A note on the Gaussian maximal function

Jonas Teuwen

ABSTRACT. In this note we give an improvement on a result first demonstrated by Pineda and Urbina [3]. In particular we present an improvement to their Lemma 1.1 which gives the boundedness of the Gaussian maximal function associated to the Ornstein-Uhlenbeck operator.

We present a proof which is at least to the author more transparant. Our main finding in this note is that our proof allows to use a larger cone and actually obtain the maximal function boundedness for a whole class of cones $\Gamma_x^{(A,a)}(\gamma)$.

1. Introduction

Maximal functions are one of the most studied objects in harmonic analysis. For instance, the classical real-valued harmonic analytic maximal function can be seen to be of the form

$$\sup_{(y,t)\in\Gamma_x} |\mathrm{e}^{-t^2\Delta}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \,\mathrm{d}\lambda.$$

(verify) In this note we are interested in gaussian harmonic analysis which is harmonic analysis with respect to the gaussian measure as opposed to the Lebesgue measure in classical real-valued harmonic analysis. As the gaussian maximal function will be of our main interest we

The gaussian maximal function will be our main interest. Formally we are looking for

Pineda and Urbina [3]

1.1. Notation. To begin, let us fix some notation. As is common, we use N to represent a positive integer. That is, $N \in \mathbf{Z}_+ = \{1, 2, 3, \dots\}$. In the same way we cast letters that denote the number of dimensions, e.g. $d \in \mathbf{R}^d$ as positive integers.

We use the capital letter T to denote a “time” endpoint, for instance, when writing $t \in (0, T]$.

2. The Mehler kernel and friends

2.1. Setting. Our setting is the one concerning the *Ornstein-Uhlenbeck* operator L which is defined as:

$$L := -\frac{1}{2}\Delta + x \cdot \nabla, \tag{1}$$

We define the Mehler kernel (see e.g., Sjögren [5]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(\mathrm{e}^{-tL})_t$. More precisely, this means:

$$\mathrm{e}^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot)u \,\mathrm{d}\gamma. \tag{2}$$

It is often more convenient to use e^{-t^2L} instead of e^{-tL} as is done in e.g., Portal [4] and we will also do so.

2.2. The Mehler kernel. For the calculation of the Mehler kernel M_t in (2) we refer to e.g., Sjögren [5] which additionally offers related results such as those related to Hermite polynomials.

Observe that the kernel M_{t^2} is invariant under the permutation $x \longleftrightarrow y$. A formula for M_t which honors this observation is:

$$M_{t^2}(x,y) = \frac{\exp\left(-\mathrm{e}^{-2t^2}\frac{|x-y|^2}{1-\mathrm{e}^{-2t^2}}\right)}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\,\frac{\exp\left(2\mathrm{e}^{-t^2}\frac{\langle x,y\rangle}{1+\mathrm{e}^{-t^2}}\right)}{(1+\mathrm{e}^{-t^2})^{\frac{d}{2}}}. \tag{3}$$

3. Some fine lemmata and definitions

3.1. *m*inimal function. We recall the lemma from [1, lemma 2.3] which first –although implicitly– appeared in [2]. For what follows it will be convenient to define a function m as:

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\} = 1 \vee \frac{1}{|x|}.$$

We use m in our next lemma.

1. Lemma. *Let a, A be strictly positive numbers. We have for x, y in $\mathbf{R}^d$ that:*

(1) If $|x-y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1+aA)m(y)$;
(2) Likewise, if $|x-y| < Am(x)$, then $m(x) \leqslant (1+A)m(y)$ and $m(y) \leqslant 2(1+A)m(x)$.

Recall that:

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}_+^d : |x-y| < At \text{ and } t \leqslant am(x)\}. \tag{4}$$

To ease the notational burden a bit, we will write Γ_x^a and mean $\Gamma_x^{(1,a)}$. Using this notation we can deduce a cone version of Lemma 1. That is:

2. Lemma. *Let a, A be strictly positive numbers. Then:*

(1) If $(y,t) \in \Gamma_x^{(A,a)}$ then $t \leqslant (1+aA)m(y)$;
(2) If $(y,t) \in \Gamma_x^{(A,a)}$ then $(x,t) \in \Gamma_y^{(1+aA,a)}$.

In what is next we will use a global/local region dichotomy and define it as follows:

1. Definition. *Given $\tau > 0$, the set N_τ is given as:*

$$N_\tau(x) := N_\tau := \{(x,y) \in \mathbf{R}^{2d} : |x-y| \leqslant \tau m(x)\}. \tag{5}$$

Sometimes it is easier to work with the set $N_\tau(B)$, which is given for $B := B_r(x)$ as:

$$N_\tau(B) := \{y \in \mathbf{R}^d : |x-y| \leqslant \tau m(x)\}. \tag{6}$$

When we partition the space into N_τ and its complement, we call the part belonging to N_τ the local region *and the part belonging to its complement the* global region*.*

The set $t \leqslant am(x)$ is used in the definition of the cones $\Gamma_x^{(A,a)}$ and we will name it D^a, that is:

$$D^a := \{(x,t) \in \mathbf{R}_+^d : t \leqslant am(x)\}. \tag{7}$$

We will write $D := D^1$ for simplicity.

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

3. LEMMA. *Let $\alpha > 0$ and $|x-y| \leqslant \alpha m(x)$. Then the Gaussians in x and y respectively are comparable. In particular this means that,*

$$e^{-|x|^2} \simeq e^{-|y|^2}.$$

REMARK. More precisely, from the proof we get the estimates (8) and (9). That is:

$$e^{-|x|^2} \leqslant e^{(1+\alpha)^2-1} e^{-|y|^2},$$

and,

$$e^{-|y|^2} \leqslant e^{(1+\alpha)^2} e^{2(1+\alpha)} e^{-|x|^2}.$$

PROOF. Let x and y be such that $|x-y| \leqslant \alpha m(x)$. By the inverse triangle inequality we get,

$$\begin{aligned} |y|^2 &\leqslant (\alpha m(x) + |x|)^2 \\ &= \alpha^2 + 2\alpha m(x)|x| + |x|^2 \\ &\leqslant \alpha^2 + 2\alpha + |x|^2. \end{aligned}$$

Therefore,

$$e^{-|x|^2} \leqslant e^{-|y|^2} e^{(1+\alpha)^2} e^{-1}. \tag{8}$$

For the reverse direction we use Lemma 1 to infer $t \leqslant (1+\alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant (1+\alpha)^2 + 2(1+\alpha) + |y|^2.$$

Therefore,

$$e^{-|y|^2} \leqslant e^{-|x|^2} e^{(1+\alpha)^2} e^{2(1+\alpha)}. \tag{9}$$

Combining we get:

$$e^{-(1+\alpha)^2} e^{-1} e^{-(1+2\alpha)} \overset{(9)}{\leqslant} \frac{e^{-|x|^2}}{e^{-|y|^2}} \overset{(8)}{\leqslant} e^{-1} e^{(1+\alpha)^2}.$$

Summarizing we thus have that,

$$e^{-|x|^2} \simeq e^{-|y|^2},$$

as required. ■

4. LEMMA. *Let x, y and z in $\mathbf{R}^d$. Set*

$$\tau = \frac{1}{2}(1+2\alpha)(1+\alpha).$$

If $|y-z| > \tau m(y)$ and $|x-y| \leqslant \alpha m(x)$ then $|x-z| > \frac{1}{2} m(x)$.

PROOF. We assume that $(y,z) \notin N_\tau$ and $(y,t) \in \Gamma_x^{(A,a)}$. Written out this gives by (5) the inequality $|y-z| > \tau m(y)$, and by (4) the inequality $|x-y| < aAm(x)$. Note that the latter inequality together with Lemma 1 yields,

$$\frac{1}{2}\frac{1}{1+\alpha} m(y) \leqslant m(x) \leqslant (1+\alpha)m(y). \tag{10}$$

Combining we get $|x-y| < \alpha(1+\alpha)m(y)$. Now we are in position to apply the triangle inequality:

$$|x-z| \geqslant |y-z| - |x-y| > \tau m(y) - \alpha(1+\alpha)m(y).$$

As we require an lower bound in terms of $m(x)$ and not $m(y)$, we again apply (10) to obtain:

$$\begin{aligned}|x-z| \geqslant |y-z|-|x-y| > \tau m(y)-\alpha m(y)\\ \geqslant \tau\frac{1}{1+\alpha}m(x)-\alpha m(x)\\ \geqslant \frac{1}{2}m(x).\end{aligned}$$

Very well, proof is done. ■

4. On-diagonal estimates

4.1. Kernel estimates. We begin with a technical lemma which will be useful on several occasions.

5. LEMMA. *Let t be in $(0,T]$ and let $\alpha > 1$. Then,*

$$\alpha e^{-T^2} \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} \leqslant \alpha. \tag{11}$$

and,

$$0 \leqslant \frac{1}{1+e^{-t^2}} - \frac{1}{1+e^{-\frac{t^2}{\alpha}}} \leqslant \frac{1}{4}\left(1-\frac{1}{\alpha}\right). \tag{12}$$

PROOF. We start with (11) and apply the mean value theorem to the function $f(\xi)=\xi^\alpha$. For $0<\xi'<\xi$ this gives that:

$$f(\xi)-f(\xi') = \alpha\hat{\xi}^{\alpha-1}(\xi-\xi') \text{ for some } \hat{\xi} \text{ in } [\xi,\xi'].$$

Picking $\xi=1$ and $\xi' = e^{-\frac{t^2}{\alpha}}$ yields:

$$\frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} = \alpha\hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[\exp\left(-\frac{t^2}{\alpha}\right),1\right]. \tag{13}$$

Combining this result with the monotonicity of $\xi \mapsto \alpha\xi^{\alpha-1}$ we obtain:

$$\alpha e^{-t^2} \leqslant \alpha\exp\left(-t^2\frac{\alpha-1}{\alpha}\right) \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}}.$$

Together with $e^{-T^2} \leqslant e^{-t^2}$ we obtain,

$$\alpha e^{-T^2} \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} \downarrow \alpha.$$

We proceed with (12). Recalling that $\alpha>1$ one can directly verify that:

$$\frac{1}{1+e^{-t^2}} - \frac{1}{1+e^{-\frac{t^2}{\alpha}}} \geqslant 0,$$

and that the function on the left-hand side is decreasing. To find an upper bound we compute the limit as t goes to 0. That is:

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t^2}\left[\frac{1}{1+e^{-t^2}} - \frac{1}{1+e^{-\frac{t^2}{\alpha}}}\right] &= \lim_{t\to 0}\frac{1}{2t}\left[\frac{2te^{-t^2}}{(1+e^{-t^2})^2} - \frac{1}{\alpha}\frac{2te^{-\frac{t^2}{\alpha}}}{(1+e^{-\frac{t^2}{\alpha}})^2}\right]\\ &= \lim_{t\to 0}\left[\frac{e^{-t^2}}{(1+e^{-t^2})^2} - \frac{1}{\alpha}\frac{e^{-\frac{t^2}{\alpha}}}{(1+e^{-\frac{t^2}{\alpha}})^2}\right]\\ &\uparrow \frac{1}{4}\left(1-\frac{1}{\alpha}\right).\end{aligned}$$

Which is as asserted and completes the proof. ■

The following lemma will be useful when transfering estimates from $M_{\frac{t^2}{\alpha}}$ to M_{t^2}. It follows from the mean value theorem applied to $\xi \mapsto \xi^\alpha$.

6. Lemma. *For $\alpha > 1$ and t in $(0,T]$ and all let x,y in $\mathbf{R}^d$ we have that:*

$$\exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-\frac{1}{2}\frac{\alpha}{\mathrm{e}^{T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right). \tag{14}$$

Proof. Let t in $(0,T]$. Applying (11) we get:

$$\exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\frac{1-\mathrm{e}^{-t^2}}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-\frac{1}{2}\frac{\alpha}{\mathrm{e}^{T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right).$$

Which is as asserted. ■

Our first lemma is about estimating $M_{\frac{t^2}{\alpha}}$ in terms of M_{t^2}.

4.1.1. *Time-scaling of the Mehler kernel.*

7. Lemma. *Let $\alpha \geqslant 2\mathrm{e}^{T^2}$, t in $(0,T]$ and x,y in $\mathbf{R}^d$. If $t|x| \leqslant C$ and $t|y| \leqslant C$ then:*

$$M_{\frac{t^2}{\alpha}}(x,y) \leqslant \alpha^{\frac{d}{2}}\mathrm{e}^{\frac{C^2}{2}}\exp\left(-\frac{\alpha}{2\mathrm{e}^{T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right)M_{t^2}(x,y). \tag{15}$$

Proof. To prove the lemma we compute $M_{\frac{t^2}{\alpha}}M_{t^2}^{-1}$. First note that (11) gives

$$\alpha^{\frac{d}{2}}\mathrm{e}^{-\frac{d}{2}T^2} \leqslant \frac{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}{(1-\mathrm{e}^{-\frac{t^2}{\alpha}})^{\frac{d}{2}}} \leqslant \alpha^{\frac{d}{2}}.$$

Combining the exponentials also gives,

$$\begin{aligned}\exp\left(-2\mathrm{e}^{-t^2}\frac{\langle x,y\rangle}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right)&\exp\left(2\mathrm{e}^{-t^2}\frac{\langle x,y\rangle}{1+\mathrm{e}^{-t^2}}\right)\\ &= \exp\left(\frac{2}{t^2}\left[\frac{1}{1+\mathrm{e}^{-t^2}}-\frac{1}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right]t^2\langle x,y\rangle\right).\end{aligned}$$

(12) Using this result and nothing that $|x+y|^2 \leqslant 2|x|^2+2|y|^2$ yields:

$$\begin{aligned}\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right)\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+\mathrm{e}^{-t^2}}\right) &\leqslant \exp\left(\frac{1}{8}\left(1-\frac{1}{\alpha}\right)t^2|x+y|^2\right)\\ &\leqslant \exp\left(\frac{1}{4}t^2|x|^2\right)\exp\left(\frac{1}{4}t^2|y|^2\right).\end{aligned}$$

From $t|x| \leqslant C$ and $t|y| \leqslant C$ we obtain that:

$$\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right)\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+\mathrm{e}^{-t^2}}\right) \leqslant \mathrm{e}^{\frac{C^2}{2}}.$$

Combining Lemma 6 and equation (14) gives is almost the final estimate.

$$\begin{aligned}\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)} &\leqslant \alpha^{\frac{d}{2}}\mathrm{e}^{\frac{C^2}{2}}\exp\left(\frac{1}{2}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right)\exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}}\right)\\ &\leqslant \alpha^{\frac{d}{2}}\mathrm{e}^{\frac{C^2}{2}}\exp\left(\frac{1}{2}\left[1-\frac{\alpha}{2\mathrm{e}^{T^2}}\right]\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right)\exp\left(-\frac{\alpha}{2\mathrm{e}^{T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right).\end{aligned}$$

Finally, we apply the assumption $\alpha \geqslant 2\mathrm{e}^{T^2}$ to obtain:

$$\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)} \leqslant \alpha^{\frac{d}{2}}\mathrm{e}^{\frac{C^2}{2}}\exp\left(-\frac{\alpha}{2\mathrm{e}^{T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right).$$

Which is as asserted. ■

4.2. On-diagonal kernel estimates on annuli. As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli C_k. For the sake of simplicity we will write $B := B_t(x)$ and mean that B is the closed ball with center x and radius t. Furthermore, we use notations such as $2B$ to mean the ball obtained from B by multiplying its radius by 2.

The C_k are given by,

$$C_k(B) := C_k = \begin{cases} 2B & \text{if } k = 0, \\ 2^{k+1}B \setminus 2^k B & \text{for } k \geqslant 1. \end{cases} \tag{16}$$

So, whenever ξ is in C_k, we get for $k \geqslant 1$:

$$2^k at < |y - \xi| \leqslant 2^{k+1}at. \tag{17}$$

While we get for $k = 0$:

$$|y - \xi| \leqslant 2at. \tag{18}$$

8. Lemma. *Given $a > 0$, let $B = B_{at}(y)$ and ξ in C_k. Furthermore, assume that $t \leqslant am(y)$ for some $A > 0$. Then we have:*

$$M_{t^2}(y, \xi) \leqslant \frac{e^{|y|^2}}{(1 - e^{-t^2})^{\frac{d}{2}}} \exp(2^{k+1}aC) \begin{cases} 1 & \textit{if } k = 0, \\ \exp\left(-\frac{a^2}{2e^{2t^2}} 4^{k+1}\right) & \textit{if } k \geqslant 1. \end{cases}$$

Proof. Let $B = B_{at}(y)$ and let C_k be as in (16). We consider the two exponentials in the Mehler kernel (3) separately. First we consider

$$\exp\left(2e^{-t^2} \frac{\langle y, \xi \rangle}{1 + e^{-t^2}}\right).$$

Using the Cauchy-Schwarz inequality we get that:

$$|\langle y, \xi \rangle| \leqslant |y - \xi||y| + |y|^2 \tag{19}$$

Next, note that

$$\frac{e^{-t^2}}{1 + e^{-t^2}} \leqslant \frac{1}{2}.$$

Together with (19) this gives for $k \geqslant 1$:

$$\begin{aligned} \exp\left(2e^{-t^2} \frac{\langle y, \xi \rangle}{1 + e^{-t^2}}\right) &\leqslant \exp(|\langle y, \xi \rangle|) \\ &\leqslant \exp(|y - \xi||y|) \exp(|y|^2) \\ &\overset{\text{(i)}}{\leqslant} \exp(2^{k+1}aC) \exp(|y|^2). \end{aligned}$$

Where (i) uses (17, 18) and $t|y| \leqslant C$. Considering the first exponential in the Mehler kernel (3) we get:

$$\exp\left(-e^{-2t^2} \frac{|y - \xi|^2}{1 - e^{-2t^2}}\right).$$

This inequality together with

$$\frac{t^2}{1 - e^{-2t^2}} \geqslant \frac{1}{2},$$

yields using (17, 18),

$$\exp\left(-e^{-2t^2} \frac{|y - \xi|^2}{1 - e^{-2t^2}}\right) \leqslant \exp\left(-\frac{a^2}{2e^{2t^2}} 4^{k+1}\right).$$

Concluding we get:

$$\exp\left(-\mathrm{e}^{-2t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right) \leqslant \begin{cases} 1 & \text{if } k=0, \\ \exp\left(-\frac{a^2}{2\mathrm{e}^{2t^2}}4^{k+1}\right) & \text{if } k \geqslant 1. \end{cases}$$

Thus, we can estimate the Mehler kernel (3) M_{t^2} on C_k from above by:

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp(2^{k+1}aC)\begin{cases} 1 & \text{if } k=0, \\ \exp\left(-\frac{a^2}{2\mathrm{e}^{2t^2}}4^{k+1}\right) & \text{if } k \geqslant 1. \end{cases}$$

Done. ∎

5. The boundedness of some non-tangential maximal operators

9. Lemma. *Let $B_t(x)$ be the Euclidean ball with radius t and center x. If γ is the normalized Gaussian measure with density $\sim \exp(-|x|^2)$ we have:*

$$\gamma(B_t(x)) \leqslant C_d \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\mathrm{e}^{-t^2}t^d. \tag{20}$$

Proof. Next, remark that for a ball $B := B_t(x)$ there holds that

$$\begin{aligned}\int_B \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi &= \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{-2\langle x,\xi-x\rangle}\,\mathrm{d}\xi \\ &\leqslant \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{2|x||\xi-x|}\,\mathrm{d}\xi \\ &\leqslant \pi^{\frac{d}{2}}\mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)).\end{aligned}$$

That is:

$$\gamma(B_t(x)) \leqslant \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)). \tag{21}$$

We will estimate the Gaussian volume of the ball $B_t(0)$. To ease the notation, define $C_d := |S_{d-1}|\pi^{-\frac{d}{2}}$ where $|S_{d-1}|$ is the surface area of the d-dimensional sphere. Using polar coordinates we then obtain:

$$\begin{aligned}\gamma(B_t(0)) &= \pi^{-\frac{d}{2}}\int_{B_t(0)} \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi \\ &= C_d\int_0^t \mathrm{e}^{-r^2}r^{d-1}\,\mathrm{d}r \\ &\leqslant C_d\mathrm{e}^{-t^2}t^d.\end{aligned}$$

Upon combining this result with (21) we obtain (20) as promised. ∎

The following lemma is a small modification of [3, lemma 1.1] with a new proof.

10. Lemma. *Let $A, a > 0$. For all x in $\mathbf{R}^d$ and all u in L^2_γ we have*

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |\mathrm{e}^{-t^2L}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u|\,\mathrm{d}\gamma. \tag{22}$$

Proof. Since we have $|x-y| < At$ and $t \leqslant am(x)$ we infer that $t|x| \leqslant a$. By Lemma 2 we also that $t|y| \leqslant 1 + aA$. From this and Lemma 8 we infer that:

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}\mathrm{e}^{A(1+aA)2^{k+1}}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\begin{cases} 1 & \text{if } k=0, \\ \exp\left(-\frac{A^2}{8\mathrm{e}^{2t^2}}4^k\right) & \text{if } k \geqslant 1. \end{cases} \tag{23}$$

Using Lemma 9 we get that

$$\gamma(B_t(x)) \leqslant C_d \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\mathrm{e}^{-t^2}t^d. \tag{24}$$

Recalling that $(y,t) \in \Gamma_x^{(A,a)}$ we have $t \leqslant am(x)$ and $|x-y| \leqslant aAm(x)$. Hence by the remark following Lemma 3 we get

$$\mathrm{e}^{|y|^2} \leqslant \mathrm{e}^{(2+aA)^2-1}\mathrm{e}^{|x|^2}.$$

Concluding we get

$$\gamma(B_t(x))\mathrm{e}^{|y|^2} \lesssim_{d,a,A} \mathrm{e}^{-t^2}t^d. \tag{25}$$

In the next step we will bound the integral on the right-hand side by the maximal function centered at x. For this we need to scale up the C_k. So,

$$(2^{k-1}-1)At \leqslant ||\xi - y| - |x-y|| \leqslant |y-\xi| \leqslant |x-y| + |\xi - y| \leqslant (2^k+1)At.$$

So, we can bound the integral on the right-hand side of (26) by

$$\int_{C_k} |u|\,\mathrm{d}\gamma \leqslant \int_{B(x,2^{k+1}At)} |u|\,\mathrm{d}\gamma$$

Next, we apply (25) to obtain that

$$\mathrm{e}^{|y|^2}\int_{C_k}|u|\,\mathrm{d}\gamma \lesssim_{d,a,A} \mathrm{e}^{-4^{k+1}A^2t^2}4^{(k+1)d}t^d \sup_{r>0} \fint_{B_r(x)} |u|\,\mathrm{d}\gamma.$$

Using (23) we can bound the maximal function:

$$\begin{aligned}
\mathrm{e}^{-t^2L}|u(y)| &= \sum_{k=0}^{\infty}\int_{C_k(B)} M_{t^2}(y,\cdot)|u|\,\mathrm{d}\gamma \\
&\leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\sum_{k=0}^{\infty}\mathrm{e}^{-\frac{A^2}{2}4^{k+1}}\mathrm{e}^{A(1+aA)2^{k+1}}\int_{C_k(B)}|u|\,\mathrm{d}\gamma \\
&\leqslant \frac{t^d}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\sum_{k=0}^{\infty}\mathrm{e}^{-\frac{A^2}{2}4^{k+1}}4^{d(k+1)}\int_{C_k(B)}|u|\,\mathrm{d}\gamma.
\end{aligned} \tag{26}$$

Considering the exponential in the sum, and collecting terms we get:

$$\begin{aligned}
-\frac{A^2}{2}4^{k+1} + A(1+aA)2^{k+1} &= -\frac{1}{2}[(A2^{k+1}-(1+aA))^2 - (1+aA)^2] \\
&= -A2^{k+1}[A2^k - (1+aA)],
\end{aligned}$$

which will be negative as long as $A2^k$ is larger than $1+aA$. That is, whenever

$$2^k > \frac{1+aA}{A} = a + \frac{1}{A}.$$

For the common choice of $A=1$ and $a=2$ this means that there has to hold that $k \geqslant 2$.

Wrapping it up, we have that:

$$\mathrm{e}^{-t^2L}|u(y)| \lesssim C_d 4^d A^d \frac{t^d}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\sum_{k=0}^{\infty}\mathrm{e}^{-2^{k+1}[2^k-(A^{-1}+a)]}\mathrm{e}^{-4^k 8A^2t^2}2^{kd}\sup_{r>0}\fint_{B_r(x)}|u|\,\mathrm{d}\gamma.$$

The function in front of the sum is bounded from below by 1 and bounded from above by $(t+1)^d$. As $t \leqslant am(x) \leqslant a$ we can bound it by $(a+1)^d$. Therefore,

$$\mathrm{e}^{-t^2L}|u(y)| \lesssim C_d 4^d A^d (1+a)^d \sum_{k=0}^{\infty}\mathrm{e}^{-2^{k+1}[2^k-(A^{-1}+a)]}\mathrm{e}^{-4^k 8A^2t^2}2^{kd}\sup_{r>0}\fint_{B_r(x)}|u|\,\mathrm{d}\gamma.$$

Which is what we wanted to prove. ∎

References

[1] Jan Maas, Jan Neerven, and Pierre Portal. "Whitney coverings and the tent spaces $T^{1,q}(\gamma)$ for the Gaussian measure". In: *Arkiv för Matematik* 50.2 (Apr. 2011), pp. 379–395.

[2] Giancarlo Mauceri and Stefano Meda. "BMO and H^1 for the Ornstein-Uhlenbeck operator". In: *Journal of Functional Analysis* 252.1 (Nov. 2007), pp. 278–313.

[3] E Pineda and W Urbina. "Non Tangential Convergence for the Ornstein-Uhlenbeck Semigroup". In: *Divulgaciones Matemáticas* 13.2 (2008), pp. 1–19.

[4] Pierre Portal. "Maximal and quadratic Gaussian Hardy spaces". Mar. 2012.

[5] Peter Sjögren. "Operators associated with the hermite semigroup — A survey". In: *The Journal of Fourier Analysis and Applications* 3.S1 (Jan. 1997), pp. 813–823.

Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

E-mail address: j.j.b.teuwen@tudelft.nl

URL: http://fa.its.tudelft.nl/~teuwen/