A note on the Gaussian maximal function

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ABSTRACT. In this note we give an improvement on a result first demonstrated by Pineda and Urbina [3]. In particular we present an improvement to their Lemma 1.1 which gives the boundedness of the Gaussian maximal function associated to the Ornstein-Uhlenbeck operator.

We present a proof which is at least to the author more transparant. Our main finding in this note is that our proof allows to use a larger cone and actually obtain the maximal function boundedness for a whole class of cones $\varGamma_x^{(A,a)}(\gamma)$.

1. Introduction

Maximal functions are one of the most studied objects in harmonic analysis. For instance, the classical real-valued harmonic analytic maximal function can be seen to be of the form

$$\sup_{(y,t)\in\varGamma_x} |e^{-t^2\Delta}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \,d\lambda.$$

$$\rho_t(s) := \frac{e^{-|s|^2/4t}}{\pi^{\frac{d}{2}}} \frac{1}{(4t)^{\frac{d}{2}}}$$

(verify) In this note we are interested in gaussian harmonic analysis which is harmonic analysis with respect to the gaussian measure as opposed to the Lebesgue measure in classical real-valued harmonic analysis. As the gaussian maximal function will be of our main interest we

The gaussian maximal function will be our main interest. Formally we are looking for

We define the *Gaussian measure* as:

$$d\gamma(x) := \frac{e^{-|x|^2}}{\pi^{\frac{d}{2}}} \,dx. \tag{1}$$

1.1. Pineda and Urbina [3]. Pineda and Urbina [3] use a different minimal function $\widetilde{m}$, namely:

$$\widetilde{m}(x) = \min\left\{\frac{1}{2}, \frac{1}{|x|}\right\},$$

and their proof does not easily extend the range of t up to 1. Using our notation they have shown that

$$\widetilde{\varGamma}_x(x) = \{(y,t) \in \mathbf{R}_+^d : |x-y| < t \leqslant \widetilde{m}(x)\}.$$

$$\sup_{(y,t)\in\widetilde{\varGamma}_x} |e^{-t^2\Delta}u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \,d\gamma.$$

Our contribution is the extension of the result Pineda and Urbina [3, Lemma 2.1] to the complete Gaussian cone We define:

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}_+^d : |x-y| < At \text{ and } t \leqslant am(x)\}. \tag{2}$$

To ease the notational burden a bit, we will write Γ_x^a and mean $\Gamma_x^{(1,a)}$. Using the following m-function:

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\} = 1 \wedge \frac{1}{|x|}. \tag{3}$$

1.2. Notation. To begin, let us fix some notation. As is common, we use N to represent a positive integer. That is, $N \in \mathbf{Z}_+ = \{1,2,3,\dots\}$. In the same way we cast letters that denote the number of dimensions, e.g. $d \in \mathbf{Z}_+$ as positive integers.

We use the capital letter T to denote a "time" endpoint, for instance, when writing $t \in (0,T]$.

1.2.1. *minimal function.* We recall the lemma from [1, lemma 2.3] which first –although implicitly– appeared in [2]. For what follows it will be convenient to define a function m as:

2. The Mehler kernel

2.1. Setting. We are concerned with the *Ornstein-Uhlenbeck* operator L which is defined as:

$$L := -\frac{1}{2}\Delta + \langle x, \nabla \rangle. \tag{4}$$

We define the Mehler kernel (see e.g., Sjögren [5]) as the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(\mathrm{e}^{-tL})_{t\geqslant 0}$. More precisely, this means:

$$\mathrm{e}^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x,\cdot)u \,\mathrm{d}\gamma. \tag{5}$$

It is often more convenient to use e^{-t^2L} instead of e^{-tL} as is done in e.g., Portal [4].

2.2. The Mehler kernel. For the calculation of the Mehler kernel M_t in (5) we refer to e.g., Sjögren [5] which additionally offers related results such as those related to Hermite polynomials.

The kernel M_t is invariant under the permutation $x \longleftrightarrow y$. A formula for M_t which honors this observation is:

$$M_t(x,y) = \frac{\exp\left(-\mathrm{e}^{-2t}\dfrac{|x-y|^2}{1-\mathrm{e}^{-2t}}\right)}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}} \frac{\exp\left(2\mathrm{e}^{-t}\dfrac{\langle x,y\rangle}{1+\mathrm{e}^{-t}}\right)}{(1+\mathrm{e}^{-t})^{\frac{d}{2}}}. \tag{6}$$

3. Some lemmata and definitions

We use m as defined in (3) in our next lemma.

1. Lemma. *Let a, A be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

(1) If $|x-y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1+aA)m(y)$;
(2) If $|x-y| < Am(x)$, then $m(x) \leqslant (1+A)m(y)$ and $m(y) \leqslant 2(1+A)m(x)$.

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. Lemma. *Let $\alpha > 0$ and $|x-y| \leqslant \alpha m(x)$. We get the equivalence:*

$$\mathrm{e}^{-\alpha^2(1+\alpha)^2}\mathrm{e}^{-2\alpha(1+\alpha)}\mathrm{e}^{-|y|^2} \leqslant \mathrm{e}^{-|x|^2} \leqslant \mathrm{e}^{\alpha^2}\mathrm{e}^{2\alpha}\mathrm{e}^{-|y|^2}.$$

PROOF. By the inverse triangle inequality we get,

$$\begin{aligned}|y|^2 &\leqslant (\alpha m(x)+|x|)^2\\ &= \alpha^2+2\alpha m(x)|x|+|x|^2\\ &\leqslant \alpha^2+2\alpha+|x|^2.\end{aligned}$$

Therefore,

$$e^{-|x|^2} \leqslant e^{-|y|^2}e^{\alpha^2}e^{2\alpha}. \tag{7}$$

For the reverse direction we use Lemma 1 to infer $m(x)\leqslant(1+\alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant \alpha^2(1+\alpha)^2+2\alpha(1+\alpha)+|y|^2.$$

Therefore,

$$e^{-|y|^2} \leqslant e^{-|x|^2}e^{\alpha^2(1+\alpha)^2}e^{2\alpha(1+\alpha)}. \tag{8}$$

Combining we get:

$$e^{-\alpha^2(1+\alpha)^2}e^{-2\alpha(1+\alpha)}e^{-|y|^2} \overset{(8)}{\leqslant} e^{-|x|^2} \overset{(7)}{\leqslant} e^{\alpha^2}e^{2\alpha}e^{-|y|^2}.$$

As required. ■

4. On-diagonal estimates

4.1. Kernel estimates. We begin with a technical lemma which will be useful on several occasions.

3. LEMMA. *Let $t>0$ and $\alpha\geqslant 2$. Then,*

$$\alpha e^{\frac{t^2}{\alpha}}e^{-t^2} \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} \leqslant \alpha, \tag{9}$$

and

$$0 \leqslant \frac{1}{t^2}\left[\frac{1}{1+e^{-t^2}}-\frac{1}{1+e^{-\frac{t^2}{\alpha}}}\right] \leqslant \frac{1}{4}\left(1-\frac{1}{\alpha}\right) \leqslant \frac{1}{8}. \tag{10}$$

PROOF. We start with (9) and apply the mean value theorem to the function $f(\xi)=\xi^\alpha$. For $0<\xi<\xi'$ this gives that:

$$f(\xi)-f(\xi')=\alpha\hat{\xi}^{\alpha-1}(\xi-\xi') \text{ for some } \hat{\xi} \text{ in } [\xi,\xi'].$$

Picking $\xi=1$ and $\xi'=e^{-\frac{t^2}{\alpha}}$ yields:

$$\frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} = \alpha\hat{\xi}^{\alpha-1} \text{ for some } \hat{\xi} \text{ in } \left[\exp\left(-\frac{t^2}{\alpha}\right),1\right]. \tag{11}$$

Combining this result with the monotonicity of $\xi\mapsto\alpha\xi^{\alpha-1}$ we obtain:

$$\alpha\exp\left(-t^2\frac{\alpha-1}{\alpha}\right) \leqslant \frac{1-e^{-t^2}}{1-e^{-\frac{t^2}{\alpha}}} \leqslant \alpha,$$

where the last bound follows from the monotonicity together with the limit as $t\downarrow 0$.

We proceed with (10). Recalling that $\alpha>2$ one can directly verify that the function

$$\frac{1}{t^2}\left[\frac{1}{1+e^{-t^2}}-\frac{1}{1+e^{-\frac{t^2}{\alpha}}}\right]$$

is non-negative and decreasing in t. To find an upper bound we compute the limit as t goes to 0. That is:

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t^2}\left[\frac{1}{1+\mathrm{e}^{-t^2}}-\frac{1}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right] &= \lim_{t\to 0}\frac{1}{2t}\left[\frac{2t\mathrm{e}^{-t^2}}{(1+\mathrm{e}^{-t^2})^2}-\frac{1}{\alpha}\frac{2t\mathrm{e}^{-\frac{t^2}{\alpha}}}{(1+\mathrm{e}^{-\frac{t^2}{\alpha}})^2}\right]\\ &= \lim_{t\to 0}\left[\frac{\mathrm{e}^{-t^2}}{(1+\mathrm{e}^{-t^2})^2}-\frac{1}{\alpha}\frac{\mathrm{e}^{-\frac{t^2}{\alpha}}}{(1+\mathrm{e}^{-\frac{t^2}{\alpha}})^2}\right]\\ &\uparrow \frac{1}{4}\left(1-\frac{1}{\alpha}\right).\end{aligned}$$

Which is as asserted and completes the proof. ■

The following lemma will be useful when transfering estimates from $M_{\frac{t^2}{\alpha}}$ to M_{t^2}. It follows from the mean value theorem applied to $\xi \mapsto \xi^\alpha$.

4. Lemma. *For $\alpha > 1$ and $t \in (0, T]$ and all let $x, y \in \mathbf{R}^d$ we have that:*

$$\exp\left(-\mathrm{e}^{-2\frac{t^2}{\alpha}}\frac{|x-y|^2}{1-\mathrm{e}^{-2\frac{t^2}{\alpha}}}\right) \leqslant \exp\left(-\frac{\alpha}{2\mathrm{e}^{2T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right). \tag{12}$$

Proof. First note that

$$\frac{1}{2\mathrm{e}^{2T^2}} \leqslant \frac{\mathrm{e}^{-2t^2}}{1+\mathrm{e}^{-t^2}} \leqslant \frac{1}{2}.$$

Therefore,

$$\begin{aligned}\exp\left(-\mathrm{e}^{-2t^2}\frac{|x-y|^2}{1-\mathrm{e}^{-2t^2}}\right) &\leqslant \exp\left(-\frac{\mathrm{e}^{-2t^2}}{1+\mathrm{e}^{-t^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right)\\ &\leqslant \exp\left(-\frac{1}{2\mathrm{e}^{2T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right).\end{aligned}$$

Applying (9) we get:

$$\begin{aligned}\exp\left(-\mathrm{e}^{-2\frac{t^2}{\alpha}}\frac{|x-y|^2}{1-\mathrm{e}^{-2\frac{t^2}{\alpha}}}\right) &= \exp\left(-\frac{1}{2\mathrm{e}^{2\frac{T^2}{\alpha}}}\frac{1-\mathrm{e}^{-t^2}}{1-\mathrm{e}^{-\frac{t^2}{\alpha}}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right)\\ &\leqslant \exp\left(-\frac{1}{2\mathrm{e}^{\frac{T^2}{\alpha}}}\frac{\alpha}{\mathrm{e}^{T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right)\\ &\leqslant \exp\left(-\frac{\alpha}{2\mathrm{e}^{2T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right).\end{aligned}$$

Which is as asserted. ■

Our first lemma is about estimating $M_{\frac{t^2}{\alpha}}$ in terms of M_{t^2}.

4.1.1. *Time-scaling of the Mehler kernel.*

5. Lemma. *Let $T > 0$, $\alpha \geqslant 4\mathrm{e}^{2T^2}$, $t \in (0, T]$ and $x, y \in \mathbf{R}^d$. Then:*

$$M_{\frac{t^2}{\alpha}}(x,y) \leqslant \alpha^{\frac{d}{2}}\mathrm{e}^{\frac{t^2}{4}|\langle x,y\rangle|}\exp\left(-\frac{\alpha}{4\mathrm{e}^{2T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right)M_{t^2}(x,y). \tag{13}$$

Proof. To prove the lemma we compute $M_{\frac{t^2}{\alpha}}M_{t^2}^{-1}$. First note that (9) gives

$$\alpha^{\frac{d}{2}}\mathrm{e}^{-\frac{d}{2}T^2} \leqslant \frac{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}}{(1-\mathrm{e}^{-\frac{t^2}{\alpha}})^{\frac{d}{2}}} \leqslant \alpha^{\frac{d}{2}}.$$

Combining the exponentials also gives,

$$\begin{aligned}\exp\left(-2\mathrm{e}^{-\frac{t^2}{\alpha}}\frac{\langle x,y\rangle}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right)\exp\left(2\mathrm{e}^{-t^2}\frac{\langle x,y\rangle}{1+\mathrm{e}^{-t^2}}\right)&\\ &=\left(\frac{2}{t^2}\left[\frac{1}{1+\mathrm{e}^{-t^2}}-\frac{1}{1+\mathrm{e}^{-\frac{t^2}{\alpha}}}\right]t^2\langle x,y\rangle\right)\\ &\overset{(10)}{\leqslant}\mathrm{e}^{\frac{t^2}{4}|\langle x,y\rangle|}.\end{aligned}$$

Combining Lemma 4 and equation (12) gives is almost the final estimate.

$$\begin{aligned}\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)}&\leqslant\alpha^{\frac{d}{2}}\mathrm{e}^{\frac{t^2}{4}|\langle x,y\rangle|}\exp\left(\mathrm{e}^{-2t^2}\frac{|x-y|^2}{1-\mathrm{e}^{-2t^2}}\right)\exp\left(-\mathrm{e}^{-2\frac{t^2}{\alpha}}\frac{|x-y|^2}{1-\mathrm{e}^{-2\frac{t^2}{\alpha}}}\right)\\ &\leqslant\alpha^{\frac{d}{2}}\mathrm{e}^{\frac{t^2}{4}|\langle x,y\rangle|}\exp\left(\frac{|x-y|^2}{1-\mathrm{e}^{-2t^2}}\right)\exp\left(-\mathrm{e}^{-2\frac{t^2}{\alpha}}\frac{|x-y|^2}{1-\mathrm{e}^{-2\frac{t^2}{\alpha}}}\right)\\ &\leqslant\alpha^{\frac{d}{2}}\mathrm{e}^{\frac{t^2}{4}|\langle x,y\rangle|}\exp\left(\left[1-\frac{\alpha}{4\mathrm{e}^{2T^2}}\right]\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right)\exp\left(-\frac{\alpha}{4\mathrm{e}^{2T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right).\end{aligned}$$

Finally, we apply the assumption $\alpha\geqslant 4\mathrm{e}^{2T^2}$ to obtain:

$$\frac{M_{\frac{t^2}{\alpha}}(x,y)}{M_{t^2}(x,y)}\leqslant\alpha^{\frac{d}{2}}\mathrm{e}^{\frac{t^2}{4}|\langle x,y\rangle|}\exp\left(-\frac{\alpha}{4\mathrm{e}^{2T^2}}\frac{|x-y|^2}{1-\mathrm{e}^{-t^2}}\right).$$

Which is as asserted. ■

4.2. An estimate on Gaussian balls.

6. LEMMA. *Let $B_t(x)$ be the Euclidean ball with radius t and center x and let γ be the Gaussian measure (1). We have the inequality:*

$$\frac{\gamma(B_t(x))}{V_d(t)}\leqslant d\pi^{-\frac{d}{2}}\mathrm{e}^{-(t-|x|)^2}. \tag{14}$$

PROOF. Next, remark that for a ball $B:=B_t(x)$ there holds that

$$\begin{aligned}\int_B\mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi&=\mathrm{e}^{-|x|^2}\int_B\mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{-2\langle x,\xi-x\rangle}\,\mathrm{d}\xi\\ &\leqslant\mathrm{e}^{-|x|^2}\int_B\mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{2|x||\xi-x|}\,\mathrm{d}\xi\\ &\leqslant\pi^{\frac{d}{2}}\mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)).\end{aligned}$$

That is:

$$\gamma(B_t(x))\leqslant\mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)). \tag{15}$$

We will estimate the Gaussian volume of the ball $B_t(0)$. To ease the notation, let S_d and V_d be the surface area and volume respectively of the d-dimensional unit sphere. Using polar coordinates we then obtain:

$$\begin{aligned}\gamma(B_t(0))&=\pi^{-\frac{d}{2}}\int_{B_t(0)}\mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi\\ &=S_d\pi^{-\frac{d}{2}}\int_0^t\mathrm{e}^{-r^2}r^{d-1}\,\mathrm{d}r\\ &\leqslant S_d t^d\pi^{-\frac{d}{2}}\mathrm{e}^{-t^2}\\ &=dV_d(t)\pi^{-\frac{d}{2}}\mathrm{e}^{-t^2}.\end{aligned}$$

Upon combining this result with (15) we obtain (14), which is as promised. ■

4.3. On-diagonal kernel estimates on annuli. As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli C_k. For the sake of simplicity we will write $B := B_t(x)$ and mean that B is the closed ball with center x and radius t. Furthermore, we use notations such as $2B$ to mean the ball obtained from B by multiplying its radius by 2.

The C_k are given by,

$$C_k(B) := C_k = (2^{k+1}-1)B \setminus (2^k-1)B. \tag{16}$$

So, whenever ξ is in $C_k(B_t(x))$, we get for $k \geqslant 0$:

$$(2^k-1)t < |y-\xi| \leqslant (2^{k+1}-1)t. \tag{17}$$

7. Lemma. *Given $A > 0$, let $B = B_{At}(y)$, $t \in (0, T]$ and $\xi \in C_k$. Then we have:*

$$M_{t^2}(y,\xi) \leqslant \frac{e^{-\beta}e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)At|y|\big) e^{\beta 2^{k+1}} e^{-\beta 4^k},$$

where $\beta = \frac{A^2}{2e^{2T^2}}$.

Proof. Let $B = B_{At}(y)$ and let C_k be as in (16). Considering the first exponential which occurs in the Mehler kernel (6) together with (17) gives for $k \geqslant 0$:

$$\begin{aligned}\exp\left(-e^{-2t^2}\frac{|y-\xi|^2}{1-e^{-2t^2}}\right) &\leqslant \exp\left(-e^{-2t^2}\frac{(2^k-1)^2A^2t^2}{1-e^{-2t^2}}\right)\\ &\overset{\text{(i)}}{\leqslant} \exp\left(-\frac{A^2}{2e^{2t^2}}(2^k-1)^2\right).\end{aligned}$$

Where (i) follows from

$$\frac{t^2}{1-e^{-2t^2}} \geqslant \frac{1}{2}.$$

Before we consider the last exponential in the Mehler kernel we note that by Cauchy-Schwarz:

$$|\langle y,\xi\rangle| \leqslant |\langle y-\xi, y\rangle| + |\langle y,y\rangle| \leqslant |y-\xi||y| + |y|^2. \tag{18}$$

Furthermore we have the estimate:

$$\frac{e^{-t^2}}{1+e^{-t^2}} \leqslant \frac{1}{2},$$

Using these we get for the last exponential in the Mehler kernel (6):

$$\begin{aligned}\exp\left(2e^{-t^2}\frac{\langle y,\xi\rangle}{1+e^{-t^2}}\right) &\leqslant \exp(|\langle y,\xi\rangle|)\\ &\overset{(18)}{\leqslant} \exp(|y-\xi||y|)e^{|y|^2}.\end{aligned}$$

Wrapping it up, we can estimate the Mehler kernel (6) M_{t^2} on C_k from above by:

$$M_{t^2}(y,\xi) \leqslant \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)At|y|\big) \exp\left(-\frac{A^2}{2e^{2t^2}}(2^k-1)^2\right).$$

Setting $\beta = \frac{A^2}{2e^{2T^2}}$ and expanding the last exponential we get:

$$M_{t^2}(y,\xi) \leqslant \frac{e^{-\beta}e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)At|y|\big) e^{\beta 2^{k+1}} e^{-\beta 4^k}.$$

Which is as claimed.

Lemma 2 gives us by using $|x-y| \leqslant \alpha t \leqslant \alpha^2 m(x)$ the following estimate:

$$e^{|y|^2} \leqslant e^{|x|^2} e^{\alpha^4} e^{2\alpha^2}$$

$$M_{t^2}(y,\xi) \leqslant \frac{e^{-\beta} e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)\alpha(1+\alpha)\big) e^{\beta 2^{k+1}} e^{-\beta 4^k}$$
$$\leqslant e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2} \frac{e^{|x|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp\big(2^{k+1}\alpha(1+\alpha)\big) e^{\beta 2^{k+1}} e^{-\beta 4^k}.$$

Which is as claimed.

■

5. The boundedness of some non-tangential maximal operators

Our theorem is a small modification of [3, lemma 1.1] with a new proof.

1. Theorem. *Let $A, a > 0$. For all x in $\mathbf{R}^d$ and all u in L^2_γ we have*

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |e^{-t^2 L} u(y)| \lesssim \sup_{r>0} \fint_{B_r(x)} |u| \, d\gamma. \tag{19}$$

Proof. First we note that $\Gamma_x^{(A,a)} \subset \Gamma_x^{(1+aA,aA)}$ as $a, A \geqslant 1$.

$$|x-y| \leqslant At \leqslant aAt$$
$$t \leqslant am(x) \leqslant aAm(x)$$

So if $y \in \Gamma_x^{(A,a)}$ then $x \in \Gamma_y^{(aA,aA)}$. So set $\alpha = aA$ and $\Gamma_x^\alpha = \Gamma_x^{(\alpha,\alpha)}$

We will prove (19) by splitting up the integration domain in annuli.

$$e^{-t^2 L}|u(y)| \leqslant \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k(B)} M_{t^2}(y,\cdot)|u| \, d\gamma.$$

More precisely, we will set $B = B(y, aAt)$ in the above and find a suitable upper bound for each integral on the right-hand side which we will denote by I_k for the sake of simplicity.

$$M_{t^2}(y,\xi) \leqslant \frac{e^{-\beta} e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)\alpha t|y|\big) e^{\beta 2^{k+1}} e^{-\beta 4^k},$$

where $\beta = \frac{\alpha^2}{2e^{2\alpha^2}}$.

Since we have $|x-y| < \alpha t$ and $t \leqslant am(x)$ we infer that $t|x| \leqslant \alpha$. By Lemma 1 we also have that $t|y| \leqslant 1+\alpha$. From this and Lemma 7 we infer that:

$$M_{t^2}(y,\xi) \leqslant e^{-\beta} e^{-\alpha(1+\alpha)} \frac{e^{|y|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} \exp\big(2^{k+1}\alpha(1+\alpha)\big) e^{\beta 2^{k+1}} e^{-\beta 4^k}, \tag{20}$$

Setting $\beta = \frac{\alpha^2}{2e^{2\alpha^2}}$. Note that β is maximal for $\alpha = \frac{1}{2}$ and after this value, β is decreasing. Setting $\lambda := \alpha(1+\alpha)$ we get:

$$M_{t^2}(y,\xi) \lesssim_\alpha e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2} \frac{e^{|x|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} e^{(\lambda+\beta)2^{k+1}} e^{-\beta 4^k}. \tag{21}$$

Where the implied constant is given by $e^{-(\alpha+\beta)} e^{\alpha^4} e^{\alpha^2}$

Or, using $\Lambda = \beta + \lambda$ we get:

$$M_{t^2}(y,\xi) \lesssim_\alpha \frac{e^{|x|^2}}{(1-e^{-t^2})^{\frac{d}{2}}} e^{\Lambda 2^{k+1}} e^{-\beta 4^k}, \tag{22}$$

Recalling Lemma 6 we get:

$$\gamma(B_t(x)) \leqslant V_d d \pi^{-\frac{d}{2}} t^d e^{-(t-|x|)^2}. \tag{23}$$

Where we abbreviate $V_d(1)$ with V_d. Recall

$$V_d \leqslant \frac{1}{\sqrt{\pi}}\left(\frac{2\pi e}{d}\right)^{\frac{d}{2}}.$$

To get,

$$\gamma(B_t(x)) \leqslant \frac{d}{\sqrt{\pi}}\left(\frac{2e}{d}\right)^{\frac{d}{2}} t^d \mathrm{e}^{-(t-|x|)^2} = C_d t^d \mathrm{e}^{-(t-|x|)^2}. \tag{24}$$

This allows us to estimate the remaining unbounded exponential in the Mehler kernel and allow a penalty up to $\mathrm{e}^{-|x|^2}$. Furthermore, we have the following estimate which will make clear how to handle the time part in the Mehler kernel:

$$\frac{t^d}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \leqslant \left(\frac{t^2}{1-\mathrm{e}^{-t^2}}\right)^{\frac{d}{2}} \leqslant \frac{a^d}{(1-\mathrm{e}^{-a^2})^{\frac{d}{2}}}.$$

Let $B' := B(x, 2^{k+1}\alpha t)$ and B as before the ball $B(y, \alpha t)$. In the next step we will bound .... by the maximal function centered at x. For this we need to scale up the C_k. So,

$$|x-\xi| \leqslant |x-y| + |\xi - y| \leqslant \alpha t + (2^{k+1}-1)\alpha t = 2^{k+1}\alpha t.$$

And set $D_k = B(2^{k+1}\alpha t)$. So, we can bound the integral on the right-hand side of (**??**) by

$$\begin{aligned}
\int_{C_k(B)} M_{t^2}(y,\cdot)|u| \, \mathrm{d}\gamma &\lesssim_\alpha \frac{\mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2} \int_{C_k(B)} |u| \, \mathrm{d}\gamma \\
&\leqslant \frac{\mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2} \int_{D_k(B)} |u| \, \mathrm{d}\gamma \\
&\leqslant (M_\gamma u)(x) \frac{\mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2} \gamma(D_k) \\
&\overset{\text{(i)}}{\leqslant} (M_\gamma u)(x) C_d \alpha^d 2^{d(k+1)} t^d \frac{\mathrm{e}^{\Lambda 2^{k+1}}\mathrm{e}^{-\beta 4^k}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} \mathrm{e}^{|x|^2} \mathrm{e}^{-(t-|x|)^2} \\
&\leqslant (M_\gamma u)(x) 2^{kd} \mathrm{e}^{\Lambda 2^{k+1}} \mathrm{e}^{-\beta 4^k} \frac{t^d \mathrm{e}^{-t^2}}{(1-\mathrm{e}^{-t^2})^{\frac{d}{2}}} C_d \mathrm{e}^{2\alpha}(2\alpha)^d
\end{aligned}$$

Where (i) uses Lemma 6 and $t|x| \leqslant a$.

We can then bound the maximal function:

$$\begin{aligned}
\mathrm{e}^{-t^2 L}|u(y)| &= \sum_{k=0}^{\infty} I_k \\
&\leqslant (M_\gamma u)(x) C_{d,a,A} \sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{\Lambda 2^{k+1}} \mathrm{e}^{-\beta 4^k}
\end{aligned}$$

Wrapping it up, we have that:

$$\mathrm{e}^{-t^2 L}|u(y)| \lesssim \fint_{B_r(x)} |u| \, \mathrm{d}\gamma.$$

With implied constant

Which is what we wanted to prove.

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} = \sum_{k=0}^{\infty} x^{kd} \mathrm{e}^{-Cx^{2k}}$$

Noting for $x \geqslant 1$ that $\exp(-Cx^{2k}) \leqslant \exp(-Ckx^2)$, thus,

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \sum_{k=0}^{\infty} x^{kd} (\mathrm{e}^{-Cx^2})^k = \sum_{k=0}^{\infty} (x^d \mathrm{e}^{-Cx^2})^k$$

Here $x = 2$, so

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \sum_{k=0}^{\infty} (2^d \mathrm{e}^{-4C})^k$$

If $2^d < \mathrm{e}^{4C}$, that is whenever $d \log 2 < 4C$, we can compute using the geometric series that

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \frac{1}{1 - 2^d \mathrm{e}^{-4C}} = \frac{\mathrm{e}^{4C}}{\mathrm{e}^{4C} - 2^d}$$

■

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