The TOV Equation and the Mass of Stars

Jonas Pleyer

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- Needs additional information to be solvable
- Thermodynamics yields this information
- Questions to answer:
  - Well defined radius?
  - Mass limits?
- Mathematically interesting: show that a solution of the differential equation has a zerovalue without knowing the solution

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5. Outlook

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Concepts

TODO include nice picture

- General relativity (GR) models large scale of measurable universe

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$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} + \left(\frac{1}{2}R + \Lambda\right) g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Heavy objects create curvature in space

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- Energy-Momentum Tensor of perfect fluid

$$T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p) \tag{2}$$

- Solve Einstein Equations (without cosm. constant)

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3}$$

- Obtain 3 distinct differential equations ($R_{33} = R_{22}$)

$$-8\pi T_0^0 = 8\pi\rho = \frac{\lambda' e^{-\lambda}}{r} + \frac{1 - e^{-\lambda}}{r^2} \tag{4}$$

$$8\pi T_1^1 = 8\pi p = \nu' \frac{e^{-\lambda}}{r} - \frac{1 - e^{-\lambda}}{r^2} \tag{5}$$

$$8\pi T_2^2 = 8\pi p = \frac{e^{-\lambda}}{2}\left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r}\right)(\nu' - \lambda')\right] \tag{6}$$

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- Use equation 4 and identify Mass $m(r)$

$$e^{-\lambda} = 1 - \frac{2}{r} \int_0^r 4\pi\rho(r') r'^2 dr' =: 1 - \frac{2m(r)}{r} \tag{7}$$

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- Divergence of Energy-Momentum Tensor is $\nabla_\mu T^{\mu\nu} = 0$. Then obtain

$$\frac{\partial p}{\partial r} = -\frac{p + \rho}{2}\frac{\partial \nu}{\partial r} \tag{8}$$

- Use equations 5, 7 and 8 to get

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2 \tag{9}$$

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2}\left(1+\frac{p}{\rho c^2}\right)\left(\frac{4\pi r^3 p}{mc^2}+1\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1} \tag{10}$$

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- Needs equation of state (EOS) $f(\rho, p, r) = 0$ to be numerically solvable

Newtonian Limit

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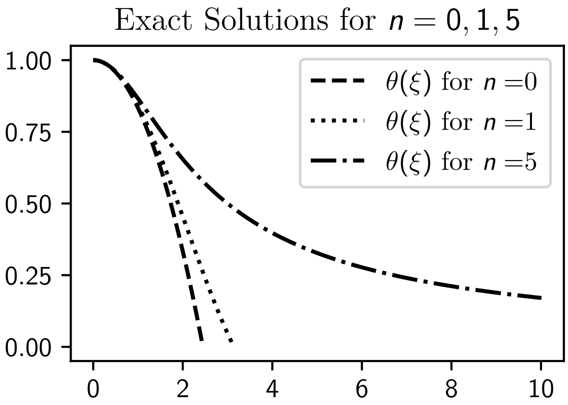
- Some exact solutions are known [Cha58]



$n = 0$	$1 - 1/6\xi^2$	$\xi_0 = \sqrt{6}$
$n = 1$	$\sin(\xi)/\xi$	$\xi_0 = \pi$
$n = 5$	$\left(1 + 1/3\xi^2\right)^{-1/2}$	$\xi_0 = \infty$

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Short Summary of main Principles

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- Partition function contains all information about N particles with position $x_i \in M$ and momentum $p_i \in T_{x_i}M$ with $V = \mathrm{vol}(M)$

$$\mathcal{Z}(T, V, N) = \int_{TM^N} \exp\left(-\frac{H(x_1, \ldots, p_N)}{k_B T}\right) \frac{dx_1 \ldots dp_N}{N! h^{3N}} \tag{14}$$

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- Calculate equation of state via internal Energy $\mathcal{U}$ and

$$p = k_B T \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial V} \qquad \rho = \frac{\mathcal{U}}{V} = \frac{k_B T^2}{V} \frac{\partial \mathcal{Z}}{\partial T} \tag{15}$$

- Partition Function for $H = \sqrt{m^2 + p^2}$

$$\mathcal{Z} = \frac{1}{N!}\left(8\pi V \left(\frac{k_B T}{hc}\right)^3 \frac{\alpha^2 K_2(\alpha)}{2}\right)^N \tag{16}$$

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$$p = \frac{N k_B T}{V} = CNmc^2 \frac{1}{K_2(\alpha)\alpha^2} \exp\left(-\alpha \frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)}\right) \tag{17}$$

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Theorem

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- Sketch of proof: take differential, use properties/representations of K_ν
- Use inverse and previous equations 15, 16 and 17 to obtain EOS

$$\rho = \frac{\mathcal{U}}{V} = p\left(1 + \alpha(p)\frac{K_1(\alpha(p)) + K_3(\alpha(p))}{2K_2(\alpha(p))}\right) \tag{18}$$

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3.2 Verifying Results
3.3 Zero Values of TOV and LE equation
3.4 TOV Hypothesis

4. Exact Results

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Comparison of TOV and LE results

TOV		LE	
EOS	$\rho = Ap^{1/\gamma}$	EOS	$p = K\rho^{\gamma}$
A	2		
$\gamma = 1 + \frac{1}{n}$	$4/3$	$n = 1/(\gamma - 1)$	3
p_0	0.5	θ_0	1
m_0	0	$d\theta_0$	0
dr	0.01	$d\xi = dr/\beta$	$\frac{0.01}{0.3355} \approx 0.0298$
$\rho_0 = Ap_0^{1/\gamma}$	$2(2)^{\frac{4}{3}} \approx 1.1892$	$\lambda = \rho_0$	$2(2)^{\frac{4}{3}} \approx 1.1892$
		$K = A^{-1/\gamma}$	$2^{-3/4} \approx 0.5946$
		$\beta^2 = \frac{(n+1)K\lambda^{1/n-1}}{4\pi}$	≈ 0.1125

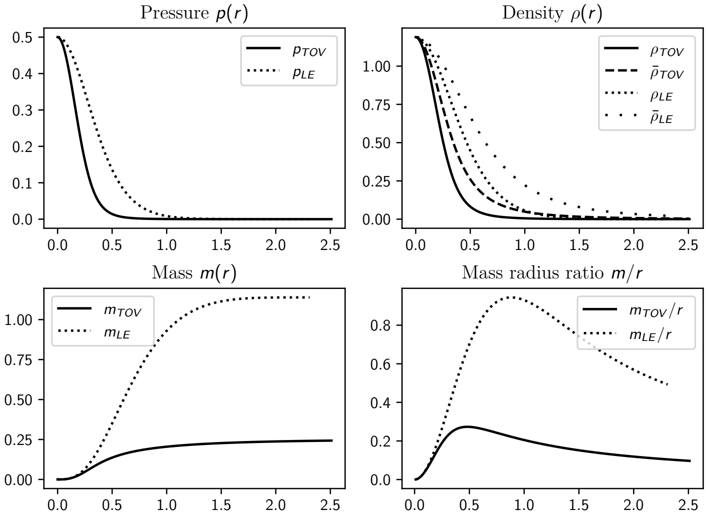

Figure: Comparison of TOV and LE results

Verifying Results

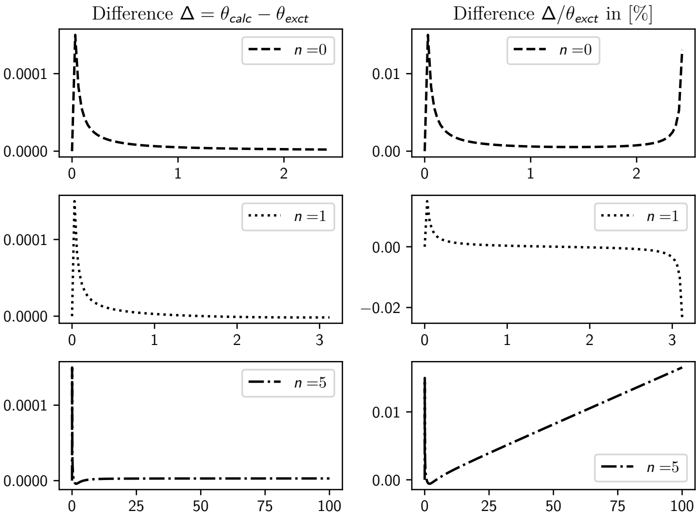

Figure: Validation of numerically calculated results

Zero Values of TOV and LE equation

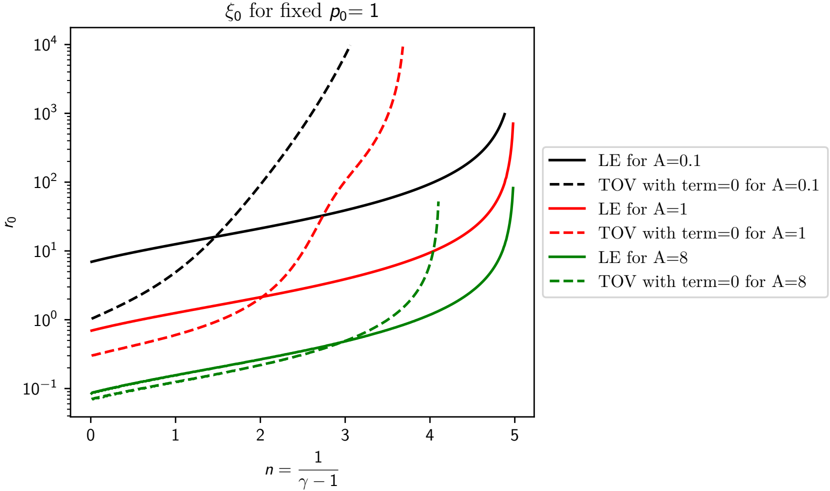

Figure: TOV and LE results for varying A parameter of $\rho = Ap^{n/(n+1)}$

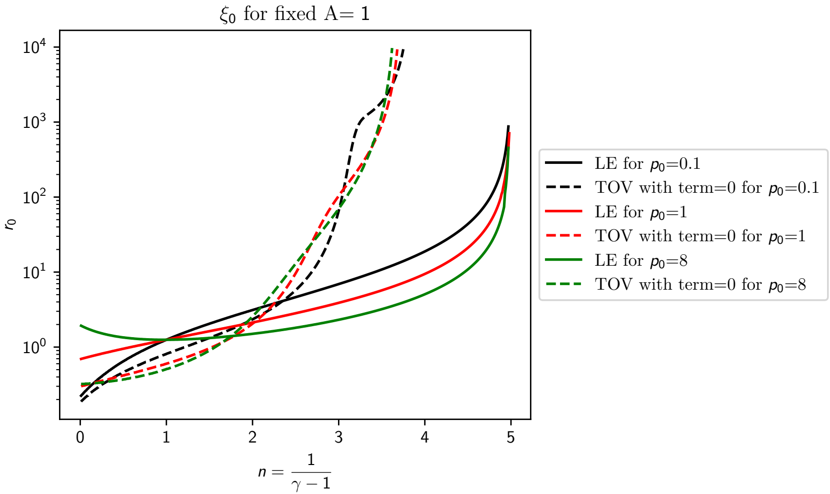

Figure: TOV and LE solutions for varying parameter p_0.

Intersection: $r = \beta\xi$ with $4\pi\beta A^{n/(n+1)} = (n+1)\lambda^{1-1/n}$ is independent of $\lambda = \rho(p_0)$.

TOV Hypothesis

Theorem

To each combination $p_0, A > 0$ there exists a $n_0 \geq 0$ such that each solution of the TOV differential equation

$$\begin{aligned} \frac{\partial m}{\partial r} &= \quad 4\pi \rho r^2 \\ \frac{\partial p}{\partial r} &= -\frac{m\rho}{r^2}\left(1+\frac{p}{\rho}\right)\left(\frac{4\pi r^3 p}{m}+1\right)\left(1-\frac{2m}{r}\right)^{-1} \end{aligned}$$

with $\rho = Ap^{\frac{n}{n+1}}$ where $n \leq n_0$ has $p(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}_{>0}$.

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4.1 New Exact Solution for n=2
4.2 Hypothesis
4.3 Limiting Case TOV

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New Exact Solution for n=2

- Found new solution for $n = 2$ by using simple power-series $\theta = \sum a_m \xi^m$

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \theta}{\partial \xi}\right) + \theta^2 = 0 \tag{19}$$

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- Apply Cauchy-Product formula and combine

$$(m+1)\sum_{m=0}^{\infty}\left((m+2)a_{m+2}\xi^m + 2a_{m+1}\xi^{m-1} + \sum_{k=0}^{m} a_{m-k}a_k\xi^m\right) = 0 \tag{20}$$

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Theorem

The odd coefficients a_{2m+1} vanish.

- Proof by induction. Start at $\partial\theta|_{\xi=0} = a_1 = 0$ is clear.

- Obtain recursive Formula for coefficients

$$a_{2m+2} = b_{m+1} = -\frac{1}{(m+2)(m+3)} \sum_{k=0}^{m} b_{m-k} b_k \tag{21}$$

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The series $\theta = \sum_{m=0}^{\infty} b_m \xi^{2m}$ *converges for* $\xi < 1$.

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Proof.

We know that $a_0 = 1$. Use L'Hospital's rule for $\xi \to 0$

$$0 = \frac{2}{\xi}\frac{\partial\theta}{\partial\xi} + \frac{\partial^2\theta}{\partial\xi^2} + \theta^2 \longrightarrow 3\left.\frac{\partial^2\theta}{\partial\xi^2}\right|_{\xi=0} + \theta|_{\xi=0} = 0 \tag{22}$$

Thus $b_1 = \theta_0/6$. By induction, we get $|b_{m+1}| \leq \frac{m\theta_0^2}{(m+2)(m+3)}$. Now use quotient criterion. □

Hypothesis

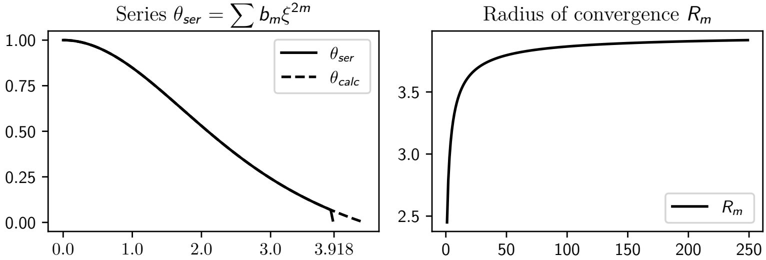

Figure: Exact solution θ and $(b_m)^{-1/(2m+2)}$ against m

Theorem

The radius of convergence R of the above calculated series is exactly the value at which $\theta(R) = 0$.

- Numerics: Calculation not optimized, reaches floating point limit

Limiting Case TOV I

- Suppose the TOV equation has a solution that continuously depends on its parameters for $r \in [0, l)$ where l may be ∞.

Limiting Case TOV II

Proof.

With $\partial m/\partial r = 4\pi A p^{1/\gamma} r^2$, define $v := m/A$

$$\frac{\partial p}{\partial r} = -\frac{p^{1/\gamma}}{r^2}\left(A + p^{1-1/\gamma}\right)\left(4\pi r^3 p + vA\right)\left(1 - \frac{2vA}{r}\right)^{-1} \qquad (24)$$

$\square$

Proof.

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- Solvability of TOV equation is complicated and subject of current research [Mar+19; BVW07]

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