1. Baxandall and Liebeck

Exercise 1.1. Prove that $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (|t|, t)$ is not differentiable at 0.

Proof. If f were differentiable at 0, then there would exist a linear function $L\colon \mathbb{R} \to \mathbb{R}^2$ and a function $\eta\colon \mathbb{R} \to \mathbb{R}^2$ such that $\lim_{h\to 0} \eta(h) = 0$ and

$$f(h) - f(0) = L(h) + h\eta(h)$$

Now $f(h) - f(0) = (|h|, h)$. Suppose $h = -\varepsilon < 0$. Then $f(h) - f(0) = (\varepsilon, -\varepsilon)$ and $L(h) + h\eta(h) = -\varepsilon L(1) - \varepsilon\eta(-\varepsilon) = -\varepsilon (L(1) + \eta(-\varepsilon))$. Thus

$$L(1) = \lim_{h\to 0-} \frac{f(h) - f(0)}{h} = \lim_{-\varepsilon\to 0, -\varepsilon<0} \frac{(\varepsilon, -\varepsilon)}{-\varepsilon} = (-1, 1)$$

And for $h = \varepsilon > 0$, we have $f(h) - f(0) = (\varepsilon, \varepsilon)$ and $L(h) + h\eta(h) = \varepsilon (L(1) + \eta(\varepsilon))$, so

$$L(1) = \lim_{h\to 0+} \frac{f(h) - f(0)}{h} \lim_{\varepsilon\to 0, \varepsilon>0} \frac{(\varepsilon, \varepsilon)}{\varepsilon} = (1, 1)$$

giving a contradiction for $L(1)$. □

Exercise 1.2 (3.3.1)**.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be $f(x_1, x_2) = x_1 + x_2$. Show that for any $p \in \mathbb{R}^2$,

$$\lim_{h\to 0} \frac{f(p+h) - f(p)}{\|h\|}$$

does not exist.

Solution. We have for $h \neq 0$,

$$\frac{f(p+h) - f(p)}{\|h\|} = \frac{p_1 + h_1 + p_2 + h_2 - p_1 - p_2}{\|h\|} = \frac{h_1 + h_2}{\sqrt{h_1^2 + h_2^2}}$$

Supposing $h_1 = h_2$, we get $\frac{2h_1}{\sqrt{2}h_1} = \frac{2}{\sqrt{2}}$ which is the limit along the linear $x = y$ in $\mathbb{R}^2$ approaching from above. However, if $h_1 = -h_2$, then the expression is 0, so the limit is 0 as well. Thus this limit does not exist.
However,

$$f(p+h) - f(p) = h_1 + h_2 = h_1 \frac{\partial f}{\partial x^1}(p) + h_2 \frac{\partial f}{\partial x^2}(p)$$

so by theorem 3.3.22, f is differentiable at p.
This illustrates that the simple definition for functions $\mathbb{R} \to \mathbb{R}$ of

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

does not generalize to $\mathbb{R}^m \to \mathbb{R}$ for $m \geq 2$.

Exercise 1.3. Prove that a linear function $L\colon \mathbb{R}^m \to \mathbb{R}$ is differentiable everywhere, and that it is equal to its own differential at all points in $\mathbb{R}^m$.

Proof. Clearly $L(p+h) - L(p) = L(p) + L(h) - L(p) = L(h)$, so $\delta_{L,p}$ is closely approximated near 0 by L, so L is differentiable everywhere, and $L_{L,p} = L$ by uniqueness, so L is its own differential.
Putting this in the context of differential geometry, we have that the differential of L is

$$dL = \sum \frac{\partial L}{\partial x^i} dx^i = \sum L(e_i)\, dx^i$$

Now, applying it to $e_i = \frac{\partial}{\partial x^i}$, we get $dL(e_i) = \sum L(e_j) dx^j \frac{\partial}{\partial x^i} = \sum L(e_j) \frac{\partial x^j}{\partial x^i} = \sum L(e_j)\delta_{i,j} = L(e_i)$. As both are linear maps $\mathbb{R}^m \to \mathbb{R}$ and agree on the standard basis, we conclude that $dL = L = L_{L,p}$, so indeed the differentials agree. □

Exercise 1.4 (3.3.3)**.** Prove that the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_1^3 + x_2^3$ is differentiable everywhere. Find the linear function $L_{f,p}$ that closely approximates $\delta_{f,p}$ near 0.

Solution. Recalling that if such an $L_{f,p}$ exists, then $L_{f,p}(e_i) = \frac{\partial f}{\partial x^i}(p)$, we find $L_{f,p}(e_1) = 3p_1^2$ and $L_{f,p}(e_2) = 3p_2^2$, so $L_{f,p}(h) = h_1 3p_1^2 + h_2 3p_2^2$. Now

$$\begin{aligned}\delta_{f,p}(h) = (h_1+p_1)^3 + (h_2+p_2)\, s - p_1^3 - p_2^3 &= h_1^3 + h_2^3 + 3\left(h_1^2 p_1 + h_2^2 p_2 + h_1 p_1^2 + h_2 p_2^2\right) \\ &= h_1^3 + h_2^3 + 3h_1^2 p_1 + 3h_2^2 p_2 + L_{f,p}(h) \\ &= L_{f,p}(h) + \|h\| \eta(h)\end{aligned}$$

where

$$\eta(h) = \begin{cases} \frac{h_1^3 + h_2^3 + 3h_1^2 p_1 + 3h_2^2 p_2}{\|h\|}, & h \neq 0 \\ 0, & h = 0 \end{cases}$$

Now, since $|h_1|, |h_2| \leq \|h\|$, we find that $\eta(h) \to 0$ as $h \to 0$, so f is differentiable.
The Jacobian $J_{f,p}$ becomes

$$J_{f,p} = \begin{pmatrix} 3p_1^2 & 3p_2^2 \end{pmatrix}$$