# AST3310 - Project 1: Modelling energy production

Candidate number: 15015
(Dated: February 21, 2021)

We have calculated the energy production is a star as a function of temperature and found that the total energy produced per second per kilo for the solar core is $\epsilon_{\text{tot}} = 2.50 \cdot 10^{-3}$ J/kg/s

## INTRODUCTION

Stars are concentrations of gas in space that radiate energy due to thermally ignited fusion reactions in their cores [1]. A fusion reaction happens when two atomic nuclei collide and fuse into a heavier element. For elements lighter than iron, the mass per nucleon in the nucleus is lower the heavier the element. This means that a fusion reaction results in a net loss in mass, where the mass lost is released as energy. In stars like our sun, the majority of the energy produced comes from the fusion of the lightest element (Hydrogen), into the next lightest element (Helium). There are many ways to get from Hydrogen to Helium and which paths the reactions take are dependent of temperature and pressure. All these paths have a net reaction of four Hydrogen nuclei into one Helium nuclei and thus they all release the same amount of energy $Q_{\text{tot}}$, where $Q_{\text{tot}} = 26.732$ MeV. However, depending on the path, different neutrinos may be produced. The energy carried by the neutrinos in practice completely escapes the star as neutrinos very rarely interact with matter. As a consequence the different paths will contribute with different amounts of energy to the star, even if their rates are the same. Their rates are not the same however, and different paths include reactions which are productive at different temperatures. In this report we will consider four different paths from Hydrogen to Helium, namely the three branches of the proton-proton chain and the dominant branch of the CNO cycle. The branches of the proton-proton chain all stem from two reactions, involving fusion of two protons into Deuterium, and then a proton and Deuterium into Helium-3. The CNO cycle on the other hand involves Carbon, Nitrogen and Oxygen in the production of Helium from Hydrogen. The CNO cycle only involves these heavier elements as a tool to produce Helium from Hydrogen, but does not produce or destroy them. Stars on the main sequence don't produce elements this heavy, and the presence of these elements in the sun comes from the fact that the sun is not a first generation star, and contains elements produced at the end of the life of stars from earlier generations. The reactions in the PP chain branches and the CNO cycle are productive at different temperatures and as a consequence some might dominate at different temperatures. In this report we are going to model the energy production for all the reactions in the PP chain and CNO cycle. We are going to compare energy rates for the PPI,PPII,PPIII and CNO cycles as the temperature varies, and look at for what energies the reactions involved in these branches take place.

## METHODS

We need to find the energy $Q$ produced for each reaction in the PP chain and CNO cycle. We can do this by finding the mass difference $\Delta m = m_{\text{in}} - m_{\text{out}}$ between the elements going in (reactants) and the elements going out (products) of a reaction, and converting mass to energy using the mass-energy equivalence

$$Q = \Delta m c^2 \tag{1}$$

where $c$ is the speed of light. To find the energy produced that contributes to the energy in the star $Q'$, we have to subtract the energy $Q_\nu$ that escapes with any neutrino that is potentially produced, so

$$Q' = Q - Q_\nu \tag{2}$$

It's worth to note here that this doesn't work individually for the last two reactions in PPIII because the energy they release is dependent on the electron excitation of the decaying atom, which is not always the same. This can be solved by calculating the energy production of the net mass input and output for both the reactions. To find the percentage of energy lost to neutrinos for each branch we can remember that each path releases the same amount of energy $Q_{\text{tot}}$, so the percentage lost to neutrinos is simply

$$P_{\nu,\text{lost}} = \frac{\Sigma Q_\nu}{Q_{\text{tot}}} = \frac{\sum Q_\nu}{26.732 \text{ MeV}}. \tag{3}$$

Having the energies, we need to know how often each reaction takes place. The rate of $r_{ik}$ is the amount of reactions per second per kilo between element $i$ and element $k$ and is given by the expression:

$$r_{ik} = \frac{n_i n_k}{\rho(1+\delta_{ik})}\lambda_{ik} \tag{4}$$

where $n_i$,$n_k$ are the number densities of the elements, $\rho$ is the mass density in the star and $\delta_{ik}$ is the Kronecker-delta function. $\lambda_{ik}$ is called the proportionality function and is a function of temperature only. This function can be formulated as a pretty complicated integral involving the energy distributions and cross sections of the elements $i$ and $k$, but for the purpose of this report we are going to use expressions for $\lambda_{ik}$ given to us in the lecture notes

[1]. $\lambda_{ik}$ has units $m^3/s$. The number densities can be found from the following expression:

$$n_i = \frac{\rho X_i}{A_i m_u} \tag{5}$$

where $X_i$ is the mass fraction and $A_i$ is the mass number of element $i$. $m_u$ is the atomic mass unit. The electron number density can be found by adding all the isotope densities times their atom number, since we assume all atoms are ionised. With what we have so far we can find the reaction rates for all the reaction. We do not need to find the rates for every single reaction however, as some rates are so fast that they are practically completely determined by the rate of the preceding reaction. An example is the two first reactions of the PP chain (PP0). The rate of the second reaction is so fast compared to the first that in practice any Deuterium produced will immediately fuse with Hydrogen into Helium-3. Thus the rate of the second reaction is equal to that of the first. This is also the case for the Beryllium decay in the PPIII chain, which happens very fast compared to the previous reaction. For the CNO cycle we only need the rate of the $p14$ reaction to calculate the energy produced by the entire cycle. We can do this because this rate is the limiting rate in the cycle. Since the CNO cycle recycles the elements involved in its reactions, and these elements are not produced in the sun, a slow rate at one point in the cycle would limit all the other reaction rates since the elements (Carbon, Nitrogen, Oxygen) simply wouldn't be present.
Some rates are not always much bigger than the rate of the previous reaction, but are bigger at some temperatures and smaller at others. In the cases where the rates are smaller, we can simply use the rates as given in expression 4 to calculate the energy production, but when they are bigger we run into the problem that they may use up more elements than are produced by the preceding reaction, which doesn't make sense. The solution is to make a test; if the following reaction rates take more elements than are produced in the preceding reaction, then the reaction rates has to be re-scaled so that they share the products from the previous reaction. This gives

$$r_{ik} \to F r_{ik} \tag{6}$$

where F is the scale factor, which can be written as

$$F = \frac{r_{\mathrm{p}}}{\sum_f n_f r_f} \tag{7}$$

where $r_{\mathrm{p}}$ is the rate of the previous reaction and the sum in the denominator is over all the following reactions, where $r_f$ is their rate and $n_f$ is how many previous reactions $p$ it takes to produce one following reaction $f$ (The reaction in PPI need two iterations of PP0 to occur, for example).

Now that we have the correct rates for all the reactions, we may use them to model the energy production of our four branches. For the CNO cycle the energy production is simply

$$\epsilon_{\mathrm{CNO}} = Q'_{\mathrm{CNO}} r_{\mathrm{p14}} \tag{8}$$

where $Q'_{\mathrm{CNO}}$ is the total energy delivered by the CNO cycle and $r_{\mathrm{p14}}$ is the limiting reaction rate, as mentioned earlier. In the following text we will use the reaction names from the lecture notes [1]. To find the energy production of the different branches in the PP chain we have to take into account that the branches share reactions. For the PPI branch we have to include the energy produced by the PP0 reactions which went into the PPI branch. Since each 33 reaction need two PP0 (pp & pd) reactions to occur, the energy produced is simply

$$\epsilon_{\mathrm{PPI}} = (2Q'_{\mathrm{PP0}} + Q'_{33}) r_{33} \tag{9}$$

Following the same principle we get

$$\epsilon_{\mathrm{PPII}} = (Q'_{\mathrm{PP0}} + Q'_{34}) r_{34} + Q'_{\mathrm{e7}} r_{\mathrm{e7}} + Q'_{17'} r_{17'} \tag{10}$$

$$\epsilon_{\mathrm{PPIII}} = (Q'_{\mathrm{PP0}} + Q'_{34}) r_{34} + (Q'_{17} + Q'_{\mathrm{decay}}) r_{17} \tag{11}$$

To get an idea of for what energies the different reactions take place, we can plot the normalised (area under curve equal to 1) Gamow Peaks of the interactions. The Gamow Peak for the reaction between two elements is the product of two exponentials, and is given by the following expression

$$\text{Gamow Peak} = \exp\left(-E/k_B T\right) \exp\left(-\sqrt{\frac{m_{ik}}{2E}} \frac{Z_i Z_k e^2 \pi}{\epsilon_0 h}\right) \tag{12}$$

where $E$ is the energy, $m_{ik} = m_i m_k/(m_i + m_k)$ is the reduced mass of the reactants and $Z_i, Z_k$ are the atom numbers of the reactants. The constants $k_B, e, \epsilon_0, h$ are in order: Boltzmann's constant, the elementary charge, the permittivity of free space and Planck's constant.
To find the total energy produced for a given density and temperature we can simply calculate the following sum:

$$\epsilon_{\mathrm{tot}} = \sum Q'_{ik} r_{ik} \tag{13}$$

## RESULTS

For all the results we have used the pressure of the solar core,

$$\rho = 1.62 \cdot 10^5 \mathrm{kg/m^3}. \tag{14}$$

| Reaction | $Q'$[MeV] | $Q_\nu$[MeV] |
|---|---|---|
| pp | 1.177 | 0.265 |
| pd | 5.494 | |
| 33 | 12.859 | |
| 34 | 1.586 | |
| e7 | 0.048 | 0.815 |
| 17' | 17.346 | |
| 17 | 0.137 | |
| decay | 11.361 | 6.711 |
| p12 | 1.943 | |
| 13 | 1.514 | 0.707 |
| p13 | 7.551 | |
| p14 | 7.296 | |
| 15 | 1.757 | 0.997 |
| p15 | 4.966 | |

TABLE I. Output energies and neutrino energies from reactions in PP chain and CNO cycle. Reaction names as in lecture notes [1], isotope masses used for calculations taken from [2]

| Branch | Energy lost to Neutrino |
|---|---|
| PPI | 1.98% |
| PPII | 4.04% |
| PPIII | 26.10% |
| CNO | 6.37% |

TABLE II. Energy percentage lost to neutrinos for all branches in PP chain, and CNO cycle.

By using equations 1&2 we can calculate the useful energy outputs of the reactions in the PP chain and the CNO cycle. In table I we have the output energies for all the reactions, as well as the energies of the neutrinos produced. In table II we have the percentage of energy lost to neutrinos for all the branches.
Using the expressions from equations 8,9,10&11 we can plot the relative energy production of all the branches for a temperature interval, as seen in figure 1.
Using the expression from equation 12 we can plot the Gamow peaks for all the reaction in the PP chain and CNO cycle for a given energy interval, which we can see in figure 2.
At last we have the total energy output per second per kilo for temperature $T = 1.57 \cdot 10^7$, calculated using equation 13:

$$\epsilon_{\text{tot}} = 2.50 \cdot 10^{-3} \text{ J/kg/s} \tag{15}$$

## DISCUSSION

Comparing the energy outputs from table I with the values in the lecture notes [1], we can see that on some values we are a few decimal points off. These differences are small enough that they're probably just the result of using a different amount of significant figures when calculating the $Q'$ values.

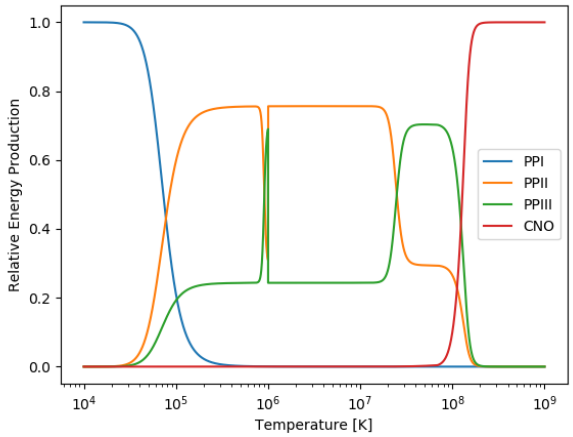


FIG. 1. Plot of relative energy production for all four branches, for $T \in [10^4, 10^9]$

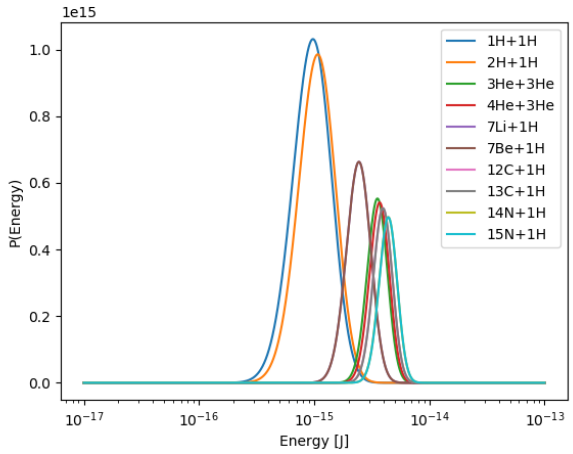


FIG. 2. Plot of normalised Gamow Peaks for the reactions in the PP chain and CNO cycle, for energies $E \in [10^{-17}, 10^{-13}]$

We know that all the branches produce the same net energy, but that some of this is lost to neutrinos which escape the sun. In table II we can see that for the PPIII branch, the energy lost to neutrinos account for more than one fourth of the total energy produced, showing that energy loss due to neutrinos can have a pretty significant impact on the energy production.
In figure 1 we have plotted the relative energy production of the branches for a range of temperatures. For temperatures lower than $T = 10^5$ K we can see that the PPI branch is the biggest contributor. For temperatures between $T = 10^5$ K and $T = 10^8$ K, the PPII and PPIII share the energy production, while for temperatures above $T = 10^8$ K the CNO completely takes over the energy production. Around $T = 10^6$ K there is some weird evolution where it looks like PPIII takes over for PPII as the biggest energy producer before very sud-

denly moving back to second place again. This is likely caused by the upper limit to Beryllium-7 electron capture in PPII, which is at work for $T < 10^6$ K (baked into the proportionality function). For temperatures below $10^6$ K, Beryllium-7 is not allowed to capture electrons at rates above a certain limit. What is happening in the plot is probably that Beryllium-7 reaches this limit a little before $T = 10^6$ K, which causes PPIII to take over energy production, as the rate of PPII no longer increases with temperature. Then when $T > 10^6$ K and the limit no longer applies, PPII's rates can increase with temperature again, which causes it to quick retake PPIII as the biggest energy producer.

The Gamow peak of two elements tells us about for which energies these elements are most likely to react. In the plot of the Gamow peaks in figure 2 we can see that of the reactions in the PP branches and the CNO cycle, the PP0 reactions take place at the lowest temperatures while the reaction in the CNO cycle take place at higher energies. This fits with the plott in figure 1, as we see that the PP chain is most productive for lower temperatures, while the CNO takes over for higher temperatures. We can see that not all reactions are included in the Gamow peak plot. This is because some reactions are decays of nuclei, and not reactions between elements, which means they do not have a Gamow peak.

The total energy produced per second per kilo, as show in 15 is more specifically the energy one kilogram of matter in the sun's core produces every second. To check if this value makes any kind of sense we can multiply is by the solar mass $M_\odot \approx 2 \cdot 10^{30}$ kg to get luminosity $L \approx 5 \cdot 10^{27}$ W, which is about one order of magnitude larger than the real solar luminosity, $L_\odot \approx 4 \cdot 10^{26}$ W. This makes some sense, considering not all the sun's mass has the same temperature and density. This means that with some idea of the temperature- and density distribution in the sun we could probably find a better estimate.

## CONCLUSION

We have attempted to calculate the energy production in a star as a function of temperature and found that the resulting plots seem to make sense. By using the functions to calculate the energy production per second per kilo for temperatures and densities like in the core of our sun, we found an estimate that also seems to make sense. What we have done in this report may be used to find a better model for a star's total energy output.

## SUMMARY

Doing this exercise I have refreshed my knowledge of classes in python, and learned that python dictionaries are pretty handy. I have also become more familiar with what is producing the energy in our sun and that the fusion reactions aren't just as straight forward as turning Hydrogen into Helium. Doing this exercise I had some trouble with calculating the energy outputs of each reaction and had some trouble with the electron-positron annihilation and how to calculate the energy gained. Turns out you somehow don't really need to take the electrons into account, because the difference in isotope masses (already including electron masses) somehow does everything for you.

## REFERENCES

1 Gudiksen, B.V., 2020, Lecture Note *AST3310: Astrophysical plasma and stellar interiors* in the course AST3310

2 Wang Meng et al 2017 *The AME2016 atomic mass evaluation (II). Tables, graphs and references* Chinese Phys. C 41 030003