1. Theory

Recall that

Definition 1.1 (Dirichlet Series)**.** Let f be an arithmetic function. Then the corresponding Dirichlet series is defined, for $s \in \mathbb{C}$, by

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Lemma 1.2.

$$0 \leq 3 + 4\cos\theta + \cos 2\theta = 2\left(1 + \cos\theta\right)^2$$

Lemma 1.3. *Let $\sigma > 1$. Then*

$$\Re\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it)\right) \geq 0$$

For the proof of the lemma, one shows that

$$\Re\left(\frac{1}{n^s}\right) = \frac{1}{n^\sigma}\cos\left(t\log n\right), \quad s = \sigma + it \tag{A_1}$$

Proof.

$$\Re\left(-\frac{\zeta'}{\zeta}(s)\right) = \Re\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}\cos\left(t\log n\right).$$

Hence

$$\Re\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it)\right) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}\left[3 + 4\cos\left(t\log n\right) + \cos\left(2t\log n\right)\right] \overset{(1.2)}{\geq} 0$$

$\square$

2. Week 1

Exercise 2.1 (E1.1. Abel summation)**.** Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ and $f\colon [1,x] \to \mathbb{C}$ be C^1. Define $A(t) = \sum_{n\le t} a_n$. Then for $x > 1$, we have

$$\sum_{n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

3. Week 2

Let $\psi(x) := \sum_{n\le x} \Lambda(n)$.

Exercise 3.1 (E2.6)**.** Show that

$$\theta(x) := \sum_{p\le x} \log p = \psi(x) + O\left(x^{\frac{1}{2}} \log^2 x\right)$$

Exercise 3.2 (E2.7)**.** Show that

$$\pi(x) = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t\log^2 t}dt + O\left(x^{\frac{1}{2}} \log x\right).$$

Proof. By Abel summation, we first find that

$$\theta(x) := \sum_{p\le x} \log p = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt$$

and from the previous exercise, we now find that

$$\pi(x) = \frac{\psi(x)}{\log x} + \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt + O\left(x^{\frac{1}{2}} \log x\right)$$

The result follows if we can show that

$$\frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt = \int_2^x \frac{\psi(t)}{t\log^2 t}dt + O\left(x^{\frac{1}{2}} \log x\right).$$

Now $\psi(t) \le \pi(t)\log t$, so

$$\begin{aligned}\left|\int_2^x \frac{\psi(t)}{t\log^2 t} - \frac{\pi(t)}{t\log x}dt\right| &\le \left|\int_2^x \frac{\pi(t)}{t\log t} - \frac{\pi(t)}{t\log x}dt\right| \\ &= \left|\int_2^x \frac{\pi(t)}{t}\frac{\log\left(\frac{x}{t}\right)}{\log x\log t}dt\right|\end{aligned}$$

∎

4. Week 3

Exercise 4.1 (E3.1)**.** Let $m \ge 0$. Show that

$$\sum_{n\le x} \log^m n = x\log^m x + O\left(x\log^{m-1} x\right).$$

Proof. Let $a_n = 1$ for all n. Then $A(x) = \lfloor x \rfloor$, so

$$\begin{aligned}\sum_{n \le x} \log^m n &= \lfloor x \rfloor \log^m x - \int_1^x m \lfloor t \rfloor \frac{1}{t} \log^{m-1} t dt \\ &= x \log^m x - (x - \lfloor x \rfloor) \log^m x - m \int_1^x \frac{\lfloor t \rfloor}{t} \log^{m-1}(t) dt\end{aligned}$$

Thus we must show that

$$\left| (x - \lfloor x \rfloor) \log^m x + m \int_1^x \frac{\lfloor t \rfloor}{t} \log^{m-1}(t) dt \right| \le C x \log^{m-1} x$$

But $\frac{\lfloor t \rfloor}{t} \log^{m-1}(t) \le \log^{m-1}(x)$ giving that the right hand term is $O\left(x \log^{m-1} x\right)$. For the left hand term, it suffices to show that $(x - \lfloor x \rfloor) \log x \le x$, but this is clear since $x - \lfloor x \rfloor \le 1$ and $\log x \le x$. $\square$

Exercise 4.2 (E3.2)**.** Let $d(n) = \sum_{d|n} 1$. Show $d(n) \le 2\sqrt{n}$. If we consider the set $D \subset \mathbb{N}$ of positive divisors of n, then we can define a bijection $D \to D$ by $k \mapsto \frac{n}{k}$. Suppose now that $d(n) > 2\sqrt{n}$. Suppose $d \mid n$ and $d \ge \sqrt{n}$. Then since $\frac{d}{n} \cdot d = n$, we must have $\frac{d}{n} \le \sqrt{n}$. This implies that under this bijection, either the source or target lies in $\{1, \dots, \lfloor \sqrt{n} \rfloor\}$. Hence $d(n) = |D| \le 2 \left|\{1, \dots, \lfloor \sqrt{n} \rfloor\}\right| \le 2\sqrt{n}$.

Exercise 4.3 (E3.3)**.** Prove that for every $\varepsilon > 0$, there exists a constant C_ε such that $d(n) \le C_\varepsilon n^\varepsilon$.
Hint:

(1) Show that $d(n_1 n_2) = d(n_1) d(n_2)$ if $(n_1, n_2) = 1$.
(2) Show that
$$\frac{d(n)}{n^\varepsilon} = \prod_{p^\alpha || n} \frac{\alpha + 1}{p^{\alpha\varepsilon}}$$
where $p^\alpha \mid\mid n$ means that α is a positive integer, $p^\alpha \mid n$ and $p^{\alpha+1} \not| n$.
(3) Split the product in 2. Into the product over those primes $p < 2^{\frac{1}{\varepsilon}}$ and the product over the rest. Show that the second product is bounded by 1.
(4) Show that the factors in the first product are less than $1 + (\varepsilon \log 2)^{-1}$.

Proof. We follow the hint:
(1) Suppose $(n_1, n_2) = 1$. Let D be the set of divisors of $n_1 n_2$, D_1 the set of divisors of n_1 and D_2 the set of divisors of n_2. Suppose $d_1 \in D_1, d_2 \in D_2$. Then $d_1 a = n_1, d_2 b = n_2$, so $d_1 d_2 ab = n_1 n_2$, hence $d_1 d_2 \in D$. We thus obtain a map $D_1 \times D_2 \to D$ sending $(d_1, d_2) \mapsto d_1 d_2$. We claim this is a bijection. Suppose $d_1 d_2 = d_1' d_2'$. If $d_1 \mid d_2'$, then $d_1 = 1$, in which case, $d_1' = 1$, and thus $d_2 = d_2'$. Suppose thus that $d_1 \ne 1$, so $d_1 \not| d_2'$. Then since $(d_1', d_2') = 1$, we have $d_1 \mid d_1'$. Similarly, $d_1' \mid d_1$. So $d_1 = d_1'$. And again $d_2 = d_2'$. This gives injectivity. For surjectivity, if $d \mid n_1 n_2$, then consider $d_1 := \frac{d}{(n_2, d)}$ and $d_2 := \frac{d}{(n_1, d)}$. Then $d_1 d_2 = d$ and $d_1 \in D_1, d_2 \in D_2$.
(2) Clearly, $n^\varepsilon = \prod_{p^\alpha || n} p^{\alpha\varepsilon}$. It thus suffices to show that $\prod_{p^\alpha || n} (\alpha + 1) = d(n)$. But if we factorize n as $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, then it is clear that the divisors corresponds precisely to tuples $(a_1, \dots, a_m)$ with $0 \le a_i \le \alpha_i$. There are precisely $\alpha_1 + 1$ choices for each a_i, giving $(\alpha_1 + 1) \cdots (\alpha_m + 1) = d(n)$ which indeed is what we wanted to show.

(3) We can split the product as

$$\frac{d(n)}{n^\varepsilon} = \underbrace{\prod_{\substack{p^\alpha || n \\ p < 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}}_{A} \cdot \underbrace{\prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}}_{B}$$

We claim that $B \leq 1$. Indeed

$$\prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}} \leq \prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \underbrace{\frac{\alpha+1}{2^\alpha}}_{\leq 1} \leq 1$$

(4) For the factors in the first product, we have $\alpha = \left\lfloor \frac{\log n}{\log p} \right\rfloor$ and $\log p < \frac{1}{\varepsilon}\log 2$, and $\alpha \leq \frac{\log n}{\log p}$, so $\frac{\log p}{\log n} \leq \frac{1}{\alpha}$

$$\varepsilon^2 \log p < \varepsilon \log 2$$

$$\frac{\alpha+1}{p^{\alpha\varepsilon}} \leq \frac{\log n + \log p}{p^{\alpha\varepsilon}\log p} \leq 1 + \frac{1}{\varepsilon \log 2} = \frac{\varepsilon \log 2 + 1}{\varepsilon \log 2}$$

What we want to bound is

$$\prod_{\substack{p^\alpha || n \\ p < 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}$$

Note here that p is bounded and as α increases, we should expect the denominator to take over. However, while α is small, we might have some large terms since p^ε might be large. All our terms are however bounded by p^ε by the looks of it? Then we would get that the product is the product is bounded by $\prod_{p<2^{\frac{1}{\varepsilon}}} \frac{\log n}{\log p}\frac{1}{p^\varepsilon}$

□

Exercise 4.4 (E3.4)**.** Show that

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

is absolutely convergent in $\Re(s) > 1$.

Proof. Fix some $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. Then choosing an $\varepsilon > 0$ with $1 + \varepsilon < \sigma$, we have that $d(n) \leq C_\varepsilon n^\varepsilon$, so

$$\sum \left| \frac{d(n)}{n^s} \right| \leq \sum C_\varepsilon \frac{n^\varepsilon}{n^\sigma} \leq C_\varepsilon \sum \frac{1}{n^{\sigma-\varepsilon}} < \infty.$$

□

Exercise 4.5 (E3.5)**.** Show that the average order of $d(n)$ is $\log n$, i.e., that

$$\frac{1}{x}\sum_{n \leq x} d(n) = \log x + o\left(\log x\right).$$

Hint: Show that

$$\sum_{n \leq x} d(n) = \sum_{a \leq x} \left[\frac{x}{a}\right]$$

where $[b]$ is the integer part of b.

Proof. We follow the hint. For each $n \in \mathbb{N}$, let D_n denote the set of positive divisors of n. Then we want to find $\left|D_1 \cup \ldots \cup D_{[x]}\right|$. Now, $\left[\frac{x}{a}\right]$ is precisely the amount of multiples of a smaller than or equal to x, i.e., the amount of numbers in between 1 and x which have a as a divisor. Hence the right hand side indeed counts the number of divisors of the numbers less than or equal to x which is precisely the left hand side. Now, recall also the bound

$$\log x + \frac{1}{x} \leq \sum_{a \leq x} \frac{1}{a} \leq \log x + 1$$

so

$$1 + \frac{1}{x \log x} \leq \frac{1}{\log x} \sum_{a \leq x} \frac{1}{a} \leq 1 + \frac{1}{\log x}.$$

In particular, taking the limit as $x \mapsto \infty$, the outer functions tend to 1, so

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{a \leq x} \frac{1}{x} = 1.$$

In particular,

$$\frac{1}{x \log x} \sum_{n \leq x} d(n) \leq \frac{1}{\log x} \sum_{a \leq x} \frac{1}{a} \to 1, \quad x \to \infty.$$

For a lower bound, we have

$$\frac{1}{\log x} \sum_{a \leq x} \frac{1}{a} - \frac{1}{x \log x} \sum_{a \leq x} \frac{1}{a} = \frac{1}{\log x} \sum_{a \leq x} \frac{x-1}{ax} \leq \frac{1}{\log x} \sum_{a \leq x} \left[\frac{x}{a}\right]$$

But

$$\frac{1}{x} + \frac{1}{x^2 \log x} \leq \frac{1}{x \log x} \sum_{a \leq x} \frac{1}{a} \leq \frac{1}{x} + \frac{1}{x \log x}$$

so letting $x \to \infty$,

$$\lim_{x \to \infty} \frac{1}{x \log x} \sum_{a \leq x} \frac{1}{a} = 0$$

Hence also

$$1 \leq \lim_{x \to \infty} \frac{1}{x \log x} \sum_{n \leq x} d(n) \leq 1$$

giving the desired result. □

Exercise 4.6 (E3.6)**.** Let

$$\chi_4(n) = \begin{cases} (-1)^{\frac{n-1}{2}}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}.$$

Show that χ_4 is a Dirichlet character modulo 4 and find $L(1, \chi_4)$. Use the value to give (yet another) proof- based on the irrationality of π - that there are infinitely many primes. Hint: Remember (or prove by playing around with arctan(1)) that

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$

Proof. We must check 3 criteria for χ_4 to be a Dirichlet character mod 4.
(i) It must be 4-periodic. Now if n is even, then $n+4$ is even, so then $\chi_4(n+4) = 0 = \chi_4(n)$.
If n is odd, then so is $n+4$, so

$$\chi_4(n+4) = (-1)^{\frac{n+4-1}{2}} = (-1)^{\frac{n-1}{2}+2} = (-1)^{\frac{n-1}{2}} = \chi_4(n).$$

So χ_4 is 4-periodic.
(ii) We must check that $\chi_4(n) = 0$ if and only if $(n,4) \neq 1$. Now, $\chi_4(n) = 0$ if and only if n is even if and only if $(n,4) \in \{2,4\}$ if and only if $(n,4) \neq 1$.
(iii) We must check that χ_4 is multiplicative. Indeed, if either n or m is even, then

$$\chi_4(nm) = 0 = \chi(n)\chi(m).$$

If both n, m are odd, then

$$\chi_4(nm) = (-1)^{\frac{nm-1}{2}} = \begin{cases} -1, & nm \equiv 3 \pmod 4 \\ 1, & nm \equiv 1 \pmod 4 \end{cases}$$

Now, if n and m are both equivalent to 3 mod 4, then their product is equivalent to 1 mod 4, which works out. If only one is equivalent to 3 mod 4, then nm is also, so it checks out, and similarly, if both are equivalent to 1 mod 4, then so is their product. Now, by definition,

$$L(1,\chi_4) := \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan(1) = \frac{\pi}{4}$$

Now, since $\chi_4 \neq \chi_0^4$, we know that $L(s,\chi_4)$ is convergent and analytic for $\Re(s) > 0$. In particular, it is continuous at $s = 1$. But for $\Re(s) > 1$, we know that $L(s,\chi_4) = \prod_p (1 - \chi_4(p)p^{-s})^{-1}$, so by continuity,

$$\frac{\pi}{4} = L(1,\chi_4) = \prod_p \left(1 - \chi_4(p)p^{-1}\right)^{-1}$$

Now, all the terms in the product are rational, so by irrationality of π, this forces there to be infinitely many primes. $\square$

Exercise 4.7 (E3.7)**.** Let $\{a_n\}$ be a sequence of complex numbers satisfying that $\sum_{n\leq x} a_n = O\left(x^\delta\right)$ for some $\delta > 0$. Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s\int_1^{\infty} \sum_{n\leq t} a_n \frac{1}{t^{s+1}} dt$$

for $\Re(s) > \delta$, and that the sum converges to an analytic function in this region.

Proof. Let $f(x) = x^s$. Then

$$\sum_{n\leq x} \frac{a_n}{n^s} = \sum_{n\leq x} a_n \frac{1}{x^s} + s\int_1^x \sum_{n\leq t} a_n \frac{1}{t^{s-1}} dt$$

when $s \neq 1$. But $\left|\sum_{n\leq x} a_n\right| \leq Cx^\delta$, so

$$\left|\sum_{n\leq x} a_n \frac{1}{x^s}\right| \leq Cx^{\delta-\sigma} \to 0, \quad x \to \infty$$

as $\delta - \sigma < 0$. Thus

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \sum_{n \le t} a_n \frac{1}{t^{s+1}} dt.$$

$\square$

5. Week 4

Exercise 5.1 (E4.1)**.** Let $K \ge 0$. Prove that

$$\log(K|t| + 4) = O(\log(|t| + 4))$$

for $t \in \mathbb{R}$. Let $c_1, c_2, c_3 > 0$. Prove that there exists a constant c_4 such that for all $t \in \mathbb{R}$,

$$c_1 + c_2 \log(|t| + 4) + c_3 \log(|2t| + 4) \le c_4 \log(|t| + 4).$$

Proof. If $0 \le K \le 1$, then $\log(K|t| + 4) \le \log(|t| + 4)$ by monotonicity of log. So assume $K > 1$. Then $\log(K|t| + 4) = \log K + \log\left(|t| + \frac{4}{K}\right) \le \log K + \log(|t| + 4)$. Now $\log(|t| + 4) > 1$, so there exists some C such that $C \log(|t| + 4) \ge \log K$. Hence $\log(K|t| + 4) = O(\log(|t| + 4))$. Since $c_1 + c_2 \log(|t| + 4) + c_3 \log(|2t| + 4)$ is a sum of terms that are all $O(\log(|t| + 4))$, so is their sum, so the conclusion holds. $\square$

Exercise 5.2 (E4.2)**.** Let $f(s)$ be a complex polynomial of degree n with complex zeroes $z_1, z_2, \ldots, z_n$. Show that

$$\frac{f'}{f}(z) = \sum_{i=1}^{n} \frac{1}{z - z_i}.$$

Consider how Lemma 6.3 is a generalization of this.

Proof. Firstly, f' is entire, so $\frac{f'}{f}$ is holomorphic on $\mathbb{C} - \{z_1, \ldots, z_n\}$. Now, by Theorem 6.1 in KomAn, there exist unique functions g_i holomorphic on $\mathbb{C} - \{z_1, \ldots, z_n\}$ such that $g_i(z_i) \ne 0$ and

$$f(z) = (z - z_i)^{n_i} g_i(z)$$

where n_i is the multiplicity of z_i. In particular, $f'(z) = n_i(z - z_i)^{n_i - 1} g_i(z) + (z - z_i)^{n_i} g_i'(z)$ which has z_i a zero of order $n_i - 1$. Hence $\frac{f'}{f}$ has z_i as a simple pole. Applying the partial fraction decomposition to $\frac{f'}{f}$ (theorem 6.12 in KomAn), we get that

$$\frac{f'}{f}(z) = \sum_{i=1}^{n} \frac{c_i}{z - z_i}$$

for certain constants c_i. Now $\lim_{z \to z_i} (z - z_i) \frac{f'}{f}(z) = n_i$. Now, f is of degree n with n distinct zeroes, so n_i must be 1 for each i.

Now let us recall Lemma 6.3:

Lemma 5.3 (6.3)**.** *Let $f \colon B \to \mathbb{C}$ be analytic, $B \subset \mathbb{C}$ open, and assume*

(1) $\{z \mid |z| \le 1\} \subset B$
(2) $|f(z)| \le M$ *when* $|z| \le 1$
(3) $f(0) \ne 0$.

Let $0 < r < R < 1$. Then for $|z| < r$,

$$\frac{f'}{f}(z) = \sum_{\substack{f(z_k)=0 \\ |z_k| \le R}} \frac{1}{z - z_k} + O\left(\log \frac{M}{|f(0)|}\right)$$

Note here that f is not required to be a polynomial. However, since f is holomorphic in B, it has an analytic representation on B, so essentially, Lemma 6.3 generalizes the representation to analytic functions. $\square$

Exercise 5.4 (E4.3)**.** Show that the Riemann zeta function $\zeta(s)$ has no zeroes for $\frac{1}{2} \le s < 1$.

Proof. Recall that for $\sigma > 0$ and $s \ne 1$, we have

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty (u - [u])\, u^{-s-1} du.$$

For $s \in [\frac{1}{2}, 1)$, $\frac{s}{s-1} \le -1$. So we wish to show that

$$s \int_1^\infty (u - [u])\, u^{-s-1} du > -1$$

But

$$s \int_1^\infty (u - [u])\, u^{-s-1} du$$

is positive since the inner function and s are both positive on $[1, \infty)$.

$\square$

Exercise 5.5 (E4.4)**.** Let χ be a Dirichlet character modulo q. Find the Dirichlet series representation for $L'(s,\chi)/L(s,\chi)$. Let χ_0 be the trivial Dirichlet character modulo q. Prove that for $\sigma > 1, t \in \mathbb{R}$,

$$R := \Re\left(-3\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} - 4\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)} - \frac{L'(\sigma+i2t,\chi^2)}{L(\sigma+i2t,\chi^2)}\right) \ge 0.$$

Proof. We want to represent $\frac{L'(s,\chi)}{L(s,\chi)}$ as a Dirichlet series. We imitate the idea for $\frac{\zeta'}{\zeta}$.

$$\begin{aligned}
\frac{L'(s,\chi)}{L(s,\chi)} &= \frac{d}{ds}\log\left(L(s,\chi)\right) \\
&= -\sum_p \frac{d}{ds}\log\left(1-\frac{\chi(p)}{p^s}\right) \\
&= -\sum_p \frac{d}{ds}\sum_{k=1}^{\infty}(-1)^{k+1}\left(-\frac{\chi(p)}{p^s}\right)^k \\
&= \sum_p\sum_{k=1}^{\infty}\frac{d}{ds}\left(\frac{\chi(p)}{p^s}\right)^k \\
&= \sum_p\sum_{k=1}^{\infty}\chi(p)^k(-k\log p)p^{-sk} \\
&= -\sum_p\sum_{k=1}^{\infty}k\log p\left(\frac{\chi(p)}{p^s}\right)^k
\end{aligned}$$

Thus We want to find $\Re\left(\left(\frac{\chi(p)}{p^s}\right)^k\right)$. We have

$$\begin{aligned}
\Re\left(\left(\frac{\chi(p)}{p^s}\right)^k\right) &= \frac{1}{2}\left[\left(\frac{\chi(p)}{p^s}\right)^k+\left(\frac{\overline{\chi(p)}}{\overline{p^s}}\right)^k\right] \\
&=
\end{aligned}$$

$$\Re\left(-\frac{L'(s,\chi)}{L(s,\chi)}\right) = \sum_p\sum_{k=1}^{\infty}k\log p\cos\left(tk\log p\right).$$

So

$\square$

Exercise 5.6 (E4.5)**.** Let $\zeta(s)$ be the Riemann zeta function. Let K be a compact subset of $\{s\in\mathbb{C}\mid\Re(s)>0\}$. Assume that $1\in K$ and that K does not contain any zeroes of ζ. Show that

$$-\frac{\zeta'}{\zeta}(s)=\frac{1}{s-1}+O(1)$$

for $s\in K-\{1\}$. Show that there exists a constant $c>0$ such that for $0<\delta<1$,

$$-\frac{\zeta'}{\zeta}(1+\delta)<\frac{1}{\delta}+c.$$

Proof. Since 1 is a simple pole of $\frac{\zeta'}{\zeta}$ and K has no other zeroes of ζ and hence neither of ζ', we have that

$$-(s-1)\frac{\zeta'}{\zeta}(s)$$

is holomorphic on K, hence bounded as K is compact. Thus

$$-\frac{\zeta'}{\zeta}(s)=\frac{1}{s-1}+O(1)$$

for $s \in K - \{1\}$. Thus for small $0 < \delta < 1$ such that $1 + \delta \in K - \{1\}$,

$$-\frac{\zeta'}{\zeta}(1+\delta) < \frac{1}{\delta} + c$$

for some $c > 0$. $\square$

Exercise 5.7 (E4.6)**.** Use partial summation (Abel summation) to show that for $\sigma > 1$,

$$-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}dx$$

where $\psi(x) = \sum_{n\leq x}\Lambda(n)$, and Λ is the von Mangoldt function.

Proof. Recall that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

for $\sigma = \Re(s) > 1$.
Let $f(x) = \frac{1}{x^s}$ and $a_n = \Lambda(n)$. Partial summation gives

$$\sum_{n\leq x}\frac{\Lambda(n)}{n^s} = \underbrace{\sum_{n\leq x}\Lambda(n)}_{\psi(x)}\frac{1}{x^s} + s\int_1^x \underbrace{\sum_{n\leq t}\Lambda(n)}_{\psi(t)}\frac{1}{t^{s+1}}dt$$

By the prime number theorem,

$$\psi(x) = x + O\left(\frac{x}{e^{c'\sqrt{\log x}}}\right)$$

so

$$\frac{\psi(x)}{x^s} \to 0, \quad x \to \infty$$

Thus

$$-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \frac{\psi(t)}{t^{s+1}}dt$$

for $\sigma > 1$. $\square$

6. Week 5

Exercise 6.1 (E5.1)**.** Show that

$$x\exp\left(-c\sqrt{\log x}\right) = O_m\left(\frac{x}{\log^m x}\right)$$

for every m, and that

$$x^{1-\varepsilon} = O_\varepsilon\left(x\exp\left(-c\sqrt{\log x}\right)\right)$$

for every $\varepsilon > 0$. Discuss what this means for the quality of the error-term in the prime number theorem.

Proof.

$$\frac{\log^m x}{e^{c\sqrt{\log x}}} = \frac{\sqrt{\log x}^{2m}}{e^{c\sqrt{\log x}}}$$

Now

Lemma 6.2. *For any $a > 0$ and any $b > 1$,*

$$\frac{x^a}{b^x} \to 0, \quad x \to \infty.$$

Let $v = \sqrt{\log x}$. Then the above reads $\frac{v^{2m}}{e^{cv}}$. Assuming $c > 0$, we find that for $v \to \infty$, $\frac{v^{2m}}{e^{cv}} \to 0$. So in fact,

$$x \exp\left(-c\sqrt{\log x}\right) = o\left(\frac{x}{\log^m x}\right)$$

Now

$$x^{1-\varepsilon} = xx^{-\varepsilon} = xe^{-\log(x)\varepsilon} \leq xe^{-c\sqrt{\log x}}.$$

Recall that we proved the following version of the prime number theorem:

Theorem 6.3 (Prime number theorem)**.** *There exists a $c' > 0$ such that*

$$\psi(x) = x + O\left(x \exp\left(-c'\sqrt{\log x}\right)\right)$$

So by the above,

$$\psi(x) = x + O_m\left(\frac{x}{\log^m(x)}\right)$$

So essentially, the error term is smaller than $\frac{x}{\log^m(x)}$ for any x but still larger than $x^{1-\varepsilon}$ for any $\varepsilon > 0$. □

Exercise 6.4 (E5.2)**.** Prove that the following two statements are equivalent:

(1) There exists a $c > 0$ such that

$$\psi(x) = x + O\left(x \exp\left(-c\sqrt{\log x}\right)\right)$$

(2) There exists a $c > 0$ such that

$$\pi(x) = li(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right)$$

where $li(x) = \int_2^x \frac{1}{\log t} dt$.

Proof. Suppose (1) is true. Then

$$\begin{aligned}\pi(x) &= \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t\log^2 t} dt + O\left(x^{\frac{1}{2}}\log x\right) \\ &= \frac{x}{\log x} + O\left(\frac{x}{\log x}\exp\left(-c\sqrt{\log x}\right)\right) + \int_2^x \frac{1}{\log^2 t} + O\left(\frac{1}{\log^2 t \exp\left(c\sqrt{\log t}\right)}\right) dt + O\left(x^{\frac{1}{2}}\log x\right)\end{aligned}$$

Now

$$\int_2^x \frac{1}{\log^2 t} dt = -\frac{t}{\log t}\bigg|_2^x + li(x)$$

giving

$$\pi(x) = li(x) + \frac{2}{\log 2} + O\left(\frac{x}{\log x}e^{-c\sqrt{\log x}}\right) + \int_2^x O\left(\frac{e^{-c\sqrt{\log t}}}{\log^2 t}\right) dt + O\left(x^{\frac{1}{2}}\log x\right)$$

All the middle terms apart from the last two are clearly $O\left(xe^{-c\sqrt{\log x}}\right)$. To take care of the last term, we use the lemma:

Lemma 6.5. *For any $a > 0$,*

$$\frac{\log x}{x^a} \to 0, \quad x \to \infty$$

Hence $x^{\frac{1}{2}} \log x = O\left(x^{\frac{3}{4}}\right) = O\left(xe^{-c'\sqrt{\log x}}\right)$.

For the last part

$$\int_2^x O\left(\frac{e^{-c\sqrt{\log t}}}{\log^2 t}\right) dt \leq$$

Note that the derivative of $xe^{-c\sqrt{\log x}}$ is

$$e^{-c\sqrt{\log x}} - c\frac{d}{dx}\left[\sqrt{\log x}\right] e^{-c\sqrt{\log x}} = e^{-c\sqrt{\log x}} - c\frac{1}{2}\frac{1}{x}\frac{1}{\sqrt{\log x}}e^{-c\sqrt{\log x}}$$

But as $x \to \infty$, this grows faster than $\frac{e^{-c\sqrt{\log x}}}{\log^2 x}$, which is what we wanted.

Now we want to show that (2) implies (1). So assume there exists a $c > 0$ such that

$$\pi(x) = li(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right).$$

Then recall that

$$\psi(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t} dt - O\left(x^{\frac{1}{2}} \log^2 x\right)$$

So

$$\psi(x) = li(x)\log x + \log x O\left(xe^{-c\sqrt{\log x}}\right) - \int_2^x \frac{li(t)}{t} dt - \int_2^x O\left(e^{-c\sqrt{\log t}}\right) dt - O\left(x^{\frac{1}{2}} \log^2 x\right)$$

Now, by repeated integration by parts, we get

$$\begin{aligned}
li(x) &= \left.\frac{t}{\log t}\right|_2^x + \int_2^x \frac{1}{\log^2 t} dt \\
&= \left.\frac{t}{\log t}\right|_2^x + \left[\left.\frac{t}{\log^2 t}\right|_2^x + 2\int_2^x \frac{1}{\log^3 t} dt\right] \\
&= \frac{t}{\log t} + \left.\frac{t}{\log^2 t}\right|_2^x + 2\left[\left.\frac{t}{\log^3 t}\right|_2^x + 3\int_2^x \frac{1}{\log^4 t} dt\right] \\
&= x\sum_{r=1}^{k-1} \frac{(r-1)!}{\log^r x} + (k-1)!\int_2^x \frac{1}{\log^k t} dt
\end{aligned}$$

□

Exercise 6.6 (E5.3)**.** Let f be a Schwartz function on the real line, and let $\hat{f}$ be its Fourier transform. Show that

$$\sum_{n\in\mathbb{Z}} f\left(\frac{v+n}{t}\right) = \sum_{n\in\mathbb{Z}} |t|\, \hat{f}\,(nt)\, e^{2\pi inv}.$$

Proof. For a Schwartz function f, we know from the Poisson summation formula that

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

where

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi ixy} f(x)dx$$

Define $g(x) = f\left(\frac{v+x}{t}\right)$. Then g is also a Schwartz function, so

$$\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-2\pi inx} f\left(\frac{v+x}{t}\right) dx$$

Let $z = \frac{v+x}{t}$. Then $dz = \frac{1}{|t|}dx$, so

$$\sum_{n\in\mathbb{Z}} f\left(\frac{v+n}{t}\right) = \sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} |t|\, e^{-2\pi in(tz-v)} f(z)dz = \sum_{n\in\mathbb{Z}} |t|\, \hat{f}(nt)e^{2\pi inv}$$

□

Exercise 6.7 (E5.4)**.** Let $\theta > \frac{1}{2}$. Prove that if for every $\varepsilon > 0$, $\psi(x) = x + O\left(x^{\theta+\varepsilon}\right)$, then the Riemann zeta function has no zeroes in $\Re(s) > \theta$. (It turns out that this is in fact an 'if and only if statement'). Think about what this implies for the Riemann hypothesis. Compare with the zerofree region provided by Theorem 6.6.

Proof. By the explicit formula, if we simply let x range among $\mathbb{R} - \mathbb{Z}$, then we have

$$O\left(x^{\theta+\varepsilon}\right) = \lim_{T\to\infty} \sum_{\substack{\zeta(\rho)=0\\ |\mathrm{im}\,\rho|\le T}} \frac{x^\rho}{\rho} + \frac{\zeta'}{\zeta}(0) + \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right),$$

however, if there is a ρ with $\Re(\rho) > \theta$, then choosing ε such that $\theta < \varepsilon < \Re(\rho)$, we get that the right hand side grows faster, giving a contradiction.

Hence the Riemann hypothesis can be reformulated as saying that for any $\varepsilon > 0$,

$$\psi(x) = x + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Now, any θ would, of course, be a very strong improvement combined with the zero-free region. This is because the zero-free region tapers off as the imaginary part grows in size, while finding a θ such that the above holds would imply, as shown, that we can shrink the critical strip to a narrower strip. □

Exercise 6.8 (E5.5)**.** Let p_n be the n th prime. Show that

$$\frac{1}{N}\sum_{n=1}^{N} \frac{p_{n+1} - p_n}{\log p_n} \to 1$$

as $N \to \infty$, and discuss how to interpret this as a statement about the average spacing between adjacent primes.

Proof. By Abel summation, we have

$$\sum_{n\le x} \frac{p_n}{\log p_n} = \sum_{n\le x} p_n \frac{1}{\log x} - \int_1^x \sum_{n\le t} p_n \frac{1}{\log t} dt$$

And similarly

$$\sum_{n\le x}\frac{p_{n+1}}{\log p_n}=\sum_{n\le x}p_{n+1}\frac{1}{\log x}-\int_1^x\sum_{n\le t}p_{n+1}\frac{1}{\log t}dt$$

Hence

$$\begin{aligned}\sum_{n\le x}\frac{p_{n+1}-p_n}{\log p_n}&=\frac{1}{\log x}\sum_{n\le x}p_{n+1}-p_n-\int_1^x\frac{1}{\log t}\sum_{n\le t}(p_{n+1}-p_n)dt\\&=\frac{p_{[x]+1}-2}{\log x}-\int_1^x\frac{p_{[t]+1}-2}{\log t}dt\end{aligned}$$

Now $\frac{p_n}{n\log n}\to 1$ as $n\to\infty$, so $\frac{p_{n+1}}{n\log n}=\frac{p_{n+1}}{(n+1)\log(n+1)}\frac{(n+1)\log(n+1)}{n\log n}\to 1$ as $n\to\infty$. So we will get the result if we can show that

$$\lim_{n\to\infty}\frac{1}{n}\int_1^n\frac{p_{[t]+1}-2}{\log t}dt=0.$$

By the PNT, we have

$$p_n\sim n\log n.$$

So

$$\lim_{N\to\infty}\frac{1}{n}\sum_{n\ge N}\frac{p_{n+1}-p_n}{\log p_n}=\lim_{N\to\infty}\sum_{n\ge N}\frac{(1+\frac{1}{n})\log(n+1)-\log n}{\log n+\log\log n}=$$

□

7. Assignment 1

Exercise 7.1 (H1.1)**.** *Proof.*

$$f*e(n)=\sum_{d|n}f(d)e(\frac{n}{d})=\sum_{d|n}f(d)\delta_{\frac{n}{d},1}=f(n)$$

and since the sets $\{d\colon d\mid n\}$ and $\left\{\frac{n}{d}\colon d\mid n\right\}$ are equal, we have

$$g*f=\sum_{d|n}g(d)f\left(\frac{n}{d}\right)=\sum_{d|n}g\left(\frac{n}{d}\right)f\left(\frac{n}{\frac{n}{d}}\right)=f*g(n)$$

□

Exercise 7.2 (H1.2)**.** *Proof.*

$$\mu*1(n)=\sum_{d|n}\mu(d)1\left(\frac{n}{d}\right)=\sum_{d|n}\mu(d)$$

If $n=p$ is a prime, we trivially have $\{d\colon d\mid n\}=\{1,p\}$, so $\sum_{d|n}\mu(d)=1-1=0=e(p)$, so it is true for n a prime.

Suppose now that $n=p_1\cdots p_s$, so $\mu(n)=(-1)^s$. We need to find out how many elements the set $D_k=\{d\mid n\colon d \text{ is a product of k distinct primes}\}$ has. But this is simply the same as choosing an unordered set of k elements from a set of s

elements. There are precisely $\binom{s}{k}$ ways to do so. Since for each $d \in D_k$, we have $\mu(d) = (-1)^k$, we find that

$$\sum_{d|n} \mu(d) = \sum_{k=1}^{s} \binom{s}{k} (-1)^k = (1-1)^s = 0.$$

Then, in particular,

$$\sum_{d|n} \mu(d)$$

Lastly, for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, it reduces to the previous case because μ is only non-zero on squarefree integers, so

$$\mu * 1(n) = \sum_{d | \frac{n}{p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1}}} \mu(d) = 0$$

since the sets $\left\{d \colon d \mid \frac{n}{p_1^{\alpha_1-1} \cdots p_k^{\alpha_k-1}}\right\}$ and $\{d \colon d \mid p_1 \cdots p_k\}$ are equal.
Thus, indeed, $\mu * 1 = e$. $\square$

Exercise 7.3 (H1.3)**.** We claim that the set of arithmetic functions with Dirichlet convolution as a binary operation is an abelian semigroup. For this, if $f, g \colon \mathbb{N} \to \mathbb{C}$, then clearly $f * g \colon \mathbb{N} \to \mathbb{C}$ too. Also, $f * g(n) = \sum_{ab=n} f(a)g(b) = \sum_{ba=n} g(b)f(a) = g * f(n)$ by commutativity of multiplication in $\mathbb{C}$. Lastly,

$$(f * g) * h(n) = \sum_{ab=n} f * g(a)h(b) = \sum_{ab=n} \sum_{cd=a} f(c)g(d)h(b) = \sum_{cdb=n} f(c)g(d)h(b)$$

and

$$f * (g * h)(n) = \sum_{ab=n} f(a)g * h(b) = \sum_{ab=n} \sum_{cd=b} f(a)g(c)h(d) = \sum_{acd=n} f(a)g(c)h(d)$$

(all of this is just Theorem 5.1.4 in the book for Introduction to Number Theory by Risager).
Now, if $f = 1 * g$ then $\mu * f = \mu * (1 * g) = (\mu * 1) * g = e * g = g * e = g$ by the above together with H1.1. Likewise, if $g = \mu * f$, then $1 * g = 1 * (\mu * f) = (1 * \mu) * f = (\mu * 1) * f = e * f = f * e = f$ again.

Exercise 7.4 (H1.4)**.** We have

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \frac{f(n)}{n^s} \right| &\le \sum_{n=1}^{\infty} \frac{Cn^k}{n^\sigma} \\ &\le \sum_{n=1}^{\infty} \frac{C}{n^{\sigma-k}} \\ &< \infty \end{aligned}$$

as $\sigma - k > 1$. Thus the series converges absolutely.

Exercise 7.5 (H1.5)**.** *Proof.* We know that L_f converges absolutely for $\sigma > 1 + k_f$ and L_g converges absolutely for $\sigma > 1 + k_g$. Assume without loss of generality that

$k_g > k_f$. Now,

$$\sum_{n=1}^{\infty}\left|\frac{\sum_{d|n} f(d)g(\frac{n}{d})}{n^s}\right| \le \sum_{n=1}^{\infty}\sum_{d|n}\frac{C_f C_g d^{k_f}\left(\frac{n}{d}\right)^{k_g}}{n^{\sigma}}$$

$$= \sum_{n=1}^{\infty} C_f C_g \sum_{d|n} d^{k_f-k_g}\frac{1}{n^{\sigma-k_g}}$$

Now, by E3.2, we have $d(n) \le 2\sqrt{n}$, so since $\sum_{d|n} d^{k_f-k_g} \le \sum_{d|n} 1 = d(n) \le 2\sqrt{n}$, we have

$$\sum_{n=1}^{\infty} C_f C_g \sum_{d|n} d^{k_f-k_g}\frac{1}{n^{\sigma-k_g}} \le \sum_{n=1}^{\infty} C_f C_g 2\sqrt{n}\frac{1}{n^{\sigma-k_g}}$$

$$= 2C_f C_g \sum_{n=1}^{\infty}\frac{1}{n^{\sigma-\left(k_g+\frac{1}{2}\right)}}$$

Hence the sum defining $L_{f*g}(s)$ is absolutely convergent for $\sigma > k_g + \frac{3}{2}$, and in this half-plane,

$$L_f(s)L_g(s) = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\frac{f(k)}{k^s}\frac{g(t)}{t^s} = \sum_{r=1}^{\infty}\sum_{d|r}\frac{f(d)g(\frac{n}{d})}{r^s} = L_{f*g}(s)$$

□

Exercise 7.6 (H1.6)**.** We have that when L_1 and L_μ are absolutely convergent, and satisfy the bounds from H1.5, we can use Cauchy summation to get $L_1(s)L_\mu(s) = L_{1*\mu}(s) = L_e(s) = 1$ which is absolutely convergent everywhere; but $L_1(s) = \zeta(s)$ and $L_\mu(s) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$, so the result follows in whenever all sums are absolutely convergent. Hence the desired equality extends (by the identity theorem) to all of $\Re(s) > 1$ since $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$ converges to a holomorphic function in this half-plane (being the uniform limit of a series of holomorphic functions).

Exercise 7.7 (H 1.7)**.** *Proof.* For $f(n) = n^w$, we have $\sigma_w(n) = f * 1(n)$. The abscissa of convergence for 1 is 1 and for f it is $1 + \Re(w)$. In some halfplane, we have $\sum_{n=1}^{\infty}\frac{\sigma_w(n)}{n^s} = L_{\sigma_w}(s) = L_f(s)L_1(s)$. Now $L_1(s) = \zeta(s)$, and

$$L_f(s) = \sum_{n=1}^{\infty}\frac{n^w}{n^s} = \sum_{n=1}^{\infty}\frac{1}{n^{s-w}} = \zeta(s-w).$$

Thus $\sum_{n=1}^{\infty}\frac{\sigma_w(n)}{n^s} = \zeta(s-w)\zeta(s)$ in some right half-plane. □

8. Assignment 2

Exercise 8.1 (H2.1)**.** Show that

$$\sum_{p\le x}\frac{1}{p} = \log\log(x) + O(1),$$

where the sum is over primes less than x.

Proof. As is the custom, we of course start by Abel summation:

$$\sum_{p\le x}\frac{1}{p} = \pi(x)\frac{1}{x} + \int_1^x \frac{\pi(t)}{t^2}dt$$

Now applying the PNT, we get

$$\pi(x)\frac{1}{x}+\int_1^x \frac{\pi(t)}{t^2}dt=\frac{1}{\log x}+O\left(e^{-c\sqrt{\log x}}\right)+\int_1^x \frac{1}{t\log t}dt+\int_1^x O\left(t^2e^{-c\sqrt{\log t}}\right)dt$$

Since

$$\int_1^x \frac{1}{t\log t}dt=\log\log t\Big|_1^x$$

we have what we needed.

$\square$

Exercise 8.2 (H2.2)**.** This exercise gives a different proof that $\zeta(s)$ has no zeros on $\Re(s)=1$.

(1) Prove that for $\sigma>1, t\in\mathbb{R}$,
$$\Re\left(3\log\zeta(\sigma)+4\log\zeta\left(\sigma+it\right)+\log\zeta\left(\sigma+2it\right)\right)\geq 0.$$
(2) Prove that $\left|\zeta(\sigma)^3\zeta\left(\sigma+it\right)^4\zeta\left(\sigma+2it\right)\right|\geq 1$.
(3) Prove that if $\zeta\left(1+it_0\right)=0$, then $\left|\zeta(\sigma)^3\zeta\left(\sigma+it_0\right)^4\zeta\left(\sigma+2it_0\right)\right|\to 0$ as $\sigma\to 1$.
(4) Conclude that $\zeta\left(1+it\right)\neq 0$ for every $t\neq 0$.

Proof. i) $\square$