p. 95:
10: Let γ, σ be two paths in the space X which begin at the point p and end at q. As in the proof of theorem (5.6), these paths induces isomorphisms γ_*, σ_* of $\pi_1(X,p)$ with $\pi_1(X,q)$. Show that σ_* is the composition of γ_* and the inner automorphism of $\pi_1(X,q)$ induced by the element $\langle\sigma^{-1}\gamma\rangle$.

Solution: The isomorphisms induces by γ and σ are given by $\gamma_*(\langle\alpha\rangle) = \langle\gamma^{-1}\alpha\gamma\rangle$ and $\sigma_*(\langle\alpha\rangle) = \langle\sigma^{-1}\alpha\sigma\rangle$, respectively.
The inner automorphism of $\pi_1(X,q)$ induced by $\langle\sigma^{-1}\gamma\rangle$ is given by $\langle\alpha\rangle \to \langle\sigma^{-1}\gamma\rangle\langle\alpha\rangle\langle\sigma^{-1}\gamma\rangle^{-1} = \langle\sigma^{-1}\gamma\rangle\langle\alpha\rangle\langle\gamma^{-1}\sigma\rangle\langle\sigma^{-1}\gamma\alpha\gamma^{-1}\sigma\rangle = \sigma_*\gamma_*^{-1}\langle\alpha\rangle$, so denoting this inner automorphism by β, we get $\langle\alpha\rangle \xrightarrow{\gamma_*} \langle\gamma^{-1}\alpha\gamma\rangle \xrightarrow{\beta} \sigma_*\gamma_*^{-1}\langle\gamma^{-1}\alpha\gamma\rangle = \sigma_*\langle\gamma\gamma^{-1}\alpha\gamma\gamma^{-1}\rangle = \sigma_*\langle\alpha\rangle$, hence $\sigma_* = \beta\circ\gamma_*$, so σ_* is the composition of γ_* and the inner automorphism of $\pi_1(X,q)$ induces by $\langle\sigma^{-1}\gamma\rangle$.

p. 102:
18: Let $\pi\colon X \to Y$ be a covering map. So each point $y \in Y$ has a neighborhood V for which $\pi^{-1}(V)$ breaks up as a union of disjoint open sets, each of which maps homeomorphically onto V under π. Call such a neighborhood 'canonical'. If α is a path in Y, show how to find points $0 = t_0 < t_1 < \ldots < t_m = 1$ such that $\alpha([t_i, t_{i+1}])$ lies in a canonical neighborhood for $0 \le i \le m-1$. Hence lift α piece by piece to a (unique) path in X which begins at any preassigned point of $\pi^{-1}(\alpha(0))$.

Solution: For each $y \in Y$, let V_y denote a canonical open neighborhood for y under π (we can assume V_y is open since π restricting to a homeomorphism also gives a homeomorphic correspondence of open sets). By assumption, $\bigcup_{y\in Y} V_y$ covers Y, so since α maps $I \to Y$, we have $\bigcup_{y\in Y}\alpha^{-1}(V_y)$ is an open covering of I, so as I is a compact metric space, there exists a Lebesgue number $\delta > 0$ for the covering. Subdivide $I = [0,1]$ into intervals $0 = t_0 < t_1 \ldots < t_m = 1$ such that $|t_{i+1} - t_i| < \delta$ for all i (for example we can choose some $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$ and let $t_i = \frac{i}{N}$ for $i = 0, \ldots, N$).

Now, by construction any $[t_i, t_{i+1}]$ lies in some $\alpha^{-1}(V_y)$, so $\alpha([t_i, t_{i+1}]) \subset V_y$ which is a canonical neighborhood.

Now, let x_0 be an arbitrary point in $\pi^{-1}(\alpha(0)) \subset X$, and suppose $\alpha([0,t_1]) \in V_{y_1}$. Then x_0 lies in some $U \subset \pi^{-1}(V_{y_1})$ such that U is mapped homeomorphically by π onto V_{y_1}. Let f be the inverse of $\pi|_U\colon U \to V_{y_1}$.
Define $\tilde{\alpha}(t) = f\circ\alpha(t)$ for $t \in [0,t_1]$. Then for $t \in [0,t_1]$, we have $\pi\circ\tilde{\alpha}(t) = \pi\circ f\circ\alpha(t) = \alpha(t)$, so $\tilde{\alpha}$ is a lift on $[0,t_1]$.

Suppose we have defined $\tilde{\alpha}$ on $[0,t_N]$ for $1 \le N < m$. Thus $\alpha|_{[0,t_N]} = \pi\circ\tilde{\alpha}$ on $[0,t_N]$. We want to define $\tilde{\alpha}$ on $[t_N, t_{N+1}]$. Suppose $\alpha([t_N, t_{N+1}]) \subset V_y$. Then $\tilde{\alpha}(t_N) \in \pi^{-1}(V_y)$, so let $\tilde{\alpha}(t_N) \in U \subset \pi^{-1}(V_y)$ such that π maps U homeomorphically onto V_y. Let g be the inverse of $\pi|_U\colon U \to V_y$. Let $\tilde{\alpha}(t) = g\circ\alpha(t)$ for $t \in [t_N, t_{N+1}]$. Since $g\circ\alpha(t_N)$ agrees with on $\tilde{\alpha}(t_N)$, the gluing lemma ensures continuity of $\tilde{\alpha}$ on all of $[0, t_{N+1}]$.

With this construction $\pi\circ\tilde{\alpha}(t) = \pi\circ g\circ\alpha(t) = \alpha(t)$ on $[t_N, t_{N+1}]$, so $\tilde{\alpha}$ is a lift on $[t_N, t_{N+1}]$.

By repeating m times, we thus have constructed $\tilde{\alpha}$ on all of $I = [0,1]$, with $\alpha = \pi\circ\tilde{\alpha}$ and thus $\tilde{\alpha}$ is a lift starting at x_0.

For uniqueness, we notice that for x_0, we had that $\alpha([0,t_1])$ lies in some V_{y_1}, so $x_0 \in \pi^{-1}(V_{y_1})$ which, by assumption, is a disjoint union of sets each homeomorphic to V_y by π. Since these are disjoint, x_0 is in precisely one of them, say U. So supposing β is another lift of α on $[0,t_1]$ starting at x_0, we get $\pi\circ\beta = \alpha = \pi\circ\tilde{\alpha}$. Since $(\pi\circ\beta)([0,t_1]) = \alpha([0,t_1]) = (\pi\circ\tilde{\alpha})([0,t_1])$ lies in some V_{y_1} and if f is the inverse of $\pi|_U\colon U \to V_{y_1}$ (which exists as π is a homeomorphism of U onto V_{y_1}), then $\beta = f\circ\pi\circ\beta = f\circ\pi\circ\tilde{\alpha} = \tilde{\alpha}$ on $[0,t_1]$. Hence there is a unique extension of $\tilde{\alpha}$ over $[0,t_1]$.
Similarly, for arbitrary $1 \le i \le m-1$, $\alpha([t_i, t_{i+1}]) \subset V_y$ for some y, so, as above, $\tilde{\alpha}(t_i) \in U \subset \pi^{-1}(V_y)$ where π maps U homeomorphically onto V_y. Suppose now β is another lift of α on $[t_i, t_{i+1}]$ starting at $\tilde{\alpha}(t_i)$. We have $\pi\circ\beta = \alpha = \pi\circ\tilde{\alpha}$ on $[t_i, t_{i+1}]$, so as this takes values in V_y, let g be the inverse of $\pi|_U\colon U \to V_y$; giving $\beta = g\circ\pi\circ\beta = g\circ\pi\circ\tilde{\alpha} = \tilde{\alpha}$ on $[t_i, t_{i+1}]$. As we showed uniqueness for the base case

$i = 0$ and this inductive step ensures that gives a unique extension $\tilde{\alpha}|_{[0,t_N]}$ there exists a unique extension $\tilde{\alpha}|_{[0,t_{N+1}]}$ of $\tilde{\alpha}|_{[0,t_N]}$ over $[t_N, t_{N+1}]$. Repeating this m times gives uniqueness of $\tilde{\alpha}$.

21: Describe the homomorphism $f_*\colon \pi_1\left(S^1,1\right) \to \pi_1\left(S^1, f(1)\right)$ induces by each of the following maps:
(a) The antipodal map $f\left(e^{i\theta}\right) = e^{i(\theta+\pi)}, 0 \le \theta \le 2\pi$.

(b) $f(e^{i\theta}) = e^{in\theta}, 0 \le \theta \le 2\pi$ where $n \in \mathbb{Z}$.

(c) $f\left(e^{i\theta}\right) = \begin{cases} e^{i\theta}, & 0 \le \theta \le \pi \\ e^{i(2\pi-\theta)}, & \pi \le \theta \le 2\pi \end{cases}.$

Solution:

In the following, let $\langle g\rangle$ denote a generator for $\pi_1\left(S^1,1\right)$ with $g\colon I \to S^1$ given by $g(t) = e^{2\pi i t}$.
For any induced homomorphism $f_*\colon \pi_1\left(S^1,1\right) \to \pi_1\left(S^1, f(1)\right)$, let $\gamma\colon I \to S^1$ be the loop $f \circ g$.
Then for any $\alpha \in \pi_1(S^1,1)$, we have that there exists an $n \in \mathbb{Z}$ such that $\langle\alpha\rangle = \langle g\rangle^n = \langle g^n\rangle$, so $f_*\langle\alpha\rangle = \langle f\circ g^n\rangle = \langle f \circ g\rangle^n = \langle\gamma\rangle^n$, so $\langle\gamma\rangle$ is a generator for the image of f_*.
Hence f_* on the generator $\langle g\rangle$, i.e. $f\circ g$, gives a complete description of the homomorphism $f_*\colon \pi_1\left(S^1,1\right) \to \pi_1\left(S^1, f(1)\right)$.

(a) We claim that the image of f_* is isomorphic to $\mathbb{Z} \cong \pi_1(S^1,-1)$.

Let $\alpha\colon I \to S^1$ be the path $\alpha(t) = e^{i\pi t}$. Then the isomorphism induces by α is given by

$$\alpha_*\langle\beta\rangle = \langle\alpha^{-1}\beta\alpha\rangle$$

which is an isomorphism of $\pi_1(S^1,1)$ and $\pi_1\left(S^1,-1\right)$. We claim that $f_*\left(\langle\beta\rangle\right) = \alpha_*\left(\langle\beta\rangle\right)$. For this, it suffices to show that it is true on the generator of $\pi_1\left(S^1,1\right)$ since f_* and α_* are homomorphisms. We want to produce a homotopy between $f\circ g$ and $\alpha^{-1}g\alpha$. Define the map

$$F(s,t) = \begin{cases} e^{i\pi(1-4s)}, & s \in \left[0, \frac{1-t}{4}\right] \\ e^{i\pi\left(\frac{4(2-t)s+4t-2}{1+t}\right)}, & s \in \left[\frac{1-t}{4}, \frac{1}{2}\right] \\ e^{i\pi(2s-1)}, & s \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

This is continuous and a homotopy between $F(s,0) = \alpha^{-1}\gamma\alpha$ and $F(s,1) = f\circ g$.

$$F(s,t) = \begin{cases} e^{i\pi(1-4s)}, & s \in \left[0, \frac{1-t}{4}\right] \\ e^{i\pi\frac{8s-4st+4t-2}{1+s}}, & s \in \left[\frac{1-t}{4}, \frac{1}{2}\right] \\ e^{i\pi(2s-1)}, & s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Hence, $f_*\colon \pi_1\left(S^1,1\right) \to \pi_1\left(S^1,-1\right)$ is the map sending a generator to a generator.

(b) We have

$$f\circ g(t) = e^{2\pi i n t} = g(t)^n, \quad t \in [0,1]$$

which is the loop that winds around S^1 counterclockwise n times.
Thus f_* maps $g \to g^n$ - or, identifying $\pi_1\left(S^1,1\right)$ with $\mathbb{Z}$, maps $1 \to n$.
This generates the image group of f_* which thus is $\langle g^n\rangle \le \pi_1\left(S^1,1\right)$.
(c)

$$\gamma(t) := f\circ g(t) = \begin{cases} e^{2\pi i t}, & t \in \left[0, \frac{1}{2}\right] \\ e^{i2\pi(1-t)}, & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

is the loop that goes counterclockwise from 1 to -1 along S^1 during $\left[0, \frac{1}{2}\right]$ and then reversing direction going from -1 to 1 clockwise during $\left[\frac{1}{2}, 1\right]$.
Now γ is homotopic to the constant loop by the homotopy $F\colon I\times I \to S^1$ given by $F(s,t) = \gamma(st)$, so since $\langle 1\rangle = \langle\gamma\rangle$ generates the image subgroup, we find that f_* is the homomorphism mapping everything to the constant loop.