Section 1.1

iii. For any category C and any object $c \in C$, show that:

1. There is a category c/C whose objects are morphisms $f\colon c \to x$ with domain c and in which a morphism from $f\colon c \to x$ to $g\colon c \to y$ is a map $h\colon x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

   commutes, i.e., so that $g = hf$.

2. There is a category C/c whose objects are morphisms $f\colon x \to c$ with codomain c and in which a morphism from $f\colon x \to c$ to $g\colon y \to c$ is a map $h\colon x \to y$ between the domains so that the triangle

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

   commutes, i.e., so that $f = gh$.

The categories c/C and C/c are called the **slice categories** of C **under** and **over** c, respectively.

Solution: Each element has identity since

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "f"] & \\ x \arrow[rr, "\mathbb{1}"'] & & x \end{tikzcd}$$

commutes for any $f\colon c \to x \in \mathrm{Hom}\,(c/C)$.
Composites exist since if each of the smaller triangles in the diagram

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "h"] & \\ x \arrow[r, "\alpha"'] & y \arrow[r, "\beta"'] & z \end{tikzcd}$$

commutes, then $h = \beta \cdot g = \beta \cdot (\alpha \cdot f) = (\beta \cdot \alpha) \cdot f$, so $\beta \cdot \alpha\colon x \to z \in \mathrm{Hom}\,(c/C)$.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[d, "f"] \arrow[dr, "g"] & \\ x \arrow[r, "\mathbb{1}_x"'] & x \arrow[r, "h"'] & y \end{tikzcd}$$

so since the outer triangles commutes, $h \cdot \mathbb{1}_x \in \mathrm{Hom}\,(c/C)$. And $\mathbb{1}_y \cdot h$ similarly.
Now if $\alpha\colon x \to y, \beta\colon y \to z, \gamma\colon z \to w$ then

$$\begin{tikzcd} & & c \arrow[dll, "f"'] \arrow[dl, "g"] \arrow[dr, "h"'] \arrow[drr, "j"] & & \\ x \arrow[r, "\alpha"'] & y \arrow[rr, "\beta"'] & & z \arrow[r, "\gamma"'] & w \end{tikzcd}$$

and clearly $(\gamma \cdot \beta) \cdot \alpha = \gamma \cdot (\beta \cdot \alpha)$.

Section 1.3

Example 1.3.7.(v) For a generic small category C, a functor $C^{op} \to \text{Set}$ is called a (set-valued) **presheaf** on C. A typical example is the functor $\mathcal{O}(X)^{op} \to \text{Set}$ whose domain is the poset $\mathcal{O}(X)$ of open subsets of a topological space X and whose value at $U \subset X$ is the set of continuous real-valued functions on U. The action on morphisms is by restriction.
So suppose we have spaces X and Y and a continuous map $f\colon X \to Y$ inducing a morphism $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$. Let $F\colon \mathcal{O}(X)^{op} \to \text{Set}$ be the given functor. Now, an object in $\mathcal{O}(X)^{op}$ is an open subset $U \subset X$, so $F(U)$ is mapped to the set of continuous real-valued functions on U, $\mathcal{C}(U, \mathbb{R}) \subset \text{Set}$. A morphism $U \to V$ means that $U \hookrightarrow V$ is a continuous inclusion. Now, if $U \hookrightarrow V$ then any function continuous on V is continuous on U, so $F(V) \subset F(U)$. In particular, we define $F(f\colon U \to V) = F(V) \hookrightarrow F(U)$.

This map is functorial since if $U \xrightarrow{f} V \xrightarrow{g} W$ then $F(gf) = F(W) \hookrightarrow F(U) = F(W) \hookrightarrow F(V) \hookrightarrow F(U) = F(f)F(g)$.

Example vi. Presheaves on the category Δ, of finite non-empty ordinals and order-preserving maps, are called **simplicial sets**. Δ is also called the **simplex category**. The ordinal $n + 1 = \{0, 1, \ldots, n\}$ may be throught of as a direct version of the topological n-simplex and, with this interpretation in mind, is typically denoted by "$[n]$ " by algebraic topologists.

Def. A **monoid** is a set M equipped with an associative binary operation $M \times M \to M$ and an identity $e \in M$ serving as a two-sided identity. In other words, a monoid is precisely a one-objects category.
A commutative monoid is then one in which $\forall a, b \in M\colon ab = ba$.

We can also state this as saying that a **monoid** is an object $M \in \text{Set}$ together with a pair of morphisms $\mu\colon M \times M \to M$ and $\eta\colon 1 \to M$ so that the following diagrams commute:

$$\begin{tikzcd}
M \times M \times M \arrow[r, "\mathbb{1}_M \times \mu"] \arrow[d, "\mu \times \mathbb{1}_M"'] & M \times M \arrow[d, "\mu"] \\
M \times M \arrow[r, "\mu"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\eta \times \mathbb{1}_M"] \arrow[dr, "\mathbb{1}_M"'] & M \times M \arrow[d, "\mu"] & M \arrow[l, "\mathbb{1}_M \times \eta"'] \arrow[dl, "\mathbb{1}_M"] \\
& M &
\end{tikzcd}$$

The first diagram gives associativity, and the second gives the η acts as a two-sided identity. Note that $\eta\colon 1 \to M$ identifies an object in M which we call η, so in the morphism $M \to M \times M$ by $\eta \times \mathbb{1}_M$ we map an object $m \in M$ to $\eta \times m \in M \times M$.

Example 1.3.2.(xi) Any commutative monoid M can be used to define a functor $M^{-}\colon \text{Fin}_* \to \text{Set}$. Writing $n_+ \in \text{Fin}_*$ for the set with n non-basepoint elements, define M^{n_+} to be M^n, the n-fold cartesian product of the set M with itself. By convention, M^{0_+} is a singleton set. For any based map $f\colon m_+ \to n_+$, define the i th component of the corresponding function $M^f\colon M^m \to M^n$ by projecting from M^m to the coordinates indexed by elements in the fiber $f^{-1}(i)$ and then multiplying these using the commutative monoid structure; if the fiber is empty, the function M^f inserts the unit element in the i th coordinate. Note each of the sets M^n itself has a basepoint, the n-tuple of unit elements, and each of the maps in the image of the functor are based. It follows that the functor M^{-} lifts along the forgetful functor $U\colon \text{Set}_* \to \text{Set}$.

There is a special property satisfied by this construction that allows one to extract the commutative monoid M from the functor $\text{Fin}_* \to \text{Set}$. This observation was used by Segal to introduce a suitable notion of "commutative monoid" into algebraic topology.

See section 1.5, exercise 2 for continuation.

Section 1.3 - Problems

i. What is a functor between groups, regarded as one-object categories?

Solution: Suppose C, D are categories with one object each representing the groups G, H, respectively.

Then each morphism represents an element of the groups.
Let $F\colon C \to D$ be a functor. Then F maps the object to the object, and if a, b are morphisms in C, then $F(ab) = FaFb$. Thus F is simply a group homomorphism between the groups.

ii. What is a functor between preorders, regarded as categories?

Solution: Suppose $(P, \leq)$ and $(Q, <)$ are preorder categories.
Let $F\colon P \to Q$ be a functor. If $a, b \in P$ and $a \leq b$ then $\exists \alpha\colon a \to b$, and $F(\alpha)\colon Fa \to Fb$, so a functor between preorders is just any map that preserves order.

v: What is the difference between a functor $C^{op} \to D$ and a functor $C \to D^{op}$? What is the difference between a functor $C \to D$ and a functor $C^{op} \to D^{op}$?

Solution: Suppose $F\colon C^{op} \to D$ is a functor. We claim that $F^{op}\colon C \to D^{op}$ defined by being equal to F on objects and for $\alpha \in \mathrm{Hom}(C)$, we have $F^{op}(\alpha) = (F(\alpha^{op}))^{op}$.
This is a functor since for a composable pair $\alpha, \beta \in \mathrm{Hom}(C)$, we have $F^{op}(\alpha\beta) = F((\alpha\beta)^{op})^{op} = F(\beta^{op}\alpha^{op})^{op} = (F(\beta^{op})F(\alpha^{op}))^{op} = F(\alpha^{op})^{op}F(\beta^{op})^{op} = F^{op}(\alpha)F^{op}(\beta)$, and $F^{op}(\mathbb{1}_c) = F(\mathbb{1}_c^{op})^{op} = \mathbb{1}_{Fc}^{op} = \mathbb{1}_{F^{op}c}^{op}$, so F^{op} is a covariant functor $C \to D^{op}$. Hence each covariant $F\colon C^{op} \to D$ induces a covariant $F^{op}\colon C \to D^{op}$. Arguing by duality, we get that each functor $C^{op} \to D$ corresponds to a functor $C \to D^{op}$.

For a functor $F\colon C \to D$, we get that if $\alpha\colon a \to b$, then $F\alpha\colon Fa \to Fb$. So defining $F^{op}\colon C^{op} \to D^{op}$ by $F^{op}(\alpha^{op}) = F(\alpha)^{op}$, we get that $F^{op}(\alpha^{op}\beta^{op}) = F^{op}((\beta\alpha)^{op}) = F(\beta\alpha)^{op} = F(\alpha)^{op}F(\beta)^{op} = F^{op}(\alpha^{op})F^{op}(\beta^{op})$, and $F^{op}(\mathbb{1}_c^{op}) = F(\mathbb{1}_c)^{op} = \mathbb{1}_{Fc}^{op} = \mathbb{1}_{F^{op}c}^{op}$.

vi (The comma category). Given functors $F\colon D \to C$ and $G\colon E \to C$, show that there is a category called the **comma category** $F \downarrow G$, which has

- as objects, triples $(d \in D, e \in E, f\colon Fd \to Ge \in C)$, and
- as morphisms $(d, e, f) \to (d', e', f')$, a pair of morphisms $(h\colon d \to d', k\colon e \to e')$ so that the square

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "Fh"] & Ge \arrow[d, "Gk"] \\ Fd' \arrow[r, "f'"] & Ge' \end{tikzcd}$$

  commutes in C, i.e., so that $f' \cdot Fh = Gk \cdot f$

Define a pair of projection functors $\mathrm{dom}\colon F \downarrow G \to D$ and $\mathrm{cod}\colon F \downarrow G \to E$.

Solution: We must firstly check that each object has an identity morphism.
Now, since each object in D and E has an identity, we have that since

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "F\mathbb{1}_d = \mathbb{1}_{Fd}"'] & Ge \arrow[d, "G\mathbb{1}_e = \mathbb{1}_{Ge}"] \\ Fd \arrow[r, "f"'] & Ge \end{tikzcd}$$

obviously commutes, the morphism $(1_{Fd}\colon Fd \to Fd, \mathbb{1}_{Ge}\colon Ge \to Ge)$ is the identity of the object $(d \in D, e \in E, f\colon Fd \to Ge \in C)$.

Now we must make sure that composition is defined.
Suppose $(d, e, f) \overset{(h,k)}{\to} (d', e', f') \overset{(h',k')}{\to} (d'', e'', f'')$. Then

$$\begin{tikzcd}
Fd \arrow[r, "f"] \arrow[d, "Fh"'] & Ge \arrow[d, "Gk"] \\
Fd' \arrow[r, "f'"] \arrow[d, "Fh'"'] & Ge' \arrow[d, "Gk'"] \\
Fd'' \arrow[r, "f''"] & Ge''
\end{tikzcd}$$

Now commutativity of each inner square gives that the outer rectangle commutes, hence $f'' \cdot Fh' \cdot Fh = Gk' \cdot Gk \cdot f$, so the square

$$\begin{tikzcd}
Fd \arrow[r, "f"] \arrow[d, "F(h' \cdot h)"'] & Ge \arrow[d, "G(k' \cdot k)"] \\
Fd'' \arrow[r, "f''"'] & Ge''
\end{tikzcd}$$

commutes by functoriality of F and G and since $h' \cdot h \in \mathrm{Hom}\,(d, d'')$ and $k' \cdot k \in \mathrm{Hom}\,(e, e'')$.

Now suppose $(d, e, f) \overset{(h\colon d \to d', k\colon e \to e')}{\to} (d', e', f')$. Then $f' \cdot \mathbb{1}_{Fd'} \cdot Fh = \mathbb{1}_{Ge'} \cdot Gk \cdot f$, so $f' \cdot F(h) = f' \cdot F(\mathbb{1}_{d'} \cdot h) = G(\mathbb{1}_{e'} \cdot k) \cdot f = G(k) \cdot f$, so $(\mathbb{1}_{d'}, \mathbb{1}_{e'}) \circ (h, k) = (h, k)$, and similarly, $(h, k) \circ (\mathbb{1}_d, \mathbb{1}_e)$. Associativity requires that

$$(h'', k'') \circ ((h', k') \circ (h, k)) = ((h'', k'') \circ (h', k')) \circ (h, k)\,.$$

This is true if

$$F(h'' \cdot (h' \cdot h)) = F((h'' \cdot h') \cdot h) \quad \text{and} \quad G(k'' \cdot (k' \cdot k)) = G((k'' \cdot k') \cdot k)$$

which is true since composition of morphisms in C is associative as it is a category.

Define $\mathrm{dom}(d, e, f) = d$ and $\mathrm{dom}\,(h\colon d \to d', k\colon e \to e') = h\colon d \to d'$. Then

$$\mathrm{dom}\,((h'\colon d' \to d'', k'\colon e' \to e'') \cdot (h\colon d \to d', k\colon e \to e')) = h' \cdot h\colon d \to d'' = \mathrm{dom}\,(h', k') \cdot \mathrm{dom}\,(h, k)\,.$$

and similarly for cod.

$$\mathrm{dom}\,(\mathbb{1}_d, \mathbb{1}_e) = \mathbb{1}_d = \mathbb{1}_{\mathrm{dom}(d,e,f)}, \quad \mathrm{cod}\,(\mathbb{1}_d, \mathbb{1}_e) = \mathbb{1}_e = \mathbb{1}_{\mathrm{cod}(d,e,f)}.$$

vii. Define functors to construct the slice categories c/C and C/c of exercise 1.1.iii as special cases of comma categories constructed in exercise 1.3.vi. What are the projection functors?

Solution: Let D be the single-object category $\mathbb{1}$ and F be the functor sending the object to c. Let $E = C$ and G be the trivial functor on C. Then an object consists of $(c, e \in C, f\colon c \to e)$, and morphisms consists of $(h\colon c \to c, k\colon e \to e') = (\mathbb{1}_c, k\colon e \to e')$.
Then dom sends any object to the underlying single object in D and any morphism to the identity in D; and cod sends any object (d, e, f) to e and (h, k) to k.

viii. Lemma 1.3.8 shows that functors preserve isos. Find an example to demonstrate that functors need not **reflect isos:** i.e., find a functor $F\colon C \to D$ and a morphism f in C so that Ff is an iso in D but f is not an iso in C.

Solution: Suppose simply $C = \{a, b\}$ with identities and a morphism $f\colon a \to b$ that is not iso. Then map it all to the category $\mathbb{1}$. Any morphism in the image is an iso as there is only one, thus solving the problem.

1.3.ix. The operators $Z(-), C(-), \mathrm{Aut}(-)$ sending a group to its center, commutator subgroup and automorphism group, respectively, all define functors on the discrete category of groups (with only identity morphisms) to Group. Are they functorial in the isomorphisms of groups, i.e., do they extend to functors $\mathrm{Group}_{\mathrm{iso}} \to \mathrm{Group}$?
Solution: Suppose $G \cong H$ by φ, so $\varphi\colon G \to H$. Then is $Z(\varphi)\colon Z(G) \to Z(H)$ functorial? If $h \in Z(G)$

then for all $g \in Z(H), \varphi(h)\varphi(\varphi^{-1}g)) = \varphi\left(h\varphi^{-1}(g)\right) = \varphi\left(\varphi^{-1}(g)h\right) = g\varphi(h)$, so $\varphi(h) \in Z(H)$ and $Z(\varphi)$.

We indeed have that if $G \cong H$ then $Z(G) \cong Z(H)$ by the restriction of φ, so $Z(\varphi)\colon Z(G) \to Z(H)$ by an iso. Furthermore, $Z(\mathbb{1}_G) = \mathbb{1}_{Z(G)}$ and $Z(\psi \cdot \varphi) = Z(\psi) \cdot Z(\varphi)$.
Now, for $C(-)$, if $G \cong H$ by φ with inverse ψ then if $x = ghg^{-1}h^{-1}$, we have $\varphi(x) = \varphi(g)\varphi(h)\varphi(g)^{-1}\varphi(h)^{-1} \in C(H)$, and clearly $\psi(\varphi(x)) = x \in C(G)$. So $C(-)$ extends to a functor $\text{Group}_{\text{iso}} \to \text{Group}$ and clearly $C(\mathbb{1}_G) = \mathbb{1}_{C(G)}$.
For $\text{Aut}(-)$, if $G \cong H$ by φ with inverse ψ, then $\text{Aut}(\varphi) = \varphi \circ -$. Functoriality: $\text{Aut}\,(\varphi \circ \psi) = (\varphi \circ \psi) \circ - = \varphi \circ (\psi \circ -) = \text{Aut}\,(\varphi) \circ \text{Aut}\,(\psi)$.

So each extends to $\text{Group}_{\text{iso}} \to \text{Group}$.

Does it extend to $\text{Group}_{\text{epi}} \to \text{Group}$?

Functoriality of $Z(-)$. If $\varphi\colon G \to H$ is a homomorphism such that if $h\varphi = k\varphi$ then $h = k$ and similarly for $\psi\colon H \to J$. I don't believe it extends here.

Section 1.4

ii. What is a natural transformation between a parallel pair of functors between groups, regarded as one-object categories?

Solution: Suppose $\alpha\colon F \implies G$ is a natural transformation where $F, G\colon BG \to BH$. So

$$\begin{array}{ccc} BH & \xrightarrow{\alpha} & BH \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ BH & \xrightarrow[\alpha]{} & BH \end{array}$$

commutes. Hence $\alpha' \cdot Ff = Gf \cdot \alpha$ for all $f \in \text{Hom}(BG)$. So a natural transformation between parallel pairs of functors between groups as one-object categories is an element $g \in \text{Hom}(BH)$, so $g \in H$, such that $g \cdot Ff = Gf \cdot g$, i.e., $Ff = g^{-1}Gfg$, so $F(-) = g^{-1}G(-)g$.

iii. What is a natural transformation between a parallel pair of functors between preorders, regarded as categories?

Solution: Suppose $\alpha\colon F \implies G$ and $F, G\colon P \to Q$ where $(P, \leq), (Q, <)$ are preorders. Then for any $f\colon c \to d$ in P, we have

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & Gc \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fd & \xrightarrow[\alpha_d]{} & Gd \end{array}$$

commutes. What this means is that $Fc < Fd < Gd$ and $Fc < Gc < Gd$, so α_c is the map $Fc \to Gc$ and α_d is the map $Fd \to Gd$.
So $\alpha_x\,(Fx) = Gx$ for all $x \in P$. Thus, for all $x \in P$, $Fx < Gx$, so $F < G$ on all of P.

iv. In the notation of example 1.4.7, prove that distinct parallel morphisms $f, g\colon c \to d$ define distinct natural transformations

$$f_*, g_*\colon C\,(-, c) \implies C(-, d) \quad \text{and} \quad f^*, g^*\colon C(d, -) \implies C(c, -)$$

by post- and pre-composition.

Solution: We must show that $(f_*)_x \neq (g_*)_x$ for some x and similarly with f^*, g^*. Now, if $h\colon y \to x$

in C, then

$$\begin{tikzcd} C(x,c) \arrow[r, "f_*"] \arrow[d, "h^*"] & C(x,d) \arrow[d, "h^*"] \\ C(y,c) \arrow[r, "f_*"] & C(y,d) \end{tikzcd}$$

If f_* and g_* define the same natural transformation, then $(f_*)_c = (g_*)_c$, but $(f_*)_c = \mathbb{1}_d f(\mathbb{1}_c) = f(\mathbb{1}_c) = f$ and $(g_*)_c = g(\mathbb{1}_c) = g$, so $f = g$, hence $(f_*)_c \neq (g_*)_c$.
Similarly for f^* and g^*.

v. Construct a canonical natural transformation $\alpha\colon F\,\mathrm{dom} \implies G\,\mathrm{cod}$ between the functors that form the boundary of the square

$$\begin{tikzcd} F \downarrow G \arrow[r, "\mathrm{cod}"] \arrow[d, "\mathrm{dom}"'] & E \arrow[d, "G"] \\ D \arrow[r, "F"'] \arrow[ur, Rightarrow, "\alpha"] & C \end{tikzcd}$$

Solution: Suppose

$$(d,e,f) \overset{(h,k)}{\to} (d',e',f')$$

so $F\,\mathrm{dom}\,(d,e,f) = Fd$ while $G\,\mathrm{cod}\,(d,e,f) = Ge$, so

$$\begin{tikzcd} Fd \arrow[r, "\alpha_d"] \arrow[d, "Fh"'] & Ge \arrow[d, "Gk"] \\ Fd' \arrow[r, "\alpha_{d'}"'] & Ge' \end{tikzcd}$$

so choosing $\alpha_d = f$ and $\alpha_{d'} = f'$, we get that the above commutes by definition of the comma category.

Definition. A functor $F\colon C \to D$ is

- **full** if $\forall x, y \in C\colon C(x,y) \to D(Fx,Fy)$ is surjective.
- **faithful** if $\forall x, y \in C$, the map $C(x,y) \to D(Fx,Fy)$ is injective;
- and **essentially surjective on objects** if for every object $d \in D$, $\exists c \in C$ such that d is isomorphic to Fc, i.e., $d \cong Fc$.

Remark: A faithful functor that is injective on objects is called an **embedding** and identifies the domain category as a subcategory of the codomain; in this case, faithfulness implies that the functor is globally injective on arrows. A full and faithful functor, called **fully faithful**, that is injective-on-objects defines a **full embedding** of the domain category into the codomain category. The domain then ddefines a **full subcategory** of the codomain.

Theorem. (Characterizing equivalences of categories) A functor defining an equivalence of categories is full, faithful and essentially surjective on objects. Assuming the axiom of choice, any functor with these properties defines an equivalence of categories.

Proof. Suppose $F\colon C \to D, G\colon D \to C$ and $\varepsilon\colon FG \cong \mathbb{1}_D$ and $\eta\colon GF \cong \mathbb{1}_C$.

Essentially surjective: let $d \in D$, then $FGd \cong d$ by ε_d, so F is essentially surjective.

Full: Suppose $Fx, Fy \in D$ and $d\colon Fx \to Fy$ is in D. Then there exists $x', y' \in D$ such that $Gx' \overset{\alpha}{\cong} x, Gy' \overset{\beta}{\cong} y$. Now

$$\begin{tikzcd} Fx \arrow[r, "\cong"] \arrow[d, "d"] & FGx' \arrow[r, "\varepsilon_{x'}"] \arrow[d, "F\left(\beta \cdot d \cdot \alpha^{-1}\right)"] & x' \arrow[d, dashed, "d'"] \\ Fy \arrow[r, "\cong"'] & FGy' \arrow[r, "\varepsilon_{y'}"'] & y' \end{tikzcd}$$

commutes, so there exists a unique map $d'\colon x' \to y'$ such that the above commutes, namely, $d' = \varepsilon_{y'} \cdot F(\beta) \cdot d \cdot F \cdot \left(\alpha^{-1}\right) \cdot \varepsilon_{x'}^{-1}$ - unique since F is a functor and hence preserves isomorphisms.

Faithful: Suppose $x, y \in C$, and $d\colon x \to y$ in C. Suppose $\exists \tilde{d}\colon x \to y$ and $F(d) = F(\tilde{d})$.
Then there exist $x', y' \in D \mid Gx' \cong x, Gy' \cong y$, so

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[d, "d;\tilde{d}"'] & GFx \arrow[d, "Gd=GFd=G\tilde{d}"] \\ y \arrow[r, "\eta_y"] & GFy \end{tikzcd}$$

Now, since $Gd = G\tilde{d}$ and η_x, η_y are isos, the map $x \to y$ making the diagram commute is uniquely determined to be $\eta_y^{-1} G d \eta_x$, so $d = \tilde{d}$.

For the opposite direction, suppose $F\colon C \to D$ is fully faithful and essentailly surjective on objects. We wish to defined $G\colon D \to C$ such that $FG \cong \mathbb{1}_D$ by ε and $GF \cong \mathbb{1}_C$ by η.
Let $x, y \in D$. Then by essential surjectivity, there must exist $a, b \in C$ such that $Fa \cong x, Fb \cong y$. By fully faithfulness, we have that $\mathrm{Hom}(a, b) \xrightarrow{F} \mathrm{Hom}(x, y)$ is bijective. Define $G(x)$ to be any element such that $G(x) \cong a$ and similarly $G(y) \cong b$. Now for any $d\colon x \to y$, we have that there exists $d'\colon a \to b$ such that $F(d') = d$, so define $G(d) = qd'p$ where p and q are isos making the domain and codomain match up with $G(x)$ and $G(y)$. This is a functor since if $\overline{d}\colon y \to z$ then since $Gy = b$, and some $c \in C$ has $Gz \cong c$, we have that there exists $d''\colon b \to c$ such that $G\overline{d} = rd''p^{-1}$ where r maps Gz to c. So $G\left(\overline{d}d\right) = rd''d'q = rd''p^{-1}pd'q = G(\overline{d})G(d)$, and $G\left(\mathbb{1}_x\right) = p^{-1}\mathbb{1}_a p = \mathbb{1}_{Gx}$.

Alternatively as in the book, we can use diagram chasing:
Suppose $F\colon C \to D$ is fully faithful and essentially surjective on objects. Then using essential surjectivity and the axiom of choice, we can, for each $d \in D$, choose a $Gd \in C$ such that $\varepsilon_d\colon FGd \cong d$. Now, suppose $\alpha\colon d \to d'$, then there is precisely one morphism $Gd \to Gd'$ making the following commute:

$$\begin{tikzcd} Gd \arrow[r, "\cong"] \arrow[d, dashed] & d \arrow[d, "\alpha"] \\ Gd' \arrow[r, "\cong"] & d' \end{tikzcd}$$

so we define $G\alpha$ as this morphism.

We must then check functriality of G :

$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"] \arrow[d, "FG\mathbb{1}_d \text{ or } F\mathbb{1}_{Gd}"'] & d \arrow[d, "\mathbb{1}_d"] \\ FGd \arrow[r, "\varepsilon_d"] & d \end{tikzcd}$$

By ε being a natural transformation, $FG\mathbb{1}_d$ makes the diagram commute, and since F is a functor, $F\mathbb{1}_{Gd} = \mathbb{1}_{FGd}$, so clearly $F\mathbb{1}_{Gd} = \varepsilon_d^{-1}\mathbb{1}_d\varepsilon$, hence, since only one morphism makes the diagram commute, $FG\mathbb{1}_d = F\mathbb{1}_{Gd}$ and since F is faithful, $G\mathbb{1}_d = \mathbb{1}_{Gd}$.

We must now check that $G\left(d'd\right) = G(d')G(d)$ if $d\colon a \to b$ and $d'\colon b \to c$. But

$$\begin{tikzcd} FGa \arrow[r, "\varepsilon_a"] \arrow[d, "FG(d'd) \text{ or } F(G(d')G(d))"'] & a \arrow[d, "d'd"] \\ FGc \arrow[r, "\varepsilon_c"'] & c \end{tikzcd}$$

so again by the same arguments, $G(d'd) = G(d')G(d)$. By full and faithfulness of F, we may define the iso $\eta_c\colon c \to GFc$ by specifying isos $F\eta_c\colon Fc \to FGFc$. Define $F\eta_c$ to be ε_{Fc}^{-1}. For any $f\colon c \to c'$, the outer triangle

$$\begin{tikzcd} Fc \arrow[r, "F\eta_c"] \arrow[d, "Ff"'] & FGFc \arrow[r, "\varepsilon_{Fc}"] \arrow[d, "FGFf"] & Fc \arrow[d, "Ff"] \\ Fc' \arrow[r, "F\eta_{c'}"'] & FGFc' \arrow[r, "\varepsilon_{Fc'}"'] & Fc' \end{tikzcd}$$

commutes. The right-hand square commutes by naturality of ε. This implies that the left-hand square commutes. Faithfulness of F tells us that $\eta_{c'} \cdot f = GFf \cdot \eta_c$, so η is a natural transformation.

Cor. For any field $\mathbb{F}$, the categories $\mathrm{Mat}_{\mathbb{F}}$ and $\mathrm{Vect}_{\mathbb{F}}^{\mathrm{fd}}$ are equivalent. We have

$$\mathrm{Mat}_{\mathbb{F}} \underset{H}{\overset{\mathbb{F}^{(-)}}{\longleftrightarrow}} \mathrm{Vect}_{\mathbb{F}}^{\mathrm{basis}} \underset{C}{\overset{U}{\longleftrightarrow}} \mathrm{Vect}_{\mathbb{F}}^{\mathrm{fd}}$$

Here U is the forgetful functor, $\mathbb{F}^{(-)}$ is the functor sending n to the vector space $\mathbb{F}^n$, equipped with the standard basis. The functor H carries a vector space to its dimension and a linear map $\varphi\colon V \to W$ to the matrix expressing the action of φ on the chosen basis of V with respect to the chosen basis of W. The functor C is defined by choosing a basis for each vector space.

Now, $\mathbb{F}^{(-)}$ is essentially surjective on objects since given a vector space V with a basis $\mathcal{B}$, it is isomorphic to $\mathbb{F}^m$ for some m and hence $(V, \mathcal{B}) \cong F(m)$.
It is full, since for any $n, m \in \mathbb{Z}_+$, we have that if $\varphi\colon \mathbb{F}^n \to \mathbb{F}^m$ is linear, there exists a map $\varphi'\colon n \to m$ in $\mathrm{Mat}_{\mathbb{F}}$ such that $\mathbb{F}^{(-)}(\varphi') = \varphi$ by definition.
It is faithful, since a linear map is uniquely determined on the basis. Thus $\mathrm{Mat}_{\mathbb{F}}$ is equivalent to $\mathrm{Vect}_{\mathbb{F}}^{\mathrm{basis}}$.
Similarly, it is clear that U is essentially surjective, since any finite-dimensional vector space has a basis. It is fully faithful since any linear map between vector spaces is uniquely determined by its values on the basis.
Hence $\mathrm{Vect}_{\mathbb{F}}^{\mathrm{basis}}$ is equivalent to $\mathrm{Vect}_{\mathbb{F}}^{\mathrm{fd}}$.

Def. A category is **connected** if any pair of objects can be connected by a finite zig-zag of morphisms.

Def. The automorphism group of an object $c \in C$ is $\mathrm{Hom}(c, c)$ in C.

Prop. Any connected groupoid is equivalent, as a category, to the automorphism group of any of its objects.

Proof: Choose an object $c \in C$ where C is a connected groupoid. Let $\mathrm{Aut}(c) = \mathrm{Hom}(c, c)$ be the automorphism group represented by a single-object category. Then define a functor $F\colon \mathrm{Aut}(c) \to C$ by mapping the object to c and any morphism in $\mathrm{Aut}(c)$ to the corresponding morphism in C. This is clearly fully faithful as it is bijective, and it is essentially surjective on objects since C is connected.

Cor. In a path-connected space X, any choice of basepoint $x \in X$ yields an isomorphic fundamental group $\pi_1(X, x)$.

Proof: Any space X has a fundamental groupoid $\Pi_1(X)$ whose objects are points in X and whose morphisms are endpoint-preserving homotopy classes of paths in X. Picking any point x, the group of automorphisms of the object $x \in \Pi_1(X)$ is exactly the fundamental group $\pi_1(X, x)$. Now the previous proposition implies that any pair of automorphism groups are equivalent, as categories, to the fundamental groupoid

$$\pi_1(X, x) \hookrightarrow \Pi_1(X) \hookleftarrow \pi_1(X, x')$$

and thus to each other. An equivalence between 1-object categories is an isomorphism of categories.

Recall: an **isomorphism of categories** is given by a pair of inverse functors $F\colon C \to D$ and $G\colon D \to C$ such that $FG = \mathbb{1}_D$ and $GF = \mathbb{1}_C$, i.e., the identity functors of D and C, respectively. An iso induces a bijection between the objects of C and D and likewise for morphisms.

And since isomorphisms of groups regarded as 1-object categories is exactly isomorphisms of groups in the usual sense, we get that all of the fundamental groups defined by choosing a basepoint in a path-connected space are isomorphic.

Def. A category C is **skeletal** if it contains just one object in each isomorphism class. The **skeleton** $\operatorname{sk} C$ of a category C is the unique (up to isomorphism) skeletal category that is equivalent to C.

An equivalence between skeletal categories is necessarily and isomorphism of categories. Suppose that $\operatorname{sk} C$ is isomorphic to $\operatorname{sk} D$ by $F\colon \operatorname{sk} C \to \operatorname{sk} D$ and $G\colon \operatorname{sk} D \to \operatorname{sk} C$.
Since the inclusion $\iota_C\colon \operatorname{sk} C \to C$ and $\iota_D\colon \operatorname{sk} D \to D$ define functors that are fully faithful and essentially surjective on object, they are equivalences of categories, so $C \cong \operatorname{sk} C \cong \operatorname{sk} D \cong D$ and hence C and D are equivalent.
Thus, two categories are equivalence if and only if their skeletons are isomorphic - so in particular, a category is always equivalent to its skeleton.

Example. The skeleton of a connected groupoid is the group of automorphisms of any of its objects.
Example. The skeleton of $\mathrm{Vect}^{\mathrm{fd}}_{\Bbbk}$ is the category $\mathrm{Mat}_{\Bbbk}$.
Def. Fin is the category of finite sets where morphisms are functions between sets. $\mathrm{Fin}_{\mathrm{iso}}$ is the maximal subgroupoid of Fin.

Example. The skeleton of $\mathrm{Fin}_{\mathrm{iso}}$ is the category whose objects are positive integers and with $\mathrm{Hom}(n, n) = \Sigma_n$, the group of permutations of n elements. The hom-sets between distinct natural numbers are all empty.

Example. (a categorification of sum orbit satiblizers) Let $X\colon BG \to \mathrm{Set}$ be a left G-set. Its **translation groupoid** $T_G X$ has elements of X as objects. A morphism $g\colon x \to y$ is an element $g \in G$ so that $g \cdot x = y$. The objects in the skeleton $\operatorname{sk} T_G X$ are the connected components in the translation groupoid. These are precisely the **orbit** of the group action, which partition X in precisely this manner.

Consider $x \in X$ as a representative of its orbit O_x. Because the translation groupoid is equivalent to its skeleton, we must have

$$\mathrm{Hom}_{\operatorname{sk} T_G X}(O_x, O_x) \cong \mathrm{Hom}_{T_G X}(x, x) =: G_x$$

the set of automorphisms of x. So $\mathrm{Hom}_{T_G X}(x, x)$ is the **stabilizer** G_x of x wrt. the G-action.
This means that any pair of elements in the same orbit must have isomorphic stabilizers. In summary, the skeleton of the translation groupoid, as a category, is the disjoint union of the stabilizer groups, indexed by the orbits of the action of G on X.
Thus $\operatorname{sk} T_G X = \bigcup_{O_x} G_x$ as a category. The set of morphisms in the translation groupoid with domain x is isomorphic to G. This set may be expressed as a disjoint union of hom-sets $\bigcup_{y \in O_x} \mathrm{Hom}_{T_G X}(x, y)$. Now, for any $g \in \mathrm{Hom}_{T_G X}(x, x)$, choose any $h\colon x \to y$. Then $\mathrm{Hom}_{T_G X}(x, y) = h\,\mathrm{Hom}_{T_G X}(x, x)$. The $\supset$ is clear, and for any $j\colon x \to y$, $j = hh^{-1}j \in h\,\mathrm{Hom}_{T_G X}(x, x)$, so $\subset$ is shown. Now, we claim that the map $g \mapsto hg$ is injective. Suppose $hg = hg'$. Then $g = g'$ by multiplying by h^{-1} on both sides. Hence $|hG_x| = |G_x|$. Thus

$$|G| = \left| \bigcup_{y \in O_x} \mathrm{Hom}_{T_G X}(x, y) \right| = \sum_{y \in O_x} |\mathrm{Hom}_{T_G X}(x, y)| = \sum_{y \in O_x} |G_x| = |O_x|\,|G_x|\,.$$

We define some concepts up to equivalence. We say a category is **essentially small** if it is equivalent to a small category, or, equivalently, if its skeleton is small. A category is **essentially discrete** if it is equivalent to a discrete category.

The following are invariant under equivalence:

- Smallness of categories
- Being a groupoid - i.e., any category equivalent to a groupoid is a groupoid.
- If $C \simeq D$ then $C^{op} \simeq D^{op}$.
- If $C \simeq D$ and $A \simeq B$ then $C \times A \simeq D \times B$.
- An arrow in C is an isomorphism iff its image under an equivalence $C \xrightarrow{\simeq} D$ is an isomorphism.

Section 1.5 - Problems

ii. Segal defined a category Γ as follows:

> Γ is the category whose objects are all finite sets, and whose morphisms from S to T are maps $\theta\colon S \to P(T)$ such that $\theta(\alpha)$ and $\theta(\beta)$ are disjoint when $\alpha \neq \beta$. The composite of $\theta\colon S \to P(T)$ and $\varphi\colon T \to P(U)$ is $\psi\colon S \to P(U)$ where $\psi(\alpha) = \bigcup_{\beta \in \theta(\alpha)} \varphi(\beta)$.

Prove that Γ is equivalent to the opposite of the category Fin_* of finite pointed sets. In particular, the functors introduced in Example 1.3.2.(xi) defined presheaves on Γ.

Solution: Can we define a functor $F\colon \Gamma \to \mathrm{Fin}_*^{op}$ that is fully faithful and essentially surjective on objects?

iv. Show taht fully faithful functors $F\colon C \to D$ both **reflect** and **create isomorphisms**. I.e., show that

1. If f is a morphism in C such that Ff is a morphism in D, then f is an isomorphism.
2. If x and y are object in C so that $Fx \cong Fy$ in D, then $x \cong y$ in C.

Solution: Suppose $f\colon c \to d$ in C so that $Ff\colon Fc \to Fd$ is an iso in D. Then there exists $g\colon Fd \to Fc$ such that $Ffg = \mathbb{1}_{Fd}$ and $gFf = \mathbb{1}_{Fc}$. But since F is full, the map $F_*\colon \mathrm{Hom}(d,c) \to \mathrm{Hom}(Fd, Fc) \ni g$ is surjective, so $\exists g' \in \mathrm{Hom}(d,c)$ such taht $Fg' = g$. Then $F(g'f) = \mathbb{1}_{Fc}$ and $F(fg') = \mathbb{1}_{Fd}$, so since F is faithful, $g'f = \mathbb{1}_c$ and $fg' = \mathbb{1}_d$, hence g' is the inverse of f.

For (ii), suppose $Fx \cong Fy$ with $d\colon Fx \to Fy$ and $e\colon Fy \to Fx$ being the isos. Since F is full, there exist $d'\colon x \to y$ and $e'\colon y \to x$ such that $Fd' = d, Fe' = e$. Now $\mathbb{1}_{Fy} = de = Fd'Fe' = F(d'e')$, and $\mathbb{1}_{Fx} = ed = F(e'd')$, so since F is faithful, $e'd' = \mathbb{1}_x$ and $d'e' = \mathbb{1}_y$.

v. Find an example to show that a faithful functor need not reflect isomorphisms.

Solution: Suppose we let $C = \mathbb{1}$ and $\mathrm{ob}\, D = \{a, b\}$ and $\mathrm{Hom}\, D = \{a \to b, b \to a, \mathbb{1}_a, \mathbb{1}_b\}$. Let $F0 = a, F1 = b$ and $F(0 \to 1) = a \to b$. Then F does not reflect $a \to b$ even though it is an isomorphism.

vii. Let G be a connected groupoid and let $\mathrm{Aut}\, G$ be the group of automorphisms at any of its objects. The inclusion $B\,\mathrm{Aut}\, G \hookrightarrow G$ defines an equivalence of categories. Construct an inverse equivalence $G \to B\,\mathrm{Aut}\, G$.

Solution: The inclusion $B\,\mathrm{Aut}\, G \hookrightarrow G$ defines an equivalence. Suppose $F\iota \simeq \mathbb{1}_{B\,\mathrm{Aut}\, G}$ and $\iota F \simeq \mathbb{1}_G$, so

$$\begin{tikzcd} F\iota * \arrow[r] \arrow[d, "F\iota g"] & * \arrow[d, "g"] \\ F\iota * \arrow[r] & * \end{tikzcd}$$

$$\begin{tikzcd} \iota F x \arrow[r] \arrow[d, "\iota F f"] & x \arrow[d, "f"] \\ \iota F y \arrow[r] & y \end{tikzcd}$$

Letting $\varepsilon\colon \iota F \implies \mathbb{1}_G$, we get that $\iota F(f) = \varepsilon_y^{-1} \cdot f \cdot \varepsilon_x$, so since the inclusion is faithful, there is precisely one morphism $\iota^{-1}\left(\varepsilon_y^{-1} \cdot f \cdot \varepsilon_x\right)$ for $F(f)$. Then

$$\begin{tikzcd} * \arrow[r] \arrow[d, "\iota^{-1}\left(\varepsilon_y^{-1} g \varepsilon_x (\iota g)\right)"'] & * \arrow[d, "g"] \\ * \arrow[r] & * \end{tikzcd}$$

ix: Show that any category that is equivalent to a locally small category is locally small.

Proof: Suppose $C \simeq D$ with $F\colon C \to D$ and $G\colon D \to C$ and $FG \simeq \mathbb{1}_D, GF \simeq \mathbb{1}_C$ and C is locally small. Then let $x', y' \in D$. By essential surjectivity on objects, $\exists x, y \in C$ such that $Fx \cong x', Fy \cong y'$.

Claim: $|\mathrm{Hom}(x', y')| \leq |\mathrm{Hom}(x, y)|$. Let $f\colon x' \to y'$. Let $\varepsilon\colon Fx \to x', \eta\colon Fy \to y'$ be isomorphisms. Then $\eta^{-1} \cdot f \cdot \varepsilon\colon Fx \to Fy$, som we claim $L = \eta_*^{-1} \circ \varepsilon^*$ is injective.
If $L(f) = L(g)$ then $\eta^{-1} f\varepsilon = \eta^{-1} g\varepsilon$ so by isomorphism, $f = g$. Thus $L\colon \mathrm{Hom}(x', y') \to \mathrm{Hom}(Fx, Fy)$ is injective, so $|\mathrm{Hom}(x', y')| \leq |\mathrm{Hom}(Fx, Fy)| = |\mathrm{Hom}(x, y)|$ where the last equality follows as F is fully faithful.

x. Characterize the categories that are equivalent to discrete categories. A category that is connected and essentially discrete is called **chaotic**.

Solution: Suppose C is a category that is equivalent to a discrete category D. Then there exists a fully faithful and essentially surjective on objects functor $F\colon C \to D$. Now, for any $x, y \in C$, we have $|\mathrm{Hom}(x, y)| = |\mathrm{Hom}(Fx, Fy)| = \delta_{Fx,Fy}$ since a discrete category only has identity morphisms. Clearly then $|\mathrm{Hom}(x, y)| = |\mathrm{Hom}(y, x)|$ and F is faithful and if $f\colon x \to y, g\colon y \to x$ then $F(gf) \in \mathrm{Hom}(Fx, Fx) = \{\mathbb{1}_{Fx}\}$, so $gf = \mathbb{1}_x$ and $fg = \mathbb{1}_y$, so $x \cong y$. Thus, C is a groupoid, and since any groupoid has a discrete skeleton, we find that the categories equivalent to discrete categories are precisely all groupoids.

Therefore if a category is a connected groupoid, it is chaotic - this simply means that $|\mathrm{Hom}(x, y)| = 1$ for all $x, y \in C$ and that each morphism is an isomorphism.

Section 1.6

Def. A **diagram** in a category C is a functor $F\colon J \to C$ whose domain is referred to as the **indexing category** of the diagram.

Formally, a diagram is just a functor, but in practice a functor is referred to as a diagram when the indexing category is smaller than the target category. There are, however, instances, such as lemma 3.7.1, where it is convenient to consider arbitrary functors as diagrams.

Functors preserve commutative diagrams.

Lemma. Suppose $f_1, \ldots, f_n$ is a composable sequence - a "path" - of morphisms in a category. If the composite $f_k f_{k-1} \ldots f_{i+1} f_i$ equals $g_m \ldots g_1$ for another composable sequence of morphisms $g_1, \ldots, g_m$ then $f_n \ldots f_1 = f_n \cdots f_{k+1} g_m \cdots g_1 f_{i-1} \cdots f_1$.

Lemma 1.6.11 and transitivity of equality imply that commutativity of an entire diagram may be checked by establishing commutativity of each minimal subdiagram in the directed graph.

Lemma. For any commutative square $\beta\alpha = \delta\gamma$ in which each of the morphisms is an isomorphism, the inverse define a commutative square $\alpha^{-1}\beta^{-1} = \gamma^{-1}\delta^{-1}$.

Def. An object $i \in C$ is **initial** if for every $c \in C$ there exists a unique morphism $i \to c$. Dually, an object $t \in C$ is **terminal** if for every $c \in C$ there exists a unique morphism $c \to t$.

Examples.

1. $\varnothing \in \mathrm{Set}$ is initial, since for any set $A \in \mathrm{Set}$, there exists precisely one subset of $\varnothing \times A$, namely the empty set. Thus the empty map is the only morphism from $\varnothing$ to any set in Set.
2. In Top, the empty and singleton spaces are, respectively, initial and terminal.

Def. A **concrete category** is a category C equipped with a faithful functor $U\colon C \to \mathrm{Set}$

Because faithful functors reflect identifications between parallel morphisms:
Lemma. If $U\colon C \to D$ is faithful, then any diagram in C whose image commutes in D also commutes in C.

This implies that a diagram in a concrete category commutes if and only if the induces diagram of underlying sets commutes.

Lemma Consider morphisms with the indicated sources and targets

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"] & b \arrow[r, "j"] \arrow[d, "h"] & c \arrow[d, "l"] \\ a' \arrow[r, "k"] & b' \arrow[r, "m"] & c' \end{tikzcd}$$

and suppose that the outer rectangle commutes. This data defines a commutative rectangle if either:

1. the right-hand square commutes and m is a monomorphism; or
2. the left-hand square commutes and f is an epimorphism.

Section 1.6 - Problems

i. Show that any map from a terminal obj in a category to an initial one is an iso. An object that is both initial and terminal is called a **zero object**.

Solution: Suppose $f: t \to i$ where t is terminal and i is initial. By definition, there exists precisely one morphism $g: i \to t$. Now, $fg \in \mathrm{Hom}(i,i) = \{\mathbb{1}_i\}$ and $gf \in \mathrm{Hom}(t,t) = \{\mathbb{1}_t\}$, so f is an isomorphism.

iii Show that any faithful functor reflects monomorphisms.

Solution: Suppose $F: C \to D$ is faithful and $Ff: Fc \to Fd$ is a monomorphism in D. If f is not a monomorphism, there exist $h, k: d \to e$ such that $hf = kf$ yet $h \neq k$. But then $F(h)F(f) = F(k)F(f)$ so since $F(f)$ is monic, we have $F(h) = F(k)$, but $F(h), F(k) \in \mathrm{Hom}(Fd, Fe)$ and since F is faithful, i.e., F_* is injective, so $h = k$.

Now, if we flip all arrow in the above proof, f becomes a map $d \to c$ and $h, k: e \to d$, so $fh = fk$. Anywhere where a property of Ff being a monomorphism was used would become the property that Ff is an epimorphism - since the graphs representing monomorphisms and epimorphisms are duals - see terminology section.

Terminology

Def. A morphism $f: x \to y$ in a category is

1. a **monomorphism** if for any parallel morphism $h, k: w \to k$, $fh = fk$ implies $h = k$
2. an **epimorphism** if for any parallel morphism $h, k: y \to z$, $hf = kf$ implies that $h = k$.

So

$$\begin{tikzcd} \bullet \arrow[dr, "h"] & & & & & \bullet \\ & \bullet \arrow[r, "f"] & \bullet & \bullet \arrow[r, "f"] & \bullet \arrow[ur, "h"] \arrow[dr, "k"'] & \\ \bullet \arrow[ur, "k"'] & & & & & \bullet \end{tikzcd}$$

where the left shows f as a monomorphism and right one shows f as an epimorphism.

Def. The category Top^2 consists of objects being ordered pairs (X, A) where X is a topological space and A is a subspace of X.

A morphism $f\colon (X,A)\to(Y,B)$ is an ordered pair (f,f') where $f\colon X\to Y$ is continuous and $fi=jf'$ where i and j are inclusions,

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ B \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

and composition is coordinatewise. Top^2 is called the category of pairs of topological spaces.
Top_* is a subcategory of Top^2.

Rotman 0.12. Given a category C, show that the following construction gives a category M. First, an object of M is a morphism of C. Next, if $f,g\in\operatorname{ob}M$, say, $f\colon A\to B$ and $g\colon C\to D$, then a morphism in M is an ordered pair (h,k) of morphisms in C such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"] & B \arrow[d, "k"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

commutes. Define composition coordinatewise:

$$(h',k')\circ(h,k)=(h'\circ h,k'\circ k).$$

Solution: Suppose we have an object $f\colon A\to B\in\operatorname{ob}M$. Then since the following commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathbb{1}_A"] & B \arrow[d, "\mathbb{1}_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

we have that $(\mathbb{1}_A,\mathbb{1}_B)\colon f\to f$.
This is an identity for f since for any map $(h,k)\colon f\to g$, where $g\colon C\to D$, we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathbb{1}_A"] \arrow[dd, bend right, dashed, "h\circ\mathbb{1}_A"'] & B \arrow[d, "\mathbb{1}_B"'] \arrow[dd, bend left, dashed, "k\circ\mathbb{1}_B"] \\ A \arrow[r, "f"] \arrow[d, "h"] & B \arrow[d, "k"'] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

gives that the outer rectangle commutes, so $(h,k)\circ(\mathbb{1}_A,\mathbb{1}_B)=(h,k)$ and similarly for the other order.

Associativity follows from associativity in C.
Composition is also well-defined from composition in C.

Def. (Rotman) A **congruence** on a category C is an equivalence relation $\sim$ on the class $\bigcup_{(A,B)}\operatorname{Hom}(A,B)$ of all morphisms in C such that:

1. $f\in\operatorname{Hom}(A,B)$ and $f\sim f'$ implies $f'\in\operatorname{Hom}(A,B)$;
2. $f\sim f', g\sim g'$, and the composite $g\circ f$ exists imply that
$$g\circ f\sim g'\circ f'.$$

Theorem 0.4 (Quotient category) Let C be a category with congruence $\sim$, and let $[f]$ denote the equivalence class of a morphism f. Define C' as follows:

$$\begin{aligned} \operatorname{ob}C' &= \operatorname{ob}C \\ \operatorname{Hom}_{C'}(A,B) &= \{[f] : f\in\operatorname{Hom}_C(A,B)\} \\ [g]\circ[f] &= [g\circ f]. \end{aligned}$$

Then C' is a category.

Proof: We must check that composition is associative: $[g]\circ([f]\circ[h]) = [g]\circ[f\circ h]\,[g\circ(f\circ h)]\,[(g\circ f)\circ h] = [g\circ f]\,[h] = ([g]\,[f])\,[h]$, where associativity inside is inherited from C.

We must check that this is well defined.
If $[g]=[g']$ and $[f]=[f']$, then g, g' have the same domain and codomain and likewise for f and f'.
Now $g\circ f \sim g'\circ f'$ by assumption on congruence, so $[g\circ f] = [g'\circ f']$.
So it is well-defined.

For identity, choose any object $A\in C'$, then $[\mathbb{1}_A]$ is an identity since if $f\colon A\to B$ then $[f]\,[\mathbb{1}_A] = [f\circ \mathbb{1}_A] = [f]$ and if $g\colon C\to A$ then $[\mathbb{1}_A]\,[g] = [\mathbb{1}_A\circ g]\,[g]$.

The category C' just constructed is called a **quotient category** of C ; one usually denotes $\mathrm{Hom}_{C'}(A,B)$ by $[A,B]$.

The most important quotient category for us is the **homotopy category**.
We will see it later, but for a lesser example:

Let $C = $ Group and $f, f'\in \mathrm{Hom}(G,H)$. Define $f\sim f'$ if there exists $a\in H$ such that $f(x) = af'(x)a^{-1}$ for all $x\in G$.

Equivalence: $f\sim f$ is clear by $a=e$.
$f\sim f'$ then $f(x) = af'(x)a^{-1}$ so $f'(x) = a^{-1}f(x)\left(a^{-1}\right)^{-1}$, so $f'\sim f$. If $f\sim f'\sim f''$ then $f(x) = af'(x)a^{-1} = abf''(x)b^{-1}a^{-1} = (ab)f''(x)(ab)^{-1}$, so $f\sim f''$.

To see congruence, if $f\in\mathrm{Hom}(G,H)$ and $f\sim f'$ then $f'(x) = af(x)a^{-1}$. Suppose $g,g'\in G$. Then $f'(gg') = af(gg')a^{-1} = af(g)a^{-1}af(g')a^{-1} = f'(g)f'(g')$, so $f'\in\mathrm{Hom}(G,H)$. If $f\sim f', g\sim g'$ and $g\circ f$ is well defined, so say, $f\colon G\to H, g\colon H\to J$. Then $f'(x) = af(x)a^{-1}, g'(x) = bg(x)b^{-1}$, so $g'\circ f'(x) = bg(f'(x))b^{-1} = bg\left(af(x)a^{-1}\right)b^{-1} = bg(a)g(f(x))g(a)^{-1}b^{-1} = (bg(a))(g\circ f)(x)(bg(a))^{-1}$ and $bg(a)\in J$, so $[g'\circ f']\sim[g\circ f]$.

Thus the quotient category is defined. If G and H are groups, then $[G,H]$ is the set of all conjugacy classes $[f]$ where $f\colon G\to H$ is a homomorphism.

Rotman 0.17. Let C and A be categories, and let $\sim$ be a congruence on C. If $T\colon C\to A$ is a functor with $T(f) = T(g)$ whenever $f\sim g$, then T defines a functor $T'\colon C'\to A$ (where C' is the quotient category) by $T'(X) = T(X)$ for every object X and $T'([f]) = T(f)$ for every morphism f.

Proof: We must show that T' is a functor.
Since T' matches X on objects, we have $T'(X)\in \mathrm{ob}\, A$.
Now, if $f\colon X\to Y$ then $T'([f]) = T(f) : TX(T'X)\to TY(T'Y)$.

Suppose $f\colon x\to y$ and $g\colon y\to z$ then $T'([f]\,[g]) = T'([fg]) = T(fg) = T(f)T(g) = T'([f])\,T'([g])$.
And $T'([\mathbb{1}_A]) = T(\mathbb{1}_A) = \mathbb{1}_{T(A)} = \mathbb{1}_{T'(A)}$.

Hence T' is a functor.

Rotman 0.20.
(i) If X is a topological space, show that $C(X)$, the set of all continuous real-valued functions on X, is a commutative ring with 1 under pointwise operations:

$$f+g\colon x\mapsto f(x)+g(x) \quad \text{and} \quad f\cdot g\colon x\mapsto f(x)g(x)$$

for all $x\in X$.
(ii) Show that $X\mapsto C(X)$ gives a contravariant functor Top $\to$ Ring.

Solution: (i) Trivial.
(ii) Let $F\colon$ Top $\to$ Ring by $F(X) = C(X)$. We let F act on morphisms as follows: suppose $f\colon X\to Y$

is a continuous map, then $F(f)\colon F(Y) \to F(X) = C(Y) \to C(X)$ by $F(f) = f^*$.

This is a functor since if $f\colon X \to Y$ and $g\colon Y \to Z$ then $F(gf) = (gf)^* = f^*g^* = F(f)F(g)$ and $F(\mathbb{1}_X) = \mathbb{1}_X^* = \mathbb{1}_{C(X)} = \mathbb{1}_{F(X)}$ because for any $g\colon X \to \mathbb{R} \in C(X)$ we have $\mathbb{1}_X^*(g) = g \circ \mathbb{1}_X = g$, so $\mathbb{1}_X^* = \mathbb{1}_{C(X)}$.

It is easy to show that homotopy is a congruence on the category Top, so it follows from theorem 0.4 (Quotient category) that there is a quotient category whose objects are topological spaces X and whose Hom sets are $\mathrm{Hom}(X,Y) = [X,Y]$ and whose composition is $[g] \circ [f] = [g \circ f]$ where $[X,Y]$ denotes the family of all homotopy classes

$$[f] = \{\text{continuous } g\colon X \to Y : g \sim f\}$$

of continuous maps $X \to Y$.

Def. The quotient category just described is called the **homotopy category**, and it is denoted by hTop

A few philosophical remarks

One might expect that the functor $C\colon$ Top $\to$ Ring of exercise 0.20 is as valuable as the homology functors. Indeed, a theorem of Gelfand and Kolmogoroff states that for X, Y compact Hausdorff, $C(X)$ and $C(Y)$ isomorphic as rings implies that X and Y are homeomorphic. However, a less accurate translation of a problem from topology to algebra is usually more interesting than a very accurate one - i.e., often a loss of information is valuable. The functor C is not as useful as other functors precisely because of the theorem of Gelfand and Kolmogoroff: the translated problem has no loss and is exactly as complicated as the original one and hence cannot be any easier to solve (one can hope only that the change in viewpoint is helpful).

Now, all the functors $T\colon$ Top $\to A$ that we shall construct, where A is some "algebraic" category, will have the property that $f \sim g$ implies $T(f) = T(g)$ - thus there is a loss of information, we collapse to the equivalence classes. This fact, aside from a natural wish to identify homotopic maps, makes homotopy valuable because it guarantees that the algebraic problem in A arising from a topological problem via T is simpler than the original problem. Furthermore, exercise 0.17 shows that every such functor gives a functor hTop $\to A$, and so the homotopy category is quite fundamental.

What are the isomorphisms in hTop? (Rotman uses equivalence for isomorphism in categories).

Suppose we have $X, Y \in \mathrm{ob}\,\mathrm{hTop}$ and morphisms $[f] \in [X,Y]\,, [g] \in [Y,X]$ such that $[f]\,[g] = [\mathbb{1}_Y]\,, [g]\,[f] = [\mathbb{1}_X]$. Thus $[f]$ is an isomorphism and any isomorphism is of this form. So

Def. (Homotopy equivalence - isomorphisms in hTop**)** A continuous map $f\colon X \to Y$ is a homotopy equivalence if there is a continuous map $g\colon Y \to X$ with $g \circ f \sim \mathbb{1}_X$ and $f \circ g \sim \mathbb{1}_Y$. Two spaces X and Y have the **same homotopy type** if there is a homotopy equivalence $f\colon X \to Y$. That is, X and Y are isomorphic, $X \cong Y$, if there is a homotopy equivalence $f\colon X \to Y$.