1. H-Spaces, H-Groups and H-Cogroups

An H-space or H-group is a space with a product that satisfies some of the laws of a group *but only up to homotopy.* An H-cogroup is a dual notion. The "H" stands for "Hopf" or for "Homotopy".

Definition 1.1 (H-Space, homotopy associativity, homotopy inverse)**.** An *H-space* is a pointed space $X \in \mathrm{Top}_*$ with base point e, together with a map

$$\cdot \colon X \times X \to X$$

sending $(x, y) \mapsto x \cdot y$ such that $e \cdot e = e$, and the maps $X \to X$ taking $x \mapsto x \cdot e$ and $x \mapsto e \cdot x$ are each homotopic rel $\{e\}$ to the identity.

It is said to be *homotopy associative* if the maps $X \times X \times X \to X$ taking (x, y, z) to $(x \cdot y) \cdot z$ and to $x \cdot (y \cdot z)$ are homotopic rel $\{(e, e, e)\}$.

It is said to have a *homotopy inverse* $\hat{-}\colon X \to X$ if $\hat{e} = e$ and the maps $X \to X$ taking x to $x \cdot \hat{x}$ and to $\hat{x} \cdot x$ are each homotopic rel $\{e\}$ to the constant map to $\{e\}$.

Definition 1.2 (H-group)**.** An *H-group* is a homotopy associative H-space with a given homotopy inverse.

There are two main examples: the first is the class of topological groups, the second is the class of "loop spaces".

Definition 1.3 (Loop space)**.** The loop space on a space X is the space

$$\Omega X = (X, *)^{(S^1, *)},$$

i.e., X^{S^1} in the pointed category. The product is concatenation of loops, and the homotopy inverse is loop reversal. ΩX is a pointed space with base point being the constant loop at $*$.

Definition 1.4 (Operations on maps)**.** If $f \colon X \to Z$ and $g \colon Y \to W$ are maps, let $f \vee g \colon X \vee Y \to Z \vee W$ be the induced map on the one-point union.
Let $\nabla \colon Z \vee Z \to Z$ be the codiagonal, i.e., the identity on both factors.
We also define $f \,\underline{\vee}\, g \colon X \vee Y \to Z$ as the composition $f \,\underline{\vee}\, g = \nabla \circ (f \vee g)$; i.e., the map which is f on X and g on Y.

Definition 1.5 (H-cogroup)**.** An *H-cogroup* is a pointed space Y and a map $\gamma \colon Y \to Y \vee Y$ such that the following three conditions are satisfied:

(1) The constant map $* \colon Y \to Y$ to the base point is a *homotopy identity*; i.e., the compositions $(* \,\underline{\vee}\, \mathrm{id}) \circ \gamma$ and $(\mathrm{id} \,\underline{\vee}\, *) \circ \gamma$ of $Y \xrightarrow{\gamma} Y \vee Y \to Y$ are both homotopic to the identity rel base point.
(2) It is homotopy associative. That is, the compositions $(\gamma \vee \mathrm{id}) \circ \gamma$ and $(\mathrm{id} \vee \gamma) \circ \gamma$ of $Y \xrightarrow{\gamma} Y \vee Y \to Y \vee Y \vee Y$ are homotopic to one another rel base point.
(3) There is a homotopy inverse $i \colon Y \to Y$. That is, $(\mathrm{id} \,\underline{\vee}\, i) \circ \gamma$ and $(i \,\underline{\vee}\, \mathrm{id}) \circ \gamma$ of $Y \xrightarrow{\gamma} Y \vee Y \to Y$ are both homotopic to the constant map to the base point rel base point.

One important class of examples is the reduced suspensions: the "coproduct" $\gamma\colon SX \to SX \vee SX$ is given by

$$\gamma(t,x) = \begin{cases} (2t,x)_1\,, & t \le \frac{1}{2} \\ (2t-1,x)_2\,, & t \ge \frac{1}{2} \end{cases}$$

where the subscripts indicate in which copy of SX in the one-point union the indicated point lies.
The homotopy inverse is just reversal of the t parameter.

Theorem 1.6. *In the pointed category:*

1. *If Y is an H-group then $[X;Y]$ is a group with multiplication induced by $(f\cdot g)(x) = f(x)\cdot g(x)$.*
2. *If X is an H-cogroup then $[X;Y]$ is a group with multiplication induced by $f * g = (f \veebar g)\circ\gamma$.*
3. *If X is an H-cogroup and Y is an H-space then the two multiplications above on $[X;Y]$ coincide and are abelian.*

Proof. We first show that $f\cdot g$ is well defined on $[X;Y]\times[X;Y]\to[X;Y]$. Suppose $f\simeq f'$ and $g \simeq g'$. Then we must show that $f(x)\cdot g(x) \simeq f'(x)\cdot g'(x)$. By assumption, $\mathrm{id} \simeq \widehat{f(x)}\cdot f'$ and $\mathrm{id}\simeq g'(x)\cdot\widehat{g(x)}$, so

$$\begin{aligned} f'(x)\cdot g'(x) &\simeq \left(f(x)\cdot\widehat{f(x)}\right)\cdot\left((f'(x)\cdot g'(x))\cdot\left(\widehat{g(x)}\cdot g(x)\right)\right) \\ &\simeq f(x)\cdot\left(\widehat{f(x)}\cdot f'(x)\right)\cdot\left(g'(x)\cdot\widehat{g(x)}\right)\cdot g(x) \\ &\simeq f(x)\cdot g(x). \end{aligned}$$

Now we check the group axioms. Associativity follows from homotopy associativity of Y. The constant map to the basepoint of Y is an identity, let us denote it by e, since $(f\cdot e)(x) = f(x)\cdot e(x) \simeq f(x) \simeq e(x)\cdot f(x) = (e\cdot f)(x)\,\mathrm{rel}\,\{e(x) = *\}$ since Y is an H-space. So there exists a map $H\colon Y\times I\to Y$ such that $H(f(x),0) = f(x)\cdot *$ ad $H(f(x),1) = f(x)$. Define $G\colon X\times I\to Y$ by $G(x,t) = H(f(x),t)$. This defines a homotopy of $f\cdot e$ with f. One can do the same for $e\cdot f\simeq f$. Lastly, given $f\in[X;Y]$, define $\widehat{f}\colon X\to Y$ by $\widehat{f}(x) = \widehat{f(x)}$. This is continuous as the composition $\widehat{-}\circ f\colon X\to Y$. Now, for each $x\in X$, $\left(f\cdot\widehat{f}\right)(x) = f(x)\cdot\widehat{f(x)} \simeq e\,\mathrm{rel}\,e$, so there exists $H\colon Y\times I\to Y$ such that $H\left(f(x)\cdot\widehat{f}(x),0\right) = f(x)\cdot\widehat{f(x)}$ and $H\left(f(x)\cdot\widehat{f(x)},1\right) = e$. Then again defining $G(x,t) = H\left(f(x)\cdot\widehat{f(x)},t\right)$ defines a homotopy from $f\cdot\widehat{f}$ to e relative $*$. Etc.

For (2), associativity is shown as follows: $(f*g)*h = [((f\veebar g)\circ\gamma)\veebar h]\circ\gamma$ which is the composition

$$X \xrightarrow{\gamma} X\vee X \xrightarrow{\gamma\vee\mathrm{id}} (X\vee X)\vee X \xrightarrow{(f\veebar g)\veebar h} Y.$$

The first composition $(\gamma\vee\mathrm{id})\circ\gamma$ is homotopic to $(\mathrm{id}\vee\gamma)\circ\gamma$ by condition (2) of an H-cogroup, and the maps $(f\veebar g)\veebar h$ and $f\veebar(g\veebar h)$ are equal, which provides the homotopy from $(f*g)*h$ to $f*(g*h)$. The other parts are similar.

For (3), we need the following lemma:

Lemma 1.7. *If X is an H-cogroup and Y an H-space, then for $f, g, h, k\colon X \to Y$, we have*
$$(f \cdot g) * (h \cdot k) = (f * h) \cdot (g * k)\,.$$

Proof. Suppose $x \in X$ is an arbitary point and that $\gamma(x) = (w, *) \in X \vee X$ (recall that we can view $X \vee X$ as $X \times \{*\} \cup \{*\} \times X$), so this amounts to $(w, *)$ just representing a point in one copy of X and $(*, w')$ would then represent a point in the other copy.
Now
$$(f \cdot g) * (h \cdot k)\,(x) = ((f \cdot g) \veebar (h \cdot k))\,(w, *) = (f \cdot g)\,(w) = f(w) \cdot g(w)$$
and
$$(f * h) \cdot (g * k)\,(x) = (f * h)(x) \cdot (g * k)(x) = f(w) \cdot g(w).$$
The case $\gamma(x) = (*, w')$ is similar. □

Returning to part (3), note first that for both product, the identity id is given by the constant map to the base point by condition (1) for an H-cogroup.
Operating in $[X; Y]$, we have
$$(\alpha \cdot \beta) * (\gamma \cdot \delta) = (\alpha * \gamma) \cdot (\beta * \delta)\,.$$
Thus
$$\alpha * \beta = (\mathrm{id} \cdot \alpha) * (\beta \cdot \mathrm{id}) = (\mathrm{id} * \beta) \cdot (\alpha * \mathrm{id}) = \beta \cdot \alpha,$$
and
$$\alpha * \beta = (\alpha \cdot \mathrm{id}) * (\mathrm{id} \cdot \beta) = (\alpha * \mathrm{id}) \cdot (\mathrm{id} * \beta) = \alpha \cdot \beta.$$
Therefore $\alpha \cdot \beta = \beta \cdot \alpha = \alpha * \beta$.

□