A brief introduction to algebraic topology

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Prerequisites: Basic topology for a few of the preliminary proofs, but I will try to convey most things visually so that we won't get caught up in technicalities on the topological side.

The goal is to get a basic visual understanding of fundamental things in algebraic topology such as path homotopies and the homotopy-lifting lemma, get a visual understanding of why $\pi_1(S^1) \cong \mathbb{Z}$, and then work on and hopefully solve the following problem:

Let A_1, A_2, A_3 be closed[1] and bounded[2] sets in $\mathbb{R}^3$. Prove that there is one plane $P \subset \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.

To show this, we wish to show the Borsuk-Ulam theorem in two dimensions aka the there-must-exist-a-pair-of-antipodal-points-on-earth-having-the-same-temperature theorem:

Theorem. (Borsuk-Ulam) *For every continuous map $f\colon S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.*[3]

Warm-up for people who don't know much group theory

Algebraic topology is about using algebra to describe topological objects, so to warm up, try the following exercise.

Exercise. Choose your favorite conic and let G denote the set of points of the conic. Now choose a point which we will call 0 in G. Then we define a binary operation as follows: for any points $A, B \in G$, we define $A * B$ as the point of intersection between the conic and the line parallel to AB passing through 0. Show that $(G, *)$ is a group.

Paths and Homotopy

Definition. A **path** in a space X is a continuous map $f\colon I \to X$ where $I = [0, 1]$.

Definition. A **homotopy** of paths in X is a continuous map $F\colon I \times I \to X$ such that $F(0, t) = x_0$ and $F(1, t) = x_1$ for all $t \in I$.
When two paths $f_0(x) = F(x, 0)$ and $f_1(x) = F(x, 1)$ are connected in this way by a homotopy F, they are said to be **homotopic**, and we denote this by $f_0 \simeq f_1$.

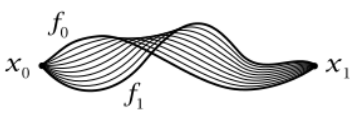

Figure 1: Here is an example of what a homotopy of paths could look like.

Exercise 1. Suppose we take any two paths f_0 and f_1 in $\mathbb{R}^n$ having the same endpoints x_0 and x_1. Show that they are homotopic.

Exercise 2. Show that the relation of homotopy on paths with fixed endpoints in any space is an

[1] A subset A of $\mathbb{R}^3$ is closed if for any $x \in \mathbb{R}^3 - A$, there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \mathbb{R}^3 - A$, i.e., such that the ball with center x and radius ε is contained in the complement of A.
[2] A subset A of $\mathbb{R}^3$ is bounded if there exists some $R \in \mathbb{R}$ such that for any $x, y \in A$, we have $\mathrm{dist}(x, y) \leq R$.
[3] S^n is the n-sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$

equivalence relation.

Thus we can denote the equivalence class of a path f under the equivalence relation of homotopy by $[f]$, which will be called the **homotopy class** of f.
So in figure 1 above, we would thus have $[f_0] = [f_1]$.

We wish to look at loops, i.e., paths with the same start and end point, and show that the set of homotopy classes of loops with a fixed start and end point has a group structure. For this, we need to define the group operation:

Definition. Given two paths $f, g\colon I \to X$ such that $f(1) = g(0)$, we can define a concatenation of the paths $f \cdot g$ by

$$f \cdot g(s) = \begin{cases} f(2s), & 0 \leq s \leq \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

This path traverses f during $\left[0, \frac{1}{2}\right]$ and then traverses g during $\left[\frac{1}{2}, 1\right]$.

Suppose we restrict our attention to paths $f\colon I \to X$ with the same start and end point, $f(0) = f(1) = x_0 \in X$. Such paths are called **loops**, and x_0 is called the **basepoint**. The set of homotopy classes $[f]$ of loops $f\colon I \to X$ at the basepoint x_0 is denoted by $\pi_1(X, x_0)$.

Algebraic topology makes use of algebra to describe topological objects, so to warm up, let's look at how we can attribute a group to any conic.

Proposition. $\pi_1(X, x_0)$ is a group with respect to the product $[f]\,[g] = [f \cdot g]$.
$\pi_1(X, x_0)$ is called the **fundamental group** of X at the basepoint x_0.

Exercise 3. Show that $\pi_1(\mathbb{R}^n, x_0) \cong 0$ for any $x_0 \in \mathbb{R}^n$ where 0 is the trivial group.

Lifting homotopies

Definition. Given a map $f\colon X \to Y$ and a map $g\colon Z \to Y$, a **lift** of f to Z is a map $h\colon X \to Z$ such that $f = g \circ h$. We say that f factors through h.
We can also state this by saying that the following diagram commutes

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[ur, "h"] \arrow[r, "f"'] & Y \end{tikzcd}$$

Homotopy-lifting lemma. *If $F\colon I \times I \to S^1$ is a map such that $F(0,t) = F(1,t) = 1$ for $0 \leq t \leq 1$, there exists a unique map $\tilde{F}\colon I \times I \to \mathbb{R}$ such that $\tilde{F}$ is a lift of F and*

$$\tilde{F}(0,t) = 0,\ 0 \leq t \leq 1.$$

This lemma can be encapsulated by stating that the following diagram commutes:

$$\begin{tikzcd} \{0\} \times I \arrow[r, "0"] \arrow[d, hook, "\iota"] & \mathbb{R} \arrow[d, "\pi"] \\ I \times I \arrow[r, "F"'] \arrow[ur, dotted, "\exists!\tilde{F}"] & S^1 \end{tikzcd}$$

All the fun stuff

Theorem. $\pi_1(S^1)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ based at $(1,0)$. I.e., $\pi_1(S^1) \cong \mathbb{Z}$.

Exercise 4. Suppose the Borsuk-Ulam theorem is false for $f\colon S^2 \to \mathbb{R}^2$. Try to create a map g using f that maps into S^1 and look at how the image of the equator of S^2 under g behaves.

Definition. A path $f\colon I \to X$ is said to be **nullhomotopic** if it is homotopic to a constant map $c\colon I \to X$, i.e. $f \simeq c$ and $c(t) = c(0)$ for all $t \in I$.

Exercise 5. Find a non-trivial loop in the image of g which is also nullhomotopic, thus giving a contradiction.

Problem: Prove the original problem using the Borsuk-Ulam theorem.

Extra problem: Also prove the following problem: Whenever S^2 is expressed as the union of three closed sets A_1, A_2 and A_3, then at least one of these sets must contain a pair of antipodal points $\{x, -x\}$. Show that 3 in this result is the best possible.

Extra exercise. Does the Borsuk-Ulam theorem hold for the torus? In other words, for every map $f\colon S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?

Extra chapter for fun - induced homomorphisms

Definition. A map $r\colon X \to X$ such that $r(X) = A$ and $r|_A = \mathbb{1}$ for a subspace $A \subset X$ is called a retraction.

Definition. A **deformation retraction** of a space X onto a subspace A is a continuous map $F\colon X \times I \to X$ such that $F(x, 0) = \mathbb{1}$ and $F(X, 1) = A$ and $F(x, t) = x$ for all $x \in A$ and $t \in I$.

Suppose now $\varphi\colon X \to Y$ is a map taking the basepoint $x_0 \in X$ to the basepoint $y_0 \in Y$. WRite this as $\varphi\colon (X, x_0) \to (Y, y_0)$. Then φ induces a homomorphism $\varphi_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $\varphi_* [f] = [\varphi \circ f]$ for any loop $f\colon I \to X$ based at x_0.

Exercise. Convince yourself that π_1 is a functor.

Proposition. If a space X retracts onto a subspace A, then the homomorphism $\iota_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ induces by the inclusion $\iota\colon A \hookrightarrow X$ is injective. If A is a deformation retract of X, then ι_* is an isomorphism.

Proposition. $\pi_1(X \times Y) \cong \pi_1(X) \cong \pi_1(Y)$ if X and Y are path-connected.

Problems. Show that there are no retractions $r\colon X \to A$ in the following cases:

1. $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
2. $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$
3. X the Möbius band and A its boundary circle.

Problem. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1\left(\mathbb{R}^3 - X\right)$.
(You may use the fact that $\pi_1\left(\bigvee_{i=1}^n S_i^1\right) = \bigoplus_{i=1}^n \mathbb{Z}$ where $\bigvee_{i=1}^n S_i^1$ denotes the one-point union of n circles - i.e., n circles joined at a single point.