ASSIGNMENT 7

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Problem 0.1. Let F be the homotopy fibre of the map $S^n \to S^n$ of degree k, for $n \ge 2$.

- (1) Show that $H^i(F) = 0$ for 0 < i < n.
- (2) Using the Serre spectral sequence, compute that

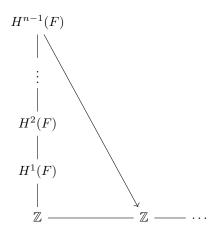
$$H^{i}(F) = \begin{cases} \mathbb{Z}, & i = 0\\ \mathbb{Z}/k, & i = 1 + m(n-1), m > 0. \\ 0, & \text{otherwise} \end{cases}$$

(3) Show that for $x, y \in H^*(F)$, if $\deg(x), \deg(y) > 0$, then $x \smile y = 0$.

Proof. (1) Since $\pi_1 S^n = 0$, the Serre spectral sequence to the homotopy fiber sequence

$$F \to S^n \to S^n$$

gives the following double complex:



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We apply the LSSS for cohomology and find that $H^i(S^n) = F_0^n$, and since $H^i(F)$ is the only nontrivial entry on the antidiagonal in degree i, and since there are no maps to kill off $H^i(F)$ for 0 < i < n-1, we obtain that $H^i(F) = H^i(S^n) = 0$ for 0 < i < n-1.

All that's missing is i = n - 1. For this, note that by the LES for the fibration, we get the following exact sequence:

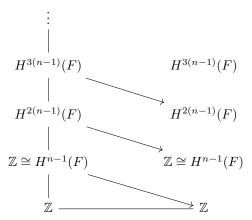
$$\underbrace{\mathbb{Z}}_{\pi_n(S^n)} \stackrel{\cdot k}{\to} \underbrace{\mathbb{Z}}_{\pi_n(S^n)} \to \pi_{n-1}(F) \to \underbrace{0}_{\pi_{n-1}(S^n)}$$

hence $\pi_{n-1}(F) \cong \operatorname{coker}\left(\mathbb{Z} \stackrel{\cdot k}{\to} \mathbb{Z}\right) \cong \mathbb{Z}/k$, and by the Hurewicz theorem, we get $H_{n-1}(F) \cong \pi_{n-1}(F) \cong \mathbb{Z}/k$. Now using the UCT, we obtain

$$0 \to \underbrace{\operatorname{Ext}(H_{n-2}(F), \mathbb{Z})}_{=0} \to H^{n-1}(F) \to \operatorname{Hom}\left(\underbrace{H_{n-1}(F)}_{=\mathbb{Z}/k}, \mathbb{Z}\right) \to 0$$

so $H^{n-1}(F) = 0$ as we wanted.

By the LSSS, the E^{∞} page has the form $E_{0,0}^{\infty} = E_{n,0}^{\infty} \cong \mathbb{Z}$, so in particular, on the E^k page, we get the following double complex:



This is the only page on which the horizontal maps can be nontrivial, so given the E^{∞} page, we conclude that the maps must be isomorphisms (including the trivial ones by just inductively shifting down by n-1 enough times). Hence we get periodicity, so

$$H^{i}(F) = \begin{cases} \mathbb{Z}, & i = 0\\ \mathbb{Z}/k, & i = 1 + m(n-1), \ m > 0 \ ,\\ 0, & \text{otherwise} \end{cases}$$

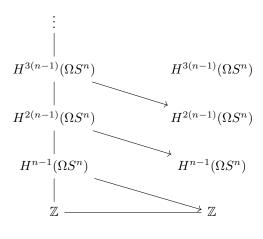
which was what we wanted to show

(3) Suppose $\deg(x)+\deg(y)=2$ so both are of degree 1, then since $H^1(F)=0$, we have x=0=y so $x\smile y=0$. Suppose we have shown it for $\deg(x)+\deg(y)\le N-1$ now. If $\deg x+\deg y=N$, then firstly we can assume $x,y\ne 0$ since otherwise $x\smile y=0$. Hence $x\in H^{1+m(n-1)}(F)$ and $y\in H^{1+m'(n-1)}$, so $x\smile y\in H^{2+(m+m')(n-1)}(F)=0$, so directly, $x\smile y=0$.

Problem 0.2. Use the path-loop fibration to deduce the cohomology ring structure of $H^*(\Omega S^n)$ when $n \geq 2$ is even.

Proof. Consider the path-loop fibration $\Omega S^n \to PS^n \to S^n$. Since S^n is simply connected, we can use the LSSS. Since $H^*(S^n)$ and $H^*(PS^n)$ are free and finitely

generated, there is no torsion, so $E_2^{s,t} = H^s(S^n) \otimes H^t(\Omega S^n)$. Thus we obtain the following E_n page:



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Again, using that E_{∞} has only $E_{\infty}^{0,0} = \mathbb{Z}$ and all other entries 0, we obtain that all the maps must be isomorphisms in this diagram.

Thus we can write $H^{k(n-1)}(\Omega S^n) \cong \mathbb{Z}a_k$ for all $k \geq 1$, where $d(a_1) = x$ and $d(a_k) = a_{k-1}x$ (we can change some a_k 's for their negatives to make signs check out if necessary) - here we also choose a_1 and then a_2 to satisfy the relation, and then a_3 , etc.

Recall now from the multiplicative structure that we have $d(xy) = (dx)y + (-1)^{p+q}x(dy)$, so

Now, $|a_1| = n - 1$ which is odd as n was assumed to be even, so $2a_1^2 = 0$ by anticommutativity, so since $H^{2(n-1)}(\Omega S^n) \cong \mathbb{Z}a_2$ is torsion-free, $a_1^2 = 0$.

Now that we have picked generators for $H^*(\Omega S^n)$, we want to see if we can reduce the generators and find relations between them.

So far, we have chosen the a_i such that $a_i \smile a_j = A(i,j)a_{i+j}$ for some integer A(i,j), simply because $a_i \smile a_j \in H^{i+j}(\Omega S^n)$ which a_{i+j} generates.

So we are interested in finding these coefficients A(i, j).

Now we have, first of all, that $a_k x = x a_k$ since $|a_k| |x|$ is even, so these commute for all k.

- $d(a_1a_{2k}) = xa_{2k} + (-1)^{2k+1}a_1a_{2k-1}x$. Now if k=1, then $a_1a_{2k-1} = a_1^2 = 0$, so $xa_2 a_1^2x$ becomes xa_2 and since $d(a_3) = xa_2$, we have $a_1a_2 = a_3$ as d is an isomorphism. Now, inductively, suppose $a_1a_{2k} = a_{2k+1}$ for $k \le N-1$. Then again $d(a_1a_{2N}) = xa_{2N} a_1a_{2N-1}x$ and by induction, either 2N-1 = 1 such that $a_1a_{2N-1} = a_1^2 = 0$ or $a_1a_{2N-1} = a_1a_1a_{2(N-1)} = 0$. Hence $d(a_1a_{2N}) = xa_{2N}$, so again $d(a_{2N+1}) = d(a_1a_{2N})$, so $a_{2N+1} = d_1a_{2N}$.
- Next we have $d_n\left(a_2^k\right)=a_1xa_2^{k-1}+a_2d\left(a_2^{k-1}\right)$. For k=1, this equals $a_1x=ka_1xa_2^{k-1}$, so we claim that $d_n\left(a_2^k\right)=ka_1xa_2^{k-1}$ in general. This is obtain by applying the inductive step to $a_2d\left(a_2^{k-1}\right)$ above to obtain $d_n\left(a_2^k\right)=a_1xa_2^{k-1}+a_2(k-1)a_1xa_2^{k-2}=a_1a_2^{k-1}xk$, as claimed. Next, inductively, we find that $a_2^{k-1}=(k-1)!a_{2k-2}$, so then $d_n(a_2^k)=k!a_1xa_{2k-2}=k!a_{2k-1}x=$

 $k!d_n(a_{2k}) = d_n(k!a_{2k})$, so again since d_n is an isomorphism, $a_2^k = k!a_{2k}$. Clearly, $a_2^{k-1} = (k-1)!a_{2k-2}$ is true for k=2, and then $d_n(a_2^k) = ka_1xa_2^{k-1} = k!a_1xa_{2k-2} = k!xa_{2k-1} = d(k!a_{2k})$ gives $a_2^k = k!a_{2k}$ inductively as desired.

Hence $H^*\left(\Omega S^n\right)\cong \Lambda_{\mathbb{Z}}\left[a\right]\otimes \Gamma_{\mathbb{Z}}\left[b\right]$ with |a|=n-1 and |b|=2n-2.