Exercise 0.1 (1)**.** Let

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

be exact and consider

$$0 \to \operatorname{Hom}_R(N'', M) \xrightarrow{g^*} \operatorname{Hom}_R(N, M) \xrightarrow{f^*} \operatorname{Hom}_R(N', M).$$

(1) Show that the induced sequence is exact.
(2) Show that even though $N' \to N$ is injective, the map $\operatorname{Hom}_R(N, M) \to \operatorname{Hom}_R(N', M)$ may fail to be surjective.

Proof. (1) Suppose $g^*(k) = g^*(h)$, so $kg = hg$ as maps $N \to M$. For any $n'' \in N''$, there exists $n \in N$ such that $g(n) = n''$, and so $k(n'') = kg(n) = hg(n) = g(n'')$ for all $n'' \in N''$, hence $k = h$. Suppose $f^*(h) = 0$, so $hf = 0$. By injectivity of f, $\ker g = \operatorname{im} f \subset \ker h$. This means that h factors through g in the sense that there exists a map $l \colon N'' \to M$ such that $h = lg$. Thus $h \in \operatorname{im} g^*$.

(2) We have $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) = 0$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$, so while $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is exact,

$$0 \to 0 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \to 0 \to 0$$

cannot be, since $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$ contains $\mathrm{id} \neq 0$, and is thus not isomorphic to the trivial ring.

$\square$