Assignment 3

Jonas Trepiakas - hvn548@alumni.ku.dk

Homework 7: Let X and Y be topological spaces. We say that X has property (*) if every compact subset of X is closed in X. Prove:

(i) If X has property (*), then X is T_1.

(ii) If X has property (*) and $f\colon Y \to X$ is a continuous map, then for every compact subset $A \subseteq Y$ we have that $f(A) \subseteq X$ is closed in X.

(iii) A finite topological space has property (*) if and only if its topology is the discrete topology.

Solution:
(i) Let $x \in X$ be any element of X. Equip $\{x\}$ with the subspace topology from X. Let $\mathcal{A}$ be any open covering of $\{x\}$ - such a covering exists since e.g. $\mathcal{A} = \{X \cap \{x\}\}$ is an open covering since X is open in X. Now since $\bigcup_{A\in\mathcal{A}} A = \{x\}$, we can choose an $A \in \mathcal{A}$ such that $x \in A$. But since $\bigcup_{A\in\mathcal{A}} A = \{x\} \subseteq A \subseteq \bigcup_{A\in\mathcal{A}}\{x\}$, we have that $\{x\} = A$, so $\{A\}$ is a finite open subcovering of $\{x\}$. Hence $\{x\}$ is compact with the subspace topology. By assumption, X has property (*), so $\{x\}$ is closed. Since $x \in X$ was arbitrary, all one-point sets are closed in X, i.e. X is T_1.

(ii) Let $A \subseteq Y$ be compact. If $f\colon Y \to X$ is continuous, then $f|_A\colon A \to X$ under the restriction is continuous since $f|_A^{-1}(U) = A \cap f^{-1}(U)$ is open in A since f is continuous for any open set $U \subseteq X$. Thus by proposition 9.12, $f(A) = f|_A(A)$ is compact in X. Assuming X has property (*), we thus find that $f(A)$ is closed in X.

(iii) Assume X is a finite topological space with property (*). We saw in (i) that X is T_1. We further claim that the one-point sets are open. Let $x \in X$. Then $\bigcup_{y\in X-\{x\}}\{y\}$ is a finite (since X is finite) union of closed sets (since X is T_1). Hence by proposition 5.2.(c), $\bigcup_{y\in X-\{x\}}\{y\}$ is closed, so by definition 5.1, its complement $X - \bigcup_{y\in X-\{x\}}\{y\} = X - (X - \{x\}) = \{x\}$ is open. Since x was arbitrary, all one-point sets are also open.
Now let A be any nonempty subset of X. Then $A = \bigcup_{a\in A}\{a\}$ is a finite union of simultaneously open and closed sets, so it is open by definition 2.1.(b). Thus the topology on X is the power set of X, so by example 2.5, the topology on X is the discrete topology.

Conversely, if X is a finite topological space equipped with the discrete topology, then every set is closed and open by example 5.4, so in particular, if A is a subset of X, then $A = \bigcup_{a\in A}\{a\}$ which is a finite union of closed sets and hence closed by proposition 5.2. So all subsets of X are closed and thus, in particular, all compact subsets of X are closed in X. So X has property (*).

Homework 8: Let $X = \mathbb{N}$ with the discrete topology, and let $X^+ = X \cup \{\infty\}$ denote the one-point compactification of X. Let Y be another topological space. Given a sequence $(y_n)_{n\in\mathbb{N}}$ in Y and a point $y \in Y$, define $f\colon X^+ \to Y$ by $f(n) = y_n$ when $n \in X$ and $f(\infty) = y$.
Show that the sequence $(y_n)_{n\in\mathbb{N}}$ converges to y in Y if and only if $f\colon X^+ \to Y$ is continuous.

Solution: Assume that $(y_n)_{n\in\mathbb{N}}$ converges to y in Y. Let $U \subseteq Y$ be any open set. Then $f^{-1}(U) = f^{-1}\left(U \cap ((y_n)_{n\in\mathbb{N}} \cup y)\right)$. Thus it suffices to show that f is continuous when restricting the range to the subspace $A = \{y_n \mid n \in \mathbb{N}\} \cup \{y\}$. Let U be an open set in A. If $y \notin U$, then $f^{-1}(U) \subseteq X$, and since the open sets of X are open in X^+ since X is open in X^+, and all subsets of X are open in X since it is equipped with the discrete topology, we have $f^{-1}(U)$ is open in X^+.
If $y \in U$, then $f^{-1}(U) = \{\infty\} \cup f^{-1}\left(U - \{y\}\right)$. We claim that the complement of $f^{-1}(U)$ is closed in X^+. The complement is contained in X since $\infty \in f^{-1}(U)$, so it is in particular closed by example 5.4. Since the complement of $f^{-1}(U)$ is closed in X^+, $f^{-1}(U)$ is open in X^+ by definition 5.1.
Thus $f\colon X^+ \to Y$ is continuous.

Assume conversely that $f\colon X^+ \to Y$ is continuous. Let U be any neighborhood of y. Then $f^{-1}(U)$ is open in X^+ and in particular contains ∞. Thus its complement is contained in X, and since X is equipped with the discrete topology, example 5.4 gives that the complement of $f^{-1}(U)$ is closed. We further claim that the complement of $f^{-1}(U)$ is bounded above by an element M.
Let $A \subseteq X$ be any infinite subset of X. Then $\bigcup_{a\in A}\{a\}$ is an open covering of A with no finite subcover, so A is not compact. Thus any infinite subset of X is not compact.
However, since all one-point sets are open, X is Hausdorff. Thus all closed subsets of X are compact by theorem 9.9. Since the complement of $f^{-1}(U)$ is a closed subspace of X, it can thus not be infinite.

Thus by the total ordering of $\mathbb{N}$, there exists a maximal element $M \in \mathbb{N}$ in the complement of $f^{-1}(U)$. Since $f^{-1}(Y - U) = f^{-1}(Y) - f^{-1}(U) = X^+ - f^{-1}(U)$, we thus have that $\mathbb{N} \cap [M+1, \infty) \subseteq f^{-1}(U)$, since $X^+ = f^{-1}(U) \cup X^+ - f^{-1}(U) = f^{-1}(U) \cup f^{-1}(Y - U)$. Thus for all $n \geq M+1$, $y_n = f(N+1) \in f(\mathbb{N} \cap [M+1, \infty)) \subseteq U$.
Since U was arbitrary, y_n converges to y by definition 5.14.

Homework 9: Investigate which of the following statements are true in general, and accordingly either give a proof or find a counterexample:

(i) If $(X, \mathcal{T}_1)$ is a connected space, and $\mathcal{T}_2$ is another topology on X which is coarser than $\mathcal{T}_1$, then $(X, \mathcal{T}_2)$ is also connected.

(ii) If $(X, \mathcal{T}_1)$ is a path-connected space, and $\mathcal{T}_2$ is another topology on X which is coarser than $\mathcal{T}_1$, then $(X, \mathcal{T}_2)$ i s also path-connected.

(iii) If $(X, \mathcal{T}_1)$ is a path-connected space, and $\mathcal{T}_2$ is another topology on X which is finer than $\mathcal{T}_1$, then $(X, \mathcal{T}_2)$ is also path-connected.

(iv) Suppose $A_n, n \in \mathbb{N}$ is a sequence of connected subsets of $\mathbb{R}^2$ (where $\mathbb{R}^2$ is equipped with the standard topology) satisfying $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} A_n$ is connected.

Solution:
(i) This is true. Assume there is a separation U, V of X where U and V are disjoint nonempty open subsets of X whose union is X. Then since $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$, U and V are also in $\mathcal{T}_1$ and thus would also form a separation of X. However, $(X, \mathcal{T}_1)$ is connected, so no such separation exists. Thus $(X, \mathcal{T}_2)$ is connected.

(ii) This is true. Let x and y be any points of X. Then by assumption there exists a path, i.e. a continuous function, $f\colon [a, b] \to X$ where X is equipped with the topology $\mathcal{T}_1$ and $f(a) = x$ and $f(b) = y$. Since a continuous function is also continuous when the range is equipped with any coarser topology by proposition 2.12, this path is also a path when X is equipped with the topology $\mathcal{T}_2$. Hence $(X, \mathcal{T}_2)$ is path connected.

(iii) This is false. For example, the function $f\colon [0, 1] \to \mathbb{R}$ given by $f(t) = x + t(y - x)$ is a path (continuous since polynomial, example 1.12) connecting any points x, y of $\mathbb{R}$ when $\mathbb{R}$ is equipped with the standard topology. Now let $\mathbb{R}_d$ denote $\mathbb{R}$ equipped with the discrete topology. Let $t \in [0, 1]$. The set $\{f(t)\}$ is open in $\mathbb{R}_d$, but $f^{-1}(\{f(t)\}) = \{t\}$ is not open in $[0, 1]$. For example the complement of $\{t\}$ is not closed in $[0, 1]$ since it has t as a limit point; or alternatively, if $\{t\}$ were open, it would contain a basis element (proposition 3.2), but any basis element is an interval by definition of the order topology, and since t has no immediate successor or precessor in $[0, 1]$, it thus contains a point different from t; since this point is not in $\{t\}$, $\{t\}$ cannot be open.

(iv) This is false. Let A_n be the sets

$$A_n = \mathbb{R}_- \times 0 \cup \mathbb{R}_+ \times 0 \cup \left\{ x \times y \mid y \in (0, \frac{1}{n}) \right\}.$$

We will show that A_n is path connected by showing that each A_n is star-shaped domain with respect to the point $(0, \frac{1}{2n})$. Then, since path-connectedness is an equivalence relation by the section following definition 8.12, it will follow that there exists a path between any points of A_n and hence that it is path-connected and then since every path-connected space is connected by theorem 8.13, it follows that A_n is connected.
For any point $x \in A_n$, define $\gamma_x\colon [0, 1] \to A_n$ to be the function

$$\gamma_x(t) = \left(0, \frac{1}{2n}\right) t + (1 - t)x.$$

For all $t \in [0, 1]$, this is clearly contained in A_n since for $t = 0$, $\gamma_x(0) = x \in A_x$ and for all other t, the y-coordinate of $\gamma_x(t)$ is in $(0, \frac{1}{n})$, so in particular $\gamma_x(t) \in \mathbb{R} \times \left(0, \frac{1}{n}\right) \subset A_n$. The function is furthermore continuous since it's a polynomial in each coordinate, $\mathbb{R} \to \mathbb{R}^2$ (example 1.12), and therefore also from the subspace $[0, 1]$. Thus it is a path. Now by the section following definition 8.12, the reverse path $\tilde{\gamma}_x\colon [0, 1] \to A_n$ by $\tilde{\gamma}_x(t) = \gamma_x(1 - t)$ is a path. Thus for any points $x, y \in A_n$, we can form the path

$\gamma(t) = \begin{cases} \gamma_x(2t) & t \leq \frac{1}{2} \\ \tilde{\gamma}_y(2t-1) & t > \frac{1}{2} \end{cases}$ which is a path connecting x and y by the section following definition 8.12.
Thus we conclude A_n to be connected for all $n \in \mathbb{N}$ and clearly $A_{n+1} \subseteq A_n$ for all n, but $\bigcap_{n\in\mathbb{N}} A_n = \mathbb{R}_- \times 0 \cup \mathbb{R}_+ \times 0$ which is clearly separable by $\{x \times y \mid x > 0\} \cap \mathbb{R}_- \times 0$ and $\{x \times y \mid x < 0\} \cap \mathbb{R}_+ \times 0$, where $\{x \times y \mid x > 0\}$ and $\{x \times y \mid x < 0\}$ are clearly open since for any point $x \times y$ in these sets, the ball of radius x is contained in the sets. For each set, taking the union over these balls, we find that they are each open.