# A note on the Gaussian maximal function

Jonas Teuwen

Abstract. In this note we give an improvement on a result first demonstrated by Pineda and Urbina [3]. In particular we present an improvement to their Lemma 1.1 which gives the boundedness of the Gaussian maximal function associated to the Ornstein-Uhlenbeck operator.

We present a proof which is at least to the author more transparant. Our main finding in this note is that our proof allows to use a larger cone and actually obtain the maximal function boundedness for a whole class of cones $\Gamma_x^{(A,a)}(\gamma)$.

## 1. Introduction

To be typed.

**1.1. Notation.** To begin, let us fix some notation. As is common, we use $N$ to represent a positive integer. That is, $N \in \mathbf{Z}_+ = \{1, 2, 3, \dots\}$. In the same way we cast letters that denote the number of dimensions, e.g. $d \in \mathbf{R}^d$ as positive integers.

We use the capital letter $T$ to denote a “time” endpoint, for instance, when writing $t \in (0, T]$.

## 2. The Mehler kernel and friends

**2.1. Setting.** Our setting is the one concerning the *Ornstein-Uhlenbeck* operator $L$ which is defined as:

$$L := -\frac{1}{2}\Delta + x \cdot \nabla, \tag{1}$$

We define the Mehler kernel (see e.g., Sjögren [5]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(\mathrm{e}^{-tL})_t$. More precisely, this means:

$$\mathrm{e}^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot)u \,\mathrm{d}\gamma. \tag{2}$$

It is often more convenient to use $\mathrm{e}^{-t^2L}$ instead of $\mathrm{e}^{-tL}$ as is done in e.g., Portal [4] and we will also do so.

**2.2. The Mehler kernel.** For the calculation of the Mehler kernel $M_t$ in (2) we refer to e.g., Sjögren [5] which additionally offers related results such as those related to Hermite polynomials.

Observe that the kernel $M_{t^2}$ is invariant under the permutation $x \longleftrightarrow y$. A formula for $M_t$ which honors this observation is:

$$M_{t^2}(x, y) = \frac{\exp\left(-\mathrm{e}^{-2t^2}\dfrac{|x-y|^2}{1-\mathrm{e}^{-2t^2}}\right)}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \frac{\exp\left(2\mathrm{e}^{-t^2}\dfrac{\langle x, y\rangle}{1+\mathrm{e}^{-t^2}}\right)}{(1+\mathrm{e}^{-t^2})^{\frac{d}{2}}}. \tag{3}$$

## 3. Some fine lemmata and definitions

**3.1. *m*inimal function.** We recall the lemma from [1, lemma 2.3] which first –although implicitly– appeared in [2]. For what follows it will be convenient to define a function $m$ as:

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\} = 1 \vee \frac{1}{|x|}.$$

We use $m$ in our next lemma.

1. Lemma. *Let $a, A$ be strictly positive numbers. We have for $x, y$ in $\mathbf{R}^d$ that:*

*(1) If $|x-y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1+aA)m(y)$;*
*(2) Likewise, if $|x-y| < Am(x)$, then $m(x) \leqslant (1+A)m(y)$ and $m(y) \leqslant 2(1+A)m(x)$.*

Recall that:

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}^d_+ : |x-y| < At \text{ and } t \leqslant am(x)\}. \tag{4}$$

To ease the notational burden a bit, we will write $\Gamma_x^a$ and mean $\Gamma_x^{(1,a)}$. Using this notation we can deduce a cone version of Lemma 1. That is:

2. Lemma. *Let $a, A$ be strictly positive numbers. Then:*

*(1) If $(y,t) \in \Gamma_x^{(A,a)}$ then $t \leqslant (1+aA)m(y)$;*
*(2) If $(y,t) \in \Gamma_x^{(A,a)}$ then $(x,t) \in \Gamma_y^{(1+aA,a)}$.*

In what is next we will use a global/local region dichotomy and define it as follows:

1. Definition. *Given $\tau > 0$, the set $N_\tau$ is given as:*

$$N_\tau(x) := N_\tau := \{(x,y) \in \mathbf{R}^{2d} : |x-y| \leqslant \tau m(x)\}. \tag{5}$$

*Sometimes it is easier to work with the set $N_\tau(B)$, which is given for $B := B_r(x)$ as:*

$$N_\tau(B) := \{y \in \mathbf{R}^d : |x-y| \leqslant \tau m(x)\}. \tag{6}$$

*When we partition the space into $N_\tau$ and its complement, we call the part belonging to $N_\tau$ the* local region *and the part belonging to its complement the* global region*.*

The set $t \leqslant am(x)$ is used in the definition of the cones $\Gamma_x^{(A,a)}$ and we will name it $D^a$, that is:

$$D^a := \{(x,t) \in \mathbf{R}^d_+ : t \leqslant am(x)\}. \tag{7}$$

We will write $D := D^1$ for simplicity.

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

3. Lemma. *Let $\alpha > 0$ and $|x-y| \leqslant \alpha m(x)$. Then the Gaussians in $x$ and $y$ respectively are comparable. In particular this means that,*

$$e^{-|x|^2} \simeq e^{-|y|^2}.$$

Remark. More precisely, from the proof we get the estimates (8) and (9). That is:

$$e^{-|x|^2} \leqslant e^{(1+\alpha)^2-1} e^{-|y|^2},$$

and,

$$e^{-|y|^2} \leqslant e^{(1+\alpha)^2} e^{2(1+\alpha)} e^{-|x|^2}.$$

Proof. Let $x$ and $y$ be such that $|x-y| \leqslant \alpha m(x)$. By the inverse triangle inequality we get,

$$\begin{aligned} |y|^2 &\leqslant (\alpha m(x) + |x|)^2 \\ &= \alpha^2 + 2\alpha m(x)|x| + |x|^2 \\ &\leqslant \alpha^2 + 2\alpha + |x|^2. \end{aligned}$$

Therefore,

$$e^{-|x|^2} \leqslant e^{-|y|^2} e^{(1+\alpha)^2} e^{-1}. \tag{8}$$

For the reverse direction we use Lemma 1 to infer $t \leqslant (1+\alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant (1+\alpha)^2 + 2(1+\alpha) + |y|^2.$$

Therefore,

$$e^{-|y|^2} \leqslant e^{-|x|^2} e^{(1+\alpha)^2} e^{2(1+\alpha)}. \tag{9}$$

Combining we get:

$$e^{-(1+\alpha)^2} e^{-1} e^{-(1+2\alpha)} \overset{(9)}{\leqslant} \frac{e^{-|x|^2}}{e^{-|y|^2}} \overset{(8)}{\leqslant} e^{-1} e^{(1+\alpha)^2}.$$

Summarizing we thus have that,

$$e^{-|x|^2} \simeq e^{-|y|^2},$$

as required. ∎

4. Lemma. *Let $x, y$ and $z$ in $\mathbf{R}^d$. Set*

$$\tau = \frac{1}{2}(1+2\alpha)(1+\alpha).$$

*If $|y-z| > \tau m(y)$ and $|x-y| \leqslant \alpha m(x)$ then $|x-z| > \frac{1}{2} m(x)$.*

Proof. We assume that $(y,z) \notin N_\tau$ and $(y,t) \in \Gamma_x^{(A,a)}$. Written out this gives by (5) the inequality $|y-z| > \tau m(y)$, and by (4) the inequality $|x-y| < aAm(x)$. Note that the latter inequality together with Lemma 1 yields,

$$\frac{1}{2}\frac{1}{1+\alpha} m(y) \leqslant m(x) \leqslant (1+\alpha)m(y). \tag{10}$$

Combining we get $|x-y| < \alpha(1+\alpha)m(y)$. Now we are in position to apply the triangle inequality:

$$|x-z| \geqslant |y-z| - |x-y| > \tau m(y) - \alpha(1+\alpha)m(y).$$

As we require an lower bound in terms of $m(x)$ and not $m(y)$, we again apply (10) to obtain:

$$\begin{aligned} |x-z| \geqslant |y-z| - |x-y| &> \tau m(y) - \alpha m(y) \\ &\geqslant \tau \frac{1}{1+\alpha} m(x) - \alpha m(x) \\ &\geqslant \frac{1}{2} m(x). \end{aligned}$$

Very well, proof is done. ∎

## 4. On-diagonal estimates

**4.1. Kernel estimates.** We begin with a technical lemma which will be useful on several occasions.

5. Lemma. *Let $t$ be in $(0,T]$ and let $\alpha > 1$. Then,*

$$\alpha e^{-T^2} \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} \leqslant \alpha. \tag{11}$$

*and,*

$$0 \leqslant \frac{1}{1+e^{-t^2}} - \frac{1}{1+e^{-\frac{t^2}{\alpha}}} \leqslant \frac{1}{4}\left(1-\frac{1}{\alpha}\right). \tag{12}$$

Proof. We start with (11) and apply the mean value theorem to the function $f(\xi)=\xi^\alpha$. For $0<\xi'<\xi$ this gives that:

$$f(\xi)-f(\xi') = \alpha\hat{\xi}^{\alpha-1}(\xi-\xi') \text{ for some } \hat{\xi} \text{ in } [\xi,\xi'].$$

Picking $\xi = 1$ and $\xi' = e^{-\frac{t^2}{\alpha}}$ yields:

$$\frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} = \alpha\hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[\exp\left(-\frac{t^2}{\alpha}\right), 1\right]. \tag{13}$$

Combining this result with the monotonicity of $\xi \mapsto \alpha\xi^{\alpha-1}$ we obtain:

$$\alpha e^{-t^2} \leqslant \alpha\exp\left(-t^2\frac{\alpha-1}{\alpha}\right) \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}}.$$

Together with $e^{-T^2} \leqslant e^{-t^2}$ we obtain,

$$\alpha e^{-T^2} \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} \downarrow \alpha.$$

We proceed with (12). Recalling that $\alpha > 1$ one can directly verify that:

$$\frac{1}{1+e^{-t^2}} - \frac{1}{1+e^{-\frac{t^2}{\alpha}}} \geqslant 0,$$

and that the function on the left-hand side is decreasing. To find an upper bound we compute the limit as $t$ goes to 0. That is:

$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t^2}\left[\frac{1}{1+e^{-t^2}} - \frac{1}{1+e^{-\frac{t^2}{\alpha}}}\right] &= \lim_{t\to 0}\frac{1}{2t}\left[\frac{2te^{-t^2}}{(1+e^{-t^2})^2} - \frac{1}{\alpha}\frac{2te^{-\frac{t^2}{\alpha}}}{(1+e^{-\frac{t^2}{\alpha}})^2}\right] \\
&= \lim_{t\to 0}\left[\frac{e^{-t^2}}{(1+e^{-t^2})^2} - \frac{1}{\alpha}\frac{e^{-\frac{t^2}{\alpha}}}{(1+e^{-\frac{t^2}{\alpha}})^2}\right] \\
&\uparrow \frac{1}{4}\left(1-\frac{1}{\alpha}\right).
\end{aligned}$$

Which is as asserted and completes the proof. ■

The following lemma will be useful when transfering estimates from $M_{\frac{t^2}{\alpha}}$ to $M_{t^2}$. It follows from the mean value theorem applied to $\xi \mapsto \xi^\alpha$.

6. Lemma. *For $\alpha > 1$ and $t$ in $(0,T]$ and all let $x,y$ in $\mathbf{R}^d$ we have that:*

$$\exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-e^{-\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-\frac{1}{2}\frac{\alpha}{e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right). \tag{14}$$

PROOF. Let $t$ in $(0,T]$. Applying (11) we get:

$$\exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-e^{-\frac{t^2}{\alpha}}}\right)\leqslant \exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-e^{-t^2}}\frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}}\right)\leqslant \exp\left(-\frac{1}{2}\frac{\alpha}{e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right).$$

Which is as asserted. ∎

Our first lemma is about estimating $M_{\frac{t^2}{\alpha}}$ in terms of $M_{t^2}$.

4.1.1. *Time-scaling of the Mehler kernel.*

7. LEMMA. *Let $\alpha \geqslant 2e^{T^2}$, $t$ in $(0,T]$ and $x,y$ in $\mathbf{R}^d$. If $t|x|\leqslant C$ and $t|y|\leqslant C$ then:*

$$M_{\frac{t^2}{\alpha}}(x,y)\leqslant \alpha^{\frac{d}{2}}e^{\frac{C^2}{2}}\exp\left(-\frac{\alpha}{2e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right)M_{t^2}(x,y). \tag{15}$$

PROOF. To prove the lemma we compute $M_{\frac{t^2}{\alpha}}M_{t^2}^{-1}$. First note that (11) gives

$$\alpha^{\frac{d}{2}}e^{-\frac{d}{2}T^2}\leqslant \frac{(1-e^{-t^2})^{\frac{d}{2}}}{(1-e^{-\frac{t^2}{\alpha}})^{\frac{d}{2}}}\leqslant \alpha^{\frac{d}{2}}.$$

Combining the exponentials also gives,

$$\begin{aligned}\exp\left(-2e^{-t^2}\frac{\langle x,y\rangle}{1+e^{-\frac{t^2}{\alpha}}}\right)&\exp\left(2e^{-t^2}\frac{\langle x,y\rangle}{1+e^{-t^2}}\right)\\ &\leqslant \exp\left(\frac{2}{t^2}\left[\frac{1}{1+e^{-t^2}}-\frac{1}{1+e^{-\frac{t^2}{\alpha}}}\right]t^2|\langle x,y\rangle|\right).\end{aligned}$$

(12) Using this result and nothing that $|x+y|^2\leqslant 2|x|^2+2|y|^2$ yields:

$$\begin{aligned}\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+e^{-\frac{t^2}{\alpha}}}\right)\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+e^{-t^2}}\right)&\leqslant \exp\left(\frac{1}{8}\left(1-\frac{1}{\alpha}\right)t^2|x+y|^2\right)\\ &\leqslant \exp\left(\frac{1}{4}t^2|x|^2\right)\exp\left(\frac{1}{4}t^2|y|^2\right).\end{aligned}$$

From $t|x|\leqslant C$ and $t|y|\leqslant C$ we obtain that:

$$\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+e^{-\frac{t^2}{\alpha}}}\right)\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+e^{-t^2}}\right)\leqslant e^{\frac{C^2}{2}}.$$

Combining Lemma 6 and equation (14) gives is almost the final estimate.

$$\begin{aligned}\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)}&\leqslant \alpha^{\frac{d}{2}}e^{\frac{C^2}{2}}\exp\left(\frac{1}{2}\frac{|x-y|^2}{1-e^{-t^2}}\right)\exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-e^{-\frac{t^2}{\alpha}}}\right)\\ &\leqslant \alpha^{\frac{d}{2}}e^{\frac{C^2}{2}}\exp\left(\frac{1}{2}\left[1-\frac{\alpha}{2e^{T^2}}\right]\frac{|x-y|^2}{1-e^{-t^2}}\right)\exp\left(-\frac{\alpha}{2e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right).\end{aligned}$$

Finally, we apply the assumption $\alpha\geqslant 2e^{T^2}$ to obtain:

$$\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)}\leqslant \alpha^{\frac{d}{2}}e^{\frac{C^2}{2}}\exp\left(-\frac{\alpha}{2e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right).$$

Which is as asserted. ∎

### 4.2. An estimate on Gaussian balls.

8. LEMMA. *Let $B_t(x)$ be the Euclidean ball with radius $t$ and center $x$. If $\gamma$ is the normalized Gaussian measure with density $\sim \exp(-|x|^2)$ we have:*

$$\gamma(B_t(x)) \leqslant S_d \pi^{-\frac{d}{2}} \mathrm{e}^{-|x|^2} \mathrm{e}^{2t|x|} \mathrm{e}^{-t^2} t^d. \tag{16}$$

PROOF. Next, remark that for a ball $B := B_t(x)$ there holds that

$$\begin{aligned}\int_B \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi &= \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{-2\langle x,\xi-x\rangle}\,\mathrm{d}\xi\\ &\leqslant \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{2|x||\xi-x|}\,\mathrm{d}\xi\\ &\leqslant \pi^{\frac{d}{2}}\mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)).\end{aligned}$$

That is:

$$\gamma(B_t(x)) \leqslant \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)). \tag{17}$$

We will estimate the Gaussian volume of the ball $B_t(0)$. To ease the notation, let $S_d$ be the surface area of the $d$-dimensional sphere. Using polar coordinates we then obtain:

$$\begin{aligned}\gamma(B_t(0)) &= \pi^{-\frac{d}{2}}\int_{B_t(0)} \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi\\ &= S_d\pi^{-\frac{d}{2}}\int_0^t \mathrm{e}^{-r^2}r^{d-1}\,\mathrm{d}r\\ &\leqslant S_d\pi^{-\frac{d}{2}}\mathrm{e}^{-t^2}t^d.\end{aligned}$$

Upon combining this result with (17) we obtain (16), which is as promised. ■

**4.3. On-diagonal kernel estimates on annuli.** As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli $C_k$. For the sake of simplicity we will write $B := B_t(x)$ and mean that $B$ is the closed ball with center $x$ and radius $t$. Furthermore, we use notations such as $2B$ to mean the ball obtained from $B$ by multiplying its radius by 2.

The $C_k$ are given by,

$$C_k(B) := C_k = (2^{k+1}-1)B \setminus (2^k-1)B. \tag{18}$$

So, whenever $\xi$ is in $C_k(B_t(x))$, we get for $k \geqslant 0$:

$$(2^k-1)t < |y-\xi| \leqslant (2^{k+1}-1)t. \tag{19}$$

9. LEMMA. *Given $A > 0$, let $B = B_{At}(y)$ and $\xi$ in $C_k$. Then we have:*

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}\mathrm{e}^{-At|y|}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp(2^{k+1}At|y|)\exp\left(-\frac{A^2}{2\mathrm{e}^{2t^2}}(2^k-1)^2\right).$$

PROOF. Let $B = B_{At}(y)$ and let $C_k$ be as in (18). For the sake of clarity we write $s := At$. Considering the first exponential which occurs in the Mehler kernel (3) together with (19) gives for $k \geqslant 0$:

$$\begin{aligned}\exp\left(-\mathrm{e}^{-2t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right) &\leqslant \exp\left(-\mathrm{e}^{-2t^2}\frac{(2^k-1)^2A^2t^2}{1-\mathrm{e}^{-2t^2}}\right)\\ &\overset{\text{(i)}}{\leqslant} \exp\left(-\frac{A^2}{2\mathrm{e}^{2t^2}}(2^k-1)^2\right).\end{aligned}$$

Where (i) follows from
$$\frac{t^2}{1-e^{-2t^2}} \geqslant \frac{1}{2}.$$
Before we consider the last exponential in the Mehler kernel we note that by Cauchy-Schwarz:

$$|\langle y,\xi\rangle| \leqslant |\langle y-\xi,\xi\rangle| + |\langle y,y\rangle| \leqslant |y-\xi||y| + |y|^2. \tag{20}$$

Furthermore we have the estimate:
$$\frac{e^{-t^2}}{1+e^{-t^2}} \leqslant \frac{1}{2},$$
So, using these we get for the last exponential in the Mehler kernel (3):
$$\begin{aligned}\exp\left(2e^{-t^2}\frac{\langle y,\xi\rangle}{1+e^{-t^2}}\right) &\leqslant \exp(|\langle y,\xi\rangle|)\\ &\overset{(20)}{\leqslant} \exp(|y-\xi||y|)e^{|y|^2}\\ &\overset{(i)}{\leqslant} \exp(2^{k+1}At|y|)e^{-At|y|}e^{|y|^2}.\end{aligned}$$
where (i) is due to $y$ and $\xi$ being in $C_k(B)$.

Wrapping it up, we can estimate the Mehler kernel (3) $M_{t^2}$ on $C_k$ from above by:
$$M_{t^2}(y,\xi) \leqslant \frac{e^{|y|^2}e^{-At|y|}}{(1-e^{-t^2})^{\frac{d}{2}}}\exp\left(2^{k+1}At|y|\right)\exp\left(-\frac{A^2}{2e^{2t^2}}(2^k-1)^2\right).$$
Which is as claimed.
$$M_{t^2}(y,\xi) \leqslant \frac{e^{|y|^2}e^{-At|y|}}{(1-e^{-t^2})^{\frac{d}{2}}}\exp\left(2^{k+1}At|y|\right)\exp\left(-\frac{A^2}{2e^{2t^2}}4^k\right)\exp\left(\frac{A^2}{e^{2t^2}}2^k\right)\exp\left(-\frac{A^2}{2e^{2t^2}}\right)$$
■

## 5. The boundedness of some non-tangential maximal operators

The following lemma is a small modification of [3, lemma 1.1] with a new proof.

10. Lemma. *Let $A,a>0$. For all $x$ in $\mathbf{R}^d$ and all $u$ in $L^2_\gamma$ we have*

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |e^{-t^2L}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u|\,d\gamma. \tag{21}$$

Proof. Since we have $|x-y|<At$ and $t\leqslant am(x)$ we infer that $t|x|\leqslant a$. By Lemma 2 we also that $t|y|\leqslant 1+aA$. From this and Lemma 9 we infer that:

$$M_{t^2}(y,\xi) \leqslant \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}}\exp\left(2^{k+1}At|y|\right)\exp\left(-\frac{A^2}{2e^{2t^2}}(2^k-1)^2\right). \tag{22}$$

Recalling Lemma 8 we get using $t|x|\leqslant a$ that

$$\gamma(B_t(x)) \leqslant S_d\pi^{-\frac{d}{2}}e^{-|x|^2}e^{2t|x|}t^d. \tag{23}$$

This allows us to estimate the remaining unbounded exponential in the Mehler kernel and allow a penalty up to $e^{-|x|^2}$. Furthermore, we have the following estimate which will make clear how to handle the time part in the Mehler kernel:
$$\frac{t^d}{(1-e^{-t^2})^{\frac{d}{2}}} \leqslant \left(\frac{t^2}{1-e^{-t^2}}\right)^{\frac{d}{2}} \leqslant \left(\frac{a^{2d^{-1}}}{1-e^{-a^2}}\right)^{\frac{d}{2}} = C_{a,d}.$$

In the next step we will bound the integral on the right-hand side by the maximal function centered at $x$. For this we need to scale up the $C_k$. So,

$$|x-\xi| \leqslant |x-y| + |\xi - y| \leqslant 2^{k+1}At.$$

So, we can bound the integral on the right-hand side of (24) by

$$\begin{aligned}
\int_{C_k} |u| \, \mathrm{d}\gamma &\leqslant \gamma(B(x, 2^{k+1}At)) \sup_{r>0} \fint_{B_r(x)} |u| \, \mathrm{d}\gamma \\
&\overset{(\mathrm{i})}{\leqslant} S_d \pi^{-\frac{d}{2}} \mathrm{e}^{2^{k+2}At|x|} 2^{d(k+1)} A^d t^d \mathrm{e}^{-|x|^2} \sup_{r>0} \fint_{B_r(x)} |u| \, \mathrm{d}\gamma \\
&\leqslant S_d \pi^{-\frac{d}{2}} \mathrm{e}^{2^{k+2}aA} 2^{d(k+1)} A^d t^d \mathrm{e}^{-|x|^2} \sup_{r>0} \fint_{B_r(x)} |u| \, \mathrm{d}\gamma.
\end{aligned}$$

Where (i) uses Lemma 8. The remark following Lemma 3 gives us by assuming $|x-y| \leqslant aAm(x)$ the following estimate:

$$\mathrm{e}^{|y|^2} \leqslant \mathrm{e}^{(1+aA)^2-1} \mathrm{e}^{|x|^2}.$$

Let $G_{d,a,A} = S_d \pi^{-\frac{d}{2}} \mathrm{e}^{(1+aA)^2-1} A^d$, then

$$\mathrm{e}^{|y|^2} \int_{C_k} |u| \, \mathrm{d}\gamma \leqslant G_{d,a,A} \mathrm{e}^{2^{k+2}aA} 2^{d(k+1)} t^d \sup_{r>0} \fint_{B_r(x)} |u| \, \mathrm{d}\gamma.$$

Using this, (22) and $t|y| \leqslant (1+aA)$ we can bound the Mehler kernel (3) from above by:

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \exp\big(2^{k+1}A(1+aA)\big) \exp\left(-\frac{A^2}{2\mathrm{e}^{2t^2}}(2^k-1)^2\right).$$

We can then bound the maximal function:
(24)

$$\begin{aligned}
\mathrm{e}^{-t^2L}|u(y)| &= \sum_{k=0}^{\infty} \int_{C_k(B)} M_{t^2}(y,\cdot)|u| \, \mathrm{d}\gamma \\
&\leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \sum_{k=0}^{\infty} \exp\left(-\frac{A^2}{2\mathrm{e}^{2a^2}}(2^k-1)^2\right) \exp\big(2^{k+1}A(1+aA)\big) \int_{C_k(B)} |u| \, \mathrm{d}\gamma \\
&\leqslant G_{d,a,A}(\mathscr{M}_\gamma u)(y) \frac{t^d}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \sum_{k=0}^{\infty} \exp\left(-\frac{A^2}{2\mathrm{e}^{2a^2}}(2^k-1)^2\right) \exp\big(2^{k+1}A(1+aA)\big) \mathrm{e}^{2^{k+2}aA} 2^{d(k+1)} \\
&\lesssim \sup_{r>0} \fint_{B_r(x)} |u| \, \mathrm{d}\gamma.
\end{aligned}$$

Wrapping it up, we have that:

$$\mathrm{e}^{-t^2L}|u(y)| \lesssim \fint_{B_r(x)} |u| \, \mathrm{d}\gamma.$$

With implied constant

$$G_{d,a,A} \left(\frac{a^{2d^{-1}}}{1-\mathrm{e}^{-a^2}}\right)^{\frac{d}{2}} \sum_{k=0}^{\infty} \exp\left(-\frac{A^2}{2\mathrm{e}^{2a^2}}(2^k-1)^2\right) \exp\big(2^{k+1}A(1+aA)\big) \mathrm{e}^{2^{k+2}aA} 2^{d(k+1)}$$

Which is what we wanted to prove. ∎

## References

[1] Jan Maas, Jan Neerven, and Pierre Portal. "Whitney coverings and the tent spaces $T^{1,q}(\gamma)$ for the Gaussian measure". In: *Arkiv för Matematik* 50.2 (Apr. 2011), pp. 379–395.

[2] Giancarlo Mauceri and Stefano Meda. "BMO and $H^1$ for the Ornstein-Uhlenbeck operator". In: *Journal of Functional Analysis* 252.1 (Nov. 2007), pp. 278–313.

[3] E. Pineda and W. Urbina. "Non Tangential Convergence for the Ornstein-Uhlenbeck Semigroup". In: *Divulgaciones Matemáticas* 13.2 (2008), pp. 1–19.

[4] Pierre Portal. "Maximal and quadratic Gaussian Hardy spaces". Mar. 2012.

[5] Peter Sjögren. "Operators associated with the hermite semigroup — A survey". In: *The Journal of Fourier Analysis and Applications* 3.S1 (Jan. 1997), pp. 813–823.

DELFT INSTITUTE OF APPLIED MATHEMATICS, DELFT UNIVERSITY OF TECHNOLOGY, P.O. BOX 5031, 2600 GA DELFT, THE NETHERLANDS

*E-mail address*: j.j.b.teuwen@tudelft.nl

*URL*: http://fa.its.tudelft.nl/~teuwen/