Problem 0.1. Show that K_5 is not planar.

Proof. We'll make use of the theorem

Theorem 0.2. $|V| - |E| + |F| = 2 - 2g$.

If K_5 is planar, then $g = 0$, so $|V| - |E| + |F| = 2$. We count the number of vertices and edges:

$$|V| = 5$$
$$|E| = \binom{5}{2} = 10.$$

Hence $|F| = 2 - 5 + 10 = 7$. Note that in a cycle of ρ^*, we must have at least 3 sides, since if $\rho^*(v, w) = (w, \rho_w(v))$, and $\rho^*(w, \rho_w(v)) = \left(\rho_w(v), \rho_{\rho_w(v)}(\rho_w(v))\right) = (v, w)$, so $\rho_w(v) = v$, but ρ_w cannot fix any vertices as it is a cyclic permutation. So if each face of K_5 must have at least 3 edges, then there must be at least $7 \cdot 3 = 21$ sides. □

Proof.

$$\sigma^*(gx, gy) = \left(gy, \sigma_{gy}(gx)\right) = \left(gy, \rho_{gy}^{(g)}(gx)\right) = \left(gy, g\rho_y g^{-1}(gx)\right) = \left(gy, gg_y(x)\right) = (gy, gz)$$

where $(x, y, z, \ldots)$ occurs in a face.
The second part of the exercise follows from noting that $\sigma^{(g^{-1})} = \rho$. □