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GLOBAL BIFURCATION OF A NONLOCAL EQUATION

MA3911 - Master’s thesis presentation

Jonas Pedersen Vean

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Contents

Introduction

Some historical background, heavily abridged:

- John Scott Russell and the observed “wave of translation", 1834 [15]
- The Korteweg–de Vries (KdV) equation, used to model waves of small amplitude and long wavelength in a shallow regime, was introduced by Joseph V. Boussinesq in 1871 [4, 12] and proven to have solitary solutions by Diederik Korteweg and Gustav de Vries in 1895 [13]
- A modification to the KdV equation to full dispersion was proposed by Gerald B. Whitham in 1967 [16] and was suggested, among other properties, to exhibit ***wave peaking*** (as in e.g. Stokes’ conjecture [2])
- Whitham’s conjecture of a ***highest wave*** to the Whitham equation became of great interest in the following decades

Introduction (Cont.)

The Whitham equation and conjecture

In steady variables the Whitham equation reads as

$$-\mu\varphi + L\varphi + \varphi^2 = 0$$

where L is the Fourier multiplier associated with the Whitham kernel.

The Whitham conjecture can be roughly stated as
“the steady wave of greatest height with wave speed μ is cusped with height $\frac{\mu}{2}$ and $1/2$–Hölder regularity at the crest"

Introduction (Cont.)

Selected summary - development of the Whitham conjecture proof

Built on the theoretical foundations due to E. N. Dancer [7], B. Buffoni and J. F. Tolland [5], to name a few.

- 2009: *Travelling waves for the Whitham equation*, Ehrnström and Kalisch [9]
- 2013: *Global bifurcation of the Whitham equation*, Ehrnström and Kalisch [8]
- 2015 (Announced, Oberwolfach): *On Whitham's conjecture*, Ehrnström and Wahlén [10], published 2019

The equation at hand

We consider the dispersive, nonlinear PDE given by

$$u_t + Lu_x + N(u,u)_x = 0,$$

where in particular we will consider $L = \Lambda^s$ and $N(u,u) = u\,\Lambda^r u$ for the *Bessel potential operator* acting as a Fourier multiplier $\mathscr{F}(\Lambda^t \varphi)(\xi) = (1+\xi^2)^{\frac{t}{2}} \mathscr{F}\varphi(\xi)$. Furthermore we will mainly look at $r, s < 0$. Our equation therefore looks like

$$u_t + (\Lambda^s u)_x + (u\,\Lambda^r u)_x = 0. \tag{1}$$

Note that for $r = 0$ we recover the fractional Korteweg–de Vries (fKdV) equation as studied by e.g. Ørke [14], Afram [1], etc. The bilinear nonlinearity in this case makes for a prototype example of a *Coifman–Meyer type* nonlinearity, see [6].

Steady version, assumptions

We impose the travelling wave ansatz $\varphi(x - \mu t) = u(t, x)$ for $\mu \geq 0$ and integrate the previous equation to obtain

$$-\mu\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi = B$$

for some integration constant B. We *artificially* set $B = 0$ (since we lack a commuting Galilean transformation), which brings us to the bifurcation problem given by

$$F(\mu, \varphi) = -\mu\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi = 0.$$

We will look for solutions that are P-periodic, even and bounded under the *ad hoc* assumption $\Lambda^r\varphi < \mu$.

The map F

Let $\mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$ denote the space of even, P-periodic functions of Hölder–Zygmund class.

We collect some properties of the map F as in the bifurcation problem.

1. $F(\mu, \cdot)$ maps even, periodic functions to even and periodic functions (well-definiteness).
2. The map $F\colon \mathbb{R} \times \mathscr{C}^t_{\text{even}}(\mathbb{S}_P) \to \mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$ is real-analytic in both arguments (see e.g. Grigis–Sjöstrand [11])

Kernel of linearization for subcritical wave speeds

Proposition

The kernel of the Fréchet derivative $\partial_\varphi F[(\mu^*, 0)] = \Lambda^s(\cdot) - \mu^* \,\mathrm{Id}$ *is one-dimensional for* $0 < \mu^* < 1$, *and furthermore is spanned by*

$$\varphi^* = \cos\left(2\pi \cdot /P\right)$$

for $\mu^* = \mu_{P,1}$. *Additionally, we have that the transversality condition holds*

$$\partial^2_{\mu,\varphi} F[(\mu^*, 0)](1, \varphi^*) \notin \mathrm{ran}(\partial_\varphi F[(\mu^*, 0)]).$$

Further properties of the linearization

Lemma (Fredholmness)

Let $r, s < 0$. The Fréchet derivative

$$\partial_\varphi F[(\mu, \varphi)] = (\Lambda^r \varphi - \mu) \, \mathrm{Id} + \varphi \, \Lambda^r(\cdot) + \Lambda^s(\cdot)$$

is a Fredholm operator of index zero when $\varphi \in \mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$ satisfies $\Lambda^r \varphi < \mu$.

The proof relies on a corollary of the Fredholm alternative [5, Theorem 2.7.6] concerning compact perturbations of homeomorphisms. The operator Λ^s is invertible between appropriate Hölder–Zygmund spaces, hence a homeomorphism. It is shown that $\varphi \, \Lambda^r(\cdot)$ is compact on $\mathscr{C}^t_{\text{even}}(\mathbb{S}_P)$.

Existence of a local bifurcation curve

Theorem (Crandall–Rabinowitz, [5, Theorem 8.3.1])

Let $F\colon \mathbb{R} \times \mathscr{C}^{(\cdot)}_{\text{even}}(\mathbb{S}_P) \to \mathscr{C}^{(\cdot)}_{\text{even}}(\mathbb{S}_P)$ be real analytic. Suppose furthermore that

(i) $\partial_\varphi F[(\mu^*, 0)]$ *is a Fredholm operator of index zero,*

(ii) $\ker(\partial_\varphi F[(\mu^*, 0)])$ *is one-dimensional and furthermore is given by*

$$\ker(\partial_\varphi F[(\mu^*, 0)]) = \{\varphi \in \mathscr{C}^{(\cdot)}_{\text{even}}(\mathbb{S}_P) \mid \varphi = t\varphi^* \text{ for some } t \in \mathbb{R}\}$$

for a given $0 \neq \varphi^* \in \mathscr{C}^{(\cdot)}_{\text{even}}(\mathbb{S}_P)$,

(iii) *the transversality condition holds:*

$$\partial^2_{\mu,\varphi} F[(\mu^*, 0)](1, \varphi^*) \notin \operatorname{ran}(\partial_\varphi F[(\mu^*, 0)]).$$

Given (i)-(iii), then $(\mu^*, 0)$ is a bifurcation point. *(Continued on next slide)*

Existence of a local bifurcation curve (Cont.)

There exists $\varepsilon > 0$ and a local branch of solutions to $F(\mu, \varphi) = 0$ given by

$$\{(\mu, \varphi) = (\mu(t), t\,\chi(t)) \mid t \in \mathbb{R}, |t| < \varepsilon\} \subset \mathbb{R} \times \mathscr{C}^{(\cdot)}_{\text{even}}(\mathbb{S}_P) \tag{2}$$

such that $\mu(0) = \mu^*$, $\chi(0) = \varphi^*$, μ and χ are both analytic on $(-\varepsilon, \varepsilon)$. In addition to this branch of solutions, there exists an open set $V \subset \mathbb{R} \times \mathscr{C}^{(\cdot)}_{\text{even}}(\mathbb{S}_P)$ such that $(\mu^*, 0) \in V$ and

$$\{(\mu, \varphi) \in V \mid F(\mu, \varphi) = 0, \varphi \neq 0\} = \{\mu(s), t\,\chi(t)) \mid 0 < |t| < \varepsilon\}.$$

The preceding setup works assuming $r, s < 0$ and $0 < \mu^* < 1$.

Bifurcation formulae

Denote $m_t(\xi) = (1+\xi^2)^{\frac{t}{2}}$. If $\mu^* = \mu_{P,k} = m_s(2\pi k/P)$, then the local bifurcation curve $(\mu(t), \varphi(t))$ takes the expansions

$$\mu(t) = \sum_{n=0}^{\infty} \mu_{2n} t^{2n}, \quad \varphi(t) = \sum_{n=0} \varphi_n t^n$$

where $\mu_0 = \mu^*$, and the first two terms of the φ-series read as

$$\varphi_1 = \cos(\tfrac{2\pi k}{P} x),$$

$$\varphi_2 = -\frac{m_r(\frac{2\pi k}{P})}{2(m_s(0) - m_s(\frac{2\pi k}{P}))} - \frac{m_r(\frac{2\pi k}{P})}{2(m_s(\frac{4\pi k}{P}) - m_s(\frac{2\pi k}{P}))} \cos(\tfrac{4\pi k}{P} x).$$

These are found by comparing orders (essentially Lyapunov–Schmidt reduction).

Super-, sub- and transcritical bifurcations

From the parametrized curve $(\mu(t), \varphi(t))$ as in the previous slide one can show

$$\mu_2 = \frac{m_{2r}(\xi) + m_r(\xi)}{2(m_s(\xi) - m_s(0))} + \frac{m_r(\xi)(m_r(\xi) + m_r(2\xi))(\xi)}{4(m_s(\xi) - m_s(2\xi))}$$

with which one can prove that there exists $P_1 < P_2$ for which

$$\mu''_{P<kP_1,k}(0) > 0, \quad \mu''_{P>kP_2,k}(0) < 0$$

with super- and subcritical bifurcations respectively. At $\mu = 1$ we obtain a transcritical bifurcation intersecting the lines of constant solutions.

A priori estimates

Recall that we call φ_1 a *supersolution* of our equation given that

$$-\mu\varphi_1 + \Lambda^s\varphi_1 + \varphi_1\,\Lambda^r\varphi_1 \leq 0,$$

and likewise we call φ_2 a *subsolution* given that

$$-\mu\varphi_2 + \Lambda^s\varphi_2 + \varphi_2\,\Lambda^r\varphi_2 \geq 0.$$

Lemma (A priori estimates)

Let I_μ be the closed interval with endpoints $\mu - 1$ and 0. Then supersolutions φ_1 and subsolutions φ_2 both satisfy

$$\inf\varphi_1 \in I_\mu \quad \textit{and} \quad \sup\varphi_2 \notin \mathrm{int}(I_\mu).$$

Furthermore, if φ is a solution, then either $\mu - 1 \leq \inf\varphi \leq 0 \leq \sup\varphi$ or $\varphi \equiv \mu - 1$ for $\mu < 1$, or either $0 \leq \inf\varphi \leq \mu - 1 \leq \sup\varphi$ or $\varphi \equiv 0$ for $\mu \geq 1$.

Due to the nonlocal nonlinearity we split into 8 separate cases for ease of control, like for instance for the infimum of the supersolution φ_1

$$\mu - \Lambda^r \varphi_1 \leq 0, \quad \inf \varphi_1 \geq 0$$

where the left estimate is to be interpreted pointwise and not uniformly. In any case we have to make use of estimates like

$$\Lambda^t \varphi \leq \sup \varphi, \quad \Lambda^t \varphi \geq \inf \varphi.$$

In the case above we use this to achieve the form

$$(\inf \varphi_1 - (\mu - 1)) \inf \varphi_1 \leq 0 \tag{3}$$

from the supersolution equation, and thus draw conclusions.

Monotonicity properties for the Bessel potential operator

Lemma (Strict monotonicity)

Let $t < 0$. If two bounded and continuous functions f and g satisfy $f \gneq g$, then $\Lambda^t f > \Lambda^t g$ holds everywhere.

Lemma (Parity preservation under Λ^t, odd monotonicity)

The operator Λ^t for $t < 0$ is a parity-preserving operator for any period $P > 0$ (including the case $P = \infty$) and furthermore satisfies $\Lambda^t f(x) > 0$ for $x \in (-P/2, 0)$ where f is P-periodic, odd and continuous with $f \gneq 0$.

Touching lemma

Lemma (Touching lemma)

Let $r, s < 0$. Let φ_1, φ_2 be continuous, bounded solutions to our equation with $\varphi_1 \geq \varphi_2$ and $\Lambda^r \varphi_1 \leq \mu$. Then either

(i) $\varphi_1 = \varphi_2$, *or*

(ii) $\varphi_1 > \varphi_2$ *when* $\Lambda^r \varphi_1 + \Lambda^r \varphi_2 < 2\mu$ *for* $r = s$, *or*

(iii) $\varphi_1(x_0) > \varphi_2(x_0)$ *when* $\Lambda^r \varphi_1 + \Lambda^r \varphi_2 < 2\mu$ *whenever* $\varphi_1(x_0) \geq 0$ *for* $r \neq s$.

Note that in the case $r = 0$ one has the bound $\varphi_1 > \varphi_2$ when $\varphi_1 + \varphi_2 < \mu$ and $\varphi_1 \gneq \varphi_2$, which is strikingly different and much easier to prove.

Proof of the touching lemma

We take the differences of the equations for the solutions φ_1, φ_2 and cleverly rewrite to obtain

$$(2\mu - (\Lambda^r \varphi_1 + \Lambda^r \varphi_2))(\varphi_1 - \varphi_2) = 2\Lambda^s(\varphi_1 - \varphi_2) + (\varphi_1 + \varphi_2)\Lambda^r(\varphi_1 - \varphi_2).$$

Need to prove that the left-hand side is strictly positive, which is only a problem when both φ_1, φ_2 are both pointwise negative.

Note that by strict monotonicity we have $\Lambda^t(\varphi_1 - \varphi_2) > 0$ for $t \in \{r, s\}$.

In the case $r = s$ we can simply bound $\varphi_1 + \varphi_2 \geq 2(\mu - 1) > -2$ when $\mu > 0$, and for the case $r \neq s$ one can only achieve a partial result.

Touching lemma for derivatives

Lemma (Touching lemma for derivatives)

Let $r, s < 0$. Let φ_1, φ_2 be even and continuously differentiable solutions to our equation, where we impose $\varphi_1 \geq \varphi_2$ and $\varphi_1' \gneq \varphi_2' \geq 0$ on $(-P/2, 0)$. Then

(i) $\varphi_1' > \varphi_2'$ *when* $\Lambda^r \varphi_1 + \Lambda^r \varphi_2 < 2\mu$ *in* $(-P/2, 0)$ *for* $r = s$*,*

(ii) $\varphi_1'(x_0) > \varphi_2'(x_0)$ *when* $\Lambda^r \varphi_1 + \Lambda^r \varphi_2 < 2\mu$ *in* $(-P/2, 0)$ *whenever* $\varphi_1(x_0) \geq 0$ *for* $r \neq s$*.*

The proof now relies on the strict positivity of

$$(2\mu - 2\Lambda^s - (\Lambda^r \varphi_1 + \Lambda^r \varphi_2))(\varphi_1' - \varphi_2') = 2\varphi_1 \Lambda^r \varphi_1' - 2\varphi_2 \Lambda^r \varphi_2' + (\varphi_1' + \varphi_2')(\Lambda^r \varphi_1 - \Lambda^r \varphi_2)$$

which is proven similarly as for the other touching lemma.

Properties of the kernel

Note that $\Lambda^t \varphi = K^t * \varphi$ in the convolutional sense.

Corollary

Let $-1 < t < 0$. Then the kernel K^t on $\mathbb{R}$ has unit integral, is smooth on $\mathbb{R} \setminus \{0\}$, is even and positive, and there exist positive constants C_t and $\tilde{C}_t$ such that

$$\begin{cases} K^t(x) \lesssim_t e^{-|x|} & |x| \geq 1, \\ K^t(x) = C_t |x|^{-t-1} + H^t(x) & |x| < 1, \end{cases} \tag{4}$$

where the regular part H^t satisfies $H^t(x) = \tilde{C}_t + O(|x|^{-t+1})$ with derivatives satisfying

$$|\mathrm{D}_x H^t(x)| = O(|x|^{-t}), \quad |\mathrm{D}_x^2 H^t(x)| = O(|x|^{-t-1}). \tag{5}$$

Furthermore, if $0 < |x| \ll 1$ we have $\mathrm{D}_x K^t(x) \gtrsim_t |x|^{-t-2}$.

Nodal property theorem

Theorem (Nodal property theorem)

Let $P \in (0, \infty]$ and $r, s < 0$. Then a P-periodic, non-constant and even solution $\varphi \in C^1(\mathbb{R})$ to our equation that is non-decreasing on $(-P/2, 0)$ satisfies

$$\varphi' > 0 \text{ and } \Lambda^r \varphi < \mu \text{ on } (-P/2, 0)$$

for any period P when $r = s$, and whenever $x \in (\tilde{x}_{r,s}, 0)$ where $\tilde{x}_{r,s}$ has $0 > \varphi(\tilde{x}_{r,s})$ and $|\varphi(\tilde{x}_{r,s})|$ small in the case $r \neq s$. Furthermore, for a solution φ as above one necessarily has $\mu > 0$. Moreover, if $\varphi \in C^2(\mathbb{R})$ and $r = s$ with $-1 < s < 0$, then $\Lambda^r \varphi < \mu$ holds everywhere and

$$\varphi''(0) < 0.$$

Furthermore, for finite periods $P < \infty$ one has $\varphi''(\pm P/2) > 0$ when $r = s$.

On the proof of the nodal property theorem

In the $r = s$ case it is sufficient to note that

$$(\mu - \Lambda^r \varphi)\varphi' = \Lambda^s \varphi' + \varphi \, \Lambda^r \varphi' > \Lambda^s \varphi' - \Lambda^r \varphi'.$$

The convexity estimates follow from integral estimates using properties of the periodized kernels K_P^s and K_P^r along with a priori estimates as for the touching lemma with derivatives. For illustration, one would arrive at

$$(\mu - \Lambda^r \varphi(0))\varphi''(0) = -\lim_{\varepsilon \searrow 0} \int_\varepsilon^{P/2} (\mathrm{D}_y K_P^s(y) + \varphi(0)\mathrm{D}_y K_P^r(y))\varphi'(y)\,\mathrm{d}y < 0$$

at the very end of the proof for the $\varphi''(0) < 0$ estimate.

The periodized kernel

We can periodize the kernel K^t by

$$K_P^t(x) = \sum_{n\in\mathbb{Z}} K^t(x+nP).$$

So for P-periodic φ we write $\Lambda^t\varphi = K_P^t * \varphi$.

Corollary

The periodized kernel K_P^t is even, P-periodic and strictly increasing on $(-P/2, 0)$.

Using the periodization one can prove the following result:

Proposition

Let $r, s < 0$. If $\mu = 1$, then the corresponding integrable and even solution $\varphi \in L^1(\mathbb{S}_P)$ for any $P \in (0, \infty]$ has to be the zero solution.

Regularity result

Theorem (Regularity)

Let $\varphi \in L^\infty(\mathbb{R})$ be an even solution to our steady equation and let $r, s < 0$. Then:

(i) *If $\Lambda^r \varphi < \mu$ uniformly on all of $\mathbb{R}$, then $\varphi \in C^\infty(\mathbb{R})$,*

(ii) *φ is smooth on any open set where $\Lambda^r \varphi < \mu$.*

The proof of (i) is based off the paralinearization theorem of Bahouri *et al.* [3, Theorem 2.87] using the following maps for $r \leq s$ and $r \geq s$ respectively

$$[u \mapsto u\,(\mu - \Lambda^{r-s} u)^{-1}] \circ [\varphi \mapsto \Lambda^s \varphi] \colon B^t_{p,q}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \hookrightarrow B^{t-s}_{p,q}(\mathbb{R}),$$

$$[u \mapsto \Lambda^{s-r} u\,(\mu - u)^{-1}] \circ [\varphi \mapsto \Lambda^r \varphi] \colon B^t_{p,q}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \hookrightarrow B^{t-r}_{p,q}(\mathbb{R}).$$

Degeneracy result

A surprising and striking result, effectively guaranteeing the nonexistence of highest waves.

Theorem

Let $r, s < 0$. Assume that φ is an even, continuous solution to our steady equation that is nondecreasing on $(-P/2, 0)$ with $\Lambda^r \varphi < \mu$ on $(-P/2, 0)$ and $\Lambda^r \varphi(0) = \mu$. Then φ is identically equal to zero, and moreover μ has to be zero.

The proof is done by contradiction by assuming the solution is not constant. In particular, we only use the strict monotonicity of Λ^t and the local smoothness result of the regularity theorem, so no need for the sharpness of $r = s$.

Preamble to global bifurcation theorem

We define the set U by

$$U = \{(\mu, \varphi) \in \mathbb{R} \times \mathscr{C}^{\alpha}_{\text{even}}(\mathbb{S}_P) \mid \Lambda^r \varphi < \mu\},$$

and the solution set S as

$$S = \{(\mu, \varphi) \in U \mid F(\mu, \varphi) = 0\}.$$

We state the final result missing for the global bifurcation theorem.

Lemma (Compactness)

Let $r, s < 0$. Then bounded and closed sets of S are compact in $\mathbb{R}_{\geq 0} \times \mathscr{C}^{\alpha}_{\text{even}}(\mathbb{S}_P)$.

Global bifurcation theorem

Theorem (Global bifurcation, [5, Theorem 3.5.1])

Let $r, s < 0$. *Whenever* $\mu_{P,1}^{(j)}(0) \neq 0$ *for some* $j \in \mathbb{N}$ *for* $\mu_{P,1}(t)$ *as in the map* $t \mapsto (\mu_{P,1}(t), \varphi_{P,1}(t))$ *of solutions per our local bifurcation formulae, then these solution curves extend to continuous, global curves of solutions* $\mathfrak{R}_P \colon \mathbb{R}_{\geq 0} \to S$, *which are locally, real-analytically reparametrizable around every* $t > 0$. *Furthermore, one of the following alternatives are true:*

(i) $\|(\mu_{P,1}(t), \varphi_{P,1}(t))\|_{\mathbb{R}\times C^\alpha(\mathbb{S}_P)} \to \infty$ *as* $t \to \infty$.

(ii) $\operatorname{dist}(\mathfrak{R}_P, \partial U) = 0$.

(iii) *The map* $t \mapsto (\mu_{P,1}(t), \varphi_{P,1}(t))$ *is* T*-periodic for some finite* $T > 0$.

Bifurcation in a cone

Define the closed cone

$$\mathscr{K} = \{\varphi \in \mathscr{C}^{\alpha}_{\text{even}}(\mathbb{S}_P) \mid \varphi \text{ nondecreasing on } (-P/2, 0)\}$$

which we use in the sense of global bifurcations in cones as in Buffoni–Tolland [5, Theorem 9.2.2] to establish the following result

Theorem

Let $-1 < r = s < 0$. Then Item (iii) in the global bifurcation theorem cannot occur, hence excluding loop branches.

The proof relies, in particular, on the nodal property theorem, degeneracy of integrable solutions with $\mu = 1$, and the regularity theorem.

Conclusion

We have established local bifurcation formulas, touching lemmata, a nodal property theorem, and have used these to extend local bifurcation curves to global continua of smooth, non-degenerative P-periodic solutions (μ, φ) of $-\mu\varphi + \Lambda^s\varphi + \varphi\,\Lambda^r\varphi = 0$ whenever $-1 < r = s < 0$.

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