A note on the Gaussian maximal function - Version 7 October 2013 + JvN additions

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ABSTRACT. This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded almost surely by the Gaussian Hardy-Littlewood maximal function. In particular this entails improvement on a result by Pineda and Urbina [3] who proved a similar result for a 'trunctated' version of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well known that the classical real-valued maximal function associated with the heat semigroup is bounded almost everywhere by the Hardy-Littlewood maximal function,

$$(1) \qquad \sup_{(y,t)\in\Gamma_x} |e^{-t\Delta}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u|\, d\lambda.$$

Here the action of *heat semigroup* $e^{-t^2\Delta}u = \rho_t * u$ is given by a convolution of u with the *heat kernel*

$$\rho_t(s) := \frac{e^{-|s|^2/4t}}{\pi^{\frac{d}{2}}} \frac{1}{(4t)^{\frac{d}{2}}}.$$

In this note we are interested in its gaussian counterpart. The change from Lebesgue measure to the gaussian measure

$$(2) \qquad d\gamma(x) := \pi^{-\frac{d}{2}} e^{-|x|^2}\, dx$$

introduces quite some intricate technical and conceptually difficulties which appears to be due to the fact that the Gaussian measure is non-doubling.

As an analogue to the Laplacian which is symmetric in L^2 with respect to the Lebesgue measure next we introduce the *Ornstein-Uhlenbeck* operator L which is symmetric with respect to the Gaussian measure:

$$(3) \qquad L := -\frac{1}{2}\Delta + \langle x, \nabla \rangle = \frac{1}{2}\nabla_\gamma^* \nabla_\gamma,$$

where ∇_γ denotes the realisation of the gradient in $L^2(\mathbf{R}^d, \gamma)$. Our main result, to be proved in (1), is the following gaussian analogue of (1):

$$(4) \qquad \sup_{(y,t)\in\Gamma_x^{(A,a)}} |e^{-tL}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u|\, d\gamma.$$

Here, $\Gamma_x^{(A,a)}$ is the Gaussian cone defined by

$$(5) \qquad \Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t)\in\mathbf{R}_+^d : |x-y| < At \text{ and } t \leqslant am(x)\},$$

where

(6)
$$m(x) := \min\left\{1, \frac{1}{|x|}\right\} = 1 \wedge \frac{1}{|x|}.$$

A slightly weaker version of the inequality (4) has been proved by Pineda and Urbina [3] who shows that

$$\sup_{(y,t) \in \widetilde{\Gamma}_x} |e^{-t\Delta} u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \, d\gamma,$$

where

$$\widetilde{\Gamma}_x(x) = \{(y,t) \in \mathbf{R}^d_+ : |x - y| < t \leqslant \widetilde{m}(x)\}$$

is the 'reduced' gaussian cone corresponding to the function

$$\widetilde{m}(x) = \min\left\{\frac{1}{2}, \frac{1}{|x|}\right\}.$$

Their proof does not seem to easily generalize the range of t from $\frac{1}{2}$ up to 1. The proof of (4) is different and, we believe, more transparent than the one presented in [3]. It has the further virtue of allowing the extension to the cones with arbitrary aperture $A > 0$ and cut-off parameter $a > 0$. This additional generality is very important and has already been used by Portal (cf. the claim made by [4, discussion preceding Lemma 2.3]) to prove the H^1-boundedness of the Riesz transform associated with L.

Before we continue, let us fix some notation. We will use without further reference notation such as $\mathbf{Z}^d$ while we implicitly imply that d is a positive integer. To avoid possible confusion, we define the *positive integers* as the set $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$.

1.0.1. *minimal function.* We recall the lemma from [1, lemma 2.3] which first –although implicitly– appeared in [2]. [1]

2. The Mehler kernel

2.1. Setting. Recall that we work with the *Ornstein-Uhlenbeck* operator L as given by (3).

We define the Mehler kernel (see e.g., Sjögren [5]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(e^{-tL})_{t \geqslant 0}$. More precisely, this means:

(7)
$$e^{-tL} u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot) u \, d\gamma.$$

It is often more convenient to use $e^{-t^2 L}$ instead of e^{-tL} as is done in e.g., Portal [4].

2.2. The Mehler kernel. For the calculation of the Mehler kernel M_t in (7) we refer to e.g., Sjögren [5] which additionally offers related results such as those related to Hermite polynomials.

The kernel M_t is invariant under the permutation $x \longleftrightarrow y$. A formula for M_t which honors this observation is:

(8)
$$M_t(x, y) = \frac{\exp\left(-e^{-2t} \frac{|x - y|^2}{1 - e^{-2t}}\right)}{(1 - e^{-t})^{\frac{d}{2}}} \frac{\exp\left(2e^{-t} \frac{\langle x, y \rangle}{1 + e^{-t}}\right)}{(1 + e^{-t})^{\frac{d}{2}}}.$$

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3. Some lemmata and definitions

We use m as defined in (6) in our next lemma.

1. LEMMA. *Let a, A be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

(1) If $|x - y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1 + aA)m(y)$;
(2) If $|x - y| < Am(x)$, then $m(x) \leqslant (1 + A)m(y)$ and $m(y) \leqslant 2(1 + A)m(x)$.

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. LEMMA. *Let $\alpha > 0$ and $|x - y| \leqslant \alpha m(x)$. We get the equivalence:*

$$e^{-\alpha^2(1+\alpha)^2} e^{-2\alpha(1+\alpha)} e^{-|y|^2} \leqslant e^{-|x|^2} \leqslant e^{\alpha^2} e^{2\alpha} e^{-|y|^2}.$$

PROOF. By the inverse triangle inequality and $m(x)|x| \leqslant 1$ we get,

$$(9) \qquad |y|^2 \leqslant (\alpha m(x) + |x|)^2 \leqslant \alpha^2 + 2\alpha + |x|^2.$$

For the reverse direction we use Lemma 1 to infer $m(x) \leqslant (1 + \alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant \alpha^2(1 + \alpha)^2 + 2\alpha(1 + \alpha) + |y|^2.$$

Combining we get:

$$(10) \qquad e^{-\alpha^2(1+\alpha)^2} e^{-2\alpha(1+\alpha)} e^{-|y|^2} \leqslant e^{-|x|^2} \leqslant e^{\alpha^2} e^{2\alpha} e^{-|y|^2}.$$

As required. ∎

4. On-diagonal estimates

4.1. Kernel estimates. We begin with a technical lemma which will be useful on several occasions.

3. LEMMA. *Let $t > 0$ and $\alpha \geqslant 1$. Then,*

$$(11) \qquad \alpha e^{\frac{t}{\alpha}} e^{-t} \leqslant \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} \leqslant \alpha,$$

$$(12) \qquad 0 \leqslant \frac{1}{t}\left[\frac{e^{-t}}{1 + e^{-t}} - \frac{e^{-\frac{t}{\alpha}}}{1 + e^{-\frac{t}{\alpha}}}\right] \leqslant \frac{1}{4}\left(1 - \frac{1}{\alpha}\right).$$

PROOF. We start with (11) and apply the mean value theorem to the function $f(\xi) = \xi^\alpha$. For $0 < \xi < \xi'$ this gives that:

$$f(\xi) - f(\xi') = \alpha \hat{\xi}^{\alpha-1}(\xi - \xi') \text{ for some } \hat{\xi} \text{ in } [\xi, \xi'].$$

Picking $\xi = 1$ and $\xi' = e^{-\frac{t}{\alpha}}$ yields:

$$(13) \qquad \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} = \alpha \hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[e^{-\frac{t^2}{\alpha}}, 1\right].$$

Combining this result with the monotonicity of $\xi \mapsto \alpha \xi^{\alpha-1}$ we obtain:

$$\alpha e^{\frac{t}{\alpha}} e^{-t} \leqslant \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} \leqslant \alpha,$$

where the last bound follows from the monotonicity together with the limit as $t \downarrow 0$. We proceed with (12). Recalling that $\alpha \geqslant 1$ one can directly verify that the function

$$\frac{1}{t}\left[\frac{1}{1 + e^{-t}} - \frac{1}{1 + e^{-\frac{t}{\alpha}}}\right]$$

is non-negative and decreasing in t. To find an upper bound we compute the limit as t goes to 0. That is:

$$\lim_{t\to 0}\frac{1}{t}\left[\frac{e^{-t}}{1+e^{-t}}-\frac{e^{-\frac{t}{\alpha}}}{1+e^{-\frac{t}{\alpha}}}\right]=\lim_{t\to 0}\left[\frac{e^{-2t}}{(1+e^{-t})^2}-\frac{1}{\alpha}\frac{e^{-2\frac{t}{\alpha}}}{(1+e^{-\frac{t}{\alpha}})^2}\right]\uparrow\frac{1}{4}\left(1-\frac{1}{\alpha}\right).$$

Which is as asserted and completes the proof. ∎

The following lemma will be useful when transfering estimates from $M_{\frac{t}{\alpha}}$ to M_t. It follows from the mean value theorem applied to $\xi\mapsto\xi^\alpha$.

4. LEMMA. *For $\alpha\geqslant 1$ and $0<t\leqslant T<\infty$ and all let $x,y\in\mathbf{R}^d$ we have that:*

(14)
$$\exp\left(-e^{-2\frac{t}{\alpha}}\frac{|x-y|^2}{1-e^{-2\frac{t}{\alpha}}}\right)\leqslant\exp\left(-\frac{\alpha}{2e^{2T}}\frac{|x-y|^2}{1-e^{-t}}\right).$$

PROOF. First note that
$$\frac{e^{-2t}}{2}\leqslant\frac{e^{-2t}}{1+e^{-t}}\leqslant\frac{1}{2}.$$

Therefore,
$$\exp\left(-e^{-2t}\frac{|x-y|^2}{1-e^{-2t}}\right)=\exp\left(-\frac{e^{-2t}}{1+e^{-t}}\frac{|x-y|^2}{1-e^{-t}}\right)$$
$$\leqslant\exp\left(-\frac{1}{2e^{2T}}\frac{|x-y|^2}{1-e^{-t}}\right).$$

Applying (11) we get:
$$\exp\left(-e^{-2\frac{t}{\alpha}}\frac{|x-y|^2}{1-e^{-2\frac{t}{\alpha}}}\right)=\exp\left(-\frac{e^{-2\frac{t}{\alpha}}}{1+e^{-\frac{t}{\alpha}}}\frac{1-e^{-t}}{1-e^{-\frac{t}{\alpha}}}\frac{|x-y|^2}{1-e^{-t}}\right)$$
$$\leqslant\exp\left(-\alpha\frac{1}{2e^{2\frac{t}{\alpha}}}\frac{e^{\frac{t}{\alpha}}}{e^t}\frac{|x-y|^2}{1-e^{-t}}\right)$$
$$\leqslant\exp\left(-\frac{\alpha}{2e^{2T}}\frac{|x-y|^2}{1-e^{-t}}\right).$$

Which is as asserted. ∎

Our first lemma is about estimating $M_{\frac{t^2}{\alpha}}$ in terms of M_{t^2}.

4.1.1. *Time-scaling of the Mehler kernel.*

5. LEMMA. *Let $T>0$, $\alpha\geqslant 1$, and $x,y\in\mathbf{R}^d$. Then:*

(15)
$$M_{\frac{t}{\alpha}}(x,y)\leqslant\alpha^{\frac{d}{2}}e^{\frac{t}{4}|\langle x,y\rangle|}\exp\left(-\frac{\alpha}{4e^{2T}}\frac{|x-y|^2}{1-e^{-t}}\right)M_t(x,y).$$

PROOF. To prove the lemma we compute $M_{\frac{t}{\alpha}}M_t^{-1}$. First note that (11) gives
$$\alpha^{\frac{d}{2}}e^{-\frac{d}{2}T}\leqslant\frac{(1-e^{-t})^{\frac{d}{2}}}{(1-e^{-\frac{t}{\alpha}})^{\frac{d}{2}}}\leqslant\alpha^{\frac{d}{2}}.$$

Combining the exponentials also gives,
$$\exp\left(-2e^{-\frac{t}{\alpha}}\frac{\langle x,y\rangle}{1+e^{-\frac{t}{\alpha}}}\right)\exp\left(2e^{-t}\frac{\langle x,y\rangle}{1+e^{-t}}\right)$$
$$=\exp\left(\frac{2}{t}\left[\frac{e^{-t}}{1+e^{-t}}-\frac{e^{-\frac{t}{\alpha}}}{1+e^{-\frac{t}{\alpha}}}\right]t\langle x,y\rangle\right)$$
$$\overset{(12)}{\leqslant}\exp\left(\frac{1}{2}\left(1-\frac{1}{\alpha}\right)t|\langle x,y\rangle|\right).$$

Combining Lemma 4 and equation (14) almost gives the final estimate.

$$\frac{M_{\frac{t}{\alpha}}(x,y)}{M_t(x,y)} \leqslant \alpha^{\frac{d}{2}} e^{\frac{t}{4}|\langle x,y\rangle|} \exp\left(e^{-2t}\frac{|x-y|^2}{1-e^{-2t}} \right) \exp\left(-e^{-2\frac{t}{\alpha}}\frac{|x-y|^2}{1-e^{-2\frac{t}{\alpha}}} \right)$$

$$\leqslant \alpha^{\frac{d}{2}} e^{\frac{t}{4}|\langle x,y\rangle|} \exp\left(\left[1 - \frac{\alpha}{4e^{2T}}\right]\frac{|x-y|^2}{1-e^{-t}} \right) \exp\left(-\frac{\alpha}{4e^{2T}}\frac{|x-y|^2}{1-e^{-t}} \right).$$

Finally, we apply the assumption $\alpha \geqslant 4e^{2T}$ to obtain:

$$\frac{M_{\frac{t}{\alpha}}(x,y)}{M_t(x,y)} \leqslant \alpha^{\frac{d}{2}} e^{\frac{t}{4}|\langle x,y\rangle|} \exp\left(-\frac{\alpha}{4e^{2T}}\frac{|x-y|^2}{1-e^{-t}} \right).$$

Which is as asserted. ∎

4.2. An estimate on Gaussian balls.

6. LEMMA. *Let $B_t(x)$ be the Euclidean ball with radius t and center x and let γ be the Gaussian measure* (2). *We have the inequality:*

$$(16) \qquad \frac{\gamma(B_t(x))}{V_d(t)} \leqslant d\pi^{-\frac{d}{2}} e^{-(t-|x|)^2}.$$

PROOF. Remark that for a ball $B := B_t(x)$ there holds that

$$\int_B e^{-|\xi|^2}\,d\xi = e^{-|x|^2}\int_B e^{-|\xi-x|^2} e^{-2\langle x,\xi-x\rangle}\,d\xi$$

$$\leqslant e^{-|x|^2}\int_B e^{-|\xi-x|^2} e^{2|x||\xi-x|}\,d\xi$$

$$\leqslant \pi^{\frac{d}{2}} e^{-|x|^2} e^{2t|x|}\gamma(B_t(0)).$$

That is:

$$(17) \qquad \gamma(B_t(x)) \leqslant e^{-|x|^2} e^{2t|x|}\gamma(B_t(0)).$$

We will estimate the Gaussian volume of the ball $B_t(0)$. To ease the notation, let S_d and V_d be the surface area and volume respectively of the d-dimensional unit sphere. Using polar coordinates we then obtain:

$$\gamma(B_t(0)) = \pi^{-\frac{d}{2}}\int_{B_t(0)} e^{-|\xi|^2}\,d\xi$$

$$= S_d\pi^{-\frac{d}{2}}\int_0^t e^{-r^2} r^{d-1}\,dr$$

$$\leqslant S_d t^d \pi^{-\frac{d}{2}} e^{-t^2}$$

$$= dV_d(t)\pi^{-\frac{d}{2}} e^{-t^2}.$$

Upon combining this result with (17) we obtain (16), which is as promised. ∎

4.3. On-diagonal kernel estimates on annuli.
As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli C_k. For the sake of simplicity we will write $B := B_t(x)$ and mean that B is the closed ball with center x and radius t. Furthermore, we use notations such as $2B$ to mean the ball obtained from B by multiplying its radius by 2.

The C_k are given by,

$$(18) \qquad C_k(B) := C_k = (2^{k+1}-1)B \setminus (2^k-1)B.$$

So, whenever ξ is in $C_k(B_{\sqrt{t}}(x))$, we get for $k \geqslant 0$:

$$(19) \qquad (2^k - 1)t < |y - \xi| \leqslant (2^{k+1} - 1)t.$$

7. LEMMA. *Given $A > 0$, let $B = B_{At}(y)$, $0 < t \leqslant T < \infty$ and $\xi \in C_k$. Then we have:*

$$M_t(y, \xi) \leqslant \frac{e^{-\beta} e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp\left((2^{k+1} - 1)At|y|\right) e^{\beta 2^{k+1}} e^{-\beta 4^k},$$

where $\beta = \frac{A^2}{2e^{2T}}$.

PROOF. Let $B = B_{At}(y)$ and let C_k be as in (18). Considering the first exponential which occurs in the Mehler kernel (8) together with (19) gives for $k \geqslant 0$:

$$\exp\left(-e^{-2t} \frac{|y - \xi|^2}{1 - e^{-2t}}\right) \leqslant \exp\left(-e^{-2t} \frac{(2^k - 1)^2 A^2 t}{1 - e^{-2t}}\right)$$

$$\overset{1}{\leqslant} \exp\left(-\frac{A^2}{2e^{2t}}(2^k - 1)^2\right).$$

Where (1) follows from

$$\frac{t}{1 - e^{-2t}} \geqslant \frac{1}{2}.$$

Before we consider the last exponential in the Mehler kernel we note that by Cauchy-Schwarz:

$$(20) \qquad |\langle y, \xi \rangle| \leqslant |\langle y - \xi, y \rangle| + |\langle y, y \rangle| \leqslant |y - \xi||y| + |y|^2.$$

Furthermore we have the estimate:

$$\frac{e^{-t}}{1 + e^{-t}} \leqslant \frac{1}{2},$$

Using these we get for the last exponential in the Mehler kernel (8):

$$\exp\left(2e^{-t} \frac{\langle y, \xi \rangle}{1 + e^{-t}}\right) \leqslant \exp(|\langle y, \xi \rangle|)$$

$$\overset{(20)}{\leqslant} \exp(|y - \xi||y|) e^{|y|^2}.$$

Wrapping it up, we can estimate the Mehler kernel (8) M_t on C_k from above by:

$$M_t(y, \xi) \leqslant \frac{e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp\left((2^{k+1} - 1)At|y|\right) \exp\left(-\frac{A^2}{2e^{2t}}(2^k - 1)^2\right).$$

Setting $\beta = \frac{A^2}{2e^{2T}}$ and expanding the last exponential we get:

$$M_t(y, \xi) \leqslant \frac{e^{-\beta} e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp\left((2^{k+1} - 1)At|y|\right) e^{\beta 2^{k+1}} e^{-\beta 4^k}.$$

Which is as claimed.

Lemma 2 gives us by using $|x - y| \leqslant \alpha t \leqslant \alpha^2 m(x)$ the following estimate:

$$e^{|y|^2} \leqslant e^{|x|^2} e^{\alpha^4} e^{2\alpha^2}$$

$$M_t(y, \xi) \leqslant \frac{e^{-\beta} e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp\left((2^{k+1} - 1)\alpha(1 + \alpha)\right) e^{\beta 2^{k+1}} e^{-\beta 4^k}$$

$$\leqslant e^{-(\alpha + \beta)} e^{\alpha^4} e^{\alpha^2} \frac{e^{|x|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp\left(2^{k+1} \alpha(1 + \alpha)\right) e^{\beta 2^{k+1}} e^{-\beta 4^k}.$$

Which is as claimed. ∎

5. The boundedness of some non-tangential maximal operators

Our theorem is a small modification of [3, lemma 1.1] with a new proof.

1. THEOREM. *Let $A, a > 0$. For all x in $\mathbf{R}^d$ and all u in L_γ^2 we have*

$$(21) \qquad \sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{-t^2 L} u(y)| \lesssim \sup_{r > 0} \fint_{B_r(x)} |u| \, d\gamma.$$

PROOF. First we note that $\Gamma_x^{(A,a)} \subset \Gamma_x^{(1+aA,aA)}$ as $a, A \geqslant 1$.

$$|x - y| \leqslant At \leqslant aAt$$
$$t \leqslant am(x) \leqslant aAm(x)$$

So if $y \in \Gamma_x^{(A,a)}$ then $x \in \Gamma_y^{(aA,aA)}$. So set $\alpha = aA$ and $\Gamma_x^\alpha = \Gamma_x^{(\alpha,\alpha)}$

We will prove (21) by splitting up the integration domain in annuli.

$$e^{-tL}|u(y)| \leqslant \sum_{k=0}^\infty I_k(y), \text{ where } I_k(y) := \int_{C_k(B)} M_t(y, \cdot)|u| \, d\gamma.$$

More precisely, we will set $B = B(y, aAt)$ in the above and find a suitable upper bound for each integral on the right-hand side which we will denote by I_k for the sake of simplicity.

$$M_t(y, \xi) \leqslant \frac{e^{-\beta} e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp\big((2^{k+1} - 1)\alpha t|y|\big) e^{\beta 2^{k+1}} e^{-\beta 4^k},$$

where $\beta = \frac{\alpha^2}{2e^{2\alpha^2}}$.

Since we have $|x - y| < \alpha t$ and $t \leqslant am(x)$ we infer that $t|x| \leqslant \alpha$. By Lemma 1 we also have that $t|y| \leqslant 1 + \alpha$. From this and Lemma 7 we infer that:

$$(22) \qquad M_t(y, \xi) \leqslant e^{-\beta} e^{-\alpha(1+\alpha)} \frac{e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp\big(2^{k+1}\alpha(1 + \alpha)\big) e^{\beta 2^{k+1}} e^{-\beta 4^k},$$

Setting $\beta = \frac{\alpha^2}{2e^{2\alpha^2}}$. Note that β is maximal for $\alpha = \frac{1}{2}$ and after this value, β is decreasing. Setting $\lambda := \alpha(1 + \alpha)$ we get:

$$(23) \qquad M_t(y, \xi) \lesssim_\alpha e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2} \frac{e^{|x|^2}}{(1 - e^{-t})^{\frac{d}{2}}} e^{(\lambda+\beta)2^{k+1}} e^{-\beta 4^k}.$$

Where the implied constant is given by $e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2}$

Or, using $\Lambda = \beta + \lambda$ we get:

$$(24) \qquad M_t(y, \xi) \lesssim_\alpha \frac{e^{|x|^2}}{(1 - e^{-t})^{\frac{d}{2}}} e^{\Lambda 2^{k+1}} e^{-\beta 4^k},$$

Recalling Lemma 6 we get:

$$(25) \qquad \gamma(B_t(x)) \leqslant V_d d \pi^{-\frac{d}{2}} t^d e^{-(t-|x|)^2}.$$

Where we abbreviate $V_d(1)$ with V_d. Recall

$$V_d \leqslant \frac{1}{\sqrt{\pi}} \left(\frac{2\pi e}{d}\right)^{\frac{d}{2}}.$$

To get,

$$(26) \qquad \gamma(B_t(x)) \leqslant \frac{d}{\sqrt{\pi}} \left(\frac{2e}{d}\right)^{\frac{d}{2}} t^d e^{-(t-|x|)^2} = C_d t^d e^{-(t-|x|)^2}.$$

This allows us to estimate the remaining unbounded exponential in the Mehler kernel and allow a penalty up to $e^{-|x|^2}$. Furthermore, we have the following estimate which will make clear how to handle the time part in the Mehler kernel:

$$\frac{t^d}{(1-e^{-t^2})^{\frac{d}{2}}} \leqslant \left(\frac{t^2}{1-e^{-t^2}}\right)^{\frac{d}{2}} \leqslant \frac{a^d}{(1-e^{-a^2})^{\frac{d}{2}}}.$$

Let $B' := B(x, 2^{k+1}\alpha t)$ and B as before the ball $B(y, \alpha t)$. In the next step we will bound .... by the maximal function centered at x. For this we need to scale up the C_k. So,

$$|x - \xi| \leqslant |x - y| + |\xi - y| \leqslant \alpha t + (2^{k+1} - 1)\alpha t = 2^{k+1}\alpha t.$$

And set $D_k = B(2^{k+1}\alpha t)$. So, we can bound the integral on the right-hand side of (**??**) by

$$\int_{C_k(B)} M_{t^2}(y, \cdot)|u| \, d\gamma \lesssim_\alpha \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1-e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} \int_{C_k(B)} |u| \, d\gamma$$

$$\leqslant \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1-e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} \int_{D_k(B)} |u| \, d\gamma$$

$$\leqslant (M_\gamma u)(x) \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1-e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} \gamma(D_k)$$

$$\overset{(1)}{\leqslant} (M_\gamma u)(x) C_d \alpha^d 2^{d(k+1)} t^d \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1-e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} e^{-(t-|x|)^2}$$

$$\leqslant (M_\gamma u)(x) 2^{kd} e^{\Lambda 2^{k+1}} e^{-\beta 4^k} \frac{t^d e^{-t^2}}{(1-e^{-t^2})^{\frac{d}{2}}} C_d e^{2\alpha} (2\alpha)^d$$

Where (1) uses Lemma 6 and $t|x| \leqslant a$.
We can then bound the maximal function:

$$e^{-t^2 L}|u(y)| = \sum_{k=0}^\infty I_k$$

$$\leqslant (M_\gamma u)(x) C_{d,a,A} \sum_{k=0}^\infty 2^{kd} e^{\Lambda 2^{k+1}} e^{-\beta 4^k}$$

Wrapping it up, we have that:

$$e^{-t^2 L}|u(y)| \lesssim \fint_{B_r(x)} |u| \, d\gamma.$$

With implied constant
   Which is what we wanted to prove.

$$\sum_{k=0}^\infty 2^{kd} e^{-C4^k} = \sum_{k=0}^\infty x^{kd} e^{-Cx^{2k}}$$

Noting for $x \geqslant 1$ that $\exp(-Cx^{2k}) \leqslant \exp(-Ckx^2)$, thus,

$$\sum_{k=0}^\infty 2^{kd} e^{-C4^k} \leqslant \sum_{k=0}^\infty x^{kd} (e^{-Cx^2})^k = \sum_{k=0}^\infty (x^d e^{-Cx^2})^k$$

Here $x = 2$, so

$$\sum_{k=0}^\infty 2^{kd} e^{-C4^k} \leqslant \sum_{k=0}^\infty (2^d e^{-4C})^k$$

If $2^d < e^{4C}$, that is whenever $d \log 2 < 4C$, we can compute using the geometric series that

$$\sum_{k=0}^{\infty} 2^{kd} e^{-C4^k} \leqslant \frac{1}{1 - 2^d e^{-4C}} = \frac{e^{4C}}{e^{4C} - 2^d}$$

∎

References

[1]   Jan Maas, Jan Neerven, and Pierre Portal. "Whitney coverings and the tent spaces T 1,q (γ) for the Gaussian measure". In: *Arkiv för Matematik* 50.2 (Apr. 2011), pp. 379–395. arXiv: 1002.4911.

[2]   Giancarlo Mauceri and Stefano Meda. "BMO and H^1 for the Ornstein-Uhlenbeck operator". In: *Journal of Functional Analysis* 252.1 (Nov. 2007), pp. 278–313.

[3]   Ebner Pineda, W. Urbina, and Wilfredo Urbina R. "Non Tangential Convergence for the Ornstein-Uhlenbeck Semigroup". In: *Divulgaciones Matemáticas* 13.2 (2008), pp. 1–19.

[4]   Pierre Portal. "Maximal and quadratic Gaussian Hardy spaces". In: (Mar. 2012). arXiv: 1203.1998.

[5]   Peter Sjögren. "Operators associated with the hermite semigroup — A survey". In: *The Journal of Fourier Analysis and Applications* 3.S1 (Jan. 1997), pp. 813–823.

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