1. Orientations

We begin by attempting to give complete rigour and detail to the definitions of orientation and the many connected theorems.

For this section, we will follow [1] and [2]

Definition 1.1 (Local Homology Group)**.** For $h_*(-)$ a homology theory and an n-manifold M, groups of the form $h_k(M, M-\{x\})$ are called *local homology groups.*

For a chart $\varphi\colon U \to \mathbb{R}^n$ on M centered at x, we get by excision that

$$h_k(M, M-\{x\}) \cong h_k(U, U-\{x\}) \xrightarrow{\varphi_*} h_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}).$$

Hence for singular homology, we obtain $H_n(M, M-\{x\}; G) \cong G$.

Definition 1.2 (Local R-orientation)**.** Let R be a commutative ring. A generator of $H_n(M, M-\{x\}; R) \cong R$ is called a *local R-orientation* of M about x.

Let $K \subset L \subset M$. The homomorphism $r^L_K\colon h_k(M, M-L) \to h_k(M, M-K)$ induced by inclusion is called restriction. We write r^L_x when $K = \{x\}$.

Proposition 1.3. *When A is a compact, convex set contained in some chart $\mathbb{R}^n \subset M$, then r^A_x is an isomorphism for each $x \in A$ and the groups are isomorphic to the coefficient group G.*

Proof. A is contained in the interior of some closed n-disk $D \subset \mathbb{R}^n \subset M$. Thus there is a commutative diagram

$$\begin{tikzcd}
h_n(M, M-A) \arrow[r] & h_n(M, M-\{x\}) \\
h_n(\mathbb{R}^n, \mathbb{R}^n - A) \arrow[r] \arrow[u, "\cong"] & h_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \arrow[u, "\cong"] \\
h_n(D, \partial D) \arrow[r, equal] \arrow[u, "\cong"] & h_n(D, \partial D) \arrow[u, "\cong"]
\end{tikzcd}$$

□

Definition 1.4 (Orientation bundle)**.** We construct a covering $\omega\colon h_k(M, M-\bullet) \to M$. Define

$$h_k(M, M-\bullet) = \bigsqcup_{x \in M} h_k(M, M-\{x\})$$

where $h_k(M, M-\{x\})$ is the fiber over x and is given the discrete topology.
Let U be an open neighborhood of x such that r^U_y is an isomorphism for each $y \in U$. Define bundle charts

$$\varphi_{x,U}\colon U \times G \to \omega^{-1}(U), \quad (y, a) \mapsto r^U_y \left(r^U_x\right)^{-1}(a).$$

We then give $h_k(M, M-\bullet)$ the topology that makes $\varphi_{x,U}$ in a homeomorphism onto an open subset. In particular, since $h_k(M, M-x)$ is given the discrete topology, this is equivalent to the map $\varphi_{x,U}(-, \alpha)$ being a homeomorphism onto an open subset for each $\alpha \in h_k(M, M-x)$. It then remains to show that the transition maps

$$\varphi^{-1}_{y,V}\varphi_{x,U}\colon (U \cap V) \times h_k(M, M-\{x\}) \to (U \cap V) \times h_k(M, M-\{y\})$$

are continuous.
Let $z \in U \cap V$, and choose W such that $z \in W \subset U \cap V$ and r_w^W is an isomorphism for each $w \in W$.
Consider the diagram

$$\begin{tikzcd}
h_k(M, M-x) & h_k(M, M-U) \arrow[l, "r_x^U"'] \arrow[r, "r_w^U"] \arrow[d, "r_W^U"] & h_k(M, M-w) \\
 & h_k(M, M-W) \arrow[ur, "r_w^W"] & h_k(M, M-V) \arrow[l, "r_W^V"] \arrow[u, "r_w^V"'] \arrow[d, "r_y^V"] \\
 & & h_k(M, M-y)
\end{tikzcd}$$

Let $\varphi_{x,U,p} \colon h_k(M, M-x) \to \omega^{-1}(p)$ be defined by

$$\varphi_{x,U,p}(y) = \varphi_{x,U}(p, y).$$

Then for $w \in U \cap V$, we have

$$\varphi_{x,U,w}^{-1}\varphi_{y,V,w} = r_y^V(r_W^V)^{-1}(r_w^W)^{-1}r_w^W r_W^U(r_x^U)^{-1} = r_y^V(r_W^V)^{-1}r_W^U r_x^U$$

Firstly, this coincides with the operation of an element of the coefficient group G since it is an isomorphism $G \to G$, and secondly, note that this does not depend on w, so the map

$$g_{x,U,y,V} \colon U \cap V \to G$$

defined by $g_{x,U,y,V}(p) = \varphi_{x,U,p}^{-1}\varphi_{y,V,p}$ is constant, hence continuous.

Thus ω is indeed a covering map.
But even moreso, the fibers are groups, so for $A \subset M$, denote by $\Gamma(A)$ the set of continuous sections over A of ω. If s and t are section, we can define $(s+t)(a) = s(a) + t(a)$. Then $s + t$ is again continuous, hence $\Gamma(A)$ is an abelian group.
Denote by $\Gamma_c(A) \subset \Gamma(A)$ the subgroup of sections with compact support, i.e., the sections which have values 0 in the fiber away from a compact set.

Proposition 1.5. *Let $z \in h_k(M, M-U)$. Then $y \mapsto r_y^U z \in h_k(M, M-y) \subset h_k(M, M-\bullet)$ is a continuous section of ω over U.*

Proof. The map $U \to U \times G$ by $y \mapsto (y, r_x^U z)$ is constant in the second coordinate, hence clearly continuous. Now composing with $\varphi_{x,U}$ gives us the section in question. □

1.1. **Homological Orientation.** If we specify to singular homology with coefficient group R, and again let M be an n-manifold and $A \subset M$, then we can define an orientation along A as follows

Definition 1.6 (R-orientation of M along A)**.** An *R-orientation of M along A* is a section $s \in \Gamma(A; R)$ of $\omega \colon H_n(M, M-\bullet; R) \to M$ such that $s(a) \in H_n(M, M-a; R) \cong R$ is a generator for each $a \in A$.

Thus s glues together the local orientations in a continuous manner.
When $A = M$, we call s an *R-orientation of M* .

Definition 1.7 (Orientation covering)**.** Let $\mathrm{Ori}(M) \subset H_n(M, M - \bullet; \mathbb{Z})$ be the subset of all generators of all fibers. Then the restriction $\mathrm{Ori}(M) \to M$ of ω gives a 2-fold covering of M, called the *orientation covering* of M.

Proposition 1.8. *The following are equivalent:*

(1) *M is orientable*
(2) *M is orientable along compact subsets.*
(3) *The orientation covering is a trivial 2-fold covering map.*
(4) *The covering $\omega\colon H_n(M, M - \bullet; \mathbb{Z}) \to M$ is a trivial covering map.*

Proof. (1) $\implies$ (2) is a subcase.
(2) $\implies$ (3). The orientation covering is trivial if and only if the covering over each component is trivial, so we may assume that M is connected. Now, if a 2-fold covering $\tilde{M} \to M$ is trivial, then $\tilde{M}$ splits as $M \times \{p, q\}$, and so $\tilde{M}$ cannot be connected. Conversely, if $\tilde{M}$ is not connected, then the covering restricted to each component must be a covering map, so the covering splits as a trivial covering.
Suppose then that $\mathrm{Ori}(M) \to M$ is non-trivial. Since $\mathrm{Ori}(M)$ is then connected, we can choose a path γ in $\mathrm{Ori}(M)$ between two points of a given fiber. The image S of such a path is compact and connected, and the covering is non-trivial over S, so by assumption (2), the orientation covering has a section s over S, but then $\gamma(0) = s\left(\omega(\gamma(0))\right) = s\left(\omega\left(\gamma(1)\right)\right) = \gamma(1)$, which gives a contradiction.
(3) $\implies$ (4).
Let $s\colon M \to \mathrm{Ori}(M) \cong M \times \{-1, 1\}$ be the section $m \mapsto (m, 1)$.
Now define a map $\varphi\colon M \times \mathbb{Z} \to H_n(M, M - \bullet; \mathbb{Z})$ by $\varphi(m, k) = ks(m)$. This is a bijective map by assumption on s being a section. It is furthermore continuous since s is continuous and since fiber-wise operations in $H_n(M, M - \bullet; \mathbb{Z})$ is continuous. Furthermore, it is also a morphism between coverings since it commutes with the projections: $\pi_M = \omega \circ \varphi$.
Lastly, one must show that it also has a continuous inverse. For this, we may take an open basis set in $M \times \mathbb{Z}$ - say $U \times \{k\}$, where $\bar{U}$ is a convex subset of $\mathbb{R}^n \subset M$. Since φ is bijective, we obtain that $\varphi(U \times \{k\}) = ks(U) = U_\alpha$ if we choose α to be the element in $H_n(M, M - U) \cong \mathbb{Z}$ which maps to k under $r_{x,U}$ for $x \in U$. And by assumption, U_α is a basis open set for the topology on $H_n(M, M - \bullet; \mathbb{Z})$.
Hence φ is a homeomorphism, and even an isomorphism of covering spaces in the sense that $\pi_M = \omega \circ \varphi$.

Note. We could also say that it is trivial since every point is in the image of some section.

(4) $\implies$ (1) : If ω is trivial, then it has a section with constant value in the set of generators.

□

1.2. **Homology in the Dimension of the Manifold.** Let M be an n-manifold and $A \subset M$ a closed subset. We will in this section use singular homology with coefficients in an abelian group G.

Proposition 1.9. *For each $\alpha \in H_n(M, M - A; G)$, the section*

$$J^A(\alpha)\colon A \to H_n(M, M - \bullet; G), \quad x \mapsto r_x^A(\alpha)$$

of ω over A is continuous and has compact support.

Proof. Choose a representative $c \in \Delta_n(M;G)$ representing α. There exists a compact set K such that c is contained in K. Suppose $A-K$ is nonempty, and let $x \in A-K$. Then the image of c under

$$\Delta_n(K;G) \to \Delta_n(M;G) \to \Delta_n(M,K;G) \to \Delta_n(M,M-x;G)$$

is zero since $K \subset M-x$. Since this image represents r_x^A, the support of $J^A(\alpha)$ is contained in $A \cap K$ which is compact.
If $A-K$ is empty, K contains A, and then the support of $J^A(\alpha)$ is a closed subset of a compact space, hence compact.

The continuity follows from the more general case of Proposition 1.5. □

Thus we obtain a homomorphism

$$J^A \colon H_n(M,M-A;G) \to \Gamma_c(A;G), \quad \alpha \mapsto \left(x \mapsto r_x^A(\alpha)\right).$$

1.2.1. *Direct Limits.*

Definition 1.10 (Directed set)**.** A *directed set* D is a partially ordered set such that, for any two elements α and β of D, there is a $\tau \in D$ with $\tau \geq \alpha$ and $\tau \geq \beta$.

Definition 1.11. Let D be a directed set and G_α an abelian group defined for each $\alpha \in D$. Suppose we are given homomorphisms $f_{\beta,\alpha} \colon G_\alpha \to G_\beta$ for each $\beta > \alpha$ in D. Assume that for all $\gamma > \beta > \alpha$ in D, we have $f_{\gamma,\beta} f_{\beta,\alpha} = f_{\gamma,\alpha}$. Such a system is called a *direct system* of abelian groups. Then $G = \lim_\to G_\alpha$ is defined to be the quotient group of the direct sum $G = \bigoplus G_\alpha$ modulo the relations $f_{\beta,\alpha}(g) \sim g$ for all $g \in G_\alpha$ and all $\beta > \alpha$.

Note. Hence the direct limit is just the colimit of the direct system.

Proposition 1.12. *Suppose we are given an abelian group A with homomorphisms $h_\alpha \colon G_\alpha \to A$ such that the cocone commutes. Since $\lim_\to G_\alpha$ is the colimit, we have a unique induced homomorphism $h \colon \lim_\to G_\alpha \to A$. Then*

1. $\operatorname{im} h = \{a \in A \mid a = h_\alpha(g)$ *for some g and α*$\} = \bigcup \operatorname{im} h_\alpha$.
2. $\ker h = \{g \in \lim_\to G_\alpha \mid \exists \alpha$ *and* $g_\alpha \in G_\alpha \colon g = i_\alpha(g_\alpha)$ *and* $h_\alpha(g_\alpha) = 0\} = \bigcup i_\alpha(\ker h_\alpha)$.

Proof. Define $h(g_\alpha) = h_\alpha(g_\alpha)$. Then if $f_{\beta,\alpha}(g_\alpha) \sim g_\alpha$, we have $h\left(g_\alpha\right) = h_\alpha\left(g_\alpha\right) = h_\beta \circ f_{\beta,\alpha}(g_\alpha) = h\left(f_{\beta,\alpha}(g_\alpha)\right)$, so h respects the equivalence relations, thus it is well-defined.
Now property (1) is clear by the way we defined h.
As for (2), note that if g represents the equivalence class of g_α and $h(g) = 0$, then $h_\alpha\left(g_\alpha\right) = 0$ which is what (2) is saying. □

Corollary 1.13. *In the situation of Proposition 1.12, $h \colon \lim_\to G_\alpha \to A$ is an isomorphism if and only if the following two statements hold true:*

1. $\forall a \in A, \exists \alpha \in D$ *and* $g_\alpha \in G_\alpha \colon h_\alpha(g_\alpha) = a$, *and*
2. *if* $h_\alpha(g_\alpha) = 0$ *then* $\exists \beta > \alpha \colon f_{\beta,\alpha}(g_\alpha) = 0$.

Theorem 1.14. *The direct limit is an exact functor. So if we have direct systems $\{A'_\alpha\}, \{A_\alpha\}$ and $\{A''_\alpha\}$ based on the same directed set, and if we have an exact sequence $A'_\alpha \to A_\alpha \to A''_\alpha$ for each α, where the maps commute with the ones defining the direct systems, then the induced sequence*

$$\lim_\to A'_\alpha \to \lim_\to A_\alpha \to \lim_\to A''_\alpha$$

is exact.

Proof. We have the following diagram, where all maps commute.

$$\begin{tikzcd}
A'_\beta \arrow[r] \arrow[d] & A_\beta \arrow[r] \arrow[d] & A''_\beta \arrow[d] \\
\lim_\to A'_\alpha \arrow[r] & \lim_\to A_\alpha \arrow[r] & \lim_\to A''_\alpha
\end{tikzcd}$$

Suppose $a \in \lim_\to A_*$ is mapped to zero in $\lim_\to A''_*$. Then there exists $g \in \lim_\to A_\alpha$ such that there exists β and $g_\beta \in A_\beta$ such that $g = i_\beta(g_\beta)$ and $h_\beta(g_\beta) = 0$.
Recall here that h_β is a homomorphism $A_\beta \to \lim_\to A^{''}_*$ and i_β is the inclusion $G_\beta \to \lim_\to G_\alpha$.
By commutativity of the diagram, there then exists $k_\beta \in A'_\beta$ such that
$i_\beta\left(d_\beta(k_\beta)\right) = d_{\lim_\to} i^{'}_\beta(k_\beta)$. Hence the kernel is contained in the image.
Now suppose let $\tilde{k} = d_{\lim_\to}(k) \in \lim_\to A_*$.
Then $\tilde{k} = i_\beta\left(d(\overline{k})\right) = d_{\lim_\to} i'_\beta\left(\overline{k}\right)$ for some $\overline{k} \in A'_\beta$.
But now

$$d_{\lim_\to}(\tilde{k}) = d_{\lim_\to} i_\beta\left(d\left(\overline{k}\right)\right) = i''_\beta d\left(d\left(\overline{k}\right)\right) = i''_\beta(0) = 0.$$

□

Theorem 1.15. *Suppose we are given two directed sets D and E. Define an order on $D \times E$ by $(\alpha, \beta) \geq (\alpha', \beta')$ if and only if $\alpha \geq \alpha'$ and $\beta \geq \beta'$. Suppose $G_{\alpha,\beta}$ is a direct system based on $D \times E$. Then the maps $G_{\alpha,\beta} \to \lim_{\to,\beta} G_{\alpha,\beta} \to \lim_{\to,\alpha}\left(\lim_{\to,\beta} G_{\alpha,\beta}\right)$ induce an isomorphism*

$$\lim_{\to,\alpha,\beta} G_{\alpha,\beta} \stackrel{\cong}{\to} \lim_{\to,\alpha}\left(\lim_{\to,\beta} G_{\alpha,\beta}\right).$$

Proof. □

Proposition 1.16. (1) *For $A \supset B$ both closed, the following diagram commutes:*

$$\begin{tikzcd}
H_n(M, M-A; G) \arrow[r] \arrow[d, "J^A"] & H_n(M, M-B; G) \arrow[d, "J^B"] \\
\Gamma_c(A, H_n(M, M-\bullet; G)) \arrow[r] & \Gamma_c\left(B, H_n(M, M-\bullet; G)\right)
\end{tikzcd}$$

(2) *For $A, B \subset M$ both closed, the sequence*

$$0 \to \Gamma_c(A \cup B, H_n(M, M-\bullet; G)) \xrightarrow{h} \Gamma_c\left(A, H_n(M, M-\bullet; G)\right) \oplus \Gamma_c\left(B, H_n(M, M-\bullet; G)\right)$$
$$\xrightarrow{k} \Gamma_c\left(A \cap B, H_n\left(M, M-\bullet; G\right)\right)$$

is exact, where h is the sum of restrictions and k is the difference of restrictions.

(3) *If $A_1 \supset A_2 \supset A_3 \supset \dots$ are all compact and $A \bigcap A_i$, then the restriction homomorphisms $\Gamma\left(A_i, H_n\left(M, M-\bullet; G\right)\right) \to \Gamma\left(A, H_n\left(M, M-\bullet; G\right)\right)$ induce an isomorphism*

$$\lim_{\to} \Gamma\left(A_i, H_n\left(M, M-\bullet; G\right)\right) \stackrel{\cong}{\to} \Gamma\left(A, H_n(M, M-\bullet; G)\right)$$

Proof. (1) Let $\alpha \in H_n(M, M-A; G)$, and denote by ι the inclusion $(M, M-A) \hookrightarrow (M, M-B)$. Then $\iota_* = r^A_B$, so $J^B\left(r^A_B(\alpha)\right)(x) = r^B_x\left(r^A_B(\alpha)\right)$. On the other hand, $J^A(\alpha)|_B(x) = J^A(\alpha)(x) = r^A_x(\alpha)$. Now, from the composition

$$(M, M-A) \hookrightarrow (M, M-B) \hookrightarrow (M, M-x)$$

we obtain by taking homology, that $r^A_x = r^B_x r^A_B$, which gives the result.

(2) Firstly, a section that is zero on both A and B is then also zero on $A \cup B$, which gives the injective part of h. Now, suppose $s - t$ is the zero section over $A \cap B$ for s a section over A and t a section over B. Then s and t agree on $A \cap B$, meaning that $s \cup t$ is well-defined and continuous, where $s \cup t$ is s on A and t on B, and $h(s \cup t) = (s, t)$. Likewise, if g is a section over $A \cup B$, then $k \circ h(g) = (g|_A)|_{A\cap B} - (g|_B)|_{A\cap B} = g|_{A\cap B} - g|_{A\cap B}$ is the zero section.

(3)

□

Theorem 1.17. *Let $A \subset M$ be closed. Then*

1. $H_i(M, M-A; G) = 0$ *for* $i > n$.
2. $J^A\colon H_n(M, M-A, G) \to \Gamma_c(A, H_n(M, M-\bullet; G))$ *is an isomorphism.*

Lemma 1.18 (The Bootstrap Lemma)**.** *Let $P_M(A)$ be a statement about compact sets A in a given n-manifold M^n. If $(i), (ii), (iii)$ hold, then $P_M(A)$ is true for all compact A in M^n.*
If M^n is separable metric, and $P_M(A)$ is defined for all closed sets A, and if $(i), (ii), (iii), (iv)$ hold, then $P_M(A)$ is true for all closed sets A in M.
For general M^n, if $P_M(A)$ is defined for all closed sets A in M, for all M^n, and if all five statement $(i) - (v)$ hold for all M^n, then $P_M(A)$ is true for all closed $A \subset M$ and all M^n.

Now note that for a given abelian group G and $g \in G$, the following maps are natural in $A \subset M$ (closed):

$$H_n(M, M-A) \cong H_n(M, M-A) \otimes \mathbb{Z} \to H_n(M, M-A) \otimes G \to H_n(M, M-A; G)$$

where the middle map is induced by the homomorphism $\mathbb{Z} \to G$ taking 1 to g.
In particular, this induces a map

$$H_n(M, M-\bullet) \to H_n(M, M-\bullet; G)$$

Lemma 1.19. *The sections $\Gamma(A; G)$ of ω over A correspond bijectively to continuous maps $\lambda\colon \operatorname{Ori}(M)|_A \to G$ with the property $\lambda \circ t = -\lambda$, where t acts on G as multiplication by -1.*

Proof. We may assume A is connected.
Let $s \in \Gamma(A; G)$ be a section of ω over A. That is, $w \circ s = \mathrm{id}_A$, and s is a map $A \to H_n(M, M-\bullet; G)$. We can define an associated map $\lambda_s\colon \operatorname{Ori}(M)|_A \to G$ by sending a generator in the fiber $x \in A$ to $s(x) \in H_n(M, M-\{x\}; G) \cong G$. If one chose the other generator, one would get the negative of the above map, so we have the relation $\lambda_s \circ t = -\lambda_s$. Subject to this relation, we obtain a well-defined map $\Gamma(A; G) \to S \subset \operatorname{Hom}(\operatorname{Ori}(M)|_A, G)$, where S is the subset for which $\lambda \circ t = -\lambda$ holds. This map is certainly injective, since the image tells us precisely the value

of s at any point in A.
It is furthermore surjective, since if $\mathrm{Ori}(M)|_A$ is connected, then S can only consist of the zero section, and if it is not connected, it consists of a map on two components on which it is constant, and the relation $\lambda \circ t = -\lambda$ then determines that is must the required values to constitute the induced map of a section. □ check

Theorem 1.20. *Suppose $A \subset M$ is a closed connected subset. Then*

(1) $H_n(M, M - A; G) = 0$ *if A is not compact.*
(2) $H_n(M, M - A; G) \cong G$ *if M is R-orientable along A and A is compact. Moreover, $H_n(M, M - A; G) \to H_n(M, M - x; G)$ is an isomorphism for each $x \in A$.*
(3) $H_n(M, M - A; G) \cong {}_2G = \{g \in G \mid 2g = 0\}$ *if M is not orientable along A and A is compact.*

Proof. (1) By Lemma 6.1, a section in $\Gamma(A; G)$ is determined by its value at a single point. By the existence of the zero section, if a section is non-zero at any point, then it is non-zero at every point. Therefore, there do not exist non-zero sections with compact support over a non-compact A, so by Theorem 1.17, $H_n(M, M - A; G) \cong \Gamma_c(A; G) \cong 0$.

(2) Since A is compact, $H_n(M, M - A; G) \cong \Gamma_c(A; G) = \Gamma(A; G)$. A section is again determined by a single point. Recall now the commutative diagram

$$\begin{tikzcd} H_n(M, M - A; G) \arrow[r, "\cong"] \arrow[d, "r_x^A"] & \Gamma(A; G) \arrow[d, "b"] \\ H_n(M, M - x; G) \arrow[r, "\cong"] & \Gamma(\{x\}; G) \end{tikzcd}$$

from Proposition 1.16, the horizontal isomorphisms following from Theorem 1.17. If M is orientable along A, there by definition exists in $\Gamma(A; G)$ an element such that its value at x is a generator. Hence b is an isomorphism, and therefore also r_x^A is an isomorphism.

(3) By Lemma 1.19, a section in $\Gamma(A; G)$ corresponds to a continuous map $\lambda\colon \mathrm{Ori}(M)|_A \to G$ with $\lambda t = -\lambda$. If M is not orientable along A, then $\mathrm{Ori}(M)|_A$ is connected and therefore λ is constant as G has the discrete topology. The relation $\lambda t = -\lambda$ shows that λ is in ${}_2G$. Now by the commutative diagram from part (2), note that since λ must be constant, firstly $\Gamma(A; G) \cong {}_2G$, and secondly,b becomes injective, so $r_x^A\colon H_n(M, M - A; G) \to H_n(M, M - x; G) \cong G$ is injective and has image ${}_2G$, so the Hom term vanishes. □

Proposition 1.21. *Let M be an n-manifold and $A \subset M$ be a closed connected subset. Then the torsion subgroup of $H_{n-1}(M, M - A; \mathbb{Z})$ is of order 2 if A is compact and M non-orientable along A, and is 0 otherwise.*

Proof. By UCT for homology,

$$\begin{aligned} \mathbb{Z}/2 \cong {}_2\mathbb{Z}/2 \cong H_n(M, M - A; \mathbb{Z}/2) &\cong H_n(M, M - A) \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}_1\left(H_{n-1}(M, M - A), \mathbb{Z}/2\right) \\ &\cong \mathrm{Tor}_1\left(H_{n-1}(M, M - A), \mathbb{Z}/2\right) \\ &\cong \{g \in H_{n-1}(M, M - A) \mid 2g = 0\}. \end{aligned}$$

where $H_n(M, M-A) \cong {}_2\mathbb{Z} = 0$, and $H_n(M, M-A; \mathbb{Z}/2) \cong {}_2\mathbb{Z}/2 \cong \mathbb{Z}/2$ both follow from Theorem 1.20.
To see that this is the whole torsions subgroup, note that for odd k,

$$\mathrm{Tor}_1\left(H_{n-1}(M, M-A), \mathbb{Z}/k\right) \cong H_n(M, M-A; \mathbb{Z}/k) \cong {}_2\mathbb{Z}/k \cong 0$$

When M is orientable along A and A is compact, we simply obtain

$$0 \to H_n(M, M-A) \otimes \mathbb{Z}/n \to H_n(M, M-A; \mathbb{Z}/n) \to \mathrm{Tor}_1\left(H_{n-1}(M, M-A), \mathbb{Z}/n\right) \to 0$$

and since $H_n(M, M-A) \cong \mathbb{Z}$ and $H_n(M, M-A; \mathbb{Z}/n) \cong \mathbb{Z}/n$ by Theorem 1.20, we find that Tor_1 vanishes for all n.
If A is non-compact, then Theorem 1.20 gives that Tor_1 trivially vanishes for all terms.

□

1.3. Fundamental Class.

Theorem 1.22. *Let M be a compact connected n-manifold. Then one of the following assertions holds:*

1. *M is orientable, $H_n(M) \cong \mathbb{Z}$, and for each $x \in M$, the restriction $H_n(M) \to H_n(M, M-x)$ is an isomorphism.*
2. *M is non-orientable and $H_n(M) = 0$.*

Proof. Special case of Theorem 1.20. □

Under the hypothesis of Theorem 1.22, the orientations of M correspond to the generators of $H_n(M)$. Such a generator will be called a *fundamental class* or *homological class/orientation* of the orientable manifold.

Definition 1.23 (Degree)**.** Let M and N be compact oriented n-manifolds. Let N be connected and suppose M has components $M_1, \ldots, M_r$. Then we have fundamental classes $z(M_j)$ for each M_j and $z(M) \in H_n(M) \cong \bigoplus_j H_n(M_j)$ is the sum of the $z(M_j)$. Now, since $H_n(N) \cong \langle z(N) \rangle \cong \mathbb{Z}$, we obtain that there exists a *degree* $d(f) \in \mathbb{Z}$ such that $f_* z(M) = d(f) z(N)$.

Lemma 1.24 (Properties)**.**
1. *The degree is a homotopy invariant.*
2. *$d(f \circ g) = d(f)d(g)$.*
3. *A homotopy equivalence has degree ± 1.*
4. *If $M = M_1 \bigsqcup M_2$, then $d(f) = d\left(f|_{M_1}\right) + d\left(f|_{M_2}\right)$.*
5. *If we pass in M or N to the opposite orientation, then the degree changes the sign.*

1.3.1. *Computations of degrees.* As usual, we can compute degrees in terms of local data of a map.
Let M and N be connected and set $K = f^{-1}(p)$. Let U be an open neighborhood of K in M. Then in particular $M - U = \overline{M - U} \subset \mathrm{int}(M - A) = M - A$, so excision gives the bottom left isomorphism in the following diagram, and the top right isomorphism follows from Theorem 1.22:

$$\begin{tikzcd}
z(M) \in \arrow[dd, mapsto] & H_n(M) \arrow[r, "f_*"] \arrow[d] & H_n(M) \arrow[d, "\cong"] & \ni z(N) \arrow[dd, mapsto] \\
 & H_n(M, M-K) \arrow[r, "f_*"] & H_n(N, N-p) & \\
z(U,K) \in & H_n(U, U-K) \arrow[r, "f_*^U"] \arrow[u, "\cong"', "i_*"] & H_n(N, N-p) \arrow[u, "="] & \ni z(N,p)
\end{tikzcd}$$

From the outer rectangle, we get $f_*^U z(U,K) = d(f)z(N,p)$, where $z(N,p)$ and $z(U,K)$ are the images of $z(N)$ and $z(M)$ under the indicated maps.
We want to show additivity of degree as in the case for spheres.
So suppose K if finite, and choose $U = \bigcup_{x\in K} U_x$ where the U_x are pair-wise disjoint open neighborhoods of x. Then

$$\bigoplus_{x\in K} H_n(U_x, U_x - x) \cong H_n(U, U-K), \quad H_n(U_x, U_x - x) \cong \mathbb{Z}.$$

The image $z(U_x, x)$ of $z(M)$ is a generator: it is the image under the following isomorphisms

$$H_n(M) \xrightarrow{\cong} H_n(M, M-x) \xrightarrow{\cong} H_n(U_x, U_x - x)$$

where the first follows from Theorem 1.22 and the second from excision. The local degree $d(f,x)$ is determined by $f_* z(U_x, x) = d(f,x)z(N,p)$, and and by additivity above, we have $d(f) = \sum_{x\in K} d(f,x)$.

Proposition 1.25. *Let M be a connected, oriented, closed n-manifold. Then there exists for each $k \in \mathbb{Z}$ a map $f\colon M^n \to S^n$ of degree k.*

Remark. If f is C^1 in a neighborhood of x, then $d(f,x)$ is the sign of the determinant of $Dg(0)$ when $Dg(x)$ is regular, where g is f in local coordinates that preserve local orientations. By this we mean that for $\varphi\colon U_x \to \mathbb{R}^n$ centered at x, $\varphi_*\colon H_n(U_x, U_x - x) \to H_n(\mathbb{R}^n, \mathbb{R}^n - 0)$ sends $z(U_x, x)$ to the standard generator. Such charts are called *positive* with respect to the given orientations.

Proof. If $f\colon M \to S^n$ has degree a and $g\colon S^n \to S^n$ degree b, then gf has degree ab. Since the proposition is true for $M = S^n$, it suffices to find f having degree ± 1. Let $\varphi\colon D^n \to M$ be an embedding. Then we have a map $f\colon M \to D^n/S^{n-1}$ which is the inverse of φ on $U = \varphi(\operatorname{int} D^n)$ and sends $M - U$ to the basepoint. This map has degree ± 1 as can be seen by choosing any neighborhood of x in the interior of U and looking at the determinant of the differential locally. □

1.4. **Manifolds with Boundary.**

Definition 1.26. For M an n-dimensional manifold with boundary, we call $z \in H_n(M, \partial M)$ a *fundamental class* if for each $x \in M - \partial M$, the restriction of z is a generator in $H_n(M, M-x)$.

Theorem 1.27. *Let M be a compact connected n-manifold with non-empty boundary. Then one of the following assertions hold:*

1. *$H_n(M, \partial M) \cong \mathbb{Z}$, and a generator of this group is a fundamental class. The image of a fundamental class under $\partial\colon H_n(M, \partial M) \to H_{n-1}(\partial M)$ is a fundamental class. The interior $M - \partial M$ is orientable.*
2. *$H_n(M, \partial M) = 0$, and $M - \partial M$ is not orientable.*

Proof. See [2, Thm 16.5.1]. We will follow that proof and only add a few extra words.
Let $\kappa\colon [0,\infty[\times\partial M \to U$ be a collar of M, i.e., a homeomorphism onto an open neighborhood U of ∂M such that $\kappa(0,x) = x$ for $x \in \partial M$ (See Milnor's h-cobordism book for existence). For simplicity of notation, we identify U with $[0,\infty)\times\partial M$ via κ ; similarly, for subsets of U. In this sense, $\partial M = 0\times\partial M$. For $A = M - ([0,1)\times\partial M) \subset M - \partial M$, we have isomorphisms

$$H_n(M,\partial M) \cong H_n(M,[0,1)\times\partial M) \cong H_n(M-\partial M,(0,1)\times\partial M) \cong \Gamma(A).$$

The first one by homotopy equivalence, the second by excision, and the third using Theorem 1.17.
Since A is connected, $\Gamma(A) \cong \mathbb{Z}$ or $\Gamma(A) \cong 0$. If $\Gamma(A) \cong \mathbb{Z}$, then $M - \partial M$ is orientable along A.
Let now $A_\varepsilon \cong A$ be the complement of $[0,\varepsilon)\times\partial M$. Since each compact subset of $M - \partial M$ is contained in some such M_ε for small enough ε, we see that $M - \partial M$ is orientable along all compact subsets, hence orientable by Proposition 1.8.
The isomorphism $H_n(M-\partial M,(0,1)\times\partial M) \cong \Gamma(A)$ from above says that there exists some $z \in H_n(M-\partial M,(0,1)\times\partial M)$ which restricts to a generator of $H_n(M-\partial M, M-\partial M-x)$ for each $x \in A$. For the corresponding element $z \in H_n(M,\partial M) \cong \mathbb{Z}$, the same assertion holds for any $x \in M - \partial M$ (simply shrink the collar to not contain x).
Lastly, we must show that ∂z is a fundamental class. Let $x \in (0,1)\times\partial M$. Consider the diagram:

$$\begin{tikzcd}
H_{n-1}(\partial M) \arrow[r, "\cong"] & H_{n-1}(\partial M \cup A, A) & H_{n-1}(\partial I \times \partial M, 1\times\partial M) \arrow[l, "\cong"'] \\
H_n(M,\partial M) \arrow[u, "\partial"] \arrow[r] \arrow[dr] & H_n(M,\partial M\cup A) \arrow[u, "\cong", "\partial"'] \arrow[d] & H_n(I\times\partial M, \partial I\times\partial M) \arrow[l, "\cong"'] \arrow[u, "\cong", "\partial"'] \\
& H_n(M, M-x) & H_n(I\times\partial M, I\times\partial M - x) \arrow[l, "\cong"']
\end{tikzcd}$$

Commutativity of the bottom left triangle tells us that the image of z under $H_n(M,\partial M) \to H_n(M,\partial M\cup A)$ gives an element whose restriction gives a generator in $H_n(M, M-x)$, but then by commutativity of the bottom right square, we get that the restriction of z transferred over by the isomorphism to $H_n(I\times\partial M,\partial I\times\partial M)$ is a generator of $H_n(I\times\partial M, I\times\partial M - x)$ at each point in $(0,1)\times\partial M$. Hence z yields a fundamental class in $H_n(I\times\partial M,\partial I\times\partial M)$.
But since z is a generator in $H_n(I\times\partial M,\partial I\times\partial M)$, the upper part shows that z is a generator in $H_{n-1}(\partial M)$, thus a fundamental class of ∂M since this characterizes fundamental classes.

□

Example 1.28. Suppose that $B\colon M \to \varnothing$ is a cobordism. We have the fundamental classes $z(B) \in H_{n+1}(B,\partial B)$ and $z(M) = \partial z_B \in H_n(M)$ (here we crucially made use of our result in Theorem 1.27). This is already a lot of information. Indeed, suppose $f\colon M \to N$ is a map which has an extension to B : $F\colon B \to N$. Then the degree of f (if defined) is zero, $d(f) = 0$, for we have $f_* z(M) = f_*\partial z(B) = F_* i_*\partial z(B) = 0$, since $i_*\partial = 0$ by the exactness of the homology sequence for the pair (B,M).

We call maps $f_\nu\colon M_\nu \to N$ *orientable bordant* if there exists a compact oriented cobordism $B\colon M_1 \to M_2$ with orientable boundary $\partial B = M_1 - M_2$ (meaning $\partial z(B) = z(M_1) - z(M_2)$) and an extension $F\colon B \to N$ of $f_1 \sqcup f_2 \colon M_1 \bigsqcup M_2 \to N$. Under these assumptions, we have $d(f_1) = d(f_2)$. This fact is called the *bordism invariance* of the degree; it generalizes the homotopy invariance.

2. Duality

Let M^n be orientable and $\vartheta_M \in \Gamma\left(M, H_n(M, M - \bullet)\right)$ an orientation.
For $K \subset M$ compact, ϑ_M restricts to to $\vartheta_K \in \Gamma\left(K, H_n(M, M-\bullet)\right) = \Gamma_c(K, H_n(M, M-\bullet)) \cong H_n(M, M-K)$, so we can regard ϑ_K as lying in $H_n(M, M-K)$. Let $\vartheta = \{\vartheta_K\}$ be the collection of all these, and we then call ϑ an orientation.

Definition 2.1. For sets $L \subset K \subset M$, we define

$$\check{H}^p(K, L; G) = \lim_{\to} \left\{ H^p(U, V; G) \mid (U, V) \supset (K, L), U, V \text{ open} \right\}.$$

This is a directed system since if (U, V) and (U', V') both contain (K, L), the $(U \cap U', V \cap V')$ also contains (K, L), and the maps induced by inclusions of nested open sets satisfy the required relation to be a directed system.

This group is naturally isomorphic to that of Cech cohomology. If K and L are spaces such as ENRs (e.g., CW-complexes or topological manifolds), then this is also naturally isomorphic to singular cohomology.

2.0.1. *Construction of the duality map.* Suppose $(K, L) \subset (U, V)$ as above (so $L \subset K$). Then note that since $K \subset U$, we have $M - U = \overline{M - U} \subset \int(M - K) = M - K$, so by excision, $H_{n-p}(U - L, U - K) \cong H_{n-p}(M - L, M - K)$. Also, $\{V, U - L\}$ is an open cover of U, hence $H_* \left(\frac{\Delta_*(V) + \Delta_*(U-L)}{\Delta_*(U-K)} \right) \cong H_*\left(U, U - K\right) \cong H_*(M, M - K)$ where the first isomorphism follows from Theorem 7.2.2 in Algtop1, and the latter follows from excision.
Now, we have a well-defined cap product

$$\Delta^p\left(U, V; G\right) \otimes \left[\frac{\Delta_n(V) + \Delta_n(U - L)}{\Delta_n(U - K)} \right] \xrightarrow{\cap} \Delta_{n-p}\left(U - L, U - K; G\right)$$

given by $f \cap (b + c) = f \cap b + f \cap c = f \cap c$, which by the above, induces a cap product

$$H^p(U, V; G) \otimes H_n(M, M - K) \to H_{n-p}(M - L, M - K; G)$$

which is natural in (K, L).
Using the same theorem from Algtop1, we find that for $\gamma \in H_n(M, M - A)$, we can represent γ by a chain $b + c + d \in \Delta_n(V) + \Delta_n(U - L) + \Delta_n(M - K)$, so for $f \in \Delta^p(U, V; G)$, we get that

$$[f] \cap \gamma = [f \cap (b + c + d)] = [f \cap c] \in H_{n-p}(M - L, M - K; G)$$

since $f \cap b = 0$ as f vanishes on V and $f \cap d$ is a chain in $M - K$.
Thus by capping with ϑ_A for A large enough to contain K, we obtain a homomorphism

$$\cap\vartheta\colon H^p\left(U, V; G\right) \to H_{n-p}(M - L, M - K; G).$$

Now, recall that the direct limit used for $\check{H}$ has the universal property that it is the colimit under the direct system. So if the above map $\cap\vartheta$ is compatible with the direct system, then we will obtain an induced homomorphism from $\check{H}$.

3. Intersection Theory

Definition 3.1 (k-disk bundle)**.** A k-disk bundle is a vector bundle whose coordinate transformations are contained in $O(k) \subset \mathrm{GL}(\mathbb{R}^k)$ and such that the local trivializations have the form $\pi^{-1}(U) \cong U \times D^k$.

Let N^n be a connected, oriented, closed n-manifold, and W^{k+n} an $(n+k)$-manifold with boundary ∂W a $(k-1)$-sphere bundle over N^n, and let $\pi\colon W^{n+k} \to N^n$ be a k-disk bundle over N.
Let us assume also that W is also oriented.

Definition 3.2. In the above situation, the *Thom class* of the disk bundle π is the class $\tau \in H^k\left(W, \partial W\right)$ given by

$$\tau = D_W\left(i_*\left[N\right]\right)$$

where $D_W : H_{n-k}(W) \to H^k(W, \partial W)$ is the inverse of the Poincaré duality isomorphism. That is,

$$D(a) \cap [M] = a.$$

Thus

$$\tau \cap [W] = i_*\left[N\right].$$

We can deformation retract the punctured disk to its boundary, giving $H^k(W, W - N) \cong H^k(W, \partial W)$, so we will sometimes regard τ as being in $H^k(W, W - N)$.

Lemma 3.3. *In the above setup, suppose $A \subset N$ is closed. Let $\tilde{A} = \pi^{-1}(A) \subset W$ and $\partial\tilde{A} = \tilde{A} \cap \partial W$. Then $\check{H}^i\left(\tilde{A}, \partial\tilde{A}\right) = 0$ for $0 < i < k$.*

Proof. Suppose first that A is compact convex subset of a Euclidean neighborhood in N. It also suffices consider the case where A is connected, so $A \cong D^n$. Consider the pullback bundle of A :

$$\begin{tikzcd} i^*(A) \arrow[r] \arrow[d] & W \arrow[d, "\pi"] \\ A \arrow[r, hook, "i"] & N \end{tikzcd}$$

Then $i^*(A) = A \times_N W \cong \pi^{-1}(A)$, so since any vector bundle over a contractible paracompact base space is trivial, we conclude that the bundle $\tilde{A} \to A$ is trivializable as $\tilde{A} \cong A \times D^k$ and $\partial\tilde{A} \cong A \times S^{k-1}$. Now the steps are as follows: calculate the homology of $A \times D^k$ and $A \times S^{k-1}$, then use UCT to obtain the cohomology, and then use the LES to find the cohomology of $\left(A \times D^k, A \times S^{k-1}\right)$.
Now... But by the Künneth theorem,

$$H_m(A \times D^k) \cong H_m(A)$$

and

$$H_m(A \times S^{k-1}) \cong H_m(A) \oplus H_{m-k+1}(A).$$

□

Lemma 3.4. *The restriction $\tau_x \in \check{H}^k(\tilde{A}, \partial\tilde{A})$ of τ, when $A = \{x\}$, is a generator.*

Proof. Note that $\left(\tilde{A}, \partial\tilde{A}\right) \cong \left(D^k, S^{k-1}\right)$.

Suppose first that $\tau_x = 0$ for some x. Let $i\colon \tilde{A} \hookrightarrow W$ be the inclusion, then we have

$$0 = i_*(0) = i_*\left(\tau_x \cap \beta\right) = \tau \cap i_*\left(\beta\right),$$

for all $\beta \in H_*\left(\tilde{A}, \partial \tilde{A}\right)$.
Let U be a neighborhood around x which is evenly covered, so $\pi^{-1}(U) \cong U \times D^k$, and $\tilde{A} = \pi^{-1}(x) \cong \{x\} \times D^k$. Let $i_x \colon \{x\} \to \{x\} \times D^k$ be the zero section, and similarly for $i_y \colon \{y\} \to \{y\} \times D^k$. Let $\gamma \colon I \to U$ be a path from x to y. Then we can define a path $F \colon I \to U \times D^k$ by $F(t) = (\gamma(t), 0)$. Then $F(t) = i_{\gamma(t)}(\gamma(t))$.
We have a pullback square as follows:

$$\begin{tikzcd}
(X, X') \arrow[r] \arrow[d, "q"] & (W, W') \arrow[d, "p"] \\
I \arrow[r] & B
\end{tikzcd}$$

Then we have an isomorphism induced by inclusions:

$$w_\# \colon H_n\left(F_c, F_c'\right) \xrightarrow{\cong} H_n(X, X') \xleftarrow{\cong} H_n(F_b, F_b')$$

In particular, $i_{c*} = i_{b*} w_\#$, so if $i_{c*}\left[x\right] = 0$, then $i_{b*}\left[y\right] = i_{b*} w_\# \left[x\right] = \pm i_{c*}\left[x\right] = 0$. Thus $\tau_y = 0$ for all y near x. Since N is connected, this implies that $\tau_y = 0$ for all $y \in N$.

For closed sets $A \subset N$, let $P_N(A)$ be the statement that $\tau_A = 0$, where $\tau_A = \tau|_{(\tilde{A}, \partial \tilde{A})}$.
Suppose now that A is a convex set in some euclidean open set in N. Then $\tilde{A}$ is also convex, so $\tau|_A \in \check{H}^k(\tilde{A}, \partial \tilde{A})$.
We have that the restriction defines an isomorphism

$$H^n\left(W, W - \tilde{A}\right) \to H^n\left(W, W - x\right)$$

for any point $x \in \tilde{A}$. Now, $\tilde{A}$ is closed, so $W - \tilde{A}$ is open, and $\overline{W - \tilde{A}} \subset \int$

$\square$

4. Thom-Pontryagin Theory

We start with an element $[f] \in \pi_{n+k}\left(S^n\right)$, so f is a pointed map $S^{n+k} \to S^n$.
Now insert a disk in place of the base point, and extend f to a map $\overline{f}$ which is constant on the next disk, taking the disk to the basepoint of S^n, and is f elsewhere. There is a deformation retract of the sphere, collapsing this disk to a point, and composing with this retract gives f. Hence we may replace f by a pointed-homotopic map which is constant in a small neighborhood of the basepoint.
Next, we can remove the base point of S^{n+k} and instead consider f as a map $\mathbb{R}^{n+k} \to S^n$ which is now constant to the base point outside some compact subset of $\mathbb{R}^{n+k}$.

By the Smooth Approximation Theorem, we can also restrict attention to smooth maps $\mathbb{R}^{n+k} \to S^n$ and smooth homotopies. Insert theorem
We regard also S^n as the one-point compactification of $\mathbb{R}^n$, denoted $\mathbb{R}^n_+ = \mathbb{R}^n \cup \{\infty\}$.
So suppose now we have a smooth map $f \colon \mathbb{R}^{n+k} \to \mathbb{R}^n_+$ as above.

If f is not null-homotopic, then it must be surjective, hence in particular the image does not have measure 0, so there exists a regular value $p \in \mathbb{R}^n \subset \mathbb{R}^n_+$. By following f by a translation in $\mathbb{R}^n$, we can assume that p is the origin $0 \in \mathbb{R}^n$ without changing the homotopy class of f.

Theorem 4.1 ([1], Thm 11.6)**.** *Let $f\colon \mathbb{R}^n \to M^m$ be a smooth map. Assume that $p \in M^m$ is a regular value, let $K = f^{-1}\{p\}$, and assume that K is compact. Then there is an open neighborhood N of K inside a tubular neighborhood of K, with normal retraction $r\colon N \to K$,and an open neighborhood $E \cong \mathbb{R}^m$ of p in M^m such that the map $r \times f\colon N \to K \times E$ is a diffeomorphism.*

Using Theorem 4.1, we find that there is a disk E^n about 0 in $\mathbb{R}^n$ and an embedding $M^k \times E^n \hookrightarrow N \subset \mathbb{R}^{nk}$ onto an open neighborhood N of M^k whose inverse $N \to M^k \times E^n$ is $r \times f$, where $r\colon N \to M^k$ is the normal retraction.

Through another homotopy of f, we can assume that E^n is the open unit disk D^n.

We will refer to an embedding $g\colon M^k \times E^n \to \mathbb{R}^{n+k}$, with M^k compact, as a "fattened k-manifold".

5. Terminology

Definition 5.1 (Neighborhood retract)**.** If $A \subset X$ and A has a neighborhood in X of which it is a retract, then A is called a *neighborhood retract* (in X).

Note. Saying that $A \hookrightarrow X$ is a cofibration is stronger than saying that A is a neighborhood retract.

6. Lemmas

Lemma 6.1. *Let $\pi\colon W \to N$ be a covering map and M a connected space. Suppose $f, g\colon M \to W$ are maps such that $\pi \circ f = \pi \circ g$ and that $f(x) = g(x)$ for some $x \in M$. Then $f = g$.*

Proof. Show that the set
$$Z = \{z \in M \mid f(z) = g(z)\}$$
is closed and open. □

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