1 Vector spaces

Problem 2.4.(8): Let V be the space of 2×2 matrices over $\mathbb{F}$. Find a basis $\{A_1, A_2, A_3, A_4\}$ for V such that $A_i^2 = A_i$ for each i.

Solution:
$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \quad \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} \quad \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}$$

Problem 2.5.(6): A littleo n linear independence and change of coordinates. Let V be the vector space over the complex numbers of all functions $\mathbb{R}\to\mathbb{C}$. Let $f_1 = 1, f_2(x) = e^{ix}, f_3(x) = e^{-ix}$.

Firstly, f_1, f_2 and f_3 are lin. indep. since if

$$af_1 + bf_2 + cf_3 = a\cdot 1 + b\cdot e^{ix} + c\cdot e^{-ix} = 0$$

then $a + i(b-c) = 0$ so $a = 0$ and $b = c$, but also $a + b + c = 0$ so $b = -c$. Hence $b = 0 = c$.
Now let $g_1 = 1, g_2(x) = \cos x, g_3(x) = \sin x$. Find an invertible 3×3 matrix P such that

$$g_j = \sum_{i=1}^{3} P_{ij} f_i.$$

Solution: We have $g_1 = 1\cdot f_1$, so $P_{1,1} = 1, P_{1,2} = P_{1,3} = 0$. Now $g_2 = \frac{1}{2}f_2 + \frac{1}{2}f_3$.
And $g_3 = \frac{1}{2}f_2 - \frac{1}{2}f_3$, so

$$P = \begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}$$

2.6.(1): Let $s < n$ and A an $s\times n$ matrix with entries in the field F. Use Theorem 4 to show that there is a non-zero X in $\mathbb{F}^{n\times 1}$ such that $AX = 0$.
Solution: We have observed that the product AX is in the row space of A which has dimension at most k. Since $s < n$, we can choose n vectors $\alpha_1, \dots, \alpha_n$ and we thus have that $A\alpha_1, \dots, A\alpha_n$ are linearly dependent, so there exists $c_1, \dots, c_n$ such that $c_1 A\alpha_1 + \ldots + c_n A\alpha_n = A(c_1\alpha_1 + \ldots + c_n\alpha_n) = 0$ where $c_1\alpha_1 + \ldots + c_n\alpha_n \neq 0$.

2.6.(3): Consider the vectors in $\mathbb{R}^4$ defined by

$$\alpha_1 = (-1, 0, 1, 2), \quad \alpha_2 = (3, 4, -2, 5), \quad \alpha_3 = (1, 4, 0, 9).$$

Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of $\mathbb{R}^4$ spanned by the three given vectors.

Solution: α_2 is a linear combination of α_1 and α_3.
We try to proceed as on page 59:
Let

$$A = \begin{pmatrix}1 & 4 & 0 & 9\\ -1 & 0 & 1 & 2\end{pmatrix}$$

which reduces to

$$\begin{aligned}\begin{pmatrix}1 & 4 & 0 & 9\\ -1 & 0 & 1 & 2\end{pmatrix} &\to \begin{pmatrix}1 & 4 & 0 & 9\\ 0 & 4 & 1 & 11\end{pmatrix}\\ &\to \begin{pmatrix}1 & 0 & -1 & -2\\ 0 & 1 & \frac{1}{4} & \frac{11}{4}\end{pmatrix}\end{aligned}$$

so $\left\{(1, 0, -1, -2), \left(0, 1, \frac{1}{4}, \frac{11}{4}\right)\right\}$ is a basis for the space spanned by α_1 and α_3.
The conditions to be a solution are then

$$Rb = 0 \iff b_j = \sum_{i=1}^{2} b_{k_i} R_{i,j}, \quad j = 1, \dots, 4$$

So $b_1 = b_{k_1}$, $b_2 = b_{k_2}$, $b_3 = -1b_1 + \frac{1}{4}b_2$, $b_4 = -2b_1 + \frac{11}{4}b_2$. That is, the subspace is the solution space for the following system of equations:

$$\begin{aligned} x_3 + x_1 - \frac{1}{4}x_2 &= 0 \\ x_4 + 2x_1 - \frac{11}{4}x_2 &= 0 \end{aligned}$$

2.6.(5): Give an explicit description of the type (2-25) for the vectors

$$\beta = (b_1, b_2, \dots, b_5)$$

in $\mathbb{R}^5$ which are linear combinations of the vectors

$$\begin{aligned} \alpha_1 &= (1, 0, 2, 1, -1) \\ \alpha_2 &= (-1, 2, -4, 2, 0) \\ \alpha_3 &= (2, -1, 5, 2, 1) \\ a_4 &= (2, 1, 3, 5, 2) \end{aligned}$$

Solution: We find the row-reduced echelon-form matrix of

$$\begin{aligned} A = \begin{pmatrix} 1 & 0 & 2 & 1 & -1 \\ -1 & 2 & -4 & 2 & 0 \\ 2 & -1 & 5 & 2 & 1 \\ 2 & 1 & 3 & 5 & 2 \end{pmatrix} &\to \begin{pmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & \frac{3}{2} & 0 \\ 0 & -1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 3 & 4 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} & 3 \\ 0 & 0 & 0 & \frac{3}{2} & 4 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 & 2 & 1 & -1 \\ 0 & 1 & -1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Thus we find $k_1 = 1, k_2 = 2, k_3 = 4, k_4 = 5$, so $b_j = \sum_{i=1}^{4} b_{k_i} R_{ij}$ gives b_1, b_2 freely chosen. Then $b_3 = 2b_1 - b_2$, and similarly b_4 and b_5 freely chosen. So the solutions are of the form
$(b_1, b_2, 2b_1 - b_2, b_3, b_4)$

2 Techniques

Lemma 2.1. *If V is a vector space, and $(v_1, \dots, v_k)$ is a linearly dependent k-tuple in V with $v_1 \neq 0$, then some v_i can be expressed as a linear combination of the preceding vectors $(v_1, \dots, v_{i-1})$.*

Lemma 2.2. *Let V be a finite-dim vec. space and $S \subset V$ a subspace in V. Then given an arbitrary basis $(\alpha_1, \dots, \alpha_n)$ for V, there exists some subset $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ such that* span $(\alpha_{i_1}, \dots, \alpha_{i_k})$ *is a complement to S.*

Proof. Use Lemma 2.1. □