## Remarks

In the text, a topological space is presumed to be $T_0$ axiomatically; thus, any time space is mentioned, one means a $T_0$ topological space.

**Def.** A topological group $G$ is a group $(G, \cdot)$ such that the underlying set of $G$ forms a topological ( $T_0$ ) space, and such that the maps $\mu\colon G \times G \to G$ by $\mu(x, y) = x \cdot y$ and $i\colon G \to G$ by $i(x) = x^{-1}$ are continuous.

**1.11.1.** If $H$ is a subgroup of $G$ and if $x, y \in \overline{H}$, then $xy \in \overline{H}$.
*Proof:* Since multiplication is continuous, we have that for any open $U \subset G$ containing $xy$, $x \in \pi_1(\mu^{-1}(U))$ and $y \in \pi_2\left(\mu^{-1}(U)\right)$, and as these are open, there exist $h_1, h_2 \in H$ with $h_1 \in \pi_1\left(\mu^{-1}(U)\right)$ and $h_2 \in \pi_2\left(\mu^{-1}(U)\right)$, so $h_1 h_2 \in U$ so $U \cap H \neq \varnothing$, hence $xy \in \overline{H}$.

Similarly, $x^{-1}$ belongs to the closure.
Thus, $\overline{H}$ is a group, and we shall call $\overline{H}$ a **closed** subgroup.

**Examples of topological groups**

**1.** The sets $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ of $n \times n$ matrices over $\mathbb{R}$ and $\mathbb{C}$ under addition with the distance of $A = (a_{ij})$ and $B = (b_{ij})$ defined by
$$d\,(A, B) = \max_{i,j} |a_{ij} - b_{ij}|$$
gives a topological group whose topological space is metrizable.
The spaces of these two groups are homeomorphic to $E_{n^2}$ and $E_{2n^2}$. They are in fact the sets of real or complex vectors with $n^2$ coordinates, and hence are vector spaces as well as groups. Another example is the set $H$ of continuous real valued functions on a compact metric space under addition.
In this example, the multiplication function on the topological group is addition, so $\mu(A, B) = A + B$ and $i(A) = -A$. Let $(A, B) \in M_n(\mathbb{R}) \times M_n(\mathbb{R})$ and take an $\varepsilon$-neighborhood of $A + B$, so $X \in B(A + B, \varepsilon)$ iff
$$\max_{i,j} |a_{ij} + b_{ij} - x_{ij}| < \varepsilon.$$
Take an $\frac{\varepsilon}{2}$-neighborhood of $A$ and $B$, then $U = B(A, \frac{\varepsilon}{2}) \times B(B, \frac{\varepsilon}{2})$ is an open set in $M_n(\mathbb{R}) \times M_n(\mathbb{R})$ containing $(A, B)$. And furthermore, if $(C, D) \in U$, then
$$d(\mu(C, D), \mu(A, B)) = \max_{i,j} |c_{ij} + d_{ij} - a_{ij} - b_{ij}| \leq \max_{i,j} |a_{ij} - c_{ij}| + \max_{i,j} |b_{ij} - d_{ij}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Thus $\mu$ is continuous.

For $i\colon M_n(\mathbb{R}) \to M_n(\mathbb{R})$, suppose $A \in M_n(\mathbb{R})$. Take any $B \in B(A, \varepsilon)$. Thus $\max_{i,j} |a_{ij} - b_{ij}| < \varepsilon$. Then
$$d\,(i(A), i(B)) = \max_{i,j} |-a_{i,j} - (-b_{i,j})| < \varepsilon.$$
Hence $i$ is continuous too.

**2.** The set of non-singular real or complex $n \times n$ matrices $GL(n, \mathbb{R}), GL(n, \mathbb{C})$ under multiplication; these are subsets of $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ induced topology. They are open subsets and are therefore locally compact and locally euclidean (**1.27**).

Suppose $\mu\colon M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is matrix multiplication. Here $M_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^{n^2}$ and can be thought of as such. A map into $\mathbb{R}^{n^2}$ is continuous if each coordinate function is continuous. Now, each coordinate function is of the form
$$\sum_{r=1}^{n} a_{ir} b_{rj}$$

which is the $i, j$ coordinate of $\mu(A, B)$. The sum and product of continuous function into $\mathbb{R}$ is continuous, so if we can show that the map $p_{ij}\colon M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is continuous where $p_{ij}(A, B) = a_{ij}$, then we are done.
Suppose $\varepsilon > 0$. Then let $B(A, \varepsilon) \times M_n(\mathbb{R})$ is an open neighborhood of $(A, B)$ and for any $(C, D) \in B(A, \varepsilon) \times M_n(\mathbb{R})$, we have

$$|p_{ij}(A, B) - p_{ij}(C, D)| = |a_{ij} - c_{ij}| < \max_{r,k} |a_{rk} - c_{rk}| < \varepsilon$$

so $p_{ij}$ is continuous.

The restriction to subspaces does not affect continuity, so $\mu\colon GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ is continuous.

**3.** Let $S$ be a compact metric space and let $G$ be the group of all homeomorphisms of $S$ onto itself topologized as a subspace of the space of continuous maps of $S$ into itself.

## 1.12 Isomorphism of topological groups

The spaces associated with two topological groups may be homeomorphic but the groups essentially different, for example, one abelian and the other not.

**Example.** The matrices

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad a, b \in \mathbb{R}$$

under addition is an abelian group with $E_2$ as space where $E_n = \underbrace{E_1 \times \ldots \times E_1}_{n \text{ times}}$ and $E_1$ is the set of all real numbers in its customary topology.
The matrices

$$\begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, \quad a, b \in \mathbb{R}$$

under multiplication forms a non-abelian group with $E_2$ as space as well.

To show that this is homeomorphic to $E_2$, define a map $f\colon \mathbb{R}^2 \to X = \left\{ \begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ by $f(x, y) = \begin{pmatrix} e^x & y \\ 0 & e^{-x} \end{pmatrix}$.
Suppose $\varepsilon > 0$. Then $f(x', y') \in B(f(x, y), \varepsilon)$ if and only if $\left|e^x - e^{x'}\right|, \left|e^{-x} - e^{-x'}\right|, |y - y'| < \varepsilon$. Now, since exp is continuous, there exists $\delta_1 > 0$ such that $|x - x'| < \delta_1 \implies \left|e^x - e^{x'}\right| < \varepsilon$ and $\delta_2 > 0$ such that $|x - x'| < \delta_2 \implies \left|e^{-x} - e^{-x'}\right| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2, \varepsilon\}$. Then supposeing $x' \in B(x, \delta)$ and $y' \in B(y, \delta)$, we get

$$|f(x', y') - f(x, y)| \leq \min\left\{\left|e^x - e^{x'}\right|, |y - y'|, \left|e^{-x} - e^{-x'}\right|\right\} < \varepsilon$$

so $f$ is continuous. Now, define $g\colon X \to \mathbb{R}^2$ by $g\left(\begin{pmatrix} x & b \\ 0 & y \end{pmatrix}\right) = (\ln(x), b)$.
Now,

$$\begin{aligned} |\ln(x) - \ln(x')| < \varepsilon &\iff \left|\ln(\frac{x}{x'})\right| < \varepsilon \\ &\iff e^{-\varepsilon} < \frac{x}{x'} < e^{\varepsilon} \\ &\iff x'e^{-\varepsilon} < x < x'e^{\varepsilon}, \quad \text{if } x' \in R_+ \\ &\iff x'(e^{-\varepsilon} - 1) < x - x' < x'\left(e^{\varepsilon} - 1\right) \end{aligned}$$

Let $A = \begin{pmatrix} e^a & b \\ 0 & e^{-a} \end{pmatrix}, B = \begin{pmatrix} e^c & d \\ 0 & e^{-c} \end{pmatrix}$.
So letting $\delta = \min\left\{\varepsilon, \left|x'\left(e^{-\varepsilon}-1\right)\right|, \left|x'\left(e^{\varepsilon}-1\right)\right|\right\}$, we have $B \in B(A,\delta)$ implies

$$d_{max}(g(B), g(A)) = \max\left\{|b-d|, |a-c|\right\} < \varepsilon.$$

Hence $g$ is also continuous, and clearly, $g \circ f = \mathbb{1}$ and $f \circ g = \mathbb{1}$, so $f$ is a homeomorphism.

If we give the space in this example (or example 1) the discrete topology, we obtain a new topological group with the same algebraic structure.

**Example 3.** In the additive group of integers, for each pair of distinct integers $h$ and $k$, let the set $\{h + nk \colon n \in \mathbb{Z}\}$ be called an open set, and let the collection of all these sets be taken as a basis for open sets.

**Example 4.** Introduce a metric into the additive group of integers, depending on the prime number $p$, defined thus:

$$d(a,b) = \frac{1}{p^n}$$

if $a \neq p$ and $p^n$ is the highest power of $p$ which is a factor of $a - b$, i.e. $n = v_p(a,b)$.

**Example 5.** Let $G$ be the integers under addition with the finite-complement topology. Algebraically $G$ is a group and it is also a topological space; however, it is not a topological group because addition is now not simultaneously continuous: suppose $\mu(n,m) = n+m$. Then $\mu^{-1}\left(\mathbb{Z} - \{0\}\right) = \{(-n,n) \mid n \in \mathbb{Z}\}^c$. Now, $\mathbb{Z}-\{0\}$ is open as the complement is finite; however, $\{(-n,n) \mid n \in \mathbb{Z}\}^c$ is not open as its complement is infinite. Hence, $\mu$ is not continuous.
It is true, however, that addition is continuous in each variable separately.
For some types of group spaces, separate continuity implies simultaneous continuity. I is not known whether this is true for a compact Hausdorff group space.

**Definition.** Two topological groups will be called isomorphic if there is a bijective correspondence between their elements which is a group isomorphism and a space homeomorphism.

## 1.13 Set products

If $G$ is a group, $A \subset G$, let $A^{-1}$ denote the inverse set $\left\{a^{-1} \colon a \in A\right\}$. If $B \subset G$, let $AB = \{ab \mid a \in A, b \in B\}$. If $B = \varnothing$ then $AB = \varnothing$.
A set $H$ in $G$ is called **invariant** if $gH = Hg$ for every $g \in G$, or equivalently, if $gHg^{-1} = H$.

**Theorem 1.** Let $G$ be a topological group and let $A \subset G$ be an open set. Then $A^{-1}$ is open.

*Proof:* Since inverses are unique, we have $i^{-1}(A) = A^{-1}$, so since $A$ is open, $A^{-1}$ is open.

In a slightly different way, suppose $x \in A^{-1}$; then $x^{-1} \in A$, so by continuity of the inverse, there exists an open set containing $x$, say $U$, such that $b \in U$ implies $b^{-1} \in A$. Hence $U^{-1} \subset A$, so $x \in U \subset A^{-1}$. Thus $A^{-1}$ is open.

**Corollary.** The map $x \mapsto x^{-1}$ is a homeomorphism.

**Lemma.** Let $G$ be a topological group, $A$ an open subset, $b$ an element. Then $Ab$ and $bA$ are open.

*Proof:* If $f \colon X \times Y \to Z$ is continuous, then $g \colon X \to Z$ defined by $g(x) = f(x, y_0)$ is continuous for any $y_0 \in Y$.
Define the map $k \colon X \to X \times Y$ by $k(x) = (x, y_0)$. Thus if $k = (k_1, k_2)$, then $k_1 = \mathbb{1}$ and $k_2 = y_0$. Since both maps are continuous, $k$ is continuous. And since $\mu$ is continuous, $\mu \circ k \colon X \to Z$ is continuous as the composition of continuous maps.

Now, define the map $k\colon G \to G \times G$ by $k(g) = (g, b)$. Then we claim that the map $r_b\colon G \to G$ by $r_b(g) = gb$ is a homeomorphism.
It is continuous since $r_b = \mu \circ k$ which is a composition of continuous maps, and it is bijective clearly. The inverse, is given by $r_{b^{-1}} = \mu \circ l$ where $l\colon G \to G \times G$ is $l(g) = \big(g, b^{-1}\big)$.

Thus it is a homeomorphism. Since $Ab = r_b(A)$, it is open. Similarly for $bA$.

**Corollary.** For each $a \in G$, the left and right translations: $x \mapsto ax, x \mapsto xa$ are homeomorphisms.

**Theorem 2.** Let $G$ be a topological group and let $A$ and $B$ be subsets. If $A$ or $B$ is open then $AB$ is open.

*Proof:* If $A$ is open, then $Ab$ is open for all $b$, so $AB = \bigcup_{b \in B} Ab$ is open. Similarly for the case where $B$ is open.

**Corollary.** Let $A$ be a closed subset of a topological group. Then $Ab$ and $bA$ are closed.

**Example.** Let $E_1$ be the additive group of real numbers and $G$ a topological group. A continuous homomorphism, $h(t)$, of $E_1$ into $G$, is called a **one-parameter group** in $G$. If $h$ is defined only on an open interval around zero satisfying $h(xy) = h(x)h(y)$ for $x, y \in E_1$ so far as it has meaning, then $h(t)$ is called a local one-parameter group in $G$. If $h(t)$ is a one-parameter group, the image of $E_1$ may consist of $e$ alone and then $h(t)$ is a trivial one-parameter group. If this is not the case and if for some $t_1 \neq 0, h(t_1) = e$, then the image of $E_1$ is homeomorphic to $S^1$. In case $h(t) = e$ only for $t = 0$, the image of $E_1$ is a bijective image of the line which may be a homeomorphism of the line or a very complicated imbedding of the line. To illustrate this, let $G$ be a torus which we obtain from the plane vector group $E_2$ by reducing mod one in both the $x$ and $y$ directions. In $E_2$ any line through the origin is a subgroup isomorphic to $E_1$ and after reduction, the line $y = ax$ is mapped onto the torus $G$ thus giving a one-parameter group in $G$. If $a$ is rational the image is a simple closed curve but if $a$ is irrational, the image is everywhere dense on the torus.

## 1.14 Products of closed sets

If $A, B$ are subgroups of a group $G$, $AB$ is not necessarily a subgroup. However, if $A$ is an invariant subgroup, i.e., $g^{-1}Ag = A, \forall g \in G$, and $B$ is a subgroup, then $AB$ is a subgroup.

The product of closed subsets, even if they are subgroups, need not be closed.
E.g.: $G = (\mathbb{R}, +)$, let $H_1 = (\mathbb{Z}, +)$ and $H_2 = \big(\big\{\pm n\sqrt{2}\colon n \in \mathbb{Z}\big\}, +\big)$. Then $H_1H_2$ is countable and a subgroup, but it is not closed as it is dense in $\mathbb{R}$.
For example, we have that $\mathbb{Z}\left[\sqrt{2}\right]$ has infinitely many units, so there exist infinitely many $m, n \in \mathbb{Z}$ such that $n^2 - 2m^2 = 1$, so $(m + \sqrt{2}n)(m - \sqrt{2}n) = 1$, and hence $\big(m - \sqrt{2}n\big) = \frac{1}{m+\sqrt{2}n}$. Suppose $m >> 0$. If $n << 0$ then $m - \sqrt{2}n$ is large, so $m + \sqrt{2}n = \frac{1}{m-\sqrt{2}n}$ is small. If $n$ is not large and negative, then $m + \sqrt{2}n$ is large, so $m - \sqrt{2}n = \frac{1}{m+\sqrt{2}n}$ is small. In particular, we can make $m \pm n\sqrt{2} \in (0, \varepsilon)$ for any $\varepsilon > 0$. Now, since $\mathbb{Z}\left[\sqrt{2}\right]$ is closed under addition, we have that this gives that the set is dense.

It will be shown later that if $A$ is a compact invariant subgroup and $B$ is a closed subgroup then $AB$ is a closed subgroup.

## 1.15 Neighborhoods of the identity

Let $G$ be a topological group and $U$ an open subset containing the identity $e$. We showed in 1.13 that $xU$ is open and clearly $x \in xU$. Conversely, if $x \in O$ and $O$ is open, then $U = x^{-1}O$ is open and contains $e$.

If a collection of open sets $\{U_\alpha\}$ is a basis for open sets at $e$ then every open set of $G$ is a union of open sets of the form $x_\alpha U_\alpha, x_\alpha \in G, U_\alpha \in \{U_\alpha\}$, and the topology of $G$ is completely determined by the basis at $e$. In particular, the collection $\{xU_\alpha\}$ is a basis for open sets at $x$ (so also i $\{U_\alpha x\}$ ).
If $U$ is a neighborhood of $e$, $U^{-1}$ is a neighborhood of $e$ and $U \cap U^{-1}$ is a symmetric neighborhood of $e$.

**Theorem.** Let $G$ be a topological group and $U$ a neighborhood of $e$. There exists a symmetric neighborhood $W$ of $e$ such that $W^2 \subset U$.

*Proof:* Let $\mu^{-1}(U) = U'$. There exists a basis element $V \times W \subset U'$ with $V, W$ open. Then by definition $e = e \cdot e \in VW \subset U$. Let $W = V \cap V^{-1} \cap W \cap W^{-1}$. Then $W^2 \subset U$ as $VW \subset U$ and it is symmetric as the intersection of symmetric neighborhoods of $e$.

**Corollary.** Let $G$ be a topological group. If $x \neq e$, there exists a neighborhood $W$ of $e$ such that $W \cap xW = \varnothing$.

*Proof:* Since $G$ is $T_0$, there either exists an open set $W_1$ containing $x$ but not $e$ or an open set $W_2$ containing $e$ but not $x$. If $W_2$ exists, there exists by the previous theorem a symmetric neighborhood $W$ of $e$ such that $W^2 \subset W_2$. Suppose $w \in W \cap xW$. Then $w = xw'$ so $x = ww'^{-1} \in W^2 \subset U$, contradiction.

Thus, suppose $W_1$ exists. Then $i^{-1}(W_1) = W_1^{-1}$ is open, and $xW_1^{-1}$ is a neighborhood around $e$ not containing $x$. By the previous theorem, there exists a symmetric neighborhood $W$ around $e$ such that $W^2 \subset xW_1^{-1}$. Suppose $w \in W \cap xW$. Then $w = xw'$ so $x = ww'^{-1} \in W^2 \subset xW_1^{-1}$, but $xW_1^{-1}$ does not contain $x$; contradiction.
**1.15.1** If $G$ is a topological group then a set $S$ of open neighborhood s $\{V\}$ which forms a basis at the identity $e$ has the following properties:

1. The intersection of all $V$ in $S$ is $\{e\}$.
2. The intersection of two sets of $S$ contains a third set of $S$.
3. Given $U \in S$ there is a $V$ in $S$ such that $VV^{-1} \subset U$.
4. If $U \in S$ and $a \in U$ then there is a $V \in S$ such that $Va \subset U$.
5. If $U \in S$ and $a \in G$, there is a $V \in S$ such that $aVa^{-1} \subset U$.

Conversely, a system of subsets of an abstract group having these properties may be used to determine a topology in $G$ as follows:
**Theorem.** Let $G$ be an abstract group in which there is given a system $S$ of subsets satisfying 1-5 above. If open sets in $G$ are defined as unions of sets of the form $Va, a \in G$, then $G$ becomes a topological space with $S$ as a basis for open sets at $e$. This is the only topology making $G$ a topological group with $S$ a basis at $e$.

*Proof:* Suppose $G$ is a topological group with $S$ the basis at $e$.
Then if $x \in G - \{e\}$, we get by the previous corollary that there exists a neighborhood $W$ of $e$ such that $W \cap xW = \varnothing$. Now, since $e \in W$, we have $x \in xW$, so $x \notin W$. As $S$ is a basis, there exist open sets $\{U_\alpha\} \subset S$ such that $W = \bigcup U_\alpha$, so $x \notin U_\alpha$ for any $\alpha$, hence $x \notin \bigcap U_\alpha \supset \bigcap_{V \in S} V$. Since $e$ is in any $V \in S$, we have $e \in \bigcap_{V \in S} V$, giving 1.
2. is clear as the intersection of two open sets at $e$ is an open set at $e$ hence contains a basis open set of $S$.

3. This follows from the theorem above. There exists open symmetric $W$ such that $WW^{-1} = W^2 \subset U$. Let $W = \bigcup_{V_\alpha \in S} V_\alpha$. Then $V_\alpha V_\alpha^{-1} \subset WW^{-1} = W^2 \subset U$.

4. Suppose $U \in S$ and $a \in U$. Then $e \in Ua^{-1}$, so there exists a $V \in S$ such that $V \subset Ua^{-1}$. Hence $Va \subset U$.

5. If $U \in S$ then $e \in U$, so $e = a^{-1}ea \in a^{-1}Ua$ and hence there exists $V \in S$ such that $V \subset a^{-1}Ua$ so $aVa^{-1} \subset U$.

Now, suppose $G$ is a group with a system $S$ of subsets satisfying 1-5. We claim $\{gU \mid g \in G, U \in S\}$ is a basis for a topology. Firstly, for any $g \in G$, $g \in gU$ for any $U \in S$ as $e \in U$, so the union of all such sets gives the whole space $G$.
Now, suppose $x \in gU \cap g'U'$. Then $e \in \left(x^{-1}g\right) U \cap \left(x^{-1}g'\right) U'$, so by the second property, there exists a $U'' \in S$ such that $e \in U'' \subset \left(x^{-1}g\right) U \cap \left(x^{-1}g'\right) U'$, so $x \in xU'' \subset gU \cap g'U'$.

This gives that $\{gU \mid g \in G, U \in S\}$ is a basis for a topology on $G$.
Suppose $e, x \in gU$. We want to find $U' \in S$ with $x \in U' \subset gU$.

## 1.16 Coset spaces

Let $G$ be a group and $H$ a subgroup. The sets $xH$ and $yH$ for $x, y \in G$ either coincide or are mutually exclusive; and $xH = yH$ iff $xy^{-1} \in H$.

**Def 1.16.1.** By the natural map $T$ of a group onto the coset space $G/H$, $H$ being a subgroup of $G$, we mean the map

$$T\colon x \to xH, \quad x \in G, xH \in G/H.$$

For any subset $U \subset H$ we have

$$T^{-1}\left(T(U)\right) = UH \subset G.$$

It will become clear as we proceed that unless the group $H$ is a closed subgroup of $G$, it will not be possible in general to have $T$ continuous and $G/H$ a topological space; for this reason, only the case where $H$ is closed will be considered.

**Theorem.** $T\colon G \to G/H$ is open, and if $xH \neq yH$, there exist neighborhoods $W_1$ and $W_2$ of $xH$ and $yH$ respectively, such that $W_1 \cap W_2 = \varnothing$.

*Proof:* Let $U \subset G$ be open. We must show that $T(U)$ is open. But since $T$ is a quotient map, $T(U)$ is open if and only if $T^{-1}\left(T(U)\right)$ is open in $G$. Now since for any $\overline{g} \in G/H$, $T^{-1}(\overline{g}) = \bigcup_{T(x)=\overline{g}} \{x\} = \bigcup_{xg^{-1} \in H} \{x\} = Hg$, we have

$$T^{-1}\left(T(U)\right) = \bigcup_{g \in U} Hg = \bigcup_{h \in H} hU$$

and since $g \mapsto hg$ is a homeomorphism $G \to G$, we have that $hU$ is open, so $T^{-1}\left(T(U)\right)$ is open. So $T$ is open.

Now, we wish to show that if $xH \neq yH$ then there exist open neighborhoods $W_1$ and $W_2$ around $xH$ and $yH$, respectively, such that $W_1 \cap W_2 = \varnothing$.
Since $xH \neq yH$, we have $x \notin yH$, so since $yH$ is closed, we have that $\tilde{W} = (G - yH)x^{-1}$ is an open set around $e$. By the theorem in 1.15, there exists a symmetric neighborhood $W$ of $e$ such that $W^2 \subset \tilde{W}$. So $Wx$ is an open set around $x$ with $Wx \cap yH = \varnothing$ since $\tilde{W}x \cap yH = \varnothing$. We claim $WxH \cap WyH = \varnothing$. Otherwise, $wxh = w'yh'$ so $(w'^{-1}w)x = y(h'h^{-1})$; but $\left(w'^{-1}w\right) \in W^2 \subset \tilde{W}$ and $h'h^{-1} \in H$ as $H$ is a subgroup. So $\tilde{W}x \cap yH \neq \varnothing$, contradiction. Hence $WxH \cap WyH = \varnothing$, and $Wx, Wy$ are open, so $T(Wx), T(Wy)$ are open and $T(Wx) \cap T(Wy) = T\left(WxH \cap WyH\right) = \varnothing$. This gives the result.

**Corollary.** A topological group $G$ and a coset-space $G/H$, with $H$ closed in $G$, are Hausdorff spaces.

**Corollary.** Suppose that $G$ is a topological group and $H$ a closed invariant (normal) subgroup. Then with the customary definition of product $(xH)(yH) = xyH$, $G/H$ becomes a topological group. The natural map of $G$ into $G/H$ is a continuous and open homomorphism.