The TOV Equation and the Mass of Stars

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- TOV equation: differential equation, describes stars in GR
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- Questions to answer:
  - Well defined radius?
  - Mass limits?
- Mathematically interesting: show that a solution of the differential equation has a zerovalue without knowing the solution

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Concepts

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$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} + \left(\frac{1}{2}R + \Lambda\right) g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Heavy objects create curvature in space

Deriving the TOV equation

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$$T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p) \tag{2}$$

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- Energy-Momentum Tensor of perfect fluid

$$T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p) \tag{2}$$

- Solve Einstein Equations (without cosm. constant)

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3}$$

- Obtain 3 distinct differential equations ($R_{33} = R_{22}$)

$$-8\pi T_0^0 = 8\pi\rho = \frac{\lambda' e^{-\lambda}}{r} + \frac{1 - e^{-\lambda}}{r^2} \tag{4}$$

$$8\pi T_1^1 = 8\pi p = \nu' \frac{e^{-\lambda}}{r} - \frac{1 - e^{-\lambda}}{r^2} \tag{5}$$

$$8\pi T_2^2 = 8\pi p = \frac{e^{-\lambda}}{2}\left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r}\right)(\nu' - \lambda')\right] \tag{6}$$

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- Use equation 4 and identify Mass $m(r)$

$$e^{-\lambda} = 1 - \frac{2}{r}\int_0^r 4\pi\rho(r')r'^2 dr' =: 1 - \frac{2m(r)}{r} \tag{7}$$

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- Divergence of Energy-Momentum Tensor is $\nabla_\mu T^{\mu\nu} = 0$. Then obtain

$$\frac{\partial p}{\partial r} = -\frac{p+\rho}{2}\frac{\partial \nu}{\partial r} \tag{8}$$

- Use equations 5, 7 and 8 to get

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2 \tag{9}$$

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2}\left(1+\frac{p}{\rho c^2}\right)\left(\frac{4\pi r^3 p}{mc^2}+1\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1} \tag{10}$$

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- Plugged in gravitational constant, speed of light $G = c = 1$
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- Ordinary differential equation
- Singular at $r = 0$
- Needs equation of state (EOS) $f(\rho, p, r) = 0$ to be numerically solvable

Newtonian Limit

- Non-relativistic Limit of 2nd TOV equation 10 is

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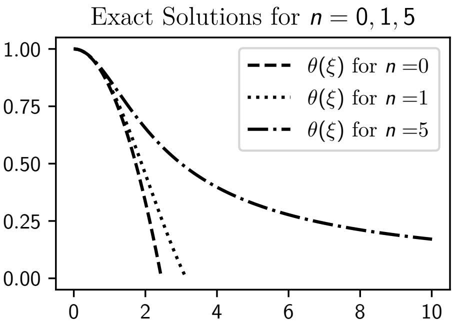
- Some exact solutions are known [Cha58]



$n = 0$	$1 - 1/6\xi^2$	$\xi_0 = \sqrt{6}$
$n = 1$	$\sin(\xi)/\xi$	$\xi_0 = \pi$
$n = 5$	$\left(1 + 1/3\xi^2\right)^{-1/2}$	$\xi_0 = \infty$

$$\rho = \lambda\theta^n \text{ and } p = K\rho^{n+1/n}$$

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$$\mathcal{Z}(T, V, N) = \int\limits_{TM^N} \exp\left(-\frac{H(x_1, \dots, p_N)}{k_B T}\right) \frac{dx_1 \dots dp_N}{N! h^{3N}} \tag{14}$$

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- Calculate equation of state via internal Energy $\mathcal{U}$ and

$$p = k_B T \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial V} \qquad \rho = \frac{\mathcal{U}}{V} = \frac{k_B T^2}{V} \frac{\partial \mathcal{Z}}{\partial T} \tag{15}$$

- Partition Function for $H = \sqrt{m^2 + p^2}$

$$\mathcal{Z} = \frac{1}{N!}\left(8\pi V \left(\frac{k_B T}{hc}\right)^3 \frac{\alpha^2 K_2(\alpha)}{2}\right)^N \tag{16}$$

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- Sketch of proof: take differential, use properties/representations of K_ν
- Use inverse and previous equations 15, 16 and 17 to obtain EOS

$$\rho = \frac{\mathcal{U}}{V} = p\left(1 + \alpha(p)\frac{K_1(\alpha(p)) + K_3(\alpha(p))}{2K_2(\alpha(p))}\right) \tag{18}$$

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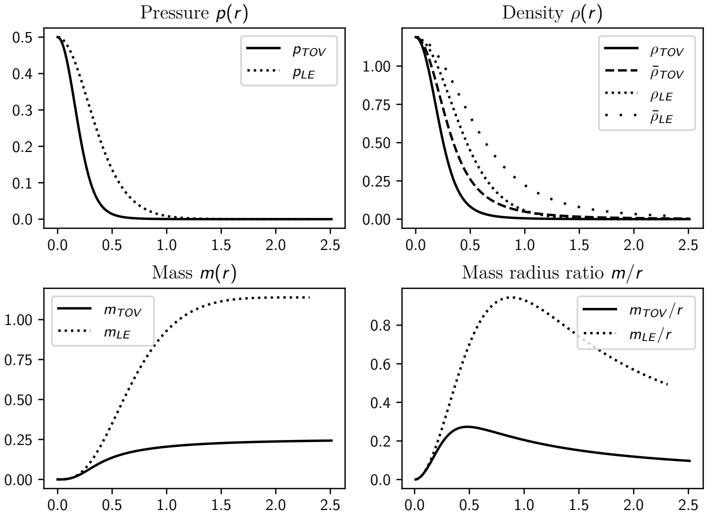

Figure: Comparison of TOV and LE results

Verifying Results

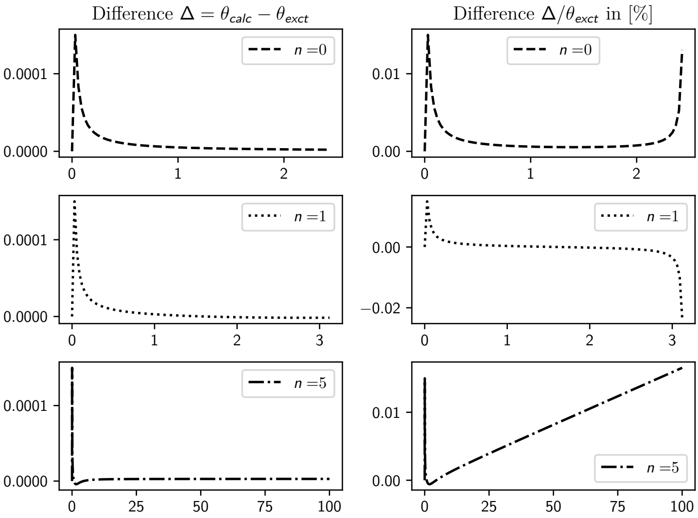

Figure: Validation of numerically calculated results

Zero Values of TOV and LE equation

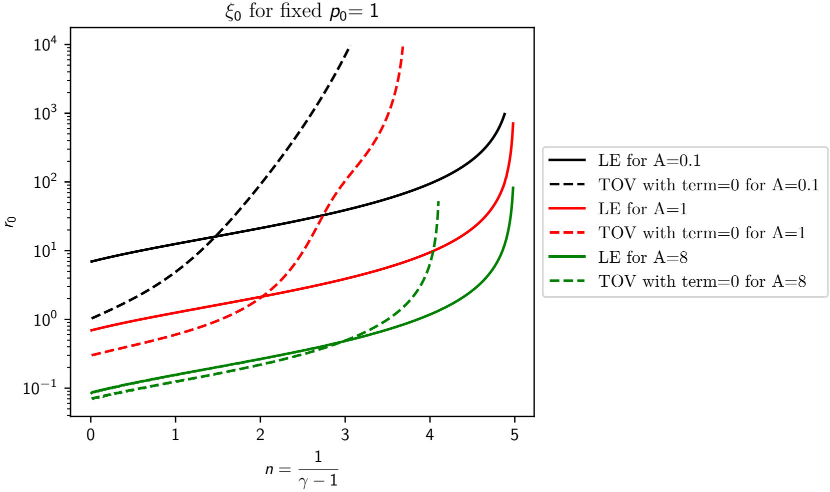

Figure: TOV and LE results for varying A parameter of $\rho = Ap^{n/(n+1)}$

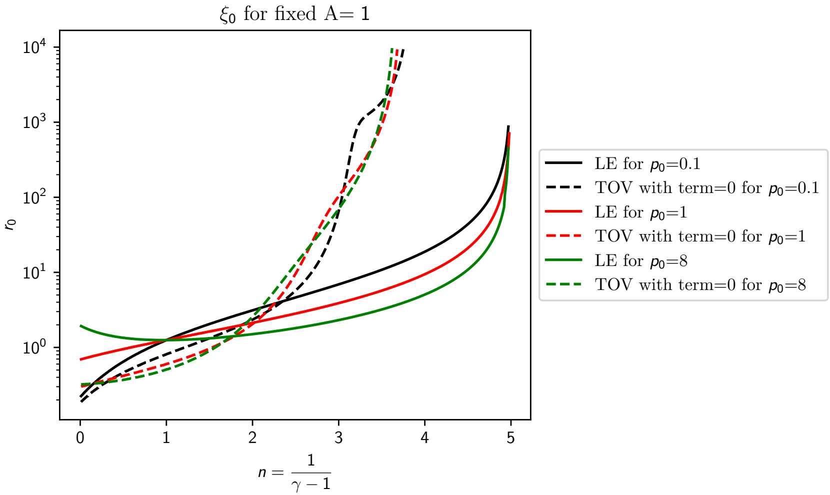

Figure: TOV and LE solutions for varying parameter p_0.

Intersection: $r = \beta\xi$ with $4\pi\beta A^{n/(n+1)} = (n+1)\lambda^{1-1/n}$ is independent of $\lambda = \rho(p_0)$.

TOV Hypothesis

Hypothesis

To each combination $p_0, A > 0$ there exists a $n_0 \geq 0$ such that each solution of the TOV differential equation

$$\begin{aligned} \frac{\partial m}{\partial r} &= \quad 4\pi\rho r^2 \\ \frac{\partial p}{\partial r} &= -\frac{m\rho}{r^2}\left(1+\frac{p}{\rho}\right)\left(\frac{4\pi r^3 p}{m}+1\right)\left(1-\frac{2m}{r}\right)^{-1} \end{aligned}$$

with $\rho = Ap^{\frac{n}{n+1}}$ where $n \leq n_0$ has $p(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}_{>0}$.

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New Exact Solution for LE at n=2

- Found new solution for $n = 2$ by using simple power-series $\theta = \sum a_m \xi^m$

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- Apply Cauchy-Product formula and combine

$$(m+1)\sum_{m=0}^{\infty}\left((m+2)a_{m+2}\xi^m + 2a_{m+1}\xi^{m-1} + \sum_{k=0}^{m} a_{m-k}a_k\xi^m\right) = 0 \tag{20}$$

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- Proof by induction. Start at $\partial\theta|_{\xi=0} = a_1 = 0$ is clear.

- Obtain recursive Formula for coefficients

$$a_{2m+2} = b_{m+1} = -\frac{1}{(m+2)(m+3)} \sum_{k=0}^{m} b_{m-k} b_k \tag{21}$$

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Proof.

We know that $a_0 = 1$. Use L'Hospital's rule for $\xi \to 0$

$$0 = \frac{2}{\xi}\frac{\partial \theta}{\partial \xi} + \frac{\partial^2 \theta}{\partial \xi^2} + \theta^2 \longrightarrow 3 \left.\frac{\partial^2 \theta}{\partial \xi^2}\right|_{\xi=0} + \theta|_{\xi=0} = 0 \tag{22}$$

Thus $b_1 = \theta_0/6$. By induction, we get $|b_{m+1}| \leq \frac{m\theta_0^2}{(m+2)(m+3)}$. Now use ratio test ("Quotientenkriterium") □

Hypothesis

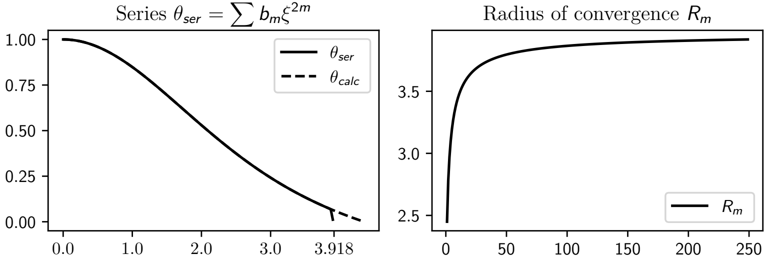

Figure: Exact solution θ and $(b_m)^{-1/(2m+2)}$ against m

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The radius of convergence R of the above calculated series is exactly the value at which $\theta(R) = 0$.

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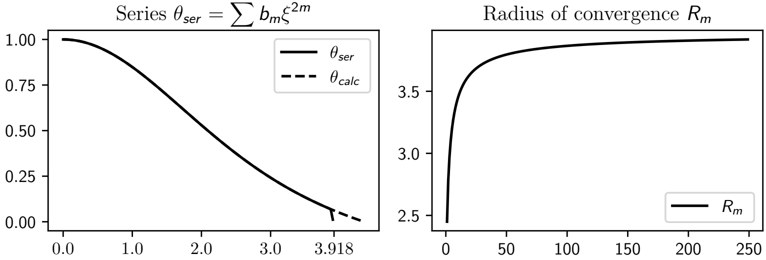

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- Suppose the TOV equation has a solution that continuously depends on its parameters for $r \in [0, l)$ where l may be ∞.

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Let p_A be a solution of the TOV equation with $\rho = Ap^{1/\gamma}$. Then

$$\lim_{A \to 0} p_A = \frac{p_0}{2\pi p_0 r^2 + 1} \tag{23}$$

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Proof.

With $\partial m/\partial r = 4\pi A p^{1/\gamma} r^2$, define $v := m/A$

$$\frac{\partial p}{\partial r} = -\frac{p^{1/\gamma}}{r^2}\left(A + p^{1-1/\gamma}\right)\left(4\pi r^3 p + vA\right)\left(1 - \frac{2vA}{r}\right)^{-1} \tag{24}$$

$\square$

Proof (Cont ...)

Then for $A = 0$*, we have*

$$\frac{\partial p}{\partial r} = -4\pi r p^2 \tag{25}$$

The solution to this differential equation is $p = \frac{p_0}{2\pi p_0 r^2 + 1}$.

Proof (Cont ...)

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- Problem: Assumption is that TOV equation is solvable

Proof (Cont ...)

Then for $A = 0$, we have

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The solution to this differential equation is $p = \frac{p_0}{2\pi p_0 r^2 + 1}$.

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