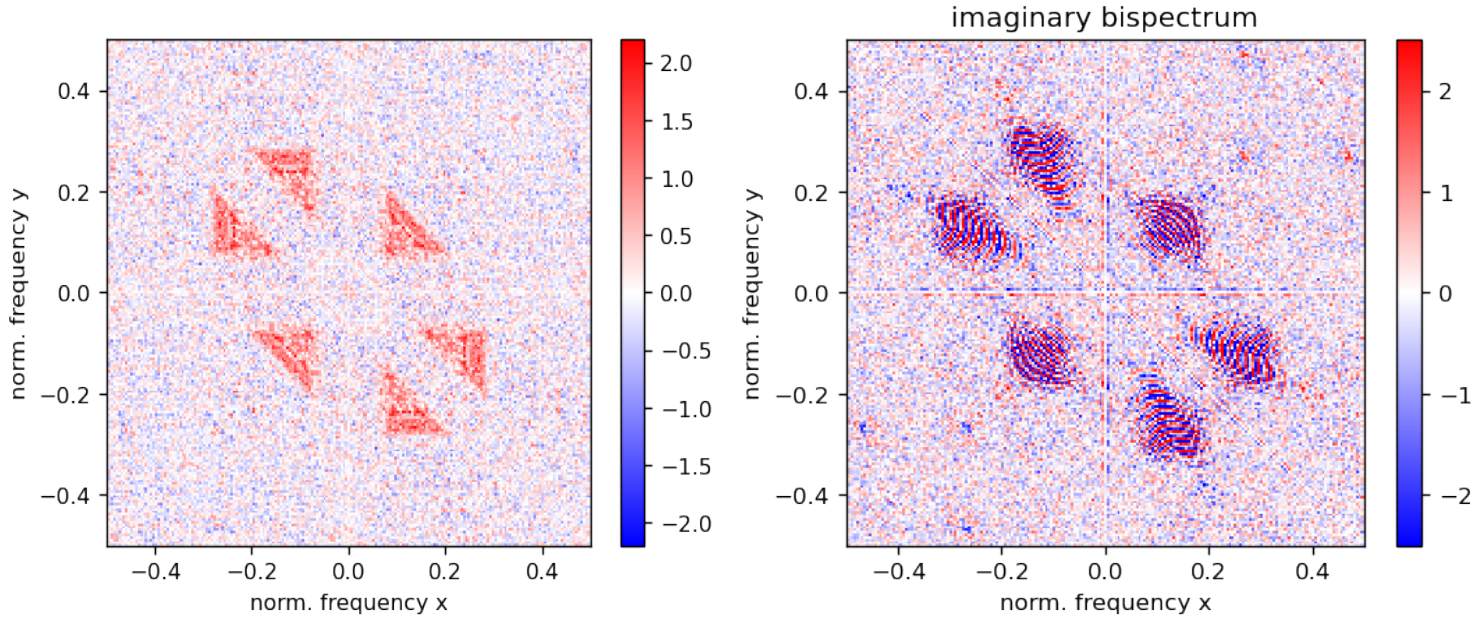
Introduction to Bispectral Analysis and High Order Correlations

Jonas Meinel

10.12.2021

How to read Bispectra? What do they mean?



Example of Higher Order Moments in Statistics

- Average

$$\mu = \langle x \rangle = \frac{1}{n} \sum_i x_i$$

- Variance

$$\mathrm{var}(x) = \sigma^2 = \langle (x - \mu)^2 \rangle$$

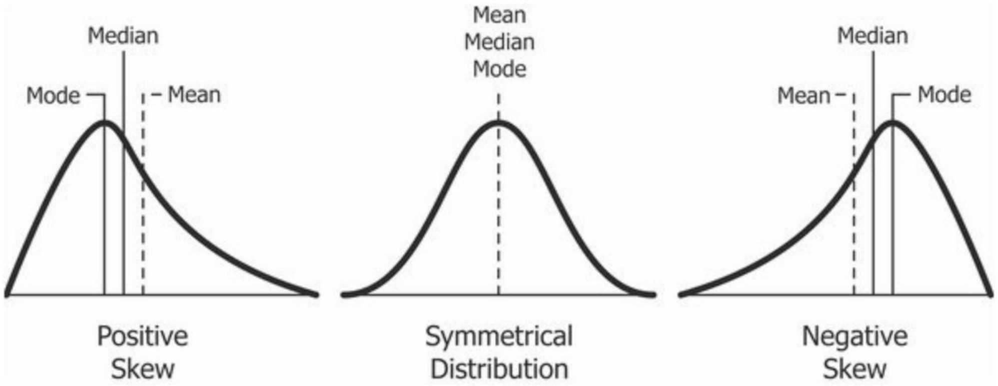
- Skewness

$$\tilde{\mu}_3 = \left\langle \left(\frac{x - \mu}{\sigma} \right)^3 \right\rangle$$

- Kurtosis

Example of Higher Order Moments in Statistics

- Average
- Variance
- Skewness
- Kurtosis



$$\tilde{\mu}_3 = \left\langle \left(\frac{x-\mu}{\sigma} \right)^3 \right\rangle$$

Example of Higher Order Moments in Statistics

- Average

$$\mu = \langle x \rangle = \frac{1}{n}\sum_i x_i$$

- Variance

$$\mathrm{var}(x) = \sigma^2 = \langle (x-\mu)^2 \rangle$$

- Skewness

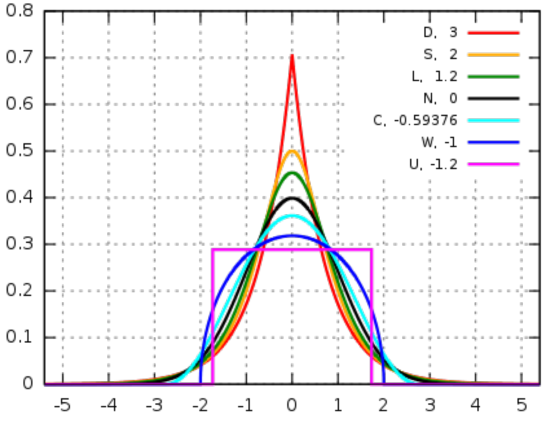
$$\tilde{\mu}_3 = \left\langle \left(\frac{x-\mu}{\sigma} \right)^3 \right\rangle$$

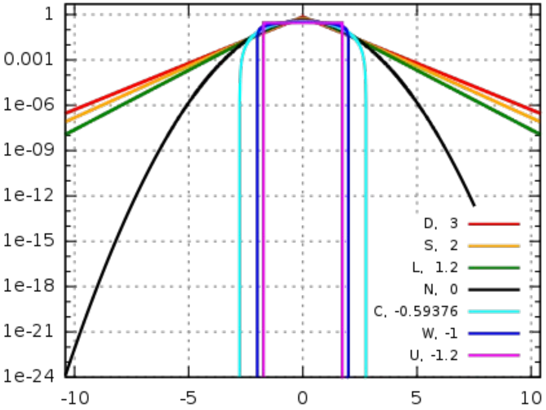
- Kurtosis

$$\tilde{\mu}_4 = \left\langle \left(\frac{x-\mu}{\sigma} \right)^4 \right\rangle = \left\langle \mathrm{var}\left(\left(\frac{x-\mu}{\sigma} \right)^2 \right) \right\rangle + 1$$

Example of Higher Order Moments in Statistics

- Average
- Variance
- Skewness
- Kurtosis





$$\tilde{\mu}_4 = \left\langle \left(\frac{x-\mu}{\sigma} \right)^4 \right\rangle = \left\langle \mathrm{var} \left(\left(\frac{x-\mu}{\sigma} \right)^2 \right) \right\rangle + 1$$

Higher Order Correlations (HOC)

- 1st order
- 2nd order
- 3rd order
- 4th order

$$
\begin{aligned}
C_1 &= \langle x(t) \rangle \\
C_2(\tau) &= \langle x(t) x(t+\tau) \rangle \\
C_3(\tau_1, \tau_2) &= \langle x(t) x(t+\tau_1) x(x+\tau_2) \rangle \\
C_4(\tau_1, \tau_2, \tau_3) &= \langle x(t) x(t+\tau_1) x(t+\tau_2) x(t+\tau_3) \rangle
\end{aligned}
$$

HOC and Higher Order Spectra (HOS)

- **Power spectrum**

$$S_{xx}(f) = \int_{-\infty}^{\infty} C_2(\tau)\mathrm{e}^{-i2\pi f\tau}\mathrm{d}\tau = \mathcal{F}_{C_2}(f),$$

- **Bispectrum**

$$B_x(f_1, f_2) = \int\int \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \, C_3(\tau_1, \tau_2) \, \mathrm{e}^{-i2\pi(f_1\tau_1 + f_2\tau_2)}.$$

- **Trispectrum: ...**

→ Why Higher order Spectra are interesting?

Motivation of Bispectra

- Very important identity of Bispectra

$$B_x(f_1, f_2) = \mathcal{F}_x(f_1)\, \mathcal{F}_x(f_2)\, \mathcal{F}_x^{-1}(f_1 + f_2),$$

- Sensitive to non-linear processes $y = ax^2 + bx$
- Conserving the relative phase of signals $B_x(f_1, f_2) \in \mathbb{C}^2$
- “Gaussian noise free” $C_{n>2} = 0$

Motivation of Bispectra

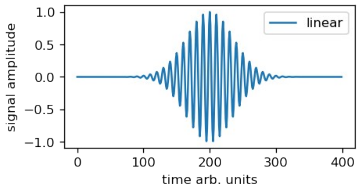
- Very important identity of Bispectrum

$$B_x(f_1, f_2) = \mathcal{F}_x(f_1)\,\mathcal{F}_x(f_2)\,\mathcal{F}_x^{-1}(f_1 + f_2),$$

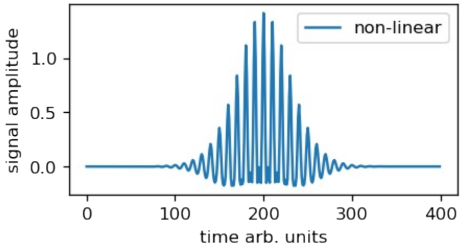
- Sensitive to non-linear processes $y = ax^2 + bx$
- Conserving the relative phase of signals $B_x(f_1, f_2) \in \mathbb{C}^2$
- "Gaussian noise free" $C_{n>2} = 0$

Example 1: Non Linear Process

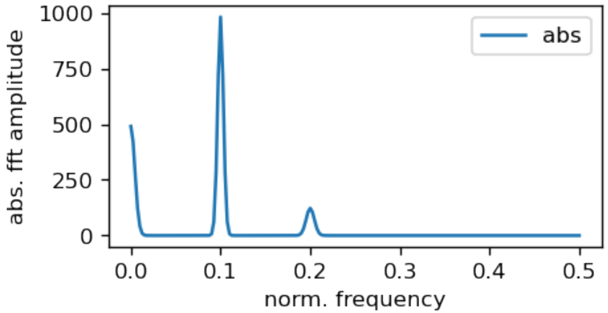
- Original signal



- Signal after non linear process



- Power spectrum of the signal

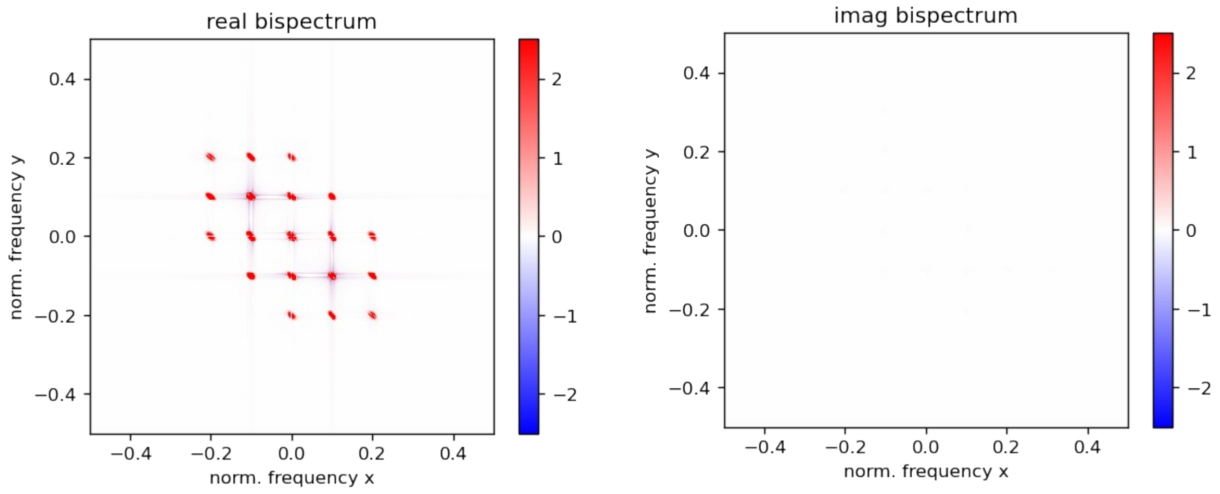


Example 1: Non Linear Process

- Quadratic Phase coupled (QPC) component

  → real peaks

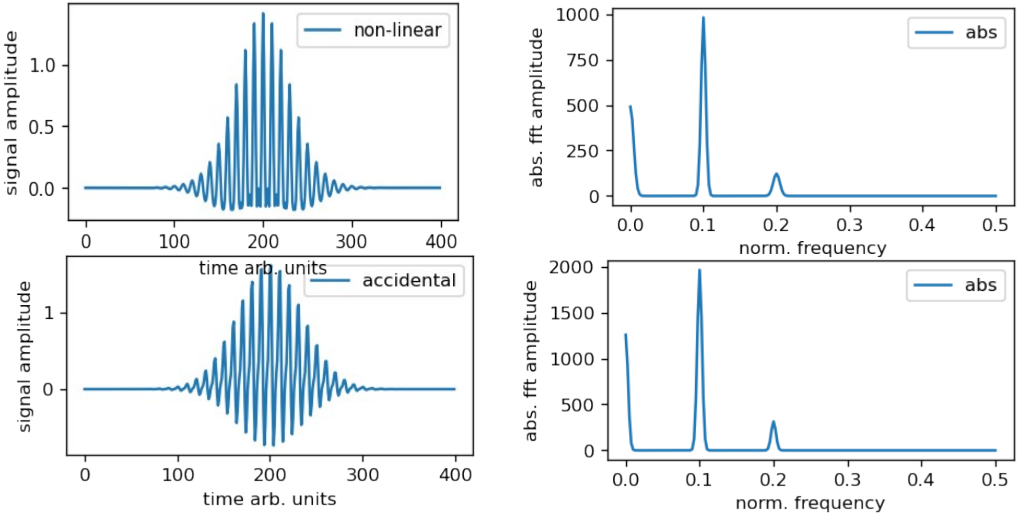
- Let’s compare to a signal not from a non linear process with the same PSD



QPC vs. accidental matched

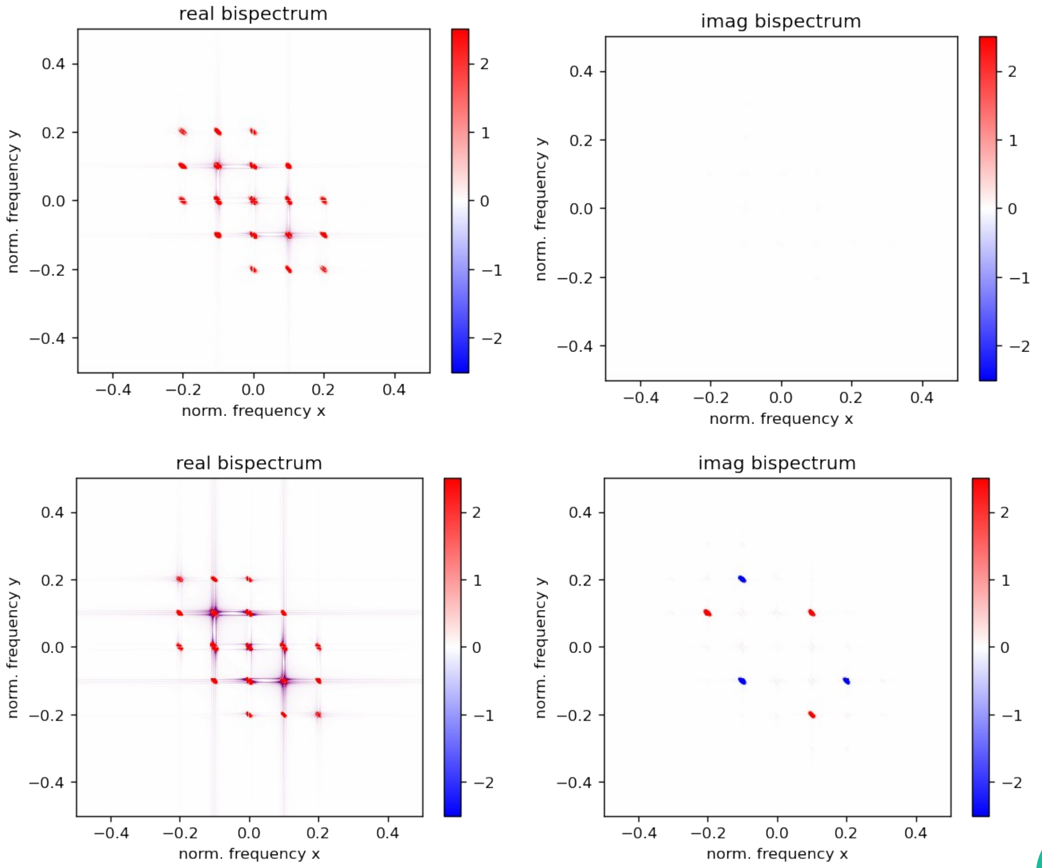
- Signal with same PSD, but 2f is phase shifted → accidentally matched

- Top: QPC
- Bottom: Accident



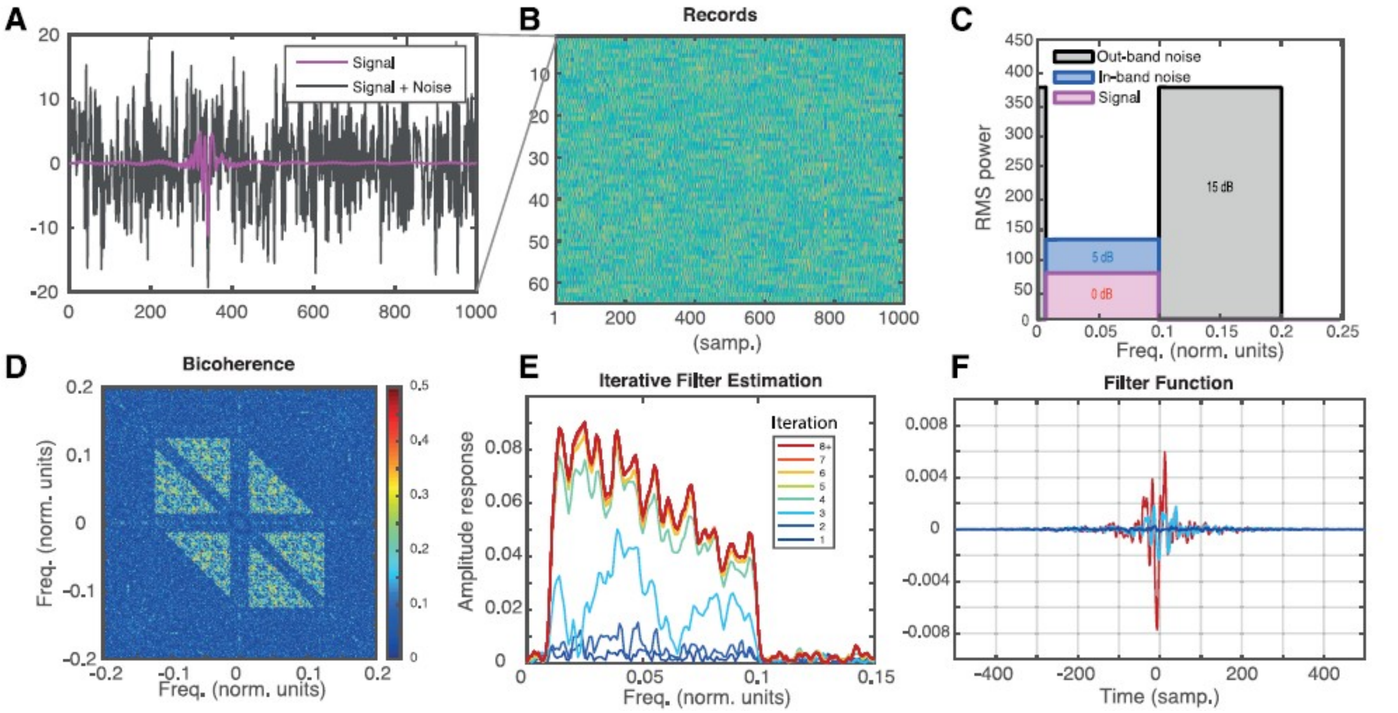
QPC vs. accidental matched - bispectrum

- Top: QPC
- Bottom: Accident

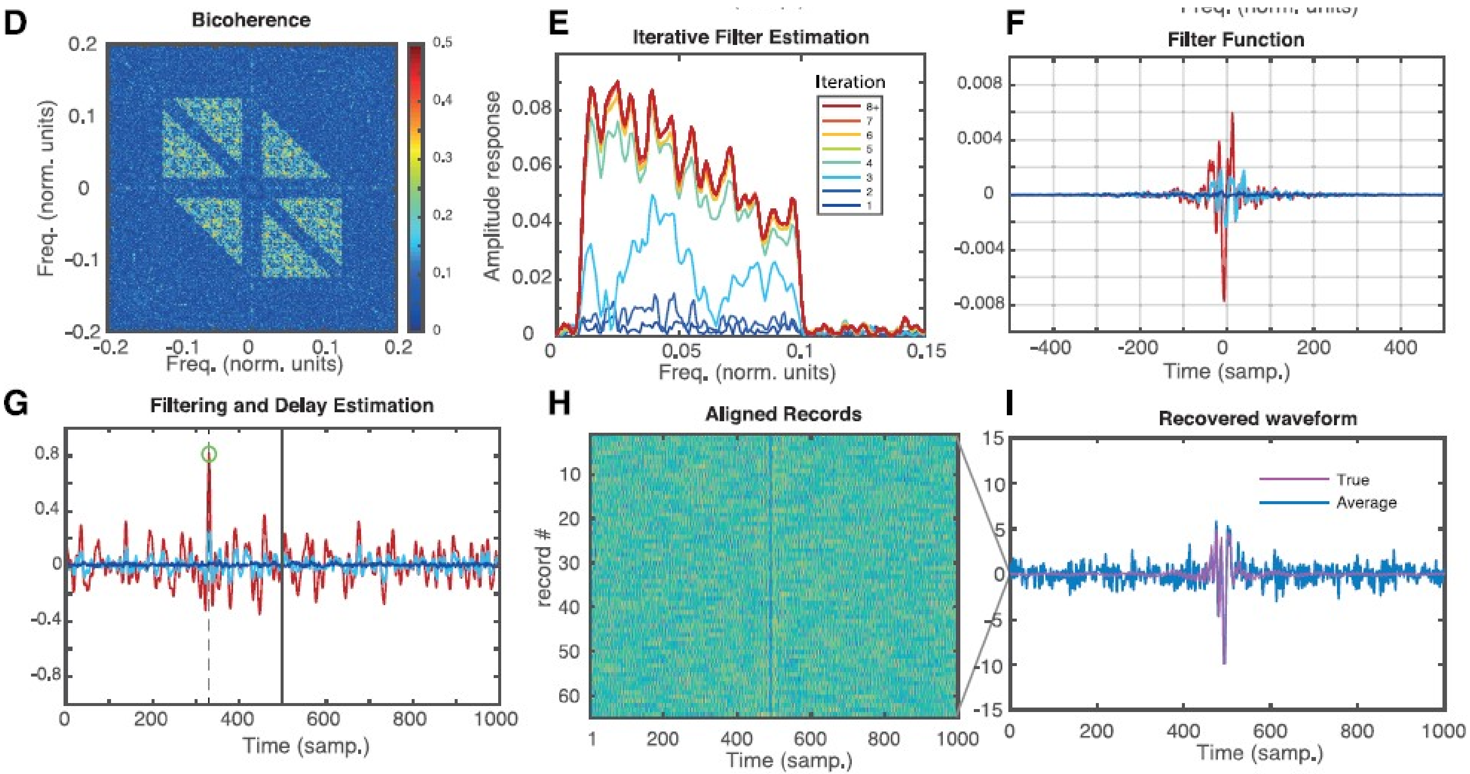


Example 2: Reconstructing a Filter for Arbitrary Signals

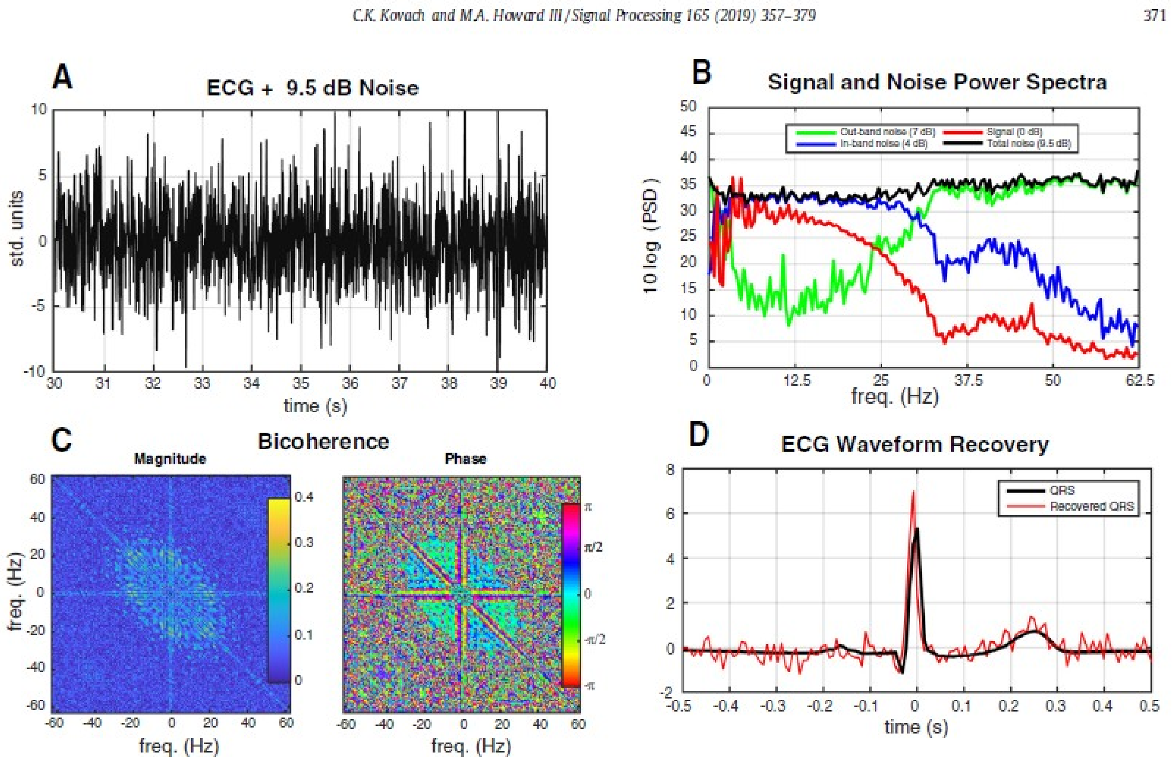
C.K. Kovach and M.A. Howard III / Signal Processing 165 (2019) 357–379 367



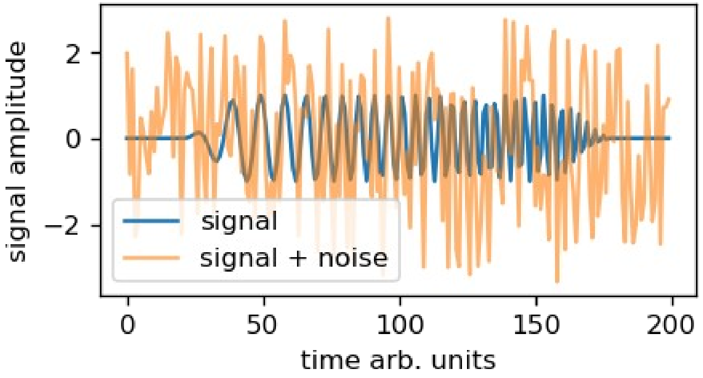
Example 2: Reconstructing a Filter for Arbitrary Signals

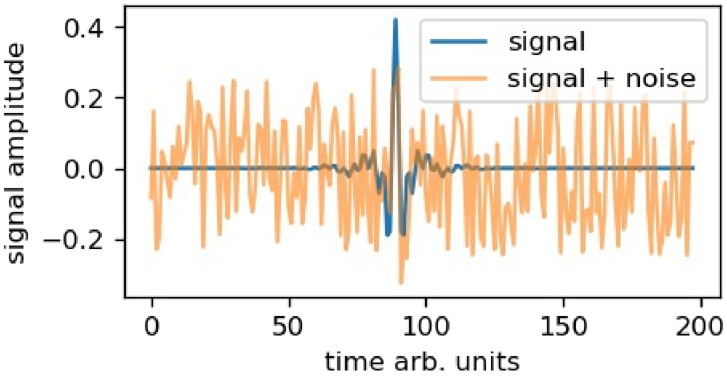


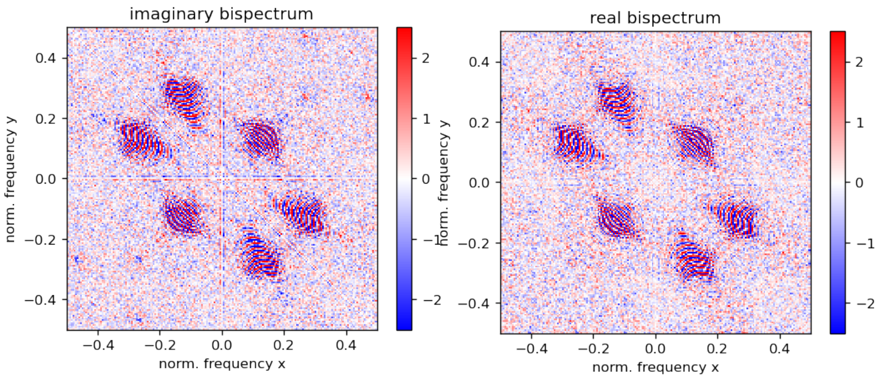
Example 2: Reconstructing a Filter for Bio Signals

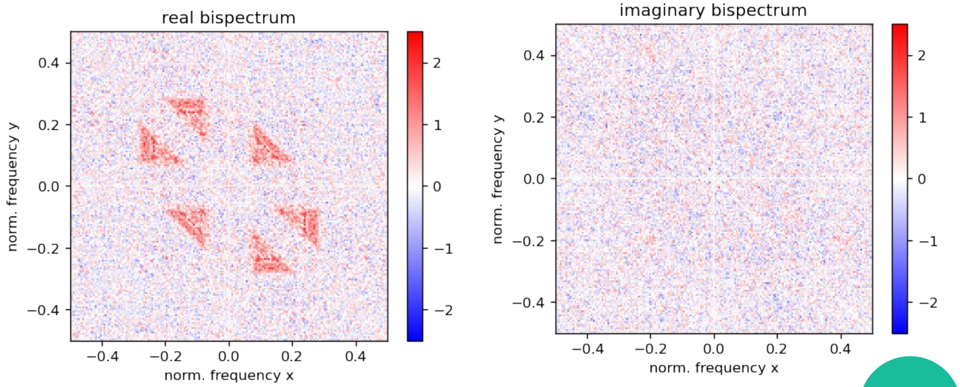


Other Examples:









Summary of Bispectral analysis

- Very important identity of Bispectrum

$$B_x(f_1, f_2) = \mathcal{F}_x(f_1)\, \mathcal{F}_x(f_2)\, \mathcal{F}_x^{-1}(f_1 + f_2),$$

- Sensitive to non-linear processes $y = ax^2 + bx$
- Conserving the relative phase of signals $B_x(f_1, f_2) \in \mathbb{C}^2$
- "Gaussian noise free" $C_{n>2} = 0$