p. 183
11: Calculate the homology groups of the following complexes:
(a) three copies of the boundary of a triangle all joined together at a vertex;
(b) two hollow tetrahedra glued together along an edge.

Solution: (a) Let K be the simplicial complex consisting of three copies of the boundary of a triangle all joined together at a vertex - so it has 9 1-simplexes constituting the edges and 7 0-simplexes constituting the vertices.
By theorem 8.2, we have that $H_0(K) \cong \mathbb{Z}$.
Now, since $|K|$ is connected, we have that $H_1(K)$ is the abelianization of the fundamental group of $|K|$. By example 1 on page 136, we have that $\pi_1(|K|) \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$, which has abelianization $\mathbb{Z}$, so $H_1(K) \cong \mathbb{Z}$.
Now, since K has no n-simplexes for $n \geq 2$, we have that $Z_n(K) = 0$ for $n \geq 2$, so $H_n(K) = 0$ for $n \geq 2$.

(b) Let K denote the simplicial complex for two hollow tetrahedra glued together along an edge.
Since $|K|$ is path-connected, the has only a single component, so $H_0(K) \cong \mathbb{Z}$ by theorem 8.2.
Now, again, since $|K|$ is connected, $H_1(K)$ is the abelianization of the fundamental group of $|K|$.
Choosing any vertex v of the common edge, and letting J be one polyhedra and L the other polyhedra, we have that $\pi_1(|J|, v) = \pi_1(|L|, v) = 0$ since each $|J| = \left|\sum^2\right| = |L|$ and $\left|\sum^2\right| \cong S^2$ by example 5 on page 181, and $\pi_1(S^2) = 0$. Thus $\pi_1(|J \cup L|, v) = 0$ by van Kampen. But this naturally also has trivial abelinization, so $H_1(K) = 0$.
Now, choosing an orientation for any two vertices of an edge in our complex $|K|$, we find that this determines an orientation on all the remaining vertices. Thus our surface is orientable, so by the last comment on page 183, $H_2(K) \cong \mathbb{Z}$. Now, since K has no n-simplexes for $n \geq 3$, we find that $H_n(K) = 0$ for all $n \geq 3$.

13: Show that any graph has the homotopy type of a bouquet of circles, and suggest a formula for the first Betti number of the graph.

Solution: Following the definition on page 3, we shall consider a graph as any connected 1-complex. Let $|K|$ be the graph with V the set of 0-simplexes and E the set of 1-simplexes. Take a maximal tree, L, of K which, by lemma 6.11, contains all the vertices of K. Then by the explanation of $G(K, L)$, any edge in $E - L$ corresponds to a cycle. So $G(K, L)$ is a free group on $|E - L|$ generators, and since $G(K, L) \cong \pi_1(|K|, v)$ for v any vertex of K, we have that $H_1(K) \cong \mathbb{Z}^{|E-L|}$. Now, removing each edge of $|E - L|$ from K, we get a tree which has the relation $|V| - |E| = 1$, so we get $|E - T| = |E| - |V| + 1$, so $H_1(K) \cong \mathbb{Z}^{|E|-|V|+1}$.
So $\beta_1 = |E| - |V| + 1$ is the first betti number.

To see the homotopy equivalence, we will prove weakened versions of propositions 0.16 and 0.17 in Hatcher which relate to CW-complexes, but work just as well for our simplicial complexes. We show the following two propositions:
Prop 1: If (K, e) is a graph K and an edge e between two distinct vertices v_0, v_1 then $K \times \{0\} \cup e \times I$ is a deformation retraction of $K \times I$.

Proof: Denote by K^0 the 0-skeleton of K. There is a retraction $r\colon I \times I \to I \times \{0\} \cup \partial I \times I$ by radial projection from $(0, 2) \in I \times \mathbb{R}$. Setting $r_t = tr + (1 - t)\mathbb{1}$ gives a deformation retraction of $I \times I$ onto $I \times \{0\} \cup \partial I \times I$, which gives rise to a deformation retraction of $K \times I$ onto $K \times \{0\} \cup \left(K^0 \cup e\right) \times I$ since $K \times I$ is obtained from $K \times \{0\} \cup \left(K^0 \cup e\right) \times I$ by attaching copies of $I \times I$.

Now, suppose X is a simplicial complex or a CW-complex such that A is an edge and hence closed in $|X|$. If we are given a map $f_0\colon X \to Y$ and on the subspace $A \subset X$ a homotopy $f_t\colon A \to Y$ of $f_0|_A$, we say a pair (X, A) has the homotopy extension property if we can always extend this given homotopy f_t to a homotopy $f_t\colon X \to Y$ of the given f_0.
Claim: A pair (X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.
Proof: The homotopy extension property for (X, A) implies that the identity $X \times \{0\} \cup A \times I \to X \times \{0\} \cup A \times I$ extends to a map $X \times I \to X \times \{0\} \cup A \times I$.
For the other direction, since A is closed in our cases in X, any two maps $X \times \{0\} \to Y$ and $A \times I \to Y$ that agree on $A \times \{0\}$ combined to a map $X \times \{0\} \cup A \times I \to Y$ whose continuity is guaranteed by the

gluing lemma. By composing $X \times \{0\} \cup A \times I \to Y$ with the retraction $X \times I \to X \times \{0\} \cup A \times I$, we get an extesion $X \times I \to Y$, so (X, A) has the homotopy extension property.

With the claim and the proposition, we thus see that for any graph K and any edge e considered as a simplicial complex of the graph, (K, e) has the homotopy extension property.

Now, the following proposition finishes the argument:
Prop 2: For any pair (K, e) for a graph K and edge e of K such that e is contractible, the quotient map $q\colon K \to K/e$ is a homotopy equivalence.

Proof: Let $f_t\colon K \to K$ be a homotopy extending a contraction of e with $f_0 = \mathbb{1}$. Since $f_t(e) \subset e$ for all t, the composition $qf_t\colon K \to K/e$ sends e to a point and hence factors as a composition $K \xrightarrow{q} K/e \to K/e$. Let the latter map be $\overline{f_t}\colon K/e \to K/e$. We have $qf_t = \overline{f_t}q$. When $t = 1$, we have $f_1(e)$ being the point e contracts to, so f_1 induces a map $g\colon K/e \to K$ with $gq = f_1$. It follows that $qg = \overline{f_1}$ since $qg(\overline{x}) = qgq(x) = qf_1(x) = \overline{f_1}q(x) = \overline{f_1}(\overline{x})$. The maps g and q are inverse homotopy equivalences since $gq = f_1 \simeq f_0 = \mathbb{1}$ via f_t and $qg = \overline{f_1} \simeq \overline{f_0} = \mathbb{1}$ via $\overline{f_t}$.

Now, we have that we can take any graph K and contract all edges with non-equal vertices. Continuing this, we eventually arrive at a wedge sum of circles. By proposition 2, we then have that any graph is homotopy equivalent to a wedge sum of circles.

From this it also follows that the fundamental group of $|K|$ is free, and looking at how we contract the edges above, we see that any loop will eventually contract to a loop, so the number of circles in the wedge sum will be precisely $|E| - |V| + 1$ as explained at first.

p. 188
20: Prove the following lemma:
If $\varphi\colon C(K) \to C(L)$ is a chain map, and $\psi\colon C(L) \to C(M)$ is a second chain map then $\psi \circ \varphi\colon C(K) \to C(M)$ is a chain map and $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*\colon H_q(K) \to H_q(M)$.

Solution: By definition, $\varphi\colon C(K) \to C(L)$ being a chain map means that for each $q \geq 0$, we have $\partial\varphi_q = \varphi_{q-1}\partial$.
Similarly, for each $q \geq 0$, we have $\partial\psi_q = \psi_{q-1}\partial$. We claim that $\partial\left(\psi_q \circ \varphi_q\right) = \left(\psi_{q-1} \circ \varphi_{q-1}\right)\partial$.
By commutativity of each square below, we get commutativity of the outer rectangle:

$$\begin{array}{ccccc} C_q(K) & \xrightarrow{\varphi_q} & C_q(L) & \xrightarrow{\psi_q} & C_q(M) \\ \downarrow\partial & & \downarrow\partial & & \downarrow\partial \\ C_{q-1}(K) & \xrightarrow{\varphi_{q-1}} & C_{q-1}(L) & \xrightarrow{\psi_{q-1}} & C_{q-1}(M) \end{array}$$

Explicitly written, we have

$$\psi_{q-1} \circ \varphi_{q-1} \circ \partial \overset{\text{first square}}{=} \psi_{q-1} \circ \partial \circ \varphi_q \overset{\text{second square}}{=} \partial \circ \psi_q \circ \varphi_q$$

So $\psi \circ \varphi\colon C(K) \to C(M)$ is a chain map.

For the induced homomorphisms, we first write down explicitly the induced homomorphism: For a chain map $\varphi\colon C(K) \to C(L)$, we have that for a q-cycle $z \in C_q(K)$, we have $\varphi_*\left([z]\right) = [\varphi_q(z)]$ is a homomorphism.
Well-definedness: We first show that φ takes q-cycles of K to q-cycles of L and boundary q-cycles of K to boundary q-cycles of L:
if z is a q-cycle of K, so $\partial z = 0$, then by φ being a chain map,

$$\partial\varphi_q(z) = \varphi_{q-1}\partial z = 0$$

so $\varphi_q(z)$ is a q-cycle of L.
Similarly, if $b \in B_q(K)$, then $b = \partial c$ for some $c \in C_{q+1}(K)$, so

$$\partial\varphi_{q+1}(c) = \varphi_q\partial c = \varphi_q(b)$$

giving $\varphi_q(b) \in B_q(K)$.

Now, suppose $[z] = [w]$ in $H_q(K)$. Then $z - w \in B_q(K)$ and so since φ_q is a homomorphism by assumption of φ being a chain map, $\varphi_q(z) - \varphi_q(w) = \varphi_q(z - w) \in B_q(L)$ as φ_q carries boundary q-cycles to boundary q-cycles by the above; so $\varphi_* ([z]) = [\varphi_q(z)] = [\varphi_q(w)] = \varphi_* ([w])$. Furthermore, it is a homomorphism, since if we let $*$ denote the group operation of $H_q(K)$ and $+$ the group operation of $C_q(K)$, we have $\varphi_* ([z] * [w]) = \varphi_* ([z + w]) = [\varphi(z + w)] = [\varphi(z) + \varphi(w)] = [\varphi(z)] * [\varphi(w)] = \varphi_* ([z]) * \varphi_* ([w])$.
So φ_* is indeed a group homomorphism.

Now, we find directly, that for any element $[z] \in H_q(K)$, we have

$$(\psi \circ \varphi)_* ([z]) = [(\psi \circ \varphi)(z)] = [\psi(\varphi(z))] = \psi_* ([\varphi(z)]) = \psi_* \circ \varphi_* ([z])$$

giving $(\psi \circ \varphi)_* = \psi_* \circ \varphi_* \colon H_q(K) \to H_q(M)$.

p. 192
25: Suppose $s, t \colon |K| \to |L|$ are simplicial, and assume we have a homomorphism $d_q \colon C_q(K) \to C_{q+1}(L)$, for each q, such that

$$d_{q-1}\partial + \partial d_q = t - s \colon C_q(K) \to C_q(L).$$

Show that s and t induce the same homomorphisms of homology groups. The collection of homomorphisms $\{d_q\}$ is called a *chain homotopy* between s and t.

Solution: For any q-cycle z of K, we have $\partial z = 0$, so in particular, $t(z) - s(z) = (d_{q-1}\partial + \partial d_q)(z) = \partial d_q(z) \in B_q(L)$, and hence $t_* ([z]) = [t(z)] = [s(z)] = s_* ([z])$ in $H_q(L)$, so $t(z)$ and $s(z)$ are homologous for all q-cycles of K, and hence t and s induce the same homomorphisms of homology groups: $H_q(K) \to H_q(L)$ for all q.