Problem 0.1. Let $T = S^1 \times S^1$ be the torus and $i\colon D^2 \hookrightarrow T$ and embedding of the unit disk that is disjoint from $S^1 \times \{s_0\}$. Define $A := \left(S^1 \times \{s_0\}\right) \cup i\left(S^1\right) \subset T$. Let $x_0 = (s_0, s_0)$ and $x_1 \in i\left(S^1\right)$.

1. Draw a picture of (X, A) and the two points x_0 and x_1.
2. Construct an explicit bijection of sets $\pi_1(T, A, x_1) \cong \mathbb{Z}^2 \sqcup \mathbb{Z}$.
3. Compute the relative homotopy groups $\pi_2(T, A, x_0)$ and $\pi_2(T, A, x_1)$.

Solution. (1)

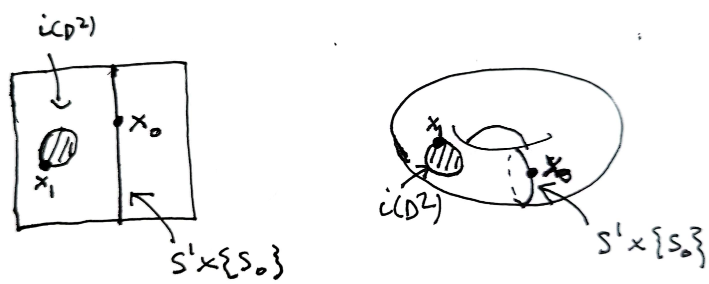

FIGURE 1. Note that in this figure, A are the parts drawn without the interior of the disk $i(D^2)$.

(2) Recall that

$$\pi_n(T, A, x_1) = \left[I^n, \partial I^n, J^{n-1}; T, A, x_1\right].$$

Thus $\pi_1(T, A, x_1)$ becomes the set of homotopy classes of maps $(I, \{0,1\}, \{1\}) \to (T, A, x_1)$. That is, the set of paths in T starting at a point in A and ending at x_1. Now, take a path γ from x_1 to x_0, and for a path $[\alpha] \in \pi_1(T, A, x_1)$, we can homotopy this path to start at x_0, so by concatinating γ with this path, we obtain a closed loop in $T \cong \mathbb{Z} \oplus \mathbb{Z}$, so to obtain a well-defined map, we homotopy α to the path in $[\alpha]$ which, when we concatinate with γ, has 0 in the second coordinate. That is, $[\gamma * \alpha] = (n, 0)$ in $\pi_1(T, x_1)$. Denote this map $k\colon \pi_1(T, A, x_1) \to \pi_1(T, x_1)$. The kernel of this map is all maps which, when concatenated with γ, become $(0,0)$ in $\pi_1(T, x_1)$. But γ can be chosen to
We have

$$\underbrace{\pi_1(A, x_1)}_{\mathbb{Z}} \xrightarrow{0} \underbrace{\pi_1(T, x_1)}_{\mathbb{Z}^2} \to \pi_1(T, A, x_1) \to \underbrace{\pi_0(A, x_1)}_{\cong \mathbb{Z}/2} \to \underbrace{\pi_0(T, x_1)}_{1}$$

(3) Using the LES of relative homotopy groups, we have that

$$\pi_2(T, x_i) \to \pi_2(T, A, x_i) \to \pi_1(A, x_i) \to \pi_1(T, x_i)$$

is exact for $i = 0, 1$. For $i = 0, 1$, $\pi_1(A, x_i) \cong \mathbb{Z}$ and $\pi_1(T, x_i) \cong \mathbb{Z}^2$, while $\pi_2(T, x_i) \cong 1$ for both $i = 0, 1$.

Problem 0.2.
1. Compute $\pi_1\left(S^1 \vee S^2\right)$ and describe the universal cover of $S^1 \vee S^2$.
2. Show that $\pi_2\left(S^1 \vee S^2\right)$ is isomorphic to $\bigoplus_{\mathbb{Z}} \mathbb{Z}$.
3. Explicitly describe the action of $\pi_1\left(S^1 \vee S^2\right)$ on $\bigoplus_{\mathbb{Z}} \mathbb{Z} \cong \pi_2\left(S^1 \vee S^2\right)$.

Solution. (1) From the LES, we have

$$\underbrace{\pi_1\left(S^2,*\right)}_{\cong 0} \to \pi_1\left(S^1 \vee S^2,*\right) \to \underbrace{\pi_1\left(S^1 \vee S^2, S^2,*\right)}_{\cong \mathbb{Z}} \to \underbrace{\pi_0\left(S^2,*\right)}_{\cong 0}$$

so we find that $\pi_1\left(S^1 \vee S^2,*\right) \cong \mathbb{Z}$. The universal cover of $S^1 \vee S^2$ is easily seen to be $\mathbb{R}$ with a copy of S^2 attached to each integer of $\mathbb{R}$.
(2) To compute $\pi_2\left(S^1 \vee S^2\right)$, it suffices to compute π_2 of its universal cover since these are isomorphic. The universal cover is $\mathbb{R}$ with S^2 attached at each integer. Since $\mathbb{R} \simeq \{*\}$, the universal cover is homotopy equivalent to $\vee_{\mathbb{Z}} S^2$. Consider the LES for the pair $\left(\prod_{\mathbb{Z}} S^2, \vee_{\mathbb{Z}} S^2\right)$:

$$\pi_3\left(\prod_{\mathbb{Z}} S^2, \bigvee_{\mathbb{Z}} S^2\right) \to \pi_2\left(\bigvee_{\mathbb{Z}} S^2\right) \to \pi_2\left(\prod_{\mathbb{Z}} S^2\right) \to \pi_2\left(\prod_{\mathbb{Z}} S^2, \bigvee_{\mathbb{Z}} S^2\right)$$

Now, $\pi_2\left(\prod_{\mathbb{Z}} S^2\right) \cong \oplus_{\mathbb{Z}} \pi_2\left(S^2\right) \cong \oplus_{\mathbb{Z}} \mathbb{Z}$ (with the inclusions $S^2 \hookrightarrow \prod_{\mathbb{Z}} S^2$ whose induced images form a basis, hence if $\pi_2\left(\prod_{\mathbb{Z}} S^2\right) \cong \pi_2\left(\bigvee_{\mathbb{Z}} S^2\right)$ under the inclusion $\bigvee_{\mathbb{Z}} S^2 \hookrightarrow \prod_{\mathbb{Z}} S^2$, then the same basis also holds for $\pi_2\left(\bigvee_{\mathbb{Z}} S^2\right)$), since $\pi_k\left(\prod_\alpha X_\alpha\right) \cong \oplus_\alpha \pi_k\left(X_\alpha\right)$. We now claim that $\pi_n\left(\prod_{\mathbb{Z}} S^2, \bigvee_{\mathbb{Z}} S^2\right) \cong 0$ for $n \leq 3$. To this end, note that on p. 8 in Hatcher, it is described that we can give products $X \times Y$, we can give these a CW structure when X and Y have a CW structure, given by a cell complex with cells the products $e_\alpha^m \times e_\beta^n$ for e_α^m ranging over the cells of X and e_β^n ranging over the cells of Y. Choosing $(s_0, s_0, \ldots, s_0, \ldots)$ as a 0-cell in $\prod_{\mathbb{Z}} S^2$ and attaching 2-cells to obtain $\bigvee_{\mathbb{Z}} S_i^2$ by S_i^2 identified with with the i th coordinate copy of S_i^2 in $\prod_{i \in \mathbb{Z}} S_i^2$ (so all other coordinates or s_0). We then precisely find that $\bigvee_{\mathbb{Z}} S^2$ constitutes the 2-skeleton for $\prod_{\mathbb{Z}} S^2$. By Corollary 4.12 in Hatcher, we then find that $\left(\prod_{\mathbb{Z}} S^2, \bigvee_{\mathbb{Z}} S^2\right)$ is 3-connected. The result follows.
(3)

Problem 0.3. Let (X, A, x_0) be a pointed pair such that the inclusion $i\colon A \hookrightarrow X$ is based nullhomotopic (the nullhomotopy preserves the basepoint). The goal is to show that for $n \geq 2$, there is an isomorphism of groups:

$$\pi_n(X, A, x_0) \cong \pi_n(X, x_0) \times \pi_{n-1}(A, x_0).$$

(1) Show that there is an exact sequence of groups

$$1 \to \pi_n(X, x_0) \xrightarrow{j_*} \pi_n\left(X, A, x_0\right) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \to 1.$$

(2) Using a based nullhomotopy $H\colon A \times [0,1] \to X$, construct a natural group morphism

$$r_*\colon \pi_n(X, A, x_0) \to \pi_n(X, x_0)$$

such that $r_* \circ j_* = 1$.
(3) Show that for any short exact sequence of groups

$$1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$$

such that α admits a retraction, there is a group isomorphism

$$B \cong A \times C.$$

Conclude the desired isomorphism.

Proof. (1) From the LES for relative homotopy groups, we obtain that

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n\left(X, A, x_0\right) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \pi_{n-1}\left(X, x_0\right)$$

is exact. For $n \geq 2$, all the sets in the exact sequence are groups and the maps are group homomorphisms. Since homotopic maps relative to the base point induce the same maps on homotopy groups, we find by assumption that $i_* = 0$. Therefore,

$$1 \xrightarrow{0} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n\left(X, A, x_0\right) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \xrightarrow{0} 1$$

is exact.

(2) Let $[f] \in \pi_n\left(X, A, x_0\right)$.

$\square$