UC Berkeley Math 228B, Spring 2019: Problem Set 6

Due April 25

Consider the traffic flow problem, described by the non-linear hyperbolic equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{1}$$

with $\rho = \rho(x,t)$ the density of cars (vehicles/km), and $u = u(x,t)$ the velocity. Assume that the velocity u is given as a function of ρ:

$$u = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{2}$$

With $u_{\max}$ the maximum speed and $0 \le \rho \le \rho_{\max}$. The flux of cars is therefore given by:

$$f(\rho) = \rho u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{3}$$

We will solve this problem using a first order finite volume scheme:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right). \tag{4}$$

For the numerical flux function, we will consider two different schemes:

Roe's Scheme

The expression of the numerical flux is given by:

$$F^R_{i+\frac{1}{2}} = \frac{1}{2}\left[f(\rho_i) + f(\rho_{i+1})\right] - \frac{1}{2}\left|a_{i+\frac{1}{2}}\right|(\rho_{i+1} - \rho_i) \tag{5}$$

with

$$a_{i+\frac{1}{2}} = u_{\max}\left(1 - \frac{\rho_i + \rho_{i+1}}{\rho_{\max}}\right). \tag{6}$$

Note that $a_{i+\frac{1}{2}}$ satisfies

$$f(\rho_{i+1}) - f(\rho_i) = a_{i+\frac{1}{2}}(\rho_{i+1} - \rho_i). \tag{7}$$

Godunov's Scheme

In this case the numerical flux is given by:

$$F^G_{i+\frac{1}{2}} = f\left(\rho\left(x_{i+\frac{1}{2}}, t^{n+}\right)\right) = \begin{cases} \min_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho), & \rho_i < \rho_{i+1} \\ \max_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho), & \rho_i > \rho_{i+1}. \end{cases} \tag{8}$$

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1. For both Roe's Scheme and Godunov's Scheme, look at the problem of a traffic light turning green at time $t = 0$. We are interested in the solution at $t = 2$ using both schemes. What do you observe for each of the schemes? Explain briefly why the behavior you get arises.

   Use the following problem parameters:

$$\begin{aligned} \rho_{\max} &= 1.0, \quad \rho_L = 0.8 \\ u_{\max} &= 1.0 \\ \Delta x &= \frac{4}{400}, \quad \Delta t = \frac{0.8\Delta x}{u_{\max}} \end{aligned} \tag{9}$$

   The initial condition at the instant when the traffic light turns green is

$$\rho(0) = \begin{cases} \rho_L, & x < 0 \\ 0, & x \geq 0 \end{cases} \tag{10}$$

   For problems 2 - 3, use only the scheme(s) which are valid models of the problem.

2. Simulate the effect of a traffic light at $x = -\frac{\Delta x}{2}$ which has a period of $T = T_1 + T_2 = 2$ units. Assume that the traffic light is $T_1 = 1$ units on red and $T_2 = 1$ units on green. Assume a sufficiently high flow density of cars (e.g. set $\rho = \frac{\rho_{\max}}{2}$ on the left boundary – giving a maximum flux), and determine the average flow, or capacity of cars over a time period T.

   The average flow can be approximated as

$$\dot{q} = \frac{1}{N_T}\sum_{n=1}^{N_T} f^n = \frac{1}{N^T}\sum_{n=1}^{N_T} \rho^n u^n, \tag{11}$$

   where N_T is the number of time steps for each period T. You should run your computation until $\dot{q}$ over a time period does not change. Note that by continuity $\dot{q}$ can be evaluated over any point in the interior of the domain (in order to avoid boundary condition effects, we consider only those points on the interior domain).

   Note: A red traffic light can be modeled by simply setting $F_{i+\frac{1}{2}} = 0$ at the position where the traffic light is located.

3. Assume now that we simulate two traffic lights, one located at $x = 0$, and the other at $x = 0.15$, both with a period T. Calculate the road capacity (= average flow) for different delay factors. That is if the first light turns green at time t, then the second light will turn green at $t + \tau$. Solve for $\tau = k\frac{T}{10}$, $k = 0, \ldots, 9$. Plot your results of capacity vs τ and determine the optimal delay τ.