Conventions for this assignment: We assume all topological spaces to be nice enough for covering theory (we can even assume locally contractible). Basepoints are assumed to be good basepoints, namely the inclusion $\{x\} \subset X$ is assumed to have the homotopy extension property. If X is a space, then ΩX denotes its loop space and there is a fiber sequence

$$\Omega X \to PX \to X$$

where PX is a contractible space.

Problem 0.1. Show that the homology of $\Omega\left(S^2 \vee S^3\right)$ is

$$H_*\left(\Omega\left(S^2 \vee S^3\right); \mathbb{Z}\right) \cong \mathbb{Z}^{F_n}$$

where F_n is the n th Fibbonaci number (We set $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$).

Proof. Using the fiber sequence

$$\Omega\left(S^2 \vee S^3\right) \to P\left(S^2 \vee S^3\right) \to S^2 \vee S^3$$

we obtain the following quadrant double complex:

$$\begin{tikzcd}
\vdots \arrow[d, dash] & \vdots & \vdots & \\
H_3(\Omega(S^2 \vee S^3)) \arrow[d, dash] & H_3(\Omega(S^2 \vee S^3)) & H_3(\Omega(S^2 \vee S^3)) & \\
H_2(\Omega(S^2 \vee S^3)) \arrow[d, dash] \arrow[dr] \arrow[ddrr] & H_2(\Omega(S^2 \vee S^3)) & H_2(\Omega(S^2 \vee S^3)) & \\
H_1(\Omega(S^2 \vee S^3)) \arrow[d, dash] \arrow[dr, "\cong"] & H_1(\Omega(S^2 \vee S^3)) & H_1(\Omega(S^2 \vee S^3)) & \\
\mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \cdots
\end{tikzcd}$$

In this complex, representing the E^2 page, we obtain that the d_2 emminating from $H_1\left(\Omega\left(S^2 \vee S^3\right)\right)$ to $\mathbb{Z}$ must be an isomorphism, since in E^∞ which represents the homology of the total space $P\left(S^2 \vee S^3\right)$ which is contractible, we have that the $\mathbb{Z}$ at $(2,1)$ vanishes, hence it must vanish in E^2 as d_2 is the only nontrivial map emminating or terminating at $(2,1)$. This gives surjectivity of this map, and the same argument on $H_1\left(\Omega\left(S^2 \vee S^3\right)\right)$, which must also vanish, gives that d_2 must be injective as well.

Next, we come to the inductive part of the diagram. Note that in E^∞, all $H_n\left(\Omega\left(S^2 \vee S^3\right)\right)$ must vanish for $n \geq 1$. Furthermore, any map in E^k for $k \geq 4$ has horizontal length greater than the greatest horizontal distance between nontrivial objects of the double complex, hence all maps in E^k, for $k \geq 4$, must be trivial, so $E^4 = E^\infty$. Hence all the homologies of $\Omega\left(S^2 \vee S^3\right)$ must vanish because of the maps d_2 and d_3. Firstly, note that d_2 maps $d_2 \colon H_i\left(\Omega\left(S^2 \vee S^3\right)\right) \to H_{i-1}\left(\Omega\left(S^2 \vee S^3\right)\right)$, and in particular, this map must be surjective since it is the only nontrivial map terminating at the homologies in the second column which all must vanish. Next note that we similarly can see that the maps d_3 must be surjective (killing off the

terms in the third column) and injective as they must kill of the objects in the 0 th column.
Hence we find that we obtain a SES

$$0 \to H_{i-1}\left(\Omega\left(S^2 \vee S^3\right)\right) \to H_{i+1}\left(\Omega\left(S^2 \vee S^3\right)\right) \to H_i\left(\Omega\left(S^2 \vee S^3\right)\right) \to 0$$

Inserting $\mathbb{Z}$ for H_0 and H_1 when $i = 1$, we obtain, since $\mathbb{Z}$ is projective, that the sequence splits and $H_2\left(\Omega\left(S^2 \vee S^3\right)\right) \cong \mathbb{Z}^2$. Assume that $H_k\left(\Omega\left(S^2 \vee S^3\right)\right) \cong \mathbb{Z}^{F_k}$ for $k \leq N-1$. Then again

$$0 \to \mathbb{Z}^{F_{N-2}} \to H_N\left(\Omega\left(S^2 \vee S^3\right)\right) \to \mathbb{Z}^{F_{N-1}} \to 0$$

Again $\mathbb{Z}^{F_{N-1}}$ is projective, so the SES splits, so

$$H_N\left(\Omega\left(S^2 \vee S^3\right)\right) \cong \mathbb{Z}^{F_{N-2}} \oplus \mathbb{Z}^{F_{N-1}} \cong \mathbb{Z}^{F_{N-2}+F_{N-1}} \cong \mathbb{Z}^{F_N}.$$

Induction now finishes the proof.

□

Problem 0.2. Let G be a finite group acting freely on a space X. There is a homotopy fiber sequence

$$X \to X/G \to BG$$

in which the action of $\pi_1(BG)$ on $H_i(X)$ is induced by the action of G on X.

(1) Show that if G acts freely on S^n via orientation preserving maps (i.e., the action on $H_n(S^n)$ is trivial), then there is an isomorphism

$$H_{k+n+1}(BG) \xrightarrow{\cong} H_k(BG)$$

for $k > 0$ ($H_*(BG)$ is periodic).
You may use without proof that n-manifolds admit n-dimensional CW structures.

(2) Compute $H_*(BC_3; \mathbb{F}_3)$. By considering the Serre spectral sequence over $\mathbb{F}_3$, show that C_3 does not acts freely on $\mathbb{RP}^{2n}$.

(3) Compute $H_*\left(B\left(C_2 \times C_2\right); \mathbb{F}_2\right)$. Is there a free action of $C_2 \times C_2$ on S^n?

Proof. (1) By assumption, the action of $\pi_1(BG)$ on $H_i(S^n)$ is induced from the action of G on X which is by orientation preserving maps, so the action on $H_i(S^n)$ is trivial.
If BG is path-connected, we can use Theorem 1.3.
Since G is finite and acts freely on S^n, the action of G on S^n is continuous, free and proper, so the quotient space S^n/G is an n-manifold, and thus admits an n-dimensional CW structure.
In particular, all homology groups of dimension greater than n vanish for S^n/G.
In the double complex, we plot $H_s(BG, H_s(S^n))$ on the (s,t) th coordinate. In particular, for $t \neq 0, n$, the coefficient group vanishes, hence $H_s(BG, H_t(S^n))$ vanishes for $t \neq 0, n$.
For $t = 0, n$, $H_s(S^n) \cong \mathbb{Z}$, so letting $H_s(BG) = H_s(BG; \mathbb{Z})$ denote the integral homology, we obtain the following diagram from the double complex using the Serre spectral sequence

$$\begin{tikzcd}
\vdots \arrow[d, dash] & & & & & & & & \\
H_n(S^n) \cong \mathbb{Z} \arrow[d, dash] & H_1(BG) \arrow[drrrr, "\cong"] & H_2(BG) \arrow[drrrr, "\cong"] & \cdots & H_k(BG) \arrow[drrrr, "\cong"] & & & & \\
\mathbb{Z} \arrow[r, dash] & H_1(BG) \arrow[r] & H_2(BG) \arrow[r, dash] & \cdots \arrow[r, dash] & H_{1+n}(BG) & H_{2+n}(BG) & H_{3+n}(BG) & \cdots & H_{k+n+1}(BG)
\end{tikzcd}$$

In the diagram, we can conclude the isomorphisms because we know that the direct sum of the objects on the diagonals $n+1+k$ for $k \geq 0$ must be 0 (as this is the $(n+k+1)$ st homology of S^n/G which vanishes), and thus $H_{k+n+1}(BG)$ on the horizontal axis must be eliminated for $k \geq 0$. The only possibilities for nontrivial map terminating at these objects are the maps drawn in the diagram, hence these are surjective.
But similarly, the domain objects of these maps must be eliminated since they lie on the diagonals for $n+k+1$ with $k \geq 0$, hence all these maps must be injective too since these are the only nontrivial maps emminating or terminating from the objects.
From this, we obtain the desired isomorphisms $H_k(BG) \cong H_{n+k+1}(BG)$ for all $k \geq 1$.

(2) Since $\mathbb{F}_3$ is a field, there is no torsion and it is abelian, so $E^2_{s,t} = H_s\left(B, H_t(F)\right)$ simplifies to $E^2_{s,t} = H_s(B) \otimes_{\mathbb{Z}} H_t(F)$.
For $C_3 = \mathbb{Z}_3$, we obtain using the free action of $\mathbb{Z}_3$ on S^1 by $z \mapsto e^{\frac{2\pi i}{3}} z$, a quotient space which is the lens space $L(3;1)$. Using (1), we now have that $H_k(B\mathbb{Z}_3) \cong H_{k+2}(B\mathbb{Z}_3)$ for all $k > 0$.
We know the homology of the lens space is

$$H_k\left(L(3;1)\right) \cong \begin{cases} \mathbb{Z}, & k = 0, 1 \\ 0, & \text{else} \end{cases}$$

so

$$H_k\left(L(3;1); \mathbb{F}_3\right) \cong \begin{cases} \mathbb{Z}/3, & k = 0, 1 \\ 0, & \text{else} \end{cases}$$

Since $\mathbb{F}_3$ is a field, the homology of $L(3;1)$ in degree k will be the direct sum of the diagonal entries on the k diagonal.
So for the double complex, we want a $\mathbb{Z}/3$ in the 0 and 1 diagonal, and trivial groups in the rest.
We obtain the following diagram, and conclude that

$$H_n\left(B\mathbb{Z}_3\right) \cong \begin{cases} \mathbb{Z}, & n = 0 \text{ or } n \text{ odd} \\ \mathbb{Z}/m, & \text{else} \end{cases}$$

for some $m \in \mathbb{N}$

$$
\begin{tikzcd}
\vdots \arrow[d, dash] \\
\mathbb{Z} \arrow[d, dash] \arrow[rrd, two heads] & H_1(B\mathbb{Z}_3) \arrow[rrd, "\cong"] & H_2(B\mathbb{Z}_3) \arrow[rrd, "\cong"] & H_3(B\mathbb{Z}_3) \\
\mathbb{Z} \arrow[r, dash] & H_1(B\mathbb{Z}_3) \arrow[r, dash] & H_2(B\mathbb{Z}_3) \arrow[r, dash] & H_3(B\mathbb{Z}_3) \arrow[r] & \cdots
\end{tikzcd}
$$

Consider now the free action of $\mathbb{Z}_3$ on S^3 giving the lens space $L(3;1,1)$. Note that

$$
H_k(L(3;1,1)) \cong \begin{cases} \mathbb{Z}, & k = 0, 3 \\ \mathbb{Z}/3, & 0 < k < 3 \text{ odd} \\ 0, & \text{else} \end{cases}
$$

so

$$
H_k\left(L(3;1,1);\mathbb{F}_3\right) \cong \begin{cases} \mathbb{Z}/3, & k = 0, 1, 3 \\ 0, & \text{else} \end{cases}
$$

So we obtain the following double complex:

$$
\begin{tikzcd}
\mathbb{Z}/3 \arrow[d, dash] \arrow[rrrd] & H_1(B\mathbb{Z}_3;\mathbb{F}_3) \arrow[rrrd] & \cdots & H_4(B\mathbb{Z}_3;\mathbb{F}_3) & H_5(B\mathbb{Z}_3;\mathbb{F}_3) \\
\mathbb{Z}/3 \arrow[r, dash] & H_1(B\mathbb{Z}_3;\mathbb{F}_3) \arrow[r, dash] & \cdots \arrow[r, dash] & H_4(B\mathbb{Z}_3;\mathbb{F}_3) \arrow[r, dash] & H_5(B\mathbb{Z}_3;\mathbb{F}_3)
\end{tikzcd}
$$

It is then clear that there are no maps that can cancel the $H_1\left(B\mathbb{Z}_3;\mathbb{F}_3\right)$ and $H_2\left(B\mathbb{Z}_3;\mathbb{F}_3\right)$ and that these terms are the only contributions to the degree 1 and 2 homology of the lens space $L(3;1,1)$, respectively. In particular, this immediately gives that $H_1\left(B\mathbb{Z}_3;\mathbb{F}_3\right) \cong \mathbb{Z}/3$ and $H_2\left(B\mathbb{Z}_3;\mathbb{F}_3\right) \cong 0$ so by the first part,

$$
H_k\left(B\mathbb{Z}_3;\mathbb{F}_3\right) \cong \begin{cases} \mathbb{Z}/3, & k = 0 \text{ or } k \text{ odd and positive} \\ 0, & \text{else} \end{cases}
$$

Now suppose that $\mathbb{Z}_3$ acted freely on $\mathbb{RP}^{2n}$. Then the quotient space, just as in the first problem, is a $2n$-manifold, hence admits a $2n$-dimensional CW structure, hence $H_k(\mathbb{RP}^{2n}/\mathbb{Z}_3)$ vanishes for $k > 2n$. Now the double complex looks like the following diagram:

$$
\begin{tikzcd}
\mathbb{Z}/3 \arrow[d, dash] \arrow[rrrd, "d_{2n}"] & \underbrace{H_1(B\mathbb{Z}/3;\mathbb{F}_3)}_{\cong\mathbb{Z}/3} \arrow[rrrd] & \underbrace{H_2(B\mathbb{Z}/3;\mathbb{F}_3)}_{\cong 0} \arrow[rrrd] & & & \\
\mathbb{Z}/3 & \underbrace{H_1(B\mathbb{Z}/3;\mathbb{F}_3)}_{\cong\mathbb{Z}/3} & \cdots & \underbrace{H_{2n+1}(B\mathbb{Z}/3;\mathbb{F}_3)}_{\cong\mathbb{Z}/3} & \underbrace{H_{2n+2}(B\mathbb{Z}/3;\mathbb{F}_3)}_{\cong 0} & \underbrace{H_{2n+3}(B\mathbb{Z}/3;\mathbb{F}_3)}_{\cong\mathbb{Z}/3}
\end{tikzcd}
$$

In particular, since the homologies of $\mathbb{RP}^{2n}/\mathbb{Z}_3$ vanish in dimensions $> 2n$, we must have that the maps into $H_{2n+k}\left(B\mathbb{Z}/3,\mathbb{F}_3\right)$ for $k > 0$ on the horizontal line must be surjective when k is odd (to cancel the $\mathbb{Z}/3$). But consider for example the map $0 \cong H_2\left(B\mathbb{Z}/3;\mathbb{F}_3\right) \to H_{2n+3}\left(B\mathbb{Z}/3;\mathbb{F}_3\right) \cong \mathbb{Z}/3$. This cannot be surjective, so we obtain a contradiction. So the assumption of $\mathbb{Z}_3$ acting freely on $\mathbb{RP}^{2n}$ must be false.

$\square$