for $0 \le f_m < 2f_N$. The DFT values for $f_m > f_N$ correspond to the values of the DFT for negative frequencies, according to the "wraparound effect" that we have already demonstrated in the Pset preamble on aliasing. The absolute value $|Y_m|$ can be computed from the DFT by

```
pspec <- abs(spec)
freq <- (0:99)/100
plot(freq,pspec)
```

Plot the absolute value $|Y_m|$ and the real and imaginary parts, that is, $\Re(Y_m)$, and $\Im(Y_m)$, of the raw DFT `spec` with the functions `Re(spec)` and `Im(spec)`. Explain how these three plots relate to each other for the signal, e.g., explain what the non-zero values of the plots signify.

(b) Perform the same computations and make the same plots for a signal that does not have an integer number of cycles in a 100-point time series, using cycle period $T = 7$.

```
time <- 0:99
xdata <- sin(2*pi*time/7)
```

Explain how the three plots of $|Y_m|$, $\Re(Y_m)$, and $\Im(Y_m)$ relate to each other for the chosen signal.

(c) The Dirichlet kernel can be visualized better by padding the time series with zeroes before the DFT. Add 900 zeroes to the time series from part (a). This has the effect of interpolating the DFT in the frequency domain.

```
xdata <- sin(2*pi*time/20)
xpad <- rep(0.0,1000)
xpad[1:100] <- xdata
specpad <- fft(xpad,inverse=TRUE)/length(xdata)
fpad <- (0:999)/1000
```

Plot `abs(specpad), Re(specpad), and Im(specpad)` against `fpad` as scatter-plots for the full frequency interval $0 \le$`fpad`< 1. Plot the same quantities as line plots for $0 \le$`fpad`< 0.1, with R plot option `type="l"` and/or command `lines(fpad,Re(specpad))`, etc. Describe what you see.

Problem 3: Statistics of the Periodogram.

(a) Generate a time series $\{x_n\}_{n=0}^{N-1} = x_0, x_1, x_2, \ldots x_{N-1}$ with $N = 1000$ with each x_n a Gaussian random variable with $\langle x \rangle = 0$ and $\langle x^2 \rangle = 1$. Compute and plot the periodogram $|Y_m|^2$ of this time series:

```
xdata <- rnorm(1000,mean=0,sd=1)
spec <- fft(xdata,inverse=TRUE)/length(xdata)
perid <- abs(spec)^2
fpad <- (0:999)/1000
plot(fpad,perid)
```

The values of the time series are uncorrelated, zero-mean, unit-variance Gaussian random variables. What is the expectation value $\langle |Y_m|^2 \rangle = \langle (Y_m)^* Y_m \rangle$? In your R-markdown file for this problem set, estimate the periodogram of several realizations of the random-Gaussian time series `xdata`, to confirm (or to adjust) your estimate of $\langle |Y_m|^2 \rangle$.

(b) Make a histogram of the periodogram values and compare it with the probability density function for the chi-squared distribution with 2 degrees of freedom. The basic histogram function in R can be applied here with the command

```
hist(perid,xlim=c(0.0,0.01),breaks=100)
```

You can adjust the breaks parameter to make the plot look better, if you wish. However, it is important to keep track of the bin width and number N of data in order to compare theory with experiment. The chi-squared probability density function (PDF) `dchisq` is used as follows

```
xx <- seq(0,20,by=0.1)
yy <- dchisq(xx,df=2)
```

The expectation value of χ_2^2 is 2, that is, equal to the `df` parameter. The x and y values of the χ_2^2 distribution would need to be rescaled to fit on the histogram graph, but if you plot the function into a separate panel of equal size, you can assess the resemblance between histogram and the statistical distribution without scaling headaches.

(c) Suppose the 1000-point time series is deterministic rather than stochastic, consisting of a single cosinusoid $x_n = A\cos(2\pi f_m n\Delta t)$ for $n = 0, 1, \ldots N-1$ and $f_m = m f_R = m/(N\Delta t)$ for integer m. What is the DFT Y_m at $f = \pm f_m$? What is the periodogram $|Y_m|^2$ for a deterministic cosinusoid with amplitude A?

(d) How large should a simple sinusoidal signal $\{y_n\}_{n=0}^{N-1} = A\cos(2\pi f n\Delta t)$ be to trigger detection when immersed in the random Gaussian series that you computed for part (a)? Recall that detection of a nonrandom peak in the periodogram must be expressed in terms of the %-confidence for nonrandomness. The sinusoid amplitude will appear as a peak in the periodogram, but will compete with all the random peaks caused by stochastic variability in the background time series. The spectrum estimate also combines the deterministic signal with a stochastic component at the targeted frequency. What are the 90th, 95th, 99th, and 99.9th percentiles for confidence of nonrandomness for a chi-squared distribution with 2 degrees of freedom? What are the amplitudes A_{90}, A_{95}, A_{99} and $A_{99.9}$ of sinusoidal signals for these confidence levels, given the Gaussian parameters in part (a), that is a random Gaussian time series with zero mean and unit standard deviation?

(e) A deterministic signal in stochastic noise becomes easier to detect if the time series is longer. How much smaller are the amplitude threshholds in part (d) if the time series has length $N = 2000$?

(f) Add phase-coherent cyclic components, that is, ordinary cosine or sine functions, with the predicted "detection" amplitudes to your 1000-poinr random-Gaussian time series and see if their amplitude in the periodogram justifies your choice. Be careful to choose cycle frequencies f that correspond to an integer number of cycles in a 1000-point time series. It is convenient to place all deterministic cycles of varying amplitudes in a single time series, but at distinct frequencies. Note that for 1000 periodogram estimates, the 95th percentile lies roughly where the 950th-ranked value is (that is, 50 values would be larger that a periodogram value at 95% confidence for nonrandomness).

(g) The periodogram estimate can be averaged over narrow bandwidths to decrease its variance. For instance, if you sum the periodogram values at three adjacent frequencies of the DFT, the summed periodograms should behave as a chi-squared random variable with 2+2+2=6 degrees of freedom (dof), combining the dof of the three individual 2-dof estimates. R commands such as

```
pgram3 <- rep(0,998)
pgram3 <- (pgram[1:998]+pgram[2:999]+pgram[3:1000])/3
plot(fpad[2:999],pgram3)
```

will compute the smoothing and synchronize the center of the smoothing interval with the frequency axis. After smoothing the periodogram, plot it against frequency. After spectral smoothing, are the phase-coherent cycles that you added to the time series in part (c) easier or harder to detect?

(h) Plot a histogram of the smoothed periodogram values and compare it with the probability density function of the chi-squared distribution with 6 degrees of freedom. Use separate plots to avoid getting into a thicket of rescaling choices.

Problem 4: Carbon Dioxide at Mauna Loa, Hawaii since 1958: Spectrum Analysis.

On the Canvas server there is a dataset of carbon-dioxide values measured monthly at Mauna Loa, Hawaii USA in csv format (`co2_maunaloa.csv`). There are four columns: "YEAR", "MONTH", "TIME", and "CO2". The column TIME for digital years and CO2 for parts-per-million CO_2 in the atmosphere, monthly-averaged from air parcels taken atop Mauna Loa. This is the same dataset used in Problem Set 6.

(a) We will be looking at spectrum analysis of the CO_2 time series after subtracting the most obvious signals, that is, the constant and the trend. For the purposes of this exercise, fit a quadratic polynomial to the CO_2 series in R and subtract the model prediction from the raw data to form the residual `co2_resid` time series. Plot the CO_2 time series and the quadratic model against time. Plot the times series of residual CO_2 against time. How has this residual changed during the lifetime of your professor?

(b) Compute and plot the periodogram for the `co2_resid` time series. Make two plots, one with a linear y-axis and one with a logarithmic y-axis, against the frequency range `xlim=c(-0.05,6.0)` cycles/year, which includes all frequencies to the Nyquist frequency $f_N = 6.0$ cycles/year. It is a good idea to specify a range for the y-axis, because very small values of the periodogram will

spread out the y-range and distort the plot. Sample R code to perform this step resembles
```
spec <- fft(co2_resid,inverse=TRUE)/length(co2_resid)
pgram <- abs(spec)^2
ymax <- 1.5*max(pgram)
ymin <- your_choice
plot(freq,pgram,log = "y",ylim=c(ymin,ymax),xlim=c(-0.05,6.0))
lines(freq,pgram)
```

How many overtone spectral peaks ($f = 2, 3, 4, 5$ cycles/year) can you identify?

(c) Smooth the periodograms to obtain spectrum estimates with 6 degrees of freedom, as with Problem 3b above, and plot with a logarithmic y axis. Are the overtone spectral peaks still prominent? Generate simple functions of inverse frequency

$$Y(f) = Af^{-\alpha}$$

for powers $\alpha = 0.5, 1, 1.5, 2$ and plot atop the periodogram. Choose A to make the visual comparison effective, but dont bother to fit an inverse-power model quantitatively. Which value of α fits the spectrum most persuasively? Use commands like
```
ff <- freq[10:npts]
y05 <- afac*ff^-0.5
```

to generate the curves, choosing values of `afac` to make the plot look well.

(d) Apply a two-point first-difference filter to the residual CO_2 time series
```
co2_diff <- rep(0,npts)
co2_diff[2:npts] <- co2_resid[2:npts] - co2_resid[1:(npts-1)]
plot(time,co2_diff,"l")
title(main="Mauna Loa Carbon Dioxide (ppm) first difference")
```

Compute and plot the periodogram of `co2_diff`. To assess the effect of the filter more clearly, compute the smoothed periodogram using three adjacent DFT values and plot the results before and after filtering on the same graph. Does the filter remove the "red" inverse frequency dependence?

(e) Pad the `co2_diff` time series with zeroes to a total of 6000 points, in order for the DFT to interpolate the spectrum estimates.
```
co2pad <- rep(0,6000)
co2pad[0:npts] <- co2_diff
dfpad <- 12.0/6000.0
fpad <- dfpad*(0:5999)
```

Compute the unsmoothed periodogram and plot its amplitude for the frequency intervals $[0.7, 1.3]$ cycles/yr, $[1.7, 2.3]$ cycles/yr, $[2.7, 3.3]$ cycles/yr, and $[3.7, 4.3]$ cycles/yr. In which of these intervals can you observe a clearly-expressed Dirichlet kernel, which intervals contain a spectral peak, but no detailed kernel pattern, and which intervals have no peak of note?