DIRECT AND SEMIDIRECT PRODUCT

1. Motivating direct product

In all fields of math, we often get a better understanding of objects by reducing them to a composition of smaller structures. Likewise, we now ask whether we can reduce groups to smaller constituents. We have seen that we can do this with finitely generated abelian groups using direct product, so how do we recognize direct product generally?

Definition 1.1 (Characteristic in)**.** A subgroup $H \leq G$ is *characteristic* in G, written $H \operatorname{char} G$, if $\sigma(H) = H$ for all $\sigma \in \operatorname{Aut}(G)$.

Recall that $HK = \{hk \mid h \in H, k \in K\}$.

Proposition 1.2. *Let G be a group, let $x, y \in G$ and let $H \leq G$. Then*

1. $xy = yx\,[x, y]$.
2. $H \trianglelefteq G$ *if and only if* $[H, G] \leq H$.
3. $\sigma\,[x, y] = [\sigma(x), \sigma(y)]$ *for any* $\sigma \in \operatorname{Aut}(G)$*,* $[G, G] \operatorname{char} G$ *and* $G/\,[G, G]$ *is abelian.*
4. $G/\,[G, G]$ *is the largest abelian quotient of* G *in the sense that if* $H \trianglelefteq G$ *and* G/H *is abelian, then* $[G, G] \leq H$*. Conversely, if* $[G, G] \leq H$*, then* $H \trianglelefteq G$ *and* G/H *is abelian. (Note, in particular, that choosing* $H = [G, G]$*, we get* $[G, G] \trianglelefteq G$*).*
5. *If* $\varphi\colon G \to A$ *is any homomorphism of* G *into an abelian group* A*, then* φ *factors through* $[G, G]$*, i.e.,* $[G, G] \leq \ker\varphi$ *and the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & G/\,[G,G] \arrow[d] \\ & A \end{tikzcd}$$

Proposition 1.3. *Let $H, K \leq G$. The number of distinct ways of writing each element of the set HK in the form hk for some $h \in H$ and $k \in K$ is $|H \cap K|$. In particular, if $H \cap K = 1$, then each element of HK can be written uniquely as a product hk for some $h \in H$ and $k \in K$.*

Theorem 1.4. *Suppose G is a group with subgroups H and K such that*

1. H *and* K *are normal in* G*, and*
2. $H \cap K = 1$.

Then $HK \cong H \times K$.

Definition 1.5 (Internal and external direct product)**.** If G is a group and H and K are normal subgroups of G with $H \cap K = 1$, we call HK the *internal direct product* of H and K. We shall (when emphasis is called for) call $H \times K$ the *external direct product* of H and K.

2. Motivating semidirect product

If we have a group G with subgroup H and K such that

- $H \trianglelefteq G$ (but K is not necessarily normal in G), and
- $H \cap K = 1$,

then HK is a subgroup of G and there is an isomorphism of <u>sets</u> $HK \cong H \times K$ given by $(h,k) \mapsto hk$. However, this isomorphism may <u>not</u> be a group isomorphism. In the group $HK \leq G$, the operation is as follows

$$\begin{aligned}(h_1k_1)(h_2k_2) &= h_1k_1h_2\left(k_1^{-1}k_1\right)k_2 \\ &= h_1\left(k_1h_2k_1^{-1}\right)k_1k_2 \\ &= h_3k_3\end{aligned} \tag{Ω}$$

where $h_3 = h_1\left(k_1h_2k_1^{-1}\right)$ and $k_3 = k_1k_2$ - note that since $H \lhd G$, $k_1h_2k_1$, so $h_3 \in H$ and $k_3 \in K$.

These calculations were predicated on the assumption that there already exists such a group G containing subgroups H and K with $H \lhd G$ and $H \cap K = 1$.

Question 2.1. Suppose we are given two (abstract) groups H and K. Can we construct a group G containing isomorpic copies of H and K such that $H \lhd G$ and $H \cap K = 1$? That will ensure that HK is an isomorphic copy of a subgroup of G and each element of HK can be written uniquely as a product hk.

Note that in (Ω), we multiplied k_1 and k_2 in K and h_1 and $k_1h_2k_1^{-1}$ in H. So if we can understand how to obtain $k_1h_2k_1^{-1}$ without reference to an operation in G, we can define the product of $(h_1,k_1)(h_2,k_2)$ to be $\left(h_1\left(k_1h_2k_1^{-1}\right), k_1k_2\right)$ whatever $k_1h_2k_1^{-1}$ is determined to mean.
Here is one thing to notice: H is normal in G, so K acts on H by conjugation:

$$k \cdot h = khk^{-1}, \quad \text{for } h \in H, k \in K.$$

Thus

$$(h_1k_1)(h_2k_2) = (h_1k_1 \cdot h_2)(k_1k_2)$$

Let us introduce some new notation

Definition 2.2 (Conjugation)**.** We define the conjugate $y^x := x^{-1}yx$. In a group, this satisfies the following properties of exponents:

$$\begin{aligned}(y_1y_2)^x &= y_1^xy_2^x \\ y^{x_1x_2} &= (y^{x_1})^{x_2}\end{aligned}$$

Thus we can rewrite

$$(h_1k_1)(h_2k_2) = \left(h_1h_2^{k_1}\right)(k_1k_2)$$

So assuming G exists, the action of K on H by conjugation gives a homomorphism $\phi\colon K \to \mathrm{Aut}(H)$. Thus multiplication in HK depends only on multiplication in H, multiplication in K and the homomorphism $\phi\colon K \to \mathrm{Aut}(H)$.

Now we assume these things and reverse the construction

Theorem 2.3. *Let H and K be groups and let $\varphi\colon K \to \mathrm{Aut}(H)$ be a homomorphism. Let h^k denote the action of k on h determined by φ for arbitrary k and h. Let G be the set of ordered pairs (h,k) with $h \in H$ and $k \in K$ and define the following multiplication on G :*

$$(h_1,k_1)(h_2,k_2) = \left(h_1h_2^{k_1}, k_1k_2\right).$$

(1) *This multiplication makes G into a group of order $|G| = |H|\,|K|$.*
(2) *The sets $\{(h, 1) \mid h \in H\}$ and $\{(1, k) \mid k \in K\}$ are subgroups of G and the maps $h \mapsto (h, 1)$ for $h \in H$ and $k \mapsto (1, k)$ for $k \in K$ are isomorphisms of these subgroups with the groups H and K, respectively:*
$$H \cong \{(h, 1) \mid h \in H\}, \quad \text{and} \quad K \cong \{(1, k) \mid k \in K\}.$$
Identifying H and K with their isomorphic copies in G described in (2), we have
(3) $H \trianglelefteq G$
(4) $H \cap K = 1$
(5) *for all $h \in H$ and $k \in K$, $khk^{-1} = h^k = \varphi(k)(h)$.*

Here, part (5) is carried out using the identifications from (2) as follows:
$$\begin{aligned}(1, k)(h, 1)(1, k)^{-1} &= ((1, k)\,(h, 1))\,(1, k)^{-1} \\ &= \left(h^k, k\right)\left(1, k^{-1}\right) \\ &= \left(h^k 1^k, 1\right) \\ &= \left(h^k, 1\right)\end{aligned}$$
so indeed $khk^{-1} = h^k$ under the identifications.

Definition 2.4 (Semidirect product)**.** Let H and K be groups and φ be a homomorphism from K into $\mathrm{Aut}(H)$. The group described in 2.3 is called the *semidirect product* of H and K with respect to φ and will be denoted by $H \rtimes_\varphi K$ (when there is no danger of confusion, we shall simply write $H \ltimes K$).

Proposition 2.5. *Let H and K be groups and let $\varphi\colon K \to \mathrm{Aut}(H)$ be a homomorphism. Then the following are equivalent:*
(1) *the identity (set) map between $H \rtimes K$ and $H \times K$ is a group homomorphism (hence an isomorphism)*
(2) *φ is the trivial homomorphism from K into* $\mathrm{Aut}(H)$
(3) *$K \trianglelefteq H \rtimes K$.*

Corollary 2.6. *When H is abelian, $H \rtimes K \cong H \oplus K = H \times K$.*

Proof. When H is abelian, $h = khk^{-1} = h^k = \varphi(k)(h)$, so $\varphi(k) = \mathrm{id}$ for all k, so $\varphi\colon K \to \mathrm{Aut}(H)$ is the trivial homomorphism. Now Proposition 2.5 gives that $H \rtimes K \cong H \times K = H \oplus K$. □