

The TOV Equation and the Mass of Stars

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- Needs additional information to be solvable
- Thermodynamics yields this information
- Questions to answer:
 - Well defined radius?
 - Mass limits?
- Mathematically interesting: show that a solution of the differential equation has a zero value without knowing the solution

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1. General Relativity
2. Thermodynamics
3. Numerical Solutions
4. Exact Results
5. Outlook

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 - 1.3 Newtonian Limit
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Concepts



- General relativity (GR) models large scale structure of measurable universe

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- Einstein equations relate curvature with mass and energy (geometrized units $G = c = 1$)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} + \left(\frac{1}{2}R + \Lambda\right) g_{\mu\nu} = 8\pi T_{\mu\nu}$$

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- Heavy objects create curvature in space

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$$T_{\mu\nu} = \text{diag}(-\rho, p, p, p) \quad (2)$$

- Solve Einstein Equations (without cosm. constant)

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \quad (3)$$

- Obtain 3 distinct differential equations ($R_{33} = R_{22}$)

$$-8\pi T_0^0 = 8\pi\rho = \frac{\lambda'e^{-\lambda}}{r} + \frac{1 - e^{-\lambda}}{r^2} \quad (4)$$

$$8\pi T_1^1 = 8\pi p = \nu' \frac{e^{-\lambda}}{r} - \frac{1 - e^{-\lambda}}{r^2} \quad (5)$$

$$8\pi T_2^2 = 8\pi p = \frac{e^{-\lambda}}{2} \left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r} \right) (\nu' - \lambda') \right] \quad (6)$$

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- Use equation 4 and identify Mass $m(r)$

$$e^{-\lambda} = 1 - \frac{2}{r} \int_0^r 4\pi\rho(r')r'^2 dr' =: 1 - \frac{2m(r)}{r} \quad (7)$$

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- Divergence of Energy-Momentum Tensor is $\nabla_\mu T^{\mu\nu} = 0$. Then obtain

$$\frac{\partial p}{\partial r} = -\frac{p + \rho}{2} \frac{\partial \nu}{\partial r} \quad (8)$$

- Use equations 5, 7 and 8 to get

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2 \quad (9)$$

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2} \left(1 + \frac{p}{\rho c^2}\right) \left(\frac{4\pi r^3 p}{mc^2} + 1\right) \left(1 - \frac{2Gm}{rc^2}\right)^{-1} \quad (10)$$

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- Needs equation of state (EoS) $f(\rho, p, r) = 0$ to be solvable

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- Obtain Lane-Emden equation [Lan70; Emd07]

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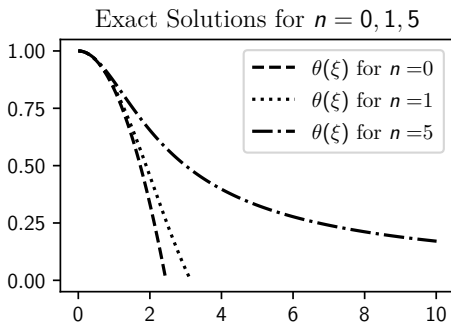
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- Some exact solutions are known [Cha58]



$n = 0$	$1 - \xi^2/6$	$\xi_0 = \sqrt{6}$
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$n = 1$	$\sin(\xi)/\xi$	$\xi_0 = \pi$
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$n = 5$	$\left(1 + \xi^2/3\right)^{-1/2}$	$\xi_0 = \infty$
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$$\rho = \lambda \theta^n \text{ and } p = K \rho^{n+1/n}$$

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Calculating an EoS

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- Partition function contains all information about N particles with position $x_i \in M$ and momentum $p_i \in T_{x_i} M$ with $V = \text{vol}(M)$

$$\mathcal{Z}(T, V, N) = \int_{TM^N} \exp\left(-\frac{H(x_1, \dots, p_N)}{k_B T}\right) \frac{dx_1 \dots dp_N}{N! h^{3N}} \quad (14)$$

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- Calculate equation of state via internal Energy \mathcal{U} and

$$p = k_B T \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial V} \quad \rho = \frac{\mathcal{U}}{V} = \frac{k_B T^2}{V} \frac{\partial \mathcal{Z}}{\partial T} \quad (15)$$

- Partition Function for $H = \sqrt{m^2 + p^2}$

$$\mathcal{Z} = \frac{1}{N!} \left(8\pi V \left(\frac{k_B T}{hc} \right)^3 \frac{\alpha^2 K_2(\alpha)}{2} \right)^N \quad (16)$$

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Theorem

The function $p(\alpha)$ above is a bijection.

- Sketch of proof: take differential, use properties/representations of K_ν
- Use inverse and previous equations 15, 16 and 17 to obtain EOS

$$\rho = \frac{\mathcal{U}}{V} = p \left(1 + \alpha(p) \frac{K_1(\alpha(p)) + K_3(\alpha(p))}{2K_2(\alpha(p))} \right) \quad (18)$$

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 - 3.1 Comparison of TOV and LE results
 - 3.2 Verifying LE Results
 - 3.3 Zero Values of TOV and LE equation
 - 3.4 TOV Hypothesis
4. Exact Results
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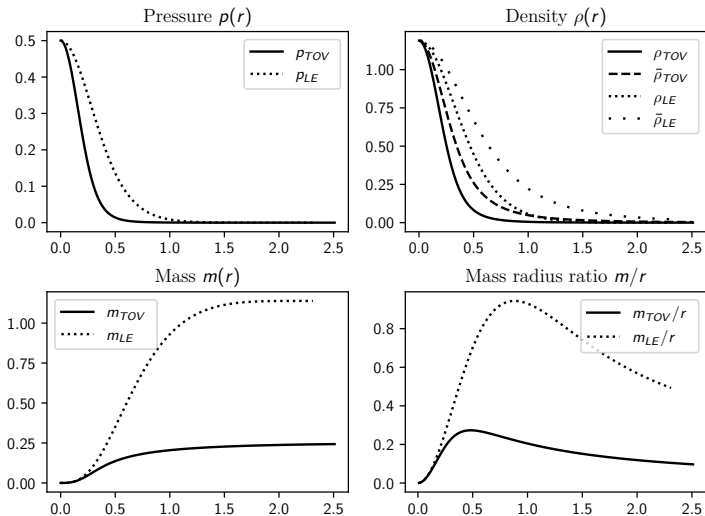


Figure: Comparison of TOV and LE results

Verifying LE Results

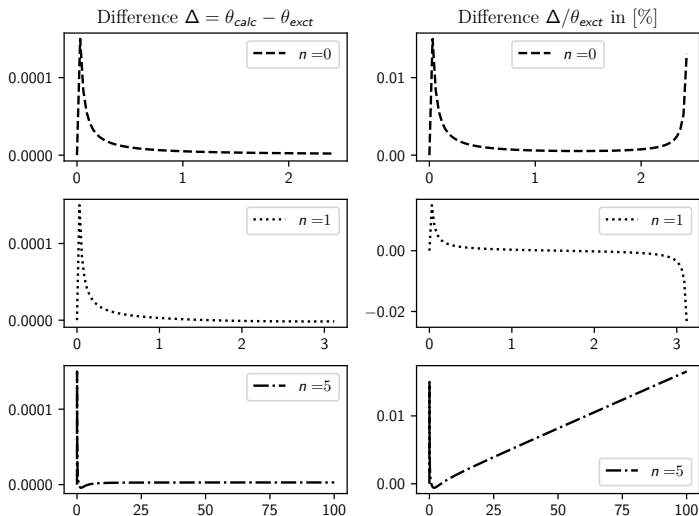


Figure: Validation of numerically calculated Lane-Emden results

Zero Values of TOV and LE equation

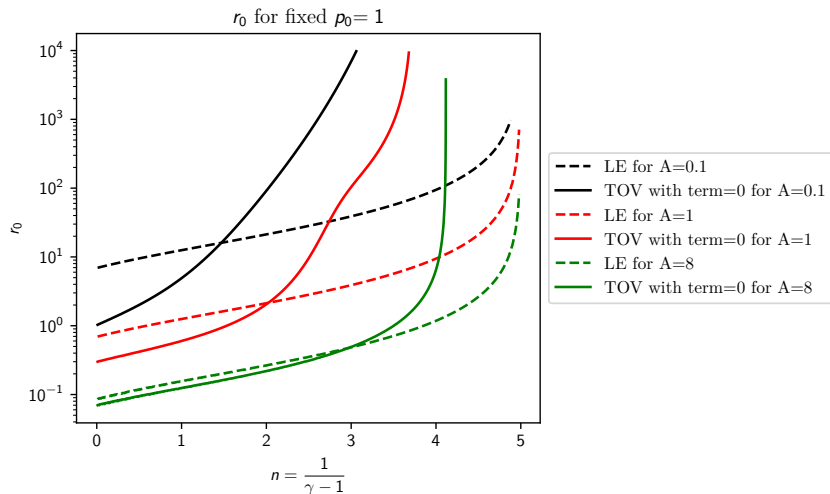


Figure: TOV and LE results for varying A parameter of $\rho = A\rho^{n/(n+1)}$

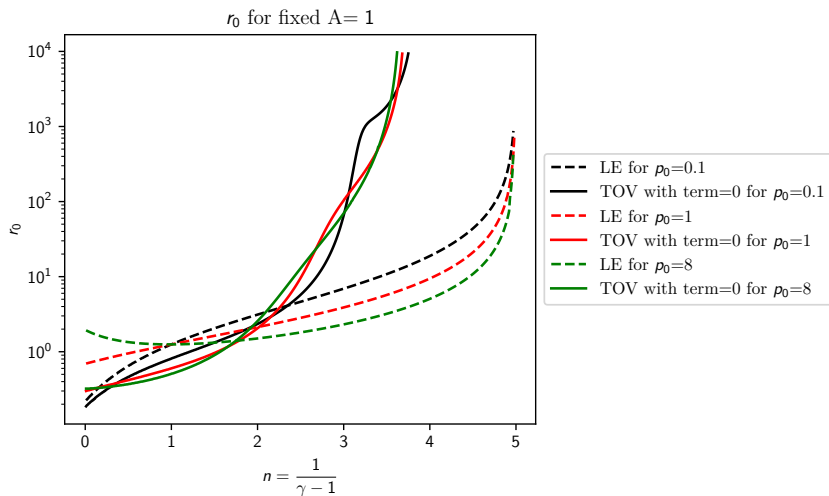


Figure: TOV and LE solutions for varying parameter p_0 .

Intersection: $r = \beta\xi$ with $4\pi\beta A^{n/(n+1)} = (n+1)\lambda^{1-1/n}$ is independent of $\lambda = \rho(p_0)$.

TOV Hypothesis

Hypothesis

Given the TOV differential equation with $\rho = Ap^{\frac{n}{n+1}}$ and $p_0, A > 0$

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2$$

$$\frac{\partial p}{\partial r} = -\frac{m\rho}{r^2} \left(1 + \frac{p}{\rho}\right) \left(\frac{4\pi r^3 p}{m} + 1\right) \left(1 - \frac{2m}{r}\right)^{-1}$$

There exists a $n_0 \geq 0$ such that all solutions with same parameters A, p_0 and smaller exponent $n < n_0$ have a $p(r_0)$ for some $r_0 > 0$.

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 - 4.1 New Exact Solution for LE at $n=2$
 - 4.2 Hypotheses
 - 4.3 Limiting Case TOV
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New Exact Solution for LE at $n=2$

- Found new solution for $n = 2$ by using simple power-series $\theta = \sum a_m \xi^m$

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \theta}{\partial \xi} \right) + \theta^2 = 0 \quad (19)$$

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The series $\theta = \sum_{m=0}^{\infty} b_m \xi^{2m}$ converges for $\xi < 1$.

Hypotheses

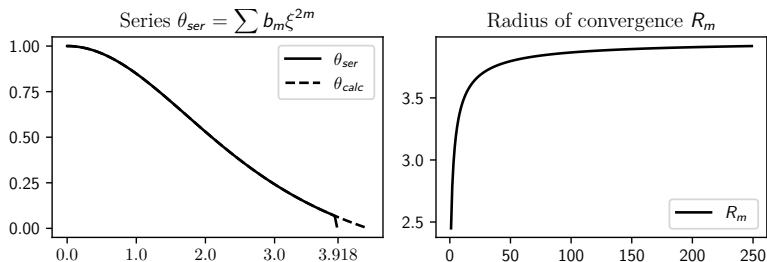


Figure: Exact solution θ and $R_m = (b_m)^{-1/(2m+2)}$ against m

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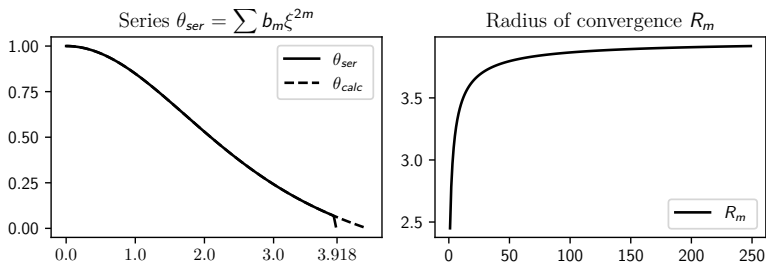


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The radius of convergence $R > 1$ and $\exists \xi_0 \geq R$ such that $p(\xi_0) = 0$.

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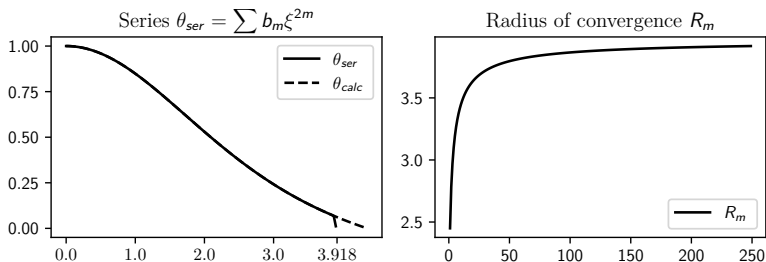


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Hypothesis

Let $n \geq 0$ and $\theta_n = \sum a_m \xi^m$ be a LE solution. Then $a_{2m+1} = 0$ for $m \in \mathbb{N}$.

Limiting Case TOV

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Proof.

With $\partial m / \partial r = 4\pi A p^{1/\gamma} r^2$, define $v := m/A$

$$\frac{\partial p}{\partial r} = -\frac{p^{1/\gamma}}{r^2} \left(A + p^{1-1/\gamma} \right) \left(4\pi r^3 p + vA \right) \left(1 - \frac{2vA}{r} \right)^{-1} \quad (22)$$



Proof (Cont ...)

Then for $A = 0$, we have

$$\frac{\partial p}{\partial r} = -4\pi r p^2 \quad (23)$$

The solution to this differential equation is $p = \frac{p_0}{2\pi p_0 r^2 + 1}$.

Proof (Cont ...)

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