**Problem 0.1** (2)**.** Show that the following spaces are topological manifolds

(1) $\mathbb{RP}^n, n \in \mathbb{N}$,
(2) $\mathbb{CP}^n, n \in \mathbb{N}$,
(3) Stiefel manifolds $V_d(\mathbb{R}^n)$: for $d \leq n$, let $V_d(\mathbb{R}^n)$ be the set of $d$-frames in $\mathbb{R}^n$, i.e., the collection of linearly independent vectors $v_1, \ldots, v_d \in \mathbb{R}^n$. This inherits a topology as a subspace of $\mathbb{R}^{nd}$.
(4) Stiefel manifolds: for $d \leq n$, let $\tilde{V_d}(\mathbb{R}^n)$ be the collection of orthonormal frames in $\mathbb{R}^n$ (with respect to the standard inner product on $\mathbb{R}^n$ ). Again, this inherits a topology as a subspace of $\mathbb{R}^{nd}$.
(5) $\mathrm{GL}_n(\mathbb{R})/B_n(\mathbb{R})$, where $B_n(\mathbb{R})$ consists of invertible upper triangular matrices.

*Proof.* (1) *Hausdorff:* let $\overline{x}, \overline{y} \in \mathbb{RP}^n$ be distinct. We have $\mathbb{RP}^n := S^n/\sim$ where $a, b \in S^n$ are identified if and only if $a = \pm b$. Hence $x \neq \pm y$. Take some neighborhood $U_x$ of $x$ which is disjoint from $-x, y, -y$ by Hausdorffness. Now let $U = U_y \cup -U_y$ where $-U_y = \{-u \mid u \in U_y\}$ which is homeomorphic to $U$ since the antipodal map is a homeomorphism. Since $\mathbb{R}^{n+1}$ is a regular space, $S^n$ is also regular, so there exist disjoint open sets $V_y, W_y$ such that $\overline{U} \subset W_y$ and $y \in V_y$. Similarly, there exist open sets $V_{-y}, W_{-y}$ such that $\overline{U} \subset W_{-y}$ and $-y \in V_{-y}$. Let $V' = V_y \cap -V_{-y}$, and then $V = V' \cup -V'$. Then $V \cap U \subset V' \cup -V' \cap (W_y \cap W_{-y}) = \varnothing$ since $V' \subset V_y$ and $-V' \subset V_{-y}$.
Furthermore, $V$ and $U$ are saturated with respect to the quotient map. I.e., $V = \pi^{-1}(\pi(V))$ and $U = \pi^{-1}(\pi(U))$, so $\pi(V), \pi(U)$ are open sets in $\mathbb{RP}^n$, and they form disjoint neighborhoods around $\overline{x}$ and $\overline{y}$.
*Second-countable:* Suppose we take the collection $\mathcal{B}$ consisting of open sets

$$\pi\left(B(x, \frac{1}{n}) \cup B\left(-x, \frac{1}{n}\right) \cap S^n\right)$$

with $n \in \mathbb{N}$ and $x \in \mathbb{Q}^{n+1} \cap S^n$. Then $\mathcal{B}$ is countable and consists of open sets in $\mathbb{RP}^n$. Now for any open set $U \subset \mathbb{RP}^n$ containing some $\overline{x} \in U$, we can take some $x \in \mathbb{Q}^{n+1} \cap S^n$ and some $n \in \mathbb{N}$ such that $\left(B\left(x, \frac{1}{n}\right) \cup B\left(-x, \frac{1}{n}\right)\right) \cap S^n \subset \pi^{-1}(U)$, hence $U$ contains some open set $V \in \mathcal{B}$ such that $x \in V \subset U$. So $\mathcal{B}$ is a countable basis for $\mathbb{RP}^n$.
*Locally-Euclidean:* Take a point $[x] = [x_1, \ldots, x_n, x_{n+1}] \in \mathbb{RP}^n$. Then some $x_i$ is non-zero. Let $U_i := \{[z_1, \ldots, z_{n+1}] : z_i \neq 0\} \subset \mathbb{RP}^n$. So $[x] \in U_i$. Now define the map $\varphi_i \colon U_i \to \mathbb{R}^n$ by $\varphi([z_1, \ldots, z_{n+1}]) = \left(\frac{z_1}{z_i}, \ldots, \hat{z_i}, \ldots, \frac{z_{n+1}}{z_i}\right)$ where $\hat{z_i}$ means that this coordinate is excluded. This is well-defined since $\frac{z_j}{z_i} = \frac{\lambda z_j}{\lambda z_i}$ for all $\lambda \neq 0$. In fact, this is a homeomorphism with inverse $\psi_i \colon \mathbb{R}^n \to U_i$ given by $\psi(y_1, \ldots, y_n) = [y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n]$. Hence we choose $(U_i, \varphi_i)$ to be one chart for $\mathbb{RP}^n$. The collection $U_1, \ldots, U_{n+1}$ covers $\mathbb{RP}^n$, hence the collection of charts $\{U_i, \varphi_i\}$ for $i = 1, \ldots, n+1$, gives an atlas for $\mathbb{RP}^n$.

(2) *Hausdorff:* The Hausdorff condition for $\mathbb{CP}^n$ is checked in the same way as for $\mathbb{RP}^n$, noting that $\mathbb{C}^{n+1}$ is a regular space.
*Second-countable:* This is also checked similarly as for $\mathbb{RP}^n$ where the collection $\mathcal{B}$ above is now taken for $x \in \mathbb{Q}[i]^{n+1} \cap S^n$.
*Locally-Euclidean:* Take the same open sets $U_i$ as for $\mathbb{RP}^n$, now as subsets of $\mathbb{CP}^n$, and define $\varphi_i$ and $\psi_i$ in the same way. It is again clear that $\varphi_i$ and $\psi_i$ are inverses of each other, hence $\varphi_i \colon U_i \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$. So composing the charts $\varphi_i$ with the

isomorphism $\mathbb{C}^n \cong \mathbb{R}^{2n}$ gives the desired map - denote this also by $\varphi_i$. Then the collection $(U_i, \varphi_i), i = 1, \ldots, n+1$ will again define an atlas for $\mathbb{CP}^n$.

(3) *Hausdorff:* Let $(v_1, \ldots, v_d), (w_1, \ldots, w_d) \in V_d(\mathbb{R}^n)$ be distinct $d$-frames. Then there exists $i$ such that $v_i \neq w_i$, hence a coordinate $j$ such that $v_{ij} \neq w_{ij}$. Take disjoint neighborhoods $V, W$ containing $v_{ij}$ and $w_{ij}$, respectively, in $\mathbb{R}$. Then $\mathbb{R}^{ij-1} \times V \times \mathbb{R}^{nd-ij} \cap V_d(\mathbb{R}^n)$ and $\mathbb{R}^{ij-1} \times W \times \mathbb{R}^{nd-ij} \cap V_d(\mathbb{R}^n)$ are disjoint neighborhoods of $(v_1, \ldots, v_d)$ and $(w_1, \ldots, w_d)$, respectively.

*Second-countable:* Every subspace of a second-countable space is second-countable, and as $\mathbb{R}^{nd}$ is second-countable, so is $V_d(\mathbb{R}^n)$.

*Locally-Euclidean:* Consider some $d$-frame $(v_1, \ldots, v_d)$ as an $n \times d$ matrix. Since the columns are linearly independent, there exists some $d \times d$ submatrix with non-vanishing determinant. By continuity of of the determinant function, this $d \times d$ matrix has an open neighborhood in $M_d(\mathbb{R}^n)$ on which the determinant function is non-vanishing. Extend this neighborhood to one on $\mathbb{R}^{nd}$ by choosing $\mathbb{R}$ for the other coordinates and then taking the product of these sets to get an open neighborhood of $(v_1, \ldots, v_d)$. On this neighborhood, the corresponding matrices have a $d \times d$ submatrix with non-vanishing determinant which means the $d$ columns are linearly independent. Thus this open set is in fact contained in $V_d(\mathbb{R}^n)$ and hence also an open set in $V_d(\mathbb{R}^n)$ as we use the subspace topology. Naturally, the chart on this open set we choose to simply be the one sending $(w_1, \ldots, w_d)$ to its coordinates in $\mathbb{R}^{nd}$. This is a bijective open map which is continuous since on any open set contained in the open set on $\mathbb{R}^{nd}$ which is the image of the chart, we still have the $d \times d$ submatrix on which the determinant function is non-vanishing, hence still linearly independent columns. Since a bijective, open continuous map is a homeomorphism, this gives a chart for $(v_1, \ldots, v_d)$.

(d) *Hausdorff:* any orthonormal frame is also linearly independent, so we can use the same open sets intersecting with $\tilde{V}_d(\mathbb{R}^n)$ as for $V_d(\mathbb{R}^n)$ above.

*Second-countable:* Every subspace of a second-countable space is second-countable, and as $\mathbb{R}^{nd}$ is second-countable, so is $\tilde{V}_d(\mathbb{R}^n)$.

*Locally-Euclidean:*

$\square$

**Problem 0.2** (3).

(1) Show that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.
(2) Show that the dimension of a connected topological manifold is unique, i.e., such a manifold $M$ cannot have dimensions $m$ and $n$ with $m \neq n$.

*Proof.* (1) If $m = n$, then $\mathbb{R}^m \cong \mathbb{R}^n$ by the identity.
Suppose $m \neq n$ and $\mathbb{R}^m \cong \mathbb{R}^n$. Then also the one point compactifications are homeomorphic, so $S^m \cong S^n$. Suppose $m > n \geq 0$. Then $\mathbb{Z} \cong H_m(S^m) \cong H_m(S^n) \cong 0$ which is a contradiction.

(2) Suppose a manifold $M$ has dimensions $m$ and $n$ with $m \neq n$. Let $p \in M$ and

choose two charts $(U,\varphi),(V,\psi)$ centered at 0 such that $\varphi(U)\subset\mathbb{R}^n$ and $\psi(V)\subset\mathbb{R}^m$, and assume $n>m$. We can take some $B(0,\varepsilon)\subset\varphi(U)$ and take its preimage under $\varphi$ to replace $U$ with an open set whose image under $\varphi$ is $B(0,\varepsilon)$, so assume without loss of generality that $\varphi(U)=B(0,\varepsilon)$. Then $\psi\circ\varphi^{-1}\colon \overline{B(0,\varepsilon)}\to\psi(U\cap V)\subset\mathbb{R}^m$ is a homeomorphism. Restricting to $S^{n-1}\cong\partial B\left(0,\frac{\varepsilon}{2}\right)$, we get an embedding $S^{n-1}\hookrightarrow\mathbb{R}^m$. But by the standard inclusion $S^m\hookrightarrow S^{n-1}$, we get an injective composite map $S^m\hookrightarrow S^{n-1}\hookrightarrow\mathbb{R}^m$, which is a contradiction by the Borsuk-Ulam theorem. □

**Problem 0.3** (4)**.**
1. Let $M$ and $N$ be two topological manifolds. Show that $M\times N$ is again a topological manifold of dimension $\dim M+\dim N$.
2. Let $M$ and $N$ be two topological manifolds of the same dimension. Show that $M\sqcup N$ is again a topological manifold.
3. Show that the connected components of a manifold are again manifolds; in other words, every manifold is written as a disjoint union of a collection of connected manifolds. Can this collection be uncountable?

*Proof.* (1) *Hausdorff:* The product of two Hausdorff spaces is Hausdorff in the product topology, so as $M,N$ are manifolds hence Hausdorff, so is $M\times N$.

*Second-countable:* Likewise, the finite product of second-countable spaces is countable, so as $M,N$ are manifolds hence second-countable, so is $M\times N$.

*Locally-Euclidean:*
Let $(p,q)\in M\times N$ be arbitrary with $m=\dim M$ and $n=\dim N$. Choose charts $(U,\varphi)$ and $(V,\psi)$ for $p$ and $q$, respectively. Then define a chart $\varphi\times\psi\colon U\times V\to\mathbb{R}^{m+n}$ by sending $\varphi\times\psi\,(u,v)=(\varphi(u)_1,\ldots,\varphi(u)_m,\psi(v)_1,\ldots,\psi(v)_n)$ where $\varphi(u)_i$ is the $i$ th coordinate of $\varphi(u)$ and $\psi(v)_j$ is the $j$ th coordinate of $\psi(v)$. Taking the product topology on $M\times N$, $\varphi\times\psi$ becomes an open embedding of the open set $U\times V$ into $\mathbb{R}^{m+n}$.

(2) We can define $M\sqcup N:=M\times\{1\}\cup N\times\{0\}$. Suppose $M$ and $N$ are $n$-dimensional manifolds. Let $\tilde{p}\in M\sqcup N$, then either $\tilde{p}=(p,0)$ with $p\in N$ or $\tilde{p}=(p,1)$ with $p\in M$. Suppose without loss of generality that $p\in N$. Take some chart $(U,\varphi)$ around $p$ in $N$. Then $\tilde{U}:=U\times[0,\frac{1}{2})\cap M\sqcup N$ is an open set around $\tilde{p}$ in $M\sqcup N$. Define a chart $\tilde{\varphi}\colon\tilde{U}\to\mathbb{R}^n$ by $\tilde{\varphi}(p,0)=\varphi(p)$. For some open set $V\subset\mathbb{R}^n$, we then have $\tilde{\varphi}^{-1}(V)=\varphi^{-1}(V)\times\{0\}=\varphi^{-1}(V)\times[0,\frac{1}{2})\cap M\sqcup N$ which is clearly open. Likewise, take some open subset $W\times\{0\}\subset\tilde{U}$. Then $\tilde{\varphi}(W\times\{0\})=\varphi(W)$ which is open since $\varphi$ is a homeomorphism on $U$ onto its open image. As $\varphi$ is bijective, so is $\tilde{\varphi}$, so $\tilde{\varphi}$ is a homeomorphism onto its open image, so $\left(\tilde{U},\tilde{\varphi}\right)$ is a chart around $\tilde{p}$ in $M\sqcup N$. As $\tilde{p}$ was arbitrary, we see that $M\sqcup N$ is a topological $n$ manifold.

*Hausdorff:* The Hausdorff condition is clear: if $\tilde{p},\tilde{q}\in M\sqcup N$ are distinct points which are both either in $M$ or $N$, suppose without loss of generality both are in $N$, then $\tilde{p}=(p,0),\tilde{q}=(q,0)$, so we can take disjoint open neighborhoods $U,V$ around $p$ and $q$ respectively using that $N$ is a manifold, and then create open sets $\tilde{U}=U\times[0,\frac{1}{2})\cap M\sqcup N,\tilde{V}=V\times[0,\frac{1}{2})\cap M\sqcup N$ which are still disjoint and open in $M \sqcup N$ containing $\tilde{p}$ and $\tilde{q}$, respectively. If instead $\tilde{p}$ and $\tilde{q}$ are in, say, $N$ and $M$, respectively, then $\tilde{p} = (p, 0)$ and $\tilde{q} = (q, 1)$, so take open neighborhoods $U, V$ around $p$ and $q$, respectively. Then $\tilde{U} = U \times [0, \frac{1}{2}) \cap M \sqcup N, \tilde{V} = V \times (\frac{1}{2}, 1] \cap M \sqcup N$ are disjoint open neighborhoods of $\tilde{p}$ and $\tilde{q}$, respectively.

*Second-countable:* Let $\mathcal{B}, \mathcal{B}'$ be countable bases for $N$ and $M$, respectively. Then $\mathcal{B} \times \{0\} \cup \mathcal{B}' \times \{1\}$ gives a countable basis for $M \sqcup N$.

(3) Suppose $M$ is a manifold and $N \subset M$ is a connected component. Let $p \in N$. Since $M$ is a manifold, there exists a chart $(U, \varphi)$ around $p$. Then there exists a ball $B(\varphi(p), \varepsilon) \subset \varphi(U)$ since $\varphi(U)$ is open. Now $\varphi^{-1}(B(\varphi(p), \varepsilon))$ is a connected open subset of $U$ containing $p$ since homeomorphisms are in particular hence preserve connectivity. Thus we in particular have that if we let $\tilde{U} := \varphi^{-1}(B(\varphi(p), \varepsilon))$, then $\left(\tilde{U}, \varphi|_{\tilde{U}}\right)$ is a chart in $M$ hence also in $N$ with $N$ inheriting the subspace topology. Thus every point $p \in N$ has a chart $U_p$ contained in $N$. Hausdorffness can be checked as follows: if $p, q \in N$, then there exist disjoint neighborhoods $V, W$ around $p$ and $q$, respectively. Now $V \cap U_p$ and $W \cap U_q$ are open disjoint neighborhoods of $p$ and $q$, respectively, in $N$. Alternatively, with $N$ inheriting the subspace topology, Hausdorffness is inherited from $M$.
For second-countability, $M$ has a countable basis $\mathcal{B}$, so since $N$ has the subspace topology, $\mathcal{B} \cap N = \{U \cap N \mid U \in \mathcal{B}\}$ is a countable basis for $N$.
Hence $N$ is a topological manifold of the same dimension as $M$.

To answer whether an uncountable collection of connected manifolds can be a manifold, we note that given an uncountable collection of connected manifolds, second-countability would imply that there exists a countable basis, suppose $\mathcal{B}$ is such a basis. Now let $\bigsqcup_{i \in I} M_i$ be the uncountable union of connected manifolds. For each $i$, choose an $x_i \in M_i$, and let $(U_i, \varphi_i)$ be a chart around $x_i$ contained in $M_i$ (whose existence follows from the previous exercise). By assumption of $\mathcal{B}$ being a basis, there exists some $B_i \in \mathcal{B}$ such that $x_i \in B_i \subset U_i$. But this defines an injective map $I \to \mathcal{B}$ where $I$ was assumed to be uncountable and $\mathcal{B}$ countable - this is a contradiction.

$\square$