1. Orientations

We begin by attempting to give complete rigour and detail to the definitions of orientation and the many connected theorems.

For this section, we will follow [1] and [2]

Definition 1.1 (Local Homology Group)**.** For $h_*(-)$ a homology theory and an n-manifold M, groups of the form $h_k(M, M-\{x\})$ are called *local homology groups.*

For a chart $\varphi\colon U \to \mathbb{R}^n$ on M centered at x, we get by excision that

$$h_k(M, M-\{x\}) \cong h_k(U, U-\{x\}) \xrightarrow{\varphi_*} h_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}).$$

Hence for singular homology, we obtain $H_n(M, M-\{x\}; G) \cong G$.

Definition 1.2 (Local R-orientation)**.** Let R be a commutative ring. A generator of $H_n(M, M-\{x\}; R) \cong R$ is called a *local R-orientation* of M about x.

Let $K \subset L \subset M$. The homomorphism $r_K^L\colon h_k(M, M-L) \to h_k(M, M-K)$ induced by inclusion is called restriction. We write r_x^L when $K = \{x\}$.

Proposition 1.3. *When A is a compact, convex set contained in some chart $\mathbb{R}^n \subset M$, then r_x^A is an isomorphism for each $x \in A$ and the groups are isomorphic to the coefficient group G.*

Proof. A is contained in the interior of some closed n-disk $D \subset \mathbb{R}^n \subset M$. Thus there is a commutative diagram

$$\begin{tikzcd}
h_n(M, M-A) \arrow[r] & h_n(M, M-\{x\}) \\
h_n(\mathbb{R}^n, \mathbb{R}^n - A) \arrow[r] \arrow[u, "\cong"] & h_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \arrow[u, "\cong"] \\
h_n(D, \partial D) \arrow[r, equal] \arrow[u, "\cong"] & h_n(D, \partial D) \arrow[u, "\cong"]
\end{tikzcd}$$

∎

Definition 1.4 (Orientation bundle)**.** We construct a covering $\omega\colon h_k(M, M-\bullet) \to M$. Define

$$h_k(M, M-\bullet) = \bigsqcup_{x\in M} h_k(M, M-\{x\})$$

where $h_k(M, M-\{x\})$ is the fiber over x and is given the discrete topology.
Let U be an open neighborhood of x such that r_y^U is an isomorphism for each $y \in U$.
Define bundle charts

$$\varphi_{x,U}\colon U \times G \to \omega^{-1}(U), \quad (y, a) \mapsto r_y^U \left(r_x^U\right)^{-1}(a).$$

We then give $h_k(M, M-\bullet)$ the topology that makes $\varphi_{x,U}$ in a homeomorphism onto an open subset. In particular, since $h_k(M, M-x)$ is given the discrete topology, this is equivalent to the map $\varphi_{x,U}(-, \alpha)$ being a homeomorphism onto an open subset for each $\alpha \in h_k(M, M-x)$. It then remains to show that the transition maps

$$\varphi_{y,V}^{-1}\varphi_{x,U}\colon (U \cap V) \times h_k(M, M-\{x\}) \to (U \cap V) \times h_k(M, M-\{y\})$$

are continuous.
Let $z \in U \cap V$, and choose W such that $z \in W \subset U \cap V$ and r_w^W is an isomorphism for each $w \in W$.
Consider the diagram

$$\begin{tikzcd}
h_k(M, M-x) & h_k(M, M-U) \arrow[l, "r_x^U"'] \arrow[r, "r_w^U"] \arrow[d, "r_W^U"] & h_k(M, M-w) \\
& h_k(M, M-W) \arrow[ur, "r_w^W"] & h_k(M, M-V) \arrow[l, "r_W^V"] \arrow[u, "r_w^V"'] \arrow[d, "r_y^V"] \\
& & h_k(M, M-y)
\end{tikzcd}$$

Let $\varphi_{x,U,p} \colon h_k(M, M-x) \to \omega^{-1}(p)$ be defined by

$$\varphi_{x,U,p}(y) = \varphi_{x,U}(p, y).$$

Then for $w \in U \cap V$, we have

$$\varphi_{x,U,w}^{-1} \varphi_{y,V,w} = r_y^V (r_W^V)^{-1} (r_w^W)^{-1} r_w^W r_W^U (r_x^U)^{-1} = r_y^V (r_W^V)^{-1} r_W^U r_x^U$$

Firstly, this coincides with the operation of an element of the coefficient group G since it is an isomorphism $G \to G$, and secondly, note that this does not depend on w, so the map

$$g_{x,U,y,V} \colon U \cap V \to G$$

defined by $g_{x,U,y,V}(p) = \varphi_{x,U,p}^{-1} \varphi_{y,V,p}$ is constant, hence continuous.

Thus ω is indeed a covering map.
But even moreso, the fibers are groups, so for $A \subset M$, denote by $\Gamma(A)$ the set of continuous sections over A of ω. If s and t are section, we can define $(s+t)(a) = s(a) + t(a)$. Then $s + t$ is again continuous, hence $\Gamma(A)$ is an abelian group.
Denote by $\Gamma_c(A) \subset \Gamma(A)$ the subgroup of sections with compact support, i.e., the sections which have values 0 in the fiber away from a compact set.

Proposition 1.5. *Let $z \in h_k(M, M-U)$. Then $y \mapsto r_y^U z \in h_k(M, M-y) \subset h_k(M, M-\bullet)$ is a continuous section of ω over U.*

Proof. The map $U \to U \times G$ by $y \mapsto (y, r_x^U z)$ is constant in the second coordinate, hence clearly continuous. Now composing with $\varphi_{x,U}$ gives us the section in question. □

1.1. **Homological Orientation.** If we specify to singular homology with coefficient group R, and again let M be an n-manifold and $A \subset M$, then we can define an orientation along A as follows

Definition 1.6 (R-orientation of M along A)**.** An *R-orientation of M along A* is a section $s \in \Gamma(A; R)$ of $\omega \colon H_n(M, M - \bullet; R) \to M$ such that $s(a) \in H_n(M, M - a; R) \cong R$ is a generator for each $a \in A$.

Thus s glues together the local orientations in a continuous manner.
When $A = M$, we call s an *R-orientation of M*.

Definition 1.7 (Orientation covering)**.** Let $\mathrm{Ori}(M) \subset H_n(M, M - \bullet; \mathbb{Z})$ be the subset of all generators of all fibers. Then the restriction $\mathrm{Ori}(M) \to M$ of ω gives a 2-fold covering of M, called the *orientation covering* of M.

Proposition 1.8. *The following are equivalent:*

1. *M is orientable*
2. *M is orientable along compact subsets.*
3. *The orientation covering is a trivial 2-fold covering map.*
4. *The covering $\omega\colon H_n(M, M - \bullet; \mathbb{Z}) \to M$ is a trivial covering map.*

Proof. (1) $\implies$ (2) is a subcase.
(2) $\implies$ (3). The orientation covering is trivial if and only if the covering over each component is trivial, so we may assume that M is connected. Now, if a 2-fold covering $\tilde{M} \to M$ is trivial, then $\tilde{M}$ splits as $M \times \{p, q\}$, and so $\tilde{M}$ cannot be connected. Conversely, if $\tilde{M}$ is not connected, then the covering restricted to each component must be a covering map, so the covering splits as a trivial covering.
Suppose then that $\mathrm{Ori}(M) \to M$ is non-trivial. Since $\mathrm{Ori}(M)$ is then connected, we can choose a path γ in $\mathrm{Ori}(M)$ between two points of a given fiber. The image S of such a path is compact and connected, and the covering is non-trivial over S, so by assumption (2), the orientation covering has a section s over S, but then $\gamma(0) = s\left(\omega(\gamma(0))\right) = s\left(\omega\left(\gamma(1)\right)\right) = \gamma(1)$, which gives a contradiction.
(3) $\implies$ (4).
Let $s\colon M \to \mathrm{Ori}(M) \cong M \times \{-1, 1\}$ be the section $m \mapsto (m, 1)$.
Now define a map $\varphi\colon M \times \mathbb{Z} \to H_n(M, M - \bullet; \mathbb{Z})$ by $\varphi(m, k) = ks(m)$. This is a bijective map by assumption on s being a section. It is furthermore continuous since s is continuous and since fiber-wise operations in $H_n(M, M - \bullet; \mathbb{Z})$ is continuous. Furthermore, it is also a morphism between coverings since it commutes with the projections: $\pi_M = \omega \circ \varphi$.
Lastly, one must show that it also has a continuous inverse. For this, we may take an open basis set in $M \times \mathbb{Z}$ - say $U \times \{k\}$, where $\bar{U}$ is a convex subset of $\mathbb{R}^n \subset M$. Since φ is bijective, we obtain that $\varphi(U \times \{k\}) = ks(U) = U_\alpha$ if we choose α to be the element in $H_n(M, M - U) \cong \mathbb{Z}$ which maps to k under $r_{x,U}$ for $x \in U$. And by assumption, U_α is a basis open set for the topology on $H_n(M, M - \bullet; \mathbb{Z})$.
Hence φ is a homeomorphism, and even an isomorphism of covering spaces in the sense that $\pi_M = \omega \circ \varphi$.

Note. We could also say that it is trivial since every point is in the image of some section.

(4) $\implies$ (1) : If ω is trivial, then it has a section with constant value in the set of generators.

□

1.2. **Homology in the Dimension of the Manifold.** Let M be an n-manifold and $A \subset M$ a closed subset. We will in this section use singular homology with coefficients in an abelian group G.

Proposition 1.9. *For each $\alpha \in H_n(M, M - A; G)$, the section*

$$J^A(\alpha)\colon A \to H_n(M, M - \bullet; G), \quad x \mapsto r_x^A(\alpha)$$

of ω over A is continuous and has compact support.

Proof. Choose a representative $c \in \Delta_n(M;G)$ representing α. There exists a compact set K such that c is contained in K. Suppose $A - K$ is nonempty, and let $x \in A - K$. Then the image of c under

$$\Delta_n(K;G) \to \Delta_n(M;G) \to \Delta_n(M,K;G) \to \Delta_n(M,M-x;G)$$

is zero since $K \subset M - x$. Since this image represents r_x^A, the support of $J^A(\alpha)$ is contained in $A \cap K$ which is compact.
If $A - K$ is empty, K contains A, and then the support of $J^A(\alpha)$ is a closed subset of a compact space, hence compact.

The continuity follows from the more general case of Proposition 1.5. □

Thus we obtain a homomorphism

$$J^A\colon H_n(M, M-A;G) \to \Gamma_c(A;G), \quad \alpha \mapsto \left(x \mapsto r_x^A(\alpha)\right).$$

1.2.1. *Direct Limits.*

Definition 1.10. Let D be a directed set and G_α an abelian group defined for each $\alpha \in D$. Suppose we are given homomorphisms $f_{\beta,\alpha}\colon G_\alpha \to G_\beta$ for each $\beta > \alpha$ in D. Assume that for all $\gamma > \beta > \alpha$ in D, we have $f_{\gamma,\beta} f_{\beta,\alpha} = f_{\gamma,\alpha}$. Such a system is called a *direct system* of abelian groups. Then $G = \lim_\to G_\alpha$ is defined to be the quotient group of the direct sum $G = \bigoplus G_\alpha$ modulo the relations $f_{\beta,\alpha}(g) \sim g$ for all $g \in G_\alpha$ and all $\beta > \alpha$.

Note. Hence the direct limit is just the colimit of the direct system.

Proposition 1.11. *Suppose we are given an abelian group A with homomorphisms $h_\alpha\colon G_\alpha \to A$ such that the cocone commutes. Since $\lim_\to G_\alpha$ is the colimit, we have a unique induced homomorphism $h\colon \lim_\to G_\alpha \to A$. Then*

1. $\operatorname{im} h = \{a \in A \mid a = h_\alpha(g) \text{ for some } g \text{ and } \alpha\} = \bigcup \operatorname{im} h_\alpha$.
2. $\ker h = \{g \in \lim_\to G_\alpha \mid \exists \alpha \text{ and } g_\alpha \in G_\alpha \colon g = i_\alpha(g_\alpha) \text{ and } h_\alpha(g_\alpha) = 0\} = \bigcup i_\alpha(\ker h_\alpha)$.

Proof. Define $h(g_\alpha) = h_\alpha(g_\alpha)$. Then if $f_{\beta,\alpha}(g_\alpha) \sim g_\alpha$, we have $h(g_\alpha) = h_\alpha(g_\alpha) = h_\beta \circ f_{\beta,\alpha}(g_\alpha) = h(f_{\beta,\alpha}(g_\alpha))$, so h respects the equivalence relations, thus it is well-defined.
Now property (1) is clear by the way we defined h.
As for (2), note that if g represents the equivalence class of g_α and $h(g) = 0$, then $h_\alpha(g_\alpha) = 0$ which is what (2) is saying. □

Corollary 1.12. *In the situation of Proposition 1.11, $h\colon \lim_\to G_\alpha \to A$ is an isomorphism if and only if the following two statements hold true:*

1. $\forall a \in A, \exists \alpha \in D$ *and* $g_\alpha \in G_\alpha \colon h_\alpha(g_\alpha) = a$, *and*
2. *if* $h_\alpha(g_\alpha) = 0$ *then* $\exists \beta > \alpha \colon f_{\beta,\alpha}(g_\alpha) = 0$.

Theorem 1.13. *The direct limit is an exact functor. So if we have direct systems $\{A'_\alpha\}, \{A_\alpha\}$ and $\{A''_\alpha\}$ based on the same directed set, and if we have an exact sequence $A'_\alpha \to A_\alpha \to A''_\alpha$ for each α, where the maps commute with the ones defining the direct systems, then the induced sequence*

$$\lim_{\to} A'_\alpha \to \lim_{\to} A_\alpha \to \lim_{\to} A''_\alpha$$

is exact.

Proof. We have the following diagram, where all maps commute.

$$\begin{array}{ccccc} A'_\beta & \longrightarrow & A_\beta & \longrightarrow & A''_\beta \\ \downarrow & & \downarrow & & \downarrow \\ \lim_\to A'_\alpha & \longrightarrow & \lim_\to A_\alpha & \longrightarrow & \lim_\to A''_\alpha \end{array}$$

Suppose $a \in \lim_\to A_*$ is mapped to zero in $\lim_\to A''_*$. Then there exists $g \in \lim_\to A_\alpha$ such that there exists β and $g_\beta \in A_\beta$ such that $g = i_\beta(g_\beta)$ and $h_\beta(g_\beta) = 0$.
Recall here that h_β is a homomorphism $A_\beta \to \lim_\to A_*^{''}$ and i_β is the inclusion $G_\beta \to \lim_\to G_\alpha$.
By commutativity of the diagram, there then exists $k_\beta \in A'_\beta$ such that $i_\beta\left(d_\beta(k_\beta)\right) = d_{\lim_\to} i_\beta^{'}(k_\beta)$. Hence the kernel is contained in the image.
Now suppose let $\tilde{k} = d_{\lim_\to}(k) \in \lim_\to A_*$.
Then $\tilde{k} = i_\beta\left(d(\overline{k})\right) = d_{\lim_\to} i'_\beta\left(\overline{k}\right)$ for some $\overline{k} \in A'_\beta$.
But now

$$d_{\lim_\to}(\tilde{k}) = d_{\lim_\to} i_\beta\left(d\left(\overline{k}\right)\right) = i''_\beta d\left(d\left(\overline{k}\right)\right) = i''_\beta(0) = 0.$$

□

Theorem 1.14. *Suppose we are given two directed sets D and E. Define an order on $D \times E$ by $(\alpha, \beta) \geq (\alpha', \beta')$ if and only if $\alpha \geq \alpha'$ and $\beta \geq \beta'$. Suppose $G_{\alpha,\beta}$ is a direct system based on $D \times E$. Then the maps $G_{\alpha,\beta} \to \lim_{\to,\beta} G_{\alpha,\beta} \to \lim_{\to,\alpha}\left(\lim_{\to,\beta} G_{\alpha,\beta}\right)$ induce an isomorphism*

$$\lim_{\to,\alpha,\beta} G_{\alpha,\beta} \stackrel{\cong}{\to} \lim_{\to,\alpha}\left(\lim_{\to,\beta} G_{\alpha,\beta}\right).$$

Proof. Todo □

Proposition 1.15. (1) *For $A \supset B$ both closed, the following diagram commutes:*

$$\begin{array}{ccc} H_n(M, M - A; G) & \longrightarrow & H_n(M, M - B; G) \\ \downarrow{\scriptstyle J^A} & & \downarrow{\scriptstyle J^B} \\ \Gamma_c(A, H_n(M, M - \bullet; G)) & \longrightarrow & \Gamma_c\left(B, H_n(M, M - \bullet; G)\right) \end{array}$$

(2) *For $A, B \subset M$ both closed, the sequence*

$$0 \to \Gamma_c(A \cup B, H_n(M, M - \bullet; G)) \xrightarrow{h} \Gamma_c\left(A, H_n(M, M - \bullet; G)\right) \oplus \Gamma_c\left(B, H_n(M, M - \bullet; G)\right)$$
$$\xrightarrow{k} \Gamma_c\left(A \cap B, H_n\left(M, M - \bullet; G\right)\right)$$

is exact, where h is the sum of restrictions and k is the difference of restrictions.

(3) *If $A_1 \supset A_2 \supset A_3 \supset \ldots$ are all compact and $A \bigcap A_i$, then the restriction homomorphisms $\Gamma\left(A_i, H_n\left(M, M - \bullet; G\right)\right) \to \Gamma\left(A, H_n\left(M, M - \bullet; G\right)\right)$ induce an isomorphism*

$$\lim_\to \Gamma\left(A_i, H_n\left(M, M - \bullet; G\right)\right) \stackrel{\cong}{\to} \Gamma\left(A, H_n(M, M - \bullet; G)\right)$$

Proof. (1) Let $\alpha \in H_n(M, M-A; G)$, and denote by ι the inclusion $(M, M-A) \hookrightarrow (M, M-B)$. Then $\iota_* = r^A_B$, so $J^B\left(r^A_B(\alpha)\right)(x) = r^B_x\left(r^A_B(\alpha)\right)$. On the other hand, $J^A(\alpha)|_B(x) = J^A(\alpha)(x) = r^A_x(\alpha)$. Now, from the composition

$$(M, M-A) \hookrightarrow (M, M-B) \hookrightarrow (M, M-x)$$

we obtain by taking homology, that $r^A_x = r^B_x r^A_B$, which gives the result.

(2) Firstly, a section that is zero on both A and B is then also zero on $A \cup B$, which gives the injective part of h. Now, suppose $s - t$ is the zero section over $A \cap B$ for s a section over A and t a section over B. Then s and t agree on $A \cap B$, meaning that $s \cup t$ is well-defined and continuous, where $s \cup t$ is s on A and t on B, and $h(s \cup t) = (s, t)$. Likewise, if g is a section over $A \cup B$, then $k \circ h(g) = (g|_A)|_{A\cap B} - (g|_B)|_{A\cap B} = g|_{A\cap B} - g|_{A\cap B}$ is the zero section.

(3)

$\square$

Theorem 1.16. *Let $A \subset M$ be closed. Then*
1. $H_i(M, M-A; G) = 0$ *for* $i > n$.
2. $J^A \colon H_n(M, M-A, G) \to \Gamma_c(A, H_n(M, M-\bullet; G))$ *is an isomorphism.*

Lemma 1.17 (The Bootstrap Lemma)**.** *Let $P_M(A)$ be a statement about compact sets A in a given n-manifold M^n. If $(i), (ii), (iii)$ hold, then $P_M(A)$ is true for all compact A in M^n.*
If M^n is separable metric, and $P_M(A)$ is defined for all closed sets A, and if $(i), (ii), (iii), (iv)$ hold, then $P_M(A)$ is true for all closed sets A in M.
For general M^n, if $P_M(A)$ is defined for all closed sets A in M, for all M^n, and if all five statement $(i) - (v)$ hold for all M^n, then $P_M(A)$ is true for all closed $A \subset M$ and all M^n.

2. Intersection Theory

Definition 2.1 (k-disk bundle)**.** A k-disk bundle is a vector bundle whose coordinate transformations are contained in $O(k) \subset \mathrm{GL}(\mathbb{R}^k)$ and such that the local trivializations have the form $\pi^{-1}(U) \cong U \times D^k$.

Let N^n be a connected, oriented, closed n-manifold, and W^{k+n} an $(n+k)$-manifold with boundary ∂W a $(k-1)$-sphere bundle over N^n, and let $\pi\colon W^{n+k} \to N^n$ be a k-disk bundle over N.
Let us assume also that W is also oriented.

Definition 2.2. In the above situation, the *Thom class* of the disk bundle π is the class $\tau \in H^k(W, \partial W)$ given by

$$\tau = D_W(i_*[N])$$

where $D_W\colon H_{n-k}(W) \to H^k(W, \partial W)$ is the inverse of the Poincaré duality isomorphism. That is,

$$D(a) \cap [M] = a.$$

Thus

$$\tau \cap [W] = i_*[N].$$

We can deformation retract the punctured disk to its boundary, giving $H^k(W, W - N) \cong H^k(W, \partial W)$, so we will sometimes regard τ as being in $H^k(W, W - N)$.

Lemma 2.3. *In the above setup, suppose $A \subset N$ is closed. Let $\tilde{A} = \pi^{-1}(A) \subset W$ and $\partial\tilde{A} = \tilde{A} \cap \partial W$. Then $\check{H}^i\left(\tilde{A}, \partial\tilde{A}\right) = 0$ for $0 < i < k$.*

Proof. Suppose first that A is compact convex subset of a Euclidean neighborhood in N. It also suffices consider the case where A is connected, so $A \cong D^n$. Consider the pullback bundle of A :

$$\begin{tikzcd} i^*(A) \arrow[r] \arrow[d] & W \arrow[d, "\pi"] \\ A \arrow[r, hook, "i"] & N \end{tikzcd}$$

Then $i^*(A) = A \times_N W \cong \pi^{-1}(A)$, so since any vector bundle over a contractible paracompact base space is trivial, we conclude that the bundle $\tilde{A} \to A$ is trivializable as $\tilde{A} \cong A \times D^k$ and $\partial\tilde{A} \cong A \times S^{k-1}$. Now the steps are as follows: calculate the homology of $A \times D^k$ and $A \times S^{k-1}$, then use UCT to obtain the cohomology, and then use the LES to find the cohomology of $\left(A \times D^k, A \times S^{k-1}\right)$.
Now... But by the Künneth theorem,

$$H_m(A \times D^k) \cong H_m(A)$$

and

$$H_m(A \times S^{k-1}) \cong H_m(A) \oplus H_{m-k+1}(A).$$

□

Lemma 2.4. *The restriction $\tau_x \in \check{H}^k(\tilde{A}, \partial\tilde{A})$ of τ, when $A = \{x\}$, is a generator.*

Proof. Note that $\left(\tilde{A}, \partial\tilde{A}\right) \cong \left(D^k, S^{k-1}\right)$.
Suppose first that $\tau_x = 0$ for some x.
Now, recall that

$$\tau_A = D_W\left(i_*\left[A\right]\right).$$

Then $\tau_x = 0$ if and only if $i_*\left[x\right] = 0$.

□

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