Exercise 3.1.ix: Show that if J has an initial object, then the limit of any functor indexed by J is the value of that functor at an initial object. Apply the dual of this result to describe the colimit of a diagram indexed by a successor ordinal.

Solution: Suppose J has initial object 1 and let $F\colon J \to C$ be a functor indexed by J. We want to show that $F(1) = \lim F$, i.e., that there exists a universal cone $\lambda\colon F(1) \implies F$.
Firstly, we define the cone λ by letting $\lambda_j = F(f_j)$ where f_j is the unique morphism $1 \to j$ in J (uniqueness by 1 being initial, and hence $|\mathrm{Hom}(1,j)| = 1$ for all $j \in J$.
Then we want to show that for any $j, j' \in J$ with a morphism $f\colon j \to j'$, the following commutes

$$\begin{tikzcd} & F(1) \arrow[dl, "F(f_j)"'] \arrow[dr, "F(f_{j'})"] & \\ Fj \arrow[rr, "Ff"] & & Fj' \end{tikzcd} \tag{1}$$

To show commutativity, we note that since $f \circ f_j$ is a map $1 \mapsto j \mapsto j'$, we have $f \circ f_j \in \mathrm{Hom}\,(1, j') = \{f_{j'}\}$, so $f \circ f_j = f_{j'}$, and since functors distribute over composition, we have

$$F(f) \circ F\,(f_j) = F(f \circ f_j) = F(f_{j'})$$

giving commutativity of (1).

To show uniqueness, suppose there exists an element $A \in J$ and a cone of F over A with components $A_j\colon A \to F(j)$ such that for any $j, j' \in J$ with a morphism $f\colon j \to j'$, the following diagram commutes

$$\begin{tikzcd} & A \arrow[dl, "A_j"'] \arrow[dr, "A_{j'}"] & \\ Fj \arrow[rr, "Ff"] & & Fj' \end{tikzcd} \tag{2}$$

Then in particular, there exists a map $A_1\colon A \to F(1)$. We claim that A_1 is the unique morphism such that

$$\begin{tikzcd} & A \arrow[d, "A_1"] \arrow[ddl, "A_j"'] \arrow[ddr, "A_{j'}"] & \\ & F(1) \arrow[dl, "Ff_j"'] \arrow[dr, "Ff_{j'}"] & \\ Fj \arrow[rr, "Ff"] & & Fj' \end{tikzcd} \tag{3}$$

commutes.
Firstly, $Ff_j \circ A_1 = A_j$ and $Ff_{j'} \circ A_1 = A_{j'}$ by (2), and $Ff \circ Ff_j = Ff_{j'}$ by (1), so (3) commutes.
Suppose there exists another morphism $B\colon A \to F(1)$ such that

$$\begin{tikzcd} & A \arrow[d, "B"] \arrow[ddl, "A_j"'] \arrow[ddr, "A_{j'}"] & \\ & F(1) \arrow[dl, "Ff_j"'] \arrow[dr, "Ff_{j'}"] & \\ Fj \arrow[rr, "Ff"] & & Fj' \end{tikzcd} \tag{4}$$

Then choosing j to be 1, and letting f be the unique morphism $f\colon 1 \to j'$, we get that the following commutes

$$\begin{tikzcd} & A \arrow[d, "B"] \arrow[ddl, "A_1"'] \arrow[ddr, "A_{j'}"] & \\ & F(1) \arrow[dl, "F\mathbb{1}_1"'] \arrow[dr, "Ff_{j'}"] & \\ F(1) \arrow[rr, "Ff"] & & Fj' \end{tikzcd} \tag{5}$$

where we can conclude $Ff_j = Ff_1$ to be $F\mathbb{1}_1$ since $f_1 \in \mathrm{Hom}\,(1,1) = \{\mathbb{1}_1\}$ as 1 is initial.
Hence we get $B = B \circ \mathbb{1}_{F(1)} = B \circ F\mathbb{1}_1 = A_1$, where the first and second equalities follow by definition of a functor. Thus we get uniqueness. So $F(1)$ is the limit of the functor F with $\{F(f_j)\}_{j\in J}$ being the legs of the universal cone $\lambda\colon F(1) \implies F$.

The dual of this result then states that if J has a terminal object, then the colimit of any functor indexed by J is the value of that functor at a terminal object.

Now, suppose we have a finite successor ordinal $\mathbb{n}$ freely generated by

$$0 \to 1 \to 2 \to \ldots \to n-1$$

Then $n-1$ is a terminal object, since by (iv) on page 5, every non-identity morphism can be uniquely factored as a composite of morphisms in the displayed graph, so if $f_j\colon j \to n-1$ is a morphism, then f_j factors uniquely as the composite $j \to j+1 \to \ldots \to n-1$ in the graph above, and hence if we call the composition $j \to j+1 \to \ldots \to n-1$ for g_j then $\mathrm{Hom}\,(j, n-1) = \{g_j\}$. In particular, g_j exists for all j (since $n-1$ must also have an identity morphism and for any other j, we have a morphism as a composition of morphisms in the graph). Since $|\mathrm{Hom}(j, n-1)| = 1$ for all j, $n-1$ is a terminal object.
Since an infinite successor ordinal, α, is a successor of the infinite ordinal, we do not run into the problem of not having a right-most object, as in for ω, instead we get

$$0 \to 1 \to 2 \to \ldots \to \alpha-1$$

for α, so $\alpha - 1$ becomes the terminal object with a completely equivalent reasoning as for the finite case.

Now the dual version states that the colimit of any functor F indexed by a successor ordinal, call it α, is the value $F\,(\alpha-1)$ where the legs of the universal cone are $F(g_j)$.

Exercise 3.2.iii: For any pair of morphisms $f\colon a \to b, g\colon c \to d$ in a locally small category C, construct the set of commutative squares $\mathrm{Sq}(f,g)$:

$$\begin{tikzcd} a \arrow[r] \arrow[d, "f"] & c \arrow[d, "g"] \\ b \arrow[r] & d \end{tikzcd}$$

from f to g as a pullback in Set.

Solution: We wish to find the set

$$\{(k,l) \in \mathrm{Hom}(b,d) \times \mathrm{Hom}(a,c) \mid l \circ f = g \circ k\}$$

since any such (k,l) define a commutative square of $\mathrm{Sq}(f,g)$ and conversely, any commutative square must define such a tuple of morphisms.

Consider the diagram

$$\mathrm{Hom}(b,d) \xrightarrow{f^*} \mathrm{Hom}(a,d) \xleftarrow[g_*]{} \mathrm{Hom}(a,c)$$

By example 3.2.11, elements of the pullback of this pair of functions are cones

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & \mathrm{Hom}(a,c) \arrow[d, "g_*"] \\ \mathrm{Hom}(b,d) \arrow[r, "f^*"] & \mathrm{Hom}(a,d) \end{tikzcd}$$

The data of this consists by example 3.2.11 of a pair of morphisms $k \in \mathrm{Hom}(a,c)$ and $l \in \mathrm{Hom}(b,d)$ such that $g \circ k = l \circ f$. I.e. the pullback is

$$\mathrm{Hom}(b,d) \times_{\mathrm{Hom}(a,d)} \mathrm{Hom}(a,c) := \{(k,l) \in \mathrm{Hom}(b,d) \times \mathrm{Hom}(a,c) \mid l \circ f = g \circ k\}$$

which is in bijective correspondence given by to the set of commutative squares $\mathrm{Sq}(f, g)$:

$$(k, l) \in \mathrm{Hom}(b, d) \times_{\mathrm{Hom}(a,d)} \mathrm{Hom}(a, c) \leftrightarrow \begin{tikzcd} a \arrow[r, "k"] \arrow[d, "f"] & c \arrow[d, "g"] \\ b \arrow[r, "l"] & d \end{tikzcd}$$

by the definition since $l \circ f = g \circ k$ for all $(k, l) \in \mathrm{Hom}(b, d) \times_{\mathrm{Hom}(a,d)} \mathrm{Hom}(a, c)$.