A note on the Gaussian maximal function - Version 9 October 2013 + JvN additions

Jonas Teuwen

Abstract. This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded almost surely by the Gaussian Hardy-Littlewood maximal function. In particular this entails improvement on a result by Pineda and Urbina [3] who proved a similar result for a 'trunctated' version of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well known that the classical real-valued maximal function associated with the heat semigroup is bounded almost everywhere by the Hardy-Littlewood maximal function,

$$\sup_{(y,t)\in\Gamma_x} |\mathrm{e}^{-t\Delta}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u|\,\mathrm{d}\lambda. \tag{1}$$

Here the action of *heat semigroup* $\mathrm{e}^{-t^2\Delta}u = \rho_t * u$ is given by a convolution of u with the *heat kernel*

$$\rho_t(s) := \frac{\mathrm{e}^{-|s|^2/4t}}{\pi^{\frac{d}{2}}}\frac{1}{(4t)^{\frac{d}{2}}}.$$

In this note we are interested in its gaussian counterpart. The change from Lebesgue measure to the gaussian measure

$$\mathrm{d}\gamma(x) := \pi^{-\frac{d}{2}}\mathrm{e}^{-|x|^2}\,\mathrm{d}x \tag{2}$$

introduces quite some intricate technical and conceptually difficulties which appears to be due to the fact that the Gaussian measure is non-doubling.

As an analogue to the Laplacian which is symmetric in L^2 with respect to the Lebesgue measure next we introduce the *Ornstein-Uhlenbeck* operator L which is symmetric with respect to the Gaussian measure:

$$L := -\frac{1}{2}\Delta + \langle x, \nabla\rangle = \frac{1}{2}\nabla_\gamma^*\nabla_\gamma, \tag{3}$$

where ∇_γ denotes the realisation of the gradient in $L^2(\mathbf{R}^d,\gamma)$. Our main result, to be proved in (1), is the following gaussian analogue of (1):

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |\mathrm{e}^{-tL}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u|\,\mathrm{d}\gamma. \tag{4}$$

Here, $\Gamma_x^{(A,a)}$ is the Gaussian cone defined by

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t)\in\mathbf{R}_+^d : |x-y| < At \text{ and } t \leqslant am(x)\}, \tag{5}$$

where

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\} = 1 \wedge \frac{1}{|x|}. \tag{6}$$

A slighly weaker version of the inequality (4) has been proved by Pineda and Urbina [3] who shows that

$$\sup_{(y,t)\in\widetilde{\Gamma}_x} |e^{-t\Delta}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \, d\gamma,$$

where

$$\widetilde{\Gamma}_x(x) = \{(y,t) \in \mathbf{R}^d_+ : |x-y| < t \leqslant \widetilde{m}(x)\}$$

is the 'reduced' gaussian cone corresponding to the function

$$\widetilde{m}(x) = \min\left\{\frac{1}{2}, \frac{1}{|x|}\right\}.$$

Their proof does not seem to easily generalize the range of t from $\frac{1}{2}$ up to 1. The proof of (4) is different and, we believe, more transparent than the one presented in [3]. It has the further virtue of allowing the extension to the cones with arbitrary aperture $A > 0$ and cut-off parameter $a > 0$. This additional generality is very important and has already been used by Portal (cf. the claim made by [4, discussion preceding Lemma 2.3]) to prove the H^1-boundedness of the Riesz transform associated with L.

Before we continue, let us fix some notation. We will use without further reference notation such as $\mathbf{Z}^d$ while we implicitly imply that d is a positive integer. To avoid possible confusion, we define the *positive integers* as the set $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$.

1.0.1. *minimal function.* We recall the lemma from [1, lemma 2.3] which first –although implicitly– appeared in [2]. [1]

2. The Mehler kernel

2.1. Setting. Recall that we work with the *Ornstein-Uhlenbeck* operator L as given by (3).

We define the Mehler kernel (see e.g., Sjögren [5]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(e^{-tL})_{t\geqslant 0}$. More precisely, this means:

$$e^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot) u \, d\gamma. \tag{7}$$

It is often more convenient to use e^{-t^2L} instead of e^{-tL} as is done in e.g., Portal [4].

2.2. The Mehler kernel. There is an abundance of literature on the Mehler kernel an its properties available, but for the present purpose Sjögren [5] will suffice. For instance, the Mehler kernel M_t of (7) is computed there. In addition it offers related results with to the Hermite polynomials.

The kernel M_t is invariant under the permutation $x \longleftrightarrow y$. A formula for M_t which honors this observation is:

$$M_t(x,y) = \frac{\exp\left(-e^{-2t}\dfrac{|x-y|^2}{1-e^{-2t}}\right)}{(1-e^{-t})^{\frac{d}{2}}} \, \frac{\exp\left(2e^{-t}\dfrac{\langle x, y\rangle}{1+e^{-t}}\right)}{(1+e^{-t})^{\frac{d}{2}}}. \tag{8}$$

[1] Nog materiaal toevoegen?

3. Some lemmata and definitions

We use m as defined in (6) in our next lemma.

1. LEMMA. *Let a, A be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

(1) If $|x-y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1+aA)m(y)$;
(2) If $|x-y| < Am(x)$, then $m(x) \leqslant (1+A)m(y)$ and $m(y) \leqslant 2(1+A)m(x)$.

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. LEMMA. *Let $\alpha > 0$ and $|x-y| \leqslant \alpha m(x)$. We get the equivalence:*

$$\mathrm{e}^{-\alpha^2(1+\alpha)^2}\mathrm{e}^{-2\alpha(1+\alpha)}\mathrm{e}^{-|y|^2} \leqslant \mathrm{e}^{-|x|^2} \leqslant \mathrm{e}^{\alpha^2}\mathrm{e}^{2\alpha}\mathrm{e}^{-|y|^2}.$$

PROOF. By the inverse triangle inequality and $m(x)|x| \leqslant 1$ we get,

$$|y|^2 \leqslant (\alpha m(x) + |x|)^2 \leqslant \alpha^2 + 2\alpha + |x|^2. \tag{9}$$

For the reverse direction we use Lemma 1 to infer $m(x) \leqslant (1+\alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant \alpha^2(1+\alpha)^2 + 2\alpha(1+\alpha) + |y|^2.$$

Combining we get:

$$\mathrm{e}^{-\alpha^2(1+\alpha)^2}\mathrm{e}^{-2\alpha(1+\alpha)}\mathrm{e}^{-|y|^2} \leqslant \mathrm{e}^{-|x|^2} \leqslant \mathrm{e}^{\alpha^2}\mathrm{e}^{2\alpha}\mathrm{e}^{-|y|^2}. \tag{10}$$

As required. ∎

4. On-diagonal estimates

4.1. Kernel estimates. Before we proceed with the technicalities we define κ and μ as:

$$\kappa = 2\left(1 + \frac{1}{\alpha}\right)^{-1}, \text{ and } \mu = 2\left(1 - \frac{1}{\alpha}\right)^{-1}.$$

such that κ and μ are conjugate exponents, which means:

$$\frac{1}{\kappa} + \frac{1}{\mu} = 1.$$

We proceed with a simple technical lemma which is given here as it will be used on several occasions.

3. LEMMA. *Let $t > 0$ and $\alpha \geqslant 1$. Then,*

$$\alpha \mathrm{e}^{-2\frac{t}{\mu}} \leqslant \frac{1-\mathrm{e}^{-t}}{1-\mathrm{e}^{-\frac{t}{\alpha}}} \leqslant \alpha, \tag{11}$$

$$0 \leqslant \frac{1}{t}\left[\frac{\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}} - \frac{\mathrm{e}^{-\frac{t}{\alpha}}}{1+\mathrm{e}^{-\frac{t}{\alpha}}}\right] \leqslant \frac{1}{2\mu}. \tag{12}$$

PROOF. We start with (11) and apply the mean value theorem to the function $f(\xi) = \xi^\alpha$. For $0 < \xi < \xi'$ this gives that:

$$f(\xi) - f(\xi') = \alpha\hat{\xi}^{\alpha-1}(\xi - \xi') \text{ for some } \hat{\xi} \text{ in } [\xi, \xi'].$$

Picking $\xi = 1$ and $\xi' = \mathrm{e}^{-\frac{t}{\alpha}}$ yields:

$$\frac{1-\mathrm{e}^{-t}}{1-\mathrm{e}^{-\frac{t}{\alpha}}} = \alpha\hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[\mathrm{e}^{-\frac{t^2}{\alpha}}, 1\right]. \tag{13}$$

Combining this result with the monotonicity of $\xi \mapsto \alpha\xi^{\alpha-1}$ we obtain:

$$\alpha e^{\frac{t}{\alpha}} e^{-t} \leqslant \frac{1-e^{-t}}{1-e^{-\frac{t}{\alpha}}} \leqslant \alpha,$$

where the last bound follows from the monotonicity together with the limit as $t \downarrow 0$. We proceed with (12). Recalling that $\alpha \geqslant 1$ one can directly verify that the function

$$\frac{1}{t}\left[\frac{1}{1+e^{-t}} - \frac{1}{1+e^{-\frac{t}{\alpha}}}\right]$$

is non-negative and decreasing in t. To find an upper bound we compute the limit as t goes to 0. That is:

$$\lim_{t\to 0}\frac{1}{t}\left[\frac{e^{-t}}{1+e^{-t}} - \frac{e^{-\frac{t}{\alpha}}}{1+e^{-\frac{t}{\alpha}}}\right] = \lim_{t\to 0}\left[\frac{e^{-2t}}{(1+e^{-t})^2} - \frac{1}{\alpha}\frac{e^{-2\frac{t}{\alpha}}}{(1+e^{-\frac{t}{\alpha}})^2}\right] \uparrow \frac{1}{2\mu}.$$

Which is as asserted and completes the proof. ∎

The following lemma will be useful when transfering estimates from $M_{\frac{t}{\alpha}}$ to M_t. It follows from the mean value theorem applied to $\xi \mapsto \xi^\alpha$.

4. Lemma. *For $\alpha \geqslant 1$ and $0 < t \leqslant T < \infty$ and all let $x, y \in \mathbf{R}^d$ we have that:*

$$\exp\left(-\frac{1}{e^{2\frac{t}{\alpha}}}\frac{|x-y|^2}{1-e^{-2\frac{t}{\alpha}}}\right) \leqslant \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\kappa}}}\frac{|x-y|^2}{1-e^{-t}}\right). \tag{14}$$

Proof. It is fruitful to note that

$$1-e^{-2t} = (1-e^{-t})(1+e^{-t})$$

holds. The same holds true for the inequality

$$\frac{1}{2e^{2t}} \leqslant \frac{e^{-2t}}{1+e^{-t}} \leqslant \frac{1}{2}.$$

Therefore,

$$\begin{aligned}\exp\left(-e^{-2t}\frac{|x-y|^2}{1-e^{-2t}}\right) &= \exp\left(-\frac{e^{-2t}}{1+e^{-t}}\frac{|x-y|^2}{1-e^{-t}}\right)\\ &\leqslant \exp\left(-\frac{1}{2e^{2t}}\frac{|x-y|^2}{1-e^{-t}}\right).\end{aligned}$$

Setting $\beta := 1+\alpha^{-1}$ and applying (11) we get:

$$\begin{aligned}\exp\left(-e^{-2\frac{t}{\alpha}}\frac{|x-y|^2}{1-e^{-2\frac{t}{\alpha}}}\right) &= \exp\left(-\frac{e^{-2\frac{t}{\alpha}}}{1+e^{-\frac{t}{\alpha}}}\frac{1-e^{-t}}{1-e^{-\frac{t}{\alpha}}}\frac{|x-y|^2}{1-e^{-t}}\right)\\ &\leqslant \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\alpha}}}\frac{1}{e^t e^{-\frac{t}{\alpha}}}\frac{|x-y|^2}{1-e^{-t}}\right)\\ &\leqslant \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\kappa}}}\frac{|x-y|^2}{1-e^{-t}}\right).\end{aligned}$$

Which is as asserted. ∎

Our first lemma is about estimating $M_{\frac{t}{\alpha}}$ in terms of M_t.

4.1.1. *Time-scaling of the Mehler kernel.*

5. LEMMA. *Let $T > 0$, $\alpha \geqslant 1$, and $x, y \in \mathbf{R}^d$. Then:*

$$M_{\frac{t}{\alpha}}(x,y) \leqslant \alpha^{\frac{d}{2}} \exp\left(\frac{t}{2\mu}|\langle x,y\rangle|\right) \exp\left(-\frac{\alpha}{4e^{2T}} \frac{|x-y|^2}{1-e^{-t}}\right) M_t(x,y). \tag{15}$$

PROOF. To prove the lemma we compute $M_{\frac{t}{\alpha}} M_t^{-1}$. First recall that (11) gives

$$\alpha e^{-\frac{t}{\mu}} \leqslant \frac{1-e^{-t}}{1-e^{-\frac{t}{\alpha}}} \leqslant \alpha.$$

Combining the exponentials also gives,

$$\begin{aligned}
\exp\left(-2e^{-\frac{t}{\alpha}} \frac{\langle x,y\rangle}{1+e^{-\frac{t}{\alpha}}}\right) \exp\left(2e^{-t} \frac{\langle x,y\rangle}{1+e^{-t}}\right) \\
&= \exp\left(\frac{2}{t}\left[\frac{e^{-t}}{1+e^{-t}} - \frac{e^{-\frac{t}{\alpha}}}{1+e^{-\frac{t}{\alpha}}}\right] t\langle x,y\rangle\right) \\
&\overset{(12)}{\leqslant} \exp\left(\frac{t}{\mu}|\langle x,y\rangle|\right).
\end{aligned}$$

Combining Lemma 4 and equation (14) almost gives the final estimate.

$$\begin{aligned}
\frac{M_{\frac{t}{\alpha}}(x,y)}{M_t(x,y)} &\leqslant \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu}|\langle x,y\rangle|\right) \exp\left(e^{-2t} \frac{|x-y|^2}{1-e^{-2t}}\right) \exp\left(-e^{-2\frac{t}{\alpha}} \frac{|x-y|^2}{1-e^{-2\frac{t}{\alpha}}}\right) \\
&\leqslant \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu}|\langle x,y\rangle|\right) \exp\left(\left[1-\frac{\alpha}{4e^{2\frac{t}{\kappa}}}\right] \frac{|x-y|^2}{1-e^{-t}}\right) \exp\left(-\frac{\alpha}{4e^{2\frac{t}{\kappa}}} \frac{|x-y|^2}{1-e^{-t}}\right).
\end{aligned}$$

Finally, we apply the assumption $\alpha \geqslant 4e^{2\frac{t}{\kappa}}$ to obtain:

$$\frac{M_{\frac{t}{\alpha}}(x,y)}{M_t(x,y)} \leqslant \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu}|\langle x,y\rangle|\right) \exp\left(-\frac{\alpha}{4e^{2\frac{t}{\kappa}}} \frac{|x-y|^2}{1-e^{-t}}\right).$$

Which is as asserted. The assumption $\alpha \geqslant 4e^{2\frac{t}{\kappa}}$ can be rephrased as the requirement that $\left(1+\frac{1}{\alpha}\right)^{-1} \log\left(\frac{\alpha}{4}\right) \geqslant t$. ∎

4.2. An estimate on Gaussian balls.

6. LEMMA. *Let $B_t(x)$ be the Euclidean ball with radius t and center x and let γ be the Gaussian measure (2). We have the inequality:*

$$\frac{\gamma(B_t(x))}{V_d(t)} \leqslant d\pi^{-\frac{d}{2}} e^{-(t-|x|)^2}. \tag{16}$$

PROOF. Remark that

$$\begin{aligned}
\int_B e^{-|\xi|^2} \, d\xi &= e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{-2\langle x,\xi-x\rangle} \, d\xi \\
&\leqslant e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{2|x||\xi-x|} \, d\xi \\
&\leqslant e^{-|x|^2} e^{2|x|t} \int_B e^{-|\xi-x|^2} \, d\xi \\
&= \pi^{\frac{d}{2}} e^{-|x|^2} e^{2t|x|} \gamma(B_t(0)).
\end{aligned}$$

So, for a ball $B := B_t(x)$ there holds that

$$\gamma(B_t(x)) \leqslant e^{-|x|^2} e^{2t|x|} \gamma(B_t(0)). \tag{17}$$

We will estimate the Gaussian volume of the ball $B_t(0)$. To save writing, let S_d and V_d be the surface area and volume respectively of the d-dimensional unit sphere. Using polar coordinates we proceed by:

$$\begin{aligned}\gamma(B_t(0)) &= \pi^{-\frac{d}{2}} \int_{B_t(0)} \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi \\ &= S_d \pi^{-\frac{d}{2}} \int_0^t \mathrm{e}^{-r^2} r^{d-1}\,\mathrm{d}r \\ &\leqslant S_d t^d \pi^{-\frac{d}{2}} \mathrm{e}^{-t^2} \\ &= d V_d(t) \pi^{-\frac{d}{2}} \mathrm{e}^{-t^2}.\end{aligned}$$

Upon combining this result with (17) we obtain (16), which is as promised. ■

4.3. On-diagonal kernel estimates on annuli. As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli C_k. For the sake of simplicity we will write $B := B_t(x)$ and mean that B is the closed ball with center x and radius t. Furthermore, we use notations such as $2B$ to mean the ball obtained from B by multiplying its radius by 2.

The C_k are given by,

$$C_k(B) := C_k = (2^{k+1}-1)B \setminus (2^k-1)B. \tag{18}$$

So, whenever ξ is in $C_k(B_t(x))$, we get for $k \geqslant 0$:

$$(2^k-1)t < |y-\xi| \leqslant (2^{k+1}-1)t. \tag{19}$$

7. Lemma. *Given $A>0$, let $B = B_{At}(y)$, $0 < t \leqslant T < \infty$ and $\xi \in C_k$. Then we have:*

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{-\beta}\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)At|y|\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k},$$

where $\beta = \frac{A^2}{2\mathrm{e}^{2T}}$.

Proof. Let $B = B_{At}(y)$ and let C_k be as in (18). Considering the first exponential which occurs in the Mehler kernel (8) together with (19) gives for $k \geqslant 0$:

$$\begin{aligned}\exp\left(-\mathrm{e}^{-2t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right) &\leqslant \exp\left(-\mathrm{e}^{-2t^2}\frac{(2^k-1)^2A^2t^2}{1-\mathrm{e}^{-2t^2}}\right) \\ &\overset{(\dagger)}{\leqslant} \exp\left(-\frac{A^2}{2\mathrm{e}^{2t^2}}(2^k-1)^2\right).\end{aligned}$$

Where (†) follows from

$$\frac{t}{1-\mathrm{e}^{-2t}} \geqslant \frac{1}{2}.$$

Before we consider the last exponential in the Mehler kernel we note that by Cauchy-Schwarz:

$$|\langle y,\xi\rangle| \leqslant |\langle y-\xi,y\rangle| + |\langle y,y\rangle| \leqslant |y-\xi||y| + |y|^2. \tag{20}$$

Furthermore we have the estimate:

$$\frac{\mathrm{e}^{-t}}{1+\mathrm{e}^{-t}} \leqslant \frac{1}{2},$$

Using these we get for the last exponential in the Mehler kernel (8) M_{t^2}:

$$\exp\left(2\mathrm{e}^{-t^2}\frac{\langle y,\xi\rangle}{1+\mathrm{e}^{-t}}\right) \leqslant \exp(|\langle y,\xi\rangle|) \overset{(20)}{\leqslant} \exp(|y-\xi||y|)\mathrm{e}^{|y|^2}.$$

Wrapping it up, we can estimate the Mehler kernel (8) M_t on C_k from above by:

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)At|y|\big)\exp\left(-\frac{A^2}{2\mathrm{e}^{2t^2}}(2^k-1)^2\right).$$

Setting $\beta = \frac{A^2}{2\mathrm{e}^{2T}}$ and expanding the last exponential we get:

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{-\beta}\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)At|y|\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k}.$$

Which is as claimed.

Lemma 2 gives us by using $|x-y| \leqslant \alpha t \leqslant \alpha^2 m(x)$ the following estimate:

$$\mathrm{e}^{|y|^2} \leqslant \mathrm{e}^{|x|^2}\mathrm{e}^{\alpha^4}\mathrm{e}^{2\alpha^2}$$

$$\begin{aligned} M_{t^2}(y,\xi) &\leqslant \frac{\mathrm{e}^{-\beta}\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)\alpha(1+\alpha)\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k} \\ &\lesssim \mathrm{e}^{-(\alpha+\beta)}\mathrm{e}^{\alpha^4}\mathrm{e}^{\alpha^2}\frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\exp\big(2^{k+1}\alpha(1+\alpha)\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k}. \end{aligned}$$

Which is as claimed. ■

5. The boundedness of some non-tangential maximal operators

Our theorem is a small modification of [3, lemma 1.1] with a new proof.

1. THEOREM. *Let $A, a > 0$. For all x in $\mathbf{R}^d$ and all u in L^2_γ we have*

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |\mathrm{e}^{-t^2L}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u|\,\mathrm{d}\gamma. \tag{21}$$

PROOF. First we note that $\Gamma_x^{(A,a)} \subset \Gamma_x^{(1+aA,aA)}$ as $a, A \geqslant 1$.

$$|x-y| \leqslant At \leqslant aAt$$
$$t \leqslant am(x) \leqslant aAm(x)$$

So if $y \in \Gamma_x^{(A,a)}$ then $x \in \Gamma_y^{(aA,aA)}$. So set $\alpha = aA$ and $\Gamma_x^{\alpha} = \Gamma_x^{(\alpha,\alpha)}$

We will prove (21) by splitting up the integration domain in annuli.

$$\mathrm{e}^{-tL}|u(y)| \leqslant \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k(B)} M_t(y,\cdot)|u|\,\mathrm{d}\gamma.$$

More precisely, we will set $B = B(y, aAt)$ in the above and find a suitable upper bound for each integral on the right-hand side which we will denote by I_k for the sake of simplicity.

$$M_t(y,\xi) \leqslant \frac{\mathrm{e}^{-\beta}\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}}\exp\big((2^{k+1}-1)\alpha t|y|\big)\mathrm{e}^{\beta 2^{k+1}}\mathrm{e}^{-\beta 4^k},$$

where $\beta = \frac{\alpha^2}{2\mathrm{e}^{2\alpha^2}}$.

Since we have $|x-y| < \alpha t$ and $t \leqslant am(x)$ we infer that $t|x| \leqslant \alpha$. By Lemma 1 we also have that $t|y| \leqslant 1+\alpha$. From this and Lemma 7 we infer that:

$$M_t(y,\xi) \leqslant e^{-\beta} e^{-\alpha(1+\alpha)} \frac{e^{|y|^2}}{(1-e^{-t})^{\frac{d}{2}}} \exp\left(2^{k+1}\alpha(1+\alpha)\right) e^{\beta 2^{k+1}} e^{-\beta 4^k}, \tag{22}$$

Setting $\beta = \frac{\alpha^2}{2e^{2\alpha^2}}$. Note that β is maximal for $\alpha = \frac{1}{2}$ and after this value, β is decreasing. Setting $\lambda := \alpha(1+\alpha)$ we get:

$$M_t(y,\xi) \lesssim_\alpha e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2} \frac{e^{|x|^2}}{(1-e^{-t})^{\frac{d}{2}}} e^{(\lambda+\beta)2^{k+1}} e^{-\beta 4^k}. \tag{23}$$

Where the implied constant is given by $e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2}$

Or, using $\Lambda = \beta + \lambda$ we get:

$$M_t(y,\xi) \lesssim_\alpha \frac{e^{|x|^2}}{(1-e^{-t})^{\frac{d}{2}}} e^{\Lambda 2^{k+1}} e^{-\beta 4^k}, \tag{24}$$

Recalling Lemma 6 we get:

$$\gamma(B_t(x)) \leqslant V_d d \pi^{-\frac{d}{2}} t^d e^{-(t-|x|)^2}. \tag{25}$$

Where we abbreviate $V_d(1)$ with V_d. Recall

$$V_d \leqslant \frac{1}{\sqrt{\pi}} \left(\frac{2\pi e}{d}\right)^{\frac{d}{2}}.$$

To get,

$$\gamma(B_t(x)) \leqslant \frac{d}{\sqrt{\pi}} \left(\frac{2e}{d}\right)^{\frac{d}{2}} t^d e^{-(t-|x|)^2} = C_d t^d e^{-(t-|x|)^2}. \tag{26}$$

This allows us to estimate the remaining unbounded exponential in the Mehler kernel and allow a penalty up to $e^{-|x|^2}$. Furthermore, we have the following estimate which will make clear how to handle the time part in the Mehler kernel:

$$\frac{t^d}{(1-e^{-t^2})^{\frac{d}{2}}} \leqslant \left(\frac{t^2}{1-e^{-t^2}}\right)^{\frac{d}{2}} \leqslant \frac{a^d}{(1-e^{-a^2})^{\frac{d}{2}}}.$$

Let $B' := B(x, 2^{k+1}\alpha t)$ and B as before the ball $B(y, \alpha t)$. In the next step we will bound .... by the maximal function centered at x. For this we need to scale up the C_k. So,

$$|x-\xi| \leqslant |x-y| + |\xi - y| \leqslant \alpha t + (2^{k+1}-1)\alpha t = 2^{k+1}\alpha t.$$

And set $D_k = B(2^{k+1}\alpha t)$. So, we can bound the integral on the right-hand side of (??) by

$$\begin{aligned}
\int_{C_k(B)} M_{t^2}(y,\cdot)|u|\,\mathrm{d}\gamma &\lesssim_\alpha \frac{\mathrm{e}^{A2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\mathrm{e}^{|x|^2}\int_{C_k(B)}|u|\,\mathrm{d}\gamma \\
&\leqslant \frac{\mathrm{e}^{A2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\mathrm{e}^{|x|^2}\int_{D_k(B)}|u|\,\mathrm{d}\gamma \\
&\leqslant (M_\gamma u)(x)\frac{\mathrm{e}^{A2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\mathrm{e}^{|x|^2}\gamma(D_k) \\
&\overset{(1)}{\leqslant} (M_\gamma u)(x)C_d\alpha^d 2^{d(k+1)}t^d\frac{\mathrm{e}^{A2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}\mathrm{e}^{|x|^2}\mathrm{e}^{-(t-|x|)^2} \\
&\leqslant (M_\gamma u)(x)2^{kd}\mathrm{e}^{A2^{k+1}}\mathrm{e}^{-\beta 4^k}\frac{t^d\mathrm{e}^{-t^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}C_d\mathrm{e}^{2\alpha}(2\alpha)^d
\end{aligned}$$

Where (1) uses Lemma 6 and $t|x| \leqslant a$.

We can then bound the maximal function:

$$\begin{aligned}
\mathrm{e}^{-t^2L}|u(y)| &= \sum_{k=0}^{\infty} I_k \\
&\leqslant (M_\gamma u)(x)C_{d,a,A}\sum_{k=0}^{\infty}2^{kd}\mathrm{e}^{A2^{k+1}}\mathrm{e}^{-\beta 4^k}
\end{aligned}$$

Wrapping it up, we have that:

$$\mathrm{e}^{-t^2L}|u(y)| \lesssim \fint_{B_r(x)}|u|\,\mathrm{d}\gamma.$$

With implied constant

Which is what we wanted to prove.

$$\sum_{k=0}^{\infty}2^{kd}\mathrm{e}^{-C4^k} = \sum_{k=0}^{\infty}x^{kd}\mathrm{e}^{-Cx^{2k}}$$

Noting for $x \geqslant 1$ that $\exp(-Cx^{2k}) \leqslant \exp(-Ckx^2)$, thus,

$$\sum_{k=0}^{\infty}2^{kd}\mathrm{e}^{-C4^k} \leqslant \sum_{k=0}^{\infty}x^{kd}(\mathrm{e}^{-Cx^2})^k = \sum_{k=0}^{\infty}(x^d\mathrm{e}^{-Cx^2})^k$$

Here $x = 2$, so

$$\sum_{k=0}^{\infty}2^{kd}\mathrm{e}^{-C4^k} \leqslant \sum_{k=0}^{\infty}(2^d\mathrm{e}^{-4C})^k$$

If $2^d < \mathrm{e}^{4C}$, that is whenever $d\log 2 < 4C$, we can compute using the geometric series that

$$\sum_{k=0}^{\infty}2^{kd}\mathrm{e}^{-C4^k} \leqslant \frac{1}{1-2^d\mathrm{e}^{-4C}} = \frac{\mathrm{e}^{4C}}{\mathrm{e}^{4C}-2^d}$$

■

References

[1] Jan Maas, Jan Neerven, and Pierre Portal. "Whitney coverings and the tent spaces T 1,q (γ) for the Gaussian measure". In: *Arkiv för Matematik* 50.2 (Apr. 2011), pp. 379–395. arXiv: 1002.4911.

[2] Giancarlo Mauceri and Stefano Meda. "BMO and H^1 for the Ornstein-Uhlenbeck operator". In: *Journal of Functional Analysis* 252.1 (Nov. 2007), pp. 278–313.

[3] Ebner Pineda, W. Urbina, and Wilfredo Urbina R. "Non Tangential Convergence for the Ornstein-Uhlenbeck Semigroup". In: *Divulgaciones Matemáticas* 13.2 (2008), pp. 1–19.

[4] Pierre Portal. "Maximal and quadratic Gaussian Hardy spaces". In: (Mar. 2012). arXiv: 1203.1998.

[5] Peter Sjögren. "Operators associated with the hermite semigroup — A survey". In: *The Journal of Fourier Analysis and Applications* 3.S1 (Jan. 1997), pp. 813–823.

DELFT INSTITUTE OF APPLIED MATHEMATICS, DELFT UNIVERSITY OF TECHNOLOGY, P.O. BOX 5031, 2600 GA DELFT, THE NETHERLANDS

E-mail address: `j.j.b.teuwen@tudelft.nl`

URL: `http://fa.its.tudelft.nl/~teuwen/`