$$
\begin{tikzcd}
B \arrow[dr, dashed] & & \\
& X \times_{X'} B' \cong \eta^* B' & B' \\
& X & X'
\end{tikzcd}
$$

M/G, let $p \in M/G$ and U an evenly covered open neighborhood of p. Then U splits into homeomorphic copies $\sqcup U_\alpha$ in M with $\pi|_{U_\alpha}\colon U_\alpha \cong U$ homeomorphisms. For $\tilde{p} \in U_\alpha$, choose a smooth chart $(V_{\tilde{p}}, \varphi_{\tilde{p}})$ contained in U_α. Since $\tilde{p} = g \cdot p$ for some g, we may as well denote these charts as $(V_{g,p}, \psi_{g,p})$. Now consider the charts $\left(\pi|_g(V_{g,p}), \psi_{g,p} \circ (\pi|_g)^{-1}\right)$. On an overlap the transition functions have the form

$$\psi_{g,p} \circ (\pi|_g)^{-1} \left(\psi_{g',p'} \circ (\pi|_{g'})^{-1}\right)^{-1} = \psi_{g,p} \circ (\pi|_g)^{-1} \, \pi|_{g'} \circ \psi^{-1}_{g',p'} = \psi_{g,p} \circ \psi^{-1}_{g',p'}$$

on the overlap, which is smooth by assumption. Hence we indeed obtain a smooth structure on M/G. In particular, the map $\pi\colon M \to M/G$ has coordinate form

$$\left(\psi_{g,p} \circ \pi|_g^{-1}\right) \pi \circ \psi_{g,p}^{-1} = \mathrm{id}$$

which is a diffeomorphism. So π is a local diffeomorphism when we equip M/G with this smooth structure.

(2) Define the functor $F\colon \mathrm{Sm}^G \to \mathrm{Sm}$ sending $M \mapsto M/G$ with the smooth structure defined in the first part of the exercise. Here, since maps $f\colon M \to N$ in Sm^G are G-equivariant, they, in particular, descend to smooth maps $\overline{f}\colon M/G \to N/G$, and we let $F(f) = \overline{f}$. Then indeed $F(\mathrm{id}_M) = \overline{\mathrm{id}_M} = \mathrm{id}_{M/G}$ and if $f\colon M \to N$ and $g\colon N \to P$, then $F\left(g \circ f\right) = \overline{g \circ f}$. But by pasting the two squares

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d] & N \arrow[r, "g"] \arrow[d] & P \arrow[d] \\
M/G \arrow[r, "\overline{f}"] & N/G \arrow[r, "\overline{g}"] & P/G
\end{tikzcd}$$

we find that $\overline{g \circ f} = \overline{g} \circ \overline{f}$. So $F\left(g \circ f\right) = F(g) \circ F(f)$.
This shows that F is indeed a functor.
We want to show that this defines a bundle theory on Sm. So suppose we have some $N \in \mathrm{Sm}^G$ and $f\colon M \to N/G$ in Sm. Now, the quotient map $N \to N/G$ is a submersion (show this), so the pullback along f exists in Sm, giving

$$\begin{tikzcd}
f^*N \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d] \\
M \arrow[r] & N/G
\end{tikzcd}$$

Lastly, we must then show that f^*N is in Sm^G. For this, note that the induced bundle f^*N is precisely the pullback which is equivalent as a fibre bundle to $M \times_{N/G} N$. But this inherits a natural action of G given by $g \cdot (m, n) = (m, g \cdot n)$. Choosing the same cover $\{U_\alpha\}$ for N as given in the condition of it being in Sm^G, i.e., $\{g \cdot U_\alpha\}$ being disjoint for all g and α, the neighborhoods $M \times U_\alpha \cap f^*N$ then satisfy the same conditions under this action of G. Lastly, the map $f^*N \cong M \times_{N/G} N \to N$ given by the projection to the N component which is the top map in the pullback diagram is naturally G-equivariant. This shows that the above diagram indeed can be made.

Now suppose we have some $P \in \mathrm{Sm}^G$ and a bundle map $P \to N$ giving the solid part of the diagram

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & M \times_{N/G} N \arrow[r] \arrow[d] & N \arrow[d] \\ P/G \arrow[r] & M \arrow[r] & N/G \end{tikzcd}$$

where the map $P \to N$ descends to the composite map $P/G \to M \to N/G$ on the bottom.

We then want to show that the dashed map exists. Let $p\colon P \to P/G$ and $q\colon f^*N \cong M \times_{N/G} N \to M$ be the projection. Let $k\colon P \to N$ be the map on the top. Let $f\colon P/G \to M$ be the map on the bottom. Define a map $h\colon P \to M \times_{N/G} N$ by $h(x) = (f(p(x)), k(p))$. Then if $l\colon M \to N/G$ denotes the map on the bottom, $l \circ f\,(p(x)) = \pi\,(k(p))$ where $\pi\colon N \to N/G$. By definition then $h(x) \in M \times_{N/G} N$. Furthermore,

$$h\,(g \cdot x) = (f\,(p\,(g \cdot x))\,, k\,(g \cdot x)) = (f\,(p\,(x))\,, g \cdot k(x)) = g \cdot (f\,(p(x))\,, k(x)) = g \cdot h(x),$$

so h is G-equivariant.

Next we must check that the bundle theory is locally trivial. That is, we must check that for any $M \in \mathrm{Sm}^G$ and any point $x \in M/G$, there exists an open neighborhood U about x such that if we let $\pi\colon U \to *$ be the unique map and $i\colon U \to M/G$ the open embedding, there exists a manifold $N \in \mathrm{Sm}^G$ such that $N/G \cong *$, and such that the pullbacks are isomorphic: $i^*M \cong \pi^*N$.
Note that these pullbacks are really

$$\begin{tikzcd} U \times_{M/G} M \cong i^*M \arrow[r] \arrow[d] & M \arrow[d, "p"] \\ U \arrow[r] & M/G \end{tikzcd}$$

But clearly if $(u, m) \in U \times_{M/G} M$, then essentially $\overline{m} = u$, so $U \times_{M/G} M \cong p^{-1}(U)$, and

$$\begin{tikzcd} U \times N \cong U \times_* N \arrow[r] \arrow[d] & N \arrow[d] \\ U \arrow[r] & * \end{tikzcd}$$

So we find that the condition is indeed equivalent to the usual one: the existence of a neighborhood U about x and a homeomorphism $p^{-1}(U) \cong U \times N$. In this case, suppose $x \in M/G$ and simply choose one of the U_α such that $x \in p\,(U_\alpha)$. Note that this is open in M/G since the $g \cdot U_\alpha$ are pairwise disjoint and g acts by homeomorphisms (G is discrete and each g has g^{-1} as inverse). Choosing $U = p\,(U_\alpha)$, we get $p^{-1}(U) = \sqcup_{g \in G} U_\alpha \cong U_\alpha \times G \cong U \times G$ where $G \in \mathrm{Sm}^G$ is precisely G considered as a smooth manifold with the trivial charts $g \mapsto *$, at each $g \in G$. Indeed then $G/G \cong *$, so this satisfies the condition above. I.e., the functor $\mathrm{Sm}^G \to \mathrm{Sm}$ is locally trivial.

Lastly, we must check gluing. Namely that for $M \in \mathrm{Sm}^G$ and some open coordinate neighborhoods $U_i, U_j, U_k \subset M/G$, with coordinate maps $g_{ij}: U_i \cap U_j \to G, g_{jk}: U_j \cap U_k \to G$ and $g_{ki}: U_k \cap U_i \to G$, the maps satisfy $g_{ik}(x) = g_{ij}(x)g_{jk}(x)$ for $x \in U_i \cap U_j \cap U_k$. As we saw above, $p^{-1}(U_i) = U_i \times G$, and we shall call this coordinate function $\varphi_i: U_i \times G \to p^{-1}(U_i)$. Let $g_{ij}(x) = \varphi_{i,x}^{-1}\varphi_{j,x}$ where $\varphi_{i,x}(y) = \varphi_i(x, y)$ is the function considered only as a function of y. But then the condition $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$ follows trivially.

This completes the proof that the functor we constructed $\mathrm{Sm}^G \to \mathrm{Sm}$ is indeed a bundle theory over Sm. □

2.2. **Change of fibres of bundles.**

Problem 2.2 (Change of fibres of bundles)**.** Let W_0 and W_1 be two smooth manifolds, and let G be a group which we assume as a simultaneous subgroup of both $\mathrm{Homeo}(W_0)$ and $\mathrm{Homeo}(W_1)$, i.e., we have injective group homomorphisms $\iota_0: G \hookrightarrow \mathrm{Homeo}(W_0)$ and $\iota_1: G \hookrightarrow (W_1)$. Given a fixed smooth manifold M, construct a bijection $\mathrm{Bun}_G^{W_0}(M) \to \mathrm{Bun}_G^{W_1}(M)$, where $\mathrm{Bun}_G^{W_i}(M)$ denotes the set of isomorphism classes of manifold bundles with fibre W_i and structure group G over the base space M.

Proof. Let $\mathcal{B} = \{B, p, M, W_0, G\} \in \mathrm{Bun}_G^{W_0}$. By Theorem 1.6, the bundle $\mathcal{B}$ is equivalent to its associated principal bundle $\tilde{\mathcal{B}} = \{B, p, M, G, G\}$ which thus represents the same isomorphism class. But by assumption, G embeds into $\mathrm{Homeo}(W_1)$, so by Theorem 1.3, also $\tilde{B}$ is equivalent to $\{B, p, M, W_1, G\} =: \mathcal{B}'$ which has the same coordinate transformations. Thus $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}'$ are equivalent. Now, seeing as equivalence of bundles is purely determined by their base space, fibre, structure group and coordinate transformations by Lemma 1.1, this gives an injective map $\mathrm{Bun}_G^{W_0} \to \mathrm{Bun}_G^{W_1}$. We can simply use the existence theorem directly. Seeing as we can do the exact same thing to obtain an injective map $\mathrm{Bun}_G^{W_1} \to \mathrm{Bun}_G^{W_0}$, we obtain a bijection by Schröder-Bernstein. □

2.3. **Associated frame bundles and structure group reductions.** I couldn't figure this one out in time.

Problem 2.3 (Associated frame bundles and structure group reductions)**.** For a rank d vector bundle $\xi: E \to M$ over a smooth manifold, we define the associated frame bundle $\mathrm{Fr}(\xi)$ as the associated $\mathrm{GL}_d(\mathbb{R})$-bundle.

(1) For M a smooth d-dimensional manifold, we define its frame bundle $\mathrm{Fr}(M)$ as the associated frame bundle of its tangent bundle TM. Show that $\mathrm{Fr}(M) \to M$ is a principal $\mathrm{GL}_d(\mathbb{R})$-bundle.

2.4. **Invertible Cobordisms and Boundaries of Compact Manifolds.**

Problem 2.4 (Invertible cobordisms and boundaries of compact manifolds)**.** Let $W_0: M_0 \rightsquigarrow \varnothing$ and $W_1: M_1 \rightsquigarrow \varnothing$ be two compact d-dimensional smooth cobordisms from compact $(d-1)$-dimensional smooth manifolds M_0 and M_1 to the empty manifold, viewed as a $(d-1)$-manifold. In other words, we have a smooth embedding $M_i \times \mathbb{R} \hookrightarrow W_i$ satisfying that $M_i \times (-\infty, 0]$ is closed, and such that their complement $W_i - (M_i \times \mathbb{R})$ is compact. We define $\mathrm{Int}(W_i)$ to be the complement of the image of $M_i \times (-\infty, t]$ for some $t \in \mathbb{R}$ (and hence any $t \in \mathbb{R}$), and observe that $\mathrm{Int}(W_i)$ is again a smooth manifold, being an open subset of W_i.

(1) Assume that in the situation of the above, $\mathrm{Int}(W_0)$ is diffeomorphic to $\mathrm{Int}\,(W_1)$. Show that M_0 and M_1 are invertibly cobordant, i.e., there exists a cobordism $M_0 \rightsquigarrow M_1$ which is invertible in the category Cob_d.
(2) Let W be a smooth, open (i.e., non-compact) d-manifold. We define a compact closure of W to be a compact cobordism $W' \colon M \rightsquigarrow \varnothing$ such that W is diffeomorphic to $\mathrm{Int}(W')$. Assume that W admits a compact closure $W' \colon M \rightsquigarrow \varnothing$. Show that the set of compact closures of W up to isomorphism of their interiors is in bijection with the set of invertible cobordisms over M.

Proof. (1)
Saying that $M_0 \rightsquigarrow M_1$ is invertible in Cob_d is precisely saying that there exists a cobordism $M_1 \rightsquigarrow M_0$ such that the composite cobordism $M_0 \rightsquigarrow M_1 \rightsquigarrow M_0$ is equivalent to the trivial cobordism $M_0 \rightsquigarrow M_0$. We will do this using the usual definition of cobordisms with boundaries. Then the problem is equivalently to show that we can find coborisms $M_0 \rightsquigarrow M_1 \rightsquigarrow M_0$ such that the composite is a product cobordism - i.e., has Morse number 0. In this case, we are dealing with closed compact manifolds W_0, W_1 such that $\partial W_0 \cong M_0$ and $\partial W_1 \cong M_1$. Furthermore, the boundaries have closed collar neighborhoods $\partial W_i \times I$, and removing some open/usual collar neighborhoods of these boundaries $\partial W_i \times [0,1)$ leaves us with compact spaces which are, by assumption, diffeomorphic. Now, take the cobordism W_0 and choose a collar neighborhood of ∂W_0: $M_0 \times [0,1]$, where M_0 is identified with $M_0 \times 0$ in W_0. By assumption, there is a diffeomorphism $W_0 - (M_0 \times [0,1]) \cong W_1 - (M_1 \times [0,1])$. Now, the diffeomorphism extends to the closure of the interiors which is also M_i since the collar is a cylinder, so we obtain a diffeomorphism $h \colon M_0 \times 1 \cong M_1 \times 1$. Without loss of generality, we can reparametrize, to get the diffeomorphism $h \colon M_0 \times 1 \to M_1 \times 0$ since the boundaries of the interiors must map to each other. Now we can glue the collars by gluing the cobordisms they represent using theorem 1.4 in Milnor's book on h-cobordisms to get a cobordism c_h which is the manifold $M_0 \times [0,1] \cup_h M_1 \times [0,1]$. This indeed now gives a cobordism $M_0 \rightsquigarrow M_1$. We can likewise obtain the cobordism $M_1 \rightsquigarrow M_0$ which is also obtained by gluing $M_1 \times [0,1]$ with $M_0 \times [0,1]$ along $M_1 \times 1$ and $M_0 \times 0$. Denote this cobordism by $c_{h'}$. We claim that $c_h c_{h'} = \mathrm{id}_{M_0}$. That is, that $c_h c_{h'}$ is a product cobordism/trivial cobordism of M_0. One way to see this is by using theorem 1.6 in Milnor's book on h-cobordisms which says that $c_h c_{h'} = c_{h'h} = c_{\mathrm{id}_{M_0}}$ which indeed is the trivial cobordism. Alternatively, each collar neighborhood has no critical values, so c_h and $c_{h'}$ both have Morse number 0, and then corollary 3.8 in Milnor's book on h-cobordisms gives that $c_h c_{h'}$ also has Morse number 0, hence is trivial by theorem 3.4 in the same book. ◻

References

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