Recall that the cohomology with integral coefficients of $K(\mathbb{Z}/3, 2)$ up to degree 6 is:

$$H^*(K(\mathbb{Z}/3,2);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & * = 0, \\ \mathbb{Z}/3, & * = 3, 5, \\ 0, & * = 1, 2, 4, 6. \end{cases}$$

With $\mathbb{F}_3$-coefficients we have

$$H^*\left(K\left(\mathbb{Z}/3, 2\right); \mathbb{F}_3\right) \cong \mathbb{F}_3\left\langle 1, a, b, a^2, ab \right\rangle \quad \text{for } * \leq 5$$

where $|a| = 2$ and $|b| = 3$.

Problem 0.1. The goal is to classify up to homotopy equivalence all CW complexes X with the following properties:

(1) X has a single cell in each dimension $0, 3, 4, 6$, and no cells in other dimensions.
(2) $H_3(X) \cong \mathbb{Z}/3$.

Let $Y = X^{(4)}$ be the 4-skeleton of X.

(1) Show that Y is uniquely determined up to homotopy equivalence.
(2) Compute the E_4-page of the cohomology spectral sequence (up to degree 6) for the fiber sequence

$$K\left(\mathbb{Z}/3, 2\right) \to \tau_{>3} Y \to Y.$$

Assuming the differential $d_3 \colon E_3^{0,5} \to E_3^{3,3}$ is non-trivial, show that

$$\pi_k(Y) \cong \begin{cases} 0, & * = 0, 1, 2, 4, 5 \\ \mathbb{Z}/3, & * = 3 \end{cases}$$

and that $\pi_6(Y)$ has at least three elements.
(3) Redo step 3 with $\mathbb{F}_3$-coefficients. Deduce that the d_3-differential in step 2 must indeed have been non-trivial.
(4) Show that X is unique up to homotopy.

Solution. (1) The Δ-chain complex for Y has the form

$$\ldots \to 0 \to \mathbb{Z} \xrightarrow{a} \mathbb{Z} \to 0 \to 0 \to \mathbb{Z} \to 0 \to \ldots$$

where $\operatorname{coker} a = \mathbb{Z}/3$, hence a must be multiplication by 3. But then taking homology, we obtain

$$\ldots \to 0 \to 0 \to \mathbb{Z}/3 \to 0 \to 0 \to \mathbb{Z} \to 0 \to \ldots$$

I.e.,

$$\tilde{H}_k(Y) \cong \begin{cases} \mathbb{Z}/3, & k = 3 \\ 0, & k \neq 3 \end{cases}$$

so Y is a Moore space $M(\mathbb{Z}/3, 3)$. We have seen, (Hatcher, example 4.34), that Moore spaces are unique up to homotopy equivalent from which the claim follows.

(2) Firstly, since X and hence Y only have cells in dimension 0 and then > 1, we obtain by the cellular approximation theorem that $\pi_1(Y) \cong \pi_1(X) \cong 0$, so that π_1 acts trivially on homology. Hence we can use the LSSS. By the UCT,

$$\tilde{H}^k(Y) \cong \begin{cases} \mathbb{Z}/3, & k = 4 \\ 0, & k \neq 4 \end{cases}$$

so we obtain a double complex as follows:

$$\begin{tikzcd}
& \vdots \arrow[d, dash] & & & \\
5 & \mathbb{Z}/3 \arrow[d, dash] \arrow[dr, dashed] & \mathbb{Z}/3 & \mathbb{Z}/3 & \\
3 & \mathbb{Z}/3 \arrow[d, dash] \arrow[drr, dashed] & \mathbb{Z}/3 & \mathbb{Z}/3 & \\
& \mathbb{Z} \arrow[rr, dash] & & \mathbb{Z}/3 \arrow[r, dash] & \cdots \\
& & 3 & 4 &
\end{tikzcd}$$

where $H^3\left(Y; H^k\left(K\left(\mathbb{Z}/3, 2\right); \mathbb{Z}\right)\right) \cong \operatorname{Hom}\left(H_3(Y), H^k\left(K\left(\mathbb{Z}/3, 2\right)\right)\right) \cong \operatorname{Hom}\left(\mathbb{Z}/3, \mathbb{Z}/3\right) \cong \mathbb{Z}/3$ for $k = 3, 5$ from the UCT and $H^4\left(Y; H^k\left(K\left(\mathbb{Z}/3, 2\right); \mathbb{Z}\right)\right) \cong \operatorname{Ext}\left(H_3(Y), H^k\left(K\left(\mathbb{Z}/3, 2\right); \mathbb{Z}\right)\right) \cong \operatorname{Ext}\left(\mathbb{Z}/3, \mathbb{Z}/3\right) \cong \mathbb{Z}/3$ again from the UCT.

Since there can only be trivial maps in E_2, the same double complex forms E_3 where we also obtain the topmost indicated dashed map as the only possible nontrivial map.
We assumed that this map is nontrivial, hence must be an isomorphism, so the E_4 page will be as follows:

$$\begin{tikzcd}
& \vdots \arrow[dd, dash] & & & \\
5 & & \mathbb{Z}/3 & \mathbb{Z}/3 & \\
3 & \mathbb{Z}/3 \arrow[d, dash] \arrow[drr, dashed] & & \mathbb{Z}/3 & \\
& \mathbb{Z} \arrow[rr, dash] & & \mathbb{Z}/3 \arrow[r, dash] & \cdots \\
& & 3 & 4 &
\end{tikzcd}$$

Since $H^3\left(\tau_{>3}Y; \mathbb{Z}\right) \cong \operatorname{Hom}\left(H_3(\tau_{>3}Y), \mathbb{Z}\right) \cong 0$ by UCT, we must have that the indicated map is injective, hence an isomorphism. Thus E^4 will look as follows:

$$\begin{array}{c|cccc}
\vdots & & & & \\
5 & & \mathbb{Z}/3 & \mathbb{Z}/3 & \\
3 & & & \mathbb{Z}/3 & \\
 & \mathbb{Z} & & & \cdots \\
\hline
 & & 3 & 4 &
\end{array}$$

From this, we can read off that $H^7(\tau_{>3}Y;\mathbb{Z}) \cong \mathbb{Z}/3$. Now, recall that $\tau_{>3}Y$ is a CW-complex, hence its homology and cohomology groups are finitely generated abelian!!!. Now

$$0 \to \operatorname{Ext}(H_6(\tau_{>3}Y),\mathbb{Z}) \to \underbrace{H^7(\tau_{>3}Y;\mathbb{Z})}_{\cong\mathbb{Z}/3} \to \operatorname{Hom}(H_7(\tau_{>3}Y),\mathbb{Z}) \to 0$$

and $\operatorname{Hom}(H_7(\tau_{>3}Y),\mathbb{Z})$ is the torsion-free part of $H_7(\tau_{>3}Y)$, so since $\mathbb{Z}/3$ surjects onto this part, it must be 0. Thus $\operatorname{Ext}(H_6(\tau_{>3}Y),\mathbb{Z}) \cong H^7(\tau_{>3}Y;\mathbb{Z}) \cong \mathbb{Z}/3$, but $\operatorname{Ext}(H_6(\tau_{>3}Y),\mathbb{Z})$ is the torsion part of $H_6(\tau_{>3}Y)$, so $H_6(\tau_{>3}Y) \cong \mathbb{Z}/3 \bigoplus (\text{free part})$, where the free part could be trivial. In any case, $H_6(\tau_{>3}Y)$ is nontrivial, while all $H_k(\tau_{>3}Y) \cong 0$ for $1 \le k \le 5$ by the UCT. Hence by Hurewicz,

$$\pi_k(\tau_{>3}Y) \cong \begin{cases} \mathbb{Z}/3 \bigoplus (\text{free part}), & k = 6 \\ 0, & 1 \le k \le 5 \end{cases}$$

Combining this with $\pi_k(Y)$ for $k \le 3$ which by Hurewicz and the previous problem is

$$\pi_k(Y) \cong \begin{cases} \mathbb{Z}/3, & k = 3 \\ 0, & 0 \le k \le 2 \end{cases}$$

we obtain

$$\pi_k(Y) \cong \begin{cases} \mathbb{Z}/3 \bigoplus (\text{free part}), & k = 6 \\ \mathbb{Z}/3, & k = 3 \\ 0, & k = 0,1,2,4,5 \end{cases}$$

(3) We have

$$H_*(Y;\mathbb{Z}/3) \cong H_*(Y;\mathbb{Z}) \otimes \mathbb{Z}/3 \cong \begin{cases} \mathbb{Z}/3, & k = 0,3 \\ 0, & \text{else} \end{cases}$$

and

$$H^*(K(\mathbb{Z}/3,2);\mathbb{Z}/3) \cong \mathbb{Z}/3\left\langle 1,a,b,a^2,ab\right\rangle, \quad * \le 5$$

was given, where $|a| = 2$ and $|b| = 3$.

By UCT,

$$0 \to \operatorname{Ext}(H_{n-1}(Y), \mathbb{Z}/3) \to H^n(Y; \mathbb{Z}/3) \to \operatorname{Hom}(H_n(Y), \mathbb{Z}/3) \to 0$$

so when $n = 3$, Ext vanishes, so $H^3(Y; \mathbb{Z}/3) \cong \operatorname{Hom}(\mathbb{Z}/3, \mathbb{Z}/3) \cong \mathbb{Z}/3$ and when $n = 4$, Hom vanishes, so $H^4(Y; \mathbb{Z}/3) \cong \operatorname{Ext}(H_3(Y), \mathbb{Z}/3) \cong \mathbb{Z}/3$. Furthermore, $H^0(Y; \mathbb{Z}/3) \cong \mathbb{Z}/3$.
Consider the double complex

$$\begin{tikzcd}
 & \vdots \arrow[d, dash] & & & \\
5 & \mathbb{Z}/3ab \arrow[d, dash] \arrow[ddr, "d_3"] \arrow[dddrr, "d_4"] & \mathbb{Z}/3abx & \mathbb{Z}/3aby & \\
4 & \mathbb{Z}/3a^2 \arrow[d, dash] \arrow[ddr] & \mathbb{Z}/3a^2x & \mathbb{Z}/3a^2y & \\
3 & \mathbb{Z}/3b \arrow[d, dash] \arrow[ddrr, "\cong"] & \mathbb{Z}/3bx & \mathbb{Z}/3by & \\
2 & \mathbb{Z}/3a \arrow[d, dash] \arrow[dr, "\cong"] & \mathbb{Z}/3ax & \mathbb{Z}/3ay & \\
 & \mathbb{Z}/3 \arrow[r, dash] & \mathbb{Z}/3x \arrow[r, dash] & \mathbb{Z}/3y \arrow[r, dash] & \cdots \\
 & & 3 & 4 &
\end{tikzcd}$$

Since $H^p(\tau_{>3}Y; \mathbb{Z}/3) \cong 0$ for $p = 2, 3$, we must have that the maps eminating from $\mathbb{Z}/3a$ and $\mathbb{Z}/3b$ are injective, hence they must be isomorphisms since any nontrivial group homomorphism $\mathbb{Z}/3 \to \mathbb{Z}/3$ is an isomorphism. Thus $d_3(a) = x$ and $d_4(b) = y$. These are both forced since all other maps eminating from these groups or terminating at them are 0-maps. Now, using the multiplicative structure, we can calculate d evaluated at the other generators. Using the Leibniz rule, we get $d_3(a^2) = d(a)a + (-1)^{|a|}ad(a) = 2ax \in \mathbb{Z}/3ax$ which still generates $\mathbb{Z}/3ax$, hence $d_3 \colon \mathbb{Z}/3a^2 \to \mathbb{Z}/3ax$ is an isomorphism. Likewise

$$d_3(ab) = xb + ad_3(b) = xb \in \mathbb{Z}/3xb$$

so since $d_3 \colon \mathbb{Z}/3ab \to \mathbb{Z}/3xb$ maps generators to generators, it is an isomorphism. Thus, the page E_4 looks as follows:

$$\begin{array}{c|cccc}
\vdots & & & & \\
5 & & \mathbb{Z}/3abx & \mathbb{Z}/3aby & \\
4 & & \mathbb{Z}/3a^2x & \mathbb{Z}/3a^2y & \\
3 & & & \mathbb{Z}/3by & \\
2 & & & \mathbb{Z}/3ay & \\
 & \mathbb{Z}/3 & & & \cdots \\
\hline
 & & 3 & 4 &
\end{array}$$

Clearly, on this and all subsequent pages, all maps are trivial, so $E_4 = E_\infty$, and we get that $H^p\left(\tau_{>3}Y;\mathbb{Z}/3\right) \cong \begin{cases} \mathbb{Z}/3, & p = 0, 6 \\ 0, & 1 \leq p \leq 5 \end{cases}$

Now, suppose that the map $\mathbb{Z}/3 \to \mathbb{Z}/3$ in the integral case from part (2) of the problem were not an isomorphism - i.e., suppose it were trivial. Then since that group was the only group of total degree 5 and since all maps from it or terminating at it on subsequent pages would be 0-maps, we would get that $H^5(\tau_{>3}Y;\mathbb{Z}) \cong \mathbb{Z}/3$. But then by UCT, and the above part of this problem, $0 = H^5\left(\tau_{>3}Y;\mathbb{Z}/3\right) \cong H^5\left(\tau_{>3}Y;\mathbb{Z}\right) \otimes \mathbb{Z}/3 \cong \mathbb{Z}/3 \otimes \mathbb{Z}/3 \neq \mathbb{Z}/3 \neq 0$ gives a contradiction. Hence the map must have been nontrivial, which was what we wanted to show.

Now proceeding as in step (2), we can conclude all the same things.

(4)
We can obtain X by attaching a 6-cell to $Y = X^{(4)}$ by its attaching map $\varphi\colon S^5 \to Y$. But note that since $\pi_5(Y) \cong 0$ as we showed above, the attaching map can by proposition 0.18 in Hatcher be assumed to be constant at the basepoint. Furthermore,
We have that $\tau_{\leq 5}Y \cong K\left(\mathbb{Z}/3, 3\right)$.
But Y is 5-connected, so we may replace it with a homotopy equivalent CW complex with cells of dimension ≥ 6 only as well as the
Suppose X and X' are two spaces satisfying the assumptions of the problem. Let Y and Y' be the 4-skeletons of X and X', respectively. We saw in problem (1), that $Y \simeq Y'$ because they were Moore spaces. Let $f\colon Y' \to Y$ be this homotopy equivalence. Let φ, φ' be the attaching maps of the 6-cell. Then $f \circ \varphi'$ is an attaching map on Y. But since $\pi_5\left(Y\right) \cong 0$, we have $f \circ \varphi' \simeq \varphi$. Now $X = Y \cup_\varphi S^5 \simeq Y \cup_{f\circ\varphi'} S^5$