

# The Mass of Stars and the TOV Equation

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2. Thermodynamics - Calculating an EoS
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# Concepts

TODO include nice picture

- General relativity (GR) models large scale of measurable universe
- Lorentzian Geometry (with indefinite metric  $g_{\mu\nu}$ )
- Einstein equations relate curvature with mass and energy

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} + \left(\frac{1}{2}R + \Lambda\right) g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Heavy objects create curvature in space

# Deriving the TOV equation

- TOV equation was first derived independently by Tolman [Tol39] and Oppenheimer with Volkoff [OV39].
- Spherically symmetric (Lorentz) metric

$$g = -e^\nu dt^2 + e^\lambda dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

- Energy-Momentum Tensor of perfect fluid

$$T_{\mu\nu} = \text{diag}(-\rho, p, p, p) \quad (2)$$

- Solve Einstein Equations (without cosm. constant)

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \quad (3)$$

- Obtain 3 distinct differential equations ( $R_{33} = R_{22}$ )

$$-8\pi T_0^0 = 8\pi\rho = \frac{\lambda'e^{-\lambda}}{r} + \frac{1 - e^{-\lambda}}{r^2} \quad (4)$$

$$8\pi T_1^1 = 8\pi p = \nu' \frac{e^{-\lambda}}{r} - \frac{1 - e^{-\lambda}}{r^2} \quad (5)$$

$$8\pi T_2^2 = 8\pi p = \frac{e^{-\lambda}}{2} \left[ \nu'' + \left( \frac{\nu'}{2} + \frac{1}{r} \right) (\nu' - \lambda') \right] \quad (6)$$

- Use equation 4 and identify Mass  $m(r)$

$$e^{-\lambda} = 1 - \frac{2}{r} \int_0^r 4\pi\rho(r')r'^2 dr' =: 1 - \frac{2m(r)}{r} \quad (7)$$

- Divergence of Energy-Momentum Tensor is  $\nabla_\mu T^{\mu\nu} = 0$ . Then obtain

$$\frac{\partial p}{\partial r} = -\frac{p + \rho}{2} \frac{\partial \nu}{\partial r} \quad (8)$$

- Use equations 5, 7 and 8 to get

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2 \quad (9)$$

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2} \left(1 + \frac{p}{\rho c^2}\right) \left(\frac{4\pi r^3 \rho}{m} + 1\right) \left(1 - \frac{2Gm}{rc^2}\right)^{-1} \quad (10)$$

- Plugged in gravitational constant, speed of light  $G = c = 1$
- Equation 9 from Mass-Definition
- Ordinary differential equation
- Singular at  $r = 0$
- Needs equation of state (EOS)  $f(\rho, p, r) = 0$  to be solvable

# Newtonian Limit

- Non-relativistic Limit of 2nd TOV equation 10 is

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2} \quad (11)$$

- Polytropic EOS  $p = K\rho^{1+1/n}$
- Transformation  $\rho = \lambda\theta^n$  and  $\xi = r/\beta$  where

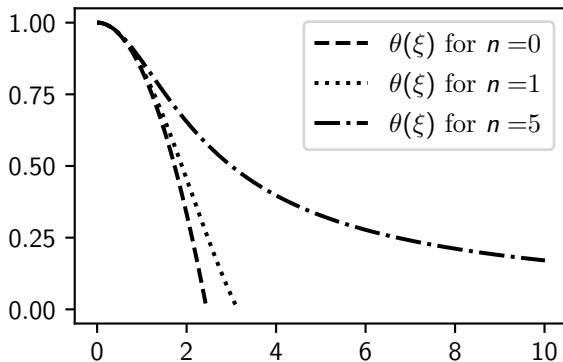
$$4\pi\beta^2 = (n+1)K\lambda^{1-1/n} \quad (12)$$

- Obtain Lane-Emden equation [Lan70] [Emd07]

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial \theta}{\partial \xi} \right) + \theta^n = 0 \quad (13)$$

- Some exact solutions are known [Cha58]



Exact Solutions for  $n = 0, 1, 5$ 

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$n = 0$	$1 - 1/6\xi^2$	$\xi_0 = \sqrt{6}$
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$n = 1$	$\sin(\xi)/\xi$	$\xi_0 = \pi$
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$n = 5$	$(1 + 1/3\xi^2)^{-1/2}$	$\xi_0 = \infty$
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# Short Summary of main Principles

- Statistical theory of manyparticle systems
- Describe macroscopic Phenomena by microscopic principles
- Partition function contains all information about  $N$  particles with position  $x_i \in M$  and momentum  $p_i \in T_{x_i}M$  with  $V = \text{vol}(M)$

$$\mathcal{Z}(T, V, N) = \int_{TM^N} \exp\left(-\frac{H(x_1, \dots, p_N)}{k_B T}\right) \frac{dx_1 \dots dp_N}{N! h^{3N}} \quad (14)$$

- Calculate equation of state via internal Energy  $\mathcal{U}$  and

$$p = k_B T \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial V} \quad \rho = \frac{\mathcal{U}}{V} = \frac{k_B T^2}{V} \frac{\partial \mathcal{Z}}{\partial T} \quad (15)$$

- Partition Function ( $K_2$  is modified Bessel function of 2nd kind)

$$\mathcal{Z} = \frac{1}{N!} \left( 8\pi V \left( \frac{k_B T}{hc} \right)^3 2\alpha K_2(\alpha) \right)^N \quad (16)$$

- With equations 15 and 16 obtain

$$p = \frac{Nk_B T}{V} = CNmc^2 \frac{1}{K_2(\alpha)\alpha^2} \exp \left( -\alpha \frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)} \right) \quad (17)$$

### Theorem

*The function  $p(\alpha)$  above is a bijection.*

- Sketch of proof: take differential, use properties/representations of  $K_\nu$
- Use inverse and previous equations 15, 16 and 17 to obtain EOS

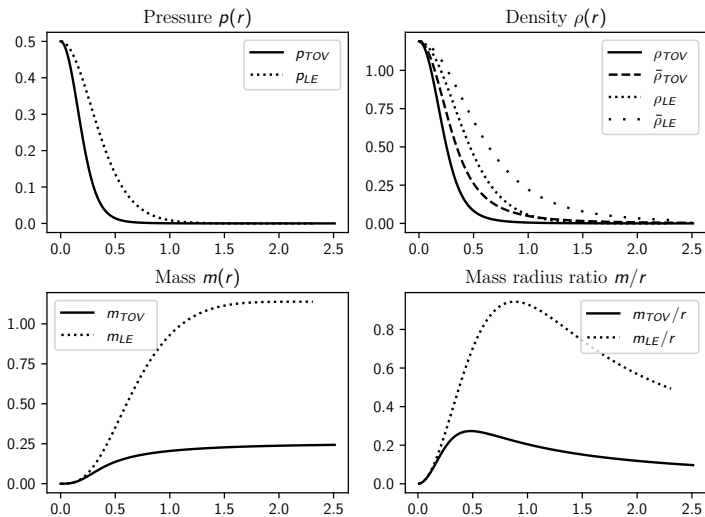
$$\rho = \frac{\mathcal{U}}{V} = p \left( 1 + \alpha(p) \frac{K_1(\alpha(p)) + K_3(\alpha(p))}{2K_2(\alpha(p))} \right) \quad (18)$$

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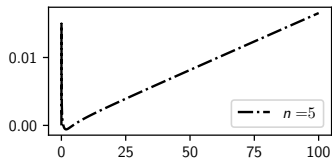
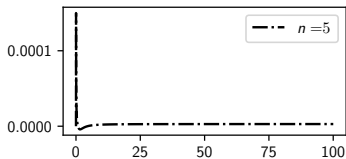
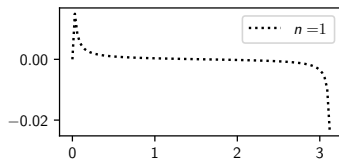
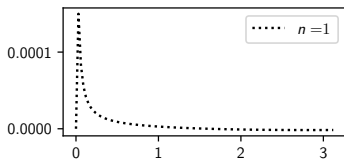
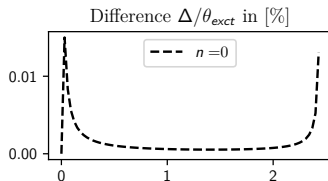
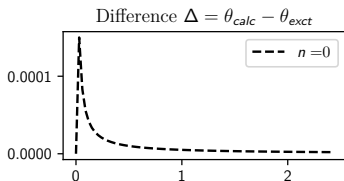
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# Comparison of TOV and LE results

TOV		LE	
EOS	$\rho = A p^{1/\gamma}$	EOS	$p = K \rho^\gamma$
$A$	2		
$\gamma = 1 + \frac{1}{n}$	4/3	$n = 1/(\gamma - 1)$	3
$p_0$	0.5	$\theta_0$	1
$m_0$	0	$d\theta_0$	0
$dr$	0.01	$d\xi = dr/\alpha$	$\frac{0.01}{0.3355} \approx 0.0298$
$\rho_0 = A p_0^{1/\gamma}$	$2(2)^{\frac{4}{3}} \approx 1.1892$	$\lambda = \rho_0$	$2(2)^{\frac{4}{3}} \approx 1.1892$
		$K = A^{-1/\gamma}$	$2^{-3/4} \approx 0.5946$
		$\alpha^2 = \frac{(n+1)K\lambda^{1/n-1}}{4\pi}$	$\approx 0.1125$



# Verifying Results



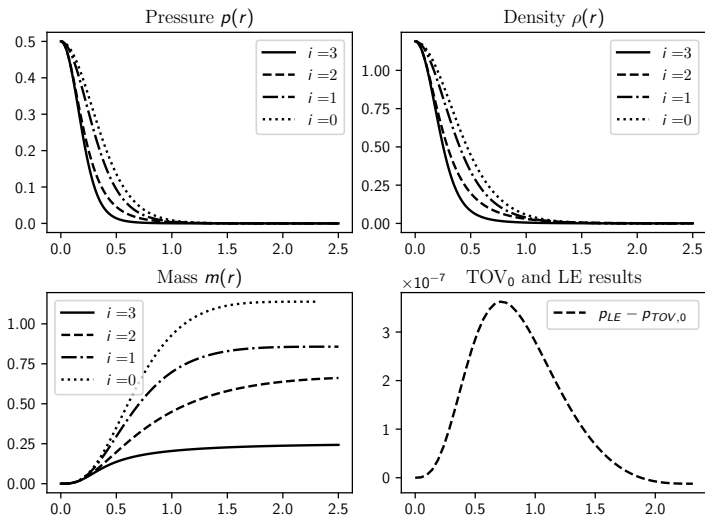


# Comparing Partial TOV and LE results

- Next, reduce terms present in TOV equation

$$\frac{\partial p}{\partial r} = - \underbrace{\frac{Gm\rho}{r^2}}_{i=0} \underbrace{\left(1 + \frac{p}{\rho c^2}\right)}_{i=1} \underbrace{\left(\frac{4\pi r^3 p}{mc^2} + 1\right)}_{i=2} \underbrace{\left(1 - \frac{2Gm}{rc^2}\right)^{-1}}_{i=3} \quad (19)$$

- We expect  $i = 0$  to be the LE equation as derived earlier



# Zero Values of TOV and LE equation

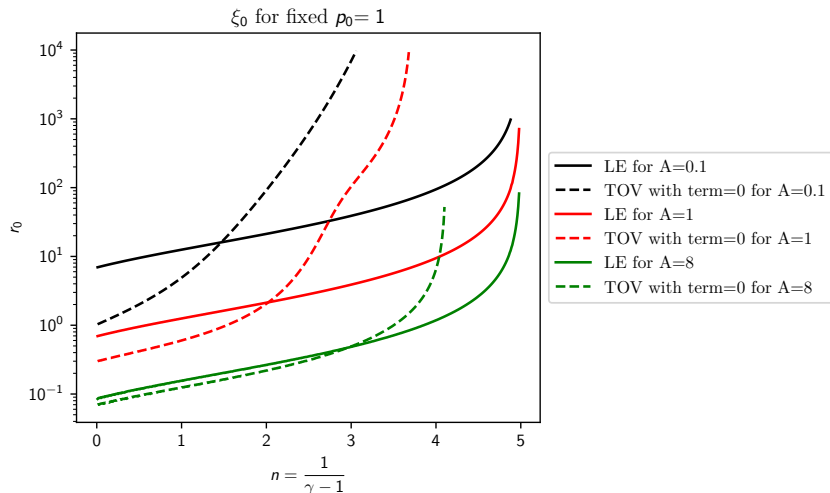


Figure: TOV and LE results for varying  $A$  parameter of  $\rho = A p^{n/(n+1)}$

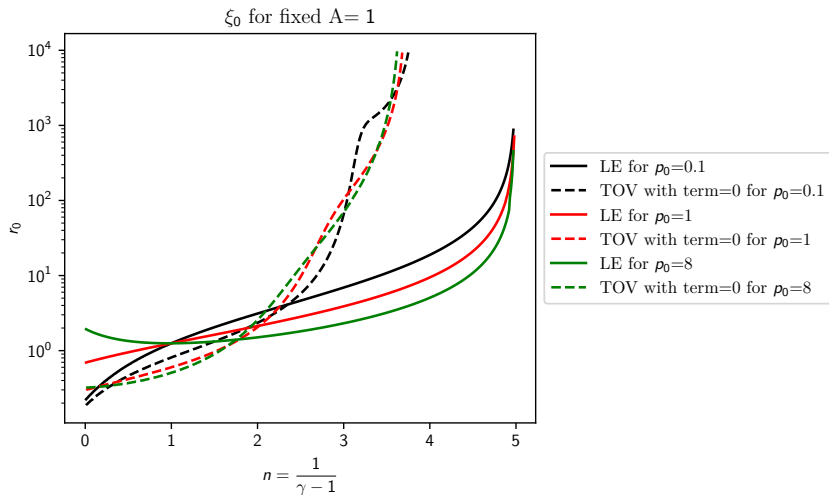


Figure: TOV and LE solutions for varying parameter  $\rho_0$ .

Intersection:  $r = \beta\xi$  with  $4\pi\beta A^{n/(n+1)} = (n+1)\lambda^{1-1/n}$  is independent of  $\lambda = \rho(p_0)$ .

# TOV Hypothesis

## Theorem

*To each combination  $p_0, A > 0$  there exists a  $n_0 \geq 0$  such that each solution of the TOV differential equation*

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2$$

$$\frac{\partial p}{\partial r} = -\frac{m\rho}{r^2} \left(1 + \frac{p}{\rho}\right) \left(\frac{4\pi r^3 p}{m} + 1\right) \left(1 - \frac{2m}{r}\right)^{-1}$$

*with  $\rho = Ap^{\frac{n}{n+1}}$  where  $n \leq n_0$  has  $p(\xi_0) = 0$  for some  $\xi_0 \in \mathbb{R}_{>0}$ .*

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# New Exact Solution for $n=2$

- Found new solution for  $n = 2$  by using simple power-series  $\theta = \sum a_m \xi^m$

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial \theta}{\partial \xi} \right) + \theta^2 = 0 \quad (20)$$

- Apply Cauchy-Product formula and combine

$$(m+1) \sum_{m=0}^{\infty} \left( (m+2)a_{m+2}\xi^m + 2a_{m+1}\xi^{m-1} + \sum_{k=0}^m a_{m-k}a_k\xi^m \right) = 0 \quad (21)$$

## Theorem

*The odd coefficients  $a_{2m+1}$  vanish.*

- Proof by induction. Start at  $\partial\theta|_{\xi=0} = a_1 = 0$  is clear.

- Obtain recursive Formula for coefficients

$$a_{2m+2} = b_{m+1} = -\frac{1}{(m+2)(m+3)} \sum_{k=0}^m b_{m-k} b_k \quad (22)$$

### Theorem

The series  $\theta = \sum_{m=0}^{\infty} b_m \xi^{2m}$  converges for  $\xi < 1$ .

### Proof.

We know that  $a_0 = 1$ . Use L'Hospital's rule for  $\xi \rightarrow 0$

$$0 = \frac{2}{\xi} \frac{\partial \theta}{\partial \xi} + \frac{\partial^2 \theta}{\partial \xi^2} + \theta^2 \longrightarrow 3 \left. \frac{\partial^2 \theta}{\partial \xi^2} \right|_{\xi=0} + \theta|_{\xi=0} = 0 \quad (23)$$

Thus  $b_1 = \theta_0/6$ . By induction, we get  $|b_{m+1}| \leq \frac{m\theta_0^2}{(m+2)(m+3)}$ . Now use quotient criterion. □



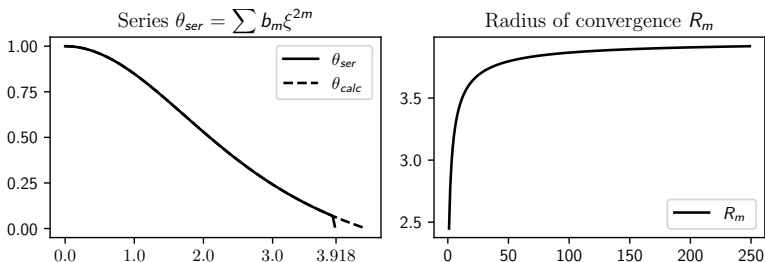


Figure: Exact solution  $\theta$  and  $(b_m)^{-1/(2m+2)}$  against  $m$

## Theorem

*The radius of convergence  $R$  of the above calculated series is exactly the value at which  $\theta(R) = 0$ .*

- Numerical problems: Calculation not very optimized, reaches floating point limit

# Limiting Case TOV I

- Suppose the TOV equation has a solution that continuously depends on its parameters for  $r \in [0, l)$  where  $l$  may be  $\infty$ .

## Limiting Case TOV II

Proof.

With  $\partial m / \partial r = 4\pi A p^{1/\gamma} r^2$ , define  $v := m/A$

$$\frac{\partial p}{\partial r} = -\frac{p^{1/\gamma}}{r^2} \left( A + p^{1-1/\gamma} \right) \left( 4\pi r^3 p + vA \right) \left( 1 - \frac{2vA}{r} \right)^{-1} \quad (25)$$

Then for  $A = 0$ , we have

$$\frac{\partial p}{\partial r} = -4\pi r p^2 \quad (26)$$

The solution to this differential equation is  $p = \frac{p_0}{2\pi p_0 r^2 + 1}$ . □

- Problem: Assumption needs that TOV equation is solvable
- No experience with 2D singular ODEs.
- Easy transformation tricks for 1D singular ODEs not working.

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# Outlook

- Plenty of information on numerical side
- Should be able to obtain exact solution for  $n \in \mathbb{N}, n > 1$  analogously to  $n = 2$  cases
- Solvability of LE equation is known and well researched eg. [QS07]
- Solvability of TOV equation is subject of current research [Mar+19; BVW07]

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