ASSIGNMENT 2

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Exercise 0.1. Given two local rings $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be local rings. A ring homomorphism $f: R \to S$ is called a local homomorphism if the image of $\mathfrak{m}$ under f is contained in $\mathfrak{n}$.

(1) Let $g: A \to B$ be a ring homomorphism between two arbitrary rings and let $\mathfrak{p} \subset B$ and $\mathfrak{q} = g^{-1}(\mathfrak{p}) \subset A$ be prime ideals. Show that g localizes to a ring homomorphism $A_{\mathfrak{q}} \to B_{\mathfrak{p}}$ to be more precise, let $\pi_A: A \to A_{\mathfrak{q}}$ and $\pi_B: B \to B_{\mathfrak{p}}$ be the natural ring homomorphisms from the rings to their localizations. You need to construct a ring homomorphism $g': A_{\mathfrak{q}} \to B_{\mathfrak{p}}$ such that $\pi_B \circ g = g' \circ \pi_A$. Show that the map g' you construct is a local homomorphism.

(2) Find a ring homomorphims between local rings which is not a local homomorphism.

Proof. (1) Consider the diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\pi_A"'] & B \arrow[d, "\pi_B"] \\ A_{\mathfrak{q}} \arrow[r, dashed, "g'"'] & B_{\mathfrak{p}} \end{tikzcd}$$

Note that by the universal property of localizations, the map g' exists if and only if $\pi_B \circ g\left((A - \mathfrak{q})\right) \subset B_{\mathfrak{p}}^{\times} = (R - \mathfrak{p})_{\mathfrak{p}}$. Let $a \in A - \mathfrak{q}$. By assumption, $g(a) \notin \mathfrak{p}$, so $\pi_B\left(g(a)\right) = \frac{g(a)}{1} \in (R - \mathfrak{p})_{\mathfrak{p}}$. This gives the existence of g'.

Next we show that g' is a local homomorphism. Note that $A_{\mathfrak{q}}$ and $B_{\mathfrak{p}}$ are local rings with unique maximal ideals $\mathfrak{q}_{\mathfrak{q}}$ and $\mathfrak{p}_{\mathfrak{p}}$, respectively. Thus, to show that g' is a local homomorphism, we must show that $g'\left(\mathfrak{q}_{\mathfrak{q}}\right) \subset \mathfrak{p}_{\mathfrak{p}}$. Explicitly,

$$\mathfrak{q}_{\mathfrak{q}} = \left\{ \frac{a}{b} \in A_{\mathfrak{q}} : a \in \mathfrak{q} \right\}$$
$$\mathfrak{p}_{\mathfrak{p}} = \left\{ \frac{a}{b} \in B_{\mathfrak{p}} : a \in \mathfrak{q} \right\}.$$

Let $x \in \mathfrak{q}_{\mathfrak{q}}$, so there exist $a \in \mathfrak{q}$ and $b \in A - \mathfrak{q}$ such that $x = \frac{a}{b}$. Then $\frac{a}{b} = \pi_A(a)\pi_A(b)^{-1}$, so $g'\left(\frac{a}{b}\right) = g' \circ \pi_A(a)\left(g' \circ \pi_A(b)\right)^{-1} = \pi_B \circ g(a)\left(\pi_B \circ g(b)\right)^{-1} = \frac{g(a)}{g(b)}$ where we can invert by $g(b)$ since $g(b) \in B - \mathfrak{p}$ as $b \in A - \mathfrak{q}$ and $\mathfrak{q} = g^{-1}(\mathfrak{p})$. Since $g(b) \in B - \mathfrak{p}$ and $g(\mathfrak{q}) \subset \mathfrak{p}$, so $g(a) \in \mathfrak{p}$, we have $\frac{g(a)}{g(b)} \in \mathfrak{p}_{\mathfrak{p}}$, so $g'(x) \in \mathfrak{p}_{\mathfrak{p}}$, hence $g\left(\mathfrak{q}_{\mathfrak{q}}\right) \subset \mathfrak{p}_{\mathfrak{p}}$.

(2) The ring $\mathbb{Z}_{(p)}$ is local and $\mathbb{Q}$ being a field is also local. However, the inclusion $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}$ is not a local homomorphism since $\frac{p}{1}$ is not mapped to 0, for example. $\square$