Gaussian estimates

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Abstract. Maximal function! An attempt! A good attempt! I hope!

1. The Mehler kernel and friends

1.1. Notation. To begin, let us fix some notation. As is common, we use N to represent a positive integer. That is, $N \in \mathbf{Z}_+ = \{1,2,3,\dots\}$. In the same way we cast letters that denote the number of dimensions, e.g. d in $\mathbf{R}^d$ as positive integers.

We use the capital letter T to denote a "time" endpoint, for instance, when writing t in $(0,T]$.

1.2. Setting. Given the Ornstein-Uhlenbeck operator L defined as:

$$L = -\frac{1}{2}\Delta + x\cdot\nabla, \tag{1}$$

We define the Mehler kernel (see e.g., Sjögren [4]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(\mathrm{e}^{-tL})_t$. More precisely, this means:

$$\mathrm{e}^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x,\cdot)u \,\mathrm{d}\gamma. \tag{2}$$

It is often more convenient to use e^{-t^2L} instead of e^{-tL} as is done in e.g., Portal [3] and we will also do so.

1.3. The Mehler kernel. For the computation of the Mehler kernel in (2) we refer to e.g., Sjögren [4] which additionally offers related results such as those concerning Hermite polynomials.

If one observes that the kernel M_{t^2} is symmetric in its arguments, a useful expression is:

$$M_{t^2}(x,y) = \frac{\exp\left(-\mathrm{e}^{-t^2}\dfrac{|x-y|^2}{1-\mathrm{e}^{-2t^2}}\right)}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\,\frac{\exp\left(\mathrm{e}^{-t^2}\dfrac{|x|^2+|y|^2}{1+\mathrm{e}^{-t^2}}\right)}{(1+\mathrm{e}^{-t^2})^{\frac{d}{2}}}. \tag{3}$$

2. Some fine lemmata and definitions

2.1. *m*inimal function. We recall the lemma from [1, lemma 2.3] which first, –although implicitly– appeared in [2]. It will be convenient to define a function m as:

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\} = 1 \vee \frac{1}{|x|}.$$

1. Lemma. *Let a,A be strictly positive numbers. We have for x, y in $\mathbf{R}^d$ that:*

(1) If $|x-y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1+aA)m(y)$;

(2) *Likewise, if* $|x-y|<Am(x)$, *then* $m(x)\leqslant(1+A)m(y)$ *and* $m(y)\leqslant 2(1+A)m(x)$.

We rewrite this lemma using the Gaussian cone $\Gamma_x^{(A,a)}$. Recall that:

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t)\in\mathbf{R}_+^d : |x-y|<At \text{ and } t\leqslant am(x)\}. \tag{4}$$

We will also write Γ_x^a to mean $\Gamma_x^{(1,a)}$. So we can infer from Lemma 1 that:

2. Lemma. *Let* a,A *be strictly positive numbers. Then:*

(1) *If* $(y,t)\in\Gamma_x^{(A,a)}$ *then* $t\leqslant(1+aA)m(y)$;
(2) *If* $(y,t)\in\Gamma_x^{(A,a)}$ *then* $(x,t)\in\Gamma_y^{(1+aA,a)}$.

We will use a global/local region dichotomy which we define as follows.

1. Definition. *Given* $\tau>0$, *the set* N_τ *is given as:*

$$N_\tau(x) := N_\tau := \{(x,y)\in\mathbf{R}^{2d} : |x-y|\leqslant\tau m(x)\}. \tag{5}$$

Sometimes it is easier to work with the set $N_\tau(B)$, *which is given for* $B:=B_r(y)$ *as:*

$$N_\tau(B) := \{y\in\mathbf{R}^d : |x-y|\leqslant\tau m(x)\}. \tag{6}$$

When we partition the space into N_τ *and its complement, we call the part belonging to* N_τ *the* local region *and the part belonging to* $\complement N_\tau$ *the* global region.

The set $t\leqslant am(x)$ is used in the definition of the cones $\Gamma_x^{(A,a)}$ and we will name it D^a, that is:

$$D^a := \{(x,t)\in\mathbf{R}_+^d : t\leqslant am(x)\}. \tag{7}$$

We will write $D:=D^1$ for simplicity.

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

3. Lemma. *Let* $(y,t)\in\Gamma_x^{(A,a)}$. *Then the Gaussians in* x *and* y *respectively are comparable. In particular this means that,*

$$\mathrm{e}^{-|x|^2}\simeq\mathrm{e}^{-|y|^2}.$$

Remark. More precisely, from the proof we get the estimates (8) and (9). That is:

$$\mathrm{e}^{-|x|^2}\leqslant\mathrm{e}^{(1+aA)^2-1}\mathrm{e}^{-|y|^2},$$

and,

$$\mathrm{e}^{-|y|^2}\leqslant\mathrm{e}^{(1+aA)^2}\mathrm{e}^{2(1+aA)}\mathrm{e}^{-|x|^2}.$$

Proof. Let $(y,t)\in\Gamma_x^{(A,a)}$. Unwrapping the definition we have

$$|x-y|<At \text{ and } t\leqslant am(x).$$

Hence, by the inverse triangle inequality we get,

$$\begin{aligned}|y|^2 &\leqslant (aAm(x)+|x|)^2\\ &= (aA)^2+2aAm(x)|x|+|x|^2\\ &\leqslant (aA)^2+2aA+|x|^2.\end{aligned}$$

Therefore,

$$\mathrm{e}^{-|y|^2}\geqslant\mathrm{e}^{-(aA)^2}\mathrm{e}^{-2aA}\mathrm{e}^{-|x|^2}. \tag{8}$$

By Lemma 1 we have $t \leqslant (1+aA)m(y)$

$$\begin{aligned}|x|^2 &\leqslant ((1+aA)m(y)+|y|)^2\\ &= ((1+aA)m(y))^2 + 2(1+aA)m(y)|y| + |y|^2\\ &\leqslant (1+aA)^2 + 2(1+aA) + |y|^2.\end{aligned}$$

Therefore,

$$\mathrm{e}^{-|x|^2} \geqslant \mathrm{e}^{-(1+aA)^2}\mathrm{e}^{-2(1+aA)}\mathrm{e}^{-|y|^2}. \tag{9}$$

Summarizing we thus have that,

$$\mathrm{e}^{-|x|^2} \simeq \mathrm{e}^{-|y|^2},$$

as required. ∎

4. Lemma. *Let x, y and z in $\mathbf{R}^d$. Set*

$$\tau = \frac{1}{2}(1+2aA)(1+aA).$$

If $|y-z| > \tau m(y)$ (i.e., $(y,z) \notin N_\tau$) and $(y,t) \in \Gamma_x^{(A,a)}$ then $|x-z| > \frac{1}{2}m(x)$ (i.e., $(x,z) \notin N_{\frac{1}{2}}$).

Proof. We assume that $(y,z) \notin N_\tau$ and $(y,t) \in \Gamma_x^{(A,a)}$. Written out this gives by (5) the inequality $|y-z| > \tau m(y)$, and by (4) the inequality $|x-y| < aAm(x)$. Note that the latter inequality together with Lemma 1 yields,

$$\frac{1}{2}\frac{1}{1+aA}m(y) \leqslant m(x) \leqslant (1+aA)m(y). \tag{10}$$

Combining we get $|x-y| < aA(1+aA)m(y)$. Now we are in position to apply the triangle inequality:

$$|x-z| \geqslant |y-z| - |x-y| > \tau m(y) - aA(1+aA)m(y).$$

As we require an lower bound in terms of $m(x)$ and not $m(y)$, we again apply (10) to obtain:

$$\begin{aligned}|x-z| \geqslant |y-z| - |x-y| &> \tau m(y) - aAm(y)\\ &\geqslant \tau\frac{1}{1+aA}m(x) - aAm(x)\\ &\geqslant \frac{1}{2}m(x).\end{aligned}$$

So we are done. ∎

3. On-diagonal estimates

3.1. Kernel estimates. We begin with a technical lemma which will be useful on several occasions.

5. Lemma. *Let t in $(0,T]$ and $\alpha > 1$. Then,*

$$\alpha\mathrm{e}^{-T^2} \leqslant \frac{1-\mathrm{e}^{-t^2}}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}} \leqslant \alpha. \tag{11}$$

Proof. Let t in $(0,T]$ and $\alpha > 1$. Applying the mean value theorem to the function $f(\xi) = \xi^\alpha$ gives, for $0 < \xi < \xi'$:

$$f(\xi) - f(\xi') = \alpha\hat{\xi}^{\alpha-1}(\xi - \xi') \text{ for some } \hat{\xi} \text{ in } [\xi, \xi'].$$

Picking $\xi = 1$ and $\xi' = \mathrm{e}^{-\frac{t^2}{\alpha}}$ gives:

$$\frac{1-\mathrm{e}^{-t^2}}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}} = \alpha\hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } [\mathrm{e}^{-\frac{t^2}{\alpha}}, 1]. \tag{12}$$

Applying this result together with the monotonicity of $\xi \mapsto \alpha\xi^{\alpha-1}$ we get:

$$\alpha\mathrm{e}^{-T^2} \leqslant \alpha\mathrm{e}^{-t^2} \leqslant \alpha\exp\left(-t^2\frac{\alpha-1}{\alpha}\right) \leqslant \frac{1-\mathrm{e}^{-t^2}}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}}.$$

Hence,

$$\alpha\mathrm{e}^{-T^2} \leqslant \frac{1-\mathrm{e}^{-t^2}}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}} \downarrow \alpha.$$

Which completes the proof. ■

The following lemma will be useful when transfering estimates from $M_{\frac{t^2}{\alpha}}$ to M_{t^2}. It follows from the mean value theorem applied to $\xi \mapsto \xi^{\alpha}$.

6. LEMMA. *For $C, T > 0, \alpha > 1, t$ in $(0, T]$ and all x, y in $\mathbf{R}^d$ we have that*

$$\exp\left(-C\frac{|x-y|^2}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-C\frac{\alpha}{\mathrm{e}^{T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right). \tag{13}$$

PROOF. Let t in $(0, T]$. Applying Lemma 5 we get:

$$\exp\left(-C\frac{|x-y|^2}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-C\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\frac{1-\mathrm{e}^{-t^2}}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-C\frac{\alpha}{\mathrm{e}^{T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right).$$

Which is as asserted. ■

Later on we will study kernel estimates of kernels related to the Mehler kernel, but our first lemma is about estimating $M_{\frac{t}{\alpha}}$ in terms of M_t.

7. LEMMA. *Let $\alpha \geqslant 2\mathrm{e}^{T^2}$, t in $(0, T]$ and x, y in $\mathbf{R}^d$. If $t|x| \lesssim 1$ and $t|y| \lesssim 1$ then:*

$$M_{\frac{t^2}{\alpha}}(x,y) \lesssim \exp\left(-\frac{\alpha}{2\mathrm{e}^{T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right)M_{t^2}(x,y), \tag{14}$$

where the implied constant does not depend on x, y and t.

REMARK. If C is the positive constant such that $t|x| \leqslant C$ and $t|y| \leqslant C$ then the proof gives that the implied constant is bounded from above by $\alpha^{\frac{d}{2}}\mathrm{e}^{\frac{\alpha}{2}C^2}$.

PROOF. To prove the lemma we compute $M_{\frac{t^2}{\alpha}}M_{t^2}^{-1}$. First note that dividing the time-parts by (11) gives the upper-bound $\alpha^{\frac{d}{2}}$. Furthermore, we can verify that

$$\frac{1}{1+\mathrm{e}^{-t^2}} - \frac{1}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}} \geqslant 0.$$

Also,

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right)\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+\mathrm{e}^{-t^2}}\right) \\
&\quad\leqslant \exp\left(\frac{1}{2}\left[\frac{1}{1+\mathrm{e}^{-t^2}} - \frac{1}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right]|x+y|^2\right) \\
&\quad= \exp\left(\frac{1}{2}\frac{1}{t^2}\left[\frac{1}{1+\mathrm{e}^{-t^2}} - \frac{1}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right]t^2|x+y|^2\right).
\end{aligned}$$

Next, as the inner most function is decreasing,

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t^2}\left[\frac{1}{1+e^{-t^2}}-\frac{1}{1+e^{-\frac{t^2}{\alpha}}}\right]&=\lim_{t\to 0}\frac{1}{2t}\left[\frac{2te^{-t^2}}{(1+e^{-t^2})^2}-\frac{1}{\alpha}\frac{2te^{-\frac{t^2}{\alpha}}}{(1+e^{-\frac{t^2}{\alpha}})^2}\right]\\&=\lim_{t\to 0}\left[\frac{e^{-t^2}}{(1+e^{-t^2})^2}-\frac{1}{\alpha}\frac{e^{-\frac{t^2}{\alpha}}}{(1+e^{-\frac{t^2}{\alpha}})^2}\right]\\&\uparrow\frac{1}{4}\left(1-\frac{1}{\alpha}\right).\end{aligned}$$

So that

$$\begin{aligned}\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+e^{-\frac{t^2}{\alpha}}}\right)&\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+e^{-t^2}}\right)\\&\leqslant\exp\left(\frac{1}{8}\left(1-\frac{1}{\alpha}\right)t^2|x+y|^2\right)\\&\leqslant\exp\left(\frac{1}{4}t^2|x|^2\right)\exp\left(\frac{1}{4}t^2|y|^2\right).\end{aligned}$$

So using Lemma 6 and equation (13) we get

$$\begin{aligned}\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)}&\leqslant\alpha^{\frac{d}{2}}\exp\left(\frac{1}{2}\frac{|x-y|^2}{1-e^{-t^2}}\right)\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+e^{t^2}}\right)\\&\quad\times\exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-e^{-\frac{t^2}{\alpha}}}\right)\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+e^{-\frac{t^2}{\alpha}}}\right)\\&\leqslant\alpha^{\frac{d}{2}}\exp\left(\frac{1}{2}\left[1-\frac{\alpha}{2e^{T^2}}\right]\frac{|x-y|^2}{1-e^{-t^2}}\right)\exp\left(-\frac{\alpha}{2e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right)\\&\quad\times\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+e^{t^2}}\right)\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+e^{-\frac{t^2}{\alpha}}}\right).\end{aligned}$$

Thus that,

$$\begin{aligned}\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)}&\leqslant\alpha^{\frac{d}{2}}\exp\left(\frac{1}{2}\left[1-\frac{\alpha}{2e^{T^2}}\right]\frac{|x-y|^2}{1-e^{-t^2}}\right)\exp\left(-\frac{\alpha}{2e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right)\\&\quad\times\exp\left(t^2\frac{|x|^2+|y|^2}{4}\right).\end{aligned}$$

For $\alpha\geqslant 2e^{T^2}$ we then obtain:

$$\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)}\leqslant\alpha^{\frac{d}{2}}\exp\left(-\frac{\alpha}{2e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right)\exp\left(\frac{1}{4}t^2|x|^2\right)\exp\left(\frac{1}{4}t^2|y|^2\right).$$

From $t|x|\lesssim 1$ and $t|y|\lesssim 1$ we infer that there exists a positive constant C such that $t|x|\leqslant C$ and $t|y|\leqslant C$.

$$\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)}\leqslant\alpha^{\frac{d}{2}}e^{\frac{C^2}{2}}\exp\left(-\frac{\alpha}{2e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right).$$

Which is as asserted. ■

REMARK. More precisely we have the estimate:

$$\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right)\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+\mathrm{e}^{-t^2}}\right)$$
$$\leqslant \exp\left(\frac{1}{4}t^2|x|^2\right)\exp\left(\frac{1}{4}t^2|y|^2\right)\exp\left(-\frac{t^2}{\alpha}\frac{1}{8}|x+y|^2\right).$$

Which then produces:

$$\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)} \leqslant \alpha^{\frac{d}{2}}\mathrm{e}^{\frac{C^2}{2}}\exp\left(-\frac{\alpha}{2\mathrm{e}^{T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right)\exp\left(-\frac{t^2}{\alpha}\frac{|x+y|^2}{8}\right).$$

3.2. On-diagonal kernel estimates on annuli. As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Thus we will decompose the space into annuli C_k. We will write $B := B_t(x)$ and assume that B is the closed ball with center x and radius t. Recall that $2B$ is the ball obtain from B by multiplying its radius by 2.

The C_k are given by,

$$C_k(B) := C_k = \begin{cases} 2B & \text{if } k=0, \\ 2^{k+1}B \setminus 2^kB & \text{for } k \geqslant 1. \end{cases} \tag{15}$$

So, whenever ξ is in C_k, we get for $k \geqslant 1$:

$$2^k at < |y-\xi| \leqslant 2^{k+1}at. \tag{16}$$

While we get for $k=0$:

$$|y-\xi| \leqslant 2at. \tag{17}$$

8. LEMMA. *Given $a>0$, let $B = B_{at}(y)$ and ξ in C_k. Furthermore, assume that $t \leqslant aAm(y)$ for some $A>0$. Then we have for $k \geqslant 1$:*

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp\left(-\frac{a^2}{2}4^{k+1}\right)\exp\left(2^{k+1}at|y|\right).$$

and for $k=0$:

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp\left(2^{k+1}at|y|\right).$$

PROOF. Let $B = B_{at}(y)$ and let C_k be as in (15). We consider the two exponentials in the Mehler kernel (3) separately. First we consider

$$\exp\left(\mathrm{e}^{-t^2}\frac{|y|^2+|\xi|^2}{1+\mathrm{e}^{-t^2}}\right).$$

Using the triangle inequality we note that:

$$|\xi|^2 \leqslant |y-\xi|^2+|y|^2+2|y-\xi||y|. \tag{18}$$

Next, note that

$$\frac{\mathrm{e}^{-t^2}}{1+\mathrm{e}^{-t^2}} \leqslant \frac{1}{2}.$$

Together with (18) this gives for $k \geqslant 1$:

$$\exp\left(\mathrm{e}^{-t^2}\frac{|y|^2+|\xi|^2}{1+\mathrm{e}^{-t^2}}\right) \leqslant \exp\left(\mathrm{e}^{-t^2}\frac{|y-\xi|^2}{1+\mathrm{e}^{-t^2}}\right)\exp(|y-\xi||y|)\exp(|y|^2)$$
$$\overset{\text{(i)}}{\leqslant} \exp\left(\mathrm{e}^{-t^2}\frac{|y-\xi|^2}{1+\mathrm{e}^{-t^2}}\right)\exp(2^{k+1}at|y|)\exp(|y|^2)$$

Where (i) uses (16) or (17). Next we consider the exponential:

$$\exp\left(\mathrm{e}^{-t^2}\frac{|y-\xi|^2}{1+\mathrm{e}^{-t^2}}\right).$$

Combining this with the first exponential in the Mehler kernel (3) we get:

$$\begin{aligned}\exp\left(-\mathrm{e}^{-t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right)\exp\left(\mathrm{e}^{-t^2}\frac{|y-\xi|^2}{1+\mathrm{e}^{-t^2}}\right)&\\ &\leqslant \exp\left(-\mathrm{e}^{-t^2}\frac{|y-\xi|^2}{1+\mathrm{e}^{-t^2}}\left[\frac{1}{1-\mathrm{e}^{-t^2}}-1\right]\right)\\ &\leqslant \exp\left(-\mathrm{e}^{-t^2}\frac{|y-\xi|^2}{1+\mathrm{e}^{-t^2}}\left[\frac{1}{1-\mathrm{e}^{-t^2}}-\frac{1-\mathrm{e}^{-t^2}}{1-\mathrm{e}^{-t^2}}\right]\right)\\ &\leqslant \exp\left(-\mathrm{e}^{-2t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right).\end{aligned}$$

Using (16) or (17) we get:

$$\exp\left(-\mathrm{e}^{-2t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right)\leqslant\begin{cases}1 & \text{if } k=0,\\ \exp\left(-\frac{a^2}{2\mathrm{e}^{2t^2}}4^{k+1}\right) & \text{if } k\geqslant 1.\end{cases}$$

Using the assumption that $t\leqslant aAm(y)$ gives that

$$\exp\left(-\frac{a^2}{2\mathrm{e}^{2t^2}}4^{k+1}\right)\leqslant\exp\left(-\frac{a^2}{2\mathrm{e}^{2a^2A^2}}4^{k+1}\right).$$

Combining we get for the Mehler kernel (3):

$$\begin{aligned}M_{t^2}(y,\xi)&\leqslant\frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp(2^{k+1}at|y|)\\ &\leqslant\frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp(2^{k+1}a^2A).\end{aligned}$$

This inequality together with

$$\frac{t^2}{1-\mathrm{e}^{-2t^2}}\geqslant\frac{1}{2},$$

yields,

$$\exp\left(-\mathrm{e}^{-t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right)\exp\left(-\mathrm{e}^{-t^2}\frac{|y-\xi|^2}{1+\mathrm{e}^{-t^2}}\right)\leqslant\exp\left(-\frac{a^2}{2\mathrm{e}^{2t^2}}4^{k+1}\right).$$

Thus, we can estimate the Mehler kernel M_{t^2} on C_k for $k\geqslant 1$ from above by:

$$M_{t^2}(y,\xi)\leqslant\frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp\left(-\frac{a^2}{2}4^{k+1}\right)\exp\left(2^{k+1}at|y|\right).$$

We are left with the case $k=0$, which can be done similarly and yields:

$$M_{t^2}(y,\xi)\leqslant\frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp\left(2^{k+1}at|y|\right).$$

Done. ■

3.3. The Ornstein-Uhlenbeck maximal function.

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