ASSIGNMENT 4 AND 5

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Exercise 0.1. If (X, x) and (Y, y) are based spaces, let $C((X, x), (Y, y))$ be the set of pairs $(f, [\lambda])$, where $f \colon X \to Y$ is a continuous map, $\lambda \colon I \to Y$ is a path from y to $f(x)$ and $[\lambda]$ denotes its homotopy class relative to ∂I.
If (Z, z) is a third based space and $(f, [\lambda]) \in C((X, x), (Y, y))$ and $(g, [\mu]) \in C((Y, y), (Z, z))$, define $(g, [\mu]) \circ_C (f, [\lambda]) = (g \circ f, [\mu * (g \circ \lambda)])$.

(i) Check that $(g, [\mu]) \circ_C (f, [\lambda])$ is an element of $C((X, x), (Z, z))$.
(ii) Explain why $\circ_C$ is associative.
(iii) Define identity elements $\mathbb{1}_{(X,x)} \in C((X, x), (X, x))$ and verify that C is in fact a category.

Proof. (i): Compositions of continuous maps are continuous, so $g \circ f \colon X \to Z$ is continuous. We must simply check that $\mu * (g \circ \lambda)$ is a path from z to $g \circ f(x)$. Since $(g, [\mu]) \in C((Y, y), (Z, z))$, μ starts at z by assumption, hence so does $\mu * (g \circ \lambda)$. The path ends at $\mu * (g \circ \lambda)(1) = (g \circ \lambda)(1) = g(\lambda(1)) = g(f(x)) = (g \circ f)(x)$ where again $\lambda(1) = f(x)$ follows from $(f, [\lambda]) \in C((X, x), (Y, y))$.
We do not have problems with well-definedness of endpoints since our homotopy classes are rel ∂I, so endpoints are unique.

(ii): Associativity of the function part follows from general associativity of functions. Associativity of the paths up to homotopy follows from lemma 1.4.2.(ii) (associativity of paths under homotopy when they are defined).

(iii): Let c_x denote the constant path at x, and let $\mathbb{1}_X$ denote the identity map $X \to X$ (which indeed is continuous). Then c_x is also a path from $\mathbb{1}_X(x) = x$ to x, so $(\mathbb{1}_X, c_x) \in C((X, x), (X, x))$. Now, with $\operatorname{ob} C = \operatorname{ob} \mathrm{Top}_*$ and the collection of morphisms between the objects (X, x) and (Y, y) being $C((X, x), (Y, y))$, we claim this forms a category under the above composition operation and designated identity morphisms.
The only thing left to check is that for an arbitrary morphism $(f, [\lambda]) \in C((X, x), (Y, y))$, we have $(f, [\lambda]) = (f, [\lambda]) \circ_C (\mathbb{1}_X, c_x) = (\mathbb{1}_Y, c_y) \circ_C (f, [\lambda])$. Now

$$(f, [\lambda]) \circ_C (\mathbb{1}_X, c_x) = (f \circ \mathbb{1}_X, [\lambda * f \circ c_x]) = (f, [\lambda])$$

since $f \circ c_x = c_{f(x)}$, so $\lambda * f \circ c_x \simeq \lambda \circ c_{f(x)} \simeq \lambda$. Similarly,

$$(\mathbb{1}_Y, c_y) \circ_C (f, [\lambda]) = (\mathbb{1}_Y \circ f, [c_y * \mathbb{1}_Y \circ \lambda]) = (f, [c_y * \lambda]) = (f, [\lambda]).$$

(iv) To check that this is well defined, suppose $[\alpha] = [\beta] \in \pi_1(X, x)$. Then

$$(f, [\lambda])_* ([\alpha]) = \lambda_* \circ f_* [\alpha] = \lambda_* [f \circ \alpha] = \left[\lambda * f \circ \alpha * \overline{\lambda}\right],$$

and similarly,

$$(f, [\lambda]) [\beta] = \left[\lambda * f \circ \beta * \overline{\lambda}\right].$$

Now $\lambda * f \circ \alpha * \overline{\lambda} \simeq \lambda * f \circ \beta * \overline{\lambda} \operatorname{rel} I$ follows from lemma 1.4.2.(i) and (v). Hence the map $(f, [\lambda])_*$ is well-defined.
Now we check functoriality of π_1. To each pointed space $(X, x) \in \operatorname{ob} C$, we associate the group $\pi_1(X, x) \in \mathrm{Ab}$. To each morphism $(f, [\lambda]) \in \operatorname{Mor}((X, x), (Y, y))$, we associate the homomorphism $(f, [\lambda])_* : \pi_1(X, x) \to \pi_1(Y, y)$ as defined above. We must check that $\pi_1(\mathbb{1}_X, c_x) = \mathbb{1}_{\pi_1(X,x)}$ and that for $(f, [\lambda]) \in \operatorname{Mor}((X, x), (Y, y))$ and $(g, [\mu]) \in \operatorname{Mor}((Y, y), (Z, z))$, we have $\pi_1((g, [\mu]) \circ_C (f, [\lambda])) = \pi_1((g, [\mu])) \circ \pi_1((f, [\lambda]))$.
Now, for any $[\alpha] \in \pi_1(X, x)$, we have

$$\begin{aligned}
\pi_1(\mathbb{1}_X, c_x)[\alpha] &= (c_x)_* \circ (\mathbb{1}_X)_* [\alpha] \\
&= (c_x)_* [\mathbb{1}_X \circ \alpha] \\
&= [c_x * \mathbb{1}_X \circ \alpha * \overline{c_x}] \\
&= [\alpha] \\
&= \mathbb{1}_{\pi_1(X,x)} [\alpha]
\end{aligned}$$

giving $\pi_1(\mathbb{1}_X, c_x) = \mathbb{1}_{\pi_1(X,x)}$, and also

$$\begin{aligned}
\pi_1((g, [\mu]) \circ_C (f, [\lambda]))[\alpha] &= \pi_1(g \circ f, [\mu * g \circ \lambda])[\alpha] \\
&= (\mu * g \circ \lambda)_* \circ (g \circ f)_* [\alpha] \\
&= (\mu * g \circ \lambda)_* [g \circ f \circ \alpha] \\
&= \left[\mu * g \circ \lambda * g \circ f \circ \alpha * \overline{\mu * g \circ \lambda}\right] \\
&= \left[\mu * g \circ \lambda * g \circ f \circ \alpha * g \circ \overline{\lambda} * \overline{\mu}\right] \\
&= \left[\mu * g \circ \left(\lambda * f \circ \alpha * \overline{\lambda}\right) * \overline{\mu}\right] \\
&= \mu_* \circ g_* \circ \lambda_* \circ f_* [\alpha] \\
&= \pi_1(g, [\mu]) \circ \pi_1(f, [\lambda])[\alpha],
\end{aligned}$$

so indeed

$$\pi_1((g, [\mu]) \circ_C (f, [\lambda])) = \pi_1((g, [\mu])) \circ \pi_1((f, [\lambda]))$$

□

Question. I'm having trouble understanding precisely what the implication of this exercise is in in a categorical framework. Is the important part that we have essentially constructed a category wherein $\operatorname{Mor}((X, x), (Y, y))$ consists precisely of those maps f which send x to the path component of y instead of only the maps which send x to y as in Top_*, and thus we have "enlarged" the morphisms in our category a bit (which essentially corresponds to the result that π_1 is independent of basepoint in a path-connected space). Put in a different way, Top_* is a subcategory of C and under the inclusion functor $\iota \colon \mathrm{Top}_* \to C$ (where a morphism $f \colon (X, x) \to (Y, y)$ is mapped to $(f, [c_y])$) , we have that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "\pi_1"] & \mathrm{Group} \\
\mathrm{Top}_* \arrow[u, "\iota"] \arrow[ur, "\pi_1"'] &
\end{tikzcd}$$

If you could shed some light onto whether there is something I'm missing, that would be really great! Thank you.

Exercise 0.2. Let $p\colon \mathbb{R} \to S^1$ denote the usual covering map $x \mapsto e^{2\pi i x}$.

(1) Prove that for any $\sigma \in \operatorname{Sin}_1\left(S^1\right)$, there exists a $\tau \in \operatorname{Sin}_1(\mathbb{R})$ such that $\sigma = p \circ \tau$. In particular, we get two numbers $d_0\tau, d_1\tau \in \operatorname{Sin}_0(\mathbb{R}), = \mathbb{R}$. Explain why the difference $d_0\tau - d_1\tau \in \mathbb{R}$ depends only on σ, and not on the choice of τ (remember that d_1 is the starting point and d_0 is the end point). Hence we may define a function
$$\varphi\colon \operatorname{Sin}_1\left(S^1\right) \to \mathbb{R}$$
by $\varphi(\sigma) = d_1\tau - d_0\tau$ if $\sigma = p \circ \tau$.

(2) Using addition as a group structure on $\mathbb{R}$, extend φ to a homomorphism $C_1\left(S^1\right) \to \mathbb{R}$. Let us use the same letter φ for this homomorphism.

(3) Prove that $\varphi\circ\partial\colon C_2\left(S^1\right) \to \mathbb{R}$ is trivial and deduce a well-defined homomorphism $H_1\left(S^1\right) \to \mathbb{R}$, sending $[c] \mapsto [\varphi(c)]$.

(4) Let $\sigma_0\colon \Delta^1 \to S^1$ be given by $\sigma_0(t_0, t_1) = e^{2\pi i t_1}$. Prove that $\sigma_0 \in Z_1\left(S^1\right)$, so that it represents a class $[\sigma_0] \in H_1\left(S^1\right)$.

(5) Prove that the homomorphism $\varphi\colon H_1\left(S^1\right) \to \mathbb{R}$ constructed above sends $[\sigma_0]$ to $1 \in \mathbb{R}$.

Proof. (i) Let $k\colon \Delta^1 \to I$ denote the homeomorphism $(t_0, t_1) \mapsto t_1$ and $k^{-1}\colon I \to \Delta^1$ denote the inverse homeomorphism $t \mapsto (1-t, t)$. Now, $\sigma \circ k^{-1}$ is a path $I \to S^1$ which we can thus lift to a path $\tilde{\sigma}\colon I \to \mathbb{R}$ such that $\sigma \circ k^{-1} = p \circ \tilde{\sigma}$, hence $\tau := \tilde{\sigma} \circ k \in \operatorname{Sin}_1(\mathbb{R})$ and satisfies $\sigma = p \circ \tau$.
Now, we know that given a lift $\tilde{\sigma}$ of $\sigma \circ k^{-1}$ and a starting point $\tilde{\sigma}(0)$ of $\tilde{\sigma}$, the endpoint $\tilde{\sigma}(1)$ is uniquely determined. Now, $d_0\tau - d_1\tau = \tau \circ \delta^0 - \tau \circ \delta^1$, where $\delta^0, \delta^1\colon \{1\} = \Delta^0 \to \Delta^1$ is given by $\delta^0(1) = (0,1)$ and $\delta^1(1) = (1,0)$. Thus $\tau \circ \delta^0(1) = \tilde{\sigma} \circ k(0,1) = \tilde{\sigma}(1)$ and $\tau \circ \delta^1(1) = \tilde{\sigma}(0)$. So $d_0\tau - d_1\tau = \tilde{\sigma}(1) - \tilde{\sigma}(0)$. We have thus reduced the question to showing that $\tilde{\sigma}(1) - \tilde{\sigma}(0)$ only depends on σ. One way to see this is that the convering space $\mathbb{R}$ inherits a metric since p is a local isometry, so distances are preserved. I.e., if $\tilde{U}$ is one of the sheets of $p^{-1}(U)$ for some evenly covered open set $U \subset S^1$ contained in one of the standard evenly covered sets U_1, U_2, U_3, U_4 corresponding to the 'right', 'upper', 'left' and 'bottom' part of S^1, respective, then $p|_{\tilde{U}}\colon \tilde{U} \to U$ is an isometry where S^1 is equipped with the metric defined by $\frac{\text{arc length}}{2\pi}$. Now we can cover S^1 by U_1, U_2, U_3, U_4 and use the Lebesgue lemma to find $\tilde{\sigma}(0) = t_0 < t_1 < \ldots < t_n = \tilde{\sigma}(1)$ such that $p \circ \tilde{\sigma}([t_{i-1}, t_i]) \subset U_{k_i}$ for all $i = 1, \ldots, n$. Then if d_{S^1} denotes the metric on S^1, and d the metric on $\mathbb{R}$, we have $t_i - t_{i-1} = d(t_{i-1}, t_i) = d_{S^1}(p \circ \tilde{\sigma}(t_i), p \circ \tilde{\sigma}(t_{i-1}))$. Hence we have
$$\tilde{\sigma}(1) - \tilde{\sigma(0)} = \sum d_{S^1}(p \circ \tilde{\sigma}(t_i), p \circ \tilde{\sigma}(t_{i-1})) = \sum d_{S^1}\left(\sigma \circ k^{-1}(t_i), \sigma \circ k^{-1}(t_{i-1})\right)$$
which in particular only depends on σ.

Remark. I am not completely sure whether the details work here, to be honest.

(ii) Equipped with addition, $(\mathbb{R}, +)$ becomes an abelian group, so lemma 5.1.2 directly gives that φ can be extended uniquely to $C_1\left(S^1\right) = \mathbb{Z}\operatorname{Sin}_1\left(S^1\right)$ (extended linearly).

(iii) It suffices to show it only for $\sigma \in \mathrm{Sin}_2\left(S^1\right)$. In this case,

$$\begin{aligned}\varphi \circ \partial\sigma &= \varphi\left(d_0\sigma - d_1\sigma + d_2\sigma\right) \\ &= \varphi d_0\sigma - \varphi d_1\sigma + \varphi d_2\sigma \\ &= d_1\tau_0 - d_0\tau_0 - (d_1\tau_1 - d_0\tau_1) + d_1\tau_2 - d_0\tau_2\end{aligned}$$

where $\tau_i \in \mathrm{Sin}_1(\mathbb{R})$ is such that $d_i\sigma = p \circ \tau_i$. Now, since

$$\begin{aligned} d_0\sigma(0,1) &= \sigma(0,0,1), \quad d_0\sigma(1,0) = \sigma(0,1,0) \\ d_1\sigma(0,1) &= \sigma(0,0,1), \quad d_1\sigma(1,0) = \sigma(1,0,0) \\ d_2\sigma(0,1) &= \sigma(0,1,0), \quad d_2\sigma(1,0) = \sigma(1,0,0), \end{aligned}$$

we can choose that $\tau_2 k^{-1}(1) = \tau_0 k^{-1}(0)$ and $\tau_2 k^{-1}(0) = \tau_1 k^{-1}(0)$; so $\tau_i k^{-1}$ gives a path lift of $d_i\sigma k^{-1}$. But $d_2\sigma k^{-1} * d_0\sigma k^{-1} \simeq d_1\sigma k^{-1} \operatorname{rel} I$ (by pulling across the singular 2-simplex), so in particular, $\tau_1 k^{-1}(1) = \tau_0 k^{-1}(1)$. But this gives

$$\begin{aligned} d_1\tau_0 - d_0\tau_0 - (d_1\tau_1 - d_0\tau_1) + d_1\tau_2 - d_0\tau_2 &= \tau_0(1,0) - \tau_0(0,1) - \tau_1(1,0) + \tau_1(0,1) + \tau_2(1,0) - \tau_2(0,1) \\ &= \tau_0 k^{-1}(0) - \tau_0 k^{-1}(1) - \tau_1 k^{-1}(0) + \tau_1 k^{-1}(1) + \tau_2 k^{-1}(0) - \tau_2 k^{-1}(1) \\ &= 0 \end{aligned}$$

Hence $\varphi \circ \partial = 0$. Thus $B_1(S^1) \subset \ker\varphi$ and hence φ factors through $H_1\left(S^1\right)$.

(iv) We have $\partial\sigma_0 = d_1\sigma_0 - d_0\sigma_0 = \sigma_0(1,0) - \sigma_0(0,1) = e^{2\pi i 0} - e^{2\pi i} = 1 - 1 = 0$, so $\sigma_0 \in Z_1\left(S^1\right)$. Hence $[\sigma_0] \in H_1\left(S^1\right)$.

(v) Define $\tau \in \mathrm{Sin}_1(\mathbb{R})$ by $\tau(t_0,t_1) = t_1$. Then $p \circ \tau(t_0,t_1) = e^{2\pi i t_1} = \sigma_0(t_0,t_1)$, so by definition, $\varphi(\sigma_0) = d_1\tau - d_0\tau = \tau(1,0) - \tau(0,1) = 0 - 1 = -1$.

Remark. Weird, I get it to be sent to -1. The definition of the maps δ^i in the notes is ambiguous, but I believe they are defined as $\delta^i = [e_1, \ldots, \hat{e_i}, \ldots, e^n]$ where $[v_1, \ldots, v_n]$ is the map $\Delta^n \to \mathbb{R}^n$ taking $\sum \lambda_j e_j \mapsto \sum \lambda_j v_j$.

$\square$