Sheet 0

Exercise 0.1 (1)**.** *Proof.* (1) Choose $\varphi\colon \mathbb{Z}/2 \to \mathbb{Z}/4$ by $1 \mapsto 2$.

(2) Any map $\varphi\colon \mathbb{Z}/m \to \mathbb{Z}/n$ in Ring must take $1 \mapsto 1$, and is uniquely determined thereby since $\varphi(k) = \varphi(1 + \ldots + 1) = \varphi(1) + \ldots + \varphi(1) = k$. Therefore, $0 = \varphi(m) = m$ in $\mathbb{Z}/n$, so $n \mid m$. And it is clear that if $n \mid m$, then $1 \mapsto 1$ is a well-defined ring homomorphism. Thus

$$\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}/m, \mathbb{Z}/n) = \begin{cases} 1 \mapsto 1, & n \mid m \\ \varnothing, & \text{otherwise} \end{cases}.$$

(3) We claim that the correspondence

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x], R) &\to R \\ \varphi &\mapsto \varphi(x) \end{aligned}$$

is bijective. Indeed, $\varphi(1) = 1$ necessarily, so $\varphi(k) = k$ for all $k \in \mathbb{Z}$, so we simply have $\varphi\left(\sum \alpha_i x^i\right) = \sum \varphi(\alpha_i)\varphi(x)^i = \sum \alpha_i \varphi(x)^i$.

(4) Suppose $(\mathbb{Q}/\mathbb{Z}, +)$ admits a ring structure with multiplication $\cdot$. Let $\frac{a}{b} \in \mathbb{Q}/\mathbb{Z}$ be the unit. Then for any $\frac{x}{y} \in \mathbb{Q}/\mathbb{Z}$

$$\frac{x}{y} = \frac{x}{y} \cdot \frac{a}{b} = \frac{x}{y} \cdot \left(\frac{1}{b} + \ldots + \frac{1}{b}\right) = \frac{x}{y} \cdot \frac{1}{b} + \ldots + \frac{x}{y} \cdot \frac{1}{b}.$$

Therefore, we must have $\frac{x}{y} \cdot \frac{1}{b} = \frac{x}{ay}$. But then

$$\frac{x}{ay} = \frac{x}{y} \cdot \frac{1}{b}$$

implies

$$\frac{bx}{ay} = \frac{x}{y} \cdot \frac{1}{b} + \ldots + \frac{x}{y} \cdot \frac{1}{b} = \frac{bx}{y} \cdot \frac{1}{b}$$

Hence we get

$$\frac{x}{y} = \frac{x}{y} \cdot \frac{a}{b} = \frac{bx}{ay}$$

and so $\frac{a}{b} = 1$ as $\frac{x}{y}$ was arbitrary. However, $1 = 0$ in $\mathbb{Q}/\mathbb{Z}$, giving $\mathbb{Q}/\mathbb{Z} = \{0\}$, which is a contradiction.

(5) By (2), $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}/m, \mathbb{Z}/n)$ is either a single map or empty, and as the empty set is not a ring, this Hom set in general does not admit a ring structure - when it does, it must be the trivial one.

□

Exercise 0.2 (2)**.** (1) A admits a unique $\mathbb{Z}$-algebra structure since any ring homomorphism $\mathbb{Z} \to A$ is uniquely determined by $1 \mapsto 1$.

(2) Take $A = \mathbb{Z}[x]$ and R to be any ring with more than one element. By exercise 1.(c), $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x], R) \cong R$, so R admits more than one R-algebra structure.

However, these structures could be isomorphic in the sense that there exists maps $\varphi, \psi\colon R \to R$ with $\varphi\psi = \mathrm{id} = \psi\varphi$ and composing one algebra structure $\mathbb{Z}[x] \to R$ with φ gives ψ and vice versa.

So we must find explicit examples which are non-isomorphic. Define $f, g\colon \mathbb{Z}[x] \to \mathbb{Z}/6$ by $f(x) = 2$ and $g(x) = 3$. Now, there is no ring homomorphism $\varphi\mathbb{Z}/6 \to \mathbb{Z}/6$ such that $\varphi \circ f = g$ since $0 = \varphi(0) = \varphi \circ f(3x) = g(3x) = 3$ gives a contradiction.

Exercise 0.3 (3)**.** This is just the 4th isomorphism theorem for ideals of rings. Define a map $\pi\colon \mathcal{A} \to \mathcal{B}$ by sending $A \to \overline{A} = A + I$.
Suppose $\pi(A) = \pi(B)$. Then for any $a \in A$, there exists $b \in B$ such that $a - b \in I \subset A \cap B$, hence $a, b \in A \cap B$. Thus $A \subset B \subset A$, so $A = B$.
Now, suppose $V \in \mathcal{B}$. Let $A = \pi^{-1}(V)$. This is an ideal containing I. If $a, b \in A$ then $\pi(a), \pi(b) \in V$ so $\pi(ab) = \pi(a)\pi(b) \in V$, hence $ab \in \pi^{-1}(V)$. Similar closure for the rest. And if $r \in R$ then $\pi(ar) = \pi(a)\pi(r) \in V$ as $\pi(a) \in V$ and V is an ideal, so $ar \in A$, hence A is an ideal. This gives surjectivity.

Exercise 0.4 (4)**.** $(2, x) \subset \mathbb{Z}[x]$ is not principle as an ideal generated over $\mathbb{Z}$. If $(2, x) = (p(x))$, then $p(x) \mid 2$ implies that that the degree of p is 0. Now let $q(x)$ be such that $p(x)q(x) = x$ and $h(x)$ such that $p(x)h(x) = 2$. Then the degree of 2 is 0 and that of q is 1. Furthermore, as $p \in (2, x)$, we must have $p(x) = 2k(x)$. But this implies $2k(x)q(x) = x$, so $2 \mid x$ in $\mathbb{Z}[x]$, which is impossible.

Exercise 0.5 (7)**.**

(1) Surjectivity amounts to finding an $f \in K[x_1, \ldots, x_n]$ such that $f(y) = k$ for some arbitrary $k \in K$. Consider the map $f(x_1, \ldots, x_n) = k + (x_1 - y_1) \ldots (x_n - y_n)$. Or the constant polynomial at k works also. Now, $\varphi_y(f + g) = (f + g)(y) = f(y) + g(y) = \varphi_y(f) + \varphi_y(g)$, and $\varphi_y(fg) = \varphi_y(f)\varphi_y(g)$ is seen likewise.

That it is a homomorphism of K-algebras (with the standard K-algebra structure) amounts to showing that $\varphi_y(k) = k$ which is clear.

(2) Let $\varphi\colon K[x_1, \ldots, x_n] \to K$ be a ring homomorphism. Let $y_i = \varphi(x_i)$. Then $\varphi\left(\sum a_I x_I\right) = \sum \varphi(a_I) y_I = \varphi_{y_I}\left(\sum a_I x_I\right)$ (for this we need φ to be a K-algebra homomorphism of with $K[x_1, \ldots, x_n]$ in the standard structure. Is there a different way of arguing?)