2.1.ii Prove that if $F \colon C \to \text{Set}$ is representable, then F preserves monomorphisms, i.e., sends every monomorphism in C to an injective function. Use the contrapositive to find a covariant set-valued functor defined on your favorite concrete category that is not representable.

Solution: Suppose F is representable by $c \in C$. Thus there is a natural isomorphism between F and $C(c, -)$. Thus

$$\begin{tikzcd} C(c,x) \arrow[r, "\cong"] \arrow[d, "f_*"'] & F(x) \arrow[d, "F(f)"] \\ C(c,y) \arrow[r, "\cong"'] & F(y) \end{tikzcd}$$

commutes for any $x, y \in C$ with $f \colon x \to y$. Now, if $f \colon x \to y$ is a monomorphism, then for any $z, w \in C$ with $g \colon z \to x$ and $h \colon w \to x$,

$$fg = fh \implies g = h.$$

This is equivalent to saying that for any $z \in C$, post-composition with f defines an injection $f_* \colon C(z,x) \to C(z,y)$. Hence, in the square above, if we denote the top isomorphism by α_x and the bottom one by α_y, then we find that $\alpha_y \circ f_* \circ \alpha_x^{-1} = F(f)$, and since isomorphisms between sets are bijective functions, and as f_* is injective, we have that $F(f) = \alpha_y \circ f_* \circ \alpha_x^{-1}$ is injective.
Taking the contrapositive, we have that if a covariant functor $F \colon C \to \text{Set}$ does not preserve monomorphisms, then F is not representable.

Consider the category $C = \mathbb{2} = \{0, 1\}$ with the single non-identity map $0 \to 1$. Now define a functor $F \colon C \to \text{Set}$ sending $0 \to \{1, 2\}$ and $1 \to \{3\}$. Then $F(0 \to 1)$ is not mono since the maps $\alpha, \beta \colon \{-1, -2\} \to \{1, 2\}$ by $\alpha(-1) = 1, \alpha(-2) = 2$ and $\beta(-1) = 2, \beta(-2) = 1$ each give that for $\gamma \colon \{1, 2\} \to \{3\}$, $\gamma\alpha = \gamma\beta$, yet $\alpha \neq \beta$, so γ is not a monomorphism.

2.1.iii: Suppose $F \colon C \to \text{Set}$ is equivalent to $G \colon D \to \text{Set}$ in the sense that there is an equivalence of categories $H \colon C \to D$ so that GH and F are naturally isomorphic.

(i) If G is representable, then is F representable?

(ii) If F is representable, then is G representable?

Solution:
We claim both (i) and (ii) are true.

(i) We have that GH and F are naturally isomorphic, so

$$\begin{tikzcd} GH(c) \arrow[r, "\cong"] \arrow[d, "GH(f)"'] & F(c) \arrow[d, "F(f)"] \\ GH(c') \arrow[r, "\cong"'] & F(c') \end{tikzcd}$$

commutes.
Suppose G is representable, so there exists some $d \in D$ such that

$$\begin{tikzcd} C(d,x) \arrow[r, "\cong"] \arrow[d, "f_*"'] & G(x) \arrow[d, "G(f)"] \\ C(d,y) \arrow[r, "\cong"'] & G(y) \end{tikzcd}$$

commutes.
Since H is one part of an equivalence of categories, it is full, faithful and essentially surjective on objects

by theorem 1.5.9, so there exists some $\tilde{c} \in C$ such that $H(\tilde{c}) \cong d$. Furthermore, by theorem 1.5.9, it is full and faithful, so for any $a, b \in C$, $|\operatorname{Hom}(a,b)| = |\operatorname{Hom}(H(a), H(b))|$, s $C(a,b) \cong C(H(a), H(b))$. Composing the two commutative squares, and using this last bijection, we find that

$$\begin{tikzcd}
C(\tilde{c}, c) \arrow[r, "\cong"] \arrow[d, "f_*"'] & C(H(\tilde{c}), H(c)) \arrow[r, "\cong"] \arrow[d, "(H(f))_*"'] & GH(c) \arrow[r, "\cong"] \arrow[d, "GH(f)"'] & F(c) \arrow[d, "F(f)"] \\
C(\tilde{c}, c') \arrow[r, "\cong"'] & C(H(\tilde{c}), H(c')) \arrow[r, "\cong"'] & GH(c') \arrow[r, "\cong"'] & F(c')
\end{tikzcd}$$

commutes, and since each square commutes, we have that the outer rectangle commutes, giving that F is represented by $\tilde{c}$.

(ii) Suppose GH and F are naturally isomorphic as before, and suppose F is representable - suppose it is represented by $c \in C$, so

$$\begin{tikzcd}
GH(c) \arrow[r, "\cong"] \arrow[d, "GH(f)"'] & F(c) \arrow[d, "F(f)"] \\
GH(c') \arrow[r, "\cong"'] & F(c')
\end{tikzcd}$$

and

$$\begin{tikzcd}
C(c, x) \arrow[r, "\cong"] \arrow[d, "f_*"'] & F(x) \arrow[d, "F(f)"] \\
C(c, y) \arrow[r, "\cong"'] & F(y)
\end{tikzcd}$$

commute.
Letting α_x and α_y denote the top and bottom isomorphism, respectively, we have

$$f_* \alpha_x^{-1} = \alpha_y^{-1} \alpha_y f_* \alpha_x^{-1} = \alpha_y^{-1} F(f) \alpha_x \alpha_x^{-1} = \alpha_y^{-1} F(f)$$

so we can rewrite it as the following diagram commuting:

$$\begin{tikzcd}
F(x) \arrow[r, "\cong"] \arrow[d, "F(f)"'] & C(c, x) \arrow[d, "f_*"] \\
F(y) \arrow[r, "\cong"'] & C(c, y)
\end{tikzcd}$$

Now let $d', \tilde{d} \in D$ be arbitrary with $g \colon d' \to \tilde{d}$ a morphism between them. As H is essentially surjective on objects and full and faithful by theorem 1.5.9., there exist $c', \tilde{c} \in C$ such that $H(c') \cong d'$ and $H(\tilde{c}) \cong \tilde{d}$. By fullness and faithfullness, $C(c', \tilde{c}) \cong C\left(d', \tilde{d}\right)$, so there exists $g' \colon c' \to \tilde{c}$ with $H(g') = g$. We thus find

$$\begin{tikzcd}
G(d') \arrow[r, "\cong"] \arrow[d, "G(g)"'] & F(c') \arrow[r, "\cong"] \arrow[d, "F(g')"] & C(c, c') \arrow[r, "\cong"] \arrow[d, "g'_*"] & C(H(c), d') \arrow[d, "g_*"] \\
G(\tilde{d}) \arrow[r, "\cong"'] & F(\tilde{c}) \arrow[r, "\cong"'] & C(c, \tilde{c}) \arrow[r, "\cong"'] & C(H(c), \tilde{d})
\end{tikzcd}$$

commutes with each square commuting. Here the last square again follows as H is part of an equivalence and hence both faithful and full, so $C(c, x) \cong C(H(c), H(x))$ for any $x \in C$.

Thus the outer square commutes, so G is represented by $G(c)$.