A note on the Gaussian maximal function

Jonas Teuwen

Abstract. In this note we give an improvement on a result first demonstrated by Pineda, Urbina, and Urbina R. [3]. In particular we present an improvement to their Lemma 1.1 which gives the boundedness of the Gaussian maximal function associated to the Ornstein-Uhlenbeck operator.

We present a proof which is at least to the author more transparant. Our main finding in this note is that our proof allows to use a larger cone and actually obtain the maximal function boundedness for a whole class of cones $\Gamma_x^{(A,a)}(\gamma)$.

1. Introduction

Maximal functions are one of the most studied objects in harmonic analysis. The classical real-valued harmonic analytic maximal function, for instance, can be seen to be of the form

$$\sup_{(y,t)\in\Gamma_x} |\mathrm{e}^{-t\Delta}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \,\mathrm{d}\lambda.$$

where the action of *heat semigroup* $\mathrm{e}^{-t^2\Delta}u$ is given by a convolution of u with the *heat kernel*

$$\rho_t(s) := \frac{\mathrm{e}^{-|s|^2/4t}}{\pi^{\frac{d}{2}}} \frac{1}{(4t)^{\frac{d}{2}}},$$

so that,

$$\mathrm{e}^{-t\Delta}u(x) = (\rho_t * u)(x).$$

In this note we are interested in its gaussian harmonic analysis counterpart. Gaussian harmonic analysis seems to be conceptually nothing more than harmonic analysis with the gaussian measure, but this is far off from reality. This change of measure introduces quite some intricate technical and conceptually difficulties which appears to be due to the fact that the Gaussian measure is non-doubling whereas the Lebesgue measure is.

The *Gaussian measure* which we will use in multitudinous occasions in this note is fixed here.

$$\mathrm{d}\gamma(x) := \pi^{-\frac{d}{2}} \mathrm{e}^{-|x|^2} \,\mathrm{d}x. \tag{1}$$

As an analogue to the Laplacian which is symmetric in L^2 with respect to the Lebesgue measure next we introduce the *Ornstein-Uhlenbeck* operator L which is symmetric with respect to the Gaussian measure.

$$L := -\frac{1}{2}\Delta + \langle x, \nabla\rangle. \tag{2}$$

In Theorem 1 we will prove the analogue of the result with respect to the Lebesgue measure.

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |\mathrm{e}^{-tL}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \,\mathrm{d}\gamma.$$

We extend the result of [3, Lemma 1.1]. Our contribution is a different proof of this result and an extension to the complete Gaussian cone (3) for all $A, a > 0$.

Furthermore, this proof justifies the claim made by [4] above Lemma 2.3 that the proof extends to general $a, A > 0$.

1.1. Pineda, Urbina, and Urbina R. [3]. We will use as Gaussian cone:

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}_+^d : |x-y| < At \text{ and } t \leqslant am(x)\}, \tag{3}$$

where,

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\} = 1 \wedge \frac{1}{|x|}. \tag{4}$$

To save writing, we write Γ_x^a to mean $\Gamma_x^{(1,a)}$. In [3] the authors use a slightly different cut-off function $\widetilde{m}$:

$$\widetilde{m}(x) = \min\left\{\frac{1}{2}, \frac{1}{|x|}\right\},$$

and their proof does not seem to easily generalize the range of t up to 1. In brief, they have used the cone

$$\widetilde{\Gamma}_x(x) = \{(y,t) \in \mathbf{R}_+^d : |x-y| < t \leqslant \widetilde{m}(x)\},$$

and have shown that,

$$\sup_{(y,t)\in\widetilde{\Gamma}_x} |\mathrm{e}^{-t\Delta}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \,\mathrm{d}\gamma.$$

Before we continue, let us fix some notation. We will use without further reference notation such as $\mathbf{Z}^d$ while we implicitly imply that d is a positive integer. To avoid possible confusion, we define the *positive integers* as the set $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$.

1.1.1. *minimal function.* We recall the lemma from [1, lemma 2.3] which first –although implicitly– appeared in [2].

2. The Mehler kernel

2.1. Setting. Recall that we work with the *Ornstein-Uhlenbeck* operator L as given by (2).

We define the Mehler kernel (see e.g., Sjögren [5]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(\mathrm{e}^{-tL})_{t\geqslant 0}$. More precisely, this means:

$$\mathrm{e}^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x,\cdot)u \,\mathrm{d}\gamma. \tag{5}$$

It is often more convenient to use e^{-t^2L} instead of e^{-tL} as is done in e.g., Portal [4].

2.2. The Mehler kernel. For the calculation of the Mehler kernel M_t in (5) we refer to e.g., Sjögren [5] which additionally offers related results such as those related to Hermite polynomials.

The kernel M_t is invariant under the permutation $x \longleftrightarrow y$. A formula for M_t which honors this observation is:

$$M_t(x,y) = \frac{\exp\left(-\mathrm{e}^{-2t}\dfrac{|x-y|^2}{1-\mathrm{e}^{-2t}}\right)}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}} \frac{\exp\left(2\mathrm{e}^{-t}\dfrac{\langle x,y\rangle}{1+\mathrm{e}^{-t}}\right)}{(1+\mathrm{e}^{-t})^{\frac{d}{2}}}. \tag{6}$$

3. Some lemmata and definitions

We use m as defined in (4) in our next lemma.

1. LEMMA. *Let a, A be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

(1) If $|x - y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1 + aA)m(y)$;
(2) If $|x - y| < Am(x)$, then $m(x) \leqslant (1 + A)m(y)$ and $m(y) \leqslant 2(1 + A)m(x)$.

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. LEMMA. *Let $\alpha > 0$ and $|x - y| \leqslant \alpha m(x)$. We get the equivalence:*

$$\mathrm{e}^{-\alpha^2(1+\alpha)^2}\mathrm{e}^{-2\alpha(1+\alpha)}\mathrm{e}^{-|y|^2} \leqslant \mathrm{e}^{-|x|^2} \leqslant \mathrm{e}^{\alpha^2}\mathrm{e}^{2\alpha}\mathrm{e}^{-|y|^2}.$$

PROOF. By the inverse triangle inequality and $m(x)|x| \leqslant 1$ we get,

$$|y|^2 \leqslant (\alpha m(x) + |x|)^2 \leqslant \alpha^2 + 2\alpha + |x|^2. \tag{7}$$

For the reverse direction we use Lemma 1 to infer $m(x) \leqslant (1 + \alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant \alpha^2(1 + \alpha)^2 + 2\alpha(1 + \alpha) + |y|^2.$$

Combining we get:

$$\mathrm{e}^{-\alpha^2(1+\alpha)^2}\mathrm{e}^{-2\alpha(1+\alpha)}\mathrm{e}^{-|y|^2} \leqslant \mathrm{e}^{-|x|^2} \leqslant \mathrm{e}^{\alpha^2}\mathrm{e}^{2\alpha}\mathrm{e}^{-|y|^2}. \tag{8}$$

As required. ∎

4. On-diagonal estimates

4.1. Kernel estimates. We begin with a technical lemma which will be useful on several occasions.

3. LEMMA. *Let $t > 0$ and $\alpha \geqslant 1$. Then,*

$$\alpha \mathrm{e}^{\frac{t}{\alpha}}\mathrm{e}^{-t} \leqslant \frac{1 - \mathrm{e}^{-t}}{1 - \mathrm{e}^{-\frac{t}{\alpha}}} \leqslant \alpha, \tag{9}$$

and

$$0 \leqslant \frac{1}{t}\left[\frac{1}{1 + \mathrm{e}^{-t}} - \frac{1}{1 + \mathrm{e}^{-\frac{t}{\alpha}}}\right] \leqslant \frac{1}{4}\left(1 - \frac{1}{\alpha}\right), \tag{10}$$

and

$$0 \leqslant \frac{1}{t}\left[\frac{\mathrm{e}^{-t}}{1 + \mathrm{e}^{-t}} - \frac{\mathrm{e}^{-\frac{t}{\alpha}}}{1 + \mathrm{e}^{-\frac{t}{\alpha}}}\right] \leqslant \frac{1}{4}\left(1 - \frac{1}{\alpha}\right). \tag{11}$$

PROOF. We start with (9) and apply the mean value theorem to the function $f(\xi) = \xi^\alpha$. For $0 < \xi < \xi'$ this gives that:

$$f(\xi) - f(\xi') = \alpha \hat{\xi}^{\alpha-1}(\xi - \xi') \text{ for some } \hat{\xi} \text{ in } [\xi, \xi'].$$

Picking $\xi = 1$ and $\xi' = \mathrm{e}^{-\frac{t}{\alpha}}$ yields:

$$\frac{1 - \mathrm{e}^{-t}}{1 - \mathrm{e}^{-\frac{t}{\alpha}}} = \alpha \hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[\mathrm{e}^{-\frac{t^2}{\alpha}}, 1\right]. \tag{12}$$

Combining this result with the monotonicity of $\xi \mapsto \alpha \xi^{\alpha-1}$ we obtain:

$$\alpha \mathrm{e}^{\frac{t}{\alpha}}\mathrm{e}^{-t} \leqslant \frac{1 - \mathrm{e}^{-t}}{1 - \mathrm{e}^{-\frac{t}{\alpha}}} \leqslant \alpha,$$

where the last bound follows from the monotonicity together with the limit as $t \downarrow 0$.

$$\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} \leqslant \frac{1}{\alpha}\frac{\mathrm{e}^{-\frac{t}{\alpha}}}{1-\mathrm{e}^{-\frac{t}{\alpha}}} \leqslant \frac{\mathrm{e}^{-\frac{t}{\alpha}}}{1-\mathrm{e}^{-t}}$$

We proceed with (10). Recalling that $\alpha \geqslant 1$ one can directly verify that the function

$$\frac{1}{t}\left[\frac{1}{1+\mathrm{e}^{-t}} - \frac{1}{1+\mathrm{e}^{-\frac{t}{\alpha}}}\right]$$

is non-negative and decreasing in t. To find an upper bound we compute the limit as t goes to 0. That is:

$$\lim_{t\to 0}\frac{1}{t}\left[\frac{1}{1+\mathrm{e}^{-t}} - \frac{1}{1+\mathrm{e}^{-\frac{t}{\alpha}}}\right] = \lim_{t\to 0}\left[\frac{\mathrm{e}^{-t}}{(1+\mathrm{e}^{-t})^2} - \frac{1}{\alpha}\frac{\mathrm{e}^{-\frac{t}{\alpha}}}{(1+\mathrm{e}^{-\frac{t}{\alpha}})^2}\right] \uparrow \frac{1}{4}\left(1-\frac{1}{\alpha}\right).$$

Which is as asserted and completes the proof. ∎

The following lemma will be useful when transfering estimates from $M_{\frac{t}{\alpha}}$ to M_t. It follows from the mean value theorem applied to $\xi \mapsto \xi^{\alpha}$.

4. Lemma. *For $\alpha \geqslant 1$ and $0 < t \leqslant T < \infty$ and all let $x, y \in \mathbf{R}^d$ we have that:*

$$\exp\left(-\mathrm{e}^{-2\frac{t}{\alpha}}\frac{|x-y|^2}{1-\mathrm{e}^{-2\frac{t}{\alpha}}}\right) \leqslant \exp\left(-\frac{\alpha}{2\mathrm{e}^{2T}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right). \tag{13}$$

Proof. First note that

$$\frac{\mathrm{e}^{-2t}}{2} \leqslant \frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-t}} \leqslant \frac{1}{2}.$$

Therefore,

$$\begin{aligned}\exp\left(-\mathrm{e}^{-2t}\frac{|x-y|^2}{1-\mathrm{e}^{-2t}}\right) &= \exp\left(-\frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-t}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right)\\ &\leqslant \exp\left(-\frac{1}{2\mathrm{e}^{2T}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right).\end{aligned}$$

Applying (9) we get:

$$\begin{aligned}\exp\left(-\mathrm{e}^{-2\frac{t}{\alpha}}\frac{|x-y|^2}{1-\mathrm{e}^{-2\frac{t}{\alpha}}}\right) &= \exp\left(-\frac{\mathrm{e}^{-2\frac{t}{\alpha}}}{1+\mathrm{e}^{-\frac{t}{\alpha}}}\frac{1-\mathrm{e}^{-t}}{1-\mathrm{e}^{-\frac{t}{\alpha}}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right)\\ &\leqslant \exp\left(-\alpha\frac{1}{2\mathrm{e}^{2\frac{t}{\alpha}}}\frac{\mathrm{e}^{\frac{t}{\alpha}}}{\mathrm{e}^{t}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right)\\ &\leqslant \exp\left(-\frac{\alpha}{2\mathrm{e}^{2T}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right).\end{aligned}$$

Which is as asserted. ∎

Our first lemma is about estimating $M_{\frac{t^2}{\alpha}}$ in terms of M_{t^2}.

4.1.1. *Time-scaling of the Mehler kernel.*

5. Lemma. *Let $T > 0$, $\alpha \geqslant 1$, and $x, y \in \mathbf{R}^d$. Then:*

$$M_{\frac{t}{\alpha}}(x,y) \leqslant \alpha^{\frac{d}{2}}\mathrm{e}^{\frac{t}{4}|\langle x,y\rangle|}\exp\left(-\frac{\alpha}{4\mathrm{e}^{2T}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right)M_t(x,y). \tag{14}$$

Proof. To prove the lemma we compute $M_{\frac{t}{\alpha}}M_t^{-1}$. First note that (9) gives

$$\alpha^{\frac{d}{2}}\mathrm{e}^{-\frac{d}{2}T} \leqslant \frac{(1-\mathrm{e}^{-t})^{\frac{d}{2}}}{(1-\mathrm{e}^{-\frac{t}{\alpha}})^{\frac{d}{2}}} \leqslant \alpha^{\frac{d}{2}}.$$

Combining the exponentials also gives,

$$\begin{aligned}
&\exp\left(-2\mathrm{e}^{-\frac{t}{\alpha}}\frac{\langle x,y\rangle}{1+\mathrm{e}^{-\frac{t}{\alpha}}}\right)\exp\left(2\mathrm{e}^{-t}\frac{\langle x,y\rangle}{1+\mathrm{e}^{-t}}\right)\\
&\qquad = \exp\left(\frac{2}{t}\left[\frac{\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}}-\frac{\mathrm{e}^{-\frac{t}{\alpha}}}{1+\mathrm{e}^{-\frac{t}{\alpha}}}\right]t\langle x,y\rangle\right)\\
&\qquad \overset{(10)}{\leqslant} \exp\left(\frac{1}{2}\left(1-\frac{1}{\alpha}\right)t|\langle x,y\rangle|\right).
\end{aligned}$$

Combining Lemma 4 and equation (13) almost gives the final estimate.

$$\begin{aligned}
\frac{M_{\frac{t}{\alpha}}(x,y)}{M_t(x,y)} &\leqslant \alpha^{\frac{d}{2}}\mathrm{e}^{\frac{t}{4}|\langle x,y\rangle|}\exp\left(\mathrm{e}^{-2t}\frac{|x-y|^2}{1-\mathrm{e}^{-2t}}\right)\exp\left(-\mathrm{e}^{-2\frac{t}{\alpha}}\frac{|x-y|^2}{1-\mathrm{e}^{-2\frac{t}{\alpha}}}\right)\\
&\leqslant \alpha^{\frac{d}{2}}\mathrm{e}^{\frac{t}{4}|\langle x,y\rangle|}\exp\left(\left[1-\frac{\alpha}{4\mathrm{e}^{2T}}\right]\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right)\exp\left(-\frac{\alpha}{4\mathrm{e}^{2T}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right).
\end{aligned}$$

Finally, we apply the assumption $\alpha \geqslant 4\mathrm{e}^{2T}$ to obtain:

$$\frac{M_{\frac{t}{\alpha}}(x,y)}{M_t(x,y)} \leqslant \alpha^{\frac{d}{2}}\mathrm{e}^{\frac{t}{4}|\langle x,y\rangle|}\exp\left(-\frac{\alpha}{4\mathrm{e}^{2T}}\frac{|x-y|^2}{1-\mathrm{e}^{-t}}\right).$$

Which is as asserted. ∎

4.2. An estimate on Gaussian balls.

6. Lemma. *Let $B_t(x)$ be the Euclidean ball with radius t and center x and let γ be the Gaussian measure (1). We have the inequality:*

$$\frac{\gamma(B_t(x))}{V_d(t)} \leqslant d\pi^{-\frac{d}{2}}\mathrm{e}^{-(t-|x|)^2}. \tag{15}$$

Proof. Remark that for a ball $B := B_t(x)$ there holds that

$$\begin{aligned}
\int_B \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi &= \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{-2\langle x,\xi-x\rangle}\,\mathrm{d}\xi\\
&\leqslant \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{2|x||\xi-x|}\,\mathrm{d}\xi\\
&\leqslant \pi^{\frac{d}{2}}\mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)).
\end{aligned}$$

That is:

$$\gamma(B_t(x)) \leqslant \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)). \tag{16}$$

We will estimate the Gaussian volume of the ball $B_t(0)$. To ease the notation, let S_d and V_d be the surface area and volume respectively of the d-dimensional unit sphere. Using polar coordinates we then obtain:

$$\begin{aligned}
\gamma(B_t(0)) &= \pi^{-\frac{d}{2}}\int_{B_t(0)} \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi\\
&= S_d\pi^{-\frac{d}{2}}\int_0^t \mathrm{e}^{-r^2}r^{d-1}\,\mathrm{d}r\\
&\leqslant S_d t^d\pi^{-\frac{d}{2}}\mathrm{e}^{-t^2}\\
&= dV_d(t)\pi^{-\frac{d}{2}}\mathrm{e}^{-t^2}.
\end{aligned}$$

Upon combining this result with (16) we obtain (15), which is as promised. ∎

4.3. On-diagonal kernel estimates on annuli. As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli C_k. For the sake of simplicity we will write $B := B_t(x)$ and mean that B is the closed ball with center x and radius t. Furthermore, we use notations such as $2B$ to mean the ball obtained from B by multiplying its radius by 2.

The C_k are given by,

$$C_k(B) := C_k = (2^{k+1}-1)B \setminus (2^k-1)B. \tag{17}$$

So, whenever ξ is in $C_k(B_{\sqrt{t}}(x))$, we get for $k \geqslant 0$:

$$(2^k-1)t < |y-\xi| \leqslant (2^{k+1}-1)t. \tag{18}$$

7. Lemma. *Given $A > 0$, let $B = B_{At}(y)$, $0 < t \leqslant T < \infty$ and $\xi \in C_k$. Then we have:*

$$M_t(y,\xi) \leqslant \frac{\mathrm{e}^{-\beta}\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)At|y|\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k},$$

where $\beta = \frac{A^2}{2\mathrm{e}^{2T}}$.

Proof. Let $B = B_{At}(y)$ and let C_k be as in (17). Considering the first exponential which occurs in the Mehler kernel (6) together with (18) gives for $k \geqslant 0$:

$$\begin{aligned}\exp\left(-\mathrm{e}^{-2t}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t}}\right) &\leqslant \exp\left(-\mathrm{e}^{-2t}\frac{(2^k-1)^2A^2t}{1-\mathrm{e}^{-2t}}\right)\\ &\overset{\text{(i)}}{\leqslant} \exp\left(-\frac{A^2}{2\mathrm{e}^{2t}}(2^k-1)^2\right).\end{aligned}$$

Where (i) follows from

$$\frac{t}{1-\mathrm{e}^{-2t}} \geqslant \frac{1}{2}.$$

Before we consider the last exponential in the Mehler kernel we note that by Cauchy-Schwarz:

$$|\langle y,\xi\rangle| \leqslant |\langle y-\xi,y\rangle| + |\langle y,y\rangle| \leqslant |y-\xi||y| + |y|^2. \tag{19}$$

Furthermore we have the estimate:

$$\frac{\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}} \leqslant \frac{1}{2},$$

Using these we get for the last exponential in the Mehler kernel (6):

$$\begin{aligned}\exp\left(2\mathrm{e}^{-t}\frac{\langle y,\xi\rangle}{1+\mathrm{e}^{-t}}\right) &\leqslant \exp(|\langle y,\xi\rangle|)\\ &\overset{(19)}{\leqslant} \exp(|y-\xi||y|)\mathrm{e}^{|y|^2}.\end{aligned}$$

Wrapping it up, we can estimate the Mehler kernel (6) M_t on C_k from above by:

$$M_t(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)At|y|\big) \exp\left(-\frac{A^2}{2\mathrm{e}^{2t}}(2^k-1)^2\right).$$

Setting $\beta = \frac{A^2}{2\mathrm{e}^{2T}}$ and expanding the last exponential we get:

$$M_t(y,\xi) \leqslant \frac{\mathrm{e}^{-\beta}\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)At|y|\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k}.$$

Which is as claimed.

Lemma 2 gives us by using $|x-y| \leqslant \alpha t \leqslant \alpha^2 m(x)$ the following estimate:

$$\mathrm{e}^{|y|^2} \leqslant \mathrm{e}^{|x|^2}\mathrm{e}^{\alpha^4}\mathrm{e}^{2\alpha^2}$$

$$
\begin{aligned}
M_t(y,\xi) &\leqslant \frac{\mathrm{e}^{-\beta}\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)\alpha(1+\alpha)\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k}\\
&\leqslant \mathrm{e}^{-(\alpha+\beta)}\mathrm{e}^{\alpha^4}\mathrm{e}^{\alpha^2}\frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}}\exp\big(2^{k+1}\alpha(1+\alpha)\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k}.
\end{aligned}
$$

Which is as claimed.

■

5. The boundedness of some non-tangential maximal operators

Our theorem is a small modification of [3, lemma 1.1] with a new proof.

1. THEOREM. *Let $A, a > 0$. For all x in $\mathbf{R}^d$ and all u in L^2_γ we have*

$$
\sup_{(y,t)\in\Gamma_x^{(A,a)}} |\mathrm{e}^{-t^2L}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u|\,\mathrm{d}\gamma. \tag{20}
$$

PROOF. First we note that $\Gamma_x^{(A,a)} \subset \Gamma_x^{(1+aA,aA)}$ as $a, A \geqslant 1$.

$$
\begin{gathered}
|x-y| \leqslant At \leqslant aAt\\
t \leqslant am(x) \leqslant aAm(x)
\end{gathered}
$$

So if $y \in \Gamma_x^{(A,a)}$ then $x \in \Gamma_y^{(aA,aA)}$. So set $\alpha = aA$ and $\Gamma_x^{\alpha} = \Gamma_x^{(\alpha,\alpha)}$

We will prove (20) by splitting up the integration domain in annuli.

$$
\mathrm{e}^{-tL}|u(y)| \leqslant \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k(B)} M_t(y,\cdot)|u|\,\mathrm{d}\gamma.
$$

More precisely, we will set $B = B(y, aAt)$ in the above and find a suitable upper bound for each integral on the right-hand side which we will denote by I_k for the sake of simplicity.

$$
M_t(y,\xi) \leqslant \frac{\mathrm{e}^{-\beta}\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)\alpha t|y|\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k},
$$

where $\beta = \frac{\alpha^2}{2\mathrm{e}^{2\alpha^2}}$.

Since we have $|x-y| < \alpha t$ and $t \leqslant am(x)$ we infer that $t|x| \leqslant \alpha$. By Lemma 1 we also have that $t|y| \leqslant 1+\alpha$. From this and Lemma 7 we infer that:

$$
M_t(y,\xi) \leqslant \mathrm{e}^{-\beta}\mathrm{e}^{-\alpha(1+\alpha)}\frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}}\exp\big(2^{k+1}\alpha(1+\alpha)\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k}, \tag{21}
$$

Setting $\beta = \frac{\alpha^2}{2\mathrm{e}^{2\alpha^2}}$. Note that β is maximal for $\alpha = \frac{1}{2}$ and after this value, β is decreasing. Setting $\lambda := \alpha(1+\alpha)$ we get:

$$
M_t(y,\xi) \lesssim_{\alpha} \mathrm{e}^{-(\alpha+\beta)}\mathrm{e}^{\alpha^4}\mathrm{e}^{\alpha^2}\frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}}\mathrm{e}^{(\lambda+\beta)2^{k+1}}\mathrm{e}^{-\beta 4^k}. \tag{22}
$$

Where the implied constant is given by $\mathrm{e}^{-(\alpha+\beta)}\mathrm{e}^{\alpha^4}\mathrm{e}^{\alpha^2}$

Or, using $\Lambda = \beta + \lambda$ we get:

$$
M_t(y,\xi) \lesssim_{\alpha} \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}}\mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k}, \tag{23}
$$

Recalling Lemma 6 we get:

$$
\gamma(B_t(x)) \leqslant V_d d\pi^{-\frac{d}{2}} t^d \mathrm{e}^{-(t-|x|)^2}. \tag{24}
$$

Where we abbreviate $V_d(1)$ with V_d. Recall

$$V_d \leqslant \frac{1}{\sqrt{\pi}}\left(\frac{2\pi e}{d}\right)^{\frac{d}{2}}.$$

To get,

$$\gamma(B_t(x)) \leqslant \frac{d}{\sqrt{\pi}}\left(\frac{2e}{d}\right)^{\frac{d}{2}} t^d \mathrm{e}^{-(t-|x|)^2} = C_d t^d \mathrm{e}^{-(t-|x|)^2}. \tag{25}$$

This allows us to estimate the remaining unbounded exponential in the Mehler kernel and allow a penalty up to $\mathrm{e}^{-|x|^2}$. Furthermore, we have the following estimate which will make clear how to handle the time part in the Mehler kernel:

$$\frac{t^d}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \leqslant \left(\frac{t^2}{1-\mathrm{e}^{-t^2}}\right)^{\frac{d}{2}} \leqslant \frac{a^d}{(1-\mathrm{e}^{-a^2})^{\frac{d}{2}}}.$$

Let $B' := B(x, 2^{k+1}\alpha t)$ and B as before the ball $B(y, \alpha t)$. In the next step we will bound .... by the maximal function centered at x. For this we need to scale up the C_k. So,

$$|x-\xi| \leqslant |x-y| + |\xi - y| \leqslant \alpha t + (2^{k+1}-1)\alpha t = 2^{k+1}\alpha t.$$

And set $D_k = B(2^{k+1}\alpha t)$. So, we can bound the integral on the right-hand side of (**??**) by

$$\begin{aligned}
\int_{C_k(B)} M_{t^2}(y,\cdot)|u| \,\mathrm{d}\gamma &\lesssim_\alpha \frac{\mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2} \int_{C_k(B)} |u| \,\mathrm{d}\gamma \\
&\leqslant \frac{\mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2} \int_{D_k(B)} |u| \,\mathrm{d}\gamma \\
&\leqslant (M_\gamma u)(x) \frac{\mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2}\gamma(D_k) \\
&\overset{\text{(i)}}{\leqslant} (M_\gamma u)(x) C_d \alpha^d 2^{d(k+1)} t^d \frac{\mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2}\mathrm{e}^{-(t-|x|)^2} \\
&\leqslant (M_\gamma u)(x) 2^{kd} \mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k} \frac{t^d \mathrm{e}^{-t^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} C_d \mathrm{e}^{2\alpha}(2\alpha)^d
\end{aligned}$$

Where (i) uses Lemma 6 and $t|x| \leqslant a$.

We can then bound the maximal function:

$$\begin{aligned}
\mathrm{e}^{-t^2 L}|u(y)| &= \sum_{k=0}^{\infty} I_k \\
&\leqslant (M_\gamma u)(x) C_{d,a,A} \sum_{k=0}^{\infty} 2^{kd}\mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k}
\end{aligned}$$

Wrapping it up, we have that:

$$\mathrm{e}^{-t^2 L}|u(y)| \lesssim \fint_{B_r(x)} |u| \,\mathrm{d}\gamma.$$

With implied constant

Which is what we wanted to prove.

$$\sum_{k=0}^{\infty} 2^{kd}\mathrm{e}^{-C4^k} = \sum_{k=0}^{\infty} x^{kd}\mathrm{e}^{-Cx^{2k}}$$

Noting for $x \geqslant 1$ that $\exp(-Cx^{2k}) \leqslant \exp(-Ckx^2)$, thus,

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \sum_{k=0}^{\infty} x^{kd} (\mathrm{e}^{-Cx^2})^k = \sum_{k=0}^{\infty} (x^d \mathrm{e}^{-Cx^2})^k$$

Here $x = 2$, so

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \sum_{k=0}^{\infty} (2^d \mathrm{e}^{-4C})^k$$

If $2^d < \mathrm{e}^{4C}$, that is whenever $d \log 2 < 4C$, we can compute using the geometric series that

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \frac{1}{1 - 2^d \mathrm{e}^{-4C}} = \frac{\mathrm{e}^{4C}}{\mathrm{e}^{4C} - 2^d}$$

■

References

[1] Jan Maas, Jan Neerven, and Pierre Portal. "Whitney coverings and the tent spaces $T^{1,q}(\gamma)$ for the Gaussian measure". In: *Arkiv för Matematik* 50.2 (Apr. 2011), pp. 379–395.

[2] Giancarlo Mauceri and Stefano Meda. "BMO and H^1 for the Ornstein-Uhlenbeck operator". In: *Journal of Functional Analysis* 252.1 (Nov. 2007), pp. 278–313.

[3] Ebner Pineda, W. Urbina, and Wilfredo Urbina R. "Non Tangential Convergence for the Ornstein-Uhlenbeck Semigroup". In: *Divulgaciones Matemáticas* 13.2 (2008), pp. 1–19.

[4] Pierre Portal. "Maximal and quadratic Gaussian Hardy spaces". Mar. 2012.

[5] Peter Sjögren. "Operators associated with the hermite semigroup — A survey". In: *The Journal of Fourier Analysis and Applications* 3.S1 (Jan. 1997), pp. 813–823.

Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

E-mail address: `j.j.b.teuwen@tudelft.nl`

URL: `http://fa.its.tudelft.nl/~teuwen/`