

# Gaussian estimates

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ABSTRACT. Maximal function! An attempt! A good attempt! I hope!

## 1. The Mehler kernel and friends

**1.1. Notation.** To begin, let us fix some notation. As is common, we use  $N$  to represent a positive integer. That is,  $N \in \mathbf{Z}_+ = \{1, 2, 3, \dots\}$ . In the same way we cast letters that denote the number of dimensions, e.g.  $d$  in  $\mathbf{R}^d$  as positive integers.

We use the capital letter  $T$  to denote a “time” endpoint, for instance, when writing  $t$  in  $(0, T]$ .

**1.2. Setting.** Given the Ornstein-Uhlenbeck operator  $L$  defined as:

$$(1) \quad L = -\frac{1}{2}\Delta + x \cdot \nabla,$$

We define the Mehler kernel (see e.g., Sjögren [4]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup  $(e^{-tL})_t$ . More precisely, this means:

$$(2) \quad e^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x, \cdot)u \, d\gamma.$$

It is often more convenient to use  $e^{-t^2L}$  instead of  $e^{-tL}$  as is done in e.g., Portal [3] and we will also do so.

**1.3. The Mehler kernel.** For the computation of the Mehler kernel in (2) we refer to e.g., Sjögren [4] which additionally offers related results such as those concerning Hermite polynomials.

If one observes that the kernel  $M_{t^2}$  is symmetric in its arguments, a useful expression is:

$$(3) \quad M_{t^2}(x, y) = \frac{\exp\left(-e^{-t^2} \frac{|x-y|^2}{1-e^{-2t^2}}\right) \exp\left(e^{-t^2} \frac{|x|^2 + |y|^2}{1+e^{-t^2}}\right)}{(1-e^{-t^2})^{\frac{d}{2}} (1+e^{-t^2})^{\frac{d}{2}}}.$$

## 2. Some fine lemmata and definitions

**2.1. minimal function.** We recall the lemma from [1, lemma 2.3] which first, –although implicitly– appeared in [2]. It will be convenient to define a function  $m$  as:

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\} = 1 \vee \frac{1}{|x|}.$$

1. LEMMA. Let  $a, A$  be strictly positive numbers. We have for  $x, y$  in  $\mathbf{R}^d$  that:

(1) If  $|x - y| < At$  and  $t \leq am(x)$ , then  $t \leq (1 + aA)m(y)$ ;

(2) Likewise, if  $|x - y| < Am(x)$ , then  $m(x) \leq (1 + A)m(y)$  and  $m(y) \leq 2(1 + A)m(x)$ .

We rewrite this lemma using the Gaussian cone  $\Gamma_x^{(A,a)}$ . Recall that:

$$(4) \quad \Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y, t) \in \mathbf{R}_+^d : |x - y| < At \text{ and } t \leq am(x)\}.$$

We will also write  $\Gamma_x^a$  to mean  $\Gamma_x^{(1,a)}$ . So we can infer from [Lemma 1](#) that:

2. LEMMA. Let  $a, A$  be strictly positive numbers. Then:

(1) If  $(y, t) \in \Gamma_x^{(A,a)}$  then  $t \leq (1 + aA)m(y)$ ;

(2) If  $(y, t) \in \Gamma_x^{(A,a)}$  then  $(x, t) \in \Gamma_y^{(1+aA,a)}$ .

We will use a global/local region dichotomy which we define as follows.

1. DEFINITION. Given  $\tau > 0$ , the set  $N_\tau$  is given as:

$$(5) \quad N_\tau(x) := N_\tau := \{(x, y) \in \mathbf{R}^{2d} : |x - y| \leq \tau m(x)\}.$$

Sometimes it is easier to work with the set  $N_\tau(B)$ , which is given for  $B := B_r(y)$  as:

$$(6) \quad N_\tau(B) := \{y \in \mathbf{R}^d : |x - y| \leq \tau m(x)\}.$$

When we partition the space into  $N_\tau$  and its complement, we call the part belonging to  $N_\tau$  the local region and the part belonging to  $\mathbb{C}N_\tau$  the global region.

The set  $t \leq am(x)$  is used in the definition of the cones  $\Gamma_x^{(A,a)}$  and we will name it  $D^a$ , that is:

$$(7) \quad D^a := \{(x, t) \in \mathbf{R}_+^d : t \leq am(x)\}.$$

We will write  $D := D^1$  for simplicity.

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

3. LEMMA. Let  $(y, t) \in \Gamma_x^{(A,a)}$ . Then the Gaussians in  $x$  and  $y$  respectively are comparable. In particular this means that,

$$e^{-|x|^2} \simeq e^{-|y|^2}.$$

REMARK. More precisely, from the proof we get the estimates (8) and (9). That is:

$$e^{-|x|^2} \leq e^{(1+aA)^2-1} e^{-|y|^2},$$

and,

$$e^{-|y|^2} \leq e^{(1+aA)^2} e^{-2(1+aA)} e^{-|x|^2}.$$

PROOF. Let  $(y, t) \in \Gamma_x^{(A,a)}$ . Unwrapping the definition we have

$$|x - y| < At \text{ and } t \leq am(x).$$

Hence, by the inverse triangle inequality we get,

$$\begin{aligned} |y|^2 &\leq (aAm(x) + |x|)^2 \\ &= (aA)^2 + 2aAm(x)|x| + |x|^2 \\ &\leq (aA)^2 + 2aA + |x|^2. \end{aligned}$$

Therefore,

$$(8) \quad e^{-|y|^2} \geq e^{-(aA)^2} e^{-2aA} e^{-|x|^2}.$$

By **Lemma 1** we have  $t \leq (1 + aA)m(y)$

$$\begin{aligned} |x|^2 &\leq ((1 + aA)m(y) + |y|)^2 \\ &= ((1 + aA)m(y))^2 + 2(1 + aA)m(y)|y| + |y|^2 \\ &\leq (1 + aA)^2 + 2(1 + aA) + |y|^2. \end{aligned}$$

Therefore,

$$(9) \quad e^{-|x|^2} \geq e^{-(1+aA)^2} e^{-2(1+aA)} e^{-|y|^2}.$$

Summarizing we thus have that,

$$e^{-|x|^2} \simeq e^{-|y|^2},$$

as required. ■

4. **LEMMA.** Let  $x, y$  and  $z$  in  $\mathbb{R}^d$ . Set

$$\tau = \frac{1}{2}(1 + 2aA)(1 + aA).$$

If  $|y - z| > \tau m(y)$  (i.e.,  $(y, z) \notin N_\tau$ ) and  $(y, t) \in \Gamma_x^{(A, a)}$  then  $|x - z| > \frac{1}{2}m(x)$  (i.e.,  $(x, z) \notin N_{\frac{1}{2}}$ ).

**PROOF.** We assume that  $(y, z) \notin N_\tau$  and  $(y, t) \in \Gamma_x^{(A, a)}$ . Written out this gives by (5) the inequality  $|y - z| > \tau m(y)$ , and by (4) the inequality  $|x - y| < aAm(x)$ . Note that the latter inequality together with **Lemma 1** yields,

$$(10) \quad \frac{1}{2} \frac{1}{1 + aA} m(y) \leq m(x) \leq (1 + aA)m(y).$$

Combining we get  $|x - y| < aA(1 + aA)m(y)$ . Now we are in position to apply the triangle inequality:

$$|x - z| \geq |y - z| - |x - y| > \tau m(y) - aA(1 + aA)m(y).$$

As we require an lower bound in terms of  $m(x)$  and not  $m(y)$ , we again apply (10) to obtain:

$$\begin{aligned} |x - z| &\geq |y - z| - |x - y| > \tau m(y) - aAm(y) \\ &\geq \tau \frac{1}{1 + aA} m(x) - aAm(x) \\ &\geq \frac{1}{2} m(x). \end{aligned}$$

So we are done. ■

### 3. On-diagonal estimates

**3.1. Kernel estimates.** We begin with a technical lemma which will be useful on several occasions.

5. **LEMMA.** Let  $t$  in  $(0, T]$  and  $\alpha > 1$ . Then,

$$(11) \quad \alpha e^{-T^2} \leq \frac{1 - e^{-t^2}}{1 - e^{-\frac{t^2}{\alpha}}} \leq \alpha.$$

**PROOF.** Let  $t$  in  $(0, T]$  and  $\alpha > 1$ . Applying the mean value theorem to the function  $f(\xi) = \xi^\alpha$  gives, for  $0 < \xi < \xi'$ :

$$f(\xi) - f(\xi') = \alpha \hat{\xi}^{\alpha-1}(\xi - \xi') \text{ for some } \hat{\xi} \text{ in } [\xi, \xi'].$$

Picking  $\xi = 1$  and  $\xi' = e^{-\frac{t^2}{\alpha}}$  gives:

$$(12) \quad \frac{1 - e^{-t^2}}{1 - e^{-\frac{t^2}{\alpha}}} = \alpha \xi^{\alpha-1} \text{ for some } \xi \text{ in } [e^{-\frac{t^2}{\alpha}}, 1].$$

Applying this result together with the monotonicity of  $\xi \mapsto \alpha \xi^{\alpha-1}$  we get:

$$\alpha e^{-T^2} \leq \alpha e^{-t^2} \leq \alpha \exp\left(-t^2 \frac{\alpha-1}{\alpha}\right) \leq \frac{1 - e^{-t^2}}{1 - e^{-\frac{t^2}{\alpha}}}.$$

Hence,

$$\alpha e^{-T^2} \leq \frac{1 - e^{-t^2}}{1 - e^{-\frac{t^2}{\alpha}}} \downarrow \alpha.$$

Which completes the proof.  $\blacksquare$

The following lemma will be useful when transferring estimates from  $M_{\frac{t^2}{\alpha}}$  to  $M_{t^2}$ . It follows from the mean value theorem applied to  $\xi \mapsto \xi^\alpha$ .

6. LEMMA. For  $C, T > 0, \alpha > 1, t$  in  $(0, T]$  and all  $x, y$  in  $\mathbf{R}^d$  we have that

$$(13) \quad \exp\left(-C \frac{|x-y|^2}{1 - e^{-\frac{t^2}{\alpha}}}\right) \leq \exp\left(-C \frac{\alpha}{e^{T^2}} \frac{|x-y|^2}{1 - e^{-t^2}}\right).$$

PROOF. Let  $t$  in  $(0, T]$ . Applying Lemma 5 we get:

$$\exp\left(-C \frac{|x-y|^2}{1 - e^{-\frac{t^2}{\alpha}}}\right) \leq \exp\left(-C \frac{|x-y|^2}{1 - e^{-t^2}} \frac{1 - e^{-t^2}}{1 - e^{-\frac{t^2}{\alpha}}}\right) \leq \exp\left(-C \frac{\alpha}{e^{T^2}} \frac{|x-y|^2}{1 - e^{-t^2}}\right).$$

Which is as asserted.  $\blacksquare$

Later on we will study kernel estimates of kernels related to the Mehler kernel, but our first lemma is about estimating  $M_{\frac{t^2}{\alpha}}$  in terms of  $M_t$ .

7. LEMMA. Let  $\alpha \geq 2e^{T^2}$ ,  $t$  in  $(0, T]$  and  $x, y$  in  $\mathbf{R}^d$ . If  $t|x| \lesssim 1$  and  $t|y| \lesssim 1$  then:

$$(14) \quad M_{\frac{t^2}{\alpha}}(x, y) \lesssim \exp\left(-\frac{\alpha}{2e^{T^2}} \frac{|x-y|^2}{1 - e^{-t^2}}\right) M_{t^2}(x, y),$$

where the implied constant does not depend on  $x, y$  and  $t$ .

REMARK. If  $C$  is the positive constant such that  $t|x| \leq C$  and  $t|y| \leq C$  then the proof gives that the implied constant is bounded from above by  $\alpha^{\frac{d}{2}} e^{\frac{\alpha}{2} C^2}$ .

PROOF. To prove the lemma we compute  $M_{\frac{t^2}{\alpha}} M_{t^2}^{-1}$ . First note that dividing the time-parts by (11) gives the upper-bound  $\alpha^{\frac{d}{2}}$ . Furthermore, we can verify that

$$\frac{1}{1 + e^{-t^2}} - \frac{1}{1 + e^{-\frac{t^2}{\alpha}}} \geq 0.$$

Also,

$$\begin{aligned} & \exp\left(-\frac{1}{2} \frac{|x+y|^2}{1 + e^{-\frac{t^2}{\alpha}}}\right) \exp\left(\frac{1}{2} \frac{|x+y|^2}{1 + e^{-t^2}}\right) \\ & \leq \exp\left(\frac{1}{2} \left[ \frac{1}{1 + e^{-t^2}} - \frac{1}{1 + e^{-\frac{t^2}{\alpha}}} \right] |x+y|^2\right) \\ & = \exp\left(\frac{1}{2} \frac{1}{t^2} \left[ \frac{1}{1 + e^{-t^2}} - \frac{1}{1 + e^{-\frac{t^2}{\alpha}}} \right] t^2 |x+y|^2\right). \end{aligned}$$

Next, as the inner most function is decreasing,

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t^2} \left[ \frac{1}{1 + e^{-t^2}} - \frac{1}{1 + e^{-\frac{t^2}{\alpha}}} \right] &= \lim_{t \rightarrow 0} \frac{1}{2t} \left[ \frac{2te^{-t^2}}{(1 + e^{-t^2})^2} - \frac{1}{\alpha} \frac{2te^{-\frac{t^2}{\alpha}}}{(1 + e^{-\frac{t^2}{\alpha}})^2} \right] \\ &= \lim_{t \rightarrow 0} \left[ \frac{e^{-t^2}}{(1 + e^{-t^2})^2} - \frac{1}{\alpha} \frac{e^{-\frac{t^2}{\alpha}}}{(1 + e^{-\frac{t^2}{\alpha}})^2} \right] \\ &\uparrow \frac{1}{4} \left( 1 - \frac{1}{\alpha} \right). \end{aligned}$$

So that

$$\begin{aligned} \exp\left(-\frac{1}{2} \frac{|x+y|^2}{1 + e^{-\frac{t^2}{\alpha}}}\right) \exp\left(\frac{1}{2} \frac{|x+y|^2}{1 + e^{-t^2}}\right) \\ \leq \exp\left(\frac{1}{8} \left(1 - \frac{1}{\alpha}\right) t^2 |x+y|^2\right) \\ \leq \exp\left(\frac{1}{4} t^2 |x|^2\right) \exp\left(\frac{1}{4} t^2 |y|^2\right). \end{aligned}$$

So using [Lemma 6](#) and equation (13) we get

$$\begin{aligned} \frac{M_{\frac{t^2}{\alpha}}(x, y)}{M_{t^2}(x, y)} &\leq \alpha^{\frac{d}{2}} \exp\left(\frac{1}{2} \frac{|x-y|^2}{1 - e^{-t^2}}\right) \exp\left(\frac{1}{2} \frac{|x+y|^2}{1 + e^{-t^2}}\right) \\ &\quad \times \exp\left(-\frac{1}{2} \frac{|x-y|^2}{1 - e^{-\frac{t^2}{\alpha}}}\right) \exp\left(-\frac{1}{2} \frac{|x+y|^2}{1 + e^{-\frac{t^2}{\alpha}}}\right) \\ &\leq \alpha^{\frac{d}{2}} \exp\left(\frac{1}{2} \left[1 - \frac{\alpha}{2e^{T^2}}\right] \frac{|x-y|^2}{1 - e^{-t^2}}\right) \exp\left(-\frac{\alpha}{2e^{T^2}} \frac{|x-y|^2}{1 - e^{-t^2}}\right) \\ &\quad \times \exp\left(\frac{1}{2} \frac{|x+y|^2}{1 + e^{-t^2}}\right) \exp\left(-\frac{1}{2} \frac{|x+y|^2}{1 + e^{-\frac{t^2}{\alpha}}}\right). \end{aligned}$$

Thus that,

$$\begin{aligned} \frac{M_{\frac{t^2}{\alpha}}(x, y)}{M_{t^2}(x, y)} &\leq \alpha^{\frac{d}{2}} \exp\left(\frac{1}{2} \left[1 - \frac{\alpha}{2e^{T^2}}\right] \frac{|x-y|^2}{1 - e^{-t^2}}\right) \exp\left(-\frac{\alpha}{2e^{T^2}} \frac{|x-y|^2}{1 - e^{-t^2}}\right) \\ &\quad \times \exp\left(t^2 \frac{|x|^2 + |y|^2}{4}\right). \end{aligned}$$

For  $\alpha \geq 2e^{T^2}$  we then obtain:

$$\frac{M_{\frac{t^2}{\alpha}}(x, y)}{M_{t^2}(x, y)} \leq \alpha^{\frac{d}{2}} \exp\left(-\frac{\alpha}{2e^{T^2}} \frac{|x-y|^2}{1 - e^{-t^2}}\right) \exp\left(\frac{1}{4} t^2 |x|^2\right) \exp\left(\frac{1}{4} t^2 |y|^2\right).$$

From  $t|x| \lesssim 1$  and  $t|y| \lesssim 1$  we infer that there exists a positive constant  $C$  such that  $t|x| \leq C$  and  $t|y| \leq C$ .

$$\frac{M_{\frac{t^2}{\alpha}}(x, y)}{M_{t^2}(x, y)} \leq \alpha^{\frac{d}{2}} e^{\frac{C^2}{2}} \exp\left(-\frac{\alpha}{2e^{T^2}} \frac{|x-y|^2}{1 - e^{-t^2}}\right).$$

Which is as asserted. ■

REMARK. More precisely we have the estimate:

$$\begin{aligned} \exp\left(-\frac{1}{2} \frac{|x+y|^2}{1+e^{-\frac{t^2}{\alpha}}}\right) \exp\left(\frac{1}{2} \frac{|x+y|^2}{1+e^{-t^2}}\right) \\ \leq \exp\left(\frac{1}{4} t^2 |x|^2\right) \exp\left(\frac{1}{4} t^2 |y|^2\right) \exp\left(-\frac{t^2}{\alpha} \frac{1}{8} |x+y|^2\right). \end{aligned}$$

Which then produces:

$$\frac{M_{\frac{t^2}{\alpha}}(x, y)}{M_{t^2}(x, y)} \leq \alpha^{\frac{d}{2}} e^{\frac{c^2}{2}} \exp\left(-\frac{\alpha}{2e^{t^2}} \frac{|x-y|^2}{1-e^{-t^2}}\right) \exp\left(-\frac{t^2}{\alpha} \frac{|x+y|^2}{8}\right).$$

**3.2. On-diagonal kernel estimates on annuli.** As is common in harmonic analysis, we often wish to decompose  $\mathbf{R}^d$  into sets on which certain phenomena are easier to handle. Thus we will decompose the space into annuli  $C_k$ . We will write  $B := B_t(x)$  and assume that  $B$  is the closed ball with center  $x$  and radius  $t$ . Recall that  $2B$  is the ball obtain from  $B$  by multiplying its radius by 2.

The  $C_k$  are given by,

$$(15) \quad C_k(B) := C_k = (2^{k+1} - 1)B \setminus (2^k - 1)B. \begin{cases} 2B & \text{if } k = 0, \\ 2^{k+1}B \setminus 2^k B & \text{for } k \geq 1. \end{cases}$$

So, whenever  $\xi$  is in  $C_k$ , we get for  $k \geq 1$ :

$$(16) \quad 2^k at < |y - \xi| \leq 2^{k+1} at.$$

While we get for  $k = 0$ :

$$(17) \quad |y - \xi| \leq 2at.$$

8. LEMMA. Given  $a > 0$ , let  $B = B_{at}(y)$  and  $\xi$  in  $C_k$ . Furthermore, assume that  $t \leq aAm(y)$  for some  $A > 0$ . Then we have for  $k \geq 1$ :

$$M_{t^2}(y, \xi) \leq \frac{e^{|y|^2}}{(1 - e^{-t^2})^{\frac{d}{2}}} \exp\left(-\frac{a^2}{2} 4^{k+1}\right) \exp(2^{k+1} at |y|).$$

and for  $k = 0$ :

$$M_{t^2}(y, \xi) \leq \frac{e^{|y|^2}}{(1 - e^{-t^2})^{\frac{d}{2}}} \exp(2^{k+1} at |y|).$$

PROOF. Let  $B = B_{at}(y)$  and let  $C_k$  be as in (15). We consider the two exponentials in the Mehler kernel (3) separately. First we consider

$$\exp\left(e^{-t^2} \frac{|y|^2 + |\xi|^2}{1 + e^{-t^2}}\right).$$

Using the triangle inequality we note that:

$$(18) \quad |\xi|^2 \leq |y - \xi|^2 + |y|^2 + 2|y - \xi||y|.$$

Next, note that

$$\frac{e^{-t^2}}{1 + e^{-t^2}} \leq \frac{1}{2}.$$

Together with (18) this gives for  $k \geq 1$ :

$$\begin{aligned} \exp\left(e^{-t^2} \frac{|y|^2 + |\xi|^2}{1 + e^{-t^2}}\right) &\leq \exp\left(e^{-t^2} \frac{|y - \xi|^2}{1 + e^{-t^2}}\right) \exp(|y - \xi||y|) \exp(|y|^2) \\ &\stackrel{(i)}{\leq} \exp\left(e^{-t^2} \frac{|y - \xi|^2}{1 + e^{-t^2}}\right) \exp(2^{k+1} at |y|) \exp(|y|^2) \end{aligned}$$

Where (i) uses (16) or (17). Next we consider the exponential:

$$\exp\left(\frac{e^{-t^2}|y-\xi|^2}{1+e^{-t^2}}\right).$$

Combining this with the first exponential in the Mehler kernel (3) we get:

$$\begin{aligned} \exp\left(-\frac{e^{-t^2}|y-\xi|^2}{1-e^{-2t^2}}\right) \exp\left(\frac{e^{-t^2}|y-\xi|^2}{1+e^{-t^2}}\right) \\ \leq \exp\left(-\frac{e^{-t^2}|y-\xi|^2}{1+e^{-t^2}} \left[\frac{1}{1-e^{-t^2}} - 1\right]\right) \\ \leq \exp\left(-\frac{e^{-t^2}|y-\xi|^2}{1+e^{-t^2}} \left[\frac{1}{1-e^{-t^2}} - \frac{1-e^{-t^2}}{1-e^{-t^2}}\right]\right) \\ \leq \exp\left(-\frac{e^{-2t^2}|y-\xi|^2}{1-e^{-2t^2}}\right). \end{aligned}$$

Using (16) or (17) we get:

$$\exp\left(-\frac{e^{-2t^2}|y-\xi|^2}{1-e^{-2t^2}}\right) \leq \begin{cases} 1 & \text{if } k = 0, \\ \exp\left(-\frac{a^2}{2e^{2t^2}}4^{k+1}\right) & \text{if } k \geq 1. \end{cases}$$

Using the assumption that  $t \leq aAm(y)$  gives that

$$\exp\left(-\frac{a^2}{2e^{2t^2}}4^{k+1}\right) \leq \exp\left(-\frac{a^2}{2e^{2a^2A^2}}4^{k+1}\right).$$

Combining we get for the Mehler kernel (3):

$$\begin{aligned} M_{t^2}(y, \xi) &\leq \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp(2^{k+1}at|y|) \\ &\leq \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp(2^{k+1}a^2A). \end{aligned}$$

This inequality together with

$$\frac{t^2}{1-e^{-2t^2}} \geq \frac{1}{2},$$

yields,

$$\exp\left(-\frac{e^{-t^2}|y-\xi|^2}{1-e^{-2t^2}}\right) \exp\left(-\frac{e^{-t^2}|y-\xi|^2}{1+e^{-t^2}}\right) \leq \exp\left(-\frac{a^2}{2e^{2t^2}}4^{k+1}\right).$$

Thus, we can estimate the Mehler kernel  $M_{t^2}$  on  $C_k$  for  $k \geq 1$  from above by:

$$M_{t^2}(y, \xi) \leq \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp\left(-\frac{a^2}{2}4^{k+1}\right) \exp(2^{k+1}at|y|).$$

We are left with the case  $k = 0$ , which can be done similarly and yields:

$$M_{t^2}(y, \xi) \leq \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp(2^{k+1}at|y|).$$

Done. ■

### 3.3. The Ornstein-Uhlenbeck maximal function.

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