

A NOTE ON THE GAUSSIAN MAXIMAL FUNCTION

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ABSTRACT. This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded almost surely by the Gaussian Hardy-Littlewood maximal function. In particular this entails improvement on a result by Pineda and Urbina [2] who proved a similar result for a ‘truncated’ version of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

1. INTRODUCTION

Maximal functions are among the most studied objects in harmonic analysis. It is well known that the classical non-tangential maximal function associated with the heat semigroup is bounded almost everywhere by the Hardy-Littlewood maximal function,

$$(1) \quad \sup_{\substack{(y,t) \in \mathbf{R}_+^{d+1} \\ |x-y| < t}} |e^{-t\Delta} u(y)| \lesssim \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \, d\lambda.$$

Here the action of *heat semigroup* $e^{-t\Delta} u = \rho_t * u$ is given by a convolution of u with the *heat kernel*

$$\rho_t(s) := \frac{e^{-|s|^2/4t}}{(4\pi t)^{\frac{d}{2}}}.$$

In this note we are interested in its Gaussian counterpart. The change from Lebesgue measure to the Gaussian measure

$$(2) \quad d\gamma(x) := \pi^{-\frac{d}{2}} e^{-|x|^2} \, dx$$

introduces quite some intricate technical and conceptual difficulties which are caused by the fact that the Gaussian measure is non-doubling. The Gaussian analogue to the Laplacian is the *Ornstein-Uhlenbeck operator* L ,

$$(3) \quad L := -\frac{1}{2}\Delta + \langle x, \nabla \rangle = \frac{1}{2}\nabla_\gamma^* \nabla_\gamma,$$

where ∇_γ denotes the realisation of the gradient in $L^2(\mathbf{R}^d, \gamma)$. Our main result, to be proved in Theorem 1, is the following Gaussian analogue of (1):

$$(4) \quad \sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma.$$

Here,

$$(5) \quad \Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}_+^{d+1} : |x-y| < At \text{ and } t \leq am(x)\}$$

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is the Gaussian cone with aperture A and cut-off parameter a , and

$$(6) \quad m(x) := \min \left\{ 1, \frac{1}{|x|} \right\}.$$

A slightly weaker version of the inequality (4) has been proved by Pineda and Urbina [2] who showed that

$$\sup_{(y,t) \in \tilde{\Gamma}_x} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| d\gamma,$$

where

$$\tilde{\Gamma}_x(x) = \{(y, t) \in \mathbf{R}_+^d : |x - y| < t \leq \tilde{m}(x)\}$$

is the ‘reduced’ Gaussian cone corresponding to the function

$$\tilde{m}(x) = \min \left\{ \frac{1}{2}, \frac{1}{|x|} \right\}.$$

Their proof does not seem to easily generalize the range of t from $\frac{1}{2}$ up to 1. Our proof of (4) is different and, we believe, more transparent than the one presented in [2]. It has the further advantage of allowing the extension to cones with arbitrary aperture $A > 0$ and cut-off parameter $a > 0$ without any additional technicalities. This additional generality is very important and has already been used by Portal (cf. the claim made in [3, discussion preceding Lemma 2.3]) to prove the H^1 -boundedness of the Riesz transform associated with L .

To save writing, let us fix some notation. The number d is a positive integer. To avoid possible confusion, we define the *positive integers* as the set $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$.

2. THE MEHLER KERNEL

The Mehler kernel (see e.g., [4]) is the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(e^{-tL})_{t \geq 0}$, that is,

$$(7) \quad e^{-tL} u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot) u d\gamma.$$

There is an abundance of literature on the Mehler kernel and its properties. We shall only use the fact, proved e.g. in the review paper [4], that the Mehler kernel is given explicitly by

$$M_t(x, y) = \frac{\exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right)}{(1 - e^{-2t})^{\frac{d}{2}}} e^{|y|^2}.$$

Note that $M_t(x, y)$ is symmetric in x and y . A formula for M_t which honors this observation is:

$$(8) \quad M_t(x, y) = \frac{\exp\left(-e^{-2t} \frac{|x - y|^2}{1 - e^{-2t}}\right) \exp\left(2e^{-t} \frac{\langle x, y \rangle}{1 + e^{-t}}\right)}{(1 - e^{-t})^{\frac{d}{2}} (1 + e^{-t})^{\frac{d}{2}}}.$$

3. SOME LEMMATA

We use m as defined in (6) in our next lemma, which is taken from [1].

1. Lemma. *Let a, A be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

- (1) *If $|x - y| < At$ and $t \leq am(x)$, then $t \leq (1 + aA)m(y)$;*
- (2) *If $|x - y| < Am(x)$, then $m(x) \leq (1 + A)m(y)$ and $m(y) \leq 2(1 + A)m(x)$.*

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. Lemma. *Let $\alpha > 0$ and $|x - y| \leq \alpha m(x)$. Then:*

$$e^{-\alpha^2 - 2\alpha|y|^2} \leq e^{|x|^2} \leq e^{\alpha^2(1+\alpha)^2 + 2\alpha(1+\alpha)|y|^2}.$$

Proof. By the inverse triangle inequality and $m(x)|x| \leq 1$ we get,

$$|y|^2 \leq (\alpha m(x) + |x|)^2 \leq \alpha^2 + 2\alpha + |x|^2.$$

This gives the first inequality. For the second we use Lemma 1 to infer $m(x) \leq (1 + \alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leq \alpha^2(1 + \alpha)^2 + 2\alpha(1 + \alpha) + |y|^2.$$

As required. ■

3.1. An estimate on Gaussian balls. Let $B = B_t(x)$ be the open Euclidean ball with radius t and center x and let γ be the Gaussian measure as defined by (2). We shall denote by S_d the surface area of the unit sphere in \mathbf{R}^d .

3. Lemma. *For all $x \in \mathbf{R}^d$ and $t > 0$ we have the inequality:*

$$(9) \quad \gamma(B_t(x)) \leq \frac{1}{2} S_d t^d e^{2t|x|} e^{-|x|^2}.$$

Proof. Remark that

$$\begin{aligned} \int_B e^{-|\xi|^2} d\xi &= e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{-2\langle x, \xi-x \rangle} d\xi \\ &\leq e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{2|x||\xi-x|} d\xi \\ &\leq e^{-|x|^2} e^{2t|x|} \int_B e^{-|\xi-x|^2} d\xi \\ &= \pi^{\frac{d}{2}} e^{-|x|^2} e^{2t|x|} \gamma(B_t(0)). \end{aligned}$$

So, there holds that

$$(10) \quad \gamma(B_t(x)) \leq e^{-|x|^2} e^{2t|x|} \gamma(B_t(0)).$$

We will estimate the Gaussian volume of the ball $B_t(0)$. Using polar coordinates we proceed for $d \geq 2$ by:

$$\begin{aligned}
\gamma(B_t(0)) &= \pi^{-\frac{d}{2}} \int_{B_t(0)} e^{-|\xi|^2} d\xi \\
&= \pi^{-\frac{d}{2}} S_d \int_0^t e^{-r^2} r^{d-1} dr \\
&\leq \frac{1}{2} \pi^{-\frac{d}{2}} S_d t^{d-2} \int_0^t 2r e^{-r^2} dr \\
&= \frac{1}{2} \pi^{-\frac{d}{2}} S_d t^{d-2} (1 - e^{-t^2}) \\
&\leq \frac{1}{2} \pi^{-\frac{d}{2}} S_d t^d,
\end{aligned}$$

where the last step uses $1 - e^{-x} \leq x$ for $x \geq 0$. The case for $d = 1$ follows by a simplified argument. Upon combining this result with (10) we obtain (9). \blacksquare

3.2. On-diagonal kernel estimates on annuli. As is common in harmonic analysis, we often wish to decompose \mathbf{R}^d into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli.

Throughout this subsection we fix $x \in \mathbf{R}^d$, constants $A, a > 0$, a pair $(y, t) \in \Gamma_x^{(A,a)}$, and we put $\alpha := Aa$. We use the notation rB to mean the ball obtained from the ball B by multiplying its radius by r .

The annuli C_k are given by:

$$(11) \quad C_k := (2^{k+1} - 1)B_{\alpha t}(x) \setminus (2^k - 1)B_{\alpha t}(x) \text{ with } k \geq 0.$$

Note that $C_0 = B_{\alpha t}(x)$. Whenever ξ is in C_k , we get for $k \geq 0$:

$$(12) \quad (2^k - 1)\alpha t < |y - \xi| \leq (2^{k+1} - 1)\alpha t.$$

On C_k we have the following bound for $M_{t^2}(y, \cdot)$:

4. Lemma. *For all $\xi \in C_k$ we have:*

$$(13) \quad M_{t^2}(y, \xi) \leq \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \exp((2^{k+1} - 1)\alpha t |y|) \exp\left(-\frac{\alpha^2}{2e^{2a^2}}(2^k - 1)^2\right),$$

Proof. Considering the first exponential which occurs in the Mehler kernel (8) together with (12) gives for $k \geq 0$:

$$\begin{aligned}
\exp\left(-e^{-2t^2} \frac{|y - \xi|^2}{1 - e^{-2t^2}}\right) &\leq \exp\left(-e^{-2t^2} \frac{(2^k - 1)^2 \alpha^2 t^2}{1 - e^{-2t^2}}\right) \\
&\stackrel{(\dagger)}{\leq} \exp\left(-\frac{\alpha^2}{2e^{2t^2}}(2^k - 1)^2\right) \stackrel{(\dagger\dagger)}{\leq} \exp\left(-\frac{\alpha^2}{2e^{2a^2}}(2^k - 1)^2\right),
\end{aligned}$$

where (\dagger) follows from $1 - e^{-x} \leq x$ for $x \geq 0$, and $(\dagger\dagger)$ uses that $t \leq am(x) \leq a$. Using the estimate $1 + x \geq 2x$ for $0 \leq x \leq 1$, for the second exponential in the Mehler kernel (8) we obtain, by (12):

$$\begin{aligned}
\exp\left(2e^{-t^2} \frac{\langle y, \xi \rangle}{1 + e^{-t^2}}\right) &\leq \exp(|\langle y, \xi \rangle|) \\
&\leq \exp(|\langle y, \xi - y \rangle|) e^{|y|^2} \\
&\leq \exp((2^{k+1} - 1)\alpha t |y|) e^{|y|^2}.
\end{aligned}$$

Combining things, we obtain the estimate in the formulation of the lemma. \blacksquare

4. THE MAIN RESULT

Our theorem is a modification of [2, lemma 1.1] with a new proof.

1. Theorem. *Let $A, a > 0$. For all $x \in \mathbf{R}^d$ and all $u \in L^2(\mathbf{R}^d, \gamma)$ we have*

$$(14) \quad \sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma.$$

Proof. We fix $x \in \mathbf{R}^d$ and $(y, t) \in \Gamma_x^{(A,a)}$. Set $\alpha = aA$. We will prove (14) by splitting up the integration domain into the annuli C_k as defined by (11):

$$(15) \quad |e^{-t^2 L} u(y)| \leq \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma.$$

From $t \leq am(x)$ we get $t|x| \leq a$, and by Lemma 1 we have $t|y| \leq 1 + \alpha$. From this and Lemma 4 we infer, for $\xi \in C_k$, that:

$$(16) \quad M_{t^2}(y, \xi) \leq \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \exp((2^{k+1} - 1)\alpha(1 + \alpha)) \exp\left(-\frac{\alpha^2}{2e^{2a^2}}(2^k - 1)^2\right).$$

Combining (16) and Lemma 2, we obtain

$$M_{t^2}(y, \xi) \leq c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}}$$

where c_k depends only on A, a , and t . Also, by (12),

$$|x - \xi| \leq |x - y| + |\xi - y| \leq (A + 2^{k+1}\alpha)t.$$

It follows that C_k is contained in $D_k := B_{(A+2^{k+1}\alpha)t}(x)$.

Let us denote the supremum on right-hand side of (14) by $M_\gamma u(x)$. Using (16), we can bound the integral on the right-hand side of (15) by

$$\begin{aligned} \int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma &\leq c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \int_{C_k} |u| \, d\gamma \\ &\leq c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \int_{D_k} |u| \, d\gamma \\ &\leq c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \gamma(D_k) M_\gamma u(x) \\ &\stackrel{(\dagger)}{\lesssim}_{A,a,d} c_k \frac{t^d}{(1 - e^{-2t^2})^{\frac{d}{2}}} e^{2((A+2^{k+1}\alpha)t)|x|} M_\gamma u(x) \\ &\stackrel{(\dagger\dagger)}{\lesssim}_{A,a,d} c_k e^{2^{k+2}\alpha a} M_\gamma u(x), \end{aligned}$$

where (\dagger) uses Lemma 3 applied to D_k and $(\dagger\dagger)$ uses that $t \leq am(x)$ implies $t|x| \leq a$ and $t \leq a$, the latter implying

$$\left(\frac{t^2}{1 - e^{-t^2}}\right)^{\frac{d}{2}} \leq \left(\frac{a^2}{1 - e^{-a^2}}\right)^{\frac{d}{2}}$$

(note that $x/(1 - e^{-x})$ is increasing).

Inserting the dependency of c_k upon k as coming from (16) and using that $t \leq a$, we can then bound the maximal function as follows:

$$|e^{-t^2 L} u(y)| = \sum_{k=0}^{\infty} I_k \lesssim_{A,a,d} M_{\gamma} u(x) \sum_{k=0}^{\infty} 2^{kd} e^{\alpha(1+\alpha)2^{k+1}} e^{-\frac{\alpha^2}{2e^2 a^2} 4^k} e^{2^{k+2} \alpha a}.$$

Evidently the sum on the right-hand side converges. ■

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