BIFURCATION OF WEAKLY DISPERSIVE PARTIAL DIFFERENTIAL EQUATIONS

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Introduction

Things that will be needed along the way:

(i) Aspects of local bifuraction theory and Banach space calculus (Kielhöfer and Buffoni–Toland respectively)

(ii) Some functional analysis, distribution theory and general information about function spaces like Hölder spaces and spaces of classical symbols $S^m_{1,0}(\mathbb{R})$.

With all of this background, we shall be able to prove existence of small-amplitude traveling solutions to the partial differential equation

$$\partial_t u + L\partial_x u + \partial_x(u^{p+1}) = 0, \quad p \in \mathbb{Z}_{\geq 1}$$

where L is a Fourier multiplier of a Bessel symbol $m(\xi) = (1+\xi^2)^{\frac{s}{2}}$ for $s < 0$.

Some Banach space calculus

Continuity of maps between Banach spaces:

Definition

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $U \subseteq X$ open. A map $F\colon U \to Y$ is called *continuous at* $x \in U$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $y \in Y$ with $\|x - y\|_X < \delta$ we have $\|F(x) - F(y)\|_Y < \varepsilon$. If F is continuous at each and every point $x \in U$ we simply call F continuous. In this case we may write $F \in C(U, Y)$ or $F \in C^0(U, Y)$.

Differentiability of maps of Banach spaces

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $U \subseteq X$ open. We say that a map $F\colon U \to Y$ is *Fréchet differentiable at* $x_0 \in U$ if there exists a linear map $A \in \mathcal{L}(X, Y)$ such that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - A(x - x_0)\|_Y}{\|x - x_0\|_X} = 0.$$

In this case we call A the Fréchet derivative of F at x_0 and write $A = \mathrm{d}F[x_0]$. If F is Fréchet differentiable at every point in X, then the map

$$\mathrm{d}F\colon X \to \mathcal{L}(X, Y)\,; x \mapsto \mathrm{d}F[x]$$

is well-defined and the evaluation $\mathrm{d}F[x_0](x)$ acts as a directional derivative of F at x_0 “along” the vector $x \in X$.

Partial derivatives of maps of Banach spaces

Definition

Let X, Y and Z be Banach spaces, $U \subseteq X \times Y$ be open in the product topology, and $F\colon U \to Z$ a function. Consider the projection maps $\pi_X(x, y) = x$, $\pi_Y(x, y) = y$, then set $U_{x_0} = \pi_X^{-1}(x_0) \cap U$ and $U_{y_0} = \pi_Y^{-1}(y_0) \cap U$ for $(x_0, y_0) \in U$. If $F(\cdot, y_0)$ has a Fréchet derivative at x_0 on U_{y_0} we denote it by $\partial_x F[(x_0, y_0)] \in \mathcal{L}(X, Z)$ and call it the *partial derivative* of F with respect to x at $(x_0, y_0) \in U$. Similarly for $y_0 \in U_{x_0}$, where F is Fréchet differentiable on U_{x_0} with $\partial_y F[(x_0, y_0)] \in \mathcal{L}(Y, Z)$.

A note on higher order derivatives

It is possible to define higher order derivatives multi-linearly on Banach spaces.

Definition

Let X and Y be Banach spaces, suppose that $F\colon U \to Y$, $U \subseteq X$ open, is continuously Fréchet differentiable on U. If $\mathrm{d}F\colon U \to \mathcal{L}(X, Y)$ is itself differentiable at $x_0 \in U$, then we say that the *second (order) Fréchet derivative* exists and is denoted by $\mathrm{d}(\mathrm{d}F)[x_0] \in \mathcal{L}(X, \mathcal{L}(X, Y))$. Higher k-order Fréchet derivatives are defined similarly when the previous order is defined and continuously differentiable, namely through a k-fold multilinear scheme: $\mathrm{d}(\mathrm{d} \cdots (\mathrm{d}F))[x_0] \in \mathcal{L}(X, \mathcal{L}(\cdots \mathcal{L}(X, Y)))$. A function that is k times continuously Fréchet differentiable on $U \subseteq X$ is said to be of class $C^k(U, Y)$.

Classifications of mappings

Throughout we will use some terms more commonly used in differential topology or similar fields.

Definition

Let X and Y be Banach spaces, $U \subseteq X$ open, $F\colon U \to Y$ a continuous function. The function F is called a *homeomorphism* if it is bijective and if F^{-1} is continuous on Y. Furthermore, if $F \in C^k(U, Y)$ is k times continuously Fréchet differentiable and bijective with $F^{-1} \in C^k(Y, U)$, then we say that F is a C^k-diffeomorphism.

Inverse Function Theorem

Theorem

Let X and Y be Banach spaces, $x_0 \in U$ be an open neighborhood of $U \subseteq X$ and let $F \in C^1(U, Y)$ such that the Fréchet derivative $\mathrm{d}F[x_0] \in \mathcal{L}(X, Y)$ is a homeomorphism. Then there exists a connected open set $\tilde{U} \subset U$ with $x_0 \in \tilde{U}$ such that $F|_{\tilde{U}} \colon \tilde{U} \to V$ for some $V \subseteq Y$ open with $F(x_0) \in V$ is a local C^1-diffeomorphism.

Remark

If one instead assumes $F \in C^k(U, Y)$, then F with the above assumptions becomes a local C^k-diffeomorphism.

Implicit Function Theorem

Theorem

Let X, Y and Z be Banach spaces and let $U \subseteq X \times Y$ be open in the product topology. Let $(X_0, y_0) \in U$. Assume $F\colon U \to Z$ is of class $F \in C^k(U, Z)$ such that $F(x_0, y_0) = 0$ and $\partial_x F[(x_0, y_0)] \in \mathcal{L}(X, Z)$ is a homeomorphism. Then there exists an open ball $B(y_0; r)$, $r > 0$, and a connected open set $V \subseteq U$ and a mapping $\phi \in C^k(B(y_0; r), X)$ such that

$$(x_0, y_0) \in V \text{ and } F(\phi(y), y) = 0 \text{ for all } y \in B(y_0; r).$$

Local Bifurcations

Our problem will go along the lines of the following:

- Want solutions $x \in X$, for a Banach space X, to $F(\lambda, x) = 0$ given that we know $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.
- Need to have a special kind of function F as in the problem above to say anything constructive about the behaviour of the solutions $x \in X$.

Definition

(Nonlinear Fredholm Operators)

Let X and Z be Banach spaces, $U \subset X$ open, $F\colon U \to Z$ Fréchet differentiable. Assume furthermore that $\mathrm{d}F[x]$, $x \in U$ satisfies

(i) $\dim\ker(\mathrm{d}F[x]) < \infty$, the kernel is finite dimensional

(ii) $\operatorname{codim}\operatorname{im}(\mathrm{d}F[x]) < \infty$

(iii) the image $\operatorname{im}(\mathrm{d}F[x])$ is closed in Z

then we call F a *nonlinear Fredholm operator* with *Fredholm index* given by the integer $\dim\ker(\mathrm{d}F[x]) - \operatorname{codim}\operatorname{im}(\mathrm{d}F[x])$.

Lyapunov–Schmidt Reduction

Considering the function $F\colon U \to Z$ for $U \subset X \times Y$ open, we consider the conditions $F(x_0, y_0) = 0$, $F \in C(U, Z)$ and $\partial_x F \in C(U, \mathcal{L}(X, Z))$. Furthermore, we assume that $F(\cdot, y_0)$ is a nonlinear Fredholm operator with respect to x for some $y_0 \in V$.

We may decompose the Banach spaces X and Z into

$$X = \ker(\partial_x F[(x_0, y_0)]) \oplus X_0 \quad \text{and} \quad Z = \mathrm{im}(\partial_x F[(x_0, y_0)]) \oplus Z_0.$$

We define projections $P\colon X \to \ker(\partial_x F[(x_0, y_0)])$ and $Q\colon Z \to Z_0$ in the natural way.

Lyapunov–Schmidt Reduction (cont.)

Theorem

(Lyapunov–Schmidt Reduction)
Let X, Y and Z be Banach spaces, $F\colon U \to Z$ as before with $U \subset X \times Y$ open, and P, Q projections onto $\ker(\partial_x F[(x_0, y_0)])$ and Z_0 respectively. Then there is an open neighborhood $\tilde{U}$ of (x_0, y_0) in $U \subset X \times Y$ such that our problem $F(x, y) = 0$ with $(x, y) \in \tilde{U}$ is equivalent to a finite-dimensional problem

$$\Phi(\xi, y) = 0 \qquad (\xi, y) \in U_0 \times V \subset \ker(\partial_x F[(x_0, y_0)]) \times Y$$

where $\Phi\colon U_0 \times V \to Z_0$ is continuous with $\Phi(\xi_0, y_0) = 0$.

Furthermore, we have that if $F\colon U \to Z$ has regularity $F \in C^k(U, Z)$, then for the function Φ we have $\Phi \in C^k(U_0 \times V, Z_0)$. Given this, we also have

$$\partial_\xi \Phi[(\xi_0, y_0)] = 0.$$

Notes on the proof of Lyapunov–Schmidt

- Define a function G based (cleverly) on the projection maps P and Q and our Fredholm operator F;
- show G is bilinear, continuous both ways and therefore a homeomorphism;
- due to the implicit function theorem on G we obtain our result.

The choice $G = (I - Q)F(Px + (I - P)x, y)$ happens to give us the resulting bifurcation function and solution curves as seen in the theorem.

The Crandall–Rabinowitz theorem

Theorem

Assume $F \in C^2(V \times U, Z)$ is a nonlinear Fredholm operator (satisfying the Lyapunov–Schmidt conditions) for $0 \in U \subset X$ and $\lambda_0 \in V \subset \mathbb{R}$ open, along with the normalized assumptions as outlined above. Furthermore, assume that

$$\ker(\partial_x F[(\lambda_0, 0)]) = \mathrm{span}\{v_0\}, \quad v_0 \in X, \quad \|v_0\|_X = 1$$

and that the second mixed partial derivatives commute and satisfy

$$\partial^2_{x\lambda} F[(\lambda_0, 0)] v_0 \notin \mathrm{im}(\partial_x F[(\lambda_0, 0)]).$$

Then there is a second, distinct solution curve $\gamma\colon (-\delta, \delta) \to V \times U$ through $\gamma(0) = (\lambda_0, 0)$ which is continuously differentiable and solves $F(\gamma(s)) = 0$ for all $s \in (-\delta, \delta)$. Finally, there are only two solutions intersecting at the bifurcation point $(\lambda_0, 0)$, namely the trivial solution line curve and γ as above.

The Korteweg–de Vries and Whitham equations

The Korteweg–de Vries (KdV) equation is a nonlinear PDE given by

$$\partial_t \eta + c_0\, \partial_x \eta + \frac{3}{2}\frac{c_0}{h_0} \eta\, \partial_x \eta + \frac{1}{6} c_0 h_0^2\, \partial_x^3 \eta = 0$$

where h_0, c_0 are constants determined by the physical constraints of the problem considered. A modified version of this equation was put forward by Gerald B. Whitham and remedies peaking and wave breaking behaviours of KdV, taking the form

$$\partial_t \eta + \frac{3}{2}\frac{c_0}{h_0} \eta\, \partial_x \eta + K_{\text{Whitham}} * \partial_x \eta = 0$$

where K_{Whitham} is a convolution kernel given by

$$K_{\text{Whitham}} = \mathcal{F}^{-1}\left(\sqrt{\frac{g\ \tanh h_0 \xi}{\xi}} \right).$$

Schwartz space, Distributions and Tempered distributions

Given what may be established about the Schwartz space, distributions and tempered distributions, consider these definitions:

Definition

(Fourier transform of tempered distributions)
Let $T \in \mathscr{S}'(\mathbb{R}^n)$. Then the Fourier transform of T denoted $\mathcal{F}\,T$ is defined formally by

$$\mathcal{F}\,T(\varphi) = T(\mathcal{F}\varphi), \quad \varphi \in \mathscr{S}(\mathbb{R}^n).$$

Definition

(Convolutions on Tempered Distributions)
Given $\psi \in \mathscr{S}(\mathbb{R}^n)$ and $f \in \mathscr{S}'(\mathbb{R}^n)$ we define the distribution $\psi * f$ by

$$\langle \psi * f, \varphi \rangle = \langle f, \tilde{\psi} * \varphi \rangle \quad \text{for } \varphi \in \mathscr{S}(\mathbb{R}^n)$$

where $\tilde{\psi}(x) = \psi(-x)$.

Hölder spaces

Definition

Let $\Omega \subseteq \mathbb{R}^n$ be open, and denote the space of bounded, continuous functions over Ω as $BC(\Omega)$, and likewise with $BC^k(\Omega)$ for k-times differentiable, bounded continuous functions. We say a function $f \in BC^k(\Omega)$ is *Hölder k-times continuously differentiable with exponent* $0 < \alpha \leq 1$ if each derivative of f up to order k has finite $C^{0,\alpha}$-norm given by

$$\|f\|_{C^{0,\alpha}(\Omega)} := \sup_{x \in \Omega} |f(x)| + \sup_{\substack{x,\, y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{\|x - y\|^\alpha}$$

$$[f]_\alpha := \sup_{\substack{x,\, y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{\|x - y\|^\alpha}.$$

Hölder spaces (cont.)

Furthermore, the norm of $C^{k,\alpha}(\Omega)$ is given by

$$\|f\|_{C^{k,\alpha}(\Omega)} = \sum_{|\beta|\leq k} \|\partial^\beta f\|_{BC(\Omega)} + \sum_{|\beta|=k} [\partial^\beta f]_\alpha.$$

The space of all Hölder continuous functions over Ω with exponent α is then the Hölder space
$C^{0,\alpha}(\Omega) = \{f \in BC(\Omega) \mid \|f\|_{C^{0,\alpha}(\Omega)} < \infty\}$.
This will be the main space we consider for bifuractions.
Exponents α strictly larger than 1 are not interesting for us.

Symbol classes

Definition

(Symbol Classes)
Let Ω be an open subset of $\mathbb{R}^n$. If $s \in \mathbb{R}$ and $0 \leq \delta < \rho \leq 1$ we let $S^s_{\rho,\delta}(\Omega \times \mathbb{R}^n)$ be the set of all functions $a(x, \xi)$ such that for any compact $K \subset \Omega$ and multi-indices α, β there exists constants $C_{K,\alpha,\beta} > 0$ such that for all $x \in K$ and $\xi \in \mathbb{R}^n$ one has

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{K,\alpha,\beta}(1 + |\xi|)^{s-\rho|\alpha|+\delta|\beta|}.$$

We call $S^s_{\rho,\delta}(\Omega \times \mathbb{R}^n)$ the *symbol class of order s*.
Of particular interest to us are the *classical symbols* given by $S^s_{1,0}(\mathbb{R}, \mathbb{R})$. An important family of such symbols are the *Bessel symbols* given by

$$m(\xi) = (1 + \xi^2)^{\frac{s}{2}}, \quad s \in \mathbb{R} \setminus \{0\}.$$

Back to the problem at hand

Our main focus will be the family of equations given by

$$\partial_t u + L\partial_x u + \partial_x(u^{p+1}) = 0, \quad p \in \mathbb{Z}_{\geq 2}.$$

Here, the Fourier multiplier L will be assumed to be a Bessel symbol on the Fourier side

$$m(\xi) = (1+\xi^2)^{\frac{s}{2}}, \quad s < 0.$$

This is a classical symbol. Note that it is also real and symmetric as a function.

We furthermore use the ansatz of traveling solutions $u(t,x) = \eta(x - ct)$ and get

$$-c\,\eta' + L\eta' + \eta^p\,\eta' = 0$$

which after integrating and normalizing becomes

$$-c\,\eta + L\eta + \eta^{p+1} = 0.$$

The wave-speed parameter $c > 0$ will be our bifurcation parameter in the analysis that follows.

Main theorem

Theorem

For a given $L > 0$ there exists a local bifurcation curve consisting of $2L$-periodic, even and continuous solutions to the weak normalized equation. Furthermore, owing to the dispersion relation $m(\xi)$ of the equation, the wave speed at the bifurcation point is given by

$$c^* = \left(1 + \frac{\pi^2}{L^2}\right)^{\frac{s}{2}}$$

where in particular as $L \to \infty$ one has $c^* \to 1$.

Crandall-Rabinowitz revisited

Let W be a Banach algebra, and let $c \in (0,1)$ be a parameter. Let $\mathcal{L}\colon W \to W$ be the Fréchet derivative at $0 \in W$ with respect to the function u of the map

$$\mathcal{J}\colon u \longmapsto -cu + Lu + u^{p+1}.$$

Suppose also that both $\mathcal{L}$ and $\partial_c \mathcal{L}$ exist and are continuous on and onto W, and that for some specific parameter $c^* \in (0,1)$ the following conditions hold:

(i) $\dim \ker(\mathcal{L}) = 1$;

(ii) $W = \ker(\mathcal{L}) \oplus \mathrm{im}(\mathcal{L})$;

(iii) $(\partial_c \mathcal{L}) \ker(\mathcal{L}) \cap \mathrm{im}(\mathcal{L}) = 0$.

Then there exists $\varepsilon > 0$ and a continuous bifurcation curve $\{(c_s, \phi_s) \mid |s| < \varepsilon\}$ with $c_s|_{s=0} = c^*$. Furthermore ϕ_0 is the vanishing solution of the normalized equation and $\{\phi_s\}_s$ are nontrivial solutions to the normalized equation with corresponding wave speeds $\{c_s\}_s$. In addition to all of this, we have for all solutions $\phi_s \in W$ that

$$\mathrm{dist}(\phi_s, \ker(\mathcal{L})) = o(s).$$

Notes on the proof

As soon as we show that the maps $\mathcal{L}$, $\partial_c \mathcal{L}$ have the listed properties and the existence of $c^* \in (0,1)$ are established, then the existence of $\{\phi_s\}_s$ is guaranteed immediately by Crandall-Rabinowitz as stated before.

Notes on the proof

Linearization of the normalized equation gives, assuming $L\psi = K * \psi$

$$\mathcal{L}\psi := \psi - \frac{1}{c} K * \psi = 0$$

where if $\psi \in L^\infty(\mathbb{R})$ we see that in the distributional sense we have

$$\hat{\psi} \left(1 - \frac{1}{c} m(\xi) \right) = 0.$$

Note that $\hat{\psi}$, $\frac{1}{c}\widehat{K * \psi}$ and $\frac{1}{c}\hat{K}$ all exist as tempered distributions in the space $\mathscr{S}'(\mathbb{R})$. Furthermore, one may establish that

$$\frac{1}{c}\widehat{K * \psi}(\varphi) = \frac{1}{c}(\hat{\psi}\hat{K})(\varphi), \qquad \text{for any } \varphi \in \mathscr{S}(\mathbb{R}).$$

as per the usual convolution theorem.

Notes on the proof

Given our equation as above, we start to examine whenever $\hat{\psi}$ vanishes. Given $c < 1$ we see that the equation

$$1 - \frac{1}{c}(1+\xi^2)^{\frac{s}{2}} = 0$$

has two solutions $\pm\xi_0$ since the Bessel function is in particular always decreasing and symmetric about $\xi = 0$. For $c = 1$ we have only one solution, namely $\xi = 0$. Lastly, for $c > 1$ we have no solutions to the above equation - which immediately implies that the distribution $\hat{\psi}(\varphi)$ has to vanish for all φ when $c > 1$. Then it turns out that the nontrivial solutions to the linearized equation are given by the functions

$$\begin{cases} \psi(x) = C, & c = 1, \\ \psi(x) = C\cos(\xi_0 x), & c < 1, \end{cases}$$

for constants $C \in \mathbb{R} \setminus \{0\}$.

Notes on the proof

We then see that in the case of $2L$-periodic and even solutions to our linearized equation we have

$$\dim \ker(\mathcal{L}) = 1 \text{ if and only if } \xi_0 = k\pi/L \text{ for } k \in \mathbb{Z}_{\geq 1}.$$

Now, choose the lowest mode of frequency $k = 1$ as above. This ensures uniqueness of c in the dispersion relation of our equation, and also allows us to establish the proposed c^* as in the theorem.

The rest of the proof involves looking at how the maps $\mathcal{L}$ and $\partial_c \mathcal{L}$ behave, which is unfortunately rather technical.

Generalizing to arbitrary classical symbols

Things of note:

- the dispersion relation $m(\xi)$ for the symbol has to be given explicitly in order to calculate c^*, thus existence of solutions have to be established per case
- given a regularizing classical symbol, there should be no problems regarding consistency of equations