Problem Set 4
EPS 528, Science of Complex Systems

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November 16, 2022

Problem 1.

Solution. Consider the usual Lorenz system

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - \beta z \tag{1}$$

with parameters $\sigma, r, \beta > 0$. We integrate (1) forward in time using the following two codes:

(lorenz.m)

```
function [ sol ] = lorenz(sigma, r, b, t0, tf, x0, y0, z0)

opts = odeset('RelTol', 1e-12, 'AbsTol', 1e-10);
func = @(t, u) [sigma * (u(2) - u(1)); r * u(1) - u(2)...
    - u(1) * u(3); u(1) * u(2) - b * u(3)];
u0 = [x0; y0; z0];
tspan = [t0 tf];
sol = ode45(func, tspan, u0, opts);
end
```

(Problem1.m)

```
cd 'C:\Users\jonas\OneDrive\Documents\MATLAB\EPS_528_HW_4'

sigma = 10;
b = 8 / 3;
r = 28;
x0 = 0;
y0 = 1;
z0 = 0;
t0 = 0;
tf = 500;
sol = lorenz(sigma, r, b, t0, tf, x0, y0, z0);
t = sol.x;
u = sol.y;

% create Lorenz map
maxima = NaN(1, length(t));
```

```
samplei = u(3, 1);
sampleafter = u(3, 2);
for i = 2 : length(t) - 1
    samplebefore = samplei;
    samplei = sampleafter;
    sampleafter = u(3, i + 1);
    if (samplei > samplebefore) && (samplei > sampleafter)
        maxima(i) = samplei;
    end
end
maxima = maxima(~(isnan(maxima)));

figure(1)
tiledlayout(3, 1)
nexttile
plot(t, u(1, :), 'Color', [0 0.4470 0.7410])
title(['Lorenz equations solution from t=', num2str(t0),...
    ' to t=', num2str(tf)], 'fontsize', 14)
xlabel('t', 'fontsize', 12)
ylabel('x(t)', 'fontsize', 12)

nexttile
plot(t, u(2, :), 'Color', [0.9290 0.6940 0.1250])
xlabel('t', 'fontsize', 12)
ylabel('y(t)', 'fontsize', 12)

nexttile
plot(t, u(3, :), 'Color', [0.4660 0.6740 0.1880])
xlabel('t', 'fontsize', 12)
ylabel('z(t)', 'fontsize', 12)

xq = (100 * round(min(maxima)) : 100 * round(max(maxima))) / 100;

figure(2)
hold on
plot(xq, xq, ':', 'Color', 'k')
scatter(maxima(1 : end - 1), maxima(2 : end),...
    '.', 'MarkerEdgeColor', [0.4940 0.1840 0.5560])
hold off
title(['Lorenz map from t=', num2str(t0),...
    ' to t=', num2str(tf)], 'fontsize', 14)
xlabel('$z_{n}$', 'interpreter', 'latex', 'fontsize', 14)
ylabel('$z_{n+1}$', 'interpreter', 'latex', 'fontsize', 14)

figure(3)
scatter3(u(1, :), u(2, :), u(3, :), 0.1, '.')
title(['Lorenz equations attractor from t=', num2str(t0),...
    ' to t=', num2str(tf)], 'fontsize', 14)
xlabel('x(t)', 'fontsize', 12)
```

```
ylabel('y(t)', 'fontsize', 12)
zlabel('z(t)', 'fontsize', 12)

cobwebdomain = zeros(1, 2 * (length(maxima) - 1));
cobwebrange = zeros(1, 2 * (length(maxima) - 1));
for i = 1 : length(maxima) - 1
    cobwebdomain(2 * i - 1 : 2 * i) = maxima(i : i + 1);
    cobwebrange(2 * i - 1 : 2 * i) = maxima(i + 1);
end

figure(4)
hold on
plot(xq, xq, ':', 'Color', 'k')
plot(cobwebdomain, cobwebrange, 'LineWidth', 0.1)
scatter(maxima(1 : end - 1), maxima(2 : end),...
    '.', 'MarkerEdgeColor', [0.4940 0.1840 0.5560])
hold off
title(['Lorenz map from t=', num2str(t0),...
    ' to t=', num2str(tf), ' (w/ cobwebs)'], 'fontsize', 14)
xlabel('$z_{n}$', 'interpreter', 'latex', 'fontsize', 14)
ylabel('$z_{n+1}$', 'interpreter', 'latex', 'fontsize', 14)
```

`lorenz` simply integrates (1) using the well-known MATLAB ODE solver `ode45` from `t0` to `tf` with initial conditions $(x(0), y(0), z(0)) =$`[x0 y0 z0]`, values of $\sigma =$`sigma`, $r =$`r`, and $\beta =$`b`, and with relative and absolute tolerances of 10^{-12} and 10^{-10}, respectively, which will hopefully give some robustness and precision to the expectedly stiff, unpredictable solution of the Lorenz equations. This is used in the script Problem1.m. For given parameters which are changed manually, this script starts by numerically integrating the Lorenz system (1) and returns the arrays `t` and `u`, where the $1 \times n$ array `t` gives the times that the solution was sampled and the $3 \times n$ array `u` gives the values of $x(t)$, $y(t)$, and $z(t)$ at each time in `t`. From there, we can generate the Lorenz map by sampling each successive maxima in $z(t)$ (i.e., `u(3,:)`). We do this by looking at groups of $z(t)$-values at adjacent times in t, i.e., `u(3,1)`, `u(3,2)`, and `u(3,3)`, then `u(3,2)`, `u(3,3)`, and `u(3,4)`, etc., all the way up to `u(3,n-2)`, `u(3,n-1)`, and `u(3,n)`. We know that each successive maximum will be such that the center $z(t_i) =$`u(3,i)` point will be greater than the first and last points in the group, i.e., $z(t_i) =$`u(3,i)>u(3,i-1)=` $z(t_{i-1})$ and $z(t_i) =$`u(3,i)>u(3,i+1)=` $z(t_{i+1})$. Hence, this how we record each maximum.

From there, for the parameters given in the problem, we plot the separate time series of $x(t)$, $y(t)$, and $z(t)$ on one figure but in three separate plots (Figure 1), then the Lorenz map as described in the problem (Figure 2), the 3D scatter phase plot for the trajectory $\mathbf{u}(t) = (x(t), y(t), z(t))$ (which reproduces the strange attractor which the Lorenz system exhibits for the parameters we needed to test) (Figure 3), and finally, the cobweb plot for the Lorenz map (also Figure 2). Even though Jun never asked for it, we do the last one because using a cobweb plot is often standard in the analysis of 1D discrete dynamical systems, and provides a powerful visual tool to investigate the qualitative behavior of the iterated functions which express these systems.

In particular, note in Figure 2 that the cobweb plot appears to densely fill up a region of the $(z_n, f(z_n))$-plane. This is a typical behavior found in chaotic systems in which trajectories are aperiodic but do not approach a fixed point. If this trajectory generated a periodic orbit, then we would expect the cobweb plot to outline a closed curve, since we know that the successive iterates would have to repeat themselves at some point. Also, the purple curve (which reconstructs the function f such that $z_{n+1} = f(z_n)$) lies directly above the line $f(z_n) = z_n$ of slope 1 and peels away from it, indicating that $|f'(z)| > 1$ at most points. $\square$

Problem 2.

Solution. Let

$$z_{n+1} = f(z_n) \tag{2}$$

be an iterated map in 1D such that $|f'(z)| > 1$ everywhere, and suppose that (2) exhibits a periodic solution of period p consisting of points $z_n, z_{n+1}, \ldots, z_{n+p}$ such that $z_{n+p} = z_n$. Equivalently, we can write our periodic solution via the iterates of f, i.e., since $f^k(z_n) = f^{k-1}(f(z_n)) = f^{k-1}(z_{n+1}) = \cdots = z_{n+k}$, $\{z_n, z_{n+1}, \ldots, z_{n+p}\} = \{z_n, f(z_n), \ldots, f^p(z_n)\}$ such that $f^p(z_n) = z_n$.

From here, suppose that we start at some perturbed starting point $z_n+\varepsilon$ for some $\varepsilon > 0$ to be determined. Then, assuming that f is at least twice differentiable on some open interval $I \supset [z_n, z_n + \varepsilon]$, via Taylor's theorem from real analysis (with the Lagrange form of the remainder), we can Taylor expand f^m about z_n for $m \in \{1, \ldots, p\}$

$$f^m(z_n + \varepsilon) = f^m(z_n) + \varepsilon \left.\frac{df^m(z)}{dz}\right|_{z=z_n} + \frac{\varepsilon^2}{2} \left.\frac{d^2 f^m(z)}{dz^2}\right|_{z=c} \tag{3}$$

and some $c \in [z_n, z_n + \varepsilon]$. By chain and product rule, we can simplify

$$\frac{df^m(z)}{dz} = m f^{m-1}(z) f'(z), \tag{4}$$

$$\begin{aligned}\frac{d^2 f^m(z)}{dz^2} &= m\frac{d}{dz}\left[f^{m-1}(z)f'(z)\right] = mf'(z)\frac{d}{dz}\left[f^{m-1}(z)\right] + mf^{m-1}(z)\frac{d}{dz}\left[f'(z)\right] \\ &= m(m-1)\left(f'(z)\right)^2 f^{m-2}(z) + mf^{m-1}(z)f''(z), \end{aligned} \tag{5}$$

and substituting (4) and (5) into (3),

$$\begin{aligned} & f^m(z_n+\varepsilon) = f^m(z_n) + \varepsilon m f^{m-1}(z_n) f'(z_n) \\ & \quad + \frac{m\varepsilon^2}{2}\left[(m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c)f''(c)\right] \\ \Rightarrow\ & f^m(z_n+\varepsilon) - f^m(z_n) - \frac{m\varepsilon^2}{2}\left[(m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c)f''(c)\right] = \\ & \quad \varepsilon m f^{m-1}(z_n) f'(z_n) \\ \Rightarrow\ & \left| f^m(z_n+\varepsilon) - f^m(z_n) - \frac{m\varepsilon^2}{2}\left[(m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c)f''(c)\right]\right| = \\ & \quad \varepsilon m \left|f^{m-1}(z_n) f'(z_n)\right| = \varepsilon m \left|f^{m-1}(z_n)\right| \left|f'(z_n)\right|. \end{aligned} \tag{6}$$

Since $|f'(z)| > 1$ everywhere, in particular, $|f'(z_n)| > 1$, such that (6) becomes

$$\left| f^m(z_n+\varepsilon) - f^m(z_n) - \frac{m\varepsilon^2}{2}\left[(m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c)f''(c)\right]\right| > \varepsilon m \left|f^{m-1}(z_n)\right|. \tag{7}$$

Finally, by the triangle inequality,

$$\begin{aligned}
&\left| f^m\left(z_n+\varepsilon\right) - f^m\left(z_n\right) - \frac{m\varepsilon^2}{2}\left[(m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c) f''(c) \right] \right| \\
&\le \left| f^m\left(z_n+\varepsilon\right) - f^m\left(z_n\right) \right| + \frac{m\varepsilon^2}{2}\left| (m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c) f''(c) \right|, \qquad (8)
\end{aligned}$$

and substituting (8) into (7),

$$\begin{aligned}
\left| f^m\left(z_n+\varepsilon\right) - f^m\left(z_n\right) \right| + \frac{m\varepsilon^2}{2}\left| (m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c) f''(c) \right| > \varepsilon m \left| f^{m-1}\left(z_n\right) \right| \\
\Rightarrow \left| f^m\left(z_n+\varepsilon\right) - f^m\left(z_n\right) \right| > \\
\varepsilon m \left| f^{m-1}\left(z_n\right) \right| - \frac{m\varepsilon^2}{2}\left| (m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c) f''(c) \right| = \\
\varepsilon m \left(\left| f^{m-1}\left(z_n\right) \right| - \frac{\varepsilon}{2}\left| (m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c) f''(c) \right| \right). \qquad (9)
\end{aligned}$$

Assume that $f^{m-1}\left(z_n\right) \neq 0$. Hence, if we choose $\varepsilon > 0$ such that

$$\begin{aligned}
\left| f^{m-1}\left(z_n\right) \right| - \frac{\varepsilon}{2}\left| (m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c) f''(c) \right| \ge \frac{\delta}{m} \\
\Rightarrow \left| f^{m-1}\left(z_n\right) \right| \ge \frac{\varepsilon}{2}\left| (m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c) f''(c) \right| + \frac{\delta}{m} \\
\Rightarrow \varepsilon \le \frac{2\left(\left| f^{m-1}\left(z_n\right) \right| - \delta/m \right)}{\max_{z\in[z_n, z_n+\varepsilon]} \left| (m-1)\left(f'(z)\right)^2 f^{m-2}(z) + f^{m-1}(z) f''(z) \right|}, \qquad (10)
\end{aligned}$$

where $0 < \delta < m\left| f^{m-1}\left(z_n\right) \right|$,[1] then (9) implies that

$$\left| f^m\left(z_n+\varepsilon\right) - f^m\left(z_n\right) \right| > \varepsilon m \left(\frac{\delta}{m} \right) = \varepsilon\delta > 0. \qquad (11)$$

Furthermore, if we take $\delta < \min_{m\in\{1,\ldots,p\}} \left\{ m\left| f^{m-1}\left(z_n\right) \right| \right\}$ and

$$\varepsilon \le \min_{m\in\{1,\ldots,p\}} \left\{ \frac{2\left(\left| f^{m-1}\left(z_n\right) \right| - \delta/m \right)}{\max_{z\in[z_n, z_n+\varepsilon]} \left| (m-1)\left(f'(z)\right)^2 f^{m-2}(z) + f^{m-1}(z) f''(z) \right|} \right\},$$

then (11) holds for *every* $m \in \{1, \ldots, p\}$, i.e.,

$$\boxed{\left| f^m\left(z_n+\varepsilon\right) - f^m\left(z_n\right) \right| > \varepsilon\delta > 0, \quad m = 1, \ldots, p.}$$

We assumed above that $f^{m-1}\left(z_n\right) \neq 0$. However, if $f^{m-1}\left(z_n\right) = 0$, then (4) is now equal to zero when $z = z_n$ (i.e., the first-order term in (3) vanishes), such that if we substitute (4) and (5) into (3),

$$\begin{aligned}
f^m\left(z_n+\varepsilon\right) = f^m\left(z_n\right) + \frac{m\varepsilon^2}{2}\left[(m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c) f''(c) \right] \\
\Rightarrow \left| f^m\left(z_n+\varepsilon\right) - f^m\left(z_n\right) \right| = \frac{m\varepsilon^2}{2}\left| (m-1)\left(f'(c)\right)^2 f^{m-2}(c) + f^{m-1}(c) f''(c) \right|. \qquad (12)
\end{aligned}$$

[1] Note that the denominator of (10) will never be zero because $|f'(z)| > 1$ always, such that $|f'(z)| \neq 0$ for any z.

Since $|f'(z)| > 1$ always, in particular, we know that $|f'(c)| > 1 \Rightarrow (f'(c))^2 > 1 \neq 0$, such that (12) will be greater than zero *unless* $f^{m-2}(c) = f^{m-1}(c) = 0$ or $f^{m-2}(c) = f''(c) = 0$. If this occurs, then the entire second-order remainder term in (3) vanishes and we are forced to let $f^m(z_n + \varepsilon) = f^m(z_n)$. But all hope is not lost. Since $|f'(z)| > 1 \neq 0$, f and hence f^m cannot be trivially constant on any open subset of $[z_n, z_n + \varepsilon]$. Thus, $f^m(x) \neq 0$ on a dense subset of $[z_n, z_n + \varepsilon]$. Thus, if $f^m(z_n) = f^m(z_n + \varepsilon)$ for some ε, we have infinitely many other, arbitrarily small choices of ε such that $f^m(z_n) \neq f^m(z_n + \varepsilon)$.

In other words, for *any* ε small enough, the iterates of $z_n + \varepsilon$ remain separated from the periodic solution generated via the iterates of z_n, since $f^m(z_n + \varepsilon) \neq f^m(z_n)$ for $m = 1, \ldots, p$. Therefore, by definition, we conclude that the periodic solution $\{z_n, f(z_n), \ldots, f^p(z_n)\}$ is unstable, as for ε small enough, $z_n + \varepsilon$ does not converge to the same periodic solution.

In particular, as we noted in Problem 1, the Lorenz map $z_{n+1} = f(z_n)$ appears to be such that $|f'(z)| > 1$ at almost every value we tested in the domain. This demonstrates that any periodic solutions for the Lorenz map must be unstable, i.e., neighboring trajectories close enough will not converge to the same orbit. However, since the Lorenz equations are chaotic, one of the identifying features of chaotic systems is the presence of dense periodic orbits. Therefore, even though we proved above that any periodic solution is unstable for the Lorenz map, there must exist *another* periodic solution which can be generated by iterating an initial condition *arbitrarily* close to z_n.

To demonstrate that $|f'(z)| > 1$ for most points in the Lorenz map, we can compute $|f'(z)|$ numerically using the data from Problem 1. We do this with the following script:

(Problem2.m)

```
cd 'C:\Users\jonas\OneDrive\Documents\MATLAB\EPS_528_HW_4'

sigma = 10;
b = 8 / 3;
r = 28;
x0 = 0;
y0 = 1;
z0 = 0;
t0 = 0;
tf = 500;
sol = lorenz(sigma, r, b, t0, tf, x0, y0, z0);
t = sol.x;
u = sol.y;

% create Lorenz map
maxima = NaN(1, length(t));
samplei = u(3, 1);
sampleafter = u(3, 2);
for i = 2 : length(t) - 1
    samplebefore = samplei;
    samplei = sampleafter;
    sampleafter = u(3, i + 1);
    if (samplei > samplebefore) && (samplei > sampleafter)
        maxima(i) = samplei;
    end
end
```

```
maxima = maxima(~(isnan(maxima)));
[maxdomain, I] = sort(maxima(1 : end - 1));
maxvalue = maxima(2 : end);
maxvalue = maxvalue(I);

fprimefirst = (maxvalue(3 : end) - maxvalue(2 : end - 1))...
    ./ (maxdomain(3 : end) - maxdomain(2 : end - 1));
fprimesecond = (maxvalue(2 : end - 1) - maxvalue(1 : end - 2))...
    ./ (maxdomain(2 : end - 1) - maxdomain(1 : end - 2));
fprime = 0.5 * (fprimefirst + fprimesecond);
x = maxdomain(2 : end - 1);
y = abs(fprime);
threshold = Inf;

xq = (100 * round(min(x)) :...
    100 * round(max(x))) / 100;

figure(1)
hold on
scatter(x(y < threshold), y(y < threshold), '.',...
    'MarkerEdgeColor', [0.4940 0.1840 0.5560])
yline(1, '--k', '$x=1$', 'interpreter', 'latex');
hold off
title(['$\left|f^{\prime}(z)\right|$', ' from t=',...
    num2str(t0), ' to t=', num2str(tf), ' with ',...
    '$\left|f^{\prime}(z)\right|<$', ' ', num2str(threshold)],...
    'interpreter', 'latex', 'fontsize', 14)
xlabel('$z$', 'interpreter', 'latex', 'fontsize', 14)
ylabel('$\left|f^{\prime}(z)\right|$',...
    'interpreter', 'latex', 'fontsize', 14)
```

For Problem2.m, we yet again start by numerically integrating the Lorenz system (1) and returning the arrays `t` and `u`. However, note that when collecting the maxima to generate $z_1, z_2, \ldots$, we follow the cobweb map in Figure 2 which of course traces out $f(z)$ out of order. Hence, we also need to sort the values of $f(z_n)$ according to the order of the corresponding values z_n. After doing this, we can use the second-order central difference approximation to the first derivative to compute $|f'(z)|$ numerically, i.e.,

$$f'(z_n) \approx \frac{1}{2}\left[\frac{f(z_{n+1}) - f(z_n)}{z_{n+1} - z_n} + \frac{f(z_n) - f(z_{n-1})}{z_n - z_{n-1}}\right]$$

and then take the absolute value. We can then plot the results, which are shown in Figure 4; we also mark a horizontal line at $|f'(z)| = 1$ to show that $|f'(z)| > 1$ for the most part. □

Problem 3.

Solution. The Lorenz maps for z_{n+1} vs. z_n, z_{n+2} vs. z_n, z_{n+3} vs. z_n, z_{n+4} vs. z_n, and z_{n+5} vs. z_n are found in Figure 5. We made these plots using the following code:

(Problem3.m)

```
cd 'C:\Users\jonas\OneDrive\Documents\MATLAB\EPS_528_HW_4'

N = 5;

sigma = 10;
b = 8 / 3;
r = 28;
x0 = 0;
y0 = 1;
z0 = 0;
t0 = 0;
tf = 500;
sol = lorenz(sigma, r, b, t0, tf, x0, y0, z0);
t = sol.x;
u = sol.y;

% create Lorenz map
maxima = NaN(1, length(t));
samplei = u(3, 1);
sampleafter = u(3, 2);
for i = 2 : length(t) - 1
    samplebefore = samplei;
    samplei = sampleafter;
    sampleafter = u(3, i + 1);
    if (samplei > samplebefore) && (samplei > sampleafter)
        maxima(i) = samplei;
    end
end
maxima = maxima(~(isnan(maxima)));

xq = (100 * round(min(maxima)) : 100 * round(max(maxima))) / 100;

figure(1)
tiledlayout(N, 1)
for i = 1 : N
    nexttile
    hold on
    plot(xq, xq, ':', 'Color', 'k')
    scatter(maxima(1 : end - i), maxima(i + 1 : end),...
        '.', 'MarkerEdgeColor', [0.4940 0.1840 0.5560])
    hold off
    if i == 1
        title(['Lorenz map from t=', num2str(t0),...
            ' to t=', num2str(tf)], 'fontsize', 14)
    end
    xlabel('$z_{n}$', 'interpreter', 'latex', 'fontsize', 14)
    ylabel(['$z_{n+', num2str(i), '}$'],...
        'interpreter', 'latex', 'fontsize', 14)
end
```

We note a geometrically-increasing pattern for the number of "peaks" in the graph as we increase k and plot z_{n+k} vs. z_n. In particular, there appear to be 2^{k-1} "peaks" in the graph of z_{n+k} vs. z_n. We can explain this as follows: Note that $z_{n+1} = z_n$ occurs when the purple curve intersects the dotted diagonal line for the topmost plot in Figure 5, because that diagonal line marks where $z_{n+1} = z_n$. However, when $z_{n+p} = z_n$, this indicates a periodic orbit of period p in the Lorenz map. Hence, for instance, if we consider the graph of z_{n+3} vs. z_n, we see that the purple curve crosses the diagonal line at $z_{n+3} = z_n$ seven times. This suggests that the Lorenz map has a total of two orbits with period 3, since two groups of three crossings each correspond to each of these orbits, while the last crossing corresponds to the same fixed point found in all of the plots.

More generally, it appears from Figure 5 that there are $2^p - 1$ crossings for each plot of z_{n+p} vs. z_n. To count the number of periodic solutions of period p and *no less* (e.g., excluding fixed points), we need to exclude all possible periodic solutions whose period divides p and is less than p. However, if p is prime, then of course, we only need to subtract out the fixed point since no other period less than p divides p, in which case there are $\frac{(2^p-1)-1}{p} = \frac{2^p-2}{p}$ solutions of period p and no less for p a prime. Either way, since there are an infinite number of primes, we deduce that the Lorenz system also has an infinite number of periodic solutions at different values of p, and since the peaks in Figure 5 appear to densely fill up the region at the center of the domain, we conclude that yes, these periodic solutions are dense on the Lorenz attractor, confirming the chaotic properties of solutions to the Lorenz equations using these parameters. However, since Figure 4 also appears to suggest that $|f'(z)| > 1$, we conclude by our work in Problem 2 that *all* of these periodic solutions are unstable.

The analysis above highlights the power of discretizing our dynamical system, since we were able to deduce all of these properties above quite easily by merely looking at the plots for the iterates. However, all of our work here was done numerically, which means that doing a more rigorous, analytic study requires much more work, and might even be impossible in some cases. □

Problem 4.

Solution. Another property of chaotic systems is the exponential divergence of neighboring trajectories, provided that these trajectories start adequately close enough from one another. Aside from the maps, we repeat everything from Problem 1, except up to $t = 100$ while comparing two trajectories that solve the same Lorenz equations, but with one starting at $(0, 1, 0)$ (denoted by $\mathbf{u}_0(t) = (x_0(t), y_0(t), z_0(t))$) and another at $(10^{-6}, 1, 0)$ (denoted by $\mathbf{u}_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t), z_\varepsilon(t))$, sort of with the definition that $\varepsilon = 10^{-6}$). We do this with the code below:

(Problem4.m)

```
cd 'C:\Users\jonas\OneDrive\Documents\MATLAB\EPS_528_HW_4'

sigma = 10;
b = 8 / 3;
r = 28;
x0 = 0;
y0 = 1;
z0 = 0;
t0 = 0;
```

```
tf = 100;
sol0 = lorenz(sigma, r, b, t0, tf, x0, y0, z0);
solp = lorenz(sigma, r, b, t0, tf, x0 + 1e-6, y0, z0);
t0 = sol0.x;
u0 = sol0.y;
tp = solp.x;
up = solp.y;

% create Lorenz map
maxima0 = NaN(1, length(t0));
maximap = NaN(1, length(tp));
samplei0 = u0(3, 1);
sampleip = up(3, 1);
sampleafter0 = u0(3, 2);
sampleafterp = up(3, 2);
for i = 2 : max(length(t0), length(tp)) - 1
    if i < length(t0)
        samplebefore0 = samplei0;
        samplei0 = sampleafter0;
        sampleafter0 = u0(3, i + 1);
        if (samplei0 > samplebefore0) && (samplei0 > sampleafter0)
            maxima0(i) = samplei0;
        end
    end
    if i < length(tp)
        samplebeforep = sampleip;
        sampleip = sampleafterp;
        sampleafterp = up(3, i + 1);
        if (sampleip > samplebeforep) && (sampleip > sampleafterp)
            maximap(i) = sampleip;
        end
    end
end
maxima0 = maxima0(~(isnan(maxima0)));
maximap = maximap(~(isnan(maximap)));

figure(1)
tiledlayout(3, 1)
nexttile
plot(t0, u0(1, :), tp, up(1, :))
title('Lorenz equation solution', 'fontsize', 14)
xlabel('t', 'fontsize', 12)
ylabel('x(t)', 'fontsize', 12)
legend('original', 'perturbed')

nexttile
plot(t0, u0(2, :), tp, up(2, :))
xlabel('t', 'fontsize', 12)
ylabel('y(t)', 'fontsize', 12)
```

```
legend('original', 'perturbed')

nexttile
plot(t0, u0(3, :), tp, up(3, :))
xlabel('t', 'fontsize', 12)
ylabel('z(t)', 'fontsize', 12)
legend('original', 'perturbed')

figure(2)
xlabel('x(t)', 'fontsize', 12)
ylabel('y(t)', 'fontsize', 12)
zlabel('z(t)', 'fontsize', 12)
hold on
scatter3(u0(1, :), u0(2, :), u0(3, :), 0.1, '.')
scatter3(up(1, :), up(2, :), up(3, :), 0.1, '.')
hold off
grid on
title('Lorenz equations attractor', 'fontsize', 14)
legend('original', 'perturbed')

if length(tp) < length(t0)
    u = u0;
    uinterp = zeros(1, length(t0));
    for i = 1 : 3
        uinterp(i, :) = interp1(tp, up(i, :), t0, 'makima');
    end
    tq = t0;
else
    u = up;
    uinterp = zeros(1, length(tp));
    for i = 1 : 3
        uinterp(i, :) = interp1(t0, u0(i, :), tp, 'makima');
    end
    tq = tp;
end
distance = 0;
for i = 1 : 3
    distance = distance + (uinterp(i, :) - u(i, :)) .^ 2;
end
distance = sqrt(distance);

A = [ones(length(tq(round(end / 100) : round(end / 6))),...
    1) tq(round(end / 100) : round(end / 6)).'];
b = log(distance(round(end / 100) : round(end / 6))).';
coeff1 = A \ b;
disp(['The MLE in the first section is ', num2str(coeff1(2))])

A = [ones(length(tq(round(end / 6) : round(35 * end / 100))),...
    1) tq(round(end / 6) : round(35 * end / 100)).'];
```

```
b = log(distance(round(end / 6) : round(35 * end / 100))).';
coeff2 = A \ b;
disp(['The MLE in the second section is ', num2str(coeff2(2))])

figure(3)
hold on
plot(tq, log(distance))
plot(tq(round(end / 100) : round(end / 6)), coeff1(1)...
    + coeff1(2) * tq(round(end / 100) : round(end / 6)), '--')
plot(tq(round(end / 6) : round(35 * end / 100)), coeff2(1)...
    + coeff2(2) * tq(round(end / 6) : round(35 * end / 100)), '--')
hold off
title('log(distance) vs. time', 'fontsize', 14)
xlabel('t', 'fontsize', 12, 'interpreter', 'latex')
ylabel(['$\log\left|\left|\mathbf{u}_{\varepsilon}(t)'...
    , '-\mathbf{u}_{0}(t)\right|\right|_{2}$'],...
     'fontsize', 12, 'interpreter', 'latex')
legend('log(distance)', ['slope: ', num2str(coeff1(2))],...
    ['slope: ', num2str(coeff2(2))])
```

This code plots the time series for $\mathbf{u}_0(t)$ and $\mathbf{u}_\varepsilon(t)$ in one figure but with three plots, each corresponding to the x-, y-, and z-components of the unperturbed and perturbed solutions simultaneously (Figure 6), as well as the 3D scatter phase plot for both trajectories $\mathbf{u}_\varepsilon(t)$ and $\mathbf{u}_0(t)$ simultaneously (Figure 7). The code above also plots $\log||\mathbf{u}_\varepsilon(t) - \mathbf{u}_0(t)||_2$ vs. time and generates an approximation for the maximal Lyapunov exponent (MLE),[2] which should be the approximate slope of the curve on the graph found in Figure 8. However, we find in our actual data that there is a different Lyapunov exponent once the trajectory lies on the strange attractor (from $t \approx 16$ to $t \approx 35$) vs. when it spirals onto the strange attractor (from $t \approx 1$ up to $t \approx 16$). The first of these MLEs is $\lambda \approx 0.12$ and the second is $\boxed{\lambda \approx 0.91}$; the second seems more meaningful because, as we can see in Figure 6, this Lyapunov exponent corresponds to once the trajectories have reached the strange attractor and henceforth exhibit chaotic trajectories.

Furthermore, since the strange attractor only takes up a bounded region of phase space, there will be some time (in this case $t \approx 35$) after which the perturbed and unperturbed trajectories cannot separate any further from each other and will essentially "saturate" at a roughly constant distance. Perhaps this distance represents the "average" distance between two randomly chosen points on the strange attractor, because once the trajectories have separated entirely, we will observe behavior like in Figure 6 where near $t \approx 35$, the original and perturbed trajectories diverge entirely in qualitative behavior, and continue to remain separated for all future times. Hence, when computing the MLE, we must only consider the log(distance) curve *before* it saturates. All of this explains exactly the behavior of the curve in Figure 8. □

Problem 5.

[2]Normally, there is a spectrum of Lyapunov exponents, one corresponding to each orthogonal direction that we get from the Jacobian of the displacement between a given trajectory and one that started very close to it. In practice, we can use a Gram-Schmidt orthogonalization procedure at each timestep to find these exponents. However, to find the MLE alone, doing what we did suffices, because for t large enough, the exponential term corresponding to the MLE will eventually come to dominate the distance between $\mathbf{u}_\varepsilon$ and $\mathbf{u}_0$.

Solution. We can repeat Problems 1 and 3 with $r = 100$ by just running Problem1.m again, except replacing `r = 28` with `r = 100`. From this, we generate Figures 9, 10, 11, and 12. While we see the faint ghost of the butterfly-shaped strange attractor in 10, we still instead have a knotted periodic orbit. In fact, if we had chosen $r \approx 99.96$, we would have seen a T(3,2) torus knot. (!)

After a seemingly chaotic transient, the system settles down to regular, periodic solutions for each component. But how might we qualitatively gather information about these periodic solutions? We can look at the iterates in Figure 11. This time, our iterate plots are discrete and the cobweb plot no longer densely fills up some region of phase space, indicating the finite periodicity of our solution. In particular, these points do not lie on the dotted diagonal line unless we consider the plot of z_{n+3} vs. z_n (or at least out of the five iterate plots we sampled for Figure 11), in which case there are three intersections at $z_n \approx 114, 124, 143$. This is completely expected, because if we look at the plot of $z(t)$ in Figure 9, it is evident that in each period, $z(t)$ has three local maxima, leading to a period-3 cycle in the Lorenz map. □

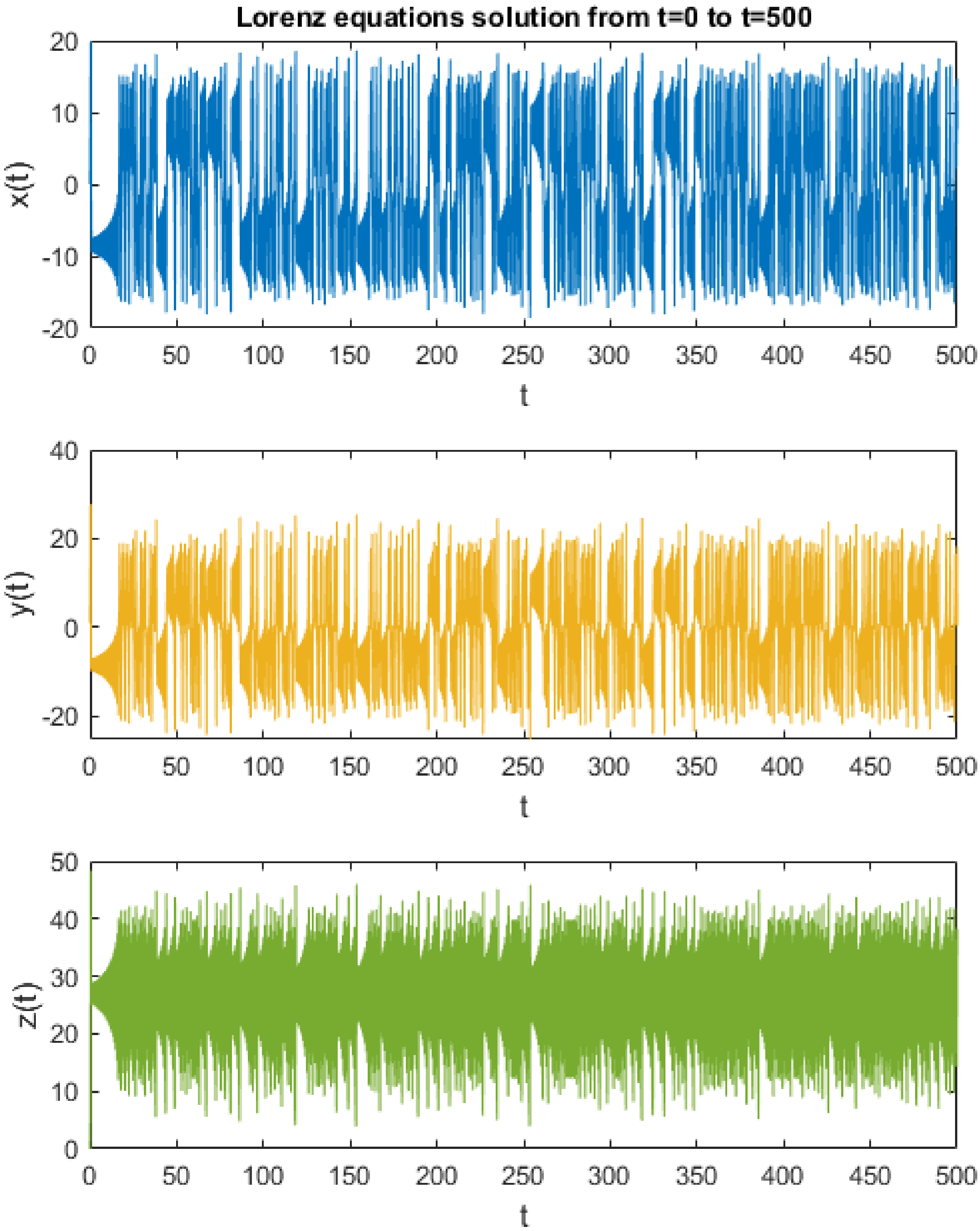


Figure 1: The $x(t)$, $y(t)$, and $z(t)$ trajectories that solve (1) from $t = 0$ to $t = 500$ using an initial condition of $(x(0), y(0), z(0)) = (0, 1, 0)$, $\sigma = 10$, $\beta = 8/3$, and $r = 28$.

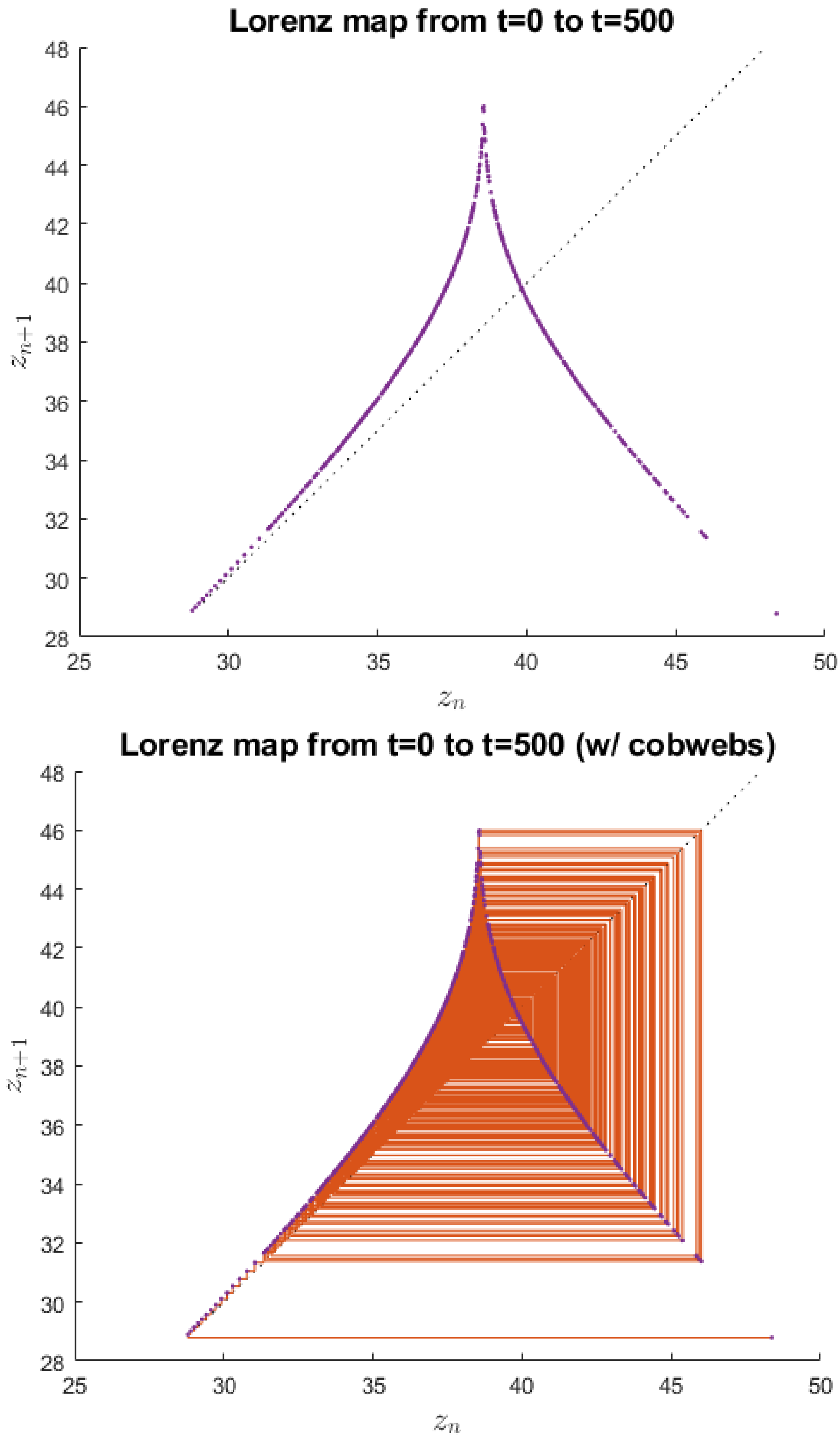


Figure 2: The Lorenz map in z for the solution to (1) from $t = 0$ to $t = 500$, i.e., the successive values for each successive maxima of $z(t)$, using the parameters and initial conditions mentioned in Figure 1.

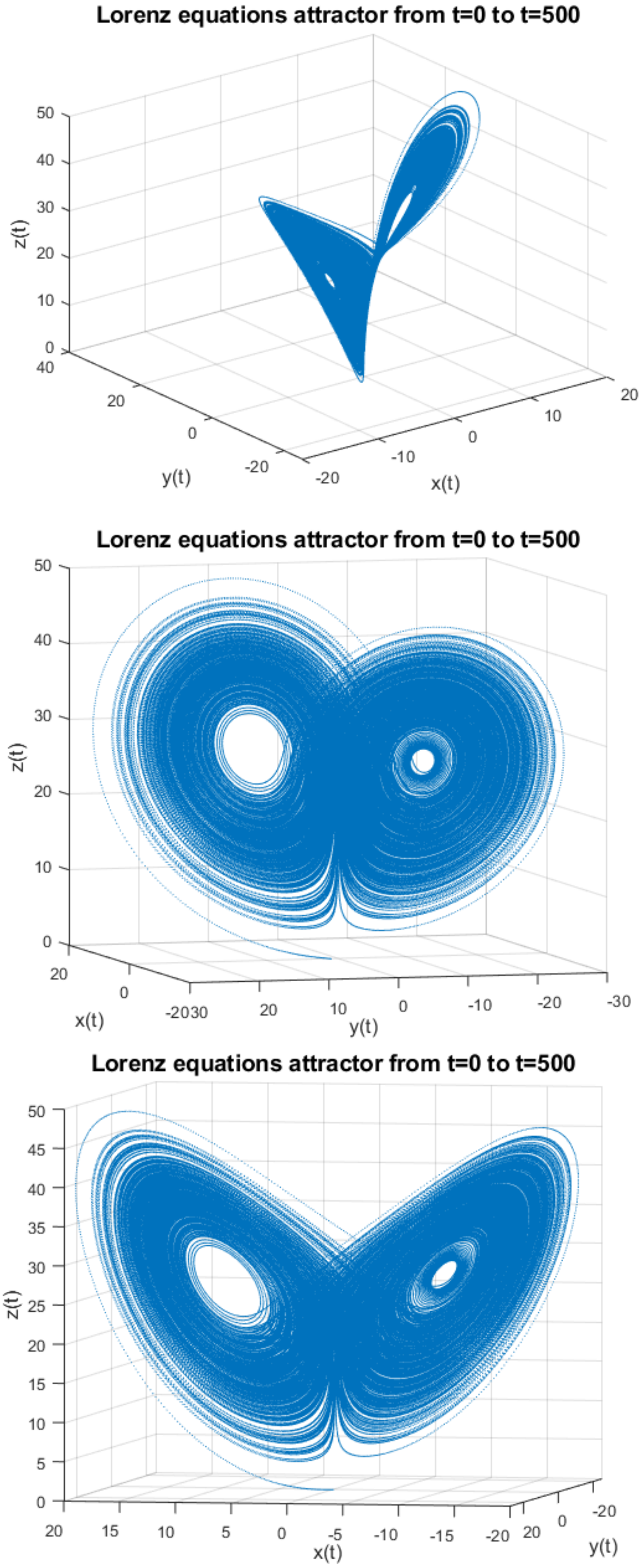
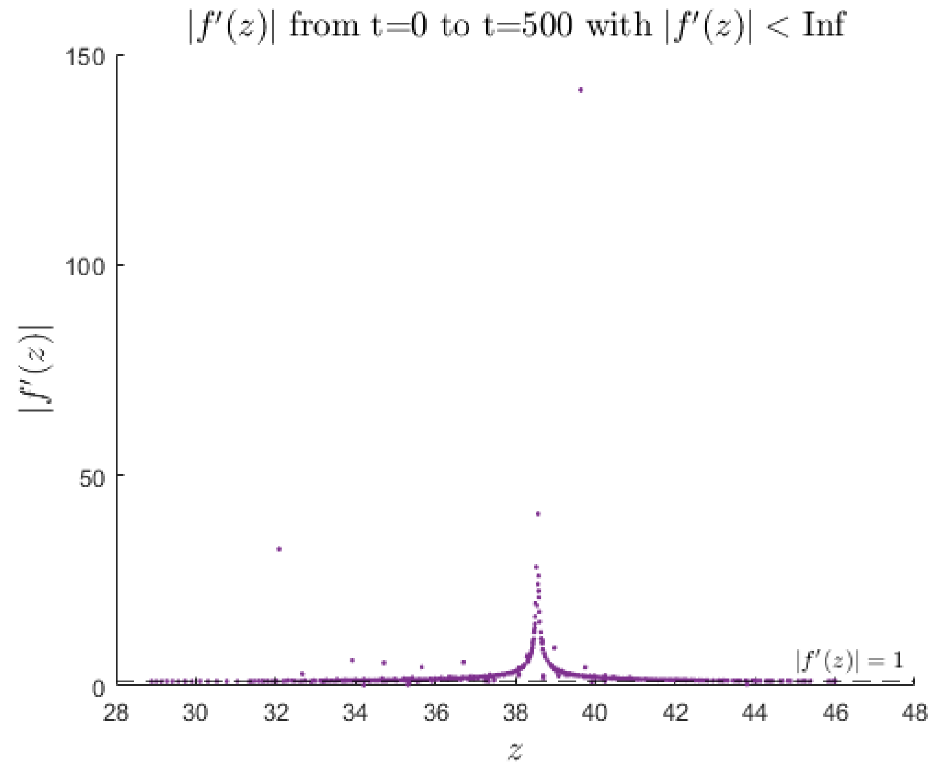


Figure 3: The trajectories for $\mathbf{u}(t) = (x(t), y(t), z(t))$ at three different vantage points in $\mathbb{R}^3$ with the parameters and initial conditions mentioned in Figure 1. Note the characteristic "butterfly" shape of the strange attractor which all trajectories approach after some amount of time.



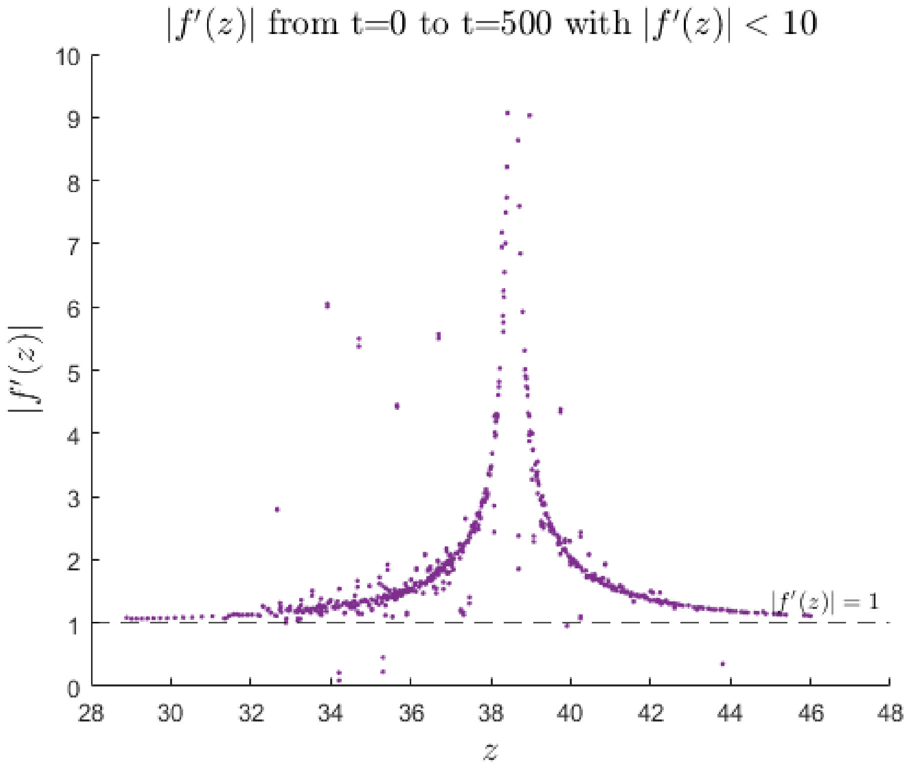


Figure 4: The plots for $|f'(z)|$ vs. z, as approximated using second-order finite differences to the Lorenz map data plotted in Figure 2. Since Figure 2 indicates a possible corner, i.e., first-derivative singularity, in $f(z)$ at $z \approx 39.4 \pm 0.2$, we should expect the graph of $|f'(z)|$ to blow up near that point in the domain. We see this behavior quite drastically in our numerical results, and hence, we plot $|f'(z)|$ both at all points and for values only below 10, such that we can better see the curve in the second case. In either case, it is clear that most of the points follow a defined curve lying above $|f'(z)| = 1$; any deviations are possibly due to numerical error, or the fact that $f(z)$ is not completely a well-defined function.

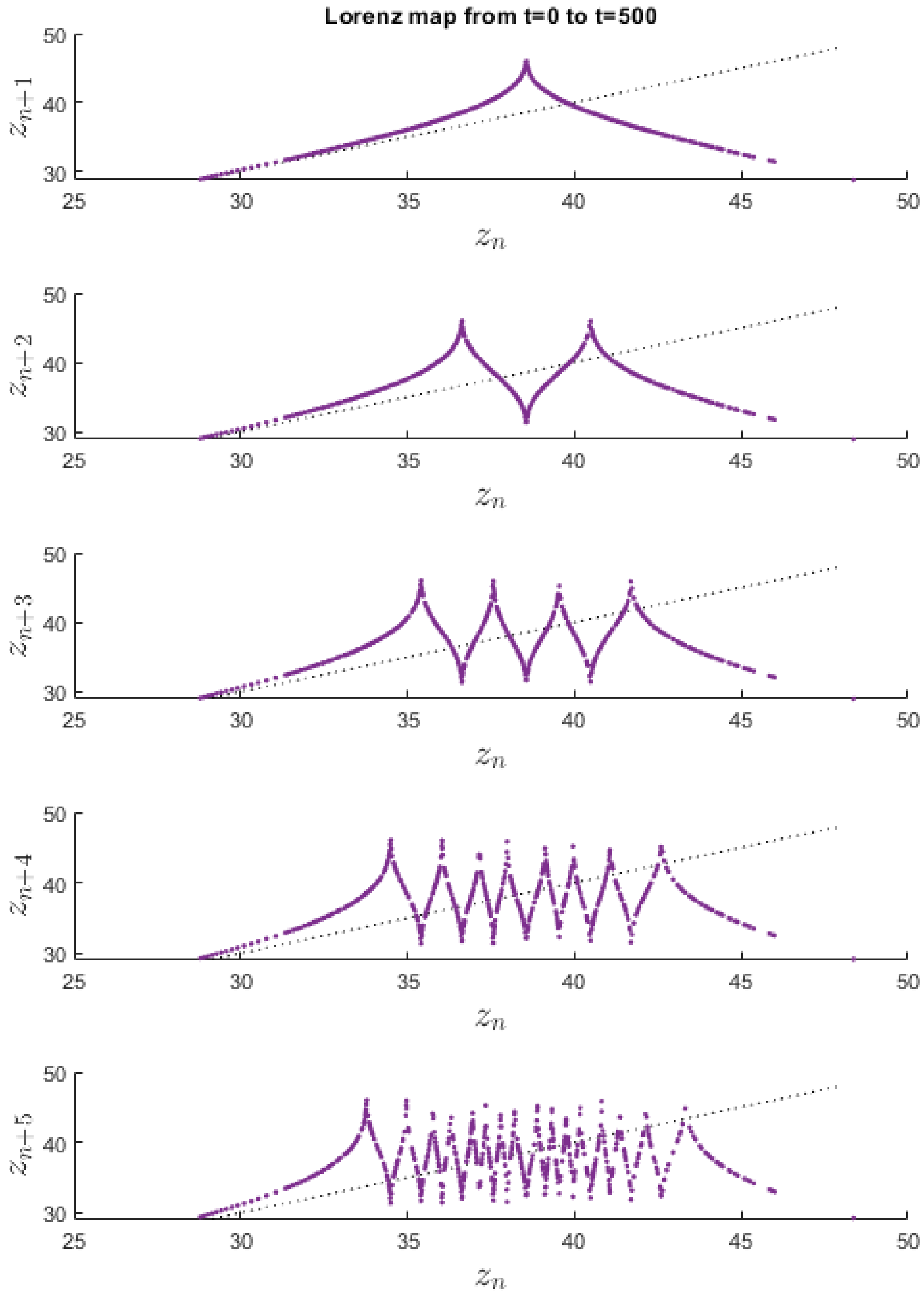


Figure 5: The Lorenz maps for z_{n+1} vs. z_n, z_{n+2} vs. z_n, z_{n+3} vs. z_n, z_{n+4} vs. z_n, and z_{n+5} vs. z_n for the iterates plotted in Figure 2.

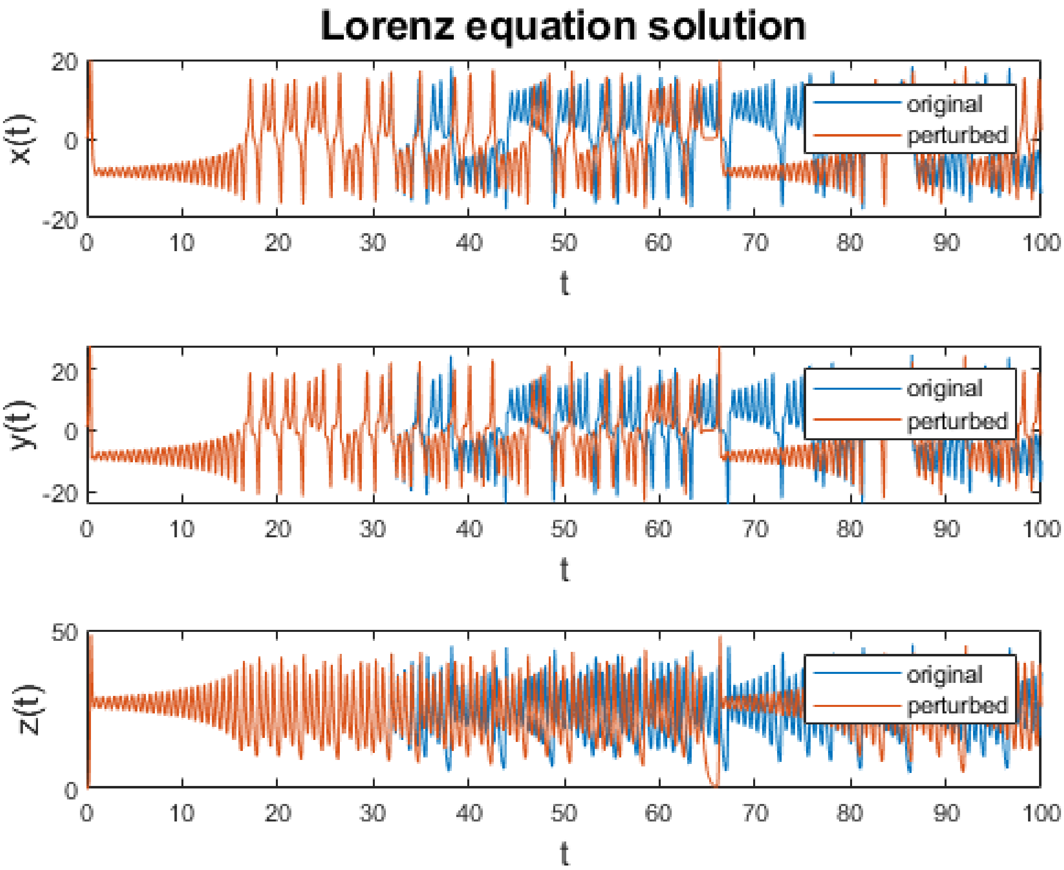


Figure 6: The unperturbed and perturbed trajectories that solve (1) from $t = 0$ to $t = 100$ using initial conditions of $\mathbf{u}_0(0) = (0, 1, 0)$ and $\mathbf{u}_\varepsilon(0) = (10^{-6}, 1, 0)$, respectively, $\sigma = 10$, $\beta = 8/3$, and $r = 28$.

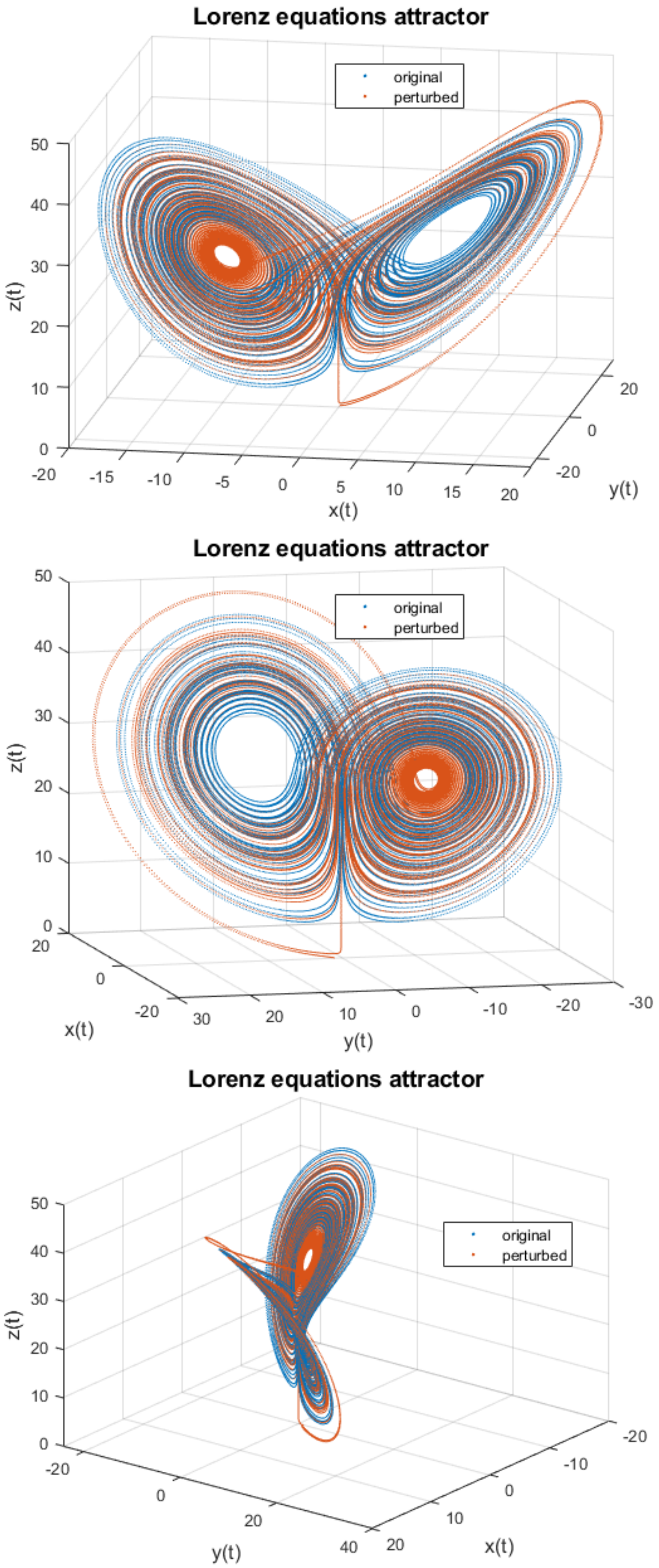


Figure 7: The trajectories for $\mathbf{u}_0(t)$ and $\mathbf{u}_\varepsilon(t)$ at three different vantage points in $\mathbb{R}^3$ with the parameters and initial conditions mentioned in Figure 6.

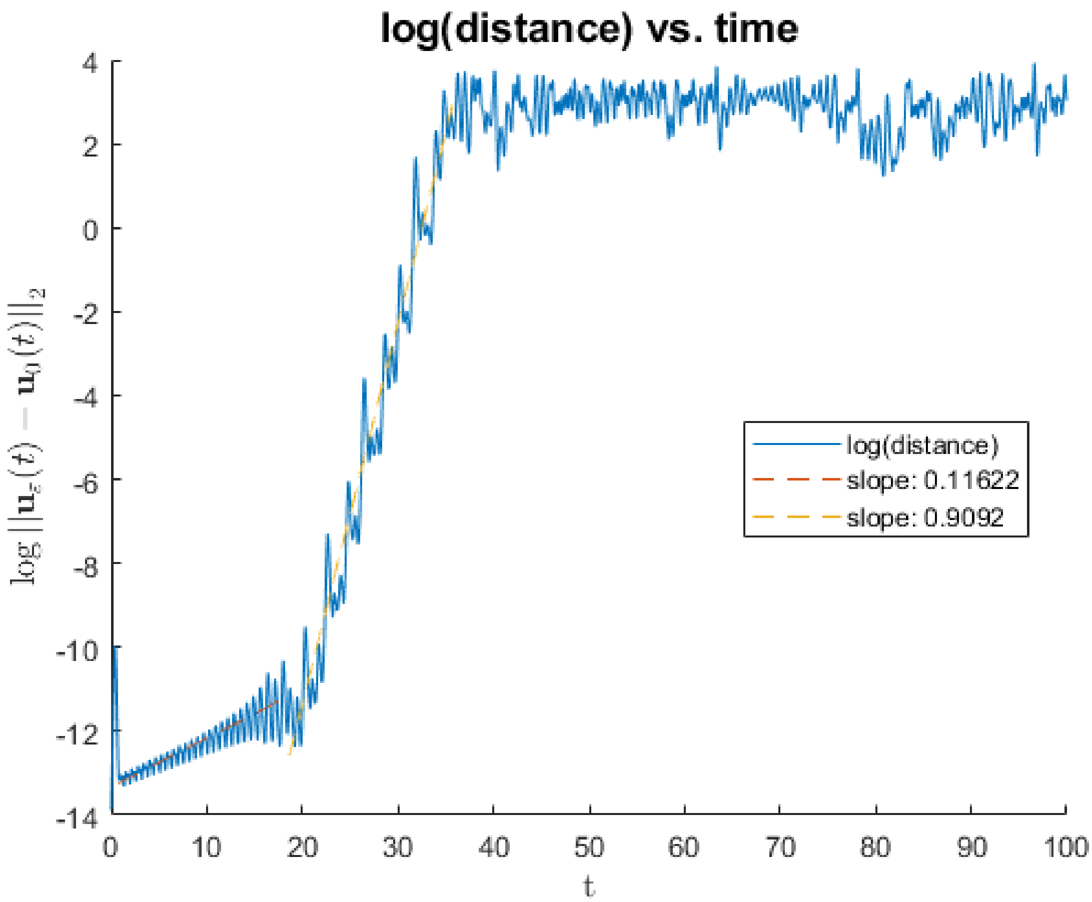


Figure 8: The plot of $\log ||\mathbf{u}_\varepsilon(t) - \mathbf{u}_0(t)||_2$ vs. t for the trajectories shown in Figures 6 and 7. The dashed lines show the different portions of the curve on which we performed linear least squares to find a best-fit line and its slope, the last of which approximates the MLE.

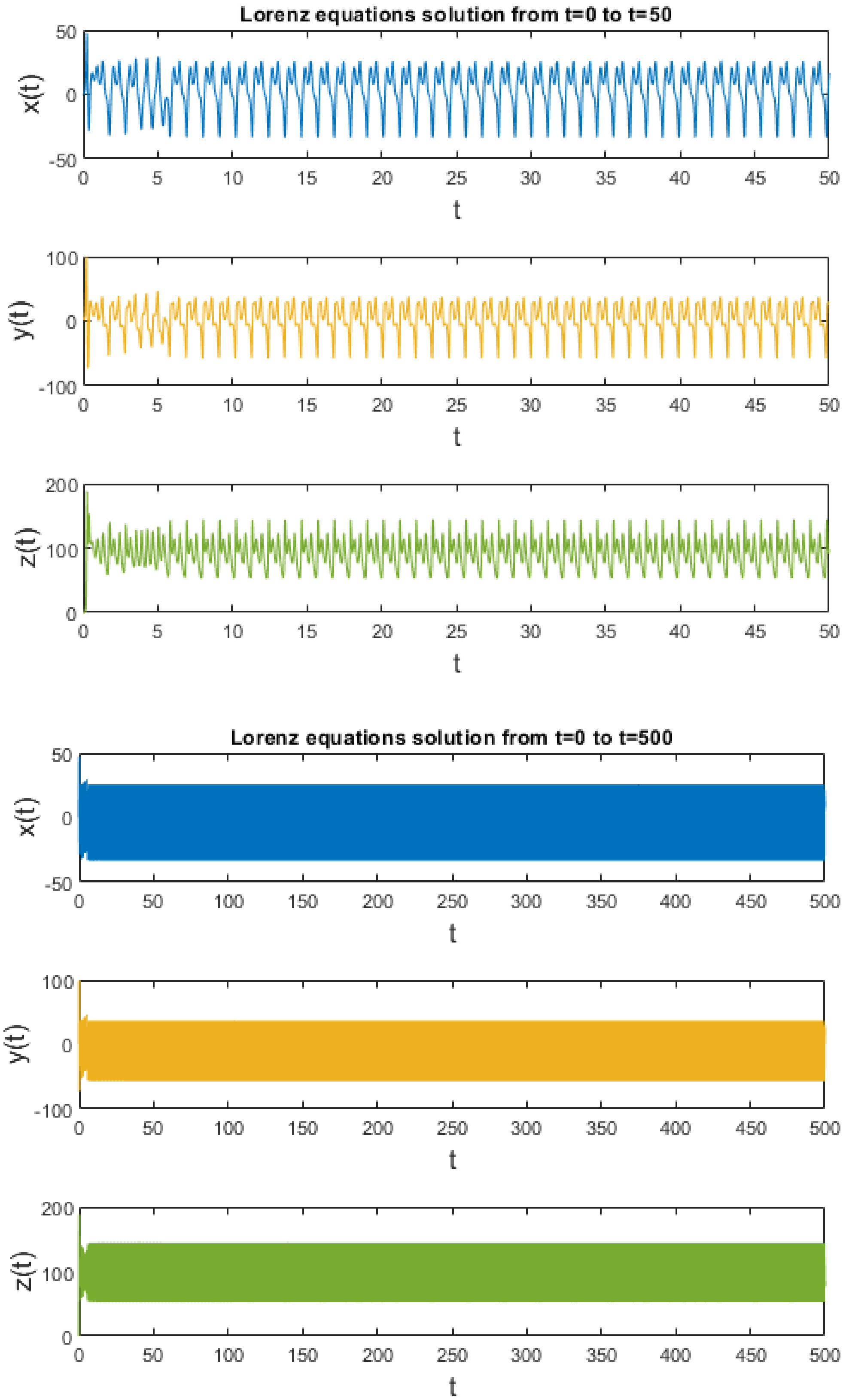


Figure 9: This is the same as Figure 1, except with $r = 100$ instead of $r = 28$ and running the simulation up to both $t = 50$ and $t = 500$. We did the latter so that one can see the evident periodicity of the solution.

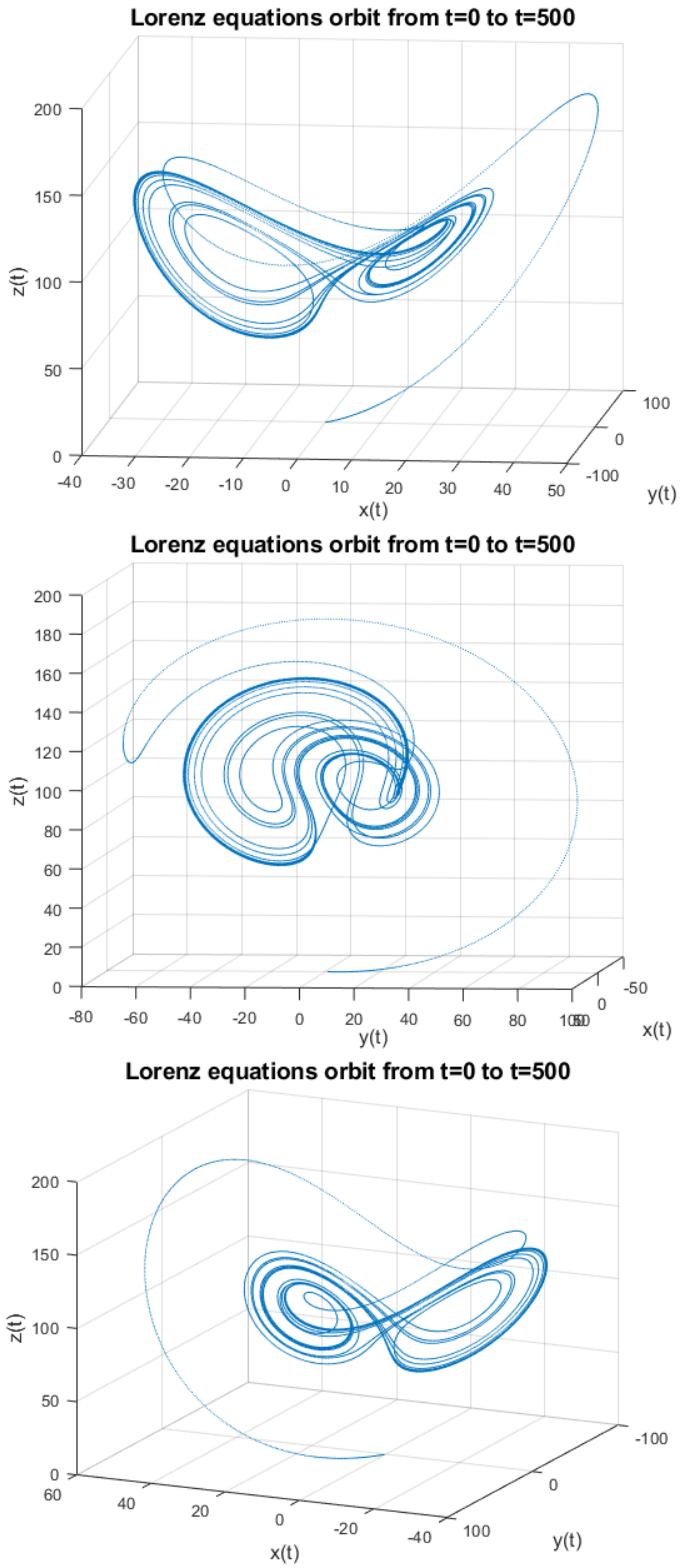


Figure 10: Figure 3 except with $r = 100$ instead of $r = 28$.

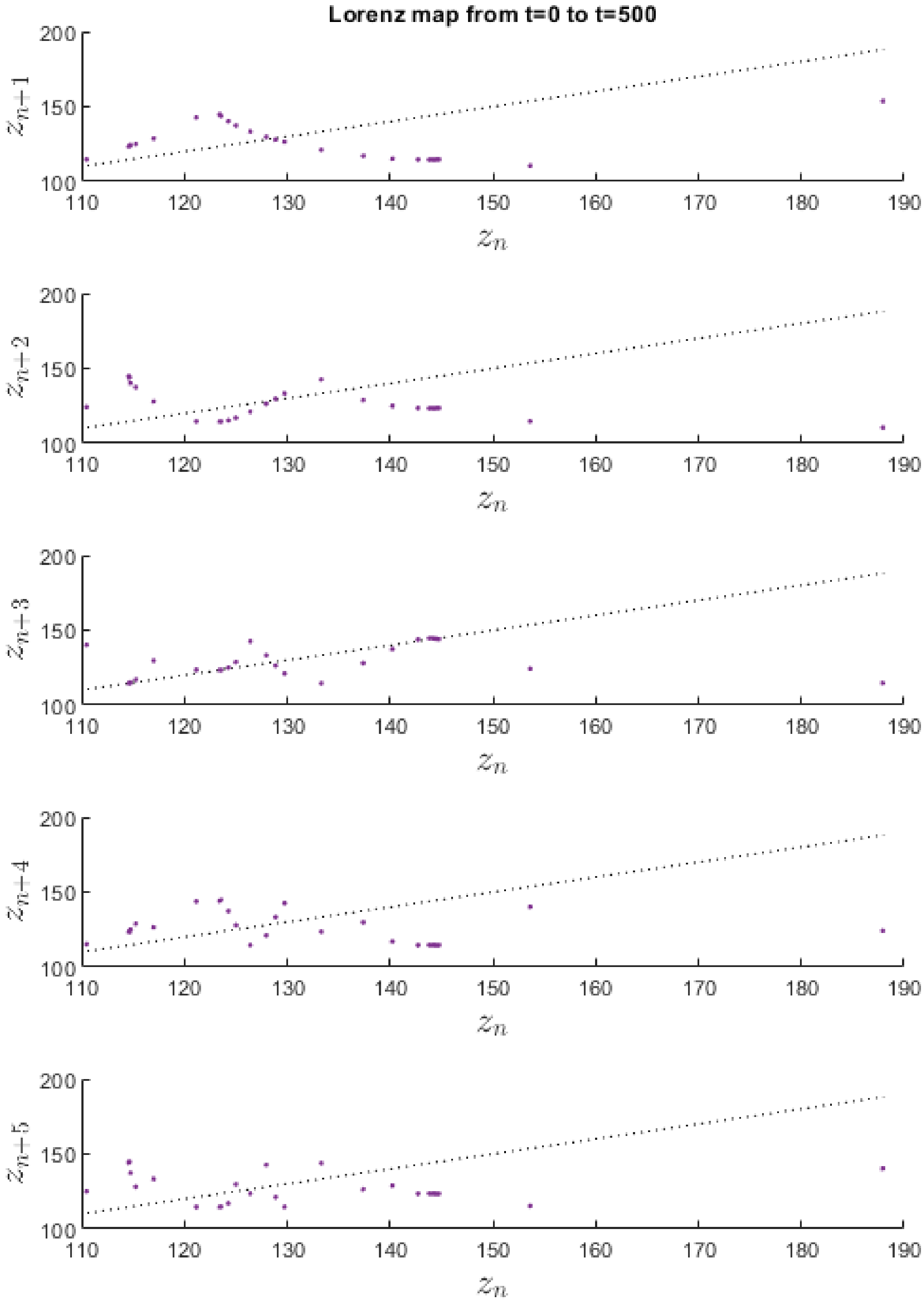


Figure 11: The Lorenz maps for z_{n+1} vs. z_n, z_{n+2} vs. z_n, z_{n+3} vs. z_n, z_{n+4} vs. z_n, and z_{n+5} vs. z_n for the iterates plotted in the topmost plot, as gathered from each successive maxima of z in Figure 9.

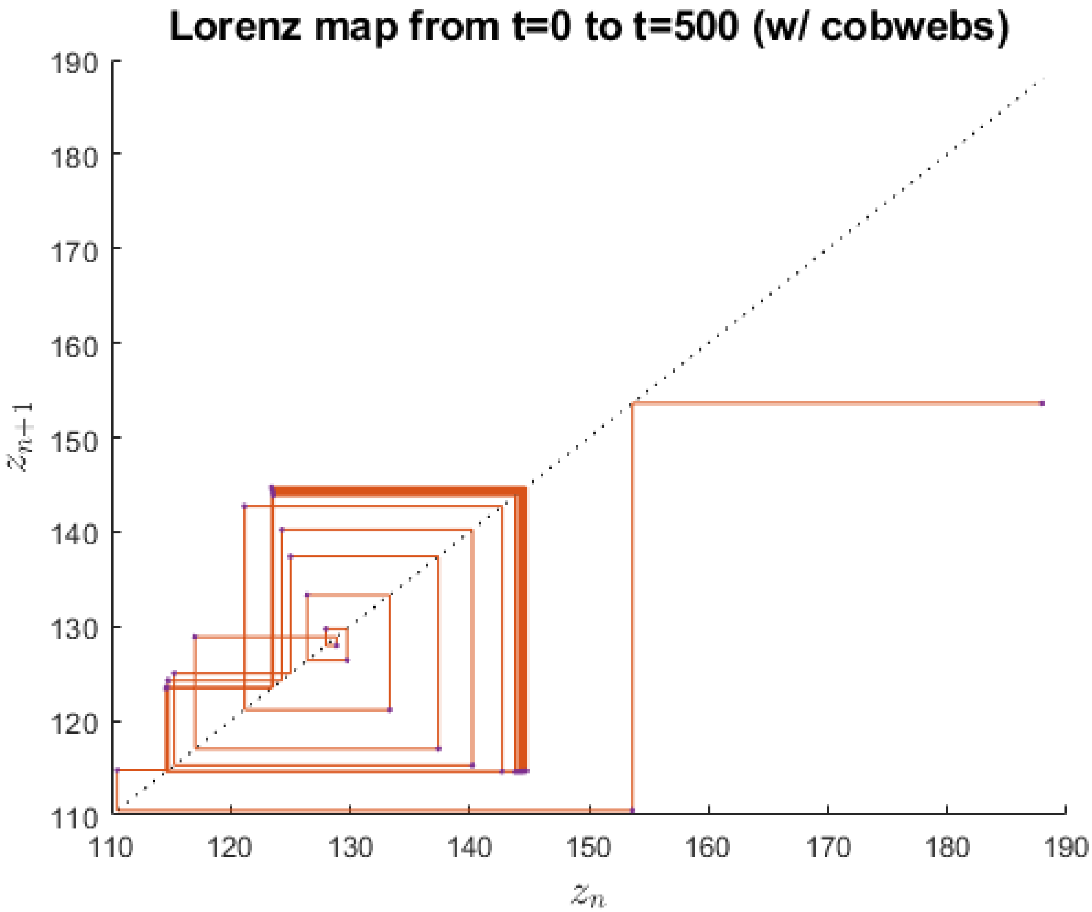


Figure 12: The cobweb plot corresponding to the trajectory in Figure 9 and the iterates plotted in the topmost plot in Figure 11.