Checking that the image of $\{(r, \sqrt{2}r) \mid r \in \mathbb{R}\}$ is dense in the torus with torus being I^2 with opposite sides identified.

Solution: We want to show that for any point (x, y) on the torus, there is a point in any neighborhood on the line with gradient $\sqrt{2}$ through the origin modulo translations consisting of integer translations in the axes.
That is, for any point $(x, y) \in I^2$ and any $\varepsilon > 0$, we must find some $r \in \mathbb{R}$ and $k, l \in \mathbb{Z}$ such that

$$\|(r, \sqrt{2}r) - (x + k, y + l)\| < \varepsilon.$$

Consider for the given x all $r = x$ modulo 1. I.e. $r = x + k$ for some $k \in \mathbb{Z}$. We wish to find in this set some $l \in \mathbb{Z}$ such that $|y + l - \sqrt{2}(x + k)| = |y - \sqrt{2}x + (l - \sqrt{2}k)| < \varepsilon$. Now, we claim that $\{l + \sqrt{2}k \mid l, k \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.
Firstly, note that $S = \{l + \sqrt{2}k \mid l, k \in \mathbb{Z}\}$ is a ring.
Now, let $r \in \mathbb{R}$ and $\varepsilon > 0$.
Then, firstly note that $\left|\sqrt{2} - 1\right| < \frac{1}{2}$.
Thus $(\sqrt{2} - 1)^n \to 0$ as $n \to \infty$. We can thus find a $N \in \mathbb{N}$ such that $\mathbb{Z}\left(\sqrt{2} - 1\right)^N \cap B(r, \varepsilon) \neq \varnothing$, but $\mathbb{Z}\left(\sqrt{2} - 1\right)^N \in S$ since S was a ring, giving $\overline{S} = \mathbb{R}$.
Thus we can find $l, k \in \mathbb{Z}$ such that $\left|y - \sqrt{2}x + l - \sqrt{2}k\right| < \varepsilon$ and hence $\|(r, \sqrt{2}r) - (x + k, y + l)\| < \varepsilon$, so $\{(r, \sqrt{2}r) \mid r \in \mathbb{R}\}$ is dense in the torus.

Problem 4.26: Give an action of $\mathbb{Z}$ on $\mathbb{E}^1 \times [0, 1]$ which has the Möbius strip as orbit space.

Solution: Define the action of $\mathbb{Z}$ on $\mathbb{E}^1 \times [0, 1]$ by $z(s, t) = (s + z, (-1)^z t)$ where $(-1)^z t$ means t if z is even and $1 - t$ if z is odd.
These clearly satisfy conditions (a) and (b) of definition 4.14 since $hg(s, t) = (s + g + h, (-1)^{h+g} t) = h\left(s + g, (-1)^g t\right) = h\left(g(s, t)\right)$.
The continuity follows as the components are continuous.

Homotopy-lifting lemma: If $F \colon I \times I \to S^1$ is a map such that $F(0, t) = F(1, t) = 1$ for $0 \leq t \leq 1$, there is a unique map $\tilde{F} \colon I \times I \to \mathbb{R}$ which satisfies

$$\pi \circ \tilde{F} = F; \text{ and}$$
$$\tilde{F}(0, t) = 0, \ 0 \leq t \leq 1.$$

Proof: Suppose $U = S^1 - \{-1\}$ and $V = S^1 - \{1\}$. Then $F^{-1}(U) \cup F^{-1}(V) = I \times I$, so $F^{-1}(U), F^{-1}(V)$ is an open cover of $I \times I$ which is a compact metric space. By Lebesgue's lemma, there exists a real number $\delta > 0$ such that any subset of $I \times I$ with diameter less than δ is contained in some member of our cover.
Divide $I \times I$ into squares of sides-lengths $< \frac{\sqrt{2}\delta}{2}$, say $s_1, s_2, \ldots, s_k$ where s_1 is the bottom-left square, s_2 is the adjecent square to the right, etc. and we go dictionary order, essentially, ending at the top-right. We built up our definition of $\tilde{F}$ over these squares one at a time (kind of like with Van-Kampen) - we start from the bottom left, s_1. Now, since $F(0, t) = 1$ for $0 \leq t \leq 1$, we must have that $F(s_1) \subset U$. $\pi|_{(-\frac{1}{2}, \frac{1}{2})}$ is a homeomorphism of $(-\frac{1}{2}, \frac{1}{2})$ and U, so let f denote its inverse. Define $\tilde{F}(s, t) = f \circ F(s, t)$ for all $(s, t) \in s_1$.
This indeed satisfies $\tilde{F}(0, t) = 0$ for all $(0, t) \in s_1$.

Now, suppose we have defined $\tilde{F}$ on all s_i for $i < N$. We want to define it on s_N.
Since s_N shares a border with s_{N-1}, $\tilde{F}$ maps this border into either $\pi^{-1}(U)$ or $\pi^{-1}(V)$ as it is connected. If $\tilde{F}$ maps it into $\pi^{-1}(U)$, it is mapped into a connected component, suppose $(n - \frac{1}{2}, n + \frac{1}{2})$. So $F(s_N) \subset U$. Then letting g be the inverse to π mapping into $(n - \frac{1}{2}, n + \frac{1}{2})$, we define $\tilde{F}(s, t) = g \circ F(s, t)$ for $(s, t) \in s_N$. If instead $\tilde{F}$ maps into $\pi^{-1}(V)$, suppose into $(n, n+1)$, so $F(s_N) \subset V$, then let h be the inverse of $\pi|_{(n,n+1)}$ and define $\tilde{F}(s, t) = h \circ F(s, t)$ for $(s, t) \in s_N$.
This gives a definition of $\tilde{F}$ over all of $I \times I$ such that $\pi \circ \tilde{F} = F$ on all of $I \times I$, $\tilde{F}$ is continuous, and since $\pi \circ \tilde{F}(0, t) = F(0, t) = 1$, we have $\tilde{F}(0, t) \in 2\pi\mathbb{Z}$, and since $\tilde{F}(0, t)$ maps $\{0\} \times I$ into a connected component and $\tilde{F}(0, 0) = 0$, we have $\tilde{F}(0, t) = 0$ for all $t \in [0, 1]$.

Theorem 5.12: Let X be a space which can be written as the union of two simply connected open sets U, V in such a way that $U \cap V$ is path-connected. Then X is simply connected.

Proof: Since a simply-connected space is path-connected by definition, we have that X is path-connected, so choose a basepoint $p \in U \cap V$. We wish to show that $\pi(X) = \pi(X, p) = 0$ (the trivial group).
Let $\alpha \in \pi(X, p)$. Now, since $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ is an open cover of I, we can find a Lebesgue number $\delta > 0$ for the covering. Suppose $0 = t_0 < t_1 < \ldots < t_n = 1$ with $|t_{i+1} - t_i| < \delta$ for all i. Then since wlog. $\alpha([t_i, t_{i+1}]) \subset U$. Now, let γ_{t_i} be a path from p to $\alpha(t_i)$ contained in U if $\alpha(t_i) \in U$ and contained in V if $\alpha(t_i) \in V$ (if both, then we choose a path contained in $U \cap V$ which is possible as we assumed $U \cap V$ is path-connected) and γ'_{t_i} be a path from $\alpha(t_{i+1})$ to p with similar containment as above. Then $\gamma_{t_i}\alpha|_{[t_i,t_{i+1}]}\gamma'_{t_i}$ is a loop in U which is simply connected, so it is homotopic to the constant loop.
Now since the starting point of γ'_{t_i} is $\alpha(t_{i+1})$ which is the endpoint of $\gamma_{t_{i+1}}$, we have that γ'_{t_i} and $\gamma_{t_{i+1}}$ run in the same component, so $\gamma'_{t_i}\gamma_{t_{i+1}} \simeq 1$, hence

$$\alpha = \alpha|_{[0,t_1]}\alpha|_{[t_1,t_2]}\ldots\alpha|_{[t_{n-1},1]} \simeq \gamma_0\alpha|_{[0,t_1]}\gamma'_0\gamma_{t_1}\alpha|_{[t_1,t_2]}\gamma'_{t_1}\ldots\gamma_{t_{n-1}}\alpha|_{[t_{n-1},1}\gamma'_{t_{n-1}} \simeq 11\ldots1 \simeq 1$$

so $\pi(X) \cong 0$.

Theorem 5.13: If G acts as a group of homeomorphisms on a simply connected space X, and if each point $x \in X$ has a neighbourhood U which satisfies $U \cap g(U) = \varnothing$ for all $g \in G - \{e\}$, then $\pi_1(X/G)$ is isomorphic to G.

Proof: Fix a point $x_0 \in X$ and, given $g \in G$, join x_0 to $g(x_0)$ by a path γ. If $\pi\colon X \to X/G$ denotes the projection, $\pi \circ \gamma$ is a loop based at $\pi(x_0)$ in X/G. Define

$$\varphi\colon G \to \pi_1(X/G, \pi(x_0))$$

by $\varphi(g) = \langle \pi \circ \gamma \rangle$. Since X is simply connected, changing γ to any other path joining x_0 to $g(x_0)$ does not affect φ.

Now, given $g, h \in G$, let γ_{g*h} denote a path from x_0 to $g * h(x_0) = g(h(x_0))$. Let γ_g denote the path from x_0 to $g(x_0)$ and γ_h denote the path from $g(x_0)$ to $h(g(x_0)) = h * g(x_0)$. Then $\gamma_{g*h} \simeq \gamma_g\gamma_h$ as X is simply connected, so $\varphi(g*h) = \langle \pi\circ\gamma_{g*h}\rangle = \langle \pi\circ\gamma_g\gamma_h\rangle = \langle(\pi\circ\gamma_g)(\pi\circ\gamma_h)\rangle = \langle\pi\circ\gamma_g\rangle\langle\pi\circ\gamma_h\rangle = \varphi(g)\varphi(h)$.

So φ is a homomorphism. It remains to show that φ is surjective and injective.

Suppose $\langle\alpha\rangle \in \pi_1(X/G, \pi(x_0))$.

We make use of the following theorems:

Interlude

Def: Given a space X, a **covering space** of X consists of a space $\tilde{X}$ and a map $p\colon \tilde{X} \to X$ satisfying the following condition:

For each point $x \in X$ there exists an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.

Such a U is called **evenly covered.**

Theorem: Given a covering space $p\colon \tilde{X} \to X$ and a map $F\colon Y \times I \to X$ and a map $\tilde{F}\colon Y \times \{0\} \to \tilde{X}$ lifting $F|_{Y\times\{0\}}$, there exists a unique map $\tilde{F}\colon Y \times I \to \tilde{X}$ lifting F and restricting to the given $\tilde{F}$ on $Y \times \{0\}$.

This theorem gives the following two corollaries for covering space $p\colon \tilde{X} \to X$:
Cor. 1: For each path $f\colon I \to X$ starting at a point $x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, there is a unique lift $\tilde{f}\colon I \to \tilde{X}$ starting at $\tilde{x}_0$.

Cor. 2: For each homotopy $f_t\colon I \to X$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde{f}_t\colon I \to \tilde{X}$ of paths starting at $\tilde{x}_0$.

The first follows by letting Y be a single point.
The second follows by letting $Y = I$, where $\tilde{F}\colon I \times \{0\} \to \tilde{X}$ is constant $\tilde{x}_0$ since all the paths start at x_0 so it lifts $F|_{Y\times\{0\}}$. Hence it gives an extension to a lifted homotopy $\tilde{F}\colon I \times I \to X$ of paths starting at $\tilde{x}_0$.

Continuing

Okay, now we claim that X is a covering space of X/G with π the covering map. Let $x \in X/G$ be arbitrary. By assumption, each point $y \in X$ has a neighborhood U which satisfies $U \cap g(U) = \varnothing$ for all $g \in G - \{e\}$. Now, π is injective on $g(U)$ for each $g \in G$, and the set $\{g(U) \mid g \in G\}$ consists of disjoint open sets which cover $\pi^{-1}(x)$ and is the preimage of $\pi(U)$. Thus X is a covering space of X/G.

Now, by corollary 1, for any $\tilde{x} \in \pi^{-1}(x)$, there is a unique lift $\tilde{\alpha}\colon I \to X$ starting at $\tilde{x}$ such that $\alpha = \pi \circ \tilde{\alpha}$. Since α is a loop, the endpoint of $\tilde{\alpha}$ must be in the orbit of x, so there exists $g \in G$ such that $g(x) = \tilde{\alpha}(1)$. Hence $\varphi(g) = \langle\alpha\rangle$. This shows surjectivity.

To show that φ is injective. Suppose $\varphi(g) \simeq \langle 1\rangle$, i.e. the constant loop. This is saying that if γ connects
x_0 and $g(x_0)$, then $\langle\pi\gamma\rangle = \langle 1\rangle$. Lifting 1, we get that there is a unique lift of $\langle 1\rangle$ starting at x_0, which is the constant map, so γ must be 1. Hence $g = e$, the identity. So φ is injective.

Examples of results:
Example 1: $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{E}^2$ with orbit space the torus. For any point $x \in \mathbb{E}^2$, we have $B(x, \frac{1}{2}) \cap gB\left(x, \frac{1}{2}\right) = \varnothing$ for all $g \in \mathbb{Z} \times \mathbb{Z}$. Thus the requirements are fulfilled, so we conclude that $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$.

Example 2: Z_2 acts on S^n with orbit space $\mathbb{P}^n$. Again, for any point, the open hemisphere with the point as the top point defines an open set that fulfills the requirement in the theorem. Thus we can conclude that $\pi_1(\mathbb{P}^n) \cong Z_2$ for $n \geq 2$.

Example 3: Z_p acts on S^3 with orbit space the Lens space $L_{p,q}$.
Recall that if p, q are relatively prime integers and g is the generator for Z_p, then the action of Z_p on S^3 is

$$g(z_0, z_1) = \left(e^{2\pi i/p} z_0, e^{2\pi i \frac{q}{p}} z_1\right)$$

Now, clearly, each iteration of applying g is an isometry of each component space of $\mathbb{C}^2$. Thus, define $d_0 = \frac{1}{2}\min\left\{\|e^{n2\pi i/p} z_0 - e^{m2\pi i/p} z_0\| \colon n, m \in \{1, \ldots, p\}\right\}$ and $d_1 = \frac{1}{2}\min\left\{\|e^{n2\pi i\frac{q}{p}} z_1 - e^{m2\pi i\frac{q}{p}} z_1\| \colon m, n \in \{1, \ldots, p\}\right\}$. Then for $(z_0, z_1) \in S^3$, we have $B(z_0, d_0) \times B(z_1, d_1)$ is an open neighborhood of (z_0, z_1) satisfying the criteria of the theorem. Hence we can conclude that $\pi_1(L(p, q)) \cong Z_p$.

Fundamental group of the Klein bottle: Not all fundamental groups are abelian. E.g., let G be the group $\langle u, t \mid u^{-1}tut\rangle$. Consider the action of G on the plane determined by

$$t(x, y) = (x + 1, y)$$
$$u(x, y) = (-x + 1, y + 1).$$

Then t is a translation parallel to the x axis, and u is a glide reflection along the line $x = \frac{1}{2}$. The hypotheses of the theorem are easily checked - for the open set, we can obviously take an open ball of radius $\frac{1}{2}$. We must also check that this is indeed an action as a group of homeomorphisms on the plane. Conditions (b) and (c) in definition 4.14 in Armstrong are clearly true.
For (a), I believe it follows simply from the fact that the only relation we have is $u^{-1}tut$, but I'm not completely sure. Nevertheless, we find that the orbit space is the unit with its sides identified as for the Klein bottle.
Therefore the fundamental group of the Klein bottle is the group G. In terms of the parallel glide reflections $a = tu, b = u$, we can recapture example 8c of section 4.4 and we have $\pi_1(K) \cong \left\{a, b \mid a^2 = b^2\right\}$.

Theorem 5.14 If X and Y are path-connected space, $\pi_1(X \times Y)$ is isomorphism to $\pi_1(X) \times \pi_1(Y)$.

The comb space: it has a 'tooth' joining $(0,0)$ to $(0,\frac{1}{2})$ and $(\frac{1}{n},0)$ to $(\frac{1}{n},\frac{1}{2})$ for $n = 1,2,3,\ldots$.

Why is there no homotopy from 1_X to the constant map at the point $p = (0,\frac{1}{2})$, i.e. to c_p, which keeps p fixed?

Suppose such a $F\colon X \times I \to X$ existed with $F(s,0) = s$ and $F(s,1) = (0,\frac{1}{2})$, and $F(p,t) = p$ for all $t \in I$.

Suppose U is an open neighborhood of $p = (0,\frac{1}{2})$. Then $\{p\} \times I \subset F^{-1}(U)$. By the tube lemma, there exists a neighborhood V of p in X such that $V \times I \subset F^{-1}(U)$. But any suppose U is the ball of radius $\varepsilon < \frac{1}{4}$. Then choose an $x = (x_1,\frac{1}{2}) \in V$ with $x_1 > 0$, then $\{x\} \times I \subset F^{-1}(U)$. But then $F(x,t) \in U$ for all t. However, in U, x is not path connected to p, so such a path cannot exist in U.

Chapter 6 - Triangulations

Let $v_0, v_1, \ldots, v_k \in \mathbb{R}^n$. The hyperplane of these points is

$$\left\{\lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_k v_k \mid \lambda_i \in \mathbb{R}, \sum \lambda_i = 1\right\}.$$

These points are in *general position* if any subset of them spans a strictly smaller hyperplane.

Suppose $v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0$ are linearly dependent.
Then there exist $\lambda_1, \ldots, \lambda_k$ such that

$$\left(-\sum \lambda_i\right) v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_k v_k = 0$$

Since some λ_i must be nonzero, we can isolate some v_i, and thus the hyperplane without v_i spans the same hyperplane as with.

Conversely, if some subset $v_{n_0}, \ldots, v_{n_r}$ spans the same hyperplane with $r < k$, then
Lemma (6.3) Let K be a simplicial complex in $\mathbb{E}^n$.

(a) $|K|$ is a closed bounded subset of $\mathbb{E}^n$, and so $|K|$ is a compact space.
(b) Each point of $|K|$ lies in the interior of exactly one simplex of K.

(c) If we take the simplexes of K separately and give their union the identification topology, we obtain exactly $|K|$.
(d) If $|K|$ is a connected space, then it is path-connected.

Proof: (a) Each simplex of K is closed and bounded. Since K is finite, (a) follows.
(b) Let $v_0, v_1, \ldots, v_k$ be the vertices of K. Then each point of the complex is of the form $\lambda_0 v_0 + \ldots + \lambda_k v_k$ with $\lambda_0, \ldots, \lambda_k \geq 0$. If the point is a vertex, it lies in the 0-simplex consisting of that vertex.
Otherwise, there exists some simplex consisting of vertices $v_{n_1}, \ldots, v_{n_r}$ such that the point lies in the simplex. Suppose this is the smallest dimension simplex that the point lies in. Then all coefficients must be positive, so it lies in the interior of this simplex.
If A and B are two simplexes whose interiors overlap, then since K is a complex, A and B are required to meet in a common face.
Claim: The only face of a simplex which contains interior points is the whole simplex itself.
Claim: Every point in a simplex has a unique expression in terms of the vertices in the simplex with nonnegative coefficients.

Proof:

Barycentric division

It is a way to refine a simplicial complex. The dimension of a simplicial complex K is the maximum of the dimensions of its simplexes, and its mesh $\mu(K)$ is the maximum of the diameters of its simplexes.

Given a simplex A of K with vertices $v_0, v_1, \ldots, v_k$ then each point x of A has a unique expression of the form $x = \lambda_0 v_0 + \ldots + \lambda_k v_k$ where $\sum \lambda_i = 1$ and $\lambda_i \geq 0$ for all i.
These λ_i are called the barycentric coordinates of x and the barycentre of A is the point

$$\hat{A} = \frac{1}{k+1}\left(v_0 + \ldots + v_k\right)$$

Define K^1, the first barycentric subdivision of K, as having the barycentres of the simplexes of K as vertices and then let a collection of barycenters $\hat{A}_0, \ldots, \hat{A}_k$ forms a k-simplex if and only if there exists some permutation σ of the intergers $0, 1, 2, \ldots, k$ such that

$$A_{\sigma(0)} < A_{\sigma(1)} < \ldots < A_{\sigma_k}$$

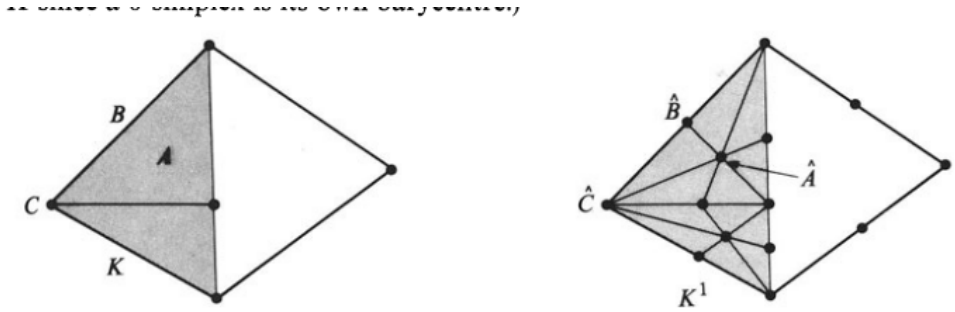


Figure 6.6

Lemma 6.4: The collection of simplexes described above forms a simplicial complex denoted by K^1 and called the first barycentric subdivision of K. It has the following properties:
(a) each simplex of K^1 is contained in a simplex of K;
(b) $\left|K^1\right| = |K|$;
(c) if the dimension of K is n, then $\mu(K^1) \leq \frac{n}{n+1}\mu(K)$.

To show (c), we observe that the diameter of a simplex is the length of its longest edge. Let σ be an edge of K with vertices $\hat{A}$ and $\hat{B}$, say, where $B < A$. Then σ is contained in A, and if the dimension of A is k we have

$$\begin{aligned}
\text{length } \sigma \leq \max_i \left\{\|\hat{A} - v_i\|\right\} &= \max_i \left\{\frac{1}{k+1}\|\sum_{j \neq i} v_j - v_i\|\right\} \\
&\leq \max_i \left\{\frac{k}{k+1} \max_j \{\|v_j - v_i\|\}\right\} \\
&= \frac{k}{k+1} \max_{i,j} \|v_j - v_i\| \\
&= \frac{k}{k+1} \text{diameter } (A)
\end{aligned}$$

Thus

$$\text{length } \sigma \leq \frac{k}{k_1}\text{diameter}(A) \leq \frac{n}{n+1}\text{diameter}(A) \leq \frac{n}{n+1}\mu(K)$$

Therefore $\mu(K^1) \leq \frac{n}{n+1}\mu(K)$.

Simplicial approximations

Def 6.5: Let K and L be simplicial complexes. A function $s\colon |K| \to |L|$ is called simplicial if it takes simplexes of K linearly onto simplexes of L.

This means that if A is a simplex of K, we require $s(A)$ to be a simplex of L ; the linearity means that if A has vertices $v_0, v_1, \ldots, v_k$ and if $x \in A$ is the point $x = \lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_k v_k$, where the λ_i are nonnegative and $\sum \lambda_i = 1$, then $s(x)$ when expressed in terms of the vertices of $s(A)$ is $s(x) = \lambda_0 s(v_0) + \lambda_1 s(v_1) + \ldots + \lambda_k s(v_k)$. Note that $s(A)$ may have lower dimension than A (we do not require s to be injective), in which case $s(v_0), \ldots, s(v_k)$ will not all be distinct.

Simplicial functions are continuous since they are linear functions between simplexes (and hence continuous) and the gluing lemma ensures continuity on all of K.

Because of its linearity on each simplex of K, a simplicial map s is completely determined once we know its effect on the vertices of K. In fact, if a function s from the vertices of K to the vertices of L has the property that if vertices $v_0, v_1, \ldots, v_k$ determine a simplex of K then $s(v_0), \ldots, s(v_k)$ determine a simplex of L, then s can be extended linearly across each simplex of K to give a simplicial map $|K| \to |L|$. In particular, an isomorphism from K to L extends in this way to a simplicial homeomorphism from the polyhedron of K to the polyhedron of L.

Def: Now let $f\colon |K| \to |L|$ be a map between polyhedra. Given a point $x \in |K|$, the point $f(x)$ lies in the interior of a unique simplex of L. Call this simplex the *carrier* of $f(x)$.

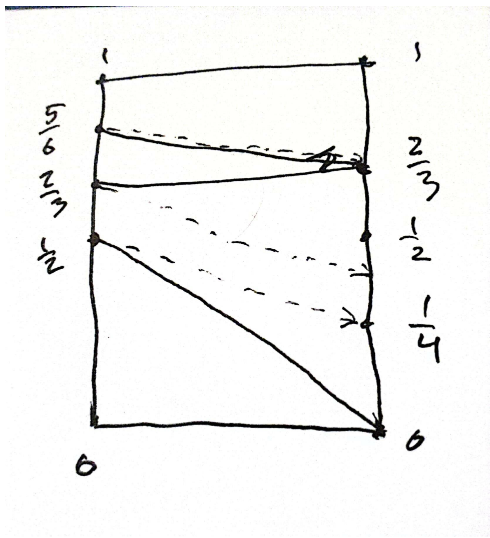
Def 6.6: A simplicial map $s\colon |K| \to |L|$ is a simplicial approximation of $f\colon |K| \to |L|$ if $s(x)$ lies in the carrier of $f(x)$ for each $x \in |K|$.

It follows immediately from the definition that s and f are homotopic, for suppose L lies in $\mathbb{E}^n$, and let $F\colon |K| \times I \to \mathbb{E}^n$ denote the straight-line homotopy defined by $F(x,t) = (1-t)s(x) + tf(x)$. Given $x \in |K|$, we know that some simplex of L contains $s(x)$ and $f(x)$ and, since a simplex is convex, all points $(1-t)(s) + tf(x)$, $0 \le t \le 1$, must also lie in this simplex. Therefore the image of F lies in $|L|$, and F is a homotopy from s to f.

Simplicial approximations do not always exist; example 6.8: Let $|K| = |L| = [0,1]$ with K having vertices at the points $0, \frac{1}{3}, 1$ and L at $0, \frac{2}{3}, 1$. Suppose the given map $f\colon |K| \to |L|$ is $f(x) = x^2$. Then f does not admit a simplicial approximation.
Here, the simplicial complexes do not include $[0,1]$ as a 1-simplex, since the intersection of this simplex with either $\left[0, \frac{1}{3}\right]$ in K or $\left[0, \frac{2}{3}\right]$ in L is not a face of $[0,1]$, whose only faces are 0 and 1.
Now, with this in mind, we show that there does not exist a simplicial approximation.
Suppose such a simplicial approximation $s\colon |K| \to |L|$ exists. Then for any vertex v of L, $s\left(f^{-1}(v)\right) = v$ since v is a 0-simplex and thus the carrier of v is v itself. Hence $s\left(f^{-1}(v)\right)$ must lie in the carrier which is v, giving the equality. So $s(0) = 0, s(1) = 1$ and $s\left(\sqrt{\frac{2}{3}}\right) = \frac{2}{3}$.
Now, s is simplicial, so it must take $\left[0, \frac{1}{3}\right]$ to a simplex in L. If $\left[0, \frac{1}{3}\right]$ were mapped to 0, then $\left[\frac{1}{3}, 1\right]$ has nothing to be mapped to: it must be mapped to $[0,1]$ to agree with $s(\frac{1}{3}) = 0$ and $s(1) = 1$, however, then s does not take the simplex $\left[\frac{1}{3}, 1\right]$ to a simplex in L as $[0,1]$ is not a simplex. Similar reasoning shows that $\left[0, \frac{1}{3}\right]$ cannot be mapped to $\frac{2}{3}$ or 1.
If $\left[0, \frac{1}{3}\right]$ were mapped to $\left[\frac{2}{3}, 1\right]$ then $s(0) = \frac{2}{3}$ by linearity, contradiction, so s must map it to $\left[0, \frac{2}{3}\right]$ and hence $s(\frac{1}{3}) = \frac{2}{3}$. However, then $s(\frac{1}{2}) \in \left[\frac{2}{3}, 1\right]$ which is not the carrier for $f(\frac{1}{2})$ whose carrier is $\left[0, \frac{2}{3}\right]$. This shows that s is not simplicial.

One can similarly show that there is no simplicial approximation for K^1. There is one for K^2, however. Map $\left[0, \frac{1}{2}\right]$ to 0, $\left[\frac{1}{2}, \frac{2}{3}\right]$ to $\left[0, \frac{2}{3}\right]$ linearly, $\left[\frac{2}{3}, \frac{5}{6}\right]$ to $\frac{2}{3}$ and $\left[\frac{5}{6}, 1\right]$ to $\left[\frac{2}{3}, 1\right]$ linearly.

Firstly, this map satisfies that $s(0) = 0, s(1) = 1$ and $s\left(\sqrt{\frac{2}{3}}\right) = \frac{2}{3}$.
Now, the below figure shows that the mapping is indeed simplicial.



So why is it impossible for K^1? Well, there we have that $\sqrt{\frac{2}{3}}$ lies in $\left[\frac{2}{3}, 1\right]$ in K, and since $s\left(\sqrt{\frac{2}{3}}\right) = \frac{2}{3}$ and $s(1) = 1$ and s maps $\left[\frac{2}{3}, 1\right]$ to a simplex, it maps $\left[\frac{2}{3}, 1\right]$ to a simplex containing both 1 and $\frac{2}{3}$. The only such simplex is $\left[\frac{2}{3}, 1\right]$, so s maps $\left[\frac{2}{3}, 1\right]$ to $\left[\frac{2}{3}, 1\right]$. Now, s maps this linearly, so we have that $\frac{2}{3} = s\left(\sqrt{\frac{2}{3}}\right) = s\left(\lambda_0 \cdot \frac{2}{3} + (1 - \lambda_0) \cdot 1\right) = \lambda_0 \cdot \frac{2}{3} + 1 - \lambda_0 = 1 - \frac{\lambda_0}{3}$. So $\lambda_0 = 1$, however
$\sqrt{\frac{2}{3}} \neq \frac{2}{3}$, contradiction.

Def: Let K be a complex and let v be a vertex of K. The *open star* of v in K is the union of the interiors of those simplexes of K which have v as a vertex. It is an open subset of $|K|$ and we denote it by $\text{star}(v, K)$.

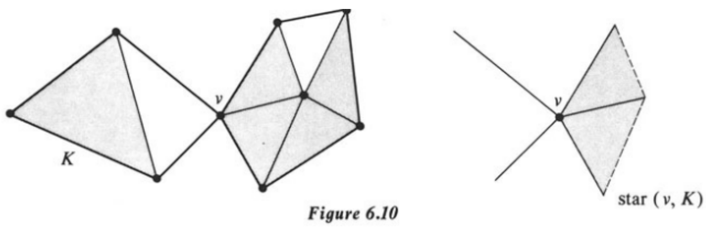


Figure 6.10

Lemma 6.9: Vertices $v_0, v_1, \ldots, v_k$ of a simplicial complex K span (i.e. are the vertices of) a simplex of K if and only if the intersection of their open stars is nonempty.

Proof: If $v_0, \ldots, v_k$ are the vertices of the simplex A of K then the whole of the interior of A lies in $\text{star}(v_i, K)$ for $0 \leq i \leq k$. Conversely, suppose that $x \in \bigcap_0^k \text{star}(v_i, K)$ and let A be the carrier of x. By the definition of an open star, each v_i must be a vertex of A, and therefore $v_0, \ldots, v_k$ span some face of A.

We use this for the following theorem:
Theorem 6.7: Simplicial approximation theorem: Let $f\colon |K| \to |L|$ be a map between polyhedra. If m is chosen large enough there is a simplicial approximation $s\colon |K^m| \to |L|$ to $f\colon |K^m| \to L$.

Proof: We first deal with the special case of the theorem where it is not necessary to chop up/refine the simplexes of K.
Suppose that for each vertex u of K, we can ind a vertex v of L satisfying the inclusion

$$f\left(\text{star}(u, K)\right) \subset \text{star}(v, L) \qquad (\Omega)$$

Define a function s from the vertices of K to those of L by choosing such a v for each u and setting $s(u) = v$. Then suppose that $u_0, \ldots, u_k$ span a simplex of K.

By construction,

$$\bigcap_0^k \text{star}(s(u_i), L) \supset \bigcap_0^k f\left(\text{star}(u_i, K)\right) = f\left(\underbrace{\bigcap_0^k \text{star}(u_i, K)}_{\neq\varnothing}\right) \tag{α}$$

where we have $\neq \varnothing$ by lemma 6.9. and the inclusion from Ω.
Now we can therefore extend s linearly over each simplex of K to give a simplicial map $s\colon |K| \to |L|$. This map s simplicially approximates f: for let $x \in |K|$ and let $u_0, \dots, u_k$ be the vertices of its carrier. Then $x \in \bigcap \text{star}(u_i, K)$, so by the α, we have $f(x) \in \bigcap \text{star}(s(u_i), L)$, so the carrier of $f(x)$ in L has the simplex spanned by $s(u_0), \dots, s(u_k)$ as a face, and consequently, it must contain the point $s(x)$ which lies in this face.

To deal with the theorem in general, we need only show that we can arrange for the inclusion (Ω) to be satisfied at the expense of replacing K by a suitable barycentric subdivision K^m.

6.3.(13): Use the simplicial approximation theorem to show that the n-sphere is simply connected for $n \geq 2$.

Solution: Suppose $\gamma\colon I \to S^n$ is a path.
Since S^n is homeomorphic to an $n+1$-simplex which is a polyhedra and I is a 1-simplex and a polyhedra, we have by the simplicial approximation theorem that there exists some $m \in \mathbb{N}$ such that $s\colon |I^m| \to \left|\Delta^{n+1}\right|$ is a simplicial approximation to $f\colon |I^m| \to \left|\Delta^{n+1}\right|$. Then s and f are homotopic. Furthermore, we can assume that γ starts at a vertex of the $n+1$-simplex. Now, s takes simplexes to simplexes, so letting $v_0, \dots, v_k$ be the vertices of I^m, $s\left([v_i, v_{i+1}]\right)$ is a simplex of the $n+1$-simplex. Linearity of s implies that the simplex is of dimension 0 or 1, i.e. s lives completely on 1-simplex faces of the $n+1$-simplex. Thus s is not onto, so γ is nullhomotopic.

14: If $k < m, n$, show that any map from S^k to S^m is null homotopic, and that the same is true of any map from S^k to $S^m \times S^n$.

Solution: Suppose $f\colon S^k \to S^m$ is a map for $k < m$. Let $g\colon |K| \to S^k$ be a triangulation of S^k and $h\colon |L| \to S^m$ a triangulation of S^m to an $k+1$ and $m+1$ simplex, resp. Then $h^{-1} \circ f \circ g$ is a map of complexes which has a simplicial approximation $s\colon |K^m| \to |L|$. But $\dim |K^m| = \dim |K| < \dim |L|$, so as s maps simplexes to simplexes of lesser or equal dimension, the dimension of the image of s is of degree at most $k+1 < m+1$, so s is not surjective. For a point $p \notin s\left(|K^m|\right)$, we have that $|L| - \{p\} \cong S^m - \{h(p)\} \cong \mathbb{R}^m$, so s is nulhomotopic as $\mathbb{R}^m$ is convex.
Now $s \simeq h^{-1} \circ f \circ g \simeq f$, so f is nulhomotopic.

Triangulating orbit spaces

Let $K \subset \mathbb{E}^n$ be a simplicial complex whose simplexes lie in $\mathbb{E}^n$. Let V denote the set of vertices of K and S the collection of those subsets of V which span simplexes of K.
The pair $\{V, S\}$ is called the **vertex scheme** of K. The set V is finite and S has the following properties:

1. Each element of V belongs to S (A vertex is a 0-simplex)
2. If X belongs to S then any nonempty subset of X belongs to S (Any face of a simplex of K is itself in K)
3. The sets in S are nonempty and have at most $m+1$ elements for some non-negative integer m.

Def. Realization means finding a simplicial complex K, and a bijection from V to the set of vertices of K, so that memebers of S correspond exactly to those sets of vertices which span simplexes.

6.14. Realization theorem. Let V be a finite nonempty set and S a collection of subsets of V which satisfies properties (a)-(c) listed above. Then $\{V, S\}$ can be realized as the vertex scheme of a simplicial complex.

Simplicial homology

Problem 8.2.2: Show that the elementary 1-cyles generates $Z_1(K)$ for any complex K.

Solution: Suppose $\lambda_1(u_1, v_1) + \ldots + \lambda_k(u_k, v_k)$ is an arbitrary element of $Z_1(K)$. Then define $G = \{(u, v) \mid u, v \text{ are 0-simplexes of } K\}$. Then $\lambda_1(u_1, v_1)+\ldots+\lambda_k(u_k, v_k) \in \langle G\rangle$ and clearly $\langle G\rangle \subset \{\lambda_1(u_1, v_1) + \ldots + \lambda_k(u_k, v_k$
Lemma 8.8: χ is a chain map.

Solution: Suppose $\sigma = (v_0, \ldots, v_k, v_{k+1}, \ldots, v_q)$ and $v_0, \ldots, v_k$ are the vertices of A. Then

$$\chi(\sigma) = \sum_{i=0}^{k}(-1)^i (v, v_0, \ldots, \hat{v}_i, \ldots, v_k, v_{k+1}, \ldots, v_q).$$

If σ does not have A as a face we set $\chi(\sigma) = \sigma$.
Now

$$\begin{aligned}
\partial\chi(\sigma) &= \partial\sum_{i=0}^{k}(-1)^i (v, v_0, \ldots, \hat{v}_i, \ldots, v_k, v_{k+1}, \ldots, v_q) \\
&= \sum_{i=0}^{k}(-1)^i (v_0, \ldots, \hat{v}_i, \ldots, v_k, \ldots, v_q) + \sum_{i=0}^{k}\sum_{j=0}^{i-1}(-1)^{i+j+1} (v, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_k, \ldots, v_q) \\
&+ \sum_{i=0}^{k}\sum_{j=i+1}^{q}(-1)^{i+j}(v, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_k, \ldots, v_q)
\end{aligned}$$

And

$$\begin{aligned}
\chi\partial\sigma &= \chi\sum_{i=0}^{q}(-1)^i (v_0, \ldots, \hat{v}_i, \ldots, v_q) \\
&= \sum_{j=0}^{k}(-1)^i (v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}, \ldots, v_q) + \sum_{i=k+1}^{q}\sum_{j=0}^{k}(-1)^{i+j} (v, v_0, \ldots, \hat{v}_j, \ldots, v_k, \ldots, \hat{v}_i, \ldots, v_q)
\end{aligned}$$