ASSIGNMENT 3

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Exercise 0.1. Let R be a Noetherian ring. Show the following

(1) For every ideal $I \subset R$, there exists $n \in \mathbb{N}$ such that $\left(\sqrt{I}\right)^n \subset I$.
(2) Every radical ideal of R is a finite intersection of prime ideals.
(3) If a radical ideal of R is irreducible, then it is a prime ideal.

Proof. (1) Since $I \subset \sqrt{I}$ are sub-R-modules of R considered as a module over itself, we find that $\sqrt{I}$ must be finitely generated, so let $\sqrt{I} = \langle x_1, \ldots, x_n \rangle$, and by assumption, there exist $\alpha_1, \ldots, \alpha_n$ such that $x_i^{\alpha_i} \in I$. Let $\alpha = \alpha_1 + \ldots + \alpha_n$. Now let $x \in \sqrt{I}$ and write $x = \sum_i c_i x_i$. Then any term in x^α will contain some x_i to the power of at least α_i by the pigeonhole principle. Since I is an ideal, the whole term is in I, so again, ideals are closed under sums, so $x^\alpha \in I$. Since x was arbitrary, we find that $\left(\sqrt{I}\right)^\alpha \subset I$.

(2) Let I be a radical ideal of R, so $\sqrt{I} = I$. By theorem 7.19 (Primary decomposition), I is the finite intersection of primary ideals, so

$$I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_\mathfrak{n}$$

where each $\mathfrak{p}_i$ is primary.

Lemma 0.2. *For an ideal $J = J_1 \cap \ldots \cap J_n$, we have*

$$\sqrt{J} = \sqrt{J_1} \cap \ldots \cap \sqrt{J_n}$$

Proof. Suppose $x \in \sqrt{J}$ so $x^i \in J = J_1 \cap \ldots \cap J_n$, then $x^i \in J_j$ for all j so $x \in \sqrt{J_j}$ for all J, so $x \in \sqrt{J_1} \cap \ldots \cap \sqrt{J_n}$. Conversely, if $x \in \sqrt{J_1} \cap \ldots \cap \sqrt{J_n}$ then there exist $\alpha_1, \ldots, \alpha_n$ such that $x^{\alpha_i} \in J_i$. Let $\alpha = \max_i \{\alpha_i\}$. Then $x^\alpha \in J_1 \cap \ldots \cap J_n = J$, so $x \in \sqrt{J}$. ☐

Hence we obtain

$$I = \sqrt{I} = \sqrt{\mathfrak{p}_1} \cap \ldots \cap \sqrt{\mathfrak{p}_n}.$$

To finish it off, we note that by Lemma 7.11, each $\sqrt{\mathfrak{p}_i}$ is prime.

(3) Suppose $I \subset R$ is a radical ideal which is irreducible. By Lemma 7.16, I is primary, and now by Lemma 7.11, $I = \sqrt{I}$ is prime. ☐

Exercise 0.3. Let $V \subset K^n$ be an affine algebraic set. Show the following.

(1) V is irreducible if and only if $\mathbb{I}(V)$ is a prime ideal.
(2) V can be written as a finite union of irreducible affine algebraic sets.
(3) There is a minimal decomposition $V = V_1 \cup \ldots \cup V_m$ of V into irreducible affine algebraic sets V_i, where $m \in \mathbb{N}_0$. This is meant in the sense that no V_i is contained in $\bigcup_{j \neq i} V_j$.

(4) The minimal decomposition $V = V_1 \cup \ldots \cup V_m$ is unique, up to reordering of $V_1, \ldots, V_m$. We call $V_1, \ldots, V_m$ the irreducible components of V.

Proof. (1) Since $V \subset K^n$ is an affine algebraic set, there exists some ideal $I \subset k[x_1, \ldots, x_n]$ such that $V = \mathbb{V}(I)$. Suppose $V = V_1 \cap V_2$ with both V_1 and V_2 being affine algebraic sets properly containing V. Then $\mathbb{I}(V) \subset \mathbb{I}(V_1) \cap \mathbb{I}(V_2)$ since any polynomial vanishing on V must vanish on both V_1 and on V_2. But now any prime ideal is irreducible, so $\mathbb{I}(V_1) = \mathbb{I}(V)$ or $\mathbb{I}(V_2) = \mathbb{I}(V)$. Suppose without loss of generality that $\mathbb{I}(V_2) = \mathbb{I}(V)$. Then $V_2 = \mathbb{V}(\mathbb{I}(V_2)) = \mathbb{V}(\mathbb{I}(V)) = V$. For this, we need to show that $\mathbb{V}(\mathbb{I}(W)) = W$ when W is an affine algebraic set. But $\mathbb{I}(\mathbb{V}(U)) \subset U$ always, so since $\mathbb{V}$ is containment-reversing, we get $W = \mathbb{V}(U) \subset \mathbb{V}(\mathbb{I}(W))$. For the opposite direction, we simply have that if $x \in \mathbb{V}(\mathbb{I}(W))$, then any $f \in \mathbb{I}(W) = \mathbb{I}(\mathbb{V}(U)) \subset U$ vanishes on x. Suppose $x \notin W = \mathbb{V}(U)$. Then there exists some $g \in U$ such that $g(x) \neq 0$. But $g \in \mathbb{I}(\mathbb{V}(U)) = \mathbb{I}(W)$ by definition which gives a contradiction. Hence $\mathbb{V}(\mathbb{I}(W)) \subset W$. Having concluded that $V = V_1$ or $V = V_2$, this shows that V is irreducible.

A faster way to see this, I suppose would be the following: If $V = V_1 \cup V_2$, then $\mathbb{I}(V_1) \cap \mathbb{I}(V_2) \subset \mathbb{I}(V)$, showing that $\mathbb{I}(V)$ is not irreducible, contradicting lemma 7.3.

Conversely, if $\mathbb{I}(V)$ is not prime, let $fg \in \mathbb{I}(V)$ such that $f, g \notin \mathbb{I}(V)$. Then $V = \mathbb{V}(\mathbb{I}(V)) \subset \mathbb{V}((f)(g)) \subset \mathbb{V}(f) \cap \mathbb{V}(g)$ using that $\mathbb{V}$ is inclusion-reversing. Now by assumption, if $V = \mathbb{V}(f)$, then f would vanish on all of V, contradicting $f \notin \mathbb{I}(V)$. Similarly for g. Hence V is shown to not be irreducible.
(2) Since V is an affine algebraic set, there exists an ideal I such that $V = \mathbb{V}(I)$. Now, $I \subset k[x_1, \ldots, x_n]$ which is Noetherian by applying Hilbert's basis theorem iteratively since a field is Noetherian (having only (0) and itself as ideals) considered as k-modules. This in particular gives us a decomposition

$$I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n$$

where each $\mathfrak{p}_i$ is primary. Hence

$$\mathbb{V}(I) = \mathbb{V}(\mathfrak{p}_1) \cup \ldots \cup \mathbb{V}(\mathfrak{p}_n).$$

To show that $\mathbb{V}(\mathfrak{p}_i)$ is irreducible, we can show that $\mathbb{I}(\mathbb{V}(\mathfrak{p}_i))$ is a prime ideal. This can be easily achieved if we may use Nullstellensatz since then $\mathbb{I}(\mathbb{V}(\mathfrak{p}_i)) = \sqrt{\mathfrak{p}_i}$ which is prime by Lemma 7.11.

(3) By part (2), V can be decomposed as $V = V_1 \cup \ldots \cup V_n$ where each V_i is an irreducible affine algebraic set. Suppose now that $V_1 \subset \bigcup_{i=2}^n V_i$. But then

$$V_1 = \bigcup_{i=2}^n (V_1 \cap V_i).$$

Now, the intersection of affine algebraic sets is still an affine algebraic set since $\mathbb{V}(I_1) \cap \mathbb{V}(I_2) = \mathbb{V}(I_1 \cup I_2)$. Similarly, a union of finitely many affine algebraic sets is also an affine algebraic set since $\mathbb{V}(I_1 \cdots I_n) = \mathbb{V}(I_1) \cap \ldots \cap \mathbb{V}(I_n)$. So by irreducibility of V_1, either $V_1 = V_1 \cap V_2$ or $V_1 = \bigcup_{i=3}^n V_1 \cap V_i$. Inductively, we obtain that for some $i \geq 2$, $V_1 = V_1 \cap V_2$, i.e., $V_1 \subset V_2$. Hence we may discard V_1 from the collection, so $V = V_2 \cup \ldots \cup V_n$. Thus if we have a collection $V = V_1 \cup \ldots \cup V_n$ such that $V_i \subset \bigcup_{j \neq i} V_j$, then we can simply discard V_i. We can continue to do so and after at most $n-1$ steps, we will obtain a minimal decomposition.

(4) Suppose
$$V = V_1 \cup \ldots \cup V_m = W_1 \cup \ldots \cup W_n$$
are two minimal decompositions. Then $W_i \subset V_1 \cup \ldots \cup V_m$, so
$$W_i = \bigcup_{j=1}^{m} V_j \cap W_i$$
By part (3), this is a union of affine algebraic sets, so we completely equivalently obtain that $W_i = V_j \cap W_i$ for some j. Hence $W_i \subset V_j$. For each i, let j_i be such that $W_i \subset V_{j_i}$. Repeating this the other way around, we obtain i_k such that $V_k \subset W_{i_k}$. Now $V_k \subset W_{i_k} \subset V_{j_{i_k}}$. So since the decomposition is minimal, we must have $k = j_{i_k}$, so $V_k = W_{i_k}$ for all k. This in particular gives an injective map $\{1, \ldots, m\} \to \{1, \ldots, n\}$, so $m \leq n$. And similarly, $W_i = V_{j_i}$ for all i, so we similarly get $n \leq n$. This implies that $m = n$ and that indeed the decompositions are the same up to reordering, namely by the reordering $\sigma \colon k \mapsto i_k$ giving $V_k = W_{\sigma(k)}$. $\square$