1. Orientations

We begin by attempting to give complete rigour and detail to the definitions of orientation and the many connected theorems.

For this section, we will follow [1] and [2]

Definition 1.1 (Local Homology Group)**.** For $h_*(-)$ a homology theory and an n-manifold M, groups of the form $h_k(M, M - \{x\})$ are called *local homology groups*.

For a chart $\varphi\colon U \to \mathbb{R}^n$ on M centered at x, we get by excision that

$$h_k(M, M - \{x\}) \cong h_k(U, U - \{x\}) \xrightarrow{\varphi_*} h_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}).$$

Hence for singular homology, we obtain $H_n(M, M - \{x\}; G) \cong G$.

Definition 1.2 (Local R-orientation)**.** Let R be a commutative ring. A generator of $H_n(M, M - \{x\}; R) \cong R$ is called a *local R-orientation* of M about x.

Let $K \subset L \subset M$. The homomorphism $r^L_K\colon h_k(M, M-L) \to h_k(M, M-K)$ induced by inclusion is called restriction. We write r^L_x when $K = \{x\}$.

Proposition 1.3. *When A is a compact, convex set contained in some chart $\mathbb{R}^n \subset M$, then r^A_x is an isomorphism for each $x \in A$ and the groups are isomorphic to the coefficient group G.*

Proof. A is contained in the interior of some closed n-disk $D \subset \mathbb{R}^n \subset M$. Thus there is a commutative diagram

$$\begin{tikzcd}
h_n(M, M - A) \arrow[r] & h_n(M, M - \{x\}) \\
h_n(\mathbb{R}^n, \mathbb{R}^n - A) \arrow[r] \arrow[u, "\cong"] & h_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \arrow[u, "\cong"] \\
h_n(D, \partial D) \arrow[r, equal] \arrow[u, "\cong"] & h_n(D, \partial D) \arrow[u, "\cong"]
\end{tikzcd}$$

□

Definition 1.4 (Orientation bundle)**.** We construct a covering $\omega\colon h_k(M, M-\bullet) \to M$. Define

$$h_k(M, M - \bullet) = \bigsqcup_{x \in M} h_k(M, M - \{x\})$$

where $h_k(M, M - \{x\})$ is the fiber over x and is given the discrete topology.
Let U be an open neighborhood of x such that r^U_y is an isomorphism for each $y \in U$. Define bundle charts

$$\varphi_{x,U}\colon U \times G \to \omega^{-1}(U), \quad (y, a) \mapsto r^U_y \left(r^U_x\right)^{-1}(a).$$

We then give $h_k(M, M-\bullet)$ the topology that makes $\varphi_{x,U}$ in a homeomorphism onto an open subset. In particular, since $h_k(M, M - x)$ is given the discrete topology, this is equivalent to the map $\varphi_{x,U}(-, \alpha)$ being a homeomorphism onto an open subset for each $\alpha \in h_k(M, M - x)$. It then remains to show that the transition maps

$$\varphi^{-1}_{y,V}\varphi_{x,U}\colon (U \cap V) \times h_k(M, M - \{x\}) \to (U \cap V) \times h_k(M, M - \{y\})$$

are continuous.
Let $z \in U \cap V$, and choose W such that $z \in W \subset U \cap V$ and r_w^W is an isomorphism for each $w \in W$.
Consider the diagram

$$\begin{tikzcd}
h_k(M, M-x) & h_k(M, M-U) \arrow[l, "r_x^U"'] \arrow[r, "r_w^U"] \arrow[d, "r_W^U"] & h_k(M, M-w) \\
& h_k(M, M-W) \arrow[ur, "r_w^W"] & h_k(M, M-V) \arrow[l, "r_W^V"] \arrow[u, "r_w^V"'] \arrow[d, "r_y^V"] \\
& & h_k(M, M-y)
\end{tikzcd}$$

Let $\varphi_{x,U,p} \colon h_k(M, M-x) \to \omega^{-1}(p)$ be defined by

$$\varphi_{x,U,p}(y) = \varphi_{x,U}(p, y).$$

Then for $w \in U \cap V$, we have

$$\varphi_{x,U,w}^{-1} \varphi_{y,V,w} = r_y^V (r_W^V)^{-1} (r_w^W)^{-1} r_w^W r_W^U (r_x^U)^{-1} = r_y^V (r_W^V)^{-1} r_W^U r_x^U$$

Firstly, this coincides with the operation of an element of the coefficient group G since it is an isomorphism $G \to G$, and secondly, note that this does not depend on w, so the map

$$g_{x,U,y,V} \colon U \cap V \to G$$

defined by $g_{x,U,y,V}(p) = \varphi_{x,U,p}^{-1} \varphi_{y,V,p}$ is constant, hence continuous.

Thus ω is indeed a covering map.
But even moreso, the fibers are groups, so for $A \subset M$, denote by $\Gamma(A)$ the set of continuous sections over A of ω. If s and t are section, we can define $(s+t)(a) = s(a) + t(a)$. Then $s + t$ is again continuous, hence $\Gamma(A)$ is an abelian group.
Denote by $\Gamma_c(A) \subset \Gamma(A)$ the subgroup of sections with compact support, i.e., the sections which have values 0 in the fiber away from a compact set.

Proposition 1.5. *Let $z \in h_k(M, M-U)$. Then $y \mapsto r_y^U z \in h_k(M, M-y) \subset h_k(M, M-\bullet)$ is a continuous section of ω over U.*

Proof. The map $U \to U \times G$ by $y \mapsto (y, r_x^U z)$ is constant in the second coordinate, hence clearly continuous. Now composing with $\varphi_{x,U}$ gives us the section in question. □

1.1. **Homological Orientation.**

If we specify to singular homology with coefficient group R, and again let M be an n-manifold and $A \subset M$, then we can define an orientation along A as follows

Definition 1.6 (R-orientation of M along A)**.** An *R-orientation of M along A* is a section $s \in \Gamma(A; R)$ of $\omega \colon H_n(M, M-\bullet; R) \to M$ such that $s(a) \in H_n(M, M-a; R) \cong R$ is a generator for each $a \in A$.

Thus s glues together the local orientations in a continuous manner.
When $A = M$, we call s an *R-orientation of M* .

Definition 1.7 (Orientation covering)**.** Let $\mathrm{Ori}(M) \subset H_n(M, M - \bullet; \mathbb{Z})$ be the subset of all generators of all fibers. Then the restriction $\mathrm{Ori}(M) \to M$ of ω gives a 2-fold covering of M, called the *orientation covering* of M.

Proposition 1.8. *The following are equivalent:*

(1) *M is orientable*
(2) *M is orientable along compact subsets.*
(3) *The orientation covering is a trivial 2-fold covering map.*
(4) *The covering $\omega\colon H_n(M, M - \bullet; \mathbb{Z}) \to M$ is a trivial covering map.*

Proof. (1) $\implies$ (2) is a subcase.
(2) $\implies$ (3). The orientation covering is trivial if and only if the covering over each component is trivial, so we may assume that M is connected. Now, if a 2-fold covering $\tilde{M} \to M$ is trivial, then $\tilde{M}$ splits as $M \times \{p, q\}$, and so $\tilde{M}$ cannot be connected. Conversely, if $\tilde{M}$ is not connected, then the covering restricted to each component must be a covering map, so the covering splits as a trivial covering.
Suppose then that $\mathrm{Ori}(M) \to M$ is non-trivial. Since $\mathrm{Ori}(M)$ is then connected, we can choose a path γ in $\mathrm{Ori}(M)$ between two points of a given fiber. The image S of such a path is compact and connected, and the covering is non-trivial over S, so by assumption (2), the orientation covering has a section s over S, but then $\gamma(0) = s\left(\omega(\gamma(0))\right) = s\left(\omega\left(\gamma(1)\right)\right) = \gamma(1)$, which gives a contradiction.
(3) $\implies$ (4).
Let $s\colon M \to \mathrm{Ori}(M) \cong M \times \{-1, 1\}$ be the section $m \mapsto (m, 1)$.
Now define a map $\varphi\colon M \times \mathbb{Z} \to H_n(M, M - \bullet; \mathbb{Z})$ by $\varphi(m, k) = ks(m)$. This is a bijective map by assumption on s being a section. It is furthermore continuous since s is continuous and since fiber-wise operations in $H_n(M, M - \bullet; \mathbb{Z})$ is continuous. Furthermore, it is also a morphism between coverings since it commutes with the projections: $\pi_M = \omega \circ \varphi$.
Lastly, one must show that it also has a continuous inverse. For this, we may take an open basis set in $M \times \mathbb{Z}$ - say $U \times \{k\}$, where $\bar{U}$ is a convex subset of $\mathbb{R}^n \subset M$. Since φ is bijective, we obtain that $\varphi(U \times \{k\}) = ks(U) = U_\alpha$ if we choose α to be the element in $H_n(M, M - U) \cong \mathbb{Z}$ which maps to k under $r_{x,U}$ for $x \in U$. And by assumption, U_α is a basis open set for the topology on $H_n(M, M - \bullet; \mathbb{Z})$.
Hence φ is a homeomorphism, and even an isomorphism of covering spaces in the sense that $\pi_M = \omega \circ \varphi$.

Note. We could also say that it is trivial since every point is in the image of some section.

(4) $\implies$ (1) : If ω is trivial, then it has a section with constant value in the set of generators.

□

1.2. **Homology in the Dimension of the Manifold.** Let M be an n-manifold and $A \subset M$ a closed subset. We will in this section use singular homology with coefficients in an abelian group G.

Proposition 1.9. *For each $\alpha \in H_n(M, M - A; G)$, the section*

$$J^A(\alpha)\colon A \to H_n(M, M - \bullet; G), \quad x \mapsto r_x^A(\alpha)$$

of ω over A is continuous and has compact support.

Proof. Choose a representative $c \in \Delta_n(M;G)$ representing α. There exists a compact set K such that c is contained in K. Suppose $A - K$ is nonempty, and let $x \in A - K$. Then the image of c under

$$\Delta_n(K;G) \to \Delta_n(M;G) \to \Delta_n(M,K;G) \to \Delta_n(M,M-x;G)$$

is zero since $K \subset M - x$. Since this image represents r_x^A, the support of $J^A(\alpha)$ is contained in $A \cap K$ which is compact.
If $A - K$ is empty, K contains A, and then the support of $J^A(\alpha)$ is a closed subset of a compact space, hence compact.

The continuity follows from the more general case of Proposition 1.5. □

Thus we obtain a homomorphism

$$J^A \colon H_n(M, M-A; G) \to \Gamma_c(A;G), \quad \alpha \mapsto \left(x \mapsto r_x^A(\alpha)\right).$$

1.3. **Direct Limits.**

Definition 1.10 (Directed set)**.** A *directed set* D is a partially ordered set such that, for any two elements α and β of D, there is a $\tau \in D$ with $\tau \geq \alpha$ and $\tau \geq \beta$.

Definition 1.11. Let D be a directed set and G_α an abelian group defined for each $\alpha \in D$. Suppose we are given homomorphisms $f_{\beta,\alpha} \colon G_\alpha \to G_\beta$ for each $\beta > \alpha$ in D. Assume that for all $\gamma > \beta > \alpha$ in D, we have $f_{\gamma,\beta} f_{\beta,\alpha} = f_{\gamma,\alpha}$. Such a system is called a *direct system* of abelian groups. Then $G = \lim_{\to} G_\alpha$ is defined to be the quotient group of the direct sum $G = \bigoplus G_\alpha$ modulo the relations $f_{\beta,\alpha}(g) \sim g$ for all $g \in G_\alpha$ and all $\beta > \alpha$.

Note. Hence the direct limit is just the colimit of the direct system.

The inclusions $G_\alpha \hookrightarrow \bigoplus G_\alpha$ induce homomorphisms $i_\alpha \colon G_\alpha \to \lim_{\to} G_\alpha$ and $i_\beta \circ f_{\beta,\alpha} = i_\alpha$. Moreover, for any $g \in G$, there is a $g_\alpha \in G_\alpha$ for some α such that $g = i_\alpha(g_\alpha)$. Also, for any index $\alpha \in D$, and element $g_\alpha \in G_\alpha$, we have $i_\alpha(g_\alpha) = 0$ if and only if there exists a $\beta \geq \alpha$ such that $f_{\beta,\alpha}(g_\alpha) = 0$. These properties characterize the direct limit:

Proposition 1.12. *Suppose we are given an abelian group A with homomorphisms $h_\alpha \colon G_\alpha \to A$ such that the cocone commutes. Since $\lim_{\to} G_\alpha$ is the colimit, we have a unique induced homomorphism $h \colon \lim_{\to} G_\alpha \to A$. Then*

1. $\operatorname{im} h = \{a \in A \mid a = h_\alpha(g) \text{ for some } g \text{ and } \alpha\} = \bigcup \operatorname{im} h_\alpha$.
2. $\ker h = \{g \in \lim_{\to} G_\alpha \mid \exists \alpha \text{ and } g_\alpha \in G_\alpha \colon g = i_\alpha(g_\alpha) \text{ and } h_\alpha(g_\alpha) = 0\} = \bigcup i_\alpha(\ker h_\alpha)$.

Proof. Define $h(g_\alpha) = h_\alpha(g_\alpha)$. Then if $f_{\beta,\alpha}(g_\alpha) \sim g_\alpha$, we have $h(g_\alpha) = h_\alpha(g_\alpha) = h_\beta \circ f_{\beta,\alpha}(g_\alpha) = h(f_{\beta,\alpha}(g_\alpha))$, so h respects the equivalence relations, thus it is well-defined.
Now property (1) is clear by the way we defined h.
As for (2), note that if g represents the equivalence class of g_α and $h(g) = 0$, then $h_\alpha(g_\alpha) = 0$ which is what (2) is saying. □

Corollary 1.13. *In the situation of Proposition 1.12, $h \colon \lim_{\to} G_\alpha \to A$ is an isomorphism if and only if the following two statements hold true:*

1. $\forall a \in A, \exists \alpha \in D$ *and* $g_\alpha \in G_\alpha \colon h_\alpha(g_\alpha) = a$, *and*
2. *if* $h_\alpha(g_\alpha) = 0$ *then* $\exists \beta > \alpha \colon f_{\beta,\alpha}(g_\alpha) = 0$.

Theorem 1.14. *The direct limit is an exact functor. So if we have direct systems $\{A'_\alpha\}, \{A_\alpha\}$ and $\{A''_\alpha\}$ based on the same directed set, and if we have an exact sequence $A'_\alpha \to A_\alpha \to A''_\alpha$ for each α, where the maps commute with the ones defining the direct systems, then the induced sequence*

$$\lim_{\rightarrow} A'_\alpha \to \lim_{\rightarrow} A_\alpha \to \lim_{\rightarrow} A''_\alpha$$

is exact.

Proof. We have the following diagram, where all maps commute.

$$\begin{tikzcd}
A'_\beta \arrow[r] \arrow[d] & A_\beta \arrow[r] \arrow[d] & A''_\beta \arrow[d] \\
\lim_\rightarrow A'_\alpha \arrow[r] & \lim_\rightarrow A_\alpha \arrow[r] & \lim_\rightarrow A''_\alpha
\end{tikzcd}$$

Suppose $a \in \lim_\rightarrow A_*$ is mapped to zero in $\lim_\rightarrow A''_*$. Then there exists $g \in \lim_\rightarrow A_\alpha$ such that there exists β and $g_\beta \in A_\beta$ such that $g = i_\beta(g_\beta)$ and $h_\beta(g_\beta) = 0$.
Recall here that h_β is a homomorphism $A_\beta \to \lim_\rightarrow A''_*$ and i_β is the inclusion $G_\beta \to \lim_\rightarrow G_\alpha$.
By commutativity of the diagram, there then exists $k_\beta \in A'_\beta$ such that
$i_\beta\left(d_\beta(k_\beta)\right) = d_{\lim_\rightarrow} i'_\beta(k_\beta)$. Hence the kernel is contained in the image.
Now suppose let $\tilde{k} = d_{\lim_\rightarrow}(k) \in \lim_\rightarrow A_*$.
Then $\tilde{k} = i_\beta\left(d(\overline{k})\right) = d_{\lim_\rightarrow} i'_\beta\left(\overline{k}\right)$ for some $\overline{k} \in A'_\beta$.
But now

$$d_{\lim_\rightarrow}(\tilde{k}) = d_{\lim_\rightarrow} i_\beta\left(d\left(\overline{k}\right)\right) = i''_\beta d\left(d\left(\overline{k}\right)\right) = i''_\beta(0) = 0.$$

□

Theorem 1.15. *Suppose we are given two directed sets D and E. Define an order on $D \times E$ by $(\alpha, \beta) \geq (\alpha', \beta')$ if and only if $\alpha \geq \alpha'$ and $\beta \geq \beta'$. Suppose $G_{\alpha,\beta}$ is a direct system based on $D \times E$. Then the maps $G_{\alpha,\beta} \to \lim_{\rightarrow,\beta} G_{\alpha,\beta} \to \lim_{\rightarrow,\alpha}\left(\lim_{\rightarrow,\beta} G_{\alpha,\beta}\right)$ induce an isomorphism*

$$\lim_{\rightarrow,\alpha,\beta} G_{\alpha,\beta} \stackrel{\cong}{\rightarrow} \lim_{\rightarrow,\alpha}\left(\lim_{\rightarrow,\beta} G_{\alpha,\beta}\right).$$

Proof. □

Proposition 1.16. (1) *For $A \supset B$ both closed, the following diagram commutes:*

$$\begin{tikzcd}
H_n(M, M-A; G) \arrow[r] \arrow[d, "J^A"] & H_n(M, M-B; G) \arrow[d, "J^B"] \\
\Gamma_c(A, H_n(M, M-\bullet; G)) \arrow[r] & \Gamma_c\left(B, H_n(M, M-\bullet; G)\right)
\end{tikzcd}$$

(2) *For $A, B \subset M$ both closed, the sequence*

$$0 \to \Gamma_c(A \cup B, H_n(M, M-\bullet; G)) \xrightarrow{h} \Gamma_c\left(A, H_n(M, M-\bullet; G)\right) \oplus \Gamma_c\left(B, H_n(M, M-\bullet; G)\right) \xrightarrow{k} \Gamma_c\left(A \cap B, H_n\left(M, M-\bullet; G\right)\right)$$

is exact, where h is the sum of restrictions and k is the difference of restrictions.

(3) *If* $A_1 \supset A_2 \supset A_3 \supset \dots$ *are all compact and* $A \bigcap A_i$*, then the restriction homomorphisms* $\Gamma(A_i, H_n(M, M-\bullet; G)) \to \Gamma(A, H_n(M, M-\bullet; G))$ *induce an isomorphism*

$$\lim_{\to} \Gamma(A_i, H_n(M, M-\bullet; G)) \xrightarrow{\cong} \Gamma(A, H_n(M, M-\bullet; G))$$

Proof. (1) Let $\alpha \in H_n(M, M-A; G)$, and denote by ι the inclusion $(M, M-A) \hookrightarrow (M, M-B)$. Then $\iota_* = r_B^A$, so $J^B\left(r_B^A(\alpha)\right)(x) = r_x^B\left(r_B^A(\alpha)\right)$. On the other hand, $J^A(\alpha)|_B(x) = J^A(\alpha)(x) = r_x^A(\alpha)$. Now, from the composition

$$(M, M-A) \hookrightarrow (M, M-B) \hookrightarrow (M, M-x)$$

we obtain by taking homology, that $r_x^A = r_x^B r_B^A$, which gives the result.

(2) Firstly, a section that is zero on both A and B is then also zero on $A \cup B$, which gives the injective part of h. Now, suppose $s - t$ is the zero section over $A \cap B$ for s a section over A and t a section over B. Then s and t agree on $A \cap B$, meaning that $s \cup t$ is well-defined and continuous, where $s \cup t$ is s on A and t on B, and $h(s \cup t) = (s, t)$. Likewise, if g is a section over $A \cup B$, then $k \circ h(g) = (g|_A)|_{A\cap B} - (g|_B)|_{A\cap B} = g|_{A\cap B} - g|_{A\cap B}$ is the zero section.

(3)

$\square$

Theorem 1.17. *Let* $A \subset M$ *be closed. Then*

(1) $H_i(M, M-A; G) = 0$ *for* $i > n$.
(2) $J^A\colon H_n(M, M-A, G) \to \Gamma_c(A, H_n(M, M-\bullet; G))$ *is an isomorphism.*

Lemma 1.18 (The Bootstrap Lemma)**.** *Let* $P_M(A)$ *be a statement about compact sets* A *in a given* n*-manifold* M^n*. If* $(i), (ii), (iii)$ *hold, then* $P_M(A)$ *is true for all compact* A *in* M^n*.*
If M^n *is separable metric, and* $P_M(A)$ *is defined for all closed sets* A*, and if* $(i), (ii), (iii), (iv)$ *hold, then* $P_M(A)$ *is true for all closed sets* A *in* M*.*
For general M^n*, if* $P_M(A)$ *is defined for all closed sets* A *in* M*, for all* M^n*, and if all five statement* $(i) - (v)$ *hold for all* M^n*, then* $P_M(A)$ *is true for all closed* $A \subset M$ *and all* M^n*.*

Now note that for a given abelian group G and $g \in G$, the following maps are natural in $A \subset M$ (closed):

$$H_n(M, M-A) \cong H_n(M, M-A) \otimes \mathbb{Z} \to H_n(M, M-A) \otimes G \to H_n(M, M-A; G)$$

where the middle map is induced by the homomorphism $\mathbb{Z} \to G$ taking 1 to g.
In particular, this induces a map

$$H_n(M, M-\bullet) \to H_n(M, M-\bullet; G)$$

Lemma 1.19. *The sections* $\Gamma(A; G)$ *of* ω *over* A *correspond bijectively to continuous maps* $\lambda\colon \mathrm{Ori}(M)|_A \to G$ *with the property* $\lambda \circ t = -\lambda$*, where* t *acts on* G *as multiplication by* -1*.*

Proof. We may assume A is connected.
Let $s \in \Gamma(A; G)$ be a section of ω over A. That is, $w \circ s = \mathrm{id}_A$, and s is a map $A \to H_n(M, M-\bullet; G)$. We can define an associated map $\lambda_s\colon \mathrm{Ori}(M)|_A \to G$ by

sending a generator in the fiber $x \in A$ to $s(x) \in H_n(M, M - \{x\}; G) \cong G$. If one chose the other generator, one would get the negative of the above map, so we have the relation $\lambda_s \circ t = -\lambda_s$. Subject to this relation, we obtain a well-defined map $\Gamma(A; G) \to S \subset \mathrm{Hom}(\mathrm{Ori}(M)|_A, G)$, where S is the subset for which $\lambda \circ t = -\lambda$ holds. This map is certainly injective, since the image tells us precisely the value of s at any point in A.
It is furthermore surjective, since if $\mathrm{Ori}(M)|_A$ is connected, then S can only consist of the zero section, and if it is not connected, it consists of a map on two components on which it is constant, and the relation $\lambda \circ t = -\lambda$ then determines that is must the required values to constitute the induced map of a section. ☐ check

Theorem 1.20. *Suppose $A \subset M$ is a closed connected subset. Then*

(1) $H_n(M, M - A; G) = 0$ *if A is not compact.*
(2) $H_n(M, M - A; G) \cong G$ *if M is R-orientable along A and A is compact. Moreover, $H_n(M, M - A; G) \to H_n(M, M - x; G)$ is an isomorphism for each $x \in A$.*
(3) $H_n(M, M - A; G) \cong {}_2G = \{g \in G \mid 2g = 0\}$ *if M is not orientable along A and A is compact.*

Proof. (1) By Lemma 6.1, a section in $\Gamma(A; G)$ is determined by its value at a single point. By the existence of the zero section, if a section is non-zero at any point, then it is non-zero at every point. Therefore, there do not exist non-zero sections with compact support over a non-compact A, so by Theorem 1.17, $H_n(M, M - A; G) \cong \Gamma_c(A; G) \cong 0$.

(2) Since A is compact, $H_n(M, M - A; G) \cong \Gamma_c(A; G) = \Gamma(A; G)$. A section is again determined by a single point. Recall now the commutative diagram

$$\begin{array}{ccc} H_n(M, M - A; G) & \xrightarrow{\cong} & \Gamma(A; G) \\ \downarrow{\scriptstyle r_x^A} & & \downarrow{\scriptstyle b} \\ H_n(M, M - x; G) & \xrightarrow{\cong} & \Gamma(\{x\}; G) \end{array}$$

from Proposition 1.16, the horizontal isomorphisms following from Theorem 1.17. If M is orientable along A, there by definition exists in $\Gamma(A; G)$ an element such that its value at x is a generator. Hence b is an isomorphism, and therefore also r_x^A is an isomorphism.

(3) By Lemma 1.19, a section in $\Gamma(A; G)$ corresponds to a continuous map $\lambda\colon \mathrm{Ori}(M)|_A \to G$ with $\lambda t = -\lambda$. If M is not orientable along A, then $\mathrm{Ori}(M)|_A$ is connected and therefore λ is constant as G has the discrete topology. The relation $\lambda t = -\lambda$ shows that λ is in ${}_2G$. Now by the commutative diagram from part (2), note that since λ must be constant, firstly $\Gamma(A; G) \cong {}_2G$, and secondly,b becomes injective, so $r_x^A\colon H_n(M, M - A; G) \to H_n(M, M - x; G) \cong G$ is injective and has image ${}_2G$, so the Hom term vanishes. ☐

Proposition 1.21. *Let M be an n-manifold and $A \subset M$ be a closed connected subset. Then the torsion subgroup of $H_{n-1}(M, M - A; \mathbb{Z})$ is of order 2 if A is compact and M non-orientable along A, and is 0 otherwise.*

Proof. By UCT for homology,

$$\begin{aligned}\mathbb{Z}/2 \cong {}_2\mathbb{Z}/2 \cong H_n(M, M-A; \mathbb{Z}/2) &\cong H_n(M, M-A) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}_1\left(H_{n-1}(M, M-A), \mathbb{Z}/2\right) \\ &\cong \operatorname{Tor}_1\left(H_{n-1}(M, M-A), \mathbb{Z}/2\right) \\ &\cong \{g \in H_{n-1}(M, M-A) \mid 2g = 0\}.\end{aligned}$$

where $H_n(M, M-A) \cong {}_2\mathbb{Z} = 0$, and $H_n(M, M-A; \mathbb{Z}/2) \cong {}_2\mathbb{Z}/2 \cong \mathbb{Z}/2$ both follow from Theorem 1.20.
To see that this is the whole torsions subgroup, note that for odd k,

$$\operatorname{Tor}_1\left(H_{n-1}(M, M-A), \mathbb{Z}/k\right) \cong H_n(M, M-A; \mathbb{Z}/k) \cong {}_2\mathbb{Z}/k \cong 0$$

When M is orientable along A and A is compact, we simply obtain

$$0 \to H_n(M, M-A) \otimes \mathbb{Z}/n \to H_n(M, M-A; \mathbb{Z}/n) \to \operatorname{Tor}_1\left(H_{n-1}(M, M-A), \mathbb{Z}/n\right) \to 0$$

and since $H_n(M, M-A) \cong \mathbb{Z}$ and $H_n(M, M-A; \mathbb{Z}/n) \cong \mathbb{Z}/n$ by Theorem 1.20, we find that Tor_1 vanishes for all n.
If A is non-compact, then Theorem 1.20 gives that Tor_1 trivially vanishes for all terms.

□

1.4. **Fundamental Class.**

Theorem 1.22. *Let M be a compact connected n-manifold. Then one of the following assertions holds:*

1. *M is orientable, $H_n(M) \cong \mathbb{Z}$, and for each $x \in M$, the restriction $H_n(M) \to H_n(M, M-x)$ is an isomorphism.*
2. *M is non-orientable and $H_n(M) = 0$.*

Proof. Special case of Theorem 1.20. □

Under the hypothesis of Theorem 1.22, the orientations of M correspond to the generators of $H_n(M)$. Such a generator will be called a *fundamental class* or *homological class/orientation* of the orientable manifold.

Definition 1.23 (Degree)**.** Let M and N be compact oriented n-manifolds. Let N be connected and suppose M has components $M_1, \ldots, M_r$. Then we have fundamental classes $z(M_j)$ for each M_j and $z(M) \in H_n(M) \cong \bigoplus_j H_n(M_j)$ is the sum of the $z(M_j)$. Now, since $H_n(N) \cong \langle z(N) \rangle \cong \mathbb{Z}$, we obtain that there exists a *degree* $d(f) \in \mathbb{Z}$ such that $f_* z(M) = d(f) z(N)$.

Lemma 1.24 (Properties)**.**

1. *The degree is a homotopy invariant.*
2. *$d(f \circ g) = d(f) d(g)$.*
3. *A homotopy equivalence has degree ± 1.*
4. *If $M = M_1 \bigsqcup M_2$, then $d(f) = d\left(f|_{M_1}\right) + d\left(f|_{M_2}\right)$.*
5. *If we pass in M or N to the opposite orientation, then the degree changes the sign.*

1.4.1. *Computations of degrees.* As usual, we can compute degrees in terms of local data of a map.
Let M and N be connected and set $K = f^{-1}(p)$. Let U be an open neighborhood of K in M. Then in particular $M - U = \overline{M - U} \subset \operatorname{int}(M - A) = M - A$, so excision gives the bottom left isomorphism in the following diagram, and the top right isomorphism follows from Theorem 1.22:

$$\begin{tikzcd}
z(M) \in \arrow[dd, mapsto] & H_n(M) \arrow[r, "f_*"] \arrow[d] & H_n(M) \arrow[d, "\cong"] & \ni z(N) \arrow[dd, mapsto] \\
& H_n(M, M-K) \arrow[r, "f_*"] & H_n(N, N-p) & \\
z(U,K) \in & H_n(U, U-K) \arrow[r, "f_*^U"] \arrow[u, "\cong"', "i_*"] & H_n(N, N-p) \arrow[u, "="] & \ni z(N,p)
\end{tikzcd}$$

From the outer rectangle, we get $f_*^U z(U,K) = d(f)z(N,p)$, where $z(N,p)$ and $z(U,K)$ are the images of $z(N)$ and $z(M)$ under the indicated maps.
We want to show additivity of degree as in the case for spheres.
So suppose K if finite, and choose $U = \bigcup_{x\in K} U_x$ where the U_x are pair-wise disjoint open neighborhoods of x. Then

$$\bigoplus_{x\in K} H_n(U_x, U_x - x) \cong H_n(U, U-K), \quad H_n(U_x, U_x - x) \cong \mathbb{Z}.$$

The image $z\left(U_x, x\right)$ of $z(M)$ is a generator: it is the image under the following isomorphisms

$$H_n(M) \xrightarrow{\cong} H_n(M, M-x) \xrightarrow{\cong} H_n(U_x, U_x - x)$$

where the first follows from Theorem 1.22 and the second from excision. The local degree $d(f,x)$ is determined by $f_* z\left(U_x, x\right) = d(f,x)z(N,p)$, and and by additivity above, we have $d(f) = \sum_{x\in K} d(f,x)$.

Proposition 1.25. *Let M be a connected, oriented, closed n-manifold. Then there exists for each $k \in \mathbb{Z}$ a map $f\colon M^n \to S^n$ of degree k.*

Remark. If f is C^1 in a neighborhood of x, then $d(f,x)$ is the sign of the determinant of $Dg(0)$ when $Dg(x)$ is regular, where g is f in local coordinates that preserve local orientations. By this we mean that for $\varphi\colon U_x \to \mathbb{R}^n$ centered at x, $\varphi_*\colon H_n\left(U_x, U_x - x\right) \to H_n\left(\mathbb{R}^n, \mathbb{R}^n - 0\right)$ sends $z\left(U_x, x\right)$ to the standard generator. Such charts are called *positive* with respect to the given orientations.

Proof. If $f\colon M \to S^n$ has degree a and $g\colon S^n \to S^n$ degree b, then gf has degree ab. Since the proposition is true for $M = S^n$, it suffices to find f having degree ± 1. Let $\varphi\colon D^n \to M$ be an embedding. Then we have a map $f\colon M \to D^n/S^{n-1}$ which is the inverse of φ on $U = \varphi\left(\operatorname{int} D^n\right)$ and sends $M - U$ to the basepoint. This map has degree ± 1 as can be seen by choosing any neighborhood of x in the interior of U and looking at the determinant of the differential locally. $\square$

1.5. **Manifolds with Boundary.**

Definition 1.26. For M an n-dimensional manifold with boundary, we call $z \in H_n\left(M, \partial M\right)$ a *fundamental class* if for each $x \in M - \partial M$, the restriction of z is a generator in $H_n\left(M, M - x\right)$.

Theorem 1.27. *Let M be a compact connected n-manifold with non-empty boundary. Then one of the following assertions hold:*

1. *$H_n\left(M, \partial M\right) \cong \mathbb{Z}$, and a generator of this group is a fundamental class. The image of a fundamental class under $\partial\colon H_n\left(M, \partial M\right) \to H_{n-1}\left(\partial M\right)$ is a fundamental class. The interior $M - \partial M$ is orientable.*
2. *$H_n\left(M, \partial M\right) = 0$, and $M - \partial M$ is not orientable.*

Proof. See [2, Thm 16.5.1]. We will follow that proof and only add a few extra words.
Let $\kappa\colon [0,\infty[\times\partial M \to U$ be a collar of M, i.e., a homeomorphism onto an open neighborhood U of ∂M such that $\kappa(0,x)=x$ for $x\in\partial M$ (See Milnor's h-cobordism book for existence). For simplicity of notation, we identify U with $[0,\infty)\times\partial M$ via κ ; similarly, for subsets of U. In this sense, $\partial M = 0\times\partial M$. For $A = M - ([0,1)\times\partial M)\subset M-\partial M$, we have isomorphisms

$$H_n\left(M,\partial M\right)\cong H_n\left(M,[0,1)\times\partial M\right)\cong H_n\left(M-\partial M,(0,1)\times\partial M\right)\cong\Gamma(A).$$

The first one by homotopy equivalence, the second by excision, and the third using Theorem 1.17.
Since A is connected, $\Gamma(A)\cong\mathbb{Z}$ or $\Gamma(A)\cong 0$. If $\Gamma(A)\cong\mathbb{Z}$, then $M-\partial M$ is orientable along A.
Let now $A_\varepsilon\cong A$ be the complement of $[0,\varepsilon)\times\partial M$. Since each compact subset of $M-\partial M$ is contained in some such M_ε for small enough ε, we see that $M-\partial M$ is orientable along all compact subsets, hence orientable by Proposition 1.8.
The isomorphism $H_n\left(M-\partial M,(0,1)\times\partial M\right)\cong\Gamma(A)$ from above says that there exists some $z\in H_n\left(M-\partial M,(0,1)\times\partial M\right)$ which restricts to a generator of $H_n\left(M-\partial M, M-\partial M-x\right)$ for each $x\in A$. For the corresponding element $z\in H_n\left(M,\partial M\right)\cong\mathbb{Z}$, the same assertion holds for any $x\in M-\partial M$ (simply shrink the collar to not contain x).
Lastly, we must show that ∂z is a fundamental class. Let $x\in(0,1)\times\partial M$. Consider the diagram:

$$\begin{tikzcd}
H_{n-1}(\partial M) \arrow[r, "\cong"] & H_{n-1}\left(\partial M\cup A, A\right) & H_{n-1}\left(\partial I\times\partial M, 1\times\partial M\right) \arrow[l, "\cong"'] \\
H_n(M,\partial M) \arrow[u, "\partial"] \arrow[r] \arrow[dr] & H_n\left(M,\partial M\cup A\right) \arrow[u, "\cong"', "\partial"] \arrow[d] & H_n\left(I\times\partial M,\partial I\times\partial M\right) \arrow[l, "\cong"'] \arrow[u, "\cong"', "\partial"] \\
 & H_n(M, M-x) & H_n\left(I\times\partial M, I\times\partial M - x\right) \arrow[l, "\cong"']
\end{tikzcd}$$

Commutativity of the bottom left triangle tells us that the image of z under $H_n\left(M,\partial M\right)\to H_n\left(M,\partial M\cup A\right)$ gives an element whose restriction gives a generator in $H_n\left(M, M-x\right)$, but then by commutativity of the bottom right square, we get that the restriction of z transferred over by the isomorphism to $H_n\left(I\times\partial M,\partial I\times\partial M\right)$ is a generator of $H_n\left(I\times\partial M, I\times\partial M-x\right)$ at each point in $(0,1)\times\partial M$. Hence z yields a fundamental class in $H_n\left(I\times\partial M,\partial I\times\partial M\right)$.
But since z is a generator in $H_n\left(I\times\partial M,\partial I\times\partial M\right)$, the upper part shows that z is a generator in $H_{n-1}(\partial M)$, thus a fundamental class of ∂M since this characterizes fundamental classes.

□

Example 1.28. Suppose that $B\colon M\to\varnothing$ is a cobordism. We have the fundamental classes $z(B)\in H_{n+1}(B,\partial B)$ and $z(M)=\partial z_B\in H_n(M)$ (here we crucially made use of our result in Theorem 1.27). This is already a lot of information. Indeed, suppose $f\colon M\to N$ is a map which has an extension to B : $F\colon B\to N$. Then the degree of f (if defined) is zero, $d(f)=0$, for we have $f_*z(M)=f_*\partial z(B)=F_*i_*\partial z(B)=0$, since $i_*\partial=0$ by the exactness of the homology sequence for the pair (B,M).

We call maps $f_\nu\colon M_\nu \to N$ *orientable bordant* if there exists a compact oriented cobordism $B\colon M_1 \to M_2$ with orientable boundary $\partial B = M_1 - M_2$ (meaning $\partial z(B) = z(M_1) - z(M_2)$) and an extension $F\colon B \to N$ of $f_1 \sqcup f_2\colon M_1 \bigsqcup M_2 \to N$. Under these assumptions, we have $d(f_1) = d(f_2)$. This fact is called the *bordism invariance* of the degree; it generalizes the homotopy invariance.

2. Duality

Let M^n be orientable and $\vartheta_M \in \Gamma\left(M, H_n(M, M - \bullet)\right)$ an orientation.
For $K \subset M$ compact, ϑ_M restricts to to $\vartheta_K \in \Gamma\left(K, H_n(M, M - \bullet)\right) = \Gamma_c(K, H_n(M, M - \bullet)) \cong H_n(M, M - K)$, so we can regard ϑ_K as lying in $H_n(M, M - K)$. Let $\vartheta = \{\vartheta_K\}$ be the collection of all these, and we then call ϑ an orientation.

Definition 2.1. For sets $L \subset K \subset M$, we define

$$\check{H}^p(K, L; G) = \lim_{\rightarrow} \left\{H^p(U, V; G) \mid (U, V) \supset (K, L), U, V \text{ open}\right\}.$$

This is a directed system since if (U, V) and (U', V') both contain (K, L), the $(U \cap U', V \cap V')$ also contains (K, L), and the maps induced by inclusions of nested open sets satisfy the required relation to be a directed system.

This group is naturally isomorphic to that of Cech cohomology. If K and L are spaces such as ENRs (e.g., CW-complexes or topological manifolds), then this is also naturally isomorphic to singular cohomology.

2.1. **Construction of the duality map.** Suppose $(K, L) \subset (U, V)$ as above (so $L \subset K$). Then note that since $K \subset U$, we have $M - U = \overline{M - U} \subset \int(M - K) = M - K$, so by excision, $H_{n-p}(U - L, U - K) \cong H_{n-p}(M - L, M - K)$. Also, $\{V, U - L\}$ is an open cover of U, hence $H_*\left(\frac{\Delta_*(V) + \Delta_*(U-L)}{\Delta_*(U-K)}\right) \cong H_*\left(U, U - K\right) \cong H_*(M, M - K)$ where the first isomorphism follows from Theorem 7.2.2 in Algtop1, and the latter follows from excision.
Now, we have a well-defined cap product

$$\Delta^p\left(U, V; G\right) \otimes \left[\frac{\Delta_n(V) + \Delta_n(U - L)}{\Delta_n(U - K)}\right] \xrightarrow{\cap} \Delta_{n-p}\left(U - L, U - K; G\right)$$

given by $f \cap (b + c) = f \cap b + f \cap c = f \cap c$, which by the above, induces a cap product

$$H^p(U, V; G) \otimes H_n(M, M - K) \to H_{n-p}(M - L, M - K; G)$$

which is natural in (K, L).
Using the same theorem from Algtop1, we find that for $\gamma \in H_n(M, M - A)$, we can represent γ by a chain $b + c + d \in \Delta_n(V) + \Delta_n(U - L) + \Delta_n(M - K)$, so for $f \in \Delta^p(U, V; G)$, we get that

$$[f] \cap \gamma = [f \cap (b + c + d)] = [f \cap c] \in H_{n-p}(M - L, M - K; G)$$

since $f \cap b = 0$ as f vanishes on V and $f \cap d$ is a chain in $M - K$.
Thus by capping with ϑ_A for A large enough to contain K, we obtain a homomorphism

$$\cap\vartheta\colon H^p\left(U, V; G\right) \to H_{n-p}(M - L, M - K; G).$$

Now, recall that the direct limit used for $\check{H}$ has the universal property that it is the colimit under the direct system. So if the above map $\cap\vartheta$ is compatible with the direct system, then we will obtain an induced homomorphism from $\check{H}$.

Suppose $i\colon (U',V') \hookrightarrow (U,V)$ is the inclusion. Let $[f] \in H^p(U,V;G)$ and represent ϑ as $b+c+d \in \Delta_n(V') + \Delta_n(U'-L) + \Delta_n(M-A)$. Then

$$[f] \cap \vartheta = [f \cap (b+c+d)] = [f \cap c] = [f \circ i \cap c] = i^*[f] \cap \vartheta$$

so the homomorphism is compatible with the morphisms in the directed system. Furthermore, choosing a different A, say A', we similarly can write ϑ as $b+c+d \in \Delta_n(V') + \Delta_n(U'-L) + \Delta_n(M-A')$, and again $f \cap d$ becomes zero in homology since it lands inside $M - A \subset M - K$. Hence the homomorphism $\cap\vartheta$ is compatible with changing A and also with the directed system maps. Thus in the direct limit, we get a natural map

$$\cap\vartheta\colon \check{H}^p(K,L;G) \to H_{n-p}(M-L, M-K;G).$$

2.2. **Properties of the duality map.** Now that we have constructed the homomorphism we are interested in, we should prove some properties that might be useful for us. In order to do this, we give its (co)chain description.
Let $\alpha \in \check{H}^p(K,L;G)$. Then by the direct limit, there exists some open neighborhood (U,V) of (K,L) such that α is represented by a p-cocycle f of (U,V). Thus $f=0$ on V and $\delta f = 0$ on U. Then extend f trivially to a cochain on all of M. We represent the orientation ϑ by a chain $a = b+c+d \in \Delta_n(V) + \Delta_n(U-L) + \Delta_n(M-K)$. Then $f \cap (b+c+d) = f\cap b + f \cap c + f \cap d$. But $f \cap b = 0$ and $f \cap d$ is a chain in $M-K$, so $\alpha \cap \vartheta$ is represented by $f \cap c$. In the special case for $L = \varnothing$, we take $V = \varnothing$, so b does not come into the consideration then.

Using the above description, we can now show the following lemma

Lemma 2.2. *The following diagram with arbitrary coefficients has exact rows and commutes:*

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \check{H}^p(K,L) & \longrightarrow & \check{H}^p(K) & \longrightarrow & \check{H}^p(L) & \longrightarrow & \check{H}^{p+1}(K,L) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_{n-p}(M-L,M-K) & \longrightarrow & H_{n-p}(M,M-K) & \longrightarrow & H_{n-p}(M,M-L) & \longrightarrow & H_{n-p-1}(M-L,M-K) & \longrightarrow & \cdots
\end{array}$$

where all vertical maps are the cap products with the orientation class ϑ.

Proof. The exactness of the top row follows from Theorem 1.14 together with the LES of a pair in singular cohomology.

Let us first check the first two squares. In the first one, if $[f] \in \check{H}^p(K,L)$, then going down, we get $[f] \cap \vartheta$ which includes into $H_{n-p}(M, M-K)$. In particular, this is represented by $f \cap c$. But in the same way, including first $[f]$ into $\check{H}^p(K)$ and the capping with ϑ is still represented by $f \cap c$ (as we discussed above), so we obtain commutativity of the first square. For the second square, suppose let $f \in \Delta^p(M;G)$ be such that $f|_U \in \Delta^p(U;G)$ represents $\alpha \in H^p(U;G)$ mapping to the class in question in $\check{H}^p(K;G)$. Let ϑ be represented by $a = b+c+d \in \Delta_n(V) + \Delta_n(U-L) + \Delta_n(M-K)$. Then $f \cap a$ is represented by $f \cap b + f \cap c$ in $H_{n-p}(M, M-K)$. But since we have the decomposition $a = 0 + b + (c+d) \in \Delta_n(\varnothing) + \Delta_n(V - \varnothing) + \Delta_n(M-L)$, we also have that in $H_n(M, M-L)$, $f \cap a$ becomes $f \cap b$.
On the other hand, we have f restricting to class which is the image of a class represented by $f|_V$. But then this simply becomes $f|_V \cap b = f \cap b$ in $H_{n-p}(M, M-L)$.

For the last square, we must check commutativity.
Let $f \in \Delta^p(M;G)$ such that $f|_V \in \Delta^p(V;G)$ represents $\alpha \in H^p(V;G)$ mapping to the class in $\check{H}^p(L;G)$ that we want to chase through the square (we can find such an f by the properties section above the lemma). Thus $\delta f = 0$ on V.
Now represent ϑ by $a = b+c+d \in \Delta_n(V)+\Delta_n(U-L)+\Delta_n(M-K)$. This is the decomposition of a appropriate to the (K,L) pair, but we can also decompose a as $a = 0+b+(c+d) \in \Delta_n(\varnothing)+\Delta_n(V-\varnothing)+\Delta_n(M-L)$ which is the decomposition with respect to the pair $(L,\varnothing)$, showing that a can be used in the definition of both the cap product for $\check{H}^p(L)$ and for $\check{H}^{p+1}(K,L)$ in question. Since ϑ is a class of $(M, M-K)$, ∂a is a chain in $M-K$. Now, starting with f representing α, going to the right gives δf and then capping with a when going down gives $\delta f \cap a$. On the other hand, going down with f first gives $f \cap a$ and going right then gives $\partial\,(f\cap a)$. Now recall that $\partial\,(f\cap a) = (\delta f)\cap a \pm f \cap \partial a$, but $f\cap\partial a$ is a chain in $M-K$, so it vanishes on passage to homology, whereby we find that the square commutes. $\square$

Lemma 2.3. *Let K and L be compact subsets of the n-manifold M with the orientation ϑ. Then the diagram with arbitrary coefficients*

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(K\cup L) \arrow[r] \arrow[d] & \check{H}^p(K)\oplus\check{H}^p(L) \arrow[r] \arrow[d] & \check{H}^p(K\cap L) \arrow[r, "\delta^*"] \arrow[d] & \check{H}^{p+1}(K\cup L) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n-p}(M, M-(K\cup L)) \arrow[r] & H_{n-p}(M, M-K)\oplus H_{n-p}(M,M-L) \arrow[r] & H_{n-p}(M, M-(K\cap L) \arrow[r, "\delta_*"] & H_{n-p-1}(M, M-(K\cup L)) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical maps are the cap products with ϑ, commutes and has exact rows.

3. Intersection Theory

Definition 3.1 (k-disk bundle)**.** A k-disk bundle is a vector bundle whose coordinate transformations are contained in $O(k) \subset \mathrm{GL}(\mathbb{R}^k)$ and such that the local trivializations have the form $\pi^{-1}(U) \cong U \times D^k$.

Let N^n be a connected, oriented, closed n-manifold, and W^{k+n} an $(n+k)$-manifold with boundary ∂W a $(k-1)$-sphere bundle over N^n, and let $\pi\colon W^{n+k} \to N^n$ be a k-disk bundle over N.
Let us assume also that W is also oriented.

Definition 3.2. In the above situation, the *Thom class* of the disk bundle π is the class $\tau \in H^k(W,\partial W)$ given by

$$\tau = D_W\left(i_*\left[N\right]\right)$$

where $D_W : H_{n-k}(W) \to H^k(W,\partial W)$ is the inverse of the Poincaré duality isomorphism. That is,

$$D(a)\cap[M] = a.$$

Thus

$$\tau \cap [W] = i_*\left[N\right].$$

We can deformation retract the punctured disk to its boundary, giving $H^k(W, W - N) \cong H^k(W,\partial W)$, so we will sometimes regard τ as being in $H^k(W, W-N)$.

Lemma 3.3. *In the above setup, suppose $A \subset N$ is closed. Let $\tilde{A} = \pi^{-1}(A) \subset W$ and $\partial\tilde{A} = \tilde{A}\cap\partial W$. Then $\check{H}^i\left(\tilde{A},\partial\tilde{A}\right) = 0$ for $0 < i < k$.*

Proof. Suppose first that A is compact convex subset of a Euclidean neighborhood in N. It also suffices consider the case where A is connected, so $A \cong D^n$. Consider the pullback bundle of A :

$$\begin{tikzcd} i^*(A) \arrow[r] \arrow[d] & W \arrow[d, "\pi"] \\ A \arrow[r, hook, "i"] & N \end{tikzcd}$$

Then $i^*(A) = A \times_N W \cong \pi^{-1}(A)$, so since any vector bundle over a contractible paracompact base space is trivial, we conclude that the bundle $\tilde{A} \to A$ is trivializable as $\tilde{A} \cong A \times D^k$ and $\partial\tilde{A} \cong A \times S^{k-1}$. Now the steps are as follows: calculate the homology of $A \times D^k$ and $A \times S^{k-1}$, then use UCT to obtain the cohomology, and then use the LES to find the cohomology of $\left(A \times D^k, A \times S^{k-1}\right)$.
Now... But by the Künneth theorem,

$$H_m(A \times D^k) \cong H_m(A)$$

and

$$H_m(A \times S^{k-1}) \cong H_m(A) \oplus H_{m-k+1}(A).$$

□

Lemma 3.4. *The restriction $\tau_x \in \check{H}^k(\tilde{A}, \partial\tilde{A})$ of τ, when $A = \{x\}$, is a generator.*

Proof. Note that $\left(\tilde{A}, \partial\tilde{A}\right) \cong \left(D^k, S^{k-1}\right)$.
Suppose first that $\tau_x = 0$ for some x. Let $i\colon \tilde{A} \hookrightarrow W$ be the inclusion, then we have

$$0 = i_*(0) = i_*\left(\tau_x \cap \beta\right) = \tau \cap i_*\left(\beta\right),$$

for all $\beta \in H_*\left(\tilde{A}, \partial\tilde{A}\right)$.
Let U be a neighborhood around x which is evenly covered, so $\pi^{-1}(U) \cong U \times D^k$, and $\tilde{A} = \pi^{-1}(x) \cong \{x\} \times D^k$. Let $i_x\colon \{x\} \to \{x\} \times D^k$ be the zero section, and similarly for $i_y\colon \{y\} \to \{y\} \times D^k$. Let $\gamma\colon I \to U$ be a path from x to y. Then we can define a path $F\colon I \to U \times D^k$ by $F(t) = (\gamma(t), 0)$. Then $F(t) = i_{\gamma(t)}(\gamma(t))$.
We have a pullback square as follows:

$$\begin{tikzcd} (X, X') \arrow[r] \arrow[d, "q"] & (W, W') \arrow[d, "p"] \\ I \arrow[r] & B \end{tikzcd}$$

Then we have an isomorphism induced by inclusions:

$$w_\#\colon H_n\left(F_c, F_c'\right) \overset{\cong}{\to} H_n(X, X') \overset{\cong}{\leftarrow} H_n(F_b, F_b')$$

In particular, $i_{c*} = i_{b*} w_\#$, so if $i_{c*}\left[x\right] = 0$, then $i_{b*}\left[y\right] = i_{b*} w_\#\left[x\right] = \pm i_{c*}\left[x\right] = 0$. Thus $\tau_y = 0$ for all y near x. Since N is connected, this implies that $\tau_y = 0$ for all $y \in N$.

For closed sets $A \subset N$, let $P_N(A)$ be the statement that $\tau_A = 0$, where $\tau_A = \tau|_{(\tilde{A}, \partial\tilde{A})}$.
Suppose now that A is a convex set in some euclidean open set in N. Then $\tilde{A}$ is also convex, so $\tau|_A \in \check{H}^k(\tilde{A}, \partial\tilde{A})$.

We have that the restriction defines an isomorphism

$$H^n\left(W, W-\tilde{A}\right) \to H^n\left(W, W-x\right)$$

for any point $x \in \tilde{A}$. Now, $\tilde{A}$ is closed, so $W-\tilde{A}$ is open, and $\overline{W-\tilde{A}} \subset \int$

$\square$

4. Thom-Pontryagin Theory

We start with an element $[f] \in \pi_{n+k}(S^n)$, so f is a pointed map $S^{n+k} \to S^n$. Now insert a disk in place of the base point, and extend f to a map $\overline{f}$ which is constant on the next disk, taking the disk to the basepoint of S^n, and is f elsewhere. There is a deformation retract of the sphere, collapsing this disk to a point, and composing with this retract gives f. Hence we may replace f by a pointed-homotopic map which is constant in a small neighborhood of the basepoint. Next, we can remove the base point of S^{n+k} and instead consider f as a map $\mathbb{R}^{n+k} \to S^n$ which is now constant to the base point outside some compact subset of $\mathbb{R}^{n+k}$.

By the Smooth Approximation Theorem, we can also restrict attention to smooth maps $\mathbb{R}^{n+k} \to S^n$ and smooth homotopies. Insert theorem
We regard also S^n as the one-point compactification of $\mathbb{R}^n$, denoted $\mathbb{R}^n_+ = \mathbb{R}^n \cup \{\infty\}$. So suppose now we have a smooth map $f\colon \mathbb{R}^{n+k} \to \mathbb{R}^n_+$ as above.
If f is not null-homotopic, then it must be surjective, hence in particular the image does not have measure 0, so there exists a regular value $p \in \mathbb{R}^n \subset \mathbb{R}^n_+$. By following f by a translation in $\mathbb{R}^n$, we can assume that p is the origin $0 \in \mathbb{R}^n$ without changing the homotopy class of f.

Theorem 4.1 ([1], Thm 11.6)**.** *Let $f\colon \mathbb{R}^n \to M^m$ be a smooth map. Assume that $p \in M^m$ is a regular value, let $K = f^{-1}\{p\}$, and assume that K is compact. Then there is an open neighborhood N of K inside a tubular neighborhood of K, with normal retraction $r\colon N \to K$,and an open neighborhood $E \cong \mathbb{R}^m$ of p in M^m such that the map $r \times f\colon N \to K \times E$ is a diffeomorphism.*

Using Theorem 4.1, we find that there is a disk E^n about 0 in $\mathbb{R}^n$ and an embedding $M^k \times E^n \hookrightarrow N \subset \mathbb{R}^{nk}$ onto an open neighborhood N of M^k whose inverse $N \to M^k \times E^n$ is $r \times f$, where $r\colon N \to M^k$ is the normal retraction.

Through another homotopy of f, we can assume that E^n is the open unit disk D^n.

We will refer to an embedding $g\colon M^k \times E^n \to \mathbb{R}^{n+k}$, with M^k compact, as a "fattened k-manifold".

5. Terminology

Definition 5.1 (Neighborhood retract)**.** If $A \subset X$ and A has a neighborhood in X of which it is a retract, then A is called a *neighborhood retract* (in X).

Note. Saying that $A \hookrightarrow X$ is a cofibration is stronger than saying that A is a neighborhood retract.

6. Lemmas

Lemma 6.1. *Let $\pi\colon W \to N$ be a covering map and M a connected space. Suppose $f, g\colon M \to W$ are maps such that $\pi \circ f = \pi \circ g$ and that $f(x) = g(x)$ for some $x \in M$. Then $f = g$.*

Proof. Show that the set

$$Z = \{z \in M \mid f(z) = g(z)\}$$

is closed and open. $\square$

References

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