EPS 659a — Problem Set 8

due Friday, November 19, 2021

Problem 1: Inverse Theory: Gravity/Density Inversion.

Vertical gravity $d(x)$ is measured above a thin buried flat sheet of rock with a 2-D density anomaly $d\rho(\xi)$ in the x ordinate.

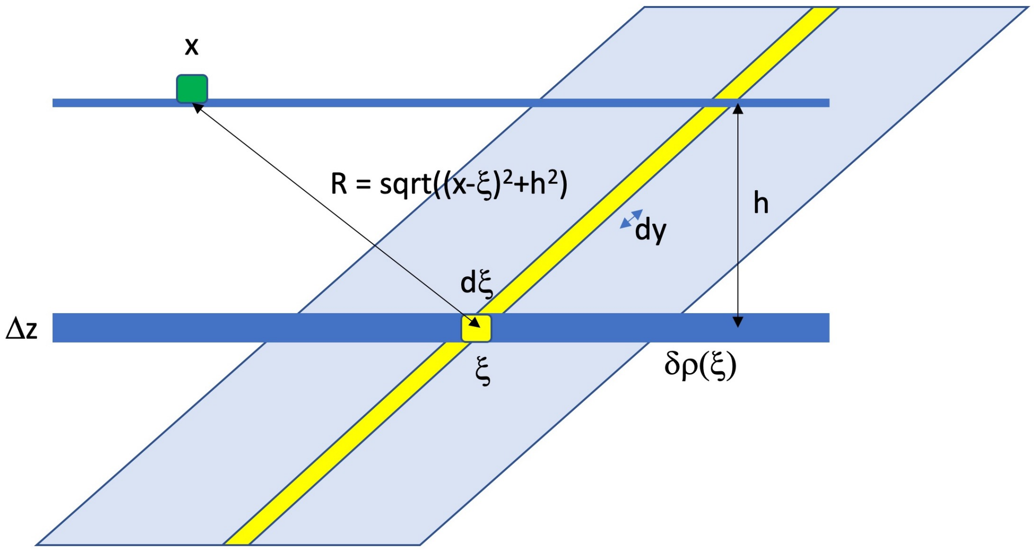

Figure 1: (Problem 1) Geometry of gravity data inversion.

The vertical gravity $d(x)$ is a linear functional of the density function $\delta\rho(\xi)$, as shown in the figure.

$$d(x) = \int_{-\infty}^{\infty} \frac{2\Gamma_o h \Delta z}{(x-\xi)^2 + h^2} \delta\rho(\xi) d\xi = \int_{-\infty}^{\infty} g(x,\xi)\delta\rho(\xi) d\xi$$

where $\Gamma_o = 6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$ is Newton's Gravitational Constant and $g(x,\xi)$ is the data representer for the observation point x. The linear integral operator can be read as an inner product on square-integrable functions on $-\infty < \xi < \infty$, according to

$$(f,g) = \int_{-\infty}^{\infty} f(\xi) g^*(\xi) d\xi$$

where the asterisk denotes complex conjugation. If the functions $f(\xi), g(\xi)$ are real-valued, the conjugation does nothing. We showed in class that the inner product matrix of data representers can be evaluated to form the "Gram matrix" $\mathbf{\Gamma}$, no direct relation to the gravitational constant, a symmetric matrix with components

$$\begin{aligned}\Gamma_{jk} = (g_j, g_k) = \int_{-\infty}^{\infty} g(x_j,\xi) g(x_k,\xi) d\xi &= \int_{-\infty}^{\infty} \frac{(2\Gamma_o h \Delta z)^2}{((x_j-\xi)^2+h^2)((x_k-\xi)^2+h^2)} d\xi \\ &= \frac{\pi}{2h}(2\Gamma_o \Delta z)^2 \left(\left(\frac{x_k - x_j}{2h} \right)^2 + 1 \right)^{-1}\end{aligned}$$

The model $\delta\rho(\xi)$ that fits the data exactly while minimizing $\|\delta\rho\|^2 = (\delta\rho, \delta\rho)$ can be expressed as a linear combination of data representers

$$\delta\rho = \sum_{k=1}^{N} \alpha_k g_k(\xi)$$

and the vector $\boldsymbol{\alpha}$ of coefficients is computed with $\boldsymbol{\alpha} = \boldsymbol{\Gamma}^{-1} \cdot \mathbf{d}$.

(a) What are the units of the data representer $g_j(\xi)$? What are the units of the Gram matrix $\boldsymbol{\Gamma}$? What are the units of the coefficient vector $\boldsymbol{\alpha}$? Hint: Use SI units. Convert kilometers to meters and express mass in kilograms, not grams. In SI units a typical silicate rock density will be $\rho \sim 2500 - 3300$ kg/m^3, with plausible $\delta\rho \sim \pm 100$ kg/m^3. In SI units, gravity has units m/s^2 and typical gravity anomalies are expressed in milligals = 10^{-5} m/s^2. So the data sets have data with values of that order.

(b) For a trio of depths $h = 5, 10, 15$ km, plot data representers $g(x, \xi)$ for $\Delta z = 1$ km and two choices of $x = 0, 10$ km, plotting for $-60 < \xi < 80$ km. You can plot all six curves on a single plot.

(c) On the Canvas server, you will find a dataset `gravity_pset8.csv` that contains four columns with position x_j, vertical gravity g_j for Model-1, vertical gravity g_j for Model-2, and the standard deviation of gravity σ_j for $j = 1, 2, \ldots N$, where $N = 30$. Given the shared observation points $\{x_j\}$ in the data set, compute the Gram matrix $\boldsymbol{\Gamma}$ and its eigenvalues for three values of $h = 5, 10, 20$ km. Plot the eigenvalues for the three cases on one comparison plot, and comment on how the eigenvalues vary with the depth of the density anomaly. You may want to plot the eigenvalues on a logarithmic scale to distinguish the cases properly. Compute and report the condition numbers of the three Gram matrices.

(d) Using the dataset, compute the best-fit model $\delta\rho(\xi)$ as a linear combination of data representers for Model 1 (the 2nd column of numbers in the `*.csv` file) for assumed layer depths $h = 5, 10, 15$ km. You should use the R function `solve(matrix,vector)` to compute the coefficient vector $\boldsymbol{\alpha}$. Plot all three estimated $\delta\rho(\xi)$ both separately and on the same graph as a function of $-100 < \xi < 100$ km.

(e) Using the dataset, compute the best-fit model $\delta\rho(\xi)$ as a linear combination of data representers for Model 2 (the 3rd column of numbers in the `*.csv` file) for assumed layer depths $h = 5, 10, 15$ km.

You should use the R function `solve(matrix,vector)` to compute the coefficient vector $\boldsymbol{\alpha}$. Plot all three estimated $\delta\rho(\xi)$ both separately and on the same graph as a function of $-100 < \xi < 100$ km.

Problem 2: Perfect continuous data. Fourier-domain solution.

In class we showed that the gravity inversion formulation was a convolution over horizontal distance ξ in the buried layer, so that the convolution theorem can be applied to data inversion. The data function $d(x)$, defined as a function of a continuous position x, can be expressed as a convolution

$$d(x) = \int_{-\infty}^{\infty} g(x-\xi)\delta\rho(\xi)d\xi = \int_{-\infty}^{\infty} \overline{g}(\lambda)\overline{\delta\rho}(\lambda)e^{-2\pi i\lambda x}d\lambda$$

also, the data function can be expressed in terms of its Fourier transform $\overline{d}(\lambda)$ with a similar integral over wavenumber

$$d(x) = \int_{-\infty}^{\infty} \overline{d}(\lambda)e^{-2\pi i\lambda x}d\lambda$$

so that we can solve for the model $\delta\rho$ in the wavenumber domain $\overline{d}(\lambda) = \overline{g}(\lambda)\overline{\delta\rho}(\lambda)$ and

$$\overline{\delta\rho}(\lambda) = \overline{d}(\lambda)/\overline{g}(\lambda)$$

In class we saw that the Fourier transform of the data representer $g(\xi) = 2\Gamma_o h\Delta z/(\xi^2 + h^2)$ is

$$\overline{g}(\lambda) = \Gamma_o \Delta z \exp(-2\pi h|\lambda|)$$

(a) Assume that $\overline{d}(\lambda) = g_o(\delta(\lambda - \lambda_o) + \delta(\lambda + \lambda_o))$, with g_o a constant gravity perturbation. What is the functional dependence of $d(x)$? What is the model function $\delta\rho$ as a function of ξ and h? Recall that the delta function $\delta(\lambda - \lambda_o)$ obeys the integral relation

$$\int_{-\infty}^{\infty} f(\lambda)\delta(\lambda - \lambda_o)d\lambda = f(\lambda_o)$$

for any function $f(\lambda)$ that is continuous at λ_o.

(b) Assume that $\overline{d}(\lambda) = g_o \exp(-3\pi h|\lambda|)$. What is the inferred model function $\delta\rho$ as a function of of ξ and h? In order to perform the inverse Fourier transform, split the integral over λ into two halves

$$\int_{-\infty}^{\infty} \overline{\delta\rho}(\lambda)\exp(-2\pi i\lambda\xi)d\lambda = \int_{-\infty}^{0} \overline{\delta\rho}(\lambda)\exp(-2\pi i\lambda\xi)d\lambda + \int_{0}^{\infty} \overline{\delta\rho}(\lambda)\exp(-2\pi i\lambda\xi)d\lambda$$

and evaluate the two integrals separately to avoid the discontinuity in the derivative of $\overline{\delta\rho}(\lambda)$ at $\lambda = 0$, which would otherwise torpedo the usual way to evaluate an integral via the anti-derivative.

(c) Assume that $\overline{d}(\lambda) = g_o \exp(-h^2\lambda^2)$, what is the inferred model function $\delta\rho$ as a function of ξ and h? In order to perform the inverse Fourier transform, use the integration formula

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp(-a\lambda^2 + b\lambda)d\lambda &= \exp\left(\tfrac{b^2}{4a}\right) \int_{-\infty}^{\infty} \exp\left(-a\left(\lambda^2 - (b/a)\lambda + \left(\tfrac{b}{2a}\right)^2\right)\right) d\lambda \\
&= \exp\left(\tfrac{b^2}{4a}\right) \int_{-\infty}^{\infty} \exp\left(-a\left(\lambda - \left(\tfrac{b}{2a}\right)\right)^2\right) d\lambda \\
&= \sqrt{\frac{\pi}{a}} \exp\left(\tfrac{b^2}{4a}\right)
\end{aligned}$$

and be mindful of the imaginary numbers.

Problem 3: Resolution of the density model. The resolution of a linear inverse problem within the domain of the model-space acknowledges that all models $\mathbf{m}$ for data $\mathbf{d}$ will effectively average the underlying properties of the system, hopefully with as narrow an average as possible. One can translate that "closest as possible" metric into a least-squares problem, in which one minimizes the integrated misfit between a linear combination of data-representer functions $g_k(\xi)$ and $\delta(\xi - \xi_o)$, the delta function centered on a solution target ξ_o. The key goal is thwarted because the delta function is not square-integrable. "Quelling" smooths over this obstacle by applying a functional operator Q that converts $\delta(\xi - \xi_o)$ into a square-integrable function. The data representers must also be quelled by the operator Q. Some quelling operators can be defined for all resolution targets ξ_o in the model domain. Other quelling operators require a different operator for each ξ_o. There are many ways to define Q to achieve a resolution estimate, but we will compare two ways: convolution quelling and multiplication quelling.

(a) In convolution quelling we convolve the function f with a narrow boxcar function

$$B(\tilde{\xi}) = \begin{cases} (2\xi_1)^{-1} & \text{for } -\xi_1 \le \tilde{\xi} \le \xi_1 \\ 0 & \text{for } |\tilde{\xi}| \ge \xi_1 \end{cases}$$

so that, for a function f, the quelled function is the local average, equivalent to a convolution of f with a boxcar function.

$$Q[f(\xi)] = f_Q(\xi) = \int_{-\xi_1}^{\xi_1} f(\tilde{\xi})B(\xi - \tilde{\xi})d\tilde{\xi} = (2\xi_1)^{-1} \int_{\xi-\xi_1}^{\xi+\xi_1} f(\tilde{\xi})d\tilde{\xi} \to f(\xi)$$

for continuous functions as the width of the local average $\xi_1 \to 0$. In this limit we can approximate the quelled data representers $Q[g_j(\xi)]$ as the unchanged data representers $g_j(\xi)$. The quelled delta function at the resolution target ξ_o is

$$Q[\delta(\xi - \xi_o)] = \delta_Q(\xi, \xi_o) = \int_{-\xi_1}^{\xi_1} \delta(\tilde{\xi} - \xi_o)B(\xi - \tilde{\xi})d\tilde{\xi} = B(\xi - \xi_o)$$

To optimize the resolution in a least-squares sense over the domain of the model space, we express the misfit to the quelled delta function for a linear combination of data representers as $\psi(\xi, \xi_o) = \sum_{j=1}^{N} \gamma_j g_j(\xi) - \delta_Q(\xi, \xi_o)$ and minimize the squared norm $\|\psi\|^2 = (\psi, \psi)$, which expands to

$$(\psi, \psi) = (\delta_Q, \delta_Q) - 2\sum_{j=1}^{N} \gamma_j(\delta_Q, g_j) + \sum_{j=1}^{N}\sum_{k=1}^{N} \gamma_j\gamma_k(g_j, g_k) = (\delta_Q, \delta_Q) - 2\boldsymbol{\gamma}^T \cdot \boldsymbol{\beta} + \boldsymbol{\gamma}^T \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{\gamma}$$

where the jth component of the N-vector $\boldsymbol{\beta}$ is (δ_Q, g_j). For this problem, use the limiting value of the convolution integral $(\delta_Q, g_j) \to g_j(\xi_o)$ as $\xi_1 \to 0$, so that $\beta_j = g_j(\xi_o)$. We extremize this functional of $\boldsymbol{\gamma}$ by setting its gradient equal to zero, so that $\boldsymbol{\Gamma} \cdot \boldsymbol{\gamma} - \boldsymbol{\beta} = 0$, leading to

$$\boldsymbol{\gamma} = \boldsymbol{\Gamma}^{-1} \cdot \boldsymbol{\beta}$$

Similar to problem (1), you should use the R function `solve(matrix,vector)` to compute the coefficient vector $\boldsymbol{\gamma}$. For the observation points x_j of the dataset `gravity_pset8.csv`, compute and plot the resolution kernel $\tilde{\delta}(\xi, \xi_o) = \sum_{j=1}^{N} \gamma_j g_j(\xi)$ for three cases $\xi_o = 0, 20, 50$ km on the domain $-100 < \xi < 100$ km. Use anomaly depth $h = 5$ km. If you plot the resolution kernel $\tilde{\delta}(\xi, \xi_o)$ against ξ in kilometers, which is more intuitive than meters in this geometry, you should multiply the function $\tilde{\delta}(\xi, \xi_o)$ by 1000 to scale it properly to mimic a function whose integral across the target ξ_o approximates unity.

(b) For the same problem, compute the resolution kernels for three cases $\xi_o = 0, 20, 50$ km on the domain $-100 < \xi < 100$ km using anomaly depth $h = 10$ km.