# A NOTE ON THE GAUSSIAN MAXIMAL FUNCTION

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Abstract. This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded almost surely by the Gaussian Hardy-Littlewood maximal function. In particular this entails improvement on a result by Pineda and Urbina [2] who proved a similar result for a 'trunctated' version of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

## 1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well-known that the classical non-tangential maximal function associated with the heat semigroup is bounded almost everywhere by the Hardy-Littlewood maximal function,

$$\sup_{\substack{(y,t)\in\mathbf{R}^{d+1}_+\\ |x-y|<t}} |\mathrm{e}^{-t\Delta}u(y)| \lesssim \sup_{r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)}|u|\,\mathrm{d}\lambda. \tag{1}$$

Here the action of *heat semigroup* $\mathrm{e}^{-t\Delta}u = \rho_t * u$ is given by a convolution of $u$ with the *heat kernel*

$$\rho_t(\xi) := \frac{\mathrm{e}^{-|\xi|^2/4t}}{(4\pi t)^{\frac{d}{2}}}, \text{ with } t>0 \text{ and } \xi\in\mathbf{R}^d.$$

In this note we are interested in its Gaussian counterpart. The change from Lebesgue measure to the *Gaussian measure*

$$\mathrm{d}\gamma(x) := \pi^{-\frac{d}{2}}\mathrm{e}^{-|x|^2}\,\mathrm{d}x \tag{2}$$

introduces quite some intricate technical and conceptual difficulties which are due to its non-doubling nature. Instead of the Laplacian, will use its Gaussian analogue, the *Ornstein-Uhlenbeck operator* $L$ which is given by,

$$L := -\frac{1}{2}\Delta + \langle x,\nabla\rangle = \frac{1}{2}\nabla_\gamma^*\nabla_\gamma, \tag{3}$$

where $\nabla_\gamma$ denotes the realisation of the gradient in $L^2(\mathbf{R}^d,\gamma)$. Our main result, to be proved in Theorem 1, is the following Gaussian analogue of (1):

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |\mathrm{e}^{-t^2L}u(y)| \lesssim \sup_{r>0}\frac{1}{\gamma(B_r(x))}\int_{B_r(x)}|u|\,\mathrm{d}\gamma. \tag{4}$$

Here,

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t)\in\mathbf{R}^{d+1}_+ \,:\, |x-y|<At \text{ and } t\leqslant am(x)\} \tag{5}$$

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is the *Gaussian cone* with aperture $A$ and cut-off parameter $a$, and

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\}. \tag{6}$$

A slighly weaker version of the inequality (4) has been proved by Pineda and Urbina [2] who showed that

$$\sup_{(y,t)\in\widetilde{\Gamma}_x} |\mathrm{e}^{-t^2L}u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))}\int_{B_r(x)} |u| \,\mathrm{d}\gamma,$$

where

$$\widetilde{\Gamma}_x(x) = \{(y,t) \in \mathbf{R}^d_+ : |x-y| < t \leqslant \widetilde{m}(x)\}$$

is the 'reduced' Gaussian cone corresponding to the function

$$\widetilde{m}(x) = \min\left\{\frac{1}{2}, \frac{1}{|x|}\right\}.$$

Our proof of (4) is much shorter and, we believe, more transparent than the one presented in [2]. It has the further advantage of allowing the extension to cones with arbitrary aperture $A > 0$ and cut-off parameter $a > 0$ without any additional technicalities. This additional generality is very important and has already been used by Portal (cf. the claim made in [3, discussion preceding Lemma 2.3]) to prove the $H^1$-boundedness of the Riesz transform associated with $L$.

## 2. The Mehler kernel

The *Mehler kernel* (see e.g., [4]) is the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(\mathrm{e}^{-tL})_{t\geqslant 0}$, that is,

$$\mathrm{e}^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x,\cdot)u \,\mathrm{d}\gamma. \tag{7}$$

There is an abundance of literature on the Mehler kernel and its properties. We shall only use the fact, proved e.g. in the review paper [4], that it is given explicitly by

$$M_t(x,y) = \frac{\exp\left(-\dfrac{|\mathrm{e}^{-t}x-y|^2}{1-\mathrm{e}^{-2t}}\right)}{(1-\mathrm{e}^{-2t})^{\frac{d}{2}}}\mathrm{e}^{|y|^2}. \tag{8}$$

Note that $M_t(x,y)$ is symmetric in $x$ and $y$. A formula for (8) honoring this observation is:

$$M_t(x,y) = \frac{\exp\left(-\mathrm{e}^{-2t}\dfrac{|x-y|^2}{1-\mathrm{e}^{-2t}}\right)}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}} \frac{\exp\left(2\mathrm{e}^{-t}\dfrac{\langle x,y\rangle}{1+\mathrm{e}^{-t}}\right)}{(1+\mathrm{e}^{-t})^{\frac{d}{2}}}. \tag{9}$$

## 3. Some lemmata

We use $m$ as defined in (6) in our next lemma, which is taken from [1].

1. **Lemma.** *Let $a, A$ be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

   *(1) If $|x-y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1+aA)m(y)$;*
   
   *(2) If $|x-y| < Am(x)$, then $m(x) \leqslant (1+A)m(y)$ and $m(y) \leqslant 2(1+A)m(x)$.*

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. **Lemma.** *Let* $\alpha > 0$ *and* $|x-y| \leqslant \alpha m(x)$. *Then:*

$$\mathrm{e}^{-\alpha^2-2\alpha}\mathrm{e}^{|y|^2} \leqslant \mathrm{e}^{|x|^2} \leqslant \mathrm{e}^{\alpha^2(1+\alpha)^2+2\alpha(1+\alpha)}\mathrm{e}^{|y|^2}.$$

*Proof.* By the inverse triangle inequality and $m(x)|x| \leqslant 1$ we get,

$$|y|^2 \leqslant (\alpha m(x) + |x|)^2 \leqslant \alpha^2 + 2\alpha + |x|^2.$$

This gives the first inequality. For the second we use Lemma 1 to infer $m(x) \leqslant (1+\alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant \alpha^2(1+\alpha)^2 + 2\alpha(1+\alpha) + |y|^2.$$

As required. ■

3.1. **An estimate on Gaussian balls.** Let $B := B_t(x)$ be the open Euclidean ball with radius $t$ and center $x$ and let $\gamma$ be the Gaussian measure as defined by (2). We shall denote by $S_d$ the surface area of the unit sphere in $\mathbf{R}^d$.

3. **Lemma.** *For all* $x \in \mathbf{R}^d$ *and* $t > 0$ *we have the inequality:*

$$\gamma(B_t(x)) \leqslant \frac{S_d}{2\pi^{\frac{d}{2}}} t^d \mathrm{e}^{2t|x|}\mathrm{e}^{-|x|^2}. \tag{10}$$

*Proof.* Remark that, with $B := B_t(x)$,

$$\begin{aligned}
\int_B \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi &= \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{-2\langle x,\xi-x\rangle}\,\mathrm{d}\xi \\
&\leqslant \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{2|x||\xi-x|}\,\mathrm{d}\xi \\
&\leqslant \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\int_B \mathrm{e}^{-|\xi-x|^2}\,\mathrm{d}\xi \\
&= \pi^{\frac{d}{2}}\mathrm{e}^{2t|x|}\mathrm{e}^{-|x|^2}\gamma(B_t(0)).
\end{aligned}$$

So, there holds that

$$\gamma(B_t(x)) \leqslant \mathrm{e}^{2t|x|}\mathrm{e}^{-|x|^2}\gamma(B_t(0)). \tag{11}$$

We will estimate the Gaussian volume of the ball $B_t(0)$. Using polar coordinates we proceed for $d \geqslant 2$ by:

$$\begin{aligned}
\gamma(B_t(0)) &= \frac{1}{\pi^{\frac{d}{2}}}\int_{B_t(0)} \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi \\
&= \frac{S_d}{\pi^{\frac{d}{2}}}\int_0^t \mathrm{e}^{-r^2} r^{d-1}\,\mathrm{d}r \\
&\leqslant \frac{S_d}{2\pi^{\frac{d}{2}}} t^{d-2}\int_0^t 2r\mathrm{e}^{-r^2}\,\mathrm{d}r \\
&= \frac{S_d}{2\pi^{\frac{d}{2}}} t^{d-2}(1-\mathrm{e}^{-t^2}) \\
&\leqslant \frac{S_d}{2\pi^{\frac{d}{2}}} t^d,
\end{aligned}$$

where the last step uses $1 - \mathrm{e}^{-t} \leqslant t$ for $t \geqslant 0$. The case for $d = 1$ follows by a simplified argument. Upon combining this result with (11) we obtain (10). ■

3.2. **On-diagonal kernel estimates on annuli.** As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli.

Throughout this subsection we fix $x \in \mathbf{R}^d$, constants $A, a \geqslant 1$, a pair $(y,t) \in \Gamma_x^{(A,a)}$. We use the notation $rB$ to mean the ball obtained from the ball $B$ by multiplying its radius by $r$.

The annuli $C_k := C_k(B_t(x))$ are given by:

$$C_k := \begin{cases} B_t(x), & k = 0, \\ 2^{k+1}B_t(x) \setminus 2^k B_t(x), & k \geqslant 1. \end{cases} \tag{12}$$

Whenever $\xi$ is in $C_k$, we get for $k \geqslant 1$:

$$2^k t < |y - \xi| \leqslant 2^{k+1} t. \tag{13}$$

On $C_k$ we have the following bound for $M_{t^2}(y, \cdot)$:

4. **Lemma.** *For all $\xi \in C_k$ for $k \geqslant 1$ we have:*

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1 - \mathrm{e}^{-2t^2})^{\frac{d}{2}}} \exp\bigl(2^{k+1} t|y|\bigr) \exp\Bigl(-\frac{4^k}{2\mathrm{e}^{2t^2}}\Bigr), \tag{14}$$

*Proof.* Considering the first exponential which occurs in the Mehler kernel (9) together with (13) gives for $k \geqslant 1$:

$$\begin{aligned} \exp\left(-\mathrm{e}^{-2t^2} \frac{|y-\xi|^2}{1 - \mathrm{e}^{-2t^2}}\right) &\leqslant \exp\left(-\frac{4^k}{\mathrm{e}^{2t^2}} \frac{t^2}{1 - \mathrm{e}^{-2t^2}}\right) \\ &\overset{(\dagger)}{\leqslant} \exp\left(-\frac{4^k}{2\mathrm{e}^{2t^2}}\right), \end{aligned}$$

where ($\dagger$) follows from $1 - \mathrm{e}^{-s} \leqslant s$ for $s \geqslant 0$. Using the estimate $1 + s \geqslant 2s$ for $0 \leqslant s \leqslant 1$, for the second exponential in the Mehler kernel (9) we obtain, by (13):

$$\begin{aligned} \exp\left(2\mathrm{e}^{-t^2} \frac{\langle y, \xi\rangle}{1 + \mathrm{e}^{-t^2}}\right) &\leqslant \exp(|\langle y, \xi\rangle|) \\ &\leqslant \exp(|\langle y, \xi - y\rangle|)\mathrm{e}^{|y|^2} \\ &\leqslant \exp\bigl(2^{k+1} t|y|\bigr)\mathrm{e}^{|y|^2}. \end{aligned}$$

Combining things, we obtain the estimate in the formulation of the lemma. ■

## 4. The main result

In this section we will prove our main theorem for which we have already made the necessary preparations in the previous sections.

1. **Theorem.** *Let $A, a > 0$. For all $x \in \mathbf{R}^d$ and all $u \in L^2(\mathbf{R}^d, \gamma)$ we have*

$$\sup_{(y,t) \in \Gamma_x^{(A,a)}} |\mathrm{e}^{-t^2 L} u(y)| \lesssim \sup_{r > 0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, \mathrm{d}\gamma. \tag{15}$$

*Proof.* We fix $x \in \mathbf{R}^d$ and $(y,t) \in \Gamma_x^{(A,a)}$. The proof of (15) is based on splitting the integration domain into the annuli $C_k$ as defined by (12) and estimating on each annulus. More explicit,

$$|\mathrm{e}^{-t^2 L}u(y)| \leqslant \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k} M_{t^2}(y,\cdot)|u(\cdot)| \,\mathrm{d}\gamma. \tag{16}$$

From $t \leqslant am(x)$ we get $t|x| \leqslant a$, and by Lemma 1 this implies $t|y| \leqslant 1 + aA$. Together with Lemma 4 we infer, for $\xi \in C_k$ and $k \geqslant 1$, that:

$$\begin{aligned} M_{t^2}(y,\xi) &\leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \exp(2^{k+1}(1+aA)) \exp\Big(-\frac{4^k}{2\mathrm{e}^{2a^2}}\Big) \\ &=: \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} c_k. \end{aligned}$$

Combining this with Lemma 2, we obtain

$$M_{t^2}(y,\xi) \lesssim_{A,a} \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} c_k. \tag{17}$$

Also, by (13),

$$|x-\xi| \leqslant |x-y| + |\xi - y| \leqslant (2^{k+1}+1)t.$$

It follows that $C_k$ is contained in $D_k := B_{(2^{k+1}+1)t}(x)$.

Let us denote the supremum on right-hand side of (15) by $M_\gamma u(x)$. Using (17), we can bound the integral on the right-hand side of (16) by

$$\begin{aligned} \int_{C_k} M_{t^2}(y,\cdot)|u(\cdot)| \,\mathrm{d}\gamma &\lesssim_{A,a} c_k \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \int_{C_k} |u| \,\mathrm{d}\gamma \\ &\leqslant c_k \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \int_{D_k} |u| \,\mathrm{d}\gamma \\ &\leqslant c_k \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \gamma(D_k) M_\gamma u(x) \\ &\overset{(\dagger)}{\leqslant} c_k \frac{S_d}{2\pi^{\frac{d}{2}}} \frac{t^d}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} (2^{k+1}+1)^d \mathrm{e}^{2(2^{k+1}+1)t|x|} M_\gamma u(x) \\ &\overset{(\ddagger)}{\leqslant} c_k \frac{S_d}{2\pi^{\frac{d}{2}}} C_a^d (2^{k+1}+1)^d \mathrm{e}^{2a(2^{k+1}+1)} M_\gamma u(x) \\ &\lesssim_{A,a} c_k S_d \widetilde{C}_a^d 2^{kd} \mathrm{e}^{2^{k+2}a} M_\gamma u(x), \end{aligned}$$

where (†) uses Lemma 3 applied to $D_k$ and (‡) uses that $t \leqslant am(x)$ implies that $t|x| \leqslant a$ and $t \leqslant a$, where the latter implying

$$\frac{t^d}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \leqslant \frac{a^d}{(1-\mathrm{e}^{-2a^2})^{\frac{d}{2}}} =: C_a^d.$$

(note that $s/(1-\mathrm{e}^{-s})$ is increasing). Similarly, for $\xi \in B_t(x)$ we obtain:

$$\int_{B_t} M_{t^2}(y,\cdot)|u(\cdot)| \,\mathrm{d}\gamma \lesssim_{A,a} S_d \widetilde{C}_a^d M_\gamma u(x).$$

Inserting the depenceny of $c_k$ upon $k$ as coming from (17), we obtain the bound:

$$\begin{aligned}|\mathrm{e}^{-t^2L}u(y)| &= I_0 + \sum_{k=1}^{\infty} I_k \\ &\lesssim_{A,a} S_d \widetilde{C}_a^d \left[1 + \sum_{k=1}^{\infty} c_k 2^{kd} \mathrm{e}^{2^{k+2}a}\right] M_\gamma u(x), \\ &\lesssim_{A,a} S_d \widetilde{C}_a^d \left[1 + \sum_{k=1}^{\infty} 2^{kd} \mathrm{e}^{2^{k+1}(1+2a+aA)} \mathrm{e}^{-\frac{4^k}{2\mathrm{e}^{2a^2}}}\right] M_\gamma u(x),\end{aligned}$$

valid for all $y \in \Gamma_x^{(A,a)}$. As the sum on the right-hand side evidently converges, we see that taking the supremum proves (15). ∎

## References


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