Gradient Flows notes

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1 Hilbert space theory

1.1 "Gradient flows" on a Hilbert space

Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space with norm $|\cdot|$. Now let $\phi : H \to \mathbf{R}$ and recall that ϕ is Fréchet differentiable at x in H if there exists a bounded operator x^* on H such that

$$\phi(x+h) - \phi(x) = x^*(h) + o(|h|)$$

If such an x^* exists, then it will be unique and will be called the gradient of ϕ at x. So, according to the Riesz representation theorem there is an unique y in H such that $\langle y, h\rangle = x^*(h)$ for all h in H. Further we have $\|x^*\| = |y|$. This y will also be called the gradient of ϕ at x. We will denote this one as $\nabla\phi(x)$. Now if ϕ is differentiable at every x in H and the map $\nabla\phi$ from H into itself is continuous, we say that ϕ is continuously differentiable and we write this as $\phi \in C^1(H, \mathbf{R})$. Further if $\nabla\phi$ is Lipschitz, then ϕ is said to be in $C^{1,1}(H, \mathbf{R})$.

Now let $\phi \in C^{1,1}(H, \mathbf{R})$ and let $\{S(t)\}_{t\in\mathbf{R}}$ be the group of operators associated with $F = \nabla\phi$. That is, we solve $\dot{u}(t) = F(u(t))$ for $t \in \mathbf{R}$ with $u(0) = x$. So, then we define $\Phi(t, x) = u_x(t)$. Now the semigroup of operators is given by $S(t)x := \Phi(t, x)$. Now clearly the orbits $t \mapsto S(t)x$ are continuously differentiable and so is the map $t \mapsto \phi(S(t)x)$. Further by the definition of S

$$\frac{d}{dt}\phi(S(t)x) = \left\langle \nabla\phi(S(t)x), \frac{d}{dt}S(t)x \right\rangle = |\nabla\phi(S(t)x)|^2 \geqslant 0$$

Hence, $t \mapsto \phi(S(t)x)$ is nondecreasing. We could as well do this with $\tilde{S}(t) = S(-t)$, then $t \mapsto \phi(\tilde{S}(t)x)$ nonincreasing. We also call this a gradient flow. In the sequel we will consider (semi)-flows associated with $-\nabla\phi$.

1.1 Lemma. *Let $\psi : H \to \mathbf{R}$ be convex and Fréchet differentiable at $x \in H$. Further let $y \in H$. Now the following statements are equivalent:*

1. *$y = \nabla\psi(x)$,*
2. *$\langle y, h\rangle + \psi(x) \leqslant \psi(x+h)$ for every $h \in H$.*

Remark: For a function $\psi : D(\psi) \subset H \to \mathbf{R}$ and every $x \in D(\psi)$ we say that $y \in H$ is a subgradient of ψ at x if

[1.1] $$\langle y, z - x\rangle + \psi(x) \leqslant \psi(z) \text{ for every } z \in D(\psi).$$

The collection of all subgradients of ψ at x is called the subdifferential of ψ at x and is denoted by $\partial\psi(x)$.

Proof. 1) $\implies$ 2): Let $x_1, x_2 \in H$. The convexity of ψ implies the convexity of $t \mapsto \psi(x_1 + tx_2)$. It follows that the difference quotient

$$t \mapsto \frac{\psi(x_2 + tx_2) - \psi(x_2)}{t}$$

is nondecreasing. We can see this by noting that $x_1 + tx_2 = \frac{t}{t'}(x_1 + t'x_2) + \frac{t'-t}{t'}x_1$. Now if we choose $x_1 = x$ and $x_2 = h$ we have by the chain rule that

[1.2] $$\begin{aligned}\langle y, h\rangle = \langle \nabla\psi(x), h\rangle &= \lim_{t\downarrow 0}\frac{\psi(x+th) - \psi(x)}{t} = \inf_{t\downarrow 0}\frac{\psi(x+th)-\psi(x)}{t} \\ &\leqslant \psi(x+h) - \psi(x)\end{aligned}$$

2) $\implies$ 1): If we replace h with th in 2) with $t > 0$ we obtain

$$\langle y, h\rangle \leqslant \frac{\psi(x+th) - \psi(x)}{t}$$

so taking the limit $t \to 0$ we get $\langle y, h\rangle \leqslant \langle \nabla\psi(x), h\rangle$. If we replace h by $-h$ we reach equality. So now if we set $h = y - \nabla\psi(x)$ we get

$$\langle y, y - \nabla\psi(x)\rangle = \langle \nabla\psi(x), y - \nabla\psi(x)\rangle$$

So we get $y = \nabla\psi(x)$. This implies 1). ∎

1.2 Corollary. *If $u \in C^1((a,b), H)$ for some $a, b \in R$, $a < b$ and $\psi : H \to \mathbf{R}$ is everywhere Fréchet differentiable and convex, then*

$$\dot{u}(t) = -\nabla\psi(u(t)), \ t \text{ in } (a,b)$$

iff

$$\frac{1}{2}\frac{d}{dt}d(u(t), z)^2 + \psi(u(t)) \leqslant \psi(z) \text{ for every } z \in H,\ t \in (a,b)$$

Proof. By the previous Lemma we have

$$\langle \dot{u}(t), z - u(t)\rangle + \psi(u(t)) \leqslant \psi(z) \text{ for every } z \in H, t \in (a,b).$$

Which is what we want since

$$\frac{d}{dt}|u(t) - z|^2 = 2\langle \dot{u}(t), u(t) - z\rangle$$

∎

We can now consider a slightly more general situation. We set $e(x) = \frac{1}{2}|x|^2$ for $x \in H$. So we now have that

[1.3] $$\nabla e(x) = x, \quad e(x-y) = \frac{1}{2}d(x,y)^2, \text{ for } x, y \in H$$

1.3 Proposition. *Let $\phi : H \to \mathbf{R}$ be everywhere Fréchet differentiable such that $\phi - \alpha e$ is convex for some $\alpha \in \mathbf{R}$. Further, let J be a nonempty interval of $\mathbf{R}$ and $u \in C^1(J, H)$. Then the following are equivalent:*

1. *$\dot{u}(t) = -\nabla\phi(u(t))$ for $t \in J$,*
2. *$\frac{1}{2}\frac{d}{dt}d(u(t), z)^2 + \frac{\alpha}{2}d(u(t), z)^2 + \phi(u(t)) \leqslant \phi(z)$ for every $z \in H$ and $t \in J$. This inequality is called the evolution variational inequality.*

Proof. Let $\psi = \phi - \alpha e$. Now 1) is equivalent to $\nabla\psi(u(t)) = -\dot{u}(t) - \alpha u(t)$. By Lemma 1.1 this is equivalent to

$$\langle -\dot{u}(t) - \alpha u(t), z - u(t)\rangle + \psi(u(t)) \leqslant \psi(z) \text{ for all } z \in H$$

Now, we can use the definition of ψ we get

$$\langle -\dot{u}(t), z - u(t)\rangle - \alpha\langle u(t), z - u(t)\rangle + \phi(u(t)) - \frac{\alpha}{2}|u(t)|^2 \leqslant \phi(z) - \frac{\alpha}{2}|z|^2 \text{ for } z \in H.$$

Grouping terms together and using that $\frac{d}{dt}|u(t) - z|^2 = 2\langle \dot{u}(t), u(t) - z\rangle$, $d(u(t), z)^2 = |u(t)|^2 - 2\langle u(t), z\rangle + |z|^2$ we get

$$\frac{1}{2}d(u(t), z)^2 + \underbrace{\frac{\alpha}{2}|u(t)|^2 - \alpha\langle u(t), z\rangle + \frac{\alpha}{2}|z|^2}_{d(u(t),z)^2} + \phi(u(t)) \leqslant \phi(z) \text{ for } z \in H.$$

■

Now it would be nice if $\phi \in C^{1,1}(H, \mathbf{R})$ there would exist an $\alpha \in \mathbf{R}$ such that $\phi - \alpha e$ is convex (ϕ is said to be α-convex). This is the case

1.4 Lemma. *Let $\psi : H \to \mathbf{R}$ be everywhere Fréchet differentiable, then ϕ is convex iff $\nabla\psi$ is monotone, that is if*

$$\langle \nabla\psi(x_1), \nabla\psi(x_2), x_1 - x_2\rangle \geqslant 0 \text{ for all } x_1, x_2 \in H.$$

Proof. $\Longrightarrow$: Let ψ be convex and let $x_1, x_2 \in H$, set $y_1 = \nabla\psi(x_1)$ and $y_2 = \nabla\psi(x_2)$. From Lemma 1.1 we obtain $\langle y_i, h\rangle + \psi(x_i) \leqslant \psi(x_i + h)$ for $i = 1, 2$ and $h \in H$. For $i = 1$, choose $h = x_2 - x_1$ and for $i = 2$ choose $h = x_1 - x_2$. Adding both inequalities

$$\langle y_1, x_2 - x_1\rangle + \langle y_2, x_1 - x_2\rangle \leqslant 0 \text{ thus } \langle y_2 - y_1, x_2 - x_1\rangle \geqslant 0$$

$\Leftarrow$: Let $\nabla\psi$ be monotone and let $x, y \in H$ and $t \in \mathbf{R}$. Define

$$\alpha(t) := \psi((1 - t)x + ty) - (1 - t)\psi(x) - t\psi(y)$$

Now $\alpha(0) = \alpha(1) = 0$ and α is differentiable

$$\alpha'(t) = \langle \nabla\psi((1 - t)x + ty), y - x\rangle + \psi(x) - \psi(y).$$

Now let $t_1 < t_2$. Note that $[(1-t_2)x + t_2 y] - [(1-t_1)x + t_1 y] = (t_2 - t_1)(y - x)$. We have

$$\begin{aligned} \alpha'(t_2) - \alpha'(t_1) = \langle \nabla\psi((1-t_2)x + t_2 y) - \nabla\psi((1-t_1)x + t_1 y) \\ [(1-t_2)x + t_2 y] - [(1-t_1)x + t_1 y]\rangle \cdot \frac{1}{t_2 - t_1} \geqslant 0. \end{aligned} \tag{1.4}$$

From this we conclude that α' is nondecreasing. So now if α had a maximum in $\xi = (0,1)$, then $\alpha'(\xi) = 0$. So by the mean value theorem there exists $\zeta \in (t, \xi)$ such that $\alpha(\xi) - \alpha(t) = \alpha'(\zeta)(\xi - t) \geqslant 0$. So α is nonincreasing on (t, ϵ). By a similar argument α is nondecreasing for $t > \xi$, which is a contradiction because then α would not be maximal in ξ. So. $\alpha(t) \geqslant 0$ this ψ is convex. ■

Now by Cauchy-Schwarz we have $\langle \nabla\psi(x_2) - \nabla\psi(x_1), x_2 - x_1\rangle \geqslant [\nabla\psi]_{\mathrm{Lip}} |x_2 - x_1|^2$ for all $x_1, x_2 \in H$. Now, for the correct α we can make $\psi - \alpha e$ convex by Lemma 1.4. We summarize this

1.5 Proposition. *Let $\phi : H \to \mathbf{R}$ be such that $\phi - \alpha e$ is convex for some $\alpha \in \mathbf{R}$. If we have that for every $\phi \in C^{1,1}(H, \mathbf{R})$, then for every $x \in H$ there is a unique function $u \in C^1(\mathbf{R}, H)$ satisfying the EVI together with $u(0) = x$. Moreover if $u_1, u_2 \in C^1(\mathbf{R}, H)$ satisfy the EVI with $J = \mathbf{R}$, then*

$$d(u_1(t), u_2(t)) \leqslant e^{-\alpha(t-s)} d(u_1(s), u_2(s))$$

for every $s < t$, $s, t \in \mathbf{R}$.

1.2 Uniqueness and a priori estimates

A function $\phi : X \to (-\infty, \infty]$ is called *proper* if its effective domain $D(\phi) := \{x \in X : \phi(x) < \infty\}$ is non-empty. A proper function is called *lower semicontinuous* (lsc) at $x \in X$ if for every sequence (x_n) converging to x we have that $\phi(x) \leqslant \liminf_n \phi(x_n)$. So ϕ is lsc at x if for every $\epsilon > 0$ there exist $\delta > 0$ such that $\phi(y) \geqslant \phi(x) - \epsilon$ for every $y \in X$ such that $d(x, y) \leqslant \delta$. A function is everywhere lsc iff for every $c \in \mathbf{R}$ we have that $\{x \in X : \phi(x) \leqslant c\}$. A lsc function on a compact metric space is bounded from below and attains its minimum.

A function $u : I \to X$ is said to be locally absolutely continuous on I, notation $u \in \mathrm{AC}_{\mathrm{loc}}(I, X)$ if $u \in \mathrm{AC}([a, b]; X)$ for every $a, b \in I$ with $a < b$ and $[a, b] \subset I$.

Recall that if u is absolutely continuous on $[a, b]$, then for every $z \in X$ the function $t \mapsto d(u(t), z)^2$ is absolutely continuous on $[a, b]$ as well.

1.6 Definition. *Let $\phi : X \to (-\infty, \infty]$ be proper and lsc, and let $\alpha \in \mathbf{R}$. If we have a function u in $C([0, \infty); X) \cap AC_{loc}((0, \infty); X)$ satisfying*

$$u(0) \in \overline{D(\phi)}, \quad u(t) \in D(\phi) \quad \textit{for every } t > 0, \tag{1.5}$$

and for every $z \in D(\phi)$

$$\frac{1}{2}\frac{d}{dt} d(u(t), z)^2 + \frac{\alpha}{2} d(u(t), z)^2 + \phi(u(t)) \leqslant \phi(z) \textit{ a.e. in } (0, \infty). \tag{☢}$$

then u is called a solution to the Evolution Variational Inequality ((☢)). The value $u(0)$ is called the initial value of u.

1.7 Theorem (A priori estimate). *Suppose u and v are solutions to ((☢)), then we have the following estimate:*

[1.6] $$d(u(t), v(t)) \leqslant e^{-\alpha(t-s)} d(u(s), v(s)) \textit{ for all } 0 \leqslant s < t < \infty$$

Proof. The function $[a, b] \ni t \mapsto \phi(u(t))$ is lsc, hence Borel and bounded from below. From (☢) we see that this function is bounded from above on $[a, b]$ by a Lebesgue integrable function, hence

$$\int_a^b |\phi(u(t))| \, dt < \infty.$$

Integrating (☢) gives us

[1.7] $$\begin{aligned} \frac{1}{2}(d(u(b), z)^2 - d(u(a), z)^2) + \frac{\alpha}{2} \int_a^b d(u(t), z)^2 \, dt + \int_a^b \phi(u(t)) \, dt \\ \leqslant (b-a)\phi(z), \text{ for every } z \in D(\phi). \end{aligned}$$

Similarly for v. We now define $g(t) := \frac{1}{2} e^{2\alpha t} d(u(t), v(t))^2$. Now $t \mapsto g(t)$ is non-increasing on $[0, \infty)$, to see this note that we want to show that the derivative must be smaller of equal to zero. Using the weak-derivative formulism we note that it is sufficient to show

[1.8] $$-\int_0^\infty g(t)\eta'(t) \, dt \leqslant 0 \text{ for every non-negative } \eta \in C_c^1(0, \infty).$$

Now let η be as in (1.8). Extend η by 0 on on the rest of the real axis. Further let $h_0 > 0$ be such that $\eta(t) = 0$ for all $-\infty < t \leqslant h_0$. We now have for $h \in (0, h_0)$

[1.9] $$-\int_0^\infty g(t)\frac{1}{h}(\eta(t) - \eta(t-h)) \, dt = \int_0^\infty \frac{1}{h}(g(t+h) - g(t))\eta(t) \, dt.$$

by substitution. Note that

$$\begin{aligned} g(t+h) - g(t) &= \frac{1}{2}[e^{2\alpha(t+h)} - e^{2\alpha t}] d(u(t+h), v(t+h))^2 \\ &+ \frac{1}{2} e^{2\alpha t}[d(u(t+h), v(t+h))^2 - d(u(t), v(t+h))^2] \\ &+ \frac{1}{2} e^{2\alpha t}[d(u(t), v(t+h))^2 - d(u(t), v(t+h))^2] \\ &= I_1 + I_2 + I_3. \end{aligned}$$

So, now if we pick $a = t$, $b = t + h$ and $z = v(t+h)$ we get from (1.7) that

[1.10] $$I_2 \leqslant \frac{1}{2} e^{2\alpha t}\left(2h\phi(v(t+h)) - \alpha \int_t^{t+h} d(u(r), v(t+h))^2 \, dr - 2\int_h^{t+h} \phi(u(r)) \, dr\right)$$

Similarly, if we replace u b v in (1.7) and set $a = t$, $b = t + h$ and $z = u(t)$ we obtain

[1.11] $$I_3 \leqslant \frac{1}{2} e^{2\alpha t}\left(2h\phi(u(t)) - \alpha \int_t^{t+h} d(v(r), u(t))^2 \, dr - 2\int_h^{t+h} \phi(v(r)) \, dr\right)$$

So, using that $\eta \geqslant 0$ we obtain that

$$\begin{aligned}
&\int \eta(t)\frac{1}{h}(g(t+h)-g(t))\,dt \\
&\quad\leqslant \int_0^\infty \frac{1}{2}e^{2\alpha t}\Bigg\{\bigg[\frac{1}{h}(e^{2\alpha h}-1)d(u(t+h),v(t+h))^2\bigg] \\
&\quad+2\bigg[\phi(v(t+h))-\frac{1}{h}\int_t^{t+h}\phi(u(r))\,dr-\frac{\alpha}{2}\frac{1}{h}\int_t^{t+h}d(u(r),v(t+h))^2\,dr\bigg] \\
&\quad+2\bigg[\phi(u(t))-\frac{1}{h}\int_t^{t+h}\phi(v(r))\,dr-\frac{\alpha}{2}\frac{1}{h}\int_t^{t+h}d(v(r),u(t))^2\,dr\bigg]\Bigg\}\,dt.
\end{aligned}$$

By the integrability of $\phi\circ v$ we have that

$$\frac{1}{h}\int_t^{t+h}\phi(u(r))\,dr \to \phi\circ u(t) \text{ as } h\to 0.$$

So, as $h \to 0$ we have

$$\begin{aligned}
-\int_0^\infty g(t)\eta'(t)\,dt &= \lim_{h\to 0} -\frac{1}{h}\int_0^\infty g(t)(\eta(t)-\eta(t-h))\,dt \\
&\leqslant \int_0^\infty \eta(t)\frac{1}{2}e^{2\alpha t}\Big[2\alpha d(u(t),v(t))^2+2\phi(v(t))-2\phi(u(t)) \\
&\quad -\alpha d(u(t),v(t))^2+2\phi(u(t))-2\phi(u(t))-\alpha d(u(t),v(t))^2\Big\} \\
&= 0
\end{aligned}$$

■

1.3 Integral formulation of EVI

1.8 Definition. *Let $\phi : X \to (-\infty,\infty]$ be proper and lsc and let $\alpha \in \mathbf{R}$. A function $u \in C([0,\infty); X)$ is called an "integral solution" if for every $0 < a < b$ the function $\phi\circ u \in L^1(a,b)$ and satisfies* (1.7).

1.9 Proposition.

1. *A solution to ☢ is an "integral solution" to ☢;*

2. *If u and v are "integral solutions" to ☢, then they satisfy the estimate of theorem 1.7. They coincide if $u(0) = v(0)$;*

3. *If u is an "integral solution" to ☢ and if $u \in Lip([a,b]; X)$ for every $0 < a < b$, then u is a solution to ☢.*

Proof. Part 1) and part 2) follow from the proof of theorem 1.7. 3): Let $z \in D(\phi)$ and $0 < a' < b'$. Further let $u \in \mathrm{Lip}([a',b']; X)$ with $\phi\circ u \in L^1(a',b')$ satisfying (1.7). Now we will show that there exist a null set N in (a',b') such that u satisfies ☢ on $(a',b')\setminus N$ and $\phi\circ u$ is bounded from above on $(a', b') \setminus N$ by a finite number C. Since $\phi \circ u \in L^1(a', b')$ and $u \in \mathrm{Lip}([a', b']; X)$ there exists N such that every $t_0 \in (a', b') \setminus N$ is a Lebesgue point of $\phi \circ u$ in (a', b') (that is this point satisfies Lebesgue's differentiation lemma) and N is a null set. Further t_0 is a point of differentiability of $t \mapsto d(u(t), z)$ in (a', b') because u is Lipschitz. This is because Lipschitz implies absolute continuity. Now we choose $a = t_0 \in (a', b') \setminus N, b = t_0 + h$ with $0 < h < b' - t_0$, so if we divide (1.7) by h and let h tend to 0, then we obtain

$$\frac{1}{2}\frac{d}{dt}d(u(t_0), z)^2 + \frac{\alpha}{2}d(u(t_0), z)^2 + \phi(u(t_0)) \leqslant \phi(z)$$

Now, set $C_1(a', b') := \max_{t \in [a', b']} d(u(t), z)$, then we get after we note that

$$\begin{aligned} |d(u(t), z)^2 - d(u(t'), z)^2| &\leqslant [d(u(t), z) + d(u(t'), z)]d(u(t), u(t')) \\ &\leqslant 2C_1[u]_{\mathrm{Lip}}|t - t'|. \end{aligned}$$

$$\phi(u(t_0)) \leqslant \phi(z) + \frac{|\alpha|}{2}C_1^2 + C_1[u]_{\mathrm{Lip}} =: C(a', b').$$

Now $(a', b') \setminus N$ is dense in (a', b') because if it were not N would contain an open interval, further u is continuous and ϕ is lsc so we get $\phi(u(t)) \leqslant C$ for every $t \in (a', b')$, hence $u(t) \in D(\phi), t \in (a', b')$. ■

1.4 "Existence" in case X is a Hilbert space

Let $(X, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with norm $| \cdot |$ and metric $d(\cdot, \cdot)$ and let $\phi : X \to (-\infty, \infty]$ be a proper lsc function such that $\phi - \alpha e$ is convex for some $\alpha \in \mathbf{R}$. In this case ϕ is said to be α-convex.

We already know that for any $x \in \overline{D(\phi)}$ there exists at most one solution u to the Evolution Variational Inequality (☢) with initial value $u(0) = x$. The goal of this section is to prove the existence of such a solution.

The proof of the existence will be done by approximating ϕ by a family of functions $(\phi_h)_{h \in I_\alpha}$ where

[1.12] $$I_\alpha := \begin{cases} (0, \infty) & \text{if } \alpha \geqslant 0, \\ (0, |\alpha|^{-1}) & \text{if } \alpha < 0. \end{cases}$$

The functions ϕ_h are usually called the *Moreau-Yosida approximations* of ϕ. These converge to ϕ as h tends to 0 and they are $\frac{\alpha}{1+\alpha h}$-convex.

1.4.1 Preliminaries

1.10 Lemma. *Let $\psi : X \to (-\infty, \infty]$ be proper, lsc and convex. Then there exists $b \in X$ and $x \in \mathbf{R}$ such that*

[1.13] $$\psi(x) \geqslant \langle b, x \rangle + c, \quad x \in X.$$

Proof. Define the epigraph of ψ by $\operatorname{epi}(\psi) := \{(x,t) \in X \times \mathbf{R} : \psi(x) \leqslant t\}$, so this are the points above the graph. Note that since ψ is proper and convex, the epigraph of ψ is non-empty and convex. We introduce the inner product $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ on $X \times \mathbf{R}$ defined by $\langle\!\langle (x_1,t_1),(x_2,t_2) \rangle\!\rangle := \langle x_1, x_2 \rangle + t_1 t_2$. Now $(X \times \mathbf{R}, \langle\!\langle \cdot,\cdot \rangle\!\rangle)$ is a Hilbert space. The subset $\operatorname{epi}(\psi)$ is closed in $X \times \mathbf{R}$ as a consequence of the lower semicontinuity of ψ.

Let $x_0 \in D(\psi)$ and $t_0 < \psi(x_0)$. Then $(x_0, t_0) \notin \operatorname{epi}(\psi)$. By the projection theorem on closed convex sets in Hilbert spaces, there exists a unique element $(\bar{x}, \bar{t}) \in \operatorname{epi}(\psi)$ satisfying

$$\langle x - \bar{x}, x_0 - \bar{x} \rangle + (t - \bar{t})(t_0 - \bar{t}) \leqslant 0 \tag{1.14}$$

for every $(x,t) \in \operatorname{epi}(\psi)$. First we choose $x = x_0$ and $t \geqslant \psi(x_0)$ in (1.14). Then we can see that $t_0 - \bar{t}$ must be non-zero. Further, if we choose $t > \bar{t}$ we can see that $t_0 - \bar{t} < 0$. Finally if we choose $x \in D(\psi)$ in (1.14) we obtain (1.13) with

$$b := \frac{1}{\bar{t} - t_0}(\bar{x} - x_0) \text{ and } c := \bar{t} - \frac{1}{\bar{t} - t_0}\langle \bar{x}, \bar{x} - x_0 \rangle.$$

Equation (1.13) trivially holds for $x \in X \setminus D(\psi)$. ∎

1.11 Lemma. *Let $\phi : X \to (-\infty, \infty]$ be proper, lsc and α-convex for some $\alpha \in \mathbf{R}$. For every $h \in I_\alpha$ and every $x \in X$ the function*

$$\psi(y) := \begin{cases} \frac{1}{2h}|y - x|^2 + \phi(y) & y \in D(\phi), \\ \infty & \textit{otherwise} \end{cases} \tag{1.15}$$

has a unique global minimizer, which we will denote by $J_h x$.

Proof. By α-convexity of ϕ and lemma 1.10 the function ϕ can be rewritten as

$$\psi(y) = \left(\alpha + \frac{1}{h}\right)\frac{1}{2}|y|^2 + \left\langle b - \frac{1}{h}x, y \right\rangle + \left(c + \frac{1}{2h}|x|^2\right) + \phi_1(y) \tag{1.16}$$

where $\phi : X \to [0, \infty]$ is proper, lsc and convex. We can see that $\alpha + \frac{1}{h} > 0$ and $\phi_1 \geqslant 0$, so ψ is bounded from below. Set $\gamma := \inf_{y \in X} \phi(y) \in \mathbf{R}$. Let $(y_n) \subset D(\psi)$ be a minimizing sequence, that is $\lim_{n\to\infty} \psi(y_n) = \lambda$. Now we claim that (y_n) is a Cauchy sequence. Suppose it is and $\bar{y}$ is its limit in X. By lower semicontinuity we obtain

$$\gamma \leqslant \psi(\bar{y}) \leqslant \liminf_{n\to\infty} \psi(y_n) = \gamma.$$

Now given $y, \hat{y} \in D(\phi)$ we have because $\psi\left(\frac{y+\hat{y}}{2}\right) \geqslant \gamma$ that

$$\psi(y) + \psi(\hat{y}) - 2\psi\left(\frac{y+\hat{y}}{2}\right) \geqslant \left(\alpha + \frac{1}{h}\right)\left[\frac{1}{2}|y|^2 + \frac{1}{2}|\hat{y}|^2 - \left|\frac{y+\hat{y}}{2}\right|^2\right] = \left(\alpha + \frac{1}{h}\right)\left|\frac{y+\hat{y}}{2}\right|^2.$$

So since $\frac{y+\hat{y}}{2} \in D(\psi)$ (by convexity) we obtain

$$\begin{aligned} |y - \hat{y}| &\leqslant 2\left(\alpha + \frac{1}{h}\right)^{-\frac{1}{2}} \sqrt{\psi(y) + \psi(\hat{y}) - 2\psi\left(\frac{y+\hat{y}}{2}\right)} \\ &\leqslant 2\left(\alpha + \frac{1}{h}\right)^{-\frac{1}{2}} \sqrt{(\psi(y) - \gamma) + (\psi(\hat{y}) - \gamma)}. \end{aligned} \tag{1.17}$$

Replacing y by y_m and $\hat{y}$ by y_m in (1.17) and noting that $\lim_{n\to\infty}\psi(y_n)=\gamma$ we can conclude that (y_n) is Cauchy. The uniqueness follows from (1.17) as well. ■

1.12 Definition. *Let $\phi : X \to (-\infty, \infty]$ be proper, lsc and α-convex for some $\alpha \in \mathbf{R}$. Set*

[1.18] $$\phi := \psi - \alpha e.$$

For $h \in I_\alpha$ and $x \in X$ set

[1.19] $$A_h x := \frac{1}{h}(x - J_h x).$$

We will now give some properties of J_h and A_h.

1.13 Lemma. *For $h \in I_\alpha$ and $x, \hat{x} \in X$ we have that*

[1.20] $$J_h x \in D(\partial\psi) \text{ and } A_h x - \alpha J_h x \in \partial\psi(J_h x),$$

[1.21] $$|J_h x - J_h \hat{x}| \leqslant \frac{1}{1+\alpha h}|x - \hat{x}|,$$

[1.22] $$|A_h x - A_h \hat{x}| \leqslant \frac{1}{h}\frac{2+\alpha h}{1+\alpha h}|x - \hat{x}|,$$

[1.23] $$\langle A_h - A_h \hat{x}, x - \hat{x}\rangle \geqslant \frac{\alpha}{1+\alpha h}|x - \hat{x}|^2.$$

Proof. (1.20). We have as in lemma 1.11 using $\psi := \phi - \alpha e$ that

$$\psi(y) = \left(\alpha + \frac{1}{h}\right)\frac{1}{2}|y|^2 - \left\langle \frac{1}{h}x, y\right\rangle + \frac{1}{2h}|x|^2 + \phi(y), \quad y \in X.$$

Set $g(y) := \frac{1}{2}(\frac{1}{h}+\alpha)|y|^2 - \langle \frac{1}{h}x, y\rangle + \frac{1}{2h}|x|^2$, $y \in X$. So $\psi = g + \phi$. Because $J_h x$ is a global minimizer of ψ, we have for every $y \in D(\phi)$ and $t \in (0,1)$ that

$$g((1-t)J_h x + ty) + \phi((1-t)J_h x + ty) \geqslant g(J_h x) + \phi(J_h x).$$

By the convexity of ϕ we have

$$-\frac{1}{t}(g((1-t)J_h x + ty) - g(J_h x)) \leqslant \phi(y) - \phi(J_h x).$$

So let $t \to 0$ we arrive at

$$-\langle \nabla g(J_h x), y - J_h x\rangle \leqslant \phi(y) - \phi(J_h x).$$

now note that $\nabla g(z) = (\frac{1}{h}+\alpha)z - \frac{1}{h}x$, $z \in X$, so now using the definition of A_h and the definition of the subdifferential of ϕ. So we obtain (1.20). (1.21). Let $x_1, x_2 \in X$. From (1.20)

$$\frac{1}{h}(x_i - J_h x_i) - \alpha J_h x_i \in \partial\phi(J_h x_i), \quad i = 1, 2.$$

$\partial\phi$ is monotone $\boxed{\text{fill in}}$ so we get

$$\left\langle \left[-\left(\frac{1}{h}+\alpha\right) J_h x_2 + \frac{1}{h} x_2 \right] - \left[-\left(\frac{1}{h}+\alpha\right) J_h x_1 + \frac{1}{h} x_1 \right], J_h x_2 - J_h x_1 \right\rangle \geqslant 0.$$

Splitting up and using Cauchy-Schwarz we obtain

$$(1+\alpha h)|J_h x_2 - J_h x_1|^2 \leqslant \langle x_2 - x_1, J_h x_2 - J_h x_1\rangle \leqslant |x_2 - x_1||J_h x_2 - J_h x_1|$$

which implies (1.21) because $1+\alpha h > 0$. (1.22) follows from (1.21) and the definition of A_h because

$$\begin{aligned} |A_h x - A_h \hat{x}| &= \frac{1}{h}|(x-\hat{x}) + (J_h x - J_h \hat{x})| \\ &\leqslant \frac{1}{h}|x-\hat{x}| + \frac{1}{h}|J_h x - J_h \hat{x}| \\ &\leqslant \frac{1}{h}|x-\hat{x}| + \frac{1}{h}\frac{1}{1+\alpha h}|x-\hat{x}| \\ &= \frac{1}{h}\frac{2+\alpha h}{1+\alpha h}|x-\hat{x}|. \end{aligned}$$

(1.23). We have that

$$(1+\alpha h)hA_h = (1+\alpha h)I - (1+\alpha h)J_h = (I-C) + \alpha h I,$$

where $C := (1+\alpha h)J_h$. By (1.21) we know that $|Cx_2 - Cx_1| \leqslant |x_2 - x_1|$ so $\langle (I-C)x_2 - (I-C)x_1, x_2 - x_1\rangle \geqslant 0$ by rearranging terms. Now

$$\begin{aligned} \langle A_h x_2 - A_h x_1, x_2 - x_1\rangle &= \frac{1}{h}\frac{1}{1+\alpha h}\langle (I-C)x_2 - (I-C)x_1, x_2 - x_1\rangle + \frac{\alpha h}{1+\alpha h}|x_2 - x_1|^2 \\ &= \frac{1}{h}\frac{1}{1+\alpha h}|x_2 - x_1|^2 - \frac{1}{h}\langle J_h x_2 - J_h x_1, x_2 - x_1\rangle + \frac{\alpha h}{1+\alpha h}|x_2 - x_1|^2 \\ &\geqslant \frac{1}{h}\frac{1}{1+\alpha h}|x_2 - x_1|^2 - \frac{1}{h}|J_h x_2 - J_h x_1||x_2 - x_1| + \frac{\alpha h}{1+\alpha h}|x_2 - x_1|^2 \\ &\geqslant \frac{1}{h}\frac{1}{1+\alpha h}|x_2 - x_1|^2 - \frac{1}{h}\frac{1}{1+\alpha h}|x_2 - x_1|^2 + \frac{\alpha h}{1+\alpha h}|x_2 - x_1|^2 \\ &= \frac{\alpha h}{1+\alpha h}|x_2 - x_1|^2. \end{aligned}$$

■

1.4.2 Moreau-Yosida approximation

1.14 Definition. *Let ϕ be proper, lsc and α-convex and let ψ as in (1.15). Further, let $h \in I_\alpha$. Then we define*

[1.24] $$\phi_h(x) := \psi(J_h x), \quad x \in X.$$

1.15 Proposition. *Let ϕ, ϕ_h be as above. Then*

[1.25] $$\phi_h(x) = \frac{h}{2}|A_h x|^2 + \phi(J_h x), \qquad x \in X.$$

$\phi_h \in C^{1,1}(X; \mathbf{R})$, $\nabla \phi_h = A_h$ and ϕ_h is $\frac{\alpha}{1+\alpha h}$-convex.

Proof. (1.25) follows from

$$\begin{aligned}
\phi_h(x) &= \psi(J_h x) \\
&= \frac{1}{2h}|J_h x - x|^2 + \phi(J_h x) \\
&= \frac{h^2}{2h}|A_h|^2 + \phi(J_h x) \\
&= \frac{h}{2}|A_h|^2 + \phi(J_h x).
\end{aligned}$$

Now we will show that $\nabla\phi(x) = A_h x$ for $x \in X$. Let $x, y \in X$. From (1.20) we know that $A_h x - \alpha J_h x \in \partial\psi(J_h x)$. So, by the definition of the subgradient we have for $z = J_h y$ that

$$\langle A_h x - \alpha J_h x, J_h y - J_h x\rangle + \psi(J_h x) \leqslant \psi(J_h y)$$

So, by rearranging we get

$$\langle A_h x - \alpha J_h x, J_h y - J_h x\rangle \leqslant \psi(J_h y) - \psi(J_h x)$$

From (1.25) and $\psi = \phi - \alpha e$ we obtain

$$\begin{aligned}
\phi_y(y) - \phi_h(x) &= \psi(J_h y) - \psi(J_h x) + \frac{\alpha}{2}|J_h y|^2 - \frac{\alpha}{2}|J_h x|^2 + \frac{h}{2}|A_h y|^2 - \frac{h}{2}|A_h x|^2 \\
&\geqslant \langle A_h x - \alpha J_h x, J_h y - J_h x\rangle + \frac{\alpha}{2}|J_h y|^2 - \frac{\alpha}{2}|J_h x|^2 + \frac{h}{2}|A_h y|^2 - \frac{h}{2}|A_h x|^2.
\end{aligned}$$

We can rewrite

[1.26] $$\langle A_h x - \alpha J_h x, J_h y - J_h x\rangle = -\langle A_h x - \alpha J_h x, x - y\rangle + \langle A_h x - \alpha J_h x, hA_h x - hA_h y\rangle.$$

By rearranging the terms we eventually obtain

$$\begin{aligned}
\phi_h(y) - \phi_h(x) - \langle A_h x, y - x\rangle &\geqslant \langle \alpha J_h x, x + y\rangle + \langle A_h x - \alpha J_h x, hA_h x - hA_h y\rangle \\
&= \alpha\langle J_h x, x - y\rangle + h\langle A_h x, A_h x - A_h y\rangle - h\alpha\langle J_h x, A_h x - A_h y\rangle \\
&+ \frac{\alpha}{2}|J_h y|^2 - \frac{\alpha}{2}|J_h x|^2 + \frac{h}{2}|A_h y|^2 - \frac{h}{2}|A_h x|^2 \\
&= \alpha\langle J_h x, x - y\rangle + h\langle A_h x, A_h x - A_h y\rangle \\
&- \alpha\langle J_h x, x - y\rangle + \alpha\langle J_h x, J_h x - J_h y\rangle \\
&+ \frac{\alpha}{2}|J_h y|^2 - \frac{\alpha}{2}|J_h x|^2 + \frac{h}{2}|A_h y|^2 - \frac{h}{2}|A_h x|^2
\end{aligned}$$

$$
\begin{aligned}
&= h\langle A_h x, A_h x - A_h y\rangle + \alpha\langle J_h x, J_h x - J_h y\rangle \\
&+ \frac{\alpha}{2}|J_h y|^2 - \frac{\alpha}{2}|J_h x|^2 + \frac{h}{2}|A_h y|^2 - \frac{h}{2}|A_h x|^2 \\
&= \frac{h}{2}|A_h x - A_h y|^2 + \frac{\alpha}{2}|J_h x - J_h y|^2.
\end{aligned}
$$

Now we switch the role of y and x and we add $\langle A_h y - A_h x, x - y\rangle$ (which is a negative term) to obtain

$$
\phi_h(x) - \phi_h(y) - \langle A_h x, x - y\rangle \geqslant \frac{h}{2}|A_h x - A_h y|^2 + \frac{\alpha}{2}|J_h x - J_h y|^2 + \langle A_h y - A_h x, x - y\rangle. \tag{1.27}
$$

Now because the LHS of the previous inequality is negative we have by (1.21), (1.22) and Cauchy-Schwarz some $M > 0$ independent on x or y such that

$$
\begin{aligned}
|\phi_h(x) - \phi_h(y) - \langle A_h x, x - y\rangle| &\leqslant \frac{h}{2}|A_h x - A_h y|^2 + \frac{\alpha}{2}|J_h x - J_h y|^2 + |\langle A_h y - A_h x, x - y\rangle| \\
&\leqslant M|x - y|^2.
\end{aligned} \tag{1.28}
$$

Hence $\nabla\phi_h(x) = A_h x$. A_h is Lipschitz because of (1.22), so we have $\phi_h \in C^{1,1}(X; \mathbf{R})$. ■

To be able to handle the case $\alpha \geqslant 0$ and $\alpha < 0$ at the same time we introduce

$$
h_\alpha := \begin{cases} 1 & \text{if } \alpha \geqslant 0, \\ \frac{1}{2|\alpha|} & \text{if } \alpha < 0. \end{cases} \tag{1.29}
$$

Then we have that

$$
1 + h\alpha \in \left[\frac{1}{2}, 1 + |\alpha|\right] \text{ for } 0 < h \leqslant h_\alpha. \tag{1.30}
$$

We use the following notation. Let $x \in D(\partial\psi)$ with $\psi := \phi - \alpha e$. The set $\{y \in X : y \in \partial\psi((x)\}$ is a non-empty closed convex set so by the projection theorem on closed convex sets in Hilbert spaces this set has a minimal element, which we denote as $(\partial\psi)^\circ x$.

1.16 Lemma.

$$
\sup_{h\in(0,h_\alpha)} |A_h x| \leqslant \infty \quad \textit{if } x \in D(\partial\psi), \tag{1.31}
$$

$$
\sup_{h\in(0,h_\alpha)} |J_h x| \leqslant \infty \quad \textit{for every } x \in X, \tag{1.32}
$$

$$
\inf_{h\in(0,h_\alpha)} \phi(J_h x) > -\infty \quad \textit{for every } x \in X. \tag{1.33}
$$

Proof. (1.31). From (1.20) and the monotonicity of $\partial\psi$ we have

$$
\langle y, x - J_h x\rangle - \langle A_h x - \alpha J_h x, x - J_h x\rangle = \langle y - A_h x + \alpha J_h x, x - J_h x\rangle \geqslant 0
$$

So

$$
\frac{1}{h}\langle y - A_h x + \alpha J_h x, x - J_h x\rangle \geqslant 0.
$$

Thus by the definition of A_h we have

$$\langle y, A_h x\rangle - |A_h x|^2 + \alpha\langle x, A_h x\rangle - \alpha h|A_h x|^2 \geqslant 0,$$

so

$$\begin{aligned}|A_h x|^2 &\leqslant \langle y, A_h x\rangle + \alpha\langle x, A_h x\rangle - \alpha h|A_h x|^2\\ &\leqslant |y||A_h x| + \alpha|x||A_h x| - \alpha h|A_h x|^2.\end{aligned}$$

Hence, by rearranging

$$(1 + h\alpha)|A_h x|^2 \leqslant (|y| + |\alpha||x|)|A_h x|.$$

Now by (1.30) and using the minimal y we have

$$|A_h x| \leqslant 2(|(\partial\psi)^\circ x| + |\alpha||x|),$$

which implies (1.31).

(1.32). Let $x \in X$ and $\hat{x} \in D(\partial\psi)$. Set $C := \sup_{h\in(0,h_\alpha)} |A_h\hat{x}|$. Using the definition of A_h, (1.21) and the previous result (1.31) we get that

$$|J_h x| \leqslant |J_h x - J_h\hat{x}| + |J_h\hat{x}| \leqslant 2|x - \hat{x}| + h|A_h\hat{x}| \leqslant 2|x - \hat{x}| + |\hat{x}| + h_\alpha C,$$

from which the result follows.

(1.33). Let $x \in X$ and $M := \sup_{h\in(0,h_\alpha)} |J_h x|$. Then by using $\psi = \phi - \alpha e$, lemma 1.10, proposition 1.15 and Cauchy-Schwarz we get

$$\phi(J_h x) = \psi(J_h x) + \frac{\alpha}{2}|J_h x|^2 \geqslant -|b|M + c - \frac{|\alpha|}{2}M^2.$$

∎

Another useful lemma

1.17 Lemma.

[1.34] $$\lim_{h\to 0} |x - J_h x| = 0 \text{ iff } x \in \overline{D(\partial\psi)},$$

[1.35] $$\sup_{h\in(0,h_\alpha)} \phi_h(x) = \infty \text{ if } x \notin \overline{D(\psi)}.$$

Proof. (1.34). Assume that $x \in \overline{D(\partial\psi)}$, so for any $\hat{x} \in D(\partial\psi)$ we have by the definition of A_h, the bounds on $1 + h\alpha$ and (1.21),

$$|x - J_h x| \leqslant |x - \hat{x}| + |\hat{x} - J_h\hat{x}| \leqslant |x - \hat{x}| + |x - J_h\hat{x}| + 2|x - \hat{x}| \leqslant 3|x - \hat{x}| + h|A_h\hat{x}|.$$

So because $|A_h x|$ is bounded by the previous lemma and the fact that we can pick $\hat{x} = x$ we obtain the result. Conversely, if $\lim_{h\to 0} |x - J_h x| = 0$ then $x \in \overline{D(\partial\psi)}$ because $J_h x \in D(\partial\psi)$.

(1.35). By proposition 1.15 and the third part of lemma 1.16 it is sufficient to check that $\sup_{h\in(0,h_\alpha)} h|A_hx|^2 = \infty$ if $x \notin \overline{D(\partial\psi)}$. Now note that

$$h|A_hx|^2 = |x - J_hx||A_hx| \geqslant d(x, \overline{D(\partial\psi)})|A_hx|$$

since $J_hx \in D(\partial\psi)$. Now $d(x, \overline{D(\partial\psi)}) > 0$ by assumtpion so it is sufficient to show that $\sup_{h\in(0,h_\alpha)} |A_hx| = \infty$ for $x \notin \overline{D(\partial\psi)}$. Set $M := \sup_{h\in(0,h_\alpha)} |A_hx| < \infty$ so then $|x - J_hx| \leqslant hM$ by the definition of A_h so by the first part we have a contradiction. ■

1.18 Proposition. *Let ϕ, ϕ_h and ψ be as above. Then*

[1.36] $$\phi_h(x) \uparrow \phi(x) \text{ for every } x \in X \text{ and } h \downarrow 0,$$

[1.37] $$D(\partial\psi) \subset D(\phi) \subset \overline{D(\partial\phi)} = \overline{D(\phi)}.$$

Proof. For $0 < h_2 < h_1 \leqslant h_\alpha$ and $x \in X$ we have

$$\begin{aligned}
\phi_{h_1}(x) &= \psi_{h_1}(J_{h_1}x)\\
&\leqslant \psi_{h_1}(J_{h_2}x)\\
&= \frac{1}{2h_1}|J_{h_2}x - x|^2 + \phi(J_{h_2}x)\\
&\leqslant \frac{1}{2h_2}|J_{h_2}x - x|^2 + \phi(J_{h_2}x)\\
&= \phi_{h_2}(x).
\end{aligned}$$

Now we will show that ϕ_h is bounded by above by ϕ. To see this note that $\phi_h(x) = \psi(J_hx) \leqslant \psi(y)$ and choosing $y = x$ we have $\phi_h(x) \leqslant \psi(x) = \phi(x)$. So, by lemma 1.17, (1.35) we have that if $x \notin \overline{D(\partial\psi)}$ then $\sup_{h\in(0,h_\alpha)} \phi_h(x) = \infty$ hence $x \notin D(\phi)$. This implies (1.36) for $x \notin \overline{D(\partial\psi)}$ and thus also the inclusion $D(\phi) \subset \overline{D(\partial\psi)}$ in (1.37). If $x \in \overline{D(\partial\psi)}$ and $h_n \in (0, h_\alpha)$, $h_n \downarrow 0$ we have by lemma 1.17, (1.34) that $\lim_{n\to\infty} |x - J_{h_n}x| = 0$ and by the lower semicontinuity of ϕ

$$\phi(x) \leqslant \liminf_{n\to\infty} \phi(J_{h_n}x) \leqslant \liminf_{n\to\infty} \phi_{h_n}(x) \leqslant \limsup_{n\to\infty} \phi_{h_n}(x) \leqslant \phi(x).$$

So we conclude that ϕ_h is decreasing in h, is bounded from above by ϕ and the limit is ϕ, so we have (1.36).

By definition we have $D(\partial\psi) \subset D(\psi) = D(\phi)$, so also $\overline{D(\partial\psi)} \subset \overline{D(\phi)}$. We already know that $D(\phi) \subset \overline{D(\partial\psi)}$ so (1.37) follows. ■

1.4.3 A quasi-contractive semigroup associated with ϕ

Let $\phi : X \to (-\infty, \infty]$ be proper, lsc and α-convex for some $\alpha \in \mathbf{R}$. Further, let $h \in (0, h_\alpha]$ and let ϕ_h be the Moreau-Yosida approximation of ϕ. We consider the abstract Cauchy problem

[1.38] $$\frac{du}{dt}(t) + A_hu(t) = 0, t \quad \in \mathbf{R},$$

together with the condition

$$u(0) = x \text{ with } x \in X. \tag{1.39}$$

In view of proposition 1.5 and proposition 1.15 this problem has exactly one solution which we will denote by $\phi_{h,x}$ or simply as ϕ_h and we set

$$S_h(t)x := u_{h,x}(t), \quad t \in \mathbf{R}, x \in X. \tag{1.40}$$

Further, this family $\{S_h(t)\}_{t\in\mathbf{R}}$ is a C_0-group of operators on X which satisfy

$$|S_h(t)x - S_h(t)y| \leqslant e^{-\frac{\alpha}{1+\alpha h}(t-s)}|S_h(s)x - S_h(s)y| \tag{1.41}$$

for $s < t$ and $x, y \in X$ since $\nabla\phi_h = A_h$ and ϕ_h is $\frac{\alpha}{1+\alpha h}$-convex. In this section we will establish the following

1.19 Theorem. *For every $x \in \overline{D(\phi)}$ and $t \geqslant 0$:*

$$S(t)x := \lim_{h\to 0} S_h(t)x \textit{ exists in } (X, |\cdot|), \tag{1.42}$$

$$S(t)x \in \overline{D(\phi)}. \tag{1.43}$$

The family of operators $\{S(t)\}_{t\geqslant 0} : \overline{D(\phi)} \to \overline{D(\phi)}$ is a C_0-semigroup satisfying

$$[S(t)]_{Lip} \leqslant e^{-\alpha t}, \quad t \geqslant 0. \tag{1.44}$$

Proof. The idea is that we prove (1.42)-(1.44) for $x \in D(\partial\psi)$ and then approximate together with the estimate (1.41). We do this in a couple of steps.

Step 1. By lemma 1.16 we can set $M_1 := \sup_{h\in(0,h_\alpha)} |A_h(x)| < \infty$. Let $T > 0$.
Claim.

$$|A_h u_h(t)| \leqslant M_1 e^{2|\alpha|T} =: M_2(\alpha, T) \text{ for } h \in (0, h_\alpha) \text{ and } t \in [0, T]. \tag{1.45}$$

To prove this take estimate (1.41) with $y = S_h x$ with $h > 0$ and $s = 0$ to obtain

$$\begin{aligned} |u_h(t) - u_h(t+h)| &\leqslant e^{-\frac{\alpha}{1+\alpha h}t}|u_h(0) - u_h(h)| \\ &\leqslant e^{2|\alpha|T}|u_h(0) - u_h(h)|. \end{aligned} \tag{1.46}$$

If we now divide by h and send h to 0 we get

$$|\dot{u}_h(t)| \leqslant e^{2|\alpha|T}|\dot{u}_h(0)| = e^{2|\alpha|T}|A_h x| \leqslant e^{2|\alpha|M} M_1. \tag{1.47}$$

So if we take $\dot{u}_h(t) = -A_h u_h(t)$ we are done.

Step 2. In this step we prove the following estimate

$$\langle A_h u_h(t) - A_{h'} u_{h'}(t), u_h(t) - u_{h'}(t)\rangle \geqslant -2|\alpha||u_h(t) - u_{h'}(t)|^2 - \lambda M_3, \tag{1.48}$$

where

$$M_3 := (8|\alpha|h_\alpha + 4)M_2^2(\alpha, T) \tag{1.49}$$

By the monotonicity of $\partial\psi$ and lemma 1.13, (1.20) we get immediately that

$$\langle (A_h u_h(t) - \alpha J_h u_h(t)) - (A_{h'} u_{h'}(t) - \alpha J_{h'} u_{h'}(t)), J_h u_h(t) - J_{h'} u_{h'}(t)\rangle \geqslant 0.$$

By rearranging terms we quickly see that

$$\langle A_h u_h(t) - A_{h'} u_{h'}(t), J_h u_h(t) - J_{h'} u_{h'}(t)\rangle \geqslant \alpha |J_h u_h(t) - J_{h'} u_{h'}(t)|^2.$$

From the definition of A_h and the definition of M_2 we obtain $\boxed{\text{HOW?????????????????????????}}$

$$|J_h u_h(t) - J_{h'} u_{h'}(t)|^2 \leqslant 2|u_h - u_{h'}|^2 + 8M_2^2 h_\alpha \lambda,$$

and finally $\boxed{\text{ROT?}}$

$$\langle A_h u_h(t) - A_{h'} u_{h'}(t), J_h u_h(t) - J_{h'} u_{h'}(t)\rangle \geqslant \langle A_h u_h(t) - A_{h'} u_{h'}(t), u_h(t) - u_{h'}(t)\rangle - 4M_2^2\lambda$$

And the claim follows $\boxed{\text{HOW????}}$

Step 3. From ACP for u_h and $u_{h'}$ and (1.48)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|u_h(t) - u_{h'}(t)|^2 &= \langle \dot{u}_h(t) - \dot{u}_{h'}(t), u_h(t) - u_{h'}(t)\rangle \\
&= -\langle A_h u_h(t) - A_{h'} u_{h'}(t), u_h(t) - u_{h'}(t)\rangle \\
&\leqslant 2|\alpha||u_h(t) - u_{h'}(t)|^2 + \lambda M_3 \\
&\leqslant \frac{2|\alpha|}{\lambda M_3 T} e^{2|\alpha|T}\lambda M_3 + \lambda M_3.
\end{aligned}$$

If we integrate we arrive at

[1.50] $$|u_h(t) - u_{h'}(t)|^2 \leqslant \lambda M_3 M_4 \text{ for some } M_4 = M_4(\alpha, T).$$

Step 4 (Convergence for $x \in D(\partial\psi)$). From (1.50) it follows that if $h_n \to 0$ $\{u_{h_n}(t)\}_{n\geqslant 1}$ is a Cauchy sequence in $(X, |\cdot|)$. So this allows us to set

[1.51] $$S(t)x := \lim_{n\to\infty} u_{h_n}(t).$$

So $S(t)x := \lim_{h\to 0} u_h(t) = \lim_{h\to 0} S_h(t)x$. Now, $T > 0$ is arbitrary, so $S(t)x$ is well-defined for every $t > 0$. From (1.51) it follows that the convergence is uniform on $[0, T]$ hence $t \mapsto S(t)x \in C([0, T]; X)$, $T > 0$. From the definition of A_h and M_2 it follows that

$$|S(t)x - J_{h_n} u_{h_n}(t)| \leqslant |S(t)x - u_{h_n}(t) + h_n A_{h_n}(y)| \leqslant |S(t)x - u_{h_n}(t)| + h_n M_1.$$

Now, $J_{h_n} u_{h_n}(t) \in D(\partial\psi)$ by lemma 1.13, (1.20), so $S(t) \in \overline{D(\partial\psi)} = \overline{D(\phi)}$ by (1.37).

Step 5 (Convergence for $x \in \overline{D(\phi)}$. Let $x \in \overline{D(\phi)}$, $\epsilon > 0$ and $T > 0$. Then for every $\hat{x} \in D(\partial\psi)$ we have

$$\begin{aligned}
|S_h(t)x - S_{h'}x| &\leqslant |S_h(t)x - S_h(t)\hat{x}| + |S_h(t)\hat{x} - S_{h'}(t)\hat{x}| + |S_{h'}(t)\hat{x} - S_{h'}(t)x| \\
&= 2e^{2|\alpha|T}|x - \hat{x}| + |S_h(t)\hat{x} - S_{h'}\hat{x}|, \quad t \in [0, T].
\end{aligned}$$

Now since $\overline{D(\partial\psi)} = \overline{D(\phi)}$ we can pick the first term smaller than $\frac{\epsilon}{2}$ and there is a $\overline{h} \in (0, h_\alpha]$ such that the last term is also smaller than $\frac{\epsilon}{2}$ for $t \in [0, T]$ and $0 < h < \lambda \leqslant \overline{h}$. So we conclude that $\lim_{h\to 0} S_h(t)x$ exists in X and we denote it by $S(t)x$, $t \geqslant 0$. By uniform continuity on $[0, T]$, $t \mapsto S(t)x$ is continuous on $[0, T]$. Property (1.44) follows from (1.41) with $s = 0$ and the fact that the limit exists. So now we prove (1.43). Let $x_n \in D(\partial\psi)$, $n \geqslant 1$ with $\lim_{n\to\infty} x_n = x$. So

$$|S(t)x - S(t)x_n| \leqslant S(t)x_n| \leqslant e^{-\alpha t}|x - x_n| \to 0.$$

So, since $S(t)x_n \in \overline{D(\phi)}$ this also holds for $S(t)x$.

Step 6 (Semigroup property). Let $x \in \overline{D(\phi)}$, $t, s \geqslant 0$, $h \in (0, h_\alpha]$. So we have

$$\begin{aligned}
|S(t+s)x - S(t)S(x)x| &\leqslant |S(t+s)x - S_h(t+s)x| + |S_h(t+s)x - S_h(t)S_h(s)x| \\
&\quad + |S_h(t)S_h(s)x - S_h(s)S(s)x| + |S_h(t)S(s)x - S(t)S(s)x| \\
&\leqslant |S(t+s)x - S_h(t+s)x| + e^{2|\alpha|T}|S_h(s) - S(s)x| \\
&\quad + |S_h(t)S_h(s)x - S(t)S(s)x| \to 0.
\end{aligned}$$

Hence $\{S(t)\}_{t\geqslant 0}$ is a semigroup of operators on $\overline{D(\phi)}$. ■

1.4.4 "Existence" theorem

Let $\phi : X \to (-\infty, \infty]$ be proper, lsc and α-convex for some $\alpha \in \mathbf{R}$. Let $\{S(t)\}_{t\geqslant 0}$ be the semigroup from theorem 1.19. We then have

1.20 Theorem. *For every $u_0 \in \overline{D(\phi)}$ the function $u : [0, \infty)$ given by $u(t) := S(t)u_0$ is a solution to (☢) with initial value u_0.*

Proof. We know from theorem 1.19 that $u \in C([0, \infty); X)$. So we still have to show that for every a, b with $0 < a < b$ the following three things hold:

1. $u \in \mathrm{AC}([a, b]; X)$,
2. $u \in D(\phi)$ for $t \in [a, b]$,
3. u satisfies for every $z \in D(\phi)$:

[1.52] $$\frac{1}{2}\frac{d}{dt}|u(t) - z|^2 + \frac{\alpha}{2}|u(t) - z|^2 + \phi(u(t)) \leqslant \phi(z) \text{ a.e. in } (a, b).$$

We first establish the following estimate, there exists $C = C(\phi, \alpha, u_0, a, b) > 0$ such that

[1.53] $$|A_h u_h(t)| \leqslant C, \quad h \in (0, h_\alpha), t \in [a, b],$$

where

[1.54] $$u_h(t) := S_h(t)u_0, \quad t \in \mathbf{R}, h \in (0, h_\alpha).$$

Recall that $u_h \in C^1(\mathbf{R}; X)$ and satisfies the ACP for A_h. From (1.41) with $x = S_h(h)u_0$, $y = u_0$, $h > 0$ we obtain after dividing by h and sending h to 0 that

$$\begin{aligned}|\dot{u}_h(t)| &\leqslant e^{-\frac{\alpha}{1+\alpha h}}|\dot{u}_h(0)| \\ &= e^{-\frac{\alpha}{1+\alpha h}}|A_h(x)| \\ &= e^{-\frac{\alpha}{1+\alpha h}}M_1 > 0,\end{aligned}$$

so we conclude that

[1.55] $$t \mapsto e^{\frac{\alpha}{1+\alpha h}}|\dot{u}_h(t)| \text{ is nonincreasing for } t \geqslant 0.$$

If we take the inner product of ACP with $te^{\frac{2\alpha}{1+\alpha h}t}\dot{u}_h(t)$ and integrate from 0 to a we get

$$\int_0^a te^{\frac{2\alpha}{1+\alpha h}}|\dot{u}_h(t)|^2\,dt + \int_0^a e^{\frac{2\alpha}{1+\alpha h}}\langle A_h u_h(t), \dot{u}_h(t)\rangle\,dt = 0.$$

So since we have by proposition 1.15 that $A_h u_h(t) = \nabla\phi_h(u_h(t))$ we also have

$$\langle A_h u_h(t), \dot{u}_h(t)\rangle = \frac{d}{dt}\phi_h(u_h(t)).$$

Using (1.55) and integration by parts we obtain

$$\begin{aligned}\frac{a^2}{2}e^{\frac{2\alpha}{1+\alpha h}a}|\dot{u}_h(a)|^2 &\leqslant \int_0^a te^{\frac{2\alpha}{1+\alpha h}t}|\dot{u}_h(t)|^2\,dt \\ &= -\int_0^a te^{\frac{2\alpha}{1+\alpha h}t}\frac{d}{dt}\phi_h(u_h(t))\,dt \\ &= -ae^{\frac{2\alpha}{1+\alpha h}a}\phi_h(u_h(a)) + \int_0^a \frac{d}{dt}(te^{\frac{2\alpha}{1+\alpha h}t})\phi_h(u_h(t))\,dt.\end{aligned}$$

Using (1.15) and $\psi = \phi - \alpha e$ we get

$$\phi_h(u_h(t)) \geqslant \phi(J_h u_h(t)) \geqslant \psi(J_h u_h(t)) - \frac{|\alpha|}{2}|J_h u_h(t)|^2, \quad t \geqslant 0.$$

By lemma 1.10 we have $a_1, b_1 \in \mathbf{R}$ depending only on ψ such that

$$\psi(J_h u_h(t)) \geqslant a_1|J_h u_h(t)| + b_1.$$

From step 2 and 3 from theorem 1.19 we obtain for $0 < h \leqslant h' \leqslant h_\alpha$

$$|J_h u_h(t) - J_{h'}u_{h'}(t)|^2 \leqslant 2\lambda M_3 M_4 + 8M_2^2 h_\alpha \lambda,$$

where $M_4 = M_4(\alpha, T)$, $T = b$. So this implies that there exists a constant
$C_1 = C_1(\phi, \alpha, u_0, a, b) > 0$ such that

[1.56] $$|J_h u_h(t)| \leqslant C_1, \quad t \in [a, b].$$

This is because $u_{h'}$ is bounded and because of the inverse triangle inequality.

So, there exists a $C_2 = C_2(\phi, \alpha, u_0, a, b) > 0$ such that

$$\phi_h(u_h(t)) \geqslant -C_2, \quad t \in [a,b], h \in (0, h_\alpha). \tag{1.57}$$

We can see this by considering the cases $a_1 \geqslant 0$ and $a_1 < 0$. Further, we have that

$$\frac{a^2}{2} e^{\frac{2\alpha}{1+\alpha h}a} |\dot{u}_h(a)|^2 + a e^{\frac{2\alpha}{1+\alpha h}a} \phi_h(u_h(a)) \leqslant \int_0^a \frac{d}{dt}(t e^{\frac{2\alpha}{1+\alpha h}t} \phi_h(u_h(t))\, dt.$$

If we now add $a e^{\frac{2\alpha}{1+\alpha h}a} C_2$ to both sides the elements on the LHS become positive, so

$$\begin{aligned}\frac{a^2}{2} e^{\frac{2\alpha}{1+\alpha h}a} |\dot{u}_h(a)|^2 + a e^{\frac{2\alpha}{1+\alpha h}a} (\phi_h(u_h(a)) + C_2) &\leqslant \left| \int_0^a \frac{d}{dt}(t e^{\frac{2\alpha}{1+\alpha h}t})(\phi_h(u_h(t)) + C_2)\, dt - a e^{\frac{2\alpha}{1+\alpha h}a} C_2 \right| \\ &\leqslant \left| \int_0^a \frac{d}{dt}(t e^{\frac{2\alpha}{1+\alpha h}t})(\phi_h(u_h(t)) + C_2)\, dt \right| + a e^{\frac{2\alpha}{1+\alpha h}a} C_2 \\ &\leqslant C_3 \left| \int_0^a \phi_h(u_h(t))\, dt + C_2 \right| + a e^{\frac{2\alpha}{1+\alpha h}a} C_2,\end{aligned}$$

where $C_3 = C_3(\alpha, a) > 0$. So we conclude

$$e^{\frac{2\alpha}{1+\alpha h}a} |\dot{u}_h(a)|^2 \leqslant C_4 \int_0^a \phi_h(u_h(t))\, dt + C_5 \tag{1.58}$$

with $C_4, C_5 > 0$.

We still need to estimate $\int_0^a \phi_h(u_h(t))\, dt$, by proposition 1.5 we have

$$\frac{1}{2}\frac{d}{dt}|u_h(t) - z|^2 + \frac{\alpha}{2}|u_h(t) - z|^2 + \phi_h(u_h(t)) \leqslant \phi_h(z), \quad h \in (0, h_\alpha),\ t \in \mathbf{R},\ z \in X. \tag{1.59}$$

From the definition of ψ an ϕ_h we see for $z \in D(\phi)$ that $\phi_h(z) \leqslant \psi(z) = \phi(z)$. Now note that $\sup_{h \in (0,h_\alpha)} \max_{t \in [a,b]} |u_h(t)| < \infty$ because of the inverse triangle inequality and (1.50). So we can conclude that there exists a C_6 such that

$$\int_0^a \phi_h(u_h(t))\, dt \leqslant C_6, \quad h \in (0, h_\alpha) \tag{1.60}$$

From (1.55), (1.58) and (1.61) we obtain that $|A_h u_h(t)| \leqslant C$ for $h \in (0, h_\alpha)$ and $t \in [a,b]$. So now we can prove 1. Now $|\dot{u}_h(t)| \leqslant C$ for $t \in [a,b]$ and $h \in (0, h_\alpha)$ we get $|u(t) - u(s)| \leqslant C|t-s|$ for $a \leqslant s, t \leqslant b$ so $u \in \mathrm{Lip}([a,b]; X) \subset \mathrm{AC}([a,b]; X)$. . From (1.27) we have for $z \in D(\phi)$ and $t \in [a,b]$:

$$\begin{aligned}\phi_h(u_h(t)) &\leqslant \phi_h(z) - \langle A_h u_h(t), z - u_h(t) \rangle - \frac{h}{2}|A_h x - A_h y|^2 - \frac{\alpha}{2}|J_h x - J_h y|^2 \\ &\leqslant \phi_h(z) - \langle A_h u_h(t), z - u_h(t) \rangle - \frac{\alpha}{2}|J_h x - J_h y|^2 \\ &\leqslant \phi(z) + |A_h u_h(t)|(|z| + |u_h(t)|) + \frac{|\alpha|}{2}(|J_h u_h(t)| + |J_h z|)^2 \\ &\leqslant \phi(z) + |A_h u_h(t)|(|z| + |u_h(t)|) + |\alpha|(|J_h u_h(t)|^2 + |J_h z|^2).\end{aligned}$$

Using the bounded on $|A_h x|, |J_h x|$ and $\sup_{h\in(0,h_\alpha)} |J_h x| < \infty$ we find a $\hat{C} = \hat{C}(\phi, \alpha, u_0, a, b) > 0$ such that

$$\phi_h(u_h(t)) \leqslant \hat{C}, \quad t \in [a,b], h \in (0, h_\alpha). \tag{1.61}$$

Now let $h_n \in (0, h_\alpha) \to 0$. Since $J_{h_n} u_{h_n}(t) \to u(t)$ for $t \in [a,b]$ we obtain by the lower semicontinuity of ϕ that

$$\phi(u(t)) \leqslant \liminf_{n\to\infty} \phi(J_{h_n}(u_{h_n}(t)) \leqslant \liminf_{n\to\infty} \phi_{h_n}(u_{h_n}(t)) \leqslant \hat{C}, \quad t \in [a,b].$$

We can now prove . Note that $t \mapsto \phi(u(t))$ is lsc, hence bounded from below which together with $\phi(u(t)) \leqslant \hat{C}$ proves that $\phi(u) \in L^\infty(a,b)$. Using (1.57) and Fatou's lemma

$$\int_t^s \phi(u(r))\, dr \leqslant \int_s^t \liminf_{n\to\infty} \phi_{h_n}(u_{h_n}(r))\, dr \leqslant \liminf_{n\to\infty} \int_s^t \phi_{h_n}(u_{h_n}(r))\, dr. \tag{1.62}$$

Integrating (1.59) on $[s,t] \subset [a,b]$, taking $z \in D(\phi)$, using theorem 1.19 and (1.62) we obtain as $h_n \to 0$:

$$\frac{1}{2}|u(t) - z|^2 - \frac{1}{2}|u(s) - z|^2 + \frac{\alpha}{2}\int_s^t |u(r) - z|^2\, dr + \int_s^t \phi(u(r)) \leqslant (t-s)\phi(z).$$

So now we can by $t - s$ and use the absolute continuity of $t \mapsto |u(t) - z|^2$ and $t \mapsto \int_s^t \phi(u(r))\, dr$ we get

$$\frac{1}{2}\frac{d}{dt}|u(t) - z|^2 + \frac{\alpha}{2}|u(t) - z|^2 + \phi(u(t)) \leqslant \phi(z) \text{ a.e. in } (a,b).$$

This completes our proof. ∎

2 Gradient flows in metric spaces

Let (X, d) be a complete metric space and let $\phi : X \to (-\infty, \infty]$ be proper and lower semicontinuous. The goal now is to establish a solution to (☢) with an arbitrary initial value $u_0 \in \overline{D(\phi)}$ under some additional assumptions which will strictly extend that α-convexity condition of the Hilbert space case.

We first reformulate the α-convexity into a more usable form for the metric space case. Note that by rearranging terms

[2.1]
$$e((1-t)y_0 + ty_1) = (1-t)e(y_0) + te(y_1) - t(1-t)e(y_0 - y_1)$$

for all $y_0, y_1 \in H$ and $t \in \mathbf{R}$. Now we can deduce that iff $\phi : H \to (-\infty, \infty]$ is α-convex then it satisfies

[2.2]
$$\phi((1-t)y_0 + ty_1) \leqslant (1-t)\phi(y_0) + t\phi(y_1) - \alpha t(1-t)e(y_0 - y_1).$$

To see this note that ϕ is α-convex iff

[2.3]
$$\phi((1-t)y_0 + ty_1) - \frac{\alpha}{2}e((1-t)y_0 + ty_1) \leqslant (1-t)\phi(y_0) + t\phi(y_1) - \frac{\alpha}{2}(1-t)e(y_0) - \frac{\alpha}{2}te(y_1).$$

Now use (2.1). Since $e(y_0 - y_1) = \frac{1}{2}d(y_0, y_1)^2$ we can see that condition (2.3) can be expressed in terms of the distance function d in the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. In lemma 1.11 we introduced the function ψ which thus can be rewritten as

[2.4]
$$\psi(y) := \begin{cases} \frac{1}{2h}d(x,y)^2 + \phi(y) & y \in D(\phi), \\ \infty & \text{otherwise} \end{cases}$$

for $h > 0$ and for $x \in X$. We can see as follows from (2.1) that if ϕ is α-convex then ψ is $\left(\frac{1}{h} + \alpha\right)$-convex: $\boxed{\text{fill in}}$

We can now formulate additional assumptions on ϕ:

$[H_1]$ There exists a $\alpha \in \mathbf{R}$ such that for every $x, y_0, y_1 \in D(\phi)$ there exists a map $\gamma : [0,1] \to D(\phi)$ satisfying $\gamma(0) = y_0$ and $\gamma(1) = y_1$ for which the following inequality holds:

[2.5]
$$\begin{aligned} \frac{1}{2h}d(x,\gamma(t))^2 + \phi(\gamma(t)) \leqslant (1-t)&\left[\frac{1}{2h}d(x,y_0)^2 + \phi(y_0)\right] \\ &+ t\left[\frac{1}{2h}d(x,y_1)^2 + \phi(y_1)\right] - \left(\frac{1}{h} + \alpha\right)\frac{1}{2}t(1-t)d(y_0,y_1)^2 \end{aligned}$$

for every $t \in [0,1]$ and for every $h \in I_\alpha$.

We further assume

$[H_2]$ There exists $x_* \in D(\phi)$, $r_* > 0$ and $m_* \in \mathbf{R}$ such that $\phi(y) \geqslant m_*$ for every $y \in X$ satisfying $d(x_*, y) \leqslant r_*$.

2.1 Lemma. *Let $\phi : X \to (-\infty, \infty]$ be proper and satisfy [H_1] and [H_2]. Also let α be as in [H_1] and x_*, r_* and m_* be as in [H_2] then we have for every $y \in X$ that*

[2.6]
$$\begin{cases} \phi(y) \geqslant m_* & \text{if } d(x_*, y) \leqslant r_*, \\ \phi(y) \geqslant c - bd(x_*, y) + \frac{1}{2}d(x_*, y)^2 & \text{if } d(x_*, y) > r_*, \end{cases}$$

where $c := \phi(x_*)$ and $b := \frac{1}{r_*}(\phi(x_* - m_*) - \frac{1}{2}\alpha_+ r_*$ with $\alpha_+ := \max(\alpha, 0)$.

Proof. The first part of (2.6) is just the second hypothesis [H_2]. So we will prove the second part. Assume that $y \in D(\phi)$ where $d(x_*, y) > r_*$. From [H_1] with $x := x_*$, $y_0 = x_*$, $y_1 := y$ and $t := \frac{r_*}{d(x_*,y)}$ we find $y_* := \gamma(t) \in D(\phi)$ independent on $h \in I_\alpha$ such that

[2.7]
$$\begin{aligned} \frac{1}{2h}d(x_*, y_*)^2 + \phi(y_*) \leqslant (1-t)\left[\frac{1}{2h}d(x_*, x_*) + \phi(x_*)\right] \\ + t\left[\frac{1}{2h}d(x_*, y)^2 + \phi(y)\right] - \left(\frac{1}{h} + \alpha\right)\frac{1}{2}t(1-t)d(x_*, y)^2 \end{aligned}$$

for every $h \in I_\alpha$. Multiplying by $h > 0$ and sending h to zero in (2.7) we get

[2.8]
$$\begin{aligned} \frac{1}{2}d(x_*, y_*)^2 &\leqslant \frac{t}{2}d(x_*, y)^2 - \frac{1}{2}t(1-t)d(x_*, y)^2 \\ &= \frac{1}{2}(t - t(1-t))d(x_*, y)^2 \\ &= \frac{t^2}{2}d(x_*, y)^2 \\ &= \frac{1}{2}r_*^2. \end{aligned}$$

Rearranging terms in (2.7) and using the non-negativity of the first term we obtain

[2.9]
$$\phi(y) \geqslant \phi(x_*) - \frac{1}{t}(\phi(x_*) - m_*) - \left(\frac{1}{h} + \alpha\right)\frac{t}{2}d(x_*, y)^2 + \frac{\alpha}{2}d(x_*, y)^2.$$

In case of $\alpha \geqslant 0$ we let h tend to ∞ so we obtain

[2.10]
$$\phi(y) \geqslant \phi(x_*) - \frac{1}{t}(\phi(x_*) - m_*) - \alpha\frac{t}{2}d(x_*, y)^2 + \frac{\alpha}{2}d(x_*, y)^2,$$

so now we can use the definition of t to obtain (2.7). In the case of $\alpha < 0$ let h tend to $\frac{1}{|\alpha|}$. ∎

It will be convenient to define the following function

[2.11]
$$\Phi(h, x; y) := \frac{1}{2h}d(x, y)^2 + \phi(y), \quad y > 0, \; x, y \in X.$$

2.2 Corollary. *Let $\phi : X \to (-\infty, \infty]$ be proper and satisfy [H_1] and [H_2], let $\alpha \in \mathbf{R}$ be as in [H_1]. Then for every $h > 0$ satisfying $\frac{1}{h} + \alpha > 0$, for every $\overline{x} \in X$, $M > 0$ there exists $\beta > 0$ and $\gamma \in \mathbf{R}$ such that*

[2.12] $\Phi(h, x; y) \geqslant \beta d(\overline{x}, y)^2 + \gamma$ *for every $x \in X$ such that $d(x, \overline{x}) \leqslant M$ and for every $y \in X$.*

Proof. We can use

$$[2.13] \qquad d(x,y)^2 \geqslant (1-\epsilon^2)d(\overline{x},y)^2 - M^2\left(\frac{1}{\epsilon^2}-1\right).$$

To see this note

$$[2.14] \qquad d(\overline{x},y) - M \leqslant d(x,y)$$

so we can square both sides to obtain

$$[2.15] \qquad d(\overline{x},y)^2 + M^2 - 2Md(\overline{x},y) \leqslant d(x,y)^2,$$

now note that $2ab \leqslant a^2+b^2$ so

$$[2.16] \qquad \frac{\epsilon}{\epsilon}2Md(\overline{x},y) \leqslant \frac{M^2}{\epsilon^2} + \epsilon^2 d(\overline{x},y)^2$$

so we obtain

$$[2.17] \qquad d^2(\overline{x},y)^2 + M^2 - \frac{M^2}{\epsilon^2} - \epsilon^2 d(x,y)^2 \leqslant d(x,y)^2.$$

So after rearranging terms we get (2.13). Similarly we have

$$[2.18] \qquad d(x_*,y)^2 \leqslant (1+\eta^2)d(\overline{x},y)^2 + \left(1+\frac{1}{\eta^2}\right)d(x_*,\overline{x})^2,$$

for $0<\epsilon,\eta<1$. ∎

So this corollary implies that $y \mapsto \Phi(h,x;y)$ is bounded from below. We define $\phi_h(x)$ as its infimum on X.

2.3 Definition. *Let $\phi : X \to (-\infty,\infty]$ be proper and satisfy [H_1] and [H_2], $h+\frac{1}{\alpha} > 0$ with $h>0$ and let α be as in [H_1].*

$$[2.19] \qquad \phi_h(x) := \inf_{y\in X} \Phi(h,x;y).$$

Remark.

- ϕ_h is a map from X to $\mathbf{R}$.

2.4 Lemma. *Let $\phi : X \to (-\infty,\infty]$ be proper, lsc and satisfy [H_1] and [H_2]. For every $h \in I_\alpha$ the function $\phi_h : X \to \mathbf{R}$ is continuous and for every $x \in \overline{D(\phi)}$ the function $X \ni y \mapsto \Phi(h,x;y)$ has a unique global minimizer in $D(\phi)$ which we will denote by $J_h x$.*

Proof. First we will show the continuity of ϕ_h. We will do this by showing that ϕ_h is upper semicontinuous and lower semicontinuous. First we will show the upper semicontinuity To this end let $(x_n)_{n\geqslant 1}$ and $\overline{x} \in X$ be such that $x_n \to \overline{x}$. Now let $y \in D(\phi)$ then we have by definition of Φ that $\phi_h(x_n) \leqslant \Phi(h,x_n;y)$ for all $n \geqslant 1$. So,

$$[2.20] \qquad \limsup_{n\to\infty} \phi_h(x_n) \leqslant \limsup_{n\to\infty} \Phi(h,x_n;y) = \Phi(h,\overline{x},y),$$

where the last equality follows from the continuity of d. So now we can take the infimum over $y \in D(\phi)$ to obtain

[2.21] $$\limsup_{n\to\infty} \phi(x_n) \leqslant \phi_h(\overline{x}) < \infty.$$

This proves the upper semicontinuity, now we can prove the lower semicontinuity. Let $(y_n)_{n\geqslant 1} \in D(\phi)$ be such that (by definition of the inf)

[2.22] $$\Phi(h, x_n; y_n) \leqslant \phi_h(x_n) + \frac{1}{n}, \quad n \geqslant 1.$$

Now by corollary 2.2 and (2.20) we have $C > 0$ such that for all $n \geqslant 1$ we have that $d(\overline{x}, y_n) \leqslant C$. We also have that $\phi_h(\overline{x}) \leqslant \Phi(h, \overline{x}; y_n)$ for $n \geqslant 1$ hence

[2.23]
$\phi_h(\overline{x}) \leqslant \liminf_{n\to\infty} \Phi(h, \overline{x}; y_n)$
now because $d(\overline{x}, y_n)$ is bounded we have
$$= \liminf_{n\to\infty} \left\{ \frac{1}{2h} d(\overline{x}, y_n)^2 - \frac{1}{h} d(x_n, \overline{x}) d(\overline{x}, y_n) + \phi(y_n) \right\}$$
because $x_n \to \overline{x}$ we have
$$= \liminf_{n\to\infty} \left\{ \frac{1}{2h} (d(\overline{x}, y_n) - d(\overline{x}, x_n))^2 + \phi(y_n) \right\}$$
$$\leqslant \left\{ \frac{1}{2h} d(x_n, y_n)^2 + \phi(y_n) \right\}$$
$$\leqslant \liminf_{n\to\infty} \phi_h(x_n),$$

hence ϕ_h is also lower semicontinous hence using the upper semicontinuity ϕ_h is continuous. Let $\overline{x} \in \overline{D(\phi)}$ and let $(y_n)_{n\geqslant 1} \subset D(\phi)$ be a minimizing sequence, that is $\lim_{n\to\infty} \Phi(h, \overline{x}, y_n) = \phi_h(\overline{x})$. We will show that in view of the lower semicontinuity of $\Phi(h, \overline{x}, \cdot)$ and the completeness of (X, d) that it is sufficient to prove that $(y_n)_n$ is a Cauchy sequence. Suppose this is true, then let its limit be $\overline{y}$. Let γ be the infimum then

$$\gamma \leqslant \Phi(h, \overline{x}; \overline{y}) \leqslant \liminf_{n\to\infty} \Phi(h, \overline{x}; y_n) = \gamma.$$

Further note that $\Phi(h, \overline{x}, \overline{y}) < \infty$ hence $\overline{y} \in D(\phi)$. In order to show that (y_n) is a Cauchy sequence we use [H_1] with $x := x_n$, $y_0 := y_n$, $y_1 := y_m$ and $t = \frac{1}{2}$ where $D(\phi) \supset x_n \to \overline{x}$. Now let $C_1 > 0$ be such that $d(x_n, \overline{x}) \leqslant C_1$ for $n \geqslant 1$. From [H_1] we obtain a $y_{n,m} \in D(\phi)$ such that

[2.24] $$\Phi(h, x_n; y_{n,m}) \leqslant \frac{1}{2}\Phi(h, x_n; y_n) + \frac{1}{2}\Phi(h, x_n; y_m) - \frac{1}{8}\left(\frac{1}{h} + \alpha\right) d^2(y_n, y_m).$$

By noting that $\Phi(h, x_n; y_{n,m}) \geqslant \phi_h(x_n)$ we can quickly deduce by rearranging terms that

[2.25] $$d^2(y_n, y_m) \leqslant 4\left(\frac{1}{h} + \alpha\right)^{-1} [(\Phi(h, x_n; y_n) - \phi_h(x_n)) + \Phi(h, x_n; y_m) - \phi_h(x_n)],$$

for $m, n \geqslant 1$. We will show that the right-hand side of (2.25) tends to 0 as $m, n \to \infty$. For this we note that by corollary 2.2 we have that $\beta d(\overline{x}, y_n)^2 + \gamma \leqslant \Phi(h, \overline{x}; y_n) \leqslant \phi_h(\overline{x}) + \frac{1}{n} \leqslant C_2$ for all $n \geqslant 1$. So it follows that

[2.26]

$$|\Phi(h, x_n; y_n) - \Phi(h, \overline{x}; y_n)| = \frac{1}{2h}|d(x_n, y_n)^2 - d(\overline{x}, y_n)^2|$$

now note that $d(x_n, y_n) - d(\overline{x}, y_n) \leqslant d(x_n, \overline{x})$ so $(d(x_n, y_n) - d(\overline{x}, y_n))(d(x_n, y_n) + d(\overline{x}, y_n)) = d(x_n, y_n)^2 - d(\overline{x}, y_n)^2 \leqslant d(x_n, \overline{x})(d(x_n, y_n) + d(\overline{x}, y_n))$ so,

$$\begin{aligned} &\leqslant \frac{1}{2h} d(x_n, \overline{x})(d(x_n, y_n) + d(\overline{x}, y_n)) \\ &\leqslant \frac{1}{2h} d(x_n, \overline{x})(d(x_n, \overline{x}) + d(\overline{x}, y_n) + d(\overline{x}, y_n) \\ &\leqslant \frac{1}{2h} d(x_n, \overline{x})(C_1 + 2C_2) \to 0 \end{aligned}$$

when $n \to \infty$. So now we have

[2.27]
$$\begin{aligned} |\Phi(h, x_n; y_n) - \phi_h(x_n)| \leqslant |\Phi(h, x_n; y_n) - \Phi(h, \overline{x}; y_n)| + |\Phi(h, \overline{x}; y_n) - \phi_h(\overline{x})| \\ + |\phi_h(\overline{x}) - \phi_h(x_n)| \to 0, \end{aligned}$$

to see this note that the first term tends to 0 by (2.26), for the second note that

[2.28]
$$|\Phi(h, x_m; y) - \phi_h(x_n)| \leqslant |\phi_h(x_m) - \phi_h(x_n)| \to 0,$$

by the continuity of ϕ_h. The last term tends to 0 by the continuity of ϕ_h.

Finally similarly to (2.26) we have

[2.29]
$$\begin{aligned} |\Phi(h, x_m; y_m) - \Phi(h, x_n; y_m)| &= \frac{1}{2h}|d(x_m, y_m)^2 - d(x_n, y_m)^2| \\ &\leqslant \frac{1}{2h} d(x_m, x_n) \cdot 2(C_1 + C_2) \to 0 \text{ as } m, n \to \infty. \end{aligned}$$

Since $|\phi_h(x_n) - \phi_h(x_m)| \to 0$ we get that the right-hand side of (2.25) tends to 0 proving that the minimizer exists. To see uniqueness repeat the argument with two minimizing sequences. ■

2.5 Definition. *Let (Y, d_Y) be a metric space and $\phi : Y \to (-\infty, \infty]$ be proper. Further, let $x \in D(\phi)$. Then*

[2.30]
$$|\partial\phi|(x) := \begin{cases} \limsup\limits_{\substack{y \to x \\ y \neq x}} \dfrac{(\phi(x) - \phi(y))^+}{d(x, y)} & \text{if } x \text{ is not isolated in } D(\phi), \\ 0 & \text{otherwise.} \end{cases}$$

Set $D(|\partial\phi|) := \{x \in D(\phi) : |\partial\phi|(x) < \infty\}$. $|\partial\phi|(x)$ is called the local slope of ϕ at x.

2.6 Proposition. *Let $\phi : X \to (-\infty, \infty]$ be proper, lsc an let it satisfy [H_1] and [H_2]. Then*

1. *if $h > 0$, $1 + h\alpha > 0$ where the α is from [H_1], $x \in \overline{D(\phi)}$, then $J_h x \in D(|\partial\phi|)$ and*

$$|\partial\phi|(J_h x) \leqslant \frac{1}{h} d(x, J_h x). \tag{2.31}$$

2. *if $h > 0$, $1 + h\alpha > 0$, $x \in \overline{D(\phi)}$ then we have*

$$\phi(J_h x) \leqslant \phi_h(x) \leqslant \phi(x), \tag{2.32}$$

if $0 < h_0 < h_1$, $1 + h_i\alpha > 0$, then,

$$\phi_{h_1}(x) \leqslant \phi_{h_0}(x), \quad x \in X \tag{2.33}$$

$$d(J_{h_0}x, x) \leqslant d(J_{h_1}x, x), \quad x \in \overline{D(\phi)}, \tag{2.34}$$

$$\phi(J_{h_1}x) \leqslant \phi(J_{h_0}x), \quad x \in \overline{D(\phi)}, \tag{2.35}$$

3. *if $x \in \overline{D(\phi)}$, then*

$$d(x, J_h x) \downarrow 0 \text{ as } h \downarrow 0, \tag{2.36}$$

$$\phi(J_h x) \uparrow \phi(x) \text{ as } h \downarrow 0, \tag{2.37}$$

$$\phi_h(x) \uparrow \phi(x) \text{ as } h \downarrow 0. \tag{2.38}$$

4.

$$\overline{D(|\partial\phi|)} = \overline{D(\phi)}. \tag{2.39}$$

Proof. 1. By definition we have

$$\begin{aligned}\phi(J_h x) - \phi(y) &= \Phi(h, x; J_h x) - \Phi(h, x; y) + \frac{1}{2h} d(x, y)^2 - \frac{1}{2h} d(x, J_h x)^2 \\ &\leqslant \frac{1}{2h}(d(x, y)^2 - d(x, J_h x)^2) \\ &\leqslant \frac{1}{2h} d(y, J_h x)(d(x, y) + d(x, J_h x))\end{aligned} \tag{2.40}$$

for every $y \in D(\phi)$. If $J_h x$ is isolated in $D(\phi)$, then $|\partial\phi|(J_h x) = 0$ and so (2.31) holds. Otherwise there exists a sequence $(y_n) \subset D(\phi)$ such that $y_n \neq J_h x$ for $n \geqslant 1$ and $y_n \to J_h$ From(2.40) we obtain

$$\begin{aligned}\frac{\phi(J_h x) - \phi(y_n)}{d(J_h x, y_n)} &\leqslant \frac{1}{2h}(d(x, y_n) + d(x, J_h x)) \\ &\leqslant \frac{1}{2h}(d(y_n, J_h x) + d(x, J_h x) + d(x, J_h x)),\end{aligned} \tag{2.41}$$

hence

$$\limsup_{n\to\infty} \frac{\phi(J_h x) - \phi(y_n)}{d(J_h x, y_n} \leqslant \frac{1}{h} d(x, J_h x), \tag{2.42}$$

so

[2.43] $$|\partial\phi|(J_hx) = \limsup_{\substack{y\to J_hx\\ x\neq J_hx}} \frac{(\phi(J_hx)-\phi(y))^+}{d(J_hx,y)} \leqslant \frac{1}{h}d(x,J_hx).$$

2. For any $x \in \overline{D(\phi)}$ we have

[2.44] $$\phi(J_hx) \leqslant \phi(J_hx) + \frac{1}{2h}d(x,J_hx)^2 = \phi_h(x) \leqslant \phi(h,x;x) = \phi(x).$$

Further, let $0 < h_0 < h_1$ with $1+\alpha h_i > 0$. (2.33) is a consequence of the definition of ϕ_h. About (2.34) we have

[2.45]
$$\begin{aligned}
\frac{1}{2h_0}d(x,J_{h_0}x)^2 + \phi(J_{h_0}x) &\leqslant \frac{1}{2h_0}d(x,J_{h_1}x)^2 + \phi(J_{h_1}x)\\
&\leqslant \frac{1}{2h_0}d(x,J_{h_1}x)^2 - \frac{1}{2h_1}d(x,J_{h_1}x)^2 + \frac{1}{2h_1}d(x,J_{h_1}x)^2 + \phi(J_{h_1}x)\\
&= \left(\frac{1}{2h_0} - \frac{1}{h_1}\right)d(x,J_{h_1}x)^2 + \Phi(h_1,x;J_{h_1}x)\\
&\leqslant \left(\frac{1}{2h_0} - \frac{1}{h_1}\right)d(x,J_{h_1}x)^2 + \frac{1}{2h_1}d(x,J_{h_0}x)^2 + \phi(J_{h_0}x).
\end{aligned}$$

Hence

[2.46] $$\left(\frac{1}{2h_0} - \frac{1}{h_1}\right)d(x,J_{h_0}x)^2 \leqslant \left(\frac{1}{2h_0} - \frac{1}{h_1}\right)d(x,J_{h_1}x)^2,$$

so (2.34) follows.

From $\Phi(h_1,x;J_{h_1}x) \leqslant \Phi(h_1,x;J_{h_0}x)$ we obtain

[2.47] $$\phi(J_{h_1}x) \leqslant \frac{1}{2h_1}\underbrace{(d(x,J_{h_0}x)^2 - d(x,J_{h_1}x)^2)}_{\leqslant 0} + \phi(J_{h_0}x) \leqslant \phi(J_{h_0}x),$$

because of (2.34)

3. Note

[2.48]
$$\begin{aligned}
d(x,J_hx)^2 &= 2h\Phi(h,x;J_hx) - 2h\phi(J_hx)\\
&\leqslant 2h\Phi(h,x,y) - 2h\phi(J_hx)\\
&= d(x,y)^2 - 2h\phi(J_hx) + 2h\phi(y)
\end{aligned}$$

For every $y \in D(\phi)$. Since $-\phi(J_hx) \leqslant -\phi(J_{h_0}x)$, $0 < h < h_0$ we obtain

$$d(x,J_hx)^2 \leqslant -2h\phi(J_{h_0}) + d(x,y)^2 + 2h\phi(y),$$

taking the lim sup yields

$$\limsup_{h\to 0} d(x,J_hx)^2 \leqslant d(x,y)^2, \text{ for all } y \in D(\phi).$$

Now since $x \in \overline{D(\phi)}$ we can take $(y_n) \subset D(\phi)$ converging to x, so then we see

[2.49] $$\limsup_{h \to 0} d(x, J_h x)^2 = 0.$$

So (2.36) follows from (2.34) $\boxed{\text{and part i) but why??}}$.

(2.37) follows from (2.32) (bounded from above by ϕ), (2.34) (increasing) and by the lower semicontinuity

[2.50] $$\phi(J_h x) \leqslant \phi(x) \leqslant \liminf_{n \to \infty} \phi(x_n),$$

where $x_n := J_{\frac{1}{n}} x$ which converges to x by (2.36). So (2.37) follows.

For (2.38) we note that by (2.32) we have $\phi_h \leqslant \phi$, from (2.33) that ϕ_h is increasing and (2.37) gives the result by noting that

[2.51] $$\phi(J_h x) \leqslant \phi_h(x) \leqslant \phi(x).$$

For 4 and (2.39) we note that it one direction is direct and for the other one we need to show

[2.52] $$\overline{D(\phi)} \subset \overline{D(|\partial\phi|)}$$

$\boxed{\text{How??}}$ ∎

2.7 Proposition. *Let $\phi : X \to (-\infty, \infty]$ be proper, lsc and satisfy [H_1] and [H_2]. Then we have*

1. *For all $x \in D(\phi)$ and x is not isolated in $D(\phi)$:*

   [2.53] $$|\partial\phi|(x) = \sup_{\substack{y \in D(\phi) \\ y \neq x}} \left(\frac{\phi(x) - \phi(y)}{d(x,y)} + \frac{\alpha}{2} d(x,y) \right)^+$$

   where α is as in [H_1].

2. *The functional $|\partial\phi| : D(\phi) \to [0, \infty]$ is lsc.*

Proof. 2. We know that

[2.54]
$$|\partial\phi|(x) = \limsup_{\substack{z \to x \\ z \in D(\phi)}} \left(\frac{\phi(x) - \phi(z)}{d(x,z)} + \frac{1}{2}\rho d(x,z) \right)^+$$

because lim sup = inf sup we have

$$\leqslant \sup_{\substack{z \neq x \\ z \in D(\phi)}} \left(\frac{\phi(x) - \phi(z)}{d(x,z)} + \frac{1}{2}\rho d(x,z) \right)^+,$$

and in particular for $\rho = \alpha$. If the right-hand side of (2.53) is equal to zero we are done. In the other case we can restrict the set on which the supremum is taken to the elements $z \in D(\phi)$ and $z \neq x$ such that

[2.55] $$\phi(x) - \phi(z) + \frac{1}{2}\alpha d(x,z)^2 > 0.$$

Now we can use [H_1] with $x, y_0 := x$ and $y_1 := z$ where z satisfies (2.55). So we get

[2.56]
$$\begin{aligned}\frac{1}{2h}d(x,\gamma(t))^2+\phi(\gamma(t)) &\leqslant (1-t)\phi(x)+t\left[\frac{1}{2h}d(x,z)^2+\phi(z)\right]-\left(\frac{1}{h}+\alpha\right)\frac{1}{2}t(1-t)d(x,z)^2\\ &=(1-t)\phi(x)+\left[\frac{1}{2h}t-\left(\frac{1}{h}+\alpha\right)\frac{1}{2}t(1-t)\right]d(x,z)^2+t\phi(z)\\ &=(1-t)\phi(x)-\frac{1}{2}t\alpha d(x,z)^2+t\phi(z)+\frac{1}{2}t^2\left(\frac{1}{h}+\alpha\right)d(x,z)^2\\ &=\phi(x)-t(\phi(x)-\phi(z)+\frac{1}{2}\alpha d(x,z)^2)+\frac{1}{2h}t^2d(x,z)^2,\end{aligned}$$

so after multiplying by h and sending h to 0 we get,

[2.57]
$$d(x,\gamma(t))^2\leqslant t^2d(x,z)^2,\qquad t\in[0,1].$$

We can now use [H_1] again with the same x, y_0, y_1 and $(\gamma(t))_{t\in[0,1]}$, so we fix $h>0$ with $1+h\alpha>0$ and we obtain by deleting the first term in [H_1] that

[2.58]
$$\begin{aligned}\phi(x)-\phi(\gamma(t)) &\geqslant -\left[\frac{1}{2h}t-\left(\frac{1}{h}+\alpha\right)+\frac{1}{2}t(1-t)\right]d(x,z)^2+t\phi(x)-t\phi(z)\\ &=\left[\frac{\phi(x)-\phi(z)}{d(x,z)}-\frac{1}{2h}d(x,z)+\left(\frac{1}{h}+\alpha\right)\frac{1}{2}(t-1)d(x,z)\right]td(x,z)\\ &=\left[\frac{\phi(x)-\phi(z)}{d(x,z)}-\frac{1}{2h}(\alpha h(1-t)-t)d(x,z)\right]td(x,z),\end{aligned}$$

for every $t\in[0,1]$. Since $h>0$ is fixed in (2.58) and z satisfies (2.55) there is $t_0\in(0,1]$ such that the right-hand side of (2.58) is positive for $t\in(0,t_0)$. So $\gamma(t)\neq x$ for $t\in(0,t_0)$. For $t\in(0,t_0)$ we divide (2.58) by $d(x,\gamma(t))$, use the sign of the right-hand side together with (2.57) we obtain

[2.59]
$$\frac{\phi(x)-\phi(\gamma(t))}{d(x,\gamma(t))}\geqslant\frac{\phi(x)-\phi(z)}{d(x,z)}-\frac{1}{2h}(\alpha h(1-t)-t)d(x,z)$$

hence,

[2.60]
$$|\partial\phi|(x)\geqslant\limsup_{t\downarrow 0}\frac{\phi(x)-\phi(\gamma(t))}{d(x,\gamma(t)}\geqslant\frac{\phi(x)-\phi(z)}{d(x,z)}-\frac{1}{2}\alpha d(x,z)$$

and so

[2.61]
$$|\partial\phi|(x)\geqslant\sup_{\substack{z\neq x\\ z\in D(\phi)}}\left(\frac{\phi(x)-\phi(z)}{d(x,z)}+\frac{1}{2}\alpha d(x,z)\right)^+.$$

2. Let $x\in D(\phi)$ and $y\neq x$, $y\in D(\phi)$. Further let $(x_n)\subset D(\phi)$ with $x_n\to x$. Then there exists $n_0\geqslant 1$ such that $x_n\neq y$ for $n\geqslant n_0$. So we have

[2.62]
$$\liminf_{n\to\infty}\sup_{\substack{z\neq x_n\\ z\in D(\phi)}}\left(\frac{\phi(x_n)-\phi(z)}{d(x_n,z)}+\frac{1}{2}\alpha d(x_n,z)\right)^+\geqslant\liminf_{n\to\infty}\left(\frac{\phi(x_n)-\phi(y)}{d(x_n,y)}+\frac{1}{2}\alpha d(x_n,y)\right)^+$$

now because ϕ is lsc,

$$\geqslant\left(\frac{\phi(x)-\phi(y)}{d(x,y)}+\frac{1}{2}\alpha d(x,y)\right)^+.$$

Taking the supremum over $y \in D(\phi)$ and $y \neq x$ we obtain using (2.53) that

[2.63] $$|\partial\phi|(x) \leqslant \liminf_{n\to\infty} |\partial\phi|(x_n).$$

This concludes the proof. ■

The following estimates will be useful in what follows

2.8 Proposition. *Let $\phi : X\ to(-\infty, \infty]$ be proper, lsc and satisfy [H_1] and [H_2]. Further let $h > 0$, $1 + h\alpha > 0$. Then*

1. *for $x \in D(\phi)$,*

   [2.64] $$d(x, J_h x)^2 \leqslant 2(1 + h\alpha)^{-1} h[\phi(x) - \phi_h(x)]$$

2. *for $x \in D(|\partial\phi|)$, then*

   [2.65] $$\phi(x) - \phi_h(x) \leqslant \frac{1}{2}(1 + h\alpha)^{-1} h|\partial\phi|^2(x),$$

   [2.66] $$|\partial\phi|(J_h x) \leqslant (1 + h\alpha)^{-1}|\partial\phi|(x),$$

   [2.67] $$\phi(x) - \phi(J_h x) \leqslant \frac{1}{2} h(1 + h\alpha)^{-2}(2 + h\alpha)|\partial\phi|^2(x),$$

3. *for $x \in \overline{D(\phi)}$,* $\boxed{\textit{Fix labels}}$

   [2.68] $$x \in D(|\partial\phi|) \text{ iff } \sup_{\substack{h>0 \\ 1+h\alpha\geqslant\frac{1}{2}}} |\partial\phi|(J_h x) < \infty, \qquad \text{iff } \sup_{\substack{h>0 \\ 1+h\alpha\geqslant\frac{1}{2}}} |\partial\frac{d(x, J_h x)}{h} < \infty,$$

4. *for $x \in D(\phi)$*

   [2.69] $$x \in D(|\partial\phi|) \text{ iff } \sup_{\substack{h>0 \\ 1+h\alpha\geqslant\frac{1}{2}}} |\partial\frac{\phi(x) - \phi_h(x)}{h} < \infty$$

5. *for $x \in D(|\partial\phi|)$*

   [2.70] $$|\partial\phi|(x) = \lim_{h\to 0} |\partial\phi|(J_h x) = \lim_{h\to 0} \frac{d(x, J_h x)}{h} = \lim_{h\to 0}\left(2\frac{\phi(x) - \phi_h(x)}{h}\right)^{\frac{1}{2}}.$$

6. *for $x \in D(|\partial\phi|)$ $|\partial\phi|(x) = 0$ iff there exist $h_0 > 0$ with $1 + h_0\alpha > 0$ such that $x = J_{h_0}x$ iff for all $h > 0$ with $1 + \alpha h > 0$: $x = J_h x$*

Proof. $\boxed{\text{Later}}$ ■

2.9 Definition. *We will denote by J_h the opertor from $\overline{D(\phi)}$ into $D(\phi)$ defined by $x \mapsto J_h x$.*

The first main result is

2.10 Theorem. *Assume that (X, d) is a complete metric space and that $\phi : X \to (-\infty, \infty]$ is proper, lsc and satisfies conditions [H_1] with $\alpha \in \mathbf{R}$ and [H_2]. Then we have for every $x \in D(|\partial\phi|)$ (☢) with α of [H_1] one unique solution u with initial condition $u(0) = x$. Further the following holds:*

[2.71] $\lim_{n\to\infty} J^n_{\frac{t}{n}} x = u(t)$ *for every* $t > 0$,

[2.72] $u(t) \in D(|\partial\phi|)$ *for every* $t > 0$,

[2.73] $u|_{[0,T]} \in Lip([0, T]; X)$ *for every* $T > 0$,

[2.74] $[0, \infty) \ni t \mapsto \phi(u(t))$ *is nonincreasing,*

[2.75] $[0, \infty) \ni t \mapsto e^{\alpha t}|\partial\phi|(u(t)$ *is nonincreasing and right-continuous,*

[2.76] $\phi(u(t)) = \lim_{n\to\infty} \phi(J^n_{\frac{t}{n}} x)$ *for every* $t > 0$,

[2.77] $$\frac{1}{2}\int_0^t |\dot{u}|^2(s)\,ds + \frac{1}{2}\int_0^t |\partial\phi|^2(u(s))\,ds + \phi(u(t)) \leqslant \phi(x) \text{ for every } t \geqslant 0.$$

Finally we set

[2.78] $$S(t)x := u(t), \quad t \geqslant 0,$$

where u is the unique solution to (☢) with initial condition $u(0) = x$. In this case $(S(t))_{t\geqslant 0}$ is a contractive C_0-semigroup of operators on $D(|\partial\phi|)$, i.e.

[2.79] $$[S(t)]_{Lip} \leqslant e^{\alpha t}, \quad t \geqslant 0.$$

Proof. *Step 1* (A variational inequality for $J_h x$).

[2.80] $$\frac{1}{2h}[d(J_h x, z)^2 - d(x, z)^2] + \frac{\alpha}{2} d(J_h x, x)^2 + \phi_h(x) \leqslant \phi(z)$$

for every $z \in D(\phi)$. Because $J_h x$ is the minimum of Φ we have for every $\hat{z} \in D(\phi)$

[2.81] $$\frac{1}{2h} d(x, J_h x)^2 + \phi(J_h x) \leqslant \frac{1}{2h} d(x, \hat{z})^2 + \phi(\hat{z}).$$

Let $z \in D(\phi)$. So if we use [H_1] with $x := x_0$, $y_0 := z$ and $y_1 := J_h x$ and substituting $\hat{z} = \gamma(t)$, $t \in (0, 1)$ in (2.81) we obtain

[2.82] $$\begin{aligned} \frac{1}{2h} d(x, \hat{z})^2 + \phi(\hat{z}) \leqslant (1 - t)\left[\frac{1}{2h} d(x, z)^2 + \phi(z)\right] \\ + t\left[\frac{1}{2h} d(x, J_h x)^2 + \phi(J_h x)\right] - \left(\frac{1}{h} + \alpha\right)\frac{1}{2} t(1 - t) d(z, J_h x)^2. \end{aligned}$$

So now we can use (2.81) we get

[2.83] $$\begin{aligned} \frac{1}{2h} d(x, J_h x)^2 + \phi(J_h x) \leqslant (1 - t)\left[\frac{1}{2h} d(x, z)^2 + \phi(z)\right] \\ + t\left[\frac{1}{2h} d(x, J_h x)^2 + \phi(J_h x)\right] - \left(\frac{1}{h} + \alpha\right)\frac{1}{2} t(1 - t) d(z, J_h x)^2, \end{aligned}$$

rearranging terms and dividing by $(1-t)$ and letting t tend to 1 we obtain

[2.84] $$\frac{1}{2h}d(x,J_hx)^2+\phi(J_hx)\leqslant\frac{1}{2h}d(x,z)^2+\phi(z)-\frac{1}{2}\left(\frac{1}{h}+\alpha\right)d(J_hx,z)^2.$$

This proves (2.80).

Step 2 (an estimate for $d(J_\gamma^m x, J_\delta^n x)^2$). Let $x\in D(\partial\phi|)$, $\gamma,\delta>0$ such that $1+\alpha\gamma>0$ and $1+\alpha\delta>0$ and let m,n be nonnegative integers. We want to estimate $d(J_\gamma^m x, J_\delta^n x)^2$ where for $n=0$ we have $J_\delta^n x := x$. The idea is to first establish the estimate in the case $m=0$ or $n=0$ and then to find a recursive identity. We will restrict ourselves to the case $\alpha\leqslant 0$. *Case $n=0$ or $m=0$; $\alpha\leqslant 0$.* We have for $x\in D(|\partial\phi|)$, $\gamma>0$, $1+\alpha\gamma>0$ and $m\geqslant 1$:

[2.85] $$d(J_\gamma^m x,x)\leqslant(m\gamma)^2(1+\alpha\gamma)^{-2m}|\partial\phi|^2(x).$$

To see this set $z=x$ in (2.80), then we obtain

[2.86] $$\frac{1}{2h}(d(J_hx,x)^2-d(x,x)^2)+\frac{\alpha}{2}d(J_hx,x)^2+\phi_h(x)\leqslant\phi(x),$$

now multiply by $2h$ and replace h by γ we get

[2.87] $$(1+\gamma\alpha)d(J_hx,x)^2\leqslant 2\gamma[\phi(x)-\phi_\gamma(x)],$$

divide by $1+\alpha\gamma$ to obtain

[2.88] $$d(J_hx,x)^2\leqslant(1+\gamma\alpha)^{-1}2\gamma[\phi(x)-\phi_\gamma(x)],$$

so now

[2.89]
$$\begin{aligned}d(J_\gamma^m x,x)&\leqslant\left(\sum_{k=1}^m d(J_\gamma^k x,J_\gamma^{k-1}x)\right)^2\\&\leqslant m\sum_{k=1}^m d(J_\gamma^k x,J_\gamma^{k-1})^2\\&\leqslant m\sum_{k=1}^m[d(J_\gamma^k x,x)+d(J_\gamma^{k-1},x)]^2\end{aligned}$$

now using (2.84) we have

$$\leqslant 2m\gamma(1+\alpha\gamma)^{-1}\sum_{k=1}^m[\phi(J_\gamma^{k-1}x)-\phi_\gamma(J_\gamma^{k-1}x)].$$

By the triangle inequality and Cauchy-Schwarz. Now we can use (2.65) to obtain

[2.90] $$d(J_\gamma^m,x)\leqslant m\gamma^2(1-\alpha\gamma)^{-2}\sum_{k=1}^m|\partial\phi|^2(J_\gamma^{k-1}).$$

Now we can use (2.66) to obtain

[2.91]
$$d(J_\gamma^m x, x)^2 \leqslant m\gamma^2(1+\alpha\gamma)^{-2}\left(\sum_{k=1}^{m}(1+\alpha\gamma)^{-2(k-1)}\right)|\partial\phi|^2(x).$$

So now we can compute the sum and use some basic estimates to get since $\alpha \leqslant 0$ that

[2.92]
$$d(J_\gamma^m, x) \leqslant m^2\gamma^2(1+\alpha\gamma)^{-2m}|\partial\phi|^2(x),$$

which proves our claim. Similarly we have for $m = 0$, $n \geqslant 1$, $\alpha \leqslant 0$, $\delta > 0$ and $1 + \alpha\delta > 0$:

[2.93]
$$d(J_\delta^n, x) \leqslant n^2\delta^2(1+\alpha\delta)^{-2n}|\partial\phi|^2(x),$$

Case $n \geqslant 1$, $m \geqslant 1$ and $\alpha \geqslant 0$ we claim that

[2.94]
$$\begin{aligned} d(J_\gamma^m x, J_\delta^n x)^2 \leqslant |\partial\phi|^2(x)\cdot \max\left\{(1+\alpha\gamma)^{-2(m+1)}, (1+\alpha\delta)^{-2(n+1)}\right\} \\ \cdot\{[m\gamma - n\delta) + (m-n)\alpha\gamma\delta]^2 + (\gamma+\delta)\cdot\min(m\gamma, n\delta)\} \end{aligned}$$

To see this let $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, $x_0 = y_0 := x$, $x_i := J_\gamma x_{i-1}$. Now use $\phi(J_h x) \leqslant \phi_h(x) \leqslant \phi(x)$ and (2.80) to obtain

[2.95]
$$\frac{1}{2\gamma}[d(x_i x, z)^2 - d(x_{i-1}, z)^2] + \frac{\alpha}{2}d(x_i, z)^2 \leqslant \phi(z)$$

[2.96]
$$\frac{1}{2\delta}[d(y_i x, \hat{z})^2 - d(y_{i-1}, \hat{z})^2] + \frac{\alpha}{2}d(y_i, \hat{z})^2 \leqslant \phi(z)$$

Now we can set $z := y_j$ in (2.95) and $\hat{z} := x_i$ in (2.96) and adding we obtain

[2.97]
$$\begin{aligned} &\frac{1}{2\gamma}[d(x_i, y_j)^2 - d(x_{i-1}, y_j)] + \frac{1}{2\delta}[d(y_j, x_i)^2 - d(y_{j-1}, x_i)^2] \\ &= \left(\frac{1}{2\gamma} + \frac{1}{2\delta} + \alpha\right)d(x_i, y_j)^2 - \frac{1}{2\gamma}d(x_{i-1}, y_j)^2 - \frac{1}{2\delta}d(y_{j-1}, x_i)^2 \\ &\leqslant 0. \end{aligned}$$

So, multiplying with $2\gamma\delta$ we get

[2.98]
$$d(x_i, y_j)^2 \leqslant \frac{\delta}{\delta + \gamma + 2\gamma\delta\alpha}d(x_{i-1}, y_j)^2 + \frac{\gamma}{\delta+\gamma+2\gamma\delta\alpha}d(y_{j-1}, x_i)^2.$$

Now we multiply (2.98) by $(1+\alpha\gamma)^i(1+\alpha\delta)^j$ and then defining (also for $i, j = 0$),

[2.99]
$$a_{i,j} := (1+\alpha\gamma)^i(1+\alpha\delta)^j d(x_i, y_j)^2,$$

so we obtain

[2.100]
$$a_{i,j} \leqslant \frac{\gamma(1+\alpha\delta)}{\delta+\gamma+2\gamma\delta\alpha}a_{i,j-1} + \frac{\delta(1+\alpha\gamma)}{\delta+\gamma+2\gamma\delta\alpha}a_{i-1,j}.$$

So now we set

[2.101] $$\hat{\gamma} := \gamma(1+\alpha\delta), \quad \hat{\delta} := \delta(1+\alpha\gamma),$$

and we arrive at

[2.102] $$a_{i,j} \leqslant \frac{\hat{\gamma}}{\hat{\gamma}+\hat{\delta}} a_{i,j-1} + \frac{\hat{\delta}}{\hat{\gamma}+\hat{\delta}} a_{i-1,j}.$$

From (2.85), (2.101) and using that $\alpha \leqslant 0$ such that $(1+\alpha\gamma)^{-1} \geqslant 1$ and $(1+\alpha\delta)^{-1} \geqslant 1$ we get

[2.103] $$a_{i,0} \leqslant |\partial\phi|^2(x) \cdot (1+\alpha\gamma)^{-m}(1+\alpha\delta)^{-2}(i\hat{g}amma)^2,$$

similarly from (2.93) we obtain

[2.104] $$a_{0,j} \leqslant |\partial\phi|^2(x) \cdot (1+\alpha\delta)^{-n}(1+\alpha\gamma)^{-2}(j\hat{\delta})^2.$$

By pairwise comparison we obtain

[2.105] $$\begin{aligned} &|\partial\phi|^2(x) \max\{(1+\alpha\gamma)^{-m}(1+\alpha\delta)^{-2}, (1+\alpha\delta)^{-n}(1+\alpha\gamma)^{-2}\} \\ &\qquad\qquad \leqslant |\partial\phi|^2(x) \max\{(1+\alpha\gamma)^{-(m+2)}, (1+\alpha\delta)^{-(n+2)}\} \end{aligned}$$

$\boxed{\text{Now use Lemma 2}}$ *Step 3* (Convergence of $J^m_{\frac{t}{m}}$). Let $x \in D(|\partial\phi|)$, $t > 0$, $\alpha \leqslant 0$ and $n_0 \in \mathbf{N}$ be such that

[2.106] $$1 + \alpha\frac{t}{n_0} > 0.$$

Now let $m, n \geqslant n_0$, then $J^m_{\frac{t}{m}} x$ and $J^n_{\frac{t}{n}}$ are well defined by lemma 2.4 and because of (2.94) with $\gamma := \frac{t}{m}$, $\delta := \frac{t}{n}$ we obtain

[2.107] $$\begin{aligned} d(J_{\frac{t}{m}} x, J_{\frac{t}{n}} x) \leqslant |\partial\phi|(x) \cdot \max\left\{\left(1+\alpha\frac{t}{m}\right)^{-(m+1)}, \left(1+\alpha\frac{t}{n}\right)^{-(n+1)}\right\} \\ \cdot \left\{\left[\frac{m-n}{mn} t^2\alpha\right]^2 + \left(\frac{t}{n}+\frac{t}{m}\right) \cdot \min\{t,t\}\right\}^{\frac{1}{2}}, \end{aligned}$$

now we use $\frac{m-n}{mn} = \left(\frac{1}{m} - \frac{1}{n}\right)^2$ to obtain

[2.108] $$\begin{aligned} d(J_{\frac{t}{m}} x, J_{\frac{t}{n}} x) \leqslant |\partial\phi|(x) \cdot t \cdot \max\left\{\left(1+\alpha\frac{t}{m}\right)^{-(m+1)}, \left(1+\alpha\frac{t}{n}\right)^{-(n+1)}\right\} \\ \cdot \left[\frac{1}{m} + \frac{1}{n} + (\alpha t)^2 \left(\frac{1}{m} - \frac{1}{n}\right)^2\right]^{\frac{1}{2}}, \end{aligned}$$

this proves that $(J^n_{\frac{t}{n}} x)_{n \geqslant n_0}$ is a Cauchy sequence in the complete space (X, d), so we can set

[2.109] $$u(t) := \lim_{n\to\infty} J^n_{\frac{t}{n}} x, \quad t > 0,$$

with the estimate

$$[2.110] \qquad d(u(t), J_{\frac{t}{n}} x) \leqslant |\partial\phi|(x) \cdot t \cdot \max\left\{e^{-\alpha t}, \left(1+\alpha\frac{t}{n}\right)^{-(n+1)}\right\} \cdot \left[\frac{1}{n} + \left(\frac{\alpha t}{n}\right)^2\right]^{\frac{1}{2}}.$$

Now we need to show that $u(t) \in D(|\partial\phi|)$. by (2.66) we have

$$[2.111] \qquad |\partial\phi(J_{\frac{t}{n}} x) \leqslant \left(1+\alpha\frac{t}{n}\right)^{-1} |\partial\phi|(x),$$

hence by induction

$$[2.112] \qquad |\partial\phi(J^n_{\frac{t}{n}} x) \leqslant \left(1+\alpha\frac{t}{n}\right)^{-n} |\partial\phi|(x).$$

By the lower semicontinuity of $|\partial\phi|$ we get

$$[2.113] \qquad |\partial\phi|(u(t)) \leqslant e^{-\alpha t}|\partial\phi|(x), \quad t > 0.$$

Step 4 (Local Lipschitz continuity of u). Let $x \in D(|\partial\phi|)$ and further set

$$[2.114] \qquad u(0) := x,$$

and for $t > 0$, $\alpha \leqslant 0$, $u(t)$ is defined by (2.110). From (2.93) with $\delta := \frac{t}{n}$, $n \geqslant n_0$ and (2.106) we have

$$[2.115] \qquad d(J^n_{\frac{t}{n}} x, x) \leqslant t\left(1+\alpha\frac{t}{n}\right)^{-n} |\partial\phi|(x),$$

so if we take the limit when n tends to infinity we have

$$[2.116] \qquad d(u(t), u(0)) \leqslant te^{-\alpha t}|\partial\phi|(x).$$

Hence u is continuous at 0. Now let $0 < s < t$, $m = n \geqslant n_0$ and $\gamma := \frac{t}{n}$, $\delta := \frac{s}{n}$. If we apply (2.94) we have

$$[2.117] \qquad d(J^n_{\frac{t}{n}} x, J^n_{\frac{s}{n}} x)^2 \leqslant |\partial\phi|^2(x)\left(1+\alpha\frac{t}{n}\right)^{-2(n+1)} \cdot \left[(t-s)^2 + \left(\frac{t}{n}+\frac{s}{n}\right)\cdot s\right],$$

if we now take the limit as n tends to infinity we obtain

$$[2.118] \qquad d(u(t), u(s)) \leqslant |\partial\phi|(x)e^{-\alpha t}|t-s|, \quad 0 < s < t,$$

now taking the limit as s tends to 0 we obtain

$$[2.119] \qquad d(u(t), u(s)) \leqslant |\partial\phi|(x)e^{-\alpha t}|t-s|, \quad 0 \leqslant s < t.$$

If $\alpha > 0$ then u is also a solution to (☢) with $\alpha = 0$, hence we obtain for any $\alpha \in \mathbf{R}$ $\boxed{\text{WHY??}}$

$$[2.120] \qquad d(u(t), u(s)) \leqslant |\partial\phi|(x)e^{-\alpha^- t}|t-s|, \quad 0 \leqslant s < t.$$

Step 5 (u is a solution to (☢)). Let $x \in D(|\partial\phi|)$ and $\alpha \in \mathbf{R}$ as in $[H_1]$. If $\alpha > 0$, then for $h > 0$, $1 + h\alpha > 0$, $J_h x$ is well defined by lemma 2.4 and satisfies (2.80) so all the estimates that follow from this hold. We have defined u in (2.110). We will prove that if u is a solution to (☢) with initial condition $u(0) = x$ and $\alpha \leqslant 0$ as above.

If $\alpha > 0$, then for every $h > 0$, $J_h x$ is well defined by lemma 2.4 and also satisfies the variational inequality (2.80) where $\alpha > 0$, hence also for $\alpha = 0$. Thus it follows from the proofs of step 2, 3 and 4 that $J_h x$ satisfies all the estimates as well with $\alpha = 0$. So $\lim_{n\to\infty} J^n_{\frac{t}{n}} x$ exists for every $t > 0$ and so we can define $u(t)$ as in (2.110) for $t > 0$ and $u(0) = x$. In this case u satisfies (2.72) and (2.73). In this case we also want to prove that u is a solution to (☢) with $\alpha > 0$. To prove this we start from (2.80) with $\alpha > 0$. From now one we will take $\alpha \in \mathbf{R}$ and distinguish between the cases $\alpha \leqslant 0$ and $\alpha > 0$ if necessary. Because of (2.73) and proposition 1.9 it is sufficient to prove that u is an "integral solution" to (☢), this means that for every $0 < a < b$, $\phi \circ u \in L^1(a, b)$ and $\phi \circ u$ satisfies the integral formulation of proposition 1.9. If follows from the continuity of u (because then we can pass limits) that if $\phi \circ u \in L^1(a, b)$ and $\phi \circ u$ satisfies the integral ☢ with $0 < a < b$ where a, b are rational numbers, then u is an "integral solution" to (☢).

Let $0 < a < b$ with a, b rational numbers. So now there exists $s > 0$ rational, $p > q > 0$ integers such that $a = qs$ and $b = ps$. Let $k_0 \in \mathbf{N}$ be such that

$$1 + \alpha \frac{s}{k_0} > 0, \tag{2.121}$$

and $k \geqslant k_0$. Then we have

$$J^{qk}_{\frac{s}{k}} x = J^{qk}_{\frac{qs}{qk}} x = J^{qk}_{\frac{a}{qk}} x \to u(a), \tag{2.122}$$

and similarly

$$J^{pk}_{\frac{s}{k}} \to u(b). \tag{2.123}$$

Now we set $h := \frac{s}{k}$, now we also have $1 + \alpha\frac{s}{k_0} > 0$, $x_m := J^m_h x$, $m \geqslant 1$ is well defined because of lemma 2.4. Further we set $x_0 := x$, for all $z \in D(\phi)$, $m \geqslant 1$ we have because $\phi(J_h x) \leqslant \phi_h(x) \leqslant \phi(x)$ and (2.80)

$$\frac{1}{2}[d(x_m, z)^2 - d(x_{m-1}, z)^2] + \frac{\alpha h}{2} d(x_m, z)^2 + h\phi(x_m) \leqslant h\phi(z). \tag{2.124}$$

So, summing (2.124) from $m := qk + 1$ to pk we have

$$\begin{aligned} \frac{1}{2}[d(x_{pk}, z)^2 - d(x_{qk}, z)^2] + \frac{\alpha}{2}\frac{s}{k} \sum_{l=qk+1}^{pk} d(x_l, z)^2 + \frac{s}{k} \sum_{l=qk+1}^{pk} \phi(x_l) \\ \leqslant \frac{s}{k} \sum_{l=qk+1}^{pk} \phi(z) \\ = (b - a)\phi(z). \end{aligned} \tag{2.125}$$

Now we take the limit of (2.125) as k tends to infinity. Note that because of (2.122) and (2.123) that $\lim_{k\to\infty} x_{pk} = u(b)$ and $\lim_{k\to\infty} x_{qk} = u(a)$. The following lemma will take care of the limit in the third and fourth term of (2.125).

2.11 Lemma. *Let x, u, s, a, b, p, q be as above and let $k \geqslant k_0$ where k_0 satisfies $1 + \alpha\frac{s}{k_0} > 0$*

1. *if $\phi : X \to \mathbf{R}$ is Lipschitz continuous on bounded subsets of X, then*

[2.126] $$\lim_{k\to\infty} \frac{s}{k} \sum_{l=qk+1}^{pk} \phi\left(J^l_{\frac{s}{k}} x\right) = \int_a^b \phi(u(t))\, dt.$$

2. *if $\phi : X \to (-\infty, \infty]$ is proper, lsc and satisfies assumptions [H_1] with $\alpha \in \mathbf{R}$ and [H_2], then $\phi \circ u$ is lsc (hence Lebesgue measurable) and $\phi \circ |_{[a,b]}$ is bounded below. Further if $C \geqslant 0$ is such that $\phi(u(t)) + C \geqslant 0$, $t \in [a, b]$, then*

[2.127] $$\int_a^b \phi(u(t)) + C \leqslant \liminf_{k\to\infty} \frac{s}{k} \sum_{l=qk+1}^{pk} \phi(J^l_{\frac{s}{k}} x) + C(b-a).$$

In particular if the rhs of (2.127) is finite, then $\phi \circ u|_{[a,b]} \in L^1(a, b)$.

Before we prove lemma 2.11 we will apply it so prove that $\phi \circ u|_{[a,b]} \in L^1(a, b)$ and satisfies the integral form of ☢. Remember that $y \mapsto d(y, z)^2$ is Lipschitz continuous on bounded subsets of X, to see this note

[2.128] $$d(y,z)^2 - d(\hat{y}, z)^2 \leqslant d(y, \hat{y})(d(y,z) + d(\hat{y}, z)).$$

We can use lemma 2.11 to prove that the third term in (2.125) converges to $\frac{\alpha}{2}\int_a^b d(u(t), z)\, dt$ as k tends to infinity. So it follows that

[2.129]
$$\liminf_{k\to\infty} \sum_{l=qk+1}^{pk} \phi(x_l) \leqslant (b-a)\phi(z) - \frac{1}{2} d(u(b), z)^2 + \frac{1}{2} d(u(a), z)^2 - \frac{\alpha}{2}\int_a^b d(u(t), z)^2\, dt < \infty.$$

It follows from lemma 2.11 that $\phi \circ u|_{[a,b]} \in L^1(a, b)$ and from (2.127) that u satisfies integral ☢. Hence u is a solution to (☢). So now we can prove lemma 2.11.

Proof (of lemma 2.11). First we prove 1. Since $u|_{[a,b]} \in C[a, b]$ we have $\phi \circ u \in u|_{[a,b]}$ and

[2.130] $$\int_a^b \phi(u(t))\, dt = \lim_{k\to\infty} \frac{s}{k} \sum_{l=qk+1}^{pk} \phi\left(u\left(l\frac{s}{k}\right)\right).$$

Note that $\{u(l\frac{s}{k}) : k \geqslant k_0, qk + 1 \leqslant l \leqslant pk\} \subset u([a, b])$ is bounded in X. From (2.110) we get

[2.131]
$$\begin{aligned} d\left(u\left(l\frac{s}{k}\right), J^l_{\frac{s}{k}} x\right) &= d\left(u\left(l\frac{s}{k}\right), J^l_{\frac{sl}{lk}} x\right) \\ &\leqslant |\partial\phi|(x) s\frac{l}{k}\left[\frac{1}{l} + \left(\frac{\alpha l\frac{s}{k}}{l}\right)^2\right]^{\frac{1}{2}} \cdot \max\left\{e^{-\alpha t}, \left(1 + \alpha\frac{s}{k}\right)^{-(l+1)}\right\} \\ &\leqslant |\partial\phi|(x)\left(\frac{ls}{k}\right) \cdot \left[\frac{1}{l} + \left(\frac{\alpha l\frac{s}{k}}{l}\right)^2\right]^{\frac{1}{2}} C(|\alpha|, b), \end{aligned}$$

for some constant $C = C(|\alpha|, b) > 0$, since $e^{-\alpha l\frac{s}{k}} \leqslant e^{|\alpha|b}$ and $\lim_{k\to\infty}\left(1 + \frac{1}{\alpha}\frac{t}{n}\right)^{-(n+1)} = e^{-\alpha t} \leqslant e^{|\alpha|b}$. Since $0 < \frac{ls}{k} \leqslant b$, it follows that $\left\{J_{\frac{s}{k}} x : k \geqslant k_0, qk + 1 \leqslant l \leqslant pk\right\}$ is bounded in X (by writing out the limit definition). Let $k \geqslant k_0$ and let $qk + 1 \leqslant l \leqslant pk$. Then in view of the Lipschitz continuity of ϕ on bounded subsets of X there exists $M > 0$ such that

[2.132]
$$\left|\phi\left(u\left(l\frac{s}{k}\right)\right) - \phi\left(J^l_{\frac{s}{k}} x\right)\right| \leqslant Md\left(u\left(l\frac{s}{k}\right), J^l_{\frac{s}{k}} x\right).$$
now using (2.131) we get

$$\leqslant M|\partial\phi|(x)\left(l\frac{s}{k}\right)\left[\frac{1}{l} + \left(\alpha\frac{ls}{k}\right)^2 \frac{1}{l^2}\right]^{\frac{1}{2}} C(|\alpha|, b)$$
$$= M|\partial\phi|(x)C(|\alpha|, b)\left[1 + (\alpha s)^2\right]^{\frac{1}{2}} s\frac{l^2}{k}.$$

Hence,

[2.133] $$\frac{s}{k}\left|\sum_{l=qk+1}^{pk} \phi\left(u\left(l\frac{s}{k}\right)\right) - \phi\left(J^l_{\frac{s}{k}} x\right)\right| \leqslant M|\partial\phi|(x)C'(|\alpha|, b, s)\frac{1}{k^2}\sum_{l=qk+1}^{pk} \sqrt{l} = O\left(\frac{1}{\sqrt{k}}\right),$$

as k tends to infinity. Therefore in view of (2.131) we find (2.126). Now we prove 2. $u \in C([a, b]; X)$ and ϕ is lsc, $[a, b]$ compact hence $\phi \circ u|_{[a,b]}$ is bounded from below. Now let $\overline{C} \geqslant 0$ be such that $\phi(u(t)) + \overline{C} \geqslant 0$ for all $t \in [a, b]$. Then $\int_a^b \phi(u(t)) + \overline{C}\, dt$ is well defined (possibly equal to ∞). We now claim that ϕ is bounded from below on $B = \left\{J^l_{\frac{s}{k}} x : k \geqslant k_0, qk \leqslant l \leqslant pk\right\}$ where x is as in theorem 2.10. And k_0 satisfies $1 + \alpha\frac{s}{k_0} > 0$ and p, q as defined above. Note that $B \subset D(\phi)$. $\boxed{\text{WHY??}}$ Suppose for contradiction that ϕ is not bounded from below on B. For $k \geqslant k_0$ let $l_k \in \mathbf{N}$ be such that $qk \leqslant l_k \leqslant pk$ and

[2.134] $$\phi_k := \phi\left(J^{l_k}_{\frac{s}{k}} x\right) = \min\left\{J^l_{\frac{s}{k}} x : qk \leqslant l \leqslant pk\right\}$$

There exists a subsequence ϕ_{j_k} tending to $-\infty$ as $k \to \infty$. Let $t_k := l_k\frac{s}{k}$, $k \geqslant k_0$. Since $t_k \in [a, b]$ there exists a subsequence of t_{j_k} which we will still denote by t_{j_k} and $\bar{t} \in [a, b]$ such that $\lim_{k\to\infty} t_{j_k} = \bar{t}$. We now claim that

[2.135] $$d\left(u(\bar{t}), J^{l_{j_k}}_{\frac{s}{j_k}} x\right) = 0.$$

Clearly we have $d(u(\bar{t}), u(t_{j_k})) \to 0$. So set $m_k = l_{j_k}$. In view of (2.110), using the same constant as in (2.131)

[2.136] $$d\left(u\left(m_k \cdot \frac{m_k s}{m_k j_k}\right), J^{m_k}_{\frac{s}{j_k}} x\right) \leqslant |\partial\phi|(x)C(|\alpha|, b) \cdot b \cdot \left(\frac{q}{j_k} + (\alpha s)^2\frac{1}{j_k^2}\right)^{\frac{1}{2}} \to 0,$$

which proves the claim. Since ϕ is lsc we have

[2.137] $$\phi(u(\bar{t})) \leqslant \liminf_{k\to\infty} \phi\left(J^{m_k}_{\frac{s}{j_k}} x\right) = \liminf_{k\to\infty} \phi_{j_k} = -\infty,$$

this is a contradiction since we know that $\phi \circ u|_{[a,b]}$ is bounded from below. Hence ϕ is bounded from below on B and so there exists $C \geqslant \overline{C} \geqslant 0$ such that $\phi(y) + C \geqslant 0$ when $y = u(t)$ for some $t \in [a, b]$ or $y \in B$.

Let $\tilde{\phi}(y) := \max\{\phi(y), -C\}$, $y \in X$. Then $\tilde{\phi} : X \to (-\infty, \infty]$ is proper, lsc and satisfies $\tilde{\phi} \geqslant -C$, $\tilde{\phi}(u(t)) = \phi(u(t))$, $t \in [a, b]$ and $\tilde{\phi}(y) = \phi(y)$, $y \in B$. Now we approximate F by Lipschitz continuous functions ϕ_n. Let $\phi_n(y) := \inf\{\tilde{\phi}(z) : nd(y, z) : z \in X\}$ where $n \geqslant 1$, $y \in X$. One can then verify $\phi_n \geqslant -C$, $\phi_n \leqslant \phi_{n+1}$, $\phi_n \uparrow \tilde{\phi}$ and $\phi_n \in \mathrm{Lip}(X; \mathbf{R})$. To see this for all $y \in X$, $n \geqslant 1$ there exists (y_n) such that

$$\begin{aligned} \phi_n(y) &\geqslant \phi(y_n) + nd(y, y_n) - \frac{1}{n} \\ &\geqslant \inf \phi + nd(y, y_n) - 1. \end{aligned} \tag{2.138}$$

So

$$nd(y, y_n) \leqslant \phi_n(y) - \inf \tilde{\phi} + 1 \leqslant \tilde{\phi}(y) - \inf \tilde{\phi} + 1. \tag{2.139}$$

so $y_n \to y$. $\boxed{\text{fix the rest}}$ For each of $n \in \mathbf{N}$ we can apply part 2 of lemma 2.11 and we get

$$\begin{aligned} \int_a^b \phi_n(u(t))\, dt &= \lim_{k\to\infty} \frac{s}{k} \sum_{l=qk+1}^{pk} \phi_n\left(J^l_{\frac{s}{k}} x\right) \\ &\leqslant \liminf_{k\to\infty} \frac{s}{k} \sum_{l=qk+1}^{pk} \tilde{\phi}\left(J^l_{\frac{s}{k}} x\right) \\ &= \liminf_{k\to\infty} \frac{s}{k} \sum_{l=qk+1}^{pk} \phi\left(J^l_{\frac{s}{k}} x\right) \\ &=: J. \end{aligned} \tag{2.140}$$

Suppose that $J < \infty$, otherwise there is nothing to prove. We have $\phi_n + C \geqslant 0$ and

$$\int_a^b \phi_n(u(t)) + C\, dt \leqslant J + C(b - a), \quad n \geqslant 1, \tag{2.141}$$

So by the monotone convergence theorem and $\tilde{\phi}(u(t)) = \phi(u(t))$ for $t \in [a, b]$ we get

$$\int_a^b \phi(u(t)) + C\, dt \leqslant J + C(b - a), \quad n \geqslant 1, \tag{2.142}$$

so we get $\phi \circ u + C \in L^1(a, b)$, hence $\phi \circ u|_{[a,b]} \in L^1(a, b)$ and $\int_a^b \phi(u(t))\, dt \leqslant J$. ∎

Step 6 (proof of (2.71)-(2.76), (2.78) and (2.79)). The function u defined above is the unique solution to (☢) to theorem 1.7. (2.71) is clear from the definition of u, (2.72) follows from (2.113) which says

$$|\partial\phi|(u(t)) \leqslant e^{-\alpha t} |\partial\phi|(x), \quad t > 0, \tag{2.143}$$

and $x \in D(|\partial\phi|)$. (2.73) follows directly from (2.120).

Let $(S(t))_{t\geqslant 0}$ be the family of operators as defined in (2.78) by $S(t)x := u(t)$, $t \geqslant 0$. Then by (2.72) $S(t)$ maps $D(|\partial\phi|)$ into itself. Clearly $S(0)$ is the identity and if $h > 0$ and $v(t) := u(t+h)$, $t \geqslant 0$ then v is a solution to (☢) with initial value $v(0) = u(h)$. So by uniqueness we have $S(t+h)x = S(t)u(h) = S(t)S(h)x$ so $S(t+h) = S(t)S(h)$ which is the semigroup property of $(S(t))_{t\geqslant 0}$. Then (2.79) follows from (1.7). Now we can prove (2.74). We have

[2.144] $$\phi\left(J^n_{\frac{t}{n}}x\right) \leqslant \phi\left(J_{\frac{t}{n}}x\right) \leqslant \phi(x), \quad n \geqslant n_0, t > 0, x \in D(|\partial\phi|) \text{ and } 1 + \alpha\frac{t}{n_0} > 0.$$

Because ϕ is lsc, we have

[2.145] $$\phi\left(\liminf_{n\to\infty} J^n_{\frac{t}{n}}x\right) \leqslant \liminf_{n\to\infty} \phi\left(J^n_{\frac{t}{n}}x\right)$$

Hence, $\phi(S(t)x) = \phi(u(t)) \leqslant \phi(x)$. If $h > 0$, then

[2.146] $$\phi(u(t+h)) = \phi(S(t+h)x) = \phi(S(t)S(h)x) \leqslant \phi(S(h)x) = \phi(u(h)),$$

which proves (2.74). Similarly we have from (2.66) for $x \in D(|\partial\phi|)$, $t > 0$ and $n \geqslant n_0$

[2.147] $$|\partial\phi|\left(J_{\frac{t}{n}}x\right) \leqslant \left(1 + \alpha\frac{t}{n}\right)^{-1} |\partial\phi|(x),$$

hence

[2.148] $$|\partial\phi|\left(J^n_{\frac{t}{n}}x\right) \leqslant \left(1 + \alpha\frac{t}{n}\right)^{-n} |\partial\phi|(x).$$

So by lower semicontinuity we obtain $|\partial\phi|(u(t)) \leqslant e^{-\alpha t}|\partial\phi|(x)$.

[2.149] $$\begin{aligned} e^{\alpha(t+h)}|\partial\phi|(u(t+h)) &= e^{\alpha(t+h)}|\partial\phi|(S(t+h)x) \\ &= e^{\alpha(t+h)}|\partial\phi|(S(t)S(h)x) \\ &\leqslant e^{\alpha(t+h)}e^{-\alpha t}|\partial\phi|(S(h)x) \\ &= e^{\alpha h}|\partial\phi|(u(h)), \end{aligned}$$

which proves the first assertion in (2.75). For the right continuity let $t_n \downarrow t$, then

[2.150] $$\begin{aligned} e^{\alpha t}|\partial\phi|(u(t)) &\leqslant \liminf_{n\to\infty} e^{\alpha t_n}|\partial\phi|(u(t_n)) \\ & e^{\alpha t}|\partial\phi|(u(t)). \end{aligned}$$

So it remains to prove (2.76). We have by the lower semicontinuity of ϕ that

[2.151] $$\phi(u(t)) \leqslant \liminf_{n\to\infty} \phi\left(J^n_{\frac{t}{n}}x\right), \quad t > 0.$$

In view of (2.53) we have for $y \in D(\phi)$ and $z \in D(|\partial\phi|)$ that

[2.152] $$\phi(y) \geqslant \phi(z) - |\partial\phi|(z)d(y,z) + \frac{\alpha}{2}d(y,z)^2.$$

If we substitute $y = u(t)$ and $z = J^n_{\frac{t}{n}} x$ in (2.151) we have

[2.153] $$\phi\left(J^n_{\frac{t}{n}} x\right) \leqslant \phi(u(t)) + |\partial\phi|\left(J^n_{\frac{t}{n}} x\right) d\left(J^n_{\frac{t}{n}} x, u(t)\right) - \frac{\alpha}{2} d\left(J^n_{\frac{t}{n}} x, u(t)\right).$$

Using (2.66) we have

[2.154] $$|\partial\phi|\left(J^n_{\frac{t}{n}} x\right) \leqslant \phi(u(t)), \quad t > 0,$$

which together with (2.151) implies (2.76).

Step 7 (proof of (2.152)). We will need the following lemma

2.12 Lemma. *Let $\phi :\to (-\infty, \infty]$ be proper, lsc and satisfy assumptions [H_1] with $\alpha \in \mathbf{R}$ and [H_2]. Let $h > 0$ be such that $1 + \alpha h > 0$. Then for any $y \in D(\phi)$ we have*

[2.155] $$\phi(y) - \phi_h(y) = \frac{1}{2} \int_0^h \frac{d(y, J_s y)^2}{s^2}\, ds.$$

Proof (of lemma 2.12). Because of the assumptions on h, $J_s y$ is well defined for $0 < s \leqslant h$ and $s \mapsto d(y, J_s y)^2$ is nondecreasing by (2.36), hence Borel measurable and so is $\frac{d(y,J_s y)^2}{s^2}$. Let $N(y) \subset (0, h)$ denote the countable sets of points of discontinuity of $s \mapsto d(y, J_s y)^2$. Because $\lim_{\overline{h} \to 0} \phi_{\overline{h}}(y) = \phi(y)$ by (2.40), hence it is sufficient to prove

[2.156] $$\phi_{\overline{h}_0}(y) - \phi_{\overline{h}_1}(y) = \frac{1}{2} \int_{\overline{h}_0}^{\overline{h}_1} \frac{d(y, J_s y)^2}{s^2}\, ds \text{ for } 0 < \overline{h}_0 < \overline{h}_1 \text{ such that } 1 + \alpha\overline{h}_1 > 0.$$

We claim that $h \mapsto \phi_h(y) \in \mathrm{Lip}[\overline{h}_0, \overline{h}_1]$. Let $h_0, h_1 \in [\overline{h}_0, \overline{h}_1]$,

[2.157] $$\begin{aligned} \phi_{h_0}(y) - \phi_{h_1}(y) &\leqslant \Phi(h_0, y; J_{h_1} y) - \Phi(h_1, y; J_{h_1} y) \\ &= \frac{1}{2h_0} d(y, J_{h_1} y)^2 - \frac{1}{2h_1} d(y, J_{h_1 y})^2 \\ &\leqslant \frac{1}{2} \frac{h_1 - h_0}{h_0 h_1} d(y, J_{h_1 y})^2. \end{aligned}$$

If we choose $h_0 < h_1$ we get by (2.36) and (2.40) that

[2.158] $$|\phi_{h_0}(y) - \phi_{h_1}(y)| \leqslant (h_1 - h_0) \frac{1}{2} \frac{1}{\overline{h}_0^2} d(y, J_{\overline{h}_1} y)^2,$$

which proves the claim that $h \mapsto \phi_h(y)$ is Lipschitz continuous. So it follows that the derivative of $h \mapsto \phi_h(y)$ exists a.e. in $(\overline{h}_0, \overline{h}_1)$ and that

[2.159] $$\phi_{\overline{h}_0}(y) - \phi_{\overline{h}_1}(y) = \int_{\overline{h}_0}^{\overline{h}_1} \frac{d}{dh} \phi_h(y)\, dh$$

we now claim that for $h \in (\overline{h}_0, \overline{h}_1) \setminus N(y)$ that

[2.160] $$\frac{d}{dh} \phi_h(y) = -\frac{1}{2} \frac{d(y, J_h y)^2}{h^2},$$

which would imply (2.156). Interchanging h_0 and h_1 in (2.156)

[2.161] $$\phi_{h_0}(y) - \phi_{h_1}(y) \geqslant \frac{1}{2}\frac{h_1 - h_0}{h_0 h_1} d(y, J_{h_0 y})^2.$$

Assuming $h_0 < h_1$ in (2.156) and (2.161) we get

[2.162] $$\frac{1}{2}\frac{1}{h_0 h_1} d(y, J_{h_0} y)^2 \leqslant \frac{\phi_{h_0}(y) - \phi_{h_1}(y)}{h_0 - h_1} \leqslant \frac{1}{2}\frac{1}{h_0 h_1} d(y, J_{h_1} y)^2.$$

Recalling that $\lim_{h\to\bar{h}} d(y, J_h y)^2 = d(y, J_{\bar{h}} y)^2$ for $\bar{h} \ni N(y)$ so we obtain (2.161) for every $h \in (\bar{h}_0, \bar{h}_1) \setminus N(y)$. ∎

In order to prove (2.77) we introduce dyadic partitions of the interval $[0, t]$, for $k \geqslant 1$ we set

[2.163] $$h_k := \frac{t}{2^k}, \quad t_i^k := i h_k, \quad 0 \leqslant 1 \leqslant 2^k,$$

and we choose $k \geqslant k_0$ where k_0 satisfies

[2.164] $$1 + \alpha h_{k_0} > 0,$$

to make sure that $J_{h_k} x$ is well defined. We can use the notation $J^0_{h_k} x = x$ and introduce the following functions associated with the above partitions where $1 \leqslant i \leqslant 2^k$

[2.165] $$\bar{u}_k(s) := \begin{cases} x, & s = 0, \\ J^i_{h_k} x, & s \in (t^k_{i-1}, t^k_i], \end{cases}$$

[2.166] $$\tilde{u}_k(s) := \begin{cases} x, & s = 0, \\ J_{s - t_{i-1}k} J^{i-1}_{h_k} x, & s \in (t^k_{i-1}, t^k_i], \end{cases}$$

[2.167] $$v_k(s) := \begin{cases} 0, & s = 0, \\ \dfrac{d(J^i_{h_k} x, J^{i-1}_{h_k} x)}{h_k}, & s \in (t^k_{i-1}, t^k_i], \end{cases}$$

and,

[2.168] $$w_k(s) := \begin{cases} 0, & s = 0, \\ \dfrac{d(\tilde{u}_k(s), J^{i-1}_{h_k} x)}{s - t^k_{i-1}} & s \in (t^k_{i-1}, t^k_i], \end{cases}$$

Clearly, v_k and w_k are non-negative real valued Borel measurable on $[0, t]$. Note that

[2.169]
$$\frac{1}{2}\int_{t^k_{i-1}}^{t^k_i} w_k^2(s)\, ds = \frac{1}{2} w_k^2(s + t^k_{i-1}$$
$$= \frac{1}{2}\int_0^{h_k} \frac{d(J^{i-1}_{h_k} x, J_s J^{i-1}_{h_k})}{s^2}\, ds$$
and by (2.155) we have
$$= \phi(J^{i-1}_{h_k} x) - \phi_{h_k}(J^{i-1}_{h_k} x).$$

Now by the definition of ϕ_{h_k} we have

[2.170] $$\phi_{h_k}(J^{i-1}_{h_k}x) = \frac{1}{2h_k}d(J^{i-1}_{h_k}x, J^i_{h_k}) + \phi(J^i_{h_k}x),$$

now noting that

[2.171] $$\frac{1}{2h_k}d(J^{i-1}_{h_k}x, J^i_{h_k}) = \frac{1}{2}\int_{t^k_{i-1}}^{t^k_i} v_k^2(s)\,ds.$$

So we obtain

[2.172] $$\frac{1}{2}\int_{t^k_{i-1}}^{t^k_i} v_k^2(s)\,ds + \frac{1}{2}\int_{t^k_{i-1}}^{t^k_i} w_k^2(s)\,ds = \phi(J^{i-1}_{h_k}x) - \phi(J^i_{h_k}x).$$

Now we can sum from 1 to 2^k to obtain

[2.173] $$\frac{1}{2}\int_0^t v_k^2(s)\,ds + \frac{1}{2}\int_0^t w_k^2(s)\,ds = \phi(x) - \phi\left(J^{2^k}_{\frac{t}{2^k}}x\right).$$

By (2.76) we have

[2.174] $$\begin{aligned}\liminf_{k\to\infty}\frac{1}{2}\int_0^t v_k^2(s)\,ds + \liminf_{k\to\infty}\frac{1}{2}\int_0^t w_k^2(s)\,ds &\leqslant \liminf_{k\to\infty}\left(\frac{1}{2}\int_0^t v_k^2(s)\,ds + \frac{1}{2}w_k^2(s)\,ds\right)\\ &= \phi(x) - \phi(u(t)).\end{aligned}$$

Now we will prove that

[2.175] $$\int_0^t |\partial\phi|^2(u(s)) \leqslant \liminf_{k\to\infty}\int_0^t w_k^2(s)\,ds,$$

and for some subsequence (j_k) that

[2.176] $$\int_0^t |\dot{u}|^2(s)\,ds \leqslant \liminf_{k\to\infty}\int_0^t v_{j_k}^2(s)\,ds.$$

Now note that (2.174), (2.175) and (2.176) imply (2.77). We will first prove that for every $s \in [0,T]$

[2.177] $$\lim_{k\to\infty} d(u(s), \overline{u}_k(s)) = \lim_{k\to\infty} d(u(s), \tilde{u}_k(s)) = 0.$$

We clearly have that $d(u(0), \overline{u}(0)) = d(u(0), \tilde{u}(0)) = 0$. Now let $s \in (0,t]$ and $\epsilon > 0$. For every $k \geqslant k_0$ there is a unique $i \in \{1,\dots,2^k\}$ such that $s \in (t^k_{i-1}, t^k_i]$, $u \in \mathrm{Lip}([0,t];X)$. So there exists $k_1 \geqslant k$ such that

[2.178] $$d(u(s), u(t^k_{i-1})) \leqslant \frac{\epsilon}{2} \text{ for } k \geqslant k_1.$$

On the other hand by (2.110) we have $C_1 = C_1(t,k_0)$ such that

[2.179] $$d(u(t^k_{i-1}, \overline{u}_k(t^k_{i-1})) = d\left(u\left(t^k_{i-1}, J_{\frac{(i-1)h_k}{i-1}}{}^{i-1}x\right)\right) \leqslant |\partial\phi|(x)C_1\frac{1}{\sqrt{i-1}}.$$

Since $\lim_{k\to\infty}(i-1)2^{-k} = \lim_{k\to\infty t_{i-1}^k} = s > 0$ we have $\lim_{k\to\infty} i(k) = \infty$ so

$$\begin{aligned} d(u(s), \overline{u}_k(s)) &\leqslant d(u(s), u(t_{i-1}^k) + d(u(t_{i-1}^k, \overline{u}_k(s)) \\ &\leqslant \frac{\epsilon}{2} + d(u(t_{i-1}^k), \overline{u}_k(t_{i-1}^k)) + d(\overline{u}_k(t_{i-1}^k), \overline{u}_k(s)) \\ &\leqslant \epsilon \text{ for } k \text{ large enough.} \end{aligned} \tag{2.180}$$

Now we estimate $d(\overline{u}_k(s), \tilde{u}_k(s))$, $s \in (0,t]$. We have $\tilde{u}_k(s) = J_{\delta_k} J_{h_k}^{i-1} x$ where $i = i(k)$ is as above and $\delta_k := s - t_{i-1}^k$. So by using (2.64) and (2.65) we get

$$\begin{aligned} d(\overline{u}_k(s), \tilde{u}_k(s)) &\leqslant d(J_{\delta_k} J_{h_k}^{i-1} x, J_{h_k}^{i-1} x) + d(J_{h_k} J_{h_k}^{i-1} x, J_{h_k}^{i-1} x) \\ &\leqslant \delta_k (1 + \alpha\delta_k)^{-1} |\partial\phi|(J_{h_k}^{i-1} x) + h_k(1 + h_k\alpha)^{-1} |\partial\phi|(J_{h_k}^{i-1} x). \end{aligned} \tag{2.181}$$

By (2.66) for $x \in D(|\partial\phi|)$, $|\partial\phi|(J_{h_k}^{i-1} x)$ is bounded. Because $0 < \delta_k \leqslant h_k \to 0$ we have that $d(\tilde{u}_k(s), \overline{u}_k(s)) \to 0$. This implies the second part of (2.177). For $s \in (t_{i-1}^k, t_i^k)$ by (2.168), (2.31) and (2.166) we have

$$w_k(s) = \frac{d(J_{\delta_k} J_{h_k}^{i-1} x, J_{h_k}^{i-1} x)}{s - t_{i-1}^k} \geqslant |\partial\phi|(J_{\delta_k} J_{h_k}^{i-1} x) = |\partial\phi|(\tilde{u}_k(s)) = |\partial\phi|(\tilde{u}_k(s)). \tag{2.182}$$

Since $|\partial\phi|$ is lsc we get by (2.177) that

$$\liminf_{k\to\infty} w_k(s) \geqslant \liminf_{k\to\infty} |\partial\phi|(\tilde{u}_k(s)) \geqslant |\partial\phi|(u(s)). \tag{2.183}$$

So by Fatou's lemma we have

$$\int_0^t |\partial\phi|^2(u(s))\,ds \leqslant \int_0^1 \liminf_{k\to\infty} w_k^2(s)\,ds \leqslant \liminf_{k\to\infty} \int_0^t w_k^2(s)\,ds. \tag{2.184}$$

This proves (2.175). Now we establish (2.176). By (2.174) there exists a constant $M = M(t) > 0$ and a subsequence j_k such that

$$\int_0^t v_{j_k}^2(s)\,ds \leqslant M, \tag{2.185}$$

so a bounded sequence has a weakly convergent subsequence which we will still denote by v_{j_k} and $v_{j_k} \to \overline{v} \in L^2(0,t)$ weakly with $\overline{v} \geqslant 0$ a.e. and

$$\int_0^t \overline{v}^2(s) \leqslant \liminf_{k\to\infty} \int_0^t v_{j_k}^2(s)\,ds. \tag{2.186}$$

Since $d(\overline{u}_k(t_{i-1}^k), \overline{u}_k(t_i^k) = \int_{t_{i-1}^k}^{t_i^k} v_k(s)\,ds$ given $0 \leqslant s_1 < s_2 \leqslant t$ we can find sequences $(s_{1,k})$ and $(s_{2,k})$ converging to s_1 and s_2 respectively such that

$$d(\overline{u}_k(s_1), \overline{u}_k(s_2)) \leqslant d(\overline{u}_k(s_{1,k}), \overline{u}_k(s_{2,k})) \leqslant \int_{s_{1,k}}^{s_{2,k}} v_k(s)\,ds, \tag{2.187}$$

in view of (2.185) and (2.177) we have

[2.188]
$$\begin{aligned} d(u(s_1), u(s_2)) &\leqslant d(u(s_1), \overline{u}_k(s_1)) + d(u(s_1), \overline{u}_k(s_2)) \\ &\leqslant d(u(s_1), \overline{u}_k(s_1)) + d(u(s_2), \overline{u}_k(s_2)) + d(\overline{u}_k(s_1), \overline{u}_k(s_2)), \end{aligned}$$

taking the limit $k \to \infty$ we obtain

[2.189]
$$d(u(s_1), u(s_2)) \leqslant \int_{s_1}^{s_2} \overline{v}(s)\, ds.$$

So the metric derivative of u, $|\dot{u}|(s)$ satisfies $|\dot{u}|(s) \leqslant \overline{v}(s)$ a.e. by Lebesgue differentiation lemma on $(0, t)$. By (2.186) we have

[2.190]
$$\int_0^t |\dot{u}|^2(s)\, ds \leqslant \int_0^t \overline{v}^2(s)\, ds \leqslant \liminf_{k\to\infty} \int_0^t v_{j_k}^2(s)\, ds,$$

which is exactly (2.176). This completes the proof of the present theorem. ■

Now we can formulate and prove the main result of this section

2.13 Theorem. *Let (X, d) be a complete metric space and let $\phi : X \to (-\infty, \infty]$ be proper, lsc. Assume that [H_1] with $\alpha \in \mathbf{R}$ and [H_2] are satisfied. Then there exists a contractive C_0-semigroup $(S(t))_{t\geqslant 0}$ on $\overline{D(\phi)}$ satisfying $[S(t)]_{Lip} \leqslant e^{-\alpha t}$, $t \geqslant 0$ such that for every $x \in \overline{D(\phi)}$ the function $u : [0, \infty) \to X$ defined by $u(t) := S(t)x$, $t \geqslant 0$ is the unique solution to (☢) with initial condition $u(0) = x$. Further the following properties of the function u hold:*

1. *$\phi \circ u(t) \leqslant \phi_{c(t)}(x)$ for every $t > 0$ such that $1 + \alpha c(t) > 0$ where*

   [2.191]
   $$c(t) := \int_0^t e^{\alpha s}\, ds,$$

2. *the map $[0, \infty) \ni t \mapsto \phi \circ u(t)$ is nonincreasing and right-continuous,*
3. *the map $[0, \infty) \ni t \mapsto e^{-2\alpha^- t} \phi \circ u(t)$ is convex,*
4. *$u(t) \in D(|\partial\phi|)$ for every $t > 0$ and*

   [2.192]
   $$\frac{t}{2} |\partial\phi|^2(u(t)) \leqslant e^{2\alpha^- t}(\phi(x) - \phi_t(x))$$

   for every $t > 0$ such that $1 + \alpha t > 0$,
5. *the map $(0, \infty) \mapsto e^{\alpha t} |\partial\phi|(u(t))$ is nonincreasing and right-continuous,*
6. [2.193]
   $$\frac{d^+}{dt} \phi \circ u(t) = -|\partial\phi|^2(u(t)) = -|\dot{u}_+|^2(t)$$

   for every $t > 0$ where $|\dot{u}_+|(t) := \lim_{s\downarrow t} \frac{d(u(t), u(s))}{s-t}$ is the right metric derivative of u at t, 7.

[2.194] $$\phi \circ u(s) - \phi \circ u(t) = \int_s^t \frac{1}{2}|\partial\phi|^2(u(r)) + \frac{1}{2}|\dot{u}|^2(r)\, dr$$

for every $0 \leqslant s < t$*,*

8. *for every* $0 < a < b$*,* $u|_{[a,b]} \in Lip([a,b]; X)$ *and*

[2.195] $$[u|_{[a,b]}]_{Lip} \leqslant |\partial\phi|(u(a))e^{\alpha^-(b-a)},$$

9.

[2.196] $$u(t) = \lim_{n\to\infty} J^n_{\frac{t}{n}} x \textit{ for every } t > 0,$$

10.

[2.197] $$\phi(u(t)) = \lim_{n\to\infty} \phi(J^n_{\frac{t}{n}} x) \textit{ for every } t > 0,$$

11. *if* $\alpha > 0$ *then* ϕ *has a unique minimizer* $\overline{x} \in D(\phi)$ *and* $d(u(t), \overline{x}) \leqslant e^{-\alpha t} d(x, \overline{x})$ *for every* $t \geqslant 0$*,*

12. *if* $\alpha = 0$*, then*

[2.198] $$d\left(u(t), J^n_{\frac{t}{n}} x\right) \leqslant \frac{t}{n}\left[\phi(x) - \phi_{\frac{t}{n}}(x)\right] \leqslant \frac{t^2}{2n^2}|\partial\phi|^2(x), \textit{ for every } t > 0.$$

Proof. *Step 1* (Extension of $(S(t))_{t\geqslant 0}$). Let $x \in \overline{D(\phi)} = \overline{D(|\partial\phi|)}$ and let $t \geqslant 0$. Let $(S(t))_{t\geqslant 0}$ be the semigroup defined in theorem 2.10. Because $S(t) : D(|\partial\phi|) \to D(|\partial\phi|)$ is Lipschitz continuous and $\overline{D(|\partial\phi|)}$ is complete, there exists a continuous extension also denoted $S(t)$ to $\overline{D(\phi)}$. Clearly $S(t) : \overline{D(\phi)} \to \overline{D(\phi)}$ is also Lipschitz continuous and satisfies $[S(t)]_{\text{Lip}} \leqslant e^{\alpha t}$ to see this let $u, v \in \overline{D(\phi)}$ and (u_n), (v_m) their approximants, then

[2.199] $$\begin{aligned} d(S(t)u, S(t)v) &\leqslant d(S(t)u, S(t)u_n) + d(S(t)u_n, S(t)v_m) + d(S(t)v_m, S(t)v) \\ &\leqslant e^{-\alpha t} d(u, v) \text{ as } n, m \to \infty. \end{aligned}$$

Let $(x_n) \subset D(|\partial\phi|)$ be such that $x_n \to x$. Then for $t, s \geqslant 0$ we have $S(t+s)x = \lim S(t+s)x_n = \lim S(t)S(s)x_n = S(t)S(s)x$. Because $S(0) = I$, $(S(t))_{t\geqslant 0}$ satisfies the semigroup property. Further, let $t_n \geqslant 0$ be such that $t_n \to t$ and let $y \in D(|\partial\phi|)$.

[2.200]
$$d(S(t)x, S(t_n)x) \leqslant d(S(t)x, S(t)y) + d(S(t)y, S(t_n)y) + d(S(t_n)y, S(t_n)x)$$
$$\leqslant (e^{-\alpha t} + e^{-\alpha t_n})d(x, y) + d(S(t)y, S(t_n)y),$$

hence $d(S(t)x, S(t_n)x) \leqslant 2e^{\alpha t} d(x, y)$, because $D(|\partial\phi|)$ is dense in $\overline{D(\phi)}$ we have $\limsup d(S(t)x, S(t_n)x) = 0$ by taking limits, so $(S(t))_{t\geqslant 0} : \overline{D(\phi)} \to \overline{D(\phi)}$ is a C_0 α-contractive semigroup on $\overline{D(\phi)}$.

Step 2. ($u(t) := S(t)x$ is an integral solution to (☢)). Let (x_n) be as in step 1 and let $u_n(t) := S(t)x_n$, $n \geqslant 1$, $u(t) := S(t)x$, $t \geqslant 0$. Because $d(u_n(t), u(t)) \leqslant e^{\alpha t} d(x, x_n)$, the sequence (u_n) converges uniformly to u on intervals $[0, T]$, $T > 0$. Let $0 < a < b$ and $z \in D(\phi)$. ϕ lsc hence

$$\phi(u(b)) \leqslant \liminf_{n\to\infty} \phi(u_n(b)), \tag{2.201}$$

so there exists $C_1 \in \mathbf{R}$ such that $\phi(u_n(b)) \geqslant \phi(u(b)) - C_1 := C$. Because $\phi \circ u_n$ is nonincreasing on $[a, b]$, thus $\phi \circ u_n(t) \geqslant C$ for $t \in [a, b]$, $n \geqslant 1$. We have

$$\int_a^b \phi(u_n(t))\, dt \leqslant \frac{1}{2} d(u_n(a), z)^2 - \frac{1}{2} d(u_n(b), z)^2 - \frac{\alpha}{2} \int_a^b d(u_n(t), z)^2\, dt + (b - a)\phi(z). \tag{2.202}$$

We can now apply Fatou's lemma and notice that $\phi \circ u$ is lower semicontinuous hence Borel measure so we obtain

$$\int_a^b \phi \circ u(t) + c\, dt \leqslant \frac{1}{2} d(u(a), z)^2 - \frac{1}{2} d(u(b), z)^2 - \frac{\alpha}{2} \int_a^b d(u(t), z)^2\, dt + (b - a)(\phi(z) + c). \tag{2.203}$$

So, $\phi \circ u \in L^1(a, b)$ and u satisfies integral ☢.

Step 3 ($u(t) := S(t)x$ is a solution to (☢)). and the proof of 1, 2 and 4). To prove that u is a solution to (☢) it is suffcient to show that $u \in \mathrm{Lip}([a, b]; X)$ for every $0 < a < b$. Recall (and remember that it is proved under the condition $\alpha \leqslant 0$) (2.120) and by the semigroup property we have

$$d(u_n(t), u_n(s)) \leqslant |\partial\phi|(u_n(a)) e^{|\alpha|(b-a)}(t - s), \tag{2.204}$$

for $0 < a \leqslant s < t \leqslant b$, $n \geqslant 1$ where u_n is defined in step 2. So if we can find $a_0 > 0$ such that for every $a \in (0, a_0)$ $|\partial\phi|(u_n(a))$ is bounded, then u will be a solution to (☢). Set

$$c(t) := \int_0^t e^{\alpha s}\, ds \tag{2.205}$$

for $t > 0$ and we choose $a_0 > 0$ such that $1 + \alpha a_0 > 0$ and $1 + \alpha c(a_0) > 0$. If $0 < a < a_0$, then $1 + \alpha a > 0$ and $1 + \alpha c(a) > 0$ too. Let $a \in (0, a_0)$. We will first establish a bound for $\phi\left(u_n\left(\frac{a}{2}\right)\right)$ and prove 1. Because u_n satisfies (☢) we obtain by multiplication of (☢) by $e^{\alpha s}$ and integrating on $[0, t]$ that

$$\int_0^t e^{\alpha t} \frac{1}{2} \frac{d}{dt} d(u_n(t), z)^2\, dt \leqslant -\int_0^t e^{\alpha t} \phi(u_n(t))\, dt - \int_0^t \frac{\alpha}{2} e^{\alpha t} d(u_n(t), z)^2\, dt + \int_0^t e^{\alpha t} \phi(z)\, dt, \tag{2.206}$$

and noting that

$$\int_0^t e^{\alpha t} \frac{1}{2} \frac{d}{dt} d(u_n(t), z)^2\, dt = \int_0^t \frac{1}{2} \frac{d}{dt} e^{\alpha t} d(u_n(t), z)^2\, dt - \int_0^t \frac{\alpha}{2} e^{\alpha t} d(u_n(t), z)^2\, dt, \tag{2.207}$$

so

$$\frac{1}{2} e^{\alpha t} d(u_n(t), z)^2 - \frac{1}{2} d(u_n(0), z)^2 + \int_0^t e^{\alpha s} \phi(u_n(s))\, ds \leqslant c(t)\phi(z), \quad z \in D(\phi). \tag{2.208}$$

Now we use the fact that $\phi \circ u_n$ is nonincreasing to get

[2.209] $$\phi \circ u_n(t) \leqslant \frac{1}{c(t)} \int_0^t e^{\alpha s} \phi(u_n(s))\, s \leqslant \phi(z) + \frac{1}{2c(t)} d(u_n(0), z)^2.$$

Now assuming that $1 + \alpha c(t) > 0$ and taking the infimum over $z \in D(\phi)$ we obtain by definition

[2.210] $$(\phi \circ u_n)(t) \leqslant \phi_{c(t)}(u_n(0)).$$

Now $\phi_{c(t)}$ is continuous and $u_n(0)x_n \to x$ so there exists $C_1(t) > 0$ indepedent of n such that $(\phi \circ u_n)(t) \leqslant C_1(t)$, since if t' is close enough to t then $(\phi \circ u_n)(t) \leqslant \phi_{c(t)}(u_n(0)) + \epsilon$. In particular there holds that

[2.211] $$\phi\left(u_n\left(\frac{a}{2}\right)\right) \leqslant C_1\left(\frac{a}{2}\right), \quad n \geqslant 1.$$

Note also that since $\phi_{c(t)}$ is continuous and ϕ is lower semicontinuous, then for $t > 0$ such that $1 + \alpha c(t) > 0$ we have

[2.212] $$\liminf_{n \to \infty} \phi \circ u_n(t) \geqslant \phi \circ \liminf_{n \to \infty} u_n(t)$$

Hence $\phi \circ u(t) \leqslant \liminf \phi \circ u_n(t) \leqslant \phi_{c(t)}(x)$. This establishes 1. Now we will find a bound for $|\partial\phi|(u_n(a))$ and for this we first prove 4 in the special case $x \in D(|\partial\phi|)$. We denote the x by y in this case and we set $v(t) := S(t)y$, $t \geqslant 0$. Let $t > 0$ be such that $1 + \alpha t > 0$. From theorem 2.10, (2.77)

[2.213] $$\frac{1}{2} \int_0^t |\partial\phi|^2(v(s))\, ds \leqslant \phi(y) - \left[\phi(v(t)) + \frac{1}{2} \int_0^t |\dot{v}|^2(s)\, ds\right],$$

because $v \in \mathrm{Lip}([0, t]; X)$ (hence absolutely continuous) we have

[2.214] $$d(v(0), v(t)) \leqslant \int_0^t |\dot{v}|(s)\, ds,$$

and by Jensen's inequality we have

[2.215] $$\begin{aligned} \frac{1}{t} d(v(0), v(t))^2 &\leqslant t \left(\int_0^t |\dot{v}|(s) \frac{ds}{t}\right)^2 \\ &\leqslant \int_0^t |\dot{v}|^2(s)\, ds. \end{aligned}$$

So there follows tht

[2.216] $$\frac{1}{2} \int_0^t |\partial\phi|^2(v(s))\, ds \leqslant \phi(y) - \left[\phi(v(t)) + \frac{1}{2t} d(y, v(t))^2\right] \leqslant \phi(y) - \phi_t(y).$$

Now we use that $[0, \infty) \ni s \mapsto e^{-2\alpha^- s}|\partial\phi|^2(v(s))$ is nonincreasing. So

[2.217] $$\begin{aligned} \frac{t}{2} e^{-2\alpha^- t}|\partial\phi|^2(v(t)) &\leqslant \frac{1}{2} \int_0^t e^{2\alpha^- s}\, ds \cdot e^{-2\alpha^- t}|\partial\phi|^2(v(t)) \\ &\leqslant \frac{1}{2} \int_0^t e^{2\alpha^- s} e^{-2\alpha^- s}|\partial\phi|^2(v(s))\, ds \\ &\leqslant \phi(y) - \phi_t(y). \end{aligned}$$

This gives 4 in the case that $y = x \in D(|\partial\phi|)$. Now we can prove a bound $|\partial\phi|(u_n(a))$. To see this choose $y = u_n\left(\frac{a}{2}\right)$, so we have $u_n(a) = S\left(\frac{a}{2}\right) = v\left(\frac{a}{2}\right)$, so

[2.218] $$\frac{a}{4}e^{-2\alpha^- \frac{a}{2}}|\partial\phi|^2(u_n(a)) \leqslant \phi\left(u_n\left(\frac{a}{2}\right)\right) - \phi_{\frac{a}{2}}\left(u_n\left(\frac{a}{2}\right)\right),$$

both terms on the righthand side are bounded, one by C_1 and the other one by continuity by $\epsilon + C_1$. So there exists $C_2 > 0$ independent of $n \geqslant 1$ such that $|\partial\phi|(u_n(a)) \leqslant C_2$, $n \geqslant 1$. So u is a solution to (☢). Now we prove that $u(t) \in D(|\partial\phi|)$ for every $t > 0$. Observe that $u_n(a) = S\left(\frac{a}{2}\right)u_n\left(\frac{a}{2}\right)$ we have

[2.219] $$\frac{a}{4}e^{-2\alpha^- a}|\partial\phi|^2(u_n(a)) \leqslant \phi\left(u_n\left(\frac{a}{2}\right)\right) - \phi_{\frac{a}{2}}\left(u_n\left(\frac{a}{2}\right)\right) \leqslant \phi_{c(\frac{a}{2})}(x_n) - \phi_{\frac{a}{2}}\left(u_n\left(\frac{a}{2}\right)\right), \quad n \geqslant 1.$$

Since $|\partial\phi|$ is lower semicontinuous we obtain

[2.220] $$|\partial\phi|^2(u(a)) \leqslant \frac{4}{a}e^{\alpha^- a}\left[\phi_{c(\frac{a}{2})} - \phi_{\frac{a}{2}}\left(u\left(\frac{a}{2}\right)\right)\right] < \infty.$$

Hence $S(a)x \in D(|\partial\phi|)$ for every $x \in \overline{D(\phi)}$ and $a > 0$ such that $1 + \alpha a > 0$ and $1 + \alpha c(a) > 0$. By induction it follows that $S(t)x \in D(|\partial\phi|)$ for every $x \in \overline{D(\phi)}$ and $t > 0$. Now we prove 2. Let $t > 0$ be such that $1 + \alpha c(t) > 0$ and let $x \in \overline{D(\phi)}$. Then $\phi(S(t)x) \leqslant \phi_{c(t)}(x) \leqslant \phi(x)$ for every $x \in \overline{D(\phi)}$ and $t > 0$. So $\phi(S(nt)x) \leqslant \phi(S((n-1)t))x) \leqslant \phi(x)$ for every $n \geqslant 1$ and $x \in \overline{D(\phi)}$. If we now use the semigroup property we obtain $\phi(S(t+h)x) = \phi(S(t)S(h)x) \leqslant \phi(S(h)x)$, $t > 0$, $h > 0$ so this proves 2.

We now prove 4. Let $t > 0$ be such that $1 + \alpha t > 0$. So there exists $h_0 > 0$ such that $1 + \alpha(t+h) > 0$ for $0 < h \leqslant h_0$. Let $x \in \overline{D(\phi)}$. Since $S(h)x \in D(|\partial\phi|)$ we have by the preceding that

[2.221] $$\frac{t}{2}|\partial\phi|^2(S(t)S(h)x) \leqslant e^{2\alpha^- t}[\phi(S(hx) - \phi_t(S(h)x)] \leqslant e^{2\alpha^- t}[\phi(x) - \phi_t(x)].$$

Choosing a sequence $h_n \downarrow 0$ we have

[2.222] $$\frac{t}{2}|\partial\phi|^2(S(t)x) \leqslant \liminf_{n\to\infty}[\phi(x) - \phi_t(S(h_n)x)] = e^{2\alpha^- t}[\phi(x) - \phi_t(x)].$$

Step 4 (proof of 5 and 8). First we prove 5. Let $h > 0$, then $S(h)x \in D(|\partial\phi|)$ by 4, hence

[2.223] $$[0, \infty) \ni t \mapsto e^{\alpha t}|\partial\phi|(u(t+h)) = e^{\alpha t}|\partial\phi(S(t)S(h)x)$$

is nonincreasing by theorem 2.10 and right-continuous since $t \mapsto e^{\alpha t}|\partial\phi|(u(t+h))$ is lower semicontinuous. This completes the proof of 5. Now we prove 8. Let $0 < a < b$ and set $v(s) := u(s+a)$, $s \geqslant 0$. Then $v(0) \in D(|\partial\phi|)$ $\boxed{\text{continue}}$ ■