AST3310 - Project 2: Modelling a star

Candidate number: 15015

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We have made a one dimensional model of a star where the luminosity, radius and mass go to zero as we reach the core. The star has a convection zone of width 16% the radius and a core reaching out to 11% of the radius.

INTRODUCTION

Observing a star directly can give information about parameters such as mass, surface temperature and luminosity. For stars that are close enough to us (i.e. the sun) we can even observe details on the stellar surface. There is a limit to how far into a star we can see however, and so if we want to know how the star looks on the inside we will have to use our knowledge of physics to model the subsurface conditions. A star can be described as a concentration of gas held together by gravity, and as the gravitational potential is a central symmetric potential it makes sense that most stars are very close to being spherical. Assuming spherical symmetry we can attempt to model a star as a one dimensional problem, i.e. as a function of radius/distance from the stellar centre. Since stars are concentrations of gas we are interested in how the pressure P, mass density ρ and temperature T change as a function of radius r. Though the pressure in a star can come from many sources, we will assume here that the only contributions to the total pressure is the gas pressure P_G from the gas itself and the radiation pressure P_{rad} from the photons produced by the fusion reactions in the star. These fusion reactions are responsible the energy output of the star, namely the luminosity L. The energy produced in the star is carried from the core to the surface through two main mechanisms; radiation and convection. Convection does not take place all over the star, but happens in regions where the radiative transport is too slow to transport all the energy by itself, making plumes of hot gas rise. These convective regions are important structures in a star and need to be implemented in our model. Since the stellar parameters we can observe are the surface parameters, we are going to model the star by starting at the stellar surface and moving step by step down into the stellar interior. To do this we need equations for how important variables we have mentioned change as we move inward. It turns out that these equations are more stable when we solve for a small change in mass as opposed to a change in radius. In this project we are going to use a set of governing equations that describe how the variables r,P,L and T change as we move into the interior of the star. We are going to attempt to find a model of star which has a core reaching out to at least 10% of the star's radius and has a convective zone near the surface of at least 15% of the star's radius. We also want r,m and L to all go to at least 5% of their values on the surface.

METHODS

Pressure and density

We have from the lecture notes [1] that the radiative pressure is a function of temperature T and is given by

$$P_{\text{rad}} = \frac{4\sigma}{3c}T^4 \tag{1}$$

where σ is the Stefan-Boltzmann constant and c denotes the speed of light. Since we are assuming an ideal gas we have the following expression for the gas pressure

$$P_G = \frac{k_B T}{\mu m_u}\rho \tag{2}$$

where k_B is Boltzmann's constant and m_u is the atomic mass unit. μ is the mean molecular weight in our star, and will be calculated later in this section. We see that the gas pressure is a function of mass density ρ. Since the total pressure is simply the sum of the pressure contributions we have

$$P(\rho, T) = P_G + P_{\text{rad}} = \frac{k_B T}{\mu m_u}\rho + \frac{4\sigma}{3c}T^4 \tag{3}$$

Doing some simple algebra we get the following expression for mass density

$$\rho(P, T) = \frac{\mu m_u}{k_B T}\left(P - \frac{4\sigma}{3c}T^4\right) = \frac{\mu m_u}{k_B T}P_G \tag{4}$$

The governing equations

The project text gives us the governing equations

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{5}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4 \rho} \tag{6}$$

$$\frac{\partial L}{\partial m} = \epsilon \tag{7}$$

$$\frac{\partial T}{\partial m} = \nabla \frac{T}{P}\frac{\partial P}{\partial m} \tag{8}$$

which describe how r, P, L and T change when we change the mass m. The equation (5) comes from our assumption that our star is a sphere, and (6) comes from the gravitational force from the mass inside radius r acting on the surface at that radius, G being the gravitational constant. Equation (7) uses our result form Project 1 where we found the luminosity per mass $\epsilon(\rho, T)$, and simply says that the luminosity of the mass shell dm is $dL = \epsilon dm$. Equation (8) comes form the definition of the temperature gradient

$$\nabla = \left(\frac{\partial \ln T}{\partial \ln P}\right) = \frac{P}{T}\frac{\partial T}{\partial P} = \frac{P}{T}\frac{\partial T}{\partial m}\frac{\partial m}{\partial P} \tag{9}$$

which we see can be rewritten to equation (8). If we know r, ρ, P, L and T at one mass shell we can use the governing equations (5),(6),(7) and (8) to calculate r, ρ, P, L and T at the next mass shell, and we can model our way into the star.

The mean molecular weight

To use the governing equations we need to know their contents. One value we don't yet know is the mean molecular weight μ, which is the mean mass of a particle given as a unit-less number by dividing with m_u. We will treat it like a constant, though in reality the mean molecular weight will vary with radius. To find μ we need to know what particles the star is composed of and their mass fractions. The lecture notes [1] gives the following formula for the mean molecular weight

$$\mu = \frac{1}{\sum(\text{particles per element} \times \text{mass fraction/nucleons})} \tag{10}$$

In our model we will assume all elements are completely ionised, so that the number of particles per element is the atom number (number of electrons) plus one (the element itself). For a star composed of Hydrogen ($^1_1$H), Helium-3 ($^3_2$He), Helium ($^4_2$He), Lithium ($^7_3$Li), Beryllium ($^7_4$Be) and Nitrogen ($^{14}_7$N) with the respective mass fractions $X, Y_{^3_2\mathrm{He}}, Y, Z_{^7_3\mathrm{Li}}, Z_{^7_4\mathrm{Be}}, Z_{^{14}_7\mathrm{N}}$ we have the following mean molecular weight

$$\frac{1}{\mu} = (1+1)X/1 + (2+1)Y_{^3_2\mathrm{He}}/3 + (2+1)Y/4 + (3+1)Z_{^7_3\mathrm{Li}}/7 + (4+1)Z_{^7_4\mathrm{Be}}/7 + (7+1)Z_{^{14}_7\mathrm{N}}/14$$

which gives a mean molecular weight of

$$\mu = \frac{1}{2X + Y_{^3_2\mathrm{He}} + 3Y/4 + 4Z_{^7_3\mathrm{Li}}/7 + 5Z_{^7_4\mathrm{Be}}/7 + 8Z_{^{14}_7\mathrm{N}}/14} \tag{11}$$

Temperature gradient

Another unknown in the governing equations is the temperature gradient ∇, which depends on what kind energy transport we have at a given point along the radius/mass. From the lecture notes [1] we have expressions for the radiative flux F_rad and the convective flux F_con

$$F_\mathrm{rad} = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla^* \tag{12}$$

$$F_\mathrm{con} = \rho c_P T\sqrt{g\delta}H_P^{-3/2}\left(\frac{l_m}{2}\right)^2(\nabla^* - \nabla_p)^{3/2} \tag{13}$$

where κ is the opacity, H_P is the scale height, g is the gravitational acceleration and c_P is the heat capacity at constant pressure. l_m is the mixing length and is a free parameter that can be written as $l_m = \alpha H_P$ where α lies between 1/2 and 2. δ is the differential $(-\partial \ln\rho/\partial \ln T)_P$. The ∇s are both temperature gradients, where ∇^* is the temperature gradient of the star itself and ∇_p is the temperature gradient of a rising parcel of hot gas. The total flux can be written as

$$F_\mathrm{rad} + F_\mathrm{con} = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla_\mathrm{stable} \tag{14}$$

where we can see that ∇_stable is the temperature gradient needed for all energy to be carried by radiation. Inserting (12) and (13) into (14) we get the following expression

$$c_P T\sqrt{g\delta}H_P^{-3/2}\left(\frac{l_m}{2}\right)^2(\nabla^* - \nabla_p)^{3/2} = \frac{16\sigma T^4}{3\kappa\rho H_P}(\nabla_\mathrm{stable} - \nabla^*)$$

rearranging so $\nabla^* - \nabla_p$ is alone of the left hand side we get

$$(\nabla^* - \nabla_p)^{3/2} = \frac{64\sigma T^3}{3\kappa\rho^2 c_P l_m^2}\sqrt{\frac{H_P}{g\delta}}(\nabla_\mathrm{stable} - \nabla^*) \tag{15}$$

From the lecture notes [1] we also have

$$(\nabla_p - \nabla_\mathrm{ad}) = \frac{64\sigma T^3}{3\kappa\rho^2 c_P l_m^2}\sqrt{\frac{H_P}{g\delta}}\left(\frac{S}{Ql_m d}\right)(\nabla^* - \nabla_p)^{1/2} \tag{16}$$

where the ∇_ad is the temperature gradient for a parcel of gas rising adiabatically. (S/Qd) is a geometric constant where S is the surface of the parcel, d is the diameter and Q is the surface normal to the velocity of the parcel. Assuming a spherical parcel with radius of half a mixing length we have $(S/Qd) = 4/l_m$. Defining

$$U = \frac{64\sigma T^3}{3\kappa\rho^2 c_P l_m^2}\sqrt{\frac{H_P}{g\delta}} \tag{17}$$

we can write $(\nabla_p - \nabla_{\mathrm{ad}})$ as

$$(\nabla_p - \nabla_{\mathrm{ad}}) = (\nabla^* - \nabla_{\mathrm{ad}}) - (\nabla^* - \nabla_p) \tag{18}$$

Defining $\xi = (\nabla^* - \nabla_p)^{1/2}$ and inserting (16) into (18) we get

$$U\frac{4}{l_m^2}\xi = (\nabla^* - \nabla_{\mathrm{ad}}) - \xi^2$$

rearranging gives us a quadratic equation for ξ

$$\xi^2 + U\frac{4}{l_m^2}\xi - (\nabla^* - \nabla_{\mathrm{ad}}) = 0 \tag{19}$$

which is a quadratic equation with the solution

$$\xi = -U\frac{2}{l_m^2} \pm \sqrt{U^2\frac{4}{l_m^4} + (\nabla^* - \nabla_{\mathrm{ad}})} \tag{20}$$

Since the temperature of a rising parcel has to decrease slower than its surroundings (or else it wouldn't rise), we have that $\nabla^* > \nabla_p$ as so $\xi > 0$, which mean only the positive solution in (20) is viable. Rewriting the quadratic equation in (19) we get an expression for ∇^*:

$$\nabla^* = \xi^2 + U\frac{4}{l_m^2}\xi + \nabla_{\mathrm{ad}} \tag{21}$$

which we can insert into equation (15) so we get

$$\begin{aligned}(\nabla^* - \nabla_p)^{3/2} &= \frac{64\sigma T^3}{3\kappa\rho^2 c_P l_m^2}\sqrt{\frac{H_P}{g\delta}}\,(\nabla_{\mathrm{stable}} - \nabla^*) \\ \xi^3 &= U/l_m^2\left(\nabla_{\mathrm{stable}} - \left(\xi^2 + U\frac{4}{l_m^2}\xi + \nabla_{\mathrm{ad}}\right)\right)\end{aligned}$$

which after some rearranging turns into the following third degree equation

$$\frac{l_m^2}{U}\xi^3 + \xi^2 + \frac{4U}{l_m^2}\xi - (\nabla_{\mathrm{stable}} - \nabla_{\mathrm{ad}}) = 0 \tag{22}$$

We see that if we can solve this equation we can find ∇^* through the expression in (21). We have some variables that we have not properly defined yet; the pressure scale height H_P is the distance over which the pressure drops by a factor $1/e$. It can be found through the following expression

$$H_P = -P\frac{\partial r}{\partial P} \tag{23}$$

We don't know how the pressure changes as a function of radius directly, but as we know the change as a function of mass we can rewrite the expression above to

$$H_P = -P\frac{\partial r}{\partial m}\frac{\partial m}{\partial P} = -P\frac{\partial r}{\partial m}\left(\frac{\partial P}{\partial m}\right)^{-1} = \frac{P}{\rho g} \tag{24}$$

where we use the definition of the gravitational acceleration $g = Gm/r^2$. As for the differential δ, it turns out that for the assumption of an ideal gas we have that $\delta = 1$. From doing some tasks in the lecture notes [1] we also find that the heat capacity at constant pressure for an ideal gas is $c_P = (5/2)k_B/\mu m_u$ which we see for our assumption of constant μ is a constant. The opacity κ we have to find by interpolating data from a pressure-temperature table. Now the only things we need to find the temperature gradient ∇^* is two of the other temperature gradients ∇_{stable} and ∇_{ad}. We have that ∇_{stable} is the temperature gradient if all energy is carried by radiation and is related to the total flux through equation (14). The total flux is also related to luminosity through the following relation

$$F_{\mathrm{rad}} + F_{\mathrm{con}} = \frac{L}{4\pi r^2} \tag{25}$$

Equating (14) and (25) we see that we can write ∇_{stable} as

$$\nabla_{\mathrm{stable}} = \frac{3L\kappa\rho H_P}{64\pi r^2\sigma T^4} = \frac{3L\kappa P}{64\pi G m\sigma T^4} \tag{26}$$

As for the adiabatic temperature gradient we find that for an ideal gas we have $\nabla_{\mathrm{ad}} = 2/5$. Now have all we need to find ∇^* is solve the third degree equation in (22), which can be done numerically. It is worth to note that the ∇^* found by solving the third degree equation is the temperature gradient when we have convection. When we don't have convection the temperature gradient is equal to ∇_{stable}. To know which one to use we need to have a criterion for when there is convection. From the lecture notes we have that this criterion is

$$\nabla > \nabla_{\mathrm{ad}} \tag{27}$$

which means that when the temperature gradient in our star is bigger than the adiabatic temperature gradient we have convection. Since the gradient in the sun is ∇_{stable} when there is no convection we have that the moment

$$\nabla_{\mathrm{stable}} > \nabla_{\mathrm{ad}} \tag{28}$$

our temperature gradient changes to the ∇^* found by solving the third degree equation in (22). We finally know everything we need to solve the governing equations. For $\nabla = \nabla_{\mathrm{stable}}$ we can even simplify the equation for T to

$$\frac{\partial T}{\partial m} = \nabla_{\text{stable}} \frac{T}{P} \frac{\partial P}{\partial m} = \nabla_{\text{stable}} \frac{T}{P} \frac{\partial P}{\partial r} \frac{\partial r}{\partial m} = -\nabla_{\text{stable}} \frac{T}{H_P} \frac{1}{4\pi r^2 \rho}$$

Inserting the expression for ∇_{stable} from equation (26) gives

$$\frac{\partial T}{\partial m} = -\frac{3\kappa L}{256\pi^2 r^4 \sigma T^3} \tag{29}$$

RESULTS

Testing parameters individually

The project text gives us a number of surface parameters to start with. For these parameters we get a star with cross section as in figure 1. In the following figures we have plotted how the cross section of the star looks if we change only one parameter. We have done this for the radius, surface temperature, surface density and surface pressure. In each of these cases we have one figure showing the cross section for a bigger value of the parameter and one for a smaller value. In figure 2 we have the cross section for changes in radius. In figure 3 we have the cross section for changes in surface temperature. In figure 4 we have the cross section for changes in surface density. Finally, in figure 5 we have the cross section for changes in surface pressure.

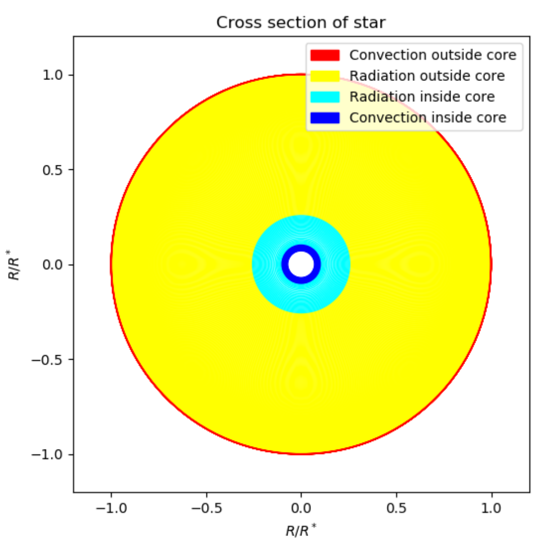
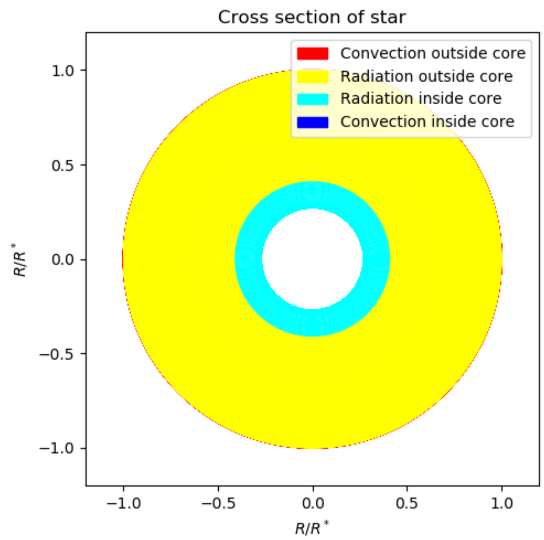
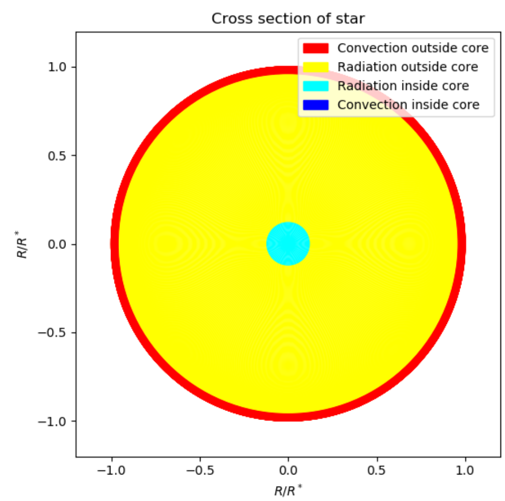


FIG. 1: Figure showing cross section of star with surface parameters as given in the project text. Upper convection zone has width of $0.01R$ and core starts at $0.26R$



(a) Star with radius $0.5R_\odot$. Upper convection zone has width of $< 0.01R$ and core starts at $0.41R$

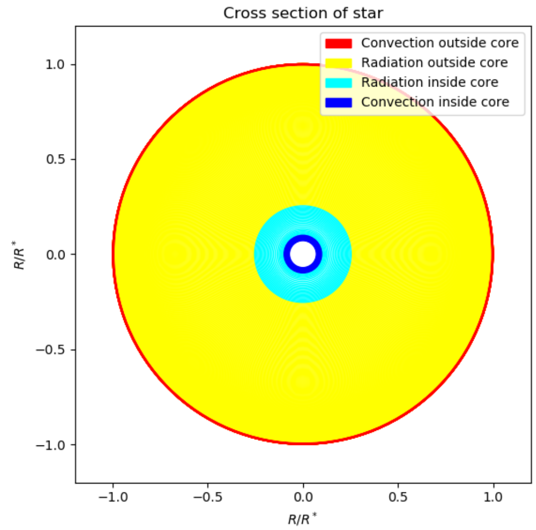


(b) Star with radius $2R_\odot$. Upper convection zone has width of $0.04R$ and core starts at $0.12R$

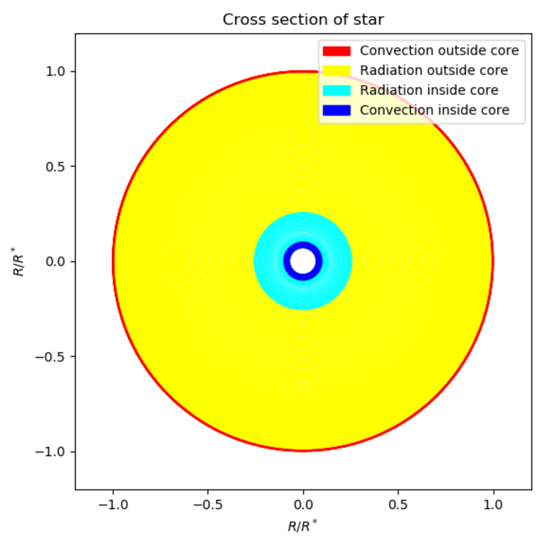
FIG. 2: Cross section of star, showing what happens when changing radius and holding all other initial parameters constant.

The best model

By adjusting just the radius and the surface temperature we have been able to find a model that fits the requirements in the project text. We have chosen a radius $R = 2R_\odot$ and a surface density $\rho_0 = 14.2 \cdot 10^{-7} \bar{\rho}_\odot$. This gives us a star with a cross section as shown in figure 6. Plotting the mass, temperature, luminosity, density and pressure as a function of radius for this star we get the plot in figure 7. Plotting the relative energy transportation of radiation and convection we get the plot in figure 8. Plotting the relative energy production from the different branches of the PP-chain and the CNO cycle we get the plot in figure 9. We have also plotted the temperature gradients as functions of radius, which is shown in figure 10.

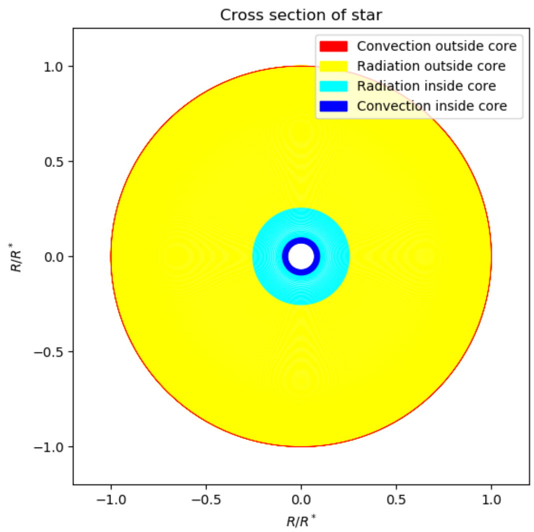


(a) Star with a surface temperature of 3000 K. Upper convection zone has width of $< 0.01R$ and core starts at $0.26R$

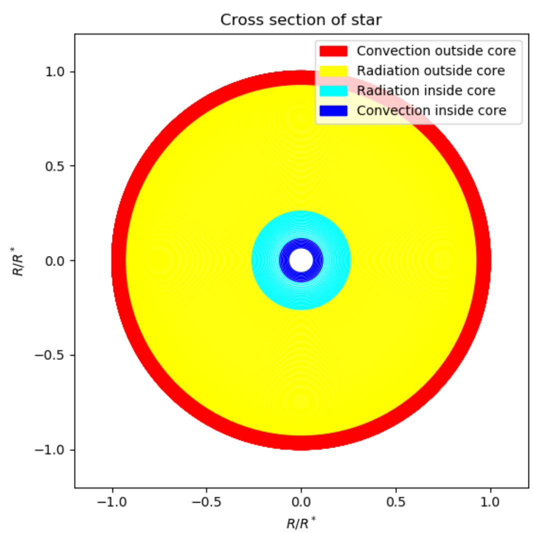


(b) Star with a surface temperature of 10000 K. Upper convection zone has width of $0.01R$ and core starts at $0.26R$

FIG. 3: Cross section of star, showing what happens when changing surface temperature and holding all other initial parameters constant

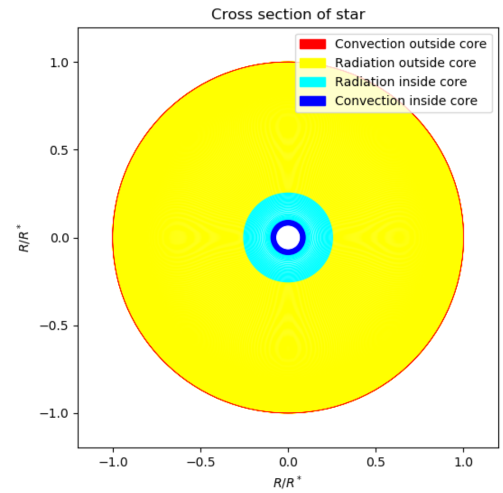


(a) Star with surface density $1.42 \cdot 10^{-7} \bar{\rho}_{\odot}$. Upper convection zone has width of $0.01R$ and core starts at $0.26R$

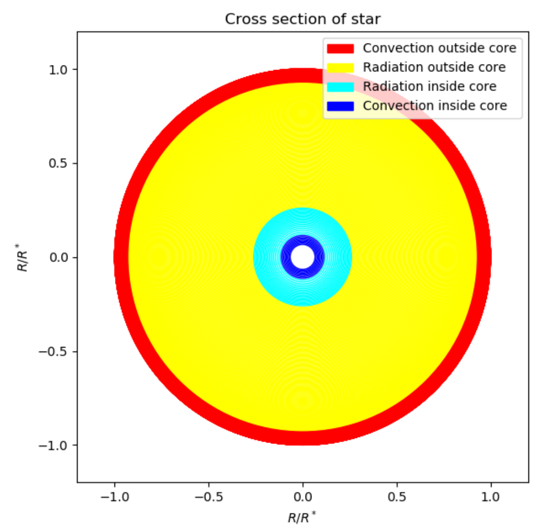


(b) Star with surface density $14.2 \cdot 10^{-7} \bar{\rho}_{\odot}$. Upper convection zone has width of $0.07R$ and core starts at $0.26R$

FIG. 4: Cross section of star, showing what happens when changing surface density and holding all other initial parameters constant



(a) Star with surface pressure $0.1P_0$. Upper convection zone has width of $0.01R$ and core starts at $0.26R$



(b) Star with surface pressure $10P_0$. Upper convection zone has width of $0.07R$ and core starts at $0.26R$

FIG. 5: Cross section of star, showing what happens when changing pressure and holding all other initial parameters constant. Here P_0 is the pressure for the initial parameters given in project text.

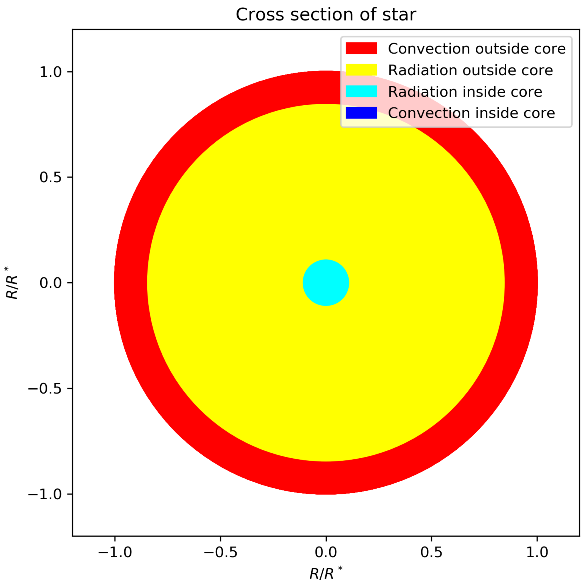


FIG. 6: Cross section with initial parameters as in the project text, except for $R = 2R_\odot$ and $\rho_0 = 14.2 \cdot 10^{-7}\bar{\rho}_\odot$. The convection zone has a width of $0.16R$ and the core starts at $0.11R$.

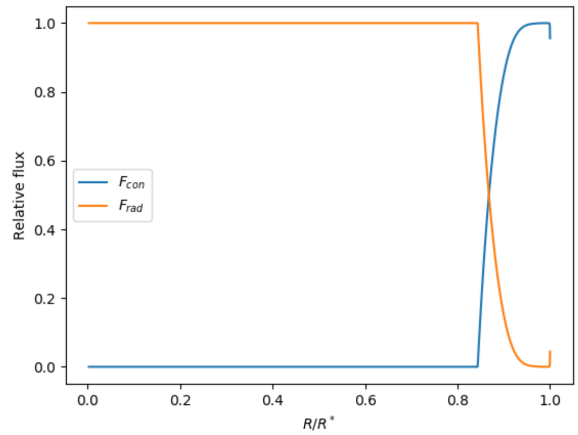


FIG. 8: Figure showing the relative energy transportation of convection and radiation as a function of radius

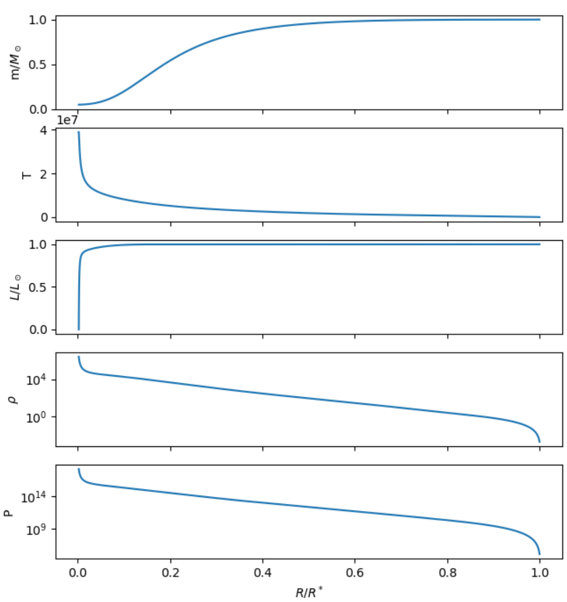


FIG. 7: Figure showing the main parameters m, T, L, ρ and P plotted as a function of radius

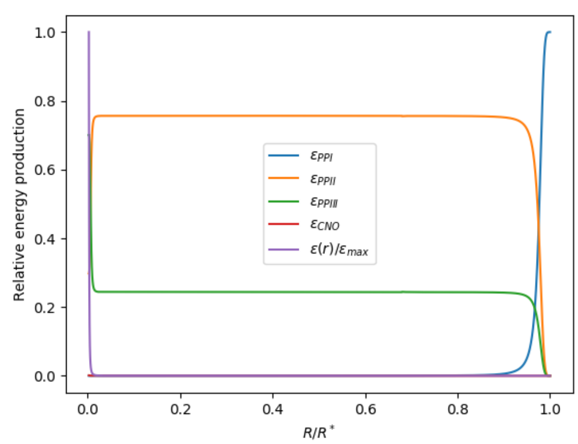


FIG. 9: Figure showing the relative energy production of branches in the PP-chain and the CNO cycle. Plot also includes $\epsilon(r)/\epsilon_{\max}$ which shows where most of the energy is produced.

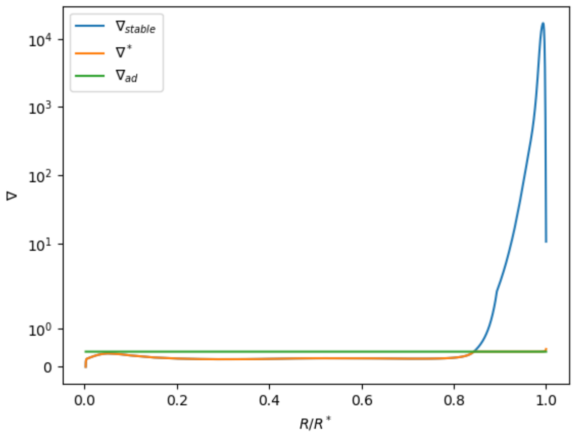


FIG. 10: Figure showing the temperature gradients plotted as a function of radius.

DISCUSSION

The star that results from the initial parameters given in the project text, of which we see a cross section in figure 1, has a outer convection zone of width $0.01R$ and a core that starts at $0.26R$. We have given the width of the convection zone and the size of the core in all the figure captions so as to be able to see which variables we had to change to get a star fitting the criteria of a convection zone width of at least $0.15R$ and a core reaching at least $0.10R$. From figure 2 we can see that decreasing the radius gives a bigger core, but also reduces the width of the outer convection zone. We are also unable to reach the centre of the star, likely due to the star being so dense that the luminosity is "used up" before we reach the centre of the star. Increasing the radius gives more positive results; though it reduces the size of the core we get a bigger convection zone. We also manage to model the star more or less the entire way to the star's centre, which is a positive. Changing the surface temperature seems to do very little about the structure of the star itself, as we can see in figure 3. Changing the surface density and changing the surface pressure gives very similar results, in fact so similar they are basically indistinguishable, as we can see in figures 4 and 5. This is probably because the pressure and density are related almost linearly, if not for the radiative pressure. This pressure is however very small as the fraction σ/c is very small, so the pressure and density are more or less proportional. In the figures we have changed both with factors of ten, so it makes sense that the cross sections are very similar. The result of lowering the surface density and pressure is that the convection zone gets smaller, which we don't want. Increasing the surface pressure density however gives a bigger convection zone and seemingly doesn't change the size of the core. The reason an increase in surface pressure or density widens the upper convection zone is probably because the temperature gradient ∇_{stable} is proportional to pressure (which we can see in equation (26), and so is also approximately proportional to density. Since we have convection if $\nabla_{\text{stable}} > \nabla_{\text{ad}} = 2/5$, increasing ∇_{stable} will make us more likely to get convection. We see that the two parameters changes that give us bigger convection zones are an increase in radius and an increase in surface density/pressure. Increasing the radius by a factor of two and the surface density by a factor of ten gives a star that fits the criteria given in the project text. Plotting the main parameters m, T, L, ρ and P we get the plot in figure 7 which shows that luminosity, mass and radius reach more or less zero, which mean that we have also managed to meet that criteria. We can see that most of the luminosity and mass are in the central parts of the star which fits with what we know about stars. In addition temperature increases exponentially when we reach the core while the pressure and density more or less increase exponentially the entire way down through the star. In our star, unlike the star created from the initial parameters given in the project text, we only have one convection zone near the surface of the star. From the plot of the relative energy flux in figure 8 we can see this; near the surface most energy is transported by convection but below the convection zone all energy is carried by radiation. Figure 9 shows the relative energy production by the different energy production chains. PPI is most active near the surface while PPII and PPIII take over when we move down into the star. At the end PPIII completely takes over. We have also plotted the total energy produced divided by the maximum energy production, which shows that basically all the energy is produced right in the centre of the star. Here, PPIII has taken over so it seems like most of the energy in the star is produced by the PPIII chain. The temperature never seems to reach a level where the CNO chain becomes active. Finally, plotting the temperature gradients in figure 10 shows that ∇_{stable} being big near the surface is what causes there to bee such a big convection zone. We could probably have experimented with changing more parameters, for example the mass and the luminosity. Maybe we could have have an even bigger convection zone, which would be cool.

CONCLUSION

We have managed to make a star that fits the criteria given in the project text. We have done this by changing around the surface parameters and figuring out which parameters gives the right sizes of the convection zone and the core. We only changed the surface density and the radius, but we probably could have changed more parameters such as mass or luminosity to get an even bigger convection zone if we had enough time.

SUMMARY

I found this project very entertaining, what I had trouble with is writing the report which always takes way too long for me to do. Solving the exercise I have also learned how to use classes in python to solve differential equations, which turns out to be pretty fun.

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[1] Gudiksen, B.V., 2020, Lecture Note *AST3310: Astrophysical plasma and stellar interiors* in the course AST3310