ASSIGNMENT 2

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Problem 0.1 (1)**.** Given a topological manifold M of dimension $d \in \mathbb{N}$, we define a smooth atlas on M as a collection of charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$, where $U_\alpha \subset M$ is open and $\varphi_\alpha \colon U_\alpha \xrightarrow{\cong} \mathbb{R}^d$ is a homeomorphism, such that the transition maps fit into diagrams

$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dl, "\varphi_\alpha"'] \arrow[dr, "\varphi_\beta"] & \\ \varphi_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "\varphi_\beta\varphi_\alpha^{-1}"'] & & \varphi_\beta(U_\alpha \cap U_\beta) \end{tikzcd}$$

where the lower map $\varphi_\beta\varphi_\alpha^{-1}$ is a smooth map between open subsets of $\mathbb{R}^d$.

(1) (2.5 pts) Show that each smooth manifold (as defined in the lecture) admits a smooth atlas.
(2) (2.5 pts) Show that any topological manifold equipped with a smooth atlas admits the structure of a smooth manifold (as defined in the lecture)

Proof. We recall the definition given in the lecture:

Definition 0.2. For a topological space X, we let $C^0_K(X)$ denote the continuous functions on X with support contained in K.

Definition 0.3 (Smooth manifold)**.** A smooth n-manifold is a topological n-manifold M together with an $\mathbb{R}$-sub-algebra $C^\infty(M) \subset C^0(M)$ such that for every point $p \in M$, there exists a chart $i \colon \mathbb{R}^n \hookrightarrow M$ sending $0 \mapsto p$ which is an open topological embedding, such that for all compact subsets $K \subset \mathbb{R}^n$, $i^* \colon C^\infty_K(M) \cong C^\infty_K(\mathbb{R}^n)$ and $i^* \colon C^0_K(M) \to C^0_K(\mathbb{R}^n)$ are $\mathbb{R}$-algebra isomorphisms where $C^\infty_K(M) = C^\infty(M) \cap C^0_K(M)$ such that

$$\begin{tikzcd} C^\infty_K(M) \arrow[r, dashed, "\cong", "i^*"'] \arrow[d, hook] & C^\infty_K(\mathbb{R}^n) \arrow[d, hook] \\ C^0_K(M) \arrow[r, "\cong"] & C^0_K(\mathbb{R}^n) \end{tikzcd}$$

commutes and such that $C^\infty(M)$ admits countable locally finite sums.

(1) Suppose we are given a smooth n-manifold as defined in the definition above. Thus our data consists of a topological manifold M and an R-sub-algebra $C^\infty(M) \subset C^0(M)$.

Let $p \in M$ be a point. By assumption, there exists a topological embedding $i_p \colon \mathbb{R}^n \hookrightarrow M$ sending $0 \mapsto p$. For each $p \in M$, let $U_p := i_p(\mathbb{R}^n)$ and $\varphi_p = i_p^{-1}$. Then $\{(U_p, \varphi_p)\}_{p \in M}$ gives an atlas for M. Now take any two charts $(U_p, \varphi_p), (U_q, \varphi_q)$ such that $U_p \cap U_q \neq \varnothing$. We must show that $\varphi_q \circ \varphi_p^{-1} \colon \varphi_p(U_p \cap U_q) \to \varphi_q(U_p \cap U_q)$ is smooth as a function between open subsets of $\mathbb{R}^n$. Smoothness is a local property,

so it suffices to check it locally at each point $x \in \varphi_p(U_p \cap U_q)$. Let x be such a point. Then we can find an open ball $B(x, \varepsilon) \subset \varphi_p(U_p \cap U_q)$, hence also the compact ball $\overline{B\left(x, \frac{\varepsilon}{2}\right)} \subset \varphi_p(U_p \cap U_q)$. Let $K = \overline{B\left(x, \frac{\varepsilon}{2}\right)}$. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function with support in $B\left(x, \frac{\varepsilon}{2}\right)$ and which has value 1 on some small open ball around x.
So $f \in C_K^\infty(\mathbb{R}^n)$. Then $(\varphi_q)^*(f) \in C_K^\infty(M)$.
Since $\varphi_p^{-1} = i$, we know that if $(\varphi_q)^*(f \cdot \pi_j) = (f \cdot \pi_j) \circ \varphi_q \in C_K^\infty(M)$ for all j, then $\varphi_q \circ \varphi_p^{-1}$ is smooth around x. Now, $\pi_j \cdot f$ is a product of two functions in $C^\infty(\mathbb{R}^n)$, and since f has support in K, the product is in $C_K^\infty(\mathbb{R}^n)$. Hence $(\varphi_q)^*(\pi_j \cdot f) \in C_K^\infty(M)$, and thus $i_p^*(\varphi_q)^*(\pi_j \cdot f) \in C_K^\infty(\mathbb{R}^n)$ and agrees with $\varphi_q \circ \varphi_p^{-1}$ in in a neighborhood of x. Therefore, $\varphi_q \circ \varphi_p^{-1}$ is smooth in a neighborhood of x. As x was arbitrary, this shows that $\varphi_q \circ \varphi_p^{-1}$ is smooth on all of $\varphi_p(U_p \cap U_q)$. Thus $\{(U_p, \varphi_p)\}_{p \in M}$ gives a smooth atlas which induces a smooth structure by taking the maximal atlas.

□

Problem 0.4 (3).

(1) Let M and N be two smooth manifolds, and let $f: M \to N$ be a smooth embedding which is a homeomorphism onto its image. Show that f is actually a diffeomorphism onto its image.
(2) Let M and N be two smooth, connected compact manifolds of the same dimension. Assume that we have an embedding $f: M \to N$. Show that f is a diffeomorphism.

Proof. (1) A smooth embedding is an injective smooth immersion. By the rank theorem, this is the same as an injective smooth map whose differential is injective. Note that a bijective local diffeomorphism is a diffeomorphism, so it suffices to show that f is a local diffeomorphism.
For this, note that since f is assumed to be a homeomorphism onto its image, its image is also an m-submanifold, hence the tangent spaces have the same dimension, so as the differential is injective, it is an isomorphism. But since the differential is an isomorphism, it in particular has non-vanishing determinant, so by the inverse function theorem, there exists some small neighborhood of every point in M which is mapped diffeomorphically into some neighborhood in $f(M)$, and as $f(M)$ is open in N, the image of the neighborhood is also open. Thus f is a local diffeomorphism.

(2)
Compact subsets of a Hausdorff space are closed, so since M is compact, $f(M) \subset N$ is compact, hence closed. However, f is also an embedding, hence a homeomorphism onto its image, so as M is open, $f(M)$ is open. As N is connected and $f(M) \neq \varnothing$, we must have $f(M) = N$. Now part (1) establishes the result. □

Problem 0.5 (5).

(1) Show that there is no smooth surjective map $f: \mathbb{R}^n \to \mathbb{R}^m$ whenever $n < m$.
(2) Let M be a connected compact manifold of dimension d, and fix a smooth map $f: M \to \mathbb{R}^{d+1}$. Show that there is a point $p \in \mathbb{R}^{d+1}$ and a line in $\mathbb{R}^{d+1}$ through p that meets $f(M)$ in finitely many points.

Proof. (1) Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ is a smooth surjective map. As $n < m$, we have that Df has rank at most n at all points, so all points of $\mathbb{R}^n$ are critical values of f. By Sard's theorem then, $f(\mathbb{R}^n)$ has measure zero in $\mathbb{R}^m$. This in particular implies, that no open set can be contained in $f(\mathbb{R}^n)$ since any open subset of $\mathbb{R}^m$ has Lebesgue measure greater than 0. But then $f(\mathbb{R}^n)$ cannot be surjective, giving us a contradiction.

(2) As before, since f is smooth and M is of dimension d, all points of M are critical points of f, so $f(M)$ has measure zero in $\mathbb{R}^{d+1}$. But furthermore, M is compact and connected, so $f(M)$ is a compact connected subset of $\mathbb{R}^{d+1}$. That is, there exists $K > 0$ such that $f(M)$ is a closed connected measure zero subset of $\overline{B(0, K)}$. $\square$