Introduction to Bispectral Analysis and High Order Correlations

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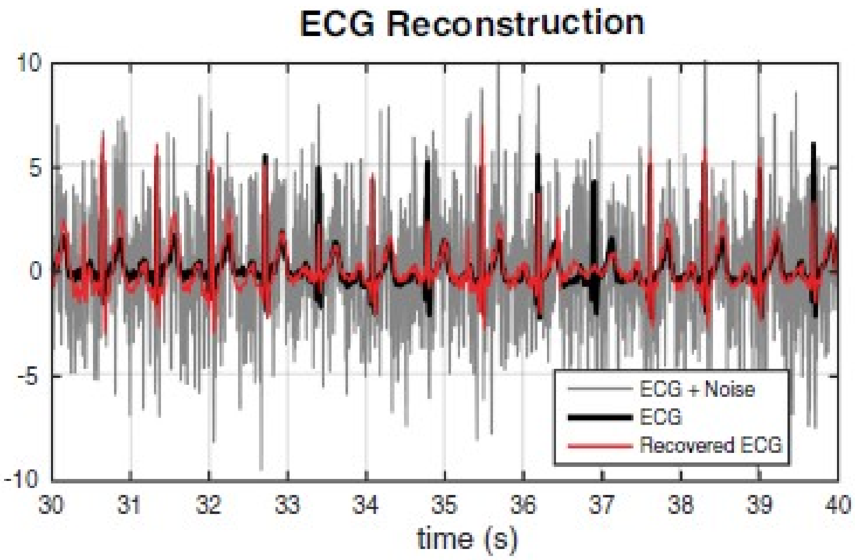
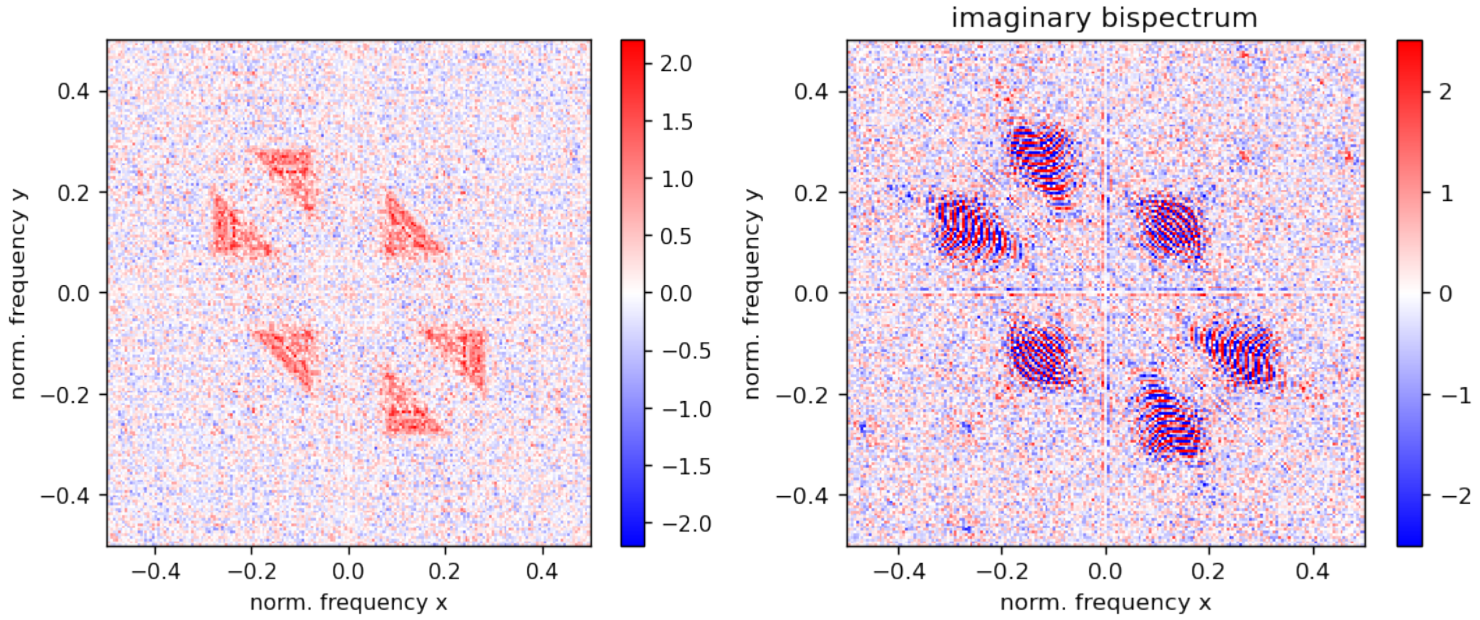
Motivation

- Reconstruction of broad-in-frequency signals with low SNR, e.g. electric/magnetic heart beat signals



How to read Bispectra? What do they mean?



Example of Higher Order Moments in Statistics

- Average

$$\mu = \langle x \rangle = \frac{1}{n} \sum_i x_i$$

- Variance

$$\mathrm{var}(x) = \sigma^2 = \langle (x - \mu)^2 \rangle$$

- Skewness

$$\tilde{\mu}_3 = \left\langle \left(\frac{x - \mu}{\sigma} \right)^3 \right\rangle$$

- Kurtosis

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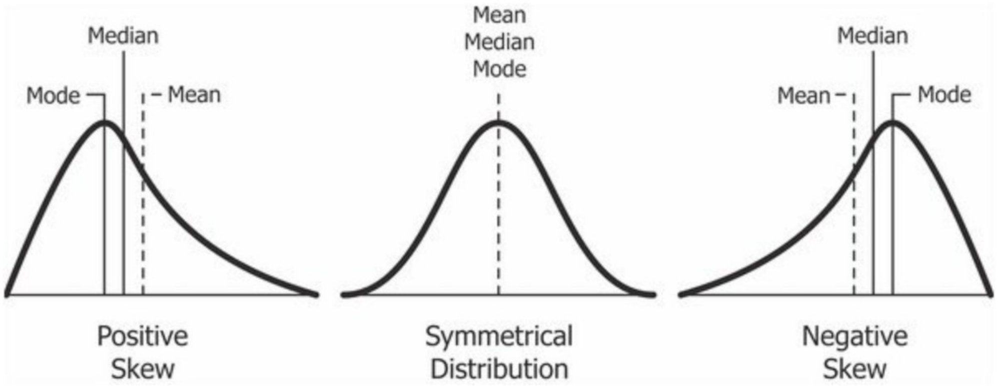

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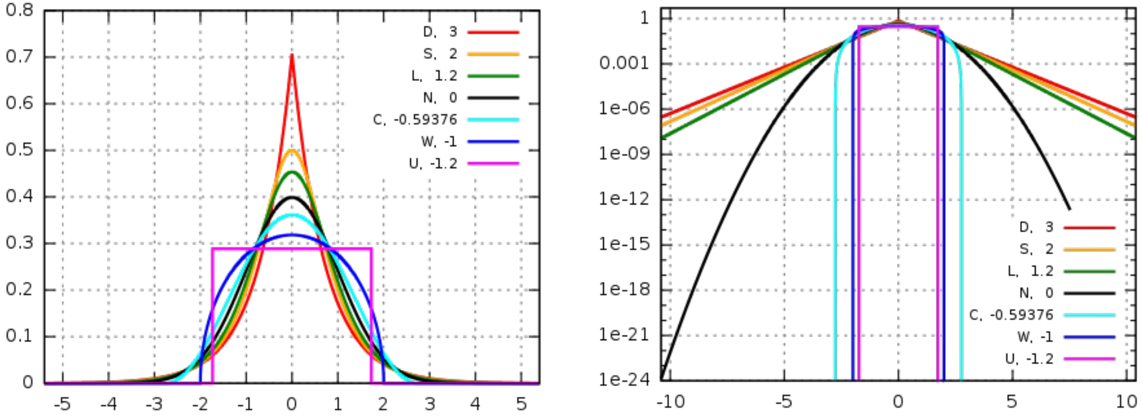

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Higher Order Correlations (HOC)

- 1st order
- 2nd order
- 3rd order
- 4th order

$$C_1 = \langle x(t) \rangle$$

$$C_2(\tau) = \langle x(t)x(t+\tau) \rangle$$

$$C_3(\tau_1, \tau_2) = \langle x(t)x(t+\tau_1)x(x+\tau_2) \rangle$$

$$C_4(\tau_1, \tau_2, \tau_3) = \langle x(t)x(t+\tau_1)x(t+\tau_2)x(t+\tau_3) \rangle$$

HOC and Higher Order Spectra (HOS)

- **Power spectrum**

$$S_{xx}(f) = \int_{-\infty}^{\infty} C_2(\tau) \mathrm{e}^{-i2\pi f\tau} \mathrm{d}\tau = \mathcal{F}_{C_2}(f),$$

- **Bispectrum**

$$B_x(f_1, f_2) = \int \int \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \, C_3(\tau_1, \tau_2) \, \mathrm{e}^{-i2\pi(f_1\tau_1 + f_2\tau_2)}.$$

- **Trispectrum: ...**

→ Why Higher order Spectra are interesting?

Motivation of Bispectra

- Most important identity of Bispectra

$$B_x(f_1, f_2) = \mathcal{F}_x(f_1)\, \mathcal{F}_x(f_2)\, \mathcal{F}_x^{-1}(f_1 + f_2),$$

- Sensitive to non-linear processes $y = ax^2 + bx$
- Conserving the relative phase of signals $B_x(f_1, f_2) \in \mathbb{C}^2$
- "Gaussian noise free" $C_{n>2} = 0$

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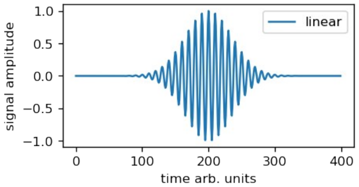
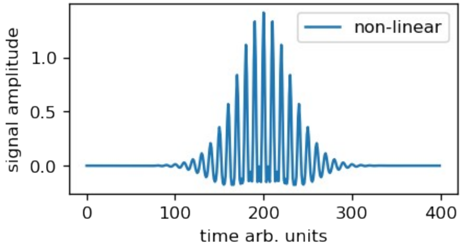
Example 1: Non Linear Process

- Original signal

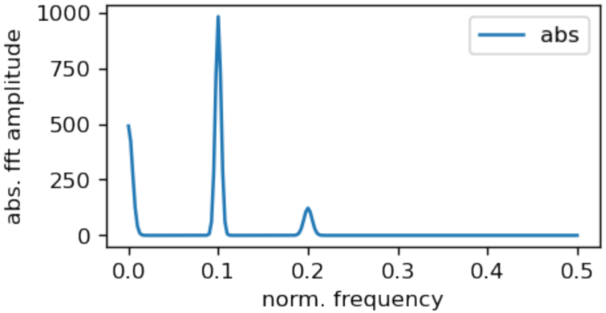


- Signal after non linear process

$$y = ax^2 + bx$$



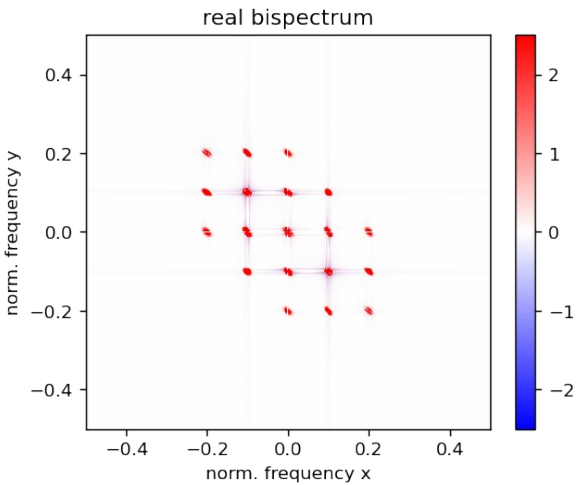
- Power spectrum of the signal after non-linear process

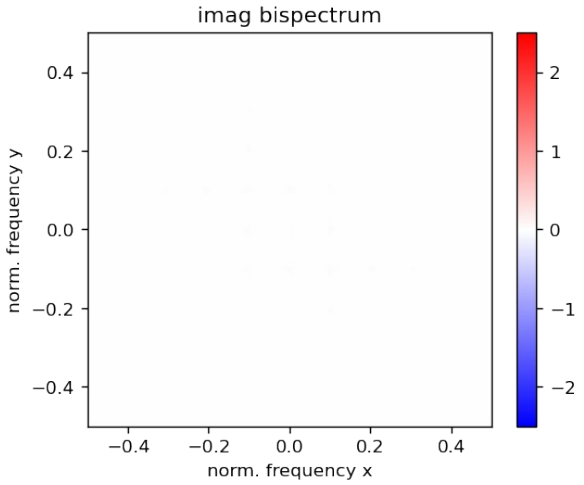


Example 1: Non Linear Process

- Quadratic Phase coupled (QPC) component

  → real peaks





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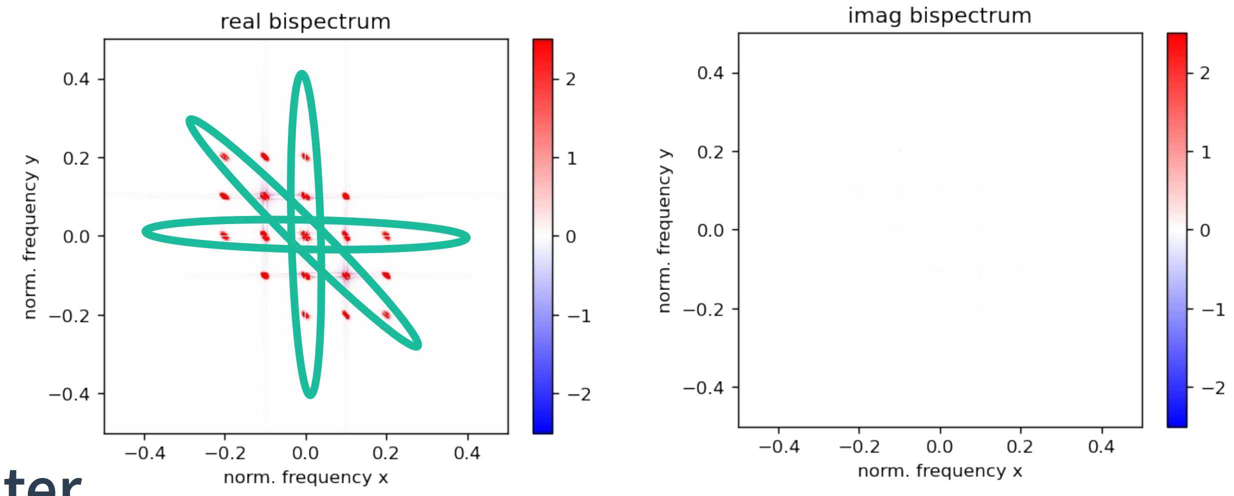
- The peaks involving f(0) are trivial → high pass filter



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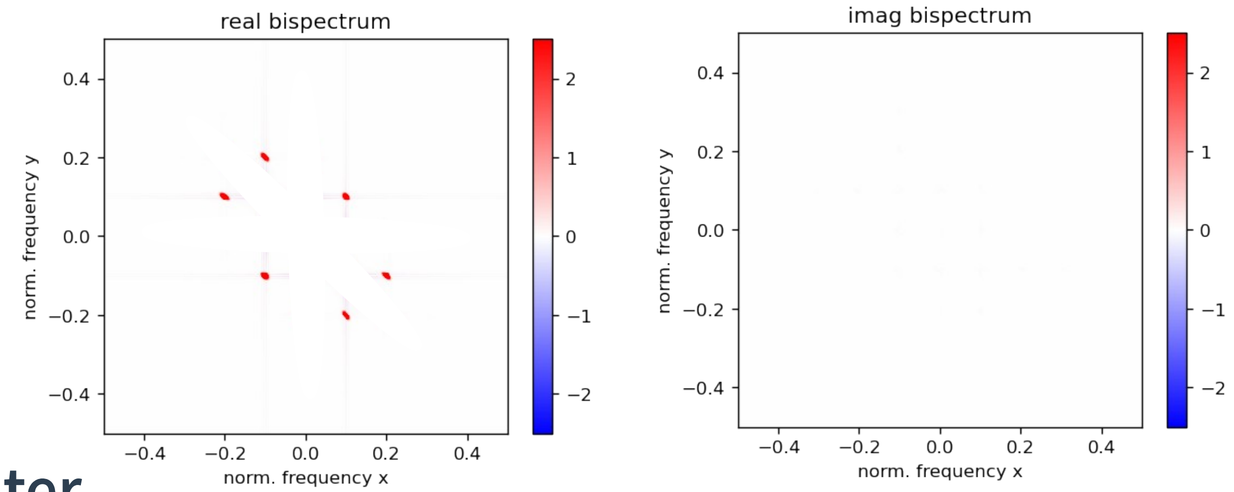

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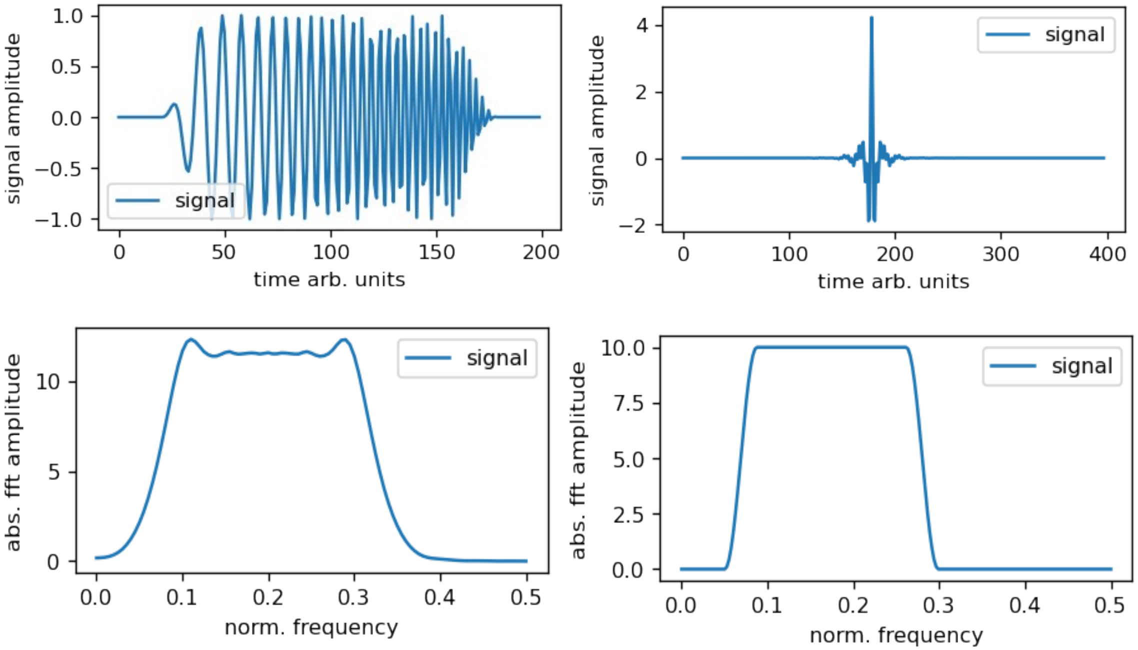
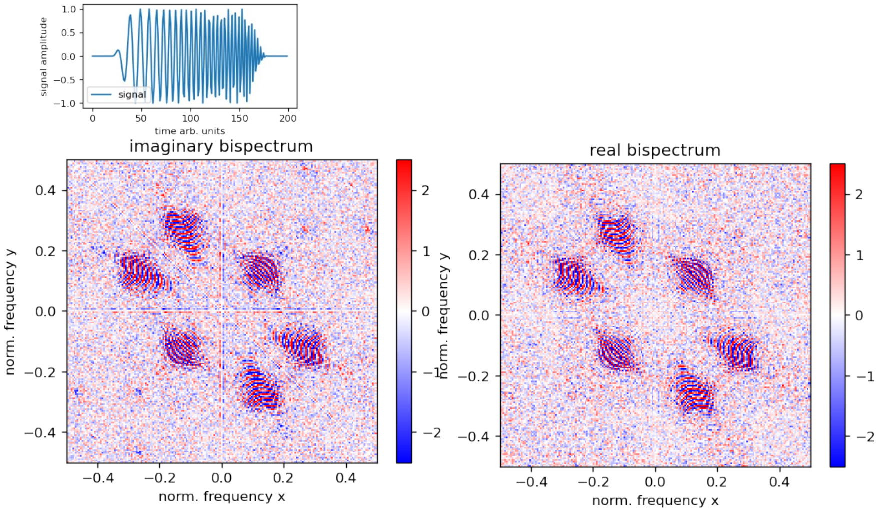
Example 2: Square Signals

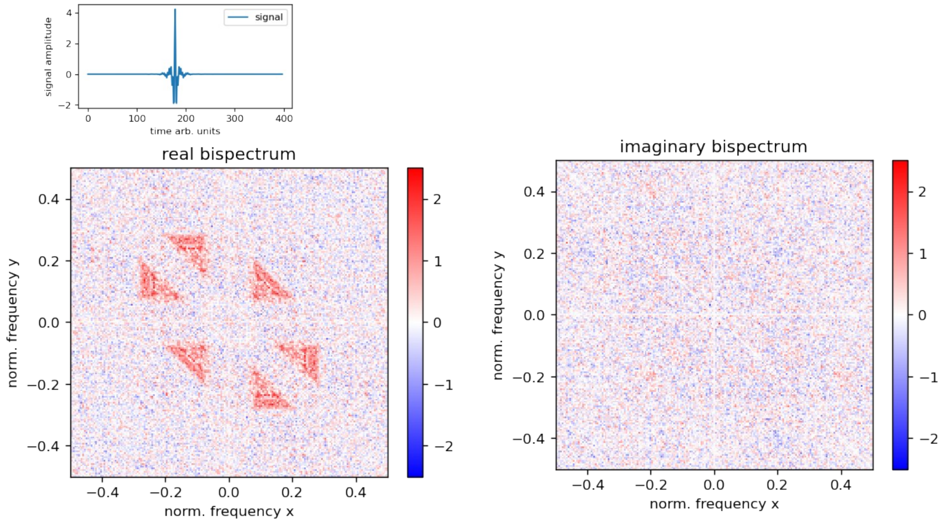
- Two rectangular signals in frequency space



Example 2: Square Signals

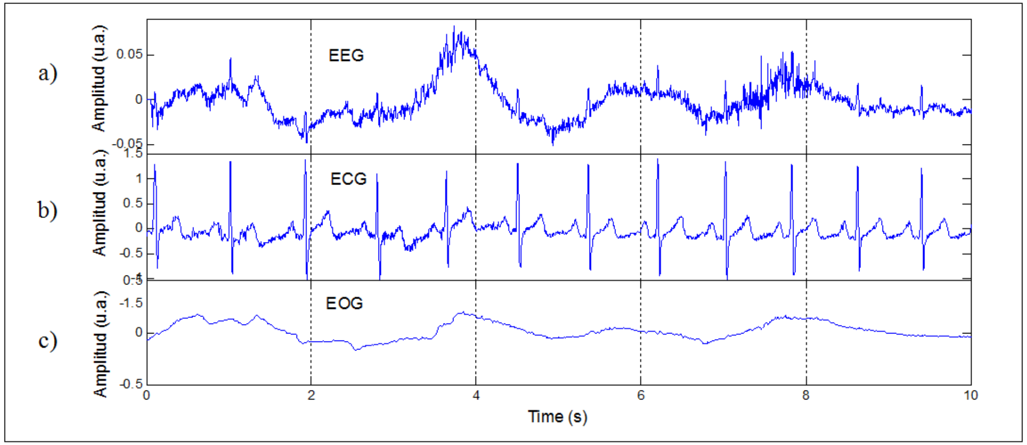
- The bispectrum is sensitive to the complex phase

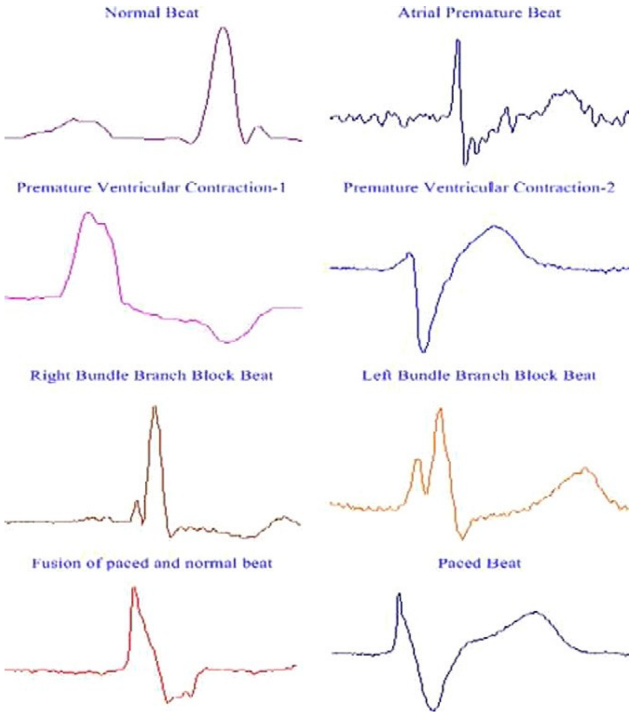




Human Electrical Signals

- Common cardiac arrhythmias (ECG)
- EEG signals

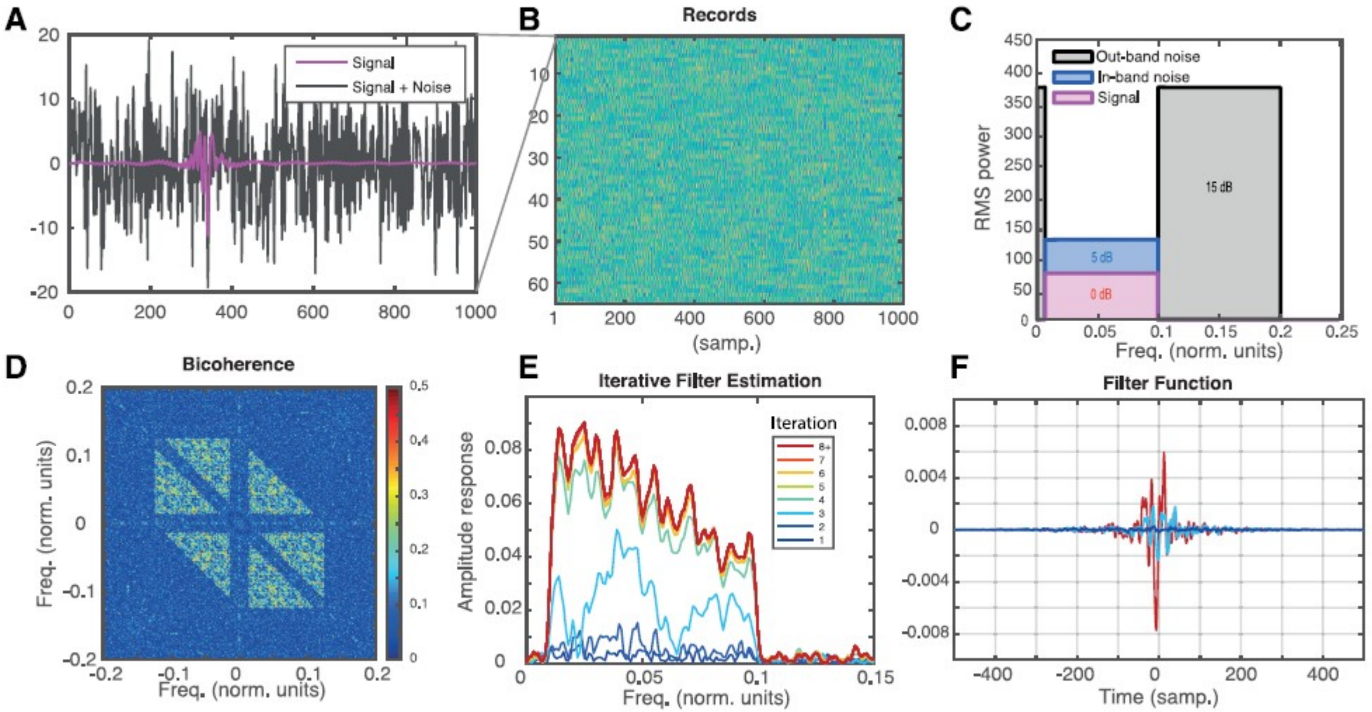




Lanata, Antonio, et al. Expert Systems with Applications 38.6 (2011): 6798-6804.
Correa, M. Agustina Garcés, and Eric Laciar Leber. Adaptive filtering applications 34 (2011): 1-26.

Example 3: Reconstructing a Filter for Arbitrary Signals

C.K. Kovach and M.A. Howard III / Signal Processing 165 (2019) 357–379 367



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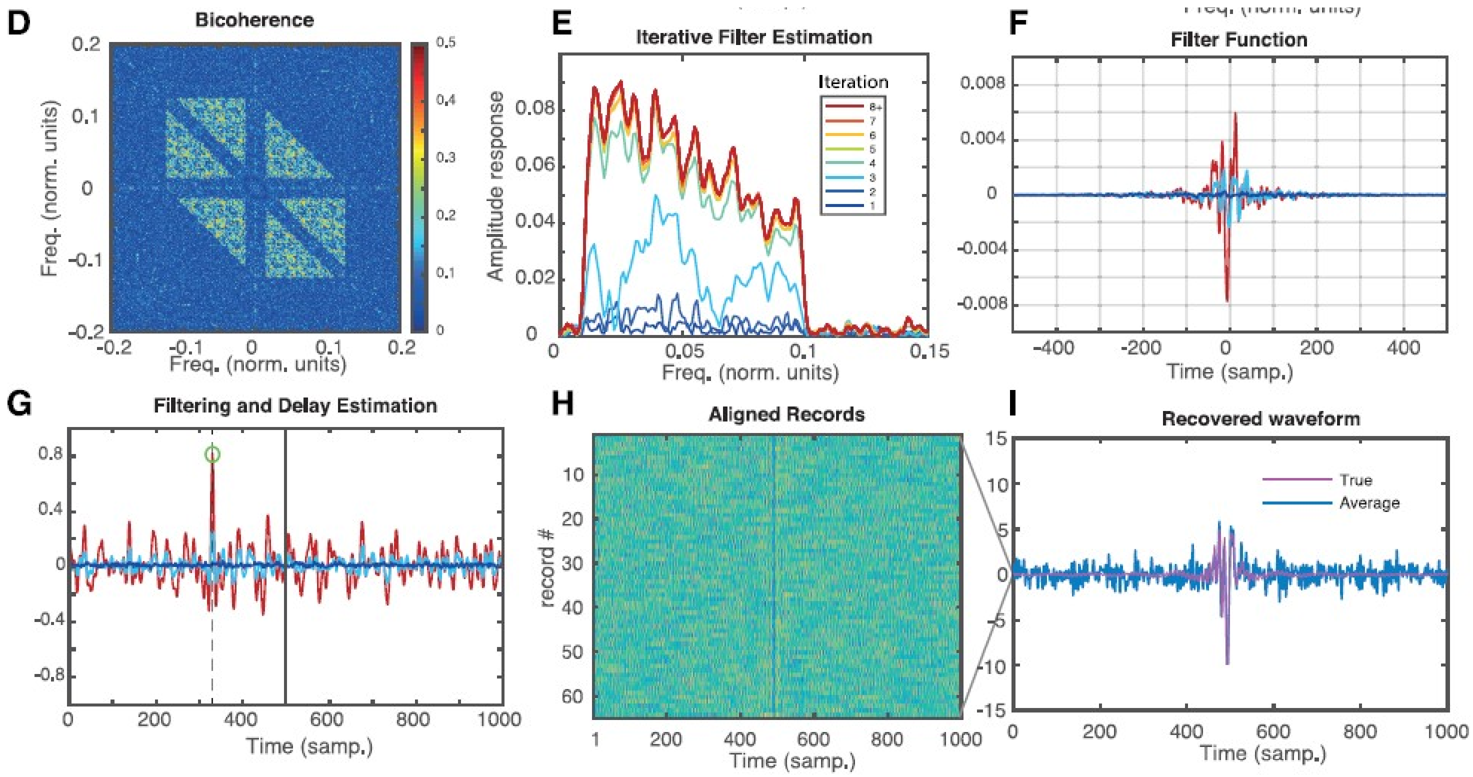
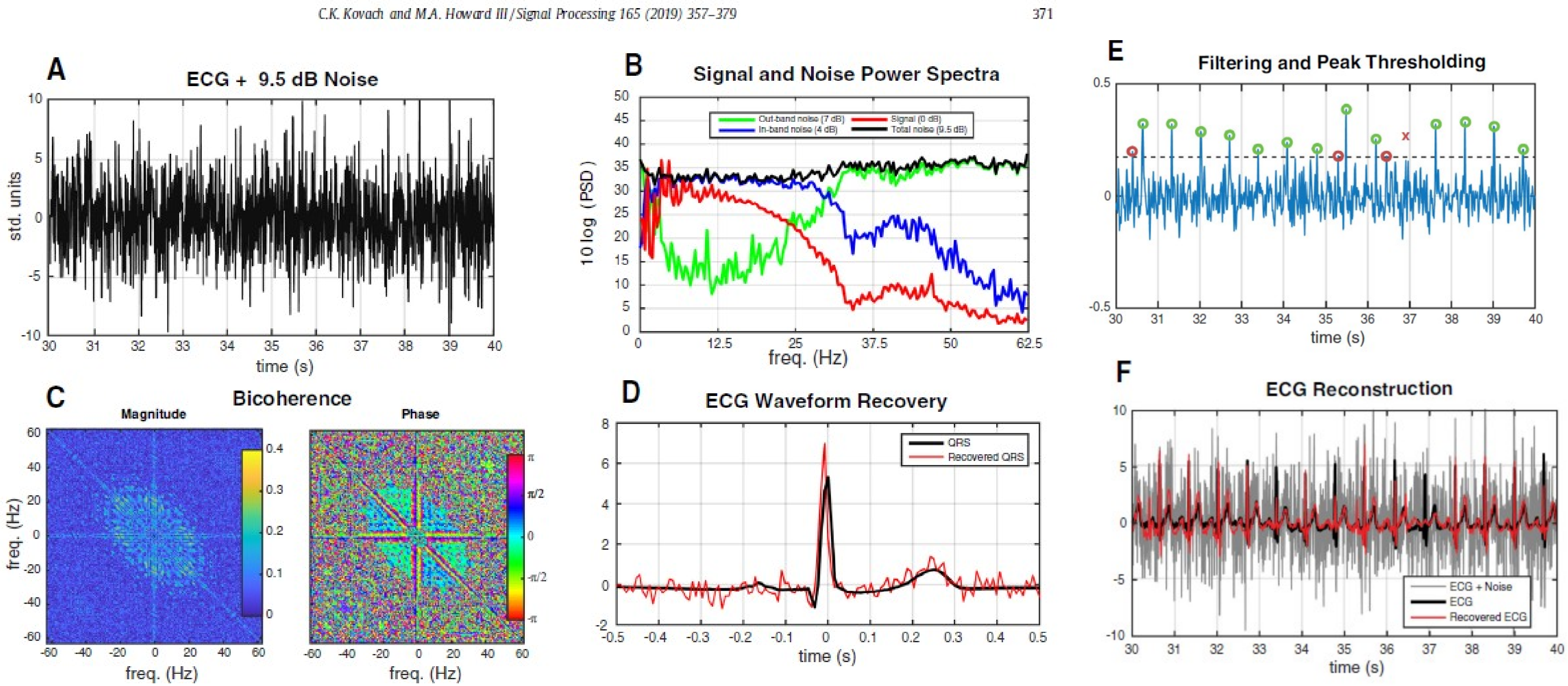


Example 4: Reconstructing a Filter for Bio Signals



Summary of Bispectral analysis

- Very important identity of Bispectrum

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Acknowledgements

- Jörg Wrachtrup & Vadim Vorobyov & the team at PI3

Questions?