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The Homotopy 2-Category of a Stable ∞-Category

Master's Thesis

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Abstract

In this thesis we investigate the structure of the homotopy 2-category of a stable ∞-category. Specifically, we show that for a cofibrantly generated stable simplicial model category the homotopy 2-category is enriched in symmetric 2-groups and admits suitable adaptations of homotopy pullbacks and pushouts, which satisfy properties analogous to the definition of a stable ∞-category. We propose to use these properties as a definition of a 2-triangulated 2-category and show that truncating it further gives rise to the classical notion of a triangulated 1-category. We show that a homotopy fiber sequence in a 2-triangulated 2-category gives rise to a long exact sequence of symmetric 2-groups in the sense of [Vit02] and that 2-triangulated 2-categories admit a short five lemma.

In the process of justifying the axioms of a 2-triangulated 2-category we show that the homotopy 2-category of a simplicial model category has a working theory of homotopy pullbacks. Specifically, the weak universal properties of adaptations of the weak comma objects employed by Riehl and Verity [RV13] in their work on ∞-cosmoi suffice to derive many standard facts about homotopy pullbacks.

In order to obtain the enrichment of 2-triangulated 2-categories in symmetric 2-groups we elaborate on the result of Dupont [Dup08] stating that the existence of 2-categorical biproducts in a 2-category makes it canonically enriched in symmetric 2-monoids and deduce the 2-group structure from the pseudogroup structure of loop objects.

Contents

Introduction

Motivation

Homotopy theory evolved out of the definition of a homotopy of paths in the early 20th century and gained momentum with the introduction of the notion of fibrations and higher homotopy groups of spheres in the 1930s. The study of homotopy lifting properties as well as weak equivalences of spheres became a central topic. This and the discoveries of examples of homotopical nature, especially in simplicial sets and chain complexes, led to the invention of model categories[1] by Quillen in his work on homotopical algebra in the 1960s. It quickly became clear that the homotopy category of a model category is in general not a good category in the sense that it lacks limits and colimits and, even worse, that limits and colimits in it in general do not give the correct notions of limits and colimits up to homotopy. While model categories provide tools to define the correct notions of homotopy limits and homotopy colimits, it is difficult to speak about homotopy-coherent diagrams. The loose idea of an infinitely homotopy-coherent category, ∞-category for short, first materialized in the form of simplicial model categories, which already appeared in the work of Quillen. Shortcomings of this model of ∞-categories made mathematicians investigate other models, prominently the one of quasicategories popularized by Boardman, Vogt, Joyal and Luri.e.

While ∞-categories allow for a formulation of homotopy-coherent diagrams and thus universal properties, a formal treatment is technically involved. Model categories provide a simpler presentation of homotopy theories, but one has do deal with fibrance issues and it is hard to express universal properties coherently. In both cases considering homotopy limits and homotopy colimits in the homotopy category is not particularly useful, since the remaining weak universal properties do not even characterize them up to equivalence.

The homotopy 2-category of an ∞-category may be used to express diagrams, which only involve maps and homotopy classes of homotopies between them. Because of this truncation it would be unreasonable to expect the homotopy 2-category to give a faithful representation of the whole homotopy theory represented by the ∞-category. Yet it might have enough flexibility to express at least some universal properties. In fact Mather did precisely this in his 1976 paper [Mat76] on *pullbacks in homotopy theory*, but unfortunately did not use the weak universal properties extensively. Homotopy 2-categories did not gain traction and can nowadays rarely be found in the literature. They resurfaced in the work of Riehl and Verity on ∞-cosmoi, in that they use the weak universal property of homotopy pullbacks in the homotopy 2-category of the ∞-category of ∞-categories to develop ∞-category theory.

In this work we show that the truncated universal properties of homotopy pullbacks in the homotopy 2-category of an ∞-category are surprisingly useful. They suffice to show that homotopy pullbacks are unique up to equivalence (cf. lemma 1.3.2), that homotopy pullbacks of equivalences are again equivalences (cf. lemma 1.3.9), that they are stable under equivalences of squares (cf. corollary 1.3.14), that an equivalence of diagrams induces an equivalence of the homotopy pullbacks (cf. proposition 1.3.15) and that they satisfy both a pasting law (cf. lemma 1.3.12) and a cancellation law (cf. proposition 1.3.16).

Our interest in the 2-categorical properties of homotopy pullbacks was a byproduct of the actual topic of this thesis, namely to find a definition of a 2-triangulated 2-category. Ordinary triangulated 1-categories often arise as homotopy 1-categories of stable ∞-categories with the distinguished triangles coming from cofiber sequences, which basically are sequences of homotopy pushouts chained together. Since homotopy pushouts in a homotopy 1-category cannot be characterized by a universal property, the definition of a triangulated 1-category has the distinguished triangles as an extra structure on the category. In contrast, as explained before, homotopy pushouts in the homotopy 2-category *are* characterized by a universal property, so cofiber sequences can be seen as a property of the homotopy 2-category rather than a structure. Thus, adapting the axioms of a stable ∞-category into a 2-categorical notion should give a definition of a 2-triangulated 2-category (cf. definition 3.3.1). We have two justifications for this axiomatization. On the one

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[1] The term model category is an abbreviation for the more lengthy “a category which models a homotopy theory”.

hand we show that a sufficiently nice stable ∞-category, modelled as a cofibrantly generated stable simplicial model category, has a homotopy 2-category, which is a 2-triangulated 2-category in our sense (cf. example 3.3.3). On the other hand we show that truncating a 2-triangulated 2-category to a 1-category by collapsing 2-cells gives rise to a triangulated category (cf. theorem 3.3.6). We also show that the structure of a 2-triangulated 2-category suffices to prove facts from the theory of stable ∞-categories in a purely 2-categorical context.

Overview

In the first chapter we investigate the structure of the homotopy 2-category of a simplicial model category. In section 1.1 we discuss terminal objects and products. In the following section 1.2 we discuss how homotopy pullbacks induce objects in the homotopy 2-category, which we call quasi-commas and which satisfy a very weak universal property. The next section 1.3 contains an explicit description of this weak universal property and all of the results mentioned in the introduction. The last section 1.4 specializes to cofibrantly generated pointed simplicial model categories. We discuss the simplicial functoriality of the loop-suspension adjunction and derive compatibility conditions in the homotopy 2-category, which state that the loop-suspension adjunction is given by the universal properties of homotopy pullbacks and homotopy pushouts.

The second chapter is an interlude on 2-category theory. In the first section 2.1 we introduce terminology and discuss 2-categorical versions of limits and colimits. In the second section 2.2 we discuss symmetric pseudomonoids in a 2-cartesian monoidal 2-category. We derive the Eckmann-Hilton argument for pseudomonoids (cf. lemma 2.2.9) from the corresponding result about monoidal categories proved in [JS93]. The next section 2.3 contains a discussion of 2-categorical generalizations of 1-categorical biproducts. There we make some essential observations and introduce conventions used in the next section 2.4, where we elaborate on Dupont's result in [Dup08] stating that a 2-category with 2-biproducts is enriched in symmetric monoidal categories. We close the chapter with a short section 2.5 on symmetric 2-groups and short exact sequences thereof.

The last chapter is devoted to the development of 2-triangulated 2-categories. Building on the results of Dupont we discuss in section 3.1 how a groupoid-enriched category with 2-biproducts and a truncated version of loop-objects is canonically enriched in symmetric 2-groups. Mimicking the usual ∞-categorical proof the next section 3.2 explains how the truncated version of fiber squares may be used to obtain a long exact sequence of symmetric 2-groups. The last section 3.3 finally contains the definition of a 2-triangulated 2-category. We discuss the relation to triangulated 1-categories and give another characterization. This characterization then allows us to adapt some proofs from stable ∞-categories in this setting.

In the appendix we elaborate on various questions which arose in the process of writing this thesis. In section A.2 we explain that one of the standard representations of homotopy pullbacks in simplicial model categories can be seen as a specific weighted limit. In section A.3 we elaborate on when a simplicial model category admits simplicial enriched functorial fibrant and cofibrant replacements. In the section A.4 we give the definition of a stable simplicial model category and show that every locally presentable stable ∞-category can be represented by one. For the reader's convenience we also included a short description of the Boardman-Vogt construction in section A.1.

Conventions

We freely use the language of simplicial sets, quasi-categories and Kan-complexes as laid out in [Lur09] or [Cis19]. For example the symbol Δ^n denotes the n-simplex in simplicial sets. Fibrations will be denoted by $\twoheadrightarrow$ and cofibrations by $\hookrightarrow$. Equivalences will be written with a $\simeq$, while the symbol $\cong$ is reserved for isomorphisms.

We also make use of the language of enriched categories as in [Kel82], often in the specific form of simplicial model categories presented in [Hov07] or [Hir03]. A strong monoidal functor $\mathcal{F} : \mathcal{V} \to \mathcal{W}$ induces a 2-functor $\mathcal{F}_* : \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{W}\text{-}\mathsf{Cat}$ to which we refer as *change of base along* $\mathcal{F}$. Some of the monoidal categories involved in this thesis are the cartesian monoidal categories of sets Set, categories Cat, groupoids Grpd, simplicial sets sSet and Kan-complexes Kan. The symbol $\pitchfork$ denotes a power, the symbol $\odot$ a copower.

For 2-category the terminology deviates for ∞-category theorists and 2-category theorists. Therefore it is important to emphasize the following conventions. With *2-category* we mean a Cat-enriched category. We will call Cat-enriched functors *strict 2-functors*, while pseudo-functors and pseudo-natural transformations will be called *weak 2-functors* and *weakly 2-natural transformations*. The universal properties of 2-limits and 2-colimits come in *strict* (up to isomorphism) and *weak* (up to equivalence) forms. The even weaker universal properties of homotopy limits and homotopy colimits in the homotopy 2-category will be indicated with the prefix *quasi*.

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1 The Homotopy 2-Category

1.1 Basic Properties

In this section we define the homotopy 2-category of a simplicial model category and discuss the universal properties of (co)final objects and (co)products in it.

Definition 1.1.1

Let $\mathcal{C}$ be a simplicial model category. The **associated 2-category** is the 2-category $\tau_*(\mathcal{C})$ obtained by change of base along $\tau : \mathsf{sSet} \longrightarrow \mathsf{Cat}$.

Restricting this construction to the full simplicial subcategory $\mathcal{C}_b$ of bifibrant objects we obtain the **homotopy 2-category** of $\mathcal{C}$ as the change of base $\mathrm{ho}_2(\mathcal{C}) := \tau_*(\mathcal{C}_b)$.

Remark 1.1.2

The full simplicial subcategory of $\mathcal{C}$ on the bifibrant objects is actually a Kan-enriched category. Hence we can use the Boardman-Vogt construction (which we recall for the reader's convenience in theorem A.1.1) to compute $\mathrm{ho}_2(\mathcal{C}) = h_*(\mathcal{C}_b)$ and note that the homotopy 2-category actually is a Grpd-category.

Since change of base commutes with taking full subcategories, we might equivalently define the homotopy 2-category as the full 2-category of $\tau_*(\mathcal{C})$ on the bifibrant objects.

In the following we discuss the universal properties of terminal objects and products in the homotopy 2-category.

Lemma 1.1.3

Let $\mathcal{C}$ be a simplicial model category.

Any cofibrant replacement $\mathcal{Q}(1)$ of the terminal object in $\mathcal{C}$ provides a weakly terminal object in $\mathrm{ho}_2(\mathcal{C})$. Similarly any fibrant replacement $\mathcal{R}(0)$ of the initial object in $\mathcal{C}$ gives a weakly initial object in $\mathrm{ho}_2(\mathcal{C})$. If the terminal respectively initial object are bifibrant in $\mathcal{C}$, the corresponding universal properties in $\mathrm{ho}_2(\mathcal{C})$ are strict.

Hence, if $\mathcal{C}$ is a pointed model category (i.e. has a strict zero object), then its homotopy 2-category has a strict zero object. In particular in this case $\mathrm{ho}_2(\mathcal{C})$ is Grpd_*-enriched.

Proof The terminal object is fibrant, hence for any bifibrant object T the canonical map

$$\mathrm{ho}_2(\mathcal{C})(T, \mathcal{Q}(1)) = \tau(\mathcal{C}(T, \mathcal{Q}(1))) \simeq \tau(\mathcal{C}(T, 1)) \cong \tau(1) = *$$

is an equivalence of categories. If 1 is already cofibrant, then 1 is a strict terminal object in $\mathrm{ho}_2(\mathcal{C})$, since for any bifibrant object T there is the canonical isomorphism $\mathrm{ho}_2(\mathcal{C})(T, 1) \cong *$.

The argument for the initial object is completely analogous.

The claim about zero objects immediately follows from the discussion of terminal and initial objects and the observation, that a zero object is necessarily bifibrant. □

Lemma 1.1.4

Let $\mathcal{C}$ be a simplicial model category.

Any cofibrant replacement $\mathcal{Q}(X \times Y)$ of a product of bifibrant objects X, Y in $\mathcal{C}$ defines a weak product in $\mathrm{ho}_2(\mathcal{C})$. Similarly any fibrant replacement $\mathcal{R}(X + Y)$ of the coproduct of bifibrant objects defines a weak coproduct in $\mathrm{ho}_2(\mathcal{C})$. If the product respectively coproduct is bifibrant in $\mathcal{C}$, its universal property is strict in $\mathrm{ho}_2(\mathcal{C})$.

Proof For any bifibrant object T we have the canonical composite equivalence

$$\begin{aligned}
\mathrm{ho}_2(\mathcal{C})(T, \mathcal{Q}(X \times Y)) &= \tau(\mathcal{C}(T, \mathcal{Q}(X \times Y))) \\
&\simeq \tau(\mathcal{C}(T, X \times Y)) \\
&\cong \tau(\mathcal{C}(T, X) \times \mathcal{C}(T, Y)) \\
&\cong \tau(\mathcal{C}(T, X)) \times \tau(\mathcal{C}(T, Y)) \\
&= \mathrm{ho}_2(\mathcal{C})(T, X) \times \mathrm{ho}_2(\mathcal{C})(T, Y).
\end{aligned}$$

defining a weak product in $\mathrm{ho}_2(\mathcal{C})$.

Analogously for any bifibrant object T we have the canonical composite equivalence

$$\begin{aligned}
\mathrm{ho}_2(\mathcal{C})(\mathcal{R}(X + Y), T) &= \tau(\mathcal{C}(\mathcal{R}(X + Y), T)) \\
&\simeq \tau(\mathcal{C}(X + Y, T)) \\
&\cong \tau(\mathcal{C}(X, T) \times \mathcal{C}(Y, T)) \\
&\cong \tau(\mathcal{C}(X, T)) \times \tau(\mathcal{C}(Y, T)) \\
&= \mathrm{ho}_2(\mathcal{C})(X, T) \times \mathrm{ho}_2(\mathcal{C})(Y, T)
\end{aligned}$$

defining a weak coproduct in $\mathrm{ho}_2(\mathcal{C})$.

The claim on strictness follows from the observation that the only obstruction to a strict universal property given by an isomorphism is the equivalence induced by cofibrant respectively fibrant replacement. □

1.2 Quasi(co)commas

One of the most important tools in homotopy theory are homotopy pullbacks and homotopy pushouts. They provide a homotopically correct notion of pullbacks and pushouts in the sense that they behave well in respect to homotopy equivalences, which is not true for usual pullbacks. A classic example is that usual pushouts are not invariant under homotopy equivalence of diagrams. The underlying spans of both pushout diagrams of spaces

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r] \arrow[d] & \mathbb{D}^2 \arrow[d] & & \mathbb{S}^1 \arrow[r] \arrow[d] & * \arrow[d] \\
\mathbb{D}^2 \arrow[r] & \mathbb{S}^2 \arrow[ul, phantom, "\ulcorner", very near start] & \text{and} & * \arrow[r] & * \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

are pointwise homotopy equivalent, yet $\mathbb{S}^2$ and $*$ are not homotopy equivalent.

Unfortunately the naive definition of a homotopy pullback as a pullback in the homotopy category fails to give the correct notion, partly because pullbacks need not even exist in the homotopy category and because the construction of the homotopy category collapses all coherence data to a mere identity of equivalence classes. The urge to find a fully coherent definition of homotopy (co)limits has lead to various incarnations of ∞-categories like quasi-categories and simplicial model categories, which allow for a fully coherent treatment of homotopy (co)limits.

In the developement of ∞-category theory the desired universal properties of a homotopy pullback in the homotopy 1- and 2-category have been studied. In both cases the universal properties are truncated versions of the proper ∞-categorical ones. However there is an important difference between both of them. To understand and appreciate it we have to briefly state these universal properties. Suppose we are given an abstract ∞-category $\mathcal{C}$.

For a cospan $Y \xrightarrow{[f]} X \xleftarrow{[g]} Z$ in a homotopy 1-category $\text{ho}(\mathcal{C})$ the homotopy pullback P will be a commutative square with the weak universal property as depicted on the right. This amounts to having for any object T a surjective canonical comparison map

$$\text{ho}(\mathcal{C})(T,P) \longrightarrow \text{ho}(\mathcal{C})(T,Y) \underset{\text{ho}(\mathcal{C})(T,X)}{\times} \text{ho}(\mathcal{C})(T,Z).$$

$$\begin{tikzcd}
T \arrow[rrd, bend left, "{[s]}"] \arrow[rd, dashed, "{\exists[p]}"] \arrow[rdd, bend right, "{[r]}"'] & & \\
& P \arrow[r, "{[z]}"] \arrow[d, "{[y]}"'] & Z \arrow[d, "{[g]}"] \\
& Y \arrow[r, "{[f]}"'] & X
\end{tikzcd}$$

Note that this universal property does not have any uniqueness property, which is due to the fact that in general we cannot deduce from an identity $[xp] = [xp']$ that $[p] = [p']$. For example this is problematic, since from the viewpoint of this universal property homotopy pullbacks need not be unique in any sense.

The situation in the homotopy 2-category $\text{ho}_2(\mathcal{C})$ is slightly better. We will not spell out the full universal property here, since we will do so explicitly in the next section. For the present discussion it suffices to note that it is formalized by requiring that for any object T the comparison functor

$$\text{ho}_2(\mathcal{C})(T,P) \longrightarrow \text{Cone}_{\text{ho}_2(\mathcal{C})}(T,\mathcal{D})$$

into an appropriate category of cones with tip T is essentially surjective on objects and full. It is worth mentioning that, while this definition seems to only entail existence properties, the fullness provides us with the desired uniqueness up to homotopy: maps with homotopic images were already homotopic. Hence this weak uniqueness property allows us to deduce the uniqueness of homotopy pullbacks up to homotopy equivalence just by using the truncated universal property. We will do so in lemma 1.3.2.

In this section we want to show that for a given simplicial model category $\mathcal{C}$ the homotopy pullbacks and homotopy pushouts actually have weak universal properties in $\text{ho}_2(\mathcal{C})$ of the form just described. A thorough treatment of the weak universal properties of general homotopy limits in homotopy n-categories of quasicategories has been carried out in [Rap22]. But instead of transferring his arguments into the simplicial enriched setting it actually turns out to be easier to adapt another approach taken by [RV13] in their study of the homotopy 2-category of quasicategories. Because in the simplicial enriched setting the homotopy limit of a diagram $\mathcal{D}$ can be computed as certain weighted limits $\lim_{\mathcal{W}} \mathcal{D}$, ideally for any object T the canonical comparison morphism

$$\mathrm{ho}_2(\mathcal{C})(T, \lim_{\mathcal{W}} \mathcal{D}) \longrightarrow \lim_{\mathcal{W}}(\mathrm{ho}_2(\mathcal{C})(T, \mathcal{D}))$$

should be essentially surjective and full. In the view of the results of [Rap22] this is slightly misleading however. The conditions posed onto the canonical comparison morphism not only depend on the homotopy level we truncate the ∞-category onto, but also on the simplicial dimension of the diagram shape and that of the weights[2]. For 0-dimensional diagrams and conical weight the universal property will give rise to an equivalence, which we already observed in lemma 1.1.4. For 1-dimensional diagrams and conical weights we can expect the comparison functor to be essentially surjective and full. Nevertheless we make the following general definitions.

Definition 1.2.1

Let $\mathcal{I}$ be a small 1-category, $\mathcal{W} : \mathcal{I} \longrightarrow \mathsf{Cat}$ be a weight, $\mathcal{C}$ be a simplicial model category and $\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{C}$ be a diagram of bifibrant objects in $\mathcal{C}_b$.

Suppose the weighted limit $\lim_{\mathcal{W}} \mathcal{D}$ exists in $\mathcal{C}$ and is a fibrant object. A cofibrant replacement $\mathcal{Q}(\lim_{\mathcal{W}} \mathcal{D}) \xrightarrow{\sim} \lim_{\mathcal{W}} \mathcal{D}$ defines a **quasi-limit weighted by** $\mathcal{W}$ in the homotopy 2-category $\mathrm{ho}_2(\mathcal{C})$, if for any bifibrant object T the canonical map

$$\begin{aligned}\mathrm{ho}_2(\mathcal{C})(T, \mathcal{Q}(\lim_{\mathcal{W}} \mathcal{D})) &= \tau(\mathcal{C}(T, \mathcal{Q}(\lim_{\mathcal{W}} \mathcal{D})))\\ &\simeq \tau(\mathcal{C}(T, \lim_{\mathcal{W}} \mathcal{D}))\\ &\cong \tau(\lim_{\mathcal{W}} \mathcal{C}(T, \mathcal{D}))\\ &\to \lim_{\mathcal{W}}(\tau(\mathcal{C}(T, \mathcal{D})))\\ &= \lim_{\mathcal{W}}(\mathrm{ho}_2(\mathcal{C})(T, \mathcal{D}))\end{aligned}$$

is essentially surjective and full.

Definition 1.2.2

Let $\mathcal{I}$ be a small 1-category, $\mathcal{W} : \mathcal{I}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ be a weight, $\mathcal{C}$ be a simplicial model category and $\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{C}$ be a diagram of cofibrant objects in $\mathcal{C}_b$.

Suppose the weighted colimit $\mathrm{colim}_{\mathcal{W}} \mathcal{D}$ exists in $\mathcal{C}$ and is a cofibrant object. A fibrant replacement $\mathrm{colim}_{\mathcal{W}} \mathcal{D} \xrightarrow{\sim} \mathcal{R}(\mathrm{colim}_{\mathcal{W}} \mathcal{D})$ defines a **quasi-colimit weighted by** $\mathcal{W}$ in the homotopy 2-category $\mathrm{ho}_2(\mathcal{C})$, if for any bifibrant object T the canonical map

$$\begin{aligned}\mathrm{ho}_2(\mathcal{C})(\mathcal{R}(\operatorname*{colim}_{\mathcal{W}} \mathcal{D}), T) &= \tau(\mathcal{C}(\mathcal{R}(\operatorname*{colim}_{\mathcal{W}} \mathcal{D}), T))\\ &\simeq \tau(\mathcal{C}(\operatorname*{colim}_{\mathcal{W}} \mathcal{D}, T))\\ &\cong \tau(\lim_{\mathcal{W}^{\mathrm{op}}} \mathcal{C}(\mathcal{D}, T))\\ &\to \lim_{\mathcal{W}^{\mathrm{op}}} \tau(\mathcal{C}(\mathcal{D}, T))\\ &= \lim_{\mathcal{W}^{\mathrm{op}}}(\mathrm{ho}_2(\mathcal{C})(\mathcal{D}, T))\end{aligned}$$

is essentially surjective and full.

The following observation is crutial for the arguments following it.

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[2] In [Rap22] weighted limits are not discussed. That the weights matter can be observed in lemma 1.2.4, where a 0-dimensional diagram gives rise to a weak universal property, which is not an equivalence.

Remark 1.2.3

In both definitions the obstruction to having a quasi-(co)limit is that for very specific diagrams $\mathcal{D}' : \mathcal{I} \longrightarrow \mathsf{Kan}$ the canonical functor

$$\tau(\lim_{\mathcal{W}} \mathcal{D}') \longrightarrow \lim_{\mathcal{W}} \tau(\mathcal{D}')$$

is essentially surjective and full. So if for every $\mathcal{I}$-shaped diagram in Kan, which may arise in this way, this map is essentially surjective and full, any homotopy 2-category of a simplicial model category admits the corresponding quasi-limits and quasi-colimits.

We show in appendix A.2 that in a simplicial model category $\mathcal{C}$ the homotopy pullback of a diagram $\mathcal{D} = X \xrightarrow{f} Y \xleftarrow{g} Z$ of fibrant objects can be computed as the simplicial weighted limit $\lim_{\mathcal{W}} \mathcal{D}$, where the weight is given by the diagram of simplicial sets $\mathcal{W} = \{0\} \to \Delta^1 \leftarrow \{1\}$. The corresponding categorical weighted limit defines a comma object, which is the reason why we decided to stick with [RV13] in calling the objects with the weak universal property *quasi-commas* as opposed to the more lengthy *quasi-homotopy-pullback*. We show in the appendix A.2 that this specific definition of a homotopy pullback amounts to computing the pullback

$$\begin{tikzcd}
X \times^h_Y Z \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^1 \pitchfork Y \arrow[d, two heads] \\
X \times Z \arrow[r, "f \times g"'] & Y \times Y
\end{tikzcd}$$

of simplicial sets, where the right arrow is given by the fibrant replacement of the diagonal morphism

$$\begin{tikzcd}
X = \Delta^0 \pitchfork X \arrow[r, "\sim"] \arrow[rr, bend left, "\Delta"] & \Delta^1 \pitchfork X \arrow[r, two heads] & \partial\Delta^1 \pitchfork X \cong X \times X
\end{tikzcd}$$

provided by applying the simplicial functor $- \pitchfork X : \mathsf{sSet}^{\mathrm{op}} \to \mathcal{C}$ to the diagram

$$\begin{tikzcd}
\partial\Delta^1 \arrow[r, hook] \arrow[dr, hook] & \Delta^1 \arrow[r, "\sim"] \arrow[d, hook, "\sim"'] & \Delta^0 \\
 & I \arrow[ur, two heads, "\sim"'] &
\end{tikzcd}$$

of simplicial sets, where I denotes the nerve of the free living isomorphism. Homotopy pushouts admit a very similar construction. We use these specific constructions to prove the desired weak universal properties by first proving the following two lemmas.

Lemma 1.2.4

Let $\mathcal{C}$ be a simplicial model category. Then its homotopy 2-category $\mathrm{ho}_2(\mathcal{C})$ has quasi-powers and quasi-copowers by $\Delta^1 = \mathbb{2}$ computed as $\mathcal{Q}(\Delta^1 \pitchfork X)$ and $\mathcal{R}(\Delta^1 \odot X)$.

Proof (adapting [RV13] Prop. 3.3.9)

By remark 1.2.3 it suffices to show that for any Kan-complex K the canonical map

$$\tau(\Delta^1 \pitchfork K) \longrightarrow \Delta^1 \pitchfork \tau(K)$$

is essentially surjective and full. The Boardman-Vogt construction A.1.1 shows that the category

$$\mathcal{A} := \tau(\Delta^1 \pitchfork K) = h(\mathrm{Hom}(\Delta^1, K))$$

is given by

$$\operatorname{Ob}\mathcal{A} = \{\text{arrows } x_0 \xrightarrow{f} x_1 \text{ in } K\}$$

$$\mathcal{A}(f,g) = \left\{ \text{equiv. classes of coherent diagrams } \begin{tikzcd} x_0 \arrow[r, "a_0"] \arrow[d, "f"'] \arrow[dr, "\alpha_0", "d" description, "\alpha_1"'] & y_0 \arrow[d, "g"] \\ x_1 \arrow[r, "a_1"'] & y_1 \end{tikzcd} \text{ in } K \right\},$$

that the category $\mathcal{B} := \mathbb{2} \pitchfork h(K)$ is given by

$$\operatorname{Ob}\mathcal{B} = \{\text{arrows } [f] : x_0 \to x_1 \text{ in } h(K)\}$$

$$\mathcal{B}([f],[g]) = \left\{ \text{comm. squares } \begin{tikzcd} x_0 \arrow[r, "{[a_0]}"] \arrow[d, "{[f]}"'] & y_0 \arrow[d, "{[g]}"] \\ x_1 \arrow[r, "{[a_1]}"'] & y_1 \end{tikzcd} \text{ in } h(K) \right\},$$

and that the canonical functor in question is defined as

$$\begin{aligned} \mathcal{A} &\longrightarrow \mathcal{B} \\ f &\longmapsto [f] \end{aligned}$$

$$\left[\begin{tikzcd} x_0 \arrow[r, "a_0"] \arrow[d, "f"'] \arrow[dr, "\alpha_0", "d" description, "\alpha_1"'] & y_0 \arrow[d, "g"] \\ x_1 \arrow[r, "a_1"'] & y_1 \end{tikzcd} \right] \longmapsto \begin{tikzcd} x_0 \arrow[r, "{[a_0]}"] \arrow[d, "{[f]}"'] & y_0 \arrow[d, "{[g]}"] \\ x_1 \arrow[r, "{[a_1]}"'] & y_1 \end{tikzcd}.$$

From this inspection one immediately gets that the functor is surjective on objects and it remains to check that it is full.

To this end consider any commutative square in $ho(K)$ as depicted on the lower left and choose a representative $[d]$ of the diagonal. Since the two triangles commute in $ho(K)$, the Boardman-Vogt construction A.1.1 supplies us with solutions to the lifting problems depicted in the lower center. The equivalence class of the resulting homotopy coherent diagram $\Delta^1 \times \Delta^1 \to K$, depicted on the lower right, gives the desired preimage.

$$\begin{tikzcd} x_0 \arrow[r, "{[a_0]}"] \arrow[d, "{[f]}"'] \arrow[dr, "{[d]}" description] & y_0 \arrow[d, "{[g]}"] \\ x_1 \arrow[r, "{[a_1]}"'] & y_1 \end{tikzcd} \rightsquigarrow \begin{tikzcd} \partial\Delta^2 \arrow[r, "{(a_0,g,d)}"] \arrow[d] & K \\ \Delta^2 \arrow[ur, dashed, "\alpha_0"'] \end{tikzcd} \quad \begin{tikzcd} \partial\Delta^2 \arrow[r, "{(f,a_1,d)}"] \arrow[d] & K \\ \Delta^2 \arrow[ur, dashed, "\alpha_1"'] \end{tikzcd} \rightsquigarrow \left[\begin{tikzcd} x_0 \arrow[r, "a_0"] \arrow[d, "f"'] \arrow[dr, "\alpha_0", "d" description, "\alpha_1"'] & y_0 \arrow[d, "g"] \\ x_1 \arrow[r, "a_1"'] & y_1 \end{tikzcd} \right]$$

□

Lemma 1.2.5

Let $\mathcal{C}$ be a simplicial model category. Then its homotopy 2-category $\mathrm{ho}_2\,\mathcal{C}$ has quasi-commas along fibrations and quasi-cocommas along cofibrations.

Explicitly the cofibrant replacement $\mathcal{Q}(X \times_Z Y)$ of the pullback of a diagram of bifibrant objects of the form depicted on the lower left computes a quasi-comma in $\mathrm{ho}_2\,\mathcal{C}$. Similarly the fibrant replacement $\mathcal{R}(B \sqcup_A C)$ of a pushout of bifibrant objects of the form depicted on the lower right computes a quasi-cocomma in $\mathrm{ho}_2\,\mathcal{C}$.

$$\begin{tikzcd}
X \times_Z Y \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads] \\
X \arrow[r] & Z
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, tail] & C \arrow[d, tail] \\
B \arrow[r] & B \sqcup_A C \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Proof (adapting [RV13] Prop. 3.3.14)

Since given any bifibrant object T the simplicial functor $\mathcal{C}(T,-)$ maps fibrations between fibrant objects to fibrations and the simplicial functor $\mathcal{C}(-,T)$ maps cofibrations between cofibrant objects to fibrations, by the remark 1.2.3 it suffices to show that given pullback square of Kan-complexes of the form

$$\begin{tikzcd}
K \times_L K' \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & K' \arrow[d, two heads, "k'"] \\
K \arrow[r, "k"'] & L
\end{tikzcd}$$

the canonical map

$$\tau(K \times_L K') \longrightarrow \tau(K) \times_{\tau(L)} \tau(K')$$

is essentially surjective and full. By the Boardman-Vogt construction A.1.1 the category $\mathcal{A} := h(K \times_L K')$ is given by

$$\begin{aligned}
\mathrm{Ob}\,\mathcal{A} &= \{(x,x') \in K_0 \times K_0' \text{ with } k(x) = k'(x') \text{ in } L\} \\
\mathcal{A}((x,x'),(y,y')) &= \left\{ \begin{array}{c} \text{equiv. classes of pairs of arrows } (f,f'), \\ \text{with } f : x \to y \in K_1,\ f' : x' \to y' \in K_1' \\ \text{and } k(f) = k'(f') \end{array} \right\},
\end{aligned}$$

where two suitable pairs of arrows (f,f') and (g,g') are equivalent if and only if there is a homotopy $\alpha : f \sim g$ in K and a homotopy $\alpha' : f' \sim g'$ in K' satisfying $k(\alpha) = k'(\alpha')$.

The category $\mathcal{B} := h(K) \times_{h(L)} h(K')$ is given by

$$\begin{aligned}
\mathrm{Ob}\,\mathcal{B} &= \{(x,x') \in K_0 \times K_0' \text{ with } k(x) = k'(x') \text{ in } L\} \\
\mathcal{A}((x,x'),(y,y')) &= \left\{ \begin{array}{c} \text{pairs of equiv. classes } ([f],[f']), \\ \text{with } [f] : x \to y \in h(K),\ [f'] : x' \to y' \in h(K') \\ \text{and } [k(f)] = h(k)([f]) = h(k')([f']) = [k'(f')] \end{array} \right\},
\end{aligned}$$

and the canonical functor is defined via

$$\begin{array}{ccc}
\mathcal{A} & \longrightarrow & \mathcal{B} \\
(x,x') & \longmapsto & (x,x') \\
[(f,f')] & \longmapsto & ([f],[f'])
\end{array} .$$

Thus the functor is the identity on objects, in particular essentially surjective. Regarding fullness we pick a pair of equivalence classes $([f],[f'])$ in $\mathcal{B}$. By assumption they coincide in $h(L)$, witnessed by a 2-simplex θ in L as depicted on the lower left. Since k' is a Kan-fibration, the lifting problem depicted in the center admits a solution θ', which amounts to a 2-simplex in K' as depicted on the lower right. But then $k'(g') = k(f)$, so the equivalence class $[(f,g')]$ is a well-defined morphism in $\mathcal{A}$ and because of $[g'] = [f']$ in $h(K')$ a preimage of $([f],[f'])$.

$$\begin{tikzcd}
& b \arrow[dd, "1"] \\
a \arrow[ur, "k(f)"] \arrow[dr, "k'(f')"'] \arrow[r, phantom, "\theta"] & {} \\
& b
\end{tikzcd}
\ \text{in } L \ \rightsquigarrow \
\begin{tikzcd}
\Lambda^2_2 \arrow[r] \arrow[d] & K' \arrow[d, two heads, "k'"] \\
\Delta^2 \arrow[r, "\theta"'] \arrow[ur, dashed, "\theta'"] & L
\end{tikzcd}
\ \rightsquigarrow \
\begin{tikzcd}
& b' \arrow[dd, "1"] \\
a' \arrow[ur, "g'"] \arrow[dr, "f'"'] \arrow[r, phantom, "\theta'"] & {} \\
& b'
\end{tikzcd}
\ \text{in } K'$$

□

Proposition 1.2.6

Let $\mathcal{C}$ be a simplicial model category. Then its homotopy 2-category $\mathrm{ho}_2(\mathcal{C})$ admits quasi-commas and quasi-cocommas of arbitrary cospans respectively spans.

Explicitly for a given cospan $X \xrightarrow{f} Z \xleftarrow{g} Y$ of bifibrant objects, the cofibrant replacement $\mathcal{Q}(P)$ of the pullback depicted on the right computes a quasi-comma of the cospan in $\mathrm{ho}_2(\mathcal{C})$.

$$\begin{tikzcd}
P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & \Delta^1 \pitchfork Z \arrow[d, two heads] \\
X \times Y \arrow[r, "{(f,g)}"'] & Z \times Z
\end{tikzcd}$$

$$\begin{tikzcd}
A \sqcup A \arrow[r, "{[f,g]}"] \arrow[d, tail] & B \sqcup C \arrow[d, tail] \\
\Delta^1 \odot A \arrow[r] & P \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

Similarly for a given span $B \xleftarrow{f} A \xrightarrow{g} C$ of bifibrant objects, the fibrant replacement $\mathcal{R}(P)$ of the pushout depicted on the left computes a quasi-cocomma of the span in $\mathrm{ho}_2(\mathcal{C})$.

Proof (adapting [RV13] Prop. 3.3.18)

We observe that for any bifibrant object T the simplicial functors $\mathcal{C}(T,-)$ respectively $\mathcal{C}(-,T)$ turn the diagrams above into a pullback diagram of Kan-complexes of the form

$$\begin{tikzcd}
K \times^h_L K' \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & \Delta^1 \pitchfork L \arrow[d, two heads] \\
K \times K' \arrow[r, "{(k,k')}"'] & L \times L,
\end{tikzcd}$$

so we again use remark 1.2.3 to reduce to the case of Kan-complexes.

The corresponding weighted limit of the homotopy categories involved can be identified with the pullback

$$\begin{tikzcd}
h(k) \downarrow h(k') \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{2} \pitchfork h(L) \arrow[d] \\
h(K) \times h(K') \arrow[r, "{(h(k),h(k'))}"'] & h(L) \times h(L),
\end{tikzcd}$$

which computes the comma category of the functors $h(k)$ and $h(k')$. We thus have to show that the canonical functor

$$h(K \times^h_L K') \longrightarrow h(k) \downarrow h(k') \qquad (\star)$$

is essentially surjective and full. To this end we consider the diagram

$$\begin{tikzcd}
h(K \times^h_L K') \arrow[dr, "①" description] \arrow[drrr, bend left] \arrow[dddr, bend right] & & & & \\
& P \arrow[rr] \arrow[dr, "③" description] \arrow[dd] & & h(\Delta^1 \pitchfork L) \arrow[dr, "②" description] \arrow[dd] & \\
& & h(k) \downarrow h(k') \arrow[rr] \arrow[dd] & & \mathbb{2} \pitchfork h(L) \arrow[dd] \\
& h(K \times K') \arrow[rr] \arrow[dr, "\cong" description] & & h(L \times L) \arrow[dr, "\cong" description] & \\
& & h(K) \times h(K') \arrow[rr] & & h(L) \times h(L)
\end{tikzcd}$$

in which the front, back and bottom faces of the cube are pullbacks by definition. By the cancellation property of pullbacks the top face is a pullback as well. Then the morphism ① is essentially surjective and full by lemma 1.2.5, while the morphism labeled ② is actually surjective on objects and full by the proof of lemma 1.2.4. Since functors, which are surjective on objects and full, can be characterized by having the right lifting property against the functors $\emptyset \to *$ and $\{0, 1\} \subseteq \mathbb{2}$, they are stable under pullback. Hence the pullback ③ of the arrow ② is surjective on objects and full. In particular the canonical functor $(\star)$ is essentially surjective on objects and full as a composite of essentially surjective and full functors. $\square$

1.3 Properties of Quasicommas

In this section we discuss the weak universal property of quasi-commas and their resulting properties. Throughout we work in a fixed `Grpd`-category $\mathcal{C}$. Although automatic, we will always state that the 2-cells in this section are invertible to avoid possible confusions. All statements readily dualize to the corresponding statements about quasi-cocommas, so we only discuss quasi-commas.

Consider a diagram $\mathcal{D} = Y \xrightarrow{f} X \xleftarrow{g} Z$ in $\mathcal{C}$. Given any object T the comma of groupoids $\mathcal{C}(T, f) \downarrow \mathcal{C}(T, g)$ can be explicitly described as the groupoid $\mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D})$ of cones over $\mathcal{D}$ with tip T, whose objects are squares of the form

$$\begin{tikzcd}
T \arrow[r, "z"] \arrow[d, "y"'] \arrow[dr, phantom, "\Nearrow\tau"] & Z \arrow[d, "g"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$

with an invertible 2-cell τ and morphisms given by pairs of invertible 2-cells

$$\begin{tikzcd}
T \arrow[r, bend left, "z"] \arrow[r, bend right, "z'"'] \arrow[r, phantom, "\Downarrow\beta"] \arrow[d, bend right, "y"'] \arrow[d, bend left, "y'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & Z \\
Y &
\end{tikzcd}
\qquad \text{satisfying} \qquad
\begin{tikzcd}
T \arrow[r, bend left, "z'"] \arrow[r, bend right, "z"'] \arrow[r, phantom, "\Uparrow\beta"] \arrow[d, "y"'] \arrow[dr, phantom, "\Nearrow\tau"] & Z \arrow[d, "g"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[r, "z'"] \arrow[d, bend right, "y"'] \arrow[d, bend left, "y'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\Nearrow\tau'"] & Z \arrow[d, "g"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}$$

$$\text{or equivalently} \qquad
\begin{tikzcd}
T \arrow[r, bend left, "z'"] \arrow[r, bend right, "z"'] \arrow[r, phantom, "\Uparrow\beta"] \arrow[d, bend right, "y'"'] \arrow[d, bend left, "y"] \arrow[d, phantom, "\overset{\alpha^{-1}}{\Rightarrow}"] \arrow[dr, phantom, "\Nearrow\tau"] & Z \arrow[d, "g"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[r, "z'"] \arrow[d, "y'"'] \arrow[dr, phantom, "\Nearrow\tau'"] & Z \arrow[d, "g"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd} .$$

Given an object T and a square
$$\begin{tikzcd}
L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow\lambda"] & Z \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}$$
the universal property of the comma-groupoid

$\mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D})$ gives rise to the canonical functor

$$\mathcal{C}(T, L) \xrightarrow{\lambda_*} \mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D})$$

$$\begin{tikzcd} T \arrow[r, "t"] & L \end{tikzcd} \longmapsto \begin{tikzcd} T \arrow[dr, "t"] & & \\ & L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Z \arrow[d] \\ & Y \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd} T \arrow[r, bend left, "t"] \arrow[r, bend right, "t'"'] \arrow[r, phantom, "\Downarrow \tau"] & L \end{tikzcd} \longmapsto \begin{tikzcd} T \arrow[dr, bend left, "t'"] \arrow[dr, bend right, "t"'] \arrow[dr, phantom, "\Nearrow \tau"] & & \\ & L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] & Z \\ & Y & \end{tikzcd}$$

We can use this to give the definitions of a quasi-comma in $\mathcal{C}$.

Definition 1.3.1

A square λ as above is a **quasi-comma** for the diagram $\mathcal{D}$ in $\mathcal{C}$, if for every object T the canonical functor

$$\mathcal{C}(T, L) \xrightarrow{\lambda_*} \mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D})$$

is essentially surjective on objects and full.

In terms of the category of cones this weak universal property amounts to the following. Essential surjectivity on objects means that any square admits a (not necessarily unique) factorization

$$\begin{tikzcd} T \arrow[r, "z"] \arrow[d, "y"'] \arrow[dr, phantom, "\Nearrow \tau"] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} = \begin{tikzcd} T \arrow[drr, bend left, "z"] \arrow[dr, "t"] \arrow[ddr, bend right, "y"'] & \Nearrow \beta & \\ \Nearrow \alpha^{-1} & L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Z \arrow[d, "g"] \\ & Y \arrow[r, "f"'] & X \end{tikzcd}$$

and we refer to this as the *1-dimensional universal property.* Fullness means that any morphism of cones of the form

$$\begin{tikzcd} T \arrow[r, "t'"] \arrow[d, "t'"'] \arrow[dr, "t"] & L \arrow[dr, bend left, "p_Z"] & \Nearrow \beta \\ L \arrow[drr, bend right, "p_Y"'] & \Nearrow \alpha^{-1} \quad L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Z \arrow[d, "g"] \\ & Y \arrow[r, "f"'] & X \end{tikzcd} = \begin{tikzcd} T \arrow[dr, "t'"] & & \\ & L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Z \arrow[d, "g"] \\ & Y \arrow[r, "f"'] & X \end{tikzcd}$$

comes from a (not necessarily unique) invertible 2-cell $\begin{tikzcd} T \arrow[r, bend left, "t"] \arrow[r, bend right, "t'"'] \arrow[r, phantom, "\Downarrow \tau"] & L \end{tikzcd}$ in the sense that the identities

$$\begin{tikzcd}
T \arrow[dr, bend left, "t"] \arrow[dr, bend right, "t'"'] \arrow[dr, phantom, "\Swarrow\tau"] & & & T \arrow[d, "t'"'] \arrow[dr, "t"] & \\
& L \arrow[d, "p_Y"'] & = & L \arrow[dr, "p_Y"'] \arrow[r, phantom, "\Swarrow\alpha"] & L \arrow[d, "p_Y"] \\
& Y & & & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[dr, bend left, "t'"] \arrow[dr, bend right, "t"'] \arrow[dr, phantom, "\Nearrow\tau"] & & & & T \arrow[r, "t'"] \arrow[dr, "t"'] & L \arrow[dr, "p_Z"] \arrow[d, phantom, "\Nearrow\beta"] & \\
& L \arrow[r, "p_Z"'] & Z & = & & L \arrow[r, "p_Z"'] & Z
\end{tikzcd}$$

hold. This is the *2-dimensional universal property.* By a mere algebraic manipulation of identities of 2-cells it is equivalent to the condition that given two factorizations

$$\begin{tikzcd}
T \arrow[drr, bend left, "z"] \arrow[dr, "t"] \arrow[ddr, bend right, "y"'] & & & & T \arrow[drr, bend left, "z"] \arrow[dr, "t'"] \arrow[ddr, bend right, "y"'] & & \\
& L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[u, phantom, "\Nearrow\beta"] \arrow[dl, phantom, "\Nearrow\alpha^{-1}"] \arrow[dr, phantom, "\Nearrow\lambda"] & Z \arrow[d, "g"] & = & & L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[u, phantom, "\Nearrow\beta'"] \arrow[dl, phantom, "\Nearrow\alpha'^{-1}"] \arrow[dr, phantom, "\Nearrow\lambda"] & Z \arrow[d, "g"] \\
& Y \arrow[r, "f"'] & X & & & Y \arrow[r, "f"'] & X
\end{tikzcd}$$

there is a (not necessarily unique) invertible 2-cell $\begin{tikzcd} T \arrow[r, bend left, "t"] \arrow[r, bend right, "t'"'] \arrow[r, phantom, "\Downarrow\tau"] & L \end{tikzcd}$ satisfying the identities

$$\begin{tikzcd}
T \arrow[dr, bend left, "t"] \arrow[dr, bend right, "t'"'] \arrow[dr, phantom, "\Swarrow\tau"] \arrow[ddr, bend right=60, "y"'] \arrow[ddr, phantom, "\Swarrow\alpha'"] & & & T \arrow[dr, "t"] \arrow[ddr, bend right, "y"'] \arrow[ddr, phantom, "\Swarrow\alpha"] & \\
& L \arrow[d, "p_Y"] & = & & L \arrow[d, "p_Y"] \\
& Y & & & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[drr, bend left, "z"] \arrow[dr, bend left, "t'"] \arrow[dr, bend right, "t"'] \arrow[dr, phantom, "\Nearrow\tau"] \arrow[drr, phantom, "\Nearrow\beta'"] & & & & T \arrow[drr, bend left, "z"] \arrow[dr, "t"'] \arrow[drr, phantom, "\Nearrow\beta"] & & \\
& L \arrow[r, "p_Z"'] & Z & = & & L \arrow[r, "p_Z"'] & Z.
\end{tikzcd}$$

This is precisely the weak universal property of homotopy pullbacks studied by [Mat76].

The following lemma is one of the advantages of considering homotopy pullbacks in the homotopy 2-category as opposed to the homotopy 1-category. We can deduce uniqueness just by using the universal property.

Lemma 1.3.2

Quasi-commas are unique up to (not necessarily unique) equivalence.

Proof Suppose we are given two quasi-commas of the same diagram

$$\begin{tikzcd}
L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow\lambda"] & Z \arrow[d] & \text{and} & L' \arrow[r, "p'_Z"] \arrow[d, "p'_Y"'] \arrow[dr, phantom, "\Nearrow\lambda'"] & Z \arrow[d] \\
Y \arrow[r] & X & & Y \arrow[r] & X
\end{tikzcd} .$$

Then the 1-dimensional universal properties of λ' and λ provide us with factorizations

$$\begin{tikzcd}
L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Z \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}
=
\begin{tikzcd}
L \arrow[dr, "r"] \arrow[drr, bend left, "p_Z"] \arrow[ddr, bend right, "p_Y"'] \arrow[dr, phantom, shift left=3ex, "\Nearrow \rho_Z"] \arrow[dr, phantom, shift right=3ex, "\Nearrow \rho_Y^{-1}"] & & \\
& L' \arrow[r, "p'_Z"] \arrow[d, "p'_Y"'] \arrow[dr, phantom, "\Nearrow \lambda'"] & Z \arrow[d] \\
& Y \arrow[r] & X
\end{tikzcd}$$

$$\begin{tikzcd}
L' \arrow[r, "p'_Z"] \arrow[d, "p'_Y"'] \arrow[dr, phantom, "\Nearrow \lambda'"] & Z \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}
=
\begin{tikzcd}
L' \arrow[dr, "s"] \arrow[drr, bend left, "p'_Z"] \arrow[ddr, bend right, "p'_Y"'] \arrow[dr, phantom, shift left=3ex, "\Nearrow \sigma_Z"] \arrow[dr, phantom, shift right=3ex, "\Nearrow \sigma_Y^{-1}"] & & \\
& L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Z \arrow[d] \\
& Y \arrow[r] & X
\end{tikzcd}$$

Inserting one into the other we obtain a morphism of cones

$$\begin{tikzcd}
L \arrow[dr, "r"] \arrow[ddrr, bend left, "p_Z"] \arrow[dddr, bend right, "p_Y"'] \arrow[dr, phantom, shift left=3ex, "\Nearrow \rho_Z"] \arrow[dr, phantom, shift right=3ex, "\Nearrow \rho_Y^{-1}"] & & & \\
& L' \arrow[dr, "s"] \arrow[dr, phantom, shift left=3ex, "\Nearrow \sigma_Z"] \arrow[dr, phantom, shift right=3ex, "\Nearrow \sigma_Y^{-1}"] \arrow[drr, bend left, "p'_Z"] \arrow[ddr, bend right, "p'_Y"'] & & \\
& & L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Z \arrow[d] \\
& & Y \arrow[r] & X
\end{tikzcd}
=
\begin{tikzcd}
L \arrow[dr, equal] & & \\
& L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Z \arrow[d] \\
& Y \arrow[r] & X
\end{tikzcd}$$

and similarly the other way around. Then the 2-dimensional universal properties of λ and λ' provide us with a (not necessarily unique) invertible 2-cells

$$\begin{tikzcd}
L \arrow[r, "r"] \arrow[rr, bend right=50, equal, "\Downarrow \eta"] & L' \arrow[r, "s"] & L
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
L' \arrow[r, "s"] \arrow[rr, bend right=50, equal, "\Downarrow \eta'"] & L \arrow[r, "r"] & L'
\end{tikzcd}
.$$

These make r and s into mutually inverse equivalences. □

Remark 1.3.3

If $\lambda \in \mathrm{Cone}_{\mathcal{C}}(L, \mathcal{D})$ is a quasi-comma with tip L and $e : L' \simeq L$ is an equivalence, then for any object T the canonical functor

$$\mathcal{C}(T, L') \simeq \mathcal{C}(T, L) \xrightarrow{\lambda_*} \mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D})$$

is essentially surjective on objects and full (as a composite of such functors) and thus renders $\lambda * e$ a quasi-comma.

We will need to know how quasi-commas behave when changing various morphisms up to an invertible 2-cell. This will give us more flexibility when working with the weak universal property. The weakest statement will be that of corollary 1.3.14, but to prove it we will need the following two lemmas.

Lemma 1.3.4

Let $\mathcal{D} = Y \xrightarrow{f} X \xleftarrow{g} Z$ and $\mathcal{D}' = Y \xrightarrow{f'} X \xleftarrow{g'} Z$ be two diagrams and

$$\begin{tikzcd}
Y \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow \Upsilon"] & X & Z \arrow[l, bend right, "g"'] \arrow[l, bend left, "g'"] \arrow[l, phantom, "\Downarrow \zeta"]
\end{tikzcd}$$

be two invertible 2-cells. A square $\lambda \in \mathrm{Cone}_{\mathcal{C}}(L, \mathcal{D})$ is a quasi-comma of the diagram $\mathcal{D}$ if and only if the square

$$\begin{tikzcd}
L \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow \lambda'"] & Z \arrow[d, "g"] \arrow[dr, phantom, "="] & L \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow \lambda"] & Z \arrow[d, bend right, "g"'] \arrow[d, bend left, "g'"] \arrow[d, phantom, "\Rightarrow \zeta"] \\
Y \arrow[r, "f"'] & X & Y \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Uparrow \Upsilon^{-1}"] & X
\end{tikzcd}$$

is a quasi-comma of the diagram $\mathcal{D}'$.

Proof For any object T the 2-cells Υ and ζ give rise to an isomorphism of categories

$$\mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D}) \xrightarrow{\cong} \mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D}')$$

$$\begin{tikzcd}
T \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow \tau"] & Z \arrow[d, "g"] \arrow[dr, phantom, "\longmapsto"] & T \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow \tau"] & Z \arrow[d, bend right] \arrow[d, bend left, "g'"] \arrow[d, phantom, "\Rightarrow \zeta"] \\
Y \arrow[r, "f"'] & X & Y \arrow[r, bend left] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Uparrow \Upsilon^{-1}"] & X
\end{tikzcd}$$

and by the commutativity of the triangle

$$\begin{tikzcd}
 & \mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D}) \arrow[dd, "\cong"] \\
\mathcal{C}(T, L) \arrow[ur, "\lambda_*"] \arrow[dr, "\lambda'_*"'] & \\
 & \mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D}')
\end{tikzcd}$$

the upper functor is essentially surjective and full if and only if the lower functor is. ☐

Lemma 1.3.5

Let $\mathcal{D} : X \to Y \leftarrow Z$ be a diagram and $(\alpha, \beta) : \lambda \cong \lambda'$ be an isomorphism in $\mathrm{Cone}_{\mathcal{C}}(L, \mathcal{D})$. Then λ is a quasi-comma of $\mathcal{D}$ if and only if λ' is.

Proof For any object T the isomorphism $\lambda \cong \lambda'$ induces an isomorphism of the canonical functors

$$\begin{tikzcd}
\mathcal{C}(T, L) \arrow[rr, bend left, "\lambda_*"] \arrow[rr, bend right, "\lambda'_*"'] \arrow[rr, phantom, "\wr\!\|"] & & \operatorname{Cone}_{\mathcal{C}}(T, \mathcal{D})
\end{tikzcd}$$

so one is essentially surjective and full if and only if the other is. □

Remark 1.3.6

We say that two squares λ and λ' with the same corners are **equivalent**, if there are invertible 2-cells $\alpha, \beta, \Upsilon, \zeta$ giving rise to the pasting identity

$$\begin{tikzcd}
L \arrow[rr, bend left, "z'"] \arrow[rr, bend right, "z"'] \arrow[rr, phantom, "\Uparrow \beta"] \arrow[dd, bend right, "y'"'] \arrow[dd, bend left, "y"] \arrow[dd, phantom, "\overset{\alpha^{-1}}{\Rightarrow}"] & & Z \arrow[dd, bend right, "g"'] \arrow[dd, bend left, "g'"] \arrow[dd, phantom, "\overset{\zeta}{\Rightarrow}"] & & L \arrow[r, "z'"] \arrow[dd, "y'"'] & Z \arrow[dd, "g'"] \\
& \Nearrow \lambda & & = & \Nearrow \lambda' & \\
Y \arrow[rr, bend left, "f"] \arrow[rr, bend right, "f'"'] \arrow[rr, phantom, "\Uparrow \Upsilon^{-1}"] & & X & & Y \arrow[r, "f'"'] & X
\end{tikzcd}$$

By combining the preceding two lemmas we find that a square is a quasi-comma, if and only if any (and hence every) equivalent square is a quasi-comma.

We will need yet another symmetry of quasi-commas.

Lemma 1.3.7

$$\begin{tikzcd}
L \arrow[r, "p_Z"] \arrow[d, "p_Y"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Z \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}$$

A square as depicted on the left is a quasi-comma, if and only if the transposed square as depicted on the right is a quasicomma.

$$\begin{tikzcd}
L \arrow[r, "p_Y"] \arrow[d, "p_Z"'] \arrow[dr, phantom, "\Nearrow \lambda^{-1}"] & Y \arrow[d] \\
Z \arrow[r] & X
\end{tikzcd}$$

Proof Denote the diagrams by $\mathcal{D} = Y \to X \leftarrow Z$ and $\mathcal{D}^* = Z \to X \leftarrow Y$. For any object T of $\mathcal{C}$ there is a canonical isomorphism of categories

$$\operatorname{Cone}_{\mathcal{C}}(T, \mathcal{D}) \xrightarrow{\cong} \operatorname{Cone}_{\mathcal{C}}(T, \mathcal{D}^*)$$

$$\begin{tikzcd}
T \arrow[r, "z"] \arrow[d, "y"'] \arrow[dr, phantom, "\Nearrow \tau"] & Z \arrow[d] & \longmapsto & T \arrow[r, "y"] \arrow[d, "z"'] \arrow[dr, phantom, "\Nearrow \tau^{-1}"] & Y \arrow[d] \\
Y \arrow[r] & X & & Z \arrow[r] & X
\end{tikzcd}$$

.

$$\begin{tikzcd}
T \arrow[r, bend left, "z"] \arrow[r, bend right, "z'"'] \arrow[r, phantom, "\Downarrow \beta"] \arrow[d, bend right, "y"'] \arrow[d, bend left, "y'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & Z & \longmapsto & T \arrow[r, bend left, "y"] \arrow[r, bend right, "y'"'] \arrow[r, phantom, "\Downarrow \alpha"] \arrow[d, bend right, "z"'] \arrow[d, bend left, "z'"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & Y \\
Y & & & Z &
\end{tikzcd}$$

The squares above induce a commutative triangle

$$\begin{tikzcd}
 & \mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D}) \arrow[d, "\cong"] \\
\mathcal{C}(T, L) \arrow[ur, "\lambda_*"] \arrow[r, "(\lambda^{-1})_*"'] & \mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D}^*)
\end{tikzcd} ,$$

in which one of the diagonal morphisms is essentially surjective on objects and full if and only if the other is. ☐

Now let us consider some examples. The first one generalizes the fact that a square with parallel isomorphisms is a pullback to the 2-categorical setting.

Example 1.3.8
Consider a square as depicted on the right, for which both of the parallel arrows f and f' are equivalences in $\mathcal{C}$. We will prove in lemma 2.1.15 that this square is a weak comma in $\mathcal{C}$ in the sense that for every object T the canonical morphism

$$\mathcal{C}(T, X') \longrightarrow \mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D})$$

is an equivalence of categories. In particular it is essentially surjective on objects and full, which makes the square a quasi-comma.

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, phantom, "\nearrow \phi"] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

In fact the converse statement holds as well as we will see in the following lemma.

Lemma 1.3.9
Consider a quasi-comma as depicted below. If f is an equivalence in $\mathcal{C}$, so is f'. In particular the square is in fact a weak comma in $\mathcal{C}$, by means of lemma 2.1.15.

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, phantom, "\nearrow \phi"] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Proof Since f is an equivalence there exists an inverse equivalence $g : Y \to X$, which can be chosen to be a right adjoint equivalence with unit η and counit ε.

The 1-dimensional universal property of ϕ provides us with a factorization

$$\begin{tikzcd}
Y' \arrow[r, equal] \arrow[d, "y"'] & Y' \arrow[dd, "y"] & & Y' \arrow[d, "y"'] \arrow[dr, "g'"] \arrow[drr, bend left, equal] \arrow[drr, phantom, "\nearrow \varepsilon'"] & & \\
Y \arrow[d, "g"'] \arrow[dr, equal] & & = & Y \arrow[dr, "g"'] \arrow[r, phantom, "\nearrow \alpha^{-1}"] & X' \arrow[r, "f'"] \arrow[d, "x"] \arrow[dr, phantom, "\nearrow \phi"] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"'] \arrow[ur, phantom, "\nearrow \varepsilon"] & Y & & & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Using this and one of the triangle identities gives us the pasting identity

$$\begin{tikzcd}
X' \arrow[d, "x"'] \arrow[dr, "f'"] \arrow[ddrrr, bend left=40, "f'"] & & & \\
X \arrow[dr, "f"] \arrow[ddrr, bend right=40, equal] \arrow[r, phantom, "\Nearrow\phi"] & Y' \arrow[d, "y"] \arrow[dr, "g'"] \arrow[drr, bend left=20, equal] \arrow[drr, phantom, "\Nearrow\varepsilon'"] & & \\
& Y \arrow[dr, "g"] \arrow[r, phantom, "\Nearrow\alpha^{-1}"] \arrow[d, phantom, "\Nearrow\eta"] & X' \arrow[r, "f'"] \arrow[d, "x"] \arrow[dr, phantom, "\Nearrow\phi"] & Y' \arrow[d, "y"] \\
& & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
X' \arrow[d, "x"'] \arrow[dr, "f'"] \arrow[ddrr, bend left=40, "f'"] & & \\
X \arrow[dr, "f"] \arrow[ddr, bend right=40, equal] \arrow[r, phantom, "\Nearrow\phi"] & Y' \arrow[d, "y"] \arrow[dr, equal] & \\
\Nearrow\eta & Y \arrow[d, "g"] \arrow[dr, equal] \arrow[dr, phantom, "\Nearrow\varepsilon"'] & Y' \arrow[d, "y"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$= \begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow\phi"] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Hence by the 2-dimensional universal property of ϕ there is an invertible 2-cell

$$\begin{tikzcd}
X' \arrow[d, "f'"'] \arrow[dr, bend left=30, equal] \arrow[dr, phantom, "\Nearrow\eta'"] & \\
Y' \arrow[r, "g'"'] & X'
\end{tikzcd}$$

turning $(f', g', \varepsilon', \eta')$ into an equivalence. □

The previous lemma seems to promote a quasi-comma to a weak comma. In the light of the following observation this is not surprising.

Remark 1.3.10

Let $\mathcal{D} = X \to Y \leftarrow Y'$ be a diagram, which admits a weak comma (P, ρ). Then any quasi-comma object (Q, ω) of $\mathcal{D}$ already is a weak comma.

The reason for this is that weak commas are quasi-commas and quasi-commas are unique up to equivalence. We fix such an equivalence $e : P \simeq Q$. For any object T this gives rise to a 2-commutative triangle of categories

$$\begin{tikzcd}
\mathcal{C}(T,P) \arrow[dr, "\rho^*"] \arrow[dd, "e_*"'] & \\
\cong & \mathrm{Cone}_{\mathcal{C}}(T,\mathcal{D}) \\
\mathcal{C}(T,Q) \arrow[ur, "\omega^*"'] &
\end{tikzcd}$$

in which the upper and left functors are equivalences of categories. But then the lower functor is an equivalence by 2 out of 3.

Example 1.3.11

Let X, Y be two objects in $\mathcal{C}$. Suppose $\mathcal{C}$ has a weakly terminal object 1 and a weak product $X \times Y$ of X and Y in the sense that for every object T the canonical functors

$$\mathcal{C}(T,1) \longrightarrow * \quad \text{and} \quad \mathcal{C}(T, X \times Y) \longrightarrow \mathcal{C}(T,X) \times \mathcal{C}(T,Y)$$

are equivalences of categories. We fix a diagram $\mathcal{D} = X \longrightarrow 1 \longleftarrow Y$.

A square as depicted on the right defines a quasi-comma of the diagram $\mathcal{D}$ if and only if P is equivalent to the weak product of X and Y.

$$\begin{tikzcd}
P \arrow[r, "p_Y"] \arrow[d, "p_X"'] \arrow[dr, phantom, "\Nearrow"] & Y \arrow[d] \\
X \arrow[r] & 1
\end{tikzcd}$$

This holds because by lemma 2.1.16 having a weak product $X \times Y$ is equivalent to the square

$$\begin{tikzcd}
X \times Y \arrow[r, "\mathrm{pr}_Y"] \arrow[d, "\mathrm{pr}_X"'] \arrow[dr, phantom, "\Nearrow"] & Y \arrow[d] \\
X \arrow[r] & 1
\end{tikzcd}$$

being a weak comma. But then by the preceding remark 1.3.10 any quasicomma as depicted on the right above actually is a weak comma and hence computes a weak product.

The weak universal property of quasi-commas also allows us to prove the pasting law for homotopy pullbacks.

Lemma 1.3.12 (Pasting Law)
Given two quasicommas as depicted below, the composite square is again a quasicomma.

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow \phi"] & Y' \arrow[r, "g'"] \arrow[d, "y"] \arrow[dr, phantom, "\Nearrow \psi"] & Z' \arrow[d, "z"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Proof Using first the 1-dimensional universal property of ψ and then that of ϕ we obtain for an arbitrary square α a factorization

$$\begin{tikzcd}
T \arrow[r, "t'"] \arrow[d, "r"'] \arrow[dr, phantom, "\Nearrow \alpha"] & Z' \arrow[d, "z"] \\
X \arrow[r, "g \circ f"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[d, "r"'] \arrow[dr, "s'"] \arrow[drr, bend left, "t'"] & \Nearrow \tau & \\
X \arrow[r, phantom, "\Nearrow \beta^{-1}"] \arrow[dr, "f"'] & Y' \arrow[r, "g'"] \arrow[d, "y"] \arrow[dr, phantom, "\Nearrow \psi"] & Z' \arrow[d, "z"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, "r'"] \arrow[drr, bend left, "s'"] \arrow[drrr, bend left=40, "t'"] \arrow[ddr, bend right, "r"'] & \Nearrow \sigma & \Nearrow \tau & \\
\Nearrow \rho^{-1} & X' \arrow[r, "f'"] \arrow[d, "x"] \arrow[dr, phantom, "\Nearrow \phi"] & Y' \arrow[r, "g'"] \arrow[d, "y"] \arrow[dr, phantom, "\Nearrow \psi"] & Z' \arrow[d, "z"] \\
& X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

This proves the 1-dimensional universal property. Regarding the 2-dimensional universal property we consider a morphism of cones

$$\begin{tikzcd}
T \arrow[r, "\widehat{r}'"] \arrow[d, "\widehat{r}'"'] \arrow[dr, "r'"] & X' \arrow[r, "f'"] \arrow[dr, phantom, "\Nearrow\omega"] & Y' \arrow[dr, bend left, "g'"] & \\
X' \arrow[dr, bend right, "x"'] \arrow[r, phantom, "\Nearrow\rho^{-1}"] & X' \arrow[r, "f'"] \arrow[d, "x"] \arrow[dr, phantom, "\Nearrow\phi"] & Y' \arrow[r, "g'"] \arrow[d, "y"] \arrow[dr, phantom, "\Nearrow\psi"] & Z' \arrow[d, "z"] \\
& X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, "\widehat{r}'"] & & & \\
& X' \arrow[r, "f'"] \arrow[d, "x"] \arrow[dr, phantom, "\Nearrow\phi"] & Y' \arrow[r, "g'"] \arrow[d, "y"] \arrow[dr, phantom, "\Nearrow\psi"] & Z' \arrow[d, "z"] \\
& X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Bringing ϕ on the left side we obtain the pasting identity

$$\begin{tikzcd}
T \arrow[r, "\widehat{r}'"] \arrow[d, "\widehat{r}'"'] \arrow[dr, "r'"] & X' \arrow[dr, "f'"] & & \\
X' \arrow[d, "f'"'] \arrow[dr, "x"] \arrow[r, phantom, "\Rightarrow\rho^{-1}"] & X' \arrow[d, "x"] \arrow[dr, "f'"] \arrow[r, phantom, "\Nearrow\omega"] & Y' \arrow[dr, "g'"] & \\
Y' \arrow[drr, bend right, "y"'] \arrow[dr, phantom, "\Nearrow\phi^{-1}"] & X \arrow[dr, "f"] \arrow[r, phantom, "\Rightarrow\phi"] & Y' \arrow[r, "g'"] \arrow[d, "y"] \arrow[dr, phantom, "\Nearrow\psi"] & Z' \arrow[d, "z"] \\
& & Y \arrow[r, "g"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, "\widehat{r}'"] & & \\
& X' \arrow[dr, "f'"] & \\
& & Y' \arrow[r, "g'"] \arrow[d, "y"'] \arrow[dr, phantom, "\Nearrow\psi"] & Z' \arrow[d, "z"] \\
& & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

and thus by the 2-dimensional universal property of ψ there is an invertible 2-cell σ, satisfying

$$\begin{tikzcd}
T \arrow[r, "\widehat{r}'"] \arrow[d, "r'"'] \arrow[dr, phantom, "\Nearrow\sigma"] & X' \arrow[d, "f'"] & \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z'
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[r, "\widehat{r}'"] \arrow[d, "r'"'] & X' \arrow[r, "f'"] \arrow[d, phantom, "\Nearrow\omega"] & Y' \arrow[d, "g'"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z'
\end{tikzcd}
\tag{$\star$}$$

as well as

$$\begin{tikzcd}
T \arrow[r, "\widehat{r}'"] \arrow[d, "r'"'] \arrow[dr, phantom, "\Nearrow\sigma"] & X' \arrow[d, "f'"] \\
X' \arrow[r, "f'"'] & Y' \arrow[d, "y"] \\
& Y
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[r, "\widehat{r}'"] \arrow[d, "r'"'] \arrow[dr, phantom, "\Nearrow\rho"] & X' \arrow[d, "x"] \arrow[dr, "f'"] & \\
X' \arrow[r, "x"] \arrow[dr, "f'"'] & X \arrow[r, phantom, "\Nearrow\phi"] \arrow[dr, "f"] \arrow[d, phantom, "\Nearrow\phi^{-1}"] & Y' \arrow[d, "y"] \\
& Y' \arrow[r, "y"'] & Y
\end{tikzcd}$$

Bringing ρ and ϕ^{-1} to the left we obtain the pasting identity

$$\begin{tikzcd}
T \arrow[r, "\widehat{r}'"] \arrow[d, "\widehat{r}'"'] \arrow[dr, "r'" description] & X' \arrow[dr, "f'"] \arrow[d, phantom, "\Nearrow \sigma"] & \\
X' \arrow[dr, "x"'] \arrow[r, phantom, "\Nearrow \rho^{-1}"] & X' \arrow[r, "f'" description] \arrow[d, "x" description] \arrow[dr, phantom, "\Nearrow \phi"] & Y' \arrow[d, "y"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, "\widehat{r}'"] & & \\
& X' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow \phi"] & Y' \arrow[d, "y"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Hence by the 2-dimensional universal property of ϕ there is an invertible 2-cell ξ satisfying

$$\begin{tikzcd}
T \arrow[dr, bend left, "\widehat{r}'"] \arrow[dr, bend right, "r'"'] \arrow[dr, phantom, "\Nearrow \xi"] & \\
& X' \arrow[d, "x"] \\
& X
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[d, "r'"'] \arrow[dr, "\widehat{r}'"] & \\
X' \arrow[dr, "x"'] \arrow[r, phantom, "\Nearrow \rho"] & X' \arrow[d, "x"] \\
& X
\end{tikzcd}$$

as well as

$$\begin{tikzcd}
T \arrow[dr, bend left, "\widehat{r}'"] \arrow[dr, bend right, "r'"'] \arrow[dr, phantom, "\Nearrow \xi"] & & \\
& X' \arrow[r, "f'"'] & Y'
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[r, "\widehat{r}'"] \arrow[dr, "r'"'] & X' \arrow[dr, "f'"] \arrow[d, phantom, "\Nearrow \sigma"] & \\
& X' \arrow[r, "f'"'] & Y'.
\end{tikzcd}$$

Postcomposing the last identity with g' and using the identity $g'\sigma = \omega$ $(\star)$ above shows that ξ is indeed a morphism of cones. ☐

For the converse we have the following partial result.[3]

Lemma 1.3.13

Consider two squares as depicted below. If the morphisms g' and g are equivalences and the composite rectangle is a quasi-comma, so is the left square.

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow \phi"] & Y' \arrow[r, "g'"] \arrow[d, "y" description] \arrow[dr, phantom, "\Nearrow \psi"] & Z' \arrow[d, "z"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Proof Since g and g' are equivalences there are inverse equivalences k and k'. As in remark 2.1.14

---

[3] As it turned out late into writing the thesis there is a cancellation law for quasi-commas. We prove it at the end of the section.

we obtain another square ζ satisfying the pasting identity

$$\begin{tikzcd}
Y' \arrow[r, "g'"] \arrow[d, "y"'] \arrow[rr, bend left=50, equal, "\Uparrow \eta'^{-1}"'] & Z' \arrow[r, "k'"] \arrow[d, "z" description] & Y' \arrow[d, "y"] & Y' \arrow[r, equal] \arrow[d, "y"'] & Y' \arrow[d, "y"] \\
Y \arrow[r, "g"'] \arrow[rr, bend right=50, equal, "\Uparrow \eta"] & Z \arrow[r, "k"'] & Y & Y \arrow[r, equal] & Y
\arrow[from=1-1, to=2-2, phantom, "\Nearrow \psi"]
\arrow[from=1-2, to=2-3, phantom, "\Nearrow \zeta"]
\arrow[from=1-3, to=2-4, phantom, "="]
\end{tikzcd} .$$

Since ζ is a square with parallel equivalences it is a quasi-comma by example 1.3.8, so by the pasting law 1.3.12 the whole rectangle

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] & Y' \arrow[r, "g'"] \arrow[d, "y" description] & Z' \arrow[r, "k'"] \arrow[d, "z" description] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "k"'] & Y
\arrow[from=1-1, to=2-2, phantom, "\Nearrow \phi"]
\arrow[from=1-2, to=2-3, phantom, "\Nearrow \psi"]
\arrow[from=1-3, to=2-4, phantom, "\Nearrow \zeta"]
\end{tikzcd}$$

is a quasi-comma. But then the equivalent square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] & Y' \arrow[r, "g'"] \arrow[d, "y" description] \arrow[rr, bend left=50, equal, "\Uparrow \eta'^{-1}"'] & Z' \arrow[r, "k'"] \arrow[d, "z" description] & Y' \arrow[d, "y"] & X' \arrow[r, "f'"] \arrow[d, "x"'] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] \arrow[rr, bend right=50, equal, "\Uparrow \eta"] & Z \arrow[r, "k"'] & Y & X \arrow[r, "f"'] & Y
\arrow[from=1-1, to=2-2, phantom, "\Nearrow \phi"]
\arrow[from=1-2, to=2-3, phantom, "\Nearrow \psi"]
\arrow[from=1-3, to=2-4, phantom, "\Nearrow \zeta"]
\arrow[from=1-4, to=2-5, phantom, "="]
\arrow[from=1-5, to=2-6, phantom, "\Nearrow \phi"]
\end{tikzcd}$$

is by remark 1.3.6 a quasi-comma as well. ☐

Corollary 1.3.14

Consider a 2-commutative cube as depicted below with all vertical maps being equivalences and the lower square being a quasi-comma. Then the upper square is a quasi-comma as well.

$$\begin{tikzcd}
L' \arrow[rr] \arrow[dr] \arrow[dd, "\simeq"] & & Z' \arrow[dr] \arrow[dd, "\simeq" near start] & \\
& Y' \arrow[rr] \arrow[dd, "\simeq" near end] & & X' \arrow[dd, "\simeq"] \\
L \arrow[rr] \arrow[dr] & & Z \arrow[dr] & \\
& Y \arrow[rr] & & X
\arrow[from=1-3, to=2-4, phantom, "\Nearrow"]
\arrow[from=1-2, to=3-3, phantom, "\Nearrow"]
\arrow[from=3-1, to=4-2, phantom, "\Nearrow"]
\arrow[from=3-3, to=2-4, phantom, "\Nearrow"]
\arrow[from=4-3, to=3-3, phantom, "\Nearrow"]
\arrow[from=4-3, to=3-4, phantom, "\Nearrow"]
\end{tikzcd}$$

Proof The top and front face assemble into a diagram as in the previous lemma 1.3.13. By means of the invertible cells left and right this composite rectangle is equivalent in the sense of remark 1.3.6 to the rectangle given by the back and lower face. By assumption the lower face is a quasi-comma and the back face is a quasi-comma by example 1.3.8. Thus by the pasting law 1.3.12 the rectangle given by the back and lower face is a quasi-comma as well. By applying remark 1.3.6 the rectangle given by the top and front square is a quasi-comma. Applying the preceding lemma 1.3.13 we find that the top face is a quasi-comma. ☐

Proposition 1.3.15

Fix quasi-commas λ and λ' of the horizontal parts of the diagram

$$\begin{tikzcd}
Y' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow \phi"] & X' \arrow[d, "y"] \arrow[dr, phantom, "\Nwarrow \psi"] & Z' \arrow[l, "g'"'] \arrow[d, "z"] \\
Y \arrow[r, "f"'] & X & Z \arrow[l, "g"]
\end{tikzcd}$$

Then there is a morphism e and invertible 2-cells making the diagram

$$\begin{tikzcd}
L' \arrow[rr] \arrow[dr] \arrow[dd, dashed, "e"'] & & Z' \arrow[dr] \arrow[dd] \arrow[dr, phantom, "\Nearrow", near end] & \\
& Y' \arrow[rr, crossing over] \arrow[dd] & & X' \arrow[dd] \\
L \arrow[rr, crossing over] \arrow[dr] \arrow[dr, phantom, "\Nearrow", near end] & & Z \arrow[dr] \arrow[dr, phantom, "\Nearrow"] & \\
& Y \arrow[rr] \arrow[ur, phantom, "\Nearrow"] & & X
\end{tikzcd}$$

commute. Furthermore, if the vertical morphisms x, y, z are equivalences, so is e.

Proof The existence of e follows from the universal property of L yielding a factorization

$$\begin{tikzcd}
L' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \lambda'"] & Z' \arrow[r, "z"] \arrow[d, "g'"] \arrow[dr, phantom, "\Nearrow \psi^{-1}"] & Z \arrow[dd, "g"] \\
Y' \arrow[r, "f'"'] \arrow[d, "y"'] \arrow[dr, phantom, "\Nearrow \phi"] & X' \arrow[dr, "x"] & \\
Y \arrow[rr, "f"'] & & X
\end{tikzcd}
=
\begin{tikzcd}
L' \arrow[r] \arrow[d] \arrow[dr, "e"] & Z' \arrow[dr, "z"] \arrow[d, phantom, "\Nearrow \zeta"] & \\
Y' \arrow[dr, "y"'] \arrow[r, phantom, "\Nearrow \Upsilon"] & L \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \lambda"] & Z \arrow[d, "g"] \\
& Y \arrow[r, "f"'] & X
\end{tikzcd} .$$

If x, y, z are equivalences they have inverse adjoint equivalences x', y', z'. Applying remark

2.1.14 thrice we obtain another cube

$$\begin{tikzcd}
L \arrow[rr] \arrow[dr] \arrow[dd, dashed, "e'"'] & & Z \arrow[dr] \arrow[dd] \arrow[dr, phantom, bend right, "\Rightarrow"] & \\
& Y \arrow[rr] \arrow[dd] \arrow[r, phantom, "\Rightarrow"] & & X \arrow[dd] \\
L' \arrow[rr] \arrow[dr] \arrow[dr, phantom, bend left, "\Rightarrow"] & & Z' \arrow[dr] \arrow[dr, phantom, bend left, "\Rightarrow"] \arrow[dr, phantom, "\Rightarrow"] \arrow[dr, phantom, bend right, "\Rightarrow"] & \\
& Y' \arrow[rr] & & X'
\end{tikzcd}$$

with the morphism e' induced as before. When stacking this new cube under the former one we obtain a composite cube. By inserting the corresponding identity 2-cells $\eta \circ \eta^{-1}$ into the vertical maps $x' \circ x$, $y' \circ y$ and $z' \circ z$ we obtain an equivalent cube of the form

$$\begin{tikzcd}
L' \arrow[rr] \arrow[dr] \arrow[dd, dashed, "e' \circ e"'] & & Z' \arrow[dr] \arrow[dd, equal] \arrow[dr, phantom, bend right, "\Rightarrow"] & \\
& Y' \arrow[rr] \arrow[dd, equal] \arrow[r, phantom, "\Rightarrow"] & & X' \arrow[dd, equal] \\
L' \arrow[rr] \arrow[dr] \arrow[dr, phantom, bend left, "\Rightarrow"] & & Z' \arrow[dr] \arrow[dr, phantom, bend right, "\Rightarrow"] & \\
& Y' \arrow[rr] & & X'
\end{tikzcd}$$

which expresses that the composite $e' \circ e$ together with the left and back 2-cells provides a factorization of the square λ' with respect to the square λ'. But by the 2-dimensional universal property of λ' this implies that there is an invertible 2-cell $e' \circ e \Rightarrow \mathrm{id}$ and thus that e' is a left-inverse to e.

By the dual statement to remark 2.1.14 there is yet another cube with induced morphism $e'' : L \to L'$, which can be stacked upon the original one and modified in a dual manner to obtain a cube of the form

$$\begin{tikzcd}
L \arrow[rr] \arrow[dr] \arrow[dd, dashed, "e \circ e''"'] & & Z \arrow[dr] \arrow[dd, equal] \arrow[dr, phantom, bend right, "\Rightarrow"] & \\
& Y \arrow[rr] \arrow[dd, equal] \arrow[r, phantom, "\Rightarrow"] & & X \arrow[dd, equal] \\
L \arrow[rr] \arrow[dr] \arrow[dr, phantom, bend left, "\Rightarrow"] & & Z \arrow[dr] \arrow[dr, phantom, bend right, "\Rightarrow"] & \\
& Y \arrow[rr] & & X
\end{tikzcd}$$

From this we can deduce analogously that e has a right-inverse given by e''. Thus, since it has both a left- and a right-inverse, e is an equivalence. $\square$

Proposition 1.3.16 (Cancellation Law)
Consider two squares as depicted below. If the right square ψ and the composite square $\phi \bullet \psi$ are quasi-commas and the quasi-comma of the diagram $X \xrightarrow{f} Y \xleftarrow{y} Y'$ exists, then the left square ϕ is a quasi-comma.

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow \phi"] & Y' \arrow[r, "g'"] \arrow[d, "y"] \arrow[dr, phantom, "\Nearrow \psi"] & Z' \arrow[d, "z"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Proof By assumption the quasi-comma L of f and y provides us with a factorization

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow \phi"] & Y' \arrow[d, "y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
X' \arrow[dr, "e"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "x"'] & \Nearrow \beta & \\
\Nearrow \alpha & L \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Y' \arrow[d, "y"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

of the square ϕ. By the pasting law 1.3.12 the composite square $\lambda \bullet \psi$ is a quasi-comma. The factorization above gives rise to a factorization

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow \phi"] & Y' \arrow[r, "g'"] \arrow[d, "y"] \arrow[dr, phantom, "\Nearrow \psi"] & Z' \arrow[d, "z"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}
=
\begin{tikzcd}
X' \arrow[dr, "e"] \arrow[drr, bend left, "f'"] \arrow[drrr, bend left=40, "g'f'"] \arrow[ddr, bend right, "x"'] & \Nearrow \beta & & \\
\Nearrow \alpha & L \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "\Nearrow \lambda"] & Y' \arrow[r, "g'"] \arrow[d, "y"] \arrow[dr, phantom, "\Nearrow \psi"] & Z' \arrow[d, "z"] \\
& X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

but since both composite squares $\phi \bullet \psi$ and $\lambda \bullet \psi$ are quasi-commas, the map e is an equivalence by lemma 1.3.2. The former factorization thus shows that the square ϕ is equivalent to the quasi-comma square λ and as a consequence is a quasi-comma square as well. $\square$

We will also need the fact that a strict 2-functor, which is a weak 2-equivalence in the sense of definition 2.1.5, preserves quasi-commas.

Lemma 1.3.17
Let $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ be a strict 2-functor, which is a weak 2-equivalence. Let $\mathcal{D} = Y \to X \leftarrow Z$ be a diagram in $\mathcal{A}$ and $\lambda \in \mathrm{Cone}_{\mathcal{A}}(L, \mathcal{D})$ be a quasi-comma square of $\mathcal{D}$. Then $\mathcal{F}(\lambda) \in \mathrm{Cone}_{\mathcal{B}}(\mathcal{F}(L), \mathcal{F}\mathcal{D})$ is a quasi-comma square of the diagram $\mathcal{F}\mathcal{D}$ in $\mathcal{B}$.

Proof Let $\tau \in \mathrm{Cone}_{\mathcal{B}}(T, \mathcal{F}\mathcal{D})$ be an arbitrary cone. Since $\mathcal{F}$ is weakly essentially surjective on objects there is an equivalence $\mathcal{F}T' \simeq T$ in $\mathcal{B}$, where T' is some object in $\mathcal{A}$. Precomposing this to our chosen cone and exploiting that the functor $\mathcal{A}(T', X) \longrightarrow \mathcal{B}(\mathcal{F}(T'), \mathcal{F}(X))$ is an equivalence gives us an equivalence of cones

$$\begin{tikzcd}
\mathcal{F}(T') \arrow[dr, "\simeq"'] \arrow[drr, bend left, "\mathcal{F}(z')"] \arrow[ddr, bend right, "\mathcal{F}(y')"'] & \Nearrow\beta & & & & \\
\Nearrow\alpha & T \arrow[r, "z"] \arrow[d, "y"'] \arrow[dr, phantom, "\Nearrow\tau"] & \mathcal{F}(Z) \arrow[d] & = & \mathcal{F}(T') \arrow[r, "\mathcal{F}(z')"] \arrow[d, "\mathcal{F}(y')"'] \arrow[dr, phantom, "\Nearrow\mathcal{F}(\tau')"] & \mathcal{F}(Z) \arrow[d] \\
 & \mathcal{F}(Y) \arrow[r] & \mathcal{F}(X) & & \mathcal{F}(Y) \arrow[r] & \mathcal{F}(X).
\end{tikzcd}$$

This provides us with an equivalence $\mathrm{Cone}_{\mathcal{B}}(T, \mathcal{F}\mathcal{D}) \simeq \mathrm{Cone}_{\mathcal{B}}(\mathcal{F}(T'), \mathcal{F}\mathcal{D})$, which fits into a commutative diagram of categories

$$\begin{tikzcd}
\mathcal{A}(T', L) \arrow[r, "\lambda_*"] \arrow[d, "\simeq"'] & \mathrm{Cone}_{\mathcal{A}}(T', \mathcal{D}) \arrow[d, "\simeq"] \\
\mathcal{B}(\mathcal{F}(T'), \mathcal{F}(L)) \arrow[r, "\mathcal{F}(\lambda)_*"] \arrow[d, "\simeq"'] & \mathrm{Cone}_{\mathcal{B}}(\mathcal{F}(T'), \mathcal{F}\mathcal{D}) \arrow[d, "\simeq"] \\
\mathcal{B}(T, \mathcal{F}(L)) \arrow[r, "\mathcal{F}(\lambda)_*"'] & \mathrm{Cone}_{\mathcal{B}}(T, \mathcal{F}\mathcal{D})
\end{tikzcd}$$

where the upper square commutes by strict functoriality of $\mathcal{F}$ and the lower commutes, since the vertical morphisms are given by precomposition, while the horizontal ones are given by postcomposition. Because the upper functor is essentially surjective on objects and full, the lower functor is as well. This shows that $\mathcal{F}(\lambda)$ is a quasi-comma in $\mathcal{B}$. $\square$

1.4 Quasiloops and Quasisuspensions

Stable ∞-categories arise from sufficiently nice pointed ∞-categories by forcing the loop-∞-endofunctor to become an equivalence, a process called *stabilization*. In this section we want to describe, how to obtain the suspension-loop adjunction $\Sigma \dashv \Omega$ in the setting of sufficiently nice pointed simplicial model categories. We first discuss the functoriality of homotopy pullbacks and homotopy pushouts before specializing to the case of loops and suspension. We also discuss how the sSet-enriched functors we construct induce adjunctions on the homotopy 1- and 2-category.

Let $\mathcal{I} = 1 \to 0 \leftarrow 2$ be the free cospan in sSet-Cat. For a simplicial model category $\mathcal{C}$ we have the sSet-enriched functor

$$\mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}) \xrightarrow{\lim} \mathcal{C}$$

taking the pullback of the diagrams. Furthermore there is an sSet-enriched functor

$$\begin{array}{ccc} \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}) & \xrightarrow{\text{fold}} & \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}) \\ (Y \xrightarrow{f} X \xleftarrow{g} Z) & \longmapsto & (Y \times Z \xrightarrow{f \times g} X \times X \leftarrow X^{\Delta^1}). \end{array}$$

We have used in section 1.2 that when restricted to cospan diagrams of fibrant objects the composite sSet-enriched functor

$$\mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}_f) \xrightarrow{\text{fold}} \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}_f) \xrightarrow{\lim} \mathcal{C}_f$$

computes the homotopy pullbacks of the diagrams.

If $\mathcal{C}$ is a pointed simplicial model category with zero object 0, there is a canonical sSet-enriched functor

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{cospan}} & \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}) \\ X & \longmapsto & (0 \to X \leftarrow 0) \end{array}$$

and the composite sSet-enriched functor

$$\widetilde{\Omega} : \mathcal{C} \xrightarrow{\text{cospan}} \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}) \xrightarrow{\text{fold}} \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}) \xrightarrow{\lim} \mathcal{C}$$

computes *loop objects*, when restricted to fibrant objects in $\mathcal{C}$.

In appendix A.3 we discuss enriched factorizations and show in corollary A.3.5 that a cofibrantly generated simplicial model category admits sSet-enriched factorizations for the model structure. Dualizing the arguments for the case of cofibrant object and homotopy pushouts we arrived at the following proposition.

Proposition 1.4.1
Let $\mathcal{C}$ be a cofibrantly generated simplicial model category with fibrant and cofibrant sSet-enriched replacement functors $\mathcal{R}$ and $\mathcal{Q}$. Denoting the free cospan and span categories with $\mathcal{I}$ and $\mathcal{J}$ respectively, there are sSet-enriched functors

$$\mathrm{holim} : \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}_b) \xrightarrow{\text{fold}} \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}_f) \xrightarrow{\lim} \mathcal{C}_f \xrightarrow{\mathcal{Q}} \mathcal{C}_b$$

and

$$\mathrm{hocolim} : \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{J}, \mathcal{C}_b) \xrightarrow{\text{cofold}} \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{J}, \mathcal{C}_c) \xrightarrow{\text{colim}} \mathcal{C}_c \xrightarrow{\mathcal{R}} \mathcal{C}_b$$

computing homotopy pullbacks and homotopy pushouts of diagrams of bifibrant objects.

If in addition $\mathcal{C}$ is pointed, then there are sSet-enriched functors

$$\Omega : \mathcal{C}_b \xrightarrow{\text{cospan}} \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{I}, \mathcal{C}_b) \xrightarrow{\text{holim}} \mathcal{C}_b$$

and

$$\Sigma : \mathcal{C}_b \xrightarrow{\text{span}} \mathrm{Fun}_{\mathsf{sSet}}(\mathcal{J}, \mathcal{C}_b) \xrightarrow{\text{hocolim}} \mathcal{C}_b$$

called the **looping** and **suspension** functors.

Since the simplicial Hom-functors preserve limits, we note that for a pointed simplicial model category $\mathcal{C}$ we have canonical isomorphisms of simplicial sets

$$\begin{aligned}\mathcal{C}(\widetilde{\Sigma}X, Y) &= \mathcal{C}(\mathrm{colim} \circ \mathrm{cofold} \circ \mathrm{span}(X), Y) \\ &\cong \lim \circ \mathrm{fold} \circ \mathrm{cospan}(\mathcal{C}(X, Y)) \\ &\cong \mathcal{C}(X, \lim \circ \mathrm{fold} \circ \mathrm{cospan}(Y)) \\ &= \mathcal{C}(X, \widetilde{\Omega}Y)\end{aligned}$$

which are sSet-natural in X and Y. In other words the simplicial functors $\widetilde{\Sigma}$ and $\widetilde{\Omega}$ are adjoint sSet-enriched functors. In appendix A.3 we in particular showed that the fibrant and cofibrant replacement equivalences are sSet-natural. This leads us to the following proposition.

Proposition 1.4.2

Let $\mathcal{C}$ be a cofibrantly generated simplicial pointed model category with fibrant and cofibrant sSet-enriched replacement functors $\mathcal{R}$ and $\mathcal{Q}$.

For bifibrant objects X and Y there are weak equivalences of Kan-complexes

$$\mathcal{C}_b(\Sigma X, Y) = \mathcal{C}_b(\mathcal{R}\widetilde{\Sigma}X, Y) \xrightarrow{\simeq} \mathcal{C}(\widetilde{\Sigma}X, Y) \cong \mathcal{C}(X, \widetilde{\Omega}Y) \xleftarrow{\simeq} \mathcal{C}_b(X, \mathcal{Q}\widetilde{\Omega}Y) = \mathcal{C}_b(X, \Omega Y) \tag{1.4.2.1}$$

which are sSet-natural in X and Y.

In particular the sSet-enriched functors Σ and Ω induce an adjunction $\Sigma \dashv \Omega$ on the homotopy 1-category $\mathrm{ho}(\mathcal{C}) = (\pi_0)_*(\mathcal{C}_b)$. Similarly the induced strict 2-functors on the homotopy 2-category $\mathrm{ho}_2(\mathcal{C}) = \tau_*(\mathcal{C}_b)$ are weakly 2-adjoint in the sense of definition 2.1.5.

Proof Basechange along $\pi_0 : \mathsf{sSet} \longrightarrow \mathsf{Set}$ turns the cospan of equivalences 1.4.2.1 into a cospan of isomorphisms

$$\mathrm{ho}(\mathcal{C})(\Sigma X, Y) \xrightarrow{\cong} (\pi_0)_*\mathcal{C}(\widetilde{\Sigma}X, Y) \cong (\pi_0)_*\mathcal{C}(X, \widetilde{\Omega}Y) \xleftarrow{\cong} \mathrm{ho}(\mathcal{C})(X, \Omega Y)$$

natural in X and Y. Reversing the isomorphism on the right gives rise to the desired natural isomorphism

$$\mathrm{ho}(\mathcal{C})(\Sigma X, Y) \cong \mathrm{ho}(\mathcal{C})(X, \Omega Y)$$

and hence the desired adjunction $\Sigma \dashv \Omega$ on $\mathrm{ho}(\mathcal{C})$. In the same way basechange along $\tau : \mathsf{sSet} \longrightarrow \mathsf{Cat}$ yields a cospan of equivalences

$$\mathrm{ho}_2(\mathcal{C})(\Sigma X, Y) \xrightarrow{\simeq} \tau_*\mathcal{C}(\widetilde{\Sigma}X, Y) \cong \tau_*\mathcal{C}(X, \widetilde{\Omega}Y) \xleftarrow{\simeq} \mathrm{ho}_2(\mathcal{C})(X, \Omega Y)$$

which are strictly 2-natural in X and Y. Choosing an inverse of the equivalence to the right gives rise to an equivalence

$$\mathrm{ho}_2(\mathcal{C})(\Sigma X, Y) \simeq \mathrm{ho}_2(\mathcal{C})(X, \Omega Y),$$

which is *weakly* 2-natural in X and Y. □

Corollary 1.4.3

In the setting of the preceding proposition 1.4.2, if $\mathcal{C}$ is *stable* in the sense that the induced adjunction $\Sigma \dashv \Omega$ on $\mathrm{ho}(\mathcal{C})$ is an adjoint 1-equivalence, then the 2-endofunctors Σ and Ω on $\mathrm{ho}_2(\mathcal{C})$ are mutually inverse weak 2-equivalences.

Proof The weak 2-adjunction $\Sigma \dashv \Omega$ on $\mathrm{ho}_2(\mathcal{C})$ comes with a weakly 2-natural unit $\eta : \mathrm{id} \Rightarrow \Omega\Sigma$ and weakly 2-natural counit $\varepsilon : \Sigma\Omega \Rightarrow \mathrm{id}$. Change of base along $\pi_0 : \mathsf{Cat} \longrightarrow \mathsf{Set}$ turns these into the unit η' and counit ε' of the adjoint 1-equivalence $\Sigma \dashv \Omega$ on $\mathrm{ho}(\mathcal{C})$ and thus in particular into 1-natural isomorphisms. That the components of η' and ε' are isomorphisms implies that the components of η and ε are equivalences. Hence η and ε are weakly 2-natural equivalences, rendering Σ and Ω mutually inverse weak 2-equivalences. □

When discussing the universal property of quasiloops in section 3.1 we impose the condition that the quasiloop 2-functor $\Omega : \mathrm{ho}_2(\mathcal{C}) \longrightarrow \mathrm{ho}_2(\mathcal{C})$ is compatible with the universal 2-cells. More precisely, similar to the case of quasicommas, for a bifibrant object X the morphism

$$\Omega X = \mathcal{Q}\widetilde{\Omega}X \xrightarrow{\simeq} \widetilde{\Omega}X \longrightarrow \Delta^1 \pitchfork X$$

corresponds to a morphism of simplicial sets

$$\Delta^1 \longrightarrow \mathcal{C}(\Omega X, X),$$

which after applying ho : sSet $\longrightarrow$ Cat gives rise to the universal 2-cell

$$\mathbb{2} \longrightarrow \mathrm{ho}_2(\mathcal{C})(\Omega X, X)$$

usually depicted as

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_X"] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} .$$

The compatibility we will impose is justified by the following lemma.

Lemma 1.4.4

Let $\mathcal{C}$ be a cofibrantly generated simplicial pointed model category with fibrant and cofibrant sSet-enriched replacement functors $\mathcal{R}$ and $\mathcal{Q}$.

For a morphism $f : X \to Y$ between bifibrant objects the universal 2-cells $\omega_X \in \mathrm{ho}_2(\mathcal{C})(\Omega X, X)$ and $\omega_Y \in \mathrm{ho}_2(\mathcal{C})(\Omega Y, Y)$ just described satisfy the following pasting identity in $\mathrm{ho}_2(\mathcal{C})$.

$$\begin{tikzcd} \Omega X \arrow[dr, "\Omega f"] & & & & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_X"] & 0 \arrow[d] & \\ & \Omega Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_Y"] & 0 \arrow[d] & = & 0 \arrow[r] & X \arrow[dr, "f"] & \\ & 0 \arrow[r] & Y & & & & Y \end{tikzcd} \tag{1.4.4.1}$$

Proof The action of the simplicial functor $\Omega : \mathcal{C}_b \longrightarrow \mathcal{C}_b$ on morphisms is illustrated by the diagram

$$\begin{tikzcd} \Omega X = \mathcal{Q}\widetilde{\Omega}X \arrow[rr, "\simeq"] \arrow[dr, "\Omega f"] & & \widetilde{\Omega}X \arrow[rr] \arrow[dd] \arrow[dr, "\widetilde{\Omega}f"] & & \Delta^1 \pitchfork X \arrow[dr, "\Delta^1 \pitchfork f"] \arrow[dd] & \\ & \Omega Y = \mathcal{Q}\widetilde{\Omega}Y \arrow[rr, "\simeq"] & & \widetilde{\Omega}Y \arrow[rr] \arrow[dd] & & \Delta^1 \pitchfork Y \arrow[dd] \\ & & 0 \arrow[rr] \arrow[dr, equal] & & X \times X \arrow[dr, "f \times f"] & \\ & & & 0 \arrow[rr] & & Y \times Y \end{tikzcd}$$

in $\mathcal{C}$, where the left square on the top commutes by the simplicial naturality of cofibrant replacement. Using the universal property of the power by Δ^1 the commutativity of the upper two squares translates into the requirement that the square of simplicial sets

$$\begin{tikzcd}
\Delta^1 \arrow[r] \arrow[d] & \mathcal{C}(\Omega X, X) \arrow[d, "f_*"] \\
\mathcal{C}(\Omega Y, Y) \arrow[r, "(\Omega f)^*"'] & \mathcal{C}(\Omega X, Y)
\end{tikzcd}$$

commutes. Applying $\tau : \mathsf{sSet} \longrightarrow \mathsf{Cat}$ gives rise to the desired pasting identity in $\mathrm{ho}_2(\mathcal{C})$. $\square$

We will need another compatibility, this time of the unit and counit of the induced 2-adjunction with the universal 2-cells. More precisely we have the following lemma.

Lemma 1.4.5

Let $\mathcal{C}$ be a cofibrantly generated simplicial pointed model category with fibrant and cofibrant replacement sSet-enriched replacement functors $\mathcal{R}$ and $\mathcal{Q}$.

For any object X the pasting identities

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "\eta_X"] \arrow[dd] & & 0 \arrow[dd] \\
& \Omega\Sigma X \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\Nearrow \omega_{\Sigma X}"] & \\
0 \arrow[rr] & & \Sigma X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \sigma_X"] & 0 \arrow[d] \\
0 \arrow[r] & \Sigma X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Omega X \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\Nearrow \sigma_{\Omega X}"] & & 0 \arrow[dd] \arrow[dl] \\
& \Sigma\Omega X \arrow[dr, "\varepsilon_X"] & \\
0 \arrow[rr] \arrow[ur] & & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_X"] & 0 \arrow[d] \\
0 \arrow[r] & X
\end{tikzcd}$$

hold true in the homotopy 2-category $\mathrm{ho}_2(\mathcal{C})$.

Proof By construction the component ε_X of the counit of the weak 2-adjunction can be realized to satisfy the mapping property

$$\begin{array}{ccccccc}
\mathrm{ho}_2(\mathcal{C})(\Sigma\Omega X, X) & \xrightarrow{\simeq} & \tau_*\mathcal{C}(\widetilde{\Sigma}\Omega X, X) & \cong & \tau_*\mathcal{C}(\Omega X, \widetilde{\Omega} X) & \xleftarrow{\simeq} & \mathrm{ho}_2(\mathcal{C})(\Omega X, \Omega X) \\
\varepsilon_X & \longmapsto & \varepsilon_X r_{\widetilde{\Sigma}\Omega X} & \longmapsto & q_{\widetilde{\Omega} X} & \longleftarrow\!\shortmid & \mathrm{id}_{\Omega X}
\end{array}$$

where $\widetilde{\Sigma}\Omega X \xrightarrow{r_{\widetilde{\Sigma}\Omega X}} \mathcal{R}\widetilde{\Sigma}\Omega X = \Sigma\Omega X$ denotes the component of the fibrant replacement and $\Omega X = \mathcal{Q}\widetilde{\Omega} X \xrightarrow{q_{\widetilde{\Omega} X}} \widetilde{\Omega} X$ denotes the component of the cofibrant replacement. This gives rise to a commutative diagram of simplicial sets

$$\begin{tikzcd}
 & & \Delta^0 \arrow[dll, "\varepsilon_X"'] \arrow[drr, "\mathrm{id}_{\Omega X}"] & & \\
\mathcal{C}(\Sigma\Omega X, X) \arrow[r, "\simeq"] \arrow[dr, "(\sigma_{\Omega X})^*"'] & \mathcal{C}(\widetilde{\Sigma}\Omega X, X) \arrow[r, "\cong"] \arrow[d] & \widetilde{\Omega}\mathcal{C}(\Omega X, X) \arrow[d] & \mathcal{C}(\Omega X, \widetilde{\Omega} X) \arrow[l, "\cong"'] \arrow[d] & \mathcal{C}(\Omega X, \Omega X) \arrow[l, "\simeq"'] \arrow[dl, "(\omega_X)_*"] \\
 & \mathcal{C}(\Delta^1 \odot \Omega X, X) \arrow[r, "\cong"] & \Delta^1 \pitchfork \mathcal{C}(\Omega X, X) & \mathcal{C}(\Omega X, \Delta^1 \pitchfork \Omega X) \arrow[l, "\cong"'] &
\end{tikzcd}$$

where the vertical morphism is the upper morphism in the pullback

$$\begin{tikzcd}
\widetilde{\Omega}\mathcal{C}(\Omega X, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^1 \pitchfork \mathcal{C}(\Omega X, X) \arrow[d] \\
\Delta^0 \arrow[r] & \mathcal{C}(\Omega X, X)
\end{tikzcd}$$

of pointed simplicial sets. Applying τ to the pentagon above proves the second pasting identity.

The first pasting identity is analogous. □

2 An Interlude on 2-Categories

In this chapter we briefly recall some standard definitions and facts from 2-category theory. We do so with the intend to fix notation and terminology for the following chapters. Moreover we prove some facts on products and pseudomonoids, which likely are folklore but hard to find in the literature.

2.1 Strict and Weak 2-Constructions

This thesis only uses the notion of *strict* 2-categories in the sense of Cat-enriched categories. We will not speak about *weak* 2-categories, commonly called bicategories. Hence, as opposed to the [nlab], we will always mean a *strict* 2-category, when we speak of a 2-category.

In 2-category theory many definitions admit variations, where strict identities are replaced by identities up to invertible natural transformations. First examples are functors and natural transformations.

Definition 2.1.1

A **weak 2-functor** (commonly called **pseudofunctor**) between 2-categories $\mathcal{C}$ and $\mathcal{D}$ consists of

- a function $\mathcal{F} : \operatorname{Ob}\mathcal{C} \longrightarrow \operatorname{Ob}\mathcal{D}$
- for any pair of objects X, Y in $\mathcal{C}$ a functor $\mathcal{F} := \mathcal{F}_{XY} : \mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY)$
- for any object X of $\mathcal{C}$ an invertible 2-cell $\phi_X : \mathrm{id}_{\mathcal{F}X} \Rightarrow \mathcal{F}(\mathrm{id}_X)$ in $\mathcal{D}$
- for any triple of objects X, Y, Z in $\mathcal{C}$ and any pair of morphisms $f : X \to Y$, $g : Y \to Z$ an invertible 2-cell $\phi_{gf} : \mathcal{F}g \circ \mathcal{F}f \Rightarrow \mathcal{F}(g \circ f)$.

satisfying the following coherence axioms.

- The ϕ_{gf} are natural in f and g, that is for every pair of 2-cells $\alpha : f \Rightarrow f'$ and $\beta : g \Rightarrow g'$ the following pasting identity holds.

$$\begin{tikzcd}
 & \mathcal{F}Y \arrow[dr, bend left, "\mathcal{F}g"] \arrow[d, phantom, "\Downarrow \phi_{gf}"] & \\
\mathcal{F}X \arrow[ur, bend left, "\mathcal{F}f"] \arrow[rr, "F(gf)" description] \arrow[rr, bend right=60, "\mathcal{F}(g'f')"'] \arrow[rr, phantom, bend right=30, "\Downarrow \mathcal{F}(\beta\alpha)"] & {} & \mathcal{F}Z
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{F}X \arrow[r, bend left, "\mathcal{F}f"] \arrow[r, bend right, "\mathcal{F}f'" description] \arrow[r, phantom, "\Downarrow \mathcal{F}\alpha"] \arrow[rr, bend right=60, "\mathcal{F}(g'f')"'] \arrow[rr, phantom, bend right=35, "\Downarrow \phi_{g'f'}"] & \mathcal{F}Y \arrow[r, bend left, "\mathcal{F}g"] \arrow[r, bend right, "\mathcal{F}g'" description] \arrow[r, phantom, "\Downarrow \mathcal{F}\beta"] & \mathcal{F}Z
\end{tikzcd}$$

- For every morphism f there are pasting identities

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, bend left, equal] \arrow[r, bend right, "\mathcal{F}\,\mathrm{id}_X" description] \arrow[r, phantom, "\Downarrow \phi_X"] \arrow[rr, bend right=60, "\mathcal{F}f"'] \arrow[rr, phantom, bend right=35, "\Downarrow \phi_{f\,\mathrm{id}}"] & \mathcal{F}X \arrow[r, "\mathcal{F}f"] & \mathcal{F}Y
\end{tikzcd}
= \mathcal{F}X \xrightarrow{Ff} \mathcal{F}Y$$

and

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\mathcal{F}f"] \arrow[rr, bend right=50, "\mathcal{F}f"'] \arrow[r, phantom, shift right=5ex, "\Downarrow \phi_{\mathrm{id}\,f}"] & \mathcal{F}Y \arrow[r, equal, bend left] \arrow[r, "\mathcal{F}\,\mathrm{id}_Y"'] \arrow[r, phantom, shift left=1.5ex, "\Downarrow \phi_X"] & \mathcal{F}Y & = & \mathcal{F}X \arrow[r, "\mathcal{F}f"] & \mathcal{F}Y .
\end{tikzcd}$$

- For any triple of morphisms $f : W \to X$, $g : X \to Y$, $h : Y \to Z$ the following pasting identity holds.

$$\begin{tikzcd}
\mathcal{F}W \arrow[r, "\mathcal{F}f"] \arrow[rr, bend right, "\mathcal{F}(gf)"'] \arrow[rrr, bend right=60, "\mathcal{F}(hgf)"'] \arrow[r, phantom, shift right=3ex, "\Downarrow \phi_{gf}"] & \mathcal{F}X \arrow[r, "\mathcal{F}g"] & \mathcal{F}Y \arrow[r, "\mathcal{F}h"] \arrow[l, phantom, shift left=6ex, "\Downarrow \phi_{h(gf)}"] & \mathcal{F}Z & = & \mathcal{F}W \arrow[r, "\mathcal{F}f"] \arrow[rrr, bend right=60, "\mathcal{F}(hgf)"'] \arrow[r, phantom, shift right=5ex, "\Downarrow \phi_{(hg)f}"] & \mathcal{F}X \arrow[r, "\mathcal{F}g"] \arrow[rr, bend right, "\mathcal{F}(hg)"'] & \mathcal{F}Y \arrow[r, "\mathcal{F}h"] \arrow[r, phantom, shift right=3ex, "\Downarrow \phi_{hg}"] & \mathcal{F}Z
\end{tikzcd}$$

A weak 2-functor is **normalized** if the 2-cells ϕ_X are identities.

A **strict 2-functor** is a weak 2-functor such that all ϕ_X and all ϕ_{gf} are identities. In other words it is a Cat-enriched functor.

Definition 2.1.2

Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \longrightarrow \mathcal{D}$ be two weak 2-functors between strict 2-categories.

A **weakly 2-natural transformation** (usually called **pseudonatural transformation**) consists of

- for any object X of $\mathcal{C}$ a morphism $\alpha_X : \mathcal{F}X \to \mathcal{G}X$
- for any morphism $f : X \to Y$ in $\mathcal{C}$ an invertible 2-cell as depicted in the diagram[4]

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\alpha_X"] \arrow[d, "\mathcal{F}f"'] \arrow[dr, phantom, "\Nearrow \alpha_f"] & \mathcal{G}X \arrow[d, "\mathcal{G}f"] \\
\mathcal{F}Y \arrow[r, "\alpha_Y"'] & \mathcal{G}Y
\end{tikzcd}$$

such that for any 2-cell $\phi : f \Rightarrow g : X \to Y$ in $\mathcal{C}$ the following pasting identity holds.

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, "\alpha_X"] \arrow[d, bend right=50, "\mathcal{F}f"'] \arrow[d, bend left=50, "\mathcal{F}g"] \arrow[d, phantom, "\overset{\mathcal{F}\phi}{\Rightarrow}"] & \mathcal{G}X \arrow[d, "\mathcal{G}g"] \arrow[dl, phantom, "\Nearrow \alpha_g"] & = & \mathcal{F}X \arrow[r, "\alpha_X"] \arrow[d, "\mathcal{F}f"'] \arrow[dr, phantom, "\alpha_f \Nearrow"] & \mathcal{G}X \arrow[d, bend right=50, "\mathcal{G}f"'] \arrow[d, bend left=50, "\mathcal{G}g"] \arrow[d, phantom, "\overset{\mathcal{G}\phi}{\Rightarrow}"] \\
\mathcal{F}Y \arrow[r, "\alpha_Y"'] & \mathcal{G}Y & & \mathcal{F}Y \arrow[r, "\alpha_Y"'] & \mathcal{G}Y
\end{tikzcd}$$

A **strictly 2-natural transformation** is a weakly 2-natural transformation such that all cells α_f are identities.

We chose to use the more lengthy *weak 2-functor* and *weakly 2-natural* as opposed to the frequently used *pseudofunctor* and *pseudonatural*, because there is a difference between *pseudo 2-limits* and

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[4] By accident the author chose to use the oplax direction for pseudonatural transformations and wants to apologize for any confusion this might cause.

weak 2-limits in the literature. To avoid possible confusion we hence only use the bywords *strict* and *weak*.

The advent of 2-cells allows us to define morphisms of natural transformations.

Definition 2.1.3

Let $\alpha, \beta : \mathcal{F} \Rightarrow \mathcal{G} : \mathcal{C} \to \mathcal{D}$ be two weakly 2-natural transformations between weak 2-functors. A **modification** $\Theta : \alpha \Rrightarrow \beta$ consists of 2-cells $\Theta_X : \alpha_X \Rightarrow \beta_X : \mathcal{F}X \to \mathcal{G}X$ in $\mathcal{D}$ for every object X of $\mathcal{C}$, such that for every 2-cell $\phi : f \Rightarrow g : X \to Y$ in $\mathcal{C}$ the following pasting identity holds.

$$\begin{tikzcd}
\mathcal{F}X \arrow[r, bend left, "\beta_X"] \arrow[d, bend right, "\mathcal{F}f"'] \arrow[d, bend left, "\mathcal{F}g"] \arrow[d, phantom, "\overset{\mathcal{F}\phi}{\Rightarrow}"] \arrow[dr, phantom, "\Nearrow \beta_g"] & \mathcal{G}X \arrow[d, bend left, "\mathcal{G}g"] \\
\mathcal{F}Y \arrow[r, bend left, "\beta_Y"] \arrow[r, bend right, "\alpha_Y"'] \arrow[r, phantom, "\Uparrow \Theta_Y"] & \mathcal{G}Y
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{F}X \arrow[r, bend left, "\beta_X"] \arrow[r, bend right, "\alpha_X"'] \arrow[r, phantom, "\Uparrow \Theta_X"] \arrow[d, bend right, "\mathcal{F}f"'] \arrow[dr, phantom, "\Nearrow \alpha_f"] & \mathcal{G}X \arrow[d, bend right, "\mathcal{G}f"'] \arrow[d, bend left, "\mathcal{G}g"] \arrow[d, phantom, "\overset{\mathcal{G}\phi}{\Rightarrow}"] \\
\mathcal{F}Y \arrow[r, bend right, "\alpha_Y"'] & \mathcal{G}Y
\end{tikzcd}$$

Example 2.1.4

Given a small 2-category $\mathcal{C}$ and a 2-category $\mathcal{D}$, strict 2-functors, strictly 2-natural transformations and modifications between them define a 2-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Similarly the weak 2-functors, weak 2-natural transformations and modifications between them define a 2-category $\mathrm{Fun}_w(\mathcal{C}, \mathcal{D})$.

Definition 2.1.5

A strict 2-functor $\mathcal{L} : \mathcal{C} \longrightarrow \mathcal{D}$ is **strictly left 2-adjoint** to a strict 2-functor $\mathcal{R} : \mathcal{D} \longrightarrow \mathcal{C}$, if there is an isomorphism

$$\mathcal{D}(\mathcal{L}X, Y) \cong \mathcal{C}(X, \mathcal{R}Y)$$

strictly 2-natural in X and Y. The strict 2-functor $\mathcal{L}$ is **weakly left 2-adjoint** to $\mathcal{R}$, if there is an equivalence

$$\mathcal{D}(\mathcal{L}X, Y) \simeq \mathcal{C}(X, \mathcal{R}Y)$$

weakly 2-natural in X and Y.

A strict 2-functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is a **strict 2-equivalence**, if it is essentially surjective on objects and induces an isomorphism of Hom-categories. It is a **weak 2-equivalence**, if it is weakly essentially surjective on objects and induces an equivalence of Hom-categories.

Remark 2.1.6

Strict / weak 2-adjunctions can be rephrased in terms of strict / weak unit and counit transformations and triangle identities, which in the weak case only have to hold up to an invertible modification.

Strict / weak 2-equivalences admit a strict / weak inverse 2-functor, which can be turned into a strict / weak 2-adjoint pair.

In the following sections we will need some constructions of 2-limits. They are commonly defined as a sort of weighted limit, however the resulting notion heavily depends on which functor category is used (strict/weak 2-functors and strict/weak 2-natural transformations) and leads to many distinct notions of 2-limits. To circumvent this pitfall we only give a slightly non-standard definition of strict and weak 2-limits taylormade for the examples in this thesis.

Definition 2.1.7

Let $\mathcal{C}$ be a 2-category. For a given 2-diagram $\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{C}$ fix a strict 2-functor $\mathrm{Cone}_{\mathcal{C}}(-, \mathcal{D}) : \mathcal{C}^{op} \longrightarrow \mathsf{Cat}$. A cone $\lambda \in \mathrm{Cone}_{\mathcal{C}}(L, \mathcal{D})$ gives rise to a strictly 2-natural transformation $\lambda_* : \mathcal{C}(-, L) \Longrightarrow \mathrm{Cone}_{\mathcal{C}}(-, \mathcal{D})$. We say that λ is a **strict 2-limit** of the diagram $\mathcal{D}$, if λ_* is an isomorphism in $\mathrm{Fun}(\mathcal{C}^{op}, \mathsf{Cat})$. It is a **weak 2-limit**, if λ_* is an equivalence in $\mathrm{Fun}(\mathcal{C}^{op}, \mathsf{Cat})$.

The definition of strict and weak 2-colimits are dual.

Since the various comparison functors λ_* are 2-natural in T, strict / weak 2-limits are unique up to isomorphism / equivalence in $\mathcal{C}$. Similarly strict / weak 2-colimits are unique up to isomorphism / equivalence.

We take the time to explicitly spell out the universal properties of some examples on which later sections heavily rely. Often we will discuss the strict versions of the 2-limits and the weak versions of the 2-colimits. For the following examples we fix a 2-category $\mathcal{C}$.

Example 2.1.8

Consider the empty diagram 2-category $\mathcal{I} = \emptyset$ and the unique diagram $\mathcal{D} : \emptyset \longrightarrow \mathcal{C}$. Take the constant 2-functor on the *free object* $\mathbb{1} \in \mathsf{Cat}$ as the 2-functor $\mathrm{Cone}_{\mathcal{C}}(-, \mathcal{D})$.

A **strictly terminal object** 1 is defined as the strict 2-limit of this diagram. For every object T in $\mathcal{C}$ the canonical comparison functor is the unique functor $\mathcal{C}(T, 1) \longrightarrow \mathbb{1}$. Requiring all of them to be isomorphisms of categories amounts to saying that for every object T there is a unique morphism $! : T \to 1$ and <u>no</u> nontrivial endo-2-cells.

A **weakly initial object** 0 is the weak 2-colimit of the diagram $\mathcal{D}$. The canonical comparison functor for an object T can be identified as the unique functor $\mathcal{C}(0, T) \longrightarrow \mathbb{1}$. Such a functor is an equivalence if and only if it is surjective on objects ($\mathbb{1}$ only has a single object) and full and faithful. Explicitly this means that for any T there exists at least one morphism $0 \to T$, which is unique up to a unique invertible 2-cell.

Example 2.1.9

We consider as diagram category the discrete category $\mathcal{I} = \mathbb{1} + \mathbb{1}$ on two objects. A diagram $\mathcal{I} \longrightarrow \mathcal{C}$ amounts to choosing two objects X and Y in $\mathcal{C}$. Take as Cone-functor the 2-functor $T \mapsto \mathcal{C}(T, X) \times \mathcal{C}(T, Y)$.

The strict 2-limit of this data defines the **strict 2-product** $X \times Y$ of X and Y. The universal cone λ amounts to two morphisms $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ in $\mathcal{C}$. With this notation the canonical comparison functor for a given object T can be identified as the functor

$$\mathcal{C}(T, X \times Y) \xrightarrow{((\mathrm{pr}_X)_*, (\mathrm{pr}_Y)_*)} \mathcal{C}(T, X) \times \mathcal{C}(T, Y).$$

Requiring this to be an isomorphism of categories means that it ought to be bijective on objects and full and faithful. Bijectivity on objects states that for every pair of morphisms $x : T \to X$ and $y : T \to Y$ there is a unique morphism $t : T \to X \times Y$ making the diagram

$$\begin{tikzcd} & T \arrow[dl, "x"'] \arrow[d, "t"] \arrow[dr, "y"] & \\ X & X \times Y \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & Y \end{tikzcd}$$

commute. Meanwhile full and faithfulness amounts to saying that for any pair of 2-cells

$$\begin{tikzcd}
& T \arrow[dl, bend right, "x"'] \arrow[dl, bend left, "x'"] \arrow[dr, bend left, "y"] \arrow[dr, bend right, "y'"'] \arrow[dl, phantom, "\alpha \Searrow"] \arrow[dr, phantom, "\Swarrow \beta"] & \\
X & & Y
\end{tikzcd}$$

there is a unique 2-cell τ with the property that

$$\begin{tikzcd}
& T \arrow[dl, bend right, "x"'] \arrow[dl, bend left, "x'"] \arrow[dl, phantom, "\alpha \Searrow"] \\
X &
\end{tikzcd}
=
\begin{tikzcd}
& & T \arrow[dl, bend right, "t"'] \arrow[dl, bend left, "t'"] \arrow[dl, phantom, "\tau \Searrow"] \arrow[ddll, bend right=40, "x"'] \arrow[ddll, bend left=40, "x'"] \\
& X \times Y \arrow[dl, "\mathrm{pr}_X"'] & \\
X & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
T \arrow[dr, bend left, "t"] \arrow[dr, bend right, "t'"'] \arrow[dr, phantom, "\Swarrow \tau"] \arrow[ddrr, bend left=40, "y"] \arrow[ddrr, bend right=40, "y'"'] & & \\
& X \times Y \arrow[dr, "\mathrm{pr}_Y"] & \\
& & Y
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[dr, bend left, "y"] \arrow[dr, bend right, "y'"'] \arrow[dr, phantom, "\Swarrow \beta"] & \\
& Y
\end{tikzcd}
.$$

Example 2.1.10

We take the same diagram category $\mathcal{I} = \mathbb{1} + \mathbb{1}$ and diagram $\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{C}$ corresponding to objects X and Y. As cone-functor we choose the 2-functor $T \mapsto \mathcal{C}(X, T) \times \mathcal{C}(Y, T)$.

The weak 2-colimit of this data defines the **weak 2-coproduct** $X + Y$ of X and Y. The universal cocone γ amounts to two morphisms $\mathrm{incl}_X : X \to X + Y$ and $\mathrm{incl}_Y : Y \to X + Y$ in $\mathcal{C}$. With these at hand we can identify the canonical comparison functor for a given object T as the functor

$$\mathcal{C}(X + Y, T) \xrightarrow{(\mathrm{incl}_X^*, \mathrm{incl}_Y^*)} \mathcal{C}(X, T) \times \mathcal{C}(Y, T).$$

To say that this is an equivalence of categories is to say that it is essentially surjective on objects and full and faithful. Essential surjectivity means that for every pair of morphisms $x : X \to T$ and $y : Y \to T$ in $\mathcal{C}$ there exist some (not necessarily unique) morphism t and invertible 2-cells χ and Υ as depicted in the diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{incl}_X"] \arrow[dr, "x"'] & X + Y \arrow[d, "t"] \arrow[dl, phantom, "\Swarrow \chi"] \arrow[dr, phantom, "\Upsilon \Searrow"] & Y \arrow[l, "\mathrm{incl}_Y"'] \arrow[dl, "y"] \\
& T &
\end{tikzcd}
.$$

Full and faithfulness amounts to saying that for any pair of 2-cells of the form

$$\begin{tikzcd}
X \arrow[dr, bend left, "t\,\mathrm{incl}_X"] \arrow[dr, bend right, "t'\,\mathrm{incl}_X"'] \arrow[dr, phantom, "\Swarrow \alpha"] & & Y \arrow[dl, bend right, "t\,\mathrm{incl}_Y"'] \arrow[dl, bend left, "t'\,\mathrm{incl}_Y"] \arrow[dl, phantom, "\beta \Searrow"] \\
& T &
\end{tikzcd}$$

there is a unique 2-cell τ with the property that

$$\begin{tikzcd}
X \arrow[dr, bend left, "t\,\mathrm{incl}_X"] \arrow[dr, bend right, "t'\,\mathrm{incl}_X"'] \arrow[dr, phantom, "\Swarrow\alpha"] & \\
& T
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[dr, "\mathrm{incl}_X"] & \\
& X+Y \arrow[dr, bend left, "t"] \arrow[dr, bend right, "t'"'] \arrow[dr, phantom, "\Swarrow\tau"] & \\
& & T
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
& & Y \arrow[dl, "\mathrm{incl}_Y"'] \\
& X+Y \arrow[dl, bend right, "t"'] \arrow[dl, bend left, "t'"] \arrow[dl, phantom, "\tau\Searrow"] & \\
T & &
\end{tikzcd}
=
\begin{tikzcd}
& Y \arrow[dl, bend right, "t\,\mathrm{incl}_Y"'] \arrow[dl, bend left, "t'\,\mathrm{incl}_Y"] \arrow[dl, phantom, "\beta\Searrow"] \\
T &
\end{tikzcd}
.$$

In a later section we will need to use the functoriality of 2-products. The following proposition takes care of that.

Proposition 2.1.11

Let $\mathcal{C}$ be a 2-category, which admits a strict 2-product $X \xleftarrow{\mathrm{pr}_X} X \times Y \xrightarrow{\mathrm{pr}_Y} Y$ of every pair of objects X and Y. Fixing the datum of a product for each pair of objects amounts to specifying a strict 2-functor

$$\begin{array}{rcl}
\mathcal{C}\times\mathcal{C} & \xrightarrow{-\times-} & \mathcal{C} \\
(X,Y) & \longmapsto & X\times Y \\
(f,g) & \longmapsto & f\times g \\
(\alpha,\beta) & \longmapsto & \alpha\times\beta
\end{array}$$

Furthermore fixing an object Y turns the projections $\mathrm{pr}_X : X\times Y \longrightarrow X$ into a strictly 2-natural transformation $\mathrm{pr}_1 : -\times Y \Longrightarrow \mathrm{id}_{\mathcal{C}}$.

Finally the unique diagonal morphisms $\Delta_X : X \longrightarrow X\times X$ given by the universal property assemble into a strictly 2-natural transformation $\Delta_\bullet : \mathrm{id}_{\mathcal{C}} \Longrightarrow (-\times -)\circ\Delta$, where

$$\begin{array}{rcl}
\mathcal{C} & \xrightarrow{\Delta} & \mathcal{C}\times\mathcal{C} \\
X & \longmapsto & (X,X)
\end{array}$$

is the strictly 2-functorial diagonal functor.

Sketch Dualize and spezialize the proof of the following proposition. ◊

Proposition 2.1.12

Let $\mathcal{C}$ be a 2-category which admits binary weak 2-coproducts of every pair of objects.

Fixing the data of

- a coproduct object $X \xrightarrow{\mathrm{incl}_X} X+Y \xleftarrow{\mathrm{incl}_Y} Y$ for every pair of objects X and Y
- a factorization

$$\begin{tikzcd}
X \arrow[r, "\mathrm{incl}_X"] \arrow[dr, "x"'] \arrow[r, phantom, shift right=4ex, "\Swarrow\chi_{(x,y)}"] & X+Y \arrow[d, "{[x,y]}"] & Y \arrow[l, "\mathrm{incl}_Y"'] \arrow[dl, "y"] \arrow[l, phantom, shift right=4ex, "\Upsilon_{(x,y)}\Searrow"] \\
& T &
\end{tikzcd}
.$$

  for every object T and any pair of morphism $x : X \longrightarrow T$ and $y : Y \longrightarrow T$.

specifies a weak 2-functor

$$\begin{array}{rcl} \mathcal{C}\times\mathcal{C} & \xrightarrow{-+-} & \mathcal{C} \\ (X,Y) & \longmapsto & X+Y \\ (f,g) & \longmapsto & f+g \\ (\alpha,\beta) & \longmapsto & \alpha+\beta \end{array}$$

whose comparison 2-cells are induced from the universal property. Furthermore fixing an object Y turns the inclusions $\mathrm{incl}_X : X \longrightarrow X+Y$ together with the chosen factorization 2-cells χ into a weakly 2-natural transformation $\mathrm{incl}_1 : \mathrm{id}_{\mathcal{C}} \Longrightarrow -+Y$. When fixing the object X the analogue statement holds true.

Finally the fixed codiagonal morphisms $\nabla_X = [\mathrm{incl}_1, \mathrm{incl}_2] : X+X \to X$ together with their chosen factorizations assemble into a weakly 2-natural transformation $\nabla_- : (-+-)\circ\Delta \Longrightarrow \mathrm{id}_{\mathcal{C}}$, where Δ denotes the strictly 2-functorial diagonal functor.

Proof The action on a pair of morphisms $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ is given by the chosen factorization

$$\begin{tikzcd}
X \arrow[r, "\mathrm{incl}_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow\chi_{(f,g)}"] & X+Y \arrow[d, dashed, "f+g"] \arrow[dr, phantom, "\Upsilon_{(f,g)}\Searrow"] & Y \arrow[l, "\mathrm{incl}_Y"'] \arrow[d, "g"] \\
X' \arrow[r, "\mathrm{incl}_{X'}"'] & X'+Y' & Y' \arrow[l, "\mathrm{incl}_{Y'}"]
\end{tikzcd}$$

where we abuse notation to abbreviate $\chi_{(\mathrm{incl}_{X'}\circ f, \mathrm{incl}_{Y'}\circ g)}$ to $\chi_{(f,g)}$.

Given 2-cells $\alpha : f \Rightarrow f' : X \to X'$ and $\beta : g \Rightarrow g' : Y \to Y'$ postcomposition with $\mathrm{incl}_{X'}$ respectively $\mathrm{incl}_{Y'}$ gives rise to 2-cells

$$\begin{tikzcd}
X \arrow[dr, bend left, "\mathrm{incl}_{X'}\circ f"] \arrow[dr, bend right, "\mathrm{incl}_{X'}\circ f'"'] \arrow[dr, phantom, "\mathrm{incl}_{X'}\circ\alpha\Swarrow"] & & Y' \arrow[dl, bend right, "\mathrm{incl}_{Y'}\circ g"'] \arrow[dl, bend left, "\mathrm{incl}_{Y'}\circ g'"] \arrow[dl, phantom, "\Searrow\mathrm{incl}_{X'}\circ\beta"] \\
& X'+Y' &
\end{tikzcd}$$

which by the extended universal property uniquely determines a 2-cell $\alpha+\beta : f+g \Longrightarrow f'+g'$. The compatibilities this 2-cell satisfies give rise to the claimed weak 2-naturality of the inclusion maps.

The unit comparison 2-cell $\phi_{X,Y}$ is the unique invertible 2-cell $\phi_{X,Y} : \mathrm{id}_{X+Y} \Rightarrow \mathrm{id}_X + \mathrm{id}_Y$ induced from comparing the canonical factorization

$$\begin{tikzcd}
X \arrow[r, "\mathrm{incl}_X"] \arrow[d, equal] & X+Y \arrow[d, equal] & Y \arrow[l, "\mathrm{incl}_Y"'] \arrow[d, equal] \\
X \arrow[r, "\mathrm{incl}_X"'] & X+Y & Y \arrow[l, "\mathrm{incl}_Y"]
\end{tikzcd}$$

and the chosen one.

For composable morphisms $X \xrightarrow{f} X' \xrightarrow{f'} X''$ and $Y \xrightarrow{g} Y' \xrightarrow{g'} Y''$ the composition comparison 2-cell $\phi_{(f'f,g'g)} : (f'+g')\circ(f+g) \Rightarrow (f'f+g'g)$ is the unique invertible 2-cells induced by the invertible 2-cells

$$\begin{tikzcd}
X \arrow[rrr, "\mathrm{incl}_X"] \arrow[dr, "f"'] \arrow[ddd, "\mathrm{incl}_X"'] & & & X+Y \arrow[d, "f+g"] \arrow[dll, phantom, "\swarrow\chi_{(f,g)}"] \\
& X' \arrow[rr, "\mathrm{incl}_{X'}"] \arrow[dr, "f'"'] & & X'+Y' \arrow[dd, "f'+g'"] \arrow[dl, phantom, "\swarrow\chi_{(f',g')}"] \\
\swarrow\chi^{-1}_{(f'f,g'g)} & & X'' \arrow[dr, "\mathrm{incl}_{X''}"'] & \\
X+Y \arrow[rrr, "f'f+g'g"'] & & & X''+Y''
\end{tikzcd}$$

and

$$\begin{tikzcd}
X+Y \arrow[d, "f+g"'] \arrow[drr, phantom, "\Upsilon_{(f,g)}\searrow"] & & & Y \arrow[lll, "\mathrm{incl}_Y"'] \arrow[dl, "g"] \arrow[ddd, "\mathrm{incl}_Y"] \\
X'+Y' \arrow[dd, "f'+g'"'] \arrow[dr, phantom, "\Upsilon_{(f',g')}\searrow"] & & Y' \arrow[ll, "\mathrm{incl}_{Y'}"'] \arrow[dl, "g'"] & \\
& Y'' \arrow[dl, "\mathrm{incl}_{Y''}"] & & \Upsilon^{-1}_{(f'f,g'g)}\searrow \\
X''+Y'' & & & X+Y \arrow[lll, "f'f+g'g"]
\end{tikzcd} .$$

The required axioms of a weak 2-functor then follow from the uniqueness of the 2-cell in the universal property. For example, given triples of composable morphisms $X \xrightarrow{f} X' \xrightarrow{f'} X'' \xrightarrow{f''} X'''$ and $Y \xrightarrow{g} Y' \xrightarrow{g'} Y'' \xrightarrow{g''} Y'''$ the associativity axiom immediately follows from the observation, that when precomposing the desired identity with incl_X both sides reduce to the 2-cell

$$\begin{tikzcd}
X \arrow[rrrr, "\mathrm{incl}_X"] \arrow[dr, "f"'] \arrow[dddd, "\mathrm{incl}_X"'] & & & & X+Y \arrow[d, "f+g"] \arrow[dlll, phantom, "\swarrow\chi_{(f,g)}"] \\
& X' \arrow[rrr, "\mathrm{incl}_{X'}"] \arrow[dr, "f'"'] & & & X'+Y' \arrow[d, "f'+g'"] \arrow[dll, phantom, "\swarrow\chi_{(f',g')}"] \\
& & X'' \arrow[rr, "\mathrm{incl}_{X''}"] \arrow[dr, "f''"'] & & X''+Y'' \arrow[dd, "f''+g''"] \arrow[dl, phantom, "\swarrow\chi_{(f'',g'')}"] \\
& \swarrow\chi^{-1}_{(f''f'f,g''g'g)} & & X''' \arrow[dr, "\mathrm{incl}_{X'''}"'] & \\
X+Y \arrow[rrrr, "f''f'f+g''g'g"'] & & & & X'''+Y'''
\end{tikzcd}$$

and that the analogous statement holds when precomposing with incl_Y.

It is left to show the weak 2-naturality of the codiagonals ∇_{-}. For a morphism $f : X \to X'$ the two factorizations

$$\begin{tikzcd}
X \arrow[r, "\mathrm{incl}_1"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\swarrow\chi_{\mathrm{id},\mathrm{id}}"] & X+X \arrow[d, "\nabla"] & X \arrow[l, "\mathrm{incl}_2"'] \arrow[dl, equal, bend left] \arrow[dl, phantom, "\Upsilon_{\mathrm{id},\mathrm{id}}\searrow"'] \\
& X \arrow[d, "f"] & \\
& X' &
\end{tikzcd}$$

and

$$\begin{tikzcd}
X \arrow[r, "\mathrm{incl}_1"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow\chi_{f,f}"] & X+X \arrow[d, "f+f"] & X \arrow[l, "\mathrm{incl}_2"'] \arrow[d, "f"] \arrow[dl, phantom, "\Upsilon_{f,f}\searrow"'] \\
X' \arrow[r, "\mathrm{incl}_1"] \arrow[dr, equal, bend right] \arrow[dr, phantom, "\swarrow\chi_{\mathrm{id},\mathrm{id}}"] & X'+X' \arrow[d, "\nabla"] & X' \arrow[l, "\mathrm{incl}_2"'] \arrow[dl, equal, bend left] \arrow[dl, phantom, "\Upsilon_{\mathrm{id},\mathrm{id}}\searrow"'] \\
& X' &
\end{tikzcd}$$

give rise to a unique invertible 2-cell

$$\begin{tikzcd}
X+X \arrow[r, "\nabla"] \arrow[d, "f+f"'] \arrow[dr, phantom, "\nearrow\phi_f"] & X \arrow[d, "f"] \\
X'+X' \arrow[r, "\nabla"'] & X'
\end{tikzcd}$$

relating the two. The 2-dimensional universal property of $X + X$ ensures that a 2-cell $\alpha : f \Rightarrow f'$ gives rise to the desired pasting identity

$$\begin{tikzcd}
X+X \arrow[r, "\nabla"] \arrow[d, bend right=60, "f+f"'] \arrow[d, bend left=60, "f'+f'"] \arrow[d, phantom, "\overset{\alpha+\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\nearrow \phi_{f'}"] & X \arrow[d, "f'"] \\
X'+X' \arrow[r, "\nabla"'] & X'
\end{tikzcd}
=
\begin{tikzcd}
X+X \arrow[r, "\nabla"] \arrow[d, "f+f"'] \arrow[dr, phantom, "\phi_f \nearrow"] & X \arrow[d, bend right=60, "f"'] \arrow[d, bend left=60, "f'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
X'+X' \arrow[r, "\nabla"'] & X'
\end{tikzcd}$$

since both sides satisfy the property of the 2-cell induced by the pair of 2-cells (α, α). $\square$

Our final example is concerned with a 2-categorification of a pullback.

Example 2.1.13

Take as diagram category the cospan category $\mathcal{I} = 1 \to 0 \leftarrow 2$. A diagram $\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{C}$ picks a cospan $Y \xrightarrow{f} X \xleftarrow{g} Z$ in $\mathcal{C}$. The cone-functor evaluates an object T at the category $\mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D})$ explicitly described at the beginning of section 1.3 in the special case of a Grpd-category. The 2-functoriality is given by precomposing morphisms and 2-cells. This 2-functor can also be explicitly described as the 2-functor $T \mapsto \mathrm{Fun}_w(\mathcal{I}, \mathsf{Cat})(\mathcal{W}, \mathcal{C}(T, \mathcal{D}-))$, where $\mathcal{W} = \mathbb{1} \xrightarrow{\mathrm{incl}_0} \mathbb{2} \xleftarrow{\mathrm{incl}_1} \mathbb{1}$.

A **weak 2-comma** is a weak 2-limit of this data. If $\mathcal{C}$ is a Grpd-category, as in the aforementioned section 1.3 the universal cone consists of a 2-commutative square with invertible cell, which satisfies the same universal properties as a quasi-comma with the additional requirement, that the 2-cell induced by the 2-dimensional universal property is unique.

We close this section by describing two ways in which weak 2-commas can arise. For that purpose the following remark is essential.

Remark 2.1.14

Consider a square ϕ as depicted on the right, in which the 2-cell ϕ is invertible and the morphisms f and f' are equivalences. Denote the inverse adjoint equivalences by g and g' with respective units $\eta : \mathrm{id} \Rightarrow gf$ and $\eta' : \mathrm{id} \Rightarrow g'f'$ and counits $\varepsilon : fg \Rightarrow \mathrm{id}$ and $\varepsilon' : f'g' \Rightarrow \mathrm{id}$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "\nearrow \phi"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

Then the invertible 2-cell

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "y"'] \arrow[dr, phantom, "\nearrow \psi"] & X \arrow[d, "x"] \\
Y' \arrow[r, "g'"'] & X'
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, equal] \arrow[dd, bend right=60, "y"'] & X \arrow[dl, "f"] \arrow[d, "x"] \arrow[dd, bend left=60, "x"] \arrow[dl, phantom, shift right=4, "\Rightarrow \varepsilon^{-1}"] \\
Y \arrow[d, "y"] \arrow[r, phantom, "\Rightarrow \phi^{-1}"] & X' \arrow[dl, "f'"'] \arrow[d, equal] \arrow[dl, phantom, shift left=4, "\Rightarrow \eta'^{-1}"] \\
Y' \arrow[r, "g'"'] & X'
\end{tikzcd}$$

satisfies the pasting identity

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] \arrow[rr, bend left=50, equal, "\Uparrow \eta^{-1}"] \arrow[dr, phantom, "\Nearrow \phi"] & Y \arrow[r, "g"] \arrow[d, "y"] \arrow[dr, phantom, "\Nearrow \psi"] & X \arrow[d, "x"] \\
X' \arrow[r, "f'"'] \arrow[rr, bend right=50, equal, "\Uparrow \eta'"'] & Y' \arrow[r, "g'"'] & X'
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "x"'] & X \arrow[d, "x"] \\
X' \arrow[r, equal] & X'
\end{tikzcd} .$$

Lemma 2.1.15

Suppose $\mathcal{C}$ is a Grpd-category. Let ϕ be a square as in the preceding remark 2.1.14. Then ϕ is a weak comma square and a weak cocomma square.

Proof We prove the universal property of a weak comma, the one of a weak cocomma follows dually.

Let $\tau \in \mathrm{Cone}_{\mathcal{C}}(T, \mathcal{D})$ be a square. By the dual version of the preceding remark 2.1.14 there is a square ψ' giving rise to the pasting identity

$$\begin{tikzcd}
T \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\Nearrow \tau"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\Nearrow \tau"] & Y \arrow[r, "g"] \arrow[d, "y"] \arrow[rr, bend left=50, equal, "\Uparrow \varepsilon"] \arrow[dr, phantom, "\Nearrow \psi'"] & X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "\Nearrow \phi"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] \arrow[rr, bend right=50, equal, "\Uparrow \varepsilon'^{-1}"'] & X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

$$=
\begin{tikzcd}
T \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\Nearrow \tau"] & Y \arrow[r, "g"] \arrow[d, "y"] \arrow[rr, bend left=50, equal, "\Uparrow \varepsilon"] \arrow[dr, phantom, "\Nearrow \psi'"] & X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "\Nearrow \phi"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"] \arrow[rr, bend right=50, equal, "\Uparrow \eta'"'] & Y' \arrow[r, "g'"] & X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

where the second identity follows from the triangle axioms for $f' \dashv g'$. This provides a factorization for the square τ and thus proves the 1-dimensional universal property.

Regarding the 2-dimensional universal property we assume that

$$\begin{tikzcd}
T \arrow[drr, bend left, "t"] \arrow[dr, "p"] \arrow[ddr, bend right, "s"'] & & \\
 & X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "\Nearrow \phi"] & Y \arrow[d, "y"] \\
 & X' \arrow[r, "f'"'] & Y'
\end{tikzcd}
\quad \Nearrow \Upsilon, \ \Nearrow \chi
=
\begin{tikzcd}
T \arrow[drr, bend left, "t"] \arrow[dr, "p"] \arrow[ddr, bend right, "s"'] & & \\
 & X \arrow[r, "f"] \arrow[d, "x"] \arrow[dr, phantom, "\Nearrow \phi"] & Y \arrow[d, "y"] \\
 & X' \arrow[r, "f'"'] & Y'
\end{tikzcd}
\quad \Nearrow \Upsilon', \ \Nearrow \chi'$$

are two given factorizations. Pasting the square ψ from remark 2.1.14 and the unit η' gives rise to the pasting identity

$$\begin{tikzcd}
T \arrow[drrr, bend left, "t"] \arrow[dr, "p"] \arrow[dddrr, bend right, "s"'] & & \Nearrow \Upsilon & \\
& X \arrow[rr, "f"] \arrow[dr, equal] & \Nearrow \eta & Y \arrow[dl, "g"] \\
& \Nearrow \chi & X \arrow[d, "x"] & \\
& & X' &
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[drrr, bend left, "t"] \arrow[dr, "p'"] \arrow[dddrr, bend right, "s"'] & & \Nearrow \Upsilon' & \\
& X \arrow[rr, "f"] \arrow[dr, equal] & \Nearrow \eta & Y \arrow[dl, "g"] \\
& \Nearrow \chi' & X \arrow[d, "x"] & \\
& & X' &
\end{tikzcd}
.$$

Bringing Υ' and η to the other side we obtain an invertible 2-cell

$$\begin{tikzcd}
T \arrow[r, "p'"] \arrow[d, "p"'] \arrow[dr, "t" description] \arrow[dr, phantom, shift left=4, "\Nearrow \Upsilon'^{-1}"] \arrow[dr, phantom, shift right=4, "\Nearrow \Upsilon"'] & X \arrow[d, "f"] \arrow[ddr, equal, bend left] & \\
X \arrow[r, "f"] \arrow[drr, equal, bend right] & Y \arrow[dr, "g"] & \Nearrow \eta^{-1} \\
& \Nearrow \eta & X
\end{tikzcd}$$

which satisfies the desired pasting identity with respect to χ and χ'. From the pasting identity

$$\begin{tikzcd}
T \arrow[r, "p'"] \arrow[d, "p"'] \arrow[dr, "t" description] \arrow[dr, phantom, shift left=4, "\Nearrow \Upsilon'^{-1}"] \arrow[dr, phantom, shift right=4, "\Nearrow \Upsilon"'] \arrow[ddrrr, bend left=60] & X \arrow[d, "f"] \arrow[ddr, equal, bend left] & & \\
X \arrow[r, "f"] \arrow[drr, equal, bend right] & Y \arrow[dr, "g"] & \Nearrow \eta^{-1} & \Nearrow \Upsilon' \\
& \Nearrow \eta & X \arrow[r, "f"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
T \arrow[r, "p'"] \arrow[d, "p"'] \arrow[dr, "t" description] \arrow[dr, phantom, shift left=4, "\Nearrow \Upsilon'^{-1}"] \arrow[dr, phantom, shift right=4, "\Nearrow \Upsilon"'] \arrow[ddrrr, bend left=60] & X \arrow[d, "f"] & & \\
X \arrow[r, "f"] \arrow[drr, equal, bend right] & Y \arrow[dr, "g"] \arrow[drr, equal, bend left] & & \Nearrow \Upsilon' \\
& \Nearrow \eta & X \arrow[r, "f"'] \arrow[ur, phantom, "\Nearrow \varepsilon"] & Y
\end{tikzcd}$$

the required pasting identity with respect to Υ and Υ' follows as well.

The 2-cell is unique, since whiskering with the equivalence f is faithful. This shows the 2-dimensional universal property. □

Lemma 2.1.16

Suppose $\mathcal{C}$ is a Grpd-category with a weakly terminal object 1. Then a span $X \xleftarrow{p} P \xrightarrow{q} Y$ constitutes a weak 2-product of X and Y in $\mathcal{C}$ if and only if the unique square depicted on the right is a weak comma.

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\Nearrow"] & Y \arrow[d] \\ X \arrow[r] & 1 \end{tikzcd}$$

Proof Let $\mathcal{D}$ be the diagram $X \to 1 \leftarrow Y$. By the universal property of the terminal object for any object T the functor

$$\begin{array}{ccc} \mathcal{C}(T,X) \times \mathcal{C}(T,Y) & \longrightarrow & \mathrm{Cone}_{\mathcal{C}}(T,\mathcal{D}) \\ (f,g) & \longmapsto & \begin{tikzcd} T \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\Nearrow"] & Y \arrow[d] \\ X \arrow[r] & 1 \end{tikzcd} \\ (\alpha,\beta) & \longmapsto & (\alpha,\beta) \end{array}$$

constitutes an isomorphism of categories. Any choice of a pair of morphisms (f,g) gives rise to a commutative triangle of the form

$$\begin{tikzcd} & \mathcal{C}(T,X) \times \mathcal{C}(T,Y) \arrow[d, "\cong"] \\ \mathcal{C}(T,L) \arrow[ur, "{(p,q)}"] \arrow[r] & \mathrm{Cone}_{\mathcal{C}}(T,\mathcal{D}) \end{tikzcd} ,$$

in which the upper functor is an equivalence if and only if the lower functor is. $\square$

2.2 Symmetric Pseudomonoids

In this section we briefly recall some facts about pseudomonoids, a 2-categorical generalization of monoid objects. They can be defined in any 2-monoidal 2-category, but we will only need and discuss them in a 2-cartesian monoidal 2-category $\mathcal{C}$. To make the following diagrams more readable, we chose to denote the monoidal unit (i.e. the terminal object) of the 2-cartesian monoidal structure by I.

Definition 2.2.1

A **pseudomonoid** in $\mathcal{C}$ is an object M together with a unit morphism $u : I \longrightarrow M$, a multiplication morphism $m : M \times M \longrightarrow M$ as well as invertible 2-cells λ, ρ, α as depicted in the following diagram

$$\begin{tikzcd}
I \times M \arrow[r, "u \times M"] \arrow[dr, "\ell"'] & M \times M \arrow[d, "m"] \arrow[dl, phantom, "\swarrow \lambda"] \arrow[dr, phantom, "\searrow \rho"] & M \times I \arrow[l, "M \times u"'] \arrow[dl, "r"] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
M^{\times 3} \arrow[r, "M \times m"] \arrow[d, "m \times M"'] \arrow[dr, phantom, "\nearrow \alpha"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"'] & M
\end{tikzcd}$$

subject to the following axioms.

Triangle Axiom

$$\begin{tikzcd}
M \times I \times M \arrow[d, "M \times u \times M"'] \arrow[dr, bend left, "M \times \ell"] \arrow[dr, phantom, "\nearrow M \times \lambda"] & \\
M^{\times 3} \arrow[r, "M \times m"] \arrow[d, "m \times M"'] \arrow[dr, phantom, "\nearrow \alpha"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"'] & M
\end{tikzcd}
=
\begin{tikzcd}
M \times I \times M \arrow[d, "M \times u \times M"'] \arrow[dr, bend left, "r \times M"] \arrow[dr, phantom, "\nearrow \rho \times M"] & \\
M^{\times 3} \arrow[r, "m \times M"'] & M \times M \arrow[d, "m"] \\
& M
\end{tikzcd}$$

Pentagon Axiom

$$\begin{tikzcd}
M^{\times 4} \arrow[rr, "M \times M \times m"] \arrow[dr, "M \times m \times M"] \arrow[dd, "m \times M \times M"'] & & M^{\times 3} \arrow[dr, "M \times m"] \arrow[d, phantom, "\Rightarrow M \times \alpha"] & \\
& M^{\times 3} \arrow[rr, "M \times m"'] \arrow[dd, "m \times M"] \arrow[dl, phantom, "\alpha \times M \Rightarrow"] \arrow[ddrr, phantom, "\Rightarrow \alpha"] & & M \times M \arrow[dd, "m"] \\
M^{\times 3} \arrow[dr, "m \times M"'] & & & \\
& M \times M \arrow[rr, "m"'] & & M
\end{tikzcd}
=
\begin{tikzcd}
M^{\times 4} \arrow[rr, "M \times M \times m"] \arrow[dd, "m \times M \times M"'] & & M^{\times 3} \arrow[dr, "M \times m"] \arrow[dd, "m \times M"'] & \\
& & & M \times M \arrow[dd, "m"] \arrow[dl, phantom, "\Rightarrow \alpha"] \\
M^{\times 3} \arrow[rr, "M \times m"] \arrow[dr, "m \times M"'] \arrow[drr, phantom, "\Rightarrow \alpha"] & & M \times M \arrow[dr, "m"] & \\
& M \times M \arrow[rr, "m"'] & & M
\end{tikzcd}$$

Pseudomonoids in the 2-category of categories are precisely the monoidal categories. On the one hand this motivates the study of pseudomonoids, since in many instances pseudomonoids in a 2-category of categories (like 2-categories or abelian categories) give the right notion of a monoidal category. On the other hand we can readily generalize statements about monoidal categories to pseudomonoids. At least in the case of pseudomonoids in a cartesian monoidal 2-category this is a formal application of the 2-Yoneda lemma, as we will justify in remark 2.2.7 and see in detail in the proof of the Eckmann-Hilton argument 2.2.9.

Remark 2.2.2

By an argument of Kelly the pasting identity

$$\begin{tikzcd}
I \times I \arrow[rr, "\cong"] \arrow[d, "u \times I"'] & & I \arrow[dd, "u"] \\
M \times I \arrow[d, "M \times u"'] \arrow[rrd, bend left, "\ell"] \arrow[r, phantom, "\Nearrow \lambda", shift right=4ex] & & \\
M \times M \arrow[rr, "m"'] & & M
\end{tikzcd}
=
\begin{tikzcd}
I \times I \arrow[rr, "\cong"] \arrow[d, "I \times u"'] & & I \arrow[dd, "u"] \\
I \times M \arrow[d, "u \times M"'] \arrow[rrd, bend left, "r"] \arrow[r, phantom, "\Nearrow \rho", shift right=4ex] & & \\
M \times M \arrow[rr, "m"'] & & M
\end{tikzcd}$$

holds. Furthermore the pasting identities

$$\begin{tikzcd}
I \times M \times M \arrow[d, "u \times M \times M"'] \arrow[rd, bend left, "l \times M"] \arrow[r, phantom, "\Nearrow \lambda \times M", shift right=4ex] & \\
M^{\times 3} \arrow[r, "M \times m"] \arrow[d, "m \times M"'] \arrow[rd, phantom, "\Nearrow \alpha"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"'] & M
\end{tikzcd}
=
\begin{tikzcd}
I \times M \times M \arrow[d, "I \times m"'] \arrow[rd, bend left, "\ell"] & \\
I \times M \arrow[d, "u \times M"'] \arrow[rd, bend left, "\ell"] \arrow[r, phantom, "\Nearrow \lambda", shift right=4ex] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"'] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \times M \times I \arrow[d, "M \times M \times u"'] \arrow[rd, bend left, "M \times r"] \arrow[r, phantom, "\Nearrow M \times \rho", shift right=4ex] & \\
M^{\times 3} \arrow[r, "M \times m"] \arrow[d, "m \times M"'] \arrow[rd, phantom, "\Nearrow \alpha"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"'] & M
\end{tikzcd}
=
\begin{tikzcd}
N \times M \times I \arrow[d, "m \times I"'] \arrow[rd, bend left, "r"] & \\
M \times I \arrow[d, "M \times u"'] \arrow[rd, bend left, "r"] \arrow[r, phantom, "\Nearrow \rho", shift right=4ex] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"'] & M
\end{tikzcd}$$

are satisfied. The proof in the particular case of monoidal categories is given in [JS93] proposition 1.1.

Often monoidal categories have a commutative tensor product in the sense that there is a natural braiding or symmetry isomorphism. The following definition is a straightforward generalization. We use cycle-notation to denote permutation isomorphisms.

Definition 2.2.3

A **braided pseudomonoid** in $\mathcal{C}$ is a pseudomonoid $(M, u, m, \lambda, \rho, \alpha)$ in $\mathcal{C}$ together with an invertible 2-cell

$$\begin{tikzcd}
M \times M \arrow[rr, "(12)"] \arrow[rd, "m"'] & \Rightarrow \beta & M \times M \arrow[ld, "m"] \\
& M &
\end{tikzcd}$$

which satisfies the following axioms.

Hexagon Axioms

$$\begin{tikzcd}
 & M^{\times 3} \arrow[rr, "(132)"] \arrow[dl, "m \times M"'] \arrow[d, "M \times m"] & & M^{\times 3} \arrow[d, "m \times M"'] \arrow[dr, "M \times m"] & \\
M \times M \arrow[drr, "m"'] \arrow[r, phantom, "\Rightarrow \alpha"] & M \times M \arrow[rr, "(12)"] \arrow[dr, "m"] & \Rightarrow \beta & M \times M \arrow[dl, "m"'] \arrow[r, phantom, "\Rightarrow \alpha"] & M \times M \arrow[dll, "m"] \\
 & & M & &
\end{tikzcd}
=
\begin{tikzcd}
M^{\times 3} \arrow[rr, "(132)"] \arrow[dr, "(12)"] \arrow[dd, bend right, "m \times M"'] & & M^{\times 3} \arrow[dd, bend left, "M \times m"] \\
\overset{\beta \times M}{\Rightarrow} & M^{\times 3} \arrow[ur, "(23)"] \arrow[dl, "m \times M"] \arrow[dr, "M \times m"'] & \overset{M \times \beta}{\Rightarrow} \\
M \times M \arrow[dr, "m"'] & \Rightarrow \alpha & M \times M \arrow[dl, "m"] \\
 & M &
\end{tikzcd}$$

$$\begin{tikzcd}
 & M^{\times 3} \arrow[rr, "(123)"] \arrow[dl, "M \times m"'] \arrow[d, "m \times M"] & & M^{\times 3} \arrow[d, "M \times m"'] \arrow[dr, "m \times M"] & \\
M \times M \arrow[drr, "m"'] \arrow[r, phantom, "\Rightarrow \alpha^{-1}"] & M \times M \arrow[rr, "(12)"] \arrow[dr, "m"] & \Rightarrow \beta & M \times M \arrow[dl, "m"'] \arrow[r, phantom, "\Rightarrow \alpha^{-1}"] & M \times M \arrow[dll, "m"] \\
 & & M & &
\end{tikzcd}
=
\begin{tikzcd}
M^{\times 3} \arrow[rr, "(123)"] \arrow[dr, "(23)"] \arrow[dd, bend right, "M \times m"'] & & M^{\times 3} \arrow[dd, bend left, "m \times M"] \\
\overset{M \times \beta}{\Rightarrow} & M^{\times 3} \arrow[ur, "(12)"] \arrow[dl, "M \times m"] \arrow[dr, "m \times M"'] & \overset{\beta \times M}{\Rightarrow} \\
M \times M \arrow[dr, "m"'] & \Rightarrow \alpha^{-1} & M \times M \arrow[dl, "m"] \\
 & M &
\end{tikzcd}$$

A braided pseudomonoid is a **symmetric pseudomonoid**, if it additionally satisfies the following axiom. In this case the braiding β is called a **symmetry** and will be denoted by σ instead.

Symmetry Axiom

$$\begin{tikzcd}
M \times M \arrow[rr, "(12)"] \arrow[ddrr, "m"'] & & M \times M \arrow[rr, "(12)"] \arrow[dd, "m"] & & M \times M \arrow[ddll, "m"] \\
 & \Rightarrow \sigma & & \Rightarrow \sigma & \\
 & & M & &
\end{tikzcd}
=
\begin{tikzcd}
M \times M \arrow[rr, equal] \arrow[dr, "m"'] & & M \times M \arrow[dl, "m"] \\
 & M &
\end{tikzcd}$$

Remark 2.2.4

If σ is a symmetry, then one of the hexagon axioms is redundant by noting that $\sigma^{-1} = \sigma$.

Furthermore, as is shown in [JS93] proposition 1.2 in the case of braided monoidal categories, the pasting identities

$$\begin{tikzcd}
I \times M \arrow[rr, "(12)"] \arrow[d, "u \times M"'] & & M \times I \arrow[d, "M \times u"] \\
M \times M \arrow[rr, "(12)"] \arrow[dr, "m"'] & \Rightarrow \beta & M \times M \arrow[dl, "m"] \\
 & M &
\end{tikzcd}
=
\begin{tikzcd}
I \times M \arrow[rr, "(12)"] \arrow[d, "u \times M"'] \arrow[ddr, bend left, "\ell"] & & M \times I \arrow[d, "M \times u"] \arrow[ddl, bend right, "r"'] \\
M \times M \arrow[dr, "m"'] \arrow[r, phantom, "\Rightarrow \lambda"] & & M \times M \arrow[dl, "m"] \arrow[l, phantom, "\Rightarrow \rho^{-1}"'] \\
 & M &
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \times I \arrow[rr, "(12)"] \arrow[d, "M \times u"'] & & I \times M \arrow[d, "u \times M"] & & M \times I \arrow[rr, "(12)"] \arrow[d, "M \times u"'] \arrow[ddr, bend left, "r"] & & I \times M \arrow[d, "u \times M"] \arrow[ddl, bend right, "\ell"'] \\
M \times M \arrow[rr, "(12)"] \arrow[dr, "m"'] & \Rightarrow \beta & M \times M \arrow[dl, "m"] & = & M \times M \arrow[dr, "m"'] & \Rightarrow \rho \qquad \Rightarrow \lambda^{-1} & M \times M \arrow[dl, "m"] \\
& M & & & & M &
\end{tikzcd}$$

are true for a braided pseudomonoid in $\mathcal{C}$.

Definition 2.2.5
A **pseudomorphism** of pseudomonoids $(M, u, m, \lambda, \rho, \alpha)$ and $(\widetilde{M}, \widetilde{u}, \widetilde{m}, \widetilde{\lambda}, \widetilde{\rho}, \widetilde{\alpha})$ in $\mathcal{C}$ consists of a morphism $f : M \to \widetilde{M}$ together with invertible 2-cells

$$\begin{tikzcd}
& I \arrow[dl, "u"'] \arrow[dr, "\widetilde{u}"] & & & M \times M \arrow[rr, "f \times f"] \arrow[d, "m"'] & \Nearrow \phi & \widetilde{M} \times \widetilde{M} \arrow[d, "\widetilde{m}"] \\
M \arrow[rr, "f"'] & \Rightarrow \phi_0 & \widetilde{M} & & M \arrow[rr, "f"'] & & \widetilde{M}
\end{tikzcd}$$

satisfying the following coherence axioms.

$$\begin{tikzcd}
I \times M \arrow[r, "I \times f"] \arrow[d, "u \times M"] \arrow[dd, bend right=60, "\ell"', "\lambda^{-1} \Rightarrow"] \arrow[dr, phantom, "\phi_0 \times f \ \Nearrow"] & I \times \widetilde{M} \arrow[d, "\widetilde{u} \times \widetilde{M}"] & & I \times M \arrow[r, "I \times f"] \arrow[dd, "\ell"'] & I \times \widetilde{M} \arrow[d, "\widetilde{u} \times \widetilde{M}"] \arrow[dd, bend right=60, "\ell"', "\widetilde{\lambda}^{-1} \Rightarrow"] \\
M \times M \arrow[r, "f \times f"] \arrow[d, "m"] \arrow[dr, phantom, "\Nearrow \phi"] & \widetilde{M} \times \widetilde{M} \arrow[d, "\widetilde{m}"] & = & & \widetilde{M} \times \widetilde{M} \arrow[d, "\widetilde{m}"] \\
M \arrow[r, "f"'] & \widetilde{M} & & M \arrow[r, "f"'] & \widetilde{M}
\end{tikzcd}$$

$$\begin{tikzcd}
M \times I \arrow[r, "f \times I"] \arrow[d, "M \times u"'] \arrow[dr, phantom, "f \times \phi_0 \ \Nearrow"] & \widetilde{M} \times I \arrow[d, "\widetilde{M} \times \widetilde{u}"] \arrow[dd, bend left=60, "r", "\overset{\widetilde{\rho}}{\Rightarrow}"'] & & M \times I \arrow[r, "f \times I"] \arrow[d, "M \times u"'] \arrow[dd, bend left=60, "r", "\overset{\rho}{\Rightarrow}"'] & \widetilde{M} \times I \arrow[dd, "r"] \\
M \times M \arrow[r, "f \times f"] \arrow[d, "m"'] \arrow[dr, phantom, "\Nearrow \phi"] & \widetilde{M} \times \widetilde{M} \arrow[d, "\widetilde{M}"] & = & M \times M \arrow[d, "m"'] & \\
M \arrow[r, "f"'] & \widetilde{M} & & M \arrow[r, "f"'] & \widetilde{M}
\end{tikzcd}$$

$$\begin{tikzcd}
M^{\times 3} \arrow[rr, "f^{\times 3}"] \arrow[dr, "M\times m"] \arrow[dd, "m\times M"'] & \arrow[r, phantom, "\Nearrow f\times\phi"] & \widetilde{M}^{\times 3} \arrow[dr, "\widetilde{M}\times\widetilde{m}"] & \\
& M\times M \arrow[rr, "f\times f"] \arrow[dd, "m"] & & \widetilde{M}\times\widetilde{M} \arrow[dd, "\widetilde{m}"] \\
M\times M \arrow[dr, "m"'] \arrow[r, phantom, "\overset{\alpha}{\Rightarrow}"] & & \Nearrow\phi & \\
& M \arrow[rr, "f"'] & & \widetilde{M}
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
M^{\times 3} \arrow[rr, "f^{\times 3}"] \arrow[dd, "m\times M"'] & & \widetilde{M}^{\times 3} \arrow[dr, "\widetilde{M}\times\widetilde{m}"] \arrow[dd, "\widetilde{m}\times\widetilde{M}"'] & \\
& \Nearrow\phi\times f & & \widetilde{M}\times\widetilde{M} \arrow[dd, "\widetilde{m}"] \\
M\times M \arrow[rr, "f\times f"] \arrow[dr, "m"'] & & \widetilde{M}\times\widetilde{M} \arrow[dr, "\widetilde{m}"] \arrow[ur, phantom, "\overset{\widetilde{\alpha}}{\Rightarrow}"] & \\
& M \arrow[rr, "f"'] & \Nearrow\phi & \widetilde{M}
\end{tikzcd}$$

If M and $\widetilde{M}$ are braided pseudomonoids with respective braidings β and $\widetilde{\beta}$ a **pseudomorphism of braided pseudomonoids** is a pseudomorphism (f, ϕ_0, ϕ), which additionally satisfies the pasting identity

$$\begin{tikzcd}
M\times M \arrow[rr, "f\times f"] \arrow[dr, "(12)"] \arrow[ddr, bend right, "m"'] & & \widetilde{M}\times\widetilde{M} \arrow[dr, "(12)"] & \\
\overset{\beta}{\Rightarrow} & M\times M \arrow[rr, "f\times f"] \arrow[d, "m"] & & \widetilde{M}\times\widetilde{M} \arrow[d, "\widetilde{m}"] \\
& M \arrow[rr, "f"'] & \Nearrow\phi & \widetilde{M}
\end{tikzcd}
=
\begin{tikzcd}
M\times M \arrow[rr, "f\times f"] \arrow[ddr, bend right, "m"'] & & \widetilde{M}\times\widetilde{M} \arrow[dr, "(12)"] \arrow[ddr, bend right, "\widetilde{m}"'] & \\
& \Nearrow\phi & \overset{\widetilde{\beta}}{\Rightarrow} & \widetilde{M}\times\widetilde{M} \arrow[d, "\widetilde{m}"] \\
& M \arrow[rr, "f"'] & & \widetilde{M}
\end{tikzcd}\ .$$

Definition 2.2.6

A **transformation** between two pseudomorphisms (f, ϕ_0, ϕ) and (g, ψ_0, ψ) of pseudomonoids M and $\widetilde{M}$ (braided or not) is a 2-cell $\chi : f \Rightarrow g$ satisfying

$$\begin{tikzcd}
M\times M \arrow[r, bend left, "g\times g"] \arrow[r, bend right, "f\times f"'] \arrow[r, phantom, "\Uparrow\chi\times\chi"] \arrow[d, "m"'] & \widetilde{M}\times\widetilde{M} \arrow[d, "\widetilde{m}"] \\
M \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Nearrow\phi"] & \widetilde{M}
\end{tikzcd}
=
\begin{tikzcd}
M\times M \arrow[r, bend left, "g\times g"] \arrow[d, "m"'] & \widetilde{M}\times\widetilde{M} \arrow[d, "\widetilde{m}"] \\
M \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow\chi"] \arrow[ur, phantom, "\Nearrow\psi"] & \widetilde{M}
\end{tikzcd}$$

as well as

$$\begin{tikzcd}
& I \arrow[dl, "u"'] \arrow[dr, "\widetilde{u}"] \arrow[d, phantom, "\Nearrow\psi_0"] & \\
M \arrow[rr, bend left, "g"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\Uparrow\chi"] & & M
\end{tikzcd}
=
\begin{tikzcd}
& I \arrow[dl, "u"'] \arrow[dr, "\widetilde{u}"] \arrow[d, phantom, "\Nearrow\phi_0"] & \\
M \arrow[rr, bend right, "f"'] & & \widetilde{M}
\end{tikzcd}\ .$$

If χ is invertible, we call it an **isotransformation**.

Remark 2.2.7

With all these definitions at hand we obtain a 2-category PsMon($\mathcal{C}$) of pseudomonoids, pseudomorphisms and transformations in $\mathcal{C}$. It has a (in general not full) sub-2-category BPsMon($\mathcal{C}$) consisting of braided pseudomonoids, morphisms of braided pseudomonoids and transformations. In the latter there is a full sub-2-category SPsMon($\mathcal{C}$) consisting of the symmetric pseudomonoids.

A cartesian monoidal 2-functor $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ between 2-categories with finite strict 2-products restricts to 2-functors as indicated in the diagram

$$\begin{tikzcd}
\mathrm{SPsMon}(\mathcal{C}) \arrow[r, dashed] \arrow[d] & \mathrm{SPsMon}(\mathcal{D}) \arrow[d] \\
\mathrm{BPsMon}(\mathcal{C}) \arrow[r, dashed] \arrow[d] & \mathrm{BPsMon}(\mathcal{D}) \arrow[d] \\
\mathrm{PsMon}(\mathcal{C}) \arrow[r, dashed] \arrow[d] & \mathrm{PsMon}(\mathcal{D}) \arrow[d] \\
\mathcal{C} \arrow[r, "\mathcal{F}"'] & \mathcal{D}
\end{tikzcd}$$

which are full and faithful, if $\mathcal{F}$ is.

Furthermore, for any 2-category $\mathcal{I}$ and a 2-category $\mathcal{C}$ with finite strict 2-products the category Fun($\mathcal{I}, \mathcal{C}$) has finite strict 2-products, which are computed objectwise. Hence there is a canonical isomorphism of 2-categories PsMon(Fun($\mathcal{I}, \mathcal{C}$)) $\cong$ Fun($\mathcal{I}$, PsMon($\mathcal{C}$)), which further restricts to BPsMon(Fun($\mathcal{I}, \mathcal{C}$)) $\cong$ Fun($\mathcal{I}$, BPsMon($\mathcal{C}$)) and SPsMon(Fun($\mathcal{I}, \mathcal{C}$)) $\cong$ Fun($\mathcal{I}$, SPsMon($\mathcal{C}$)).

Applying these facts to the 2-Yoneda embedding justifies the claim, that statements about monoidal categories generalize to pseudomonoids in a 2-category with finite products.

Remark 2.2.8

If $(M, m, u, \lambda, \rho, \alpha)$ and $(N, n, v, \lambda', \rho', \alpha')$ are pseudomonoids in $\mathcal{C}$, then the product $M \times N$ has a canonical structure of a pseudomonoid with multiplication and unit of the form

$$M \times N \times M \times N \xrightarrow{(23)} M^2 \times N^2 \xrightarrow{m \times n} X \times Y \qquad \text{and} \qquad I \xrightarrow{(u,v)} M \times N$$

and coherence 2-cells given by

$$\begin{tikzcd}
I \times M \times N \arrow[r, "\cong"] \arrow[dr] \arrow[ddrr, bend right, "\ell"'] & I \times I \times M \times N \arrow[r, "u \times v \times M \times N"] \arrow[d, "(23)"'] & M \times N \times M \times N \arrow[d, "(23)"] \\
 & I \times M \times I \times N \arrow[r, "u \times M \times v \times N"] \arrow[dr, "\ell \times \ell"'] \arrow[r, phantom, shift right=6ex, "\Swarrow \lambda \times \lambda'"] & M^2 \times N^2 \arrow[d, "m \times n"] \\
 & & M \times N
\end{tikzcd}$$

$$\begin{tikzcd}
M\times N\times M\times N \arrow[d, "(23)"'] & M\times N\times I\times I \arrow[l, "M\times N\times u\times v"'] \arrow[d, "(23)"] & M\times N\times I \arrow[l, "\cong"'] \arrow[dl] \arrow[ddll, bend left, "r"] \\
M^2\times N^2 \arrow[d, "m\times n"'] \arrow[dr, phantom, "\Searrow \rho\times\rho'"] & M\times I\times N\times I \arrow[l, "M\times u\times N\times v"'] \arrow[dl, "r\times r"] & \\
M\times N & &
\end{tikzcd}$$

$$\begin{tikzcd}
(M\times N)^3 \arrow[r, "(45)"] \arrow[d, "(23)"'] \arrow[dr, "(2453)"] & M\times N\times M^2\times N^2 \arrow[r, "M\times N\times m\times n"] \arrow[d, "(243)"'] & M\times N\times M\times N \arrow[d, "(23)"] \\
M^2\times N^2\times M\times N \arrow[r, "(345)"] \arrow[d, "m\times n\times M\times N"'] & M^3\times N^3 \arrow[r, "M\times m\times N\times n"] \arrow[d, "m\times M\times n\times N"'] \arrow[dr, phantom, "\Nearrow \alpha\times\alpha'"] & M^2\times N^2 \arrow[d, "m\times n"] \\
M\times N\times M\times N \arrow[r, "(23)"'] & M^2\times N^2 \arrow[r, "m\times n"'] & M\times N
\end{tikzcd}$$

If both M and N are braided with respective braidings β and β', then

$$\begin{tikzcd}
M\times N\times M\times N \arrow[r, "(13)(24)"] \arrow[d, "(23)"'] & M\times N\times M\times N \arrow[d, "(23)"] \\
M^2\times N^2 \arrow[r, "(12)(34)"] \arrow[dr, "m\times n"'] \arrow[dr, phantom, shift left=4ex, "\Rightarrow \beta\times\beta'"] & M^2\times N^2 \arrow[d, "m\times n"] \\
& M\times N
\end{tikzcd}$$

defines a braiding for the product $M\times N$.

Lemma 2.2.9 (Eckmann Hilton Argument)
Let $(M, m, u, \lambda, \rho, \alpha)$ be a pseudomonoid in $\mathcal{C}$. Suppose we are given a diagram

$$\begin{tikzcd}
M \arrow[r, "u\times M"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3ex, "\Swarrow\phi"] & M\times M \arrow[d, "n"] & M \arrow[l, "M\times u"'] \arrow[dl, equal] \arrow[dl, phantom, shift right=3ex, "\psi\Searrow"] \\
& M &
\end{tikzcd}$$

in the 2-category $\mathrm{PsMon}(\mathcal{C})$ of pseudomonoids in $\mathcal{C}$. Then there is an invertible 2-cell β, which turns $(M, m, u, \lambda, \rho, \alpha, \beta)$ into a braided pseudomonoid. Furthermore, there are invertible 2-cells α' and β' making $(M, n, u, \phi, \psi, \alpha', \beta')$ a braided pseudomonoid together with an isomorphism

$$(M, m, u, \lambda, \rho, \alpha, \beta) \cong (M, n, u, \phi, \psi, \alpha', \beta')$$

in $\mathrm{BPsMon}(\mathcal{C})$.

Finally, if $(M, m, u, \lambda, \rho, \alpha, \beta)$ is a braided pseudomonoid and n is a morphism of braided pseudomonoids, then β and β' are symmetries.

Proof For every object X of $\mathcal{C}$ the strict 2-functor $\mathcal{C}(X,-) : \mathcal{C} \longrightarrow \mathsf{Cat}$ is cartesian monoidal, hence preserves pseudomonoids. Since $\mathrm{PsMon}(\mathsf{Cat}) \cong \mathsf{MonCat}$ the statements follow in MonCat by [JS93] proposition 5.2, 5.3 and 5.4 as well as remark 5.1. We note that by the uniqueness of the induced braiding this construction is 2-natural in X. This means that the statements above hold for the 2-functors in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{MonCat})$. By full and faithfullness of the the functor

$$\mathrm{PsMon}(\mathcal{C}) \longrightarrow \mathrm{PsMon}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Cat})) \cong \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{MonCat})$$

induced from the 2-Yoneda embedding, they thus hold in $\mathcal{C}$. □

2.3 Semi-Weak 2-Biproducts

In this section we want to study a 2-categorical analogue of the 1-categorical notion of biproducts: products, which coincide with coproducts. It is unfortunate that the word biproduct has different meanings in 1- and 2-category theory. Since we did not adopt the Australian *bicategorical* terminology for weak 2-categorical notions, we use the prefix *bi-* in the 1-categorical sense (for limits coinciding with colimits).

In [Dup08] 2-categorical biproducts have been studied in the setting of Grpd-categories. One important consequence of their existence in a Grpd-category $\mathcal{C}$ is that they provide a canonical enrichment of $\mathcal{C}$ in symmetric monoidal groupoids.

The present and following sections are devoted to fill in some details omitted in [Dup08]. We need to do so because we will need the explicit description of this enrichment in the following chapter. We will elaborate on it in the next section, while this section serves a preparatory purpose. We provide the definition of 2-biproducts, explain why we can assume them to be strictly 2-functorial and finish with fixing conventions.

To even start talk about 2-biproducts we will need a 2-categorical analogue of a zero object.

Definition 2.3.1

Let $\mathcal{C}$ be a 2-category and assume that it admits both a weakly initial object 0 and a weakly terminal object 1.

We say that $\mathcal{C}$ has a **weak zero-object**, if it satisfies one of the following equivalent conditions.

(i) Any (hence all) of the canonical morphisms $0 \to 1$ is an equivalence in $\mathcal{C}$.

(ii) The weakly initial object 0 is a weakly terminal object.

(iii) The weakly terminal object 1 is a weakly initial object.

(iv) There is an object Z, which is both weakly initial and weakly terminal.

Proof The equivalence of these conditions follow immediately from the universal properties of weakly initial and weakly terminal objects. ☐

Remark 2.3.2

In the situation of the preceding definition, if the category $\mathcal{C}$ has a strictly terminal object and a weak zero-object, then the terminal object is itself a weak zero object, which satisfies the strictly terminal and weakly initial universal property. We refer to this specific choice of a zero object as a **semi-weak zero object**.

We would like to have a similar description for what we will call weak 2-biproducts. More specifically, we would like to have a canonical comparison morphisms from coproduct to product. In analogy to the 1-categorical case there is an essentially unique morphism induced from the universal property of the coproduct and product. We will prove that the collection of these canonical morphisms give rise to a weakly 2-natural transformation. However, to do so we will need the following two compatibilities of coproducts with zero objects.

Lemma 2.3.3
Let $\mathcal{C}$ be a 2-category with a weak zero object and binary weak 2-coproducts. Then for any morphism $f : X \to T$ and any object Y there is an invertible 2-cell

$$\begin{tikzcd}
X+Y \arrow[dd, "{[f,0]}"'] \arrow[dr, "{[\mathrm{id},0]}"] & \\
\arrow[r, phantom, "\overset{\psi_f}{\Rightarrow}"] & X \arrow[dl, "f"] \\
T &
\end{tikzcd}$$

and for any 2-cell $\alpha : f \Rightarrow f' : X \to T$ these 2-cells satisfy the pasting identity

$$\begin{tikzcd}
X+Y \arrow[dd, bend right, "{[f,0]}"'{name=A}] \arrow[dd, bend left, "{[f',0]}"{name=B}] \arrow[dr, bend left, "{[\mathrm{id},0]}"] & \\
& X \arrow[dl, bend left, "f'"] \\
T &
\arrow[from=A, to=B, phantom, "\overset{[\alpha,0]}{\Rightarrow}"]
\end{tikzcd}
\overset{\psi_{f'}}{\Rightarrow}
=
\begin{tikzcd}
X+Y \arrow[dd, "{[f,0]}"'] \arrow[dr, bend left, "{[\mathrm{id},0]}"] \arrow[dr, phantom, "\overset{\psi_f}{\Rightarrow}"'] & \\
& X \arrow[dl, bend right, "f"'{name=C}] \arrow[dl, bend left, "f'"{name=D}] \\
T &
\arrow[from=C, to=D, phantom, "\Downarrow\alpha"]
\end{tikzcd}
\tag{2.3.3.1}$$

Proof The 2-cell ψ_f is the unique invertible 2-cell obtained from the two given factorizations

$$\begin{tikzcd}
X \arrow[r, "\mathrm{incl}_X"] \arrow[dr, "f"'] & X+Y \arrow[d, "{[f,0]}"] \arrow[dl, phantom, "\swarrow\chi_{f,0}"] \arrow[dr, phantom, "\Upsilon_{f,0}\searrow"] & Y \arrow[l, "\mathrm{incl}_Y"'] \arrow[dl, "0"] \\
& T &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r, "\mathrm{incl}_X"] \arrow[dr, equal] \arrow[ddr, bend right, "f"'] & X+Y \arrow[d, "{[\mathrm{id},0]}"] \arrow[dl, phantom, "\swarrow\chi_{\mathrm{id},0}"] \arrow[dr, phantom, "\Upsilon_{\mathrm{id},0}\searrow"] & Y \arrow[l, "\mathrm{incl}_Y"'] \arrow[dl, "0"] \arrow[ddl, bend left, "0"] \\
& X \arrow[d, "f"] \arrow[r, phantom, "\searrow 0_f"] & {} \\
& T &
\end{tikzcd},$$

where $0_f : f0 \Rightarrow 0$ is the comparison isomorphism given by our fixed choices of zero morphisms.

For a 2-cell $\alpha : f \Rightarrow f' : X \to T$ the desired pasting identity follows from the universal property of $X+Y$. Indeed we find that precomposing it with incl_X gives rise to the pasting identity

$$\begin{tikzcd}
X \arrow[d, "\mathrm{incl}_X"'] \arrow[dr, equal, bend left] \arrow[dr, phantom, "\chi_{\mathrm{id},0}\nearrow"'] & \\
X+Y \arrow[r, "{[\mathrm{id},0]}"] \arrow[r, phantom, "\psi_{f'}\Rightarrow"', shift right=3] \arrow[d, bend right, "{[f,0]}"'{name=A}] \arrow[d, bend left, "{[f',0]}"{name=B}] & X \arrow[dl, bend left, "f'"] \\
T &
\arrow[from=A, to=B, phantom, "\overset{[\alpha,0]}{\Rightarrow}"]
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[dl, bend right, "\mathrm{incl}_X"'] \arrow[dd, bend left, "f'"] \\
X+Y \arrow[dr, "{[f',0]}"] \arrow[dr, bend right, "{[f,0]}"'] \arrow[r, phantom, "\overset{\chi_{f',0}}{\Rightarrow}"] & {} \\
\arrow[r, phantom, "\nearrow{[\alpha,0]}"] & T
\end{tikzcd}
=
\begin{tikzcd}
& X \arrow[dl, bend right, "\mathrm{incl}_X"'] \arrow[dd, bend right, "f"'{name=C}] \arrow[dd, bend left, "f'"{name=D}] \\
X+Y \arrow[dr, bend right, "{[f,0]}"'] \arrow[r, phantom, "\overset{\chi_{f,0}}{\Rightarrow}"] & {} \\
& T
\arrow[from=C, to=D, phantom, "\overset{\alpha}{\Rightarrow}"]
\end{tikzcd}.$$

Similarly precomposing it with incl_Y amounts to the pasting identity

$$\begin{tikzcd}
& Y \arrow[dl, bend right, "\mathrm{incl}_Y"'] \arrow[d, "0"] \arrow[dd, bend left=60, "0"] \arrow[dl, phantom, shift left=3, "\overset{\Upsilon_{\mathrm{id},0}}{\Rightarrow}"] \\
X+Y \arrow[r, "{[\mathrm{id},0]}"] \arrow[dr, "{[f',0]}"] \arrow[dr, bend right=60, "{[f,0]}"'] \arrow[dr, phantom, shift left=3, "\overset{\psi_{f'}}{\Rightarrow}"] \arrow[dr, phantom, shift right=3, "{[\alpha,0]}\Rrightarrow"] & X \arrow[d, "f'"] \arrow[r, phantom, "\overset{0_{f'}}{\Rightarrow}"] & {} \\
& T
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[d, "\mathrm{incl}_Y"'] \arrow[dr, bend left=40, "0"] \arrow[dr, phantom, "\overset{\Upsilon_{f',0}}{\Rightarrow}"] \\
X+Y \arrow[r, bend left=20, "{[f',0]}"] \arrow[r, bend right=40, "{[f,0]}"'] \arrow[r, phantom, "\overset{[\alpha,0]}{\Rrightarrow}"] & T
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[d, "\mathrm{incl}_Y"'] \arrow[dd, bend left=80, "0"] \\
X+Y \arrow[d, "f"'] \arrow[r, phantom, "\overset{\Upsilon_{f,0}}{\Rightarrow}"] & {} \\
T
\end{tikzcd}$$

$$= \begin{tikzcd}
& Y \arrow[dl, bend right, "\mathrm{incl}_Y"'] \arrow[d, "0"] \arrow[dd, bend right=50, "0"'] \arrow[dd, bend left=70, "0"] \\
X+Y \arrow[dr, bend right=40, "{[f,0]}"'] \arrow[r, phantom, "\overset{\Upsilon_{f,0}}{\Rightarrow}\quad\overset{0_f^{-1}}{\Rightarrow}"] & X \arrow[d, bend right, "f"'] \arrow[d, bend left, "f'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[r, phantom, "\overset{0_{f'}}{\Rightarrow}"] & {} \\
& T
\end{tikzcd}
=
\begin{tikzcd}
& Y \arrow[dl, bend right, "\mathrm{incl}_Y"'] \arrow[d, "0"] \arrow[dd, bend left=70, "0"] \arrow[dl, phantom, shift left=3, "\Rightarrow\Upsilon_{\mathrm{id},0}"] \\
X+Y \arrow[r, "{[\mathrm{id},0]}"] \arrow[dr, bend right=40, "{[f,0]}"'] \arrow[dr, phantom, "\overset{\psi_f}{\Rrightarrow}"] & X \arrow[d, bend right, "f"'] \arrow[d, bend left, "f'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \arrow[r, phantom, "\overset{0_{f'}}{\Rightarrow}"] & {} \\
& T
\end{tikzcd} .$$

This shows that the desired identity (2.3.3.1) indeed holds. □

In the same way we can show the following lemma.

Lemma 2.3.4

Let $\mathcal{C}$ be a 2-category with a weak zero object and binary weak 2-coproducts. Then for any pair of morphisms $f : X \to X'$ and $g : Y \to Y'$ there is an invertible 2-cell

$$\begin{tikzcd}
X+Y \arrow[dr, bend left, "{[f,0]}"] \arrow[dd, "f+g"'] \\
& X' \arrow[l, phantom, "\Rrightarrow\phi_{f,g}"] \\
X'+Y' \arrow[ur, bend right, "{[\mathrm{id},0]}"']
\end{tikzcd}$$

such that for any pair of 2-cells $\alpha : f \Rightarrow f' : X \to X'$ and $\beta : g \Rightarrow g' : Y \to Y'$ the pasting identity

$$\begin{tikzcd}
X+Y \arrow[dr, bend left, "{[f',0]}"] \arrow[dd, bend right=50, "f+g"'] \arrow[dd, bend left=50, "f'+g'"] \arrow[dd, phantom, "\overset{\alpha+\beta}{\Rightarrow}"] \\
& X' \arrow[l, phantom, "\overset{\phi_{f',g'}}{\Rightarrow}"] \\
X'+Y' \arrow[ur, bend right, "{[\mathrm{id},0]}"']
\end{tikzcd}
=
\begin{tikzcd}
X+Y \arrow[dr, bend left=50, "{[f',0]}"] \arrow[dr, bend right=20, "{[f,0]}"'] \arrow[dr, phantom, shift left=2, "\Rrightarrow{[\alpha,0]}"] \arrow[dd, "f+g"'] \\
& X \arrow[l, phantom, shift right=3, "\overset{\phi_{f,g}}{\Rightarrow}"] \\
T \arrow[ur, bend right, "{[\mathrm{id},0]}"']
\end{tikzcd}
\tag{2.3.4.1}$$

is satisfied.

Equipped with these lemmas, we can show that there is a canonical comparison morphism from coproduct to product, giving rise to a weakly 2-natural transformation.

Lemma 2.3.5

Let $\mathcal{C}$ be a 2-category with a weak zero-object. For every pair of objects we fix a zero morphism $0 = 0_{XY} : X \to 0 \to Y$.

Assume that it has binary weak 2-coproducts $- + -$ and binary weak 2-products $- \times -$ given in a weakly 2-functorial way (i.e. with fixed choices of factorizations).

Then there is a canonical weakly 2-natural transformation $I : - + - \Longrightarrow - \times - : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$.

Proof By Power's coherence result [Pow89] we may assume that the zero object is semi-weak and that the products are strictly described.

Given objects X and Y of $\mathcal{C}$ we let $I_{X,Y} : X + Y \longrightarrow X \times Y$ be the unique morphism induced from the requirements that

$$\begin{tikzcd}
 & X+Y \arrow[dl, "{[\mathrm{id},0]}"'] \arrow[d, "I_{X,Y}"] \arrow[dr, "{[0,\mathrm{id}]}"] & \\
X & X \times Y \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & Y
\end{tikzcd}$$

commutes. Given morphisms $f : X \to X'$ and $g : Y \to Y'$ we get a unique invertible 2-cell

$$\begin{tikzcd}
X+Y \arrow[r, "I_{X,Y}"] \arrow[d, "f+g"'] \arrow[dr, phantom, "\nearrow I_{f,g}"] & X \times Y \arrow[d, "f \times g"] \\
X'+Y' \arrow[r, "I_{X',Y'}"'] & X' \times Y'
\end{tikzcd}$$

induced from the universal property of the product $X' \times Y'$ by requiring that postcomposing it with $\mathrm{pr}_{X'}$ yields the pasting identity

$$\begin{tikzcd}
X+Y \arrow[r, "I_{X,Y}"] \arrow[rr, bend left, "{[\mathrm{id},0]}"] \arrow[d, "f+g"'] \arrow[dr, phantom, "\nearrow I_{f,g}"] & X \times Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, "f \times g"] & X \arrow[d, "f"] & = & X+Y \arrow[r, "{[\mathrm{id},0]}"] \arrow[d, "f+g"'] \arrow[dr, "{[f,0]}" description] & X \arrow[d, "f"] \arrow[dl, phantom, "\nearrow \psi_f", near start] \\
X'+Y' \arrow[r, "I_{X',Y'}"'] \arrow[rr, bend right, "{[\mathrm{id},0]}"'] & X' \times Y' \arrow[r, "\mathrm{pr}_{X'}"'] & X' & & X'+Y' \arrow[r, "{[\mathrm{id},0]}"'] \arrow[ur, phantom, "\nearrow \phi_{f,g}", near start] & X'
\end{tikzcd}$$

and that postcomposition with pr'_Y gives an analogous pasting identity. Here $\phi_{f,g}$ and ψ_f are the 2-cells induced by the universal property of $- + -$ as described in lemma 2.3.3 and lemma 2.3.4.

Given two 2-cells $\alpha : f \Rightarrow f' : X \to X'$ and $\beta : g \Rightarrow g' : Y \to Y'$ the universal property of the product $X' \times Y'$ together with the compatibilities of ϕ, ψ with α, β described in loc. cit. let us deduce that the weak 2-naturality of $I_{-,-}$. Indeed postcomposing the desired pasting identity with $\mathrm{pr}_{X'}$ results in an identity by computing

$$\begin{tikzcd}[row sep=large]
X+Y \arrow[r, "I_{X,Y}"] \arrow[rr, bend left, "{[\mathrm{id},0]}"] \arrow[d, bend right=50, "f+g"'] \arrow[d, bend left=50, "f'+g'"] \arrow[d, phantom, "\overset{\alpha+\beta}{\Rightarrow}"] \arrow[dr, phantom, "\Nearrow I_{f',g'}"] & X\times Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, "f'\times g'"] & X \arrow[d, "f'"] \\
X'+Y' \arrow[r, "I_{X',Y'}"'] \arrow[rr, bend right, "{[\mathrm{id},0]}"'] & X'\times Y' \arrow[r, "\mathrm{pr}_{X'}"'] & X'
\end{tikzcd}
=
\begin{tikzcd}[row sep=huge, column sep=huge]
X+Y \arrow[r, "{[\mathrm{id},0]}"] \arrow[dr, "{[f',0]}" description] \arrow[d, bend right=50, "f+g"'] \arrow[d, bend left=50, "f'+g'"] \arrow[d, phantom, "\overset{\alpha+\beta}{\Rightarrow}"] \arrow[r, phantom, shift right=4ex, "\Nearrow\psi_{f'}"] & X \arrow[d, "f'"] \\
X'+Y' \arrow[r, "{[\mathrm{id},0]}"'] \arrow[r, phantom, shift left=4ex, "\Nearrow\phi_{f',g'}"] & X'
\end{tikzcd}$$

$$=
\begin{tikzcd}[row sep=huge, column sep=huge]
X+Y \arrow[r, "{[\mathrm{id},0]}"] \arrow[d, "f+g"'] \arrow[dr, bend left=20, "{[f',0]}" description] \arrow[dr, bend right=20, "{[f,0]}" description] \arrow[dr, phantom, "\Nearrow{[\alpha,0]}"] \arrow[r, phantom, shift right=4ex, "\Nearrow\psi_{f'}"] & X \arrow[d, "f'"] \\
X'+Y' \arrow[r, "{[\mathrm{id},0]}"'] \arrow[r, phantom, shift left=4ex, "\Nearrow\phi_{f,g}"] & X'
\end{tikzcd}
=
\begin{tikzcd}[row sep=huge, column sep=huge]
X+Y \arrow[r, "{[\mathrm{id},0]}"] \arrow[d, "f+g"'] \arrow[dr, "{[f,0]}" description] \arrow[r, phantom, shift right=4ex, "\Nearrow\psi_{f}"] & X \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "f'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
X'+Y' \arrow[r, "{[\mathrm{id},0]}"'] \arrow[r, phantom, shift left=4ex, "\Nearrow\phi_{f,g}"] & X'
\end{tikzcd}$$

$$=
\begin{tikzcd}[row sep=large]
X+Y \arrow[r, "I_{X,Y}"] \arrow[rr, bend left, "{[\mathrm{id},0]}"] \arrow[d, "f+g"'] \arrow[dr, phantom, "\Nearrow I_{f,g}"] & X\times Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, "f\times g"] & X \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "f'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\
X'+Y' \arrow[r, "I_{X',Y'}"'] \arrow[rr, bend right, "{[\mathrm{id},0]}"'] & X'\times Y' \arrow[r, "\mathrm{pr}_{X'}"'] & X'
\end{tikzcd}
=
\begin{tikzcd}[row sep=large]
X+Y \arrow[r, "I_{X,Y}"] \arrow[rr, bend left, "{[\mathrm{id},0]}"] \arrow[d, "f+g"'] \arrow[dr, phantom, "\Nearrow I_{f,g}"] & X\times Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, bend right=50, "f\times g"'] \arrow[d, bend left=50, "f'\times g'"] \arrow[d, phantom, "\overset{\alpha\times\beta}{\Rightarrow}"] & X \arrow[d, "f'"] \\
X'+Y' \arrow[r, "I_{X',Y'}"'] \arrow[rr, bend right, "{[\mathrm{id},0]}"'] & X'\times Y' \arrow[r, "\mathrm{pr}_{X'}"'] & X'
\end{tikzcd}$$

and similarly for postcomposing with $\mathrm{pr}_{Y'}$. ☐

With this canonical comparison transformation at hand we can make the following definition.

Definition 2.3.6

Let $\mathcal{C}$ be a 2-category with a weak zero-object, binary weak 2-coproducts and binary weak 2-products. Suppose all of these are given in a weakly 2-functorial way.

Then $\mathcal{C}$ has **weak 2-biproducts**, if it satisfies the following equivalent conditions.

(i) The canonical comparison transformation $I_{-,-} : - + - \Longrightarrow - \times -$ is a weakly 2-natural equivalence, i.e. every component $I_{X,Y} : X + Y \to X \times Y$ is an equivalence in $\mathcal{C}$.

(ii) Every weak 2-coproduct $X + Y$ becomes a weak 2-product with respect to the projection morphisms $X \xleftarrow{[\mathrm{id},0]} X + Y \xrightarrow{[0,\mathrm{id}]} Y$.

(iii) Every weak 2-product $X \times Y$ becomes a weak 2-coproduct with respect to the inclusion morphisms $X \xrightarrow{(\mathrm{id},0)} X \times Y \xleftarrow{(0,\mathrm{id})} Y$.

Proof The equivalence of these conditions is [Dup08] Prop. 225. ☐

Remark 2.3.7

The pseudonaturality of $I_{-,-}$ allows us to assume without loss of generality that the weak 2-biproduct is strictly 2-functorial. The reason is that due to the coherence theorem of Power [Pow89] a 2-category $\mathcal{C}$ with weak products is weakly 2-equivalent to a 2-category $\mathcal{C}'$ with strict 2-products. If $\mathcal{C}$ has weak 2-coproducts then (due to the weak 2-equivalence) $\mathcal{C}'$ will have weak 2-coproducts. Similarly, if $\mathcal{C}$ has weak 2-biproducts, so does $\mathcal{C}'$. Now, since $\mathcal{C}'$ has weak 2-biproducts and strict 2-products, the canonical comparison transformation $I_{-,-} : - + - \Longrightarrow - \times -$ is a weakly 2-natural equivalence, making the strict 2-functor $- \times -$ a weak 2-left-adjoint to the constant diagram 2-functor and as such a model for weak 2-coproducts. We will refer to this specific model of the weak 2-biproducts as **semi-weak 2-biproducts** and denote the strict 2-functor given by the product by $- \oplus -$. As explained in proposition 2.1.12 we obtain a weakly 2-natural transformation $\nabla_- : (- \oplus -) \circ \Delta \Longrightarrow \mathrm{id}_{\mathcal{C}'}$ adjusted to this specific strict 2-functor.

In the following section we will deal with rather large diagram involving n-ary biproducts. For the sake of readability we declare the following conventions. To do this we assume that we work in a 2-category $\mathcal{C}$ with semi-weak zero object 0 and binary semi-weak 2-biproducts, denoted by $\oplus$. We fix for every pair of objects a zero morphism $0_{X,Y} : X \to Y$.

Convention 2.3.8

In most diagrams we will omit all instances of $\oplus$ from our notation, meaning that we abbreviate $X \oplus Y$ to XY and powers $X \oplus X \oplus X$ to X^3. The author is aware that this convention applied to morphisms clashed with the unual convention that fg denotes the composite $f \circ g$ and not the sum $f \oplus g$. However in the context of the following section, there will be no need to label composite morphisms, so no ambiguity ariese. To avoid unnecessary subscribts, we write X instead of id_X, which means that XfY has to be understood as $\mathrm{id}_X \oplus f \oplus \mathrm{id}_Y$.

Furthermore we omit all instances of associativity, which may be formally justified by applying MacLane's coherence theorem to the strictly cartesian monoidal structure given by $\oplus$. This is to say that we consider the objects $X(YZ)$ and $(XY)Z$ to be *equal.* Note that the coherence theorem would allow us to consider for example $0X$ and X to be equal as well. Yet for the sake of recognizing 2-naturality in the following pasting diagrams more easily, we refrain from doing so.

By the strict universal property of the terminal object 0 all three morphisms

$$\mathrm{pr}_1,\ \mathrm{pr}_2,\ \nabla : 00 \longrightarrow 0$$

agree. Similarly the strict universal property of the respective 2-products enforces the equality of the following families of morphisms.

$$\begin{aligned}
\mathrm{pr}_1 Y,\ X \,\mathrm{pr}_2 : \quad & X0Y \quad \longrightarrow \quad XY \\
\mathrm{pr}_1 YZ,\ X \,\mathrm{pr}_2 Z : \quad & X0YZ \quad \longrightarrow \quad XYZ \\
X \,\mathrm{pr}_1 Z,\ XY \,\mathrm{pr}_2 : \quad & XY0Z \quad \longrightarrow \quad XYZ \\
\mathrm{pr}_1 0Y,\ X \,\mathrm{pr}_1 Y,\ X\nabla Y,\ X \,\mathrm{pr}_2 Y,\ X0 \,\mathrm{pr}_2 : \quad & X00Y \quad \longrightarrow \quad X0Y
\end{aligned}$$

These equalities will become very relevant in the following pasting arguments, since they allow us to recognize the same morphism as part of several natural transformations at once. This makes it hard to choose the right label for it in the diagram though, so to further simplify notation we introduce the following convention, backed by the fact that all obvious candidates for the morphisms in question agree.

Convention 2.3.9

We will omit the labels of those morphisms given by biproducts of identities and our chosen zero morphisms. Furthermore we will omit the labels of those morphisms listed above.

For example there is one obvious choice for the morphism $X0Y \to 0ZY$. The morphism $X0 \to 0X$

could a priori mean two things: the biproduct of two zero morphisms or a twist. However by our convention it will mean the biproduct of the unique terminal zero morphism with the chosen initial zero morphism 0_X.

A twist from $X \oplus Y$ to $Y \oplus X$ will always be indicated by $XY \xrightarrow{(12)} YX$. Similarly we will indicate permutations of n-ary biproducts by their defining cycles, as for example demonstrated by $XYZ \xrightarrow{(132)} YZX$.

As explained in proposition 2.1.12 a 2-coproduct $X + Y$ comes with invertible 2-cells $\chi_{-,-}$ and $\Upsilon_{-,-}$ parameterized over the cocones of the discrete diagram on X and Y. In the special case of the semi-weak biproduct $X \oplus X$ we will denote the 2-cells

$$\begin{tikzcd}
0 \oplus X \arrow[r, "\cong", "\mathrm{pr}"'] \arrow[drr, "\mathrm{pr}"'] & X \arrow[r, "\mathrm{incl}_2"] \arrow[dr, equal] & X \oplus X \arrow[d, "\nabla", "\Swarrow \Upsilon_{\mathrm{id},\mathrm{id}} \,|\, \chi_{\mathrm{id},\mathrm{id}} \Searrow"] & X \arrow[l, "\mathrm{incl}_1"'] \arrow[dl, equal] & X \oplus 0 \arrow[l, "\cong"', "\mathrm{pr}"] \arrow[dll, "\mathrm{pr}"] \\
 & & X & &
\end{tikzcd}$$

by

$$\begin{tikzcd}
0X \arrow[r] \arrow[dr] & XX \arrow[d, "\nabla", "\Swarrow \lambda_X"', "\rho_X \Searrow"] & X0 \arrow[l] \arrow[dl] \\
 & X &
\end{tikzcd}.$$

As in loc. cit. a morphism $f : X \to Y$ induces an invertible 2-cell

$$\begin{tikzcd}
X \oplus X \arrow[r, "\nabla"] \arrow[d, "f \oplus f"'] \arrow[dr, phantom, "\Nearrow \phi_f"] & X \arrow[d, "f"] \\
Y \oplus Y \arrow[r, "\nabla"'] & Y
\end{tikzcd}$$

whose inverse we will denote by

$$\begin{tikzcd}
XX \arrow[r, "ff"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\Nearrow \mu_f"] & YY \arrow[d, "\nabla"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}.$$

Convention 2.3.10

As a last convention we will sometimes omit the symbol $\Rightarrow$ from the 2-diagrams to reduce clutter. By convention all 2-cells in such diagrams will be oriented from left to right i.e. either $\Rightarrow$ or $\Nearrow$.

To further improve readibility we drop the subscripts of λ, ρ and μ, when the objects involved are understood.

We conclude this section by making some elementary observations about the cells λ, ρ and μ, which will be essential in the arguments of the following section.

The weak universal property of the weakly initial object $0 \oplus 0$ enforces the pasting identities

$$\begin{tikzcd}
 & 0X \arrow[d] \arrow[dd, bend left=70, "\mathrm{pr}"] & \\
00 \arrow[ur] \arrow[r] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\Nearrow \mu"] & XX \arrow[d, "\nabla"] \arrow[r, phantom, "\overset{\lambda}{\Rightarrow}"] & {} \\
0 \arrow[r] & X &
\end{tikzcd}
=
\begin{tikzcd}
00 \arrow[r] \arrow[d, "\mathrm{pr}"'] & 0X \arrow[d, "\mathrm{pr}"] \\
0 \arrow[r] & X
\end{tikzcd}
\tag{2.3.10.1}$$

$$\begin{tikzcd}
 & X0 \arrow[d] \arrow[dd, bend left=70, "\mathrm{pr}"] & \\
00 \arrow[ur] \arrow[r] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\Nearrow \mu"] & XX \arrow[d, "\nabla"] \arrow[r, phantom, "\overset{\rho}{\Rightarrow}"] & {} \\
0 \arrow[r] & X &
\end{tikzcd}
=
\begin{tikzcd}
00 \arrow[r] \arrow[d, "\mathrm{pr}"'] & X0 \arrow[d, "\mathrm{pr}"] \\
0 \arrow[r] & X
\end{tikzcd}
\tag{2.3.10.2}$$

In particular we obtain the pasting identities

$$\begin{tikzcd}
00 \arrow[r] \arrow[d, "\mathrm{pr}"'] & 0X \arrow[r] \arrow[dr, "\mathrm{pr}"'] & XX \arrow[d, "\nabla"] \arrow[dl, phantom, "\overset{\lambda^{-1}}{\Nearrow}"] \\
0 \arrow[rr] & & X
\end{tikzcd}
=
\begin{tikzcd}
00 \arrow[r] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\Nearrow \mu"] & XX \arrow[d, "\nabla"] \\
0 \arrow[r] & X
\end{tikzcd}
=
\begin{tikzcd}
00 \arrow[r] \arrow[d, "\mathrm{pr}"'] & X0 \arrow[r] \arrow[dr, "\mathrm{pr}"'] & XX \arrow[d, "\nabla"] \arrow[dl, phantom, "\overset{\rho^{-1}}{\Nearrow}"] \\
0 \arrow[rr] & & X
\end{tikzcd}
\tag{2.3.10.3}$$

$$\begin{tikzcd}
00 \arrow[r, "\mathrm{pr}"] \arrow[d] & 0 \arrow[dd] \\
0X \arrow[d] \arrow[dr, "\mathrm{pr}"] \arrow[dr, phantom, "\Nearrow \lambda"', shift right=2] & \\
XX \arrow[r] & X
\end{tikzcd}
=
\begin{tikzcd}
00 \arrow[r, "\nabla"] \arrow[d] \arrow[dr, phantom, "\Nearrow \mu^{-1}"] & 0 \arrow[d] \\
XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
=
\begin{tikzcd}
00 \arrow[r, "\mathrm{pr}"] \arrow[d] & 0 \arrow[dd] \\
X0 \arrow[d] \arrow[dr, "\mathrm{pr}"] \arrow[dr, phantom, "\Nearrow \rho"', shift right=2] & \\
XX \arrow[r] & X
\end{tikzcd}
\tag{2.3.10.4}$$

We furthermore have the pasting identity

$$\begin{tikzcd}
X00X \arrow[r] \arrow[d] & XX0X \arrow[d] \arrow[dr] \arrow[dr, phantom, "\overset{XX\lambda}{\Nearrow}"', shift right=3] & \\
X0XX \arrow[r] \arrow[dr] \arrow[dr, phantom, "\overset{\rho^{-1}XX}{\Nearrow}", shift left=3] & XXXX \arrow[r, "XX\nabla"'] \arrow[d, "\nabla XX"] & XXX \arrow[d, "\nabla X"] \\
 & XXX \arrow[r, "X\nabla"'] & XX
\end{tikzcd}
=
\begin{tikzcd}
X00X \arrow[r] \arrow[d] \arrow[dr] & XX0X \arrow[dr] & \\
X0XX \arrow[dr] & X0X \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\overset{X\lambda}{\Nearrow}"', shift right=3] \arrow[dr, phantom, "\overset{\rho^{-1}X}{\Nearrow}", shift left=3] & XXX \arrow[d, "X\nabla"] \\
 & XXX \arrow[r, "\nabla X"'] & XX
\end{tikzcd}
\quad ,
\tag{2.3.10.5}$$

which can be deduced from the strict 2-functoriality of the biproduct via the calculation

$$\begin{tikzcd}
X00X \arrow[r] \arrow[d] & XX0X \arrow[d] \arrow[dr] & \\
X0XX \arrow[r] \arrow[dr] \arrow[r, phantom, shift right=4, "\overset{\rho^{-1}XX}{\Nearrow}"] & XXXX \arrow[r, "XX\nabla"'] \arrow[d, "\nabla XX"] \arrow[ur, phantom, "\overset{XX\lambda}{\Nearrow}"] & XXX \arrow[d, "\nabla X"] \\
 & XXX \arrow[r, "X\nabla"'] & XX
\end{tikzcd}
=
\begin{tikzcd}
X00X \arrow[r] \arrow[d] \arrow[dr] & XX0X \arrow[d, "\nabla 0X"] \arrow[dr] \arrow[drr, bend left] \arrow[dl, phantom, "\overset{\rho^{-1}0X}{\Nearrow}"] & & \\
X0XX \arrow[dr] & X0X \arrow[d] & XXXX \arrow[r, "XX\nabla"'] \arrow[dl, "\nabla XX"] \arrow[u, phantom, "\overset{XX\lambda}{\Nearrow}"] & XXX \arrow[d, "\nabla X"] \\
 & XXX \arrow[rr, "X\nabla"'] & & XX
\end{tikzcd}$$

$$= \begin{tikzcd}
X00X \arrow[r] \arrow[d] \arrow[dr] & XX0X \arrow[d, "\nabla 0X"] \arrow[dr] \arrow[dl, phantom, "\overset{\rho^{-1}0X}{\Nearrow}"] & \\
X0XX \arrow[dr] & X0X \arrow[d] \arrow[dr] & XXX \arrow[d, "\nabla X"] \\
 & XXX \arrow[r, "X\nabla"'] \arrow[ur, phantom, "\overset{X\lambda}{\Nearrow}"] & XX
\end{tikzcd}
=
\begin{tikzcd}
X00X \arrow[r] \arrow[d] \arrow[dr] & XX0X \arrow[dr] & \\
X0XX \arrow[dr] & X0X \arrow[r] \arrow[d] \arrow[dr] & XXX \arrow[d, "X\nabla"] \arrow[dl, phantom, "\overset{\rho^{-1}X}{\Nearrow}"] \\
 & XXX \arrow[r, "\nabla X"'] \arrow[ur, phantom, "\overset{X\lambda}{\Nearrow}"] & XX
\end{tikzcd}$$

Regarding the symmetry, we note that, by definition of the symmetry and the strict universal property of the biproducts, diagrams of the form

$$\begin{tikzcd}
X0 \arrow[rr, "(12)"] \arrow[dr, "\mathrm{pr}_1"'] & & 0X \arrow[dl, "\mathrm{pr}_2"] \\
 & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
XYZ \arrow[r, "(13)"] \arrow[d, "\mathrm{pr}_{23}"'] & ZYX \arrow[d, "\mathrm{pr}_{12}"] \\
YZ \arrow[r, "(12)"'] & ZY
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \arrow[dl, "tst"'] \arrow[dr, "tst"] & \\
XYX \arrow[rr, "(13)"'] & & XYX
\end{tikzcd}
\tag{2.3.10.6}$$

commute. Furthermore the permutation $XYZ \xrightarrow{(132)} YZX$ can be identified with the transposition $X(YZ) \xrightarrow{(12)} (YZ)X$. In particular we can use its 2-naturality to obtain the pasting identity

$$\begin{tikzcd}
X00 \arrow[r, "(132)"] \arrow[d] & 00X \arrow[d] \\
XXX \arrow[r, "(132)"] \arrow[d, "X\nabla"'] & XXX \arrow[d, "\nabla X"] \\
XX \arrow[r, "(12)"] & XX
\end{tikzcd}
=
\begin{tikzcd}
 & X00 \arrow[rrr, "(132)"] \arrow[dl] \arrow[dr, "X\nabla"] & & & 00X \arrow[dl, "\nabla X"'] \arrow[dr] & \\
XXX \arrow[dr, "X\nabla"'] & \overset{X\mu^{-1}}{\Rightarrow} & X0 \arrow[r, "(12)"] \arrow[dl] & 0X \arrow[dr] & \overset{\mu X}{\Rightarrow} & XXX \arrow[dl, "\nabla X"] \\
 & XX \arrow[rrr, "(12)"'] & & & XX &
\end{tikzcd} .
\tag{2.3.10.7}$$

Equipped with these conventions and observations we can now prove that every object in a 2-category with semi-weak 2-biproducts is canonically equipped with the structure of a pseudomonoid. This is the content of the next section.

2.4 An Enrichment in Symmetric 2-Monoids

Lemma 2.4.1

Let $\mathcal{C}$ be a 2-category with semi-weak zero object and semi-weak 2-biproducts.

Then every object X has a canonical structure of a symmetric pseudomonoid with identity $0 : 0 \longrightarrow X$, multiplication $\nabla : X \oplus X \longrightarrow X$ and canonical symmetry.

Proof It is evident that the unitor 2-cells λ and ρ are supposed to be those given by the diagram

$$\begin{tikzcd}
0X \arrow[r] \arrow[dr, "\mathrm{pr}"'] & XX \arrow[d, "\nabla"] \arrow[dl, phantom, "\Swarrow\lambda"] \arrow[dr, phantom, "\rho\Searrow"] & X0 \arrow[l] \arrow[dl, "\mathrm{pr}"] \\
& X &
\end{tikzcd}$$

The weak universal property of the biproduct lets us define the invertible 2-cell σ in the diagram

$$\begin{tikzcd}
XX \arrow[rr, "(12)"] \arrow[dr, "\nabla"'] & \arrow[d, phantom, "\Rightarrow \sigma"] & XX \arrow[dl, "\nabla"] \\
& X &
\end{tikzcd}$$

by asserting that

$$\begin{tikzcd}
X0 \arrow[r] & XX \arrow[rr, "(12)"] \arrow[dr, "\nabla"'] & \arrow[d, phantom, "\Rightarrow \sigma"] & XX \arrow[dl, "\nabla"] \\
& & X &
\end{tikzcd}
=
\begin{tikzcd}
& XX \arrow[dr, "(12)"] & \\
X0 \arrow[ur] \arrow[r, "(12)"] \arrow[d] \arrow[drr] \arrow[dr, phantom, "\rho"] & 0X \arrow[r] \arrow[dr] \arrow[r, phantom, shift right=4, "\lambda^{-1}"] & XX \arrow[d, "\nabla"] \\
XX \arrow[rr, "\nabla"'] & & X
\end{tikzcd}
\tag{2.4.1.1}$$

$$\begin{tikzcd}
0X \arrow[r] & XX \arrow[rr, "(12)"] \arrow[dr, "\nabla"'] & \arrow[d, phantom, "\sigma"] & XX \arrow[dl, "\nabla"] \\
& & X &
\end{tikzcd}
=
\begin{tikzcd}
& XX \arrow[dr, "(12)"] & \\
0X \arrow[ur] \arrow[r, "(12)"] \arrow[d] \arrow[drr] \arrow[dr, phantom, "\lambda"] & X0 \arrow[r] \arrow[dr] \arrow[r, phantom, shift right=4, "\rho^{-1}"] & XX \arrow[d, "\nabla"] \\
XX \arrow[rr, "\nabla"'] & & X
\end{tikzcd}
\tag{2.4.1.2}$$

It is easy to verify the symmetry axiom by using the weak universal property of $X \oplus X$. It suffices to recognize the pasting identity

$$\begin{tikzcd}
X0 \arrow[r] & XX \arrow[r, "\cong"] \arrow[dr, "\nabla"'] \arrow[r, phantom, shift right=5, "\sigma"] & XX \arrow[d, "\nabla"] \arrow[r, "\cong"] \arrow[r, phantom, shift right=5, "\sigma"] & XX \arrow[dl, "\nabla"] \\
& & X &
\end{tikzcd}
=
\begin{tikzcd}
X0 \arrow[rr, "(12)"] \arrow[d] \arrow[ddrr] & & 0X \arrow[rr, "(12)"] \arrow[d] \arrow[ddr, phantom, ""] \arrow[dd, bend right=60] \arrow[dd, bend left=60] & & X0 \arrow[d] \arrow[ddll] \\
XX \arrow[ddrr, phantom, ""] \arrow[r, phantom, "\rho"] \arrow[drr, "\nabla"'] & \arrow[r, phantom, "\lambda^{-1}"] & XX \arrow[d, "\nabla"] \arrow[r, phantom, "\lambda"] & \arrow[r, phantom, "\rho^{-1}"] & XX \arrow[dll, "\nabla"] \\
& & X & &
\end{tikzcd}$$

and the analogous one for precomposing with the other inclusion.

To define the associator we recall that $X \oplus X \oplus X$ has the weak universal property of a ternary weak 2-coproduct. Hence we might let α be the unique invertible 2-cell

$$\begin{tikzcd}
XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"'] \arrow[dr, phantom, "\Rightarrow \alpha"] & XX \arrow[d, "\nabla"] \\
XX \arrow[r, "\nabla"'] & X
\end{tikzcd}$$

induced by the requirements that

$$\begin{tikzcd}
X00 \arrow[d] & \\
XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"'] \arrow[dr, phantom, "\alpha"] & XX \arrow[d, "\nabla"] \\
XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
=
\begin{tikzcd}
X00 \arrow[rr] \arrow[d] \arrow[dr] & & XXX \arrow[d, "X\nabla"] \\
X0X \arrow[d] \arrow[dr, "\rho X"'] & X0 \arrow[r] \arrow[d] \arrow[ur, phantom, "X\mu"] & XX \arrow[d, "\nabla"] \\
XXX \arrow[r, "\nabla X"'] & XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
\tag{2.4.1.3}$$

$$\begin{tikzcd}
0X0 \arrow[d] & \\
XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"'] \arrow[dr, phantom, "\alpha"] & XX \arrow[d, "\nabla"] \\
XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
=
\begin{tikzcd}
0X0 \arrow[r] \arrow[dd] \arrow[dr] \arrow[ddr] & XX0 \arrow[r] \arrow[dr, "X\rho^{-1}"] & XXX \arrow[d, "X\nabla"] \\
 & 0X \arrow[r] \arrow[ddr, "\lambda^{-1}"] & XX \arrow[dd, "\nabla"] \\
0XX \arrow[d] \arrow[dr, "\lambda X"] & X0 \arrow[d] \arrow[dr, "\rho"] & \\
XXX \arrow[r, "\nabla X"'] & XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
\tag{2.4.1.4}$$

$$\begin{tikzcd}
00X \arrow[d] & \\
XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"'] \arrow[dr, phantom, "\alpha"] & XX \arrow[d, "\nabla"] \\
XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
=
\begin{tikzcd}
00X \arrow[r] \arrow[dd] \arrow[dr, "\mu^{-1}X"'] & X0X \arrow[r] \arrow[dr, "X\lambda^{-1}"] & XXX \arrow[d, "X\nabla"] \\
 & 0X \arrow[r] \arrow[d] & XX \arrow[d, "\nabla"] \\
XXX \arrow[r, "\nabla X"'] & XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
\tag{2.4.1.5}$$

Note further that by using the equations 2.3.10.3 and 2.3.10.4 on the equations 2.4.1.3 and 2.4.1.5 respectively, the weak universal property of the biproduct $X \oplus 0 \oplus X$ gives us the pasting identity

$$\begin{tikzcd}
X0X \arrow[d] & \\
XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"'] \arrow[dr, phantom, "\alpha"] & XX \arrow[d, "\nabla"] \\
XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
=
\begin{tikzcd}
X0X \arrow[r] \arrow[d] \arrow[dr, "X\lambda^{-1}", "\rho X"'] & XXX \arrow[d, "X\nabla"] \\
XXX \arrow[r, "\nabla X"'] & XX \arrow[d, "\nabla"] \\
 & X
\end{tikzcd}
\tag{2.4.1.6}$$

This makes the triangle axiom immediate, since

$$\begin{tikzcd}
X0X \arrow[d] \arrow[dr] \arrow[dr, phantom, "X\lambda" near start, shift right=3] & \\
XXX \arrow[r, "X\nabla"'] \arrow[d, "\nabla X"'] \arrow[dr, phantom, "\alpha"] & XX \arrow[d, "\nabla"] \\
XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
=
\begin{tikzcd}
X0X \arrow[dd] \arrow[dr] \arrow[ddrr, bend right] \arrow[ddrr, bend left=40] & & \\
& XXX \arrow[dr, "X\nabla"] \arrow[dr, phantom, "X\lambda^{-1}", shift right=4] \arrow[dr, phantom, "X\lambda", shift left=5] & \\
XXX \arrow[rr, "\nabla X"'] \arrow[r, phantom, "\rho X"] & & XX \arrow[d, "\nabla"] \\
& & X
\end{tikzcd}
=
\begin{tikzcd}
X0X \arrow[d] \arrow[dr] \arrow[dr, phantom, "\rho X" near start, shift right=3] & \\
XXX \arrow[r, "\nabla X"'] & XX \arrow[d, "\nabla"] \\
& X
\end{tikzcd}
\qquad (2.4.1.7)$$

Note that inverting that by inverting the 2-cells and pasting α we also get the following pasting identity, which we will need later.

$$\begin{tikzcd}
X0X \arrow[r] \arrow[dr, "\rho^{-1}X"] & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "\nabla"] \\
& XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
=
\begin{tikzcd}
X0X \arrow[r] \arrow[dr, "X\lambda^{-1}"] & XXX \arrow[d, "X\nabla"] & \\
& XX \arrow[r, "\nabla"] & X
\end{tikzcd}
\qquad (2.4.1.8)$$

We will also need the pasting identity

$$\begin{tikzcd}
0X0 \arrow[d] \arrow[dr] & & \\
0XX \arrow[r] \arrow[dr, bend right, "\lambda^{-1}X"] & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"] \arrow[dr, phantom, "\alpha"] & XX \arrow[d, "\nabla"] \\
& XX \arrow[r, "\nabla"'] & X
\end{tikzcd}
=
\begin{tikzcd}
0X0 \arrow[r] \arrow[d] \arrow[dr] & XX0 \arrow[r] \arrow[dr, "X\rho^{-1}"] & XXX \arrow[d, "X\nabla"] \\
X0 \arrow[d] \arrow[drr] \arrow[drr, phantom, "\rho X", shift right=4] & 0X \arrow[r] \arrow[dr, "\lambda^{-1}"] & XX \arrow[d, "\nabla"] \\
XX \arrow[rr, "\nabla"'] & & X
\end{tikzcd}
\qquad (2.4.1.9)$$

To check the pentagon axiom, we use the weak universal property of the fourfold biproduct $X \oplus X \oplus X \oplus X$. We may check the first and fourth inclusion at once. In this situation the left side of the pentagon axiom reduces as

$$\begin{tikzcd}
X00X \arrow[d] & & & \\
XXXX \arrow[rr, "XX\nabla"] \arrow[dr, "X\nabla X"] \arrow[dd, "\nabla XX"'] & & XXX \arrow[dr, "X\nabla"] \arrow[d, phantom, "X\alpha"] & \\
& XXX \arrow[rr, "X\nabla"'] \arrow[dd, "\nabla X"] \arrow[l, phantom, "\alpha X"] & & XX \arrow[dd, "\nabla"] \\
XXX \arrow[dr, "\nabla X"'] & & \alpha & \\
& XX \arrow[rr, "\nabla"'] & & X
\end{tikzcd}
\overset{\text{①}}{=}
\begin{tikzcd}
X00X \arrow[rrr] \arrow[dd] \arrow[ddr] \arrow[dr] \arrow[drr] & & & XX0X \arrow[r] \arrow[dr, "XX\lambda^{-1}"] & XXXX \arrow[d, "XX\nabla"] \\
& & XXXX \arrow[dr, "X\nabla X"] \arrow[dr, phantom, "X\mu X", shift right=4] & X0X \arrow[r] \arrow[d] \arrow[d, phantom, "X\mu^{-1}X", shift right=4] & XXX \arrow[d, "X\nabla"] \\
X0XX \arrow[d] \arrow[dr] \arrow[dr, phantom, "\rho XX", shift right=3] & X0X \arrow[rr] \arrow[d] & & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"'] \arrow[dr, phantom, "\alpha"] & XX \arrow[d, "\nabla"] \\
XXXX \arrow[r, "\nabla XX"'] & XXX \arrow[rr, "\nabla X"'] & & XX \arrow[r, "\nabla"'] & X
\end{tikzcd}$$

$$=\quad\begin{tikzcd}
X00X \arrow[r] \arrow[d] \arrow[dr] & XX0X \arrow[r] \arrow[ddr, "XX\lambda^{-1}"] & XXXX \arrow[dd, "XX\nabla"] & \\
X0XX \arrow[d] \arrow[drr, "\rho XX"'] & X0X \arrow[dr] & & \\
XXXX \arrow[rr, "\nabla XX"'] & & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"'] \arrow[dr, phantom, "\alpha"] & X \arrow[d, "\nabla"] \\
& & XX \arrow[r, "\nabla"'] & X
\end{tikzcd}$$

$$\overset{②}{=}\quad\begin{tikzcd}
X00X \arrow[r] \arrow[d] \arrow[dr] & XX0X \arrow[r] \arrow[dr, "XX\lambda^{-1}"] & XXXX \arrow[d, "XX\nabla"] \\
X0XX \arrow[d, "\rho XX"] \arrow[dr] & X0X \arrow[r] \arrow[d, "\rho X"] \arrow[dr, "X\lambda^{-1}"] & XXX \arrow[d, "X\nabla"] \\
XXXX \arrow[r, "\nabla XX"'] & XXX \arrow[r, "\nabla X"'] & XX \arrow[d, "\nabla"] \\
& & X
\end{tikzcd}$$

by ① applying the defining equations 2.4.1.3 and 2.4.1.5 and then ② pasting in the equation 2.4.1.6.

At the same time the right side of the pentagon axiom reduces as

$$\begin{tikzcd}
X00X \arrow[d] & & \\
XXXX \arrow[r, "XX\nabla"'] \arrow[d, "\nabla XX"] & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"] \arrow[r, phantom, shift right=5ex, "\alpha"] & XX \arrow[dd, "\nabla"] \\
XXX \arrow[r, "X\nabla"'] \arrow[d, "\nabla X"'] & XX \arrow[dr, "\nabla"] & \\
XX \arrow[rr, "\nabla"'] \arrow[r, phantom, shift left=4ex, "\alpha"] & & X
\end{tikzcd}$$

$$\overset{①}{=}\quad\begin{tikzcd}
X00X \arrow[r] \arrow[d] & XX0X \arrow[r] \arrow[d, "XX\lambda"] \arrow[dr, "XX\lambda^{-1}"] & XXXX \arrow[d, "XX\nabla"] & \\
X0XX \arrow[r] \arrow[d, "\rho XX"] \arrow[dr, "\rho^{-1}XX"] & XXXX \arrow[r, "XX\nabla"'] \arrow[d, "\nabla XX"] & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"] \arrow[r, phantom, shift right=5ex, "\alpha"] & XX \arrow[dd, "\nabla"] \\
XXXX \arrow[r, "\nabla XX"'] & XXX \arrow[r, "X\nabla"'] \arrow[d, "\nabla X"'] & XX \arrow[dr, "\nabla"] & \\
& XX \arrow[rr, "\nabla"'] \arrow[r, phantom, shift left=4ex, "\alpha"] & & X
\end{tikzcd}$$

$$\overset{②}{=}\quad\begin{tikzcd}
X00X \arrow[r] \arrow[d] \arrow[dr] & XX0X \arrow[r] \arrow[dr, "XX\lambda^{-1}"] & XXXX \arrow[d, "XX\nabla"] & \\
X0XX \arrow[d, "\rho XX"] \arrow[dr] & X0X \arrow[r] \arrow[d] \arrow[dr, "X\lambda"', "\rho^{-1}X"] & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"] \arrow[r, phantom, shift right=5ex, "\alpha"] & XX \arrow[dd, "\nabla"] \\
XXXX \arrow[r, "\nabla XX"'] & XXX \arrow[r, "X\nabla"'] \arrow[d, "\nabla X"'] & XX \arrow[dr, "\nabla"] & \\
& XX \arrow[rr, "\nabla"'] \arrow[r, phantom, shift left=4ex, "\alpha"] & & X
\end{tikzcd}$$

$$\overset{③}{=}\quad\begin{tikzcd}
X00X \arrow[r] \arrow[d] \arrow[dr] & XX0X \arrow[r] \arrow[dr, "XX\lambda^{-1}"] & XXXX \arrow[d, "XX\nabla"] \\
X0XX \arrow[d, "\rho XX"] \arrow[dr] & X0X \arrow[r] \arrow[d, "\rho X"] \arrow[dr, "X\lambda^{-1}"] & XXX \arrow[d, "X\nabla"] \\
XXXX \arrow[r, "\nabla XX"'] & XXX \arrow[r, "\nabla X"'] & XX \arrow[d, "\nabla"] \\
& & X
\end{tikzcd}$$

by ① inserting identities to the left and top, ② using equation 2.3.10.5 and ③ the triangle axiom 2.4.1.7 and its cousin 2.4.1.8.

The second and third inclusions are analogous. We check the former. The left side of the pentagon axiom reduces as

$$\begin{tikzcd}
0X00 \arrow[d] & & & \\
XXXX \arrow[rr, "XX\nabla"] \arrow[dr, "X\nabla X"] \arrow[ddd, "\nabla XX"'] & & XXX \arrow[dr, "X\nabla"] \arrow[d, phantom, "X\alpha"] & \\
& XXX \arrow[rr, "X\nabla"'] \arrow[dd, "\nabla X"] \arrow[ddl, phantom, "\alpha X"] \arrow[ddrr, phantom, "\alpha"] & & XX \arrow[dd, "\nabla"] \\
XXX \arrow[dr, "\nabla X"'] & & & \\
& XX \arrow[rr, "\nabla"'] & & X
\end{tikzcd}
=
\begin{tikzcd}
0X00 \arrow[r] \arrow[d] & XX00 \arrow[r] & XXXX \arrow[r, "XX\nabla"] \arrow[dd, "X\nabla X"'] \arrow[ddr, phantom, "X\alpha"] & XXX \arrow[dd, "X\nabla"] \\
0X0X \arrow[d] & & & \\
XXXX \arrow[rr, "X\nabla X"] \arrow[d, "\nabla XX"'] \arrow[drr, phantom, "\alpha X"] & & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"'] \arrow[dr, phantom, "\alpha"] & XX \arrow[d, "\nabla"] \\
XXX \arrow[rr, "\nabla X"'] & & XX \arrow[r, "\nabla"'] & X
\end{tikzcd}$$

$$\overset{\textcircled{1}}{=}
\begin{tikzcd}
0X00 \arrow[rr] \arrow[d] & & XX00 \arrow[rr] \arrow[d] & & XX0X \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "XX\lambda^{-1}"] \arrow[d, phantom, "X\rho X"] & XXXX \arrow[d, "XX\nabla"] \\
0X0X \arrow[rr] \arrow[d] \arrow[dr] \arrow[drrr] & & XX0X \arrow[rr] \arrow[drr] \arrow[drr, phantom, "X\rho^{-1}X"] & & XXXX \arrow[r, "X\nabla X"'] \arrow[d, "X\nabla X"] & XXX \arrow[d, "X\nabla"] \\
0XXX \arrow[d] \arrow[drr] \arrow[dr, phantom, "\lambda XX"] & X0X \arrow[dr] \arrow[drrr] \arrow[drr, phantom, "\rho X"] & & 0XX \arrow[r] \arrow[dr] \arrow[dr, phantom, "\lambda^{-1}X"] & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"] \arrow[dr, phantom, "\alpha"] & XX \arrow[d, "\nabla"] \\
XXXX \arrow[rr, "\nabla XX"'] & & XXX \arrow[rr, "\nabla X"'] & & XX \arrow[r, "\nabla"'] & X
\end{tikzcd}$$

$$\overset{\textcircled{2}}{=}
\begin{tikzcd}
0X00 \arrow[rrr] \arrow[dd] \arrow[ddr] \arrow[drr] & & & XX0X \arrow[r] \arrow[dr] \arrow[dr, phantom, "XX\lambda^{-1}"] & XXXX \arrow[d, "XX\nabla"] \\
& & 0X0 \arrow[r] \arrow[ddd] \arrow[dr] & XX0 \arrow[r] \arrow[dr] \arrow[dr, phantom, "X\rho^{-1}"] & XXX \arrow[d, "X\nabla"] \\
0X0X \arrow[d] \arrow[dr] & X00 \arrow[d] \arrow[dr] & & 0X \arrow[r] \arrow[ddr] \arrow[dr, phantom, "\lambda^{-1}"] & XX \arrow[dd, "\nabla"] \\
0XXX \arrow[d] \arrow[dr] \arrow[d, phantom, "\lambda XX"] & X0X \arrow[d] \arrow[dr] \arrow[d, phantom, "\rho X"] & X0 \arrow[d] \arrow[drr] \arrow[dr, phantom, "\rho"] & & \\
XXXX \arrow[r, "\nabla XX"'] & XXX \arrow[r, "\nabla X"'] & XX \arrow[rr, "\nabla"'] & & X
\end{tikzcd}$$

by ① using 2.4.1.4 and ② cancelling $X\rho^{-1}X$ with $X\rho X$ cells and applying 2.4.1.9.

Meanwhile the right side reduces as

$$\begin{tikzcd}
0X00 \arrow[d] & & & \\
0XXX \arrow[r] \arrow[d] \arrow[rd, "\lambda^{-1}XX", "\lambda XX"'] & XXXX \arrow[r, "XX\nabla"] \arrow[d, "\nabla XX"] & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"] \arrow[rd, phantom, "\alpha"] & XX \arrow[dd, "\nabla"] \\
XXXX \arrow[r, "\nabla XX"'] & XXX \arrow[r, "X\nabla"'] \arrow[d, "\nabla X"'] \arrow[rd, phantom, "\alpha"] & XX \arrow[rd, "\nabla" description] & \\
 & XX \arrow[rr, "\nabla"'] & & X
\end{tikzcd}$$

$$\overset{\textcircled{1}}{=}\quad\begin{tikzcd}
0X00 \arrow[d] & & & \\
0X0X \arrow[r] \arrow[d, "0X\lambda"] \arrow[rd, "0X\lambda^{-1}"] & 0XXX \arrow[r] \arrow[d, "0X\nabla"] & XXXX \arrow[d, "XX\nabla"] & \\
0XXX \arrow[r, "0X\nabla"'] \arrow[d] \arrow[rd, "\lambda XX"] & 0XX \arrow[r] \arrow[rd, "\lambda^{-1}X"] & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"] \arrow[rd, phantom, "\alpha"] & XX \arrow[dd, "\nabla"] \\
XXXX \arrow[r, "\nabla XX"'] & XXX \arrow[r, "X\nabla"] \arrow[d, "\nabla X"'] \arrow[rd, phantom, "\alpha"] & XX \arrow[rd, "\nabla" description] & \\
 & XX \arrow[rr, "\nabla"'] & & X
\end{tikzcd}$$

$$\overset{\textcircled{2}}{=}\quad\begin{tikzcd}
0X00 \arrow[rr] \arrow[rrd] \arrow[dd] \arrow[ddr] & & 0X0X \arrow[r] \arrow[rd] & XX0X \arrow[r] \arrow[rd, "XX\lambda^{-1}"] & XXXX \arrow[d, "XX\nabla"] \\
 & & 0X0 \arrow[r] \arrow[d] & 0XX \arrow[r] \arrow[dd] & XXX \arrow[ddl, "\nabla X" description, "\lambda^{-1}X"'] \arrow[dd, "X\nabla"] \\
0X0X \arrow[d] \arrow[rd] & X00 \arrow[r] \arrow[d] & 0X \arrow[rd] & & \\
0XXX \arrow[d] \arrow[rd, "\lambda XX"'] & X0X \arrow[rr] \arrow[d] \arrow[rd, phantom, "X\lambda"] & & XX \arrow[r, phantom, "\alpha"] \arrow[rd, "\nabla" description] \arrow[d, phantom, "\alpha"] & XX \arrow[d, "\nabla"] \\
XXXX \arrow[r, "\nabla XX"'] & XXX \arrow[rr, "\nabla X"'] \arrow[rru, "X\nabla" description] & & XX \arrow[r, "\nabla"'] & X
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0X00 \arrow[rrr] \arrow[dd] \arrow[ddr] \arrow[drr] & & & XX0X \arrow[r] \arrow[dr, "XX\lambda^{-1}"'] & XXXX \arrow[d, "XX\nabla"] \\
 & & & 0X0 \arrow[r] \arrow[dr] \arrow[dd] & XX0 \arrow[r] \arrow[dr, "X\rho^{-1}"'] & XXX \arrow[d, "X\nabla"] \\
\overset{\textcircled{3}}{=} & 0X0X \arrow[d] \arrow[dr] & X00 \arrow[d] \arrow[dr] & & 0X \arrow[r] \arrow[ddr, "\lambda^{-1}"] & XX \arrow[dd, "\nabla"] \\
 & 0XXX \arrow[d] \arrow[dr, "\lambda XX"'] & X0X \arrow[d] \arrow[dr, "\rho X"] & X0 \arrow[d] \arrow[drr, "\rho"] & & \\
 & XXXX \arrow[r, "\nabla XX"'] & XXX \arrow[r, "\nabla X"'] & XX \arrow[rr, "\nabla"'] & & X
\end{tikzcd}$$

by ①+② using the 2-functoriality of the biproduct and ③ applying 2.4.1.9 and 2.4.1.7.

Now only the hexagon axiom is left. For the first (and analogously the third) inclusion we find that

$$\begin{tikzcd}
 & X00 \arrow[rr, "(132)"] \arrow[d] & & 00X \arrow[d] & & & & X00 \arrow[rrrr, "(132)"] \arrow[dl] \arrow[dr] & & & & 00X \arrow[dl] \arrow[dr] & \\
 & XXX \arrow[rr, "(132)"] \arrow[dl, "\nabla X"'] \arrow[d, "X\nabla"] & & XXX \arrow[d, "\nabla X"'] \arrow[dr, "X\nabla"] & & \overset{\textcircled{1}}{=} & XXX \arrow[rr, phantom, "X\mu^{-1}"] \arrow[d, "\nabla X"'] \arrow[drr, "X\nabla"] & & X0 \arrow[rr, "(12)"] \arrow[d] & & 0X \arrow[rr, phantom, "\mu X"] \arrow[d] & & XXX \arrow[dll, "\nabla X"'] \arrow[d, "X\nabla"] \\
XX \arrow[r, phantom, "\alpha"] \arrow[drr, "\nabla"'] & XX \arrow[rr, "(12)", "\sigma"'] \arrow[dr, "\nabla"'] & & XX \arrow[dl, "\nabla"] \arrow[r, phantom, "\alpha"] & XX \arrow[dll, "\nabla"] & & XX \arrow[rr, phantom, "\alpha"] \arrow[drrr, "\nabla"'] & & XX \arrow[rr, "(12)", "\sigma"'] \arrow[dr, "\nabla"'] & & XX \arrow[dl, "\nabla"] \arrow[rr, phantom, "\alpha"] & & XX \arrow[dlll, "\nabla"] \\
 & & X & & & & & & & X & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & X00 \arrow[rrrr, "(132)"] \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & & & & 00X \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\overset{\textcircled{2}}{=} & X0X \arrow[d] \arrow[ddrr, "\rho X"] & X0 \arrow[ddr] & XXX \arrow[dd, "X\mu"', "X\mu^{-1}", "X\nabla" description] & X0 \arrow[rr, "(12)"] \arrow[ddl] \arrow[dddr, "\rho"'] & & 0X \arrow[dddl, "\lambda"'] \arrow[ddr] & XXX \arrow[dd, "\mu X"', "\mu^{-1}X", "\nabla X" description] & 0X \arrow[ddl] & X0X \arrow[ddll, "X\lambda^{-1}"'] \arrow[d] \\
 & XXX \arrow[drr, "\nabla X"'] & & & & & & & & XXX \arrow[dll, "X\nabla"] \\
 & & & XX \arrow[drr, "\nabla"'] & & & & XX \arrow[dll, "\nabla"] & & \\
 & & & & & X & & & &
\end{tikzcd}$$

$\overset{③}{=}$

$$\begin{tikzcd}
 & X00 \arrow[rrrr, "(132)"] \arrow[dl] \arrow[drr, "(12)"] & & & & 00X \arrow[dr] & \\
X0X \arrow[dr, "(12)"] \arrow[ddd] \arrow[ddddr] & & & 0X0 \arrow[urr, "(23)"'] \arrow[dll] \arrow[drr] \arrow[ddl] \arrow[ddr] & & & X0X \arrow[ddd] \arrow[ddddl] \\
 & 0XX \arrow[ddd] & & & & XX0 \arrow[ur, "(23)"] \arrow[ddd] & \\
 & & X0 \arrow[rr, "(12)"] \arrow[ddl] \arrow[dddr] & & 0X \arrow[dddl] \arrow[ddr] & & \\
XXX \arrow[dr, "\nabla X"'] \arrow[uur, phantom, "\rho X"] & & & & & & XXX \arrow[dl, "X\nabla"] \arrow[uul, phantom, "X\lambda^{-1}"] \\
 & XX \arrow[drr, "\nabla"'] \arrow[uurrr, phantom, "\rho"] & & & & XX \arrow[dll, "\nabla"] \arrow[uulll, phantom, "\lambda"] & \\
 & & & X & & &
\end{tikzcd}$$

$\overset{④}{=}$

$$\begin{tikzcd}
 & X00 \arrow[rrrr, "(132)"] \arrow[dl] \arrow[drr] & & & & 00X \arrow[dr] & \\
X0X \arrow[dr, "(12)"] \arrow[ddd] \arrow[ddddr, bend right] & & & 0X0 \arrow[urr] \arrow[dll] \arrow[drr] \arrow[ddl] \arrow[ddr] & & & X0X \arrow[ddd] \arrow[ddddl, bend left] \\
 & 0XX \arrow[d] \arrow[ddd, bend right=60] \arrow[ddd, bend left=60] \arrow[d, bend right=30, phantom, "\lambda^{-1}X"] \arrow[d, bend left=30, phantom, "\lambda X"] & & & & XX0 \arrow[ur, "(23)"] \arrow[d] \arrow[ddd, bend right=60] \arrow[ddd, bend left=60] \arrow[d, bend right=30, phantom, "X\rho^{-1}"] \arrow[d, bend left=30, phantom, "X\rho"] & \\
 & XXX \arrow[dd, "\nabla X"] & X0 \arrow[rr, "(12)"] \arrow[ddl] \arrow[dddr] & & 0X \arrow[dddl] \arrow[ddr] & XXX \arrow[dd, "X\nabla"] & \\
XXX \arrow[dr, "\nabla X"'] \arrow[uur, phantom, "\rho X"] & & & & & & XXX \arrow[dl, "X\nabla"] \arrow[uul, phantom, "X\lambda^{-1}"] \\
 & XX \arrow[drr, "\nabla"'] \arrow[uurrr, phantom, "\rho"] & & & & XX \arrow[dll, "\nabla"] \arrow[uulll, phantom, "\lambda"] & \\
 & & & X & & &
\end{tikzcd}$$

$\overset{⑤}{=}$

$$\begin{tikzcd}
X00 \arrow[rr, "(132)"] \arrow[dr, "(12)"'] \arrow[dd] & & 00X \arrow[dd] \\
 & 0X0 \arrow[ur, "(23)"'] \arrow[dd] & \\
XXX \arrow[dr, "(12)"] \arrow[dd, "\nabla X"'] & & XXX \arrow[dd, "X\nabla"] \\
 & XXX \arrow[ur, "(23)"] \arrow[dl, "\nabla X"] \arrow[dr, "X\nabla"'] \arrow[dd, phantom, "\alpha"] & \\
XX \arrow[dr, "\nabla"'] \arrow[uuur, phantom, "\sigma X"] & & XX \arrow[dl, "\nabla"] \arrow[uuul, phantom, "X\sigma"] \\
 & X &
\end{tikzcd}$$

by ① using 2.3.10.7, ② applying the defining equations 2.4.1.3 and 2.4.1.5, ③+④ cancelling and inserting identities and finally ⑤ recognizing the defining identities 2.4.1.1, 2.4.1.2 and 2.4.1.4.

The center inclusion of the hexagon axiom is more involved. One side computes as

$$\begin{tikzcd}
 & 0X0 \arrow[rrrr, "(132)"] \arrow[d] & & & & X00 \arrow[d] & \\
 & XXX \arrow[rrrr, "(132)"] \arrow[dl, "\nabla X"'] \arrow[dr, "X\nabla"] & & & & XXX \arrow[dl, "\nabla X"'] \arrow[dr, "X\nabla"] & \\
XX \arrow[drrr, "\nabla"'] & \alpha & XX \arrow[rr, "(12)"] \arrow[dr, "\nabla"'] & \sigma & XX \arrow[dl, "\nabla"] & \alpha & XX \arrow[dlll, "\nabla"] \\
 & & & X & & &
\end{tikzcd}$$

$$\overset{①}{=}$$

$$\begin{tikzcd}
0X0 \arrow[rrrr, "(132)"] \arrow[dd] \arrow[ddr] \arrow[ddrr] \arrow[drr] & & & & X00 \arrow[d] \arrow[ddr] & \\
 & & XX0 \arrow[rr, "(132)"] \arrow[dr] \arrow[ddr, "X\rho^{-1}"'] & & X0X \arrow[d] \arrow[ddr] \arrow[dr] & \\
0XX \arrow[d] \arrow[ddrr] & X0 \arrow[r, "(12)"] \arrow[ddr] \arrow[ddrr] & 0X \arrow[dr, "\lambda^{-1}"'] \arrow[ddr] & XXX \arrow[r, "(132)"] \arrow[d, "X\nabla"] & XXX \arrow[dr, "\nabla X"'] \arrow[r, phantom, "\rho X"] & XXX \arrow[d, "X\nabla"] \\
XXX \arrow[drr, "\nabla X"'] & \lambda X & \rho & XX \arrow[rr, "(12)"] \arrow[d, "\nabla"] & \sigma & XX \arrow[dll, "\nabla"] \\
 & & XX \arrow[r, "\nabla"'] & X & &
\end{tikzcd}$$

$$\overset{②}{=}$$

$$\begin{tikzcd}
0X0 \arrow[rrrr, "(132)"] \arrow[ddd] \arrow[dddr] \arrow[dddrr] \arrow[drr] & & & & X00 \arrow[d] \arrow[ddr] & \\
 & & XX0 \arrow[rr, "(132)"] \arrow[dr] & & X0X \arrow[d] \arrow[dd, bend right, "\rho^{-1}X"'] \arrow[dd, bend left, "\rho X"] \arrow[dr] & \\
 & & & XX \arrow[dr] & XXX \arrow[d, "\nabla X"] & XXX \arrow[dl, "X\nabla"] \arrow[l, phantom, "X\lambda^{-1}"] \\
0XX \arrow[d] \arrow[ddr, "\lambda X"] & X0 \arrow[r, "(12)"] \arrow[dd] \arrow[ddr] & 0X \arrow[r, "(12)"] \arrow[ur] \arrow[d] \arrow[dd, bend right, "\lambda^{-1}"'] \arrow[dd, bend left, "\lambda"] & X0 \arrow[r, "(12)"] \arrow[ddl] & XX \arrow[ddll, "\nabla"] & \\
XXX \arrow[dr, "\nabla X"'] & \rho & XX \arrow[d, "\nabla"] & \rho^{-1} & & \\
 & XX \arrow[r, "\nabla"'] & X & & &
\end{tikzcd}$$

$$
= \quad
\begin{tikzcd}
0X0 \arrow[rr, "(132)"] \arrow[d] & & X00 \arrow[d] \\
0XX \arrow[rr, "(132)"] \arrow[d] \arrow[dr] & & XX0 \arrow[d] \arrow[dl] \\
XXX \arrow[r, "\nabla X"'] \arrow[ur, phantom, "\lambda X"] & XX \arrow[d, "\nabla"] & XXX \arrow[l, "X\nabla"] \arrow[ul, phantom, "X\lambda^{-1}"'] \\
& X &
\end{tikzcd}
$$

by ① using 2.4.1.4 and 2.4.1.3 and then ② using the 2-naturality of the permutations and 2.4.1.2. The other side of the hexagon axiom reduces as

$$
\begin{tikzcd}
0X0 \arrow[rr, "(132)"] \arrow[dr, "(12)"] \arrow[dd] & & X00 \arrow[dd] \\
& X00 \arrow[ur, "(23)"] \arrow[dd] & \\
XXX \arrow[dr, "(12)"] \arrow[dd, "\nabla X"'] & & XXX \arrow[dd, "X\nabla"] \\
& XXX \arrow[ur, "(23)"] \arrow[dl, "\nabla X"] \arrow[dr, "X\nabla"'] \arrow[l, phantom, "\sigma X"] \arrow[r, phantom, "X\sigma"] & \\
XX \arrow[dr, "\nabla"'] \arrow[rr, phantom, "\alpha"] & & XX \arrow[dl, "\nabla"] \\
& X &
\end{tikzcd}
\overset{\textcircled{1}}{=}
\begin{tikzcd}
& 0X0 \arrow[r, "(12)"] \arrow[d] \arrow[dl, bend right] & X00 \arrow[r, "(23)"] \arrow[d] & X00 \arrow[d] \arrow[dr, bend left] & \\
& 0XX \arrow[r, "(12)"] \arrow[dl] \arrow[dd] & X0X \arrow[r, "(23)"] \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & XX0 \arrow[dd] \arrow[dr] & \\
XXX \arrow[dr, "\nabla X"'] \arrow[r, phantom, "\lambda X"] & & XXX \arrow[dl, "\nabla X"] \arrow[dr, "X\nabla"'] \arrow[l, phantom, "\rho^{-1}X"] \arrow[r, phantom, "X\lambda"] & & XXX \arrow[dl, "X\nabla"] \arrow[l, phantom, "X\rho^{-1}"] \\
& XX \arrow[dr, "\nabla"'] \arrow[rr, phantom, "\alpha"] & & XX \arrow[dl, "\nabla"] & \\
& & X & &
\end{tikzcd}
$$

$$
\overset{\textcircled{2}}{=} \quad
\begin{tikzcd}
0X0 \arrow[r, "(12)"] \arrow[d] & X00 \arrow[r, "(12)"] \arrow[d] & X00 \arrow[d] \\
0XX \arrow[r, "(23)"] \arrow[d] \arrow[dr] & X0X \arrow[r, "(23)"] \arrow[d] & XX0 \arrow[d] \arrow[dl] \\
XXX \arrow[r, "\nabla X"'] \arrow[ur, phantom, "\lambda X"] & XX \arrow[d, "\nabla"] & XXX \arrow[l, "X\nabla"] \arrow[ul, phantom, "X\rho^{-1}"'] \\
& X &
\end{tikzcd}
$$

by ① using the defining equations 2.4.1.1 and 2.4.1.2 and ② inserting the equation 2.4.1.6. That the both sides of the hexagon axiom coincide is now immediate from the observation 2.3.10.3.

Finally we haved checked all axioms of a symmetric pseudomonoid. □

Lemma 2.4.2

Let $\mathcal{C}$ be a 2-category with semi-weak zero object and semi-weak 2-biproducts.

Then every morphisms $f : X \to Y$ can be promoted to a pseudomorphism of symmetric pseudomonoids (with respect to the pseudomonoid structure defined in lemma 2.4.1).

Proof The weak universal property of the zero object and the weak 2-naturality of the codiagonal described in proposition 2.1.12 involve 2-cells

$$\begin{tikzcd}
& 0 \arrow[dl] \arrow[dr] \arrow[d, phantom, "\omega"] & & & XX \arrow[r, "ff"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\mu"] & YY \arrow[d, "\nabla"] \\
X \arrow[rr, "f"'] & {} & Y & \text{and} & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The desired pasting identities

$$\begin{tikzcd}
0X \arrow[r, "0f"] \arrow[d] \arrow[dr, phantom, "f\omega"] & 0Y \arrow[d] \arrow[dd, bend left=60, "\lambda"'] & & 0X \arrow[r, "0f"] \arrow[d] \arrow[dd, bend left=60, "\lambda"'] & 0Y \arrow[dd, bend left=60] \\
XX \arrow[r, "ff"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\mu"] & YY \arrow[d, "\nabla"] & = & XX \arrow[d, "\nabla"'] & \\
X \arrow[r, "f"'] & Y & & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X0 \arrow[r, "f0"] \arrow[d] \arrow[dr, phantom, "f\omega"] & Y0 \arrow[d] \arrow[dd, bend left=60, "\rho"'] & & X0 \arrow[r, "f0"] \arrow[d] \arrow[dd, bend left=60, "\rho"'] & Y0 \arrow[dd, bend left=60] \\
XX \arrow[r, "ff"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\mu"] & YY \arrow[d, "\nabla"] & = & XX \arrow[d, "\nabla"'] & \\
X \arrow[r, "f"'] & Y & & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

are a mere rewriting of the defining property of the 2-cell μ (cf. proposition 2.1.12).

For the associativity and symmetry axioms we again use the weak universal property of the ternary biproduct. However, instead of checking the resulting four pasting identities directly, we check them after precomposing certain invertible 2-cells. For example we compute

$$\begin{tikzcd}
X0X \arrow[rr, "f0f"] \arrow[d] \arrow[drr, phantom, "f\omega f"] & & Y0Y \arrow[d] & & & X0X \arrow[r, "f0f"] \arrow[dr] \arrow[dd] \arrow[ddr, bend right] & Y0Y \arrow[dr] \arrow[d, phantom, "f\omega f"] & \\
XXX \arrow[rr, "fff"] \arrow[dr, "X\nabla"] \arrow[dd, "\nabla X"'] & & YYY \arrow[dr, "Y\nabla"] \arrow[d, phantom, "f\mu"] & & & & XXX \arrow[r, "fff"] \arrow[d, "X\nabla"] \arrow[dl, phantom, "X\lambda^{-1}"] & YYY \arrow[d, "Y\nabla"] \arrow[dl, phantom, "f\mu"] \\
{} \arrow[r, phantom, "\alpha"] & XX \arrow[rr, "ff"'] \arrow[dd, "\nabla"] \arrow[drr, phantom, "\mu"] & {} & YY \arrow[dd, "\nabla"] & = & XXX \arrow[r, "\nabla X"'] \arrow[u, phantom, "\rho X"] & XX \arrow[r, "ff"'] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\mu"] & YY \arrow[d, "\nabla"] \\
XX \arrow[dr, "\nabla"'] & & & & & & X \arrow[r, "f"'] & Y \\
& Y \arrow[rr, "f"'] & & Y & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
& X0X \arrow[r, "f0f"] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\rho X"'] & Y0Y \arrow[r] \arrow[dr] \arrow[dr, phantom, "Y\lambda^{-1}"] & YYY \arrow[d, "Y\nabla"] \\
= & XXX \arrow[r, "\nabla X"'] & XX \arrow[r, "ff"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\mu"] & YY \arrow[d, "\nabla"] \\
& & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
& X0X \arrow[r, "f0f"] \arrow[d] \arrow[dr, phantom, "f\omega f"] & Y0Y \arrow[r] \arrow[d] \arrow[ddr, bend left] \arrow[r, phantom, "Y\lambda^{-1}"', shift right=3] & YYY \arrow[dd, "Y\nabla"] \\
& XXX \arrow[r, "fff"] \arrow[dr, "\nabla X"'] \arrow[dr, phantom, "\mu f"] & YYY \arrow[dr, "\nabla Y"'] \arrow[r, phantom, "\rho Y"] & \\
= & & XX \arrow[r, "ff"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\mu"] & YY \arrow[d, "\nabla"] \\
& & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
& X0X \arrow[rr, "f0f"] \arrow[d] \arrow[drr, phantom, "f\omega f"] & & Y0Y \arrow[d] & \\
& XXX \arrow[rr, "fff"'] \arrow[dd, "\nabla X"'] \arrow[ddrr, phantom, "\mu f"] & & YYY \arrow[dd, "\nabla Y"'] \arrow[dr, "Y\nabla"] & \\
= & & & & YY \arrow[dd, "\nabla"] \arrow[dl, phantom, "\alpha"] \\
& XX \arrow[rr, "ff"] \arrow[dr, "\nabla"'] \arrow[drr, phantom, "\mu"] & & YY \arrow[dr, "\nabla"] & \\
& & X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

The center inclusion is similar, although more complicated, so we break it into two parts. The left side of the axiom reduces as

$$\begin{tikzcd}
0X0 \arrow[rr, "0f0"] \arrow[d] \arrow[drr, phantom, "\omega f\omega"] & & 0Y0 \arrow[d] & & \\
XXX \arrow[rr, "fff"] \arrow[dr, "X\nabla"] \arrow[dd, "\nabla X"'] \arrow[drr, phantom, "f\mu"] & & YYY \arrow[dr, "Y\nabla"] & & \\
& XX \arrow[rr, "ff"'] \arrow[dd, "\nabla"] \arrow[ddrr, phantom, "\mu"] \arrow[dl, phantom, "\alpha"] & & YY \arrow[dd, "\nabla"] & = \\
XX \arrow[dr, "\nabla"'] & & & & \\
& X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

$$\begin{tikzcd}
0X0 \arrow[r] \arrow[rrrr, bend left=50, "0f0"] \arrow[dd] \arrow[dddr] \arrow[drr] \arrow[r, phantom, "\omega f0", shift left=6] & XX0 \arrow[rr] \arrow[rrr, bend left, "ff0"] \arrow[ddrr] \arrow[rr, phantom, "ff\omega", shift left=6] & & XXX \arrow[r, "fff"] \arrow[dd, "X\nabla"] \arrow[d, phantom, "X\rho^{-1}X"'] \arrow[ddr, phantom, "f\mu"] & YYY \arrow[dd, "Y\nabla"] \\
& & 0X \arrow[dr] \arrow[dddr] \arrow[ddr, phantom, "\lambda^{-1}X"] & & \\
0XX \arrow[dd] \arrow[ddr] \arrow[ddr, phantom, "\lambda XX"'] & & & XX \arrow[r, "ff"] \arrow[dd, "\nabla"] \arrow[ddr, phantom, "\mu"] & YY \arrow[dd, "\nabla"] \\
& X0 \arrow[d] \arrow[drr] \arrow[d, phantom, "\rho X", shift left=4] & & & \\
XXX \arrow[r, "\nabla X"'] & XX \arrow[rr, "\nabla"'] & & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
& & & 0Y0 \arrow[dr] \arrow[drr, bend left] & & \\
& 0X0 \arrow[urr, bend left, "0f0"] \arrow[r] \arrow[dd] \arrow[dddr] \arrow[drr] & XX0 \arrow[rr, "ff0"] \arrow[ddrr] \arrow[u, phantom, "\omega f0"] & & YY0 \arrow[r] \arrow[ddr] \arrow[dr, phantom, "Y\rho^{-1}Y"] & YYY \arrow[dd, "Y\nabla"] \\
& & & 0X \arrow[dr] \arrow[dddr] \arrow[ddr, phantom, "\lambda^{-1}X"] & & \\
= & 0XX \arrow[dd] \arrow[ddr] \arrow[ddr, phantom, "\lambda XX"'] & & & XX \arrow[r, "ff"] \arrow[dd, "\nabla"] \arrow[ddr, phantom, "\mu"] & YY \arrow[dd, "\nabla"] \\
& & X0 \arrow[d] \arrow[drr] \arrow[d, phantom, "\rho X", shift left=4] & & & \\
& XXX \arrow[r, "\nabla X"'] & XX \arrow[rr, "\nabla"'] & & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$= \begin{tikzcd}
0X0 \arrow[r, "0f0"] \arrow[ddd] \arrow[ddddr] \arrow[ddrr] & 0Y0 \arrow[rr] \arrow[drr] & & YY0 \arrow[r] \arrow[dddr] \arrow[dr, phantom, "Y\rho^{-1}Y"] & YYY \arrow[ddd, "Y\nabla"] \\
& & & 0Y \arrow[ddr] & \\
& & 0X \arrow[ur, "0f"] \arrow[dr] \arrow[dddr, bend right] & \omega f & \\
0XX \arrow[dd] \arrow[ddr] & & & XX \arrow[r, "ff"] \arrow[dd, "\nabla"] \arrow[ddr, phantom, "\mu"] & YY \arrow[dd, "\nabla"] \\
{} \arrow[r, phantom, "\lambda XX"] & X0 \arrow[d] \arrow[drr] \arrow[dr, phantom, "\rho X"] & \lambda^{-1}X & & \\
XXX \arrow[r, "\nabla X"'] & XX \arrow[rr, "\nabla"'] & {} & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$$= \begin{tikzcd}
0X0 \arrow[r, "0f0"] \arrow[dd] \arrow[dddr] \arrow[drr] & 0Y0 \arrow[rr] \arrow[ddrr] & & YY0 \arrow[r] \arrow[ddr] \arrow[dr, phantom, "Y\rho^{-1}Y"] & YYY \arrow[dd, "Y\nabla"] \\
& & 0X \arrow[dr, "0f"'] \arrow[dddr] & & \\
0XX \arrow[dd] \arrow[ddr] & & & 0Y \arrow[r] \arrow[ddr] \arrow[dr, phantom, "\lambda^{-1}Y"] & YY \arrow[dd, "\nabla"] \\
{} \arrow[r, phantom, "\lambda XX"] & X0 \arrow[d] \arrow[drr] \arrow[dr, phantom, "\rho X"] & & & {} \\
XXX \arrow[r, "\nabla X"'] & XX \arrow[rr, "\nabla"'] & {} & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The right side reduces as

$$\begin{tikzcd}
0X0 \arrow[r, "0f0"] \arrow[d] \arrow[dr, phantom, "\omega f\omega"] & 0Y0 \arrow[d] & \\
XXX \arrow[r, "fff"'] \arrow[dd, "\nabla X"'] \arrow[ddr, phantom, "\mu f"] & YYY \arrow[dd, "\nabla Y"'] \arrow[dr, "Y\nabla"] \arrow[dddr, phantom, "\alpha"] & \\
& & YY \arrow[dd, "\nabla"] \\
XX \arrow[r, "ff"] \arrow[dr, "\nabla"'] \arrow[drr, phantom, "\mu"] & YY \arrow[dr, "\nabla"'] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
=
\begin{tikzcd}
0X0 \arrow[r, "0f0"] \arrow[d] \arrow[dr, phantom, "0f\omega"] & 0Y0 \arrow[r] \arrow[d] \arrow[dddr] \arrow[ddrr] & YY0 \arrow[rr] \arrow[ddrr] \arrow[dr, phantom, "Y\rho^{-1}"] & & YYY \arrow[dd, "Y\nabla"] \\
0XX \arrow[r, "0ff"] \arrow[d] \arrow[dr, phantom, "\omega ff"] & 0YY \arrow[d] \arrow[dddr, bend left] & & {} & \\
XXX \arrow[r, "fff"] \arrow[dd, "\nabla X"'] \arrow[ddr, phantom, "\mu f"] & YYY \arrow[ddr, "\nabla Y"'] \arrow[r, phantom, "\lambda Y"] & {} & 0Y \arrow[r] \arrow[dddr] \arrow[ddr, phantom, "\lambda^{-1}"] & YY \arrow[ddd, "\nabla"] \\
& & Y0 \arrow[d] \arrow[ddrr] & & \\
XX \arrow[rr, "ff"'] \arrow[d, "\nabla"'] \arrow[drr, phantom, "\mu"] & {} & YY \arrow[drr, "\nabla"'] \arrow[r, phantom, "\rho"] & {} & {} \\
X \arrow[rrrr, "f"'] & & {} & & Y
\end{tikzcd}$$

$$= \begin{tikzcd}
0X0 \arrow[rr, "0f0"] \arrow[d] \arrow[rrd, phantom, "0f\omega"] & & 0Y0 \arrow[rr] \arrow[d] \arrow[ddrr] \arrow[ddrrr] & & YY0 \arrow[rr] \arrow[ddrr] \arrow[drr, phantom, "Y\rho^{-1}"] & & YYY \arrow[dd, "Y\nabla"] \\
0XX \arrow[rr, "0ff"] \arrow[d] \arrow[ddrr, bend left] & & 0YY \arrow[ddrr, bend left] & & & & \\
XXX \arrow[drr, "\nabla X"'] & \lambda Y & & & Y0 \arrow[d] \arrow[ddrr] & 0Y \arrow[r] \arrow[ddr] \arrow[dr, phantom, "\lambda^{-1}"] & YY \arrow[dd, "\nabla"] \\
& & XX \arrow[rr, "ff"'] \arrow[d, "\nabla"'] \arrow[drr, phantom, "\mu"] & & YY \arrow[drr, "\nabla"'] & \rho & \\
& & X \arrow[rrrr, "f"'] & & & & Y
\end{tikzcd}$$

$$= \begin{tikzcd}
0X0 \arrow[rr, "0f0"] \arrow[d] \arrow[ddrr] & & 0Y0 \arrow[rr] \arrow[ddrr] \arrow[ddrrr] & & YY0 \arrow[rr] \arrow[ddrr] \arrow[drr, phantom, "Y\rho^{-1}"] & & YYY \arrow[dd, "Y\nabla"] \\
0XX \arrow[d] \arrow[ddrr, bend left] & & & & & & \\
XXX \arrow[drr, "\nabla X"'] & \lambda Y & X0 \arrow[rr, "f0"] \arrow[d] \arrow[drr, phantom, "f\omega"] & & Y0 \arrow[d] \arrow[ddrr] & 0Y \arrow[r] \arrow[ddr] \arrow[dr, phantom, "\lambda^{-1}"] & YY \arrow[dd, "\nabla"] \\
& & XX \arrow[rr, "ff"'] \arrow[drr, "\nabla"'] & & YY \arrow[drr, "\nabla"'] \arrow[d, phantom, "\mu"] & \rho & \\
& & & & X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

$$= \begin{tikzcd}
0X0 \arrow[rr, "0f0"] \arrow[d] \arrow[drr] \arrow[drrr] & & 0Y0 \arrow[rr] \arrow[drr] & & YY0 \arrow[rr] \arrow[drr] \arrow[drr, phantom, "Y\rho^{-1}Y"] & & YYY \arrow[d, "Y\nabla"] \\
0XX \arrow[d] \arrow[drr] \arrow[drr, phantom, "\lambda XX"] & & X0 \arrow[d] \arrow[drr] \arrow[dr, phantom, "\rho X"] & 0X \arrow[r, "0f"'] \arrow[dr] & 0Y \arrow[rr] \arrow[drr] \arrow[drr, phantom, "\lambda^{-1}Y"] & & YY \arrow[d, "\nabla"] \\
XXX \arrow[rr, "\nabla X"'] & & XX \arrow[rr, "\nabla"'] & & X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

For the symmetry it suffices to check the first inclusion, the second inclusion is analogous.

We compute

$$\begin{tikzcd}
X0 \arrow[rr, "f0"] \arrow[d] \arrow[drr, phantom, "f\omega"] & & Y0 \arrow[d] \arrow[dr, "(12)"] & \\
XX \arrow[rr, "ff"] \arrow[dr, "(12)"] \arrow[ddr, bend right, "\nabla"'] \arrow[ddr, phantom, "\sigma"] & & YY \arrow[dr, "(12)"] & 0Y \arrow[d] \\
& XX \arrow[rr, "ff"] \arrow[d, "\nabla"] \arrow[drr, phantom, "\mu"] & & YY \arrow[d, "\nabla"] \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}
=
\begin{tikzcd}
X0 \arrow[rr, "f0"] \arrow[dr, "(12)"] \arrow[dd] \arrow[dddr] & & Y0 \arrow[dr, "(12)"] & \\
& 0X \arrow[rr, "0f"] \arrow[dd] \arrow[dr] \arrow[d, phantom, "\rho"'] \arrow[dd, phantom, "\lambda^{-1}"] & & 0Y \arrow[d] \arrow[dl, phantom, "\omega f"] \\
XX \arrow[dr, "\nabla"'] & & XX \arrow[r, "ff"] \arrow[dl, "\nabla"] \arrow[dr, phantom, "\mu"] & YY \arrow[d, "\nabla"] \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

$$=
\begin{tikzcd}
X0 \arrow[rr, "f0"] \arrow[dr, "(12)"] \arrow[dd] \arrow[dddr] & & Y0 \arrow[dr, "(12)"] & \\
& 0X \arrow[rr, "0f"] \arrow[dd] \arrow[d, phantom, "\rho"'] & & 0Y \arrow[d] \arrow[dd, bend right=60] \arrow[dd, phantom, "\lambda^{-1}"'] \\
XX \arrow[dr, "\nabla"'] & & & YY \arrow[d, "\nabla"] \\
& X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

$$=
\begin{tikzcd}
X0 \arrow[r, "f0"] \arrow[d] \arrow[dr, phantom, "f\omega"] & Y0 \arrow[r, "(12)"] \arrow[d] \arrow[ddr, bend left] \arrow[dr, phantom, "\rho"] & 0Y \arrow[dd] \arrow[dr] \arrow[dr, phantom, "\lambda^{-1}"'] & \\
XX \arrow[r, "ff"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\mu"] & YY \arrow[dr, "\nabla"] & & YY \arrow[dl, "\nabla"] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd}
=
\begin{tikzcd}
X0 \arrow[r, "f0"] \arrow[d] \arrow[dr, phantom, "f\omega"] & Y0 \arrow[d] \arrow[dr, "(12)"] & \\
XX \arrow[r, "ff"'] \arrow[dd, "\nabla"'] \arrow[ddr, phantom, "\mu"] & YY \arrow[dr, "(12)"] \arrow[ddr, bend right, "\nabla"'] \arrow[ddr, phantom, "\sigma"] & 0Y \arrow[d] \\
& & YY \arrow[d, "\nabla"] \\
X \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

□

Lemma 2.4.3
Let $\mathcal{C}$ be a 2-category with semi-weak zero object and semi-weak 2-biproducts.

Then every 2-cell $\chi : f \Rightarrow g : X \to Y$ is compatible with the pseudomorphism structure defined in lemma 2.4.2.

Proof The desired pasting identities

$$\begin{tikzcd}
XX \arrow[r, bend left, "gg"] \arrow[r, bend right, "ff"'] \arrow[r, phantom, "\Uparrow \chi\chi"] \arrow[d, "\nabla_X"'] & YY \arrow[d, "\nabla_Y"] \\
X \arrow[r, bend right, "f"'] \arrow[ur, phantom, "\nearrow \mu_f"] & Y
\end{tikzcd}
=
\begin{tikzcd}
XX \arrow[r, bend left, "gg"] \arrow[d, "\nabla_X"'] \arrow[dr, phantom, "\nearrow \mu_g"] & YY \arrow[d, "\nabla_Y"] \\
X \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Uparrow \chi"] & Y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& 0 \arrow[dl] \arrow[dr] & \\
X \arrow[rr, bend left, "g"] \arrow[rr, bend right, "f"'] \arrow[rr, phantom, "\Uparrow \chi"] \arrow[ur, phantom, "\nearrow \omega"] & & Y
\end{tikzcd}
=
\begin{tikzcd}
& 0 \arrow[dl] \arrow[dr] & \\
X \arrow[rr, "f"'] \arrow[ur, phantom, "\nearrow \omega"] & & Y
\end{tikzcd}$$

are simply a restatement of the weak 2-naturality of the codiagonal (cf. proposition 2.1.12) and of the weak universal property of the zero object. ☐

Lemma 2.4.4

Let $\mathcal{C}$ be a 2-category with semi-weak zero object and semi-weak 2-biproducts.

For an object X the biproduct $X \oplus X$ admits two pseudomonoid structures, one given by lemma 2.4.1, which we will denote by $(\widehat{\lambda}, \widehat{\rho}, \widehat{\alpha}, \widehat{\sigma})$, and one given by remark 2.2.8, denoted as $(\lambda\lambda, \rho\rho, \alpha\alpha, \sigma\sigma)$. The two canonical pseudomorphism structures are isomorphic in $\mathrm{SPsMon}(\mathcal{C})$.

Proof By the weak universal property of the biproduct there is a unique 2-cell

$$\begin{tikzcd}
XXXX \arrow[r, equal] \arrow[d, "(23)"'] & (XX)(XX) \arrow[dd, "\widehat{\nabla}"] \\
XXXX \arrow[d, "\nabla\nabla"'] \arrow[r, phantom, "\Rightarrow \tau"] & \\
XX \arrow[r, equal] & XX
\end{tikzcd}$$

induced by the requirements that

$$\begin{tikzcd}
XX00 \arrow[r] \arrow[d, "(23)"'] & XXXX \arrow[r, equal] \arrow[d, "(23)"'] & XXXX \arrow[dd, "\widehat{\nabla}"] \\
X0X0 \arrow[r] & XXXX \arrow[d, "\nabla\nabla"'] \arrow[r, phantom, "\tau"] & \\
& XX \arrow[r, equal] & XX
\end{tikzcd}
=
\begin{tikzcd}
XX00 \arrow[rr] \arrow[d, "(23)"'] \arrow[ddrr] & & XXXX \arrow[dd, "\widehat{\nabla}"] \arrow[dl, phantom, "\widehat{\rho}^{-1}"] \\
X0X0 \arrow[d] \arrow[drr] & & \\
XXXX \arrow[rr, "\nabla\nabla"'] \arrow[ur, phantom, "\rho\rho"] & & XX
\end{tikzcd}$$

as well as

$$\begin{tikzcd}
00XX \arrow[r] \arrow[d, "(23)"'] & XXXX \arrow[r, equal] \arrow[d, "(23)"'] & XXXX \arrow[dd, "\widehat{\nabla}"] \\
0X0X \arrow[r] & XXXX \arrow[d, "\nabla\nabla"'] \arrow[r, phantom, "\tau"] & \\
& XX \arrow[r, equal] & XX
\end{tikzcd}
=
\begin{tikzcd}
00XX \arrow[rr] \arrow[d, "(23)"'] \arrow[ddrr] & & XXXX \arrow[dd, "\widehat{\nabla}"] \arrow[dl, phantom, "\widehat{\lambda}^{-1}"] \\
0X0X \arrow[d] \arrow[drr] & & \\
XXXX \arrow[rr, "\nabla\nabla"'] \arrow[ur, phantom, "\lambda\lambda"] & & XX
\end{tikzcd}$$

Since the morphism is the identity on objects, the unit cell τ_0 is the identity. This makes the two unit axioms true by the very definition of τ. It remains to check that τ is compatible with the two associator 2-cells and the two symmetry 2-cells.

For the symmetry we find that

$$\begin{tikzcd}
XX00 \arrow[r] \arrow[dd, "(23)"'] & XXXX \arrow[rr, equal] \arrow[dd, "(23)"'] \arrow[dr, "(13)(24)"] & & XXXX \arrow[d, "(13)(24)"] \\
& & XXXX \arrow[r, equal] \arrow[d, "(23)"] & XXXX \arrow[dd, "\widehat{\nabla}"] \\
X0X0 \arrow[r] & XXXX \arrow[r, "(12)(34)"] \arrow[dr, "\nabla\nabla"'] \arrow[dr, phantom, "\sigma\sigma", shift left=3ex] & XXXX \arrow[d, "\nabla\nabla"] \arrow[r, phantom, "\tau"] & {} \\
& & XX \arrow[r, equal] & XX
\end{tikzcd}$$

$$= \begin{tikzcd}
& XX00 \arrow[dl, bend right, "(23)"'] \arrow[d, "(1243)"] \arrow[ddr, bend left, "(13)(24)"] & \\
X0X0 \arrow[r, "(12)(34)"] \arrow[dd] \arrow[ddr] \arrow[ddr, phantom, "\rho\rho"', shift right=3ex] & 0X0X \arrow[d] \arrow[dd, bend right=60] \arrow[dd, bend left=60] \arrow[d, phantom, "\lambda^{-1}\lambda^{-1}"', shift right=4ex] \arrow[d, phantom, "\lambda", shift left=3ex] & \\
& XXXX \arrow[d, "\nabla\nabla"] & 00XX \arrow[ul, "(23)"'] \arrow[dl] \arrow[d] \arrow[dl, phantom, "\widehat{\lambda}^{-1}", shift right=3ex] \\
XXXX \arrow[r, "\nabla\nabla"'] & XX & XXXX \arrow[l, "\widehat{\nabla}"]
\end{tikzcd}
= \begin{tikzcd}
& XX00 \arrow[dl, bend right, "(23)"'] \arrow[d] \arrow[dr, bend left, "(13)(24)"] \arrow[dd, bend right=60] \arrow[dd, bend left=60] \arrow[d, phantom, "\widehat{\rho}^{-1}"', shift right=3ex] \arrow[d, phantom, "\widehat{\rho}", shift left=3ex] & \\
X0X0 \arrow[d] \arrow[dr] \arrow[dr, phantom, "\rho\rho"', shift right=3ex] & XXXX \arrow[d, "\widehat{\nabla}"] & 00XX \arrow[dl] \arrow[d] \arrow[dl, phantom, "\widehat{\lambda}^{-1}", shift left=3ex] \\
XXXX \arrow[r, "\nabla\nabla"'] & XX & XXXX \arrow[l, "\widehat{\nabla}"]
\end{tikzcd}$$

$$= \begin{tikzcd}
XX00 \arrow[r] & XXXX \arrow[r, equal] \arrow[d, "(23)"'] & XXXX \arrow[dr, "(13)(24)"] \arrow[ddr, bend right, "\widehat{\nabla}"'] & \\
& XXXX \arrow[d, "\nabla\nabla"'] \arrow[r, phantom, "\tau"] & {} \arrow[r, phantom, "\widehat{\sigma}"] & XXXX \arrow[d, "\widehat{\nabla}"] \\
& XX \arrow[rr, equal] & & XX
\end{tikzcd}$$

by inserting the defining equations for $\sigma\sigma$, τ and $\widehat{\sigma}$. The other inclusion is analogous, hence the weak universal property of the biproduct asserts the desired pasting identity.

It is left to show the compatibility of the associator 2-cells. We show the required pasting identity by considering the first and second inclusions of XX into $XXXXXX$, the third inclusion is analogous to the first.

For the first inclusion we have that

$$\begin{tikzcd}
XX0000 \arrow[r] & XXXXXX \arrow[rr, equal] \arrow[d, "(23)"'] & & XXXXXX \arrow[dd, "\widehat{\nabla}XX"'] \arrow[dr, "XX\widehat{\nabla}"] & \\
& XXXXXX \arrow[d, "\nabla\nabla XX"'] & \tau XX & & XXXX \arrow[ddd, "\widehat{\nabla}"] \\
& XXXX \arrow[rr, equal] \arrow[dr, "(23)"'] & & XXXX \arrow[ddr, "\widehat{\nabla}"] & \widehat{\alpha} \\
& & XXXX \arrow[dr, "\nabla\nabla"'] & \tau & \\
& & & XX \arrow[r, equal] & XX
\end{tikzcd}$$

$$\begin{tikzcd}
& XX0000 \arrow[r] \arrow[d, "(23)"'] & XX00XX \arrow[r] \arrow[d, "(23)"'] \arrow[ddr] & XXXXXX \arrow[dd, "\widehat{\nabla}XX"] \arrow[dr, "XX\widehat{\nabla}"] & \\
& X0X000 \arrow[r] \arrow[d] & X0X0XX \arrow[d] \arrow[dr] & \widehat{\rho}^{-1}XX & XXXX \arrow[d, "\widehat{\nabla}"] \\
= & XXXX00 \arrow[r] & XXXXXX \arrow[r, "\nabla\nabla XX"'] & XXXX \arrow[r, "\widehat{\nabla}"'] \arrow[d, "(23)"'] & XX \arrow[dd, equal] \\
& & \rho\rho XX & XXXX \arrow[d, "\nabla\nabla"'] & \tau \\
& & & XX \arrow[r, equal] & XX
\end{tikzcd}$$

$$\begin{tikzcd}
& XX0000 \arrow[r] \arrow[d, "(23)"'] & XX00XX \arrow[r] \arrow[d, "(23)"'] \arrow[ddr] \arrow[drr, "XX\widehat{\lambda}^{-1}"] & XXXXXX \arrow[dr, "XX\widehat{\nabla}"] & \\
& X0X000 \arrow[r] \arrow[d] & X0X0XX \arrow[d] \arrow[dr] & & XXXX \arrow[d, "\widehat{\nabla}"] \\
= & XXXX00 \arrow[r] & XXXXXX \arrow[r, "\nabla\nabla XX"'] & XXXX \arrow[r, "\widehat{\nabla}"'] \arrow[d, "(23)"'] & XX \arrow[dd, equal] \\
& & \rho\rho XX & XXXX \arrow[d, "\nabla\nabla"'] & \tau \\
& & & XX \arrow[r, equal] & XX
\end{tikzcd} \tag{2.4.4.1}$$

while at the same time

$$\begin{tikzcd}
XX0000 \arrow[r] \arrow[d, "(2453)"'] & XXXXXX \arrow[rr, equal] \arrow[d, "(2453)"'] \arrow[dr, "(45)"] & & XXXXXX \arrow[dr, "XX\widehat{\nabla}"] \arrow[d, phantom, "XX\tau"] & \\
X00X00 \arrow[r] & XXXXXX \arrow[r, "(234)"] \arrow[d, "\nabla X\nabla X"'] \arrow[drr, "X\nabla X\nabla"] & XXXXXX \arrow[r, "XX\nabla\nabla"] & XXXX \arrow[r, equal] \arrow[d, "(23)"'] & XXXX \arrow[dd, "\widehat{\nabla}"] \\
& XXXX \arrow[r, phantom, "\alpha\alpha"] \arrow[drr, "\nabla\nabla"'] & & XXXX \arrow[d, "\nabla\nabla"'] \arrow[r, phantom, "\tau"] & \\
& & & XX \arrow[r, equal] & XX
\end{tikzcd}$$

$$= \quad \begin{tikzcd}
XX0000 \arrow[r] \arrow[d, "(2453)"'] & XX00XX \arrow[r] \arrow[ddl, "(2453)"'] & XXXXXX \arrow[rr, equal] \arrow[d, "(45)"] \arrow[ddl, "(2453)"'] & & XXXXXX \arrow[dd, "XX\widehat{\nabla}"] \\
X00X00 \arrow[d] & & XXXXXX \arrow[dr, "XX\nabla\nabla"] & XX\tau & \\
X0XX0X \arrow[r] \arrow[d] \arrow[drr, "X\lambda^{-1}X\lambda^{-1}"] \arrow[dr, phantom, "\rho X\rho X"] & XXXXXX \arrow[ur, "(243)"'] \arrow[dr, bend left, "X\nabla X\nabla"] & & XXXX \arrow[r, equal] \arrow[dl, "(23)"] & XXXX \arrow[dd, "\widehat{\nabla}"] \\
XXXXXX \arrow[rr, "\nabla X\nabla X"'] & & XXXX \arrow[dr, "\nabla\nabla"'] & \tau & \\
& & & XX \arrow[r, equal] & XX
\end{tikzcd}$$

$$= \quad \begin{tikzcd}
XX0000 \arrow[r] \arrow[d, "(2453)"'] & XX00XX \arrow[rr] \arrow[d, "(45)"] \arrow[ddl, "(2453)"'] \arrow[dddrrr, bend left] & & XXXXXX \arrow[dddr, "XX\widehat{\nabla}"] \arrow[d, phantom, "XX\widehat{\lambda}^{-1}"] & \\
X00X00 \arrow[d] & XX0X0X \arrow[dr, "XX\lambda\lambda"] \arrow[ddrr, bend left] \arrow[ddrr, bend right, "XX\lambda^{-1}\lambda^{-1}"'] & & & \\
X0XX0X \arrow[ur, "(234)"'] \arrow[d] \arrow[dr, "\rho X\rho X"] & & XXXXXX \arrow[dr, "XX\nabla\nabla"] & & \\
XXXXXX \arrow[r, "\nabla X\nabla X"'] & XXXX \arrow[rr, "(23)"'] \arrow[drr, equal] & & XXXX \arrow[r, equal] \arrow[d, "(23)"'] & XXXX \arrow[dd, "\widehat{\nabla}"] \\
& & & XXXX \arrow[d, "\nabla\nabla"'] \arrow[r, phantom, "\tau"] & \\
& & & XX \arrow[r, equal] & XX
\end{tikzcd} \tag{2.4.4.2}$$

By inspection the 2-cells in 2.4.4.1 and 2.4.4.2 are equal. This shows that the required pasting identity for the associators holds true when precomposed with the first (and analogously the third inclusion). It is left to show that this is true when precomposed with the second inclusion. There we have that one side computes as

$$\begin{tikzcd}
00XX00 \arrow[r] & XXXXXX \arrow[rr, equal] \arrow[d, "(23)"'] & & XXXXXX \arrow[dd, "\widehat{\nabla}XX"'] \arrow[dr, "XX\widehat{\nabla}"] & \\
& XXXXXX \arrow[d, "\nabla\nabla XX"'] & \tau XX & & XXXX \arrow[ddd, "\widehat{\nabla}"] \\
& XXXX \arrow[rr, equal] \arrow[dr, "(23)"'] & & XXXX \arrow[ddr, "\widehat{\nabla}"] & \widehat{\alpha} \\
& & XXXX \arrow[dr, "\nabla\nabla"'] & \tau & \\
& & & XX \arrow[r, equal] & XX \\
& & \lambda\lambda XX & &
\end{tikzcd}$$

$$\begin{tikzcd}
& 00XX00 \arrow[r] \arrow[d, "(23)"'] & 00XXXX \arrow[r] \arrow[d, "(23)"'] \arrow[ddr] & XXXXXX \arrow[dd, "\widehat{\nabla}XX"] \arrow[dr, "XX\widehat{\nabla}"] & \\
& 0X0X00 \arrow[r] \arrow[d] & 0X0XXX \arrow[d] \arrow[dr] & \widehat{\lambda}^{-1}XX & XXXXXX \arrow[dd, "\widehat{\nabla}"] \\
= & XXXX00 \arrow[r] \arrow[dr, "\nabla\nabla 00"'] & XXXXXX \arrow[r, "\nabla\nabla XX"'] & XXXX \arrow[d, "(23)"'] \arrow[dr, "\widehat{\nabla}"] & \widehat{\alpha} \\
& & XX00 \arrow[ur] & XXXX \arrow[d, "\nabla\nabla"'] \arrow[r, phantom, "\tau"] & XX \\
& & & XX \arrow[ur, equal] &
\end{tikzcd}$$

$$
= \begin{tikzcd}
00XX00 \arrow[r] \arrow[d, "(23)"'] & 00XXXX \arrow[r, "00\widehat{\nabla}"] \arrow[ddr] & 00XX \arrow[dr] \arrow[dddr, "\widehat{\lambda}^{-1}"] & \\
0X0X00 \arrow[d] \arrow[dr, "\lambda\lambda 00"] & & & XXXX \arrow[dd, "\widehat{\nabla}"] \\
XXXX00 \arrow[r, "\nabla\nabla 00"'] & XX00 \arrow[r] \arrow[d, "(23)"'] \arrow[drr, bend right, "\widehat{\rho}^{-1}"] & XXXX \arrow[dr, "\widehat{\nabla}"] & \\
& X0X0 \arrow[d] \arrow[drr, "\rho\rho"] & & XX \arrow[d, equal] \\
& XXXX \arrow[rr, "\nabla\nabla"'] & & XX
\end{tikzcd}
\tag{2.4.4.3}
$$

while the other side computes as

$$
\begin{tikzcd}
00XX00 \arrow[r] \arrow[d, "(23)(45)"'] & XXXXXX \arrow[rr, equal] \arrow[d, "(23)(45)"'] \arrow[dr, "(45)"] & \arrow[r, phantom, "XX\tau"] & XXXXXX \arrow[dr, "XX\widehat{\nabla}"] & \\
0X00X0 \arrow[r] & XXXXXX \arrow[r, "(23)"] \arrow[d, "\nabla X\nabla X"'] \arrow[drr, "X\nabla X\nabla"] & XXXXXX \arrow[r, "XX\nabla\nabla"] & XXXX \arrow[r, equal] \arrow[d, "(23)"'] & XXXX \arrow[dd, "\widehat{\nabla}"] \\
& XXXX \arrow[drr, "\nabla\nabla"'] & \alpha\alpha & XXXX \arrow[d, "\nabla\nabla"'] \arrow[r, phantom, "\tau"] & \\
& & & XX \arrow[r, equal] & XX
\end{tikzcd}
$$

$$
= \begin{tikzcd}
00XX00 \arrow[r] \arrow[dd, "(23)(45)"'] & XXXX00 \arrow[rr] \arrow[dd, "(23)(45)"'] & & XXXXXX \arrow[rr, equal] \arrow[dd, "(23)(45)"'] \arrow[dr, "(45)"] & & XXXXXX \arrow[dd, "XX\widehat{\nabla}"] \\
& & & & XXXXXX \arrow[dr, "XX\nabla\nabla"] \arrow[r, phantom, "XX\tau"] & \\
0X00X0 \arrow[r] \arrow[d] \arrow[drr] \arrow[ddr] & XX0XX0 \arrow[rr] \arrow[drrr, bend right, "X\rho^{-1}X\rho^{-1}"] & & XXXXXX \arrow[ur, "(23)"] \arrow[dr, "X\nabla X\nabla"] & & XXXX \arrow[dl, "(23)"] \arrow[ddd, "\widehat{\nabla}"] \\
0XX0XX \arrow[dd] \arrow[ddr, "\lambda X\lambda X"] & & 0X0X \arrow[rr] \arrow[ddrr, "\lambda^{-1}\lambda^{-1}"] & & XXXX \arrow[dd, "\nabla\nabla"] \arrow[r, phantom, "\tau"] & \\
& X0X0 \arrow[d] \arrow[drrr, "\rho\rho"] & & & & \\
XXXXXX \arrow[r, "\nabla X\nabla X"'] & XXXX \arrow[rrr, "\nabla\nabla"'] & & & XX \arrow[r, equal] & XX
\end{tikzcd}
$$

$$\begin{tikzcd}
00XX00 \arrow[r] \arrow[dd] \arrow[dddrr] & XXXX00 \arrow[rrr, equal] \arrow[dr] \arrow[dddrrr, bend right] \arrow[dddrrr, bend left] & & & XXXXXX \arrow[ddd, "XX\widehat{\nabla}", "XX\widehat{\rho}^{-1}"' near start] \\
& & XXX0X0 \arrow[dr, "XX\rho\rho"] \arrow[ddrr, bend right] \arrow[ddrr, bend left] & & \\
0X00X0 \arrow[dd] \arrow[ddr] \arrow[ddrr] & & & XXXXXX \arrow[dr, "XX\nabla\nabla", "XX\rho^{-1}\rho^{-1}"'] & \\
& & 00XX \arrow[rr] \arrow[d] \arrow[dddrr] & & XXXX \arrow[ddd, "\widehat{\nabla}"] \\
0XX0XX \arrow[dd] \arrow[ddr] & X0X0 \arrow[dd] \arrow[ddr] & 0X0X \arrow[d] \arrow[dd, bend right=60, "\lambda^{-1}\lambda^{-1}"] \arrow[dd, bend left=60, "\lambda\lambda"'] & \widehat{\lambda}^{-1} & \\
\lambda X\lambda X & \rho\rho & XXXX \arrow[d, "\nabla\nabla"] & & \\
XXXXXX \arrow[r, "\nabla X\nabla X"'] & XXXX \arrow[r, "\nabla\nabla"'] & XX \arrow[rr, equal] & & XX
\end{tikzcd}$$

(2.4.4.4)

The last pasted 2-cell of equation 2.4.4.4 reduces to the last 2-cell of equation 2.4.4.3, so the desired pasting identity holds after precomposing the second inclusion. This finishes the proof of the associativity axiom and thus of the proposition. ☐

2.5 Symmetric 2-Groups

We end the interlude on 2-category theory with a brief discussion of symmetric 2-groups. They are a natural categorification of abelian groups and we will be interested in short exact sequences of them. We begin with the definition.

Definition 2.5.1

A **symmetric 2-group** (synonymously symmetric categorical group or symmetric Picard groupoid) is a symmetric monoidal groupoid $(\mathcal{A}, \otimes, 1)$, such that every object is **invertible** with respect to the tensor product. Specifically, for every object $X \in \mathcal{A}$ there exists some object $X^* \in \mathcal{A}$ such that $X \otimes X^* \cong 1$.

Small symmetric 2-groups, strong monoidal functors and monoidal transformations form a 2-category $\mathsf{S2Grp}$.

When discussing 2-groups it is often stressed that the tensor inverses X^* can be forced to satisfy additional coherences giving rise to so called *dualities*. Using this coherences one actually gets a contravariant endofunctor $(-)^* : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$. We will not need these coherences, so we refrain from introducing them in more detail.

A curious thing about symmetric 2-groups is that the 2-category $\mathsf{S2Grp}$ has naturally occuring semi-weak 2-biproducts as opposed to weak 2-biproducts. This is because the products are strictly described (being computed as in $\mathsf{SMonCat}$) and have the weak universal property of a coproduct (as remarked in [Dup08] section 6.1.1). For the semiweak zero object the claim is immediate: The trivial groupoid $\mathbb{1}$ is both a strict terminal object and a weakly initial object with essentially unique initial functors sending the unique object to the monoidal unit of the target. We will denote this semiweak zero 2-group by $\mathbf{0}$.

In a later section we want to discuss a 2-categorical analogue of the Puppe sequences arising from fiber and cofiber sequences. To do so we will need a notion of exact sequence of symmetric 2-groups. In [Vit02] multiple equivalent definitions are given. They require a notion of kernel or cokernel. Since colimits of categories are harder to describe than limits of categories, we choose to use the characterization involving the kernel.

Definition 2.5.2

Let $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ be a morphism of symmetric 2-groups. The **kernel** of $\mathcal{F}$ is the comma groupoid $\ker \mathcal{F} := 0 \downarrow \mathcal{F}$ as depicted on the right with tensor product of $f : 1 \to \mathcal{F}X$ and $g : 1 \to \mathcal{F}Y$ given by

$$1 \cong 1 \otimes 1 \xrightarrow{f \otimes g} \mathcal{F}X \otimes \mathcal{F}Y \cong \mathcal{F}(X \otimes Y).$$

The tensor inverse of a morphism $f : 1 \to \mathcal{F}X$ is given by the inverse of the dual morphism $f^* : \mathcal{F}(X^*) \cong (\mathcal{F}X)^* \to 1^* = 1$.

$$\begin{tikzcd}
\ker \mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \mathcal{A} \arrow[d, "\mathcal{F}"] \\
0 \arrow[r, "\mathrm{const}_I"'] & \mathcal{B}
\end{tikzcd}$$

Remark 2.5.3

[Vit02] only gives an explicit description of the kernel in section 2 and proves that it has the universal property of the comma, without calling it such. For our purposes it suffices to note that the kernel is the comma in $\mathsf{S2Grp}$ and that it is computed as the comma in Cat.

In 1-dimensional algebra a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of abelian groups is exact, if the image of f equals the kernel of g. This can be rephrased as saying that the composite $g \circ f$ is the zero morphism and that the induced morphism $A \to \ker g$ is surjective. The following definition is a straightforward categorification of this.

Definition 2.5.4
A sequence $\mathcal{A} \xrightarrow{\mathcal{F}} \mathcal{B} \xrightarrow{\mathcal{G}} \mathcal{C}$ of symmetric 2-groups is **exact in** $\mathcal{B}$, if $\mathcal{G} \circ \mathcal{F}$ is a zero-morphism and the canonical functor $\mathcal{A} \longrightarrow \ker \mathcal{G}$ is essentially surjective on objects and full.

Remark 2.5.5
It is noteworthy that the preceding definition makes sense for pointed groupoids as well. In fact, since the kernel of symmetric 2-groups and pointed groupoids are computed as commas in Cat, we can say that a sequence of symmetric 2-groups is exact if and only if the underlying sequence of pointed groupoids is exact.

There is a canonical functor

$$\begin{array}{rcl} \mathsf{S2Grp} & \xrightarrow{\pi_0} & \mathsf{Ab} \\ \mathcal{A} & \longmapsto & \operatorname{Ob}\mathcal{A}/_{\cong} \\ \mathcal{F} & \longmapsto & \mathcal{F} \end{array}$$

sending a symmetric 2-group to the abelian group of isomorphism classes of objects. This functor sends isocommas to pullbacks and essentially surjective functors to surjections. This hints towards the following lemma.

Lemma 2.5.6
Let $\mathcal{A} \xrightarrow{\mathcal{F}} \mathcal{B} \xrightarrow{\mathcal{G}} \mathcal{C}$ be an exact sequence of symmetric 2-groups. Then the induced sequence $\pi_0(\mathcal{A} \xrightarrow{\mathcal{F}} \mathcal{B} \xrightarrow{\mathcal{G}} \mathcal{C})$ is an exact sequence of abelian groups.

Proof Confer [Vit02] proposition 3.1. □

3 Towards 2-Triangulated Categories

3.1 An Enrichment in Symmetric 2-Groups

We have seen that the homotopy 2-category $\mathrm{ho}_2(\mathcal{C})$ of a stable ∞-category $\mathcal{C}$ admits weakly 2-adjoint strict 2-functors $\Omega : \mathrm{ho}_2(\mathcal{C}) \longrightarrow \mathrm{ho}_2(\mathcal{C})$ and $\Sigma : \mathrm{ho}_2(\mathcal{C}) \longrightarrow \mathrm{ho}_2(\mathcal{C})$, which are mutually inverse weak 2-equivalences.

In this section we want to study the properties of these functors in an abstract homotopy 2-category. We begin with the following definition.

Definition 3.1.1

Let $\mathcal{C}$ be a Grpd-category with weak zero object, weak 2-products and quasi-commas.

We say that $\mathcal{C}$ **has quasi-loops**, if there is a strict 2-functor $\Omega : \mathcal{C} \longrightarrow \mathcal{C}$ together with specified quasi-comma squares

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_X"] & 0 \arrow[d] \\
0 \arrow[r] & X
\end{tikzcd}$$

for each object X, such that for every morphisms $f : X \to Y$ the pasting identity

$$\begin{tikzcd}
\Omega X \arrow[dr, "\Omega f"] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[drr, phantom, "\Nearrow"] \arrow[ddr, phantom, "\Nearrow"] & & & & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_X"] & 0 \arrow[d] \arrow[ddr, bend left] \arrow[ddr, phantom, "\Nearrow"] & \\
& \Omega Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_Y"] & 0 \arrow[d] & = & 0 \arrow[r] \arrow[drr, bend right] \arrow[drr, phantom, "\Nearrow"] & X \arrow[dr, "f"] & \\
& 0 \arrow[r] & Y & & & & Y
\end{tikzcd}$$

holds. We call the squares ω_X a **quasi-loop square** and the top left object ΩX a **quasi-loop object**.

Remark 3.1.2

Intuitively we would want the functor Ω to be given by universal property of the quasi-commas. By the weak universal property of the quasi-comma ΩY the condition above determines the morphism Ωf up to a not necessarily unique invertible 2-cell.

In this section we want to discuss how such quasi-loops give rise to symmetric pseudo-groups in $\mathcal{C}$. Since we only discussed pseudomonoids in a cartesian monoidal 2-category with respect to strict products, we assumme without loss of generality that the products are strict and that the zero-object is semi-weak. We can do so by Power's coherence result [Pow89], since the existence of quasi-commas and hence particular quasi-loop squares is invariant under weak 2-equivalences. We might loose the strict 2-functoriality of Ω in doing so, but luckily it is not required for the following arguments.

So let us assume that $\mathcal{C}$ has strict 2-products and a semi-weak zero object. It is immediate from the weak universal property of the quasi-loop squares, that a quasi-loop object is very close to being a symmetric pseudo-group. Indeed we can induce a *multiplication morphism*

$$\begin{tikzcd}
\Omega X\times\Omega X \arrow[r,"\mathrm{pr}_2"] \arrow[d,"\mathrm{pr}_1"'] & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\omega"] & 0 \arrow[dd] \\
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\omega"] & 0 \arrow[dr] & \\
0 \arrow[rr] & & X
\end{tikzcd}
=
\begin{tikzcd}
\Omega X\times\Omega X \arrow[dr,"m"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\omega"] & 0 \arrow[d] \\
& 0 \arrow[r] & X
\end{tikzcd},$$

a *unit morphism*

$$\begin{tikzcd}
0 \arrow[r, equal] \arrow[d, equal] & 0 \arrow[d] \\
0 \arrow[r] & X
\end{tikzcd}
=
\begin{tikzcd}
0 \arrow[dr,"u"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\omega"] & 0 \arrow[d] \\
& 0 \arrow[r] & X
\end{tikzcd}$$

(any initial morphism will do, so we assume u is the chosen initial morphism) and an *inverting morphism*

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\omega^{-1}"] & 0 \arrow[d] \\
0 \arrow[r] & X
\end{tikzcd}
=
\begin{tikzcd}
\Omega X \arrow[dr,"i"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\omega"] & 0 \arrow[d] \\
& 0 \arrow[r] & X
\end{tikzcd}.$$

We obtain invertible 2-cells

$$\begin{tikzcd}
\Omega X \arrow[r,"{(u,\mathrm{id})}"] \arrow[dr, equal] \arrow[dr, phantom, "\Swarrow\lambda", shift left=4] & \Omega X\times\Omega X \arrow[d,"m"] & \Omega X \arrow[l,"{(\mathrm{id},u)}"'] \arrow[dl, equal] \arrow[dl, phantom, "\Searrow\rho"', shift right=4] \\
& \Omega X &
\end{tikzcd}$$

from the pasting identity

$$\begin{tikzcd}
\Omega X \arrow[dr,"{(u,\mathrm{id})}"] \arrow[ddrrr, bend left] \arrow[dddrr, bend right] & & & \\
& \Omega X\times\Omega X \arrow[dr,"m"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\omega"] & 0 \arrow[d] \\
& & 0 \arrow[r] & X
\end{tikzcd}
=
\begin{tikzcd}
\Omega X \arrow[dr,"{(u,\mathrm{id})}"] \arrow[drr, equal, bend left] \arrow[drrr, bend left] \arrow[dd] & & & \\
& \Omega X\times\Omega X \arrow[r] \arrow[d] & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\omega"] & 0 \arrow[dd] \\
0 \arrow[r,"u"] \arrow[dr, equal, bend right] & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\omega"] & 0 \arrow[dr] & \\
& 0 \arrow[rr] & & X
\end{tikzcd}
=
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow\omega"] & 0 \arrow[d] \\
0 \arrow[r] & X
\end{tikzcd}$$

and the analogous one for (id, u). The universal propery gives rise to an invertible 2-cell

$$\begin{tikzcd}
\Omega X \times \Omega X \times \Omega X \arrow[r, "\Omega X \times m"] \arrow[d, "m \times \Omega X"'] \arrow[dr, phantom, "\nearrow \alpha"] & \Omega X \times \Omega X \arrow[d, "m"] \\
\Omega X \times \Omega X \arrow[r, "m"'] & \Omega X
\end{tikzcd}$$

since pasting ω to the lower left composite reduces as

$$\begin{tikzcd}
\Omega X \times \Omega X \times \Omega X \arrow[dr, "m \times \Omega X"] \arrow[ddrrr, bend left] \arrow[dddrr, bend right] & & & \\
& \Omega X \times \Omega X \arrow[dr, "m"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow \omega"] & 0 \arrow[d] \\
& & 0 \arrow[r] & X
\end{tikzcd}
=
\begin{tikzcd}
\Omega X \times \Omega X \times \Omega X \arrow[dr] \arrow[drrr, bend left] \arrow[dddr, bend right] & & & \\
& \Omega X \times \Omega X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \Omega X \arrow[r] \arrow[d] & 0 \arrow[dd] \arrow[ddl, phantom, "\nearrow \omega"] \\
& \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow \omega"] & 0 \arrow[dr] & \\
& 0 \arrow[rr] & & X
\end{tikzcd}$$

$$=
\begin{tikzcd}
\Omega X \times \Omega X \times \Omega X \arrow[d, "\mathrm{pr}_{12}"'] \arrow[dr, "m \times \Omega X"] \arrow[drr, bend left, "\mathrm{pr}_3"] \arrow[drrr, bend left] & & & \\
\Omega X \times \Omega X \arrow[dr, "m"] \arrow[ddr, bend right] & \Omega X \times \Omega X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \Omega X \arrow[r] \arrow[d] & 0 \arrow[dd] \arrow[ddl, phantom, "\nearrow \omega"] \\
& \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow \omega"] & 0 \arrow[dr] & \\
& 0 \arrow[rr] & & X
\end{tikzcd}
=
\begin{tikzcd}
\Omega X \times \Omega X \times \Omega X \arrow[rr, "\mathrm{pr}_3"] \arrow[dr, "\mathrm{pr}_2"] \arrow[dd, "\mathrm{pr}_1"'] & & \Omega X \arrow[r] \arrow[d] & 0 \arrow[ddd] \\
& \Omega X \arrow[r] \arrow[d] & 0 \arrow[ddr] \arrow[r, phantom, "\nearrow \omega"] & {} \\
\Omega X \arrow[r] \arrow[d] & 0 \arrow[drr] \arrow[r, phantom, "\nearrow \omega"] & {} & \\
0 \arrow[rrr] \arrow[ur, phantom, "\nearrow \omega"] & & & X
\end{tikzcd}$$

and so does pasting ω to the upper right composite. Finally

However, by the lack of faithfulness of the defining functor of a quasicomma, we cannot expect these 2-cells to satisfy the axioms of a pseudomonoid. Still the multiplication and unit map make the quasiloop object ΩX an abelian group object in the homotopy category.

Lemma 3.1.3

Let $\mathcal{C}$ be a Grpd-category with quasi-loops. Then each quasi-loop object ΩX is a group object in the homotopy category $\mathrm{ho}(\mathcal{C})$, with group structure given by the morphisms (m, u, i) just described.

Proof The 2-cells λ, ρ, α are witnesses for the commutativity of the required diagrams to make $(\Omega X, m, u)$ a monoid object in $\mathrm{ho}(\mathcal{C})$. We are left to show that the inverting morphism i does provide inverses for the multiplication. To this end we note that there is an invertible

2-cell

$$\begin{tikzcd}
\Omega X \arrow[r, "\Delta"] \arrow[d] & \Omega X \times \Omega X \arrow[r, "i \times \Omega X"] \arrow[d, phantom, "\Nearrow \iota"] & \Omega X \times \Omega X \arrow[d, "m"] \\
0 \arrow[rr, "u"'] & & \Omega X
\end{tikzcd}$$

induced from the pasting identity

$$\begin{tikzcd}
\Omega X \arrow[dr, "\Delta"] \arrow[rrrrddd, bend left] \arrow[rrrdddd, bend right] & & & & \\
& \Omega X \times \Omega X \arrow[dr, "i \times \Omega X"] \arrow[rrrdd, bend left] \arrow[rrddd, bend right] & & & \\
& & \Omega X \times \Omega X \arrow[dr, "m"] \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\
& & & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega"] & 0 \arrow[d] \\
& & & 0 \arrow[r] & X
\end{tikzcd}
=
\begin{tikzcd}
\Omega X \arrow[dr, "\Delta"] \arrow[rr, equal, bend left] \arrow[rrrrdd, bend left] \arrow[rdd, equal, bend right] \arrow[rrdddd, bend right] & & & & \\
& \Omega X \times \Omega X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, "i \times \Omega X"] & \Omega X \arrow[dr, equal] \arrow[rrd, bend left] & & \\
& \Omega X \arrow[dr, "i"] \arrow[rdd, bend right] & \Omega X \times \Omega X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega"] & 0 \arrow[dd] \\
& & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega"] & 0 \arrow[dr] & \\
& & 0 \arrow[rr] & & X
\end{tikzcd}$$

$$=
\begin{tikzcd}
\Omega X \arrow[dr, "\Delta"] \arrow[rr, equal, bend left] \arrow[rrrd, bend left] \arrow[rdd, equal, bend right] \arrow[rddd, bend right] & & & \\
& \Omega X \times \Omega X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega"] & 0 \arrow[dd] \\
& \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega^{-1}"] & 0 \arrow[dr] & \\
& 0 \arrow[rr] & & X
\end{tikzcd}
=
\begin{tikzcd}
\Omega X \arrow[d] \\
0 \arrow[d] \\
\Omega X
\end{tikzcd}
=
\begin{tikzcd}
\Omega X \arrow[dr] \arrow[rrrdd, bend left] \arrow[rrddd, bend right] & & & \\
& 0 \arrow[dr] \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\
& & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega"] & 0 \arrow[d] \\
& & 0 \arrow[r] & X
\end{tikzcd}$$

This invertible 2-cell ι witnesses the fact that the morphism i provides left inverses for the monoid ΩX in $\mathrm{ho}(\mathcal{C})$. That it provides right inverses follows analogously. ☐

If we assume that the Grpd-category $\mathcal{C}$ has weak 2-biproducts, then its homotopy category $\mathrm{ho}(\mathcal{C})$ has biproducts in the 1-categorical sense. Thus every object has a canonical structure of an abelian monoid in $\mathrm{ho}(\mathcal{C})$, given by the 1-categorical universal property of the biproduct. Every morphism in $\mathrm{ho}(\mathcal{C})$ is a morphism of abelian monoids with respect to this canonical monoid structures, in particular the multiplication m of a quasi-loop object ΩX is compatible with this monoid structure. By the 1-categorical Eckmann-Hilton argument we deduce that the chosen multiplication map $m : \Omega X \oplus \Omega X \to \Omega X$ and the codiagonal $\nabla : \Omega X \oplus \Omega X \to \Omega X$ coincide in $\mathrm{ho}(\mathcal{C})$ and hence that ΩX is in fact an abelian group object. If additionally every object X is equivalent to some quasi-loop object $\Omega X'$, as is the case when $\Omega : \mathcal{C} \longrightarrow \mathcal{C}$ is a weak 2-equivalence, then every object X has a canonical structure of an abelian group object. In this case $\mathrm{ho}(\mathcal{C})$ canonically enriched in abelian groups, i.e. additive. We have proven the following corollary.

Corollary 3.1.4

Let $\mathcal{C}$ be a Grpd-category with finite weak 2-biproducts, which admits quasi-loops. Then every quasi-loop object is an abelian group object in $\mathrm{ho}(\mathcal{C})$. Furthermore, if $\Omega : \mathcal{C} \longrightarrow \mathcal{C}$ is a weak 2-equivalence, then $\mathrm{ho}(\mathcal{C})$ is Ab-enriched.

We have the following immediate consequence.

Corollary 3.1.5

Let $\mathcal{C}$ be a Grpd-category with finite weak 2-biproducts, which admits quasi-loops with $\Omega : \mathcal{C} \longrightarrow \mathcal{C}$ being a weak 2-equivalence. Then $\mathcal{C}$ is 2-additive in the sense of [Dup08], i.e. S2Grp-enriched with weak 2-biproducts.

Proof By the previous corollary we have that $\mathrm{ho}(\mathcal{C})$ is additive and since $\mathcal{C}$ has weak 2-biproducts the claim follows from [Dup08] corollary 243. ☐

This is the first ingredient of the definition of a 2-triangulated 2-category. We will investigate the other ingredients in the following section. The rest of this section, however, is devoted to show the somewhat surprising fact that a quasi-loop object indeed forms a symmetric pseudomonoid in $\mathcal{C}$, as long as $\mathcal{C}$ admits semi-weak 2-biproducts.

The idea is the same as in the 1-categorical case. With the results of the preceding section we can show that the pre-pseudomonoid structure of the quasi-loop object ΩX is compatible with the symmetric pseudomonoid structure of the abstract object ΩX given by the weak 2-biproduct. Then we apply the 2-categorical Eckmann-Hilton argument.

Proposition 3.1.6

Let $\mathcal{C}$ be a Grpd-category with finite semi-weak 2-biproducts and quasiloops. Then every quasi-loop object ΩX has a symmetric pseudomonoid structure in $\mathcal{C}$.

Proof We note that the diagram

$$\begin{tikzcd}
\Omega X \arrow[r, "{(u, \mathrm{id})}"] \arrow[dr, equal] & \Omega X \oplus \Omega X \arrow[d, "m"] \arrow[dl, phantom, "\Swarrow \lambda"] \arrow[dr, phantom, "\Searrow \rho"] & \Omega X \arrow[l, "{(\mathrm{id}, u)}"'] \arrow[dl, equal] \\
 & \Omega X &
\end{tikzcd}$$

is a diagram in $\mathrm{SPsMon}(\mathcal{C})$, in the sense that the objects are equipped with their canonical symmetric pseudomonoid structure provided by lemma 2.4.1, the morphisms are pseudomorphisms of symmetric pseudomonoids by lemma 2.4.2 and the 2-cells are transformations of pseudomonoids by lemma 2.4.3. Since the canonical pseudomonoid structure on the object $\Omega X \oplus \Omega X$ and the product pseudomonoid-structure agree by lemma 2.4.4, by the Eckmann-Hilton argument 2.2.9 there are invertible 2-cells α', σ' making $(\Omega X, m, u, \lambda, \rho, \alpha', \sigma')$ a symmetric pseudomonoid. ☐

It is worth mentioning that this argument does not show that the pre-pseudomonoid structure $(\Omega X, m, u, \lambda, \alpha, \rho)$ described above gives a pseudomonoid. Indeed the Eckmann-Hilton argument provides us with a new associator 2-cell and we have to discard the already chosen 2-cell α to obtain a pseudomonoid. This is an artifact of the non-uniqueness of the weak universal property of quasi-loop objects. In other words the point of the above proposition is that we can choose cells λ, ρ, α and σ, such that ΩX becomes a pseudomonoid, not that every choice gives rise to a pseudomonoid.

The proposition gives another way to deduce the enrichement in symmetric 2-groups of corollary 3.1.5, since the Hom-2-functor $\mathcal{C}(T, -)$ preserves 2-products and thus symmetric pseudomonoids.

We thus have an explicit description of the symmetric 2-group structure of the Hom-category $\mathcal{C}(R, \Omega X)$ for an object X. The following observation will become important later.

Remark 3.1.7

For any object T the explicit description of the symmetric 2-group structure of $\mathcal{C}(T, \Omega X)$ tells us that the tensor-inverse of a morphism $f : T \to \Omega X$ is given by postcomposing with the inverting morphism $i : \Omega X \to \Omega X$ described above.

In particular, given a square α with a chosen factorization

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha"] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} = \begin{tikzcd} T \arrow[dr, "f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega"] & 0 \arrow[d] \\ & 0 \arrow[r] & X \end{tikzcd}$$

its transpose

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha^{-1}"] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd} = \begin{tikzcd} T \arrow[dr, "f"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega^{-1}"] & 0 \arrow[d] \\ & 0 \arrow[r] & X \end{tikzcd} = \begin{tikzcd} T \arrow[dr, "f"] \arrow[ddrrr, bend left] \arrow[dddrr, bend right] & & & \\ & \Omega X \arrow[dr, "i"] \arrow[ddrr, bend left] \arrow[ddr, bend right] & & \\ & & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega"] & 0 \arrow[d] \\ & & 0 \arrow[r] & X \end{tikzcd}$$

classifies the tensor inverse of the factorization.

There is of cause a dual definition, which we spell out for the reader's convenience and to introduce terminology.

Definition 3.1.8

Let $\mathcal{C}$ be a Grpd-category with weak zero object, weak 2-coproducts and quasi-cocommas.

We say that $\mathcal{C}$ **has quasi-suspensions**, if there is a strict 2-functor $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ together with specified quasi-cocomma squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \sigma_X"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

for each object X, such that for every morphisms $f : X \to Y$ the pasting identity

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \sigma_X"] & 0 \arrow[d] \arrow[ddr, bend left] & & & X \arrow[dr, "f"] \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[r, phantom, "\Nearrow"] & & \\
0 \arrow[r] \arrow[drr, bend right] & \Sigma X \arrow[dr, "\Sigma f"] \arrow[r, phantom, "\Nearrow"] \arrow[d, phantom, "\Nearrow"] & & = & \Nearrow & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \sigma_Y"] & 0 \arrow[d] \\
 & & \Sigma Y & & & 0 \arrow[r] & \Sigma Y
\end{tikzcd}$$

holds. We call the squares σ_X a **quasi-suspension square** and the top left object ΣX a **quasi-suspension object**.

All of the statements of this section readily dualize by passing to $\mathcal{C}^{\mathrm{op}}$ and then applying Power's coherence result [Pow89] if necessary.

Usually loops and suspension come as an adjoint pair, with unit and counit induced from the universal properties. We thus make the following definition.

Definition 3.1.9

Let $\mathcal{C}$ be a Grpd-category with weak zero object, weak 2-products and weak 2-coproducts. It has a **quasi-loop quasi-suspension adjunction**, if it admits quasi-loops, quasi-suspensions such that the strict 2-functors $\Omega : \mathcal{C} \to \mathcal{C}$ and $\Sigma : \mathcal{C} \to \mathcal{C}$ form an weakly 2-adjoint pair $\Sigma \dashv \Omega$ with the additional requirement that the unit and counit satisfy the pasting identities

$$\begin{tikzcd}
X \arrow[rr] \arrow[dd] \arrow[dr, "\eta_X"] & \Uparrow & 0 \arrow[dd] & & & \\
\Rightarrow & \Omega\Sigma X \arrow[ur] \arrow[dl] & & & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \sigma_X"] & 0 \arrow[d] \\
 & & \Nearrow \omega_{\Sigma X} & = & 0 \arrow[r] & \Sigma X \\
0 \arrow[rr] & & \Sigma X & & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Omega X \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\Nearrow \sigma_{\Omega X}"] & & 0 \arrow[dd] \arrow[dl] & & & \\
 & \Sigma\Omega X \arrow[dr] \arrow[r, phantom, "\Rightarrow"] & & = & \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_X"] & 0 \arrow[d] \\
0 \arrow[rr] \arrow[ur] & \Uparrow & X & & 0 \arrow[r] & X
\end{tikzcd}$$

for every object X.

3.2 A Puppe Sequence of Symmetric 2-Groups

It is a standard fact in homotopy theory that by iteratively taking homotopy fibers of a morphism one can obtain a long exact sequence, often called Puppe-sequence or fiber sequence. This section is devoted to showing that in an abstract homotopy 2-category with zero object and quasi-commas the same procedure gives rise to a long exact sequence of pointed groupoids. If the homotopy 2-category is enriched in symmetric 2-groups this sequence will in fact be a long exact sequence of symmetric 2-groups.

Definition 3.2.1

Let $\mathcal{C}$ be a Grpd-category with weak zero object. A **quasi-fiber square** for a morphism $f : X \to Y$ is a quasi-comma square of the form

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\Nearrow \kappa"] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} K \arrow[r] \arrow[d, "k"'] \arrow[dr, phantom, "\Nearrow \kappa"] & 0 \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd} .$$

The following lemma will be the crucial ingredient in obtaining a long exact sequence out of quasi-fiber squares.

Lemma 3.2.2

Let $\mathcal{C}$ be a Grpd-category with weak zero object. Consider a square in $\mathcal{C}$ as depicted on the right. It is a quasi-comma if and only if for any object T the induced sequence

$$\mathcal{C}(T,P) \xrightarrow{k_*} \mathcal{C}(T,X) \xrightarrow{f_*} \mathcal{C}(T,Y)$$

of pointed groupoids is exact.

$$\begin{tikzcd} K \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\Nearrow \kappa"] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

Proof By definition the square λ is a quasi-comma if and only if for every object T the functor $\mathcal{C}(T,K) \longrightarrow \mathcal{C}(T,0) \downarrow \mathcal{C}(T,f) = \mathbb{1} \downarrow f_*$ is essentially surjective and full. But by definition $\mathbb{1} \downarrow f_* = \ker f_*$ is the kernel of f_*, so this is equivalent to the exactness of the sequence above. □

This lemma readily dualizes.

Lemma 3.2.3

Let $\mathcal{C}$ be a Grpd-category with weak zero object. Consider a square in $\mathcal{C}$ as depicted on the right. It is a quasi-cocomma if and only if for any object T the induced sequence

$$\mathcal{C}(C,T) \xrightarrow{c^*} \mathcal{C}(Y,T) \xrightarrow{f^*} \mathcal{C}(X,T)$$

of pointed groupoids is exact.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \gamma"] & Y \arrow[d, "c"] \\ 0 \arrow[r] & C \end{tikzcd}$$

Remark 3.2.4

Since for a functor of groupoids $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ the commas $\mathbb{1} \downarrow \mathcal{F}$ and $\mathcal{F} \downarrow \mathbb{1}$ are isomorphic, both lemmas above also hold true for squares with the positions of 0 and X respectively 0 and Y interchanged.

Proposition 3.2.5

Let $\mathcal{C}$ be a Grpd-category with weak zero object, quasi-commas and quasi-loops.

Any quasi-fiber square α for a morphism $f : X \to Y$ induces a staircase of quasi-fiber squares of the form

$$\begin{tikzcd}
\Omega K \arrow[r] \arrow[d, "\Omega k"'] \arrow[dr, phantom, "\Nearrow \delta"] & 0 \arrow[d] & \\
\Omega X \arrow[r, "\Omega f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \gamma"] & \Omega Y \arrow[r] \arrow[d, "d"] \arrow[dr, phantom, "\Nearrow \beta"] & 0 \arrow[d] \\
0 \arrow[r] & K \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha"] & X \arrow[d, "f"] \\
 & 0 \arrow[r] & Y
\end{tikzcd}
\qquad \text{resp.} \qquad
\begin{tikzcd}
\Omega K \arrow[r, "\Omega k"] \arrow[d] \arrow[dr, phantom, "\Nearrow \delta"] & \Omega X \arrow[r] \arrow[d, "\Omega f"] \arrow[dr, phantom, "\Nearrow \gamma"] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega Y \arrow[r, "d"] \arrow[d] \arrow[dr, phantom, "\Nearrow \beta"] & K \arrow[r] \arrow[d, "k"] \arrow[dr, phantom, "\Nearrow \alpha"] & 0 \arrow[d] \\
 & 0 \arrow[r] & X \arrow[r, "f"'] & Y
\end{tikzcd} .$$

Proof Starting with the square α we can form a quasi-fiber of k to obtain a diagram of the form

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "d'"'] \arrow[dr, phantom, "\Nearrow \beta'"] & 0 \arrow[d] \\
K \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha"] & X \arrow[d, "f"] \\
0 \arrow[r] & Y
\end{tikzcd} .$$

By the pasting property of quasi-commas 1.3.12, the outer rectangle is a quasi-comma i.e. a quasi-loop object of Y. Thus there are invertible 2-cells and an equivalence e giving rise to the pasting identity

$$\begin{tikzcd}
\Omega Y \arrow[dr, "e"] \arrow[drr, bend left] \arrow[dddr, bend right] & & & & & \\
 & P \arrow[r] \arrow[d, "d'"'] \arrow[dr, phantom, "\Nearrow \beta'"] & 0 \arrow[d] \arrow[dd, bend left=60, "\Rightarrow"'] & & \Omega Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_Y"] & 0 \arrow[d] \\
 & K \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha"] & X \arrow[d, "f"] & {} \arrow[u, phantom, "="] & 0 \arrow[r] & Y \\
 & 0 \arrow[r] & Y & & &
\end{tikzcd} .$$

Setting $d = d' \circ e$ and whiskering β' gives us the first two squares α and β in the staircase. Note that β still is a quasi-comma square by lemma 1.3.3, since e is an equivalence and β' is a quasi-comma square.

By forming a quasi-fiber of d and repeating the same argument we obtain a diagram of the form

$$\begin{tikzcd}
\Omega X \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\Nearrow \gamma'"] & \Omega Y \arrow[r] \arrow[d, "d"'] \arrow[dr, phantom, "\Nearrow \beta"] & 0 \arrow[d] \\
0 \arrow[r] & K \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha"] & X \arrow[d, "f"] \\
& 0 \arrow[r] & Y
\end{tikzcd}$$

in which all three squares are quasi-commas. We still have to check that the morphism f' can be chosen to be Ωf. To this end we note that f' satisfies the pasting identity

$$\begin{tikzcd}
\Omega X \arrow[dr, "f'"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \Omega Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_Y"] & 0 \arrow[d] \\
& 0 \arrow[r] & Y
\end{tikzcd}
=
\begin{tikzcd}
\Omega X \arrow[r, "f'"] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\Nearrow \gamma'"] & \Omega Y \arrow[r] \arrow[d, "d"'] \arrow[dr, phantom, "\Nearrow \beta"] & 0 \arrow[d] \arrow[dd, bend left=60, "\Rightarrow"'] \\
0 \arrow[r] \arrow[dr, equal] & K \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha"] & X \arrow[d, "f"] \\
& 0 \arrow[r] & Y
\end{tikzcd}$$

$$=
\begin{tikzcd}
\Omega X \arrow[rr] \arrow[d] \arrow[drr, phantom, "\Nearrow \omega_X"] & & 0 \arrow[d] \arrow[ddd, bend left, "\Rightarrow"'] \\
0 \arrow[rr] \arrow[dr] \arrow[ddr, equal, bend right] & \Uparrow & X \arrow[dd, "f"] \\
& K \arrow[ur] \arrow[d] \arrow[r, phantom, "\Nearrow \alpha"] & {} \\
& 0 \arrow[r] & Y
\end{tikzcd}
=
\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_X"] & 0 \arrow[d] \arrow[ddr, bend left] & \\
0 \arrow[r] \arrow[drr, bend right] & X \arrow[dr, "f"] & \Nearrow \\
& \Nearrow & Y
\end{tikzcd}$$

where we first used the definition of β and the strict finality of 0 and then used the definition of γ'. Since Ωf is uniquely determined by this universal property up to non-unique iso, there exists some isomorphism $f' \cong \Omega f$. Appending this invertible 2-cell to the square γ' gives us the desired square γ.

Finally we can apply the same discussion to the two squares

$$\begin{tikzcd}
\Omega X \arrow[r, "\Omega f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \gamma"] & \Omega Y \arrow[r] \arrow[d, "d"'] \arrow[dr, phantom, "\Nearrow \beta"] & 0 \arrow[d] \\
0 \arrow[r] & K \arrow[r, "k"'] & X
\end{tikzcd}$$

to obtain the desired quasi-fiber square δ and hence the desired staircase. □

We can repeat the process of taking quasi-fiber squares and identifying the occuring morphisms indefinitely. More precisely we can inductively apply the proposition to already constructed quasi-fiber squares of the form

$$\begin{tikzcd}
\Omega^n K \arrow[r, "\Omega^n k"] \arrow[d] \arrow[dr, phantom, "\Nearrow \kappa"] & \Omega^n X \arrow[d, "\Omega^n f"] \\
0 \arrow[r] & \Omega^n Y
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
\Omega^n K \arrow[r] \arrow[d, "\Omega^n k"'] \arrow[dr, phantom, "\Nearrow \kappa"] & 0 \arrow[d] \\
\Omega^n X \arrow[r, "\Omega^n f"'] & \Omega^n Y
\end{tikzcd}$$

to extend the staircase of quasi-fiber squares three steps further. Applying lemma 3.2.2 to the infinite staircase obtained this way we arrive at the following corollary.

Corollary 3.2.6

Let $\mathcal{C}$ be a Grpd-category with weak zero object, quasi-commas and quasi-loops. For any object T a quasi-fiber square as in the preceding proposition induces a long exact sequence of pointed groupoids of the form

$$\begin{tikzcd}
& & \cdots \arrow[dll, "\Omega^n d_*"] \\
\mathcal{C}(T,\Omega^n K) \arrow[r, "\Omega^n k_*"] & \mathcal{C}(T,\Omega^n X) \arrow[r, "\Omega^n f_*"] & \mathcal{C}(T,\Omega^n Y) \arrow[dll, "\Omega^{n-1} d_*"] \\
\cdots & & \cdots \arrow[dll, "\Omega d_*"] \\
\mathcal{C}(T,\Omega K) \arrow[r, "\Omega k_*"] & \mathcal{C}(T,\Omega X) \arrow[r, "\Omega f_*"] & \mathcal{C}(T,\Omega Y) \arrow[dll, "d_*"] \\
\mathcal{C}(T,K) \arrow[r, "k_*"] & \mathcal{C}(T,X) \arrow[r, "f_*"] & \mathcal{C}(T,Y).
\end{tikzcd}$$

In the light of corollary 3.1.5, since by remark 2.5.5 a sequence of symmetric 2-groups is exact if and only if the underlying sequence of pointed groupoids is, we also obtain the following corollary.

Corollary 3.2.7

Let $\mathcal{C}$ be a Grpd-category with finite weak 2-biproducts, quasi-commas and quasi-loops such that Ω is a weak 2-equivalence.

Given a quasi-fiber square the induced long exact sequence in the preceding corollary 3.2.6 is a long exact sequence of symmetric 2-groups.

In fact, since the weak equivalence Ω^{-1} preserves quasi-commas, we can apply it to the given quasi-fiber square and apply the previous corollary to it. Inductively the long exact sequence extends to the right to a long exact sequence of symmetric 2-groups of the form

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners, "\Omega d_*"] \\
\mathcal{C}(T,\Omega K) \arrow[r, "\Omega k_*"] & \mathcal{C}(T,\Omega X) \arrow[r, "\Omega f_*"] & \mathcal{C}(T,\Omega Y) \arrow[dll, rounded corners, "d_*"] \\
\mathcal{C}(T,K) \arrow[r, "k_*"] & \mathcal{C}(T,X) \arrow[r, "f_*"] & \mathcal{C}(T,Y) \arrow[dll, rounded corners, "\Omega^{-1} d_*"] \\
\mathcal{C}(T,\Omega^{-1} K) \arrow[r, "\Omega^{-1} k_*"] & \mathcal{C}(T,\Omega^{-1} X) \arrow[r, "\Omega^{-1} f_*"] & \mathcal{C}(T,\Omega^{-1} Y) \arrow[dll, rounded corners, "\Omega^{-2} d_*"] \\
\cdots & & .
\end{tikzcd}$$

Remark 3.2.8

At this point we should point out, that quite often the Puppe-sequences feature alternating signs. Specifically we would expect the long exact sequence to look like

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners, "(-1)^n \Omega^n d_*"] \\
\mathcal{C}(T,\Omega^n K) \arrow[r, "(-1)^n \Omega^n k_*"] & \mathcal{C}(T,\Omega^n X) \arrow[r, "(-1)^n \Omega^n f_*"] & \mathcal{C}(T,\Omega^n Y) \arrow[dll, rounded corners, "(-1)^{n-1} \Omega^{n-1} d_*"] \\
\cdots & & \cdots \arrow[dll, rounded corners, "-\Omega d_*"] \\
\mathcal{C}(T,\Omega K) \arrow[r, "-\Omega k_*"] & \mathcal{C}(T,\Omega X) \arrow[r, "-\Omega f_*"] & \mathcal{C}(T,\Omega Y) \arrow[dll, rounded corners, "d_*"] \\
\mathcal{C}(T,K) \arrow[r, "k_*"] & \mathcal{C}(T,X) \arrow[r, "f_*"] & \mathcal{C}(T,Y).
\end{tikzcd}$$

This is not a point of concern though. Since kernel and essential image are invariant under changing signs we can interchange any morphism in a short exact sequence of symmetric 2-groups with its negative and still obtain a short exact sequence. So it is possible to deduce one short exact sequence from the other.

The reason for the discrepancy is, that in our proof of corollary 3.2.6 we choose quasifiber squares one at a time to obtain the long exact sequence. We did not identify the 2-cells of these squares, so no signs appear.

However the signs naturally appear when constructing the long exact sequence in a slightly different manner. In the setting of the preceding corollary the strict 2-functor Ω preserves quasi-fiber squares, since it is a weak 2-equivalence. If we start with a quasi-fiber square with the zero in the bottom left we can apply proposition 3.2.5 to obtain the first few quasi-fiber squares of the staircase, which span a 2×3-grid. Applying Ω to this partial staircase gives us another partial staircase, which also spans a 2×3-grid. Hence, to combine them we need to transpose the latter to obtain a 3×2-grid. By remark 3.1.7 this introduces the signs, when identifying the first square of the transposed staircase with the last square of the original one. Repeating this argument inductively gives rise to the long exact sequence of symmetric 2-groups with alternating signs as depicted above.

Again the statements dualize. Applying $\mathcal{C}(-, X)$ to a staircase of quasi-cofiber squares gives rise to a long exact sequence of symmetric 2-groups.

3.3 2-Triangulated Categories

In this section we want to finally give a definition of a 2-triangulated 2-category based on the structure we found on the homotopy 2-category of a stable ∞-category. The definition we give is very close to the characterization of a stable ∞-category as a pointed ∞-category with ∞-pullbacks and ∞-pushouts for which Σ respectively Ω is an equivalence.

Definition 3.3.1

A **2-triangulated 2-category** is a Grpd-category $\mathcal{C}$ with

(i) a weak zero object and binary weak 2-biproducts (cf. definition 2.3.1 and definition 2.3.6)

(ii) quasicommas and quasicocommas (cf. definition 1.3.1)

(iii) quasi-loops and quasi-suspensions (cf. definition 3.1.1)

such that

- the strict 2-functors Σ and Ω are a quasi-loop quasi-suspension adjunction (cf. definition 3.1.9) and
- the strict 2-functors Σ and Ω are mutually inverse weak 2-equivalences.

Remark 3.3.2

While not explicitly stated in the definition of a 2-triangulated 2-category, corollary 3.1.5 shows that a 2-triangulated 2-category is canonically enriched in symmetric 2-groups, i.e. is 2-additive in the sense of [Dup08].

This is the reason we decided to add the axiom of having 2-biproducts. If we just assumed having quasi-commas and quasi-cocommas, we would get quasi-products and quasi-coproducts, meaning 2-products and 2-coproducts whose 2-dimensional universal property involves a non-unique 2-cells. The author does not know yet, whether having these would already imply having weak 2-products and weak 2-coproducts, nor whether having these would suffice for the following arguments.

With a definition of 2-triangulated categories at hand we should show that the class of all of them is non-empty. Luckily the work we have done so far provides us with the following example.

Example 3.3.3

Let $\mathcal{C}$ be a stable cofibrantly generated simplicial model category. Then its homotopy 2-category $\mathrm{ho}_2(\mathcal{C})$ is 2-triangulated.

Indeed it has a (even strict) zero object. It has weak products and weak coproducts by lemma 1.1.4 and since the homotopy 1-category is additive the components of the canonical weakly natural transformation $- + - \Rightarrow - \times -$ provided by lemma 2.3.5 are equivalences. This shows that it has weak biproducts. It has quasi-commas and quasi-cocommas by proposition 1.2.6. Furthermore $\mathrm{ho}_2(\mathcal{C})$ has quasi-loops and quasi-suspensions by proposition 1.4.2 with compatibility conditions provided by lemma 1.4.4 and lemma 1.4.5. Corollary 1.4.3 shows that the strict 2-functors Σ and Ω are mutually inverse weak 2-equivalences.

Example 3.3.4

If $\mathcal{C}$ is a 2-triangulated 2-category, so is $\mathcal{C}^{\mathrm{op}}$.

Now let us compare this definition with the ordinary definition of a *triangulated category*, as given for example in the following axiomatization due to [May05].

Definition 3.3.5
An additive category $\mathcal{A}$ is **triangulated**, if it is equipped with an autoequivalence $\Sigma : \mathcal{A} \longrightarrow \mathcal{A}$ called the **suspension** and a class of **distinguished triangles** consisting of triples of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

satisfying the following axioms.

(T1)
- For any object X the triangle $X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow \Sigma X$ is distinguished.
- Any morphism $f : X \to Y$ fits into a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$.
- Any triangle isomorphic to a distinguished triangle is distinguished.

(T2) For a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

the rotated triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is distinguished as well.

(T3) For given distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{f'} Y/X \xrightarrow{f''} \Sigma X$$
$$Y \xrightarrow{g} Z \xrightarrow{g'} Z/Y \xrightarrow{g''} \Sigma Y$$
$$X \xrightarrow{h} Z \xrightarrow{h'} Z/X \xrightarrow{h''} \Sigma X$$

with $h = g \circ f$ there exists another distinguished triangle

$$Y/X \xrightarrow{j} Z/X \xrightarrow{j'} Z/Y \xrightarrow{j''} \Sigma(Y/X)$$

such that the braid diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "h"] \arrow[dr, "f"'] & & Z \arrow[rr, bend left, "g'"] \arrow[dr, "h'"] & & Z/Y \arrow[rr, dashed, bend left, "j''"] \arrow[dr, "g''"] & & \Sigma(Y/X) \\
& Y \arrow[ur, "g"] \arrow[dr, "f'"'] & & Z/X \arrow[ur, dashed, "j'"] \arrow[dr, "h''"] & & \Sigma Y \arrow[ur, "\Sigma f'"'] & \\
& & Y/X \arrow[ur, dashed, "j"] \arrow[rr, bend right, "f''"'] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd} \tag{3.3.5.1}$$

commutes in $\mathcal{A}$.

There is a crucial difference between our definition of a 2-triangulated category and the classical notion of a triangulated category. In the latter case the distinguished triangles are an extra datum, while in the former case we only have to assume some properties of the 2-category. We close this chapter by showing that the homotopy category of a 2-triangulated category is a triangulated category, thus demonstrating that 2-triangulated categories can be seen as a generalization of trianglulated categories. The proof we give is the one of theorem 1.1.2.14 in [Lur17], adapted to

May's axioms for triangulated categories and only using the 2-categorical properties at hand.

Theorem 3.3.6
Let $\mathcal{C}$ be a 2-triangulated category. Then its homotopy category $\mathrm{ho}(\mathcal{C})$ is canonically triangulated with distinguished triangles those sequences

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

for which there exists a diagram

$$\begin{tikzcd}
X \arrow[r, "\widetilde{f}"] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha"] & Y \arrow[r] \arrow[d, "\widetilde{g}"'] \arrow[dr, phantom, "\Nearrow \beta"] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r, "\widetilde{h}"'] & W
\end{tikzcd}$$

such that both squares α and β are quasi-cocommas and $[\widetilde{f}] = [f]$, $[\widetilde{g}] = [g]$ and $[e] \circ [\widetilde{h}] = [h]$ in $\mathrm{ho}(\mathcal{C})$, where $e : W \to \Sigma X$ is the essentially unique induced equivalence.

Proof By corollary 3.1.4 the homotopy category $\mathrm{ho}(\mathcal{C})$ is additive. Moreover the weak 2-equivalence Σ on $\mathcal{C}$ induces an equivalence Σ on $\mathrm{ho}(\mathcal{C})$.

Regarding (T1) we note that the identity fits into a diagram of quasi-cocommas of the form

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \sigma"] & 0 \arrow[d] \\
0 \arrow[r, equal] & 0 \arrow[r] & \Sigma X
\end{tikzcd}$$

giving rise to the desired distinguished triangle $X \xrightarrow{\mathrm{id}} X \to 0 \to \Sigma X$. Similarly a morphism $f : X \to Y$ induces a diagram of quasi-cocommas

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha"] & Y \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\Nearrow \beta"] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r, "\widetilde{h}"'] & W
\end{tikzcd}$$

by taking quasi-cofibers two times. This amounts to the desired distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$.

An isomorphism of triangles

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"', "\cong"] & Y \arrow[r, "g"] \arrow[d, "y"', "\cong"] & Z \arrow[r, "h"] \arrow[d, "z"', "\cong"] & \Sigma X \arrow[d, "\Sigma x"', "\cong"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'
\end{tikzcd}$$

in $\mathrm{ho}(\mathcal{C})$ amounts to a homotopy commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "\Nearrow \phi"] & Y \arrow[r, "g"] \arrow[d, "y"'] \arrow[dr, phantom, "\Nearrow \psi"] & Z \arrow[r, "h"] \arrow[d, "z"'] \arrow[dr, phantom, "\Nearrow \eta"] & \Sigma X \arrow[d, "\Sigma x"'] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & \Sigma X'
\end{tikzcd}$$

in $\mathcal{C}$ in which all vertical maps are equivalences. If we assume the upper triangle to be distinguished this extends to a diagram of cubes of the form

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd, "\simeq"'] & & Y \arrow[rr] \arrow[dr] \arrow[dd, "\simeq"] \arrow[dr, phantom, "\Nearrow"] & & 0 \arrow[dr] \arrow[dd, equal] \arrow[dr, phantom, "\Nearrow"] & \\
& 0 \arrow[rr] \arrow[dd, equal] \arrow[dr, phantom, "\Nearrow"] & & Z \arrow[rr] \arrow[dd, "\simeq"] \arrow[dr, phantom, "\Nearrow"] & & \Sigma X \arrow[dd, "\simeq"] \\
X' \arrow[rr] \arrow[dr] \arrow[dr, phantom, "\Nearrow"] & & Y' \arrow[rr] \arrow[dr] \arrow[dr, phantom, "\Nearrow"] & & 0 \arrow[dr] \arrow[dr, phantom, "\Nearrow"] & \\
& 0 \arrow[rr] & & Z' \arrow[rr] & & \Sigma X'
\end{tikzcd}$$

in which the top squares are quasi-cocommas by assumption and the vertical maps are equivalences. By corollary 1.3.14 the lower squares are quasi-cocommas as well. This shows that the lower triangle is distinguished as well.

For the axiom (T2) we first have to note that for a diagram of quasi-cocommas

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\Nearrow \alpha"] & 0 \arrow[d] \\
Y \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\Nearrow \beta"] & Z \arrow[d, "h"] \\
0 \arrow[r] & \Sigma X
\end{tikzcd}$$

the transposed diagram of quasi-cocommas

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha^{-1}"] & Y \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\Nearrow \beta^{-1}"] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r, "h"'] & \Sigma X
\end{tikzcd}$$

represents by remark 3.1.7 a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h} \Sigma X.$$

Now suppose we are given a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, which is without loss of generality represented by a diagram of quasi-cocommas of the form

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha^{-1}"] & Y \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\Nearrow \beta^{-1}"] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r, "h"'] & \Sigma X
\end{tikzcd}$$

As in the proof of proposition 3.2.5 we can extend this diagram to a staircase of quasi-cocommas of the form

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \alpha"] & Y \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\Nearrow \beta"] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r, "h"] \arrow[d] \arrow[dr, phantom, "\Nearrow \gamma"] & \Sigma X \arrow[d, "\Sigma f"] \\
 & 0 \arrow[r] & \Sigma Y
\end{tikzcd}$$

Then the transpose of the two right quasi-cocommas gives rise to the desired distinguished triangle $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$.

It remains to check (T3). To do this we note that by using proposition 1.3.15 a distinguished triangle is determined by the first morphism up to a (not necessarily unique) isomorphism of distinguished triangles. Since we may replace any distinguished triangle in the diagram (3.3.5.1) by an isomorphic one, it thus suffices to show that there is *some* triple of distinguished triangles with the given first maps satisfying (T3).

To this end we consider the diagram of quasi-cocommas

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & Y \arrow[r, "g"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Nearrow"] & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & 0 \arrow[d] & \\
0 \arrow[r] & Y/X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & Z/X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & 0 \arrow[d] \\
 & 0 \arrow[r] & Z/Y \arrow[r] & Y' \arrow[r] & (Y/X)'
\end{tikzcd}$$

from which we can derive equivalences

$$X' \simeq \Sigma X, \qquad Y' \simeq \Sigma Y, \qquad (Y/X)' \simeq \Sigma(Y/X)$$

in $\mathcal{C}$ and distinguished triangles

$$\begin{aligned}
&X \xrightarrow{f} Y \xrightarrow{f'} Y/X \to \Sigma X \\
&Y \xrightarrow{g} Z \to Z/Y \to \Sigma Y \\
&X \xrightarrow{g \circ f} Z \to Z/X \to \Sigma X \\
&Y/X \to Z/X \to Z/Y \to \Sigma(Y/X).
\end{aligned}$$

The commutativity of the braid diagram in $\mathrm{ho}(\mathcal{C})$ then follows from the 2-commutativity of the diagram of quasi-cocommas by noting that under the chosen equivalences the morphisms $X' \to Y' \to (Y/X)'$ can be identified as $\Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma f'} \Sigma(Y/X)$. □

Another way to characterize stable ∞-categories is as those pointed ∞-categories with finite ∞-limits and ∞-colimits, for which ∞-pullback squares and ∞-pushout squares coincide. We can show that a similar characterization can be given for 2-triangulated 2-categories. On the one hand we have the following proposition.

Proposition 3.3.7

Let $\mathcal{C}$ be a 2-triangulated 2-category. Then a square is a quasi-comma if and only if it is a quasi-cocomma.

Proof (adaptation of a proof given in Cisinski's lecture on *Derived Categories* in WS2021/22)

Since being 2-triangulated is a self-dual notion it suffices to show that being a quasi-comma implies being a quasi-cocomma or vice versa.

We first show that quasi-fiber squares are quasi-cofiber squares. To this end we take a quasi-fiber square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \tau"] & Y \arrow[d, "t"] \\
0 \arrow[r] & T
\end{tikzcd}$$

and a quasi-cofiber square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \zeta"] & Y \arrow[d, "z"] \\
0 \arrow[r] & Z
\end{tikzcd} .$$

The latter fits into a diagram of quasi-cofiber squares

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \zeta"] & Y \arrow[r] \arrow[d, "z"] \arrow[dr, phantom, "\Nearrow \alpha"] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r, "s"'] & \Sigma X
\end{tikzcd}$$

while the former fits into a diagram of fiber squares

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\Nearrow \tau"] & Y \arrow[r] \arrow[d, "t"] \arrow[dr, phantom, "\Nearrow \beta"] & 0 \arrow[d] \\
0 \arrow[r] & T \arrow[r, "p"'] & \Sigma X
\end{tikzcd}$$

obtained as in remark 3.2.8 up to equivalence as the upper left two squares of the staircase proposition 3.2.5 of the quasi-fiber square $\Sigma(\tau^{-1})$. Note that the composite square $\tau \bullet \beta$ is equivalent to the quasi-loop square

$$\begin{tikzcd}
\Omega\Sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \omega_{\Sigma X}"] & 0 \arrow[d] \\
0 \arrow[r] & \Sigma X
\end{tikzcd}$$

which itself is equivalent to the quasi-suspension square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \sigma_X"] & 0 \arrow[d] \\
0 \arrow[r] & \Sigma X
\end{tikzcd}$$

by means of the unit of the quasi-loop quasi-suspension adjunction. Thus the composite square $\tau \bullet \beta$ is a quasi-cocomma square.

Invoking the weak universal property of the quasi-cofiber square ζ and then of the quasi-cofiber square α yields a 2-commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] \arrow[dd, equal] & & Y \arrow[rr] \arrow[dr, "z"] \arrow[dd, equal] & & 0 \arrow[dr] \arrow[dd, equal] & \\
& 0 \arrow[rr] \arrow[dd, equal] \arrow[ur, Rightarrow, bend left, shorten=4mm, "\zeta"'] & & Z \arrow[rr, "s"] \arrow[dd, dashed, "e"] \arrow[ur, Rightarrow, bend left, shorten=4mm, "\alpha"'] & & \Sigma X \arrow[dd, dashed, "k"] \\
X \arrow[rr] \arrow[dr] & & Y \arrow[rr] \arrow[dr, "t"] & & 0 \arrow[dr] & \\
& 0 \arrow[rr] \arrow[ul, Rightarrow, bend left, shorten=4mm] \arrow[ur, Rightarrow, bend left, shorten=4mm, "\tau"] \arrow[uu, Rightarrow, phantom] & & T \arrow[rr, "p"'] \arrow[ur, Rightarrow, bend left, shorten=4mm, "\beta"] & & \Sigma X
\end{tikzcd}$$

in which the top composite square and bottom composite square are quasi-cocomma squares and the relevant vertical maps are equivalences. Thus the map k is an equivalence.

The squares ζ and α give rise to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{z} Z \xrightarrow{s} \Sigma X$$

in $\mathrm{ho}(\mathcal{C})$. Note that $\mathrm{ho}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ admits a triangulated structure coming from quasi-fiber squares in $\mathcal{C}$. We do <u>not</u> claim that it agrees with the triangulated structure on $\mathrm{ho}(\mathcal{C})$ given by quasi-cofiber squares. Still the quasi-fiber squares τ and β gives rise to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{t} T \xrightarrow{p} \Sigma X$$

in the triangulated category $\mathrm{ho}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. The 2-commutativity of the diagram above shows that we have a morphism of (not necessarily distinguished) triangles

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "z"] \arrow[d, equal] & Z \arrow[r, "s"] \arrow[d, "e"] & \Sigma X \arrow[d, "\cong"] \\
X \arrow[r, "f"'] & Y \arrow[r, "t"'] & T \arrow[r, "p"'] & \Sigma X
\end{tikzcd}$$

in $\mathrm{ho}(\mathcal{C})$. For any object A applying $\mathrm{ho}(\mathcal{C})(A, -) := [A, -]$ to this diagram gives a morphism of long exact sequences of abelian groups

$$\begin{tikzcd}
\cdots \arrow[r] & {[A, X]} \arrow[r] \arrow[d, equal] & {[A, Y]} \arrow[r] \arrow[d, equal] & {[A, Z]} \arrow[r] \arrow[d, "e_*"] & {[A, \Sigma X]} \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & {[A, X]} \arrow[r] & {[A, Y]} \arrow[r] & {[A, T]} \arrow[r] & {[A, \Sigma X]} \arrow[r] & \cdots
\end{tikzcd}$$

since the upper triangle was distinguished in $\mathrm{ho}(\mathcal{C})$ and the lower triangle was distinguished in $\mathrm{ho}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. By the 5-lemma the map e_* is an isomorphism of abelian groups and since A was chosen arbitrarily the Yoneda-lemma implies that e is an isomorphism in $\mathrm{ho}(\mathcal{C})$, hence an equivalence in $\mathcal{C}$. But this implies that τ is a quasi-cofiber square, because ζ is.

For a general square
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "\nearrow \phi"] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$
we can form the quasi-fiber squares γ and γ' of f and f' and the universal property of γ' provides us with a 2-commutative cube

$$\begin{tikzcd}
Q \arrow[rr] \arrow[dr, "g"] \arrow[dd, "q"'] & & 0 \arrow[dr] \arrow[dd, equal] \arrow[d, phantom, "\gamma"] & \\
& X \arrow[rr, "f"] \arrow[dd, "x"] & {} & Y \arrow[dd, "y"] \\
Q' \arrow[rr] \arrow[dr, "g'"'] & & 0 \arrow[dr] \arrow[d, phantom, "\gamma'"] & \\
& X' \arrow[rr, "f'"'] & {} & Y'
\end{tikzcd}$$

in which the top and bottom faces are quasi-fiber squares.

If the front face is a quasi-comma square then the composite square of the front and top face is as well. The composite square of the bottom and back square is equivalent to that and thus is a quasi-comma square as well. Since the bottom square is a quasi-comma square, the back face is so by the cancellation property 1.3.16. But now the top, bottom and back faces are quasi-fiber squares, because they have a zero object in some corner, and therefore are quasi-cofiber squares and in particular quasi-cocomma squares. Thus the composite square of back and bottom is a quasi-cocomma square, so is the composite square of top and front and by the cancellation property 1.3.16 the front face ϕ is. ☐

Remark 3.3.8

Note that we only used the triangulations of the underlying homotopy 1-categories for the application of the 5-lemma. Assuming a 5-lemma for symmetric 2-groups, which has been proven in [BV02], we could also derive the preceding proposition immediately from the long exact sequence of symmetric 2-groups provided by corollary 3.2.6.

On the other hand we have the following lemma.

Lemma 3.3.9

Let $\mathcal{C}$ be a Grpd-category with weak zero object, binary weak 2-biproducts, quasi-commas and quasi-cocommas and a quasi-loop quasi-suspension adjunction. If quasi-comma squares and quasi-cocomma squares coincide, then $\mathcal{C}$ is 2-triangulated.

Proof Since quasi-comma squares and quasi-cocomma squares coincide, the compatibility conditions of the quasi-loop quasi-suspension adjunction (cf. definition 3.1.9) enforce that the components of the unit and of the counit of the adjunction are equivalences. Hence Σ and Ω are mutually inverse weak 2-equivalences. ☐

We finish the chapter on 2-triangulated 2-categories by proving the following two corollaries of the preceding proposition.

Corollary 3.3.10 (Short 5-Lemma)
Let $\mathcal{C}$ be a 2-triangulated 2-category. Suppose we are given a 2-commutative cube as depicted below, for which the top and bottom faces are quasi-(co)fiber squares and the maps x and z are equivalences. Then the morphism y is an equivalence.

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "f"] \arrow[dd, "x"'] & & 0 \arrow[dr] \arrow[dd, equal] \arrow[dr, phantom, "\Nearrow", shift right=3] & \\
& Y \arrow[rr, "g"] \arrow[dd, "y"] & & Z \arrow[dd, "z"] \\
X' \arrow[rr] \arrow[dr, "f'"'] \arrow[dr, phantom, "\Nearrow", shift left=3] & \arrow[ur, phantom, "\Nearrow"] & 0 \arrow[dr] \arrow[dr, phantom, "\Nearrow\Nearrow\Nearrow", shift left=3] & \\
& Y' \arrow[rr, "g'"'] & & Z'
\end{tikzcd}$$

Proof By assumption and lemma 1.3.8 the top, bottom, back and right faces are quasi-(co)comma squares. Thus the composite square of the back and bottom square is a quasi-(co)comma square. Thus the composite square of the top and front face is a quasi-(co)comma square. By the cancellation property 1.3.16 for quasi-cocomma squares the front face is a quasi-cocomma square. But since quasi-cocomma squares and quasi-comma squares in $\mathcal{C}$ coincide by the preceding proposition 3.3.7, it is a quasi-comma square as well. But then y is an equivalence by lemma 1.3.9. ☐

Corollary 3.3.11
Let $\mathcal{C}$ be a 2-triangulated 2-category and $f : X \to Y$ a morphism. The following assertions are equivalent.

(i) The morphism f is an equivalence.

(ii) The quasi-fiber F of f is zero.

(iii) The quasi-cofiber C of f is zero.

Proof We prove (i) ⇔ (ii), (i) ⇔ (iii) is formally dual.

$$\begin{tikzcd}
F \arrow[r] \arrow[d, "z"'] \arrow[dr, phantom, "\Nearrow \phi"] & X \arrow[d, "f"] \\
0 \arrow[r] & Y
\end{tikzcd}$$

Consider the quasi-fiber square depicted above. If f is an equivalence, then by lemma 1.3.9 the morphism z is, showing that F is zero.

Conversely, if F is zero, the morphism z is an equivalence. Since quasi-comma squares and quasi-cocomma squares coincide, the quasi-fiber square is a quasi-cocomma square. Therefore f is an equivalence by lemma 1.3.9. ☐

This finishes the chapter on 2-triangulated 2-categories.

Outlook

We have found a definition of 2-triangulated 2-categories and have seen that a cofibrantly generated stable simplicial model category has a 2-triangulated homotopy 2-category. Instead of giving another summary of the results of this thesis, we would like to give the following perspectives.

- We used cofibrantly generated simplicial model categories to obtain a strict homotopy 2-category as well as strict loop-suspension 2-functors. These model the restricted class of locally presentable stable ∞-categories, as we discuss in appendix A.4. However, for quasi-categories there are several other ways to obtain a strict homotopy 2-category as is explained on MathOverflow [MO]. It might be worth to look into whether these give a faster and more general way to obtain 2-triangulated 2-categories as defined in this thesis.
- Similarly it would be interesting to investigate whether one can obtain homotopy 2-categories and the corresponding notions of homotopy limits and homotopy colimits directly from a model category, without simplicial enrichment or cofibrant generation. This would give a nice way to interpolate between model category theory and higher category theory, simplifiying proofs in model categories via a truncated version of ∞-categorical reasoning.
- In the spirit of the previous bullet, if there is an easy way to define the homotopy 2-category of a model category, looking into the homotopy 2-category of spectra may be illuminating. For example one could hope for a weakly monoidal structure, which may be useful for a coarse treatment of ring spectra, before applying heavier tools like model categories of diagram spectra or the infinity category of spectra for a detailed analysis. One could use this monoidal structure to define homology and cohomology theories valued in symmetric 2-groups and it would be interesting to connect this to the cohomology theory defined in [BCC93].

  Another example worth looking at would be the derived 2-category of a ring, i.e. the homotopy 2-category of its derived ∞-category. Again it might be illuminating to investigate the weak monoidal structure of the derived tensor product.
- Besides the convenient setup for 2-triangulations, homotopy 2-categories have other benefits. For example [Arl20] gives a 2-categorical treatment of Brown representability, which comes without any connectivity conditions. We would like to connect this with the work done in this thesis.

Much more can be said, but we hope that at the very least this thesis demonstrated the usefulness of homotopy 2-categories. We hope that they will be investigated more thoroughly and will be used more often.

A Appendix

A.1 The Boardman Vogt Construction

By general abstract nonsense the nerve functor $N : \mathsf{Cat} \longrightarrow \mathsf{sSet}$ has a cartesian monoidal left adjoint functor $\tau : \mathsf{sSet} \longrightarrow \mathsf{Cat}$. When restricted to quasi-categories, this functor admits an explicit description, sometimes called the *Boardman Vogt construction*. To give this description we need to fix some terminology.

Fix a quasi-category X. An *object* in X is a morphism of simplicial sets $\Delta^0 \to X$. An *arrow* in X is a morphism of simplicial sets $\Delta^1 \to X$. Precomposing with the two inclusions $\Delta^0 \to \Delta^1$ gives the *source* and *target* of the arrow. The *identity morphism* on an object x of X is the constant morphism on x, i.e. the morphism $1_x : \Delta^1 \to \Delta^0 \xrightarrow{x} X$.

A morphism $\Lambda^2_1 \to X$ describes a *composable pair* of arrows, a morphism $\partial\Delta^2 \to X$, a (not necessarily commutative) *triangle* and a morphism $\Delta^2 \to X$ a *commutative triangle* in X. The three inclusions $d_i : \Delta^1 \to \Delta^2$ give rise to the second, composite and first arrow. For a triangle with first arrow f, second arrow g and composite arrow h we write $(f, g, h) : \partial\Delta^2 \to X$ for the triangle.

Two arrows $f, g : \Delta^1 \to X$ with same source s and target t are *homotopic*, if one of the lifting problems

$$\begin{tikzcd}
\partial\Delta^2 \arrow[r, "(1_s, f, g)"] \arrow[d] & X & \partial\Delta^2 \arrow[r, "(f, 1_t, g)"] \arrow[d] & X & \partial\Delta^2 \arrow[r, "(1_s, g, f)"] \arrow[d] & X & \partial\Delta^2 \arrow[r, "(g, 1_t, g)"] \arrow[d] & X \\
\Delta^2 \arrow[ur, dashed] & & \Delta^2 \arrow[ur, dashed] & & \Delta^2 \arrow[ur, dashed] & & \Delta^2 \arrow[ur, dashed] &
\end{tikzcd}$$

admits a solution. In [Cis19] lemma 1.6.4 it is shown that these lifting problems are equivalent to each other and that being homotopic defines an equivalence relation $\sim$ on the set of arrows in X, i.e. on the Hom-set $\mathsf{sSet}(\Delta^1, X)$. It is also shown that any two composites h and h' for a composable pair of arrows f and g are homotopic.

Theorem A.1.1 (Boardman Vogt)
For a quasicategory X the 1-category with set of objects $\mathsf{sSet}(\Delta^0, X)$, set of arrows $\mathsf{sSet}(\Delta^1, X)/\sim$ and composition law given by homotopy classes of composite arrows (as described above) is welldefined.

Furthermore this construction is functorial in X and describes a functor $h : \mathsf{qCat} \longrightarrow \mathsf{Cat}$, which is naturally isomorphic to the restriction $\tau|_{\mathsf{qCat}} : \mathsf{qCat} \longrightarrow \mathsf{Cat}$.

Proof Confer [Cis19] theorem 1.6.6. The claim on functoriality follows from the fact that by construction postcomposition with morphisms of simplicial sets is compatible with the equivalence condition $\sim$. ☐

Remark A.1.2
The explicit description of h yields another argument to deduce that the functor τ is cartesian monoidal, when restricted to quasi-categories.

A.2 Homotopy Pullbacks as Weighted Limits

Let $\mathcal{C}$ be a simplicial model category. Let $\mathcal{D} = X \xrightarrow{f} Y \xleftarrow{g} Z$ be a diagram of fibrant objects in $\mathcal{C}$. One standard model for the homotopy pullback of this diagram is the pullback

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^1 \pitchfork Y \arrow[d, "{(\mathrm{ev}_0, \mathrm{ev}_1)}"] \\
X \times Z \arrow[r, "f \times g"'] & Y \times Y
\end{tikzcd}$$

or equivalently the equalizer

$$P = \mathrm{eq}\left(\; X \times Z \times (\Delta^1 \pitchfork Y) \underset{(f \times g) \circ \pi_{12}}{\overset{(\mathrm{ev}_0, \mathrm{ev}_1) \circ \pi_3}{\rightrightarrows}} Y \times Y \; \right).$$

Let $\mathcal{I}$ be the diagram category $\{0\} \xrightarrow{a} \{1\} \xleftarrow{b} \{2\}$ for pullbacks and $\mathcal{W} = \Delta^0 \xrightarrow{\mathrm{incl}_0} \Delta^1 \xleftarrow{\mathrm{incl}_1} \Delta^0$ a diagram in sSet. The $\mathcal{W}$-weighted limit of $\mathcal{D}$ can be explicitly computed as the equalizer

$$\lim_{\mathcal{W}} \mathcal{D} = \mathrm{eq}\left(\; \prod_{i \in \mathcal{I}} \mathcal{W}(i) \pitchfork \mathcal{D}(i) \underset{t}{\overset{s}{\rightrightarrows}} \prod_{i \to j \in \mathcal{I}} \mathcal{W}(i) \pitchfork \mathcal{D}(j) \; \right)$$

where s and t are given by the requirements that the diagrams

$$\begin{tikzcd}
\prod\limits_{i \in \mathcal{I}} \mathcal{W}(i) \pitchfork \mathcal{D}(i) \arrow[r, "s"] \arrow[d, "\pi_j"'] & \prod\limits_{i \to j \in \mathcal{I}} \mathcal{W}(i) \pitchfork \mathcal{D}(j) \arrow[d, "\pi_f"] & & \prod\limits_{i \in \mathcal{I}} \mathcal{W}(i) \pitchfork \mathcal{D}(i) \arrow[r, "s"] \arrow[d, "\pi_i"'] & \prod\limits_{i \to j \in \mathcal{I}} \mathcal{W}(i) \pitchfork \mathcal{D}(j) \arrow[d, "\pi_f"] \\
\mathcal{W}(j) \pitchfork \mathcal{D}(j) \arrow[r, "\mathcal{W}(f) \pitchfork \mathcal{D}(j)"'] & \mathcal{W}(i) \pitchfork \mathcal{D}(j) & \text{and} & \mathcal{W}(i) \pitchfork \mathcal{D}(i) \arrow[r, "\mathcal{W}(i) \pitchfork \mathcal{D}(f)"'] & \mathcal{W}(i) \pitchfork \mathcal{D}(j)
\end{tikzcd}$$

commute. With the specific shape of $\mathcal{I}$, $\mathcal{W}$ and $\mathcal{D}$ and using the canonical isomorphisms $\Delta^0 \pitchfork T \cong T$ this reduces to computing the equalizer

$$\lim_{\mathcal{W}} \mathcal{D} = \mathrm{eq}\left(\; X \times (\Delta^1 \pitchfork Y) \times Z \underset{t}{\overset{s}{\rightrightarrows}} Y \times Y \; \right)$$

where s and t are induced from the requirements

$$\begin{tikzcd}
\Delta^1 \pitchfork Y \arrow[r, "\mathrm{ev}_0"] & Y & & X \arrow[r, "a"] & Y \\
X \times (\Delta^1 \pitchfork Y) \times Z \arrow[r, "s"] \arrow[u, "\pi_1"] \arrow[d, "\pi_1"'] & Y \times Y \arrow[u, "\pi_a"'] \arrow[d, "\pi_b"] & \text{and} & X \times (\Delta^1 \pitchfork Y) \times Z \arrow[r, "t"] \arrow[u, "\pi_0"] \arrow[d, "\pi_2"'] & Y \times Y \arrow[u, "\pi_a"'] \arrow[d, "\pi_b"] \\
\Delta^1 \pitchfork Y \arrow[r, "\mathrm{ev}_1"'] & Y & & Z \arrow[r, "b"'] & Y
\end{tikzcd} .$$

Therefore the equalizer diagrams of P and $\lim_{\mathcal{W}} \mathcal{D}$ agree up to a permutation of the product, which shows $P = \lim_{\mathcal{W}} \mathcal{D}$.

A.3 Enriched Functorial Factorization Systems

In this section fix a complete and cocomplete symmetric monoidal closed category $\mathcal{V}$ with monoidal unit I and initial element $\emptyset$. There is a canonical adjunction

$$\begin{tikzcd}\mathsf{Cat} \arrow[r, bend left, "F_{\mathcal{V}}"] \arrow[r, phantom, "\perp"] & \mathcal{V}{-}\mathsf{Cat} \arrow[l, bend left, "(-)_1"]\end{tikzcd}$$

where

$$\begin{array}{rcl} F_{\mathcal{V}} : \mathsf{Cat} & \longrightarrow & \mathcal{V}{-}\mathsf{Cat} \\ \mathcal{C} & \longmapsto & \left\{ \begin{array}{l} \mathrm{Ob}\, F_{\mathcal{V}}(\mathcal{C}) = \mathrm{Ob}\, \mathcal{C} \\ F_{\mathcal{V}}(X,Y) = \coprod\limits_{\mathcal{C}(X,Y)} I \end{array} \right. \end{array}$$

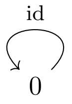
forms the free $\mathcal{V}$-category and

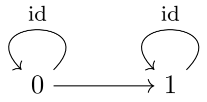
$$\begin{array}{rcl} (-)_1 : \mathcal{V}{-}\mathsf{Cat} & \longrightarrow & \mathsf{Cat} \\ \mathcal{C} & \longmapsto & \left\{ \begin{array}{l} \mathrm{Ob}\, \mathcal{C}_1 = \mathrm{Ob}\, \mathcal{C} \\ \mathcal{C}_1(X,Y) = \mathcal{V}(I, \mathcal{C}(X,Y)) \end{array} \right. \end{array}$$

forms the underlying 1-category (cf. [Bor94] proposition 6.4.7).

Definition A.3.1

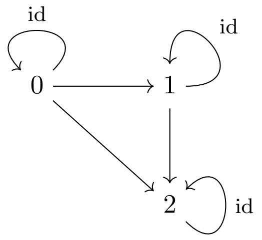
The **free object** is the category $\mathbb{1}$ completely described by the graph on the right.





The **free arrow** is the category $\mathbb{2}$ completely described by the graph on the left.

The **free composite** is the category $\mathbb{3}$ completely described by the graph on the right.



There are obvious functors

$$p_k : \mathbb{1} \longrightarrow \mathbb{2} \quad \text{and} \quad p_k : \mathbb{1} \longrightarrow \mathbb{3}$$

picking the k-th object, as well as functors

$$d_i : \mathbb{2} \longrightarrow \mathbb{3}$$

picking the arrow opposite to the i-th object.

A functor $\mathbb{1} \longrightarrow \mathcal{C}$ amounts to a choice of an object in $\mathcal{C}$, a functor $\mathbb{2} \longrightarrow \mathcal{C}$ to a choice of a morphism and a functor $\mathbb{3} \longrightarrow \mathcal{C}$ to a choice of a composable pair of morphisms (and their composite). Via the functor $F_{\mathcal{V}}$ the $\mathcal{V}$-categories $F_{\mathcal{V}}(\mathbb{1})$, $F_{\mathcal{V}}(\mathbb{2})$ and $F_{\mathcal{V}}(\mathbb{3})$ inherit the same universal properties in the 2-category of $\mathcal{V}$-categories. We abuse notation and write $\mathbb{1}$, $\mathbb{2}$, $\mathbb{3}$ for the corresponding free $\mathcal{V}$-categories.

For any $\mathcal{V}$-category $\mathcal{C}$ the functoriality of $F_{\mathcal{V}}$ and $\mathrm{Fun}_{\mathcal{V}}(-,\mathcal{C})$ gives rise to $\mathcal{V}$-functors

$$\mathrm{Fun}_{\mathcal{V}}(\mathbb{2},\mathcal{C}) \xrightarrow{\text{dom,codom}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{1},\mathcal{C}) = \mathcal{C}$$

given by precomposition of p_0 and p_1 respectively. Similarly we have canonical $\mathcal{V}$-functors

$$\mathrm{Fun}_{\mathcal{V}}(\mathbb{3},\mathcal{C}) \xrightarrow{\text{dom,interm,codom}} \mathcal{C}$$

picking the domain, intermediate and codomain of the composite arrow by precomposing p_0, p_1 and p_2. Finally precomposing with d_0, d_1 and d_2 gives rise to $\mathcal{V}$-functors

$$\mathrm{Fun}_{\mathcal{V}}(\mathbb{3},\mathcal{C}) \xrightarrow{\text{first,comp,second}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{2},\mathcal{C})$$

picking the first arrow, the composite or the second arrow of a chosen pair of composable arrows.

Definition A.3.2
A **$\mathcal{V}$-functorial factorization** on a $\mathcal{V}$-category $\mathcal{C}$ is a section $\mathrm{Fun}_{\mathcal{V}}(\mathbb{2},\mathcal{C}) \xrightarrow{\text{fact}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{3},\mathcal{C})$ of the $\mathcal{V}$-functor $\mathrm{Fun}_{\mathcal{V}}(\mathbb{3},\mathcal{C}) \xrightarrow{\text{comp}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{2},\mathcal{C})$.

If $\mathcal{C}$ is a $\mathcal{V}$-category with initial object $\mathbf{0}$ there is a constant $\mathcal{V}$-functor

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathrm{const}_{\mathbf{0}}} & \mathcal{C} \\ X & \longmapsto & \mathbf{0} \\ \mathcal{C}(X,Y) & \longrightarrow & \mathcal{C}(\emptyset,\emptyset) = * \end{array}$$

and a canonical $\mathcal{V}$-natural transformation

$$I \longrightarrow \mathcal{C}(\mathrm{const}_{\mathbf{0}}(X), \mathrm{id}(X)) = \mathcal{C}(\mathbf{0}, X) = *.$$

Together they specify an ordinary 1-functor

$$\mathbb{2} \longrightarrow (\mathrm{Fun}_{\mathcal{V}}(\mathcal{C},\mathcal{C}))_1,$$

hence by adjunction a $\mathcal{V}$-functor

$$\mathbb{2} \longrightarrow \mathrm{Fun}_{\mathcal{V}}(\mathcal{C},\mathcal{C})$$

and thus again by adjunction a $\mathcal{V}$-functor

$$\mathcal{C} \xrightarrow{\text{init}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{2},\mathcal{C}).$$

By construction we have that the composite

$$\mathcal{C} \xrightarrow{\text{init}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{2},\mathcal{C}) \xrightarrow{\text{codom}} \mathcal{C}$$

is the identity $\mathcal{V}$-functor.

Remark A.3.3
Assume that $\mathcal{C}$ is a $\mathcal{V}$-category with an initial object and a $\mathcal{V}$-functorial factorization.

Then we can form the composite $\mathcal{V}$-functor

$$\mathcal{Q} : \mathcal{C} \xrightarrow{\text{init}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{2},\mathcal{C}) \xrightarrow{\text{fact}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{3},\mathcal{C}) \xrightarrow{\text{interm}} \mathcal{C}$$

as well as

$$\mathcal{C} \xrightarrow{\text{init}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{2},\mathcal{C}) \xrightarrow{\text{fact}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{3},\mathcal{C}) \xrightarrow{\text{second}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{2},\mathcal{C}),$$

which by adjunction gives rise to a $\mathcal{V}$-functor

$$\mathrm{tf} : \mathbb{2} \longrightarrow \mathrm{Fun}_{\mathcal{V}}(\mathcal{C},\mathcal{C}).$$

Note that the composite

$$\mathcal{C} \xrightarrow{\text{init}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{2}, \mathcal{C}) \xrightarrow{\text{fact}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{3}, \mathcal{C}) \xrightarrow{\text{second}} \mathrm{Fun}_{\mathcal{V}}(\mathbb{2}, \mathcal{C}) \xrightarrow{\text{codom}} \mathcal{C}$$

is the identity $\mathcal{V}$-functor. In the light of the following theorem and corollary we think of the $\mathcal{V}$-functor $\mathcal{Q}$ as a *cofibrant replacement* $\mathcal{V}$-functor and of tf as the $\mathcal{V}$-natural transformation $\mathcal{Q} \Rightarrow \mathrm{id}$ whose components are the *trivial fibrations* of the factorization.

Theorem A.3.4

Suppose $\mathcal{V}$ is a monoidal model category in which all objects are cofibrant. Let $\mathcal{C}$ be a $\mathcal{V}$-model category, which is $\mathcal{V}$-bicomplete and permits the small object argument.

Then there exist $\mathcal{V}$-functorial factorizations for the model structure.

Proof Confer [Rie14] Theorem 13.2.1, Remark 13.2.2 and Corollary 13.2.3. □

Corollary A.3.5

Let $\mathcal{C}$ be a cofibrantly generated simplicial model category. Then $\mathcal{C}$ has sSet-enriched functorial factorizations for the simplicial model structure.

Proof In the Quillen-model structure on sSet all objects are cofibrant. A cofibrantly generated model category permits the small object argument. Thus the claim follows from the previous theorem A.3.4. □

A.4 Stable Simplicial Model Categories

We offer a brief discussion of modelling stable ∞-categories as stable simplicial model categories.

Definition A.4.1

A simplicial model category $\mathcal{C}$ is **stable**, if its underlying 1-category is a stable model category, that is if it is pointed and the loop-functor $\Omega : \mathrm{ho}(\mathcal{C}) \longrightarrow \mathrm{ho}(\mathcal{C})$ is an equivalence of categories.

We now want to discuss how stable simplicial model categories relate to stable quasi-categories. We restrict ourselves to locally presentable stable quasi-categories, which includes all small stable quasi-categories.

We first recall that by [Lur09] every locally presentable quasi-category is the localization of a combinatorial model category. By [Dug00] the latter is Quillen-equivalent to a combinatorial simplicial model category, in which all objects are cofibrant.

If we consider a pointed, locally presentable quasi-category, we will obtain a combinatorial simplicial model category $\mathcal{C}$, in which the canoncical morphism $\emptyset \to *$ is a trivial cofibration.

The simplicial slice category $*/\mathcal{C}$, whose Hom-object from $x : * \to X$ to $y : * \to Y$ are given by the pullback

$$\begin{tikzcd}
(*/\mathcal{C})(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(X,Y) \arrow[d, "x^*"] \\
* \arrow[r, "y"'] & \mathcal{C}(*,Y)
\end{tikzcd}$$

of simplicial sets, comes with an simplicial enriched adjunction

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "(-)_+"] \arrow[r, phantom, "\perp"] & */\mathcal{C} \arrow[l, bend left, "\mathcal{U}"]
\end{tikzcd} \qquad (\star)$$

which is by definition of the model structure on the slice a Quillen-adjunction. The slice category is pointed and a combinatorial simplicial model category, since the underlying 1-category is still locally presentable and cofibrantly generated[5]. That the Quillen-adjunction is in fact a Quillen-equivalence is due to the following lemma.

Lemma A.4.2

Let $\mathcal{C}$ be a simplicial model category, in which the unique morphism $\emptyset \longrightarrow *$ is a trivial cofibration. Then the simplicial adjunction $(\star)$ defines an simplicial Quillen-equivalence.

Proof By definition of an enriched Quillen equivalence it suffices, that the adjunction of underlying categories is a Quillen equivalence. By definition of the enriched model structure on an enriched slice category it is clear that $\mathcal{U}$ preserves fibrations and trivial fibrations, hence the adjunction constitutes an (enriched) Quillen adjunction. It remains to check that a morphism is a weak equivalence, if and only if its adjoint morphism is a weak equivalence.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & * \arrow[d] \\
X \arrow[r] & X_+ \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

To do this we first note that $\emptyset \longrightarrow *$ being a trivial cofibration forces the unit of the adjunction to be a trivial cofibration: Given an object $X \in \mathcal{C}$ the canonical map $\eta_X : X \longrightarrow X_+$ is a trivial cofibration in $\mathcal{C}$ as the pushout of a trivial cofibration depicted on the right. Hence for a weak equivalence $f : X_+ \longrightarrow (Y, y)$ in $*/\mathcal{C}$, with X cofibrant in $\mathcal{C}$ and (Y, y) fibrant in $*/\mathcal{C}$, its adjoint map $f\eta_X : X \longrightarrow X_+ \longrightarrow Y$ is a weak equivalence in $\mathcal{C}$.

---

[5] We might loose the property of having only cofibrant objects though.

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[dr, equal] & Y_+ \arrow[d] & * \arrow[l] \arrow[dl, "y"] \\
 & Y &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & X_+ \arrow[d, "f_+"] \\
Y \arrow[r, "\eta_Y"'] & Y_+
\end{tikzcd}$$

The counit of the adjunction can be identified as the canonical map $\varepsilon_{(Y,y)} : Y_+ \longrightarrow (Y, y)$. It is a weak equivalence in $*/\mathcal{C}$, since it is a weak equivalence in $\mathcal{C}$ by applying 2/3 in the defining diagram on the left. Given a weak equivalence $f : X \longrightarrow Y$ in $\mathcal{C}$, where X is cofibrant in $\mathcal{C}$ and (Y, y) is fibrant in $*/\mathcal{C}$, its adjoint morphism is given by the composite $\varepsilon_{(Y,y)} f_+ : X_+ \longrightarrow Y_+ \longrightarrow (Y, y)$. Hence it suffices to check, that the morphism f_+ is a weak equivalence, if f is. This immediately follows from the naturality of η by applying 2/3, as depicted in the square on the left.

□

Finally, if we start with a locally presentable stable quasi-category, since passing through all these equivalences induces an equivalence of the respective homotopy 1-categories, the resulting simplicial model category will be stable. We have have proven.

Theorem A.4.3

Let C be a locally presentable stable quasi-category. Then C is equivalent to the simplicial localization of a combinatorial stable simplicial model category.

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Erklärung

Ich habe die Arbeit selbstständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt. Außerdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind und dass ich von den in §26 Abs. 6 vorgesehenen Rechtsfolgen Kenntnis habe.

Regensburg, den 12. September 2022 Jonas Linßen