**Exercise 0.1** (2)**.** Let $R$ be a Noetherian ring. Show that every ideal of $R$ is a finite intersection of irreducible ideals.

*Proof.* Let $\mathcal{A}$ be the set of ideals of $R$ which are not a finite intersection of irreducible ideals. Suppose $\{X_i\} = \mathcal{X} \subset \mathcal{A}$ is a chain with respect to inclusion. Then since $R$ is Noetherian, this chain stabilizes, hence it has an upper bound. Thus $\mathcal{A}$ has a maximal element, call it $M$. Now, $M$ is not a finite intersection of irreducible ideals, so in particular, $M$ is not irreducible, so write $M = I \cap J$ where $I$ and $J$ contain $M$ properly. But then $I, J \notin \mathcal{A}$, so they are finite intersections of irreducible ideals. However, then their intersection is also a finite intersection of irreducible ideals. □

**Exercise 0.2** (3)**.** Let $R$ be a ring. Show that an ideal $\mathfrak{B}$ of $R$ is a prime ideal if and only if for all ideals $I$ and $J$ of $R$, $IJ \subset \mathfrak{B}$ implies $I \subset \mathfrak{B}$ or $J \subset \mathfrak{B}$.

*Proof.* If $\mathcal{B}$ is a prime ideal, then suppose $IJ \subset \mathcal{B}$, and suppose $I \not\subset \mathcal{B}$. Then there exists some $i \in I$ such that $i \notin \mathcal{B}$ but $iJ \subset \mathcal{B}$. Since $\mathcal{B}$ is prime, we must have that $J \subset \mathcal{B}$.
Conversely, suppose that for all $I, J$, $IJ \subset \mathcal{B}$ implies $I \subset \mathcal{B}$ or $J \subset \mathcal{B}$. Let $a, b \in R$ such that $ab \in \mathcal{B}$. Then
$(a)(b) \subset \mathcal{B}$, so either $(a) \subset \mathcal{B}$ or $(b) \subset \mathcal{B}$, so either $a \in \mathcal{B}$ or $b \in \mathcal{B}$. So $\mathcal{B}$ is prime. □

**Exercise 0.3** (4)**.** Let $R_S$ be the localization of the integral domain $R$ by a multiplicative subset $S$ which does not contain 0. Let $\mathfrak{U}$ be a primary ideal of $R$. Show that the extension $\mathfrak{U}R_S$ is a primary ideal of $R_S$ and that $R \cap \mathfrak{U}R_S = \mathfrak{U}$.

*Proof.* Let $\tau\colon R \to R_S = (R - S)^{-1} R$. Recall that $\mathfrak{U}R_S = (\tau(\mathfrak{U}))$. Suppose $ab \in \mathfrak{U}R_S$. Then there exists $\sum \alpha_i u_i \in \mathfrak{U}$ such that $ab = \sum \alpha_i \tau(u_i) = \sum \alpha_i \frac{u_i}{1}$. We can write $a = \frac{x}{y}$ and $b = \frac{v}{w}$. Then $xv \in \mathfrak{U}$, so either $x \in \mathfrak{U}$, in which case $\frac{x}{y} \in \mathfrak{U}R_S$, or $v^n \in \mathfrak{U}$, in which case $\left(\frac{v}{w}\right)^n \in \mathfrak{U}R_S$ since $S$ is multiplicative, so $w^n \in S$. Thus $a \in \mathfrak{U}R_S$ or $b^n \in \mathfrak{U}R_S$. Hence $\mathfrak{U}R_S$ is primary in $R_S$.
Now recall that $R \cap \mathfrak{U}R_S$ denote the contraction of $\mathfrak{U}R_S$ along $\tau$, i.e., $R \cap \mathfrak{U}R_S = \tau^{-1}(\mathfrak{U}R_S)$. Clearly, $\mathfrak{U} \subset R \cap \mathfrak{U}R_S$. Conversely, suppose $a \in \tau^{-1}(\mathfrak{U}R_S)$. Then $\frac{a}{1} \in \mathfrak{U}R_S$, so $\frac{a}{1} = \sum \alpha_i \frac{u_i}{1}$. So there exists some $r \in R - S$ such that $ra = r \sum \alpha_i u_i$. So since $r \neq 0$ as $0 \notin S$, we have $a = \sum \alpha_i u_i$ since $R$ is an integral domain. Thus $a \in \mathfrak{U}$. □

**Exercise 0.4** (5)**.** Let $R$ be a ring and $\mathfrak{U}$ a $\mathfrak{B}$-primary ideal. Show the following.

(1) For all $x \in \mathfrak{U}$, we have $(\mathfrak{U}\colon x) = R$.
(2) For all $x \in R - \mathfrak{U}$, we have that $(\mathfrak{U}\colon x)$ is $\mathfrak{B}$-primary.
(3) For all $x \in R - \mathfrak{B}$, we have that $(\mathfrak{U}\colon x) = \mathfrak{U}$.
(4) If $R$ is Noetherian, then there is some $x \in R - \mathfrak{U}$ such that $(\mathfrak{U}\colon x) = \mathfrak{B}$.

*Proof.* So $\sqrt{\mathfrak{U}} = \mathfrak{B}$ by assumption.

(1) Since $\mathfrak{U}$ is an ideal, $x\mathfrak{U} \subset \mathfrak{U}$ for all $x \in R$, so for all $x \in \mathfrak{U}$, $(\mathfrak{U}\colon x) = R$.
(2) Suppose $a \in \sqrt{(\mathfrak{U}\colon x)}$, so $a^n x \in \mathfrak{U}$. Since $\mathfrak{U}$ is primary, either $x \in \mathfrak{U}$, which we have assumed it is not, or $a^{nm} \in \mathfrak{U}$ for some $m$. Hence the latter must be true. So $a \in \sqrt{\mathfrak{U}} = \mathfrak{B}$.
Hence $\sqrt{(\mathfrak{U} : x)} \subset \mathfrak{B}$. Conversely, suppose $a \in \mathfrak{B}$, so $a^n \in \mathfrak{U}$. So for any $x \in R - \mathfrak{U}$, we have $a^n x \in \mathfrak{U}$, so $a \in \sqrt{(\mathfrak{U} : x)}$.

(3) Suppose $x \in R - \mathfrak{B}$. Now if $y \in \mathfrak{U}$ then $xy \in \mathfrak{U}$, so $y \in (\mathfrak{U} : x)$. Conversely, suppose $y \in (\mathfrak{U} : x)$. So $xy \in \mathfrak{U}$, so since $x \notin \mathfrak{B} = \sqrt{\mathfrak{U}}$, we must have that $y \in \mathfrak{U}$.
(4) Let $n$ be minimal such that $\mathfrak{B}^n \subset \mathfrak{U}$. Let $x \in \mathfrak{B}^{n-1} - \mathfrak{U}$. We claim $(\mathfrak{U} : x) = \mathfrak{B}$. For $b \in \mathfrak{B}$, we have $bx \in \mathfrak{B}^n \subset \mathfrak{U}$, so $b \in (\mathfrak{U} : x)$. Conversely, if $bx \in \mathfrak{U}$, then since $x \notin U$, we have $b^m \in \mathfrak{U}$, so $b \in \sqrt{U} = \mathfrak{B}$. $\square$