Assignment 5

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39: Prove that the product of two path-connected spaces is path-connected.

Solution: Let X, Y be path-connected spaces. Let $(x,y),(x',y') \in X \times Y$. By path-connectedness of X and Y, choose paths $\alpha\colon [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = x'$ and $\beta\colon [0,1] \to Y$ such that $\beta(0) = y$ and $\beta(1) = y'$.
By theorem 3.13., the map $\gamma\colon [0,1] \to X \times Y$ by $\gamma(t) = (\alpha(t), \beta(t))$ is continuous, and hence a path from $\gamma(0) = (\alpha(0), \beta(0)) = (x,y)$ to $\gamma(1) = (\alpha(1), \beta(1)) = (x', y')$.
As (x,y) and (x',y') were arbitrary, we have that $X \times Y$ is path-connected.

40: If A and B are path-connected subsets of a space, and if $A \cap B$ is nonempty, prove that $A \cup B$ is path-connected.

Solution: Let $x, y \in A \cup B$. If x, y are both in A (resp, B), then by path connectedness of A (B), there exists a path connecting x and y contained in $A \subset A \cup B$ (resp, $B \subset A \cup B$) which induces a path into $A \cup B$ by composing with the inclusions $A \to A \cup B$ (resp, $B \to A \cup B$).

Assume therefore without loss of generality that $x \in A$ and $y \in B$.
Let $z \in A \cap B$. By the aforementioned case, we can find a path $\alpha\colon [0,1] \to A$ such that $\alpha(0) = x$ and $\alpha(1) = z$ and a path $\beta\colon [0,1] \to B$ such that $\beta(0) = z$ and $\beta(1) = y$.
Now define the function $\gamma\colon [0,1] \to A \cup B$ by

$$\gamma(t) = \begin{cases} \alpha(2t), & t \in \left[0, \frac{1}{2}\right] \\ \beta(2t-1), & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We have that γ is continuous on $[0, \frac{1}{2})$ where it is α and on $(\frac{1}{2}, 0]$ where it is β. On $\frac{1}{2}$, $\alpha(2t)$ and $\beta(2t-1)$ agree, so by the following gluing lemma, as $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ are closed, we find that γ is a continuous function and hence a path between $\gamma(0) = \alpha(0) = x$ and $\gamma(1) = \beta(1) = y$.

As x and y were arbitrary, $A \cup B$ is path-connected.
Gluing Lemma: If X and Y are closed in $X \cup Y$, and if $f\colon X \to Z$ and $g\colon Y \to Z$ are continuous and agree on $X \cap Y$, then the map $f \cup g\colon X \cup Y \to Z$ defined by $f \cup g(x) = \begin{cases} f(x), & x \in X \\ g(x), & x \in Y \end{cases}$ is continuous.

Proof: Let $C \subset Z$ be closed. Then $f^{-1}(C)$ is closed in X and hence closed in $X \cup Y$ since X is closed in $X \cup Y$.
Similarly, $g^{-1}(C)$ is closed in $X \cup Y$. Thus $(f \cup g)^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ is closed in $X \cup Y$, so $f \cup g$ is continuous.

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7: Describe each of the following spaces:
(a) the cylinder with each of its boundary circles identified to a point;
(b) the torus with the subset consisting of a meridianal and a longitudinal circle identified to a point;
(c) S^2 with the equator identified to a point;
(d) $\mathbb{R}^2$ with each of the circles centre the origin and of integer radius identified to a point.

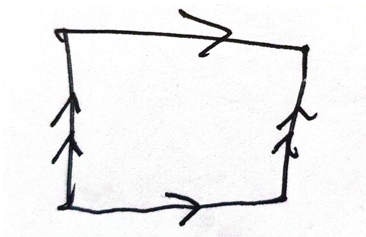
Solution:
(a) We claim that the cylinder with each of its boundary circles identified to a point is a closed ball.

Solution: Consider a cylinder $X \subset \mathbb{R}^3$ given by $X = S^1 \times I$. Let $i\colon S^1 \to S^2$ be the inclusion sending $(x,y) \to (x,y,0)$. Now define the map $f\colon X \to S^2$ by

$$f(x,s) = [i(x) + (0,0,1-2s)]\,[1 - |1-2s|]$$

This is a continuous surjective map. X is compact as the product of compact spaces and S^2 is Hausdorff as a subspace of $\mathbb{R}^3$.
Thus f is an identification map by corollary 4.4, and by theorem 4.2.(a), we get that S^2 is isomorphic with Y_* which is the cylinder with the boundary circles identified to points since f is injective everywhere except at $S^1 \times \{0,1\}$ where it collapses each S^1 $\{0\}$ and $S^1 \times \{1\}$ to a point.

(b) We claim that the resulting space is homeomorphic to S^2.
We know that the identification space for the torus is a rectangle with opposite sides identified as shown below

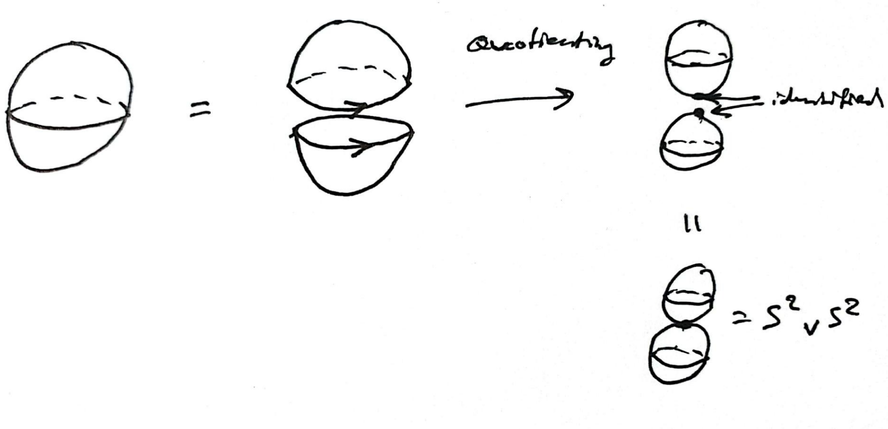


Meridean circles correspond to vertical lines on the rectangle and longitudinal circles correspond to horizontal lines. By simply "cutting" along the meridian and longitudinal circles which we collapse, we can without loss of generality assume that the lines on the rectangle which we collapse are the borders of the rectangle.
From here we have two reasonings which give the result:
(1) The resulting space is a CW-complex with a 2-cell attached to a 0-cell which is S^2.

(2) Alternatively, we can note that the resulting space is a single point adjoined with $(0,1) \times (0,1)$. Now, as I^2 is compact as the product of compact spaces, so is the identification space. Furthermore, it is clearly Hausdorff and hence is the one-point compactification of $(0,1) \times (0,1)$. Now, as $(0,1) \times (0,1)$ is homeomorphic with $\mathbb{R}^2$ and one point compactifications are unique up to homeomorphism that differs only on the appended point, we have that the obtained identification space from the torus is the one-point compactification of $\mathbb{R}^2$ which is precisely S^2 (by stereographic projection).

(c) We claim that it is the wedge sum of two 2-spheres, $S^2 \vee S^2$ - i.e. a 2-sphere attached to another 2-sphere by a common point, or equivalently, the quotienting of two disjoint 2-spheres by identifying a point on each sphere.

To see this, suppose we divide the 2-sphere into two closed hemispheres - a northern and a southern one. The borders of the hemispheres are identified, and furthermore, the borders of the hemispheres are precisely the equator. So collapsing each border to a point, we find two separate CW-complexes of a 0-cell with a 2-cell attached, which is a sphere - this is simply $S^2 = D^2/\partial D^2$. The 0-cells are identified, giving the quotient of two disjoint 2-spheres by identifying a point on each sphere - i.e., we get $S^2 \vee S^2$.



(d) We claim that this is an infinite sequence of 2-spheres, where each 2-sphere is attached to two others at its north and south pole by a common point.
To see this, let X denote the identification space $\mathbb{R}^2$ with each circle centre the origin and of integer

radius identified to its own point.
Now, define maps $\varphi_n \colon A_n \to S^1 \times I$ where A_n is the closed annulus

$$A_n = \left\{ x \in \mathbb{R}^2 \mid n \leq \|x\| \leq n+1 \right\},$$

by

$$\varphi_n(x,y) = \left(\frac{(x,y)}{\|(x,y)\|}, \|(x,y)\| - n \right).$$

φ_n is bijective, A_n is compact and $S^1 \times I$ is Hausdorff, hence φ_n is a homeomorphism.
Now, composing for each $n \geq 1$, φ_n with the translation $S^1 \times I \to S^1\,[n, n+1]$ by $(z,t) \to (z, t+n)$, which is a homeomorphism. Now, A_0 is the closed disk with the boundary identified, which is the standard method of creating S^2. As we just showed, $\bigcup_{n \geq 1} A_n$ is homeomorphic to the infinite cylinder with base S^1: $S^1 \times [1, \infty)$. Under this map, collapsing a circle with integer radius n corresponds to collapsing the circle $S^1 \times \{n\}$, which, by considering the infinite cylinder as $S^1 \times [1, \infty) = \bigcup_{n \in \mathbb{N}} S^1 \times [n, n+1]$, gives by part (a) and theorem 4.2, that X is homeomorphic with $S^1 \times [0, \infty)$ with each $S^1 \times \{n\}$ identified to a point for each $n \in \mathbb{N}_0$.