

The TOV Equation and the Mass of Stars

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Motivation

- TOV equation: differential equation, describes stars in GR
- Needs additional information to be solvable
- Thermodynamics yields this information
- Questions to answer:
 - Well defined radius?
 - Mass limits?
- Mathematically interesting: show that a solution of the differential equation has a zero value without knowing the solution

Table of Contents

1. General Relativity
2. Thermodynamics - Calculating an EoS
3. Numerical Solutions
4. Exact Results
5. Outlook

Table of Contents

1. General Relativity
 - 1.1 Concepts
 - 1.2 Deriving the TOV equation
 - 1.3 Newtonian Limit
2. Thermodynamics - Calculating an EoS
3. Numerical Solutions
4. Exact Results
5. Outlook

Concepts

TODO include nice picture

- General relativity (GR) models large scale structure of measurable universe
- Lorentzian Geometry (with indefinite metric $g_{\mu\nu}$)
- Einstein equations relate curvature with mass and energy (geometrized units $G = c = 1$)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} + \left(\frac{1}{2}R + \Lambda\right) g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Heavy objects create curvature in space

Deriving the TOV equation

- TOV equation was first derived independently by Tolman [Tol39] and Oppenheimer with Volkoff [OV39].
- Spherically symmetric (Lorentz) metric

$$g = -e^\nu dt^2 + e^\lambda dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

- Energy-Momentum Tensor of perfect fluid

$$T_{\mu\nu} = \text{diag}(-\rho, p, p, p) \quad (2)$$

- Solve Einstein Equations (without cosm. constant)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \quad (3)$$

- Obtain 3 distinct differential equations ($R_{33} = R_{22}$)

$$-8\pi T_0^0 = 8\pi\rho = \frac{\lambda'e^{-\lambda}}{r} + \frac{1 - e^{-\lambda}}{r^2} \quad (4)$$

$$8\pi T_1^1 = 8\pi p = \nu' \frac{e^{-\lambda}}{r} - \frac{1 - e^{-\lambda}}{r^2} \quad (5)$$

$$8\pi T_2^2 = 8\pi p = \frac{e^{-\lambda}}{2} \left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r} \right) (\nu' - \lambda') \right] \quad (6)$$

- Use equation 4 and identify Mass $m(r)$

$$e^{-\lambda} = 1 - \frac{2}{r} \int_0^r 4\pi\rho(r')r'^2 dr' =: 1 - \frac{2m(r)}{r} \quad (7)$$

- Divergence of Energy-Momentum Tensor is $\nabla_\mu T^{\mu\nu} = 0$. Then obtain

$$\frac{\partial p}{\partial r} = -\frac{p + \rho}{2} \frac{\partial \nu}{\partial r} \quad (8)$$

- Use equations 5, 7 and 8 to get

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2 \quad (9)$$

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2} \left(1 + \frac{p}{\rho c^2}\right) \left(\frac{4\pi r^3 p}{mc^2} + 1\right) \left(1 - \frac{2Gm}{rc^2}\right)^{-1} \quad (10)$$

- Plugged in gravitational constant, speed of light $G = c = 1$
- Equation 9 from Mass-Definition
- Ordinary differential equation
- Singular at $r = 0$
- Needs equation of state (EOS) $f(\rho, p, r) = 0$ to be numerically solvable

Newtonian Limit

- Non-relativistic Limit of 2nd TOV equation 10 is

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2} \quad (11)$$

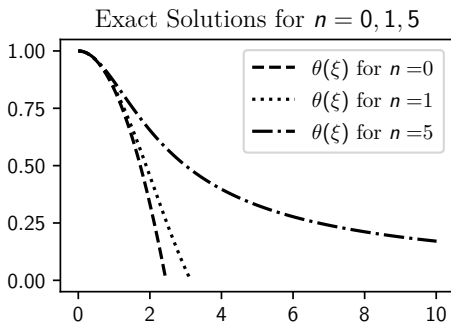
- Ansatz: Polytropic EOS $p = K\rho^{1+1/n}$
- Transformation $\rho = \lambda\theta^n$ and $\xi = r/\beta$ where

$$4\pi\beta^2 = (n+1)K\lambda^{1-1/n} \quad (12)$$

- Obtain Lane-Emden equation [Lan70] [Emd07]

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \theta}{\partial \xi} \right) + \theta^n = 0 \quad (13)$$

- Some exact solutions are known [Cha58]



$n = 0$	$1 - 1/6\xi^2$	$\xi_0 = \sqrt{6}$
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$n = 1$	$\sin(\xi)/\xi$	$\xi_0 = \pi$
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$n = 5$	$\left(1 + 1/3\xi^2\right)^{-1/2}$	$\xi_0 = \infty$
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$$\rho = \lambda\theta^n \text{ and } p = K\rho^{n+1/n}$$

Table of Contents

1. General Relativity
2. Thermodynamics - Calculating an EoS
 - 2.1 Short Summary of main Principles
3. Numerical Solutions
4. Exact Results
5. Outlook

Short Summary of main Principles

- Statistical theory of manyparticle systems
- Describe macroscopic Phenomena by microscopic principles
- Partition function contains all information about N particles with position $x_i \in M$ and momentum $p_i \in T_{x_i}M$ with $V = \text{vol}(M)$

$$\mathcal{Z}(T, V, N) = \int_{TM^N} \exp\left(-\frac{H(x_1, \dots, p_N)}{k_B T}\right) \frac{dx_1 \dots dp_N}{N! h^{3N}} \quad (14)$$

- Calculate equation of state via internal Energy \mathcal{U} and

$$p = k_B T \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial V} \quad \rho = \frac{\mathcal{U}}{V} = \frac{k_B T^2}{V} \frac{\partial \mathcal{Z}}{\partial T} \quad (15)$$

- Partition Function for $H = \sqrt{m^2 + p^2}$

$$\mathcal{Z} = \frac{1}{N!} \left(8\pi V \left(\frac{k_B T}{hc} \right)^3 \frac{\alpha^2 K_2(\alpha)}{2} \right)^N \quad (16)$$

- K_2 is modified Bessel function of 2nd kind
- Substitution $\alpha = mc^2/k_B T$
- With equations 15 and 16 obtain

$$p = \frac{Nk_B T}{V} = CNmc^2 \frac{1}{K_2(\alpha)\alpha^2} \exp \left(-\alpha \frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)} \right) \quad (17)$$

Theorem

The function $p(\alpha)$ above is a bijection.

- Sketch of proof: take differential, use properties/representations of K_ν
- Use inverse and previous equations 15, 16 and 17 to obtain EOS

$$\rho = \frac{\mathcal{U}}{V} = p \left(1 + \alpha(p) \frac{K_1(\alpha(p)) + K_3(\alpha(p))}{2K_2(\alpha(p))} \right) \quad (18)$$

Table of Contents

1. General Relativity
2. Thermodynamics - Calculating an EoS
3. Numerical Solutions
 - 3.1 Comparison of TOV and LE results
 - 3.2 Verifying Results
 - 3.3 Zero Values of TOV and LE equation
 - 3.4 TOV Hypothesis
4. Exact Results
5. Outlook

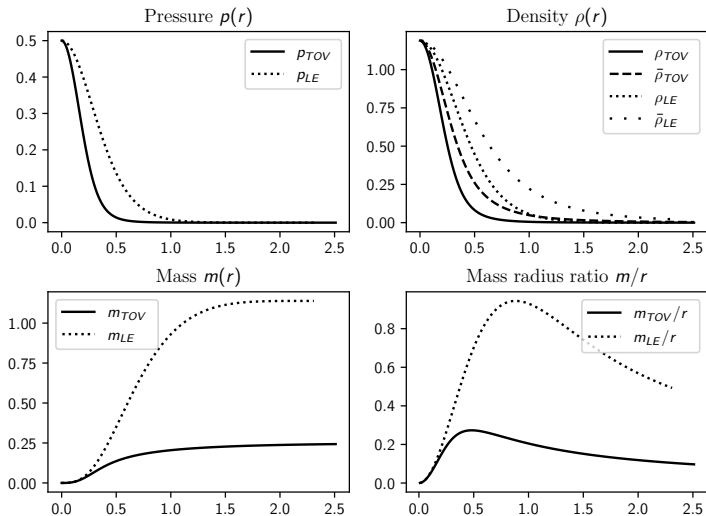


Figure: Comparison of TOV and LE results

Verifying Results

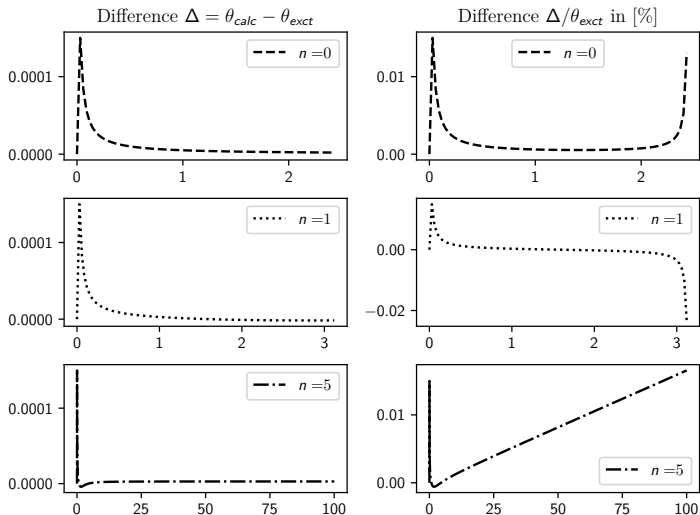


Figure: Validation of numerically calculated results

Zero Values of TOV and LE equation

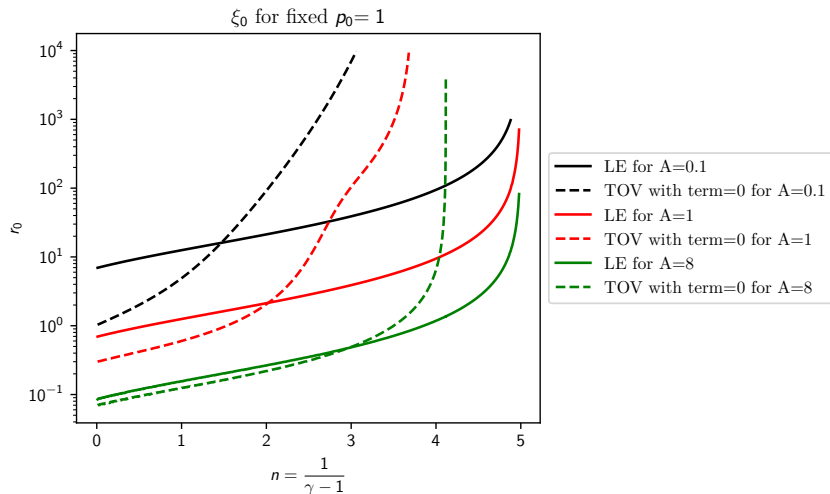


Figure: TOV and LE results for varying A parameter of $\rho = A\rho^{n/(n+1)}$

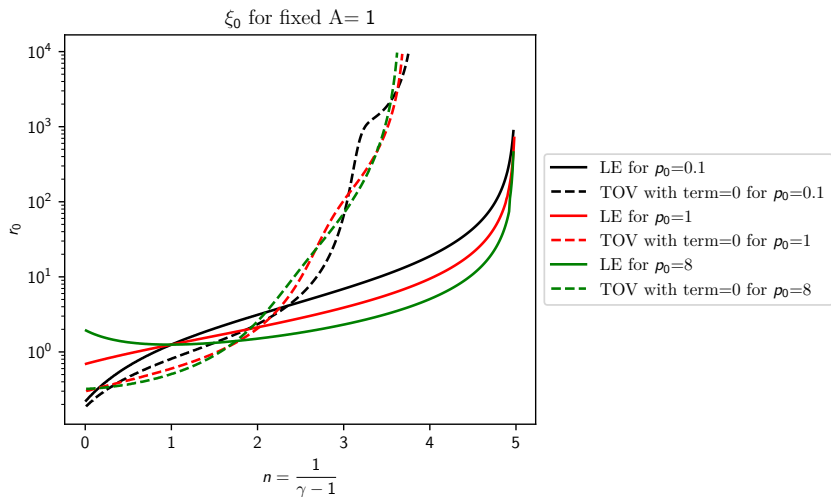


Figure: TOV and LE solutions for varying parameter ρ_0 .

Intersection: $r = \beta\xi$ with $4\pi\beta A^{n/(n+1)} = (n+1)\lambda^{1-1/n}$ is independent of $\lambda = \rho(p_0)$.

TOV Hypothesis

Hypothesis

To each combination $p_0, A > 0$ there exists a $n_0 \geq 0$ such that each solution of the TOV differential equation

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2$$

$$\frac{\partial p}{\partial r} = -\frac{m\rho}{r^2} \left(1 + \frac{p}{\rho}\right) \left(\frac{4\pi r^3 p}{m} + 1\right) \left(1 - \frac{2m}{r}\right)^{-1}$$

with $\rho = Ap^{\frac{n}{n+1}}$ where $n \leq n_0$ has $p(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}_{>0}$.

Table of Contents

1. General Relativity
2. Thermodynamics - Calculating an EoS
3. Numerical Solutions
4. Exact Results
 - 4.1 New Exact Solution for LE at $n=2$
 - 4.2 Hypothesis
 - 4.3 Limiting Case TOV
5. Outlook

New Exact Solution for LE at $n=2$

- Found new solution for $n = 2$ by using simple power-series $\theta = \sum a_m \xi^m$

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \theta}{\partial \xi} \right) + \theta^2 = 0 \quad (19)$$

- Apply Cauchy-Product formula and combine

$$(m+1) \sum_{m=0}^{\infty} \left((m+2)a_{m+2}\xi^m + 2a_{m+1}\xi^{m-1} + \sum_{k=0}^m a_{m-k}a_k\xi^m \right) = 0 \quad (20)$$

Theorem

The odd coefficients a_{2m+1} vanish.

- Proof by induction. Start at $\partial\theta|_{\xi=0} = a_1 = 0$ is clear.

- Obtain recursive Formula for coefficients

$$a_{2m+2} = b_{m+1} = -\frac{1}{(m+2)(m+3)} \sum_{k=0}^m b_{m-k} b_k \quad (21)$$

Theorem

The series $\theta = \sum_{m=0}^{\infty} b_m \xi^{2m}$ converges for $\xi < 1$.

Proof.

We know that $a_0 = 1$. Use L'Hospital's rule for $\xi \rightarrow 0$

$$0 = \frac{2}{\xi} \frac{\partial \theta}{\partial \xi} + \frac{\partial^2 \theta}{\partial \xi^2} + \theta^2 \longrightarrow 3 \left. \frac{\partial^2 \theta}{\partial \xi^2} \right|_{\xi=0} + \theta|_{\xi=0} = 0 \quad (22)$$

Thus $b_1 = \theta_0/6$. By induction, we get $|b_{m+1}| \leq \frac{m\theta_0^2}{(m+2)(m+3)}$. Now use ratio test ("Quotientenkriterium") □

Hypothesis

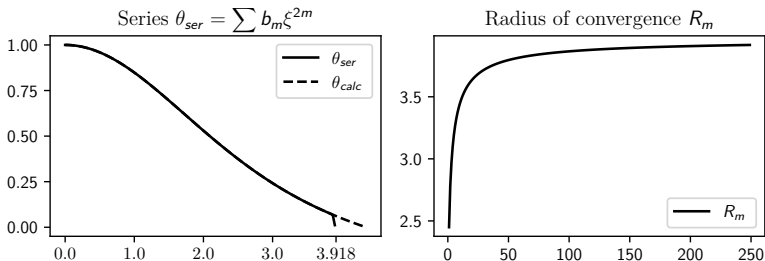


Figure: Exact solution θ and $(b_m)^{-1/(2m+2)}$ against m

Hypothesis

The radius of convergence R of the above calculated series is exactly the value at which $\theta(R) = 0$.

- Numerics: Calculation not optimized, reaches floating point limit

Limiting Case TOV

- Suppose the TOV equation has a solution that continuously depends on its parameters for $r \in [0, l)$ where l may be ∞ .

Theorem

Let p_A be a solution of the TOV equation with $\rho = Ap^{1/\gamma}$. Then

$$\lim_{A \rightarrow 0} p_A = \frac{p_0}{2\pi p_0 r^2 + 1} \quad (23)$$

Proof.

With $\partial m / \partial r = 4\pi A p^{1/\gamma} r^2$, define $v := m/A$

$$\frac{\partial p}{\partial r} = -\frac{p^{1/\gamma}}{r^2} \left(A + p^{1-1/\gamma} \right) \left(4\pi r^3 p + vA \right) \left(1 - \frac{2vA}{r} \right)^{-1} \quad (24)$$



Proof (Cont ...)

Then for $A = 0$, we have

$$\frac{\partial p}{\partial r} = -4\pi r p^2 \quad (25)$$

The solution to this differential equation is $p = \frac{p_0}{2\pi p_0 r^2 + 1}$.

- Problem: Assumption is that TOV equation is solvable
- Easy transformation tricks for 1D singular ODEs not working.

Table of Contents

1. General Relativity
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5. Outlook

Outlook

- Plenty of information on numerical side
- Current work is (hopefully) well documented
- Should be able to obtain exact solution for $n \in \mathbb{N}, n > 1$ analogously to $n = 2$ cases
- Solvability of LE equation is known and well researched eg. [QS07]
- Solvability of TOV equation is complicated and subject of current research [Mar+19; BVW07]
- Mass limit of $M/R < 4/9$ already known for border of star. Maybe possible to prove inside as well.
- Improve mass limits

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