

1. COHOMOLOGY AND HOMOLOGICAL ALGEBRA

1.1. Cohomology in terms of Homological Algebra. Recall the Universal Coefficient Theorem for Cohomology:

Theorem 1.1 (Universal Coefficient Theorem for Cohomology). *Let R be a ring and A an R -module. Let C_* be a complex of projective R -modules such that the subcomplex of boundaries B_* is also a complex of projective modules.*

(1) *For all n , there is a SES*

$$0 \rightarrow \text{Ext}_R^1(H_{n-1}(C_*), A) \xrightarrow{\lambda_n} H^n(\text{Hom}_R(C_*, A)) \xrightarrow{\mu_n} \text{Hom}_R(H_n(C_*), A) \rightarrow 0$$

where both λ_n and μ_n are natural in C_ and A .*

(2) *If R is a PID, then the SES in (1) is split, but it is not always naturally split.*

Also recall the basic properties:

Lemma 1.2. *For a finitely generated H , we have*

- $\text{Ext}(H \oplus H', G) \cong \text{Ext}(H, G) \oplus \text{Ext}(H', G)$
- $\text{Ext}(H, G) = 0$ if H is free.
- $\text{Ext}(\mathbb{Z}/n, G) \cong G/nG$.

Corollary 1.3. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_{n-1}$, then $H^n(\text{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z})) \cong (H_n/T_n) \oplus T_{n-1}$.*

Proof. By the Universal Coefficient theorem for cohomology, we have that

$$H^n(\text{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z})) \cong \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(C_*), \mathbb{Z}) \oplus \text{Hom}_{\mathbb{Z}}(H_n(C_*), \mathbb{Z})$$

Now, $\text{Ext}_{\mathbb{Z}}^1(H_{n-1}(C_*), \mathbb{Z}) \cong T_{n-1}$ and $\text{Hom}_{\mathbb{Z}}(H_n(C_*), \mathbb{Z}) \cong H_n/T_n$. □

Proposition 1.4. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G .*

Proof. Suppose $\alpha: C_* \rightarrow C'_*$ is the chain map such that $\alpha_*: H_n(C_*) \rightarrow H_n(C'_*)$ is an isomorphism for all n . Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}(H_{n-1}(C), G) & \longrightarrow & H^n(C; G) & \xrightarrow{h} & \text{Hom}(H_n(C), G) \longrightarrow 0 \\ & & (\alpha_*)^* \uparrow \cong & & \alpha^* \uparrow & & (\alpha_*)^* \uparrow \cong \\ 0 & \longrightarrow & \text{Ext}(H_{n-1}(C'), G) & \longrightarrow & H^n(C'; G) & \xrightarrow{h} & \text{Hom}(H_n(C'), G) \longrightarrow 0 \end{array}$$

which follows from naturality of the Universal Coefficient theorem. Then by the 5-lemma, we obtain that α^* is an isomorphism also. □

1.2. Cohomology of Spaces. Define $S^{-n}(X; A) := \text{Hom}_{\mathbb{Z}}(S_n(X), A)$, so $S^*(X; A)$ is a chain complex. We define $H^n(X; A) := H_{-n}(S^*(X; A))$, called *singular cohomology of X with coefficients in A* .

Thus an n -cochain $\varphi \in S^{-n}(X; G)$ assigns to each n -simplex $\sigma: \Delta^n \rightarrow X$ a value $\varphi(\sigma) \in G$. Since the n -simplices form a basis for $S_n(X)$, these values can be chosen

arbitrarily, hence n -cochains are exactly equivalent to functions from singular n -simplices to G .

The *coboundary map* $\delta: S^{-n}(X; G) \rightarrow S^{-(n+1)}(X; G)$ is the dual ∂^* , so for a cochain $\varphi \in S^{-n}(X; G)$, its coboundary $\delta\varphi$ is the composition $\delta\varphi = \partial^*\varphi = \varphi \circ \partial$, i.e., the composition $C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\varphi} G$.

Hence for a singular $(n+1)$ -simplex $\sigma: \Delta^{n+1} \rightarrow X$, we have

$$\delta\varphi(\sigma) = \sum_i (-1)^i \varphi(\sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_{n+1}]}) .$$

Since δ^2 is the dual of $\partial^2 = 0$, we have $\delta^2 = 0$ also, so $H^n(X; G)$ can be defined as above.

Note. For a cochain $\varphi \in S^{-n}(X; G)$ to be a cocycle means that $\delta\varphi = \varphi\partial = 0$, i.e., it means that φ vanishes on boundaries.

1.2.1. H^0 : When $n = 0$, there is no Ext term and so $H^0(X; G) \cong \text{Hom}(H_0(X), G)$. This can also be seen from definitions: singular 0-simplices are just points of X , so a cochain in $S^0(X; G)$ is an arbitrary function $\varphi: X \rightarrow G$, not necessarily continuous. For this to be a cocycle, we must have $0 = \delta\varphi = \varphi \circ \partial$. Evaluating this at some 1-simplex $[v_0, v_1]$, we find that $\varphi(v_1) - \varphi(v_0) = 0$, so φ is simply constant on each path component. Thus $H^0(X; G)$ is just the set of all functions from the path components of X into G , which is the same as the set of all group homomorphisms from $S_0(X)$ into G , i.e., $H^0(X; G) \cong \text{Hom}(H_0(X), G)$.

1.2.2. H^1 : Likewise, $\text{Ext}(H_0(X), G) = 0$ since $H_0(X)$ is free, so by the Universal Coefficient theorem, $H^1(X; G)$ is isomorphic to $\text{Hom}(H_1(X), G)$ which, when X is path-connected, is isomorphic to $\text{Hom}(\pi_1(X), G)$ since G is abelian.

1.2.3. *Reduced Cohomology Groups.* Reduced cohomology groups $\tilde{H}^n(X; G)$ can be defined by dualizing the augmented chain complex $\dots \rightarrow C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$, then taking \ker / im . This gives $\tilde{H}^n(X; G) = H^n(X; G)$ for $n > 0$, and the Universal Coefficient theorem gives $\tilde{H}^0(X; G) = \text{Hom}(\tilde{H}_0(X), G)$.

We can also describe $\tilde{H}^0(X; G) \cong \text{Hom}(\tilde{H}_0(X), G)$ more explicitly by using the above interpretation of $H^0(X; G)$ as functions $X \rightarrow G$ which are constant on path-components. Recall that $\varepsilon: C_0(X) \rightarrow \mathbb{Z}$ sends each singular 0-simplex σ to 1, so ε^* sends a homomorphism $\varphi: \mathbb{Z} \rightarrow G$ to $C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\varphi} G$ which sends $\sigma \mapsto \varphi(1)$ for all 0-simplices. That is, $\varepsilon^*\varphi$ is a constant function $X \rightarrow G$, and since $\varphi(1)$ can be any element of G , the image of ε^* consists of precisely the constant functions. Thus $\tilde{H}^0(X; G)$ is all functions $X \rightarrow G$ that are constant on path-components modulo the functions that are constant on all of X .

1.2.4. *Relative Groups and the LES of a Pair.* To define relative groups $H^n(X, A; G)$ for a pair (X, A) , we first dualize the SES

$$0 \rightarrow S_n(A) \xrightarrow{i} S_n(X) \xrightarrow{j} S_n(X, A) \rightarrow 0$$

by applying $\text{Hom}(-, G)$ to get

$$0 \leftarrow S^n(A; G) \xleftarrow{i^*} S^n(X; G) \xleftarrow{j^*} S^n(X, A; G) \leftarrow 0 \quad (\Omega)$$

where $S^n(X, A; G) := \text{Hom}(C_n(X, A), G)$.

To see exactness of the dual sequence, we note the following: i^* restricts cochains on X to cochains on A , so for a function from singular n -simplices in X to G , the image of this function under i^* is obtained by restricting the domain of the function to singular n -simplices in A . Every function from singular n -simplices in A to G can be extended to all singular n -simplices in X , for example, by assigning the value 0 to all singular n -simplices not in A , so i^* is surjective. The kernel consists of cochains taking the value 0 on all singular n -simplices in A . Such cochains are the same as homomorphisms $C_n(X, A) = C_n(X)/C_n(A) \rightarrow G$, so the kernel of i^* is exactly $C^n(X, A; G) = \text{Hom}(C_n(X, A), G)$, giving the desired exactness.

Note. Note that we can view $C^n(X, A; G)$ as the functions from singular n -simplices in X to G that vanish on simplices in A , since the basis for $C_n(X)$ consisting of singular n -simplices in X is the disjoint union of the simplices with image contained in A and the simplices with image no contained in A .

Relative coboundary maps $\delta: C^n(X, A; G) \rightarrow C^{n+1}(X, A; G)$ are obtained as restrictions of the absolute δ 's.

Since the maps i^* and j^* commute with δ (since i and j commute with ∂), the maps i^* and j^* induce chain maps $i^*: S^*(X; G) \rightarrow S^*(A; G)$ and $j^*: S^*(X, A; G) \rightarrow S^*(X; G)$, and in particular, since $i^*j^* = 0$, this is a SES of chain complexes, hence induces a LES of cohomology groups (Thm. 6.5.5, AlgTop1):

$$\dots \rightarrow H_n(S^*(X, A; G)) \xrightarrow{j^*} H_n(S^*(X; G)) \xrightarrow{i^*} H_n(S^*(A; G)) \xrightarrow{\delta} H^{n+1}(X, A; G) \rightarrow \dots$$

By similar reasoning, one obtains a LES of reduced cohomology groups for a pair (X, A) with A nonempty. In particular, if A is a point, we find $\tilde{H}^n(X; G) \cong H^n(X, x_0; G)$.

More generally, there is a LES for a triple (X, A, B) coming from the SES

$$0 \leftarrow C^n(A, B; G) \xleftarrow{i^*} C^n(X, B; G) \xleftarrow{j^*} C^n(X, A; G) \leftarrow 0.$$

1.2.5. Duality between Connecting Homomorphisms. There is a duality between $\delta: H^n(A; G) \rightarrow H^{n+1}(X, A; G)$ and $\partial: H_{n+1}(X, A) \rightarrow H_n(A)$ as depicted in the following diagram:

$$\begin{array}{ccc} H^n(A; G) & \xrightarrow{\delta} & H^{n+1}(X, A; G) \\ \downarrow h & & \downarrow h \\ \text{Hom}(H_n(A), G) & \xrightarrow{\partial^*} & \text{Hom}(H_{n+1}(X, A), G) \end{array}$$

Proof. Recall that the connecting homomorphisms were defined by the diagrams

$$\begin{array}{ccc} & S^{n+1}(X; G) & \longleftarrow S^{n+1}(X, A; G) \\ & \uparrow & \nearrow \\ S^n(A; G) & \longleftarrow & S^n(X; G) \end{array}$$

and

$$\begin{array}{ccc}
& S_{n+1}(X) & \longrightarrow S_{n+1}(X, A) \\
& \uparrow & \nearrow \text{dashed} \\
S_n(A) & \xleftarrow{\quad} S^n(X; G) &
\end{array}$$

where the dashed arrows are only there when the chain and cochain groups are replaced by homology and cohomology groups.

To see that $h\delta = \partial^*h$, let $\alpha \in H^n(A; G)$ be represented by a cocycle $\varphi \in S^n(A; G)$. To compute $\delta(\alpha)$, first extend φ to $\bar{\varphi} \in S^n(X; G)$ by letting $\bar{\varphi}$ be 0 on all cochains not contained in A . Then composing $\bar{\varphi}$ with $\partial: S_{n+1}(X) \rightarrow S_n(X)$ to get a cochain $\delta\bar{\varphi} = \bar{\varphi}\partial \in S^{n+1}(X; G)$, and since $\varphi \in S^n(A; G)$ was a cocycle, we have that $\delta\varphi = 0$, so $\bar{\varphi} \circ \partial$ is actually in $S^{n+1}(X, A; G)$ and represents $\delta(\alpha) \in H^{n+1}(X, A; G)$. Now applying h to $\bar{\varphi}\partial$ simply restricts the domain of $\bar{\varphi}\partial$ to relative cycles in $S_{n+1}(X, A)$, i.e., $(n+1)$ -chains in X whose boundary lies in A . On such chains $\bar{\varphi}\partial = \varphi\partial$ since the extension of φ to $\bar{\varphi}$ has no effect here. Thus $h\delta(\alpha)$ is represented by $\varphi\partial$.

On the other hand, let us consider $\partial^*h(\alpha)$. Recall that $S^n(X; A) = \text{Hom}_{\mathbb{Z}}(S_n(X), A)$, so φ is a homomorphism $S_n(X) \rightarrow A$. Applying h to φ then restricts the domain to n -cycles in A . Then applying ∂^* composes with the map which sends a relative $(n+1)$ -cycle in X to its boundary in A . Thus $\partial^*h(\alpha)$ is represented by $\varphi\partial$ just as $h\delta(\alpha)$ was, hence the square commutes. \square

1.2.6. Induced Homomorphisms. If we have a chain map $f_{\#}: S_*(X) \rightarrow S_*(Y)$ induced by $f: X \rightarrow Y$, then we also have dual cochain maps $f^{\#}: S^*(Y; G) \rightarrow S^*(X; G)$. The relation $f_{\#}\partial = \partial f_{\#}$ dualizes to $\delta f^{\#} = f^{\#}\delta$, so $f^{\#}$ is a cochain map and hence induced homomorphisms $f^*: H^n(Y; G) \rightarrow H^n(X; G)$.

In the relative case, a map $f: (X, A) \rightarrow (Y, B)$ induces $f^*: H^n(Y, B; G) \rightarrow H^n(X, A; G)$ by the same reasoning.

Since f induces a map between short exact sequences of cochain complexes, it induces a map between long exact sequences of cohomology groups with commuting squares.

The properties $(fg)^{\#} = g^{\#}f^{\#}$ and $\text{id}^{\#} = \text{id}$ imply $(fg)^* = g^*f^*$ and $\text{id}^* = \text{id}$, so $X \mapsto H^n(X; G)$ and $(X, A) \mapsto H^n(X, A; G)$ are contravariant functors.

The algebraic Universal Coefficient theorem applies also to the relative cohomology since the relative groups $C_n(X, A)$ are free, and there is a naturality statement also: a map $f: (X, A) \rightarrow (Y, B)$ induces a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Ext}(H_{n-1}(X, A), G) & \longrightarrow & H^n(X, A; G) & \xrightarrow{h} & \text{Hom}(H_n(X, A), G) \longrightarrow 0 \\
& & \uparrow (f_*)^* & & \uparrow f^* & & \uparrow (f_*)^* \\
0 & \longrightarrow & \text{Ext}(H_{n-1}(Y, B), G) & \longrightarrow & H^n(Y, B; G) & \xrightarrow{h} & \text{Hom}(H_n(Y, B), G) \longrightarrow 0
\end{array}$$

1.2.7. Axioms for Cohomology. These are exactly dual to the axioms for homology. Restricting attention to CW complexes, a (reduced) *cohomology theory* is a sequence of contravariant functors \tilde{h}^n from CW complexes to abelian groups, together with natural coboundary homomorphisms $\delta: \tilde{h}^n(A) \rightarrow \tilde{h}^{n+1}(X/A)$ for CW pairs (X, A) , satisfying the following axioms:

- (1) If $f \simeq g: X \rightarrow Y$, then $f^* = g^*: \tilde{h}^n(Y) \rightarrow \tilde{h}^n(X)$.
 (2) For each CW pair (X, A) , there is a LES

$$\dots \xrightarrow{\delta} \tilde{h}^n(X/A) \xrightarrow{q^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \xrightarrow{q^*} \dots$$

- (3) For a wedge sum $X = \bigvee_{\alpha} X_{\alpha}$ with inclusions $i_{\alpha}: X_{\alpha} \hookrightarrow X$, the product map $\prod_{\alpha} i_{\alpha}^*: \tilde{h}^n(X) \rightarrow \prod_{\alpha} \tilde{h}^n(X_{\alpha})$ is an isomorphism for each n .

1.2.8. Simplicial Cohomology. If X is a Δ -complex and $A \subset X$ is a subcomplex, then the simplicial chain groups $\Delta_n(X, A)$ dualize to simplicial cochain groups $\Delta^n(X, A; G) := \text{Hom}(\Delta_n(X, A), G)$, and the resulting cohomology groups are by definition the simplicial cohomology groups $H_{\Delta}^n(X, A; G)$. Since the inclusions $\Delta_n(X, A) \hookrightarrow C_n(X, A)$ induce isomorphisms $H_{\Delta}^n(X, A) \cong H_n(X, A)$, then Corollary 1.3 implies that the dual maps $S^n(X, A; G) \rightarrow \Delta^n(X, A; G)$ also induce isomorphisms $H^n(X, A; G) \cong H_{\Delta}^n(X, A; G)$.

Exercise 1.5. (1) Directly from the definitions, compute the simplicial cohomology groups of $S^1 \times S^1$ with \mathbb{Z} and \mathbb{Z}_2 coefficients, using the product Δ -complex structure.
 (2) Do the same for \mathbb{RP}^2 and the Klein bottle.

Solution. We are asked to do it directly from definitions, so let us track through each step: we can give $S^1 \times S^1 =: T$ the Δ -complex structure indicated Figure 1.

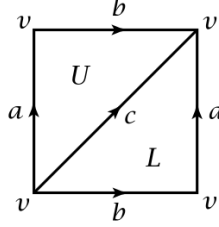


FIGURE 1.

$$\text{Then } \Delta_n(S^1 \times S^1) \cong \begin{cases} \mathbb{Z}^2, & n = 2 \\ \mathbb{Z}^3, & n = 1 \\ \mathbb{Z}, & n = 0 \\ 0, & \text{else} \end{cases} \text{ And in particular, we get a chain complex}$$

$$0 \rightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \rightarrow 0$$

so dualizing, we get that since $\text{Hom}(\mathbb{Z}^k, \mathbb{Z}) \cong \mathbb{Z}^k$, then the dual complex becomes

$$0 \leftarrow \mathbb{Z}^2 \xleftarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}^*} \mathbb{Z}^3 \xleftarrow{0} \mathbb{Z} \leftarrow 0.$$

By linear algebra, we know that the dual of a matrix is given by its transpose, so this chain complex becomes

$$0 \leftarrow \mathbb{Z}^2 \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \leftarrow \mathbb{Z}^3 \xleftarrow{0} \mathbb{Z} \leftarrow 0.$$

From this, we can read off the homology groups of the chain complex. In degree 2, $\ker \delta = \mathbb{Z}^2$, so

$$H^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}^2 / (U = L, U = -L) \cong 0.$$

For H^1 , the kernel of the transposed matrix is $\ker \delta = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$ which is thus

isomorphic to \mathbb{Z} . Since the boundary map is 0, we find that $H^1(S^1 \times S^1, \mathbb{Z}) \cong \mathbb{Z}$. Lastly, $H^0(S^1 \times S^1, \mathbb{Z}) \cong \mathbb{Z}$ and $H^k = 0$ for all other k .

Note that this agrees with what Poincaré duality tells us.

If instead, we wanted to calculate $H^*(S^1 \times S^1; \mathbb{Z}_2)$, then since $\text{Hom}(\mathbb{Z}^k, \mathbb{Z}_2) \cong \mathbb{Z}_2^k$, we find that the dual chain complex takes the form

$$0 \leftarrow \mathbb{Z}_2^2 \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \leftarrow \mathbb{Z}_2^3 \xleftarrow{0} \mathbb{Z}_2 \leftarrow 0$$

In this case, the image in degree three is $\langle (1, 1) \rangle$, so $H^2(\mathbb{RP}^2, \mathbb{Z}_2) \cong \mathbb{Z}_2^2 / \{(1, 1) = (0, 0)\} \cong$

\mathbb{Z}_2 . The kernel in degree 1 also now becomes $\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}_2^2$, so $H^1(\mathbb{RP}^2, \mathbb{Z}_2) \cong$

\mathbb{Z}_2^2 , and clearly, $H^0(\mathbb{RP}^2, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

(2) We use the Δ -complexes for \mathbb{RP}^2 and the Klein bottles shown in Figure 2.

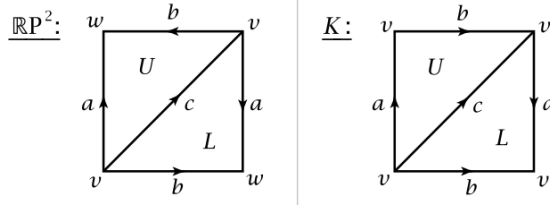


FIGURE 2.

In this case, the Δ -chain complex for \mathbb{RP}^2 becomes

$$0 \rightarrow \mathbb{Z}^2 \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} \rightarrow \mathbb{Z}^3 \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix} \rightarrow \mathbb{Z}^2 \rightarrow 0$$

The dual complex becomes

$$0 \leftarrow \mathbb{Z}^2 \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \leftarrow \mathbb{Z}^3 \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} \leftarrow \mathbb{Z}^2 \leftarrow 0$$

We thus find that $H^2(\mathbb{RP}^2; \mathbb{Z}) = 0$, $H^1(\mathbb{RP}^2; \mathbb{Z}) \cong \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle / \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \cong \mathbb{Z}$, and $H^0(\mathbb{RP}^2; \mathbb{Z}) \cong \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle \cong \mathbb{Z}$.

2. A HOMOTOPY CONSTRUCTION OF COHOMOLOGY

Theorem 2.1. *There are natural bijections $T: [X, K(G, n)]_* \rightarrow H^n(X; G)$ for all CW complexes X and all $n > 0$, with G any abelian group. Such a T has the form $T([f]) = f^*(\alpha)$ for a certain distinguished class $\alpha \in H^n(K(G, n); G)$.*

Definition 2.2 (Fundamental Class). A class $\alpha \in H^n(K(G, n); G)$ with the property stated in Theorem 2.1 is called a *fundamental class*.

Note. The theorem also holds with $[X, K(G, n)]_*$ replaced by $[X, K(G, n)]$, the non-basepointed homotopy classes.

When $n > 1$, every map $X \rightarrow K(G, n)$ can be homotoped to take basepoint to basepoint and every homotopy between basepoint-preserving maps can be homotoped to be basepoint-preserving since $K(G, n)$ is simply-connected.

When $n = 1$, $[X, K(G, n)]_* = [X, K(G, n)]$ for abelian G according to an exercise for section 4.A in Hatcher.

For $n = 0$, $H^0(X; G) = [X, K(G, 0)]$ and $\tilde{H}^0(X; G) = [X, K(G, 0)]_*$.

The main two steps in the proof, will be the following two assertions:

- (1) The functors $h^n(X) = [X, K(G, n)]_*$ define a reduced cohomology theory on the category of based CW complexes.
- (2) If a reduced cohomology theory h^* defined on CW complexes has coefficient group $h^n(S^0)$ which are zero for $n \neq 0$, then there are natural isomorphisms $h^n(X) \cong \tilde{H}^n(X; h^0(S^0))$ for all CW complexes X and all n .

Towards proving (1), we will study a more general question: When does a sequence of spaces K_n define a cohomology theory by setting $h^n(X) = [X, K_n]_*$? Note that since $[X, K_n]_*$ is trivial when X is a point, this will be a reduced cohomology theory.

The first part to address is putting a group structure on the set $[X, K]_*$. When $X = S^n$, we have $[S^n, K]_* = \pi_n(K)$, which has a group structure when $n > 0$. The definition of this

2.1. Fibrations. By convention, a fibration will be a map $p: E \rightarrow B$ having the homotopy lifting property with respect to all spaces. I.e., fibrations will mean Hurewicz fibrations.

Proposition 2.3. *For a fibration $p: E \rightarrow B$, the fibers $F_b = p^{-1}(b)$ over each path component of B are all homotopy equivalent.*

Proof. From a path $\gamma: I \rightarrow B$, we can construct a homotopy $g: F_{\gamma(0)} \times I \rightarrow E$ by $g(x, t) = \gamma(t)$. The inclusion $F_{\gamma(0)} \hookrightarrow E$ provides a lift $\tilde{g}_0: F_{\gamma(0)} \times \{0\} \rightarrow E$, so we

have the following diagram:

$$\begin{array}{ccc} F_{\gamma(0)} \times \{0\} & \xrightarrow{\tilde{g}_0} & E \\ \downarrow & \searrow \tilde{g} & \downarrow p \\ F_{\gamma(0)} \times I & \xrightarrow{g} & B \end{array}$$

so since p is a fibration, there exists a lift $\tilde{g}: F_{\gamma(0)} \times I \rightarrow E$ making the diagram commute. Hence $p \circ \tilde{g} = g$ maps $F_{\gamma(0)}$ to $\gamma(t)$ for all t , hence $\tilde{g}(F_{\gamma(0)} \times \{t\}) \subset F_{\gamma(t)}$ for all $t \in I$. In particular, let L_γ be the composition $F_{\gamma(0)} \hookrightarrow F_{\gamma(0)} \times \{1\} \xrightarrow{\tilde{g}} F_{\gamma(1)}$. The association $\gamma \mapsto L_\gamma$ has the following basic properties:

- (1) If $\gamma \simeq \gamma' \text{ rel } \partial I$, then $L_\gamma \simeq L_{\gamma'}$. In particular, the homotopy class of L_γ is independent of the choice of the lifting \tilde{g}_t of g_t .
- (2) For a composition of paths $\gamma\gamma'$, $L_{\gamma\gamma'}$ is homotopic to the composition $L_{\gamma'}L_\gamma$.

Note. From these statement, it follows that L_γ is a homotopy equivalence with homotopy inverse $L_{\bar{\gamma}}$.

Note also that it is true in general that a fibration has the homotopy lifting property for pairs $(X \times I, X \times \partial I)$. This is because $(I \times I, I \times \{0\} \cup \partial I \times I) \cong (I \times I, I \times \{0\})$ which naturally has the property that any map on $I \times \{0\}$ can be extended to $I \times I$, and hence $X \times (I \times I, I \times \{0\} \cup \partial I \times I) = (X \times I \times I, X \times I \times \{0\}) \cup X \times \partial I \times I$ also has this same property which is equivalent to the pair $(X \times I, X \times \partial I)$ having the homotopy extension property.

Now, to prove (a), let $\gamma: I \times I \rightarrow B$ be a homotopy from γ to γ' . This determines a family $g: F_{\gamma(0)} \times I \times I \rightarrow B$ with $g(s, t, F_{\gamma(0)}) = \gamma(s, t)$. Now we want to define a map $G: F_{\gamma(0)} \times I \times I \rightarrow E$. We start by letting $G|_{F_{\gamma(0)} \times \{0\} \times \{t\}}$ be equal to the composition $F_{\gamma(0)} \hookrightarrow F_{\gamma(0)} \times \{1\} \xrightarrow{\tilde{g}} F_{\gamma(1)}$, and similarly $G|_{F_{\gamma(0)} \times \{1\} \times \{t\}}$ but with γ' . Denote $G|_{F_{\gamma(0)} \times \{s\} \times \{t\}}$ by $G_{s,t}$. Then $G_{0,t} = L_\gamma$ and $G_{1,t} = L_{\gamma'}$. Let $G|_{F_{\gamma(0)} \times \{s\} \times \{0\}}$ be given by the inclusion $F_{\gamma(0)} \hookrightarrow E$ for all s . Then using the homotopy lifting property for the pair $(F_{\gamma(0)} \times I, F_{\gamma(0)} \times \partial I)$, we can extend G to a map $F_{\gamma(0)} \times I \times I \rightarrow E$. Letting now $t = 1$, we get a homotopy $F_{\gamma(0)} \times I \rightarrow B$ from $G_{F_{\gamma(0)} \times \{0\} \times \{1\}} = G_{0,1} = L_\gamma$ to $G_{1,1} = L_{\gamma'}$.

For (b), if \tilde{g}_t and \tilde{g}'_t define L_γ and $L_{\gamma'}$, respectively, then we obtain a lift defining $L_{\gamma\gamma'}$ by taking \tilde{g}_{2t} for $0 \leq t \leq \frac{1}{2}$ and $\tilde{g}'_{2t-1}L_\gamma$ for $\frac{1}{2} \leq t \leq 1$. \square

Definition 2.4 (Fiber-preserving). Given fibrations $p_i: E_i \rightarrow B$, $i = 1, 2$, a map $f: E_1 \rightarrow E_2$ is called *fiber-preserving* if $p_1 = p_2 f$.

Definition 2.5 (Fiber homotopy equivalence). A fiber-preserving map $f: E_1 \rightarrow E_2$ is a *fiber homotopy equivalence* if there is a fiber-preserving map $g: E_2 \rightarrow E_1$ such that both compositions fg and gf are homotopic to the identity through fiber-preserving maps.

Proposition 2.6. *Given a fibration $p: E \rightarrow B$ and a homotopy $f_t: A \rightarrow B$, the pullback fibrations $f_0^*(E) \rightarrow A$ and $f_1^*(E) \rightarrow A$ are fiber homotopy equivalent.*

Remark. This is meant in the sense that there exists a fiber homotopy equivalence $f_0^*(E) \rightarrow f_1^*(E)$ with respect to these fibrations.

Corollary 2.7. *A fibration $E \rightarrow B$ over a contractible base B is fiber homotopy equivalent to a product fibration $B \times F \rightarrow B$.*

2.1.1. Pathspace Constructions.

Definition 2.8 (Construction of). Given a map $f: A \rightarrow B$, let

$$E_f := \{(a, \gamma) \mid a \in A, \gamma: (I, \{0\}) \rightarrow (B, f(a))\}.$$

We topologize E_f as a subspace of $A \times B^I$ where B^I has the compact-open topology.

Proposition 2.9. *The map $p: E_f \rightarrow B$ with $p(a, \gamma) = \gamma(1)$ is a fibration.*

Note. We have a natural embedding of A as the set of pairs $(a, \gamma) \in E_f$ with γ being the constant path at $f(a)$. Then E_f deformation retracts onto this subspace by restricting all the paths γ to shorter and shorter initial segments. The map $p: E_f \rightarrow B$ restricts to f on the subspace A , so we have factored a map $f: A \rightarrow B$ as the composition $A \hookrightarrow E_f \rightarrow B$ of a homotopy equivalence and a fibration.

Definition 2.10 (Homotopy fiber). The fiber F_f of $E_f \rightarrow B$ is called the *homotopy fiber* of f . It consists of all pairs (a, γ) with $a \in A$ and γ a path in B from $f(a)$ to a basepoint $b_0 \in B$.

Remark. If $f: A \rightarrow B$ is the inclusion of a subspace, then E_f is the space of paths in B starting at points of A . In this case, a map $f: (I^{i+1}, \partial I^{i+1}, J^i) \rightarrow (B, A, x_0)$ can be regarded as a map $(I^i, \partial I^i) \rightarrow (F_f, \gamma_0)$ where γ_0 is the constant path at x_0 and F_f is the fiber of E_f over x_0 .

Therefore, $\pi_{i+1}(B, A, x_0)$ can be identified with $\pi_i(F_f, \gamma_0)$, so the LES of homotopy groups of the pairs (B, A) and of the fibration $E_f \rightarrow B$ can be identified (since also $E_f \simeq A$).

Definition 2.11. An important special case of the above construction is when f is the inclusion of the basepoint $\{b_0\} \hookrightarrow B$. Then E_f is the space PB of paths in B starting at b_0 , and $p: PB \rightarrow B$ sends each path to its endpoint. The fiber $p^{-1}(b_0)$ is the loop space ΩB consisting of all loops in B based at b_0 .

Since PB is contractible by progressively truncating paths, the LES of homotopy groups for the path fibration $PB \rightarrow B$ yields another proof that $\pi_n(X, x_0) \cong \pi_{n-1}(\Omega X, x_0)$ for all n .

Theorem 2.12 (Milnor 1959, [?]). *The loop space of a CW complex is homotopy equivalent to a CW complex.*

Proposition 2.13. *If $p: E \rightarrow B$ is a fibration, then the inclusion $E \hookrightarrow E_p$ is a fiber homotopy equivalence. In particular, the homotopy fibers of p are homotopy equivalent to the actual fibers.*

Proof. Using that p is a fibration, we apply the HLP to the homotopy $g: E_p \times I \rightarrow B$ given by $g((e, \gamma), t) = \gamma(t)$, with the initial lift $\tilde{g}_0: E_p \times \{0\} \rightarrow E$ given by $\tilde{g}_0((e, \gamma), 0) = e$. This is indeed a lift on $E_p \times \{0\}$ since $g((e, \gamma), 0) = \gamma(0) = p(e) = p \circ \tilde{g}_0((e, \gamma), 0)$. Given the homotopy $\tilde{g}: E_p \times I \rightarrow E$, we can now construct a homotopy $h: E_p \times I \rightarrow E_p$ by $h((e, \gamma), t) = (\tilde{g}((e, \gamma), t), \gamma \circ \varphi_{[t, 1]})$, where $\varphi_{[t, 1]}: [0, 1] \rightarrow [t, 1]$ reparametrizes $[0, 1]$ to $[t, 1]$. Since $p \circ \tilde{g} = g$, we have that $p \circ \tilde{g}((e, \gamma), t) = g((e, \gamma), t) = \gamma(t) = \gamma \circ \varphi_{[t, 1]}(0)$, so h is indeed a map $E_p \times I \rightarrow E_p$. Also, since the endpoints of the γ are unchanged, $h|_{E_p \times \{t\}}$ is fiber-preserving: $p(e, \gamma) = \gamma(1) = \gamma \circ \varphi_{[t, 1]}(1) = p \circ h|_{E_p \times \{t\}}(e, \gamma)$.

Next note that $h_0(e, \gamma) = (\tilde{g}((e, \gamma), 0), \gamma) = (e, \gamma)$, so $h_0 = \text{id}$. Also, $h_1(e, \gamma) = (\tilde{g}((e, \gamma), 1), \gamma(1))$ which belongs to the elements of E_p for which the path is constant - this is precisely what we identified E with, so $h_1(E_p) \subset E$. Also $h((e, c_{p(e)}), t) = (\tilde{g}((e, c_{p(e)}), t), c_{p(e)} \circ \varphi_{[t, 1]})$ which likewise belongs to E , so $h(E \times I) \subset E$. Let i denote the inclusion $E \hookrightarrow E_p$. Then $i \circ h_1 \simeq \text{id}$ via h and $h_1 i \simeq \text{id}$ via h , so i is a fiber homotopy equivalence.

Now, recall that a map $f: E_1 \rightarrow E_2$ is fiber preserving if $p_1 = p_2 f$, or, in other words, if $f(p_1^{-1}(b)) \subset p_2^{-1}(b)$. Let $q: E_p \rightarrow B$ be one fibration and $p: E \rightarrow B$ the other. Then let $F_{p,b}$ be the fiber above b for p and $F_{q,b}$ the fiber above b for q .

Since then since i and h_1 are fiber-preserving, they restrict to maps $F_{p,b} \rightarrow F_{q,b}$ and $F_{q,b} \rightarrow F_{p,q}$, respectively, whose compositions are homotopic to the identity, so $F_{p,q} \simeq F_{q,b}$ for each $b \in B$, showing that the fibers and homotopy fibers of p are homotopy equivalent. \square

We have seen that loopspaces occur as fibers of fibrations $PB \rightarrow B$ with contractible total space PB . Here is something of a converse:

Proposition 2.14. *If $F \rightarrow E \rightarrow B$ is a fibration or fiber bundle with E contractible, then there is a weak homotopy equivalence $F \rightarrow \Omega B$.*

Proof. Composing a contraction of E with a projection $p: E \rightarrow B$, we obtain for each point $x \in E$ a path γ_x in B from $p(x)$ to a basepoint $b_0 = p(x_0)$, where x_0 is the point to which E contracts. This yields a map $E \rightarrow PB, x \mapsto \overline{\gamma_x}$ whose composition with the fibration $PB \rightarrow B$ sending a path in B starting at b_0 to its endpoint, is p . By restriction, this then gives a map $F \rightarrow \Omega B$, where $F = p^{-1}(b_0)$, and the LES of homotopy groups for $F \rightarrow E \rightarrow B$ maps to the LES for $\Omega B \rightarrow PB \rightarrow B$ (by section 6.6 in the AlgTop1 notes) Since E and PB are contractible, the five-lemma implies that $F \rightarrow \Omega B$ is a weak homotopy equivalence. \square

2.1.2. Puppe Sequence/Fibration Sequence. Given a fibration $p: E \rightarrow B$ with fiber $F = p^{-1}(b_0)$, we know that the inclusion of F into the homotopy fiber F_p is a homotopy equivalence. Recall that F_p consists of pairs (e, γ) with $e \in E$ and γ a path in B from $p(e)$ to b_0 . The inclusion $F \hookrightarrow E$ extends to a map $i: F_p \rightarrow E, i(e, \gamma) = e$ which is obviously a fibration. In fact, it is the pullback via p of the path fibration $PB \rightarrow B$ which sends each path to its endpoint.

This allows us to iterate, taking the homotopy fiber F_i with its map to F_p , and so on, as in the first row of the following diagram:

$$\begin{array}{ccccccccc}
 \dots & \longrightarrow & F_j & \longrightarrow & F_i & \xrightarrow{j} & F_p & \xrightarrow{i} & E & \xrightarrow{p} & B \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \parallel & & \parallel \\
 \dots & \longrightarrow & \Omega E & \xrightarrow{\Omega p} & \Omega B & \longrightarrow & F & \longrightarrow & E & \xrightarrow{p} & B
 \end{array}$$