A note on the Gaussian maximal function - Version 17 October 2013 + JvN additions

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ABSTRACT. This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded almost surely by the Gaussian Hardy-Littlewood maximal function. In particular this entails improvement on a result by Pineda and Urbina [3] who proved a similar result for a 'trunctated' version of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well known that the classical real-valued maximal function associated with the heat semigroup is bounded almost everywhere by the Hardy-Littlewood maximal function,

$$\sup_{(y,t)\in\Gamma_x} |e^{-t\Delta}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \, d\lambda. \tag{1}$$

Here the action of *heat semigroup* $e^{-t^2\Delta}u = \rho_t * u$ is given by a convolution of u with the *heat kernel*

$$\rho_t(s) := \frac{e^{-|s|^2/4t}}{\pi^{\frac{d}{2}}} \frac{1}{(4t)^{\frac{d}{2}}}.$$

In this note we are interested in its gaussian counterpart. The change from Lebesgue measure to the gaussian measure

$$d\gamma(x) := \pi^{-\frac{d}{2}} e^{-|x|^2} \, dx \tag{2}$$

introduces quite some intricate technical and conceptually difficulties which appears to be due to the fact that the Gaussian measure is non-doubling.

As an analogue to the Laplacian which is symmetric in L^2 with respect to the Lebesgue measure next we introduce the *Ornstein-Uhlenbeck* operator L which is symmetric with respect to the Gaussian measure:

$$L := -\frac{1}{2}\Delta + \langle x, \nabla \rangle = \frac{1}{2}\nabla_\gamma^* \nabla_\gamma, \tag{3}$$

where ∇_γ denotes the realisation of the gradient in $L^2(\mathbf{R}^d, \gamma)$. Our main result, to be proved in (1), is the following gaussian analogue of (1):

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |e^{-tL}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \, d\gamma. \tag{4}$$

Here, $\Gamma_x^{(A,a)}$ is the Gaussian cone defined by

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}_+^d : |x-y| < At \text{ and } t \leqslant am(x)\}, \tag{5}$$

where

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\} = 1 \wedge \frac{1}{|x|}. \tag{6}$$

A slighly weaker version of the inequality (4) has been proved by Pineda and Urbina [3] who shows that

$$\sup_{(y,t)\in\widetilde{\Gamma}_x} |\mathrm{e}^{-t\Delta}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u|\,\mathrm{d}\gamma,$$

where

$$\widetilde{\Gamma}_x(x) = \{(y,t) \in \mathbf{R}^d_+ : |x-y| < t \leqslant \widetilde{m}(x)\}$$

is the 'reduced' gaussian cone corresponding to the function

$$\widetilde{m}(x) = \min\left\{\frac{1}{2}, \frac{1}{|x|}\right\}.$$

Their proof does not seem to easily generalize the range of t from $\frac{1}{2}$ up to 1. The proof of (4) is different and, we believe, more transparent than the one presented in [3]. It has the further virtue of allowing the extension to the cones with arbitrary aperture $A > 0$ and cut-off parameter $a > 0$. This additional generality is very important and has already been used by Portal (cf. the claim made by [4, discussion preceding Lemma 2.3]) to prove the H^1-boundedness of the Riesz transform associated with L.

Before we continue, let us fix some notation. We will use without further reference notation such as $\mathbf{Z}^d$ while we implicitly imply that d is a positive integer. To avoid possible confusion, we define the *positive integers* as the set $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$.

1.0.1. *minimal function.* We recall the lemma from [1, lemma 2.3] which first –although implicitly– appeared in [2]. $\boxed{\text{Nog materiaal toevoegen?}}$

2. The Mehler kernel

2.1. Setting. Recall that we work with the *Ornstein-Uhlenbeck* operator L as given by (3).

We define the Mehler kernel (see e.g., Sjögren [5]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(\mathrm{e}^{-tL})_{t\geqslant 0}$. More precisely, this means:

$$\mathrm{e}^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x,\cdot)u\,\mathrm{d}\gamma. \tag{7}$$

It is often more convenient to use e^{-t^2L} instead of e^{-tL} as is done in e.g., Portal [4].

2.2. The Mehler kernel. There is an abundance of literature on the Mehler kernel an its properties available, but for the present purpose Sjögren [5] will suffice. For instance, the Mehler kernel M_t of (7) is computed there. In addition it offers related results with to the Hermite polynomials.

The kernel M_t is invariant under the permutation $x \longleftrightarrow y$. A formula for M_t which honors this observation is:

$$M_t(x,y) = \frac{\exp\left(-\mathrm{e}^{-2t}\dfrac{|x-y|^2}{1-\mathrm{e}^{-2t}}\right)}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}} \frac{\exp\left(2\mathrm{e}^{-t}\dfrac{\langle x,y\rangle}{1+\mathrm{e}^{-t}}\right)}{(1+\mathrm{e}^{-t})^{\frac{d}{2}}}. \tag{8}$$

3. Some lemmata and definitions

We use m as defined in (6) in our next lemma.

1. Lemma. *Let a, A be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

(1) If $|x - y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1 + aA)m(y)$;
(2) If $|x - y| < Am(x)$, then $m(x) \leqslant (1 + A)m(y)$ and $m(y) \leqslant 2(1 + A)m(x)$.

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. Lemma. *Let $\alpha > 0$ and $|x - y| \leqslant \alpha m(x)$. We get the equivalence:*

$$e^{-\alpha^2(1+\alpha)^2} e^{-2\alpha(1+\alpha)} e^{-|y|^2} \leqslant e^{-|x|^2} \leqslant e^{\alpha^2} e^{2\alpha} e^{-|y|^2}.$$

Proof. By the inverse triangle inequality and $m(x)|x| \leqslant 1$ we get,

$$|y|^2 \leqslant (\alpha m(x) + |x|)^2 \leqslant \alpha^2 + 2\alpha + |x|^2. \tag{9}$$

For the reverse direction we use Lemma 1 to infer $m(x) \leqslant (1 + \alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant \alpha^2(1 + \alpha)^2 + 2\alpha(1 + \alpha) + |y|^2.$$

Combining we get:

$$e^{-\alpha^2(1+\alpha)^2} e^{-2\alpha(1+\alpha)} e^{-|y|^2} \leqslant e^{-|x|^2} \leqslant e^{\alpha^2} e^{2\alpha} e^{-|y|^2}. \tag{10}$$

As required. ∎

4. On-diagonal estimates

4.1. Kernel estimates. Before we proceed with the technicalities we define κ and μ as:

$$\kappa = 2\left(1 + \frac{1}{\alpha}\right)^{-1}, \text{ and } \mu = 2\left(1 - \frac{1}{\alpha}\right)^{-1}.$$

such that κ and μ are conjugate exponents, which means:

$$\frac{1}{\kappa} + \frac{1}{\mu} = 1.$$

We proceed with a simple technical lemma which is given here as it will be used on several occasions.

3. Lemma. *Let $t > 0$ and $\alpha \geqslant 1$. Then,*

$$\alpha e^{-2\frac{t}{\mu}} \leqslant \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} \leqslant \alpha, \tag{11}$$

$$0 \leqslant \frac{1}{t}\left[\frac{e^{-\frac{t}{\alpha}}}{1 + e^{-\frac{t}{\alpha}}} - \frac{e^{-t}}{1 + e^{-t}}\right] \leqslant \frac{1}{2\mu}. \tag{12}$$

Proof. We start with (11) and apply the mean value theorem to the function $f(\xi) = \xi^\alpha$. For $0 < \xi < \xi'$ this gives that:

$$f(\xi) - f(\xi') = \alpha\hat{\xi}^{\alpha-1}(\xi - \xi') \text{ for some } \hat{\xi} \text{ in } [\xi, \xi'].$$

Picking $\xi = 1$ and $\xi' = e^{-\frac{t}{\alpha}}$ yields:

$$\frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} = \alpha\hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[e^{-\frac{t}{\alpha}}, 1\right]. \tag{13}$$

Combining this result with the monotonicity of $\xi \mapsto \alpha\xi^{\alpha-1}$ we obtain:

$$\alpha e^{-2\frac{t}{\mu}} \leqslant \frac{1-e^{-t}}{1-e^{-\frac{t}{\alpha}}} \leqslant \alpha,$$

where the last bound follows from the monotonicity together with the limit as $t \downarrow 0$. We proceed with (12). Recalling that $\alpha \geqslant 1$ one can directly verify that the function

$$t \mapsto \frac{1}{t}\left[\frac{e^{-\frac{t}{\alpha}}}{1+e^{-\frac{t}{\alpha}}} - \frac{e^{-t}}{1+e^{-t}}\right]$$

is non-negative and decreasing. To find an upper bound we compute the limit as t goes to 0. That is:

$$\lim_{t\to 0}\frac{1}{t}\left[\frac{e^{-\frac{t}{\alpha}}}{1+e^{-\frac{t}{\alpha}}} - \frac{e^{-t}}{1+e^{-t}}\right] = \lim_{t\to 0}\left[\frac{1}{\alpha}\frac{e^{-2\frac{t}{\alpha}}}{(1+e^{-\frac{t}{\alpha}})^2} - \frac{e^{-2t}}{(1+e^{-t})^2}\right] \uparrow \frac{1}{2\mu}.$$

Which is as asserted and completes the proof. ∎

The following lemma will be useful when transfering estimates from $M_{\frac{t}{\alpha}}$ to M_t. It follows from the mean value theorem applied to $\xi \mapsto \xi^\alpha$.

4. Lemma. *For $\alpha \geqslant 1$ and $0 < t \leqslant T < \infty$ and all let $x, y \in \mathbf{R}^d$ we have that:*

$$\exp\left(-\frac{1}{e^{2\frac{t}{\alpha}}}\frac{|x-y|^2}{1-e^{-2\frac{t}{\alpha}}}\right) \leqslant \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\kappa}}}\frac{|x-y|^2}{1-e^{-t}}\right). \tag{14}$$

Proof. Noting that

$$1-e^{-2t} = (1-e^{-t})(1+e^{-t})$$

we can quickly deduce the inequality:

$$\frac{e^{-2t}}{1-e^{-2t}} \geqslant \frac{1}{2e^{2t}}\frac{1}{1-e^{-t}}.$$

Which leads to:

$$\exp\left(-e^{-2t}\frac{|x-y|^2}{1-e^{-2t}}\right) \leqslant \exp\left(-\frac{1}{2e^{2t}}\frac{|x-y|^2}{1-e^{-t}}\right).$$

Using the substitution $t \to t/\alpha$ we remark that

$$\begin{aligned}
\exp\left(-e^{-2\frac{t}{\alpha}}\frac{|x-y|^2}{1-e^{-2\frac{t}{\alpha}}}\right) &\leqslant \exp\left(-\frac{1}{2e^{2\frac{t}{\alpha}}}\frac{|x-y|^2}{1-e^{-\frac{t}{\alpha}}}\right)\\
&\leqslant \exp\left(-\frac{1}{2e^{2\frac{t}{\alpha}}}\frac{1-e^{-t}}{1-e^{-\frac{t}{\alpha}}}\frac{|x-y|^2}{1-e^{-t}}\right)\\
&\overset{(11)}{\leqslant} \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\alpha}}}\frac{1}{e^{2\frac{t}{\mu}}}\frac{|x-y|^2}{1-e^{-t}}\right)\\
&\leqslant \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\kappa}}}\frac{|x-y|^2}{1-e^{-t}}\right),
\end{aligned}$$

to finish the proof. ∎

Our first proposition will be useful when we want transfer bounds on M_t for large values of t where the potential plays a smaller role compared to the diffusion to a bound on $M_{t'}$ for t' close to 0 where diffusion plays a much smaller role.

4.1.1. *Time-scaling of the Mehler kernel.*

1. PROPOSITION. *Let $\alpha \geqslant 1$, and $x, y \in \mathbf{R}^d$. Assume that $\alpha \geqslant 4\mathrm{e}^{2\frac{t}{\kappa}}$, then:*

$$M_{\frac{t}{\alpha}}(x,y) \leqslant \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu}|\langle x,y\rangle|\right) \exp\left(-\frac{\alpha}{4\mathrm{e}^{2\frac{t}{\kappa}}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right) M_t(x,y). \tag{15}$$

PROOF. To prove the lemma we compute $M_{\frac{t}{\alpha}} M_t^{-1}$. First recall that (11) gives

$$\alpha \mathrm{e}^{-2\frac{t}{\mu}} \leqslant \frac{1-\mathrm{e}^{-t}}{1-\mathrm{e}^{-\frac{t}{\alpha}}} \leqslant \alpha.$$

This implies that

$$\frac{(1-\mathrm{e}^{-t})^{\frac{d}{2}}}{(1-\mathrm{e}^{-\frac{t}{\alpha}})^{\frac{d}{2}}} \leqslant \alpha^{\frac{d}{2}}.$$

Compute the bounds for the exponentials in the product $M_{\frac{t}{\alpha}} M_t^{-1}$ separately.

$$\begin{aligned}
\exp\left(2\mathrm{e}^{-\frac{t}{\alpha}}\frac{\langle x,y\rangle}{1+\mathrm{e}^{-\frac{t}{\alpha}}}\right) &\exp\left(-2\mathrm{e}^{-t}\frac{\langle x,y\rangle}{1+\mathrm{e}^{-t}}\right) \\
&= \exp\left(\frac{2}{t}\left[\frac{\mathrm{e}^{-\frac{t}{\alpha}}}{1+\mathrm{e}^{-\frac{t}{\alpha}}} - \frac{\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}}\right] t\langle x,y\rangle\right) \\
&\overset{(12)}{\leqslant} \exp\left(\frac{t}{\mu}|\langle x,y\rangle|\right).
\end{aligned}$$

We proceed using Lemma 4.

$$\begin{aligned}
\frac{M_{\frac{t}{\alpha}}(x,y)}{M_t(x,y)} &\leqslant \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu}|\langle x,y\rangle|\right) \exp\left(\mathrm{e}^{-2t}\frac{|x-y|^2}{1-\mathrm{e}^{-2t}}\right) \exp\left(-\mathrm{e}^{-2\frac{t}{\alpha}}\frac{|x-y|^2}{1-\mathrm{e}^{-2\frac{t}{\alpha}}}\right) \\
&\overset{(14)}{\leqslant} \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu}|\langle x,y\rangle|\right) \exp\left(\left[1-\frac{\alpha}{4\mathrm{e}^{2\frac{t}{\kappa}}}\right]\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right) \exp\left(-\frac{\alpha}{4\mathrm{e}^{2\frac{t}{\kappa}}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right).
\end{aligned}$$

Finally, we apply the assumption $\alpha \geqslant 4\mathrm{e}^{2\frac{t}{\kappa}}$ to obtain:

$$\frac{M_{\frac{t}{\alpha}}(x,y)}{M_t(x,y)} \leqslant \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu}|\langle x,y\rangle|\right) \exp\left(-\frac{\alpha}{4\mathrm{e}^{2\frac{t}{\kappa}}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right).$$

Which is as asserted. The assumption $\alpha \geqslant 4\mathrm{e}^{2\frac{t}{\kappa}}$ can be rephrased as the requirement that $\frac{\kappa}{2}\log\left(\frac{\alpha}{4}\right) \geqslant t$. ∎

4.2. An estimate on Gaussian balls.

5. LEMMA. *Let $B_t(x)$ be the Euclidean ball with radius t and center x and let γ be the Gaussian measure (2). We have the inequality:*

$$\frac{\gamma(B_t(x))}{V_d(t)} \leqslant d\pi^{-\frac{d}{2}}\mathrm{e}^{-(t-|x|)^2}. \tag{16}$$

PROOF. Remark that

$$\begin{aligned}
\int_B \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi &= \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{-2\langle x,\xi-x\rangle}\,\mathrm{d}\xi \\
&\leqslant \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{2|x||\xi-x|}\,\mathrm{d}\xi \\
&\leqslant \mathrm{e}^{-|x|^2}\mathrm{e}^{2|x|t}\int_B \mathrm{e}^{-|\xi-x|^2}\,\mathrm{d}\xi \\
&= \pi^{\frac{d}{2}}\mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)).
\end{aligned}$$

So, for a ball $B := B_t(x)$ there holds that

$$\gamma(B_t(x)) \leqslant \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)). \tag{17}$$

We will estimate the Gaussian volume of the ball $B_t(0)$. To save writing, let S_d and V_d be the surface area and volume respectively of the d-dimensional unit sphere. Using polar coordinates we proceed by:

$$\begin{aligned}\gamma(B_t(0)) &= \pi^{-\frac{d}{2}}\int_{B_t(0)} \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi\\ &= S_d\pi^{-\frac{d}{2}}\int_0^t \mathrm{e}^{-r^2}r^{d-1}\,\mathrm{d}r\\ &\leqslant S_d t^d \pi^{-\frac{d}{2}}\mathrm{e}^{-t^2}\\ &= dV_d(t)\pi^{-\frac{d}{2}}\mathrm{e}^{-t^2}.\end{aligned}$$

Upon combining this result with (17) we obtain (16), which is as promised. ■

4.3. On-diagonal kernel estimates on annuli. As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli C_k. For the sake of simplicity we will write $B := B_t(x)$ and mean that B is the closed ball with center x and radius t. Furthermore, we use notations such as $2B$ to mean the ball obtained from B by multiplying its radius by 2.

The C_k are given by,

$$C_k(B) := C_k = (2^{k+1}-1)B \setminus (2^k-1)B. \tag{18}$$

So, whenever ξ is in $C_k(B_t(x))$, we get for $k \geqslant 0$:

$$(2^k-1)t < |y-\xi| \leqslant (2^{k+1}-1)t. \tag{19}$$

6. Lemma. *Given $\alpha \geqslant 1$, let $B = B_{\alpha t}(y)$, $0 < t \leqslant T < \infty$ and $\xi \in C_k$. Then we have:*

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)\alpha t|y|\big)\exp(-\beta(2^k-1)^2),$$

where $\beta = \frac{\alpha^2}{2\mathrm{e}^{2T^2}}$.

Proof. Let $B = B_{\alpha t}(y)$ and let C_k be as in (18). Considering the first exponential which occurs in the Mehler kernel (8) together with (19) gives for $k \geqslant 0$:

$$\begin{aligned}\exp\left(-\mathrm{e}^{-2t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right) &\leqslant \exp\left(-\mathrm{e}^{-2t^2}\frac{(2^k-1)^2\alpha^2t^2}{1-\mathrm{e}^{-2t^2}}\right)\\ &\overset{(\dagger)}{\leqslant} \exp\left(-\frac{\alpha^2}{2\mathrm{e}^{2t^2}}(2^k-1)^2\right).\end{aligned}$$

Where (†) follows from

$$\frac{t}{1-\mathrm{e}^{-2t}} \geqslant \frac{1}{2}.$$

Before we consider the last exponential in the Mehler kernel we note that by Cauchy-Schwarz:

$$|\langle y,\xi\rangle| \leqslant |\langle y-\xi,y\rangle| + |\langle y,y\rangle| \leqslant |y-\xi||y| + |y|^2. \tag{20}$$

Furthermore we have the estimate:

$$\frac{\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}} \leqslant \frac{1}{2},$$

Using these we get for the last exponential in the Mehler kernel (8):

$$\begin{aligned}\exp\left(2e^{-t^2}\frac{\langle y,\xi\rangle}{1+e^{-t}}\right) &\leqslant \exp(|\langle y,\xi\rangle|)\\ &\overset{(20)}{\leqslant} \exp((2^{k+1}-1)\alpha t|y|)e^{|y|^2}\\ &\leqslant \exp(|y-\xi||y|)e^{|y|^2}.\end{aligned}$$

Gluing things, we thus can estimate the Mehler kernel (8) on C_k from above by:

$$M_{t^2}(y,\xi)\leqslant \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)\alpha t|y|\big)\exp\left(-\frac{\alpha^2}{2e^{2t^2}}(2^k-1)^2\right).$$

Setting $\beta=\frac{\alpha^2}{2e^{2T^2}}$ and expanding the last exponential we get:

$$M_{t^2}(y,\xi)\leqslant \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)\alpha t|y|\big)\exp(-\beta(2^k-1)^2).$$

Which is as claimed. ∎

5. The boundedness of some non-tangential maximal operators

Our theorem is a small modification of [3, lemma 1.1] with a new proof.

1. THEOREM. *Let $A,a>0$. For all x in $\mathbf{R}^d$ and all u in L^2_γ we have*

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}}|e^{-t^2L}u(y)|\lesssim \sup_{r>0}\fint_{B_r(x)}|u|\,d\gamma. \tag{21}$$

PROOF. Set $\alpha=aA$. By Lemma 1, if $y\in\Gamma_x^{(A,a)}$ then $x\in\Gamma_y^{(\alpha,1+\alpha)}$. Furthermore we set $\Gamma_x^\alpha=\Gamma_x^{(\alpha,1+\alpha)}$. We will prove (21) by splitting up the integration domain into the annuli $(C_k(B))_k:=(C_k)_{k=0}^\infty$.

$$e^{-t^2L}|u(y)|\leqslant\sum_{k=0}^\infty I_k(y),\text{ where } I_k(y):=\int_{C_k}M_{t^2}(y,\cdot)|u|\,d\gamma.$$

Where we use $B=B(y,\alpha t)$ in C_k in the above and find a suitable upper bound for each integral on the right-hand side which we will denote by I_k for the sake of simplicity.

We proceed with applying Lemma 6. From $t\leqslant\alpha m(x)$ we get $t|x|\leqslant\alpha$ and additionally by Lemma 1 we get $t|y|\leqslant 1+\alpha$. From this and we infer that:

$$M_{t^2}(y,\xi)\leqslant \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)\alpha(1+\alpha)\big)\exp(-\beta(2^k-1)^2). \tag{22}$$

Setting $\beta=\frac{\alpha^2}{2e^{2T^2}}$. Note that β is maximal for $\alpha=\frac{1}{2}$ and after this value, β is decreasing. Setting $\lambda:=\alpha(1+\alpha)$ we get:

$$M_{t^2}(y,\xi)\lesssim_\alpha \frac{e^{|x|^2}}{(1-e^{-t^2})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)\lambda\big)\exp(-\beta(2^k-1)^2). \tag{23}$$

$\boxed{\text{Implied constant lemma 2}}$ Writing c_k for the constants in the exponentials we get:

$$M_{t^2}(y,\xi)\lesssim_\alpha \frac{e^{|x|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \tag{24}$$

Recalling Lemma 5 we get:

$$\gamma(B_t(x))\leqslant V_d d\pi^{-\frac{d}{2}}t^d e^{-(t-|x|)^2}. \tag{25}$$

Where we abbreviate $V_d(1)$ with V_d. Recall

$$V_d \leqslant \frac{1}{\sqrt{\pi}}\left(\frac{2\pi e}{d}\right)^{\frac{d}{2}}.$$

To get,

$$\gamma(B_t(x)) \leqslant \frac{d}{\sqrt{\pi}}\left(\frac{2e}{d}\right)^{\frac{d}{2}} t^d \mathrm{e}^{-(t-|x|)^2} = C_d t^d \mathrm{e}^{-(t-|x|)^2}. \tag{26}$$

This allows us to estimate the remaining unbounded exponential in the Mehler kernel and allow a penalty up to $\mathrm{e}^{-|x|^2}$. Furthermore, we have the following estimate which will make clear how to handle the time part in the Mehler kernel:

$$\frac{t^d}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \leqslant \left(\frac{t^2}{1-\mathrm{e}^{-t^2}}\right)^{\frac{d}{2}} \leqslant \frac{a^d}{(1-\mathrm{e}^{-a^2})^{\frac{d}{2}}}.$$

Let $B' := B(x, 2^{k+1}\alpha t)$ and B as before the ball $B(y, \alpha t)$. In the next step we will bound .... by the maximal function centered at x. For this we need to scale up the C_k. So,

$$|x-\xi| \leqslant |x-y| + |\xi - y| \leqslant \alpha t + (2^{k+1}-1)\alpha t = 2^{k+1}\alpha t.$$

And set $D_k = B(2^{k+1}\alpha t)$. So, we can bound the integral on the right-hand side of (**??**) by

$$\begin{aligned}
\int_{C_k(B)} M_{t^2}(y,\cdot)|u| \,\mathrm{d}\gamma &\lesssim_\alpha c_k \frac{1}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2} \int_{C_k(B)} |u| \,\mathrm{d}\gamma \\
&\leqslant c_k \frac{1}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2} \int_{D_k(B)} |u| \,\mathrm{d}\gamma \\
&\leqslant c_k (M_\gamma u)(x) \frac{1}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2} \gamma(D_k) \\
&\overset{(\dagger)}{\leqslant} c_k C_d (M_\gamma u)(x) \alpha^d 2^{d(k+1)} t^d \frac{1}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2} \mathrm{e}^{-(t-|x|)^2} \\
&\leqslant c_k C_d (M_\gamma u)(x) 2^{kd} \frac{t^d \mathrm{e}^{-t^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{2\alpha} (2\alpha)^d
\end{aligned}$$

Where (†) uses Lemma 5 and $t|x| \leqslant a$.

We can then bound the maximal function:

$$\begin{aligned}
\mathrm{e}^{-t^2 L}|u(y)| &= \sum_{k=0}^{\infty} I_k \\
&\leqslant (M_\gamma u)(x) C_{d,a,A} \sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{A 2^{k+1}} \mathrm{e}^{-\beta 4^k}
\end{aligned}$$

Wrapping it up, we have that:

$$\mathrm{e}^{-t^2 L}|u(y)| \lesssim \fint_{B_r(x)} |u| \,\mathrm{d}\gamma.$$

With implied constant

Which is what we wanted to prove.

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} = \sum_{k=0}^{\infty} x^{kd} \mathrm{e}^{-Cx^{2k}}$$

Noting for $x \geqslant 1$ that $\exp(-Cx^{2k}) \leqslant \exp(-Ckx^2)$, thus,

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \sum_{k=0}^{\infty} x^{kd} (\mathrm{e}^{-Cx^2})^k = \sum_{k=0}^{\infty} (x^d \mathrm{e}^{-Cx^2})^k$$

Here $x = 2$, so

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \sum_{k=0}^{\infty} (2^d \mathrm{e}^{-4C})^k$$

If $2^d < \mathrm{e}^{4C}$, that is whenever $d \log 2 < 4C$, we can compute using the geometric series that

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \frac{1}{1 - 2^d \mathrm{e}^{-4C}} = \frac{\mathrm{e}^{4C}}{\mathrm{e}^{4C} - 2^d}$$

■

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