A note on Gaussian maximal functions

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Abstract

This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded pointwise by the Gaussian Hardy-Littlewood maximal function. In particular this entails an extension on a result by Pineda and Urbina [1] who proved a similar result for a fixed 'truncated' version of the non-tangential maximal function and parameters. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

Keywords: Ornstein-Uhlenbeck operator, Ornstein-Uhlenbeck semigroup, Mehler kernel, Gaussian maximal function, Gaussian Hardy-Littlewood inequality

1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well-known that the classical non-tangential maximal function associated with the heat semigroup is bounded pointwise by the Hardy-Littlewood maximal function, for every $x \in \mathbf{R}^d$,

$$\sup_{\substack{(y,t)\in\mathbf{R}^{d+1}_+ \\ |x-y|<t}} |\mathrm{e}^{t^2\Delta}u(y)| \lesssim \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \,\mathrm{d}\lambda, \text{ with } u \in C_{\mathrm{c}}(\mathbf{R}^d), \tag{1}$$

with λ the Lebesgue measure on $\mathbf{R}^d$ (cf. [2, Proposition II 2.1.]). Here the action of *heat semigroup* $\mathrm{e}^{t\Delta}u = \rho_t * u$ is given by a convolution of u with the *heat kernel*

$$\rho_t(\xi) := \frac{\mathrm{e}^{-|\xi|^2/4t}}{(4\pi t)^{\frac{d}{2}}}, \text{ with } t > 0 \text{ and } \xi \in \mathbf{R}^d.$$

In this note we are interested in its Gaussian counterpart. The change from Lebesgue measure to the *Gaussian measure*

$$\mathrm{d}\gamma(x) := \pi^{-\frac{d}{2}} \mathrm{e}^{-|x|^2} \,\mathrm{d}\lambda(x) \tag{2}$$

introduces quite some intricate technical and conceptual difficulties which are due to its non-doubling nature. Instead of the Laplacian, will use its Gaussian analogue, the *Ornstein-Uhlenbeck operator* L which is given by,

$$L := \frac{1}{2}\Delta - \langle x, \nabla \rangle = -\frac{1}{2}\nabla^* \nabla, \tag{3}$$

where ∇^* denotes the adjoint of ∇ with respect to the measure $\mathrm{d}\gamma$. Our main result, to be proved in Theorem 1, is the following Gaussian analogue of (1):

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |\mathrm{e}^{t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \,\mathrm{d}\gamma. \tag{4}$$

Here,

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}_+^{d+1} \,:\, |x-y| < At \text{ and } t \leqslant am(x)\} \tag{5}$$

is the *Gaussian cone* with aperture A and cut-off parameter a, and

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\}. \tag{6}$$

If we were to consider the boundedness of the centered Gaussian Hardy-Littlewood maximal function, we could combine a weak type $(1,1)$ estimate such as

$$\gamma(\{x \in \mathbf{R}^d : M_\gamma u(x) > \alpha\}) \lesssim \frac{1}{\alpha}\|u\|_{L^1(\mathbf{R}^d;\gamma)},$$

where $M_\gamma u(x)$ is the right-hand side of (4), with the Marcinkiewicz interpolation theorem and the $L^\infty(\mathbf{R}^d;\gamma)$-boundedness to obtain the $L^p(\mathbf{R}^d;\gamma)$-boundedness. Unfortunately, it is unkown to the author whether we have such a weak type $(1,1)$ result, but a natural strategy is to consider the local and global part separately. The weak type $(1,1)$ estimate for the local part has been shown in [3, Lemma 3.2] and the global part will be investigated in a forthcoming paper.

A slightly weaker version of the inequality (4) has been proved by Pineda and Urbina [1] who showed that

$$\sup_{(y,t)\in\widetilde{\Gamma}_x} |\mathrm{e}^{t^2 L}u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \,\mathrm{d}\gamma,$$

where

$$\widetilde{\Gamma}_x(x) = \{(y,t) \in \mathbf{R}^d_+ : |x-y| < t \leqslant \widetilde{m}(x)\}$$

is the 'reduced' Gaussian cone corresponding to the function

$$\widetilde{m}(x) = \min\left\{\frac{1}{2}, \frac{1}{|x|}\right\}.$$

Our proof of (4) is much shorter than the one presented in [1]. It has the further advantage of allowing the extension to cones with arbitrary aperture $A > 0$ and cut-off parameter $a > 0$ without any additional technicalities. This additional generality is very important and has already been used by Portal (cf. the claim made in [4, discussion preceding Lemma 2.3]) to prove the H^1-boundedness of the Riesz transform associated with L.

2. The Mehler kernel

The *Mehler kernel* (see e.g., [5]) is the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(\mathrm{e}^{tL})_{t\geqslant 0}$, that is,

$$\mathrm{e}^{tL}u(x) = \int_{\mathbf{R}^d} M_t(x,\cdot)u \,\mathrm{d}\gamma. \tag{7}$$

There is an abundance of literature on the Mehler kernel and its properties. We shall only use the fact, proved e.g. in the survey paper [5], that it is given explicitly by

$$M_t(x,y) = \frac{\exp\left(-\dfrac{|\mathrm{e}^{-t}x-y|^2}{1-\mathrm{e}^{-2t}}\right)}{(1-\mathrm{e}^{-2t})^{\frac{d}{2}}}\mathrm{e}^{|y|^2}. \tag{8}$$

Note that the symmetry of the semigroup e^{tL} allows us to conclude that $M_t(x,y)$ is symmetric in x and y as well. A formula for (8) honoring this observation is:

$$M_t(x,y) = \frac{\exp\left(-\mathrm{e}^{-2t}\dfrac{|x-y|^2}{1-\mathrm{e}^{-2t}}\right)}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}} \frac{\exp\left(2\mathrm{e}^{-t}\dfrac{\langle x,y\rangle}{1+\mathrm{e}^{-t}}\right)}{(1+\mathrm{e}^{-t})^{\frac{d}{2}}}. \tag{9}$$

3. Some lemmata

We use m as defined in (6) in our next lemma, which is taken from [6, Lemma 2.3].

1 Lemma. *Let a, A be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

1. *If $|x-y| < At$ and $t \leqslant am(x)$, then $t \leqslant a(1+aA)m(y)$;*
2. *If $|x-y| < Am(x)$, then $m(x) \leqslant (1+A)m(y)$ and $m(y) \leqslant 2(1+A)m(x)$.*

The next lemma, taken from [7, Proposition 2.1(i)], will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone. For the reader's convenience, we include a short proof.

2 Lemma. *Let $\alpha > 0$ and $|x-y| \leqslant \alpha m(x)$. Then:*

$$\mathrm{e}^{-\alpha^2-2\alpha}\mathrm{e}^{|y|^2} \leqslant \mathrm{e}^{|x|^2} \leqslant \mathrm{e}^{\alpha^2(1+\alpha)^2+2\alpha(1+\alpha)}\mathrm{e}^{|y|^2}.$$

Proof. By the triangle inequality and $m(x)|x| \leqslant 1$ we get,

$$|y|^2 \leqslant (\alpha m(x) + |x|)^2 \leqslant \alpha^2 + 2\alpha + |x|^2.$$

This gives the first inequality. For the second we use Lemma 1 to infer $m(x) \leqslant (1+\alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant \alpha^2(1+\alpha)^2 + 2\alpha(1+\alpha) + |y|^2.$$

As required. ■

3.1. An estimate on Gaussian balls

Let $B := B_t(x)$ be the open Euclidean ball with radius t and center x and let γ be the Gaussian measure as defined by (2). We shall denote by S_d the surface area of the unit sphere in $\mathbf{R}^d$.

3 Lemma. *For all $x \in \mathbf{R}^d$ and $t > 0$ we have the inequality:*

$$\gamma(B_t(x)) \leqslant \frac{S_d}{\pi^{\frac{d}{2}}}\frac{t^d}{d}\mathrm{e}^{2t|x|}\mathrm{e}^{-|x|^2}. \tag{10}$$

Proof. Remark that, with $B := B_t(x)$,

$$\begin{aligned}\int_B \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi &= \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{-2\langle x,\xi-x\rangle}\,\mathrm{d}\xi \\ &\leqslant \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{2|x||\xi-x|}\,\mathrm{d}\xi \\ &\leqslant \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\int_B \mathrm{e}^{-|\xi-x|^2}\,\mathrm{d}\xi \\ &= \pi^{\frac{d}{2}}\mathrm{e}^{2t|x|}\mathrm{e}^{-|x|^2}\gamma(B_t(0)).\end{aligned}$$

So, there holds that

$$\gamma(B_t(x)) \leqslant \mathrm{e}^{2t|x|}\mathrm{e}^{-|x|^2}\gamma(B_t(0)). \tag{11}$$

Proceeding by noting that

$$\gamma(B_t(0)) \leqslant \pi^{-\frac{d}{2}}|B_t(0)| \leqslant \pi^{-\frac{d}{2}}t^d\frac{S_d}{d},$$

and combining this with the previous calculation yields

$$\gamma(B_t(x)) \leqslant \frac{S_d}{\pi^{\frac{d}{2}}}\frac{t^d}{d}\mathrm{e}^{2t|x|}\mathrm{e}^{-|x|^2}.$$

This completes the proof. ■

3.2. Off-diagonal kernel estimates on annuli

As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli.

Throughout this subsection we fix $x \in \mathbf{R}^d$, constants $A, a \geqslant 1$, a pair $(y,t) \in \Gamma_x^{(A,a)}$. We use the notation rB to mean the ball obtained from the ball B by multiplying its radius by r.

The annuli $C_k := C_k(B_t(x))$ are given by:

$$C_k := \begin{cases} 2B_t(y), & k = 0, \\ 2^{k+1}B_t(y) \setminus 2^kB_t(y), & k \geqslant 1. \end{cases} \tag{12}$$

Whenever ξ is in C_k, we get for $k \geqslant 1$:

$$2^kt \leqslant |y-\xi| < 2^{k+1}t. \tag{13}$$

On C_k we have the following bound for $M_{t^2}(y,\cdot)$:

4 Lemma. *For all $\xi \in C_k$ for $k \geqslant 1$ we have:*

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \exp\bigl(2^{k+1}t|y|\bigr) \exp\Bigl(-\frac{4^k}{2\mathrm{e}^{2t^2}}\Bigr), \tag{14}$$

Proof. Considering the first exponential which occurs in the Mehler kernel (9) together with (13) gives for $k \geqslant 1$:

$$\begin{aligned} \exp\left(-\mathrm{e}^{-2t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right) &\leqslant \exp\left(-\frac{4^k}{\mathrm{e}^{2t^2}}\frac{t^2}{1-\mathrm{e}^{-2t^2}}\right) \\ &\overset{(\dagger)}{\leqslant} \exp\left(-\frac{4^k}{2\mathrm{e}^{2t^2}}\right), \end{aligned}$$

where $(\dagger)$ follows from $1-\mathrm{e}^{-s} \leqslant s$ for $s \geqslant 0$. Using the estimate $1+s \geqslant 2s$ for $0 \leqslant s \leqslant 1$, for the second exponential in the Mehler kernel (9) we obtain, by (13):

$$\begin{aligned} \exp\left(2\mathrm{e}^{-t^2}\frac{\langle y,\xi\rangle}{1+\mathrm{e}^{-t^2}}\right) &\leqslant \exp(|\langle y,\xi\rangle|) \\ &\leqslant \exp(|\langle y,\xi-y\rangle|)\mathrm{e}^{|y|^2} \\ &\leqslant \exp\bigl(2^{k+1}t|y|\bigr)\mathrm{e}^{|y|^2}. \end{aligned}$$

Combining things, we obtain the estimate in the formulation of the lemma. ■

4. The main result

In this section we will prove our main theorem for which we have already made the necessary preparations in the previous sections.

1 Theorem. *Let $A, a > 0$. For all $x \in \mathbf{R}^d$ and all $u \in C_c(\mathbf{R}^d)$ we have*

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |\mathrm{e}^{t^2L}u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))}\int_{B_r(x)} |u|\,\mathrm{d}\gamma, \tag{15}$$

where the implicit constant only depends on A, a and d.

Proof. We fix $x \in \mathbf{R}^d$ and $(y,t) \in \Gamma_x^{(A,a)}$. The proof of (15) is based on splitting the integration domain into the annuli C_k as defined by (12) and estimating on each annulus. More explicit,

$$|\mathrm{e}^{t^2 L}u(y)| \leqslant \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k} M_{t^2}(y,\cdot)|u(\cdot)| \,\mathrm{d}\gamma. \tag{16}$$

We have $t \leqslant am(x) \leqslant a$ and, by Lemma 1, $t|y| \leqslant a(1+aA)$. Together with Lemma 4 we infer, for $\xi \in C_k$ and $k \geqslant 1$, that:

$$\begin{aligned} M_{t^2}(y,\xi) &\leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \exp(2^{k+1}a(1+aA)) \exp\Big(-\frac{4^k}{2\mathrm{e}^{2a^2}}\Big) \\ &=: \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} c_k. \end{aligned}$$

Combining this with Lemma 2, we obtain

$$M_{t^2}(y,\xi) \lesssim_{A,a} \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} c_k. \tag{17}$$

Also, by (13),

$$|x-\xi| \leqslant |x-y| + |\xi - y| \leqslant (2^{k+1}+A)t.$$

Let K be the smallest integer such that $2^{k+1} \geqslant A$ whenever $k \geqslant K$. Then it follows that C_k for $k \geqslant K$ is contained in $B_{2^{k+2}t}(x)$ and for $k < K$ is contained in $B_{2At}(x)$. We set,

$$D_k := D_k(x) = \begin{cases} B_{2^{k+2}t}(x) & \text{if } k \geqslant K, \\ B_{2At}(x) & \text{elsewhere.} \end{cases}$$

Let us denote the supremum on right-hand side of (15) by $M_\gamma u(x)$. Using (17), we can bound the integral on the right-hand side of (16) by

$$\begin{aligned} \int_{C_k} M_{t^2}(y,\cdot)|u(\cdot)| \,\mathrm{d}\gamma &\lesssim_{A,a} c_k \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \int_{C_k} |u| \,\mathrm{d}\gamma \\ &\leqslant c_k \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \int_{D_k} |u| \,\mathrm{d}\gamma \\ &\leqslant c_k \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \gamma(D_k) M_\gamma u(x), \end{aligned}$$

where we pause for a moment to compute a suitable bound for $\gamma(D_k)$. As above we have both $t|x| \leqslant am(x)|x| \leqslant a$ and $t \leqslant a$. Together with Lemma 3 applied to D_k for $k \geqslant K$ we obtain:

$$\begin{aligned}\gamma(D_k)\mathrm{e}^{|x|^2} &\lesssim_A C^d \frac{S_d}{d} t^d 2^{kd} \mathrm{e}^{2^{k+3}t|x|} \mathrm{e}^{-|x|^2} \mathrm{e}^{|x|^2} \\ &\lesssim_{A,a,d} t^d 2^{kd} \mathrm{e}^{2^{k+3}a}.\end{aligned}$$

Similarly, for $k < K$:

$$\gamma(D_k)\mathrm{e}^{|x|^2} \lesssim_{A,a,d} t^d \mathrm{e}^{2Aa}.$$

Using the bound $t \leqslant a$, we can infer that

$$\frac{t^d}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \leqslant \frac{a^d}{(1-\mathrm{e}^{-2a^2})^{\frac{d}{2}}} \lesssim_{a,d} 1.$$

(note that $s/(1-\mathrm{e}^{-s})$ is increasing). Combining these computations with where we left off we get for $k \geqslant K$,

$$\int_{C_k} M_{t^2}(y,\cdot)|u(\cdot)| \,\mathrm{d}\gamma \lesssim_{A,a,d} c_k 2^{kd} \mathrm{e}^{2^{k+2}a} M_\gamma u(x),$$

while we get for $k < K$,

$$\int_{C_k} M_{t^2}(y,\cdot)|u(\cdot)| \,\mathrm{d}\gamma \lesssim_{A,a,d} c_k M_\gamma u(x).$$

Similarly, for $\xi \in 2B_t(x)$ we obtain:

$$I_0 := \int_{2B_t} M_{t^2}(y,\cdot)|u(\cdot)| \,\mathrm{d}\gamma \lesssim_{A,a,d} M_\gamma u(x).$$

Inserting the dependency of c_k upon k as coming from (17), we obtain the bound:

$$\begin{aligned}|\mathrm{e}^{t^2 L}u(y)| &= I_0 + \sum_{k=1}^{K-1} I_k + \sum_{k=K}^{\infty} I_k \\ &\lesssim_{A,a,d} \left[1 + \sum_{k=1}^{K-1} c_k + \sum_{k=K}^{\infty} c_k 2^{kd} \mathrm{e}^{2^{k+2}a}\right] M_\gamma u(x), \\ &\lesssim_{A,a,d} \left[1 + \sum_{k=1}^{K-1} \mathrm{e}^{-\frac{4^k}{2\mathrm{e}^{2a^2}}} + \sum_{k=K}^{\infty} 2^{kd} \mathrm{e}^{2^{k+1}(1+2a+aA)} \mathrm{e}^{-\frac{4^k}{2\mathrm{e}^{2a^2}}}\right] M_\gamma u(x),\end{aligned}$$

valid for all $(y,t) \in \Gamma_x^{(A,a)}$. As the sum on the right-hand side evidently converges, we see that taking the supremum proves (15). ■

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