ASSIGNMENT 4

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Exercise 0.1 (2)**.** Which of the following modules are flat over the corresponding rings? Justify your answer

(1) $R = \mathbb{C}[x, y]$ and the module is $I = (x, y) \subset R$.
(2) $R = \mathbb{C}[x]/(x^2)$ and the module is $I = (x) \subset R$.
(3) $R = \mathbb{C}[x]$ and the module is the ring $\mathbb{C}[y]$ considered as an R-module by ring homomorphism $\mathbb{C}[x] \to \mathbb{C}[y] : x \mapsto y^2$.
(4) $R = \mathbb{C}[x]$ and the module is the ring $\mathbb{C}[x, y]/(xy)$ considered as an R-module by ring homomorphism $\mathbb{C}[x] \to \mathbb{C}[x, y]/(xy)$: $x \mapsto x$.

Solution. (1) We claim that $I = (x, y)$ is not a flat $R = \mathbb{C}[x, y]$ module. Firstly, $\mathbb{C}[x, y]$ is Noetherian by Hilbert's basis theorem since $\mathbb{C}$ is, and it is also local: we claim that $(x, y) = I$ is precisely the maximal ideal. Firstly, it is maximal because $\mathbb{C}[x, y]/(x, y) \cong \mathbb{C}$ is a field. Now if $M \subset \mathbb{C}[x, y]$ is a maximal ideal, then $1 \notin M$, so for any $f \in M$, we have that $f(x, y) = \sum_{i+j \geq 1} \alpha_{ij} x^i y^j \in (x, y)$. Thus $M \subset (x, y)$, so M is not maximal unless $M = (x, y)$. Therefore (x, y) is the only maximal ideal. Now, I is finitely generated as a $\mathbb{C}[x, y]$-module with generators x and y, hence proposition 9.15 applies. Since (x, y) is not free since it in particular is a proper submodule of R, we have that it is not flat.

(2) We claim that $I = (x) \subset \mathbb{C}[x]/(x^2)$ is indeed flat. This can be seen since $\mathbb{C}[x]/(x^2) = \mathbb{C} \oplus (x) \cong \mathbb{C} \oplus \mathbb{C}$, so since (x) is a direct summand of $R = \mathbb{C}[x]/(x^2)$, it is flat by proposition 9.13 and the fact that R is itself flat by example 9.2.

(3) **I will give two solutions since I'm not sure whether I may use that over a PID, a module is flat iff it is torsion-free** Suppose there is a relation $\sum a_i y^i = 0$ in $\mathbb{C}[y]$ where $a_i \in \mathbb{C}[x]$. However, then taking the maximal degree of x^j in a_j for y^j the maximal degree of y in the relation, we find that $a_j = 0$. But this contradicts y^j being the maximal degree. Hence $a_i = 0$ for all i. But this shows that the relation is trivial. Now remark 9.21 tells us that $\mathbb{C}[y]$ is flat cosidered as a $\mathbb{C}[x]$ module by the homomorphism $\mathbb{C}[x] \to \mathbb{C}[y]$ by $x \mapsto y^2$.

The other solution is the following: Since $R = \mathbb{C}[x]$ is a PID, we immediately find that $\mathbb{C}[y]$ is flat if and only if it is torsion-free considered as a $\mathbb{C}[x]$-module by restriction of scalars along $x \mapsto y^2$. Suppose $f(y) \in \mathbb{C}[y]$ is such that for $g(x) \in R$, $g(x)f(y) = 0$, i.e., $g(y^2)f(y) = 0$ in $\mathbb{C}[y]$. However, this forces f or g to be 0, so we find that $\mathbb{C}[y]$ is torsion-free as a $\mathbb{C}[x]$ module under the ring-homomorphism $x \mapsto y^2$. Thus $\mathbb{C}[y]$ is a flat $\mathbb{C}[x]$-module by the ring homomorphism $\mathbb{C}[x] \to \mathbb{C}[y]$ sending $x \mapsto y^2$.

(4) We note that if a module has torsion, this gives an non-trivial relation since

$am = 0$ with $a \neq 0$ and $m \neq 0$ admitting a genuinely trivial reparametrization implies $a = 0$, contradiction. Hence proposition 9.20 gives that if a module has torsion, then it cannot be flat. $\mathbb{C}[x,y]/(xy)$ is clearly not a flat $\mathbb{C}[x]$-module under the homomorphism $\mathbb{C}[x] \to \mathbb{C}[x,y]/(xy)$ sending $x \mapsto x$ since y is nonzero in $\mathbb{C}[x,y]/(xy)$ however, $x \cdot y := xy = 0$, hence $\mathbb{C}[x,y]/(xy)$ is not torsion-free over $\mathbb{C}[x]$.