

A note on the Gaussian maximal function

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Abstract

This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded almost surely by the Gaussian Hardy-Littlewood maximal function. In particular this entails improvement on a result by Pineda and Urbina [1] who proved a similar result for a ‘truncated’ version of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

Keywords: ornstein-uhlenbeck, ornstein-uhlenbeck semigroup

1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well-known that the classical non-tangential maximal function associated with the heat semigroup is bounded almost everywhere by the Hardy-Littlewood maximal function,

$$\sup_{\substack{(y,t) \in \mathbf{R}_+^{d+1} \\ |x-y| < t}} |e^{-t\Delta} u(y)| \lesssim \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \, d\lambda. \quad (1)$$

Here the action of *heat semigroup* $e^{-t\Delta} u = \rho_t * u$ is given by a convolution of u with the *heat kernel*

$$\rho_t(\xi) := \frac{e^{-|\xi|^2/4t}}{(4\pi t)^{\frac{d}{2}}}, \text{ with } t > 0 \text{ and } \xi \in \mathbf{R}^d.$$

In this note we are interested in its Gaussian counterpart. The change from Lebesgue measure to the *Gaussian measure*

$$d\gamma(x) := \pi^{-\frac{d}{2}} e^{-|x|^2} dx \quad (2)$$

introduces quite some intricate technical and conceptual difficulties which are due to its non-doubling nature. Instead of the Laplacian, will use its Gaussian analogue, the *Ornstein-Uhlenbeck operator* L which is given by,

$$L := -\frac{1}{2}\Delta + \langle x, \nabla \rangle = \frac{1}{2}\nabla_\gamma^* \nabla_\gamma, \quad (3)$$

where ∇_γ denotes the realisation of the gradient in $L^2(\mathbf{R}^d, \gamma)$. Our main result, to be proved in Theorem 1, is the following Gaussian analogue of (1):

$$\sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma. \quad (4)$$

Here,

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y, t) \in \mathbf{R}_+^{d+1} : |x - y| < At \text{ and } t \leq am(x)\} \quad (5)$$

is the *Gaussian cone* with aperture A and cut-off parameter a , and

$$m(x) := \min \left\{ 1, \frac{1}{|x|} \right\}. \quad (6)$$

A slightly weaker version of the inequality (4) has been proved by Pineda and Urbina [1] who showed that

$$\sup_{(y,t) \in \tilde{\Gamma}_x} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma,$$

where

$$\tilde{\Gamma}_x(x) = \{(y, t) \in \mathbf{R}_+^d : |x - y| < t \leq \tilde{m}(x)\}$$

is the ‘reduced’ Gaussian cone corresponding to the function

$$\tilde{m}(x) = \min \left\{ \frac{1}{2}, \frac{1}{|x|} \right\}.$$

Our proof of (4) is much shorter and, we believe, more transparent than the one presented in [1]. It has the further advantage of allowing the extension to cones with arbitrary aperture $A > 0$ and cut-off parameter $a > 0$ without any additional technicalities. This additional generality is very important and has already been used by Portal (cf. the claim made in [2, discussion preceding Lemma 2.3]) to prove the H^1 -boundedness of the Riesz transform associated with L .

2. The Mehler kernel

The *Mehler kernel* (see e.g., [3]) is the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(e^{-tL})_{t \geq 0}$, that is,

$$e^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot) u \, d\gamma. \quad (7)$$

There is an abundance of literature on the Mehler kernel and its properties. We shall only use the fact, proved e.g. in the review paper [3], that it is given explicitly by

$$M_t(x, y) = \frac{\exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right)}{(1 - e^{-2t})^{\frac{d}{2}}} e^{|y|^2}. \quad (8)$$

Note that $M_t(x, y)$ is symmetric in x and y . A formula for (8) honoring this observation is:

$$M_t(x, y) = \frac{\exp\left(-e^{-2t} \frac{|x - y|^2}{1 - e^{-2t}}\right)}{(1 - e^{-t})^{\frac{d}{2}}} \frac{\exp\left(2e^{-t} \frac{\langle x, y \rangle}{1 + e^{-t}}\right)}{(1 + e^{-t})^{\frac{d}{2}}}. \quad (9)$$

3. Some lemmata

We use m as defined in (6) in our next lemma, which is taken from [4].

1 Lemma. *Let a, A be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

1. *If $|x - y| < At$ and $t \leq am(x)$, then $t \leq (1 + aA)m(y)$;*
2. *If $|x - y| < Am(x)$, then $m(x) \leq (1 + A)m(y)$ and $m(y) \leq 2(1 + A)m(x)$.*

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2 Lemma. *Let $\alpha > 0$ and $|x - y| \leq \alpha m(x)$. Then:*

$$e^{-\alpha^2 - 2\alpha} e^{|y|^2} \leq e^{|x|^2} \leq e^{\alpha^2(1+\alpha)^2 + 2\alpha(1+\alpha)} e^{|y|^2}.$$

Proof. By the inverse triangle inequality and $m(x)|x| \leq 1$ we get,

$$|y|^2 \leq (\alpha m(x) + |x|)^2 \leq \alpha^2 + 2\alpha + |x|^2.$$

This gives the first inequality. For the second we use Lemma 1 to infer $m(x) \leq (1 + \alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leq \alpha^2(1 + \alpha)^2 + 2\alpha(1 + \alpha) + |y|^2.$$

As required. ■

3.1. An estimate on Gaussian balls

Let $B := B_t(x)$ be the open Euclidean ball with radius t and center x and let γ be the Gaussian measure as defined by (2). We shall denote by S_d the surface area of the unit sphere in \mathbf{R}^d .

3 Lemma. *For all $x \in \mathbf{R}^d$ and $t > 0$ we have the inequality:*

$$\gamma(B_t(x)) \leq \frac{S_d}{2\pi^{\frac{d}{2}}} t^d e^{2t|x|} e^{-|x|^2}. \quad (10)$$

Proof. Remark that, with $B := B_t(x)$,

$$\begin{aligned} \int_B e^{-|\xi|^2} d\xi &= e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{-2\langle x, \xi-x \rangle} d\xi \\ &\leq e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{2|x||\xi-x|} d\xi \\ &\leq e^{-|x|^2} e^{2t|x|} \int_B e^{-|\xi-x|^2} d\xi \\ &= \pi^{\frac{d}{2}} e^{2t|x|} e^{-|x|^2} \gamma(B_t(0)). \end{aligned}$$

So, there holds that

$$\gamma(B_t(x)) \leq e^{2t|x|} e^{-|x|^2} \gamma(B_t(0)). \quad (11)$$

We will estimate the Gaussian volume of the ball $B_t(0)$. Using polar coordi-

nates we proceed for $d \geq 2$ by:

$$\begin{aligned}
\gamma(B_t(0)) &= \frac{1}{\pi^{\frac{d}{2}}} \int_{B_t(0)} e^{-|\xi|^2} d\xi \\
&= \frac{S_d}{\pi^{\frac{d}{2}}} \int_0^t e^{-r^2} r^{d-1} dr \\
&\leq \frac{S_d}{2\pi^{\frac{d}{2}}} t^{d-2} \int_0^t 2re^{-r^2} dr \\
&= \frac{S_d}{2\pi^{\frac{d}{2}}} t^{d-2} (1 - e^{-t^2}) \\
&\leq \frac{S_d}{2\pi^{\frac{d}{2}}} t^d,
\end{aligned}$$

where the last step uses $1 - e^{-t} \leq t$ for $t \geq 0$. The case for $d = 1$ follows by a simplified argument. Upon combining this result with (11) we obtain (10). \blacksquare

3.2. On-diagonal kernel estimates on annuli

As is common in harmonic analysis, we often wish to decompose \mathbf{R}^d into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli.

Throughout this subsection we fix $x \in \mathbf{R}^d$, constants $A, a \geq 1$, a pair $(y, t) \in \Gamma_x^{(A, a)}$. We use the notation rB to mean the ball obtained from the ball B by multiplying its radius by r .

The annuli $C_k := C_k(B_t(x))$ are given by:

$$C_k := \begin{cases} 2B_t(x), & k = 0, \\ 2^{k+1}B_t(x) \setminus 2^k B_t(x), & k \geq 1. \end{cases} \quad (12)$$

Whenever ξ is in C_k , we get for $k \geq 1$:

$$2^k t < |y - \xi| \leq 2^{k+1} t. \quad (13)$$

On C_k we have the following bound for $M_{t^2}(y, \cdot)$:

4 Lemma. *For all $\xi \in C_k$ for $k \geq 1$ we have:*

$$M_{t^2}(y, \xi) \leq \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \exp(2^{k+1}t|y|) \exp\left(-\frac{4^k}{2e^{2t^2}}\right), \quad (14)$$

Proof. Considering the first exponential which occurs in the Mehler kernel (9) together with (13) gives for $k \geq 1$:

$$\begin{aligned} \exp\left(-e^{-2t^2} \frac{|y - \xi|^2}{1 - e^{-2t^2}}\right) &\leq \exp\left(-\frac{4^k}{e^{2t^2}} \frac{t^2}{1 - e^{-2t^2}}\right) \\ &\stackrel{(\dagger)}{\leq} \exp\left(-\frac{4^k}{2e^{2t^2}}\right), \end{aligned}$$

where (\dagger) follows from $1 - e^{-s} \leq s$ for $s \geq 0$. Using the estimate $1 + s \geq 2s$ for $0 \leq s \leq 1$, for the second exponential in the Mehler kernel (9) we obtain, by (13):

$$\begin{aligned} \exp\left(2e^{-t^2} \frac{\langle y, \xi \rangle}{1 + e^{-t^2}}\right) &\leq \exp(|\langle y, \xi \rangle|) \\ &\leq \exp(|\langle y, \xi - y \rangle|)e^{|y|^2} \\ &\leq \exp(2^{k+1}t|y|)e^{|y|^2}. \end{aligned}$$

Combining things, we obtain the estimate in the formulation of the lemma. ■

4. The main result

In this section we will prove our main theorem for which we have already made the necessary preparations in the previous sections.

1 Theorem. *Let $A, a > 0$. For all $x \in \mathbf{R}^d$ and all $u \in L^2(\mathbf{R}^d, \gamma)$ we have*

$$\sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma, \quad (15)$$

where the implicit constant only depends on A, a and d .

Proof. We fix $x \in \mathbf{R}^d$ and $(y, t) \in \Gamma_x^{(A,a)}$. The proof of (15) is based on splitting the integration domain into the annuli C_k as defined by (12) and estimating on each annulus. More explicit,

$$|e^{-t^2 L} u(y)| \leq \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma. \quad (16)$$

We have $t \leq am(x) \leq a$ and, by Lemma 1, $t|y| \leq 1 + aA$. Together with Lemma 4 we infer, for $\xi \in C_k$ and $k \geq 1$, that:

$$\begin{aligned} M_{t^2}(y, \xi) &\leq \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \exp(2^{k+1}(1 + aA)) \exp\left(-\frac{4^k}{2e^{2a^2}}\right) \\ &=: \frac{e^{|y|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} c_k. \end{aligned}$$

Combining this with Lemma 2, we obtain

$$M_{t^2}(y, \xi) \lesssim_{A,a} \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} c_k. \quad (17)$$

Also, by (13),

$$|x - \xi| \leq |x - y| + |\xi - y| \leq (2^{k+1} + A)t.$$

Let K be the smallest integer such that $2^{k+1} \geq A$ whenever $k \geq K$. Then it follows that C_k for $k \geq K$ is contained in $B_{2^{k+2}t}(x)$ and for $k < K$ is contained in $B_{2At}(x)$. We set,

$$D_k := D_k(x) = \begin{cases} B_{2^{k+2}t}(x) & \text{if } k \geq K, \\ B_{2At}(x) & \text{elsewhere.} \end{cases}$$

Let us denote the supremum on right-hand side of (15) by $M_\gamma u(x)$. Using (17), we can bound the integral on the right-hand side of (16) by

$$\begin{aligned} \int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma &\lesssim_{A,a} c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \int_{C_k} |u| \, d\gamma \\ &\leq c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \int_{D_k} |u| \, d\gamma \\ &\leq c_k \frac{e^{|x|^2}}{(1 - e^{-2t^2})^{\frac{d}{2}}} \gamma(D_k) M_\gamma u(x), \end{aligned}$$

where we pause for a moment to compute a suitable bound for $\gamma(D_k)$. As above we have both $t|x| \leq am(x)|x| \leq a$ and $t \leq a$. Together with Lemma 3 applied to D_k for $k \geq K$ we obtain:

$$\begin{aligned} \gamma(D_k) e^{|x|^2} &\leq_A C^d S_d t^{d^2} 2^{kd} e^{2^{k+2}t|x|} e^{-|x|^2} e^{|x|^2} \\ &\lesssim_{A,a,d} t^d 2^{kd} e^{2^{k+2}a}. \end{aligned}$$

Similarly, for $k < K$:

$$\gamma(D_k)e^{|x|^2} \lesssim_{A,a,d} t^d e^{2Aa}.$$

Using the bound $t \leq a$, we can infer that

$$\frac{t^d}{(1 - e^{-2t^2})^{\frac{d}{2}}} \leq \frac{a^d}{(1 - e^{-2a^2})^{\frac{d}{2}}} \lesssim_{a,d} 1.$$

(note that $s/(1 - e^{-s})$ is increasing). Combining these computations with where we left off we get for $k \geq K$,

$$\int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma \lesssim_{A,a,d} c_k 2^{kd} e^{2^{k+2}a} M_\gamma u(x),$$

while we get for $k < K$,

$$\int_{C_k} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma \lesssim_{A,a,d} c_k M_\gamma u(x).$$

Similarly, for $\xi \in 2B_t(x)$ we obtain:

$$I_0 := \int_{2B_t} M_{t^2}(y, \cdot) |u(\cdot)| \, d\gamma \lesssim_{A,a,d} M_\gamma u(x).$$

Inserting the dependency of c_k upon k as coming from (17), we obtain the bound:

$$\begin{aligned} |e^{-t^2 L} u(y)| &= I_0 + \sum_{k=1}^{K-1} I_k + \sum_{k=K}^{\infty} I_k \\ &\lesssim_{A,a,d} \left[1 + \sum_{k=1}^{K-1} c_k + \sum_{k=K}^{\infty} c_k 2^{kd} e^{2^{k+2}a} \right] M_\gamma u(x), \\ &\lesssim_{A,a,d} \left[1 + \sum_{k=1}^{K-1} e^{-\frac{4^k}{2e^{2a^2}}} + \sum_{k=K}^{\infty} 2^{kd} e^{2^{k+1}(1+2a+aA)} e^{-\frac{4^k}{2e^{2a^2}}} \right] M_\gamma u(x), \end{aligned}$$

valid for all $y \in \Gamma_x^{(A,a)}$. As the sum on the right-hand side evidently converges, we see that taking the supremum proves (15). \blacksquare

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