

A NOTE ON THE GAUSSIAN MAXIMAL FUNCTION

JONAS TEUWEN

ABSTRACT. This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded almost surely by the Gaussian Hardy-Littlewood maximal function. In particular this entails improvement on a result by Pineda and Urbina [2] who proved a similar result for a ‘truncated’ version of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

1. INTRODUCTION

Maximal functions are among the most studied objects in harmonic analysis. It is well-known that the classical non-tangential maximal function associated with the heat semigroup is bounded almost everywhere by the Hardy-Littlewood maximal function,

$$(1) \quad \sup_{\substack{(y,t) \in \mathbf{R}_+^{d+1} \\ |x-y| < t}} |e^{-t\Delta} u(y)| \lesssim \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \, d\lambda.$$

Here the action of *heat semigroup* $e^{-t\Delta} u = \rho_t * u$ is given by a convolution of u with the *heat kernel*

$$\rho_t(\xi) := \frac{e^{-|\xi|^2/4t}}{(4\pi t)^{\frac{d}{2}}}, \text{ with } t > 0 \text{ and } \xi \in \mathbf{R}^d.$$

In this note we are interested in its Gaussian counterpart. The change from Lebesgue measure to the *Gaussian measure*

$$(2) \quad d\gamma(x) := \pi^{-\frac{d}{2}} e^{-|x|^2} dx$$

introduces quite some intricate technical and conceptual difficulties which are due to its non-doubling nature. Instead of the Laplacian, will use its Gaussian analogue, the *Ornstein-Uhlenbeck operator* L which is given by,

$$(3) \quad L := -\frac{1}{2}\Delta + \langle x, \nabla \rangle = \frac{1}{2}\nabla_\gamma^* \nabla_\gamma,$$

where ∇_γ denotes the realisation of the gradient in $L^2(\mathbf{R}^d, \gamma)$. Our main result, to be proved in Theorem 1, is the following Gaussian analogue of (1):

$$(4) \quad \sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma.$$

Here,

$$(5) \quad \Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}_+^{d+1} : |x-y| < At \text{ and } t \leq am(x)\}$$

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is the *Gaussian cone* with aperture A and cut-off parameter a , and

$$(6) \quad m(x) := \min \left\{ 1, \frac{1}{|x|} \right\}.$$

A slightly weaker version of the inequality (4) has been proved by Pineda and Urbina [2] who showed that

$$\sup_{(y,t) \in \tilde{\Gamma}_x} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| d\gamma,$$

where

$$\tilde{\Gamma}_x(x) = \{(y, t) \in \mathbf{R}_+^d : |x - y| < t \leq \tilde{m}(x)\}$$

is the ‘reduced’ Gaussian cone corresponding to the function

$$\tilde{m}(x) = \min \left\{ \frac{1}{2}, \frac{1}{|x|} \right\}.$$

Our proof of (4) is much shorter and, we believe, more transparent than the one presented in [2]. It has the further advantage of allowing the extension to cones with arbitrary aperture $A > 0$ and cut-off parameter $a > 0$ without any additional technicalities. This additional generality is very important and has already been used by Portal (cf. the claim made in [3, discussion preceding Lemma 2.3]) to prove the H^1 -boundedness of the Riesz transform associated with L .

2. THE MEHLER KERNEL

The *Mehler kernel* (see e.g., [4]) is the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(e^{-tL})_{t \geq 0}$, that is,

$$(7) \quad e^{-tL} u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot) u d\gamma.$$

There is an abundance of literature on the Mehler kernel and its properties. We shall only use the fact, proved e.g. in the review paper [4], that it is given explicitly by

$$(8) \quad M_t(x, y) = \frac{\exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right)}{(1 - e^{-2t})^{\frac{d}{2}}} e^{|y|^2}.$$

Note that $M_t(x, y)$ is symmetric in x and y . A formula for (8) honoring this observation is:

$$(9) \quad M_t(x, y) = \frac{\exp\left(-e^{-2t} \frac{|x - y|^2}{1 - e^{-2t}}\right)}{(1 - e^{-t})^{\frac{d}{2}}} \frac{\exp\left(2e^{-t} \frac{\langle x, y \rangle}{1 + e^{-t}}\right)}{(1 + e^{-t})^{\frac{d}{2}}}.$$

3. SOME LEMMATA

We use m as defined in (6) in our next lemma, which is taken from [1].

1. Lemma. *Let a, A be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

- (1) *If $|x - y| < At$ and $t \leq am(x)$, then $t \leq (1 + aA)m(y)$;*
- (2) *If $|x - y| < Am(x)$, then $m(x) \leq (1 + A)m(y)$ and $m(y) \leq 2(1 + A)m(x)$.*

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. Lemma. *Let $\alpha > 0$ and $|x - y| \leq \alpha m(x)$. Then:*

$$e^{-\alpha^2 - 2\alpha} e^{|y|^2} \leq e^{|x|^2} \leq e^{\alpha^2(1+\alpha)^2 + 2\alpha(1+\alpha)} e^{|y|^2}.$$

Proof. By the inverse triangle inequality and $m(x)|x| \leq 1$ we get,

$$|y|^2 \leq (\alpha m(x) + |x|)^2 \leq \alpha^2 + 2\alpha + |x|^2.$$

This gives the first inequality. For the second we use Lemma 1 to infer $m(x) \leq (1 + \alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leq \alpha^2(1 + \alpha)^2 + 2\alpha(1 + \alpha) + |y|^2.$$

As required. ■

3.1. An estimate on Gaussian balls. Let $B := B_t(x)$ be the open Euclidean ball with radius t and center x and let γ be the Gaussian measure as defined by (2). We shall denote by S_d the surface area of the unit sphere in \mathbf{R}^d .

3. Lemma. *For all $x \in \mathbf{R}^d$ and $t > 0$ we have the inequality:*

$$(10) \quad \gamma(B_t(x)) \leq \frac{t^d}{\Gamma(\frac{d}{2})} e^{2t|x|} e^{-|x|^2}.$$

Proof. Remark that, with $B := B_t(x)$,

$$\begin{aligned} \int_B e^{-|\xi|^2} d\xi &= e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{-2\langle x, \xi-x \rangle} d\xi \\ &\leq e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{2|x||\xi-x|} d\xi \\ &\leq e^{-|x|^2} e^{2t|x|} \int_B e^{-|\xi-x|^2} d\xi \\ &= \pi^{\frac{d}{2}} e^{-|x|^2} e^{2t|x|} \gamma(B_t(0)). \end{aligned}$$

So, there holds that

$$(11) \quad \gamma(B_t(x)) \leq e^{-|x|^2} e^{2t|x|} \gamma(B_t(0)).$$

We will estimate the Gaussian volume of the ball $B_t(0)$. Using polar coordinates we proceed for $d \geq 2$ by:

$$\begin{aligned} \gamma(B_t(0)) &= \pi^{-\frac{d}{2}} \int_{B_t(0)} e^{-|\xi|^2} d\xi \\ &= \pi^{-\frac{d}{2}} S_d \int_0^t e^{-r^2} r^{d-1} dr \\ &\leq \frac{1}{2} \pi^{-\frac{d}{2}} S_d t^{d-2} \int_0^t 2r e^{-r^2} dr \\ &= \frac{1}{2} \pi^{-\frac{d}{2}} S_d t^{d-2} (1 - e^{-t^2}) \\ &\leq \frac{1}{2} \pi^{-\frac{d}{2}} S_d t^d, \end{aligned}$$

where the last step uses $1 - e^{-t} \leq t$ for $t \geq 0$. The case for $d = 1$ follows by a simplified argument. The surface area of the d -dimensional sphere S_d given by:

$$S_d = 2 \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})},$$

hence,

$$\gamma(B_t(0)) \leq \frac{t^d}{\Gamma(\frac{d}{2})},$$

Upon combining this result with (11) we obtain (10). \blacksquare

3.2. On-diagonal kernel estimates on annuli. As is common in harmonic analysis, we often wish to decompose \mathbf{R}^d into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli.

Throughout this subsection we fix $x \in \mathbf{R}^d$, constants $A, a = 1$, a pair $(y, t) \in \Gamma_x$. We use the notation rB to mean the ball obtained from the ball B by multiplying its radius by r .

The annuli $C_k := C_k(B_t(x))$ are given by:

$$(12) \quad C_k := (2^{k+1} - 1)B_t(x) \setminus (2^k - 1)B_t(x) \text{ with } k \geq 0.$$

Note that $C_0 = B_t(x)$. Whenever ξ is in C_k , we get for $k \geq 0$:

$$(13) \quad (2^k - 1)t < |y - \xi| \leq (2^{k+1} - 1)t.$$

On C_k we have the following bound for $M_{t^2}(y, \cdot)$:

4. Lemma. *For all $\xi \in C_k$ we have:*

$$(14) \quad M_{t^2}(y, \xi) \leq \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \exp((2^{k+1} - 1)t|y|) \exp\left(-\frac{1}{2e^{2a^2}}(2^k - 1)^2\right),$$

Proof. Considering the first exponential which occurs in the Mehler kernel (9) together with (13) gives for $k \geq 0$:

$$\begin{aligned} \exp\left(-e^{-2t^2} \frac{|y - \xi|^2}{1 - e^{-2t^2}}\right) &\leq \exp\left(-e^{-2t^2} \frac{(2^k - 1)^2 t^2}{1 - e^{-2t^2}}\right) \\ &\stackrel{(\dagger)}{\leq} \exp\left(-\frac{1}{2e^{2t^2}}(2^k - 1)^2\right) \stackrel{(\ddagger)}{\leq} \exp\left(-\frac{1}{2e^2}(2^k - 1)^2\right), \end{aligned}$$

where (\dagger) follows from $1 - e^{-s} \leq s$ for $s \geq 0$, and (\ddagger) uses that $t \leq m(x) \leq 1$. Using the estimate $1 + s \geq 2s$ for $0 \leq s \leq 1$, for the second exponential in the Mehler kernel (9) we obtain, by (13):

$$\begin{aligned} \exp\left(2e^{-t^2} \frac{\langle y, \xi \rangle}{1 + e^{-t^2}}\right) &\leq \exp(|\langle y, \xi \rangle|) \\ &\leq \exp(|\langle y, \xi - y \rangle|) e^{|y|^2} \\ &\leq \exp((2^{k+1} - 1)t|y|) e^{|y|^2}. \end{aligned}$$

Combining things, we obtain the estimate in the formulation of the lemma. \blacksquare

4. THE MAIN RESULT

In this section we will prove our main theorem for which we have already made the necessary preparations in the previous sections.

1. Theorem. *Let $A, a > 0$. For all $x \in \mathbf{R}^d$ and all $u \in L^2(\mathbf{R}^d, \gamma)$ we have*

$$(15) \quad \sup_{(y,t) \in \Gamma_x} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma.$$

Proof. We fix $x \in \mathbf{R}^d$ and $(y, t) \in \Gamma_x$. The proof of (15) is based on splitting the integration domain into the annuli C_k as defined by (12) and estimating on each annulus. More explicit,

$$(16) \quad |e^{-t^2 L} u(y)| \leq \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma.$$

From $t \leq m(x)$ we get $t|x| \leq 1$, and by Lemma 1 this implies $t|y| \leq 2$. Together with Lemma 4 we infer, for $\xi \in C_k$, that:

$$\begin{aligned} M_{t^2}(y, \xi) &\leq \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} e^{-1} e^{2^{k+1}} \exp\left(-\frac{1}{2e^2}(2^k - 1)^2\right) \\ &=: \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} e^{-1} c_k. \end{aligned}$$

Combining this with Lemma 2, we obtain

$$(17) \quad M_{t^2}(y, \xi) \lesssim \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} c_k.$$

Also, by (13),

$$|x - \xi| \leq |x - y| + |\xi - y| \leq 2^{k+1}t.$$

It follows that C_k is contained in $D_k := B_{2^{k+1}t}(x)$.

Let us denote the supremum on right-hand side of (15) by $M_\gamma u(x)$. Using (17), we can bound the integral on the right-hand side of (16) by

$$\begin{aligned} \int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma &\lesssim c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \int_{C_k} |u| \, d\gamma \\ &\leq c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \int_{D_k} |u| \, d\gamma \\ &\leq c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \gamma(D_k) M_\gamma u(x) \\ &\stackrel{(\dagger)}{\leq} c_k \frac{2^{kd}}{\Gamma(\frac{d}{2})} \frac{2^{\frac{d}{2}} (2t^2)^{\frac{d}{2}}}{(1 - e^{-2t^2})^{\frac{d}{2}}} e^{2^{k+2}t|x|} M_\gamma u(x) \\ &\stackrel{(\ddagger)}{\leq} c_k \frac{2^{kd} 2^{\frac{d}{2}} C^d}{\Gamma(\frac{d}{2})} e^{2^{k+2}} M_\gamma u(x) \\ &\lesssim \frac{C^d}{\Gamma(\frac{d}{2})} 2^{kd} e^{3 \cdot 2^{k+1}} \exp\left(-\frac{1}{2e^2}(2^k - 1)^2\right) M_\gamma u(x) \end{aligned}$$

C not necessarily the same at all instances. where (†) uses Lemma 3 applied to D_k and (‡) uses that $t \leq m(x)$ implies $t|x| \leq 1$ and $t \leq 1$, the latter implying

$$\left(\frac{2t^2}{1 - e^{-2t^2}} \right)^{\frac{d}{2}} \leq \left(\frac{2}{1 - e^{-2}} \right)^{\frac{d}{2}} =: C^d.$$

(note that $s/(1 - e^{-s})$ is increasing). Where $C \approx 2.313035 \dots$.

Inserting the dependency of c_k upon k as coming from (17) and using that $t \leq 1$, we obtain the bound:

$$\begin{aligned} |e^{-t^2 L} u(y)| &= \sum_{k=0}^{\infty} I_k \\ &\lesssim \frac{C^d}{\Gamma(\frac{d}{2})} M_{\gamma} u(x) \sum_{k=0}^{\infty} 2^{kd} e^{3 \cdot 2^{k+1}} \exp\left(-\frac{1}{2e^2} (2^k - 1)^2\right) \\ &\lesssim \frac{C^d}{\Gamma(\frac{d}{2})} M_{\gamma} u(x) \sum_{k=0}^{\infty} 2^{kd} \exp\left(-\frac{4^k}{2e^2}\right) \exp\left(\left[3 - \frac{1}{2e^2}\right] 2^{k+1}\right) \end{aligned}$$

Set $\alpha = \frac{1}{2e^2}$

$$|e^{-t^2 L} u(y)| \lesssim \frac{C^d}{\Gamma(\frac{d}{2})} M_{\gamma} u(x) \sum_{k=0}^{\infty} 2^{kd} e^{3 \cdot 2^{k+1}} e^{-\alpha(2^k - 1)^2}$$

We split the last summation in to parts, one where $k < d$ and one where $k \geq d$. We begin with $k \geq d$.

$$\begin{aligned} &\sum_{k=d}^{\infty} c_k (A + 2^{k+1})^d e^{2(A+2^{k+1})a} \\ &\lesssim_{a,A} \sum_{k=d}^{\infty} (A + 2^{k+1})^d e^{2^{k+1}(1+2a+aA)} \exp\left(-\frac{4^k}{8e^{2a^2}}\right) \\ &\leq \sum_{k=d}^{\infty} (A + 2^{k+1})^k e^{2^{k+1}(1+2a+aA)} \exp\left(-\frac{4^k}{8e^{2a^2}}\right) \\ &\lesssim_{a,A} 1. \end{aligned}$$

While for $k < d$ we have:

$$\begin{aligned} &\sum_{k=0}^{d-1} c_k (A + 2^{k+1})^d e^{2(A+2^{k+1})a} \\ &\lesssim_{a,A} \sum_{k=0}^{d-1} (A + 2^{k+1})^d e^{2^{k+1}(1+2a+aA)} \exp\left(-\frac{4^k}{8e^{2a^2}}\right) \\ &\leq (A + 2^d)^d \sum_{k=0}^{d-1} e^{2^{k+1}(1+2a+aA)} \exp\left(-\frac{4^k}{8e^{2a^2}}\right) \\ &\lesssim_{a,A} d 2^{d^2}. \end{aligned}$$

Considering the sum

$$\sum_{k=0}^{d-1} 2^{(k+1)d} e^{2^{k+1}(1+2a+aA)} \exp\left(-\frac{4^k}{8e^{2a^2}}\right)$$

We proceed with estimating the argument of this sum as function of s after setting $s = 2^{k+1}$, $\alpha = 1 + 2a + aA$ and $B = \frac{1}{32}e^{-2a^2}$, which is smaller than $\frac{1}{32}$ and $\alpha \leq 4aA$

$$s \mapsto s^d e^{\alpha s} e^{-Bs^2}.$$

To find its maximum we need to solve $-2s^2B + s\alpha + d = 0$. As a maximum we obtain:

$$\left(\frac{\alpha + \sqrt{\alpha^2 + 8dB}}{B}\right)^d \exp\left(\frac{1}{8}\frac{\alpha^2}{B}\right) \exp\left(\frac{1}{8}\frac{\sqrt{\alpha^2 + 8dB}}{B}\right) \frac{e^{-\frac{1}{2}d}}{4^d}$$

Smaller than

$$(\alpha + 4\sqrt{4(aA)^2 + 2dB})^d \exp\left(\frac{2(aA)^2}{B}\right) \exp\left(\frac{1}{4}\frac{\sqrt{4(aA)^2 + 2dB}}{B}\right) \frac{e^{-\frac{1}{2}d}}{(4B)^d}$$

Smaller than

$$(aA + \sqrt{4(aA)^2 + 2dB})^d \exp\left(\frac{2(aA)^2}{B}\right) \exp\left(\frac{1}{4}\frac{\sqrt{4(aA)^2 + 2dB}}{B}\right) \frac{e^{-\frac{1}{2}d}}{B^d}$$

Smaller than

$$C_{A,a}^d d^{\frac{d}{2}} \exp\left(\alpha_{A,a} \sqrt{d}\right) e^{-\frac{1}{2}d}$$

Say $A^2 < d/4$, then smaller than a constant only depending on a and A times

$$C_a'^d d^{\frac{d}{2}} e^{\frac{1}{4}d^2} \exp\left(4e^{2a^2} \sqrt{\frac{d}{2}}\right) e^{-\frac{1}{2}d}$$

where C_a' only depends on a Say $A^2 > d/4$ then smaller than a constant only depending on a and A .

Concluding we get that

$$\sum_{k=0}^{\infty} c_k (A + 2^{k+1})^d e^{2(A+2^{k+1})a} \lesssim_{a,A} d C_a^d d^{\frac{d}{2}} e^{\frac{1}{4}d^2} \exp\left(4e^{2a^2} \sqrt{\frac{d}{2}}\right) e^{-\frac{1}{2}d}$$

$$|e^{-t^2 L} u(y)| = \sum_{k=0}^{\infty} I_k \lesssim_{A,a} M_{\gamma} u(x) \frac{1}{\Gamma(\frac{d}{2})} C_{A,a}^d d^{\frac{d}{2}+1} e^{\alpha \sqrt{d}} e^{-\frac{1}{2}d}$$

Also

$$\Gamma\left(\frac{d}{2}\right) \sim \sqrt{\frac{4\pi}{d}} \frac{d^{\frac{d}{2}}}{(2e)^{\frac{d}{2}}}$$

$$\frac{1}{\Gamma(\frac{d}{2})} \sim \sqrt{\frac{d}{4\pi}} \frac{(2e)^{\frac{d}{2}}}{d^{\frac{d}{2}}}$$

So

$$\begin{aligned}
|e^{-t^2 L} u(y)| &= \sum_{k=0}^{\infty} I_k \lesssim_{A,a} M_{\gamma} u(x) \frac{1}{\Gamma(\frac{d}{2})} C_{A,a}^d d^{\frac{d}{2}+1} e^{\alpha\sqrt{d}} e^{-\frac{1}{2}d} \\
&\sim M_{\gamma} u(x) \sqrt{\frac{d}{4\pi}} \frac{(2e)^{\frac{d}{2}}}{d^{\frac{d}{2}}} C_{A,a}^d d^{\frac{d}{2}+1} e^{\alpha\sqrt{d}} e^{-\frac{1}{2}d} \\
&= M_{\gamma} u(x) \sqrt{\frac{d}{4\pi}} C_{A,a}^d d e^{\alpha\sqrt{d}} e^{-\frac{1}{2}d}
\end{aligned}$$

valid for all $y \in \Gamma_x^{(A,a)}$. As the sum on the right-hand side evidently converges, we see that taking the supremum proves (15). \blacksquare

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DELFT INSTITUTE OF APPLIED MATHEMATICS, DELFT UNIVERSITY OF TECHNOLOGY, P.O.
 BOX 5031, 2600 GA DELFT, THE NETHERLANDS
E-mail address: j.j.b.teuwen@tudelft.nl
URL: <http://fa.its.tudelft.nl/~teuwen/>