

A note on the Gaussian maximal function - Version 8 October 2013 + JvN additions

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ABSTRACT. This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded almost surely by the Gaussian Hardy-Littlewood maximal function. In particular this entails improvement on a result by Pineda and Urbina [3] who proved a similar result for a ‘truncated’ version of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well known that the classical real-valued maximal function associated with the heat semigroup is bounded almost everywhere by the Hardy-Littlewood maximal function,

$$(1) \quad \sup_{(y,t) \in \Gamma_x} |e^{-t\Delta} u(y)| \lesssim \sup_{r>0} \int_{B_r(x)} |u| \, d\lambda.$$

Here the action of *heat semigroup* $e^{-t^2\Delta} u = \rho_t * u$ is given by a convolution of u with the *heat kernel*

$$\rho_t(s) := \frac{e^{-|s|^2/4t}}{\pi^{\frac{d}{2}}} \frac{1}{(4t)^{\frac{d}{2}}}.$$

In this note we are interested in its gaussian counterpart. The change from Lebesgue measure to the gaussian measure

$$(2) \quad d\gamma(x) := \pi^{-\frac{d}{2}} e^{-|x|^2} dx$$

introduces quite some intricate technical and conceptual difficulties which appears to be due to the fact that the Gaussian measure is non-doubling.

As an analogue to the Laplacian which is symmetric in L^2 with respect to the Lebesgue measure next we introduce the *Ornstein-Uhlenbeck* operator L which is symmetric with respect to the Gaussian measure:

$$(3) \quad L := -\frac{1}{2}\Delta + \langle x, \nabla \rangle = \frac{1}{2}\nabla_\gamma^* \nabla_\gamma,$$

where ∇_γ denotes the realisation of the gradient in $L^2(\mathbf{R}^d, \gamma)$. Our main result, to be proved in (1), is the following gaussian analogue of (1):

$$(4) \quad \sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{-tL} u(y)| \lesssim \sup_{r>0} \int_{B_r(x)} |u| \, d\gamma.$$

Here, $\Gamma_x^{(A,a)}$ is the Gaussian cone defined by

$$(5) \quad \Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y, t) \in \mathbf{R}_+^d : |x - y| < At \text{ and } t \leq am(x)\},$$

where

$$(6) \quad m(x) := \min \left\{ 1, \frac{1}{|x|} \right\} = 1 \wedge \frac{1}{|x|}.$$

A slightly weaker version of the inequality (4) has been proved by Pineda and Urbina [3] who shows that

$$\sup_{(y,t) \in \tilde{\Gamma}_x} |e^{-t\Delta} u(y)| \lesssim \sup_{r>0} \int_{B_r(x)} |u| d\gamma,$$

where

$$\tilde{\Gamma}_x(x) = \{(y, t) \in \mathbf{R}_+^d : |x - y| < t \leq \tilde{m}(x)\}$$

is the ‘reduced’ gaussian cone corresponding to the function

$$\tilde{m}(x) = \min \left\{ \frac{1}{2}, \frac{1}{|x|} \right\}.$$

Their proof does not seem to easily generalize the range of t from $\frac{1}{2}$ up to 1. The proof of (4) is different and, we believe, more transparent than the one presented in [3]. It has the further virtue of allowing the extension to the cones with arbitrary aperture $A > 0$ and cut-off parameter $a > 0$. This additional generality is very important and has already been used by Portal (cf. the claim made by [4, discussion preceding Lemma 2.3]) to prove the H^1 -boundedness of the Riesz transform associated with L .

Before we continue, let us fix some notation. We will use without further reference notation such as \mathbf{Z}^d while we implicitly imply that d is a positive integer. To avoid possible confusion, we define the *positive integers* as the set $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$.

1.0.1. *minimal function.* We recall the lemma from [1, lemma 2.3] which first –although implicitly– appeared in [2].¹

2. The Mehler kernel

2.1. Setting. Recall that we work with the Ornstein-Uhlenbeck operator L as given by (3).

We define the Mehler kernel (see e.g., Sjögren [5]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(e^{-tL})_{t \geq 0}$. More precisely, this means:

$$(7) \quad e^{-tL} u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot) u d\gamma.$$

It is often more convenient to use $e^{-t^2 L}$ instead of e^{-tL} as is done in e.g., Portal [4].

2.2. The Mehler kernel. There is an abundance of literature on the Mehler kernel and its properties available, but for the present purpose Sjögren [5] will suffice. For instance, the Mehler kernel M_t of (7) is computed there. In addition it offers related results with to the Hermite polynomials.

The kernel M_t is invariant under the permutation $x \leftrightarrow y$. A formula for M_t which honors this observation is:

$$(8) \quad M_t(x, y) = \frac{\exp\left(-e^{-2t} \frac{|x - y|^2}{1 - e^{-2t}}\right) \exp\left(2e^{-t} \frac{\langle x, y \rangle}{1 + e^{-t}}\right)}{(1 - e^{-t})^{\frac{d}{2}} (1 + e^{-t})^{\frac{d}{2}}}.$$

¹Nog materiaal toevoegen?

3. Some lemmata and definitions

We use m as defined in (6) in our next lemma.

1. LEMMA. *Let a, A be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

- (1) *If $|x - y| < At$ and $t \leq am(x)$, then $t \leq (1 + aA)m(y)$;*
- (2) *If $|x - y| < Am(x)$, then $m(x) \leq (1 + A)m(y)$ and $m(y) \leq 2(1 + A)m(x)$.*

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. LEMMA. *Let $\alpha > 0$ and $|x - y| \leq \alpha m(x)$. We get the equivalence:*

$$e^{-\alpha^2(1+\alpha)^2} e^{-2\alpha(1+\alpha)} e^{-|y|^2} \leq e^{-|x|^2} \leq e^{\alpha^2} e^{2\alpha} e^{-|y|^2}.$$

PROOF. By the inverse triangle inequality and $m(x)|x| \leq 1$ we get,

$$(9) \quad |y|^2 \leq (\alpha m(x) + |x|)^2 \leq \alpha^2 + 2\alpha + |x|^2.$$

For the reverse direction we use Lemma 1 to infer $m(x) \leq (1 + \alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leq \alpha^2(1 + \alpha)^2 + 2\alpha(1 + \alpha) + |y|^2.$$

Combining we get:

$$(10) \quad e^{-\alpha^2(1+\alpha)^2} e^{-2\alpha(1+\alpha)} e^{-|y|^2} \leq e^{-|x|^2} \leq e^{\alpha^2} e^{2\alpha} e^{-|y|^2}.$$

As required. ■

4. On-diagonal estimates

4.1. **Kernel estimates.** Before we proceed with the technicalities we define κ and μ as:

$$\kappa = 2\left(1 + \frac{1}{\alpha}\right)^{-1}, \text{ and } \mu = 2\left(1 - \frac{1}{\alpha}\right)^{-1}.$$

such that κ and μ are conjugate exponents, which means:

$$\frac{1}{\kappa} + \frac{1}{\mu} = 1.$$

We proceed with a simple technical lemma which is given here as it will be used on several occasions.

3. LEMMA. *Let $t > 0$ and $\alpha \geq 1$. Then,*

$$(11) \quad \alpha e^{-2\frac{t}{\mu}} \leq \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} \leq \alpha,$$

$$(12) \quad 0 \leq \frac{1}{t} \left[\frac{e^{-t}}{1 + e^{-t}} - \frac{e^{-\frac{t}{\alpha}}}{1 + e^{-\frac{t}{\alpha}}} \right] \leq \frac{1}{2\mu}.$$

PROOF. We start with (11) and apply the mean value theorem to the function $f(\xi) = \xi^\alpha$. For $0 < \xi < \xi'$ this gives that:

$$f(\xi) - f(\xi') = \alpha \hat{\xi}^{\alpha-1} (\xi - \xi') \text{ for some } \hat{\xi} \text{ in } [\xi, \xi'].$$

Picking $\xi = 1$ and $\xi' = e^{-\frac{t}{\alpha}}$ yields:

$$(13) \quad \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} = \alpha \hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[e^{-\frac{t}{\alpha}}, 1 \right].$$

Combining this result with the monotonicity of $\xi \mapsto \alpha \xi^{\alpha-1}$ we obtain:

$$\alpha e^{\frac{t}{\alpha}} e^{-t} \leq \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} \leq \alpha,$$

where the last bound follows from the monotonicity together with the limit as $t \downarrow 0$. We proceed with (12). Recalling that $\alpha \geq 1$ one can directly verify that the function

$$\frac{1}{t} \left[\frac{1}{1 + e^{-t}} - \frac{1}{1 + e^{-\frac{t}{\alpha}}} \right]$$

is non-negative and decreasing in t . To find an upper bound we compute the limit as t goes to 0. That is:

$$\lim_{t \rightarrow 0} \frac{1}{t} \left[\frac{e^{-t}}{1 + e^{-t}} - \frac{e^{-\frac{t}{\alpha}}}{1 + e^{-\frac{t}{\alpha}}} \right] = \lim_{t \rightarrow 0} \left[\frac{e^{-2t}}{(1 + e^{-t})^2} - \frac{1}{\alpha} \frac{e^{-2\frac{t}{\alpha}}}{(1 + e^{-\frac{t}{\alpha}})^2} \right] \uparrow \frac{1}{2\mu}.$$

Which is as asserted and completes the proof. ■

The following lemma will be useful when transferring estimates from $M_{\frac{t}{\alpha}}$ to M_t . It follows from the mean value theorem applied to $\xi \mapsto \xi^\alpha$.

4. LEMMA. For $\alpha \geq 1$ and $0 < t \leq T < \infty$ and all let $x, y \in \mathbf{R}^d$ we have that:

$$(14) \quad \exp\left(-\frac{1}{e^{2\frac{t}{\alpha}}} \frac{|x-y|^2}{1 - e^{-2\frac{t}{\alpha}}}\right) \leq \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\alpha}}} \frac{|x-y|^2}{1 - e^{-t}}\right).$$

PROOF. It is fruitful to note that

$$1 - e^{-2t} = (1 - e^{-t})(1 + e^{-t})$$

holds. The same holds true for the inequality

$$\frac{1}{2e^{2t}} \leq \frac{e^{-2t}}{1 + e^{-t}} \leq \frac{1}{2}.$$

Therefore,

$$\begin{aligned} \exp\left(-e^{-2t} \frac{|x-y|^2}{1 - e^{-2t}}\right) &= \exp\left(-\frac{e^{-2t}}{1 + e^{-t}} \frac{|x-y|^2}{1 - e^{-t}}\right) \\ &\leq \exp\left(-\frac{1}{2e^{2t}} \frac{|x-y|^2}{1 - e^{-t}}\right). \end{aligned}$$

Setting $\beta := 1 + \alpha^{-1}$ and applying (11) we get:

$$\begin{aligned} \exp\left(-e^{-2\frac{t}{\alpha}} \frac{|x-y|^2}{1 - e^{-2\frac{t}{\alpha}}}\right) &= \exp\left(-\frac{e^{-2\frac{t}{\alpha}}}{1 + e^{-\frac{t}{\alpha}}} \frac{1 - e^{-t}}{1 - e^{-\frac{t}{\alpha}}} \frac{|x-y|^2}{1 - e^{-t}}\right) \\ &\leq \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\alpha}}} \frac{1}{e^t e^{-\frac{t}{\alpha}}} \frac{|x-y|^2}{1 - e^{-t}}\right) \\ &\leq \exp\left(-\frac{\alpha}{2e^{2\frac{t}{\alpha}}} \frac{|x-y|^2}{1 - e^{-t}}\right). \end{aligned}$$

Which is as asserted. ■

Our first lemma is about estimating $M_{\frac{t}{\alpha}}$ in terms of M_t .

4.1.1. Time-scaling of the Mehler kernel.

5. LEMMA. Let $T > 0$, $\alpha \geq 1$, and $x, y \in \mathbf{R}^d$. Then:

$$(15) \quad M_{\frac{t}{\alpha}}(x, y) \leq \alpha^{\frac{d}{2}} \exp\left(\frac{t}{2\mu} |\langle x, y \rangle|\right) \exp\left(-\frac{\alpha}{4e^{2T}} \frac{|x-y|^2}{1-e^{-t}}\right) M_t(x, y).$$

PROOF. To prove the lemma we compute $M_{\frac{t}{\alpha}} M_t^{-1}$. First recall that (11) gives

$$\alpha e^{-\frac{t}{\mu}} \leq \frac{1-e^{-t}}{1-e^{-\frac{t}{\alpha}}} \leq \alpha.$$

Combining the exponentials also gives,

$$\begin{aligned} \exp\left(-2e^{-\frac{t}{\alpha}} \frac{\langle x, y \rangle}{1+e^{-\frac{t}{\alpha}}}\right) \exp\left(2e^{-t} \frac{\langle x, y \rangle}{1+e^{-t}}\right) \\ = \exp\left(\frac{2}{t} \left[\frac{e^{-t}}{1+e^{-t}} - \frac{e^{-\frac{t}{\alpha}}}{1+e^{-\frac{t}{\alpha}}} \right] t \langle x, y \rangle\right) \\ \stackrel{(12)}{\leq} \exp\left(\frac{t}{\mu} |\langle x, y \rangle|\right). \end{aligned}$$

Combining Lemma 4 and equation (14) almost gives the final estimate.

$$\begin{aligned} \frac{M_{\frac{t}{\alpha}}(x, y)}{M_t(x, y)} &\leq \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu} |\langle x, y \rangle|\right) \exp\left(e^{-2t} \frac{|x-y|^2}{1-e^{-2t}}\right) \exp\left(-e^{-2\frac{t}{\alpha}} \frac{|x-y|^2}{1-e^{-2\frac{t}{\alpha}}}\right) \\ &\leq \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu} |\langle x, y \rangle|\right) \exp\left(\left[1 - \frac{\alpha}{4e^{2\frac{t}{\kappa}}}\right] \frac{|x-y|^2}{1-e^{-t}}\right) \exp\left(-\frac{\alpha}{4e^{2\frac{t}{\kappa}}} \frac{|x-y|^2}{1-e^{-t}}\right). \end{aligned}$$

Finally, we apply the assumption $\alpha \geq 4e^{2\frac{t}{\kappa}}$ to obtain:

$$\frac{M_{\frac{t}{\alpha}}(x, y)}{M_t(x, y)} \leq \alpha^{\frac{d}{2}} \exp\left(\frac{t}{\mu} |\langle x, y \rangle|\right) \exp\left(-\frac{\alpha}{4e^{2\frac{t}{\kappa}}} \frac{|x-y|^2}{1-e^{-t}}\right).$$

Which is as asserted. The assumption $\alpha \geq 4e^{2\frac{t}{\kappa}}$ can be rephrased as the requirement that $\left(1 + \frac{1}{\alpha}\right)^{-1} \log\left(\frac{\alpha}{4}\right) \geq t$. \blacksquare

4.2. An estimate on Gaussian balls.

6. LEMMA. Let $B_t(x)$ be the Euclidean ball with radius t and center x and let γ be the Gaussian measure (2). We have the inequality:

$$(16) \quad \frac{\gamma(B_t(x))}{V_d(t)} \leq d \pi^{-\frac{d}{2}} e^{-(t-|x|)^2}.$$

PROOF. Remark that

$$\begin{aligned} \int_B e^{-|\xi|^2} d\xi &= e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{-2\langle x, \xi-x \rangle} d\xi \\ &\leq e^{-|x|^2} \int_B e^{-|\xi-x|^2} e^{2|x||\xi-x|} d\xi \\ &\leq e^{-|x|^2} e^{2|x|t} \int_B e^{-|\xi-x|^2} d\xi \\ &= \pi^{\frac{d}{2}} e^{-|x|^2} e^{2t|x|} \gamma(B_t(0)). \end{aligned}$$

So, for a ball $B := B_t(x)$ there holds that

$$(17) \quad \gamma(B_t(x)) \leq e^{-|x|^2} e^{2t|x|} \gamma(B_t(0)).$$

We will estimate the Gaussian volume of the ball $B_t(0)$. To save writing, let S_d and V_d be the surface area and volume respectively of the d -dimensional unit sphere. Using polar coordinates we proceed by:

$$\begin{aligned}\gamma(B_t(0)) &= \pi^{-\frac{d}{2}} \int_{B_t(0)} e^{-|\xi|^2} d\xi \\ &= S_d \pi^{-\frac{d}{2}} \int_0^t e^{-r^2} r^{d-1} dr \\ &\leq S_d t^d \pi^{-\frac{d}{2}} e^{-t^2} \\ &= dV_d(t) \pi^{-\frac{d}{2}} e^{-t^2}.\end{aligned}$$

Upon combining this result with (17) we obtain (16), which is as promised. \blacksquare

4.3. On-diagonal kernel estimates on annuli. As is common in harmonic analysis, we often wish to decompose \mathbf{R}^d into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli C_k . For the sake of simplicity we will write $B := B_t(x)$ and mean that B is the closed ball with center x and radius t . Furthermore, we use notations such as $2B$ to mean the ball obtained from B by multiplying its radius by 2.

The C_k are given by,

$$(18) \quad C_k(B) := C_k = (2^{k+1} - 1)B \setminus (2^k - 1)B.$$

So, whenever ξ is in $C_k(B_{\sqrt{t}}(x))$, we get for $k \geq 0$:

$$(19) \quad (2^k - 1)t < |y - \xi| \leq (2^{k+1} - 1)t.$$

7. LEMMA. *Given $A > 0$, let $B = B_{At}(y)$, $0 < t \leq T < \infty$ and $\xi \in C_k$. Then we have:*

$$M_{t^2}(y, \xi) \leq \frac{e^{-\beta} e^{|y|^2}}{(1 - e^{-t^2})^{\frac{d}{2}}} \exp((2^{k+1} - 1)At^2|y|) e^{\beta 2^{k+1}} e^{-\beta 4^k},$$

where $\beta = \frac{A^2}{2e^{2T}}$.

PROOF. Let $B = B_{At}(y)$ and let C_k be as in (18). Considering the first exponential which occurs in the Mehler kernel (8) together with (19) gives for $k \geq 0$:

$$\begin{aligned}\exp\left(-e^{-2t} \frac{|y - \xi|^2}{1 - e^{-2t}}\right) &\leq \exp\left(-e^{-2t} \frac{(2^k - 1)^2 A^2 t}{1 - e^{-2t}}\right) \\ &\stackrel{1}{\leq} \exp\left(-\frac{A^2}{2e^{2t}} (2^k - 1)^2\right).\end{aligned}$$

Where (1) follows from

$$\frac{t}{1 - e^{-2t}} \geq \frac{1}{2}.$$

Before we consider the last exponential in the Mehler kernel we note that by Cauchy-Schwarz:

$$(20) \quad |\langle y, \xi \rangle| \leq |\langle y - \xi, y \rangle| + |\langle y, y \rangle| \leq |y - \xi||y| + |y|^2.$$

Furthermore we have the estimate:

$$\frac{e^{-t}}{1 + e^{-t}} \leq \frac{1}{2},$$

Using these we get for the last exponential in the Mehler kernel (8):

$$\begin{aligned} \exp\left(2e^{-t} \frac{\langle y, \xi \rangle}{1 + e^{-t}}\right) &\leq \exp(|\langle y, \xi \rangle|) \\ &\stackrel{(20)}{\leq} \exp(|y - \xi||y|)e^{|y|^2}. \end{aligned}$$

Wrapping it up, we can estimate the Mehler kernel (8) M_t on C_k from above by:

$$M_t(y, \xi) \leq \frac{e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp((2^{k+1} - 1)At|y|) \exp\left(-\frac{A^2}{2e^{2t}}(2^k - 1)^2\right).$$

Setting $\beta = \frac{A^2}{2e^{2t}}$ and expanding the last exponential we get:

$$M_t(y, \xi) \leq \frac{e^{-\beta} e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp((2^{k+1} - 1)At|y|) e^{\beta 2^{k+1}} e^{-\beta 4^k}.$$

Which is as claimed.

Lemma 2 gives us by using $|x - y| \leq at \leq \alpha^2 m(x)$ the following estimate:

$$e^{|y|^2} \leq e^{|x|^2} e^{\alpha^4} e^{2\alpha^2}$$

$$\begin{aligned} M_t(y, \xi) &\leq \frac{e^{-\beta} e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp((2^{k+1} - 1)\alpha(1 + \alpha)) e^{\beta 2^{k+1}} e^{-\beta 4^k} \\ &\leq e^{-(\alpha + \beta)} e^{\alpha^4} e^{\alpha^2} \frac{e^{|x|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp(2^{k+1}\alpha(1 + \alpha)) e^{\beta 2^{k+1}} e^{-\beta 4^k}. \end{aligned}$$

Which is as claimed. ■

5. The boundedness of some non-tangential maximal operators

Our theorem is a small modification of [3, lemma 1.1] with a new proof.

1. THEOREM. Let $A, a > 0$. For all x in \mathbf{R}^d and all u in L_γ^2 we have

$$(21) \quad \sup_{(y, t) \in \Gamma_x^{(A, a)}} |e^{-t^2 L} u(y)| \lesssim \sup_{r > 0} \int_{B_r(x)} |u| \, d\gamma.$$

PROOF. First we note that $\Gamma_x^{(A, a)} \subset \Gamma_x^{(1+aA, aA)}$ as $a, A \geq 1$.

$$\begin{aligned} |x - y| &\leq At \leq aAt \\ t &\leq am(x) \leq aAm(x) \end{aligned}$$

So if $y \in \Gamma_x^{(A, a)}$ then $x \in \Gamma_y^{(aA, aA)}$. So set $\alpha = aA$ and $\Gamma_x^\alpha = \Gamma_x^{(\alpha, \alpha)}$

We will prove (21) by splitting up the integration domain in annuli.

$$e^{-tL} |u(y)| \leq \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k(B)} M_t(y, \cdot) |u| \, d\gamma.$$

More precisely, we will set $B = B(y, aAt)$ in the above and find a suitable upper bound for each integral on the right-hand side which we will denote by I_k for the sake of simplicity.

$$M_t(y, \xi) \leq \frac{e^{-\beta} e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp((2^{k+1} - 1)\alpha t|y|) e^{\beta 2^{k+1}} e^{-\beta 4^k},$$

where $\beta = \frac{\alpha^2}{2e^{2a^2}}$.

Since we have $|x - y| < \alpha t$ and $t \leq am(x)$ we infer that $t|x| \leq \alpha$. By [Lemma 1](#) we also have that $t|y| \leq 1 + \alpha$. From this and [Lemma 7](#) we infer that:

$$(22) \quad M_t(y, \xi) \leq e^{-\beta} e^{-\alpha(1+\alpha)} \frac{e^{|y|^2}}{(1 - e^{-t})^{\frac{d}{2}}} \exp(2^{k+1} \alpha(1 + \alpha)) e^{\beta 2^{k+1}} e^{-\beta 4^k},$$

Setting $\beta = \frac{\alpha^2}{2e^{2\alpha^2}}$. Note that β is maximal for $\alpha = \frac{1}{2}$ and after this value, β is decreasing. Setting $\lambda := \alpha(1 + \alpha)$ we get:

$$(23) \quad M_t(y, \xi) \lesssim_\alpha e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2} \frac{e^{|x|^2}}{(1 - e^{-t})^{\frac{d}{2}}} e^{(\lambda+\beta)2^{k+1}} e^{-\beta 4^k}.$$

Where the implied constant is given by $e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2}$

Or, using $\Lambda = \beta + \lambda$ we get:

$$(24) \quad M_t(y, \xi) \lesssim_\alpha \frac{e^{|x|^2}}{(1 - e^{-t})^{\frac{d}{2}}} e^{\Lambda 2^{k+1}} e^{-\beta 4^k},$$

Recalling [Lemma 6](#) we get:

$$(25) \quad \gamma(B_t(x)) \leq V_d d \pi^{-\frac{d}{2}} t^d e^{-(t-|x|)^2}.$$

Where we abbreviate $V_d(1)$ with V_d . Recall

$$V_d \leq \frac{1}{\sqrt{\pi}} \left(\frac{2\pi e}{d} \right)^{\frac{d}{2}}.$$

To get,

$$(26) \quad \gamma(B_t(x)) \leq \frac{d}{\sqrt{\pi}} \left(\frac{2e}{d} \right)^{\frac{d}{2}} t^d e^{-(t-|x|)^2} = C_d t^d e^{-(t-|x|)^2}.$$

This allows us to estimate the remaining unbounded exponential in the Mehler kernel and allow a penalty up to $e^{-|x|^2}$. Furthermore, we have the following estimate which will make clear how to handle the time part in the Mehler kernel:

$$\frac{t^d}{(1 - e^{-t^2})^{\frac{d}{2}}} \leq \left(\frac{t^2}{1 - e^{-t^2}} \right)^{\frac{d}{2}} \leq \frac{a^d}{(1 - e^{-a^2})^{\frac{d}{2}}}.$$

Let $B' := B(x, 2^{k+1} \alpha t)$ and B as before the ball $B(y, \alpha t)$. In the next step we will bound by the maximal function centered at x . For this we need to scale up the C_k . So,

$$|x - \xi| \leq |x - y| + |\xi - y| \leq \alpha t + (2^{k+1} - 1) \alpha t = 2^{k+1} \alpha t.$$

And set $D_k = B(2^{k+1}at)$. So, we can bound the integral on the right-hand side of (??) by

$$\begin{aligned}
\int_{C_k(B)} M_{t^2}(y, \cdot) |u| \, d\gamma &\lesssim_\alpha \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1 - e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} \int_{C_k(B)} |u| \, d\gamma \\
&\leq \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1 - e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} \int_{D_k(B)} |u| \, d\gamma \\
&\leq (M_\gamma u)(x) \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1 - e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} \gamma(D_k) \\
&\stackrel{(1)}{\leq} (M_\gamma u)(x) C_d \alpha^d 2^{d(k+1)} t^d \frac{e^{\Lambda 2^{k+1}} e^{-\beta 4^k}}{(1 - e^{-t^2})^{\frac{d}{2}}} e^{|x|^2} e^{-(t-|x|)^2} \\
&\leq (M_\gamma u)(x) 2^{kd} e^{\Lambda 2^{k+1}} e^{-\beta 4^k} \frac{t^d e^{-t^2}}{(1 - e^{-t^2})^{\frac{d}{2}}} C_d e^{2\alpha} (2\alpha)^d
\end{aligned}$$

Where (1) uses [Lemma 6](#) and $t|x| \leq a$.

We can then bound the maximal function:

$$\begin{aligned}
e^{-t^2 L} |u(y)| &= \sum_{k=0}^{\infty} I_k \\
&\leq (M_\gamma u)(x) C_{d,a,A} \sum_{k=0}^{\infty} 2^{kd} e^{\Lambda 2^{k+1}} e^{-\beta 4^k}
\end{aligned}$$

Wrapping it up, we have that:

$$e^{-t^2 L} |u(y)| \lesssim \int_{B_r(x)} |u| \, d\gamma.$$

With implied constant

Which is what we wanted to prove.

$$\sum_{k=0}^{\infty} 2^{kd} e^{-C 4^k} = \sum_{k=0}^{\infty} x^{kd} e^{-C x^{2k}}$$

Noting for $x \geq 1$ that $\exp(-C x^{2k}) \leq \exp(-C k x^2)$, thus,

$$\sum_{k=0}^{\infty} 2^{kd} e^{-C 4^k} \leq \sum_{k=0}^{\infty} x^{kd} (e^{-C x^2})^k = \sum_{k=0}^{\infty} (x^d e^{-C x^2})^k$$

Here $x = 2$, so

$$\sum_{k=0}^{\infty} 2^{kd} e^{-C 4^k} \leq \sum_{k=0}^{\infty} (2^d e^{-4C})^k$$

If $2^d < e^{4C}$, that is whenever $d \log 2 < 4C$, we can compute using the geometric series that

$$\sum_{k=0}^{\infty} 2^{kd} e^{-C 4^k} \leq \frac{1}{1 - 2^d e^{-4C}} = \frac{e^{4C}}{e^{4C} - 2^d}$$

■

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