1: Choose a representation for each $P_i = (a_{i1}, a_{i2}, a_{i3})$ and for each $Q_i = (b_{i1}, b_{i2}, b_{i3})$. Since P_1, P_2 and P_3 (resp. Q_1, Q_2 and Q_3) do not lie on a line, three points lying in the corresponding subset in $\mathbb{A}^3$ for P_1, P_2 and P_3 span $\mathbb{A}^3$, so choosing these as basis for $\mathbb{A}^3$, we can define a linear map sending $P_i \to Q_i$. This is clearly invertible with the linear map sending $Q_i \to P_i$, and sends $0 \to 0$, so it induces a projective change of coordinates. And since $T(\lambda P_i) = \lambda T(P_i) = \lambda Q_i$, we have that it maps the line through P_i to the line through Q_i, and thus that the induced map $\mathbb{P}^2 \to \mathbb{P}^2$ maps the points $P_i \to Q_i$.

Similarly, if $P_1, \ldots, P_{n+1}$ are points in $\mathbb{P}^n$ not lying on a hyperplane, points on the represented lines in $\mathbb{A}^{n+1}$ are linearly independent, so they define a basis for $\mathbb{A}^3$. We can again define a map sending $P_i \to Q_i$ which is invertible with $Q_i \to P_i$ and $0 \to 0$. Thus it induces a projective change of coordinates mapping $P_i \to Q_i$.

2. Duals:
(a) We first show that $(P^*)^* = P$.
As stated in the problem, assigning $[a_1 : \ldots : a_{n+1}] \in \mathbb{P}^n$ to $\Lambda = \mathbb{V}(a_1x_1 + \ldots + a_{n+1}x_{n+1})$ sets up a one-to-one correspondence between $\{\text{hyperplanes in } \mathbb{P}^n\}$ and $\mathbb{P}^n$ - since $(a_1, \ldots, a_{n+1})$ is determined by Λ up to scaling since it must be perpendicular to the vector $(x_1, \ldots, x_n)$ and thus can be freely scaled.

Now, $P^* = \mathbb{V}(a_1 : \ldots : a_{n+1})$ which we assign to $[a_1 : \ldots : a_{n+1}] = P$ by construction.

For $(\Lambda^*)^* = \Lambda$, note that $\Lambda^* = [a_1 : \ldots : a_{n+1}]$, so $(\Lambda^*)^* = \mathbb{V}(a_1x_1 + \ldots + a_{n+1}x_{n+1}) = \Lambda$.

(b) Suppose $P = [a_1 : \ldots : a_{n+1}]$ and $\Lambda = \mathbb{V}(l_1x_1 + \ldots + l_{n+1}x_{n+1})$. Then $P \in \Lambda$ if and only if $l_1a_1 + \ldots + l_{n+1}a_{n+1} = 0$ if and only if $\Lambda^* = [l_1 : \ldots : l_{n+1}] \in \mathbb{V}(a_1x_1 + \ldots + a_{n+1}x_{n+1}) = P^*$.

3: Write $P_i = [p_{i1} : p_{i2} : p_{i3}]$. Then the lines passing through P_i in $\mathbb{P}^2$ is precisely the hyperplane corresponding to P_i, i.e. P_i^*.
Now, by problem 2.(b), $\forall i \colon P_i \notin \Lambda \iff \forall i \colon \Lambda^* \notin P_i^* \iff \Lambda^* \notin P_1^* \cup \ldots \cup P_r^*$.
Thus, if we can show that there exists a point Λ^* not in the union of hyperplanes $P_1^* \cup \ldots \cup P_r^*$, then this gives the existence of the line $\Lambda = (\Lambda^*)^*$ not passing through any of the points P_i.

Now

$$P_1^* \cup \ldots \cup P_r^* = \mathbb{V}(p_{11}x + p_{12}y + p_{13}z) \cup \ldots \cup \mathbb{V}(p_{r1}x + p_{r2}y + p_{r3}z) = \mathbb{V}(\Pi_i (p_{i1}x + p_{i2}y + p_{i3}z))$$

So if there does not exist a point Λ^*, then

$$\mathbb{P}^2 = \mathbb{V}(\Pi_i (p_{i1}x + p_{i2}y + p_{i3}z))$$

Consider the polynomial $f \in k[x, y, z]$ given by $f = \Pi_i (p_{i1}x + p_{i2}y + p_{i3}z)$. We have $\mathbb{V}(f) = \mathbb{P}^2$ if and only if $V(f) = \mathbb{A}^3$, so f is constant zero. Thus one of the points (p_{i1}, p_{i2}, p_{i3}) must be 0, however then $P_i = [0 : 0 : 0] \notin \mathbb{P}^2$, contradiction. Thus such a Λ^* exists, and hence for all i, we have $P_i \notin \Lambda$, so Λ is a line in $\mathbb{P}^2$ not passing through any of the P_i.

4:
(a) We must give an inverse to $v_{1,3} \colon \mathbb{P}^1 \to \mathbb{P}^3$ by $[s : r] \to [s^3 : s^2t : st^2 : t^3]$.
Let Y denote the image under $v_{1,3}$. Consider the map $\varphi \colon Y \to \mathbb{P}^1$ by

$$\begin{cases} [s^3 : s^2t], & s \neq 0 \text{ (i.e. in } U_1) \\ [st^2 : t^3], & t \neq 0 \text{ (i.e. in } U_4) \end{cases}$$

This map covers its image by definition and also agrees on $U_1 \cap U_4$ since then $[s^3 : s^2t] = [s : t] = [st^2 : t^3]$.

Now for an arbitrary element in the image Y, there exist $s, t \in k$ such that $v_{1,3}[s : t] = [s^3 : s^2t : st^2 : t^3]$ represents the element and hence

$$v_{1,3} \circ \varphi([s^3 : s^2t : st^2 : t^3]) = \begin{cases} v_{1,3}[s^3 : s^2t] & s \neq 0 \\ v_{1,3}[st^2 : t^3] & t \neq 0 \end{cases} = v_{1,3}[s : t] = [s^3 : s^2t : st^2 : t^3]$$

and

$$\varphi \circ v_{1,3}\left[s:t\right] = \varphi\left[s^3 : s^2t : st^2 : t^3\right] = \begin{cases} \left[s^3 : s^2t\right] & s \neq 0 \\ \left[st^2 : t^3\right] & t \neq 0 \end{cases} = \left[s:t\right]$$

where either $s \neq 0$ or $t \neq 0$ since $[0:0] \notin \mathbb{P}^2$.
Hence φ is the inverse to $v_{1,3}$, so $v_{1,3}$ is an isomorphism onto its image.

(b) By writing out relations such as $z_1z_6 - z_2a_3 = 0$ which are clear by definition, we find the matrix

$$A = \begin{pmatrix} z_1 & z_3 & z_4 \\ z_2 & z_3 & z_7 \\ z_3 & z_8 & z_9 \\ z_4 & z_9 & z_{10} \end{pmatrix}$$

Every 2×2 minor in this matrix vanishes, while some 1×1 minor does not, so the rank of A is 1. Consider

$$Y = \{[x_1 : \ldots : x_{10}] : \ \operatorname{rank} A \leq 1\}$$

Now, by definition $v_{2,3}(\mathbb{P}^3) \subset Y$.
Furthermore, for an arbitrary $[z_1 : \ldots : z_{10}] \in Y$, either z_1, z_5, z_8 or z_{10} is nonzero since otherwise all entries would be 0, but $[0 : \ldots : 0] \notin \mathbb{P}^9$.
If $z_1 \neq 0$ then $v_{2,3}\left([z_1 : z_2 : z_3 : z_4]\right) = [z_1 : \ldots : z_{10}]$. If $z_5 \neq 0$, then $v_{2,3}\left([z_2 : z_5 : z_6 : z_7]\right) = [z_1 : \ldots : z_{10}]$.
If $z_8 \neq 0$ then $v_{2,3}\left([z_3 : z_6 : z_8 : z_9]\right) = [z_1 : \ldots : z_{10}]$.
If $z_{10} \neq 0$, then $v_{2,3}\left([z_4 : z_7 : z_9 : z_{10}]\right) = [z_1 : \ldots : z_{10}]$, so $Y \subset v_{2,3}\left(\mathbb{P}^3\right)$. Hence Y is precisely the image of $v_{2,3}$.

(c) Since $z_1 = x_1^2, z_3 = x_1x_3, z_7 = x_2x_4$ and $z_9 = x_3x_4$, we have

$$v_{3,2}^{-1}\left(\mathbb{V}\left(z_1 + 4z_3 - 2z_7 + 5z_9\right)\right) = \mathbb{V}\left(x_1^2 - 4x_1x_3 - 2x_2x_4 + 5x_3x_4\right)$$

5:

(a)

By the comment in the beginning on lecture note 22, we have that $\Gamma_h(\overline{X}) \cong \Gamma_h(\overline{Y})$, so $\Gamma(C(\overline{X})) = \Gamma\left(C(\overline{Y})\right)$. Hence $C(\overline{X}) \cong C(\overline{Y})$ by the lemma on lecture note 8. Thus there exist morphisms $\varphi\colon C(\overline{X}) \to C(\overline{Y})$ and $\psi\colon C\left(\overline{Y}\right) \to C\left(\overline{X}\right)$ with $\varphi \circ \psi = \mathbb{1}$ and $\psi \circ \varphi = \mathbb{1}$.

There exist polynomials $T_1, \dots, T_{n+1} \in k\left[x_1, \dots, x_{n+1}\right]$ such that $\varphi(P) = (T_1(P), \dots, T_{n+1}(P))$ and polynomials $S_1, \dots, S_{n+1} \in k\left[x_1, \dots, x_{n+1}\right]$ such that $\psi(P) = (S_1(P), \dots, S_{n+1}(P))$. The image $\varphi\left(\overline{X} \cap U_{n+1}\right)$ is isomorphic to Y as this is $\varphi(\overline{X}) \cap \varphi(U_{n+1}) = \overline{Y} \cap \varphi(U_{n+1})$.

Restricting to P having final coordinate 1 then gives us an isomorphism $X \to Y$ with the polynomials $T_1, \dots, T_n$ and $S_1, \dots, S_n$.

As $\overline{X}$ and $\overline{Y}$ are equivalent, there exists $G\colon \mathbb{P}^n \to \mathbb{P}^n$ restricting to an isomorphism $\overline{X} \to \overline{Y}$.

(b) We have $V(x-y^2) \simeq V(x)$ as affine plane curves. However, their projective closures are $Y = \mathbb{V}(xz-y^2)$ and $\mathbb{V}(x)$. Now $\Gamma_h(Y) = k\left[x,y,z\right]/(xz-y^2)$ and $\Gamma_h(\mathbb{V}(x)) = k\left[x,y,z\right]/(x) \simeq k\left[y,z\right]$ Now, $k\left[x,y,z\right]/(xz-y^2)$ is not UFD while $k\left[y,z\right]$ is UFD, so $\Gamma_h(Y)$ and $\Gamma_h(\mathbb{V}(x))$ are not isomorphic. However, by page 8 on lecture note 21, we have that if $\overline{X}$ and $\overline{Y}$ are projectively equivalent, then $\Gamma_h(\overline{X})$ and $\Gamma_h(\overline{Y})$ are isomorphic, so by contraposition, $\overline{X}$ and $\overline{Y}$ are not projectively equivalent.