

# A note on the Gaussian maximal function - Version 8 October 2013 + JvN additions

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ABSTRACT. This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded almost surely by the Gaussian Hardy-Littlewood maximal function. In particular this entails improvement on a result by Pineda and Urbina [3] who proved a similar result for a ‘truncated’ version of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

## 1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well known that the classical real-valued maximal function associated with the heat semigroup is bounded almost everywhere by the Hardy-Littlewood maximal function,

$$(1) \quad \sup_{(y,t) \in \Gamma_x} |e^{-t\Delta} u(y)| \lesssim \sup_{r>0} \int_{B_r(x)} |u| \, d\lambda.$$

Here the action of *heat semigroup*  $e^{-t^2\Delta} u = \rho_t * u$  is given by a convolution of  $u$  with the *heat kernel*

$$\rho_t(s) := \frac{e^{-|s|^2/4t}}{\pi^{\frac{d}{2}}} \frac{1}{(4t)^{\frac{d}{2}}}.$$

In this note we are interested in its gaussian counterpart. The change from Lebesgue measure to the gaussian measure

$$(2) \quad d\gamma(x) := \pi^{-\frac{d}{2}} e^{-|x|^2} dx$$

introduces quite some intricate technical and conceptual difficulties which appears to be due to the fact that the Gaussian measure is non-doubling.

As an analogue to the Laplacian which is symmetric in  $L^2$  with respect to the Lebesgue measure next we introduce the *Ornstein-Uhlenbeck* operator  $L$  which is symmetric with respect to the Gaussian measure:

$$(3) \quad L := -\frac{1}{2}\Delta + \langle x, \nabla \rangle = \frac{1}{2}\nabla_\gamma^* \nabla_\gamma,$$

where  $\nabla_\gamma$  denotes the realisation of the gradient in  $L^2(\mathbf{R}^d, \gamma)$ . Our main result, to be proved in (1), is the following gaussian analogue of (1):

$$(4) \quad \sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{-tL} u(y)| \lesssim \sup_{r>0} \int_{B_r(x)} |u| \, d\gamma.$$

Here,  $\Gamma_x^{(A,a)}$  is the Gaussian cone defined by

$$(5) \quad \Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y, t) \in \mathbf{R}_+^d : |x - y| < At \text{ and } t \leq am(x)\},$$

where

$$(6) \quad m(x) := \min \left\{ 1, \frac{1}{|x|} \right\} = 1 \wedge \frac{1}{|x|}.$$

A slightly weaker version of the inequality (4) has been proved by Pineda and Urbina [3] who shows that

$$\sup_{(y,t) \in \tilde{\Gamma}_x} |e^{-t\Delta} u(y)| \lesssim \sup_{r>0} \int_{B_r(x)} |u| d\gamma,$$

where

$$\tilde{\Gamma}_x(x) = \{(y, t) \in \mathbf{R}_+^d : |x - y| < t \leq \tilde{m}(x)\}$$

is the ‘reduced’ gaussian cone corresponding to the function

$$\tilde{m}(x) = \min \left\{ \frac{1}{2}, \frac{1}{|x|} \right\}.$$

Their proof does not seem to easily generalize the range of  $t$  from  $\frac{1}{2}$  up to 1. The proof of (4) is different and, we believe, more transparent than the one presented in [3]. It has the further virtue of allowing the extension to the cones with arbitrary aperture  $A > 0$  and cut-off parameter  $a > 0$ . This additional generality is very important and has already been used by Portal (cf. the claim made by [4, discussion preceding Lemma 2.3]) to prove the  $H^1$ -boundedness of the Riesz transform associated with  $L$ .

Before we continue, let us fix some notation. We will use without further reference notation such as  $\mathbf{Z}^d$  while we implicitly imply that  $d$  is a positive integer. To avoid possible confusion, we define the *positive integers* as the set  $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$ .

1.0.1. *minimal function.* We recall the lemma from [1, lemma 2.3] which first –although implicitly– appeared in [2].<sup>1</sup>

## 2. The Mehler kernel

**2.1. Setting.** Recall that we work with the Ornstein-Uhlenbeck operator  $L$  as given by (3).

We define the Mehler kernel (see e.g., Sjögren [5]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup  $(e^{-tL})_{t \geq 0}$ . More precisely, this means:

$$(7) \quad e^{-tL} u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot) u d\gamma.$$

It is often more convenient to use  $e^{-t^2 L}$  instead of  $e^{-tL}$  as is done in e.g., Portal [4].

**2.2. The Mehler kernel.** There is an abundance of literature on the Mehler kernel and its properties available, but for the present purpose Sjögren [5] will suffice. For instance, the Mehler kernel  $M_t$  of (7) is computed there. In addition it offers related results with to the Hermite polynomials.

The kernel  $M_t$  is invariant under the permutation  $x \leftrightarrow y$ . A formula for  $M_t$  which honors this observation is:

$$(8) \quad M_t(x, y) = \frac{\exp\left(-e^{-2t} \frac{|x - y|^2}{1 - e^{-2t}}\right) \exp\left(2e^{-t} \frac{\langle x, y \rangle}{1 + e^{-t}}\right)}{(1 - e^{-t})^{\frac{d}{2}} (1 + e^{-t})^{\frac{d}{2}}}.$$

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### 3. Some lemmata and definitions

We use  $m$  as defined in (6) in our next lemma.

1. LEMMA. *Let  $a, A$  be strictly positive real numbers and  $t > 0$ . We have for  $x, y \in \mathbf{R}^d$  that:*

- (1) *If  $|x - y| < At$  and  $t \leq am(x)$ , then  $t \leq (1 + aA)m(y)$ ;*
- (2) *If  $|x - y| < Am(x)$ , then  $m(x) \leq (1 + A)m(y)$  and  $m(y) \leq 2(1 + A)m(x)$ .*

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. LEMMA. *Let  $\alpha > 0$  and  $|x - y| \leq am(x)$ . We get the equivalence:*

$$e^{-\alpha^2(1+\alpha)^2} e^{-2\alpha(1+\alpha)} e^{-|y|^2} \leq e^{-|x|^2} \leq e^{\alpha^2} e^{2\alpha} e^{-|y|^2}.$$

PROOF. By the inverse triangle inequality and  $m(x)|x| \leq 1$  we get,

$$(9) \quad |y|^2 \leq (am(x) + |x|)^2 \leq \alpha^2 + 2\alpha + |x|^2.$$

For the reverse direction we use Lemma 1 to infer  $m(x) \leq (1 + \alpha)m(y)$ . Proceeding as before we obtain:

$$|x|^2 \leq \alpha^2(1 + \alpha)^2 + 2\alpha(1 + \alpha) + |y|^2.$$

Combining we get:

$$(10) \quad e^{-\alpha^2(1+\alpha)^2} e^{-2\alpha(1+\alpha)} e^{-|y|^2} \leq e^{-|x|^2} \leq e^{\alpha^2} e^{2\alpha} e^{-|y|^2}.$$

As required. ■

### 4. On-diagonal estimates

4.1. **Kernel estimates.** Before we proceed with the technicalities we define  $\kappa$  and  $\mu$  as:

$$\kappa = 2\left(1 + \frac{1}{\alpha}\right)^{-1}, \text{ and } \mu = 2\left(1 - \frac{1}{\alpha}\right)^{-1}.$$

such that  $\kappa$  and  $\mu$  are conjugate exponents, which means:

$$\frac{1}{\kappa} + \frac{1}{\mu} = 1.$$

We proceed with a simple technical lemma which is given here as it will be used on several occasions.

3. LEMMA. *Let  $t > 0$  and  $\alpha \geq 1$ . Then,*

$$(11) \quad \alpha e^{-2\frac{t}{\mu}} \leq \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} \leq \alpha,$$

$$(12) \quad 0 \leq \frac{1}{t} \left[ \frac{e^{-t}}{1 + e^{-t}} - \frac{e^{-\frac{t}{\alpha}}}{1 + e^{-\frac{t}{\alpha}}} \right] \leq \frac{1}{2\mu}.$$

PROOF. We start with (11) and apply the mean value theorem to the function  $f(\xi) = \xi^\alpha$ . For  $0 < \xi < \xi'$  this gives that:

$$f(\xi) - f(\xi') = \alpha \hat{\xi}^{\alpha-1} (\xi - \xi') \text{ for some } \hat{\xi} \text{ in } [\xi, \xi'].$$

Picking  $\xi = 1$  and  $\xi' = e^{-\frac{t}{\alpha}}$  yields:

$$(13) \quad \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} = \alpha \hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[ e^{-\frac{t}{\alpha}}, 1 \right].$$

Combining this result with the monotonicity of  $\xi \mapsto \alpha \xi^{\alpha-1}$  we obtain:

$$\alpha e^{\frac{t}{\alpha}} e^{-t} \leq \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} \leq \alpha,$$

where the last bound follows from the monotonicity together with the limit as  $t \downarrow 0$ . We proceed with (12). Recalling that  $\alpha \geq 1$  one can directly verify that the function

$$\frac{1}{t} \left[ \frac{1}{1 + e^{-t}} - \frac{1}{1 + e^{-\frac{t}{\alpha}}} \right]$$

is non-negative and decreasing in  $t$ . To find an upper bound we compute the limit as  $t$  goes to 0. That is:

$$\lim_{t \rightarrow 0} \frac{1}{t} \left[ \frac{e^{-t}}{1 + e^{-t}} - \frac{e^{-\frac{t}{\alpha}}}{1 + e^{-\frac{t}{\alpha}}} \right] = \lim_{t \rightarrow 0} \left[ \frac{e^{-2t}}{(1 + e^{-t})^2} - \frac{1}{\alpha} \frac{e^{-2\frac{t}{\alpha}}}{(1 + e^{-\frac{t}{\alpha}})^2} \right] \uparrow \frac{1}{2\mu}.$$

Which is as asserted and completes the proof. ■

The following lemma will be useful when transferring estimates from  $M_{\frac{t}{\alpha}}$  to  $M_t$ . It follows from the mean value theorem applied to  $\xi \mapsto \xi^\alpha$ .

4. LEMMA. For  $\alpha \geq 1$  and  $0 < t \leq T < \infty$  and all let  $x, y \in \mathbf{R}^d$  we have that:

$$(14) \quad \exp\left(-\frac{1}{e^{2\frac{t}{\alpha}}} \frac{|x-y|^2}{1 - e^{-2\frac{t}{\alpha}}}\right) \leq \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\alpha}}} \frac{|x-y|^2}{1 - e^{-t}}\right).$$

PROOF. It is fruitful to note that

$$1 - e^{-2t} = (1 - e^{-t})(1 + e^{-t})$$

holds. The same holds true for the inequality

$$\frac{1}{2e^{2t}} \leq \frac{e^{-2t}}{1 + e^{-t}} \leq \frac{1}{2}.$$

Therefore,

$$\begin{aligned} \exp\left(-e^{-2t} \frac{|x-y|^2}{1 - e^{-2t}}\right) &= \exp\left(-\frac{e^{-2t}}{1 + e^{-t}} \frac{|x-y|^2}{1 - e^{-t}}\right) \\ &\leq \exp\left(-\frac{1}{2e^{2t}} \frac{|x-y|^2}{1 - e^{-t}}\right). \end{aligned}$$

Setting  $\beta := 1 + \alpha^{-1}$  and applying (11) we get:

$$\begin{aligned} \exp\left(-e^{-2\frac{t}{\alpha}} \frac{|x-y|^2}{1 - e^{-2\frac{t}{\alpha}}}\right) &= \exp\left(-\frac{e^{-2\frac{t}{\alpha}}}{1 + e^{-\frac{t}{\alpha}}} \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} \frac{|x-y|^2}{1 - e^{-t}}\right) \\ &\leq \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\alpha}}} \frac{1}{e^t e^{-\frac{t}{\alpha}}} \frac{|x-y|^2}{1 - e^{-t}}\right) \\ &\leq \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\alpha}}} \frac{|x-y|^2}{1 - e^{-t}}\right). \end{aligned}$$

Which is as asserted. ■

Our first lemma is about estimating  $M_{\frac{t}{\alpha}}$  in terms of  $M_t$ .

#### 4.1.1. Time-scaling of the Mehler kernel.

5. LEMMA. Let  $T > 0$ ,  $\alpha \geq 1$ , and  $x, y \in \mathbf{R}^d$ . Then:

$$(15) \quad M_{\frac{t}{\alpha}}(x, y) \leq \alpha^{\frac{d}{2}} \exp\left(\frac{t}{2\mu} |\langle x, y \rangle|\right) \exp\left(-\frac{\alpha}{4e^{2T}} \frac{|x-y|^2}{1-e^{-t}}\right) M_t(x, y).$$

PROOF. To prove the lemma we compute  $M_{\frac{t}{\alpha}} M_t^{-1}$ . First recall that (11) gives

$$\alpha e^{-\frac{t}{\mu}} \leq \frac{1-e^{-t}}{1-e^{-\frac{t}{\alpha}}} \leq \alpha.$$

Combining the exponentials also gives,

$$\begin{aligned} \exp\left(-2e^{-\frac{t}{\alpha}} \frac{\langle x, y \rangle}{1+e^{-\frac{t}{\alpha}}}\right) \exp\left(2e^{-t} \frac{\langle x, y \rangle}{1+e^{-t}}\right) \\ = \exp\left(\frac{2}{t} \left[ \frac{e^{-t}}{1+e^{-t}} - \frac{e^{-\frac{t}{\alpha}}}{1+e^{-\frac{t}{\alpha}}} \right] t \langle x, y \rangle\right) \\ \stackrel{(12)}{\leq} \exp\left(\frac{t}{\mu} |\langle x, y \rangle|\right). \end{aligned}$$

Combining Lemma 4 and equation (14) almost gives the final estimate.

$$\begin{aligned} \frac{M_{\frac{t}{\alpha}}(x, y)}{M_t(x, y)} &\leq \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu} |\langle x, y \rangle|\right) \exp\left(e^{-2t} \frac{|x-y|^2}{1-e^{-2t}}\right) \exp\left(-e^{-2\frac{t}{\alpha}} \frac{|x-y|^2}{1-e^{-2\frac{t}{\alpha}}}\right) \\ &\leq \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu} |\langle x, y \rangle|\right) \exp\left(\left[1 - \frac{\alpha}{4e^{2\frac{t}{\kappa}}}\right] \frac{|x-y|^2}{1-e^{-t}}\right) \exp\left(-\frac{\alpha}{4e^{2\frac{t}{\kappa}}} \frac{|x-y|^2}{1-e^{-t}}\right). \end{aligned}$$

Finally, we apply the assumption  $\alpha \geq 4e^{2\frac{t}{\kappa}}$  to obtain:

$$\frac{M_{\frac{t}{\alpha}}(x, y)}{M_t(x, y)} \leq \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu} |\langle x, y \rangle|\right) \exp\left(-\frac{\alpha}{4e^{2\frac{t}{\kappa}}} \frac{|x-y|^2}{1-e^{-t}}\right).$$

Which is as asserted. The assumption  $\alpha \geq 4e^{2\frac{t}{\kappa}}$  can be rephrased as the requirement that  $\left(1 + \frac{1}{\alpha}\right)^{-1} \log\left(\frac{\alpha}{4}\right) \geq t$ .  $\blacksquare$

#### 4.2. An estimate on Gaussian balls.

6. LEMMA. Let  $B_t(x)$  be the Euclidean ball with radius  $t$  and center  $x$  and let  $\gamma$  be the Gaussian measure (2). We have the inequality:

$$(16) \quad \frac{\gamma(B_t(x))}{V_d(t)} \leq d\pi^{-\frac{d}{2}} e^{-(t-|x|)^2}.$$

PROOF. Remark that

$$\begin{aligned} \int_B e^{-|\xi|^2} d\xi &= e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{-2\langle x, \xi-x \rangle} d\xi \\ &\leq e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{2|x||\xi-x|} d\xi \\ &\leq e^{-|x|^2} e^{2|x|t} \int_B e^{-|\xi-x|^2} d\xi \\ &= \pi^{\frac{d}{2}} e^{-|x|^2} e^{2t|x|} \gamma(B_t(0)). \end{aligned}$$

So, for a ball  $B := B_t(x)$  there holds that

$$(17) \quad \gamma(B_t(x)) \leq e^{-|x|^2} e^{2t|x|} \gamma(B_t(0)).$$

We will estimate the Gaussian volume of the ball  $B_t(0)$ . To save writing, let  $S_d$  and  $V_d$  be the surface area and volume respectively of the  $d$ -dimensional unit sphere. Using polar coordinates we proceed by:

$$\begin{aligned}\gamma(B_t(0)) &= \pi^{-\frac{d}{2}} \int_{B_t(0)} e^{-|\xi|^2} d\xi \\ &= S_d \pi^{-\frac{d}{2}} \int_0^t e^{-r^2} r^{d-1} dr \\ &\leq S_d t^d \pi^{-\frac{d}{2}} e^{-t^2} \\ &= dV_d(t) \pi^{-\frac{d}{2}} e^{-t^2}.\end{aligned}$$

Upon combining this result with (17) we obtain (16), which is as promised.  $\blacksquare$

**4.3. On-diagonal kernel estimates on annuli.** As is common in harmonic analysis, we often wish to decompose  $\mathbf{R}^d$  into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli  $C_k$ . For the sake of simplicity we will write  $B := B_t(x)$  and mean that  $B$  is the closed ball with center  $x$  and radius  $t$ . Furthermore, we use notations such as  $2B$  to mean the ball obtained from  $B$  by multiplying its radius by 2.

The  $C_k$  are given by,

$$(18) \quad C_k(B) := C_k = (2^{k+1} - 1)B \setminus (2^k - 1)B.$$

So, whenever  $\xi$  is in  $C_k(B_{\sqrt{t}}(x))$ , we get for  $k \geq 0$ :

$$(19) \quad (2^k - 1)t < |y - \xi| \leq (2^{k+1} - 1)t.$$

7. LEMMA. *Given  $A > 0$ , let  $B = B_{At}(y)$ ,  $0 < t \leq T < \infty$  and  $\xi \in C_k$ . Then we have:*

$$M_{t^2}(y, \xi) \leq \frac{e^{-\beta} e^{|y|^2}}{(1 - e^{-t^2})^{\frac{d}{2}}} \exp((2^{k+1} - 1)At^2|y|) e^{\beta 2^{k+1}} e^{-\beta 4^k},$$

where  $\beta = \frac{A^2}{2e^{2T}}$ .

PROOF. Let  $B = B_{At}(y)$  and let  $C_k$  be as in (18). Considering the first exponential which occurs in the Mehler kernel (8) together with (19) gives for  $k \geq 0$ :

$$\begin{aligned}\exp\left(-e^{-2t^2} \frac{|y - \xi|^2}{1 - e^{-2t^2}}\right) &\leq \exp\left(-e^{-2t^2} \frac{(2^k - 1)^2 A^2 t^2}{1 - e^{-2t^2}}\right) \\ &\stackrel{(\dagger)}{\leq} \exp\left(-\frac{A^2}{2e^{2t^2}} (2^k - 1)^2\right).\end{aligned}$$

Where  $(\dagger)$  follows from

$$\frac{t}{1 - e^{-2t}} \geq \frac{1}{2}.$$

Before we consider the last exponential in the Mehler kernel we note that by Cauchy-Schwarz:

$$(20) \quad |\langle y, \xi \rangle| \leq |\langle y - \xi, y \rangle| + |\langle y, y \rangle| \leq |y - \xi||y| + |y|^2.$$

Furthermore we have the estimate:

$$\frac{e^{-t}}{1 + e^{-t}} \leq \frac{1}{2},$$

Using these we get for the last exponential in the Mehler kernel (8)  $M_{t^2}$ :

$$\begin{aligned} \exp\left(2e^{-t^2} \frac{\langle y, \xi \rangle}{1 + e^{-t}}\right) &\leq \exp(|\langle y, \xi \rangle|) \\ &\stackrel{(20)}{\leq} \exp(|y - \xi||y|)e^{|y|^2}. \end{aligned}$$

Wrapping it up, we can estimate the Mehler kernel (8)  $M_t$  on  $C_k$  from above by:

$$M_{t^2}(y, \xi) \leq \frac{e^{|y|^2}}{(1 - e^{-t^2})^{\frac{d}{2}}} \exp((2^{k+1} - 1)At|y|) \exp\left(-\frac{A^2}{2e^{2t^2}}(2^k - 1)^2\right).$$

Setting  $\beta = \frac{A^2}{2e^{2t}}$  and expanding the last exponential we get:

$$M_{t^2}(y, \xi) \leq \frac{e^{-\beta}e^{|y|^2}}{(1 - e^{-t^2})^{\frac{d}{2}}} \exp((2^{k+1} - 1)At|y|)e^{\beta 2^{k+1}}e^{-\beta 4^k}.$$

Which is as claimed.

**Lemma 2** gives us by using  $|x - y| \leq at \leq \alpha^2 m(x)$  the following estimate:

$$e^{|y|^2} \leq e^{|x|^2}e^{\alpha^4}e^{2\alpha^2}$$

$$\begin{aligned} M_t(y, \xi) &\leq \frac{e^{-\beta}e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp((2^{k+1} - 1)\alpha(1 + \alpha))e^{\beta 2^{k+1}}e^{-\beta 4^k} \\ &\leq e^{-(\alpha+\beta)}e^{\alpha^4}e^{\alpha^2} \frac{e^{|x|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp(2^{k+1}\alpha(1 + \alpha))e^{\beta 2^{k+1}}e^{-\beta 4^k}. \end{aligned}$$

Which is as claimed. ■

## 5. The boundedness of some non-tangential maximal operators

Our theorem is a small modification of [3, lemma 1.1] with a new proof.

1. THEOREM. Let  $A, a > 0$ . For all  $x$  in  $\mathbf{R}^d$  and all  $u$  in  $L_\gamma^2$  we have

$$(21) \quad \sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \int_{B_r(x)} |u| \, d\gamma.$$

PROOF. First we note that  $\Gamma_x^{(A,a)} \subset \Gamma_x^{(1+aA, aA)}$  as  $a, A \geq 1$ .

$$\begin{aligned} |x - y| &\leq At \leq aAt \\ t &\leq am(x) \leq aAm(x) \end{aligned}$$

So if  $y \in \Gamma_x^{(A,a)}$  then  $x \in \Gamma_y^{(aA, aA)}$ . So set  $\alpha = aA$  and  $\Gamma_x^\alpha = \Gamma_x^{(\alpha, \alpha)}$

We will prove (21) by splitting up the integration domain in annuli.

$$e^{-tL}|u(y)| \leq \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k(B)} M_t(y, \cdot) |u| \, d\gamma.$$

More precisely, we will set  $B = B(y, aAt)$  in the above and find a suitable upper bound for each integral on the right-hand side which we will denote by  $I_k$  for the sake of simplicity.

$$M_t(y, \xi) \leq \frac{e^{-\beta}e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp((2^{k+1} - 1)\alpha t|y|)e^{\beta 2^{k+1}}e^{-\beta 4^k},$$

where  $\beta = \frac{\alpha^2}{2e^{2\alpha^2}}$ .

Since we have  $|x - y| < \alpha t$  and  $t \leq am(x)$  we infer that  $t|x| \leq \alpha$ . By [Lemma 1](#) we also have that  $t|y| \leq 1 + \alpha$ . From this and [Lemma 7](#) we infer that:

$$(22) \quad M_t(y, \xi) \leq e^{-\beta} e^{-\alpha(1+\alpha)} \frac{e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp(2^{k+1} \alpha(1 + \alpha)) e^{\beta 2^{k+1}} e^{-\beta 4^k},$$

Setting  $\beta = \frac{\alpha^2}{2e^{2\alpha^2}}$ . Note that  $\beta$  is maximal for  $\alpha = \frac{1}{2}$  and after this value,  $\beta$  is decreasing. Setting  $\lambda := \alpha(1 + \alpha)$  we get:

$$(23) \quad M_t(y, \xi) \lesssim_\alpha e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2} \frac{e^{|x|^2}}{(1 - e^{-t})^{\frac{d}{2}}} e^{(\lambda+\beta)2^{k+1}} e^{-\beta 4^k}.$$

Where the implied constant is given by  $e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2}$

Or, using  $\Lambda = \beta + \lambda$  we get:

$$(24) \quad M_t(y, \xi) \lesssim_\alpha \frac{e^{|x|^2}}{(1 - e^{-t})^{\frac{d}{2}}} e^{\Lambda 2^{k+1}} e^{-\beta 4^k},$$

Recalling [Lemma 6](#) we get:

$$(25) \quad \gamma(B_t(x)) \leq V_d d \pi^{-\frac{d}{2}} t^d e^{-(t-|x|)^2}.$$

Where we abbreviate  $V_d(1)$  with  $V_d$ . Recall

$$V_d \leq \frac{1}{\sqrt{\pi}} \left( \frac{2\pi e}{d} \right)^{\frac{d}{2}}.$$

To get,

$$(26) \quad \gamma(B_t(x)) \leq \frac{d}{\sqrt{\pi}} \left( \frac{2e}{d} \right)^{\frac{d}{2}} t^d e^{-(t-|x|)^2} = C_d t^d e^{-(t-|x|)^2}.$$

This allows us to estimate the remaining unbounded exponential in the Mehler kernel and allow a penalty up to  $e^{-|x|^2}$ . Furthermore, we have the following estimate which will make clear how to handle the time part in the Mehler kernel:

$$\frac{t^d}{(1 - e^{-t^2})^{\frac{d}{2}}} \leq \left( \frac{t^2}{1 - e^{-t^2}} \right)^{\frac{d}{2}} \leq \frac{a^d}{(1 - e^{-a^2})^{\frac{d}{2}}}.$$

Let  $B' := B(x, 2^{k+1} \alpha t)$  and  $B$  as before the ball  $B(y, \alpha t)$ . In the next step we will bound .... by the maximal function centered at  $x$ . For this we need to scale up the  $C_k$ . So,

$$|x - \xi| \leq |x - y| + |\xi - y| \leq \alpha t + (2^{k+1} - 1) \alpha t = 2^{k+1} \alpha t.$$



And set  $D_k = B(2^{k+1}at)$ . So, we can bound the integral on the right-hand side of (??) by

$$\begin{aligned}
\int_{C_k(B)} M_{t^2}(y, \cdot) |u| \, d\gamma &\lesssim_\alpha \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1 - e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} \int_{C_k(B)} |u| \, d\gamma \\
&\leq \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1 - e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} \int_{D_k(B)} |u| \, d\gamma \\
&\leq (M_\gamma u)(x) \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1 - e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} \gamma(D_k) \\
&\stackrel{(1)}{\leq} (M_\gamma u)(x) C_d \alpha^d 2^{d(k+1)} t^d \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1 - e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} e^{-(t-|x|)^2} \\
&\leq (M_\gamma u)(x) 2^{kd} e^{\Lambda 2^{k+1}} e^{-\beta 4^k} \frac{t^d e^{-t^2}}{(1 - e^{-t^2})^{\frac{d}{2}}} C_d e^{2\alpha} (2\alpha)^d
\end{aligned}$$

Where (1) uses [Lemma 6](#) and  $t|x| \leq a$ .

We can then bound the maximal function:

$$\begin{aligned}
e^{-t^2 L} |u(y)| &= \sum_{k=0}^{\infty} I_k \\
&\leq (M_\gamma u)(x) C_{d,a,A} \sum_{k=0}^{\infty} 2^{kd} e^{\Lambda 2^{k+1}} e^{-\beta 4^k}
\end{aligned}$$

Wrapping it up, we have that:

$$e^{-t^2 L} |u(y)| \lesssim \int_{B_r(x)} |u| \, d\gamma.$$

With implied constant

Which is what we wanted to prove.

$$\sum_{k=0}^{\infty} 2^{kd} e^{-C 4^k} = \sum_{k=0}^{\infty} x^{kd} e^{-C x^{2k}}$$

Noting for  $x \geq 1$  that  $\exp(-C x^{2k}) \leq \exp(-C k x^2)$ , thus,

$$\sum_{k=0}^{\infty} 2^{kd} e^{-C 4^k} \leq \sum_{k=0}^{\infty} x^{kd} (e^{-C x^2})^k = \sum_{k=0}^{\infty} (x^d e^{-C x^2})^k$$

Here  $x = 2$ , so

$$\sum_{k=0}^{\infty} 2^{kd} e^{-C 4^k} \leq \sum_{k=0}^{\infty} (2^d e^{-4C})^k$$

If  $2^d < e^{4C}$ , that is whenever  $d \log 2 < 4C$ , we can compute using the geometric series that

$$\sum_{k=0}^{\infty} 2^{kd} e^{-C 4^k} \leq \frac{1}{1 - 2^d e^{-4C}} = \frac{e^{4C}}{e^{4C} - 2^d}$$

■

## References

- [1] Jan Maas, Jan Neerven, and Pierre Portal. “Whitney coverings and the tent spaces  $T_{1,q}(\gamma)$  for the Gaussian measure”. In: *Arkiv för Matematik* 50.2 (Apr. 2011), pp. 379–395. arXiv: [1002.4911](#).

- [2] Giancarlo Mauceri and Stefano Meda. “BMO and  $H^1$  for the Ornstein-Uhlenbeck operator”. In: *Journal of Functional Analysis* 252.1 (Nov. 2007), pp. 278–313.
- [3] Ebner Pineda, W. Urbina, and Wilfredo Urbina R. “Non Tangential Convergence for the Ornstein-Uhlenbeck Semigroup”. In: *Divulgaciones Matemáticas* 13.2 (2008), pp. 1–19.
- [4] Pierre Portal. “Maximal and quadratic Gaussian Hardy spaces”. In: (Mar. 2012). arXiv: 1203.1998.
- [5] Peter Sjögren. “Operators associated with the hermite semigroup — A survey”. In: *The Journal of Fourier Analysis and Applications* 3.S1 (Jan. 1997), pp. 813–823.

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