# A NOTE ON THE GAUSSIAN MAXIMAL FUNCTION - VERSION 20 OCTOBER 2013 + JVN ADDITIONS

JONAS TEUWEN

Abstract. This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded almost surely by the Gaussian Hardy-Littlewood maximal function. In particular this entails improvement on a result by Pineda and Urbina [2] who proved a similar result for a 'trunctated' version of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

## 1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well known that the classical non-tangential maximal function associated with the heat semigroup is bounded almost everywhere by the Hardy-Littlewood maximal function,

$$\sup_{\substack{(y,t)\in\mathbf{R}^{d+1}_+\\ |x-y|<t}} |\mathrm{e}^{-t\Delta}u(y)| \lesssim \sup_{r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)}|u|\,\mathrm{d}\lambda. \tag{1}$$

Here the action of *heat semigroup* $\mathrm{e}^{-t\Delta}u = \rho_t * u$ is given by a convolution of $u$ with the *heat kernel*

$$\rho_t(s) := \frac{\mathrm{e}^{-|s|^2/4t}}{(4\pi t)^{\frac{d}{2}}}.$$

In this note we are interested in its Gaussian counterpart. The change from Lebesgue measure to the Gaussian measure

$$\mathrm{d}\gamma(x) := \pi^{-\frac{d}{2}}\mathrm{e}^{-|x|^2}\,\mathrm{d}x \tag{2}$$

introduces quite some intricate technical and conceptual difficulties which are caused by the fact that the Gaussian measure is non-doubling. The Gaussian analogue to the Laplacian is the *Ornstein-Uhlenbeck operator* $L$,

$$L := -\frac{1}{2}\Delta + \langle x,\nabla\rangle = \frac{1}{2}\nabla_\gamma^*\nabla_\gamma, \tag{3}$$

where $\nabla_\gamma$ denotes the realisation of the gradient in $L^2(\mathbf{R}^d,\gamma)$. Our main result, to be proved in Theorem 1, is the following Gaussian analogue of (1):

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |\mathrm{e}^{-t^2L}u(y)| \lesssim \sup_{r>0}\frac{1}{\gamma(B_r(x))}\int_{B_r(x)}|u|\,\mathrm{d}\gamma. \tag{4}$$

*Date*: October 22, 2013.

Here,

$$\Gamma_x^{(A,a)} := \Gamma_x^{(A,a)}(\gamma) := \{(y,t) \in \mathbf{R}_+^{d+1} \,:\, |x-y| < At \text{ and } t \leqslant am(x)\} \tag{5}$$

is the Gaussian cone with aperture $A$ and cut-off parameter $a$, and

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\}. \tag{6}$$

A slighly weaker version of the inequality (4) has been proved by Pineda and Urbina [2] who showed that

$$\sup_{(y,t)\in\widetilde{\Gamma}_x} |\mathrm{e}^{-t^2L}u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \,\mathrm{d}\gamma,$$

where

$$\widetilde{\Gamma}_x(x) = \{(y,t) \in \mathbf{R}_+^d : |x-y| < t \leqslant \widetilde{m}(x)\}$$

is the 'reduced' Gaussian cone corresponding to the function

$$\widetilde{m}(x) = \min\left\{\frac{1}{2}, \frac{1}{|x|}\right\}.$$

Their proof does not seem to easily generalize the range of $t$ from $\frac{1}{2}$ up to 1. Our proof of (4) is different and, we believe, more transparent than the one presented in [2]. It has the further advantage of allowing the extension to cones with arbitrary aperture $A > 0$ and cut-off parameter $a > 0$ without any additional technicalities. This additional generality is very important and has already been used by Portal (cf. the claim made in [3, discussion preceding Lemma 2.3]) to prove the $H^1$-boundedness of the Riesz transform associated with $L$.

To save writing, let us fix some notation. The number $d$ is a positive integer. To avoid possible confusion, we define the *positive integers* as the set $\mathbf{Z}_+ = \{1, 2, 3, \dots\}$.

## 2. The Mehler kernel

The Mehler kernel (see e.g., [4]) is the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup $(\mathrm{e}^{-tL})_{t\geqslant 0}$, that is,

$$\mathrm{e}^{-tL}u(x) = \int_{\mathbf{R}^d} M_t(x,\cdot)u \,\mathrm{d}\gamma. \tag{7}$$

There is an abundance of literature on the Mehler kernel and its properties. We shall only use the fact, proved e.g. in the review paper [4], that the Mehler kernel is given explicitly by

$$M_t(x,y) = \frac{\exp\left(-\dfrac{|\mathrm{e}^{-t}x-y|^2}{1-\mathrm{e}^{-2t}}\right)}{(1-\mathrm{e}^{-2t})^{\frac{d}{2}}}\mathrm{e}^{|y|^2}.$$

Note that $M_t(x,y)$ is symmetric in $x$ and $y$. A formula for $M_t$ which honors this observation is:

$$M_t(x,y) = \frac{\exp\left(-\mathrm{e}^{-2t}\dfrac{|x-y|^2}{1-\mathrm{e}^{-2t}}\right)}{(1-\mathrm{e}^{-t})^{\frac{d}{2}}}\frac{\exp\left(2\mathrm{e}^{-t}\dfrac{\langle x,y\rangle}{1+\mathrm{e}^{-t}}\right)}{(1+\mathrm{e}^{-t})^{\frac{d}{2}}}. \tag{8}$$

## 3. Some lemmata

We use $m$ as defined in (6) in our next lemma, which is taken from [1].

1. **Lemma.** *Let $a, A$ be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbf{R}^d$ that:*

*(1) If $|x-y| < At$ and $t \leqslant am(x)$, then $t \leqslant (1+aA)m(y)$;*
*(2) If $|x-y| < Am(x)$, then $m(x) \leqslant (1+A)m(y)$ and $m(y) \leqslant 2(1+A)m(x)$.*

The next lemma will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone.

2. **Lemma.** *Let $\alpha > 0$ and $|x-y| \leqslant \alpha m(x)$. Then:*

$$\mathrm{e}^{-\alpha^2-2\alpha}\mathrm{e}^{|y|^2} \leqslant \mathrm{e}^{|x|^2} \leqslant \mathrm{e}^{\alpha^2(1+\alpha)^2+2\alpha(1+\alpha)}\mathrm{e}^{|y|^2}.$$

*Proof.* By the inverse triangle inequality and $m(x)|x| \leqslant 1$ we get,

$$|y|^2 \leqslant (\alpha m(x) + |x|)^2 \leqslant \alpha^2 + 2\alpha + |x|^2.$$

This gives the first inequality. For the second we use Lemma 1 to infer $m(x) \leqslant (1+\alpha)m(y)$. Proceeding as before we obtain:

$$|x|^2 \leqslant \alpha^2(1+\alpha)^2 + 2\alpha(1+\alpha) + |y|^2.$$

As required. ∎

### 3.1. An estimate on Gaussian balls.

Let $B = B_t(x)$ be the open Euclidean ball with radius $t$ and center $x$ and let $\gamma$ be the Gaussian measure as defined by (2). We shall denote by $S_d$ the surface area of the unit sphere in $\mathbf{R}^d$.

3. **Lemma.** *For all $x \in \mathbf{R}^d$ and $t > 0$ we have the inequality:*

$$\gamma(B_t(x)) \leqslant \frac{1}{2} S_d t^d \mathrm{e}^{2t|x|}\mathrm{e}^{-|x|^2}. \tag{9}$$

*Proof.* Remark that

$$\begin{aligned}\int_B \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi &= \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{-2\langle x,\xi-x\rangle}\,\mathrm{d}\xi \\ &\leqslant \mathrm{e}^{-|x|^2}\int_B \mathrm{e}^{-|\xi-x|^2}\mathrm{e}^{2|x||\xi-x|}\,\mathrm{d}\xi \\ &\leqslant \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\int_B \mathrm{e}^{-|\xi-x|^2}\,\mathrm{d}\xi \\ &= \pi^{\frac{d}{2}}\mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)).\end{aligned}$$

So, there holds that

$$\gamma(B_t(x)) \leqslant \mathrm{e}^{-|x|^2}\mathrm{e}^{2t|x|}\gamma(B_t(0)). \tag{10}$$

We will estimate the Gaussian volume of the ball $B_t(0)$. Using polar coordinates we proceed for $d \geqslant 2$ by:

$$\begin{aligned}\gamma(B_t(0)) &= \pi^{-\frac{d}{2}} \int_{B_t(0)} \mathrm{e}^{-|\xi|^2}\,\mathrm{d}\xi \\ &= \pi^{-\frac{d}{2}} S_d \int_0^t \mathrm{e}^{-r^2} r^{d-1}\,\mathrm{d}r \\ &\leqslant \frac{1}{2}\pi^{-\frac{d}{2}} S_d t^{d-2} \int_0^t 2r\mathrm{e}^{-r^2}\,\mathrm{d}r \\ &= \frac{1}{2}\pi^{-\frac{d}{2}} S_d t^{d-2}(1-\mathrm{e}^{-t^2}) \\ &\leqslant \frac{1}{2}\pi^{-\frac{d}{2}} S_d t^{d},\end{aligned}$$

where the last step uses $1-\mathrm{e}^{-x} \leqslant x$ for $x \geqslant 0$. The case for $d = 1$ follows by a simplified argument. Upon combining this result with (10) we obtain (9). ■

3.2. **On-diagonal kernel estimates on annuli.** As is common in harmonic analysis, we often wish to decompose $\mathbf{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli.

Throughout this subsection we fix $x \in \mathbf{R}^d$, constants $A, a > 0$, a pair $(y,t) \in \Gamma_x^{(A,a)}$, and we put $\alpha := Aa$. We use the notation $rB$ to mean the ball obtained from the ball $B$ by multiplying its radius by $r$.

The annuli $C_k$ are given by:

$$C_k := (2^{k+1}-1)B_{\alpha t}(x) \setminus (2^k-1)B_{\alpha t}(x) \text{ with } k \geqslant 0. \tag{11}$$

Note that $C_0 = B_{\alpha t}(x)$. Whenever $\xi$ is in $C_k$, we get for $k \geqslant 0$:

$$(2^k-1)\alpha t < |y-\xi| \leqslant (2^{k+1}-1)\alpha t. \tag{12}$$

On $C_k$ we have the following bound for $M_{t^2}(y,\cdot)$:

4. **Lemma.** *For all $\xi \in C_k$ we have:*

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)\alpha t|y|\big) \exp\Big(-\frac{\alpha^2}{2\mathrm{e}^{2a^2}}(2^k-1)^2\Big), \tag{13}$$

*Proof.* Considering the first exponential which occurs in the Mehler kernel (8) together with (12) gives for $k \geqslant 0$:

$$\begin{aligned}\exp\left(-\mathrm{e}^{-2t^2}\frac{|y-\xi|^2}{1-\mathrm{e}^{-2t^2}}\right) &\leqslant \exp\left(-\mathrm{e}^{-2t^2}\frac{(2^k-1)^2\alpha^2 t^2}{1-\mathrm{e}^{-2t^2}}\right) \\ &\overset{(\dagger)}{\leqslant} \exp\left(-\frac{\alpha^2}{2\mathrm{e}^{2t^2}}(2^k-1)^2\right) \overset{(\dagger\dagger)}{\leqslant} \exp\left(-\frac{\alpha^2}{2\mathrm{e}^{2a^2}}(2^k-1)^2\right),\end{aligned}$$

where ($\dagger$) follows from $1-\mathrm{e}^{-x} \leqslant x$ for $x \geqslant 0$, and ($\dagger\dagger$) uses that $t \leqslant am(x) \leqslant a$. Using the estimate $1 + x \geqslant 2x$ for $0 \leqslant x \leqslant 1$, for the second exponential in the Mehler kernel (8) we obtain, by (12):

$$\begin{aligned}\exp\left(2\mathrm{e}^{-t^2}\frac{\langle y,\xi\rangle}{1+\mathrm{e}^{-t^2}}\right) &\leqslant \exp(|\langle y,\xi\rangle|) \\ &\leqslant \exp(|\langle y,\xi-y\rangle|)\mathrm{e}^{|y|^2} \\ &\leqslant \exp\big((2^{k+1}-1)\alpha t|y|\big)\mathrm{e}^{|y|^2}\end{aligned}$$

Combining things, we obtain the estimate in the formulation of the lemma. ■

## 4. The main result

Our theorem is a small modification of [2, lemma 1.1] with a new proof.

1. **Theorem.** *Let $A, a > 0$. For all $x \in \mathbf{R}^d$ and all $u \in L^2(\mathbf{R}^d, \gamma)$ we have*

$$\sup_{(y,t)\in\Gamma_x^{(A,a)}} |\mathrm{e}^{-t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, \mathrm{d}\gamma. \tag{14}$$

*Proof.* We fix $x \in \mathbf{R}^d$ and $(y,t) \in \Gamma_x^{(A,a)}$. Set $\alpha = aA$. We will prove (14) by splitting up the integration domain into the annuli $C_k$ as defined by (11):

$$|\mathrm{e}^{-t^2 L} u(y)| \leqslant \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k} M_{t^2}(y,\cdot)|u(\cdot)| \, \mathrm{d}\gamma. \tag{15}$$

From $t \leqslant am(x)$ we get $t|x| \leqslant a$, and by Lemma 1 we have $t|y| \leqslant 1 + \alpha$. From this and Lemma 4 we infer, for $\xi \in C_k$, that:

$$M_{t^2}(y,\xi) \leqslant \frac{\mathrm{e}^{|y|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \exp\big((2^{k+1}-1)\alpha(1+\alpha)\big) \exp\Big(-\frac{\alpha^2}{2\mathrm{e}^{2a^2}}(2^k-1)^2\Big). \tag{16}$$

Combining (16) and Lemma 2, we obtain

$$M_{t^2}(y,\xi) \leqslant c_k \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}}$$

where $c_k$ depends only on $A$, $a$, and $t$.

Also, by (12),

$$|x-\xi| \leqslant |x-y| + |\xi - y| \leqslant (A + 2^{k+1}\alpha)t.$$

It follows that $C_k$ is contained in $D_k := B_{(A+2^{k+1}\alpha)t}(x)$.

Let us denote the supremum on right-hand side of (14) by $M_\gamma u(x)$. Using (16), we can bound the integral on the right-hand side of (15) by

$$\begin{aligned}
\int_{C_k} M_{t^2}(y,\cdot)|u(\cdot)| \, \mathrm{d}\gamma &\leqslant c_k \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \int_{C_k} |u| \, \mathrm{d}\gamma \\
&\leqslant c_k \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \int_{D_k} |u| \, \mathrm{d}\gamma \\
&\leqslant c_k \frac{\mathrm{e}^{|x|^2}}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \gamma(D_k) M_\gamma u(x) \\
&\overset{(\dagger)}{\lesssim}_{A,a,d} c_k \frac{t^d}{(1-\mathrm{e}^{-2t^2})^{\frac{d}{2}}} \mathrm{e}^{2((A+2^{k+1}\alpha)t)|x|} M_\gamma u(x) \\
&\overset{(\dagger\dagger)}{\lesssim}_{A,a,d} c_k \mathrm{e}^{2^{k+2}\alpha a} M_\gamma u(x),
\end{aligned}$$

where (†) uses Lemma 3 applied to $D_k$ and (††) uses that $t \leqslant am(x)$ implies $t|x| \leqslant a$ and $t \leqslant a$, the latter implying

$$\Big(\frac{t^2}{1-\mathrm{e}^{-t^2}}\Big)^{\frac{d}{2}} \leqslant \Big(\frac{a^2}{1-\mathrm{e}^{-a^2}}\Big)^{\frac{d}{2}}$$

(note that $x/(1-\mathrm{e}^{-x})$ is increasing).

Inserting the depenceny of $c_k$ upon $k$ as coming from (16) and using that $t \leqslant a$, we can then bound the maximal function as follows:

$$|\mathrm{e}^{-t^2 L} u(y)| = \sum_{k=0}^{\infty} I_k \lesssim_{A,a,d} M_\gamma u(x) \sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{\alpha(1+\alpha)2^{k+1}} \mathrm{e}^{-\frac{\alpha^2}{2\mathrm{e}^2 a^2} 4^k} \mathrm{e}^{2^{k+2}\alpha a}$$

Evidently the sum on the right-hand side converges. ■

(Tot hier alles gecontroleerd)

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} = \sum_{k=0}^{\infty} x^{kd} \mathrm{e}^{-Cx^{2k}}$$

Noting for $x \geqslant 1$ that $\exp(-Cx^{2k}) \leqslant \exp(-Ckx^2)$, thus,

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \sum_{k=0}^{\infty} x^{kd} (\mathrm{e}^{-Cx^2})^k = \sum_{k=0}^{\infty} (x^d \mathrm{e}^{-Cx^2})^k$$

Here $x = 2$, so

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \sum_{k=0}^{\infty} (2^d \mathrm{e}^{-4C})^k$$

If $2^d < \mathrm{e}^{4C}$, that is whenever $d \log 2 < 4C$, we can compute using the geometric series that

$$\sum_{k=0}^{\infty} 2^{kd} \mathrm{e}^{-C4^k} \leqslant \frac{1}{1 - 2^d \mathrm{e}^{-4C}} = \frac{\mathrm{e}^{4C}}{\mathrm{e}^{4C} - 2^d}$$

## References


1. Jan Maas, Jan Neerven, and Pierre Portal, *Whitney coverings and the tent spaces $T^{1,q}(\gamma)$ for the Gaussian measure*, Arkiv för Matematik **50** (2011), no. 2, 379–395.
2. Ebner Pineda and Wilfredo R. Urbina, *Non Tangential Convergence for the Ornstein-Uhlenbeck Semigroup*, Divulgaciones Matemáticas **13** (2008), no. 2, 1–19.
3. Pierre Portal, *Maximal and quadratic Gaussian Hardy spaces*, (2012).
4. Peter Sjögren, *Operators associated with the hermite semigroup — A survey*, The Journal of Fourier Analysis and Applications **3** (1997), no. S1, 813–823.



Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

*E-mail address*: j.j.b.teuwen@tudelft.nl

*URL*: http://fa.its.tudelft.nl/~teuwen/