INRODUCTION TO SMOOTH MANIFOLDS - LEE

1. Smooth Manifolds

1.1. **Topological Manifolds.**

Definition 1.1 (Topological manifold). Suppose M is a topological space. We say that M is a topological manifold of dimension n or a topological n-manifold if it is a second-countable Hausdorff space which is locally Euclidean of dimension n, where the last criterion means that each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

Exercise 1.2. Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to *any* open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Solution. Suppose $p \in U$ is homeomorphic to an open subset $\hat{U} \subset \mathbb{R}^n$ by a map $\varphi \colon U \to \hat{U}$. Then there exists some $\varepsilon > 0$ such that $B\left(\varphi(p), \varepsilon\right) \subset \varphi(U)$, so $p \in \varphi^{-1}\left(B\left(\varphi(p)\right), \varepsilon\right) \subset U$, and as φ is a homeomorphism, $\varphi^{-1}\left(B\left(\varphi(p), \varepsilon\right)\right)$ is open and $\varphi|_{\varphi^{-1}(B(\varphi(p),\varepsilon))} \colon \varphi^{-1}\left(B\left(\varphi(p), \varepsilon\right)\right) \to B\left(\varphi(p), \varepsilon\right)$ is a homeomorphism.
We claim that $B\left(0, r\right) \approx \mathbb{R}^n$.
First we show that $[0, r) \approx [0, \infty)$.
Define $f(x) = \frac{x}{r-x}$. This is continuous. Define $g(x) = \frac{rx}{1+x}$. Then g is a continuous map $[0, \infty) \to [0, r)$ and

$$f \circ g(x) = \frac{rx}{(r - \frac{rx}{1+x})(1+x)} = \frac{rx}{r + rx - rx} = x$$

and

$$g \circ f(x) = \frac{r\frac{x}{r-x}}{1 + \frac{x}{r-x}} = \frac{rx}{r - x + x} = x$$

so f is a homeomorphism.

Now define a map $\gamma \colon B\left(0, r\right) \to \mathbb{R}^n$ by

$$\gamma\left(x\right) = \frac{x}{r - \|x\|}$$

and $\alpha \colon \mathbb{R}^n \to B\left(0, r\right)$ by

$$\alpha(x) = \frac{xr}{1 + \|x\|}$$

Then

$$\gamma \circ \alpha(x) = \frac{xr}{(1 + \|x\|)\left(r - \|\frac{xr}{1+\|x\|}\|\right)} = \frac{xr}{r + \|x\|r - \|x\|r} = x$$

and

$$\alpha \circ \gamma(x) = \frac{xr}{(r - \|x\|)\left(1 + \|\frac{x}{r-\|x\|}\|\right)} = x$$

As γ is continuous and α is continuous, we see that $B\left(0, r\right) \approx \mathbb{R}^n$.

The converse is obviously trivial. $\qquad\square$

If M is a topological manifold, we often abbreviate the dimension of M as $\dim M$. Informally, one sometimes writes "Let M^n be a manifold" as shorthand for "Let M be a manifold of dimension n ".

It is important to note that every topological manifold has, by definition, a specific, well-defined dimension. Thus, we do not consider spaces of mixed dimension, such as the disjoint union of a plane and a line, to be

manifolds at all.

In chapter 17, we will use the theory of de Rham cohomology to prove the following theorem, which shows that the dimension of a (nonempty) topological manifold is, in fact, a topological invariant:

Theorem 1.3 (Topological invariance of dimension). *A nonempty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.*

Note. The empty set satisfies the definition of a topological n-manifold for every n. For the most part, we will ignore this special case but because it is useful in certain contexts to allow the empty manifold, we choose not to exclude it from the definition.

The basic example of a topological n-manifold is $\mathbb{R}^n$ itself. It is Hausdorff because it is a metric space, and it is second-countable because the set of all open balls with rational centers and rational radii is a countable basis for its topology.

Coordinate charts. Let M be a topological n-manifold. A **coordinate chart** (or just a **chart**) on M is a pair (U, φ) where U is an open subset of M and $\varphi \colon U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subset \mathbb{R}^n$.

By definition of a topological manifold, each point $p \in M$ is contained in the domain of some chart (U, φ). If $\varphi(p) = 0$, we say that the chart is **centered at** p.

Given a chart (U, φ), we call the set U a **coordinate domain**, or a **coordinate neighborhood** of each of its points. If, in addition, $\varphi(U)$ is an open ball in $\mathbb{R}^n$, then U is called a **coordinate ball**; if $\varphi(U)$ is an open cube, U is a **coordinate cube.** The map φ is called a **(local)** coordinate map, and the component functions $(x^1, \ldots, x^n)$ of φ, defined by $\varphi(p) = (x^1(p), \ldots, x^n(p))$ are called **local coordinates** on U.

Exercise 1.4. Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Proof. Suppose $[x], [y] \in \mathbb{RP}^n$ are distinct linear subspaces. Then $(\varphi_{n+1}([x]), 1) = (x^1, \ldots, x^n, 1) \neq (y^1, \ldots, y^n, 1) = (\varphi_{n+1}([y]), 1)$. There must be some i such that $x^i \neq y^i$, so take some open basis element $\tilde{U}_{x,i}$ around x^i and $\tilde{V}_{y,i}$ around y^i that are disjoint. Take some basis element $\tilde{U}_{x,1} \times \ldots \times \tilde{U}_{x,n}$ and $\tilde{V}_{y,1} \times \ldots \times \tilde{V}_{y,n}$ around $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$, respectively.

Claim: $U = \pi^{-1}\left(\pi\left(\tilde{U}_{x,1} \times \ldots \times \tilde{U}_{x,n} \times \{1\}\right)\right)$ is open. Suppose $(z^1, \ldots, z^{n+1}) \in U$ so $[z^1, \ldots, z^{n+1}] \in \pi\left(\tilde{U}_{x,1} \times \ldots \times \tilde{U}_{x,n} \times \{1\}\right)$. Then $z^{n+1} \neq 0$ and $z^i = u_{x,i} z^{n+1}$. Now $\mu_i(x) = \frac{x^i}{x^{n+1}} \colon \tilde{U}_i \to \mathbb{R}$ is a continuous map and $U' = \mu_1^{-1}\left(\tilde{U}_{x,1}\right) \cap \ldots \cap \mu_n^{-1}\left(\tilde{U}_{x,n}\right)$ is an open set around $\left(\frac{z^1}{z^{n+1}}, \ldots, \frac{z^n}{z^{n+1}}\right)$, so as φ_{n+1} is a homeomorphism, $\varphi_{n+1}^{-1}(U')$ is an open set around $[z^1, \ldots, z^{n+1}]$, so $\pi^{-1}\left(\varphi_{n+1}^{-1}(U')\right)$ is an open set around $(z^1, \ldots, z^{n+1})$ which, by construction, is contained in U.

Thus $\pi^{-1}\left(\pi\left(\tilde{U}_{x,1} \times \ldots \times \tilde{U}_{x,n}\right) \times \{1\}\right)$ and $\pi^{-1}\left(\pi\left(\tilde{V}_{y,1} \times \ldots \times \tilde{V}_{y,n}\right) \times \{1\}\right)$ are disjoint open sets around $(x^1, \ldots, x^n, 1)$ and $(y^1, \ldots, y^n, 1)$, respectively. Their images under φ_{n+1} are then open sets around $[x]$ and $[y]$, respectively. $\qquad \square$

Proof. Let $[x], [y] \in \mathbb{RP}^n$ be distinct. Then $\frac{x}{\|x\|} \neq \frac{y}{\|y\|}$. Define $\rho \colon \mathbb{R}^{n+1} - \{0\} \to S^n$ by $\rho(x) = \frac{x}{\|x\|}$. Since S^n is Hausdorff, we can find open sets U, V around $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$, respectively. Since ρ is continuous, $\rho^{-1}(U)$ and $\rho^{-1}(V)$ are open and contain x and y. We claim that they are saturated with respect to $\pi \colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$. Suppose $w \in \pi^{-1}\left(\pi\left(\rho^{-1}(U)\right)\right)$. Then $[w] \in \pi\left(\rho^{-1}(U)\right)$. So there exists $u \in U$ such that $\pi\left(\rho^{-1}(u)\right) = [w]$. Then clearly, $u = \frac{w}{\|w\|}$, so $w = u\|w\| \in \rho^{-1}(U)$. Hence $\pi\left(\rho^{-1}(U)\right)$ and $\pi\left(\rho^{-1}(V)\right)$ are open sets around $[x]$ and $[y]$, respectively. $\qquad \square$

Now, define an action on S^n by $\mathbb{Z}_2 \times S^n \to S^n$ by $(0, x) = x$ and $(1, x) = -x$. Then the orbit space is second countable since the quotient map is open, so as the orbit space is $\mathbb{RP}^n$, we are done.

Alternatively, for the Hausdorff condition, we can also say that $\mathbb{Z}_2$ acts properly on S^n and since if a group G acts properly on a Hausdorff space X then X/G is Hausdorff, we conclude that $\mathbb{RP}^n$ is Hausdorff.

Exercise 1.5. Show that $\mathbb{RP}^n$ is compact.

Proof. Let $[x] \in \mathbb{RP}^n$. Then $\frac{x}{\|x\|} \in S^n$, so $[x] = \pi(x) = \pi\left(\frac{x}{\|x\|}\right) \in \pi(S^n)$, so $\pi|_{S^n}\colon S^n \to \mathbb{RP}^n$ is surjective, and since the image of a compact space under a continuous map is compact, we find that $\mathbb{RP}^n$ is compact.

In conclusion, $\mathbb{RP}^n$ is a compact n-manifold. $\qquad\square$

Example 1.6 (Product Manifolds)**.** Suppose $M_1, \ldots, M_k$ are topological manifolds of dimensions $n_1, \ldots, n_k$, respectively. The product space $M_1 \times \ldots \times M_k$ is shown to be a topological manifold of dimension $n_1 + \ldots + n_k$ as follows. It is Hausdorff and second-countable as subspaces of such spaces inherit these properties. Now, given any point $(p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k$, we can choose a coordinate chart (U_i, φ_i) for each M_i with $p_i \in U_i$. The product map

$$\varphi_1 \times \ldots \times \varphi_k \colon U_1 \times \ldots \times U_k \to \mathbb{R}^{n_1 + \ldots + n_k}$$

is a homeomorphism onto its image, which is a product open subset of $\mathbb{R}^{n_1 + \ldots + n_k}$. Thus $M_1 \times \ldots \times M_k$ is a topological manifold of dimension $n_1 + \ldots + n_k$, with charts of the form $(U_1 \times \ldots \times U_k, \varphi_1 \times \ldots \times \varphi_k)$.

Example 1.7 (Torus)**.** For a positive integer n, the n-**torus** is the product $\mathbb{T}^n = \underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}} = \Pi_{i=1}^n S^1$.

By example 1.8, $\mathbb{T}^n$ is a topological n-manifold.

Locally compact Hausdorff spaces. A topological space X is said to be **locally compact** if every point has a neighborhood contained in a compact subset of X. If X is Hausdorff, this property has two equivalent formulations that are often more useful, as the next exercise shows. A subset of X is said to be **precompact in** X if its closure in X is compact.

Exercise 1.8. For a Hausdorff space X, show that the following are equivalent:

    (1) X is locally compact.
    (2) Each point of X has a precompact neighborhood.
    (3) X has a basis of precompact open subsets.

Proof. (1) $\implies$ (2): A subset of a compact set is compact. Suppose $x \in X$. Then it has a neighborhood U contained in a compact subset C of X, so in particular, since compact subsets of Hausdorff spaces are closed, $\overline{U} \subset C$. So U is a precompact neighborhood of x in X.

(2) $\implies$ (3) : For each point $x \in X$, let U_x denote a precompact neighborhood of x in X. We claim that if $\mathcal{B}_x = \{U_x \cap V \mid V \text{ open in } X\}$, then $\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$ is a basis for X.
Suppose U is open in X. Then $U = \bigcup_{x \in U} U_x \cap U$ which is in the topology generated by $\mathcal{B}$; hence the topologies are equal, so $\mathcal{B}$ is a basis of precompact open subsets.

(3) $\implies$ (1) : clear. $\qquad\square$

Exercise 1.9. Prove that every open or closed subspace of a locally compact Hausdorff space is itself a locally compact Hausdorff space.

Proof. Suppose $X \subset Z$ where Z is a locally compact Hausdorff space, and X is an open subspace. Let $x \in X$, then x has a neighborhood U which is contained in a compact subset C of Z. But then $U \cap X$ is an open set in X which is contained in $C \cap X$ which is compact. Hence X is locally compact.
If X were closed instead, then let $x \in X$. By exercise A.55, x has a precompact neighborhood U in Z, so $U \cap Z$ is an open neighborhood of x in X, and since Z is Hausdorff, $\overline{U}$ which is compact is closed, so $\overline{U} \cap Z$ is closed and compact and contains $U \cap Z$ which contains x. Hence X is locally compact Hausdorff. $\qquad\square$

Topological properties of Manifolds.

Lemma 1.10. *Every topological manifold has a countable basis of precompact coordinate balls.*

Proof. Let M be a topological m-manifold. First, we consider the special case in which M can be covered by a single chart. Suppose $\varphi\colon M \to \hat{U} \subset \mathbb{R}^n$ is a global coordinate map, and let $\mathcal{B}$ be the collection of all open balls $B_r(x) \subset \mathbb{R}^n$ such that r is rational, x has rational coordinates, and $B_{r'}(x) \subset \hat{U}$ for some $r' > r$. Each such

ball is precompact in $\hat{U}$, and it is easy to check that $\mathcal{B}$ is a countable basis for the topology of $\hat{U}$. Now, because φ is a homeomorphism, it follows that the collection of sets of the form $\varphi^{-1}(B)$ for $B \in \mathcal{B}$ is a countable basis for the topology of M, consisting of precompact coordinate balls, with the restriction of φ as coordinate maps.

Now let M be an arbitrary n-manifold. By definition, each point of M is in the domain of a chart. Because every open cover of a second-countable space has a countable subcover, M is covered by countably many charts $\{(U_i, \varphi_i)\}$. By the argument in the preceding parahraph, each coordinate domain U_i has a countable basis of coordinate balls that are precompact in U_i, and the union of these countable bases is a countable basis for the topology of M. If $V \subset U_i$ is one of these balls, then the closure of V in U_i is compact, and because M is Hausdorff, it is closed in M. It follows that the closure of V in M is the same as its closure in U_i, so V is precompact in M as well. $\hfill\square$

Connectivity.

Proposition 1.11. *Let M be a topological manifold. Then*

(1) *M is locally path-connected.*
(2) *M is connected if and only if it is path-connected.*
(3) *The components of M are the same as its path components.*
(4) *M has countably many components, each of which is an open subset of M and a connected topological manifold.*

Proof. The equivalence of $(1), (2)$ and (3) is clear.
Now, by Lemma 1.10, there exists a countable basis of precompact coordinate balls. $\hfill\square$

Local Compactness and Paracompactness.

Proposition 1.12 (Manifolds Are Locally Compact)**.** *Every topological manifold is locally compact.*

Definition 1.13. Let M be a topological space. A collection $\mathcal{A}$ of subsets of M is said to be **locally finite** in M if every point of M has a neighborhood that intersects only finitely many elements of A.

Definition 1.14. Let $\mathcal{A}$ be a collection of subsets of the space M. A collection $\mathcal{B}$ of subsets of M is said to be a **refinement** of $\mathcal{A}$ if for each element B of $\mathcal{B}$, there exists an element A of $\mathcal{A}$ containing B.

Definition 1.15 (Paracompactness)**.** A space M is **paracompact** if every open covering $\mathcal{A}$ of M has a locally finite open refinement $\mathcal{B}$ that covers M.

Lemma 1.16. *Suppose $\mathcal{A}$ is a locally finite collection of subsets of a topological space M.*

(1) *The collection $\{\overline{X} \colon X \in \mathcal{A}\}$ is also locally finite.*
(2) *$\overline{\bigcup_{x \in \mathcal{A}} X} = \bigcup_{X \in \mathcal{A}} \overline{X}$.*

Proof. For (1), suppose for contradiction that there exists an $x \in M$ such that for every neighborhood U of x, U intersects infinitely many elements of the collection $\{\overline{X} \colon X \in \mathcal{A}\}$. Since each elements of this collection is closed, it contains its limit points, and hence x is a limit point of infinitely many $X \in \mathcal{A}$, but this means that for any neighborhood U of x, this neighborhood intersects infinitely many elements of $\mathcal{A}$; contradiciton.

For (2), note that $\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{A}} X}$ is clear. Now let $x \in \overline{\bigcup_{X \in \mathcal{A}} X}$. As M is locally finite, there exists some neighborhood U of x such that U only intersects a finite subset of $\mathcal{A}$, say $X_1, \ldots, X_n$. Now, suppose there exist some neighborhoods $V_1, \ldots, V_n$ of x such that $V_i \cap X_i = \varnothing$. Then $U \cap V_1 \cap \ldots \cap V_n$ is a neighborhood of x that intersects none of the X_i, for $i = 1, \ldots, n$, and hence none of the subsets in $\mathcal{A}$. But then $x \notin \overline{\bigcup_{X \in \mathcal{A}} X}$, so there must exist some X_i for $i \in \{1, \ldots, n\}$ such that all neighborhoods of x intersect X_i. Hence $x \in \overline{X_i} \subset \bigcup_{X \in \mathcal{A}} \overline{X}$. $\hfill\square$

Definition 1.17 (Exhaustion by compact sets)**.** A sequence $(K_i)_{i=1}^{\infty}$ of compact subsets of a topological space X is called an **exhaustion of X by compact sets** if $X = \bigcup_i K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$ for each i.

Proposition 1.18. *A second-countable, locally compact Hausdorff space admits an exhaustion by compact sets.*

Proof. By exercise 1.8, X has a basis of precompact open subsets, say $\mathcal{B}$. Since X is second-countable, we can replace $\mathcal{B}$ by a countable subcover $\mathcal{B} = \{B_i\}_{i \in \mathbb{Z}_+}$.

Now define a sequence of sets as follows: Let $A_1 = \overline{B_1}$. Suppose we have defined A_n for $n \leq N$. We wish to define A_{N+1}. Now, $A_1 \cup \ldots \cup A_N$ is the finite union of compact sets and hence compact as well, so there exists some finite subcollection $\{B_{n_1}, \ldots, B_{n_k}\} \subset \mathcal{B}$ such that $A_1 \cup \ldots \cup A_N \subset \bigcup_{i=1}^k B_{n_i}$. Then define $A_{N+1} = \bigcup_{i=1}^k \overline{B_{n_i}} \cup \bigcup_{j=1}^{N+1} \overline{B_j}$. Then A_{N+1} is compact and contains A_N. Furthermore, $X = \bigcup_i B_i \subset \bigcup_i A_i$ since $B_i \subset A_i$. Hence $\{A_i\}$ is an exhaustion of X by compact sets. $\qquad\square$

Theorem 1.19 (Manifolds Are Paracompact). *Every topological manifold is paracompact. In fact, given a topological manifold M, an open cover $\mathcal{A}$ of M, and any basis $\mathcal{B}$ for the topology of M, there exists a countable, locally finite open refinement of $\mathcal{A}$ consisting of elements of $\mathcal{B}$.*

Proof. Since manifolds are second-countable, locally compact Hausdorff spaces, they admit an exhaustion by compact sets according to proposition 1.18, so let $\mathcal{K} = \{K_i\}$ be such an exhaustion. Let $\mathcal{A}$ be an open cover of M and $\mathcal{B}$ a basis for the topology of M. Define $V_i := K_i - \operatorname{Int} K_i$ which is a closed (compact sets in a Hausdorff space are closed, so this difference is closed) subset of a compact space and hence compact. Hence $K_i \subset V_i$ but $K_i \not\subset V_j$ when $i < j$, and $\bigcup_{i \in \mathbb{Z}_+} V_i = M$. Suppose $x \in V_n$. Then $V_n \subset \operatorname{Int} K_{n+1}$, so in particular, $x \in M - \bigcup_{i=n+1}^\infty V_i$ which is an open set, call it U'. Now, since $\bigcup_{i=1}^n V_i$ is compact, we can cover it by some $A_1, \ldots, A_m \in \mathcal{A}$. Each of these can be written as a union of elements of $\mathcal{B}$, so $\bigcup_{i=1}^n V_i = \bigcup_{j \in J_n \subset \mathbb{Z}_+} B_j$ where J_n is a finite subset. Then U' intersects at most all of these B_j, and hence only a finite number. $\qquad\square$

Fundamental Groups of Manifolds.

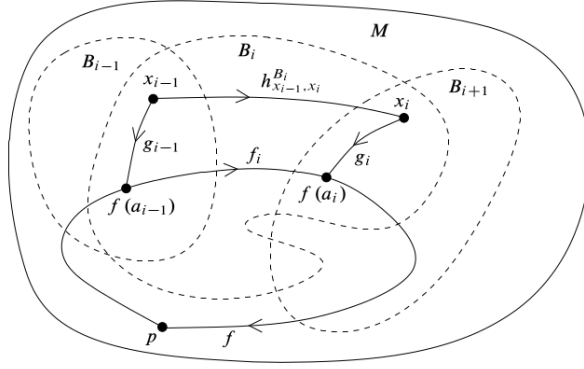**Proposition 1.20.** *The fundamental group of a topological manifold is countable.*

Proof. Let M be a topological manifold. By Lemma 1.10, there is a countable collection $\mathcal{B}$ of precompact coordinate balls covering M. By 1.11, each nonempty intersection $B_i \cap B_j$ for $B_i, B_j \in \mathcal{B}$ has countable many components, each of which is a connected topological manifold, and thus path-connected. Thus, the collection $\{B_i \cap B_j \mid B_i, B_j \in \mathcal{B}\}$ is countable as well. Suppose for each component in an element of $\{B_i \cap B_j \mid B_i, B_j \in \mathcal{B}\}$, we choose a point and let the collection of all these points be denoted by $\mathcal{A}$ - hence $\mathcal{A}$ is in particular countable. Now, for each $B \in \mathcal{B}$ and any two $x, x' \in \mathcal{A}$ with $x, x' \in B$, let $h^B_{x,x'}$ denote a path from x to x' in B.

Since fundamental groups are invariant under change of basepoint in a path-connected space, we may reduce the question to choosing basepoints in $\mathcal{A}$. Let $p \in \mathcal{A}$. Define a **special loop** to be a loop composed of a finite product of paths of the form $h^B_{x,x'}$. The set of such loops is countable. Thus if we can show that any loop in M is homotopy equivalent to a special loop, we are done.

Suppose $f \colon I \to M$ is a loop based at p. Then $\{f^{-1}(B) \mid B \in \mathcal{B}\}$ is an open cover of I, so there exists a finite subcover, say $[0, t_1] = f^{-1}(B_1), \ldots, [t_{i-1}, t_i] = f^{-1}(B_i), \ldots, [t_{n-1}, 1] = f^{-1}(B_n)$. Then $f(t_i) \in B_i \cap B_{i+1}$. Let α_i be a path from $f(t_i)$ to the $x_i \in \mathcal{A} \cap (B_i \cap B_{i+1})$ which lies in the same component as $f(t_i)$. Let f_i be the restriction of f to $[t_{i-1}, t_i]$ reparametrized to have domain I. Then

$$
\begin{aligned}
f &\simeq f_1 \cdot \ldots \cdot f_n \\
&\simeq f_1 \cdot \alpha_1 \cdot \alpha_1^{-1} \cdot f_2 \cdot \ldots \cdot \alpha_{n-1} \cdot \alpha_{n-1}^{-1} \cdot f_n \\
&\simeq h^{B_1}_{p, x_1} \cdot h^{B_2}_{x_1, x_2} \cdot \ldots \cdot h^{B_n}_{x_{n-1}, p}
\end{aligned}
$$

$\qquad\square$

Smooth Structures. We have the foundations to study topological properties of topological manifolds, however, we want to introduce certain properties from calculus such as derivatives to manifolds.

However, derivatives in the typical sense don't make sense since they are not invariant under homeomorphisms. For example, the map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(u,v) = \left(u^{\frac{1}{3}}, v^{\frac{1}{3}}\right)$ is a homeomorphism, but while $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x$ is differentiable everywhere, the map $f \circ \varphi$ is not differentiable at the origin.

To make sense of derivatives of real-valued functions, curves, or maps between manifolds, we need to introduce a new kind of manifold called a *smooth manifold*. It will be a topological manifold with some extra structure in addition to its topology.

The definition will be based on the calculus of maps between Euclidean spaces, so let us begin by reviewing some basic terminology about such maps.

Definition 1.21 (Smooth maps). If U and V are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F\colon U \to V$ is said to be **smooth** (or C^∞, or **infinitely differentiable**) if each of its component functions has continuous partial derivatives of all orders.

Definition 1.22 (Diffeomorphism). If a smooth map $F\colon U \to V$ is bijetive and has a smooth inverse map, it is called a **diffeomorphism**.

To see what additional structure on a topological manifold might be appropriate for discerning which maps are smooth, consider an arbitrary topological n-manifold M. Each point in M is in the domain of a coordinate map $\varphi\colon U \to \hat{U} \subset \mathbb{R}^n$.

A plausible definition of a smooth function on M would be to say that $f\colon M \to \mathbb{R}$ is smooth if and only if the composite function $f \circ \varphi^{-1}\colon \hat{U} \to \mathbb{R}$ is smooth in the sense of ordinary calculus. But this will make sense only if this property is independent of the choice of coordinate chart. Since smoothness is not a homeomorphism-invariant property, we consider the collection of all smooth charts as a new kind of structure on M.

With this in mind, we describe the construction:

Definition 1.23 (Transition maps). Let now M be a topological n-manifold. If (U, φ) and (V, ψ) are two charts such that $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ is called the **transition map from φ to ψ**.

It is a composition of homeomorphisms and therefore a homeomorphism.

Definition 1.24 (Smoothly compatible). Two charts (U, φ) and (V, ψ) are said to be **smoothly compatible** if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

Definition 1.25 (Atlas). An **atlas for** M is a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a **smooth atlas** if any two charts in $\mathcal{A}$ are smoothly compatible with each other.

To show that an atlas is smooth, we thus need to verify that each transition map $\psi \circ \varphi^{-1}$ is smooth whenever (U, φ) and (V, ψ) are charts in $\mathcal{A}$; then it follows that $\psi \circ \varphi^{-1}$ is a diffeomorphism because the transition map $\varphi \circ \psi^{-1}$ is clearly its inverse and also a transition map, hence smooth. Therefore $\psi \circ \varphi^{-1}$ is smooth and

has a smooth inverse, and is bijective on $\varphi\left(U \cap V\right) \to \psi\left(U \cap V\right)$ since it is a homeomorphism.

Remark. Alternatively, given two particular charts (U, φ) and (V, ψ), it is often easier to show that they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth and injective with nonsingular Jacobian at each point which gives that $\psi \circ \varphi^{-1}$ is a diffeomorphism onto its image by Corollary A.20.

Our plan now is to define a "smooth structure" on M by defining a smooth atlas, and to define a function $f \colon M \to \mathbb{R}$ to be smooth if and only if $f \circ \varphi^{-1}$ is smooth in the sense of ordinary calculus for each coordinate chart (U, φ) in the atlas. There is one problem though: in general, there will be many possible atlases that give the "same" smooth structure, in that they all determine the same collection of smooth functions on M. For example, consider the following pair of atlases on $\mathbb{R}^n$:

$$\mathcal{A}_1 = \left\{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})\right\}$$
$$\mathcal{A}_2 = \left\{\left(B_1(x), \mathrm{id}_{B_1(x)}\right) : x \in \mathbb{R}^n\right\}.$$

Although these are different smooth atlases, a function $f \colon \mathbb{R}^n \to \mathbb{R}$ is smooth with respect to either atlas if and only if it is smooth in the sense of ordinary calculus.

We could choose to define a smooth structure as an equivalence class of smooth atlases under an appropriate equivalence relation. However, it is more straightforward to make the following definition:

Definition 1.26 (Maximal smooth atlas)**.** A smooth atlas $\mathcal{A}$ on M is **maximal** if it is not properly contained in any larger smooth atlas.
(Such a smooth atlas is also said to be **complete**)

Now we can define the main concept of this chapter.

Definition 1.27 (Smooth structure)**.** If M is a topological manifold, a **smooth structure on** M is a maximal smooth atlas.
Smooth structures are also called **differentiable structures** or C^∞ **structures** by some authors.

Definition 1.28 (Smooth manifold)**.** A **smooth manifold** is a pair $(M, \mathcal{A})$ where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.
We also use the term **smooth manifold structure** to mean a manifold topology together with a smooth structure.

Remark. It is not always possible to find a smooth structure on a given topological manifold: there exist topological manifolds that admit no smooth structure at all (the first example was a compact 10-dimensional manifold found in 1960 by Michel Kervaire).

Proposition 1.29. *Let M be a topological manifold.*

(1) *Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the **smooth structure determined by** $\mathcal{A}$.*
(2) *Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.*
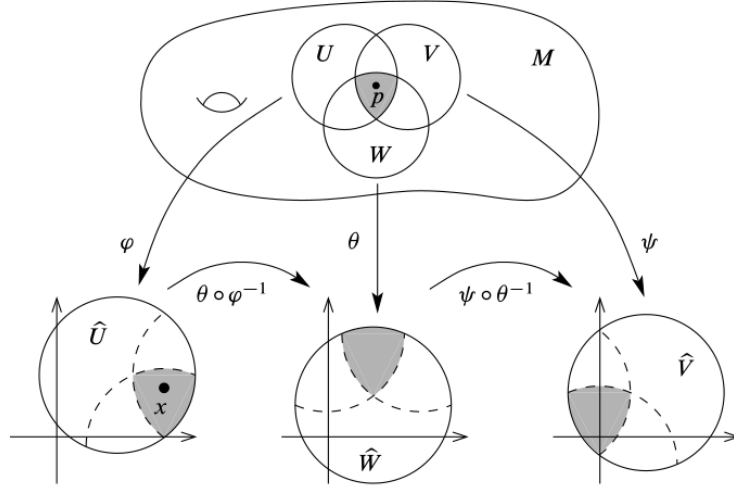
Proof. (1): Let $\mathcal{C}$ be the collection of all smooth atlases containing $\mathcal{A}$, i.e., $\forall C \in \mathcal{C} \colon \mathcal{A} \subset C$.

We claim that $\mathcal{B} = \bigcup_{C \in \mathcal{C}} C$ is a smooth atlas. To show this, let $(U, \varphi), (V, \psi) \in \mathcal{B}$. We must show that the two charts are smoothly compatible. Suppose $U \cap V \neq \varnothing$. Let $p \in U \cap V$. There exist smooth atlases $C, C' \in \mathcal{C}$ such that $(U, \varphi) \in C$ and $(V, \psi) \in C'$. Each of these contain $\mathcal{A}$ by definition, and since $\mathcal{A}$ is a smooth atlas, there exists some chart $(W, \theta) \in \mathcal{A}$ such that $p \in W$. By C being a smooth atlas and $p \in U \cap W$, the map $\varphi \circ \theta^{-1}$ is a diffeomorphism, and since C' is smooth too, the map $\theta \circ \psi^{-1}$ is a diffeomorphism. Thus the composition $\varphi \circ \psi^{-1} = \varphi \circ \theta^{-1} \circ \theta \circ \psi^{-1}$ is a diffeomorphism. Similarly, $\psi \circ \varphi^{-1}$ is a diffeomorphism. Thus $\mathcal{B}$ is a smooth atlas.

To check maximality, suppose $\mathcal{B}'$ is a smooth atlas containing $\mathcal{B}$. Then $\mathcal{B}'$ in particular contains $\mathcal{A}$, so $\mathcal{B}' \in \mathcal{C}$, so $\mathcal{B}' \subset \mathcal{B}$. Hence $\mathcal{B} = \mathcal{B}'$.

Suppose now $\mathcal{B}'$ is another maximal smooth atlas containing $\mathcal{A}$. Then in particular $\mathcal{B}' \in \mathcal{C}$, so $\mathcal{B}' \subset \mathcal{B}$, and by maximality of $\mathcal{B}'$, we have $\mathcal{B}' = \mathcal{B}$.

(2): Suppose $\mathcal{A}, \mathcal{A}'$ are smooth structures that determine the same smooth structure on M, say $\mathcal{B}$. Then $\mathcal{A}, \mathcal{A}' \subset \mathcal{B}$, so in particular, any chart of $\mathcal{A}$ is smoothly compatible with any chart of $\mathcal{A}'$, so $\mathcal{A} \cup \mathcal{A}'$ is a smooth atlas.

Conversely, if $\mathcal{A} \cup \mathcal{A}'$ is a smooth atlas, then by (a), $\mathcal{A}$ is contained in a unique maximal smooth atlas, call it $\mathcal{B}$, and similarly, $\mathcal{A}'$ is contained in a unique maximal smooth atlas, call it $\mathcal{B}'$, both of which was the set of charts which are smoothly compatible with every chart in $\mathcal{A}$ and $\mathcal{A}'$, respectively. I.e., $\mathcal{A} \cup \mathcal{A}' \subset \mathcal{B}, \mathcal{B}'$, so by uniqueness, $\mathcal{B} = \mathcal{B}'$. $\square$

Example 1.30. If a topological manifold M can be covered by a single chart, the smooth compatibility condition is trivially satisfied, so any such chart automatically determines a smooth structure on M.

Remark. If we replace the requirement that the carts be smoothly compatible by the weaker requirement that each transition map $\psi \circ \varphi^{-1}$ (and its inverse) be of class C^k, we obtain the definition of a C^k **structure**. Similarly, if we require that each transition map be real-analytic (expressible as a convergent power series in a neighborhood of each point), we obtain the definition of a **real-analytic** structure, also called a C^ω structure. If M has even dimension $n = 2m$, we can identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ and require that the transition maps be complex-analytic; this determines a **complex-analytic structure**. A manifold endowed with one of these structures is called a C^k **manifold, real-analytic manifold**, or **complex manifold**, respectively. (Note that a C^0 manifold is just a topological manifold).

Local Coordinate Representations.

Definition 1.31. If M is a smooth manifold, any chart (U, φ) contained in the given maximal smooth atlas is called a **smooth chart**, and the corresponding coordinate map φ is called a **smooth coordinate map**. It is useful also to introduce the terms **smooth coordinate domain** or **smooth coordinate neighborhood**

for the domain of a smooth coordinate chart. A **smooth coordinate ball** means a smooth coordinate domain whose image under a smooth coordinate map is a ball in Euclidean space. A **smooth coordinate cube** is defined similarly.

Definition 1.32 (Regular coordinate ball)**.** A set $B \subset M$ is a **regular coordinate ball** if there is a smooth coordinate ball $B' \supset \overline{B}$ and a smooth coordinate map $\varphi \colon B' \to \mathbb{R}^n$ such that for some positive real numbers $r < r'$,

$$\varphi(B) = B_r(0), \quad \varphi\left(\overline{B}\right) = \overline{B}_r(0), \quad \text{and} \quad \varphi\left(B'\right) = B_{r'}(0).$$

Because $\overline{B}$ is homeomorphic to $\overline{B}_r(0)$, it is compact, and thus every regular coordinate ball is precompact in M.

Proposition 1.33. *Every smooth manifold has a countable basis of regular coordinate balls.*

Proof. Let M be a smooth manifold of dimension n. Suppose first that M can be covered by one chart (U, φ). The balls at rational coordinates of $\mathbb{R}^n$ with rational radii are a basis for the topology of $\varphi(U)$ and also $\mathbb{R}^n$, so in particular, the preimage of these balls under φ is a basis for the topology on M.
Suppose now that we have a collection of charts $\{U_\alpha, \varphi_\alpha\}$ covering M. By the single chart case, each of these charts has a countable basis of regular coordinate balls, and since $\bigcup_\alpha U_\alpha$ covers covers M, and thus there is a countable union of countable bases, which is countable and a basis for M.
We must also argue that the closure of a regular coordinate ball in U_α is the same as its closure in M. Its closure is compact in U_α as the image under φ of its closure is compact and φ is a homeomorphism. So it is compact in M too and since M is Hausdorff, the closure in U_α is closed in M too, hence the closures are the same. $\square$

Note. We can represent a point $p \in M$ by its coordinates in a chart (U, φ): $\left(x^1, \ldots, x^n\right) = \varphi(p)$, and think of this n-tuple as *being* the point p. We typically express this by saying "$\left(x^1, \ldots, x^n\right)$ is the (local) coordinate representation for p " or "$p = \left(x^1, \ldots, x^n\right)$ in local coordinates".
Another way to look at it is that by means of our identification $U \leftrightarrow \hat{U}$, we can think of φ as the identity map and suppress it from notation. One needs to remember that the identification is in general only local, and depends heavily on the choice of coordinate chart.

Examples of Smooth Manifolds.

Example 1.34 (0-Dimensional Manifolds)**.** A topological manifold M of dimension 0 is just a countable discrete space. For each point $p \in M$, the only neighborhood of p that is homeomorphic to an open subset of $\mathbb{R}^0$ is $\{p\}$ itself, and there is exactly one coordinate map $\varphi \colon \{p\} \to \mathbb{R}^0$. Thus, the set of all charts on M trivially satisfies the smooth compatibility condition, and each 0-dimensional manifold has a unique smooth structure.

Example 1.35 (Euclidean Spaces)**.** For each nonnegative integer n, the Euclidean space $\mathbb{R}^n$ is a smooth n-manifold with the smooth structure determined by the atlas consisting of the single chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$. We call this the **standard smooth structure on** $\mathbb{R}^n$ and the resulting coordinate map **standard coordinates**. Unless we explicitly specify otherwise, we always use this smooth structure on $\mathbb{R}^n$. With respect to this smooth structure, the smooth coordinate charts for $\mathbb{R}^n$ are exactly those charts (U, φ) such that φ is a diffeomorphism (in the sense of ordinary calculus) from U to another open subset $\hat{U} \subset \mathbb{R}^n$.

Example 1.36 (A different smooth structure on $\mathbb{R}$)**.** Consider the homeomorphism $\psi \colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x) = x^3.$$

The atlas consisting of the single chart $(\mathbb{R}, \psi)$ defines a smooth structure on $\mathbb{R}$ that is different from the standard smooth structure because the transition map $\mathrm{id}_{\mathbb{R}} \circ \psi^{-1}(y) = y^{\frac{1}{3}}$ is not smooth at the origin. Using similar ideas, it is not hard to construct many distinct smooth structures on any given positive-dimensional topological manifold, as long as it has one smooth structure to begin with.

Example 1.37 (Finite-Dimensional Vector Spaces)**.** Let V be a finite-dimensional real vector space. Any norm on V determines a topology, which is independent of the choice of norm. With this topology, V is a

topological n-manifold, and has a natural smooth structure defined as follows. Each ordered basis $(e_1, \ldots, e_n)$ for V defines a basis isomorphism $e \colon \mathbb{R}^n \to V$ by

$$e(x) = \sum_{i=1}^{n} x^i e_i.$$

This map is a homeomorphism [Lee, p 638], so (V, e^{-1}) is a chart. If $(\tilde{e}_1, \ldots, \tilde{e}_n)$ is any other basis and $\tilde{e}(x) = \sum_j x^j \tilde{e}_j$ is the corresponding isomorphism, then there is some invertible matrix $\left(A_i^j\right)$ such that $e_i = \sum_j A_i^j \tilde{e}_j$ for each i. The transition map between the two charts is then given by $\tilde{e}^{-1} \circ e(x) = \tilde{x}$, where $\tilde{x} = \left(\tilde{x}^1, \ldots, \tilde{x}^n\right)$ is determined by

$$\sum_{j=1}^{n} \tilde{x}^j \tilde{E}_j = \sum_{i=1}^{n} x^i E_i = \sum_{i,j=1}^{n} x^i A_i^j \tilde{E}_j.$$

It follows that $\tilde{x}^j = \sum_i A_i^j x^i$. Thus, the map sending x to $\tilde{x}$ is an invertible linear map and hence a diffeomorphism, so any two such charts are smoothly compatible. The collection of all such charts thus defines a smooth structure, called the **standard smooth structure on V**.

Example 1.38 (Open submanifolds). Let M be a smooth n-manifold and let $U \subset M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subset U\}.$$

Every point $p \in U$ is contained in the domain of some chart (W, φ) for M; if we set $V = W \cap U$, then $(V, \varphi|_V)$ is a chart in $\mathcal{A}_U$ whose domain contains p. Therefore U is covered by the domains of charts in $\mathcal{A}_U$, and it is easy to verify that this is a smooth atlas for U. Thus any open subset of M is itself a smooth n-manifold in a natural way. Endowed with this smooth structure, we call any open subset an **open submanifold of M**. (We will define a more general class of submanifolds in chapter 5)

Example 1.39 (Spheres). The n-sphere $S^n \subset \mathbb{R}^{n+1}$ is a topological n-manifold. We put a smooth structure on S^n as follows. For each $i = 1, \ldots, n+1$, let $\left(U_i^\pm, \varphi_i^\pm\right)$ denote the graph coordinate charts $\varphi_i^\pm \colon U_i^\pm \cap S^n \to B^n$ given by

$$\varphi_i^\pm \left(x^1, \ldots, x^{n+1}\right) = \left(x^1, \ldots, \hat{x^i}, \ldots, x^{n+1}\right),$$

where $U_i^+ = \left\{\left(x^1, \ldots, x^{n+1}\right) \in \mathbb{R}^{n+1} \colon x^i > 0\right\}$ and U_i^- is defined similarly with $x^i < 0$.

For any indices i and j, the transition map $\varphi_i^\pm \circ \left(\varphi_j^\pm\right)^{-1}$ is easily computed. In the case $i < j$, we get

$$\varphi_i^\pm \circ \left(\varphi_j^\pm\right)^{-1} (u_1, \ldots, u_n) = \left(u_1, \ldots, \hat{u_i}, \ldots, \pm\sqrt{1 - |u|^2}, \ldots, u^n\right)$$

with the square root in the j th position. A similar formula holds for $i > j$. When $i = j$, we get $\varphi_i^+ \circ \left(\varphi_i^+\right)^{-1} = \varphi_i^- \circ \left(\varphi_i^-\right)^{-1} = \mathrm{id}_{B^n}$. Thus, the collection of charts $\left\{\left(U_i^\pm, \varphi_i^\pm\right)\right\}$ is a smooth atlas, and so defines a smooth structure on S^n. We call this its **standard smooth structure**.

Example 1.40 (Level sets). We can generalize the preceding example as follows. Suppose $U \subset \mathbb{R}^n$ is an open subset and $\Phi \colon U \to \mathbb{R}$ is a smooth function. For any $c \in \mathbb{R}$, the set $\Phi^{-1}(c)$ is called a **level set of Φ**. Choose some $c \in \mathbb{R}$, let $M = \Phi^{-1}(c)$, and suppose it happens that the total derivative $D\Phi(a)$ is nonzero for each $a \in \Phi^{-1}(c)$. Because $D\Phi(a)$ is a row matrix whose entries are the partial derivatives $\left(\frac{\partial \Phi}{\partial x^1}(a), \ldots, \frac{\partial \Phi}{\partial x^n}(a)\right)$, for each a there is some i such that $\frac{\partial \Phi}{\partial x^i}(a) \neq 0$.

It follows from the implicit function theorem (Theorem A.23) that there is a neighborhood U_0 of a such that $M \cap U_0$ can be expressed as a graph of an equation of the form

$$x^i = f\left(x^1, \ldots, \hat{x^i}, \ldots, x^n\right),$$

for some smooth real-valued f defined on an open subset of $\mathbb{R}^{n-1}$. Therefore, arguing just as in the case of the n-sphere, we see that M is a topological manifold of dimension $(n-1)$, and has a smooth structure such that each of the graph coordinate charts associated with a choice of f as above is a smooth chart.

Example 1.41 (Projective Spaces). The n-dimensional real projective space $\mathbb{RP}^n$ is a topological n-manifold. Let us check that the coordinate charts $\varphi_i \colon U_i \to \mathbb{R}^n$, where $U_i = \pi\left(\tilde{U}_i\right)$ with $\tilde{U}_i \subset \mathbb{R}^{n+1} - \{=\}$ is the set where $x^i \neq 0$, and

$$\varphi_i \left[x^1, \ldots, x^{n+1}\right] = \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right)$$

are smoothly compatible. Assume $i > j$. THen

$$\varphi_j \circ \varphi_i^{-1}\left(u^1, \ldots, u^n\right) = \left(\frac{u^1}{u^j}, \ldots, \frac{u^{j-1}}{u^j}, \frac{u^{j+1}}{u^j}, \ldots, \ldots, \frac{u^{i-1}}{u^j}, \frac{1}{u^j}, \frac{u^i}{u^j}, \ldots, \frac{u^n}{u^j}\right)$$

which is a diffeomorphism from $\varphi_i\left(U_i \cap U_j\right)$ to $\varphi_j\left(U_i \cap U_j\right)$.

Example 1.42 (Smooth Product Manifold). If $M_1, \ldots, M_k$ are smooth manifolds of dimension $n_1, \ldots, n_k$, $M_1 \times \ldots \times M_k$ is a topological manifold of dimension $n_1 + \ldots + n_k$, with charts of the form $(U_1 \times \ldots \times U_k, \varphi_1 \times \ldots \times \varphi_k)$. Any two charts are smoothly compatible because

$$(\psi_1 \times \ldots \times \psi_k) \circ (\varphi_1 \times \ldots \times \varphi_k)^{-1} = \left(\psi_1 \circ \varphi_1^{-1}\right) \times \ldots \times \left(\psi_k \circ \varphi_k^{-1}\right)$$

is a smooth map. This defines a natural smooth manifold structure on the product, called the **product smooth manifold structure**.

For example, this yields a smooth manifold structure on the n-torus $T^n = S^1 \times \ldots \times S^1$.

Lemma 1.43 (Smooth Manifold Chart Lemma). *Let M be a set, and suppose we are given a collection $\{U_\alpha\}$ of subsets of M together with maps $\varphi_\alpha \colon U_\alpha \to \mathbb{R}^n$, such that the following properties are satisfied:*

(1) For each α, φ_α is a bijection between U_α and an open subset $\varphi_\alpha\left(U_\alpha\right) \subset \mathbb{R}^n$.
(2) For each α and β, the sets $\varphi_\alpha\left(U_\alpha \cap U_\beta\right)$ and $\varphi_\beta\left(U_\alpha \cap U_\beta\right)$ are open in $\mathbb{R}^n$.
(3) Whenever $U_\alpha \cap U_\beta \neq \varnothing$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} \colon \varphi_\alpha\left(U_\alpha \cap U_\beta\right) \to \varphi_\beta\left(U_\alpha \cap U_\beta\right)$ is smooth.
(4) Countably many of the sets U_α cover M.
(5) Whenever p, q are distinct points in M, either there exists some U_α containing both p and q or there exist disjoint sets U_α, U_β with $p \in U_\alpha$ and $q \in U_\beta$.

Then M has a unique smooth manifold structure such that each $(U_\alpha, \varphi_\alpha)$ is a smooth chart.

Example 1.44 (Grassmannian). Checking Hausdorff condition: Suppose $P, P' \subset V$ are k-dimensional subspaces and V is an n-dimensional space.

Exercise 1.45. Show that every smooth manifold with boundary has a countable basis consisiting of regular coordinate balls and half-balls.

Solution. Suppose first M is covered by some chart (U, φ). If φ maps into $\mathbb{R}^n$, then $\partial M = \varnothing$, and we are done. Hence suppose $\varphi \colon U \to \mathbb{H}^n$, and $\varphi(x) = 0$ without loss of generality. Then since U is open, there exist some $r' > r > 0$ such that

$$\overline{B}_r(0) \cap \mathbb{H}^n \subset B_{r'}(0) \cap \mathbb{H}^n \subset \varphi(U).$$

Let $B = \varphi^{-1}\left(B_r(0) \cap \mathbb{H}^n\right)$ and $B' = \varphi^{-1}\left(B_{r'}(0) \cap \mathbb{H}^n\right)$. Since φ is a homeomorphism, we have $\overline{B} = \overline{\varphi^{-1}\left(B_r(0) \cap \mathbb{H}^n\right)} = \varphi^{-1}\left(\overline{B_r(0) \cap \mathbb{H}^n}\right)$, hence $\varphi\left(\overline{B}\right) = \overline{B}_r(0) \cap \mathbb{H}^n$. Thus B is a regular coordinate half-ball at x contained in U.

Let $\mathcal{B}$ be the collection of all preimages $\varphi^{-1}\left(B_s(t)\right)$ and $\varphi^{-1}\left(B_s(t) \cap \mathbb{H}^n\right)$ with $s \in \mathbb{Q}$ and $t \in \mathbb{Q}^n$. Then $\mathcal{B}$ is a countable basis for M.

Now suppose M is an arbitrary n-manifold with boundary. Then there is a countable collection of charts that cover M and each domain has by the above a countable collection of regular coordinate balls and half-balls as a basis, and since each domain is open, these are also open in M. Further, suppose B is a regular coordinate ball in some domain of a chart U_α. We must also argue that the closure in U_α of B is the same as its closure in M. Now, its closure in U_α is compact, and since M is Hausdorff, compact subsets are closed, so the closure of B in U_α is also closed in M. Thus the closures coincide.

Remark. The idea for this problem is that each point in M is contained in a chart (U, φ). Since U is a manifold with the induced topology, we simply prove it for the case of M being coverable with a single chart first. Then we have that the topology for U as a subspace has a countable basis of regular coordinate balls and half-balls. Now since a countable collection of charts cover M (by second-countability), we wish to

conclude that the union of the collections of regular coordinate balls and half-balls for the individual chart domains is a countable basis of regular coordinate balls and half-balls. For this, we need to be careful that the topological properties carry over. Openness carries over as each chart domain is open. And closures carry over since the closure is compact in U, and hence also in M and since M is Hausdorff, it is closed in M as well. Hence the closures coincide as well. We thus see that Heine-Borel for $\mathbb{R}^n$ is important for us here.

1.2. Applications of partitions of unity.

Definition 1.46 (Bump functions). If M is a topological space, $A \subset M$ is a closed subset, and $U \subset M$ is an open subset containing A, a continuous function $\psi: M \to \mathbb{R}$ is called a **bump function for A supposed in** U if $0 \le \psi \le 1$ on M, $\psi = 1$ on A, and $\operatorname{supp} \psi \subset U$.

Proposition 1.47 (Existence of smooth bump functions). *Let M be a smooth manifold with or without boundary. For any closed subset $A \subset M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.*

Proof. First, the much simpler proof from the book:
Let $U = U_0$ and $U_1 = M - A$. Then there exists a smooth partition of unity subordinate to $\{U_0, U_1\}$. Since $\psi_1 = 0$ on A, we have $\psi_0 = \sum \psi_i = 1$ on A, so ψ_0 has the desired properties. $\qquad \square$

PROBLEMS

1.3. Chapter 1.

Problem 1.48. Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \ne 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the **line with two origins**)

Proof. Let $\pi: X \to M$ be the quotient map. We first show that π is open. Let $U \subset X$ be open. Then $\pi(U)$ is open iff $\pi^{-1}(p(U))$ is open. Define $\varphi: X \to X$ by $\varphi(x, y) = (x, -y)$. Then

$$\pi^{-1}(p(U)) = U \cup \varphi(U - \{(0, 1), (0, -1)\})$$

which is open as φ is a homeomorphism. Hence π is an open map.
We can define a map $f: M \to \mathbb{R}$ by $f(\overline{x}) = \pi_1(\pi^{-1}(x))$ where π_1 here is the projection onto the first coordinate (this is the induced map in the diagram below). So for $\pi(x, 1) \in M$, we have $f(\pi(x, 1)) = x$. We claim that f also is an open map. Firstly, since $f \circ \pi: X \to \mathbb{R}$ is simply the projection onto the first coordinate and π is a quotient map, we have that f is continuous. Now, further, since projections are open and since π is open, we must have that f is open too.

Now we show that M is locally Euclidean.
Suppose $\overline{x} \in M$. Then we can assume without loss of generality that $\overline{x} = \pi(x, 1)$ ($\pi(x, -1)$ is similar). Then $\pi_1((x - \varepsilon, x + \varepsilon) \times \{1\})$ is an open set around $f(\overline{x})$.
To show that it is second-countable, note that since π is a quotient map, the preimage of any open subset of M is open in X, but the topology of X has basis of all balls in $\mathbb{R}^2$ with rational coordinates and rational radii, so the images of these under π which is open are also open and hence form a basis for M.
To see that M is not Hausdorff, it is trivial to check that $(0, 1)$ and $(0, -1)$ do not have disjoint neighborhoods. $\qquad \square$

Exercise 1.49. A topological space is said to be σ**-compact** if it can be expressed as a union of countably many compact subspaces. Show that a locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.

Proof. Suppose M is a locally Euclidean Hausdorff space. If we can show that any open subset of M is contained in a compact subset, then second-countability together with M being locally Euclidean would imply σ-compactness.

Since any manifold has a countable basis of precompact coordinate balls, we can let $\mathcal{B}$ be such a basis. Then $\overline{\mathcal{B}} = \left\{ \overline{B} \mid B \in \mathcal{B} \right\}$ is a countable collection of compact balls which cover M.

Conversely, suppose M is σ-compact. Let $\mathcal{A} = \{A_\alpha \colon \alpha \in J\}$ be a countable collection of compact subspaces covering M. Let $x \in M$ and U_x be a neighborhood of x which by local Euclidean is homeomorphic by φ_x to an open subset $\varphi_x(U_x) \subset \mathbb{R}^n$. Then the preimage under φ_x of all balls with rational coordinates and rational radii is a basis for the induced topology of U_x - which are also open in M -; let us denote this collection of coordinate balls by $\mathcal{B}_x$. Now, suppose $A_\alpha \in \mathcal{A}$. Then $\bigcup_{x \in A_\alpha} U_x$ covers A_α and since A_α is compact, there exists a finite subcollection $U_{x_1}, \ldots, U_{x_n}$. Now suppose $V \subset M$ is open and $x \in V$. Then there exists $A_\alpha \in \mathcal{A}$ such that $x \in A_\alpha$. Hence there exists i such that $x \in U_{x_i}$. Since $V \cap U_{x_i}$ is an open subset of U_{x_i}, there exists a coordinate ball B such that $x \in B \subset V \cap U_{x_i}$.

Thus we see that if $\mathcal{B}_\alpha = \bigcup_{i=1}^n \mathcal{B}_{x_i}$, then $\bigcup_{\alpha \in J} \bigcup_{B \in \mathcal{B}_\alpha} B$ is a basis for M which is countable. $\qquad\square$

Exercise 1.50. Suppose M is a locally Euclidean Hausdorff space. Show that M is second-countable if and only if it is paracompact and has countably many connected components.

Proof. ($\implies$) : is clear.

$\qquad\square$

Exercise 1.51 (1-6)**.** Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, show that it has uncountably many distinct ones.

Exercise 1.52 (Lee, 1-8)**.** By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of S^1 as lying in $\mathbb{C}$. Show that for any angle function θ on $U \subset S^1$ with $U \neq S^1$, (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.

Proof. Suppose $e^{i\alpha} \in S^1 - U$. Then Arg_α is a continuous angle function on U, and any other angle function on U is of the form $\text{Arg}_\alpha + 2\pi n$ for $n \in \mathbb{Z}$. Now, Arg_α is a smooth homeomorphism of $S^1 - \left\{ e^{i\alpha} \right\}$ onto $(\alpha - 2\pi, \alpha)$, so any angle function is a smooth homeomorphism of $U \subset S^1 - \left\{ e^{i\alpha} \right\}$ onto some subspace diffeomorphic to $(\alpha - 2\pi, \alpha)$. Now, $\theta^{-1} \colon \theta(U) \to U$ is given by $\theta^{-1}(t) = e^{it}$, so $\varphi_1 \circ \theta^{-1}(t) = \cos t$ and $\varphi_2 \circ \theta^{-1}(t) = \sin t$ which are both smooth on all of $\mathbb{R}$. Hence $\theta^{-1} \colon \theta(U) \to U$ is smooth, so θ is a diffeomorpism and hence (U, θ) is a smooth chart. $\qquad\square$

Exercise 1.53 (Complex projective n-space)**.** Let $U \subset \mathbb{C}^{n+1} - \{0\}$ be open. Then $\pi^{-1}(\pi(U)) = p^{-1}(p(U))$ where $p \colon \mathbb{C}^{n+1} - \{0\} \to S^n$ is the map $x \mapsto \frac{x}{\|x\|}$. Since if $\pi(x) \in \pi(U)$ then there exists $y \in U$ such that for some $\lambda \in \mathbb{R}_+$ we have $x = \lambda y$ and hence $\frac{x}{\|x\|} = \frac{\lambda y}{\|\lambda y\|} = \frac{y}{\|y\|}$, so $p(x) = p(y) \in p(U)$. Conversely, if $p(x) \in p(U)$ then $\frac{y}{\|y\|} = \frac{x}{\|x\|}$ for some $y \in U$, so $x = y \frac{\|x\|}{\|y\|}$ giving $\pi(x) = \pi(y)$.

Since p is continuous, π is open. Since $S^n \approx \mathbb{C}^{n+1} - \{0\}$ and p is also open, we further have that since S^n is compact, so is $\mathbb{CP}^n$.

1.4. **Chapter 2.**

Proposition 1.54 (2.15 (Properties of Diffeomorphisms))**.**
- *The restriction of a diffeo to an open submanifold with or without boundary is a diffeo onto its image.*
- *"Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.*

Proof. Suppose $F \colon M \to N$ is a diffeomorphism, and let $U \subset M$ be an open subset inheriting the structure from M. Let $F(U)$ inherit the structure from N. Then for any $p \in U$, there exists a chart (W, φ) around p in M and (V, ψ) around $F(p)$ in N such that $\psi \circ F \circ \varphi^{-1}$ is diffeomorphic. Since $\psi|_{V \cap V} \circ F|_U \circ (\varphi|_{U \cap W})^{-1}$ is also smooth, so $F|_U$ is a diffeomorphism onto its image.

The second part is clear since the composition of diffeomorphisms is a diffeomorphism. $\qquad\square$

Exercise 1.55 (2-3)**.** For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(1) $p_n \colon S^1 \to S^1$ is the **nth power map** for $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.

(2) $\alpha \colon S^n \to S^n$ is the **antipodal map** $\alpha(x) = -x$.

(3) $F \colon S^3 \to S^2$ is given by $F(w, z) = (z\overline{w} + w\overline{z}, iw\overline{z} - iz\overline{w}, z\overline{z} - w\overline{w})$ where we think of S^3 as the subset $\left\{ (w, z) \colon |w|^2 + |z|^2 = 1 \right\} \subset \mathbb{C}^2$.

Proof. (1) : First, let us consider the map $p_2(z) = z^2$. The we can subdivide S^1 into the following sets: $V_1 = \left\{ e^{it} \mid t \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right\}, V_2 = \left\{ e^{it} \mid t \in (0, \pi) \right\}, V_3 = \left\{ e^{it} \mid t \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right) \right\}, V_4 = \left\{ e^{it} \mid t \in (\pi, 2\pi) \right\}$. Let $\theta_1 \colon U_1 = S^1 - \{(-1, 0)\} \to (-\pi, \pi)$ and $\theta_2 \colon U_2 = S^1 - \{(1, 0)\} \to (0, 2\pi)$ be argument functions. Then (U_1, θ_1) and (U_2, θ_2) area smooth charts constituting and atlas for S^1. Suppose $z \in V_1$. Then a coordinate representation for $p_2(z)$ is $\theta_1 \circ p_2 \circ \theta_1^{-1}(t) 2t$ where $\theta_1 \circ p_2 \circ \theta_1^{-1} \colon \theta_1 \left(V_1 \cup V_3 \cup V_4 \cap \left(S^1 - \{(-1, 0)\} \right) \right) = \left(-\pi, \frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \pi \right) \to \mathbb{R}$. Similarly, a coordinate representation of p_2 for $z \in V_2$ with respect to the charts θ_2, θ_2^{-1} is also $t \mapsto 2t$ on an open set of $\mathbb{R}$ which is smooth. And the exact same can be shown for z in V_3 or V_4.

Now consider the case p_n for an arbitrary integer $n \geq 3$. Let $V_1 = \left\{ e^{it} \mid t \in \left(-\frac{\pi}{n}, \frac{\pi}{n} \right) \right\}$ and then for $j \in \{2, \ldots, 2n\}$, let $V_j = e^{(j-1)i\frac{\pi}{n}} V_1$. Then p_n maps each V_j to either $S^1 - \{(-1, 0)\}$ or $S^1 - \{(1, 0)\}$; where V_j is mapped to the former if j is odd and V_j is mapped to the latter if j is even. Now, V_{n+1} is precisely $\left\{ e^{it} \mid t \in \left(\pi - \frac{\pi}{n}, \pi + \frac{\pi}{n} \right) \right\} \subset S^1 - \{(1, 0)\}$ and contains $(-1, 0)$.

We first treat the cases $z \in V_1$ and $z \in V_{n+1}$. If $z \in V_1$, then $p_n(z) \neq (-1, 0)$, so $\theta_1 \circ p_n \circ \theta_1^{-1}(t) = nt$ for $t \in \theta_1 \left(p_n^{-1} \left(S^1 - \{(-1, 0)\} \right) \cap S^1 - \{((-1, 0))\} \right)$ which is an open subset of $\mathbb{R}$ and $t \mapsto nt$ is smooth on any open subset of $\mathbb{R}$. If instead $z \in V_{n+1}$ then $p_n(z) \neq (1, 0)$, so using θ_2 instead as a coordinate representation, we get also that p_n is represented by $t \mapsto tn$ on V_{n+1}.

For all other V_j, we have that if j is even, then θ_2 works as coordinate representation giving $t \mapsto nt$ again, and if j is odd, then θ_1 gives $t \mapsto nt$ as coordinate representation. Hence p_n is smooth.

(2) For any point $z \in S^n$, there exists coordinate z^i such that $z^i \neq 0$. Then let

$$w = \left(z^1, \ldots, z^{i-1}, \hat{z^i}, z^{i+1}, \ldots, z^{n+1} \right) \in D^n$$

. Then the coordinate representation for α is

$$\varphi_i^+ \circ \alpha \circ \left(\varphi_i^+ \right)^{-1} (w) = -w,$$

as a map $D^n \to \mathbb{R}^n$, which is smooth.

(3) : I will only prove this if I will need it at some point. $\qquad \square$

Problem 1.56. Show that the inclusion map $\overline{B^n} \hookrightarrow \mathbb{R}^n$ is smooth when $\overline{B^n}$ is regarded as a smooth manifold with boundary.

Proof. The inclusion on the interior is clearly smooth as it has the identity as a coordinate representation. For $x \in S^{n-1}$, let $\left(V, \pi \circ \sigma^{-1} \right)$ be a smooth chart around x. Then $\iota \circ \sigma \circ \pi^{-1} \left(\pi \circ \sigma^{-1}(x) \right) = \iota(x) = x$ is smooth as it is constant, so as ι is represented by the identity on $\pi \circ \sigma^{-1}(V)$, i is smooth. $\qquad \square$

Problem 1.57 $(2 - 10)$. For any topological space M, let $C(M)$ denote the algebra of continuous functions $f \colon M \to \mathbb{R}$. Given a continuous map $F \colon M \to N$, define $F^* \colon C(N) \to C(M)$ by $F^*(f) = f \circ F$.

(1) Show that F^* is a linear map.

(2) Suppose M and N are smooth manifolds. Show that $F \colon M \to N$ is smooth if and only if $F^* \left(C^\infty(N) \right) \subset C^\infty(M)$.

(3) Suppose $F \colon M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

Proof. (1) :

$$\begin{aligned}
F^*(\lambda f + g) &= (\lambda f + g) \circ F \\
&= \lambda f \circ F + g \circ f \\
&= \lambda F^*(f) + F^*(g)
\end{aligned}$$

(2) : Suppose F is smooth. Then since compositions of smooth maps are smooth, we have that $F^*\left(C^\infty(N)\right) \subset C^\infty(M)$.

Conversely, suppose F is continuous and $F^*\left(C^\infty(N)\right) \subset C^\infty(M)$. $\qquad\square$

Problem 1.58 (Linear maps between finite-dimensional normed vector spaces are smooth)**.** Suppose M and N are real normed vector spaces of dimension m and n, respectively. Let $T\colon M \to N$ be a linear map. Then T is smooth when M and N are regarded with the standard smooth structure on normed finite-dimensional vector spaces.

Proof. Let $u_1, \ldots, u_m$ be a basis for M and $v_1, \ldots, v_n$ a basis for N. By example 1.24 in Lee, $E\colon \mathbb{R}^m \to M$ by

$$E(x) = x^i u_i$$

is a homeomorphism, so $\left(M, E^{-1}\right)$ is a chart, and likewise $F\colon N \to \mathbb{R}^n$ by

$$F(v) = v^i e_i$$

is a homeomorphism, so (N, F) is a chart. Let the matrix of T with respect to these bases be given by

$$A = \left(A_i^j\right)$$

so that

$$T(u_i) = \sum_{k=1}^n A_k^i v_k.$$

Now, the coordinate representation of T with respect to these charts is

$$F \circ T \circ E(x) = F \circ T\left(\sum_i x^i u_i\right) = F\left(\sum_i x^i \sum_k A_k^i v_k\right) = \sum_{i,j,k} x^i A_k^i e_k$$

The k th coordinate function for this function is

$$\sum_{i,j} x^i A_k^i$$

which is a polynomial in $\mathbb{R}\left[x^1, \ldots, x^m\right]$ and hence smooth as a function $\mathbb{R}^m \to \mathbb{R}$. Thus T is smooth. $\qquad\square$

Section 3.

Proposition 1.59 (3.13. The tangent space to a vector space)**.** *Suppose V is a finite-dimensional vector space with its standard smooth manifold structure. For each point $a \in V$, the map $v \mapsto D_v|_a$ is a canonical isomorphism from V to $T_a V$, such that for any linear map $L\colon V \to W$, the following diagram commutes:*

$$
\begin{array}{ccc}
V & \xrightarrow{\ \cong\ } & T_a V \\
{\scriptstyle L}\downarrow & & \downarrow{\scriptstyle dL_a} \\
W & \xrightarrow{\ \cong\ } & T_{La} W
\end{array}
$$

Proof. We have that $D_v|_a$ is a derivation at a, and clearly $v \mapsto D_v|_a$ is injective.

For surjectivity, we must modify the proof of proposition 3.2, since V is any real normed finite-dimensional vector space.

However, V is isomorphic to $\mathbb{R}^n$ for some n, by an isomorphism φ. Then if $\varphi\colon \mathbb{R}^n \to V$, we have that for any $f \in C^\infty(V)$, $f \circ \varphi \in C^\infty(\mathbb{R}^n)$ since φ is smooth and compositions of smooth maps are smooth. Then

$$f \circ \varphi(x) = f \circ \varphi(a) + \sum_{i=1}^n \frac{\partial\left(f \circ \varphi\right)}{\partial x^i}(a)\left(x^i - a^i\right) + R(x)$$

so

$$f(y) = f(b) + \sum_{i=1}^{n} \frac{\partial f \circ \varphi}{\partial x^i}(a)\left(y^i - b^i\right) + R(y)$$

$$= f(b) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial f}{\partial z^j}(b) \frac{\partial \varphi^j}{\partial x^i}(a)\left(y^i - b^i\right) + R(y)$$

Hence

$$w(f) = \sum_{i,j} \frac{\partial f}{\partial z^j}(b)\frac{\partial \varphi^j}{\partial x^i}(a)v^i = D_u|_b f$$

where $u = \sum_j \frac{\partial \varphi^j}{\partial x^i}(a)v^i e_j$.

$\square$

Problem 1.60 (3-1). .

Proof. Suppose F is constant on a component U of M and $p \in U$. Then
$$dF_p(v)f = v\left(f \circ F\right) = 0$$
by lemma 3.4 as $f \circ F$ is constant.

Conversely, suppose dF_p is zero for all $p \in U$. Since U is connected and locally path-connected because it is locally Euclidean, we have that U is path-connected. Now, suppose F is not constant on all of U. Then there must exist a smooth chart (V, φ) such that $x, y \in V$ and $F(x) \neq F(y)$. Suppose no such chart exists. We can cover U by a countable number of charts on which F is constant, and F must take at least two values on U. Hence the image $F(U)$ is a discrete set of at least two points, but also path-connected, which is impossible since $F(U)$ is Hausdorff. Hence let (V, φ) be a chart with $x, y \in V$ and $F(x) \neq F(y)$. Then let $\gamma\colon I \to N$ be a path contained in V connecting x and y. Then $\alpha = \varphi \circ F \circ \gamma$ is a smooth curve $I \to \mathbb{R}^n$, and by the intermediate value theorem, there exists $t_0 \in (0, 1)$ such that

$$0 \neq \alpha'(t_0) = \frac{\partial\left(\varphi \circ F\right)^k}{\partial y^j}(\gamma(t_0))\frac{d\gamma^j}{dt}(t_0) = dF\left(\frac{\partial}{\partial y^j}\Big|_{\gamma(t_0)}\right)\varphi^k \frac{d\gamma^i}{dt}(t_0) = 0.$$

$\square$

Appendix A. Review of Calculus

Total and Partial Derivatives.

Definition A.1 (Total derivative). Let V and W be finite-dimensional vector spaces, which we may assume to be endowed with norms. If $U \subset V$ is an open subset and $a \in U$, a map $F \colon U \to W$ is said to be **differentiable at** a if there exists a linear map $L \colon V \to W$ such that

$$\lim_{v \to 0} \frac{|F(a+v) - F(a) - Lv|}{|v|} = 0 \tag{1}$$

The norm in the numerator is that of W, while the norm in the denominator is that of V.

If F is differentiable at a, the linear map L satisfying the above equation is denoted by $DF(a)$ and is called the **total derivative of** F **at** a. Condition (1) can also be written

$$F(a+v) = F(a) + DF(a)v + R(v)$$

where the remainder term $R(v) = F(a+v) - F(a) - DF(a)v$ satisfies $\frac{|R(v)|}{|v|} \to 0$ as $v \to 0$. Thus the total derivative represents the "best linear approximation" to $F(a+v) - F(a)$ near a.

Proposition A.2 (Chain Rule for Total Derivative). *Suppose V, W, X are finite-dimensional vector spaces, $U \subset V$ and $\tilde{U} \subset W$ open subsets, and $F \colon U \to \tilde{U}$ and $G \colon \tilde{U} \to X$ maps. If F is differentiable at $a \in U$ and G is differentiable at $F(a) \in \tilde{U}$, then $G \circ F$ is differentiable at a, and*

$$D\left(G \circ F\right)(a) = DG\left(F(a)\right) \circ DF(a).$$

Partial Derivatives. We now specialize to maps between Euclidean spaces.

Definition A.3 (Jacobian). The matrix $\left(\frac{\partial F^i}{\partial x^j}\right)$ of partial derivatives is called the **Jacobian matrix of** F, and its determinant is called the **Jacobian determinant of** F.

Definition A.4 (C^k or k times continuous differentiability). If $F \colon U \to \mathbb{R}^m$ is a function for which each partial derivative exists at each point in U and the functions $\frac{\partial F^i}{\partial x^j} \colon U \to \mathbb{R}$ so defined are all continuous, then F is said to be of **class** C^1 or **continuously differentiable**.

In general, if $U \subset \mathbb{R}^n$ is an open subset and $k \geq 0$, a function $F \colon U \to \mathbb{R}^m$ is said to be of **class** C^k of k **times continuously differentiable** if all the partial derivatives of F of order less than or equal to k exist and are continuous functions on U.

Definition A.5 (smoothness or class C^∞). A function that is of class C^k for every $k \geq 0$ is said to be of **class** C^∞, **smooth** or **infinitely differentiable**.

Proposition A.6. *Suppose $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open subsets and $F \colon U \to V$ is a diffeomorphism. Then $m = n$, and for each $a \in U$, the total derivative $DF(a)$ is invertible, with $DF(a)^{-1} = D\left(F^{-1}\right)(F(a))$.*

Proof. As F is a diffeomorphism, we have $\mathrm{id}_U = F^{-1} \circ F$ and $\mathrm{id}_V = F \circ F^{-1}$. Since both F and F^{-1} are smooth, we have

$$I_n = D\left(\mathrm{id}_U\right) = D\left(F^{-1}\right)(F(a)) \, DF(a) \implies DF(a)^{-1} = D\left(F^{-1}\right)(F(a))$$

and we also have

$$I_m = D\left(\mathrm{id}_V\right) = D\left(F\right)\left(F^{-1}(a)\right) D\left(F^{-1}\right)(a) = DF(b)D\left(F^{-1}\right)(F(b))$$

This shows that $DF(a)$ is invertible with inverse $D\left(F^{-1}\right)(F(a))$. Hence $n = m$. $\qquad \square$

Definition A.7 (Smooth on A). If $A \subset \mathbb{R}^n$ is an *arbitrary* subset, a function $F \colon A \to \mathbb{R}^m$ is said to be **smooth on** A if it admits a smooth extension to an open neighborhood of each point, or, more precisely, if for every $x \in A$, there exists an open subset $U_x \subset \mathbb{R}^n$ containing x and a smooth function $\tilde{F} \colon U_x \to \mathbb{R}^m$ that agrees with F on $U_x \cap A$.

The notion of diffeomorphism extends to arbitrary subsets in the obvious way: given arbitrary subsets $A, B \subset \mathbb{R}^n$, a **diffeomorphism from** A **to** B is a smooth bijective map $f \colon A \to B$ with a smooth inverse.

Proposition A.8. *Let $U \subset \mathbb{R}^n$ be open, and suppose $F \colon U \to \mathbb{R}^m$ is differentiable at $a \in U$. Then all of the partial derivatives of F at a exist, and $DF(a)$ is the linear map whose matrix is the Jacobian of F at a :*

$$DF(a) = \left(\frac{\partial F^j}{\partial x^i}(a) \right).$$

Exercise A.9. Suppose $U \subset \mathbb{R}^n$ is open. Show that a function $F \colon U \to \mathbb{R}^m$ is differentiable at $a \in U$ if and only if each of its component functions $F^1, \ldots, F^m$ is differentiable at a. Show that if this is the case, then

$$DF(a) = \begin{pmatrix} DF^1(a) \\ \vdots \\ DF^m(a) \end{pmatrix}$$

Proof. The direction $\implies$ is clear.
Now suppose each F^i is differentiable at a. Thus for each i, we have

$$\lim_{v \to 0} \frac{F^i(a+v) - F^i(a) - DF^i(a)v}{|v|} = 0.$$

In particular,

$$\lim_{v \to 0} \frac{F(a+v) - F(a)}{|v|} - \frac{1}{|v|} \begin{pmatrix} DF^1(a) \\ \vdots \\ DF^m(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

which gives the result. $\qquad \square$

Proposition A.10. *Let $U \subset \mathbb{R}^n$ be open. If $F \colon U \to \mathbb{R}^m$ is of class C^1, then it is differentiable at each point of U.*

Corollary A.11 (The Chain Rule for Partial Derivatives). *Let $U \subset \mathbb{R}^n$ and $\tilde{U} \subset \mathbb{R}^m$ be open subsets, and let $x = \left(x^1, \ldots, x^n \right)$ denote the standard coordinates on U and $y = \left(y^1, \ldots, y^m \right)$ those on $\tilde{U}$.*

    (1) *A composition of C^1 functions $F \colon U \to \tilde{U}$ and $G \colon \tilde{U} \to \mathbb{R}^p$ is again of class C^1, with partial derivatives given by*

$$\frac{\partial \left(G^i \circ F \right)}{\partial x^j}(x) = \sum_{k=1}^m \frac{\partial G^i}{\partial y^k} \left(F(x) \right) \frac{\partial F^k}{\partial x^j}(x).$$

    (2) *If F and G are smooth, then $G \circ F$ is smooth.*

Proof. (1): The composition is C^1 since compositions of differentiable maps are differentiable. The partial derivative $\frac{\partial \left(G^i \circ F \right)}{\partial x^j}(x)$ is the i, j entry in $D \left(G \circ F \right)(x) = DG \left(F(x) \right) DF(x)$ which is

$$\sum_{r=1}^m \left(DG \left(F(x) \right) \right)_{ir} \left(DF(x) \right)_{rj} = \sum_{r=1}^m \frac{\partial G^i}{\partial y^r} \left(F(x) \right) \frac{\partial F^r}{\partial x^j}(x).$$

(2) We have shown that if F, G are C^1, then the composition is C^1. Suppose we have shown that $G \circ F$ is C^k. Then in (1), we have shown that $D \left(G \circ F \right)(x)$ is a composition of smooth functions and hence C^k, so $G \circ F$ is C^{k+1}. By induction, $G \circ F$ is C^∞. $\qquad \square$

Definition A.12 (Directional derivative). Suppose $f \colon U \to \mathbb{R}$ is a smooth real-valued function on an open subset $U \subset \mathbb{R}^n$, and $a \in U$. For each vector $v \in \mathbb{R}^n$, we define the **directional derivative of f in the direction v at** a to be the number

$$D_v f(a) = \left. \frac{d}{dt} \right|_{t=0} f(a + tv)$$

Since $D_v f(a)$ is the ordinary derivative of the composite function $t \mapsto a + tv \mapsto f(a + tv)$, by the chain rule, it can be written more concretely as

$$D_v f(a) = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(a) = Df(a)v$$

Theorem A.13 (Differentiation Under an Integral Sign [see theorem 12.5 in Schilling). *] Let $U \subset \mathbb{R}^n$ be an open subset, let $a, b \in \mathbb{R}$ and let $f \colon U \times [a, b] \to \mathbb{R}$ be a continuous function such that the partial derivatives $\frac{\partial f}{\partial x^i} \colon U \times [a, b] \to \mathbb{R}$ exist and are continuous on $U \times [a, b]$ for $i = 1, \ldots, n$. Define $F \colon U \to \mathbb{R}$ by*

$$F(x) = \int_a^b f(x, t) dt.$$

Then F is of class C^1, and its partial derivatives can be computed by differentiating under the integral sign:

$$\frac{\partial F}{\partial x^i}(x) = \int_a^b \frac{\partial f}{\partial x^i}(x, t)\, dt.$$

Proposition A.14. *Suppose $D \subset \mathbb{R}^n$ is a domain of integration and $F \colon D \to \mathbb{R}^k$ is a bounded continuous vector-valued function. Then*

$$\left| \int_D F dV \right| \leq \int_D |F|\, dV.$$

Proposition A.15 (Lipschitz Estimate for C^1 Functions). *Let $U \subset \mathbb{R}^n$ be an open subset, and suppose $F \colon U \to \mathbb{R}^m$ is of class C^1. Then F is Lipschitz continuous on every compact convex subset $K \subset U$. The Lipschitz constant can be taken to be $\sup_{x \in K} |DF(x)|$.*

Proof. Since $|DF(x)|$ is continuous, it is bounded on K. Let $M = \sup_{x \in K} |DF(x)|$. For arbitrary $a, b \in K$, we have $a + t(b - a) \in K$ for all $t \in I$ because K is convex. By the fundamental theorem of calculus applied to each component of F, together with the chain rule,

$$F(b) - F(a) = \int_0^1 \frac{d}{dt} F(a + t(b - a))\, dt$$

$$= \int_0^1 DF(a + t(b - a))(b - a) dt$$

Thus

$$|F(b) - F(a)| \leq \int_0^1 |DF(a + t(b - a))|\, |b - a|\, dt$$

$$= \int_0^1 M\, |b - a|\, dt = M\, |b - a|.$$

$\square$

Corollary A.16. *If $U \subset \mathbb{R}^n$ is an open subset and $F \colon U \to \mathbb{R}^m$ is of class C^1, then f is locally Lipschitz continuous.*

The Inverse and Implicit Function Theorems. The last two theorems in this appendix are central results about smooth functions. They say that under certain hypotheses, the local behavior of a smooth function is modeled by the behavior of its total derivative.

Definition A.17 (Contraction). Let X be a metric space. A map $G \colon X \to Y$ is said to be a **contraction** if there is a constant $\lambda \in (0, 1)$ such that $d(G(x), G(y)) \leq \lambda d(x, y)$ for all $x, y \in X$.

Clearly contractions are continuous.

Lemma A.18 (Contraction Lemma). *Let X be a nonempty complete metric space. Every contraction $G \colon X \to X$ has a unique fixed point.*

Proof. Uniqueness is clear since if x and y are fixed points, then $d(x, y) = d(G(x), G(y)) \leq \lambda d(x, y) < d(x, y)$, contradiction.

Now, choose any point $x \in X$. Define a sequence as follows: $\alpha_i = G(\alpha_i - 1)$ and $\alpha_1 = G(x)$ with $x = \alpha_0$. Then let $\varepsilon > 0$. Since

$$d(\alpha_{i+1}, \alpha_i) \leq \lambda d(\alpha_i, \alpha_{i-1})$$

we have

$$d(\alpha_{n+1}, \alpha_n) \leq \lambda^n d(G(x), x)$$

Thus if $j \geq i$, we have

$$d\left(\alpha_i, \alpha_j\right) \leq d\left(\alpha_i, \alpha_{i+1}\right) + d\left(\alpha_{i+1}, \alpha_{i+2}\right) + \ldots + d\left(\alpha_{j-1}, \alpha_j\right)$$

$$\leq \lambda^i d\left(G(x), x\right) \sum_{k=0}^{j-i} \lambda^k$$

$$= d\left(G(x), x\right) \lambda^i \frac{\lambda^{j-i+1} - 1}{\lambda - 1} \to 0, \quad \text{as } i, j \to \infty$$

So we see that $(\alpha_i)_{i \in \mathbb{Z}_0^+}$ is a Cauchy sequence. As X is complete, this sequence converges. So $\alpha_i \to x_0$. Suppose now $G(x_0) \neq x_0$ and let $\varepsilon = |G(x_0) - x_0| > 0$. Let $N \in \mathbb{N}$ such that $|x_0 - \alpha_n| < \frac{\varepsilon}{2}$ for all $n \geq N$. Then

$$\varepsilon = |G(x_0) - x_0|$$

$$= |G(x_0) - \alpha_{N+1} + \alpha_{N+1} - x_0|$$

$$\leq |G(x_0) - G\left(\alpha_N\right)| + |\alpha_{N+1} - x_0|$$

$$< \lambda |x_0 - \alpha_N| + \frac{\varepsilon}{2}$$

$$< \varepsilon$$

Hence $G(x_0) = x_0$, so x_0 is a fixed point. $\qquad \square$

Theorem A.19 (Inverse Function Theorem). *Suppose U and V are open subsets of $\mathbb{R}^n$, and $F \colon U \to V$ is a smooth function. If $DF(a)$ is invertible at some point $a \in U$, then there exist connected neighborhoods $U_0 \subset U$ of a and $V_0 \subset V$ of $F(a)$ such that $F|_{U_0} \colon U_0 \to V_0$ is a diffeomorpism.*

Proof. We begin by making some simple modifications to the function F to streamline the proof. First, the function

$$F_1(x) = F(x + a) - F(a)$$

is smooth on a neighborhood of 0 and satisfies $F_1(0) = 0$ and $DF_1(0) = DF(a)$; clearly, F is a diffeomorphism on a conected neighborhood of a if and only if F_1 is a diffeomorphism on a connected neighborbood of 0. Second, the function $F_2 = DF_1(0)^{-1} \circ F_1$ is smooth on the same neighborhood of 0 and satisfies $F_2(0) = 0$ and $DF_2(0) = I_n$; and if F_2 is a diffeomorphism in a neighborhood of 0, then so is F_1 and therefore also F. Henceforth, replacing F by F_2, we assume that F is defined in a neighborhood U of 0, $F(0) = 0$ and $DF(0) = I_n$. Because the determinant of $DF(x)$ is a continuous function of x, by shrinking U if necessary, we may assume that $DF(x)$ is invertible for each $x \in U$.

Let $H(x) = x - F(x)$ for $x \in U$. Then $DH(0) = I_n - I_n = 0$. Because the matrix entries of $DH(x)$ are continuous functions of x, there is a number $\delta > 0$ such that $B_\delta(0) \subset U$ and for all $x \in \overline{B_\delta}(0)$, we have $|DH(x)| \leq \frac{1}{2}$. If $x, x' \in \overline{B_\delta}$, the Lipschitz estimate for smooth functions implies

$$|H(x') - H(x)| \leq \frac{1}{2} |x' - x| \qquad (\alpha)$$

In particular, taking $x' = 0$, this implies

$$H(x) \leq \frac{1}{2} |x| \qquad (\omega)$$

Since $x' - x = F(x') - F(x) + H(x') - H(x)$, it follows that

$$|x' - x| \leq |F(x') - F(x)| + |H(x') - H(x)| \leq |F(x') - F(x)| + \frac{1}{2} |x' - x| \,,$$

and rearranging gives

$$|x' - x| \leq 2 |F(x') - F(x)| \qquad (\gamma)$$

for all $x, x' \in \overline{B_\delta}(0)$. In particular, this shows that F is injective on $\overline{B_\delta}(0)$.
Now let $y \in B_{\frac{\delta}{2}}(0)$ be arbitrary. We will show that $\exists! x \in B_\delta(0) \mid F(x) = y$. Let $G(x) = y + H(x) = y + x - F(x)$, so that $G(x) = x$ if and only if $F(x) = y$. If $|x| \leq \delta$, (ω) implies

$$|G(x)| \leq |y| + |H(x)| < \frac{\delta}{2} + \frac{1}{2} |x| \leq \delta, \qquad (\beta)$$

so G maps $\overline{B_\delta}(0)$ to itself. It then follows from (α) that

$$|G(x) - G(x')| = |H(x) - H(x')| \leq \frac{1}{2}|x - x'|$$

so G is a contraction. Since $\overline{B_\delta}(0)$ is a complete metric space, the contraction lemma implies that G has a unique fixed point $x \in \overline{B_\delta}(0)$. From (α), $|x| = |G(x)| < \delta$, so in fact, $x \in B_\delta(0)$, thus proving the claim.

Let $V_0 = B_{\frac{\delta}{2}}(0)$ and $U_0 = B_\delta(0) \cap F^{-1}(V_0)$. Then U_0 is open in $\mathbb{R}^n$, and the argument above shows that $F \colon U_0 \to V_0$ is bijective, so $F^{-1} \colon V_0 \to U_0$ exists. Substituting $x = F^{-1}(y)$ and $x' = F^{-1}(y')$ into (γ) shows that F^{-1} is continuous. Thus $F \colon U_0 \to V_0$ is a homeomorphism, and it follows that U_0 is connected because V_0 is.

Lastly, we must show F^{-1} is smooth. If we knew it were smooth, then $A.6$ would give that $D\left(F^{-1}\right)(y) = DF(x)^{-1}$ where $x = F^{-1}(y)$. We first show that F^{-1} is differentiable at each point of V_0 with total derivative given by this formula.

Let $y \in V_0$ be arbitrary, and set $x = F^{-1}(y)$ and $L = DF(x)$. We need to show that

$$\lim_{y' \to y} \frac{F^{-1}(y') - F^{-1}(y) - L^{-1}(y' - y)}{|y' - y|} = 0.$$

Given $y' \in V_0 - \{y\}$, write $x' = F^{-1}(y') \in U_0 - \{x\}$. Then

$$\frac{F^{-1}(y') - F^{-1}(y) - L^{-1}(y' - y)}{|y' - y|} = L^{-1}\left(\frac{L(x' - x) - (y' - y)}{|y' - y|}\right)$$

$$= \frac{|x' - x|}{|y' - y|} L^{-1}\left(-\frac{F(x') - F(x) - L(x' - x)}{|x' - x|}\right).$$

The factor $\frac{|x' - x|}{|y' - y|}$ is bounded by (γ) and because L^{-1} is linear and therefore bounded, the norm of the second factor is bounded by a constant multiple of

$$\frac{|F(x') - F(x) - L(x' - x)|}{|x' - x|}.$$

As $y' \to y$, it follows that $x' \to x$ by continuity of F^{-1}, and then the above goes to zero because $L = DF(x)$ and F is differentiable. This completes the proof that F^{-1} is differentiable.

By proposition $A.8$, the partial derivatives of F^{-1} are defined at each point $y \in V_0$. The formula $D\left(F^{-1}\right)(y) = DF\left(F^{-1}(y)\right)^{-1}$ implies that the matrix-valued function $y \mapsto D\left(F^{-1}\right)(y)$ can be written as the composition

$$y \overset{F^{-1}}{\mapsto} F^{-1}(x) \overset{DF}{\mapsto} DF\left(F^{-1}(y)\right) \overset{i}{\mapsto} DF\left(F^{-1}(y)\right)^{-1} \tag{η}$$

where i is matrix inversion. In this composition, F^{-1} is continuous, DF is smooth because its component functions are partial derivatives of F ; and i is smooth because Cramer's rule expresses the entries of an inverse matrix as rational functions of the entries of the matrix. Thus $D\left(F^{-1}\right)$ is continuous, and so the partial derivatives of F^{-1} are continuous, so F^{-1} is of class C^1.

Now assume by induction that we have shown that F^{-1} is of class C^k. This means that each of the functions in (η) is of class C^k. Because $D\left(F^{-1}\right)$ is a composition of C^k functions, it is itself C^k; this implies that the partial derivatives of F^{-1} are of class C^k, so F^{-1} itself is of class C^{k+1}. Continuing by induction, we conclude that F^{-1} is smooth. $\square$

Corollary A.20. *Suppose $U \subset \mathbb{R}^n$ is an open subset, and $F \colon U \to \mathbb{R}^n$ is a smooth function whose Jacobian determinant is nonzero at every point in U.*

(1) *F is an open map.*
(2) *If F is injective, then $F \colon U \to F(U)$ is a diffeomorphism.*

Proof. (a): By the Inverse Function Theorem ($A.19$), since $DF(a)$ is has nonsingular determinant at every point in U, it is invertible at every point $a \in U$, so for any point $a \in U$, there exist connected neighborhoods $U_0 \subset U$ of a and $V_0 \subset \mathbb{R}^n$ of $F(a)$ such that $F|_{U_0} \colon U_0 \to V_0$ is a diffeomorphism. In particular, $F(U_0) = V_0$. Now, let W be an open subset of U. Then for any $x \in W$, we can find $W_x \subset W$ such that $\bigcup_{x \in W} W_x = W$,

and such that $F(W_x) = V_x$ where V_x is a connected open neighborhood. Thus $F(W) = F\left(\bigcup_{x \in W} W_x\right) = \bigcup_{x \in W} F(W_x)$ which is open. So F is an open map.

(b): It remains to show that F^{-1} is smooth. The inverse $F^{-1} \colon F(U) \to U$ exists since F is bijective. On a neighborhood of each point $F(a) \in F(U)$, it is equal to the inverse of $F|_{U_a}$, so it is smooth. $\square$

The next two examples illustrate the use of the preceding corollary, as well as the remark in the section on smooth structures.

Example A.21 (Polar Coordinates)**.** Let's tackle the mystery of **polar coordinates**: the polar coordinates (r, θ) in the plane are defined implicitly by the relations $x = r \cos \theta, y = r \sin \theta$. The map $F \colon (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ defined by $F(r, \theta) = (r \cos \theta, r \sin \theta)$ is smooth and has Jacobian

$$DF(r, \theta) = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

which has determinant $r \cos^2 \theta + r \sin^2 \theta = r$ which is nonzero everywhere in the domain. Thus, Corollary A.20 shows that the restriction of F to any open subset on which it is injective is a diffeomorphism onto its image. One such subset is $\{(r, \theta) : r > 0, -\pi < \theta < \pi\}$, which is mapped bijectively by F onto the complement of the nonpositive part of the x-axis.

Example A.22 (Spherical Coordinates)**.** Similarly, **spherical coordinates** on $\mathbb{R}^3$ are functions (ρ, φ, θ) defined by the relations

$$x = \rho \sin \varphi \cos \theta$$
$$y = \rho \sin \varphi \sin \theta$$
$$z = \rho \cos \varphi.$$

Geometrically, ρ is the distance from the origin, φ is the angle from the positive z-axis, and θ is the angle from the $x > 0$ half of the (x, z)-plane. If we define $G \colon (0, \infty) \times (0, \pi) \times \mathbb{R} \to \mathbb{R}^3$ by $G(\rho, \varphi, \theta) = (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi)$, we get the Jacobian

$$JG(\rho, \varphi, \theta) = \begin{pmatrix} \sin \varphi \cos \theta & \rho \cos \varphi \cos \theta & -\rho \sin \varphi \sin \theta \\ \sin \varphi \sin \theta & \rho \cos \varphi \sin \theta & \rho \sin \varphi \cos \theta \\ \cos \varphi & -\rho \sin \varphi & 0 \end{pmatrix}$$

which has determinant

$$\cos \varphi \det \begin{pmatrix} \rho \cos \varphi \cos \theta & -\rho \sin \varphi \sin \theta \\ \rho \cos \varphi \sin \theta & \rho \sin \varphi \cos \theta \end{pmatrix} + \rho \sin \varphi \det \begin{pmatrix} \sin \varphi \cos \theta & -\rho \sin \varphi \sin \theta \\ \sin \varphi \sin \theta & \rho \sin \varphi \cos \theta \end{pmatrix}$$
$$= \cos \varphi \rho^2 \left[\cos \varphi \sin \varphi \cos^2 \theta + \cos \varphi \sin \varphi \sin^2 \theta \right] + \sin \varphi \rho \left[\sin^2 \varphi \cos^2 \theta + \sin^2 \varphi \sin^2 \theta \right]$$
$$= \cos^2 \varphi \rho^2 \sin \varphi + \rho^2 \sin^3 \varphi$$
$$= \rho^2 \sin \varphi$$

which is nonzero as $\rho \in (0, \infty)$ and $\sin \varphi \in (0, 1]$. Thus, the restriction of G to any open subset on which it is injective is a diffeomorphism onto its image. One such subset is

$$\{(\rho, \varphi, \theta) : \rho > 0, 0 < \varphi < \pi, -\pi < \theta < \pi\}.$$

Notice how much easier it is to argue this way than to try to construct an inverse map explicitly out of inverse trigonometric functions.

The next result is one of the most important consequences of the inverse function theorem. It gives conditions under which a level set of a smooth function is locally the graph of a smooth function.

Theorem A.23 (Implicit Function Theorem)**.** *Let $U \subset \mathbb{R}^n \times \mathbb{R}^k$ be an open subset, and let $(x, y) = \left(x^1, \dots, x^n, y^1, \dots, y^k\right)$ denote the standard coordinates on U. Suppose $\Phi \colon U \to \mathbb{R}^k$ is a smooth function, $(a, b) \in U$, and $c = \Phi(a, b)$. If the $k \times k$ matrix*

$$\left(\frac{\partial \Phi^i}{\partial y^j}(a, b) \right)$$

is nonsingular, then there exist neighborhoods $V_0 \subset \mathbb{R}^n$ of a and $W_0 \subset \mathbb{R}^k$ of b and a smooth function $F \colon V_0 \to W_0$ such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F, that is, $\Phi(x,y) = c$ for $(x,y) \in V_0 \times W_0$ if and only if $y = F(x)$.

Proof. Consider the smooth function $\Psi \colon U \to \mathbb{R}^n \times \mathbb{R}^k$ defined by $\Psi(x,y) = (x, \Phi(x,y))$. Its total derivative at (a,b) is

$$D\Psi(a,b) = \begin{pmatrix} I_n & 0 \\ \dfrac{\partial \Phi^i}{\partial x^j}(a,b) & \dfrac{\partial \Phi^i}{\partial x^j}(a,b) \end{pmatrix},$$

which is nonsingular because it is block lower triangular and the two blocks on the main diagonal are nonsingular. Thus, by the inverse funciton theorem, there exist connected neighborhoods U_0 of (a,b) and Y_0 of (a,c) such that $\Psi \colon U_0 \to Y_0$ is a diffeomorphism. Shrinking U_0 and Y_0 if necessary, we may assume that $U_0 = V \times W$ is a product neighborhood.

Writing $\Psi^{-1}(x,y) = (A(x,y), B(x,y))$ for some smooth functions A and B, we compute

$$\begin{aligned} (x,y) = \Psi\left(\Psi^{-1}(x,y)\right) &= \Psi\left(A(x,y), B(x,y)\right) \\ &= \left(A(x,y), \Phi\left(A(x,y), B(x,y)\right)\right) \end{aligned} \tag{ω}$$

hence $A(x,y) = x$, so Ψ^{-1} has the form $\Psi^{-1}(x,y) = (x, B(x,y))$.

Now let $V_0 = \{x \in V \colon (x,c) \in Y_0\}$ and $W_0 = W$, and define $F \colon V_0 \to W_0$ by $F(x) = B(x,c)$. Comparing the second components in (ω) yields

$$c = \Phi\left(x, B(x,c)\right) = \Phi\left(x, F(x)\right)$$

whenever $x \in V_0$, so the graph of F is contained in $\Phi^{-1}(c)$. Conversely, suppose $(x,y) \in V_0 \times W_0$ and $\Phi(x,y) = c$. Then $\Psi(x,y) = (x, \Phi(x,y)) = (x,c)$, so

$$(x,y) = \Psi^{-1}(x,c) = (x, B(x,c)) = (x, F(x)),$$

which implies that $y = F(x)$. This completes the proof. $\qquad\square$

References

[Lee] John Lee. Introduction to Smooth Manifolds, 2nd edition