A note on the Gaussian maximal function

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ABSTRACT. In this note we give an improvement on a result first demonstrated by Pineda and Urbina [3]. In particular we present an improvement to their Lemma 1.1 which gives the boundedness of the Gaussian maximal function associated to the Ornstein-Uhlenbeck operator.

We present a proof which is at least to the author more transparant. Our main finding in this note is that our proof allows to use a larger cone and actually obtain the maximal function boundedness for a whole class of cones $\Gamma_x^{(A,a)}(\gamma)$.

1. Introduction

To be typed.

1.1. Notation. To begin, let us fix some notation. As is common, we use N to represent a positive integer. That is, $N \in \mathbf{Z}_+ = \{1, 2, 3, \dots\}$. In the same way we cast letters that denote the number of dimensions, e.g. $d \in \mathbf{R}^d$ as positive integers.

We use the capital letter T to denote a "time" endpoint, for instance, when writing $t \in (0, T]$.

2. The Mehler kernel and friends

2.1. Setting. Our setting is the one concerning the *Ornstein-Uhlenbeck* operator L which is defined as:

$$L := -\frac{1}{2}\Delta + x \cdot \nabla, \tag{1}$$

We define the Mehler kernel (see e.g., Sjögren [5]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(\mathrm{e}^{-tL})_t$. More precisely, this means:

$$\mathrm{e}^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot)u \, \mathrm{d}\gamma. \tag{2}$$

It is often more convenient to use e^{-t^2L} instead of e^{-tL} as is done in e.g., Portal [4] and we will also do so.

2.2. The Mehler kernel. For the calculation of the Mehler kernel M_t in (2) we refer to e.g., Sjögren [5] which additionally offers related results such as those related to Hermite polynomials.

Observe that the kernel M_{t^2} is invariant under the permutation $x \longleftrightarrow y$. A formula for M_t which honors this observation is:

$$M_{t^2}(x, y) = \frac{\exp\left(-\mathrm{e}^{-2t^2}\dfrac{|x-y|^2}{1-\mathrm{e}^{-2t^2}}\right)}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \frac{\exp\left(2\mathrm{e}^{-t^2}\dfrac{\langle x, y\rangle}{1+\mathrm{e}^{-t^2}}\right)}{(1+\mathrm{e}^{-t^2})^{\frac{d}{2}}}. \tag{3}$$

3. Some fine lemmata and definitions

3.1. $m$inimal function. We recall the lemma from [1, lemma 2.3] which first –although implicitly– appeared in [2]. For what follows it will be convenient to define a function m as:

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\} = 1 \vee \frac{1}{|x|}.$$

We use m in our next lemma.

1. Lemma. *Let a, A be strictly positive numbers. We have for x, y in $\mathbf{R}^d$ that:*

(1) If $|x-y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1+aA)m(y)$;
(2) Likewise, if $|x-y| < Am(x)$, then $m(x) \leqslant (1+A)m(y)$ and $m(y) \leqslant 2(1+A)m(x)$.

Recall that:

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}_+^d : |x-y| < At \text{ and } t \leqslant am(x)\}. \tag{4}$$

To ease the notational burden a bit, we will write Γ_x^a and mean $\Gamma_x^{(1,a)}$. Using this notation we can deduce a cone version of Lemma 1. That is:

2. Lemma. *Let a, A be strictly positive numbers. Then:*

(1) If $(y,t) \in \Gamma_x^{(A,a)}$ then $t \leqslant (1+aA)m(y)$;
(2) If $(y,t) \in \Gamma_x^{(A,a)}$ then $(x,t) \in \Gamma_y^{(1+aA,a)}$.

In what is next we will use a global/local region dichotomy and define it as follows:

1. Definition. *Given $\tau > 0$, the set N_τ is given as:*

$$N_\tau(x) := N_\tau := \{(x,y) \in \mathbf{R}^{2d} : |x-y| \leqslant \tau m(x)\}. \tag{5}$$

Sometimes it is easier to work with the set $N_\tau(B)$, which is given for $B := B_r(x)$ as:

$$N_\tau(B) := \{y \in \mathbf{R}^d : |x-y| \leqslant \tau m(x)\}. \tag{6}$$

When we partition the space into N_τ and its complement, we call the part belonging to N_τ the local region *and the part belonging to its complement the* global region*.*

The set $t \leqslant am(x)$ is used in the definition of the cones $\Gamma_x^{(A,a)}$ and we will name it D^a, that is:

$$D^a := \{(x,t) \in \mathbf{R}_+^d : t \leqslant am(x)\}. \tag{7}$$

We will write $D := D^1$ for simplicity.

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

3. Lemma. *Let $\alpha > 0$ and $|x-y| \leqslant \alpha m(x)$. Then the Gaussians in x and y respectively are comparable. In particular this means that,*

$$e^{-|x|^2} \simeq e^{-|y|^2}.$$

Remark. More precisely, from the proof we get the estimates (8) and (9). That is:

$$e^{-|x|^2} \leqslant e^{(1+\alpha)^2-1} e^{-|y|^2},$$

and,

$$e^{-|y|^2} \leqslant e^{(1+\alpha)^2} e^{2(1+\alpha)} e^{-|x|^2}.$$

PROOF. Let x and y be such that $|x-y| \leqslant \alpha m(x)$. By the inverse triangle inequality we get,

$$\begin{aligned} |y|^2 &\leqslant (\alpha m(x)+|x|)^2 \\ &= \alpha^2 + 2\alpha m(x)|x| + |x|^2 \\ &\leqslant \alpha^2 + 2\alpha + |x|^2. \end{aligned}$$

Therefore,

$$e^{-|x|^2} \leqslant e^{-|y|^2} e^{(1+\alpha)^2} e^{-1}. \tag{8}$$

For the reverse direction we use Lemma 1 to infer $t \leqslant (1+\alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant (1+\alpha)^2 + 2(1+\alpha) + |y|^2.$$

Therefore,

$$e^{-|y|^2} \leqslant e^{-|x|^2} e^{(1+\alpha)^2} e^{2(1+\alpha)}. \tag{9}$$

Combining we get:

$$e^{-(1+\alpha)^2} e^{-1} e^{-(1+2\alpha)} \overset{(9)}{\leqslant} \frac{e^{-|x|^2}}{e^{-|y|^2}} \overset{(8)}{\leqslant} e^{-1} e^{(1+\alpha)^2}.$$

Summarizing we thus have that,

$$e^{-|x|^2} \simeq e^{-|y|^2},$$

as required. ∎

4. LEMMA. *Let x, y and z in $\mathbf{R}^d$. Set*

$$\tau = \frac{1}{2}(1+2\alpha)(1+\alpha).$$

If $|y-z| > \tau m(y)$ and $|x-y| \leqslant \alpha m(x)$ then $|x-z| > \frac{1}{2} m(x)$.

PROOF. We assume that $(y,z) \notin N_\tau$ and $(y,t) \in \Gamma_x^{(A,a)}$. Written out this gives by (5) the inequality $|y-z| > \tau m(y)$, and by (4) the inequality $|x-y| < aAm(x)$. Note that the latter inequality together with Lemma 1 yields,

$$\frac{1}{2}\frac{1}{1+\alpha} m(y) \leqslant m(x) \leqslant (1+\alpha)m(y). \tag{10}$$

Combining we get $|x-y| < \alpha(1+\alpha)m(y)$. Now we are in position to apply the triangle inequality:

$$|x-z| \geqslant |y-z| - |x-y| > \tau m(y) - \alpha(1+\alpha)m(y).$$

As we require an lower bound in terms of $m(x)$ and not $m(y)$, we again apply (10) to obtain:

$$\begin{aligned} |x-z| \geqslant |y-z| - |x-y| > \tau m(y) &- \alpha m(y) \\ &\geqslant \tau \frac{1}{1+\alpha} m(x) - \alpha m(x) \\ &\geqslant \frac{1}{2} m(x). \end{aligned}$$

Very well, proof is done. ∎

4. On-diagonal estimates

4.1. Kernel estimates. We begin with a technical lemma which will be useful on several occasions.

5. Lemma. *Let t be in $(0,T]$ and let $\alpha > 1$. Then,*

$$\alpha e^{-T^2} \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} \leqslant \alpha. \tag{11}$$

and,

$$0 \leqslant \frac{1}{1+e^{-t^2}} - \frac{1}{1+e^{-\frac{t^2}{\alpha}}} \leqslant \frac{1}{4}\left(1-\frac{1}{\alpha}\right). \tag{12}$$

Proof. We start with (11) and apply the mean value theorem to the function $f(\xi) = \xi^\alpha$. For $0 < \xi' < \xi$ this gives that:

$$f(\xi) - f(\xi') = \alpha\hat{\xi}^{\alpha-1}(\xi - \xi') \text{ for some } \hat{\xi} \text{ in } [\xi, \xi'].$$

Picking $\xi = 1$ and $\xi' = e^{-\frac{t^2}{\alpha}}$ yields:

$$\frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} = \alpha\hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[\exp\left(-\frac{t^2}{\alpha}\right), 1\right]. \tag{13}$$

Combining this result with the monotonicity of $\xi \mapsto \alpha\xi^{\alpha-1}$ we obtain:

$$\alpha e^{-t^2} \leqslant \alpha \exp\left(-t^2\frac{\alpha-1}{\alpha}\right) \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}}.$$

Together with $e^{-T^2} \leqslant e^{-t^2}$ we obtain,

$$\alpha e^{-T^2} \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} \downarrow \alpha.$$

We proceed with (12). Recalling that $\alpha > 1$ one can directly verify that:

$$\frac{1}{1+e^{-t^2}} - \frac{1}{1+e^{-\frac{t^2}{\alpha}}} \geqslant 0,$$

and that the function on the left-hand side is decreasing. To find an upper bound we compute the limit as t goes to 0. That is:

$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t^2}\left[\frac{1}{1+e^{-t^2}} - \frac{1}{1+e^{-\frac{t^2}{\alpha}}}\right] &= \lim_{t\to 0}\frac{1}{2t}\left[\frac{2te^{-t^2}}{(1+e^{-t^2})^2} - \frac{1}{\alpha}\frac{2te^{-\frac{t^2}{\alpha}}}{(1+e^{-\frac{t^2}{\alpha}})^2}\right] \\
&= \lim_{t\to 0}\left[\frac{e^{-t^2}}{(1+e^{-t^2})^2} - \frac{1}{\alpha}\frac{e^{-\frac{t^2}{\alpha}}}{(1+e^{-\frac{t^2}{\alpha}})^2}\right] \\
&\uparrow \frac{1}{4}\left(1-\frac{1}{\alpha}\right).
\end{aligned}$$

Which is as asserted and completes the proof. ∎

The following lemma will be useful when transfering estimates from $M_{\frac{t^2}{\alpha}}$ to M_{t^2}. It follows from the mean value theorem applied to $\xi \mapsto \xi^\alpha$.

6. Lemma. *For $\alpha > 1$ and t in $(0,T]$ and all let x, y in $\mathbf{R}^d$ we have that:*

$$\exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-e^{-\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-\frac{1}{2}\frac{\alpha}{e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right). \tag{14}$$

PROOF. Let t in $(0,T]$. Applying (11) we get:

$$\exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-e^{-\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-e^{-t^2}}\frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-\frac{1}{2}\frac{\alpha}{e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right).$$

Which is as asserted. ∎

Our first lemma is about estimating $M_{\frac{t^2}{\alpha}}$ in terms of M_{t^2}.

4.1.1. *Time-scaling of the Mehler kernel.*

7. LEMMA. *Let $\alpha \geqslant 2e^{T^2}$, t in $(0,T]$ and x, y in $\mathbf{R}^d$. If $t|x| \leqslant C$ and $t|y| \leqslant C$ then:*

$$M_{\frac{t^2}{\alpha}}(x,y) \leqslant \alpha^{\frac{d}{2}} e^{\frac{C^2}{2}} \exp\left(-\frac{\alpha}{2e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right) M_{t^2}(x,y). \tag{15}$$

PROOF. To prove the lemma we compute $M_{\frac{t^2}{\alpha}} M_{t^2}^{-1}$. First note that (11) gives

$$\alpha^{\frac{d}{2}} e^{-\frac{d}{2}T^2} \leqslant \frac{(1-e^{-t^2})^{\frac{d}{2}}}{(1-e^{-\frac{t^2}{\alpha}})^{\frac{d}{2}}} \leqslant \alpha^{\frac{d}{2}}.$$

Combining the exponentials also gives,

$$\begin{aligned}\exp\left(-2e^{-t^2}\frac{\langle x,y\rangle}{1+e^{-\frac{t^2}{\alpha}}}\right)\exp\left(2e^{-t^2}\frac{\langle x,y\rangle}{1+e^{-t^2}}\right) \\ = \exp\left(\frac{2}{t^2}\left[\frac{1}{1+e^{-t^2}} - \frac{1}{1+e^{-\frac{t^2}{\alpha}}}\right] t^2\langle x,y\rangle\right).\end{aligned}$$

(12) Using this result and nothing that $|x+y|^2 \leqslant 2|x|^2 + 2|y|^2$ yields:

$$\begin{aligned}\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+e^{-\frac{t^2}{\alpha}}}\right)\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+e^{-t^2}}\right) &\leqslant \exp\left(\frac{1}{8}\left(1-\frac{1}{\alpha}\right)t^2|x+y|^2\right) \\ &\leqslant \exp\left(\frac{1}{4}t^2|x|^2\right)\exp\left(\frac{1}{4}t^2|y|^2\right).\end{aligned}$$

From $t|x| \leqslant C$ and $t|y| \leqslant C$ we obtain that:

$$\exp\left(-\frac{1}{2}\frac{|x+y|^2}{1+e^{-\frac{t^2}{\alpha}}}\right)\exp\left(\frac{1}{2}\frac{|x+y|^2}{1+e^{-t^2}}\right) \leqslant e^{\frac{C^2}{2}}.$$

Combining Lemma 6 and equation (14) gives is almost the final estimate.

$$\begin{aligned}\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)} &\leqslant \alpha^{\frac{d}{2}} e^{\frac{C^2}{2}} \exp\left(\frac{1}{2}\frac{|x-y|^2}{1-e^{-t^2}}\right)\exp\left(-\frac{1}{2}\frac{|x-y|^2}{1-e^{-\frac{t^2}{\alpha}}}\right) \\ &\leqslant \alpha^{\frac{d}{2}} e^{\frac{C^2}{2}} \exp\left(\frac{1}{2}\left[1-\frac{\alpha}{2e^{T^2}}\right]\frac{|x-y|^2}{1-e^{-t^2}}\right)\exp\left(-\frac{\alpha}{2e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right).\end{aligned}$$

Finally, we apply the assumption $\alpha \geqslant 2e^{T^2}$ to obtain:

$$\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)} \leqslant \alpha^{\frac{d}{2}} e^{\frac{C^2}{2}} \exp\left(-\frac{\alpha}{2e^{T^2}}\frac{|x-y|^2}{1-e^{-t^2}}\right).$$

Which is as asserted. ∎

4.2. On-diagonal kernel estimates on annuli. As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli C_k. For the sake of simplicity we will write $B := B_t(x)$ and mean that B is the closed ball with center x and radius t. Furthermore, we use notations such as $2B$ to mean the ball obtained from B by multiplying its radius by 2.

The C_k are given by,

$$C_k(B) := C_k = \begin{cases} 2B & \text{if } k = 0, \\ 2^{k+1}B \setminus 2^k B & \text{for } k \geqslant 1. \end{cases} \tag{16}$$

So, whenever ξ is in C_k, we get for $k \geqslant 1$:

$$2^k at < |y - \xi| \leqslant 2^{k+1} at. \tag{17}$$

While we get for $k = 0$:

$$|y - \xi| \leqslant 2at. \tag{18}$$

8. Lemma. *Given $a > 0$, let $B = B_{at}(y)$ and ξ in C_k. Furthermore, assume that $t \leqslant am(y)$ for some $A > 0$. Then we have:*

$$M_{t^2}(y, \xi) \leqslant \frac{e^{|y|^2}}{(1 - e^{-t^2})^{\frac{d}{2}}} \exp(2^{k+1} aC) \begin{cases} 1 & \text{if } k = 0, \\ \exp\left(-\dfrac{a^2}{2e^{2t^2}} 4^{k+1}\right) & \text{if } k \geqslant 1. \end{cases}$$

Proof. Let $B = B_{at}(y)$ and let C_k be as in (16). We consider the two exponentials in the Mehler kernel (3) separately. First we consider

$$\exp\left(2e^{-t^2} \frac{\langle y, \xi\rangle}{1 + e^{-t^2}}\right).$$

Using the Cauchy-Schwarz inequality we get that:

$$|\langle y, \xi\rangle| \leqslant |y - \xi||y| + |y|^2 \tag{19}$$

Next, note that

$$\frac{e^{-t^2}}{1 + e^{-t^2}} \leqslant \frac{1}{2}.$$

Together with (19) this gives for $k \geqslant 1$:

$$\begin{aligned} \exp\left(2e^{-t^2} \frac{\langle y, \xi\rangle}{1 + e^{-t^2}}\right) &\leqslant \exp(|\langle y, \xi\rangle|) \\ &\leqslant \exp(|y - \xi||y|) \exp(|y|^2) \\ &\overset{\text{(i)}}{\leqslant} \exp(2^{k+1} aC) \exp(|y|^2). \end{aligned}$$

Where (i) uses (17, 18) and $t|y| \leqslant C$. Considering the first exponential in the Mehler kernel (3) we get:

$$\exp\left(-e^{-2t^2} \frac{|y - \xi|^2}{1 - e^{-2t^2}}\right).$$

This inequality together with

$$\frac{t^2}{1 - e^{-2t^2}} \geqslant \frac{1}{2},$$

yields using (17, 18),

$$\exp\left(-e^{-2t^2} \frac{|y - \xi|^2}{1 - e^{-2t^2}}\right) \leqslant \exp\left(-\frac{a^2}{2e^{2t^2}} 4^{k+1}\right).$$

Concluding we get:

$$\exp\left(-\mathrm{e}^{-2t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right)\leqslant\begin{cases}1 & \text{if } k=0,\\ \exp\left(-\frac{a^2}{2\mathrm{e}^{2t^2}}4^{k+1}\right) & \text{if } k\geqslant 1.\end{cases}$$

Thus, we can estimate the Mehler kernel (3) M_{t^2} on C_k from above by:

$$M_{t^2}(y,\xi)\leqslant\frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp\left(2^{k+1}aC\right)\begin{cases}1 & \text{if } k=0,\\ \exp\left(-\frac{a^2}{2\mathrm{e}^{2t^2}}4^{k+1}\right) & \text{if } k\geqslant 1.\end{cases}$$

Done. ∎

5. The boundedness of some non-tangential maximal operators

9. Lemma. *Let $B_t(x)$ be the Euclidean ball with radius t and center x. If γ is the normalized Gaussian measure with density $\sim\exp(-|x|^2)$ we have:*

$$\gamma(B_t(x))\leqslant C_d\mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\mathrm{e}^{-t^2}t^d. \tag{20}$$

Proof. Next, remark that for a ball $B:=B_t(x)$ there holds that

$$\begin{aligned}\int_B \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi &= \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{-2\langle x,\xi-x\rangle}\,\mathrm{d}\xi\\ &\leqslant \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{2|x||\xi-x|}\,\mathrm{d}\xi\\ &\leqslant \pi^{\frac{d}{2}}\mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)).\end{aligned}$$

That is:

$$\gamma(B_t(x))\leqslant \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)). \tag{21}$$

We will estimate the Gaussian volume of the ball $B_t(0)$. To ease the notation, define $C_d:=|S_{d-1}|\pi^{-\frac{d}{2}}$ where $|S_{d-1}|$ is the surface area of the d-dimensional sphere. Using polar coordinates we then obtain:

$$\begin{aligned}\gamma(B_t(0)) &= \pi^{-\frac{d}{2}}\int_{B_t(0)}\mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi\\ &= C_d\int_0^t \mathrm{e}^{-r^2}r^{d-1}\,\mathrm{d}r\\ &\leqslant C_d\mathrm{e}^{-t^2}t^d.\end{aligned}$$

Upon combining this result with (21) we obtain (20) as promised. ∎

The following lemma is a small modification of [3, lemma 1.1] with a new proof.

10. Lemma. *Let $A,a>0$. For all x in $\mathbf{R}^d$ and all u in L^2_γ we have*

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}}|\mathrm{e}^{-t^2L}u(y)|\lesssim\sup_{r>0}\fint_{B_r(x)}|u|\,\mathrm{d}\gamma. \tag{22}$$

Proof. Since we have $|x-y|<At$ and $t\leqslant am(x)$ we infer that $t|x|\leqslant a$. By Lemma 2 we also that $t|y|\leqslant 1+aA$. From this and Lemma 8 we infer that:

$$M_{t^2}(y,\xi)\leqslant\frac{\mathrm{e}^{|y|^2}\mathrm{e}^{A(1+aA)2^{k+1}}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\begin{cases}1 & \text{if } k=0,\\ \exp\left(-\frac{A^2}{8\mathrm{e}^{2t^2}}4^k\right) & \text{if } k\geqslant 1.\end{cases} \tag{23}$$

Recalling Lemma 9 we get that

$$\gamma(B_t(x)) \leqslant C_d e^{-|x|^2} e^{2t|x|} e^{-t^2} t^d. \tag{24}$$

This allows us to estimate the remaining unbounded exponential in the Mehler kernel and allow a penalty up to $e^{-|x|^2}$.

The remark following Lemma 3 gives us by assuming $|x-y| \leqslant aAm(x)$ the following estimate:

$$e^{|y|^2} \leqslant e^{(1+aA)^2-1} e^{|x|^2}.$$

Using (23) we can bound the maximal function:

$$\begin{aligned} e^{-t^2 L}|u(y)| &= \sum_{k=0}^{\infty} \int_{C_k(B)} M_{t^2}(y,\cdot)|u| \, d\gamma \\ &\leqslant \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \sum_{k=0}^{\infty} e^{-\frac{A^2}{2}4^{k+1}} e^{A(1+aA)2^{k+1}} \int_{C_k(B)} |u| \, d\gamma. \end{aligned} \tag{25}$$

Considering the exponential in the sum, and collecting terms we get:

$$\begin{aligned} -\frac{A^2}{2}4^{k+1} + A(1+aA)2^{k+1} &= -\frac{1}{2}[(A2^{k+1} - (1+aA))^2 - (1+aA)^2] \\ &= -A2^{k+1}[A2^k - (1+aA)], \end{aligned}$$

which will be negative as long as $A2^k$ is larger than $1+aA$. That is, whenever

$$2^k > \frac{1+aA}{A} = a + \frac{1}{A}.$$

For the common choice of $A = 1$ and $a = 2$ this means that there has to hold that $k \geqslant 2$.

In the next step we will bound the integral on the right-hand side by the maximal function centered at x. For this we need to scale up the C_k. So,

$$(2^k - 1)At \leqslant ||\xi - y| - |x-y|| \leqslant |y-\xi| \leqslant |x-y| + |\xi - y| \leqslant (2^{k+1}+1)At.$$

So, we can bound the integral on the right-hand side of (25) by

$$\int_{C_k} |u| \, d\gamma \leqslant \int_{B(x,2^{k+1}2At)} |u| \, d\gamma \leqslant \gamma(B(x, 2^{k+1}2At)) \sup_{r>0} \fint_{B_r(x)} |u| \, d\gamma.$$

Next, we apply Lemma 9 to obtain that

$$\int_{C_k} |u| \, d\gamma \leqslant e^{-|x|^2} e^{2At|x|} \gamma(B(0, 2^{k+1}2At)) \sup_{r>0} \fint_{B_r(x)} |u| \, d\gamma.$$

Recall that $t|x| \leqslant a$ so that,

$$\int_{C_k} |u| \, d\gamma \leqslant e^{-|x|^2} e^{2aA} \gamma(B(0, 2^{k+1}2At)) \sup_{r>0} \fint_{B_r(x)} |u| \, d\gamma.$$

Thus, we note that

$$\gamma(B(0, 2^{k+1}2At)) \leqslant C_d e^{-4^k 8A^2 t^2} 2^{kd} 4^d A^d t^d.$$

Wrapping it up, we have that:

$$e^{-t^2 L}|u(y)| \lesssim C_d 4^d A^d \frac{t^d}{(1-e^{-t^2})^{\frac{d}{2}}} \sum_{k=0}^{\infty} e^{-2^{k+1}[2^k - (A^{-1}+a)]} e^{-4^k 8A^2 t^2} 2^{kd} \sup_{r>0} \fint_{B_r(x)} |u| \, d\gamma.$$

The function in front of the sum is bounded from below by 1 and bounded from above by $(t+1)^d$. As $t \leqslant am(x) \leqslant a$ we can bound it by $(a+1)^d$. Therefore,

$$e^{-t^2 L}|u(y)| \lesssim C_d 4^d A^d (1+a)^d \sum_{k=0}^{\infty} e^{-2^{k+1}[2^k - (A^{-1}+a)]} e^{-4^k 8A^2 t^2} 2^{kd} \sup_{r>0} \fint_{B_r(x)} |u| \, d\gamma.$$

Which is what we wanted to prove. ■

References

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