Assignment 2

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17: Let $\mathbb{R}_{fc}$ denote the set of all real numbers with the finite-complement topology, and define $f\colon \mathbb{R} \to \mathbb{R}_{fc}$ by $f(x) = x$. Show that f is continuous, but is not a homeomorphism.

Solution: Let U be any open set in $\mathbb{R}_{fc}$. Then $f^{-1}(U) = U$ by definition of f, so the first part of the problem can be reformulated as: show that the finite-complement topology is coarser than the standard topology on $\mathbb{R}$.

Now, let U be open in $\mathbb{R}_{fc}$ and let $x \in U$ be any point.
By definition, the complement of U in $\mathbb{R}$ is a finite set, say $\{x_1, \dots, x_n\}$ and $x \neq x_i$ for all $i = 1, \dots, n$. Let $\delta_i = |x - x_i| > 0$. Then $\delta = \min_i \{\delta_i\} > 0$, and we claim that $B(x, \delta) \subset U$ where the open ball is taken in the standard metric on $\mathbb{R}$.

Let $y \in B(x, \delta)$. Then

$$|y - x_i| \geq \underbrace{|x - x_i|}_{\geq \delta} - \underbrace{|y - x|}_{< \delta} > 0,$$

so $y \in U$. Hence $B(x, \delta) \subset U$. Since x was arbitrary, we can find such a $\delta_x = \delta$ for any x, so we can write $U \subset \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} B(x, \delta_x) \subset U$, and thus $U = \bigcup_{x \in U} B(x, \delta_x)$, so U is open in the standard topology on $\mathbb{R}$. Therefore f is continuous.

To see that this is not a homomorphism, note that $(0, \infty)$ is open in the standard topology as the union $\bigcup_{x \in (0,\infty)} B(x, |x|)$, however, $(0, \infty)^c = \mathbb{R} - (0, \infty) = (-\infty, 0]$ is not finite, so $(0, \infty)$ is not open in $\mathbb{R}_{fc}$, and since $f((0, \infty)) = (0, \infty)$, we find that f does not map open sets of $\mathbb{R}$ in the standard topology to open sets in $\mathbb{R}_{fc}$, hence f^{-1} is not continuous, so f is not a homeomorphism.

21: Define the maps $f\colon I^n = [-1, 1]^n \to D^n = \{x \mid \|x\| \leq 1\}$ and $g\colon D^n \to I^n$, with the euclidean norm, by

$$f(x) = \begin{cases} \frac{\max_i |x_i|}{\|x\|} x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

and

$$g(x) = \begin{cases} \frac{\|x\|}{\max_i |x_i|} x, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

f is continuous on $[-1, 1]^n - \{0\}$ since it is the composition of continuous functions there - where $\max_i |x_i|$ is the maximal norm, and g is continuous on $D^n - \{0\}$ since it is the composition of continuous functions there as well.

For $x = 0$, let $\varepsilon > 0$. Then let $\delta = \varepsilon$. For $\|x\| < \delta$, we have

$$\left|\max_i |x_i|\right|^2 \leq \sum_{i=1}^n x_i^2$$

so $|\max_i |x_i|| \leq \|x\|$, and thus

$$\|f(y_1, \dots, y_n)\| = \left|\max_i |x_i|\right| \leq \|x\| < \delta = \varepsilon.$$

Thus f is continuous at 0 too.
For given $\varepsilon > 0$, we can for g choose $\delta = \frac{\varepsilon}{\sqrt{n}}$, where we then get for $0 < \|x\| < \delta$ that since $\sum_{i=1}^n x_i^2 \leq n \max_i \{x_i^2\}$, we have

$$\|g(x_1, \dots, x_n)\| = \frac{1}{\max_i |x_i|} \sum_{i=1}^n x_i^2 = \sqrt{\frac{\sum_{i=1}^n x_i^2}{\max_i x_i^2}} \sqrt{\sum_{i=1}^n x_i^2} < \sqrt{n} \frac{\varepsilon}{\sqrt{n}} = \varepsilon.$$

Thus g is continuous at 0 as well.
Since $f \circ g = \mathbb{1}_{D^n}$ and $g \circ f = \mathbb{1}_{I^n}$, we have that f is a homeomorphism.

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28: If A, B are disjoint closed subsets of a metric space, find disjoint open sets U, V such that $A \subset U$ and $B \subset V$.

Solution: By lemma 2.14, we have that since A, B are disjoint closed subsets of a metric space X, there exists a continuous real-valued function on X which takes value 1 on A and -1 on B and values strictly in $(-1, 1)$ on $X - (A \cup B)$.
Let f denote this function. Then $A \subset f^{-1}((-\infty, 0)) = U$ and $B \subset f^{-1}((0, \infty)) = V$. Furthermore, since f is continuous and $(-\infty, 0)$ and $(0, \infty)$ are open, we have that $U = f^{-1}((-\infty, 0))$ and $V = f^{-1}((0, \infty))$ are open. Assume $y \in U \cap V$. Then $f(y) \in (-\infty, 0) \cap (0, \infty) = \varnothing$, so $U \cap V = \varnothing$. So U, V are open disjoint sets such that $A \subset U$ and $B \subset V$.