1. Curves, Surfaces and Hyperbolic Geometry

1.1. **Simple closed curves.** There is a bijective correspondence

$$\left\{\begin{array}{c}\text{Nontrivial}\\ \text{conjugacy classes}\\ \text{in } \pi_1(S)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Nontrivial free}\\ \text{homotopy classes of oriented}\\ \text{closed curves in } S\end{array}\right\}$$

Definition 1.1 (Primitive and multiple elements)**.** An element g of a group G is *primitive* if there does not exist any $h \in G$ so that $g = h^k$ for $|k| > 1$. The property of being a primitive is a conjugacy class invariant. In particular, it makes sense to say that a closed curve in a surface is primitive.

A closed curve in S is a multiple if it is a map $S^1 \to S$ that factors through the map $S^1 \overset{\times n}{\to} S^1$ for $n > 1$, i.e., there exists a map $\tilde{\alpha}\colon S^1 \to S$ such that the following diagram commutes:

$$\begin{tikzcd} S^1 \arrow[r, "\times n"] \arrow[rr, bend left, dashed, "\tilde{\alpha}"] & S^1 \arrow[r, "\alpha"] & S \end{tikzcd}$$

Definition 1.2 (Lifts)**.** We make a distinction between lifts: let $p\colon \tilde{S} \to S$ be a covering space. By a *lift* of a closed curve α to $\tilde{S}$ we will always mean the image of a lift $\mathbb{R} \to \tilde{S}$ of the map $\alpha \circ \pi$ where $\pi\colon \mathbb{R} \to S^1$ is the usual covering map. I.e., a lift of $\alpha\colon S^1 \to S$ is a map $\tilde{\alpha}\colon \mathbb{R} \to \tilde{S}$ such that the following diagram commutes

$$\begin{tikzcd} & & \tilde{S} \arrow[d, "p"] \\ \mathbb{R} \arrow[urr, "\tilde{\alpha}"] \arrow[r, "\pi"'] & S^1 \arrow[r, "\alpha"'] & S \end{tikzcd}$$

A lift is different from a *path lift* which is a proper subset of a lift. Namely, it would be the restriction of $\tilde{\alpha}$ to some interval of $\mathbb{R}$ of length 2π if the covering map π is of the form $t \mapsto e^{it}$.

Now suppose $p\colon \tilde{S} \to S$ is the universal cover and α is a simple closed curve in S that is not a multiple of another closed curve. In this case, there is a bijective correspondence between cosets in $\pi_1(S)$ of the infinite cyclic subgroup $\langle\alpha\rangle$ and the lifts of α. This can be seen as follows: first choose a basepoint $\alpha(1) = x_0 \in S$ and some $\tilde{x_0} \in p^{-1}(x_0)$. There exists a unique lift $\tilde{\alpha}$ of α such that

$$\begin{tikzcd} & & \tilde{S} \arrow[d, "p"] \\ \mathbb{R} \arrow[urr, "\tilde{\alpha}"] \arrow[r] & S^1 \arrow[r, "\alpha"] & S \end{tikzcd}$$

commutes and such that $\tilde{\alpha}(0) = \tilde{x} \in p^{-1}(\alpha \circ \pi(0))$ for some specific $\tilde{x}$ [Bredon, Cor. 4.2]. But the set $p^{-1}(\alpha \circ \pi(0))$ is in bijective correspondence with the loops in $\pi_1(S)$ by the path lifting lemma. Now, under which path lifts are the lifts the same? The lifts of α to two points $\tilde{x}, \tilde{y} \in p^{-1}(\alpha \circ \pi(0))$ will be the same if $\alpha^k \cdot \tilde{x} = \tilde{y}$ where $\cdot$ denotes the monodromy action of $\pi_1(S)$ on the fiber. Now, there exist γ_x and γ_y in $\pi_1(S)$ such that $\gamma_x \cdot \tilde{x_0} = \tilde{x}$ and $\gamma_y \cdot \tilde{x_0} = \tilde{y}$, so $\alpha^k\gamma_x = \gamma_y$. Hence the lifts corresponding to γ_x and γ_y are the same if and only if $\alpha^k\gamma_x = \gamma_y$ for some k, i.e. if and only if $\gamma_x = \gamma_y$ in $\pi_1(S)/\langle\alpha\rangle$.

As usual, the group $\pi_1(S)$ acts on the set of lifts of α by deck transformations, and this action agrees with the usual left action of $\pi_1(S)$ on the cosets of $\langle\alpha\rangle$. The stabilizer of the lift corresponding to the coset $\gamma\langle\alpha\rangle$ is the cyclic group $\langle\gamma\alpha\gamma^{-1}\rangle$. See figure 1.

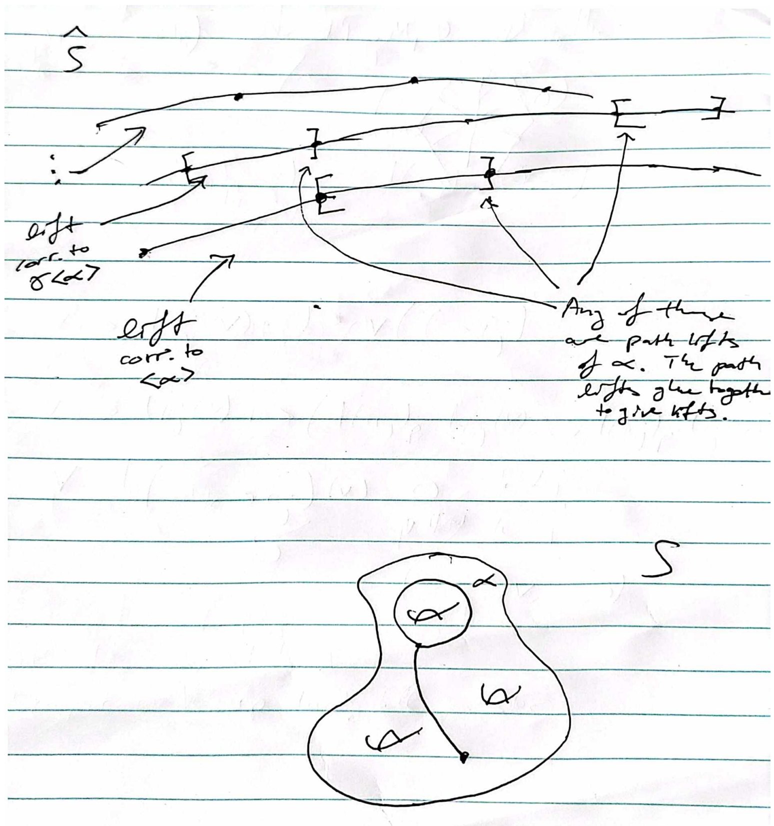

FIGURE 1.

Theorem 1.3. *When S admits a hyperbolic metric and α is a primitive element of $\pi_1(S)$, we have a bijective correspondence*

$$\left\{\begin{array}{c}\text{Elements of the conjugacy}\\ \text{class of } \alpha \text{ in } \pi_1(S)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Lifts to } \tilde{S} \text{ of the}\\ \text{closed curve } \alpha\end{array}\right\}$$

More precisely, we claim that the map which sends the lift given by the coset $\gamma\langle\alpha\rangle$ to $\gamma\alpha\gamma^{-1}$ is bijective and well-defined.

Proof. To show that it is well-defined, suppose $\gamma\langle\alpha\rangle$ and $\beta\langle\alpha\rangle$ give the same lift. Then $\gamma = \beta\alpha^k$. So in particular,

$$\gamma\alpha\gamma^{-1} = \beta\alpha^k\alpha\alpha^{-k}\beta^{-1} = \beta\alpha\beta^{-1}$$

so they do correspond to the same element of the conjugacy class $[\alpha]$. It is clear that this is a surjective map. Now suppose that $\gamma\alpha\gamma^{-1} = \beta\alpha\beta^{-1}$. Then

$\beta^{-1}\gamma\alpha\left(\beta^{-1}\gamma\right)^{-1}=\alpha$, so in particular, $\beta^{-1}\gamma \in C_{\pi_1(S)}(\alpha)$ which is a cyclic group generated by, say, θ. But then $\theta^l=\alpha$ since α is trivially in the centralizer of α; however, α is primitive, so l must be ± 1, but then α generates the centralizer of α, $C_{\pi_1(S)}(\alpha)=\langle\alpha\rangle$, and hence $\gamma=\beta\alpha^l$, so $\gamma\langle\alpha\rangle=\beta\langle\alpha\rangle$. □

Remark. If α is any multiple, then we still have a bijective correspondence between elements of the conjugacy class of α and the lifts of α. However, if α is not primitive and not a multiple, then there are more lifts of α than there are conjugates. Indeed, if $\alpha=\beta^k$, where $k>1$, then $\beta\langle\alpha\rangle\neq\langle\alpha\rangle$ while $\beta\alpha\beta^{-1}=\alpha$.

Example 1.4. The above correspondence does not hold for the torus T^2 because each closed curve has infinitely many lifts, while each element of $\pi_1\left(T^2\right)\approx\mathbb{Z}^2$ is its own conjugacy class because $\pi_1\left(T^2\right)$ is abelian.

Geodesic representatives.

Proposition 1.5. *Let S be a hyperbolic surface. If α is a closed curve in S that is not homotopic into a neighborhood of a puncture, then α is homotopic to a unique geodesic closed curve γ.*

Corollary 1.6. *For compact hyperbolic surfaces, there is a bijective correspondence:*

$$\left\{\begin{array}{c}\textit{Conjugacy classes}\\ \textit{in } \pi_1(S)\end{array}\right\}\longleftrightarrow\left\{\begin{array}{c}\textit{Oriented geodesic}\\ \textit{closed curves in } S\end{array}\right\}$$

Simple closed curves.

Definition 1.7 (Simple curves)**.** A closed curve in S is *simple* if it is topologically embedded, i.e., if the map $S^1\to S$ is injective.

By [Bredon, Thm 11.8], any closed curve α can be approximated (arbitrarily close) by a smooth closed curve which is homotopic to α. Moreover, if α is simple, then the smooth approximation can be chosen to be simple. Smooth curves are advantageous because we can make use of notions such as transversality.
Simple closed curves are also natural to study because they represent primitive elements of $\pi_1(S)$.

Proposition 1.8. *Let α be a simple closed curve in a surface S. If α is not null homotopic, then each element of the corresponding conjugacy class in $\pi_1(S)$ is primitive.*

Example: simple closed curves on the torus.

Proposition 1.9. *The nontrivial homotopy classes of oriented simple closed curves in T^2 are in bijective correspondence with the set of primitive elements of $\pi_1\left(T^2\right)\approx\mathbb{Z}^2$ which is the set of elements $(p,q)\in\mathbb{Z}^2$ such that either $(p,q)=(0,\pm 1)$ or $(p,q)=(\pm 1,0)$ or $\gcd(p,q)=1$.*

Closed geodesics.

Proposition 1.10. *Let S be a hyperbolic surface. Let α be a closed curve in S not homotopic into a neighborhood of a puncture. Let γ be the unique geodesic in the free homotopy class of α guaranteed by proposition 1.5. If α is simple, then γ is simple.*

2. Glossary

Definition 2.1 (Equivariant maps)**.** Suppose a group G acts on spaces X and Y, and let $f\colon X \to Y$ be a map. Then f is said to be equivariant if $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$ and all $g \in G$.

References

[Bredon] Glen E. Bredon Geometry and Topology
[Hirsch] Morris W. Hirsch Differential Topology
[Kosinski] Antoni A. Kosinski Differential Manifolds
[Milnor] John W. Milnor Topology from the differentiable viewpoint
[Thurston] William P. Thurston Three-dimensional Geometry and Topology