

# A note on Gaussian maximal functions

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## Abstract

This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded pointwise by the Gaussian Hardy-Littlewood maximal functions. In particular this entails an extension on a result by Pineda and Urbina [1] who proved a similar result for a ‘truncated’ version with fixed parameters of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

*Keywords:* Ornstein-Uhlenbeck semigroup, Mehler kernel, Gaussian maximal function, Gaussian cones, admissible cones

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## 1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well-known that the classical non-tangential maximal function associated with the heat semigroup is bounded pointwise by the Hardy-Littlewood maximal function, for every  $x \in \mathbf{R}^d$ , i.e.,

$$\sup_{\substack{(y,t) \in \mathbf{R}_+^{d+1} \\ |x-y| < t}} |e^{t^2 \Delta} u(y)| \lesssim \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| d\lambda, \text{ with } u \in C_c(\mathbf{R}^d), \quad (1)$$

with  $\lambda$  the Lebesgue measure on  $\mathbf{R}^d$  (cf. [2, Proposition II 2.1.]). Here the action of *heat semigroup*  $e^{t\Delta} u = \rho_t * u$  is given by a convolution of  $u$  with the *heat kernel*

$$\rho_t(\xi) := \frac{e^{-|\xi|^2/4t}}{(4\pi t)^{\frac{d}{2}}}, \text{ with } t > 0 \text{ and } \xi \in \mathbf{R}^d.$$

In this note we are interested in its Gaussian counterpart. The change from Lebesgue measure to the *Gaussian measure*

$$d\gamma(x) := \pi^{-\frac{d}{2}} e^{-|x|^2} d\lambda(x) \quad (2)$$

introduces quite some intricate technical and conceptual difficulties which are due to its non-doubling nature. Instead of the Laplacian, we will use its Gaussian analogue, the *Ornstein-Uhlenbeck operator*  $L$  which is given by

$$L := \frac{1}{2}\Delta - \langle x, \nabla \rangle = -\frac{1}{2}\nabla^* \nabla, \quad (3)$$

where  $\nabla^*$  denotes the adjoint of  $\nabla$  with respect to the measure  $d\gamma$ . Our main result, to be proved in Theorem 1, is the following Gaussian analogue of (1):

$$\sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| d\gamma. \quad (4)$$

Here,

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}_+^{d+1} : |x-y| < At \text{ and } t \leq am(x)\} \quad (5)$$

is the *Gaussian cone* with aperture  $A$  and cut-off parameter  $a$ , and

$$m(x) := \min \left\{ 1, \frac{1}{|x|} \right\}. \quad (6)$$

As shown in [3, Theorem 2.19] the centered Gaussian Hardy-Littlewood maximal function is of weak-type  $(1,1)$  and is  $L^p(\gamma)$ -bounded for  $1 < p \leq \infty$ . In fact, the same result holds when the Gaussian measure  $\gamma$  is replaced by any Radon measure  $\mu$ . Furthermore, if  $\mu$  is doubling, then these results even hold for the *uncentered* Hardy-Littlewood maximal function. For the Gaussian measure  $\gamma$  the uncentered weak-type  $(1,1)$  result is known to fail for  $d > 1$ . Nevertheless, the uncentered Hardy-Littlewood maximal function for  $\gamma$  is  $L^p$ -bounded for  $1 < p \leq \infty$  [4].

A slightly weaker version of the inequality (4) has been proved by Pineda and Urbina [1] who showed that

$$\sup_{(y,t) \in \tilde{\Gamma}_x} |e^{t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| d\gamma,$$

where

$$\tilde{\Gamma}_x(x) = \{(y, t) \in \mathbf{R}_+^d : |x - y| < t \leq \tilde{m}(x)\}$$

is the ‘reduced’ Gaussian cone corresponding to the function

$$\tilde{m}(x) = \min\left\{\frac{1}{2}, \frac{1}{|x|}\right\}.$$

Our proof of (4) is shorter than the one presented in [1]. It has the further advantage of allowing the extension to cones with arbitrary aperture  $A > 0$  and cut-off parameter  $a > 0$  without any additional technicalities. This additional generality is important and has already been used by Portal (cf. the claim made in [5, discussion preceding Lemma 2.3]) to prove the  $H^1$ -boundedness of the Riesz transform associated with  $L$ .

## 2. The Mehler kernel

The *Mehler kernel* (see e.g., [6]) is the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup  $(e^{tL})_{t \geq 0}$ , that is,

$$e^{tL}u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot) u \, d\gamma. \quad (7)$$

There is an abundance of literature on the Mehler kernel and its properties. We shall only use the fact, proved e.g. in the survey paper [6], that it is given explicitly by

$$M_t(x, y) = \frac{\exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right)}{(1 - e^{-2t})^{\frac{d}{2}}} e^{|y|^2}. \quad (8)$$

Note that the symmetry of the semigroup  $e^{tL}$  allows us to conclude that  $M_t(x, y)$  is symmetric in  $x$  and  $y$  as well. A formula for (8) honoring this observation is:

$$M_t(x, y) = \frac{\exp\left(-e^{-2t} \frac{|x - y|^2}{1 - e^{-2t}}\right)}{(1 - e^{-t})^{\frac{d}{2}}} \frac{\exp\left(2e^{-t} \frac{\langle x, y \rangle}{1 + e^{-t}}\right)}{(1 + e^{-t})^{\frac{d}{2}}}. \quad (9)$$

### 3. Some lemmata

We use  $m$  as defined in (6) in our next lemma, which is taken from [7, Lemma 2.3].

**1 Lemma.** *Let  $a, A$  be strictly positive real numbers and  $t > 0$ . We have for  $x, y \in \mathbf{R}^d$  that:*

1. *If  $|x - y| < At$  and  $t \leq am(x)$ , then  $t \leq a(1 + aA)m(y)$ ,*
2. *If  $|x - y| < Am(x)$ , then  $m(x) \leq (1 + A)m(y)$  and  $m(y) \leq 2(1 + A)m(x)$ .*

The next lemma, taken from [8, Proposition 2.1(i)], will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone. For the reader's convenience, we include a short proof.

**2 Lemma.** *Let  $\alpha > 0$  and  $|x - y| \leq \alpha m(x)$ . Then:*

$$e^{-\alpha^2 - 2\alpha} e^{|y|^2} \leq e^{|x|^2} \leq e^{\alpha^2(1+\alpha)^2 + 2\alpha(1+\alpha)} e^{|y|^2}.$$

*Proof.* By the triangle inequality and  $m(x)|x| \leq 1$  we get,

$$|y|^2 \leq (\alpha m(x) + |x|)^2 \leq \alpha^2 + 2\alpha + |x|^2.$$

This gives the first inequality. For the second we use Lemma 1 to infer  $m(x) \leq (1 + \alpha)m(y)$ . Proceeding as before we obtain

$$|x|^2 \leq \alpha^2(1 + \alpha)^2 + 2\alpha(1 + \alpha) + |y|^2,$$

which finishes the proof. ■

#### 3.1. An estimate on Gaussian balls

Let  $B := B_t(x)$  be the open Euclidean ball with radius  $t$  and center  $x$  and let  $\gamma$  be the Gaussian measure as defined by (2). We shall denote by  $S_d$  the surface area of the unit sphere in  $\mathbf{R}^d$ .

**3 Lemma.** *For all  $x \in \mathbf{R}^d$  and  $t > 0$  we have the inequality:*

$$\gamma(B_t(x)) \leq \frac{S_d}{\pi^{\frac{d}{2}}} \frac{t^d}{d} e^{2t|x|} e^{-|x|^2}. \quad (10)$$

*Proof.* Remark that, with  $B := B_t(x)$ ,

$$\begin{aligned}
\int_B e^{-|\xi|^2} d\xi &= e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{-2\langle x, \xi-x \rangle} d\xi \\
&\leq e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{2|x||\xi-x|} d\xi \\
&\leq e^{-|x|^2} e^{2t|x|} \int_B e^{-|\xi-x|^2} d\xi \\
&= \pi^{\frac{d}{2}} e^{2t|x|} e^{-|x|^2} \gamma(B_t(0)).
\end{aligned}$$

So, there holds that

$$\gamma(B_t(x)) \leq e^{2t|x|} e^{-|x|^2} \gamma(B_t(0)). \quad (11)$$

We proceed by noting that

$$\gamma(B_t(0)) \leq \pi^{-\frac{d}{2}} |B_t(0)| \leq \pi^{-\frac{d}{2}} t^d \frac{S_d}{d},$$

and combine this with the previous calculation to obtain

$$\gamma(B_t(x)) \leq \frac{S_d}{\pi^{\frac{d}{2}}} \frac{t^d}{d} e^{2t|x|} e^{-|x|^2}.$$

This completes the proof. ■

### 3.2. Off-diagonal kernel estimates on annuli

As is common in harmonic analysis, we often wish to decompose  $\mathbf{R}^d$  into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli.

Throughout this subsection we fix  $x \in \mathbf{R}^d$ , constants  $A, a \geq 1$ , and a pair  $(y, t) \in \Gamma_x^{(A, a)}$ . We use the notation  $rB$  to mean the ball obtained from the ball  $B$  by multiplying its radius by  $r$ .

The annuli  $C_k := C_k(B_t(y))$  are given by:

$$C_k := \begin{cases} 2B_t(y), & k = 0, \\ 2^{k+1}B_t(y) \setminus 2^k B_t(y), & k \geq 1. \end{cases} \quad (12)$$

So, whenever  $\xi$  is in  $C_k$ , we get for  $k \geq 1$  that

$$2^k t \leq |y - \xi| < 2^{k+1} t. \quad (13)$$

On  $C_k$  we have the following bound for  $M_{t^2}(y, \cdot)$ :

**4 Lemma.** *For all  $\xi \in C_k$  for  $k \geq 1$  we have:*

$$M_{t^2}(y, \xi) \leq \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \exp(2^{k+1}t|y|) \exp\left(-\frac{4^k}{2e^{2t^2}}\right), \quad (14)$$

*Proof.* Considering the first exponential which occurs in the Mehler kernel (9) together with (13) gives for  $k \geq 1$ :

$$\begin{aligned} \exp\left(-e^{-2t^2} \frac{|y - \xi|^2}{1 - e^{-2t^2}}\right) &\leq \exp\left(-\frac{4^k}{e^{2t^2}} \frac{t^2}{1 - e^{-2t^2}}\right) \\ &\stackrel{(\dagger)}{\leq} \exp\left(-\frac{4^k}{2e^{2t^2}}\right), \end{aligned}$$

where  $(\dagger)$  follows from  $1 - e^{-s} \leq s$  for  $s \geq 0$ . Using the estimate  $1 + s \geq 2s$  for  $0 \leq s \leq 1$ , we find for the second exponential in the Mehler kernel (9), by (13) that

$$\begin{aligned} \exp\left(2e^{-t^2} \frac{\langle y, \xi \rangle}{1 + e^{-t^2}}\right) &\leq \exp(|\langle y, \xi \rangle|) \\ &\leq \exp(|\langle y, \xi - y \rangle|)e^{|y|^2} \\ &\leq \exp(2^{k+1}t|y|)e^{|y|^2}. \end{aligned}$$

Combining these estimates we obtain (14), as required. ■

#### 4. The main result

In this section we will prove our main theorem as mentioned in (4) for which the necessary preparations have already been made.

**1 Theorem.** *Let  $A, a > 0$ . For all  $x \in \mathbf{R}^d$  and all  $u \in C_c(\mathbf{R}^d)$  we have*

$$\sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma, \quad (15)$$

where the implicit constant only depends on  $A, a$  and  $d$ .

*Proof.* We fix  $x \in \mathbf{R}^d$  and  $(y, t) \in \Gamma_x^{(A,a)}$ . The proof of (15) is based on splitting the integration domain into the annuli  $C_k$  as defined by (12) and estimating on each annulus. More explicit,

$$|e^{t^2 L} u(y)| \leq \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma. \quad (16)$$

We have  $t \leq am(x) \leq a$  and, by Lemma 1,  $t|y| \leq a(1 + aA)$ . Together with Lemma 4 we infer, for  $\xi \in C_k$  and  $k \geq 1$ , that

$$\begin{aligned} M_{t^2}(y, \xi) &\leq \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \exp(2^{k+1}a(1 + aA)) \exp\left(-\frac{4^k}{2e^{2a^2}}\right) \\ &=: \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} c_k. \end{aligned}$$

Combining this with Lemma 2, we obtain

$$M_{t^2}(y, \xi) \lesssim_{A,a} \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} c_k. \quad (17)$$

Also, by (13) we get

$$|x - \xi| \leq |x - y| + |\xi - y| \leq (2^{k+1} + A)t.$$

Let  $K$  be the smallest integer such that  $2^{k+1} \geq A$  whenever  $k \geq K$ . Then it follows that  $C_k$  for  $k \geq K$  is contained in  $B_{2^{k+2}t}(x)$  and for  $k < K$  is contained in  $B_{2At}(x)$ . We set

$$D_k := D_k(x) = \begin{cases} B_{2^{k+2}t}(x) & \text{if } k \geq K, \\ B_{2At}(x) & \text{elsewhere.} \end{cases}$$

Let us denote the supremum on right-hand side of (15) by  $M_\gamma u(x)$ . Using (17), we can bound the integral on the right-hand side of (16) by

$$\begin{aligned} \int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma &\lesssim_{A,a} c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \int_{C_k} |u| \, d\gamma \\ &\leq c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \int_{D_k} |u| \, d\gamma \\ &\leq c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \gamma(D_k) M_\gamma u(x), \end{aligned}$$

where we pause for a moment to compute a suitable bound for  $\gamma(D_k)$ . As above we have both  $t|x| \leq am(x)|x| \leq a$  and  $t \leq a$ . Together with Lemma 3 applied to  $D_k$  for  $k \geq K$  we obtain:

$$\begin{aligned} \gamma(D_k) e^{|x|^2} &\lesssim_A C^d \frac{S_d}{d} t^{d^2} 2^{kd} e^{2^{k+3}t|x|} e^{-|x|^2} e^{|x|^2} \\ &\lesssim_{A,a,d} t^d 2^{kd} e^{2^{k+3}a}. \end{aligned}$$

Similarly, for  $k < K$ :

$$\gamma(D_k)e^{|x|^2} \lesssim_{A,a,d} t^d e^{2Aa}.$$

Using the bound  $t \leq a$ , we can infer that

$$\frac{t^d}{(1 - e^{-2t^2})^{\frac{d}{2}}} \leq \frac{a^d}{(1 - e^{-2a^2})^{\frac{d}{2}}} \lesssim_{a,d} 1.$$

(note that  $s/(1 - e^{-s})$  is increasing). Combining these computations with the ones above for  $k \geq K$  we get

$$\int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma \lesssim_{A,a,d} c_k 2^{kd} e^{2^{k+2}a} M_\gamma u(x),$$

while for  $k < K$  we get

$$\int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma \lesssim_{A,a,d} c_k M_\gamma u(x).$$

Similarly, for  $\xi \in 2B_t(x)$  we obtain:

$$I_0 := \int_{2B_t} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma \lesssim_{A,a,d} M_\gamma u(x).$$

Inserting the dependency of  $c_k$  upon  $k$  as coming from (17), we obtain the bound:

$$\begin{aligned} |e^{t^2 L} u(y)| &= I_0 + \sum_{k=1}^{K-1} I_k + \sum_{k=K}^{\infty} I_k \\ &\lesssim_{A,a,d} \left[ 1 + \sum_{k=1}^{K-1} c_k + \sum_{k=K}^{\infty} c_k 2^{kd} e^{2^{k+2}a} \right] M_\gamma u(x), \\ &\lesssim_{A,a,d} \left[ 1 + \sum_{k=1}^{K-1} e^{-\frac{4^k}{2e^{2a^2}}} + \sum_{k=K}^{\infty} 2^{kd} e^{2^{k+1}(1+2a+aA)} e^{-\frac{4^k}{2e^{2a^2}}} \right] M_\gamma u(x), \end{aligned}$$

valid for all  $(y, t) \in \Gamma_x^{(A,a)}$ . As the sum on the right-hand side evidently converges, we see that taking the supremum proves (15).  $\blacksquare$

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