ASSIGNMENT 3

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Suppose firstly that $y \in W^\circ \cap U^\circ$. Since $V = U \oplus W$, we can write an arbitrary $x \in V$ as $u + w$ uniquely, and then $y(x) = y(u + w) = y(u) + y(w) = 0 + 0 = 0$ by assumption on y annihilating W and U. But then taking the contraposition of lemma 3.4.(1), we get $y = 0$, so $U^\circ \cap W^\circ = \{0\}$, hence $U^\circ \oplus W^\circ$ is indeed a direct sum.

The inclusion $U^\circ \oplus W^\circ \subset V'$ follows by definition, so suppose conversely that $y \in V'$. Then we can define linear functionals $y_U \in U^\circ$ and $y_W \in W^\circ$ as follows: $y_U(u + w) = y(w)$ and $y_W(u + w) = y(u)$ for $u \in U$ and $w \in W$.

There is no problem of being well defined since if $u + w = u' + w'$ then $u - u' = w - w' \in U \cap W = \{0\}$.

Since $y_U\left(\lambda\left(u + w\right) + \left(u' + w'\right)\right) = y_U\left(\lambda u + u' + \lambda w + w'\right) = y\left(\lambda w + w'\right) = \lambda y(w) + y(w') = \lambda y_U(u + w) + y_U(u' + w')$, we find that y_U is linear, and repeating for y_W, we find that y_W is also linear. So $y_U, y_W \in V'$.

Furthermore, if $u \in U$, then the unique way of writing it as a direct sum of an element from U and an element from W is $u + 0$, so $y_U(u) = y_U(u + 0) = y(0) = 0$ since y is linear. Hence $y_U \in U^\circ$, and similarly, $y_W \in W^\circ$.

Lastly, for all $x \in V$, write $x = u + w$ for $u \in U$ and $w \in W$. Then $y(x) = y(u + w) = y(u) + y(w) = y_W(u + w) + y_U(u + w) = \left(y_U + y_W\right)(x)$, hence $y = y_U + y_W$, so $y \in U^\circ \oplus W^\circ$ giving us the other inclusion.