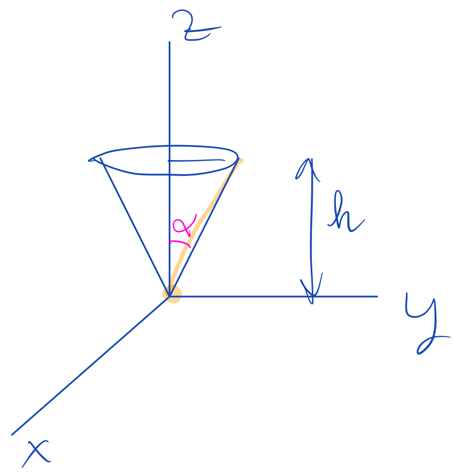
PROBLEMS LIST 3 - EXERCISES 3,7,8,21,26

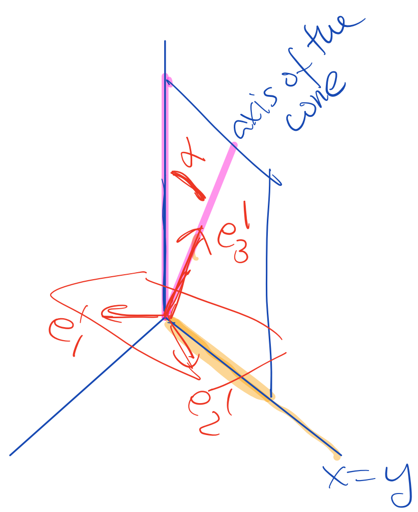
3. Obtain a parametrization of the circular cone of height h, semi-aperture α, and apex O, located in the first octant, tangent to the coordinate axis Oz along a generatrix, and such that the axis of the cone orthogonally projects onto the plane $z=0$ in the line $x=y$, $z=0$.



parametrization:

$$\begin{cases} x(t,s) = s\,\mathrm{tg}\alpha\cos t \\ y(t,s) = s\,\mathrm{tg}\alpha\sin t \\ z(t,s) = s \end{cases}, \quad t\in[0,2\pi],\ s\in[0,h]$$

New reference system:



e_3': $\mathrm{pr}_{z=0}(e_3') \approx (1,1,0)$

hence: $e_3' \approx (1,1,c)$

$\mathrm{angle}((0,0,1),(1,1,c)) = \alpha$

$$\Rightarrow \cos\alpha = \frac{c}{\sqrt{2+c^2}}$$

$$\Rightarrow (2+c^2)\cdot\cos^2\alpha = c^2$$

$$\Rightarrow c^2 = \frac{2\cos^2\alpha}{1-\cos^2\alpha} = 2\,\mathrm{ctg}^2\alpha$$

$$\Rightarrow \begin{cases} e_3' = (1,1,\sqrt{2}\,\mathrm{ctg}\alpha)/\sqrt{2+2\,\mathrm{ctg}^2\alpha} \\ e_1' = (-1,1,0)/\sqrt{2} \\ e_2' = e_3' \times e_1' \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \\ \hline 1 \end{pmatrix} = \left(\begin{array}{ccc|c} \uparrow & \uparrow & \uparrow & 0 \\ e_1' & e_2' & e_3' & 0 \\ \downarrow & \downarrow & \downarrow & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} x' \\ y' \\ z' \\ \hline 1 \end{pmatrix}$$

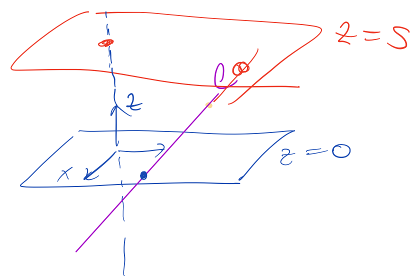
we know: cone in S' :
$$\begin{cases} x'(t,s) = s \operatorname{tg}\alpha \cos t \\ y'(t,s) = s \operatorname{tg}\alpha \sin t \\ z'(t,s) = s \end{cases}$$

we want: cone in S

$$\begin{pmatrix} x(t,s) \\ y(t,s) \\ z(t,s) \end{pmatrix} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ e_1' & e_2' & e_3' \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} s \operatorname{tg}\alpha \cos t \\ s \operatorname{tg}\alpha \sin t \\ s \end{pmatrix},$$

$s \in [0, h]$
$t \in [0, 2\pi]$

7. Obtain a parametrization of the surface that consists of all points belonging to any parallel line to the plane $z = 0$ which intersects axis Oz and the line ℓ defined by the parametric equation $(x, y, z) = (1, 1, 0) + \lambda(1, 1, 1)$, $\lambda \in \mathbb{R}$.



for $z=s$:

line through $\underbrace{(0,0,s)}_{P_1}$ and $\ell \cap \{z=s\}$

parametrization of ℓ:

$(x,y,z) = (1,1,0) + \lambda(1,1,1)$

$\ell \cap \{z=s\}$:

$z=s \Leftrightarrow \lambda = s$: $\underbrace{(1+s, 1+s, s)}_{P_2}$

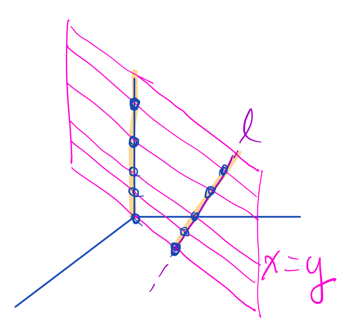
$$(x,y,z) = \underbrace{(0,0,s)}_{\text{point}} + t\underbrace{(1+s, 1+s, 0)}_{\text{direction vector} = \overrightarrow{P_1P_2} = P_2 - P_1}$$

$$\begin{cases} x(t,s) = t(1+s) \\ y(t,s) = t(1+s) \\ z(t,s) = s \end{cases} \qquad s,t \in \mathbb{R}$$

Which surface is it?

observe that $x=y$

in fact, it is the plane $x=y$:

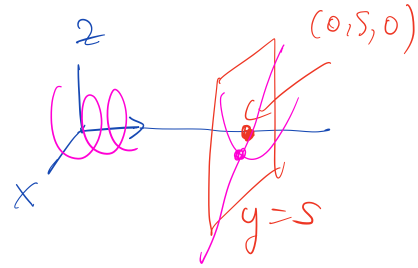


8. A *helicoid* is the trace of a moving line, touching a circular helix and its axis, and parallel to a plane perpendicular to the axis of the helix. Assume, for a start, that the axis of the helix is the coordinate axis Oy and that the helix is right-handed.

(a) Give a parametrization of the helicoid.

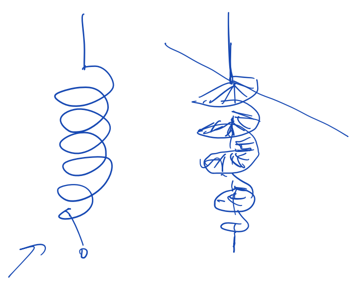
(b) Obtain the implicit equation of the helicoid.

(c) Find a parametrization of the helicoid corresponding to a right-handed helix with axis Ox, radius 3 and pitch 2.



(a) Helix: $\begin{cases} x(t) = R\sin t \\ y(t) = bt \\ z(t) = R\cos t \end{cases}$ $t \in \mathbb{R}$

$y = s$: line through $\begin{cases} (0,s,0) = P_1 \\ (R\sin t, \underbrace{bt}_{=s}, R\cos t) \Rightarrow \end{cases}$



$\Rightarrow \left(R\sin s/b,\ s,\ R\cos s/b\right) = P_2$

direction vector:

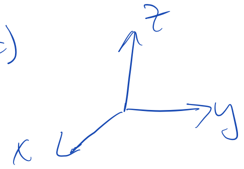
$\overrightarrow{P_1P_2} = \left(R\sin\frac{s}{b}, 0, R\cos\frac{s}{b}\right)$

$\Rightarrow \begin{cases} x(t,s) = 0 + t\,R\sin s/b \\ y(t,s) = s + t\cdot 0 \\ z(t,s) = 0 + t\cdot R\cos s/b \end{cases}$ $s,t \in \mathbb{R}$

(b) Implicit equation:

$s = y \Rightarrow \begin{cases} x = t\,R\sin y/b \\ z = t\,R\cos y/b \end{cases} \Rightarrow x\cdot\cos\frac{y}{b} = z\sin\frac{y}{b}$

(c)

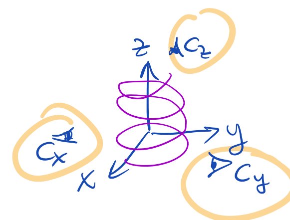


$\begin{cases} x \Leftrightarrow y \\ y \Leftrightarrow z \\ z \Leftrightarrow x \end{cases}$

(c) $\longleftrightarrow$ (a)

21. Can we express a circular helix as the intersection of two surfaces? Which?

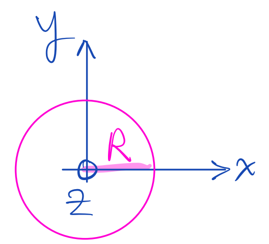
Helix with axis OZ: $\begin{cases} x(t) = R\cos t \\ y(t) = R\sin t \\ z(t) = bt \end{cases}$ $t \in \mathbb{R}$



C_z:

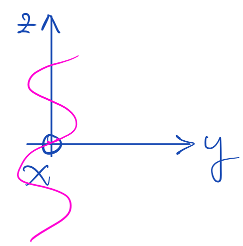
$$x^2 + y^2 = R^2$$

straight circular cylinder



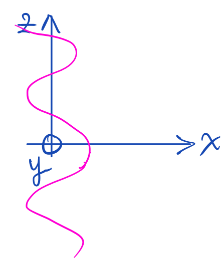
C_x:

$$y = R\sin t = R\sin\left(\frac{z}{b}\right) \quad :$$



C_y:

$$x = R\cos t = R\cos\left(\frac{z}{b}\right)$$



Then:

helix $\subseteq C_x, C_y, C_z \Rightarrow$ helix $\subseteq C_x \cap C_z$, helix $\subseteq C_y \cap C_z$, helix $\subseteq C_x \cap C_y$

But:

$C_x \cap C_z$ contains point not belonging to the helix:

if $y = R\sin(z/b)$, $x^2 + y^2 = z^2$, for $z = k$ we have 2 points $x = \pm\sqrt{z^2 - y^2}$, and only one belongs to the helix!

$C_x \cap C_y =$ helix:

if $y = R\sin(z/b)$ and $x = R\cos(z/b)$, then for $z = k$ we get only one point, $(R\cos\frac{k}{b}, R\sin\frac{k}{b}, k)$ and it belongs to the helix.

26. Constructing a torus from a circle. Parameterize the circle C intersection of the sphere $x^2+y^2+z^2=1$ and the plane $x+y=1$. Find the tangent vector to C at a generic point in terms of the parameter used in the previous parametrization. Parameterize now the circle C' of radius r centered at a generic point of C and lying on the plane perpendicular to C that goes through the center of C'.

- Parametrization of C: $\left.\begin{array}{l} x^2+y^2+z^2=1 \\ x+y=1 \end{array}\right\}$

Projecting cylinder C_y:

$y=1-x \Rightarrow x^2+(1-x)^2+z^2=1 \Rightarrow 2x^2-2x+z^2=0 \Rightarrow 2(x^2-x)+z^2=0 \Rightarrow 2(x^2-2\frac{1}{2}x+\frac{1}{4}-\frac{1}{4})+z^2=0 \Rightarrow$

$\Rightarrow 2\left(x-\frac{1}{2}\right)^2+z^2=\frac{1}{2} \Rightarrow \frac{(x-1/2)^2}{1/4}+\frac{z^2}{1/2}=1 \rightsquigarrow$ ellipse: $\begin{cases} x-1/2=\frac{1}{2}\cos t \\ z=\frac{1}{\sqrt{2}}\sin t \end{cases}, t\in[0,2\pi]$

$$\left.\begin{array}{l} \Rightarrow \begin{cases} x(t)=\frac{1}{2}\cos t+1/2 \\ z(t)=\frac{1}{\sqrt{2}}\sin t \end{cases} \\ \Rightarrow \quad y(t)=1-x(t)=-\frac{1}{2}\cos t+1/2 \end{array}\right\} t\in[0,2\pi] \rightsquigarrow \text{parametrization of } C$$

- tangent vector: $\gamma'(t)=\left(-\frac{1}{2}\sin t, +\frac{1}{2}\sin t, \frac{1}{\sqrt{2}}\cos t\right)$

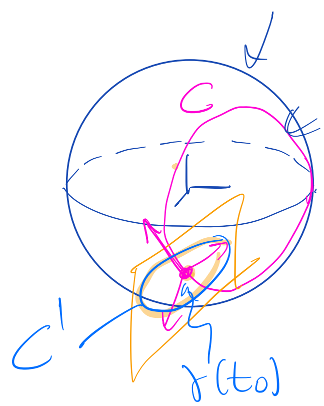
- New reference system: fix $t_0\in[0,2\pi]$ for a generic point $\gamma(t_0)$ of C

S': $O'=\gamma(t_0)$

$e_3'=\gamma'(t_0)=\left(-\frac{1}{2}\sin t_0, \frac{1}{2}\sin t_0, \frac{1}{\sqrt{2}}\cos t_0\right)/\|\ldots\|$

$e_1'\perp e_3'$: $e_1'=\left(\frac{1}{2}\sin t_0, \frac{1}{2}\sin t_0, 0\right)/\|\ldots\|$

$e_2'=e_3'\times e_1'=\ldots$

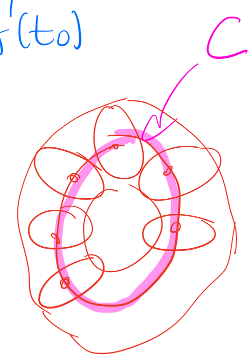


Param. of C' in S':

$$\begin{cases} x'(s)=r\cos s \\ y'(s)=r\sin s \\ z'(s)=0 \end{cases}, s\in[0,2\pi)$$

Param. of C' in S:

$$\begin{pmatrix} x(s) \\ y(s) \\ z(s) \\ \hline 1 \end{pmatrix}=\left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\quad}^{\text{in terms of } t_0}} & \\ \uparrow & \uparrow & \uparrow & \\ e_1' & e_2' & e_3' & \gamma(t_0) \\ \downarrow & \downarrow & \downarrow & \\ \hline 0 & 0 & 0 & 1 \end{array}\right)\begin{pmatrix} r\cos s \\ r\sin s \\ 0 \\ \hline 1 \end{pmatrix}, s\in[0,2\pi)$$



Parametrization of torus:

$$\begin{pmatrix} x(t,s) \\ y(t,s) \\ z(t,s) \\ \hline 1 \end{pmatrix}=\left(\begin{array}{ccc|c} \multicolumn{3}{c|}{\overbrace{\quad}^{\text{replace } t_0 \text{ with } t}} & \\ \uparrow & \uparrow & \uparrow & \\ e_1' & e_2' & e_3' & \gamma(t) \\ \downarrow & \downarrow & \downarrow & \\ \hline 0 & 0 & 0 & 1 \end{array}\right)\begin{pmatrix} r\cos s \\ r\sin s \\ 0 \\ \hline 1 \end{pmatrix}, s\in[0,2\pi), t\in[0,2\pi]$$

[Question: which is the projecting cylinder C_z of C?]

It is the plane $x+y=1$ and the circle can be parameterized as:

$$\boxed{\begin{cases} x = t \\ y = 1-t \\ z = \pm\sqrt{1-x^2-y^2} = \pm\sqrt{2t(1-t)} \end{cases} \quad , t \in [0,1]}$$

$x = t,\ y = 1-t \longrightarrow C_z$

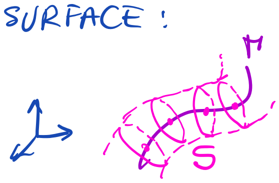
With this parametrization, the tangent vector does not exist for $(0,1,0), (1,0,0)$:

$$\begin{cases} x' = 1 \\ y' = -1 \\ z' = \pm \dfrac{2(1-2t)}{2\sqrt{2t(1-t)}} = \pm \dfrac{1-2t}{\sqrt{2t(1-t)}} \end{cases} \longrightarrow \nexists \text{ if } t \in \{0,1\}$$

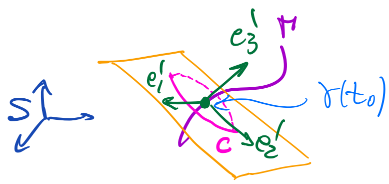
HINTS FOR LAB EXERCISE
CONSTRUCTING A "TUBULAR" SURFACE FROM A CURVE

3D curve, $\Gamma : \gamma(t) = (x(t), y(t), z(t))$, $t \in I$

TUBULAR SURFACE:



<u>IDEA</u>: Fix a point of Γ, $\gamma(t_0)$:



S': $o' = \gamma(t_0)$

$$\left\{ \begin{array}{l} e'_3 \approx \gamma'(t_0) \\ e'_1 \perp e'_3 \\ e'_2 = e'_3 \times e'_1 \end{array} \right\} \text{ for example, the Frenet trihedron is a possible basis.}$$

Parametrization of C in S':

$$\begin{cases} x'(s) = R\cos s \\ y'(s) = R\sin s \\ z'(s) = 0 \end{cases}, \quad s \in [0, 2\pi)$$

Parametrization of C in S:

$$\begin{pmatrix} x(s) \\ y(s) \\ z(s) \\ \hline 1 \end{pmatrix} = \left(\begin{array}{ccc|c} \uparrow & \uparrow & \uparrow & x(t_0) \\ e'_1 & e'_2 & e'_3 & y(t_0) \\ \downarrow & \downarrow & \downarrow & z(t_0) \\ \hline 0 & 0 & 0 & 1 \end{array} \right) \begin{pmatrix} R\cos \\ R\sin s \\ 0 \\ \hline 1 \end{pmatrix}, \quad s \in [0, 2\pi)$$

Parametrization of the tubular surface:

$$\begin{pmatrix} x(t,s) \\ y(t,s) \\ z(t,s) \\ \hline 1 \end{pmatrix} = \overset{\text{replace } t_0 \text{ with } t}{\left(\begin{array}{ccc|c} \uparrow & \uparrow & \uparrow & x(t) \\ e'_1 & e'_2 & e'_3 & y(t) \\ \downarrow & \downarrow & \downarrow & z(t) \\ \hline 0 & 0 & 0 & 1 \end{array} \right)} \begin{pmatrix} R\cos \\ R\sin s \\ 0 \\ \hline 1 \end{pmatrix}, \quad s \in [0, 2\pi), \ t \in I$$