Homework 2

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1:
(a) Let $f+g \in I+J$ with $f \in I, g \in J$ and let $h \in k[x_1, \dots, x_n]$. Then $h(f+g) = \underbrace{hf}_{\in I} + \underbrace{hg}_{\in J}$ where $hf \in I$ since I is an ideal and $hg \in J$ since J is an ideal. Commutativity in $k[x_1, \dots, x_n]$ ensures that this is two-sided. It is clearly a ring and thus an ideal.

(b)
$(\subset)$: We have $I \cup J \subset I+J$ since $0 \in I, J$, so $V(I+J) \subset V(I \cup J) = V(I) \cap V(J)$.

$(\supset)$: For any $a \in V(I) \cap V(J)$, we have for any $f \in I$ and $g \in J$ that $f(a) = 0 = g(a)$, so $0 = f(a) + g(a) = (f+g)(a)$ and thus $a \in V(f+g)$. Therefore $V(I) \cap V(J) \subset V(f+g)$.

2:
(a) We first show that $(y-x^2)$ is a prime ideal in $k[x,y]$. We claim $k[x,y]/(y-x^2) \cong k[x]$ which is an integral domain and thus it would follow that $(y-x^2)$ is a prime ideal.
Proof of claim: Let $F \in k[x,y]$ with $F(x,y) = \sum_{i,j} a_{ij} x^i y^j$. Then $\pi(F) = \sum_{i,j} a_{ij} x^{i+2j} \in k[x]$, so π here is surjective and has kernel $(y-x^2)$. The result then follows from the first isomorphism theorem.
Now by problem 3.(d) underneath, any prime ideal is a radical ideal, so by Hilbert's Nullstellensatz, since $\mathbb{C}$ is closed, $I\left(V\left(y-x^2\right)\right) = \sqrt{(y-x^2)} = \left(y-x^2\right)$. By proposition 1 in section 1.5, Fulton, we then have that $V\left(y-x^2\right)$ is irreducible.

(b) We have that $x^2 = y^4$ implies $x = \pm y^2$. hence we find $0 = y^4 - x^2y^2 + xy^2 - x^3 = y^4 - y^6 \pm y^4 \mp y^6$, so $0 = 2y^4 - 2y^6 = 2y^4(1-y^2)$ and hence $y \in \{0, \pm 1\}$.
For $x = -y^2$, we have the other condition satisfied trivially.

Thus the irreducible components are

$$v(x+y^2), (1,1), (1,-1).$$

3:
(a) Assume $a^n \in I, b^m \in I$. Then

$$(a+b)^{n+m} = \sum_{i=0}^{n+m} a^i b^{n+m-i}$$

If $i \geq n$, then $a^i \in I$, so $a^i b^{n+m-i} \in I$. If $i < n$, then $n+m-i \geq n+m-i = m$, so $b^{n+m-i} \in I$ and hence $a^i b^{n+m-i} \in I$.

(b) We have $0 \in \sqrt{0}$ as $0 \in I$.
Let $f, g \in \sqrt{I}$ with $f^k, g^j \in I$. Then $f^{2k}, (-g)^{2j} \in I$, so by a, $(f-g)^{2(k+j)} \in I$, so $f - g \in \sqrt{I}$, so $\sqrt{I}$ is closed under subtraction.
now $(fg)^{kj} = f^{kj} g^{kj} = \left(f^k\right)^j \left(g^j\right)^k \in I$, so $fg \in \sqrt{I}$. Hence $\sqrt{I}$ is a ring. Let $f \in \sqrt{I}$ with $f^k \in I$, and $r \in R$. Then $(fr)^k = f^k r^k \in I$, so $fr \in \sqrt{I}$, and $(rf)^k = r^k f^k \in I$, so $rf \in \sqrt{I}$, hence $\sqrt{I}$ is an ideal.

(c) Let $f^k \in \sqrt{I}$. Then there exists an $l \in \mathbb{Z}_+$ such that $f^{kl} = \left(f^k\right)^l \in I$, and hence $f \in \sqrt{I}$. Therefore $\sqrt{I}$ is a radical ideal.

(d) Let P be a prime ideal. Let $r^k \in P$. By definition of prime ideal, we thus have that since $rr^{k-1} \in P$, either r or r^{k-1} is in P. If $r \in P$, we are done. Assume $r \notin P$. Then $r^{k-1} \in P$ and we repeat the procedure; subtracting 1 from the exponent of r each time. After $k-1$ turns, we will find $r \in P$ or $r \in P$, contradicting $r \notin P$. Hence $r \in P$, so P is a radical ideal as r was arbitrary.

4: Let X, Y be algebraic sets.
(a) We claim $I(X \cup Y) = I(X) \cap I(Y)$.
Proof: $(\subset)$: Let $f \in I(X \cup Y)$. Then for all $x \in X \cup Y$, $f(x) = 0$, and thus since $X, Y \subset X \cup Y$, we have for all $x \in X$ and for all $y \in Y$, $f(x) = 0 = f(y)$, so $I(X \cup Y) \subset I(X) \cap I(Y)$.
$(\supset)$: Let $f \in I(X) \cap I(Y)$. Then for all $x \in X$ and all $y \in Y$, we have $f(x) = 0 = f(y)$, so since for any $z \in X \cup Y$, $z \in X$ or $z \in Y$, we have that for all $z \in X \cup Y$, $f(z) = 0$.

(b) This is false: Let $X = \{(x,y) \mid y=0\} = V(y)$ and $Y = \left\{(x,y) \mid y = x^2\right\} = V(y-x^2)$. Then $I\left(X \cap Y\right) = I\left((0,0)\right)$ which is all functions that vanish on $(0,0)$.
However, $I(X) = (y)$ and $I(Y) = \left(y - x^2\right)$, so $I(X) + I(Y) = \left(y, y-x^2\right) = \left(x^2\right) \neq (x,y) = I(X \cap Y)$.

5: Let $I \subset R$ be any ideal. We wish to show that it is finitely generated.
Choose any $x_0 \in I$ and let I_0 be the ideal generated by x_0. If $I = I_0$, we are done. Otherwise, choose $x_1 \in I - I_0$ and let I_1 be the ideal generated by x_1 and x_0. Generally, if $I_{n-1} \neq I$, then choose $x_n \in I - I_{n-1}$ and let $I_n = (x_0, \ldots, x_n)$. Thus we get an ascending chain of ideals:

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \ldots$$

By assumption this ascending chain ends; say it ends with I_N. Then by construction, I_N must equal I and hence

$$I = (x_0, \ldots, x_N)\,.$$

Thus I is finitely generated, i.e. Noetherian, and since I was arbitrary, all ideals in R are finitely generated, so R is Noetherian.