ASSIGNMENT 3

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Exercise 0.1. (1) Let $G \times X \to X$ be a continuous left properly discontinuous action. Prove that the action is free.

(2) Let $r \in \mathbb{R}$ be an irrational number, and let $G = \left(\mathbb{Z}^2, +\right)$ act on $X = \mathbb{R}$ as $(n, m).t = t + n + rm$. Prove that this action is free, but not properly discontinuous. Compare with the statement of exercise 3.

Solution. (i) Suppose it is not free, so there exists $g \in G - \{e\}$ such that $g.x = x$ for some $x \in X$. If $G \times X \to X$ is properly discontinuous, then there exists $V \subset X$ with $x \in V$ such that $V \cap gV = \varnothing$ for all $g \in G - \{e\}$. But then $x = g.x \in gV$ and $x \in V$, so $x \in V \cap gV = \varnothing$, contradiction. Hence $G \times X \to X$ can not be properly discontinuous. Taking the contraposition gives the desired result.
(ii)
Suppose $t = (n, m).t = t + n + rm$, so $n + rm = 0$. But if $m \neq 0$, then $r = -\frac{n}{m} \in \mathbb{Q}$, contradicting irrationality of r, so $m = 0$ and hence $n = 0$, giving that the action is free.
Now, suppose there exists an open neighborhood V of 0 such that $(n, m)V \cap V \neq \varnothing$ implies $(n, m) = (0, 0)$. Since V is open, we can find some basis open ball $B(0, \delta) \subset V$. But by Dirichlet's approximation theorem, we can find integers n, m with $m \geq 1$ such that $|n + rm| < \delta$, which implies that $n + rm \in (n, m)V \cap B(0, \delta) \subset (n, m)V \cap V$, giving $(0, 0) = (n, m)$, but $m \geq 1$, contradiction.

On the comparing this to exercise 3 part, since this action is not properly discontinuous, it means that the quotient map $\mathbb{R} \to \mathbb{R}/G$ cannot be a covering map since 0, for example, has no evenly covered neighborhood(if it had, then this neighborhood would split into homeomorphic disjoint copies in $\mathbb{R}$ which would satisfy the requirement of the action being properly discontinuous at 0). Exercise 3 doesn't apply since neither our group is finite nor is our space compact.

Exercise 0.2. Let X and B be Hausdorff path-connected spaces, and let $p \colon X \to B$ be a local homeomorphism, i.e., for all $x \in X$, there is a neighborhood $U \subset X$ of x such that $p(U)$ is open in B, and the restriction

$$p|_U : U \to p(U)$$

is a homeomorphism. Also, suppose that for all $b \in B$, $p^{-1}(\{b\})$ is finite, of the same cardinality for all points. Show that p is a covering map.

Solution. Let $F = \left\{1, \ldots, n := \left|p^{-1}(\{b\})\right|\right\} \subset \mathbb{N}$. In particular, $p^{-1}(\{b\})$ is non-empty for all $b \in B$ since it has the same cardinality for all $b \in B$ and so if it were empty, then $p^{-1}(B) = \varnothing$, contradicting p being a function. Now let $b \in B$ and let $p^{-1}(\{b\}) = \{x_1, \ldots, x_n\}$. For each pair x_i, x_j, using the Hausdorff property, we can choose open sets $U_{i,j}$ and $U_{j,i}$ around x_i and x_j, respectively, such that $U_{i,j} \cap U_{j,i} = \varnothing$. Then $V_i = \bigcap_j U_{i,j}$ is a non-empty open set around x_i for each

i and $V_i \cap V_j = \bigcap_{k,l} U_{i,k} \cap U_{j,l} = \varnothing$. By intersecting each V_i with some open neighborhood around x_i which p maps homeomorphically onto its image, we can assume that $p|_{V_i} : V_i \to p(V_i)$ is a homeomorphism for each i. Now, each V_i contains x_i, so $b \in \bigcap_i p(V_i) =: W$ which is open, so in particular define $W_i = p^{-1}(W) \cap V_i$ which is mapped homeomorphically onto W for each i under $p|_{W_i}$. Again we have $W_i \cap W_j$ for distinct i and j, and thus W is an evenly covered neighborhood of b. In particular, we can define $\varphi\colon p^{-1}(W) \to W \times F$ by $\varphi(x) = (p(x), \sum_i i\delta_x (W_i))$ where $\delta_x(W_i)$ is 1 if $x \in W_i$ and 0 otherwise. This is clearly a homeomorphism since it is bijective and continuous obviously, and supposing $(w, m) \in W \times F$ and U is an open neighborhood of $\varphi^{-1}(w, m)$, then $\varphi\left(U \cap W_{\sum_i i\delta_{\varphi^{-1}(w,m)}(W_i)}\right)$ is an open neighborhood of (w, m) (φ is a homeomorphism onto its image restricted to this set) which is mapped to an open neighborhood of $\varphi^{-1}(w, m)$ contained in U under φ^{-1}.
Thus shows that W is an evenly covered neighborhood of b.

Remark. I realize that I haven't used path-connectedness, so I'm wondering where that should've been used.

To show that the action is not properly discontinuous, we show something stronger:

Lemma 0.3. *If $G \neq \{e\}$ acts properly discontinuously on $\mathbb{R}$, then $G \approx \mathbb{Z}$.*

Proof. We have $\pi_1(\mathbb{R}/G) \approx G$. Let $\alpha \in \pi_1(\mathbb{R}/G, x_0) - \{e\}$. Then α lifts to a path $\tilde{\alpha}\colon I \to \mathbb{R}$ starting at some $\tilde{x_0} \in p^{-1}(x_0)$ where $p\colon \mathbb{R} \to \mathbb{R}/G$ is the covering map.

Firstly, we show that, for any two $y, z \in p^{-1}(x_0)$, $(y, z) \cap p^{-1}(x_0)$ is finite. For the map $\mathbb{R} \to \mathbb{R}$ by $x \mapsto g.x$ for some $g \in G$ is a homeomorphism, so letting V be an open neighborhood around $\tilde{x_0}$ for which $gV \cap V \neq \varnothing$ implies $g = e$, we would get that there exist infinitely many elements $g \in G$ for which $gV \subset (y, z)$ and for two distinct such g, g', $gV \cap g'V = \varnothing$ since otherwise $g^{-1}g'V \cap V \neq \varnothing$ and hence $g^{-1}g' = e$ so $g = g'$. Thus this would imply that there is an infinite disjoint union of open sets gV of the same measure (since $x \mapsto gx$ is a homeomorphism) contained in (y, z). Since (y, z) has finite measure, this implies that gV has measure 0, contradicting it being open.

This gives a total ordering on $p^{-1}(x_0)$ inherited from $\mathbb{R}$ with finitely many elements in $p^{-1}(x_0)$ between any two elements in $p^{-1}(x_0)$.
Let $\tilde{x_0} \in p^{-1}(x_0)$ and $\tilde{x_0}'$ be its successor in $p^{-1}(x_0)$. We claim that the image, γ, under p of the path $\tilde{\gamma}\colon I \to \mathbb{R}$ connecting $\tilde{x_0}$ to $\tilde{x_0}'$ linearly generates $\pi_1(\mathbb{R}/G)$ as a cyclic group. Suppose $\alpha \in \pi_1(\mathbb{R}/G)$ and lift it to a path $\tilde{\alpha}$ starting at $\tilde{x_0}$. Let $\tilde{x_n} = \tilde{\alpha}(1)$. By uniqueness of lifts, we may assume by homotopy that this is the straight line path between the points $\tilde{x_0}$ and $\tilde{x_n}$, and suppose $\tilde{\alpha}^{-1}\left(p^{-1}(x_0)\right) = \{0, t_1, \ldots, t_{n-1}, 1\}$.
Now, since any loop at x_0 lifts to a path between two points in $p^{-1}(x_0)$, we thus see that there are precisely two simple loops in $\pi_1(\mathbb{R}/G)$ corresponding to the projections of the path going from $\tilde{x_i}$ to $\tilde{x_{i+1}}$ and the path going from $\tilde{x_i}$ to $\tilde{x_{i-1}}$. In particular, since the inverse of one of them is again simple, these loops are each other's inverse.
Thus, letting $\gamma \in \pi_1(\mathbb{R}/G, x_0)$ denote one of these paths, we see that $p \circ \tilde{\alpha} = \gamma^{k_1} * \ldots * \gamma^{k_n}$ where $k_i = \pm 1$. So $\alpha \in \langle\gamma\rangle$. Thus $\mathbb{Z} \approx \langle\gamma\rangle = \pi_1(\mathbb{R}/G) \approx G$.

$\square$