Formalisation of the Yoneda-Lemma in Naproche

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1 Categories

Signature 1. An arrow is a notion.

Signature 2. A collection of arrows is a notion.

Let f, g, h denote arrows. Let C, D denote collection of arrows.

Signature 3. $f \epsilon C$ is an atom.

Axiom 4. $C = D \iff (f \epsilon C \iff f \epsilon D)$.

Signature 5. $s[f]$ is an arrow.

Signature 6. $t[f]$ is an arrow.

Signature 7. $g \circ_c f$ is an arrow.

Definition 8. A category is a collection of arrows C such that
(for every arrow f such that $f \epsilon C$ we have
$s[f] \epsilon C$ and $t[f] \epsilon C$ and $t[s[f]] = s[f]$ and $s[t[f]] = t[f]$ and

$$\begin{tikzcd} s[f] \arrow[r, "{s[f]}"] \arrow[dr, "f"'] & s[f] \arrow[d, "f"] \\ & t[f] \end{tikzcd}$$

and

$$\begin{tikzcd} s[f] \arrow[r, "f"] \arrow[dr, "f"'] & t[f] \arrow[d, "{t[f]}"] \\ & t[f] \end{tikzcd} \;)$$

and (for each arrow f, g such that $f, g \epsilon C$ we have
$(t[f] = s[g] \implies$ (there is an arrow h such that $h \epsilon C$ and

$$\begin{tikzcd} s[f] \arrow[r, "f"] \arrow[dr, "h"'] & s[f] \arrow[d, "g"] \\ & t[g] \end{tikzcd}$$

and for every arrow k such that $k \epsilon C$ and $g \circ_c f = k$ we have $h = k$)))
and for all arrows f, g, h such that $f, g, h \epsilon C$ and $t[f] = s[g]$ and $t[g] = s[h]$

$$\begin{tikzcd} s[f] \arrow[r, "f"] \arrow[d, "(g \circ_c f)"'] & t[f] \arrow[d, "(h \circ_c g)"] \\ t[g] \arrow[r, "h"] & t[h] \end{tikzcd} \;.$$

2 Construction of SET

Definition 9. An sset is a set x such that x is an element of some set.

Let f, g, h denote functions.

Signature 10. $Cod(f)$ is a notion.

Axiom 11. $Cod(f)$ is a set.

Axiom 12. Let $x \in Dom(f)$. $f(x) \in Cod(f)$.

Axiom 13. **(Ext)** Let f, g be functions and $Dom(f) = Dom(g)$ and $Cod(f) = Cod(g)$. Let $f(x) = g(x)$ for every element x of $Dom(f)$. $f = g$.

Definition 14. Let $Cod(f) = Dom(g)$.
$g \circ f$ is the function h such that $Dom(h) = Dom(f)$ and $Cod(h) = Cod(g)$ and $h(x) = g(f(x))$ for every element x of $Dom(f)$.

Lemma 15. Let $Cod(f) = Dom(g)$ and $Cod(g) = Dom(h)$.
$h \circ (g \circ f) = (h \circ g) \circ f$.

Axiom 16. Every function is an arrow.

Axiom 17. $s[f]$ is a function such that
$Dom(s[f]) = Dom(f) = Cod(s[f])$.

Axiom 18. $s[f](y) = y$ for every element y of $Dom(f)$.

Axiom 19. $t[f]$ is a function such that
$Dom(t[f]) = Cod(f) = Cod(t[f])$.

Axiom 20. $t[f](y) = y$ for every element y of $Cod(f)$.

Definition 21. $SET = \{$function $f \mid Dom(f)$ is an sset and $Cod(f)$ is an sset $\}$.

Axiom 22. SET is a set.

Axiom 23. If $Dom(f), Cod(f)$ are sset then f is setsized.

Lemma 24. Let $f, g \in SET$ and $Cod(f) = Dom(g)$. $g \circ f \in SET$.

Axiom 25. SET is a collection of arrows.

Axiom 26. Let f be an arrow. $f \mathrel{\epsilon} SET \iff f \in SET$.

Axiom 27. Let $f \mathrel{\epsilon} SET$. $s[f] = Dom(f)$ and $t[f] = Cod(f)$.

Axiom 28. Let $f, g \in SET$ and $Cod(f) = Dom(g)$. $g \circ f = g \circ_c f$.

Lemma 29. Let $f \in SET$. $s[f] \in SET$ and $t[f] \in SET$.

Theorem 30. SET is a category.

Proof. Let us show that for every arrow f such that $f \mathrel{\epsilon} SET$ we have $s[f] \mathrel{\epsilon} SET$ and $t[f] \mathrel{\epsilon} SET$ and $t[s[f]] = s[f]$ and $s[t[f]] = t[f]$ and $f \circ_c s[f] = f$ and $t[f] \circ_c f = f$.

Proof. Let f be an arrow such that $f \mathrel{\epsilon} SET$.

$s[f] \, \epsilon \, SET$ and $t[f] \, \epsilon \, SET$.

$$(t[s[f]](y) = s[f](y)$$

for every $y \in Dom(f))$ and

$$s[t[f]](y) = t[f](y)$$

for any $y \in Cod(f)$.

We have

$$((f \circ s[f])(y) = f(s[f](y)) = f(y)$$

and

$$(t[f] \circ f)(y) = t[f](f(y)) = f(y))$$

for every $y \in Dom(f)$. Hence $f \circ s[f] = f$ and $t[f] \circ f = f$. Thus

$$\begin{tikzcd} s[f] \arrow[r, "s[f]"] \arrow[dr, "f"'] & s[f] \arrow[d, "f"] \\ & t[f] \end{tikzcd}$$

and

$$\begin{tikzcd} s[f] \arrow[r, "f"] \arrow[dr, "f"'] & t[f] \arrow[d, "t[f]"] \\ & t[f] \end{tikzcd} .$$

End.

For each arrow f, g such that $f, g \, \epsilon \, SET$ we have

$(t[f] = s[g] \implies$ (there is an arrow h such that $h \, \epsilon \, SET$ and

$$\begin{tikzcd} s[f] \arrow[r, "f"] \arrow[dr, "h"'] & s[f] \arrow[d, "g"] \\ & t[g] \end{tikzcd}$$

and for every arrow k such that $k \, \epsilon \, SET$ and $g \circ_c f = k$ we have $h = k$)).

Let us show that for all arrows f, g, h such that $f, g, h \, \epsilon \, SET$ and t[f]=s[g] and t[g]=s[h] we have $h \circ_c (g \circ_c f) = (h \circ_c g) \circ_c f$.

Proof. Let f, g, h be arrows such that $f, g, h \, \epsilon \, SET$ and t[f]=s[g] and t[g]=s[h]. $h \circ (g \circ f) = (h \circ g) \circ f$. Therefore

$$\begin{tikzcd} s[f] \arrow[r, "f"] \arrow[d, "(g \circ_c f)"'] & t[f] \arrow[d, "(h \circ_c g)"] \\ t[g] \arrow[r, "h"] & t[h] \end{tikzcd}.$$

End.

□

3 Bijections

Signature 31. Let Q, R be sets. A bijection between Q and R is a notion.

Axiom 32. Let Q, R be sets. Let f be a function such that $Dom(f) = Q$ and $Cod(f) = R$. Let g be a function such that $Dom(g) = R$ and $g(y) \in Q$ for any element y of R. Let $f(g(y)) = y$ for all elements y of $Dom(g)$. Let $g(f(x)) = x$ for all elements x of $Dom(f)$. Then f is a bijection between Q and R.

4 Functors

Signature 33. A functor is a notion.

Signature 34. Let F be a functor. Let f be an arrow. $F[f]$ is an arrow.

Definition 35. Let C, D be categories. A functor from C to D is a functor F such that (for all arrows f such that $f \epsilon C$ we have $F[f] \epsilon D$ and

$$F[s[f]] = s[F[f]]$$

and

$$F[t[f]] = t[F[f]])$$

and for all arrows f, g such that $f, g \epsilon C$ and $t[f] = s[g]$ we have

$$\begin{tikzcd} F[s[f]] \arrow[r, "F[f]"] \arrow[dr, "F[g \circ_c f]"'] & F[t[f]] \arrow[d, "F[g]"] \\ & F[t[g]] \end{tikzcd}.$$

5 Construction of the Hom Functor

Let C denote a category.

Signature 36. Let $c, x \epsilon C$. $Hom[C, c, x]$ is a collection of arrows such that $f \epsilon C$ for any arrow f such that $f \epsilon Hom[C, c, x]$.

Axiom 37. Let $c, x \epsilon C$. Let h be an arrow.

$$h \epsilon Hom[C, c, x] \iff (s[h] = c \text{ and } t[h] = x).$$

Definition 38. A locally small category is a category C such that $Hom[C, c, f]$ is an element of SET for all arrows c, f such that $c, f \epsilon C$.

Let C denote a locally small category.

Axiom 39. Let $c, x \epsilon C$. Let h be an arrow.

$$h \in Dom(Hom[C, c, x]) \iff h \epsilon Hom[C, c, x].$$

Axiom 40. Let $c, f \epsilon C$.

$$Dom(Hom[C, c, f]) = Hom[C, c, s[f]]$$

and

$$Cod(Hom[C, c, f]) = Hom[C, c, t[f]]$$

and

$$Hom[C, c, f](h) = f \circ_c h$$

for each arrow h such that $h \in Hom[C, c, s[f]]$.

Axiom 41. Let $c, x \epsilon C$. Any element h of $Dom(Hom[C, c, x])$ is an arrow.

Lemma 42. **(funct)** Let $c, f, g \epsilon C$ and $t[f] = s[g]$.

$$Hom[C, c, g] \circ Hom[C, c, f] = Hom[C, c, g \circ_c f].$$

Proof. $f \circ_c h \in Hom[C, c, s[g]]$ for each arrow h such that $h \in Hom[C, c, s[f]]$.

Proof. Let h be an arrow such that $h \in Hom[C, c, s[f]]$. $f \circ_c h$ is an arrow e such that $s[e] = c$ and $t[e] = t[f]$. End.

(h is an arrow and

$$(Hom[C, c, g] \circ Hom[C, c, f])(h)$$

$$= Hom[C, c, g](Hom[C, c, f](h))$$

$$= Hom[C, c, g](f \circ_c h) = g \circ_c (f \circ_c h)$$

$$= (g \circ_c f) \circ_c h = Hom[C, c, g \circ_c f](h))$$

for each element h of $Hom[C, c, s[f]]$. Therefore the thesis (by Ext). □

Definition 43. Let $c \epsilon C$. $HomF[C, c]$ is a functor such that for each arrow f such that $f \epsilon C$ we have

$$HomF[C, c][f] = Hom[C, c, f].$$

Theorem 44. Let $c \epsilon C$. $HomF[C, c]$ is a functor from C to SET.

Proof. For all arrows f such that $f \epsilon C$ we have

$$HomF[C, c][f] \epsilon SET$$

and

$$HomF[C, c][s[f]] = s[HomF[C, c][f]]$$

and

$$HomF[C, c][t[f]] = t[HomF[C, c][f]].$$

Proof. Let $f \epsilon C$.

$$HomF[C, c][s[f]] = Hom[C, c, s[f]] =$$

$$Dom(Hom[C, c, f]) = s[Hom[C, c, f]]$$
$$= s[HomF[C, c][f]].$$

End.

For all arrows f, g such that $f, g \epsilon C$ and $t[f] = s[g]$ we have

$$HomF[C, c][g \circ_c f] = HomF[C, c][g] \circ_c HomF[C, c][f].$$

Proof. Let $f, g \epsilon C$ and $t[f] = s[g]$.

$$Hom[C, c, g] \circ Hom[C, c, f] = Hom[C, c, g \circ_c f]$$

(by funct). End.

□

6 Natural Transformations

Let C, D denote categories.

Signature 45. A transformation is a notion.

Signature 46. Let F, G be functors from C to D. Let α be a transformation. $T[C, D, F, G, \alpha]$ is a collection of arrows.

Signature 47. Let F, G be functors from C to D. Let α be a transformation. Let $d \epsilon C$. $T[C, D, F, G, \alpha, d]$ is an arrow.

Axiom 48. Let F, G be functors from C to D. Let α be a transformation.

$$T[C, D, F, G, \alpha, d] \mathrel{\epsilon} T[C, D, F, G, \alpha]$$

for every arrow d such that $d \mathrel{\epsilon} C$.

Axiom 49. Let F, G be functors from C to D. Let α be a transformation. For any arrow f such that $f \mathrel{\epsilon} T[C, D, F, G, \alpha]$ there exists an arrow d such that $d \mathrel{\epsilon} C$ and $T[C, D, F, G, \alpha, d] = f$.

Definition 50. Let F, G be functors from C to D. A natural transformation from F to G over C and D is a transformation α such that
(for any arrow d such that $d \mathrel{\epsilon} C$ we have

$$T[C, D, F, G, \alpha, d] \mathrel{\epsilon} D)$$

and
(for any arrow d such that $d \mathrel{\epsilon} C$ we have

$$s[T[C, D, F, G, \alpha, d]] = F[d])$$

and
(for any arrow d such that $d \mathrel{\epsilon} C$ we have

$$t[T[C, D, F, G, \alpha, d]] = G[d])$$

and for any arrow f such that $f \mathrel{\epsilon} C$ we have

$$\begin{tikzcd}
F[s[f]] \arrow[r, "F[f]"] \arrow[d, swap, "{T[C,D,F,G,\alpha,s[f]]}"] & F[t[f]] \arrow[d, "{T[C,D,F,G,\alpha,t[f]]}"] \\
G[s[f]] \arrow[r, "G[f]"] & G[t[f]]
\end{tikzcd}.$$

Definition 51. Let F, G be functors from C to D.

$$Nat[C, D, F, G] = \{ \text{ transformation } \alpha \mid \alpha \text{ is a natural transformation from } F \text{ to } G \text{ over } C \text{ and } D \ \}.$$

Axiom 52. Let F, G be functors from C to D. $Nat[C, D, F, G]$ is a set.

Axiom 53. Let F, G be functors from C to D. Let $\alpha, \beta \in Nat[C, D, F, G]$.

$$\alpha = \beta \iff T[C, D, F, G, \alpha] = T[C, D, F, G, \beta].$$

7 Yoneda

7.1 Construction of the bijection

Let C denote a locally small category.

Signature 54. Ψ is a notion.

Axiom 55. Let F be a functor from C to SET. Let $c \epsilon C$. Ψ is a function and $Dom(\Psi) = F[s[c]]$ and $\Psi(x)$ is a transformation for every element x of $Dom(\Psi)$.

Axiom 56. Let F be a functor from C to SET. Let $c, d \epsilon C$. Let $x \in Dom(\Psi)$. $T[C, SET, HomF[C, c], F, \Psi(x), d]$ is a function and

$$Dom(T[C, SET, HomF[C, c], F, \Psi(x), d]) = Hom[C, c, d]$$

and

$$Cod(T[C, SET, HomF[C, c], F, \Psi(x), d]) = F[d].$$

Axiom 57. **(PsiDef)** Let F be a functor from C to SET. Let $c, d \epsilon C$ and $f \epsilon Hom[C, c, d]$. Let $x \in Dom(\Psi)$.

$$T[C, SET, HomF[C, c], F, \Psi(x), d](f) = F[f](x).$$

Signature 58. Φ is a notion.

Axiom 59. Let F be a functor from C to SET. Let $c \epsilon C$. Φ is a function and

$$Dom(\Phi) = Nat[C, SET, HomF[C, c], F]$$

and

$$Cod(\Phi) = F[s[c]].$$

Axiom 60. **(PhiDef)** Let F be a functor from C to SET. Let $c \epsilon C$. Let $\alpha \in Nat[C, SET, HomF[C, c], F]$.

$$\Phi(\alpha) = T[C, SET, HomF[C, c], F, \alpha, s[c]](s[c]).$$

7.2 Result

Lemma 61. **(Yoneda)** Let F be a functor from C to SET. Let $c \epsilon C$ and $s[c] = c$.

$$\Phi \text{ is a bijection between } Nat[C, SET, HomF[C, c], F] \text{ and } F[c].$$

Proof. Let us show that for every element x of $Dom(\Psi)$

$\Psi(x)$ is a natural transformation from $HomF[C, c]$ to F over C and SET.

Proof. Let $x \in Dom(\Psi)$.

Let us show that for any arrow d such that $d \epsilon C$ and s[d]=d we have

$$T[C, SET, HomF[C, c], F, \Psi(x), d] \epsilon SET.$$

Proof. Let $d \epsilon C$ and s[d]=d.

$$Hom[C, c, d] = Dom(T[C, SET, HomF[C, c], F, \Psi(x), d])$$

and

$$F[d] = Cod(T[C, SET, HomF[C, c], F, \Psi(x), d]).$$

End.

Let us show that for any arrow d such that $d \epsilon C$ we have

$$s[T[C, SET, HomF[C, c], F, \Psi(x), d]] = HomF[C, c][d].$$

Proof. Let $d \epsilon C$.

$$s[T[C, SET, HomF[C, c], F, \Psi(x), d]] =$$

$$Dom(T[C, SET, HomF[C, c], F, \Psi(x), d]) =$$
$$Hom[C, c, d] = HomF[C, c][d].$$

End.

Let us show that for any arrow d such that $d \epsilon C$ we have

$$t[T[C, SET, HomF[C, c], F, \Psi(x), d]] = F[d].$$

Proof. Let $d \epsilon C$.

$$t[T[C, SET, HomF[C, c], F, \Psi(x), d]] =$$

$$Cod(T[C, SET, HomF[C, c], F, \Psi(x), d]) = F[d].$$

End.

Let us show that for any arrow g such that $g \epsilon C$ we have

$$\begin{tikzcd} Hom[C,c,s[g]] \arrow[r, "{T[C,SET,HomF[C,c],F,\Psi(x),s[g]]}"] \arrow[d, "{HomF[C,c][g]}"'] & F[s[g]] \arrow[d, "{F[g]}"] \\ Hom[C,c,t[g]] \arrow[r, "{T[C,SET,HomF[C,c],F,\Psi(x),t[g]]}"'] & F[t[g]] \end{tikzcd} .$$

Proof. Let $g \epsilon C$.

$$(T[C, SET, HomF[C,c], F, \Psi(x), t[g]] \circ HomF[C,c][g])(f)$$

$$= T[C, SET, HomF[C,c], F, \Psi(x), t[g]](Hom[C,c,g](f))$$
$$= T[C, SET, HomF[C,c], F, \Psi(x), t[g]](g \circ_c f)$$
$$= F[g \circ_c f](x) = (F[g] \circ_c F[f])(x)$$

for all arrows f such that $f \epsilon Dom(HomF[C,c][g])$.

$$(F[g] \circ T[C, SET, HomF[C,c], F, \Psi(x), s[g]])(f)$$

$$= F[g](T[C, SET, HomF[C,c], F, \Psi(x), s[g]](f))$$
$$= F[g](F[f](x)) = (F[g] \circ_c F[f])(x)$$

for all arrows f such that $f \epsilon Dom(T[C, SET, HomF[C,c], F, \Psi(x), s[g]])$.

End.

QED.

Let us show that for every element x of $Dom(\Psi)$

$$\Phi(\Psi(x)) = x.$$

Proof. Let $x \in Dom(\Psi)$.

$$\Phi(\Psi(x)) = T[C, SET, HomF[C,c], F, \Psi(x), s[c]](s[c]) =$$

$$F[s[c]](x) = s[F[c]](x) = x.$$

QED.

Let us show that for every element α of $Nat[C, SET, HomF[C,c], F]$

$$\Psi(\Phi(\alpha)) = \alpha.$$

Proof.

Let $\alpha \in Nat[C, SET, HomF[C,c], F]$.

For all arrows d, f such that $d \epsilon C$ and $f \epsilon Hom[C,c,d]$ we have

$$T[C, SET, HomF[C,c], F, \Psi(T[C, SET, HomF[C,c], F, \alpha, c](s[c])), d](f)$$

$$= F[f](T[C, SET, HomF[C,c], F, \alpha, c](s[c]))$$

(by PsiDef).

For all arrows d, f such that $d \epsilon C$ and $f \epsilon Hom[C,c,d]$ we have

$$\begin{array}{ccc} Hom[C,c,c] & \xrightarrow{T[C,SET,HomF[C,c],F,\alpha,c]} & F[c] \\ {\scriptstyle Hom[C,c,f]}\downarrow & & \downarrow{\scriptstyle F[f]} \\ Hom[C,c,d] & \xrightarrow[T[C,SET,HomF[C,c],F,\alpha,d]]{} & F[d] \end{array} .$$

Hence

$$F[f](T[C,SET,HomF[C,c],F,\alpha,c](s[c]))$$
$$= T[C,SET,HomF[C,c],F,\alpha,d](f)$$

for all arrows d, f such that $d \epsilon C$ and $f \epsilon Hom[C,c,d]$.

Therefore

$$T[C,SET,HomF[C,c],F,\Psi(T[C,SET,HomF[C,c],F,\alpha,c](s[c])),d]$$
$$= T[C,SET,HomF[C,c],F,\alpha,d]$$

for all arrows d such that $d \epsilon C$.

$$\Psi(\Phi(\alpha)) = \alpha \iff$$
$$T[C,SET,HomF[C,c],F,\Psi(\Phi(\alpha)),d] = T[C,SET,HomF[C,c],F,\alpha,d]$$

for all arrows d such that $d \epsilon C$. Thus we have the thesis. QED.

□