Do not choose a bad title

... at least not too bad.

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Concepts

TODO include nice picture

- General relativity (GR) models large scale of measurable universe
- Lorentzian Geometry (with indefinitive metric $g_{\mu\nu}$)
- Einstein equations relate curvature with mass and energy

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} + \left(\frac{1}{2}R + \Lambda\right) g_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Heavy objects create curvature in space

Deriving the TOV equation

- TOV equation was first derived independently by Tolman [Tol39] and Oppenheimer with Volkoff [OV39].
- Spherically symmetric (Lorentz) metric

$$g = -e^{\nu} dt^2 + e^{\lambda} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \tag{1}$$

- Energy-Momentum Tensor of perfect fluid

$$T_{\mu\nu} = \mathrm{diag}(-\rho, p, p, p) \tag{2}$$

- Solve Einstein Equations (without cosm. constant)

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{3}$$

- Obtain 3 distinct differential equations ($R_{33} = R_{22}$)

$$-8\pi T_0^0 = 8\pi\rho = \frac{\lambda' e^{-\lambda}}{r} + \frac{1 - e^{-\lambda}}{r^2} \tag{4}$$

$$8\pi T_1^1 = 8\pi p = \nu' \frac{e^{-\lambda}}{r} - \frac{1 - e^{-\lambda}}{r^2} \tag{5}$$

$$8\pi T_2^2 = 8\pi p = \frac{e^{-\lambda}}{2} \left[\nu'' + \left(\frac{\nu'}{2} + \frac{1}{r}\right)(\nu' - \lambda')\right] \tag{6}$$

- Use equation 4 and identify Mass $m(r)$

$$e^{-\lambda} = 1 - \frac{2}{r} \int_0^r 4\pi\rho(r')r'^2 dr' =: 1 - \frac{2m(r)}{r} \tag{7}$$

- Divergence of Energy-Momentum Tensor is $\nabla_\mu T^{\mu\nu} = 0$. Then obtain

$$\frac{\partial p}{\partial r} = -\frac{p + \rho}{2} \frac{\partial \nu}{\partial r} \tag{8}$$

- Use equations 5, 7 and 8 to get

$$\frac{\partial m}{\partial r} = 4\pi\rho r^2 \tag{9}$$

$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2}\left(1+\frac{p}{\rho c^2}\right)\left(\frac{4\pi r^3 p}{mc^2}+1\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1} \tag{10}$$

- Plugged in gravitational constant, speed of light $G = c = 1$
- Equation 9 from Mass-Definition
- Ordinary differential equation
- Singular at $r = 0$
- Needs equation of state (EOS) $f(\rho, p, r) = 0$ to be solvable

Newtonian Limit

- Non-relativistic Limit of 2nd TOV equation 10 is

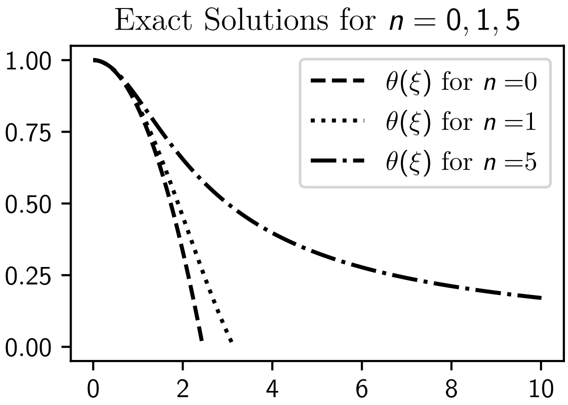
$$\frac{\partial p}{\partial r} = -\frac{Gm\rho}{r^2} \tag{11}$$

- Polytropic EOS $p = K\rho^{1+1/n}$
- Transformation $\rho = \lambda\theta^n$ and $\xi = r/\beta$ where

$$4\pi\beta^2 = (n+1)K\lambda^{1-1/n} \tag{12}$$

- Obtain Lane-Emden equation [Lan70] [Emd07]

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\theta}{\partial\xi}\right) + \theta^n = 0 \tag{13}$$



$n = 0$	$1 - 1/6\xi^2$	$\xi_0 =$	$\sqrt{6}$
$n = 1$	$\sin(\xi)/\xi$	$\xi_0 =$	π
$n = 5$	$\left(1 + 1/3\xi^2\right)^{-1/2}$	$\xi_0 =$	∞

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Short Summary of main Principles

- Statistical theory of manyparticle systems
- Describe macroscopic Phenomena by microscopic principles
- Partition function contains all information about N particles with position $x_i \in M$ and momentum $p_i \in T_{x_i}M$ with $V = \mathrm{vol}(M)$

$$\mathcal{Z}(T, V, N) = \int\limits_{TM^N} \exp\left(-\frac{H(x_1, \ldots, p_N)}{k_B T}\right) \frac{dx_1 \ldots dp_N}{N! h^{3N}} \tag{14}$$

- Calculate equation of state via internal Energy $\mathcal{U}$ and

$$p = k_B T \frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial V} \qquad \rho = \frac{\mathcal{U}}{V} = \frac{k_B T^2}{V} \frac{\partial \mathcal{Z}}{\partial T} \tag{15}$$

- Partition Function (K_2 is modified Bessel function of 2nd kind)

$$\mathcal{Z} = \frac{1}{N!}\left(8\pi V \left(\frac{k_B T}{hc}\right)^3 2\alpha K_2(\alpha)\right)^N \tag{16}$$

- With equations 15 and 16 obtain

$$p = \frac{N k_B T}{V} = C N m c^2 \frac{1}{K_2(\alpha)\alpha^2} \exp\left(-\alpha \frac{K_1(\alpha) + K_3(\alpha)}{2K_2(\alpha)}\right) \tag{17}$$

Theorem

The function $p(\alpha)$ above is a bijection.

- Sketch of proof: take differential, use properties/representations of K_ν
- Use inverse and previous equations 15, 16 and 17 to obtain EOS

$$\rho = \frac{\mathcal{U}}{V} = p\left(1 + \alpha(p)\frac{K_1(\alpha(p)) + K_3(\alpha(p))}{2K_2(\alpha(p))}\right) \tag{18}$$

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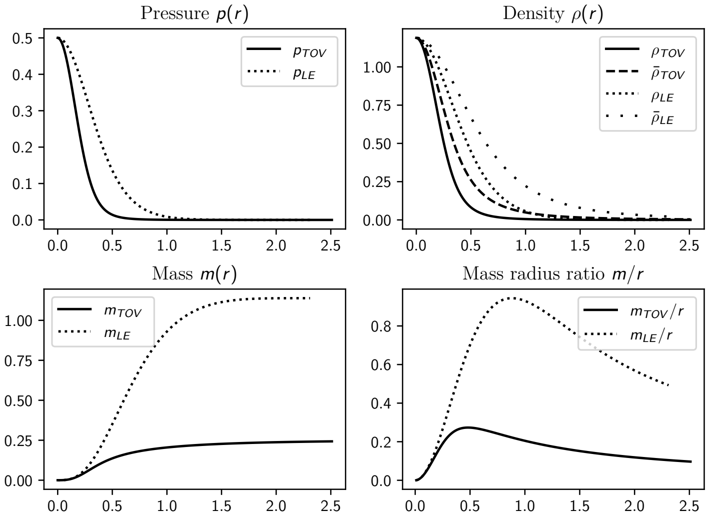
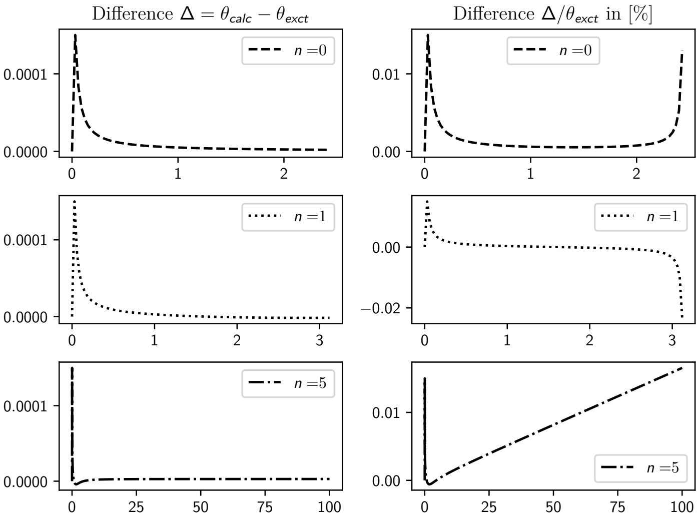
Comparison of TOV and LE results

TOV		LE	
EOS	$\rho = Ap^{1/\gamma}$	EOS	$p = K\rho^{\gamma}$
A	2		
$\gamma = 1 + \frac{1}{n}$	$4/3$	$n = 1/(\gamma - 1)$	3
p_0	0.5	θ_0	1
m_0	0	$d\theta_0$	0
dr	0.01	$d\xi = dr/\alpha$	$\frac{0.01}{0.3355} \approx 0.0298$
$\rho_0 = Ap_0^{1/\gamma}$	$2(2)^{\frac{4}{3}} \approx 1.1892$	$\lambda = \rho_0$	$2(2)^{\frac{4}{3}} \approx 1.1892$
		$K = A^{-1/\gamma}$	$2^{-3/4} \approx 0.5946$
		$\alpha^2 = \frac{(n+1)K\lambda^{1/n-1}}{4\pi}$	≈ 0.1125



Pressure $p(r)$
p_{TOV}
p_{LE}
Density $\rho(r)$
ρ_{TOV}
$\bar{\rho}_{TOV}$
ρ_{LE}
$\bar{\rho}_{LE}$
Mass $m(r)$
m_{TOV}
m_{LE}
Mass radius ratio m/r
m_{TOV}/r
m_{LE}/r
0.0 0.5 1.0 1.5 2.0 2.5
0.0 0.1 0.2 0.3 0.4 0.5
0.00 0.25 0.50 0.75 1.00
0.0 0.2 0.4 0.6 0.8

Verifying Results

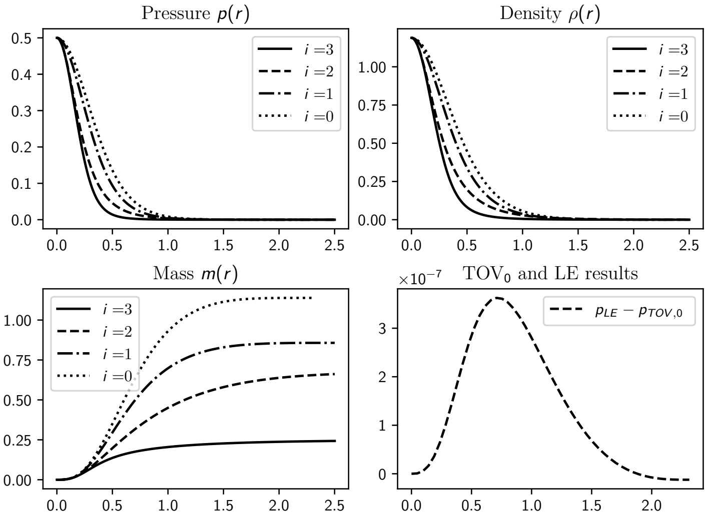


Comparing Partial TOV and LE results

- Next, reduce terms present in TOV equation

$$\frac{\partial p}{\partial r} = \underbrace{\underbrace{\underbrace{\underbrace{-\frac{Gm\rho}{r^2}}_{i=0}\left(1+\frac{p}{\rho c^2}\right)}_{i=1}\left(\frac{4\pi r^3 p}{mc^2}+1\right)}_{i=2}\left(1-\frac{2Gm}{rc^2}\right)^{-1}}_{i=3} \tag{19}$$

- We expect $i = 0$ to be the LE equation as derived earlier



Pressure $p(r)$
Density $\rho(r)$
Mass $m(r)$
$\times 10^{-7}$ TOV$_0$ and LE results
$i=3$
$i=2$
$i=1$
$i=0$
$p_{LE} - p_{TOV,0}$

Zero Values of TOV and LE equation

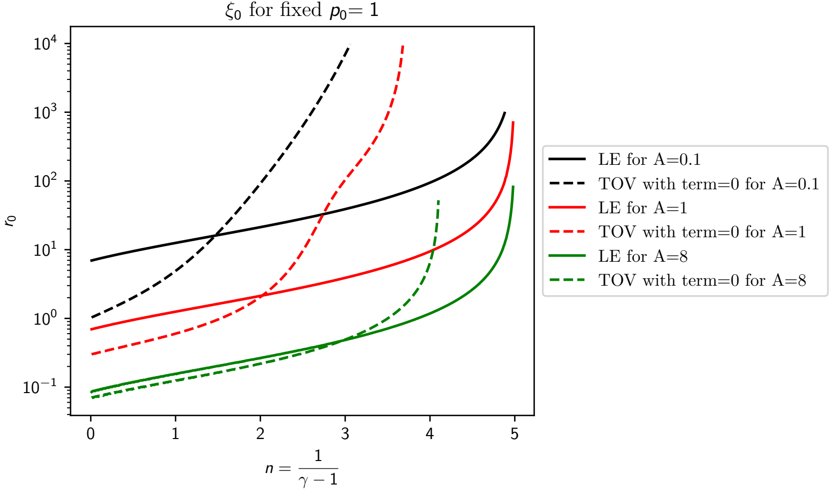


Figure: TOV and LE results for varying A parameter of $\rho = Ap^{n/(n+1)}$

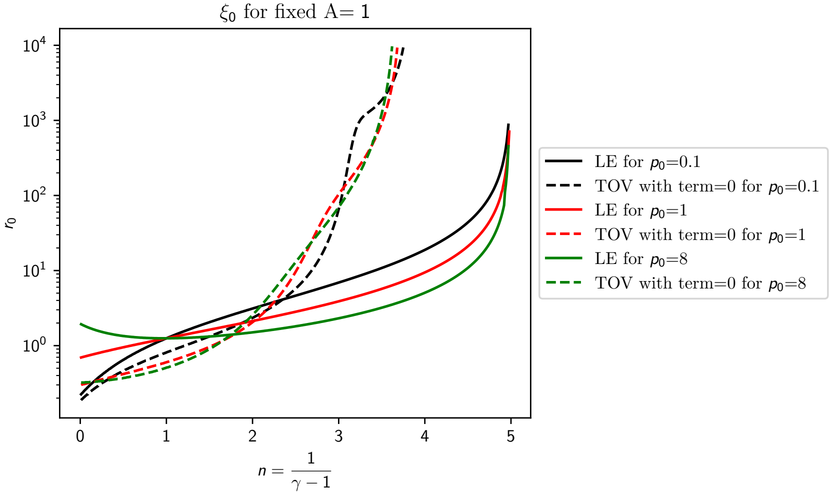


Figure: TOV and LE solutions for varying parameter p_0.

Intersection: $r = \beta\xi$ with $4\pi\beta A^{n/(n+1)} = (n+1)\lambda^{1-1/n}$ is independent of $\lambda = \rho(p_0)$.

Hypothesis

Theorem

To each combination $p_0, A > 0$ *there exists a* $n_0 \geq 0$ *such that each solution of the TOV differential equation*

$$\begin{aligned} \frac{\partial m}{\partial r} &= \quad 4\pi\rho r^2 \\ \frac{\partial p}{\partial r} &= -\frac{m\rho}{r^2}\left(1+\frac{p}{\rho}\right)\left(\frac{4\pi r^3 p}{m}+1\right)\left(1-\frac{2m}{r}\right)^{-1} \end{aligned}$$

with $\rho = Ap^{\frac{n}{n+1}}$ *where* $n \leq n_0$ *has* $p(\xi_0) = 0$ *for some* $\xi_0 \in \mathbb{R}_{>0}$.

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New Exact Solution for n=2

- Found new solution for $n = 2$ by using simple power-series $\theta = \sum a_m \xi^m$

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial \theta}{\partial \xi}\right) + \theta^2 = 0 \tag{20}$$

- Apply Cauchy-Product formula and combine

$$(m+1)\sum_{m=0}^{\infty}\left((m+2)a_{m+2}\xi^m + 2a_{m+1}\xi^{m-1} + \sum_{k=0}^{m} a_{m-k}a_k\xi^m\right) = 0 \tag{21}$$

Theorem

The odd coefficients a_{2m+1} vanish.

- Proof by induction. Start at $\partial\theta|_{\xi=0} = a_1 = 0$ is clear.

- Obtain recursive Formula for coefficients

$$a_{2m+2} = b_{m+1} = -\frac{1}{(m+2)(m+3)} \sum_{k=0}^{m} b_{m-k} b_k \tag{22}$$

Theorem

The series $\theta = \sum_{m=0}^{\infty} b_m \xi^{2m}$ converges for $\xi < 1$.

Proof.

We know that $a_0 = 1$. Use L'Hospital's rule for $\xi \to 0$

$$0 = \frac{2}{\xi}\frac{\partial\theta}{\partial\xi} + \frac{\partial^2\theta}{\partial\xi^2} + \theta^2 \longrightarrow 3\left.\frac{\partial^2\theta}{\partial\xi^2}\right|_{\xi=0} + \theta|_{\xi=0} = 0 \tag{23}$$

Thus $a_2 = \theta_0/6$. By induction, we get $|a_{m+2}| \leq \frac{m\theta_0^2}{(m+2)(m+3)}$. Now use quotient criterion. □

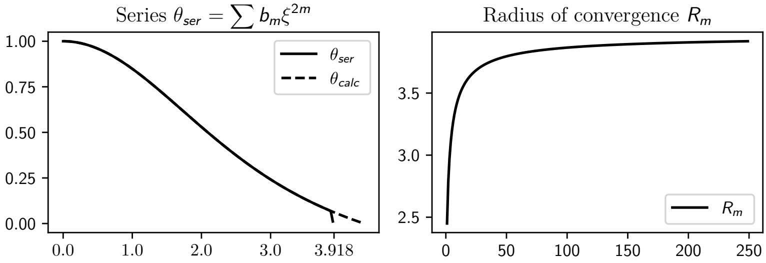

Figure: Exact solution θ and $(b_m)^{-1/(2m+2)}$ against m

Theorem

The radius of convergence R of the above calculated series is exactly the value at which $\theta(R) = 0$.

- Numerical problems: Calculation not very optimized, reaches floating point limit

Limiting Case TOV

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Outlook

- Plenty of information on numerical side
- Should be able to obtain exact solution for $n \in \mathbb{N}, n > 1$ analogously to $n = 2$ cases
- Solvability of LE equation is known

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