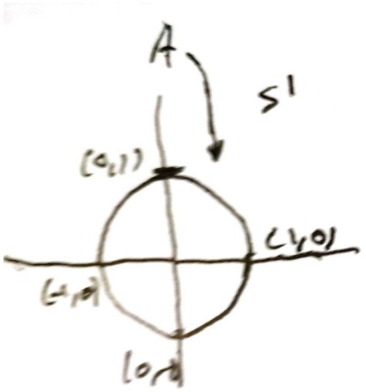
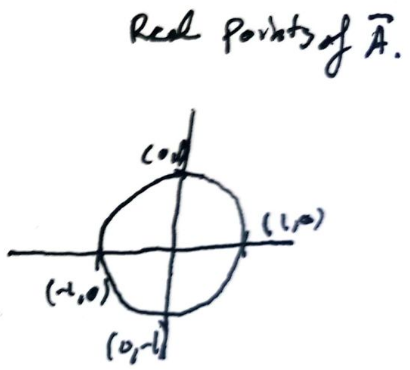
1: Let $\mathcal{U} = \{X - Z \colon Z \subset X \text{ is an algebraic subset}\}$.
(a) By the remark on page 6 on lecture note 10, it suffices to check that the union of finitely many algebraic sets is an algebraic set, that the intersection of an arbitrary collection of algebraic sets is an algebraic set, and finally that $\varnothing$ and X are in $\mathcal{U}$.
The first two requirements follow from page 7 on lecture note 1, where we have that arbitrary intersections of algebraic sets are algebraic sets and finite unions of algebraic sets are algebraic sets: $\bigcap_{i\in I} V(S_i) = V\left(\bigcup_{i\in I} S_i\right)$ with I arbitrary, and $\bigcup_{i=1}^{n} V(S_i) = V\left(S_1 \dots S_n\right)$.
The last requirement follows since $X = V(0)$ and hence $\varnothing = X - X \in \mathcal{U}$. Similarly, $\varnothing = V(1)$, so $X = X - \varnothing \in \mathcal{U}$.

(b) By the lemma on page 1 of lecture note 9, we have that for any algebraic subset $Z \subset Y$, we have $\varphi^{-1}(Z)$ is an algebraic subset of X. Equivalently, the preimage of any closed subset of Y is a closed subset of X which by the remark on page 2 of lecture note 11 is equivalent to φ being continuous with respect to the Zariski topology on both spaces.

(c) By Hilbert's basis theorem, there exist $f_1, \dots, f_n \in k\left[x_1, \dots, x_n\right]$ such that $Z = V\left(f_1, \dots, f_n\right) = V(f_1) \cap \dots \cap V(f_n)$. The set $V(f_i) = \{(x_1, \dots, x_n) \in \mathbb{A}^n \mid f_i\left(x_1, \dots, x_n\right) = 0\} = f_i^{-1}\left(\{0\}\right)$. Considering $\mathbb{C}^n$ and k with the classical topology and using that f_i is a polynomial and thus continuous, we find that since $\{0\}$ is a singleton and thus closed, that $f_i^{-1}(\{0\}) = V\left(f_i\right)$ is closed in the classical topology on $\mathbb{A}^n \cong \mathbb{C}^n$.
Therefore the Zariski topology is coarser than the classical topology.

2:
(a) Let $A = \{(\sin t, \cos t) : t \in \mathbb{R}\} \subset \mathbb{A}^2_{\mathbb{C}}$. We have $x^2 + y^2 - 1 \in I(A) \subset \mathbb{C}\left[x, y\right]$, so $\overline{A} = V\left(I\left(A\right)\right) \subset V\left(x^2 + y^2 - 1\right)$.
Now we can note for example that $x^2 + y^2 - 1 \in \mathbb{C}[y][x]$ is Eisenstein at $(y-1)$ which is prime since it is irreducible since it is linear; hence $x^2 + y^2 - 1$ is irreducible, so $V(x^2 + y^2 - 1)$ is irreducible since it is infinite, by corollary 1 chapter 1.6, Fulton. Now by corollary 2 in chapter 1.6, Fulton, we have that any non-empty proper algebraic subset of $V\left(x^2 + y^2 - 1\right)$ must be either a finite union of points or a finite union of irreducible plane curves.
However, if say $\overline{A} = V\left(g_1, \dots, g_k\right)$, and $V(g)$ for $g = g_i$ for some $i \in \{1, \dots, k\}$, is an irreducible plane curve contained in $V(x^2+y^2-1)$, then $(x^2+y^2-1) \subset (g)$. And since x^2+y^2-1 is irreducible, we must have that g is either a unit or associated to x^2+y^2-1. If g were unit, then $\overline{A} = V\left(g_1, \dots, g_k\right) \subset V(g) = \varnothing$, contradiciton. Thus g must be associated to x^2+y^2-1. So $(g) = \left(x^2+y^2-1\right)$. Since g was arbitrary of the g_i, we get $(g_1, \dots, g_k) = (x^2+y^2-1)$, so $\overline{A} = V\left(x^2+y^2-1\right) = \left\{\left(\sqrt{1-z^2}, z\right) \mid z \in \mathbb{C}\right\} \cup \left\{\left(-\sqrt{1-z^2}, z\right) \mid z \in \mathbb{C}\right\}$.



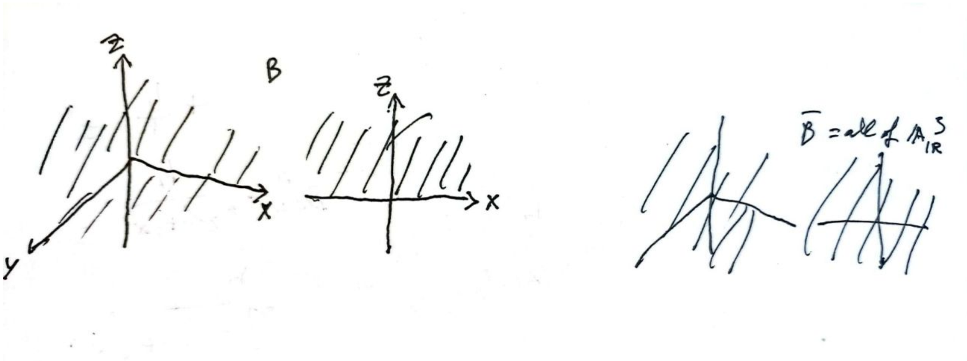


(b) Let $B = \{(x, y, z) : z > 0\} \subset \mathbb{A}^3_{\mathbb{R}}$. We have by the first lemma on lecture note 11 that $\overline{B} = V\left(I\left(B\right)\right)$. Now, assume $f \in I(B)$. Then, fixing $x_0, y_0 \in \mathbb{R}$, we have a map $\varphi \in k\left[t\right]$ by $\varphi(t) = f\left(x_0, y_0, t\right)$ which, by assumption, vanishes on $\mathbb{R}_+$. Thus $\mathbb{R}_+ \subset V\left(\varphi\right)$, but by problem 1.8 in Fulton, we have that the algebraic subsets of $\mathbb{A}^1(\mathbb{R})$ are the finite subsets, together with $\mathbb{A}^1(\mathbb{R})$ itself. Since $\mathbb{R}_+$ is infinite, we thus have $V(\varphi) = \mathbb{A}^1(\mathbb{R})$.

Hence, since x_0, y_0 were arbitrary, we have that for any x_0, y_0, $V(\varphi) = \mathbb{A}^1(\mathbb{R})$ so

$$V(f) = \bigcup_{(x_0,y_0)\in\mathbb{A}^2(\mathbb{R})} \{x_0\} \times \{y_0\} \times V\left(f\left(x_0, y_0, t\right)\right) = \mathbb{A}^3\left(\mathbb{R}\right).$$

Hence $I(B) = (0)$. Thus $V\left(I\left(B\right)\right) = \mathbb{A}^3_{\mathbb{R}}$.



(c) We have $C = \{(x, y, z) : x = y \vee x = z\} = \{(x, y, z)\colon x = y\} \cup \{(x, y, z)\colon x = z\} = V(x - y) \cup V\left(x - z\right) = V\left((x - y)(x - z)\right)$.
Now, since the closure of C is the smallest algebraic subset containing C, and C itself is an algebraic subset, we have $\overline{C} = C$.

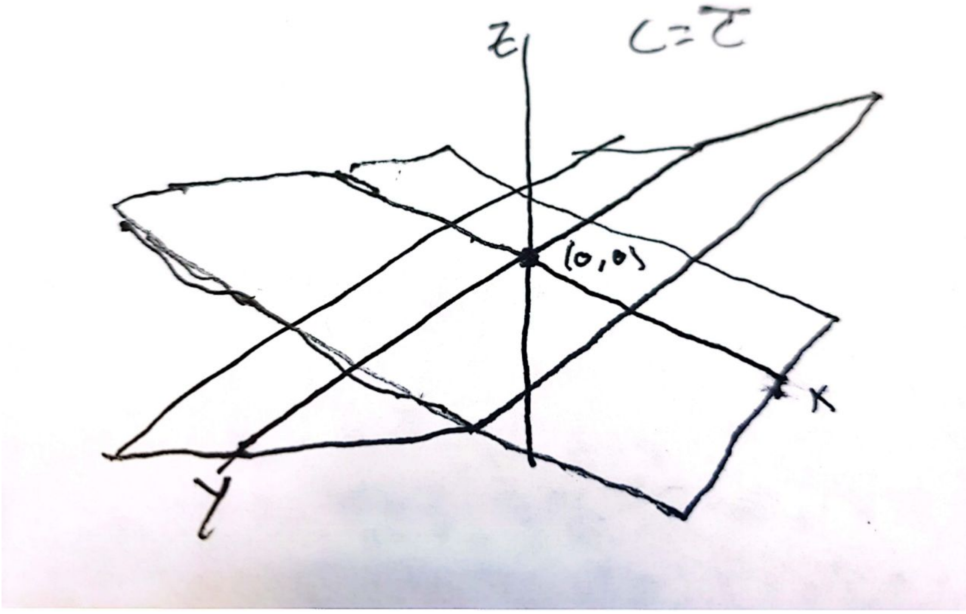


Figure 1: This is the union of the two plane $x = y$ and $x = z$.

3:
(a) Let $\varphi^*\colon \Gamma(\mathbb{A}^2) = k\,[u,v] \to k\,[x,y] = \Gamma\left(\mathbb{A}^2\right)$ be the pullback. Then $\varphi^*(u)(x,y) = (u)\,(\varphi(x,y)) = (u)(xy,y) = xy$ and $\varphi^*(v)(x,y) = (v)(\varphi(x,y)) = (v)\,(xy,y) = y$, so $\varphi^* f(u,v) = f(xy,y)$.

(b) We have φ dominant if and only if φ^* is injective. Now φ^* is a homomorphism and $\varphi^*(f(u,v)) = 0$ if and only if $f(xy,y) = 0$. Since xy and y are linearly independent, we have that φ^* is injective and hence φ is dominant.
The image of φ is $\{(xy,y) \mid x,y \in k\}$.

(c) Since φ is dominant, $V\left(I\left(\varphi(\mathbb{A}^2)\right)\right) = \mathbb{A}^2$. If $\varphi\left(\mathbb{A}^2\right)$ where closed, then $\varphi\left(\mathbb{A}^2\right) = V\left(I\left(\varphi\left(\mathbb{A}^2\right)\right)\right) = \mathbb{A}^2$ which is a contradiction, so $\varphi\left(\mathbb{A}^2\right)$ is not closed.
Since φ is a morphism, it is continuous with respect to the Zariski topology, so if $\varphi(\mathbb{A}^2)$ were open, its preimage would be open, however, the preimage $\mathbb{A}^2$ is closed since it is the hypersurface of the zero function. Thus the image is neither closed nor open in the Zariski topology.

4:
(a) Suppose A is dense in X and $\overline{f} \in \Gamma(X)$ is a polynomial functions such that $\overline{f}(P) = 0$ for all $P \in A$. Then since $P \in A \subset X$, we have $f(P) = \overline{f}(P) = 0$ for all $P \in A$, hence $f \in V(A)$ so $X = I\,(V\,(A)) \subset V(f)$ and hence $X \subset V(f) = V\,(f) \cap X = V\left(\overline{f}\right)$, so $\overline{f} \in I_X(X)$ and thus $\overline{f} = 0 \in \Gamma(X)$.

(b) We have $\varphi\colon X \to Y$ is dominant if and only if $I\,(\varphi(X)) = I(Y)$ if and only if $\overline{\varphi(X)} = V\,(I\,(\varphi(X))) = V\,(I\,(Y)) = Y$ since Y is an algebraic set - where the first equality follows from the first lemma on lecture note 11. The second "if" follows from $I\,(V\,(I\,(X))) = I(X)$ by (9) in chapter 1.3, Fulton.

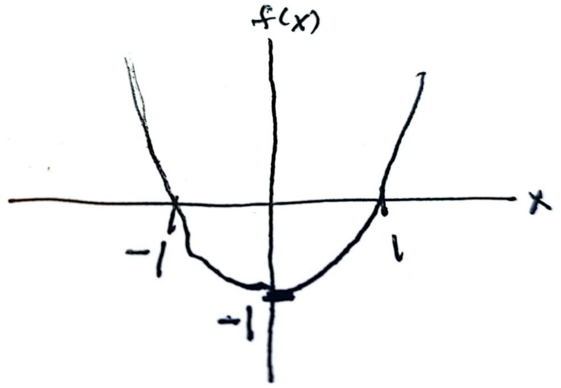
(c) We claim the intersection of dense subsets is not always dense. Take for example $X = \mathbb{R}$. Then $A = \mathbb{Q} \subset \mathbb{R}$ and $B = \mathbb{R} - \mathbb{Q}$ are both dense subsets. However, $A \cap B = \varnothing$ which is not dense in $\mathbb{R}$.

(d) Let X be an irreducible algebraic set and $U \subset X$ a non-empty open subset. We claim that $X = (X - U) \cup \overline{U}$.
Proof: Let $x \in X$. If $x \notin X - U$ then $x \in U$ and hence $x \in \overline{U}$ since $\overline{U}$ is the smallest algebraic subset containing U.
Conversely, suppose $x \in (X - U) \cup \overline{U}$. If $x \in X - U$ then $x \in X$. Assume thus that $x \notin X - U$. Then $x \in \overline{U}$. Now $U \subset X \implies I(U) \supset I(X) \implies \overline{U} = V\,(I\,(U)) \subset V\,(I\,(X)) = X$ since X is algebraic. Hence $x \in \overline{U} \implies x \in X$.

Now we note that $\overline{U}$ is, as shown above, a subset of X and by definition closed. And since U is open, $X - U$ is closed. Thus $X = (X - U) \cup \overline{U}$ is the union of two algebraic sets. Since X is irreducible, $X = X - U$ or $X = \overline{U}$. Now since U is non-emtpy, $X \neq X - U$, so $X = \overline{U}$. Hence U is dense in X.

5:
(a) Let $f(x) = x^2 - 1 \in \Gamma\left(\mathbb{A}^1\right) = k[x]$. We have

$$G(f) = \left\{(a_1,a_2) \in \mathbb{A}^2 \colon a_1 \in \mathbb{A}^1, a_2 = f(a_1)\right\} = \left\{(x, f(x)) \colon x \in \mathbb{A}^1\right\}.$$



(b) We have $G(f) = V\left(x_{n+1} - f(x_1, \ldots, x_n)\right) \subset \mathbb{A}^{n+1}$ where $x_{n+1} - f\left(x_1, \ldots, x_n\right)$ is polynomial, since it is the difference of two polnomials since f is a polynomial.

(c) We define the morphisms $\varphi \colon X \to G(f)$ by $\varphi(a_1, \ldots, a_n) = (a_1, \ldots, a_n, f(a_1, \ldots, a_n))$ and $pr_X : G(f) \to X$ by $pr_X\left(a_1, \ldots, a_{n+1}\right) = (a_1, \ldots, a_n)$. Clearly, $pr_X \circ \varphi = \mathbb{1}_X$ and $\varphi \circ pr_X = \mathbb{1}_{G_f}$. Furthermore, φ is a morphism since $\varphi(P) = (x_1(P), x_2(P), \ldots, x_n(P), f(P))$ for all $P \in X$, and $x_1, \ldots, x_n, f \in k\left[x_1, \ldots, x_n\right]$. The projection map is also a morphism by the first example on lecture note 8. Thus $G(f)$ is isomorphic to X.