5.3: Is $2^{10} \cdot \left(2^{11}-1\right)$ perfect?

Solution: If it were perfect, then by theorem 5.1.2, $2^{11}-1$ would have to be prime. Now, since $\varphi(23) = 22 = 2 \cdot 11$ and $(2, 23) = 1$, we have $2^{22} - 1 \equiv 0 \pmod{23}$, so if $2^{11}-1$ is prime, 23 must divide $2^{11}+1$, i.e. 2 must be a primitive root modulo 23. However, we find $2^5 = 32 \equiv 9 \to -5 \to -10 \to 3 \to 6 \to 12 \to 1 \equiv 2^{11} \pmod{23}$, so $\operatorname{ord}(2) = 11$, and hence 2 is not a primitive root.

Exercise 5.4: Show that $\sum_{d|n} |\mu(d)| = \Pi_{p|n} 2$.

Solution: Assume $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$. Then

$$\sum_{d|n} |\mu(d)| = \sum_{d|p_1 p_2 \dots p_k} |\mu(d)| = \sum_{d|p_1 \dots p_k} 1 = \tau(p_1 \dots p_k) \overset{prop.5.1.1}{=} \Pi_{i=1}^{k} 2 = \Pi_{p|n} 2.$$

Exercise 5.5:
Assume that there exists a complex sequence $s_k = \sigma_k + it_k$ with $\sigma_k \to \infty$ when $k \to \infty$ satisfying that $D_f(s_k) = 0$ for all k sufficiently large. We will show that this implies that $f(n) = 0$ for all n.

(i) Assume that f is not identically zero, and choose $N \in \mathbb{N}$ minimal such that $f(N) \neq 0$. Then for all k sufficiently large. we have

$$\begin{aligned} 0 &= \sum_{n=N}^{\infty} f(n) n^{-s_k} \\ \iff f(N) &= -N^{s_k} \sum_{n=N+1}^{\infty} f(n) n^{-s_k}.. \end{aligned}$$

(ii) Let $c > \sigma_a$. Then for k sufficiently large, we have

$$\begin{aligned} |f(N)| &\leq -N^{\sigma_k} \sum_{n=N+1}^{\infty} |f(n)| \left| n^{-(\sigma_k - c)} \right| \left| n^{-c} \right| \\ &\leq -N^{\sigma_k} (N+1)^{-(\sigma_k - c)} \sum_{n=N+1}^{\infty} |f(n)| \, n^{-c}. \end{aligned}$$

(iii) Since $\left(\frac{N}{N+1}\right)^{\sigma_k} \to 0$ for $\sigma_k \to \infty$ which is equivalent to letting $k \to \infty$ by assumption. Since this is true for all k sufficiently large, we let $k \to \infty$ and find $|f(N)| \leq 0$ since $\sum_{n=N+1}^{\infty} |f(n)| n^{-c} < \infty$ as $c > \sigma_a$, hence $f(n) = 0$ for all $n \in \mathbb{N}$.

Exercise 5.6: We will show that the sum of reciprocal of primes, $\sum_{p \text{ prime}} p^{-1}$, diverges.
Let p_n denote the n'th prime. Assume that $\sum_{n=1}^{\infty} p_n^{-1}$ is convergent with sum l.

(i) By assumption, since $\sum_{n=1}^{\infty} p_n^{-1} = l$, there must exist $N \in \mathbb{N}$ such that

$$\left| \sum_{n>N} p_n^{-1} \right| = \left| l - \sum_{n=1}^{N} p_n^{-1} \right| \leq \frac{1}{2}.$$

(ii) Now

$$\sum_{k=1}^{\infty} \left| \sum_{n>N} p_n^{-1} \right|^k \leq \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 1,$$

so $\sum_{k=1}^{\infty} \left(\sum_{n>N} p_n^{-1}\right)^k$ is absolutely convergent.

(iii) Let $W = p_1 \dots p_N$. For $r \in \mathbb{N}$ consider $Wr + 1$. Since all $p_i \mid W$, we have $p_i \nmid Wr + 1$.

(iv) Now, by Cauchy multiplication, we have

$$\left(\sum_{n>N} p_n^{-1}\right)^k = \sum_{n_1,\ldots,n_k>N} (p_{n_1}\cdots p_{n_k})^{-1}$$
$$= \sum_{\substack{n=q_1\ldots q_k \\ q_i \text{ prime} \\ q_i>p_N}} \frac{1}{n}..$$

(v) Now, for the sum

$$\sum_{r=1}^{\infty} \frac{1}{Wr+1},$$

we find that it is a sum of reciprocals of numbers whose prime factors are all greater than p_N, so each term is contained in the series

$$\sum_{\substack{n=q_1\ldots q_k \\ q_i \text{ prime} \\ q_i>p_N}} \frac{1}{n}$$

for some k (by the fundamental theorem of arithmetic), so

$$\sum_{r=1}^{\infty} \frac{1}{Wr+1} \leq \sum_{k=1}^{\infty} \sum_{\substack{n=q_1\ldots q_k \\ q_i \text{ prime} \\ q_i>p_N}} \frac{1}{n}$$
$$= \sum_{k=1}^{\infty} \left(\sum_{n>N} p_n^{-1}\right)^k.$$

(vi) Now

$$\infty = \frac{1}{W}\sum_{r=1}^{\infty} \frac{1}{r+1}$$
$$= \sum_{r=1}^{\infty} \frac{1}{Wr+W}$$
$$\leq \sum_{r=1}^{\infty} \frac{1}{Wr+1}$$
$$\leq \sum_{k=1}^{\infty} \left(\sum_{n>N} p_n^{-1}\right)^k.$$

This contradicts (ii), so we are done.

Exercise 5.7: Let $g(n)$ be the sum of primitive nth roots of 1, i.e.

$$g(n) = \sum_{\substack{\zeta^n=1 \\ \zeta^m\neq 1 \\ \text{for } 0<m<n}} \zeta.$$

Claim: $\mu(n) = g(n)$.
Proof: We have

$$g(n) = \sum_{\substack{\zeta^n=1 \\ \zeta^m\neq 1 \\ 0<m<n}} \zeta = \sum_{\substack{\gcd(k,n)=1 \\ 0<k<n}} \zeta_n^k.$$

We prove that g is multiplicative first.
Assume $n = st$ where $(s,t) = 1$. Let $a, b \in \mathbb{Z}$ such that $(a,s) = 1 = (b,t)$. Then $e^{\frac{2\pi i}{s}a}e^{\frac{2\pi i}{t}b} = e^{\frac{2\pi i(at+bs)}{n}}$. We claim $\gcd(at+bs, n) = 1$. If $p \mid at+bs \mid n$, then $p \mid s$ or $p \mid t$. Assume wlog $p \mid s$. Then $p \mid at$, but since $(s,t) = 1$, $p \mid a$, however $(a,s) = 1$. Contradiction. So $\gcd(at+bs, n) = 1$.

Conversely, if $(k, st) = 1$ then since $(s, t) = 1$, write $us + vt = 1$, then $s(uk) + t(vk) = k$, so $e^{\frac{2\pi i(suk+tvk)}{st}} = e^{\frac{2\pi i}{t}uk}e^{\frac{2\pi i}{s}vk}$. Now, $(k, st) = 1$ so $(k, t) = 1$ and $(u, t) = 1$ since $(u, t) \mid 1$. Similarly $(s, vk) = 1$.
Thus we have

$$g(st) = \sum_{\substack{\gcd(k,st)=1 \\ 0<k<st}} \zeta_{st}^k = \sum_{\substack{\gcd(k,s)=1 \\ 0<k<s}} \zeta_s^k \cdot \sum_{\substack{\gcd(k,t)=1 \\ 0<k<t}} \zeta_t^k = g(s)g(t).$$

Now, firstly we have for any prime p that $g(p) = \sum_{\substack{\gcd(k,p)=1 \\ 0<k<p}} \zeta_p^k = \zeta_p + \zeta_p^2 + \zeta_p^3 + \ldots + \zeta_p^{p-1} = \frac{\zeta_p^p - \zeta_p}{\zeta_p - 1} = \frac{1-\zeta_p}{\zeta_p - 1} = -1$.
It thus just remains to show that for any $\alpha \geq 2$, $g(p^\alpha) = 0$. Now

$$\begin{aligned} g(p^\alpha) &= \sum_{\substack{\gcd(k,p^\alpha) \\ 0<k<p^\alpha}} \zeta_{p^\alpha}^k \\ &= \sum_{k=0}^{p^\alpha - 1} \zeta_{p^\alpha}^k - \sum_{n=0}^{\alpha-1} \zeta_{p^\alpha}^k \\ &= \frac{\zeta_{p^\alpha}^{p^\alpha} - 1}{\zeta_{p^\alpha - 1}} - \frac{\zeta_{p^\alpha}^{p^\alpha} - 1}{\zeta_{p^\alpha} - 1} \\ &= 0. \end{aligned}$$

Combining these 3 results we find $g(n) = \mu(n)$.
Exercise 5.9: Prove Gottschalck's theorem: Let n be a k-perfect number such that 2 divides n precisely m times. Then $2^m \left(2^{m+1} - 1\right)$ divides kn.

Solution: By proposition 5.1.1, we have have for $n = 2^m r$,

$$nk = \sigma(2^m r) \stackrel{\text{multiplicative}}{=} \sigma(2^m)\sigma(r) \stackrel{5.1.1}{=} (2^{m+1} - 1)\sigma(r).$$

Now, since $2^m \nmid 2^{m+1} - 1$ but $2^m \mid n$, we must have $2^m \mid \sigma(r)$, thus we get $nk = (2^{m+1} - 1)2^m \sigma(r)'$ from which the result follows.
7.7: Prove theorem 7.1.5 by showing - using propositions 7.1.1 and 7.1.2, combined with (7.6) - that

$$|c_n - c_m| \leq \sqrt{2} \sum_{k=m}^{n} \frac{1}{2^k},$$

and conclude that c_n is a Cauchy sequence.

Solution: We proceed by induction. We have

$$|c_{n+1} - c_n| = \left| \frac{p_{n+1}q_n - p_n q_{n+1}}{q_n q_{n+1}} \right| \leq \frac{\sqrt{2}}{2^n} = \sqrt{2} \sum_{k=n}^{n} \frac{1}{2^k}.$$

Now we claim that for $m > n$, $|c_m - c_n| \leq \sqrt{2} \sum_{k=n}^{m-1} \frac{1}{2^k}$ which is stronger than what we wanted for $m > n$.
It is true when $m = n + 1$. Assume it is true for $m = N > n$. Then for $m = N + 1$,

$$|c_m - c_n| \leq |c_{N+1} - c_N| + |c_N - c_n| \leq \frac{\sqrt{2}}{2^N} + \sqrt{2} \sum_{k=n}^{N-1} \frac{1}{2^k} = \sqrt{2} \sum_{k=n}^{N} \frac{1}{2^k}.$$

Hence the result follows when $n \neq m$. If $n = m$, it is trivial.
Exercise 6.6 Show that the average value of $r_2(n)$ equals π, i.e. that

$$\frac{1}{n} \sum_{m=1}^{n} r_2(m) \to \pi \quad \text{as } n \to \infty.$$

Solution: Let $N(n)$ denote the number of lattice points inside the circle of radius n. Then

$$N(r) = \sum_{n=0}^{r^2} r_2(n)$$

And geometrically, we can say that on average, if we place a random unit square or disc of area 1, we would expect it to cover 1 lattice point, so since the number of unit discs that can fit inside a circle of radius r approaches its area as r grows, we get that $N(r) \approx \pi r^2$ for r large, so

$$\pi = \lim_{r\to\infty} \frac{N(r)}{r^2} = \frac{1}{r^2}\sum_{n=0}^{r^2} r_2(n).$$

Since $\frac{1}{r^2}\sum_{n=0}^{r^2} r_2(n)$ is a monotone (positive) subsequence of $\frac{1}{r}\sum_{n=0}^{r} r_2(n)$, we also have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n} r_2(m) = \pi.$$

Exercise from class: Write $5+i$ as a product of irreducibles in $\mathbb{Z}[i]$.

Solution: $N(5+i) = 25^2 + 1 = 26 = 2 \cdot 13 = (1+i)(1-i)(2+3i)(2-3i)$. Now we simply guess and find $(1-i)(2+3i) = 5+i$.

$r_2(n) = 4 \cdot u * \chi_4(n)$ where $\chi_4(n) = \begin{cases} 0 & (n,2) > 0 \\ (-1)^{\frac{n-1}{2}} & \text{otherwise} \end{cases}.$