Problem 0.1. It can be shown that $\pi_{13}(S^6) = \mathbb{Z}/60$. Let $X = S^6 \cup e^{14}$ be obtained by attaching a 14-cell to a S^6 along a generator in $\pi_{13}\left(S^6\right)$.

(1) Calculate $\pi_6(X)$.
(2) Calculate $\pi_{13}(X)$

Solution. Attaching a 14-cell along a generator in $\pi_{13}(S^6) = \mathbb{Z}/60$ trivializes this homotopy group, so $\pi_{13}(X) \cong 0$.
By cellular approximation, $\pi_i(X) = 0$ for $i < 6$, so by Hurewicz, $\pi_6(X) \cong H_6(X)$, and it is clear that $H_6(X) \cong \mathbb{Z}$ by cellular homology.

Problem 0.2. Compute $H_*\left(\Omega\left(S^3 \vee S^3\right)\right)$.

Solution. We have the homotopy fibration $\Omega\left(S^3 \vee S^3\right) \to P\left(S^3 \vee S^3\right) \to S^3 \vee S^3$, since the base space is simply-connected, we obtain the following by LSSS:

$$
\begin{tikzcd}
\vdots \arrow[d, dash] & & \\
H_3(\Omega(S^3 \vee S^3)) \arrow[d, dash] & 0 & \\
H_2(\Omega(S^3 \vee S^3)) \arrow[d, dash] & (\mathbb{Z}\oplus\mathbb{Z})\otimes(\mathbb{Z}\oplus\mathbb{Z}) \cong \mathbb{Z}^4 \arrow[uul, "\cong"'] & \\
0 \cong H_1(\Omega(S^3 \vee S^3)) \arrow[d, dash] & 0 \arrow[uul, "\cong"'] & \\
\mathbb{Z} \arrow[r, dash] & \mathbb{Z}\oplus\mathbb{Z} \arrow[uul, "\cong"'] \arrow[r, dash] & \cdots
\end{tikzcd}
$$

from which we can deduce that

$$H_n\left(\Omega\left(S^3 \vee S^3\right)\right) \cong \begin{cases} \mathbb{Z}^n, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Problem 0.3. True or false? Briefly justify your answer:

(1) If an n-dimensional CW complex is n-connected, then it is contractible.
(2) If X is a simply connected CW complex with finitely many cells, then ΩX is weakly equivalent to a CW complex with finitely many cells.
(3) If X is an n-truncated CW complex, then its suspension ΣX is $(n+1)$-truncated.

Solution. (a) By the Hurewicz theorem, the first nontrivial homology group equals the first nontrivial homotopy group, so if π_k is nontrivial for some $k > n$, then H_k is also nontrivial, but since there are no cells of dimension $> n$, we conclude that $\pi_k = 0$ for all k. Now the inclusion of a point induces a weak homotopy equivalence, hence it is a homotopy equivalence by Whitehead's theorem, so the space is contractible.

(b) False. If this were true, then the homology groups of ΩX would be nontrivial only in finitely many dimensions, but we have just seen that $H_n\left(\Omega\left(S^3 \vee S^3\right)\right) \cong \mathbb{Z}^n$

in all even dimensions.

(c) False: S^1 is 2-truncated, but S^2 is not 3-truncated since $\pi_3\left(S^2\right) \cong \mathbb{Z}$ (and $S^2 = \Sigma S^1$).

Problem 0.4. Let X be a simply connected space with homology

$$H_n(X) \cong \mathbb{Z}/n \quad \text{for all } n \geq 1.$$

(1) Show that $\pi_k(X)$ is finite for all k.
(2) Show that there are infinitely many $k \in \mathbb{N}$ with $\pi_k(X) \ncong 0$.

Proof. (1) Let P be the set of all primes. Then $\mathcal{F}_P$ is the set of all finite abelian groups. Now by theorem 1.7 in Hatcher's spectral sequences text, since X is simply-connected and $H_n(X) \in \mathcal{C} = \mathcal{F}_p$ for all $n > 0$, we have $\pi_n(X) \in \mathcal{F}_p$ for all $n > 0$.

(2) Suppose for contradiction that there are only finitely many nontrivial homotopy groups. Since these are also finite abelian, this means that there is a maximal finite collection P of primes such that each $p \in P$ divides the order of some nontrivial homotopy group of X. Let now $p \notin P$. Then by assumption, $\pi_k \in \mathcal{C} = \mathcal{F}_P$ for all $k < p$, so the Hurewicz homomorphism $h\colon \pi_p(X) \to H_p(X) \cong \mathbb{Z}/p$ is an isomorphism mod $\mathcal{C} = \mathcal{F}_P$. Hence the cokernel is in $\mathcal{F}_P$. But $\mathbb{Z}/p \notin \mathcal{F}_P$ by assumption, so the image of h must be all of $\mathbb{Z}/p$. But then by the first isomorphism theorem, we obtain $\pi_p(X)/\ker h \cong \mathbb{Z}/p$, so in particular,

$$|\pi_p(X)| \cong |\ker h|\,|\mathbb{Z}/p| = p\,|\ker h| > 1$$

so $\pi_p(X)$ is nontrivial, and also has order a multiple of p, so $p \in P$, which is a contradiction.

□

Problem 0.5 (Whitehead tower computation)**.** Let $\mathbb{RP}^1 \subset \mathbb{RP}^\infty$ be the inclusion. Define $X = \mathbb{RP}^\infty/\mathbb{RP}^1$ by collapsing $\mathbb{RP}^1$ to a point.

(1) Show that X is simply connected, that

$$H^*(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & * = 0,2 \\ \mathbb{Z}/2, & * > 3 \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

and that as a graded ring, $H^*(X;\mathbb{Z}) \cong \mathbb{Z}\left[a\right]/2a^2$ with $|a| = 2$.
(2) Show that there is a homotopy fiber sequence

$$S^1 \to \tau_{>2}X \to X$$

and use the cohomological Leray-Serre spectral sequence to compute $H^*\left(\tau_{>2}X;\mathbb{Z}\right)$.
(3) Show that $\tau_{>2}X$ is homotopy equivalent to a finite CW complex.
(4) Is there a finite CW complex with the same homotopy groups as X? Briefly justify your answer.

Solution. (1) Since $\mathbb{RP}^1$ is a subcomplex of $\mathbb{RP}^\infty$, we can use the LES associated to this pair. In homotopy groups, we get

$$\underbrace{\pi_1\left(\mathbb{RP}^1\right)}_{\cong 0} \to \underbrace{\pi_1\left(\mathbb{RP}^\infty\right)}_{\cong 0} \to \pi_1\left(\mathbb{RP}^\infty, \mathbb{RP}^1\right) \to \underbrace{\pi_0\left(\mathbb{RP}^1\right)}_{\cong 0}$$

hence we find $0 \cong \pi_1\left(\mathbb{RP}^\infty, \mathbb{RP}^1\right) \cong \pi_1\left(\mathbb{RP}^\infty/\mathbb{RP}^1\right)$. Next, we again get $H^n\left(\mathbb{RP}^\infty, \mathbb{RP}^1\right) \cong H^n(\mathbb{RP}^\infty/\mathbb{RP}^1)$.
Recall that

$$H^*\left(\mathbb{RP}^\infty\right) \cong \mathbb{Z}\left[\alpha\right]/(2\alpha)$$

where $|\alpha| = 2$, and

$$H^n(\mathbb{RP}^1) \cong \begin{cases} \mathbb{Z}, & n = 0, 1 \\ 0, & \text{else} \end{cases}$$

Now, $X = \mathbb{RP}^\infty/\mathbb{RP}^1$ has the cell structure of $\mathbb{RP}^\infty$ with one cell in each dimension, but with the 1-cell collapsed, so the singular chain complex becomes

$$\ldots \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \xrightarrow{0} \ldots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \mathbb{Z}$$

where the rightmost $\mathbb{Z}$ is in degree 0. Dualizing, we get that, since $\mathrm{Hom}\left(\mathbb{Z}, \mathbb{Z}\right) \cong \mathbb{Z}$, and a map $f\colon \mathbb{Z} \to \mathbb{Z}$ induces the map $f^*\colon \mathbb{Z} \to \mathbb{Z}$, we obtain

$$\mathbb{Z} \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \ldots$$

so

$$H^n\left(X; \mathbb{Z}\right) \cong \begin{cases} \mathbb{Z}, & n = 0, 2 \\ \mathbb{Z}/2, & n > 2 \text{ even} \\ 0, & \text{else} \end{cases}$$

The quotient map $\varphi\colon \mathbb{RP}^\infty \to X$ induces a map $\varphi^*\colon H^*(X; \mathbb{Z}) \to H^*\left(\mathbb{RP}^\infty; \mathbb{Z}\right) \cong \mathbb{Z}\left[\alpha\right]/\left(2\alpha\right)$ which is an isomorphism in degrees $\neq 2$ as it is the identity outside this degree. In degree 2, we want to show that it maps the generator 1 for $\mathbb{Z} = H^2\left(X; \mathbb{Z}\right)$ to a generator $a \in H^2\left(\mathbb{RP}^\infty; \mathbb{Z}\right) \cong \mathbb{Z}/2$. But the map $H^*(X; \mathbb{Z}) \to H^*\left(\mathbb{RP}^\infty; \mathbb{Z}\right)$ is the induces map from the LES of the pair $\left(\mathbb{RP}^\infty, \mathbb{RP}^1\right)$. Note that

$$H^*\left(\mathbb{RP}^1; \mathbb{Z}\right) \cong \mathbb{Z}\left[y\right]/\left(y^2\right)$$

with $|y| = 1$. Thus

$$0 \to \underbrace{H^1\left(\mathbb{RP}^1; \mathbb{Z}\right)}_{\cong \mathbb{Z}y} \to H^2(X; \mathbb{Z}) \to H^2(\mathbb{RP}^\infty; \mathbb{Z}) \to \underbrace{H^2\left(\mathbb{RP}^1\right)}_{\cong 0}$$

is exact, so so φ^* maps the image of y to 0. So we obtain

$$H^2(X; \mathbb{Z})$$

On chain complexes, we have

$$\begin{array}{ccccccccccc}
\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \cdots \\
\downarrow{\scriptstyle \varphi^*=\mathrm{id}} & & \downarrow{\scriptstyle \varphi^*=0} & & \downarrow{\scriptstyle \varphi^*=\mathrm{id}} & & \downarrow{\scriptstyle \varphi^*=\mathrm{id}} & & \downarrow{\scriptstyle \varphi^*=\mathrm{id}} & & \\
\mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \cdots
\end{array}$$

Thus in cohomology, the map in degree 2 becomes the map $\mathbb{Z} \xrightarrow{1 \mapsto 1} \mathbb{Z}/2$. So letting x be a generator for $H^2(X; \mathbb{Z})$, we get $\varphi^*(x) = a \in \mathbb{Z}\left[a\right]/(2a)$. Hence $\varphi^*(x^n) = a^n$ which is a generator of $H^{2n}\left(\mathbb{RP}^\infty; \mathbb{Z}\right)$, so we find that

$$H^*(X; \mathbb{Z}) \cong \mathbb{Z}\left[x\right]/\left(2x^2\right)$$

(2) In general, we have a homotopy fiber sequence

$$K\left(\pi_2(X), 1\right) \to \tau_{>2}X \to \tau_{>1}X \cong X.$$

Now, $H_1(X) \cong ab(\pi_1(X)) \cong 0$ by construction, and so

$$0 \to \underbrace{\mathrm{Ext}(H_1(X), \mathbb{Z})}_{\cong 0} \to \mathbb{Z} \to \mathrm{Hom}\left(H_2(X), \mathbb{Z}\right) \to 0$$

and since $\mathrm{Ext}\left(H_2(X), \mathbb{Z}\right) \cong 0$, $H_2(X)$ has no torsion, so $H_2(X) \cong \mathbb{Z}$ which is the first nontrivial homology group, so $\pi_2(X) \cong H_2(X) \cong \mathbb{Z}$. Thus $K\left(\pi_2(X), 1\right) = K\left(\mathbb{Z}, 1\right) \cong S^1$.
The cohomological Leray-Serre spectral sequence now gives

$$\begin{tikzcd}
\vdots \arrow[d, dash] & & & \\
\mathbb{Z}a \arrow[d, dash] \arrow[dr] & \mathbb{Z}ax \arrow[dr] & \mathbb{Z}/2ax^2 \arrow[dr] & \mathbb{Z}/2ax^3 \\
\mathbb{Z} \arrow[r, dash] & \mathbb{Z}x \arrow[r, dash] & \mathbb{Z}/2x^2 \arrow[r, dash] & \mathbb{Z}/2x^3 \\
 & 2 & 4 & 6
\end{tikzcd}$$

Now, the homotopy group of $\tau_{>2}X$ is trivial in dimension 1 and 2, so the map $\mathbb{Z}a \to \mathbb{Z}x$ must be an isomorphism. Furthermore, we may assume it maps a to x by a change of sign. Now we get $d\left(ax\right) = d(a)x + ad(x) = x^2$. Hence the map $\mathbb{Z}ax \to \mathbb{Z}/2x^2$ is a surjection, so the kernel is $2\mathbb{Z} \cong \mathbb{Z}$. Likewise, $d\left(ax^k\right) = x^{k+1} + ad(x^k)$ and $d(x^k) = xd(x^{k-1}) = \ldots = x^{k-1}d(x) = 0$, so $d(ax^k) = x^{k+1}$. Hence all the maps are surjective, and in the cases where we have maps $\mathbb{Z}/2 \to \mathbb{Z}/2$, it is therefore an isomorphism. Thus,

$$H^k(\tau_{>2}X) \cong \begin{cases} \mathbb{Z}, & k = 0, 3 \\ 0, & \text{else} \end{cases}$$

(3) By UCT,

$$H_k\left(\tau_{>2}X\right) \cong \begin{cases} \mathbb{Z}, & k = 0, 3 \\ 0, & \text{else} \end{cases}$$

so by uniqueness of Moore spaces, $\tau_{>2}X \simeq S^3$.

(4) No, a finite CW complex can only have finitely many nonzero cohomology groups, while X has infinitely many.