The TOV Equation and the Mass of Stars

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Motivation

- TOV equation: differential equation, describes stars in GR
- Needs additional information to be solvable
- Thermodynamics yields this information
- Questions to answer:
  - Well defined radius?
  - Mass limits?
- Mathematically interesting: show that a solution of the differential equation has a zerovalue without knowing the solution

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Concepts

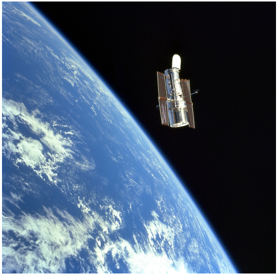


Figure: Third servicing mission of the Hubble Telescope 1999 [NAS99].

- General relativity (GR) models large scale structure of measurable universe
- Lorentzian Geometry (with indefinitive metric $g_{\mu\nu}$)
- Einstein equations relate curvature with mass and energy (geometrized units $G = c = 1$)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} + \left(\frac{1}{2}R + \Lambda\right) g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Heavy objects create curvature in space

Deriving the TOV equation

- TOV equation first derived independently by Tolman and Oppenheimer with Volkoff [Tol39; OV39].
- Spherically symmetric (Lorentz) metric

$$g = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 \left(d\vartheta^2 + \sin^2 \vartheta d\phi^2 \right) \tag{1}$$

- Energy-Momentum Tensor of perfect fluid

$$T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p) \tag{2}$$

- Solve Einstein Equations (without cosm. constant)

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3}$$

- Obtain 3 distinct differential equations ($R_{33} = R_{22}$)

$$-8\pi T_0^0 = 8\pi\rho = \frac{\lambda' e^{-\lambda}}{r} + \frac{1-e^{-\lambda}}{r^2} \quad (4)$$

$$8\pi T_1^1 = 8\pi p = \nu' \frac{e^{-\lambda}}{r} - \frac{1-e^{-\lambda}}{r^2} \quad (5)$$

$$8\pi T_2^2 = 8\pi p = \frac{e^{-\lambda}}{2}\left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r}\right)(\nu' - \lambda')\right] \quad (6)$$

- Use equation 4 and identify Mass $m(r)$

$$e^{-\lambda} = 1 - \frac{2}{r}\int_0^r 4\pi\rho(r')r'^2 dr' =: 1 - \frac{2m(r)}{r} \quad (7)$$

- Divergence of Energy-Momentum Tensor is $\nabla_\mu T^{\mu\nu} = 0$. Then obtain

$$\frac{\partial p}{\partial r} = -\frac{p+\rho}{2}\frac{\partial \nu}{\partial r} \quad (8)$$

- Use equations 5, 7 and 8 to get

$$\frac{\partial m}{\partial r} = 4\pi \rho r^2 \tag{9}$$

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2}\left(1+\frac{p}{\rho c^2}\right)\left(\frac{4\pi r^3 p}{mc^2}+1\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1} \tag{10}$$

- Plugged in gravitational constant, speed of light $G = c = 1$
- Equation 9 from Mass-Definition
- Ordinary differential equation
- Singular at $r = 0$
- Needs equation of state (EoS) $f(\rho, p, r) = 0$ to be solvable

Newtonian Limit

- Non-relativistic Limit of 2nd TOV equation 10 is

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2} \tag{11}$$

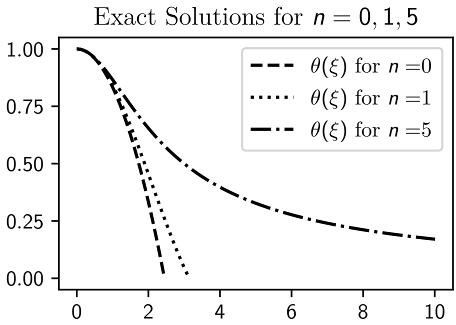
- Ansatz: Polytropic EoS $p = K\rho^{1+1/n}$
- Transformation $\rho = \lambda\theta^n$ and $\xi = r/\beta$ where

$$4\pi\beta^2 = (n+1)K\lambda^{1-1/n} \tag{12}$$

- Obtain Lane-Emden equation [Lan70; Emd07]

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\theta}{\partial\xi}\right) + \theta^n = 0 \tag{13}$$

- Some exact solutions are known [Cha58]



$n = 0$	$1 - \xi^2/6$	$\xi_0 =$	$\sqrt{6}$
$n = 1$	$\sin(\xi)/\xi$	$\xi_0 =$	π
$n = 5$	$\left(1 + \xi^2/3\right)^{-1/2}$	$\xi_0 =$	∞

$$\rho = \lambda\theta^n \text{ and } p = K\rho^{n+1/n}$$

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Calculating an EoS

- Statistical theory of manyparticle systems
- Describe macroscopic phenomena by microscopic principles
- Partition function contains all information about N particles with position $x_i \in M$ and momentum $p_i \in T_{x_i}M$ with $V = \mathrm{vol}(M)$

$$\mathcal{Z}(T, V, N) = \int\limits_{TM^N} \exp\left(-\frac{H(x_1, \ldots, p_N)}{k_B T}\right) \frac{dx_1 \ldots dp_N}{N! h^{3N}} \tag{14}$$

- Calculate equation of state via internal Energy $\mathcal{U}$ and

$$p = k_B T \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial V} \qquad \rho = \frac{\mathcal{U}}{V} = \frac{k_B T^2}{V} \frac{\partial \mathcal{Z}}{\partial T} \tag{15}$$

- Partition Function for $H = \sqrt{m^2 + p^2}$

$$\mathcal{Z} = \frac{1}{N!}\left(8\pi V\left(\frac{k_B T}{hc}\right)^3 \frac{\alpha^2 K_2(\alpha)}{2}\right)^N \tag{16}$$

- K_2 is modified Bessel function of 2nd kind
- Substitution $\alpha = mc^2/k_B T$
- With equations 15 and 16 obtain

$$p = \frac{N k_B T}{V} = CNmc^2 \frac{1}{K_2(\alpha)\alpha^2} \exp\left(-\alpha \frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)}\right) \tag{17}$$

Theorem

The function $p(\alpha)$ above is a bijection.

- Sketch of proof: take differential, use properties/representations of K_ν
- Use inverse and previous equations 15, 16 and 17 to obtain EOS

$$\rho = \frac{\mathcal{U}}{V} = p\left(1 + \alpha(p)\frac{K_1(\alpha(p)) + K_3(\alpha(p))}{2K_2(\alpha(p))}\right) \tag{18}$$

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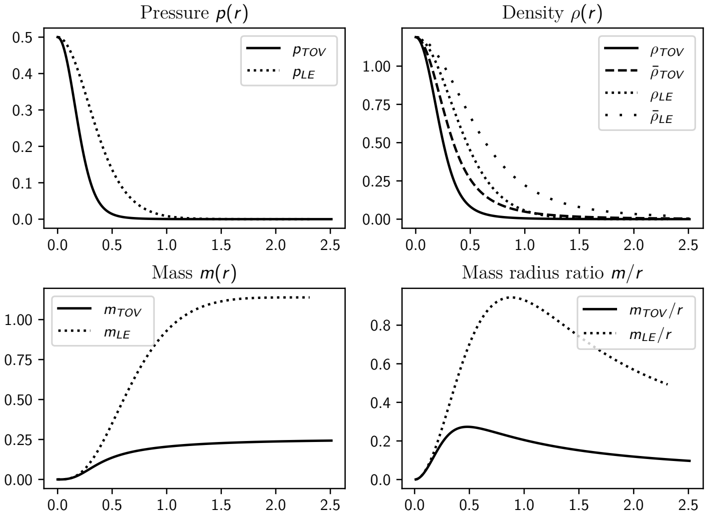

Figure: Comparison of TOV and LE results

Verifying LE Results

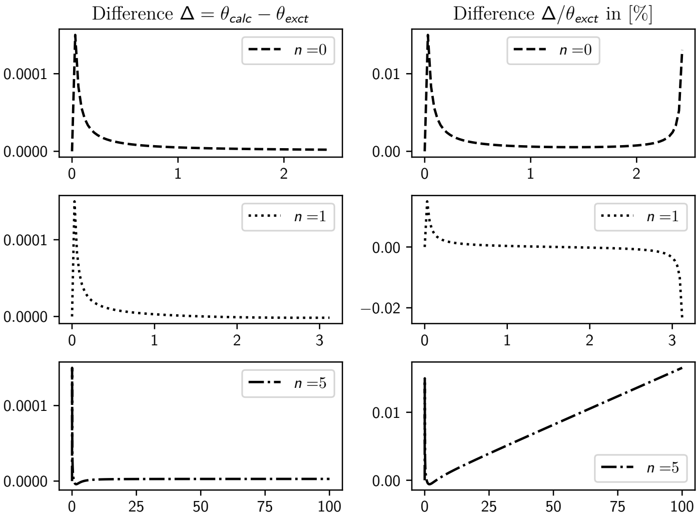

Figure: Validation of numerically calculated Lane-Emden results

Zero Values of TOV and LE equation

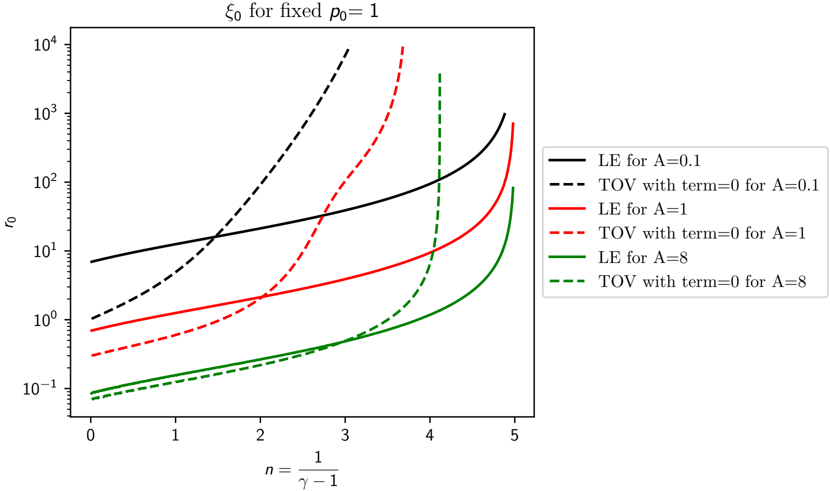

Figure: TOV and LE results for varying A parameter of $\rho = Ap^{n/(n+1)}$

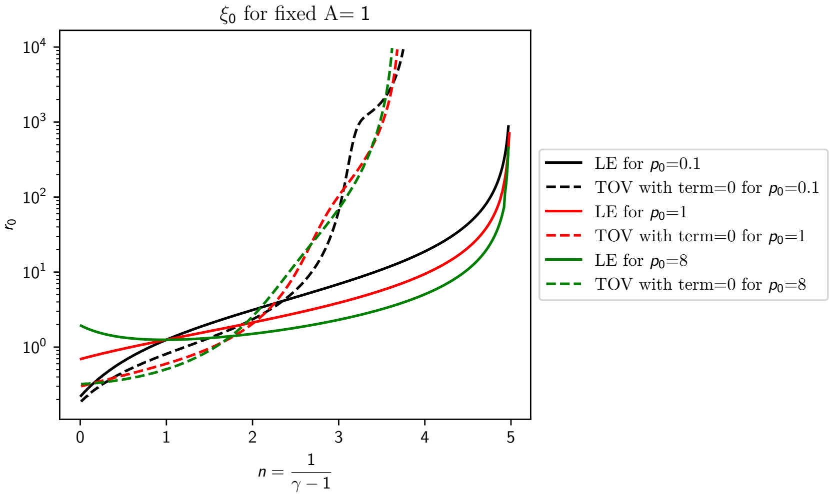

Figure: TOV and LE solutions for varying parameter p_0.

Intersection: $r = \beta\xi$ with $4\pi\beta A^{n/(n+1)} = (n+1)\lambda^{1-1/n}$ is independent of $\lambda = \rho(p_0)$.

TOV Hypothesis

Hypothesis

Given the TOV differential equation with $\rho = Ap^{\frac{n}{n+1}}$ *and* $p_0, A > 0$

$$\begin{aligned} \frac{\partial m}{\partial r} &= \quad 4\pi\rho r^2 \\ \frac{\partial p}{\partial r} &= -\frac{m\rho}{r^2}\left(1+\frac{p}{\rho}\right)\left(\frac{4\pi r^3 p}{m}+1\right)\left(1-\frac{2m}{r}\right)^{-1} \end{aligned}$$

There exists a $n_0 \geq 0$ *such that all solutions with same parameters* A, p_0 *and smaller exponent* $n < n_0$ *have a* $p(r_0)$ *for some* $r_0 > 0$.

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New Exact Solution for LE at n=2

- Found new solution for $n = 2$ by using simple power-series $\theta = \sum a_m \xi^m$

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \theta}{\partial \xi}\right) + \theta^2 = 0 \tag{19}$$

- Apply Cauchy-Product formula
- Obtain recursive Formula for coefficients

$$a_{2m+2} = b_{m+1} = -\frac{1}{(m+2)(m+3)} \sum_{k=0}^{m} b_{m-k} b_k \tag{20}$$

Theorem

The odd coefficients a_{2m+1} vanish.

Theorem

The series $\theta = \sum_{m=0}^{\infty} b_m \xi^{2m}$ converges for $\xi < 1$.

Hypotheses

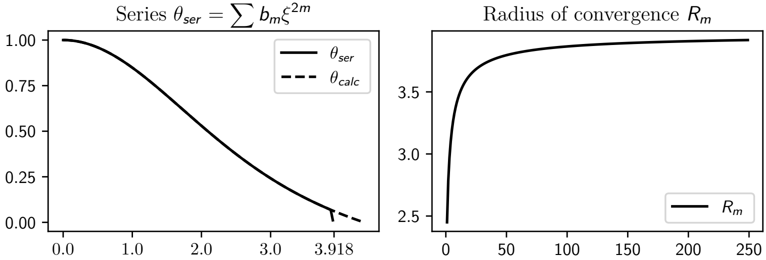

Figure: Exact solution θ and $R_m = (b_m)^{-1/(2m+2)}$ against m

Hypothesis

The radius of convergence $R > 1$ and $\exists \xi_0 \geq R$ such that $p(\xi_0) = 0$.

Hypothesis

Let $n \geq 0$ and $\theta_n = \sum a_m \xi^m$ be a LE solution. Then $a_{2m+1} = 0$ for $m \in \mathbb{N}$.

Limiting Case TOV

- Suppose the TOV equation has a solution that continuously depends on its parameters for $r \in [0, l)$ where l may be ∞.

Theorem

Let p_A be a solution of the TOV equation with $\rho = Ap^{1/\gamma}$. Then

$$\lim_{A \to 0} p_A = \frac{p_0}{2\pi p_0 r^2 + 1} \tag{21}$$

Proof.

With $\partial m/\partial r = 4\pi A p^{1/\gamma} r^2$, define $v := m/A$

$$\frac{\partial p}{\partial r} = -\frac{p^{1/\gamma}}{r^2}\left(A + p^{1-1/\gamma}\right)\left(4\pi r^3 p + vA\right)\left(1 - \frac{2vA}{r}\right)^{-1} \tag{22}$$

$\square$

Proof (Cont ...)

Then for $A = 0$, we have

$$\frac{\partial p}{\partial r} = -4\pi r p^2 \tag{23}$$

The solution to this differential equation is $p = \frac{p_0}{2\pi p_0 r^2 + 1}$.

- Problem: Assume that TOV equation is solvable
- Simple transformation tricks for 1D singular ODEs not working.

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Outlook

- Plenty of information on numerical side
- Should be able to obtain exact solution for $n \in \mathbb{N}, n > 1$ analogously to $n = 2$ cases
- Solvability of LE equation is known and well researched eg. [QS07]
- Solvability of TOV equation is complicated and subject of current research [Mar+19; BVW07]
- Mass limit of $M/R < 4/9$ already known for border of star. Maybe possible to prove inside as well.
- Improve mass limits

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