

# 1. THEORY

Recall that

**Definition 1.1** (Dirichlet Series). Let  $f$  be an arithmetic function. Then the corresponding Dirichlet series is defined, for  $s \in \mathbb{C}$ , by

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

**Lemma 1.2.**

$$0 \leq 3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2$$

**Lemma 1.3.** Let  $\sigma > 1$ . Then

$$\Re \left( -3 \frac{\zeta'}{\zeta}(\sigma) - 4 \frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it) \right) \geq 0$$

For the proof of the lemma, one shows that

$$\Re \left( \frac{1}{n^s} \right) = \frac{1}{n^\sigma} \cos(t \log n), \quad s = \sigma + it \quad (A_1)$$

*Proof.*

$$\Re \left( -\frac{\zeta'}{\zeta}(s) \right) = \Re \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} \cos(t \log n).$$

Hence

$$\Re \left( -3 \frac{\zeta'}{\zeta}(\sigma) - 4 \frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it) \right) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} [3 + 4 \cos(t \log n) + \cos(2t \log n)] \stackrel{(1.2)}{\geq} 0$$

□

## 2. WEEK 1

**Exercise 2.1** (E1.1. Abel summation). Let  $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$  and  $f: [1, x] \rightarrow \mathbb{C}$  be  $C^1$ . Define  $A(t) = \sum_{n \leq t} a_n$ . Then for  $x > 1$ , we have

$$\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

## 3. WEEK 2

Let  $\psi(x) := \sum_{n \leq x} \Lambda(n)$ .

**Exercise 3.1** (E2.6). Show that

$$\theta(x) := \sum_{p \leq x} \log p = \psi(x) + O\left(x^{\frac{1}{2}} \log^2 x\right)$$

**Exercise 3.2** (E2.7). Show that

$$\pi(x) = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t \log^2 t} dt + O\left(x^{\frac{1}{2}} \log x\right).$$

*Proof.* By Abel summation, we first find that

$$\theta(x) := \sum_{p \leq x} \log p = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt$$

and from the previous exercise, we now find that

$$\pi(x) = \frac{\psi(x)}{\log x} + \frac{1}{\log x} \int_2^x \frac{\pi(t)}{t} dt + O\left(x^{\frac{1}{2}} \log x\right)$$

The result follows if we can show that

$$\frac{1}{\log x} \int_2^x \frac{\pi(t)}{t} dt = \int_2^x \frac{\psi(t)}{t \log^2 t} dt + O\left(x^{\frac{1}{2}} \log x\right).$$

Now  $\psi(t) \leq \pi(t) \log t$ , so

$$\begin{aligned} \left| \int_2^x \frac{\psi(t)}{t \log^2 t} - \frac{\pi(t)}{t \log x} dt \right| &\leq \left| \int_2^x \frac{\pi(t)}{t \log t} - \frac{\pi(t)}{t \log x} dt \right| \\ &= \left| \int_2^x \frac{\pi(t)}{t} \frac{\log\left(\frac{x}{t}\right)}{\log x \log t} dt \right| \end{aligned}$$

□

## 4. WEEK 3

**Exercise 4.1** (E3.1). Let  $m \geq 0$ . Show that

$$\sum_{n \leq x} \log^m n = x \log^m x + O\left(x \log^{m-1} x\right).$$

*Proof.* Let  $a_n = 1$  for all  $n$ . Then  $A(x) = \lfloor x \rfloor$ , so

$$\begin{aligned} \sum_{n \leq x} \log^m n &= \lfloor x \rfloor \log^m x - \int_1^x m \lfloor t \rfloor \frac{1}{t} \log^{m-1} t dt \\ &= x \log^m x - (x - \lfloor x \rfloor) \log^m x - m \int_1^x \frac{\lfloor t \rfloor}{t} \log^{m-1}(t) dt \end{aligned}$$

Thus we must show that

$$\left| (x - \lfloor x \rfloor) \log^m x + m \int_1^x \frac{\lfloor t \rfloor}{t} \log^{m-1}(t) dt \right| \leq Cx \log^{m-1} x$$

But  $\frac{\lfloor t \rfloor}{t} \log^{m-1}(t) \leq \log^{m-1}(x)$  giving that the right hand term is  $O(x \log^{m-1} x)$ . For the left hand term, it suffices to show that  $(x - \lfloor x \rfloor) \log x \leq x$ , but this is clear since  $x - \lfloor x \rfloor \leq 1$  and  $\log x \leq x$ .  $\square$

**Exercise 4.2** (E3.2). Let  $d(n) = \sum_{d|n} 1$ . Show  $d(n) \leq 2\sqrt{n}$ . If we consider the set  $D \subset \mathbb{N}$  of positive divisors of  $n$ , then we can define a bijection  $D \rightarrow D$  by  $k \mapsto \frac{n}{k}$ . Suppose now that  $d(n) > 2\sqrt{n}$ . Suppose  $d | n$  and  $d \geq \sqrt{n}$ . Then since  $\frac{d}{n} \cdot d = n$ , we must have  $\frac{d}{n} \leq \sqrt{n}$ . This implies that under this bijection, either the source or target lies in  $\{1, \dots, \lfloor \sqrt{n} \rfloor\}$ . Hence  $d(n) = |D| \leq 2|\{1, \dots, \lfloor \sqrt{n} \rfloor\}| \leq 2\sqrt{n}$ .

**Exercise 4.3** (E3.3). Prove that for every  $\varepsilon > 0$ , there exists a constant  $C_\varepsilon$  such that  $d(n) \leq C_\varepsilon n^\varepsilon$ .

Hint:

- (1) Show that  $d(n_1 n_2) = d(n_1) d(n_2)$  if  $(n_1, n_2) = 1$ .
- (2) Show that

$$\frac{d(n)}{n^\varepsilon} = \prod_{p^\alpha || n} \frac{\alpha + 1}{p^{\alpha \varepsilon}}$$

where  $p^\alpha || n$  means that  $\alpha$  is a positive integer,  $p^\alpha | n$  and  $p^{\alpha+1} \nmid n$ .

- (3) Split the product in 2. Into the product over those primes  $p < 2^{\frac{1}{\varepsilon}}$  and the product over the rest. Show that the second product is bounded by 1.
- (4) Show that the factors in the first product are less than  $1 + (\varepsilon \log 2)^{-1}$ .

*Proof.* We follow the hint:

(1) Suppose  $(n_1, n_2) = 1$ . Let  $D$  be the set of divisors of  $n_1 n_2$ ,  $D_1$  the set of divisors of  $n_1$  and  $D_2$  the set of divisors of  $n_2$ . Suppose  $d_1 \in D_1, d_2 \in D_2$ . Then  $d_1 a = n_1, d_2 b = n_2$ , so  $d_1 d_2 ab = n_1 n_2$ , hence  $d_1 d_2 \in D$ . We thus obtain a map  $D_1 \times D_2 \rightarrow D$  sending  $(d_1, d_2) \mapsto d_1 d_2$ . We claim this is a bijection. Suppose  $d_1 d_2 = d'_1 d'_2$ . If  $d_1 | d'_2$ , then  $d_1 = 1$ , in which case,  $d'_1 = 1$ , and thus  $d_2 = d'_2$ . Suppose thus that  $d_1 \nmid d'_2$ . Then since  $(d'_1, d'_2) = 1$ , we have  $d_1 | d'_1$ . Similarly,  $d'_1 | d_1$ . So  $d_1 = d'_1$ . And again  $d_2 = d'_2$ . This gives injectivity. For surjectivity, if  $d | n_1 n_2$ , then consider  $d_1 := \frac{d}{(n_2, d)}$  and  $d_2 := \frac{d}{(n_1, d)}$ . Then  $d_1 d_2 = d$  and  $d_1 \in D_1, d_2 \in D_2$ .

(2) Clearly,  $n^\varepsilon = \prod_{p^\alpha || n} p^{\alpha \varepsilon}$ . It thus suffices to show that  $\prod_{p^\alpha || n} (\alpha + 1) = d(n)$ . But if we factorize  $n$  as  $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ , then it is clear that the divisors corresponds precisely to tuples  $(a_1, \dots, a_m)$  with  $0 \leq a_i \leq \alpha_i$ . There are precisely  $\alpha_i + 1$  choices for each  $a_i$ , giving  $(\alpha_1 + 1) \cdots (\alpha_m + 1) = d(n)$  which indeed is what we wanted to show.

(3) We can split the product as

$$\frac{d(n)}{n^\varepsilon} = \underbrace{\prod_{\substack{p^\alpha || n \\ p < 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}} \cdot \underbrace{\prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}$$

$A \qquad \qquad \qquad B$

We claim that  $B \leq 1$ . Indeed

$$\prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}} \leq \prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \underbrace{\frac{\alpha+1}{2^\alpha}}_{\leq 1} \leq 1$$

(4) For the factors in the first product, we have  $\alpha = \left\lfloor \frac{\log n}{\log p} \right\rfloor$  and  $\log p < \frac{1}{\varepsilon} \log 2$ , and  $\alpha \leq \frac{\log n}{\log p}$ , so  $\frac{\log p}{\log n} \leq \frac{1}{\alpha}$

$$\varepsilon^2 \log p < \varepsilon \log 2$$

$$\frac{\alpha+1}{p^{\alpha\varepsilon}} \leq \frac{\log n + \log p}{p^{\alpha\varepsilon} \log p} \leq 1 + \frac{1}{\varepsilon \log 2} = \frac{\varepsilon \log 2 + 1}{\varepsilon \log 2}$$

What we want to bound is

$$\prod_{\substack{p^\alpha || n \\ p < 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}$$

Note here that  $p$  is bounded and as  $\alpha$  increases, we should expect the denominator to take over. However, while  $\alpha$  is small, we might have some large terms since  $p^\varepsilon$  might be large. All our terms are however bounded by  $p^\varepsilon$  by the looks of it? Then we would get that the product is the product is bounded by  $\prod_{p < 2^{\frac{1}{\varepsilon}}} \frac{\log n}{\log p} \frac{1}{p^\varepsilon}$  □

**Exercise 4.4** (E3.4). Show that

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

is absolutely convergent in  $\Re(s) > 1$ .

*Proof.* Fix some  $s = \sigma + it \in \mathbb{C}$  with  $\sigma > 1$ . Then choosing an  $\varepsilon > 0$  with  $1 + \varepsilon < \sigma$ , we have that  $d(n) \leq C_\varepsilon n^\varepsilon$ , so

$$\sum \left| \frac{d(n)}{n^s} \right| \leq \sum C_\varepsilon \frac{n^\varepsilon}{n^\sigma} \leq C_\varepsilon \sum \frac{1}{n^{\sigma-\varepsilon}} < \infty.$$

□

**Exercise 4.5** (E3.5). Show that the average order of  $d(n)$  is  $\log n$ , i.e., that

$$\frac{1}{x} \sum_{n \leq x} d(n) = \log x + o(\log x).$$

Hint: Show that

$$\sum_{n \leq x} d(n) = \sum_{a \leq x} \left[ \frac{x}{a} \right]$$

where  $[b]$  is the integer part of  $b$ .

*Proof.* We follow the hint. For each  $n \in \mathbb{N}$ , let  $D_n$  denote the set of positive divisors of  $n$ . Then we want to find  $|D_1 \cup \dots \cup D_{[x]}|$ . Now,  $\left[\frac{x}{a}\right]$  is precisely the amount of multiples of  $a$  smaller than or equal to  $x$ , i.e., the amount of numbers in between 1 and  $x$  which have  $a$  as a divisor. Hence the right hand side indeed counts the number of divisors of the numbers less than or equal to  $x$  which is precisely the left hand side. Now, recall also the bound

$$\log x + \frac{1}{x} \leq \sum_{a \leq x} \frac{1}{a} \leq \log x + 1$$

so

$$1 + \frac{1}{x \log x} \leq \frac{1}{\log x} \sum_{a \leq x} \frac{1}{a} \leq 1 + \frac{1}{\log x}.$$

In particular, taking the limit as  $x \mapsto \infty$ , the outer functions tend to 1, so

$$\lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{a \leq x} \frac{1}{a} = 1.$$

In particular,

$$\frac{1}{x \log x} \sum_{n \leq x} d(n) \leq \frac{1}{\log x} \sum_{a \leq x} \frac{1}{a} \rightarrow 1, \quad x \rightarrow \infty.$$

For a lower bound, we have

$$\frac{1}{\log x} \sum_{a \leq x} \frac{1}{a} - \frac{1}{x \log x} \sum_{a \leq x} \frac{1}{a} = \frac{1}{\log x} \sum_{a \leq x} \frac{x-1}{ax} \leq \frac{1}{\log x} \sum_{a \leq x} \left[\frac{x}{a}\right]$$

But

$$\frac{1}{x} + \frac{1}{x^2 \log x} \leq \frac{1}{x \log x} \sum_{a \leq x} \frac{1}{a} \leq \frac{1}{x} + \frac{1}{x \log x}$$

so letting  $x \rightarrow \infty$ ,

$$\lim_{x \rightarrow \infty} \frac{1}{x \log x} \sum_{a \leq x} \frac{1}{a} = 0$$

Hence also

$$1 \leq \lim_{x \rightarrow \infty} \frac{1}{x \log x} \sum_{n \leq x} d(n) \leq 1$$

giving the desired result.  $\square$

**Exercise 4.6** (E3.6). Let

$$\chi_4(n) = \begin{cases} (-1)^{\frac{n-1}{2}}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}.$$

Show that  $\chi_4$  is a Dirichlet character modulo 4 and find  $L(1, \chi_4)$ . Use the value to give (yet another) proof- based on the irrationality of  $\pi$  - that there are infinitely many primes. Hint: Remember (or prove by playing around with  $\arctan(1)$ ) that

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$

*Proof.* We must check 3 criteria for  $\chi_4$  to be a Dirichlet character mod 4.

(i) It must be 4-periodic. Now if  $n$  is even, then  $n+4$  is even, so then  $\chi_4(n+4) = 0 = \chi_4(n)$ .

If  $n$  is odd, then so is  $n+4$ , so

$$\chi_4(n+4) = (-1)^{\frac{n+4-1}{2}} = (-1)^{\frac{n-1}{2}+2} = (-1)^{\frac{n-1}{2}} = \chi_4(n).$$

So  $\chi_4$  is 4-periodic.

(ii) We must check that  $\chi_4(n) = 0$  if and only if  $(n, 4) \neq 1$ . Now,  $\chi_4(n) = 0$  if and only if  $n$  is even if and only if  $(n, 4) \in \{2, 4\}$  if and only if  $(n, 4) \neq 1$ .

(iii) We must check that  $\chi_4$  is multiplicative. Indeed, if either  $n$  or  $m$  is even, then

$$\chi_4(nm) = 0 = \chi(n)\chi(m).$$

If both  $n, m$  are odd, then

$$\chi_4(nm) = (-1)^{\frac{nm-1}{2}} = \begin{cases} -1, & nm \equiv 3 \pmod{4} \\ 1, & nm \equiv 1 \pmod{4} \end{cases}$$

Now, if  $n$  and  $m$  are both equivalent to 3 mod 4, then their product is equivalent to 1 mod 4, which works out. If only one is equivalent to 3 mod 4, then  $nm$  is also, so it checks out, and similarly, if both are equivalent to 1 mod 4, then so is their product. Now, by definition,

$$L(1, \chi_4) := \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan(1) = \frac{\pi}{4}$$

Now, since  $\chi_4 \neq \chi_0^4$ , we know that  $L(s, \chi_4)$  is convergent and analytic for  $\Re(s) > 0$ . In particular, it is continuous at  $s = 1$ . But for  $\Re(s) > 1$ , we know that  $L(s, \chi_4) = \prod_p (1 - \chi_4(p)p^{-s})^{-1}$ , so by continuity,

$$\frac{\pi}{4} = L(1, \chi_4) = \prod_p (1 - \chi_4(p)p^{-1})^{-1}$$

Now, all the terms in the product are rational, so by irrationality of  $\pi$ , this forces there to be infinitely many primes.  $\square$

**Exercise 4.7** (E3.7). Let  $\{a_n\}$  be a sequence of complex numbers satisfying that  $\sum_{n \leq x} a_n = O(x^\delta)$  for some  $\delta > 0$ . Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \sum_{n \leq t} a_n \frac{1}{t^{s+1}} dt$$

for  $\Re(s) > \delta$ , and that the sum converges to an analytic function in this region.

*Proof.* Let  $f(x) = x^s$ . Then

$$\sum_{n \leq x} \frac{a_n}{n^s} = \sum_{n \leq x} a_n \frac{1}{x^s} + s \int_1^x \sum_{n \leq t} a_n \frac{1}{t^{s-1}} dt$$

when  $s \neq 1$ . But  $\left| \sum_{n \leq x} a_n \right| \leq Cx^\delta$ , so

$$\left| \sum_{n \leq x} a_n \frac{1}{x^s} \right| \leq Cx^{\delta-\sigma} \rightarrow 0, \quad x \rightarrow \infty$$

as  $\delta - \sigma < 0$ . Thus

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \sum_{n \leq t} a_n \frac{1}{t^{s+1}} dt.$$

□

## 5. WEEK 4

**Exercise 5.1** (E4.1). Let  $K \geq 0$ . Prove that

$$\log(K|t| + 4) = O(\log(|t| + 4))$$

for  $t \in \mathbb{R}$ . Let  $c_1, c_2, c_3 > 0$ . Prove that there exists a constant  $c_4$  such that for all  $t \in \mathbb{R}$ ,

$$c_1 + c_2 \log(|t| + 4) + c_3 \log(|2t| + 4) \leq c_4 \log(|t| + 4).$$

*Proof.* If  $0 \leq K \leq 1$ , then  $\log(K|t| + 4) \leq \log(|t| + 4)$  by monotonicity of  $\log$ . So assume  $K > 1$ . Then  $\log(K|t| + 4) = \log K + \log(|t| + \frac{4}{K}) \leq \log K + \log(|t| + 4)$ . Now  $\log(|t| + 4) > 1$ , so there exists some  $C$  such that  $C \log(|t| + 4) \geq \log K$ . Hence  $\log(K|t| + 4) = O(\log(|t| + 4))$ . Since  $c_1 + c_2 \log(|t| + 4) + c_3 \log(|2t| + 4)$  is a sum of terms that are all  $O(\log(|t| + 4))$ , so is their sum, so the conclusion holds. □

**Exercise 5.2** (E4.2). Let  $f(s)$  be a complex polynomial of degree  $n$  with complex zeroes  $z_1, z_2, \dots, z_n$ . Show that

$$\frac{f'}{f}(z) = \sum_{i=1}^n \frac{1}{z - z_i}.$$

Consider how Lemma 6.3 is a generalization of this.

*Proof.* Firstly,  $f'$  is entire, so  $\frac{f'}{f}$  is holomorphic on  $\mathbb{C} - \{z_1, \dots, z_n\}$ . Now, by Theorem 6.1 in KomAn, there exist unique functions  $g_i$  holomorphic on  $\mathbb{C} - \{z_1, \dots, z_n\}$  such that  $g_i(z_i) \neq 0$  and

$$f(z) = (z - z_i)^{n_i} g_i(z)$$

where  $n_i$  is the multiplicity of  $z_i$ . In particular,  $f'(z) = n_i(z - z_i)^{n_i-1} g_i(z) + (z - z_i)^{n_i} g_i'(z)$  which has  $z_i$  a zero of order  $n_i - 1$ . Hence  $\frac{f'}{f}$  has  $z_i$  as a simple pole.

Applying the partial fraction decomposition to  $\frac{f'}{f}$  (theorem 6.12 in KomAn), we get that

$$\frac{f'}{f}(z) = \sum_{i=1}^n \frac{c_i}{z - z_i}$$

for certain constants  $c_i$ . Now  $\lim_{z \rightarrow z_i} (z - z_i) \frac{f'}{f}(z) = c_i$ . Now,  $f$  is of degree  $n$  with  $n$  distinct zeroes, so  $n_i$  must be 1 for each  $i$ .

Now let us recall Lemma 6.3:

**Lemma 5.3** (6.3). Let  $f: B \rightarrow \mathbb{C}$  be analytic,  $B \subset \mathbb{C}$  open, and assume

- (1)  $\{z \mid |z| \leq 1\} \subset B$
- (2)  $|f(z)| \leq M$  when  $|z| \leq 1$
- (3)  $f(0) \neq 0$ .

Let  $0 < r < R < 1$ . Then for  $|z| < r$ ,

$$\frac{f'}{f}(z) = \sum_{\substack{f(z_k)=0 \\ |z_k| \leq R}} \frac{1}{z - z_k} + O\left(\log \frac{M}{|f(0)|}\right)$$

Note here that  $f$  is not required to be a polynomial. However, since  $f$  is holomorphic in  $B$ , it has an analytic representation on  $B$ , so essentially, Lemma 6.3 generalizes the representation to analytic functions.  $\square$

**Exercise 5.4** (E4.3). Show that the Riemann zeta function  $\zeta(s)$  has no zeroes for  $\frac{1}{2} \leq s < 1$ .

*Proof.* Recall that for  $\sigma > 0$  and  $s \neq 1$ , we have

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty (u - [u]) u^{-s-1} du.$$

For  $s \in [\frac{1}{2}, 1)$ ,  $\frac{s}{s-1} \leq -1$ . So we wish to show that

$$s \int_1^\infty (u - [u]) u^{-s-1} du > -1$$

But

$$s \int_1^\infty (u - [u]) u^{-s-1} du$$

is positive since the inner function and  $s$  are both positive on  $[1, \infty)$ .  $\square$

**Exercise 5.5** (E4.4). Let  $\chi$  be a Dirichlet character modulo  $q$ . Find the Dirichlet series representation for  $L'(s, \chi)/L(s, \chi)$ . Let  $\chi_0$  be the trivial Dirichlet character modulo  $q$ . Prove that for  $\sigma > 1, t \in \mathbb{R}$ ,

$$R := \Re \left( -3 \frac{L'(\sigma, \chi_0)}{L(\sigma, \chi_0)} - 4 \frac{L'(\sigma + it, \chi)}{L(\sigma + it, \chi)} - \frac{L'(\sigma + i2t, \chi^2)}{L(\sigma + i2t, \chi^2)} \right) \geq 0.$$

*Proof.* We want to represent  $\frac{L'(s, \chi)}{L(s, \chi)}$  as a Dirichlet series. We imitate the idea for  $\frac{\zeta'}{\zeta}$ .



$$\begin{aligned}
\frac{L'(s, \chi)}{L(s, \chi)} &= \frac{d}{ds} \log(L(s, \chi)) \\
&= - \sum_p \frac{d}{ds} \log \left( 1 - \frac{\chi(p)}{p^s} \right) \\
&= - \sum_p \frac{d}{ds} \sum_{k=1}^{\infty} (-1)^{k+1} \left( -\frac{\chi(p)}{p^s} \right)^k \\
&= \sum_p \sum_{k=1}^{\infty} \frac{d}{ds} \left( \frac{\chi(p)}{p^s} \right)^k \\
&= \sum_p \sum_{k=1}^{\infty} \chi(p)^k (-k \log p) p^{-sk} \\
&= - \sum_p \sum_{k=1}^{\infty} k \log p \left( \frac{\chi(p)}{p^s} \right)^k
\end{aligned}$$

Thus We want to find  $\Re \left( \left( \frac{\chi(p)}{p^s} \right)^k \right)$ . We have

$$\begin{aligned}
\Re \left( \left( \frac{\chi(p)}{p^s} \right)^k \right) &= \frac{1}{2} \left[ \left( \frac{\chi(p)}{p^s} \right)^k + \left( \overline{\frac{\chi(p)}{p^s}} \right)^k \right] \\
&=
\end{aligned}$$

$$\Re \left( -\frac{L'(s, \chi)}{L(s, \chi)} \right) = \sum_p \sum_{k=1}^{\infty} k \log p \cos(tk \log p).$$

So

□

**Exercise 5.6** (E4.5). Let  $\zeta(s)$  be the Riemann zeta function. Let  $K$  be a compact subset of  $\{s \in \mathbb{C} \mid \Re(s) > 0\}$ . Assume that  $1 \in K$  and that  $K$  does not contain any zeroes of  $\zeta$ . Show that

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1)$$

for  $s \in K - \{1\}$ . Show that there exists a constant  $c > 0$  such that for  $0 < \delta < 1$ ,

$$-\frac{\zeta'}{\zeta}(1+\delta) < \frac{1}{\delta} + c.$$

*Proof.* Since 1 is a simple pole of  $\frac{\zeta'}{\zeta}$  and  $K$  has no other zeroes of  $\zeta$  and hence neither of  $\zeta'$ , we have that

$$-(s-1) \frac{\zeta'}{\zeta}(s)$$

is holomorphic on  $K$ , hence bounded as  $K$  is compact. Thus

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1)$$

for  $s \in K - \{1\}$ . Thus for small  $0 < \delta < 1$  such that  $1 + \delta \in K - \{1\}$ ,

$$-\frac{\zeta'}{\zeta}(1 + \delta) < \frac{1}{\delta} + c$$

for some  $c > 0$ . □

**Exercise 5.7** (E4.6). Use partial summation (Abel summation) to show that for  $\sigma > 1$ ,

$$-\frac{\zeta'}{\zeta}(s) = s \int_1^\infty \frac{\psi(x)}{x^{s+1}} dx$$

where  $\psi(x) = \sum_{n \leq x} \Lambda(n)$ , and  $\Lambda$  is the von Mangoldt function.

*Proof.* Recall that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

for  $\sigma = \Re(s) > 1$ .

Let  $f(x) = \frac{1}{x^s}$  and  $a_n = \Lambda(n)$ . Partial summation gives

$$\sum_{n \leq x} \frac{\Lambda(n)}{n^s} = \underbrace{\sum_{n \leq x} \Lambda(n)}_{\psi(x)} \frac{1}{x^s} + s \int_1^x \underbrace{\sum_{n \leq t} \Lambda(n)}_{\psi(t)} \frac{1}{t^{s+1}} dt$$

By the prime number theorem,

$$\psi(x) = x + O\left(\frac{x}{e^{c'\sqrt{\log x}}}\right)$$

so

$$\frac{\psi(x)}{x^s} \rightarrow 0, \quad x \rightarrow \infty$$

Thus

$$-\frac{\zeta'}{\zeta}(s) = s \int_1^\infty \frac{\psi(t)}{t^{s+1}} dt$$

for  $\sigma > 1$ . □

## 6. WEEK 5

**Exercise 6.1** (E5.1). Show that

$$x \exp\left(-c\sqrt{\log x}\right) = O_m\left(\frac{x}{\log^m x}\right)$$

for every  $m$ , and that

$$x^{1-\varepsilon} = O_\varepsilon\left(x \exp\left(-c\sqrt{\log x}\right)\right)$$

for every  $\varepsilon > 0$ . Discuss what this means for the quality of the error-term in the prime number theorem.

*Proof.*

$$\frac{\log^m x}{e^{c\sqrt{\log x}}} = \frac{\sqrt{\log x}^{-2m}}{e^{c\sqrt{\log x}}}$$

Now

**Lemma 6.2.** *For any  $a > 0$  and any  $b > 1$ ,*

$$\frac{x^a}{b^x} \rightarrow 0, \quad x \rightarrow \infty.$$

Let  $v = \sqrt{\log x}$ . Then the above reads  $\frac{v^{2m}}{e^{cv}}$ . Assuming  $c > 0$ , we find that for  $v \rightarrow \infty$ ,  $\frac{v^{2m}}{e^{cv}} \rightarrow 0$ . So in fact,

$$x \exp(-c\sqrt{\log x}) = o\left(\frac{x}{\log^m x}\right)$$

Now

$$x^{1-\varepsilon} = xx^{-\varepsilon} = xe^{-\log(x)\varepsilon} \leq xe^{-c\sqrt{\log x}}.$$

Recall that we proved the following version of the prime number theorem:

**Theorem 6.3** (Prime number theorem). *There exists a  $c' > 0$  such that*

$$\psi(x) = x + O\left(x \exp(-c'\sqrt{\log x})\right)$$

So by the above,

$$\psi(x) = x + O_m\left(\frac{x}{\log^m(x)}\right)$$

So essentially, the error term is smaller than  $\frac{x}{\log^m(x)}$  for any  $x$  but still larger than  $x^{1-\varepsilon}$  for any  $\varepsilon > 0$ .  $\square$

**Exercise 6.4** (E5.2). Prove that the following two statements are equivalent:

- (1) There exists a  $c > 0$  such that

$$\psi(x) = x + O\left(x \exp(-c\sqrt{\log x})\right)$$

- (2) There exists a  $c > 0$  such that

$$\pi(x) = li(x) + O\left(x \exp(-c\sqrt{\log x})\right)$$

$$\text{where } li(x) = \int_2^x \frac{1}{\log t} dt.$$

*Proof.* Suppose (1) is true. Then

$$\begin{aligned} \pi(x) &= \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t \log^2 t} dt + O\left(x^{\frac{1}{2}} \log x\right) \\ &= \frac{x}{\log x} + O\left(\frac{x}{\log x} \exp(-c\sqrt{\log x})\right) + \int_2^x \frac{1}{\log^2 t} + O\left(\frac{1}{\log^2 t \exp(c\sqrt{\log t})}\right) dt + O\left(x^{\frac{1}{2}} \log x\right) \end{aligned}$$

Now

$$\int_2^x \frac{1}{\log^2 t} dt = -\frac{t}{\log t} \Big|_2^x + li(x)$$

giving

$$\pi(x) = li(x) + \frac{2}{\log 2} + O\left(\frac{x}{\log x} e^{-c\sqrt{\log x}}\right) + \int_2^x O\left(\frac{e^{-c\sqrt{\log t}}}{\log^2 t}\right) dt + O\left(x^{\frac{1}{2}} \log x\right)$$

All the middle terms apart from the last two are clearly  $O\left(xe^{-c\sqrt{\log x}}\right)$ . To take care of the last term, we use the lemma:

**Lemma 6.5.** For any  $a > 0$ ,

$$\frac{\log x}{x^a} \rightarrow 0, \quad x \rightarrow \infty$$

Hence  $x^{\frac{1}{2}} \log x = O\left(x^{\frac{3}{4}}\right) = O\left(xe^{-c'\sqrt{\log x}}\right)$ .

For the last part

$$\int_2^x O\left(\frac{e^{-c\sqrt{\log t}}}{\log^2 t}\right) dt \leq$$

Note that the derivative of  $xe^{-c\sqrt{\log x}}$  is

$$e^{-c\sqrt{\log x}} - c \frac{d}{dx} \left[ \sqrt{\log x} \right] e^{-c\sqrt{\log x}} = e^{-c\sqrt{\log x}} - c \frac{1}{2} \frac{1}{x} \frac{1}{\sqrt{\log x}} e^{-c\sqrt{\log x}}$$

But as  $x \rightarrow \infty$ , this grows faster than  $\frac{e^{-c\sqrt{\log x}}}{\log^2 x}$ , which is what we wanted.

Now we want to show that (2) implies (1). So assume there exists a  $c > 0$  such that

$$\pi(x) = li(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right).$$

Then recall that

$$\psi(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt - O\left(x^{\frac{1}{2}} \log^2 x\right)$$

So

$$\psi(x) = li(x) \log x + \log x O\left(xe^{-c\sqrt{\log x}}\right) - \int_2^x \frac{li(t)}{t} dt - \int_2^x O\left(e^{-c\sqrt{\log t}}\right) dt - O\left(x^{\frac{1}{2}} \log^2 x\right)$$

Now, by repeated integration by parts, we get

$$\begin{aligned} li(x) &= \frac{t}{\log t} \Big|_2^x + \int_2^x \frac{1}{\log^2 t} dt \\ &= \frac{t}{\log t} \Big|_2^x + \left[ \frac{t}{\log^2 t} \Big|_2^x + 2 \int_2^x \frac{1}{\log^3 t} dt \right] \\ &= \frac{t}{\log t} + \frac{t}{\log^2 t} \Big|_2^x + 2 \left[ \frac{t}{\log^3 t} \Big|_2^x + 3 \int_2^x \frac{1}{\log^4 t} dt \right] \\ &= x \sum_{r=1}^{k-1} \frac{(r-1)!}{\log^r x} + (k-1)! \int_2^x \frac{1}{\log^k t} dt \end{aligned}$$

□

**Exercise 6.6** (E5.3). Let  $f$  be a Schwartz function on the real line, and let  $\hat{f}$  be its Fourier transform. Show that

$$\sum_{n \in \mathbb{Z}} f\left(\frac{v+n}{t}\right) = \sum_{n \in \mathbb{Z}} |t| \hat{f}(nt) e^{2\pi i n v}.$$

*Proof.* For a Schwartz function  $f$ , we know from the Poisson summation formula that

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n),$$

where

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi i x y} f(x) dx$$

Define  $g(x) = f\left(\frac{v+x}{t}\right)$ . Then  $g$  is also a Schwartz function, so

$$\sum_{n \in \mathbb{Z}} g(n) = \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} e^{-2\pi i n x} f\left(\frac{v+x}{t}\right) dx$$

Let  $z = \frac{v+x}{t}$ . Then  $dz = \frac{1}{|t|} dx$ , so

$$\sum_{n \in \mathbb{Z}} f\left(\frac{v+n}{t}\right) = \sum_{n \in \mathbb{Z}} g(n) = \sum_{n \in \mathbb{Z}} |t| e^{-2\pi i n (tz-v)} f(z) dz = \sum_{n \in \mathbb{Z}} |t| \hat{f}(nt) e^{2\pi i n v}$$

□

**Exercise 6.7** (E5.4). Let  $\theta > \frac{1}{2}$ . Prove that if for every  $\varepsilon > 0$ ,  $\psi(x) = x + O(x^{\theta+\varepsilon})$ , then the Riemann zeta function has no zeroes in  $\Re(s) > \theta$ . (It turns out that this is in fact an 'if and only if statement'). Think about what this implies for the Riemann hypothesis. Compare with the zero-free region provided by Theorem 6.6.

*Proof.* By the explicit formula, if we simply let  $x$  range among  $\mathbb{R} - \mathbb{Z}$ , then we have

$$O(x^{\theta+\varepsilon}) = \lim_{T \rightarrow \infty} \sum_{\substack{\zeta(\rho)=0 \\ |\Im \rho| \leq T}} \frac{x^\rho}{\rho} + \frac{\zeta'}{\zeta}(0) + \frac{1}{2} \log \left(1 - \frac{1}{x^2}\right),$$

however, if there is a  $\rho$  with  $\Re(\rho) > \theta$ , then choosing  $\varepsilon$  such that  $\theta < \varepsilon < \Re(\rho)$ , we get that the right hand side grows faster, giving a contradiction.

Hence the Riemann hypothesis can be reformulated as saying that for any  $\varepsilon > 0$ ,

$$\psi(x) = x + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Now, any  $\theta$  would, of course, be a very strong improvement combined with the zero-free region. This is because the zero-free region tapers off as the imaginary part grows in size, while finding a  $\theta$  such that the above holds would imply, as shown, that we can shrink the critical strip to a narrower strip. □

**Exercise 6.8** (E5.5). Let  $p_n$  be the  $n$ th prime. Show that

$$\frac{1}{N} \sum_{n=1}^N \frac{p_{n+1} - p_n}{\log p_n} \rightarrow 1$$

as  $N \rightarrow \infty$ , and discuss how to interpret this as a statement about the average spacing between adjacent primes.

*Proof.* a

□