1:

(a) We have that the polynomial map $\varphi\colon X \to Y$ induces a homomorphism $\varphi^*\colon \Gamma(Y) \to \Gamma(X)$. Now by the last page on lecture note 12, the homomorphism φ^* extends to a well-defined map $\mathcal{O}_Q(Y) \to \mathcal{O}_P(X)$. This map is in particular also a homomorphism, as if $\frac{f}{g}, \frac{f'}{g'} \in \mathcal{O}_Q(Y)$, then

$$\varphi^*\left(\frac{f}{g}+\frac{f'}{g'}\right) = \varphi^*\left(\frac{fg'+f'g}{gg'}\right) = \frac{\varphi^*\left(fg'+f'g\right)}{\varphi^*\left(gg'\right)} = \frac{\varphi^*(f)\varphi^*(g')+\varphi^*(f')\varphi^*(g)}{\varphi^*(g)\varphi^*(g')} = \frac{\varphi^*(f)}{\varphi^*(g)}+\frac{\varphi^*(f')}{\varphi^*(g')} = \varphi^*\left(\frac{f}{g}\right)+\varphi^*\left(\frac{f'}{g'}\right)$$

and

$$\varphi^*\left(\frac{f}{g}\cdot\frac{f'}{g'}\right) = \varphi^*\left(\frac{ff'}{gg'}\right) = \frac{\varphi^*(ff')}{\varphi^*\left(gg'\right)} = \frac{\varphi^*(f)\varphi^*(f')}{\varphi^*(g)\varphi^*(g')} = \frac{\varphi^*(f)}{\varphi^*(g)}\cdot\frac{\varphi^*(f')}{\varphi^*(g')} = \varphi^*(\frac{f}{g})\cdot\varphi^*\left(\frac{f'}{g'}\right),$$

where in both cases we used from the lecture notes that $\varphi^*\left(\frac{f}{g}\right) = \frac{\varphi^*(f)}{\varphi^*(g)}$ for any $\frac{f}{g} \in \mathcal{O}_Q(Y)$.

We claim this homomorphism is an isomorphism.
Injectivity:

$$\varphi^*\left(\frac{f}{g}\right) = 0 \iff \frac{\varphi^*(f)}{\varphi^*(g)} = \frac{0}{1} \iff \varphi^*(f) = 0,$$

and as φ is an isomorphism, so is $\varphi^*\colon \Gamma(Y) \to \Gamma(X)$, so $\varphi^*(f) = 0 \iff f = 0$. Hence $\frac{f}{g} = \frac{0}{g} = 0$, so $\varphi^*\colon \mathcal{O}_Q(Y) \to \mathcal{O}_P(X)$ is injective.

Surjectivity: if $\frac{f}{g} \in \mathcal{O}_P(X)$, then $\varphi^*(g)(Q) = g(\varphi(Q)) = g(P) \neq 0$, so $\varphi^*(\frac{f}{g}) = \frac{\varphi^*(f)}{\varphi^*(g)} \in \mathcal{O}_Q(Y)$.

(b) By definition, we have

$$I_P(x,y) = \dim_k\left(\frac{\mathcal{O}_P(\mathbb{A}^2)}{\left(\frac{x}{1},\frac{y}{1}\right)}\right).$$

Let $\frac{f'}{g'} \in \mathcal{O}_P\left(\mathbb{A}^2\right)$ be any element such that f' does not vanish at P.

Let $\frac{f}{g} \in \mathcal{O}_P(\mathbb{A}^2)$ be arbitrary with $f, g \in \Gamma(\mathbb{A}^2) = k\left[x,y\right]$ and $g(P) \neq 0$. We claim that $\frac{f}{g} - c\frac{f'}{g'} \in (\frac{x}{1},\frac{y}{1})$ for some $c \in k$.
Now for $c = \frac{-f_0g_0'}{f_0'g_0}$, we get $\frac{f}{g} - c\frac{f'}{g'} = \frac{fg'-cf'g}{gg'} = \frac{f_0g_0'+(fg'-f_0g_0')-cf_0'g_0-(cf'g-cf_0'g_0)}{gg'} = \frac{\left(fg'-f_0g_0'\right)-c\left(f'g-f_0'g_0\right)}{gg'}$, and as $fg'-f_0g_0' \in (x,y)$ and $f'g - f_0'g_0 \in (x,y)$, and gg' does not vanish at P since neither g nor g' vanish at P and $\Gamma\left(\mathbb{A}^2\right) = k\left[x,y\right]$ is an integral domain, we have also $\frac{1}{gg'} \in \mathcal{O}_P\left(\mathbb{A}^2\right)$, so $\frac{f}{g} - c\frac{f'}{g'} = \frac{\left(fg'-f_0g_0'\right)-c\left(f'g-f_0'g_0\right)}{gg'} = \frac{1}{gg'}\cdot\left[\left(fg'-f_0g_0'\right)-c\left(f'g-f_0'g_0\right)\right] \in \left(\frac{x}{1},\frac{y}{1}\right) \subset \mathcal{O}_P(\mathbb{A}^2)$, since $\left(\frac{x}{1},\frac{y}{1}\right)$ is an ideal.
Thus we have that $\mathcal{O}_P\left(\mathbb{A}^2\right)/\left(\frac{x}{1},\frac{y}{1}\right) = \operatorname{span}\left(\frac{f'}{g'}\right)$, so $I_P(x,y) = \dim_K\left(\frac{\mathcal{O}_P\left(\mathbb{A}^2\right)}{\left(\frac{x}{1},\frac{y}{1}\right)}\right) = \dim_K\left(\operatorname{span}\left(\frac{f'}{g'}\right)\right) = 1$.

(c) P is smooth in $V(f)$ and $V(g)$, so $f_1 \neq 0 \neq g_1$. Now, for $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ being the translation $\varphi(x,y) = (x,y)+P$, we have that since the tangent lines of $V(f)$ and $V(g)$ at P are distinct, we have $TC_{(0,0)}V\left(\varphi^*f\right) = V\left(\varphi^*f_1\right) \neq V\left(\varphi^*g_1\right) = TC_{(0,0)}V\left(\varphi^*f\right)$.
Since $V(f)$ and $V(g)$ have distinct tangent lines at P, we have $I_P(f,g) = mult_{(0,0)}(\varphi^*f)mult_{(0,0)}(\varphi^*g)$. Now, since φ is a translation, φ^*f and φ^*g are still smooth at $(0,0)$, so $mult_{(0,0)}\left(\varphi^*f\right) = 1 = mult_{(0,0)}\left(\varphi^*g\right)$ (not 0 as both vanish at $(0,0)$).

2:

(a) x^2-y and y^2-x^3 have no common factors as $y^2-x^3+y(x^2-y) = -x^3+yx^2 = x^2(y-x)$ and (x^2-y) have no common factors.
P is a root of both, and we now have $TC_PV(x^2-y) = V(y)$ and $TC_PV(y^2-x^3) = V(y^2) = V(y)$, so proceeding by the algorithm, we find letting $f = x^2-y$ and $g = y^2-x^3$ that $f(x,0) = x^2$ and $g(x,0) = -x^3$, so let $h = g+xf = y^2-x^3+x\left(x^2-y\right) = y^2-xy$. Then $I_P(f,g) = I_P(f,h)$ and also $TC_PV(h) = V(y^2-xy) = V(y)\cup V(y-x)$, so $V(y)$ is still in common. Now $f(x,0) = x^2$ and $h(x,0) = 0$, so proceeding by (6) case 1, we have $I_P(f,h) = I_P(y,f)+I_P(y-x,f)$. Now writing f of the form $g = Ax^m+By$ with $A(P) \neq 0$, we find $m = 2$, and thus $I_P(y,g) = a\cdot m = 1\cdot 2 = 2$ where a is the exponent of y in the expression for h. Now, $V(y-x)$ and $V(f) = V(x^2-y)$ have no common tangent

cone lines, so $I_P(y-x,f) = mult_P(y-x)mult_P(f) = 1\cdot 1 = 1$. Hence $I_P(f,g) = 2+1 = 3$.

(b) We have $I_P(x-y^2, x+y^2) \stackrel{7}{=} I_P(x-y^2, 2x) \stackrel{7}{=} I_P(-y^2, x) \stackrel{6}{=} 2I_P(-y,x) \stackrel{1.b}{=} 2\cdot 1 = 2$.
In the last part we also used that $\left(\frac{-y}{1}, \frac{x}{1}\right) = \left(\frac{y}{1}, \frac{x}{1}\right)$.

(c) Let $f = x^3+xy$ and $g = 3x^2y+xy^2$. Since $V(x) \subset V(x(x^2+y)) = V(f)$ and $V(x) \subset V(x(3xy+y^2)) = V(g)$, and $V(x)$ passes through $P = (0,0)$, we have by property (1) that $I_P(f,g) = \infty$.

(d) Let $f = x+y+y^2x$ and $g = x+y+x^2-y^2+y^3$.
We have that f and g share no common factors and both vanish at P.
$TC_PV(f) = V(x+y)$ and $TC_PV(g) = V(x+y)$.
Now let $z = x+y$. Then $x = z-y$, so $f = x+y+y^2x = z+y^2(z-y)$ and $g = x+y+x^2-y^2+y^3 = z+(z-y)^2-y^2+y^3$. Then $TC_PV(f) = V(z)$ and $TC_PV(g) = V(x+y) = V(z)$ so viewing $f,g \in k\,[y,z]$, we have $f(y,0) = -y^3$ and $g(y,0) = y^3$.

Now let $h = g+f = z+(z-y)^2-y^2+y^3+z+y^2(z-y) = 2z+(z-y)^2-y^2+y^2z = 2z+z^2-2yz+y^2z = z\left(2+z-2y+y^2\right)$. Then $I_P(f,g) = I_P(f,h)$ by property (7).
Again f and h share no common factors and both vanish at P.
Now $TC_PV(h) = V(2z) = V(z)$. We find $h(y,0) = 0$, so proceeding by (6) case 1, we find that writing f as $Ay^m + Bz$ with $A(P) \neq 0$, we have $m = 3$, so $I_P(f,h) = I_P(f,z) + I_P(f, 2+z-2y+y^2)$. As $m = 3$, we find $I_P(f,z) = a\cdot m = 3$ where a is the maximal exponent of z such that z^a divides h. Now, as $P \notin V(2+z-2y+y^2)$, we have $I_P\left(f, 2+z-2y+y^2\right) = 0$ by property (2). Hence $I_P(f,g) = 3$.

3: Let $g = A+yB$ where $y \nmid A$, so $A = g(x,0)$, then by property (7), we have $I_P(y,g) = I_P(y, A+yB) = I_P(y, A+yB-yB) = I_P(y, g(x,0))$ which becomes the exponent of the smallest term of $g(x,0)$ by property (5) since $TC_P(y) = V(y)$ and $TC_PV(g(x,0)) = V(x)$ are distinct lines. Hence $I_P(y, g+h)$ is the smallest exponent of $g(x,0)+h(x,0)$. Now, writing $g(x,0) = \sum_{n=0}^{\infty} a_n x^n$ and $h(x,0) = \sum_{n=0}^{\infty} b_n x^n$, we find that if the smallest exponent of $g(x,0)+h(x,0)$ is m then $0 \neq a_m + b_m$, so either a_m or b_m is greater than 0 and hence the smallest exponent of either $g(x,0)$ or $h(x,0)$ - which is equal to $I_P(y,g)$ and $I_P(y,h)$ respectively - is smaller than or equal to m.
Thus we have

$$I_P(y, g+h) \geq \min\{I_P(y,g), I_P(y,h)\}.$$

(b) Let $f = xy$, $g = x$ and $h = y$. Then $I_P(f,g) = \infty = I_P(f,h)$. But $I_P(f, g+h) = I_P(xy, x+y)$, now, $TC_PV(xy) = V(xy) = V(x) \cup V(y)$ and $TC_PV(x+y) = V(x+y)$, so xy and $x+y$ do not share any tangent cone lines. By property (5) we thus get $I_P(f, g+h) = mult_P(xy)mult_P(x+y) = 2\cdot 1 = 2 < \infty$.
Thus we find

$$I_P(f, g+h) = 2 < \infty = \min\{I_P(f,g), I_P(f,h)\}$$

4. Nodes: Assume $P = (0,0)$. Then P has multiplicity 2 in $V(f)$ if f_2 is the smallest nonzero term. Write $f_2 = ax^2 + cyx + by^2$.
Now, solving this for x, we find

$$x = \frac{-cy \pm \sqrt{c^2y^2 - 4aby^2}}{2a}.$$

f_2 factors into two lines if and only if x has two distinct roots here which means $c^2y^2 - 4aby^2 \neq 0$. Now, $f_{xy} = c$, $f_{xx} = 2a$ and $f_{yy} = 2b$, so $c^2y^2 - 4aby^2 \neq 0 \iff c^2y^2 \neq 4aby^2 \stackrel{y\neq 0}{\iff} c^2 \neq 4ab \iff f_{xy}(P)^2 \neq f_{xx}(P)f_{yy}(P)$ (where $y \neq 0$ since if $y = 0$ then $f_2 = ax^2$, but then the tangent cone of f at P does not have two distinct lines).
If P is not $(0,0)$, let φ be the translation $(x,y) \to (x,y)+P$. Then P has multiplicity 2 in $V(f)$ if and only if $(0,0)$ has multiplicity 2 in $V(\varphi^* f)$, and $\varphi^* f = f((x,y)+P)$ has the same derivatives evaluated at $(0,0)$ as f has evaluated at P by the rule of differentiating compositions, so we get the same result.

5:
(a) We have

$$I_P(f,L) \geq mult_P(f)mult_P(L).$$

Since the tangent cone to $V(f)$ is $V(L)$ which is the tangent cone to L as well (L has no constant term as it vanishes at $(0,0)$), we have that the inequality above is strict by property (5) of I_P, so $I_P(f,L) > mult_P(f)mult_P(L)$.

Let $\varphi\colon \mathbb{A}^2 \to \mathbb{A}^2$ be the translation $(x, y) \to (x, y)+P$. By definition then $mult_P(f) = mult_{(0,0)}(\varphi^* f), mult_P(L) = mult_{(0,0)}(\varphi^* L)$, and since $P \in V(f) \cap V(L)$, we have $(0, 0) \in V(\varphi^* f) \cap V(\varphi^* L)$, hence $mult_{(0,0)}(\varphi^* f), mult_{(0,0)}(\varphi^* L) \geq 1$. Since P is a point of multiplicity 2 in $V(f)$, we have that the lowest term of $\varphi^* f$ is of degree 2, hence we in particular get

$$I_P(f, L) > \underbrace{mult_P(f)}_{=2}\underbrace{mult_P(L)}_{=1} = 2$$

Thus $I_P(f, L) \geq 3$.

(b) Suppose $V(f)$ has a cusp at $P = (0, 0)$ with $I_P(f, y) = 3$.
Now, P is a point of multiplicity 2 in $V(f)$, so f_2 is the lowest degree term of f; let $f_2 = ax^2 + bxy + cy^2$. Then $TC_P V(f) = V(f_2) = V(y)$, so $a = 0$, and hence if $f_3 = \gamma x^3 + \ldots$, we have $I_P(f, y) = I_P(f(x, 0), y) = I_P(\gamma x^3 + x^4 B, y)$, and as $TC_P V\left(\gamma x^3 + x^4 B\right) = V(x)$ and $TC_P V(y) = V(y)$, we get $3 = I_P(f, y) = I_P(f(x, 0), y) = I_P\left(\gamma x^3 + x^4 B, y\right) = mult_P\left(\gamma x^3 + x^4 B\right) mult_P(y) = mult_P(\gamma x^3 + x^4 B)$. Hence $\gamma \neq 0$, so $f_{xxx}(P) = 6\gamma \neq 0$.

Now suppose $f_{xxx}(P) \neq 0$. Since $TC_P V(f) = V(L)$ and as L is a line through $P = (0, 0)$ we have $TC_P V(L) = V(L)$. But as $f_{xxx}(P) \neq 0$, we have that if we write $f_3 = ax^3 + bx^2y + cxy^2 + dy^3$ then $a \neq 0$, so $mult_P(f(x, 0)) \leq 3$ hence by (a), $mult_P(f(x, 0)) = 3$, so as $mult_P(L) = 1$, and as $TC_P V(f(x, 0)) \not\supset V(y)$, we have $I_P(f(x, 0), L) = mult_P(f(x, 0))mult_P(L) = 3$ by property (5).

(c) Assume P is a cusp, so $I_P(f, L) = 3$ for a L giving the line $V(L) = TC_P V(f)$.
Assuming k is algebraically closed, we can write $f = f_1^{n_1} \ldots f_r^{n_r}$ and then by corollary 3 in section 1.6, Fulton, we have that $V(f_1) \cup \ldots \cup V(f_r)$ is the composition of $V(f)$ into irreducible components and $f \in \sqrt{(f)} = I(V(f)) = (f_1 \ldots f_r)$. In particular, assume $f = hg$ where h and g vanish at P.
Then $3 = I_P(f, L) = I_P(g, L) + I_P(h, L)$, so we can assume without loss of generality that $I_P(g, L) \in \{0, 1\}$. Now $I_P(g, L) = 1$ implies that the tangent line to g at P is not L. But then $TC_P V(f) = TC_P V(gh) = TC_P V(h) \cup TC_P V(g)$ is not just $V(L)$, contradiction. Thus $I_P(g, L) = 0$, so $P \notin V(g) \cap V(L)$ and hence $P \notin V(g)$, contradiction. Hence $V(f)$ has only one irreducible component that passes through P.