ASSIGNMENT 7

JONAS TREPIAKAS

Problem 0.1. Let F be the homotopy fibre of the map $S^n \to S^n$ of degree k, for $n \geq 2$.

(1) Show that $H^i(F) = 0$ for $0 < i < n$.
(2) Using the Serre spectral sequence, compute that
$$H^i(F) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/k, & i = 1 + m(n-1),\ m > 0 \\ 0, & \text{otherwise} \end{cases}.$$
(3) Show that for $x, y \in H^*(F)$, if $\deg(x), \deg(y) > 0$, then $x \smile y = 0$.

Proof. (1) Since $\pi_1 S^n = 0$, the Serre spectral sequence to the homotopy fiber sequence
$$F \to S^n \to S^n$$
gives the following double complex:

$$\begin{tikzcd}
H^{n-1}(F) \arrow[d, dash] \arrow[rdddd] & & \\
\vdots \arrow[d, dash] & & \\
H^2(F) \arrow[d, dash] & & \\
H^1(F) \arrow[d, dash] & & \\
\mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \cdots \\
 & n &
\end{tikzcd}$$

We apply the LSSS for cohomology and find that $H^i(S^n) = F_0^n$, and since $H^i(F)$ is the only nontrivial entry on the antidiagonal in degree i, and since there are no maps to kill off $H^i(F)$ for $0 < i < n-1$, we obtain that $H^i(F) = H^i(S^n) = 0$ for $0 < i < n-1$.
All that's missing is $i = n-1$. For this, note that by the LES for the fibration, we get the following exact sequence:
$$\underbrace{\mathbb{Z}}_{\pi_n(S^n)} \xrightarrow{\cdot k} \underbrace{\mathbb{Z}}_{\pi_n(S^n)} \to \pi_{n-1}(F) \to \underbrace{0}_{\pi_{n-1}(S^n)}$$

hence $\pi_{n-1}(F) \cong \operatorname{coker}\left(\mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z}\right) \cong \mathbb{Z}/k$, and by the Hurewicz theorem, we get $H_{n-1}(F) \cong \pi_{n-1}(F) \cong \mathbb{Z}/k$. Now using the UCT, we obtain

$$0 \to \underbrace{\operatorname{Ext}\left(H_{n-2}(F), \mathbb{Z}\right)}_{=0} \to H^{n-1}(F) \to \underbrace{\operatorname{Hom}\left(\underbrace{H_{n-1}(F)}_{=\mathbb{Z}/k}, \mathbb{Z}\right)}_{=0} \to 0$$

so $H^{n-1}(F) = 0$ as we wanted.
(2)
By the LSSS, the E^{∞} page has the form $E^{\infty}_{0,0} = E^{\infty}_{n,0} \cong \mathbb{Z}$, so in particular, on the E^k page, we get the following double complex:

$$\begin{tikzcd}
\vdots \arrow[d, dash] & \\
H^{3(n-1)}(F) \arrow[d, dash] \arrow[dr] & H^{3(n-1)}(F) \\
H^{2(n-1)}(F) \arrow[d, dash] \arrow[dr] & H^{2(n-1)}(F) \\
\mathbb{Z} \cong H^{n-1}(F) \arrow[d, dash] \arrow[dr] & \mathbb{Z} \cong H^{n-1}(F) \\
\mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

This is the only page on which the horizontal maps can be nontrivial, so given the E^{∞} page, we conclude that the maps must be isomorphisms (including the trivial ones by just inductively shifting down by $n-1$ enough times). Hence we get periodicity, so

$$H^i(F) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/k, & i = 1 + m(n-1),\ m > 0 \\ 0, & \text{otherwise} \end{cases},$$

which was what we wanted to show.

(3) Suppose $\deg(x) + \deg(y) = 2$ so both are of degree 1, then since $H^1(F) = 0$, we have $x = 0 = y$ so $x \smile y = 0$. Suppose we have shown it for $\deg(x) + \deg(y) \leq N - 1$ now. If $\deg x + \deg y = N$, then firstly we can assume $x, y \neq 0$ since otherwise $x \smile y = 0$. Hence $x \in H^{1+m(n-1)}(F)$ and $y \in H^{1+m'(n-1)}$, so $x \smile y \in H^{2+(m+m')(n-1)}(F) = 0$, so directly, $x \smile y = 0$.

□

Problem 0.2. Use the path-loop fibration to deduce the cohomology ring structure of $H^*(\Omega S^n)$ when $n \geq 2$ is even.

Proof. Consider the path-loop fibration $\Omega S^n \to PS^n \to S^n$. Since S^n is simply connected, we can use the LSSS. Since $H^*(S^n)$ and $H^*(PS^n)$ are free and finitely generated, there is no torsion, so $E_2^{s,t} = H^s(S^n) \otimes H^t(\Omega S^n)$. Thus we obtain the following E_n page:

$$
\begin{tikzcd}
\vdots \arrow[d, dash] & \\
H^{3(n-1)}(\Omega S^n) \arrow[d, dash] \arrow[dr] & H^{3(n-1)}(\Omega S^n) \\
H^{2(n-1)}(\Omega S^n) \arrow[d, dash] \arrow[dr] & H^{2(n-1)}(\Omega S^n) \\
H^{n-1}(\Omega S^n) \arrow[d, dash] \arrow[dr] & H^{n-1}(\Omega S^n) \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \\
 & n
\end{tikzcd}
$$

Again, using that E_∞ has only $E_\infty^{0,0} = \mathbb{Z}$ and all other entries 0, we obtain that all the maps must be isomorphisms in this diagram.

Thus we can write $H^{k(n-1)}(\Omega S^n) \cong \mathbb{Z}a_k$ for all $k \geq 1$, where $d(a_1) = x$ and $d(a_k) = a_{k-1}x$ (we can change some a_k 's for their negatives to make signs check out if necessary) - here we also choose a_1 and then a_2 to satisfy the relation, and then a_3, etc.

Recall now from the multiplicative structure that we have $d(xy) = (dx)y + (-1)^{p+q}x(dy)$, so

Now, $|a_1| = n - 1$ which is odd as n was assumed to be even, so $2a_1^2 = 0$ by anticommutativity, so since $H^{2(n-1)}(\Omega S^n) \cong \mathbb{Z}a_2$ is torsion-free, $a_1^2 = 0$.

Now that we have picked generators for $H^*(\Omega S^n)$, we want to see if we can reduce the generators and find relations between them.

So far, we have chosen the a_i such that $a_i \smile a_j = A(i,j)a_{i+j}$ for some integer $A(i,j)$, simply because $a_i \smile a_j \in H^{i+j}(\Omega S^n)$ which a_{i+j} generates.

So we are interested in finding these coefficients $A(i,j)$.

Now we have, first of all, that $a_k x = x a_k$ since $|a_k||x|$ is even, so these commute for all k.

- $d(a_1 a_{2k}) = x a_{2k} + (-1)^{2k+1} a_1 a_{2k-1} x$. Now if $k = 1$, then $a_1 a_{2k-1} = a_1^2 = 0$, so $x a_2 - a_1^2 x$ becomes $x a_2$ and since $d(a_3) = x a_2$, we have $a_1 a_2 = a_3$ as d is an isomorphism. Now, inductively, suppose $a_1 a_{2k} = a_{2k+1}$ for $k \leq N - 1$. Then again $d(a_1 a_{2N}) = x a_{2N} - a_1 a_{2N-1} x$ and by induction, either $2N - 1 = 1$ such that $a_1 a_{2N-1} = a_1^2 = 0$ or $a_1 a_{2N-1} = a_1 a_1 a_{2(N-1)} = 0$. Hence $d(a_1 a_{2N}) = x a_{2N}$, so again $d(a_{2N+1}) = d(a_1 a_{2N})$, so $a_{2N+1} = d_1 a_{2N}$.
- Next we have $d_n\left(a_2^k\right) = a_1 x a_2^{k-1} + a_2 d\left(a_2^{k-1}\right)$. For $k = 1$, this equals $a_1 x = k a_1 x a_2^{k-1}$, so we claim that $d_n\left(a_2^k\right) = k a_1 x a_2^{k-1}$ in general. This is obtain by applying the inductive step to $a_2 d\left(a_2^{k-1}\right)$ above to obtain $d_n\left(a_2^k\right) = a_1 x a_2^{k-1} + a_2(k-1)a_1 x a_2^{k-2} = a_1 a_2^{k-1} x k$, as claimed. Next, inductively, we find that $a_2^{k-1} = (k-1)! a_{2k-2}$, so then $d_n(a_2^k) = k! a_1 x a_{2k-2} = k! a_{2k-1} x =$

$k!d_n\left(a_{2k}\right) = d_n\left(k!a_{2k}\right)$, so again since d_n is an isomorphism, $a_2^k = k!a_{2k}$. Clearly, $a_2^{k-1} = (k-1)!a_{2k-2}$ is true for $k = 2$, and then $d_n(a_2^k) = ka_1xa_2^{k-1} = k!a_1xa_{2k-2} = k!xa_{2k-1} = d\left(k!a_{2k}\right)$ gives $a_2^k = k!a_{2k}$ inductively as desired.

Hence $H^*\left(\Omega S^n\right) \cong \Lambda_{\mathbb{Z}}\left[a\right] \otimes \Gamma_{\mathbb{Z}}\left[b\right]$ with $|a| = n-1$ and $|b| = 2n-2$.

$\square$