The TOV Equation and the Mass of Stars

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March 3, 2021

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- Questions to answer:
  - Well defined radius?
  - Mass limits?
- Mathematically interesting: show that a solution of the differential equation has a zerovalue without knowing the solution

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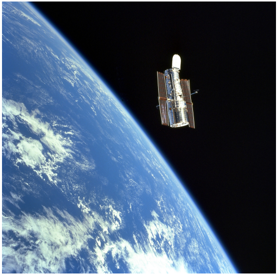
Concepts



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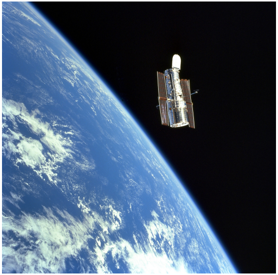

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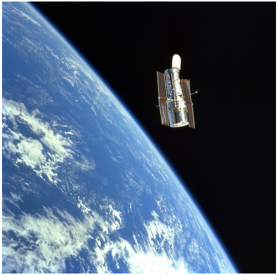

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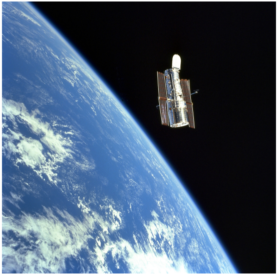

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- Heavy objects create curvature in space

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- Solve Einstein Equations (without cosm. constant)

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3}$$

- Obtain 3 distinct differential equations ($R_{33} = R_{22}$)

$$-8\pi T_0^0 = 8\pi\rho = \frac{\lambda' e^{-\lambda}}{r} + \frac{1 - e^{-\lambda}}{r^2} \tag{4}$$

$$8\pi T_1^1 = 8\pi p = \nu' \frac{e^{-\lambda}}{r} - \frac{1 - e^{-\lambda}}{r^2} \tag{5}$$

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- Divergence of Energy-Momentum Tensor is $\nabla_\mu T^{\mu\nu} = 0$. Then obtain

$$\frac{\partial p}{\partial r} = -\frac{p + \rho}{2}\frac{\partial \nu}{\partial r} \tag{8}$$

- Use equations 5, 7 and 8 to get

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2 \tag{9}$$

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2}\left(1+\frac{p}{\rho c^2}\right)\left(\frac{4\pi r^3 p}{mc^2}+1\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1} \tag{10}$$

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- Needs equation of state (EoS) $f(\rho, p, r) = 0$ to be solvable

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$$4\pi\beta^2 = (n+1)K\lambda^{1-1/n} \tag{12}$$

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- Obtain Lane-Emden equation [Lan70; Emd07]

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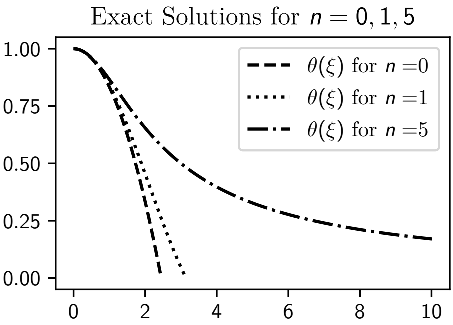
- Some exact solutions are known [Cha58]



$n = 0$	$1 - \xi^2/6$	$\xi_0 =$	$\sqrt{6}$
$n = 1$	$\sin(\xi)/\xi$	$\xi_0 =$	π
$n = 5$	$\left(1 + \xi^2/3\right)^{-1/2}$	$\xi_0 =$	∞

$$\rho = \lambda\theta^n \text{ and } p = K\rho^{n+1/n}$$

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$$\mathcal{Z}(T, V, N) = \int\limits_{TM^N} \exp\left(-\frac{H(x_1, \ldots, p_N)}{k_B T}\right) \frac{dx_1 \ldots dp_N}{N! h^{3N}} \tag{14}$$

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- Calculate equation of state via internal Energy $\mathcal{U}$ and

$$p = k_B T \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial V} \qquad \rho = \frac{\mathcal{U}}{V} = \frac{k_B T^2}{V} \frac{\partial \mathcal{Z}}{\partial T} \tag{15}$$

- Partition Function for $H = \sqrt{m^2 + p^2}$

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Theorem

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- Sketch of proof: take differential, use properties/representations of K_ν
- Use inverse and previous equations 15, 16 and 17 to obtain EOS

$$\rho = \frac{\mathcal{U}}{V} = p\left(1 + \alpha(p)\frac{K_1(\alpha(p)) + K_3(\alpha(p))}{2K_2(\alpha(p))}\right) \tag{18}$$

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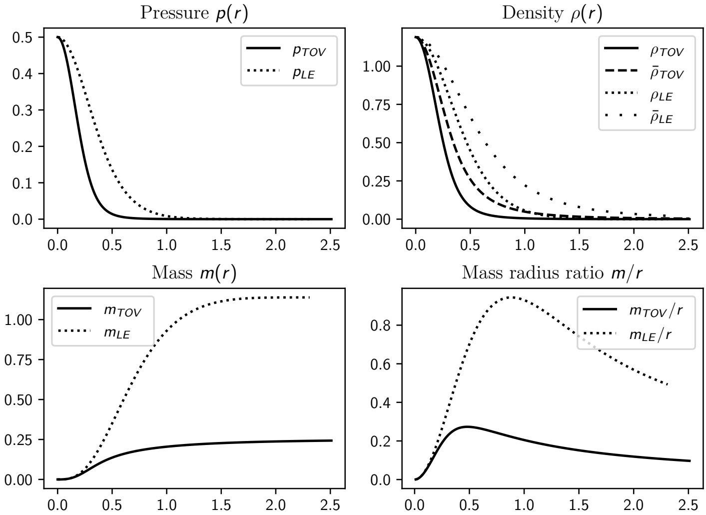

Figure: Comparison of TOV and LE results

Verifying LE Results

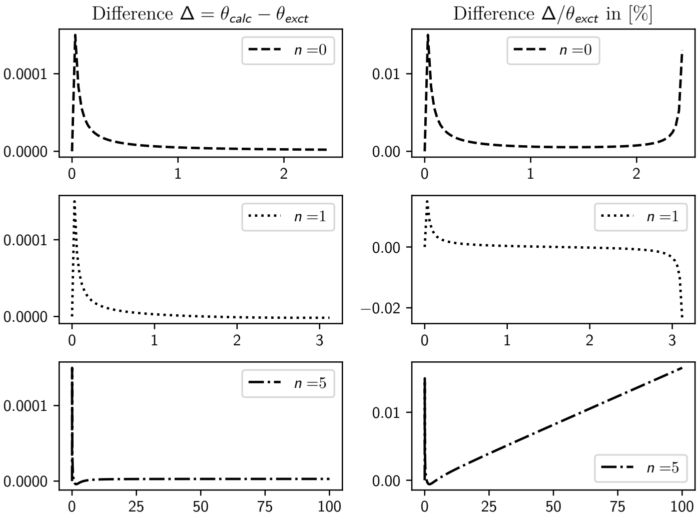

Figure: Validation of numerically calculated Lane-Emden results

Zero Values of TOV and LE equation

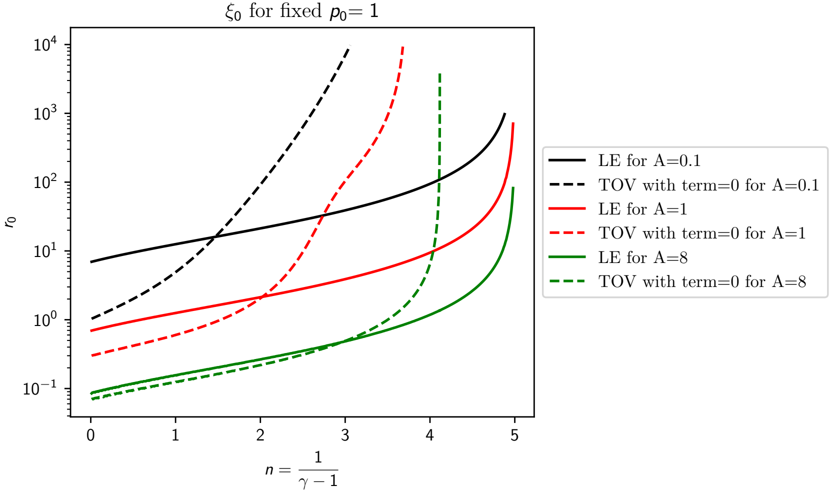

Figure: TOV and LE results for varying A parameter of $\rho = Ap^{n/(n+1)}$

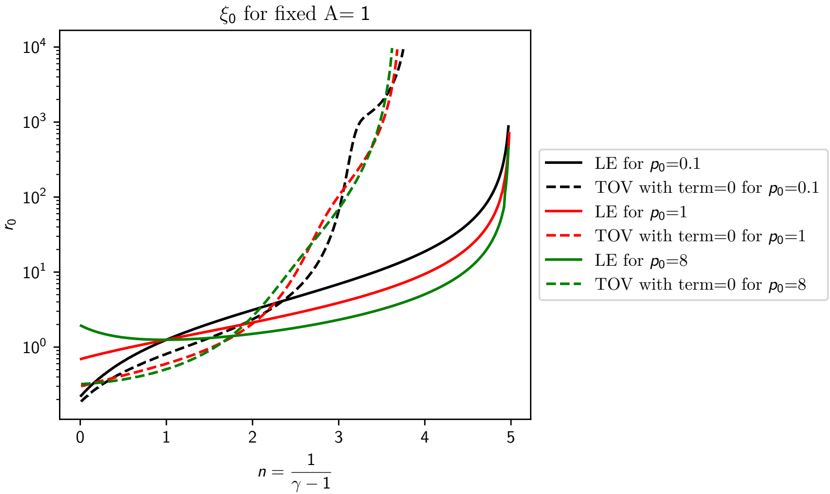

Figure: TOV and LE solutions for varying parameter p_0.

Intersection: $r = \beta\xi$ with $4\pi\beta A^{n/(n+1)} = (n+1)\lambda^{1-1/n}$ is independent of $\lambda = \rho(p_0)$.

TOV Hypothesis

Hypothesis

Given the TOV differential equation with $\rho = Ap^{\frac{n}{n+1}}$ *and* $p_0, A > 0$

$$\begin{aligned}\frac{\partial m}{\partial r} &= \quad 4\pi\rho r^2 \\ \frac{\partial p}{\partial r} &= -\frac{m\rho}{r^2}\left(1+\frac{p}{\rho}\right)\left(\frac{4\pi r^3 p}{m}+1\right)\left(1-\frac{2m}{r}\right)^{-1}\end{aligned}$$

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(i) *There exists a solution for some* $n_0 \geq 0$

(ii) *All solutions with same parameters* A, p_0 *and smaller exponent* $n < n_0$ *have a* $p(r_0)$ *for some* $r_0 > 0$.

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New Exact Solution for LE at n=2

- Found new solution for $n = 2$ by using simple power-series $\theta = \sum a_m \xi^m$

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \theta}{\partial \xi}\right) + \theta^2 = 0 \tag{19}$$

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The series $\theta = \sum_{m=0}^{\infty} b_m \xi^{2m}$ converges for $\xi < 1$.

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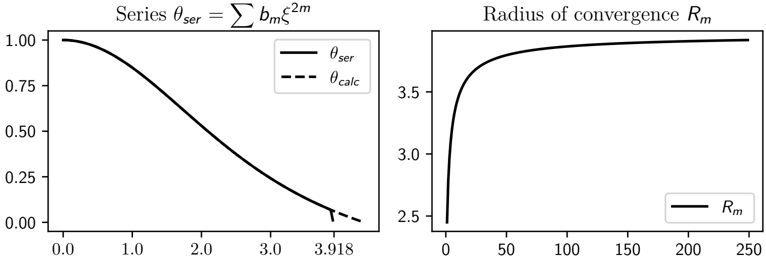

Figure: Exact solution θ and $R_m = (b_m)^{-1/(2m+2)}$ against m

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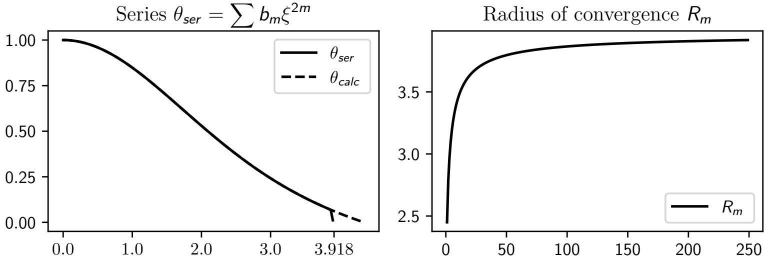

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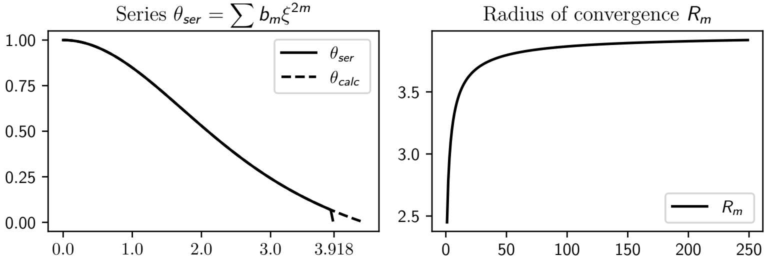

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Hypothesis

Let $n \geq 0$ and $\theta_n = \sum a_m \xi^m$ be a LE solution. Then $a_{2m+1} = 0$ for $m \in \mathbb{N}$.

Limiting Case TOV

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Proof.

With $\partial m/\partial r = 4\pi A p^{1/\gamma} r^2$, define $v := m/A$

$$\frac{\partial p}{\partial r} = -\frac{p^{1/\gamma}}{r^2}\left(A + p^{1-1/\gamma}\right)\left(4\pi r^3 p + vA\right)\left(1 - \frac{2vA}{r}\right)^{-1} \tag{22}$$

□

Proof (Cont ...)

Then for $A = 0$, we have

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