MAPPING CLASS GROUPS, BRAID GROUPS AND GEOMETRIC REPRESENTATIONS

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Contents

1. For the University of Bonn

This is a highly shortened version of the thesis which is originally 52 pages long. I have chosen the part which concerns the categorical framework into which geometric representations of braid groups can be put. In particular, my problem and goal in this thesis was to determine whether all the geometric representations in [3] arise via Proposition 3.3 on a suitably chosen monoidal category and Yang-Baxter operator. The conclusion is that they indeed all arise in such a fashion. In particular, we show that the Birman-Hilden embedding can also be obtained from a Yang-Baxter operator in the category of decorated surfaces whereas it is usually obtained in the category of bidecorated surfaces.

2. Introduction

In this thesis, we develop the theory of mapping class groups of surfaces and its connection to braid groups.
The goal of this thesis is to study connections between these groups. In particular, we study geometric representations which are homomorphisms from braid groups into the mapping class group of some surface.
Our goal is to place these representations in the more general categorical framework of monoidal categories - primarily following Harr, Vistrup and Wahl [2] and Wahl and Randal-Williams [4] where many representations are induced by so called Yang-Baxter operators on certain monoidal categories of surfaces.

Using a classification of geometric representations of braid groups on non-orientable surfaces by Stukow and Szepietowski [3], we show that each such geometric representation is obtained by an appropriate choice of monoidal category of surfaces and choice of Yang-Baxter operator - leaving certain cases out.

2.1. **Thesis Summary.** We show that general monoidal categories that come equipped with a so called Yang-Baxter element induce a geometric representation of the braid group. We construct certain categories of surfaces with Yang-Baxter elements which induce different geometric representations of the braid group, amongst which we recover the Birman-Hilden embedding.

We then turn our attention to geometric representations of the braid group on non-orientable surfaces and once again recover previously constructed representations in a new light.

3. Braided monoidal categories

3.1. Yang-Baxter operators.

Definition 3.1 (Yang-Baxter operator)**.** Let $T\colon \mathcal{A} \to \mathcal{V}$ be a functor from a category $\mathcal{A}$ to a monoidal category $\mathcal{V}$. A *Yang-Baxter operator on* T is a natural family of isomorphisms

$$y = y_{A,B}\colon TA \otimes TB \xrightarrow{\sim} TB \otimes TA$$

such that the following diagram commutes.

$$\begin{tikzcd}
(TA \otimes TB) \otimes TC \arrow[r, "a"] \arrow[d, "y\otimes TC"'] & TA \otimes (TB \otimes TC) \arrow[r, "TA\otimes y"] & TA \otimes (TC \otimes TB) \arrow[r, "a^{-1}"] & (TA \otimes TC) \otimes TB \arrow[d, "y\otimes TB"] \\
(TB \otimes TA) \otimes TC \arrow[d, "a"'] & & & (TC \otimes TA) \otimes TB \arrow[d, "a"] \\
TB \otimes (TA \otimes TC) \arrow[d, "TB\otimes y"'] & & & TC \otimes (TA \otimes TB) \arrow[d, "TC\otimes y"] \\
TB \otimes (TC \otimes TA) \arrow[r, "a^{-1}"'] & (TB \otimes TC) \otimes TA \arrow[r, "y\otimes TA"'] & (TC \otimes TB) \otimes TA \arrow[r, "a"'] & TC \otimes (TB \otimes TA)
\end{tikzcd}$$

Remark. When $\mathcal{A} = \mathbb{1}$, we say that y is a Yang-Baxter operator on $X = T(\mathcal{A}) \in \mathcal{V}$ if it is a Yang-Baxter operator on $T\colon \mathbb{1} = \mathcal{A} \to \mathcal{V}$.

Let $(\mathcal{X}, \otimes, I)$ be a monoidal category with $\tau \in \mathrm{Aut}_{\mathcal{X}}(X \otimes X)$ a Yang-Baxter operator in $\mathcal{X}$. Suppose $\mathcal{X}$ acts on a category $\mathcal{M}$ via a functor $\mathcal{M} \times \mathcal{X} \to \mathcal{M}$ which we also denote by $\otimes$. Then there is an action of the braid groupoid $\alpha_\tau\colon \mathcal{M} \times B \to \mathcal{M}$ given on objects by $\alpha_\tau(A, n) = A \otimes X^{\otimes n}$ and determined on morphisms by $\alpha_\tau(f, \sigma_i) = f \otimes \mathrm{id}_{X^{\otimes i-1}} \otimes \tau \otimes \mathrm{id}_{X^{\otimes n-i-1}}$.

Example 3.2. If $\mathcal{X} = (\mathcal{X}, \otimes, \mathcal{I})$ admits a braiding b, then $\tau = b_{X,X} \in \mathrm{Aut}_{\mathcal{X}}(X \otimes X)$ is a Yang-Baxter operator for any object X. The thing that needs verifying here is that the big Yang-Baxter diagram in definition 3.1 is satisfied, but this follows directly from proposition **??**.

Likewise, for any functor $T\colon \mathcal{A} \to \mathcal{V}$ into a braided tensor category $\mathcal{V}$, we obtain a Yang-Baxter operator as

$$y_{A,B} = c_{TA,TB}\colon TA \otimes TB \xrightarrow{\sim} TB \otimes TA.$$

In particular, we obtain a Yang-Baxter operator on the inclusion functor $\iota\colon \mathbb{1} \to \mathcal{B}$ identifying $\mathbb{1}$ with the braid of a single string. We will denote this Yang-Baxter operator by z.

We want to show that the category of strong monoidal functors from the braid groupoid into $\mathcal{X}$ is equivalent to a naturally defined category of Yang-Baxter operators in $\mathcal{X}$.

Proposition 3.3. *For any strict monoidal category $\mathcal{V}$ and any Yang-Baxter τ on an element $X \in V$, there exists a unique strict monoidal functor $\Phi_{X,\tau}\colon \mathcal{B} \to \mathcal{V}$ such that $\Phi_{X,\tau} \circ z = y$.*

Proof and construction. Define $\Phi_{X,\tau}\colon \mathcal{B} \to \mathcal{V}$ on objects by $\Phi_{X,\tau}(n) = X^{\otimes n}$. For $0 \leqslant i < n$, define

$$y_i = X^{\otimes(i-1)} \otimes y \otimes X^{\otimes(n-i-1)}\colon X^{\otimes n} \to X^{\otimes n}.$$

These satisfy the braid group relations. Thus we obtain a monoid homomorphism $\Phi_{X,\tau,n}\colon \mathcal{B}_n \to \mathcal{V}\left(X^{\otimes n}, X^{\otimes n}\right)$ taking σ_i to y_i for all $0 \leqslant i < n$. Clearly $\Phi_{X,\tau}$ is the unique strict monoidal functor with these properties. $\square$

Remark. In particular, $\Phi_{X,\tau,n}\colon \mathcal{B}_n \to \mathrm{Aut}_{\mathcal{V}}\left(X^{\otimes n}\right)$ for all n.

As we said in the beginning, our goal is to get geometric representations from Yang-Baxter operators, so given our construction of $\Phi_{X,\tau}$, we might hope to find a category $\mathcal{V}$ such that its objects are surfaces and $\mathrm{Aut}_{\mathcal{V}}\left(X^{\otimes n}\right)$ might correspond to a mapping class group. Indeed, this is what we shall do now in two different cases: the category of decorated surfaces and the category of bidecorated surfaces.

3.2. Braided monoidal category of decorated and bidecorated surfaces.

Definition 3.4 (Decorated surface)**.** A decorated surface is a pair (S, I) where S is a compact connected surface with at least one boundary component and $I\colon\ [-1,1] \hookrightarrow \partial S$ is a parametrised interval in its boundary.

Definition 3.5 ($\mathcal{M}_1$)**.** Let $\mathcal{M}_1$ denote the groupoid where the objects are decorated surfaces and morphisms are isotopy classes of diffeomorphisms/homeomorphisms restricting to the identity on a neighborhood of I.

Remark. In particular, $\mathrm{Aut}_{\mathcal{M}_1}(S) = \mathrm{Mod}(S)$.

We now construct a braided monoidal structure on $\mathcal{M}_1$: given decorated surfaces (S_1, I_1) and (S_2, I_2), define $(S_1, I_1) \otimes (S_2, I_2) := (S_1 \natural S_2, I_1 \natural I_2)$ to be the surface obtained by gluing S_1 and S_2 along the right half-interval $I_1^+ \in \partial S_1$ and the left half-interval $I_2^- \in \partial S_2$, defining $I_1 \natural I_2 = I_1^- \cup I_2^+$.
Furthermore, we define the unit object to be $I := \left(D^2, I\right)$. For it to be a strict unit, we define $\left(S_1 \natural D^2, I_1 \natural I\right) := (S_1, I_1)$ and $\left(D^2 \natural S_2, I \natural I_2\right) := (S_2, I_2)$.

Note that this category is not strict as associativity is not strict, but there is a way to construct an equivalent category which is strict [1, Section 3].

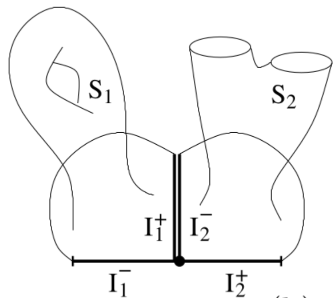
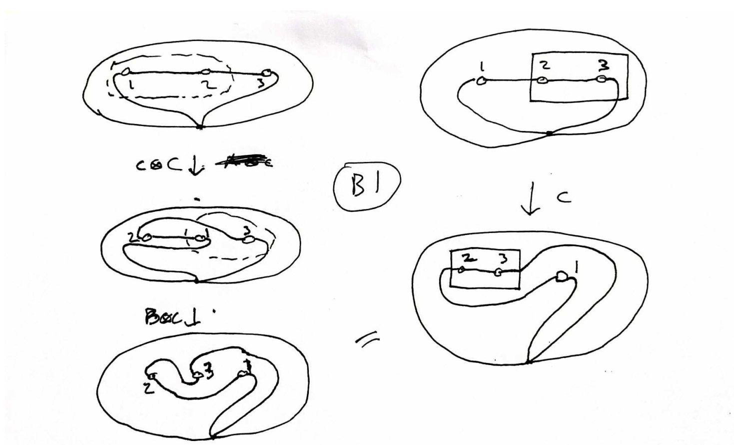
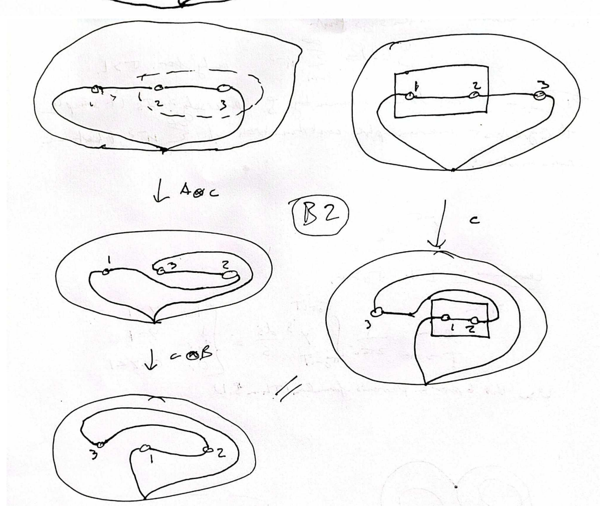


FIGURE 1. $S_1 \# S_2$ [4, Figure 2]

We define a braiding c on $(S_1 \natural S_2, I_1 \natural I_2)$ as the half-Dehn twist which satisfies that $c\colon (S_1 \natural S_2, I_1 \natural I_2) \xrightarrow{\sim} (S_2 \natural S_1, I_2 \natural I_1)$ is a natural isomorphism because it has the opposite half-Dehn twist as the inverse. It is natural because the induced map will simply be the one induced by the naturality square.
The B1 and B2 diagrams can be verified pictorially as follows:





This is precisely the requirements needed in Proposition 3.3, so we obtain a monoidal functor $\Phi\colon \mathcal{B} \to \mathcal{M}_1$ such that $\Phi \circ z = y$ where $y \in \mathrm{Aut}_{\mathcal{M}_1}(S\natural S) = \mathrm{Mod}(S\natural S)$ is the induced Yang-Baxter operator on some decorated surface S from the braiding. So again, we obtain a geometric representation $\Phi_n\colon \mathcal{B}_n \to \mathrm{Mod}\left(S^{\natural n}\right)$

We will also consider a different monoidal category of surfaces. Informally, a bidecorated surface is a surface with two intervals marked in its boundary.
To give a precise definition, we first define certain surfaces X_i that will be convenient for the monoidal structure, we set $X_1 = D^2 \subset \mathbb{C}$ to be the unit disk, and then define embeddings $\iota_1^0, \iota_1^1\colon I \to X_1$ by

$$\iota_1^0(t) = e^{i\left(\frac{\pi}{4} + t\frac{\pi}{2}\right)} \quad \text{and} \quad \iota_1^1(t) = e^{i\left(5\frac{\pi}{4} + t\frac{\pi}{2}\right)}.$$

We denote by $\overline{\iota_1^i}\colon I \to X_1$ the reverse map $t \mapsto \iota_1^i(1-t)$ for $i = 0, 1$. Then we recursively define X_{m+1} for $m \geqslant 1$ by

$$X_{m+1} := \frac{X_m \sqcup X_1}{\iota_m^i(t) \sim \overline{\iota_1^i}} \quad \text{for } t \in \left[\frac{1}{2}, 1\right]$$

and we define

$$\iota_{m+1}^i(t) = \begin{cases} \iota_m^i(t), & \text{if } t \leqslant \frac{1}{2} \\ \iota_1^i(t), & \text{else} \end{cases}.$$

In this process, the marked intervals will live in different boundary components every second time. The process for each of the two situations is illustrated below in figures 2 and 3.

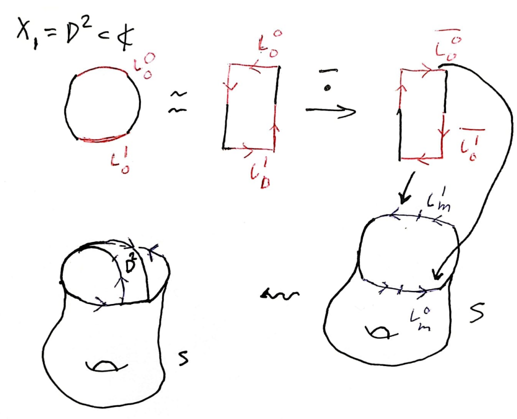


FIGURE 2. Marked intervals in single boundary components.

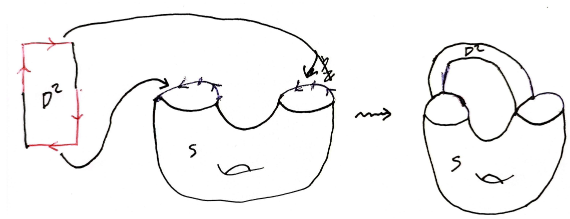


FIGURE 3. Marked intervals in different boundary components.

Lemma 3.6. *For $m \geqslant 1$, $X_m \approx S_{g,r}$ where*

$$(g,r) = \begin{cases} \left(\frac{m}{2}-1, 2\right), & m \text{ even} \\ \left(\frac{m-1}{2}, 1\right), & m \text{ odd} \end{cases}.$$

Proof. Firstly, X_m is clearly connected, and we have

$$\chi\left(X_m\right) = \chi\left(X_{m-1}\right) - 1$$

since, for example, the Δ-structure on our surface X_m can be chosen to be that for X_{m-1} with the boundary subdivided into four 2-simplices with four vertices, adding an additional two 2-simplices and then a disk. By induction, we then get $\chi\left(X_m\right) = \chi\left(X_1\right) - (m-1) = 2 - m$.
Now, by the classification of surfaces with boundary and genus, we simply need to know how many boundary components X_{m+1} has. But as can be seen from the figures, if m is odd, we will have one boundary component, while if m is even, we will have two boundary components. □

Definition 3.7 (Bidecorated surface)**.** A bidecorated surface is a tuple (S, m, φ) where S is a surface, $m \geqslant 1$ is an integer, and

$$\varphi\colon \partial X_m \sqcup \left(\sqcup_k S^1\right) \xrightarrow{\sim} \partial S$$

is a homeomorphism, giving a parametrization of the boundary of S. We think of (S, m, φ) as a surface with two parametrized arcs

$$I_0 := \varphi \circ \iota_m^0 \quad \text{and} \quad I_1 := \varphi \circ \iota_m^1$$

in its boundary, and k additional parametrized boundaries.

Definition 3.8 ($\mathcal{M}_2$)**.** Let $\mathcal{M}_2$ denote the monoidal groupoid where objects are bidecorated surfaces together with a formal unit U. The Hom set between two bidecorated surfaces (S, m, φ) and (S', m', φ') is empty if $m \neq m'$ or S and S' are nonhomeomorphic. Otherwise, the Hom set consists of all mapping classes of homeomorphisms that preserve the boundary parametrizations:

$$\mathrm{Hom}_{\mathcal{M}_2}\left((S, m, \varphi), \left(S', m, \varphi'\right)\right) = \pi_0 \mathrm{Homeo}_\partial\left(S, S'\right) = \pi_0\left\{f \in \mathrm{Homeo}\left(S, S'\right) \mid f \circ \varphi = \varphi'\right\}$$

where $\mathrm{Homeo}\left(S, S'\right)$ has the compact-open topology, and $\mathrm{Homeo}_\partial\left(S, S'\right)$ the subspace topology.

Remark. We again obtain that $\mathrm{Aut}_{\mathcal{M}_2}\left((S, m, \varphi)\right) = \mathrm{Mod}(S)$.

The monoidal structure $\natural^2$ on $\mathcal{M}_2$ is defined as follows. The object U is by definition a unit, and for the remaining objects, we define

$$(S, m, \varphi) \natural^2 (S', m', \varphi') := \left(\frac{S \sqcup S'}{I_i(t) \sim \overline{I}_i'(t), t \in \left[\frac{1}{2}, 1\right]}, m + m', \varphi \natural^2 \varphi' \right)$$

for $i = 0, 1$, and where

$$\varphi \natural^2 \varphi' \colon \partial X_{m+m'} \sqcup (\sqcup_{k+k'} S^1) \hookrightarrow \partial (S \natural^2 S')$$

is obtained using the canonical identification $\partial X_{m+m'} \approx \left(\partial X_{n - \iota_m\left(\frac{1}{2}, 1\right)}\right) \cup \left(\partial X_{m'} - \iota_{m'}\left(0, \frac{1}{2}\right)\right)$.
Now we will construct a Yang-Baxter element in $\mathcal{M}_2$ as follows. Let $D^{\natural^2 m} = D_1 \natural^2 \ldots \natural^2 (D_i \natural^2 D_{i+1}) \natural^2 \ldots \natural^2 D_m$, where subscripts are used to enumerate the disks. The underlying surface, by construction, will be X_m. Let a_i denote the isotopy class of a curve in the interior $D_i \natural^2 D_{i+1} \approx S^1 \times I$ that is parallel to its boundary components, see figure 4.

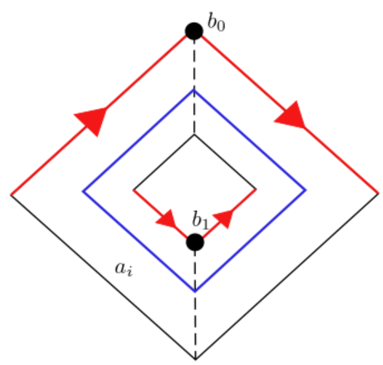


FIGURE 4. The curve a_i in $D_i \natural D_{i+1}$ [2, Figure 4]

Lemma 3.9. *The curves $a_1, \ldots, a_{m-1}$ form a chain in $D^{\natural^2 m}$.*

Proof. The curve a_i has image contained in $D_i \natural^2 D_{i+1}$, so it can only intersect a_{i-1} and a_{i+1} nontrivially. So it suffices to look at the subsurface of $D^{\natural^2 m}$ corresponding to $D_i \natural^2 D_{i+1} \natural^2 D_{i+2}$.

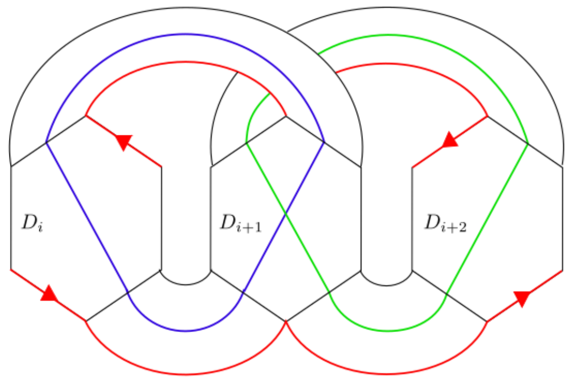


FIGURE 5. Intersection of a_i and a_{i+1} in $D_i \natural^2 D_{i+1} \natural^2 D_{i+2}$. [2, Figure 5]

$\square$

Now by Lemma ?? and Proposition ?? (the braid relation), we get that the braid group relations hold for the Dehn twists $T_i \in \mathrm{Aut}_{\mathcal{M}_2}\left(D^{\natural^2 m}\right)$ where T_i is the Dehn twist along the curve a_i in $D^{\natural^2 m}$, i.e.,

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \forall i \\ T_i T_j &= T_j T_i && \text{for } |i-j|>1 \end{aligned}$$

Hence the same relations hold for the inverses T_i^{-1}.

If we add a disk to either side of $D^{\natural^2 m}$, we get

$$T_i \natural^2 \mathrm{id}_D = T_i \quad \text{and} \quad \mathrm{id}_D \natural^2 T_i = T_{i+1}$$

in $\mathrm{Aut}_{\mathcal{M}_2}\left(D^{\natural^2 m+1}\right)$. Hence this gives the relation

$$\left(T_1^{-1}\natural^2 \mathrm{id}_D\right)\left(\mathrm{id}_D \natural^2 T_1^{-1}\right)\left(T_1^{-1}\natural^2 id_D\right) = \left(\mathrm{id}_D \natural^2 T_1^{-1}\right)\left(T_1^{-1}\natural^2 \mathrm{id}_D\right)\left(\mathrm{id}_D \natural^2 T_1^{-1}\right)$$

in $\mathrm{Aut}_{\mathcal{M}_2}\left(D^{\natural^2 3}\right)$, meaning that T_1^{-1} is a Yang-Baxter element. This yields a monoidal functor

$$\Phi = \Phi_{D,T_1^{-1}} \colon (\mathcal{B},\otimes) \to \left(\mathcal{M}_2, \natural^2\right)$$

uniquely determined up to monoidal natural isomorphism by $\Phi(n) = D^{\natural^2 n}$ and $\Phi_{D,T_1^{-1},n}(\sigma_1) = D^{\natural^2 i-1}\natural^2 T_1^{-1}\natural^2 D^{\natural^2 n-i-1} = T_i^{-1} \in \mathrm{Aut}_{\mathcal{M}_2}\left(D^{\natural^2 m}\right) = \pi_0\mathrm{Homeo}_\partial(X_m)$. Seeing that this is exactly the setup for the Birman-Hilden theorem, we note that the homomorphisms

$$\Phi_m = \Phi_{D,T_1^{-1},m} \colon B_m \to \mathrm{Aut}_{\mathcal{M}_2}\left(D^{\natural^2 m}\right) \approx \mathrm{Aut}_{\mathcal{M}_2}(X_m) \approx \begin{cases} \mathrm{Mod}\left(S_{\frac{m}{2}-1,2}\right), & m \text{ even} \\ \mathrm{Mod}\left(S_{\frac{m-1}{2},1}\right), & m \text{ odd} \end{cases}.$$

recover the Birman-Hilden embeddings from section ??.

4. Geometric Representations of the Braid Group on non-orientable surfaces

We will now look at the classified geometric representations of braid groups on non-orientable surfaces as presented in [3]. In particular, we will explore how the representations fit in our categorical framework from the previous section.

We start by noting some facts about non-orientable surfaces.

A connected orientable (respectively nonorientable) surface of genus g with b boundary components will be denoted by $S_{g,b}$ (respectively $N_{g,b}$).
Now, recall that the Möbius band (which is also called a crosscap), is the mapping cylinder on the map $z \mapsto z^2$ (see figure 6) We will denote the Möbius band by M.

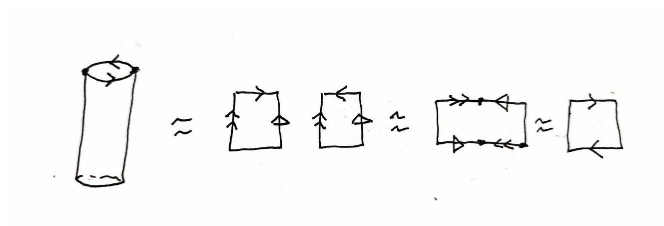


Figure 6. Möbius band

In this case, any of the two curves making up the half-circles which get identified can be cut along, rendering a connected surface. Thus gluing on a Möbius strip along the boundary, we obtain a surface of one higher genus.

We can then obtain $N_{g,b}$ from $S_{0,g+b}$ by gluing g Möbius bands along g distinct boundary components of $S_{0,g+b}$.
Note that by the classification of surfaces

$$N_{g,b} \approx \left(\mathbb{RP}^2\right)^{\#g} - \sqcup_b \mathring{D} \approx \left(\mathbb{RP}^2 - \mathring{D}\right)^{\natural g} - \sqcup_{b-1} \mathring{D}$$

where $\mathbb{RP}^2 - \mathring{D}$ is a Möbius band.
Note also that $\mathbb{RP}^2 \# \mathbb{RP}^2 \approx K$, the Klein bottle, see figure 7.

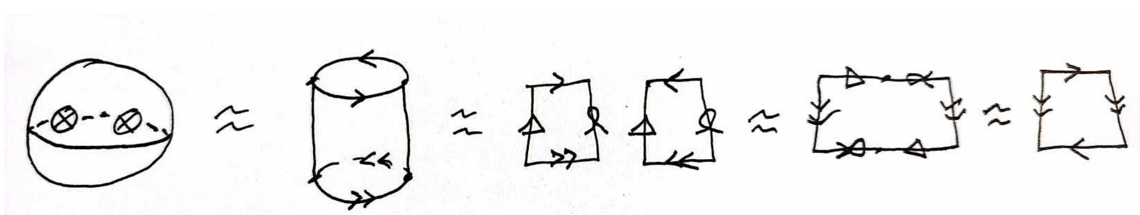


Figure 7. Klein-bottle as the sphere with two crosscaps

Lemma 4.1. *$N_{g,1} \approx S_{n,1} \natural M$ for $g = 2n + 1$.*

Proof. For $g = 2n+1$ with $n \geqslant 0$, we have (see 8 for a visual argument for $T^2\#M \approx K\#M$ - to see why the top-right surface is $K - D^2$, take the usual Klein-bottle, remove a disk such that it is embeddable in $\mathbb{R}^3$, and then enlarge the hole.)

$$\begin{aligned} N_{2n+1,1} \approx \left(\mathbb{RP}^2\right)^{\#2n+1} - \mathring{D} &\approx K^{\#n}\#M \approx K^{\#n-1}\#K\#M \\ &\approx K^{\#n-1}\#T^2\#M \\ &\approx \dots \\ &\approx \left(T^2\right)^{\#n}\#M \\ &\approx S_{n,0}\#M \\ &\approx S_{n,1}\natural M \end{aligned}$$

□

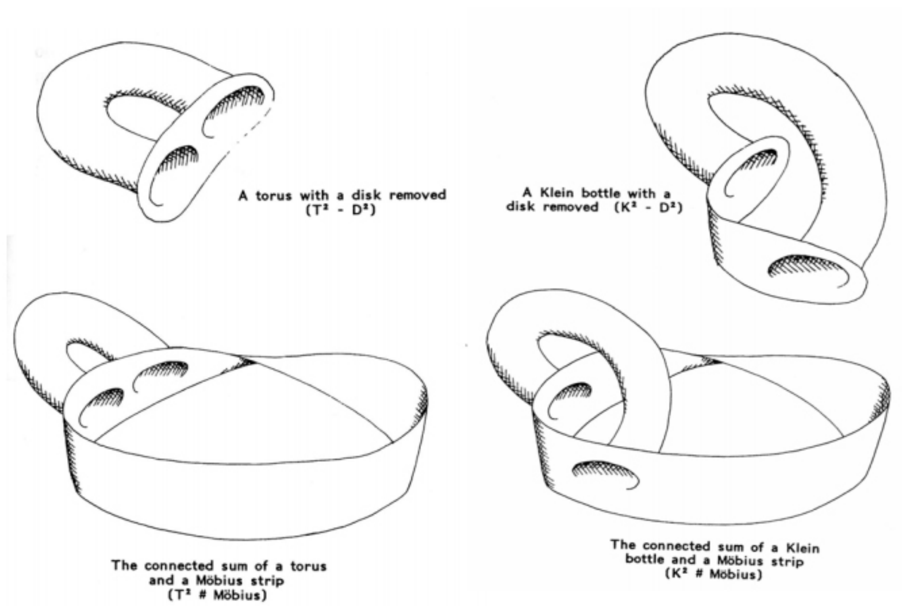


FIGURE 8. A hemeomorphism between $T^2\#M$ and $K\#M$ [5, Figure 5.7]

Hence for g odd, we have an embedding $B_g \hookrightarrow N_{g,1}$ by the Birman-Hilden embedding into the $S_{n,1}$ summand. A similar thing can be done for the even case.

We will now introduce different types of representations of the braid group on non-orientable surfaces.

Definition 4.2. We call a curve two-sided (resp. one-sided) if its regular neighborhood is an annulus (resp a Möbius band).

4.0.1. *The standard twist representation.*

Lemma 4.3. *Take a chain of two-sided curves $C = (a_1, \dots, a_{n-1})$. If we fix an orientation of a regular neighborhood of the union of the curves a_i, then C determines the standard twist representation $\rho_C \colon B_n \to \mathrm{PMod}(S)$ defined by*

$$\rho_C(\sigma_i) = t_{a_i}, \quad i = 1, \dots, n-1,$$

where t_{a_i} is the right-handed Dehn twist about a_i with respect to the orientation.

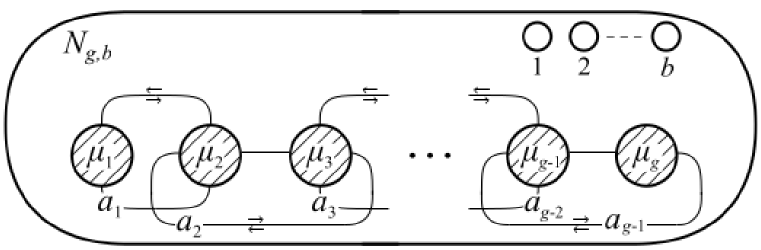


FIGURE 9. Standard chain of non-separating curves in $N_{g,b}$ [3, Figure 1]

Question 4.4. One might wonder whether the standard twist representation corresponds to an induced representation from a Yang-Baxter operator on some category of surfaces.

To answer this question, the intuitive monoidal category to look at given Figure 9 is $\mathcal{M}_1$ with a Yang-Baxter operator on the Möbius band with the parametrized interval as depicted on the left in Figure 10 and the Yang-Baxter operator given by the homeomorphism which is the Dehn twist about the curve in $M \natural M$ depicted on the right side in Figure 10.
Continuing in this manner, we find that $M^{\natural g}$ is as depicted in Figure 11 which is homeomorphic to $N_{g,1}$, and we see that the loops precisely coincide with those from the standard chain depicted in Figure 9.
Now, note that the Yang-Baxter equation in this case again is satisfied because of the braid relation for Dehn twists, Proposition **??**. Thus we obtain a monoidal functor $\Phi_{M,std} \colon \mathcal{B} \to \mathcal{M}_1$ with $\Phi_{M,std,g} \colon \mathcal{B}_n \to \mathrm{Mod}(N_{g,1})$ the standard twist representation.

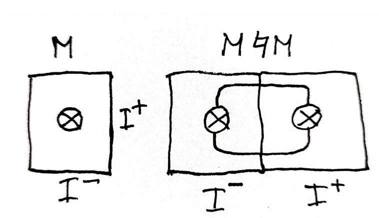


FIGURE 10. mobius-decorated.jpg

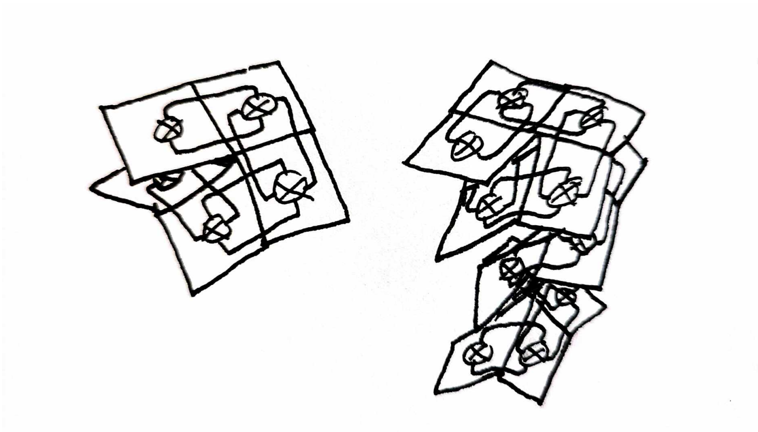


FIGURE 11. The standard chain in $M^{\natural g} \approx N_{g,1}$.

Recall that the Birman-Hilden embedding was obtained as an induced geometric representation from a Yang-Baxter operator in the category of bidecorated surfaces. The following proposition shows that the standard twist representation, in fact, is also the Birman-Hilden embedding now obtained from a Yang-Baxter operator in the category of decorated surfaces and in a seemingly different form.

Proposition 4.5. *For $b \geqslant 1$ and g odd, the standard twist representation $\rho_C\colon B_g \to \operatorname{Mod}(N_{g,b})$ is the same as the Birman-Hilden embedding $B_g \hookrightarrow S_{\frac{g-1}{2},b-1}\#M$ into the orientable factor.*

Proof. To make use of the visualization from Figure 8, we need the steps to go from $K - D^2$ as a sphere with two crosscaps and a disk removed to the depiction in Figure 8.
Suppose we look at the standard chain of non-separating curves in $N_{3,b}$, so we have curves a_1 and a_2 as in Figure 9. Now, since $N_{3,1} = K\#M$, we can decompose the loops a_1 and a_2 and follow it through the homeomorphisms for $K - D$ and for $M - D$ as in in Figure 12 and Figure 13.
After the transformations, we reglue the surfaces and twist the tube from the Klein bottle around to obtain a torus as in Figure 14.
Now, noting that

$$N_{2g+1,b} \approx \left(\left(\mathbb{RP}^2 - \mathring{D}\right) \natural \left(\mathbb{RP}^2 - \mathring{D}\right)\right)^{\natural g} \natural M - \bigsqcup_{b-1} \mathring{D} = K^{\natural g} \natural M - \bigsqcup_{b-1} \mathring{D} = \left(K^{\natural g}\right)$$

we find that for $N_{2g+1,1}$ with $g > 1$, we get a similar picture as for $N_{3,1}$, depicted in Figure 15, where we can also twist the tube around.
In conclusion, the loops really correspond to a standard chain in S_g, so the standard twist representation $\rho_C\colon B_{2g+1} \to \operatorname{Mod}\left(N_{2g+1,1}\right) \approx \operatorname{Mod}\left(S_{g,1}\natural M\right)$ corresponds to the Birman-Hilden embedding into the $S_{g,1}$ factor.

□

Note. In the case where we do not have boundary components, we can simple remove a disk, perform the operations from Proposition 4.5 and then reglue. Thus we can actually extend the proposition to the case of $b = 0$ as well.

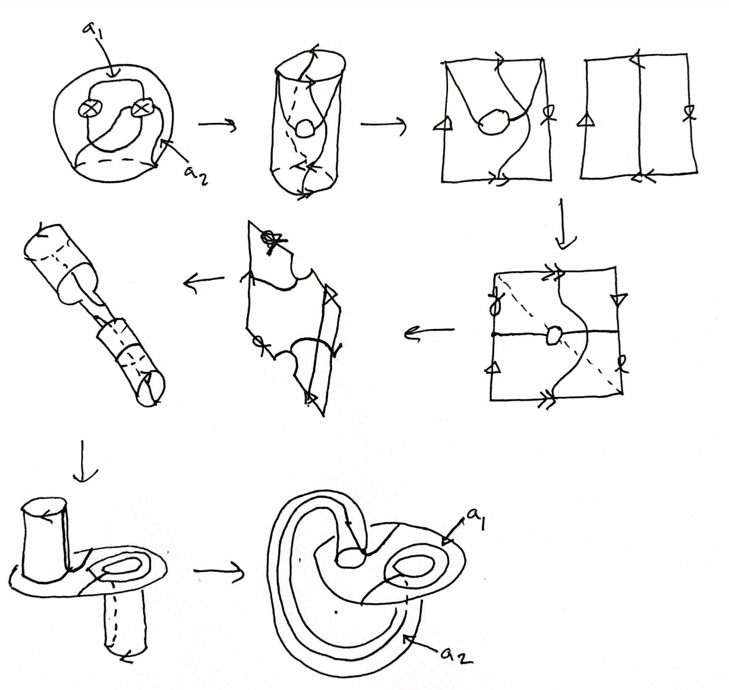


FIGURE 12. The loop a_1 and the arc a_2 throughout the homeomorphisms for $K - D^2$

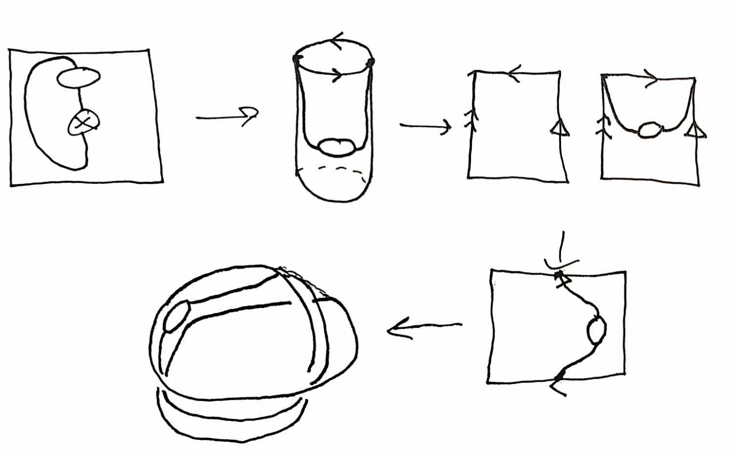


FIGURE 13. The arc from a_2 throughout the homeomorphisms for $M - D^2$

Question 4.6. What happens in the case where g is even?

Question 4.7. If g is odd and $g \geqslant 5$, the standard chain can be extended by adding a curve a_g passing once through each of the first $g - 1$ crosscaps. This extended

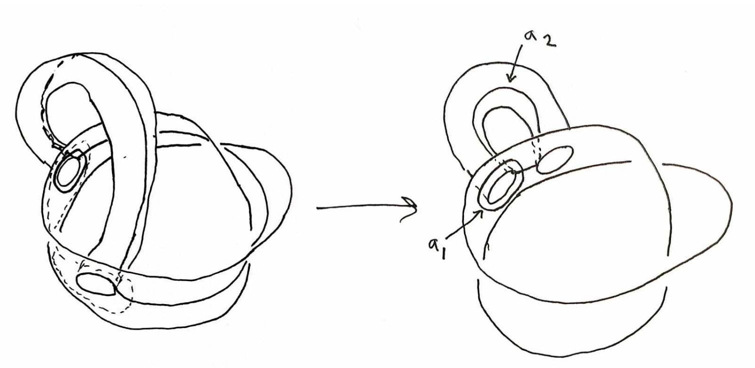


FIGURE 14. The connected sum of K and M by regluing.

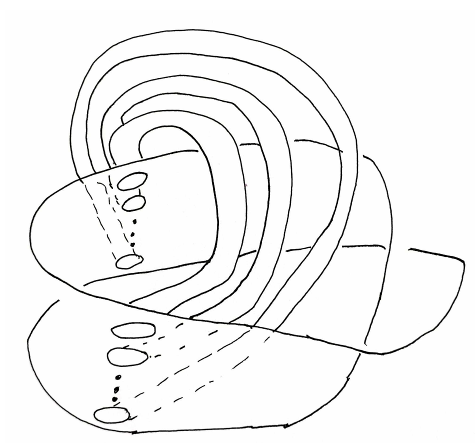


FIGURE 15. The case of $K^{\natural g}\natural M$

chain also determines a standard twist representation $\rho_{C'}\colon \mathcal{B}_{g+1} \to \mathrm{PMod}\,(N_{g,b})$. Does it correspond to something we know as well?

4.0.2. *The crosscap transposition representation.* Let $N = N_{g,b}$ be nonorientable. A sequence $C = (a_1, \dots, a_{n-1})$ of separating curves in N is called *a chain of separating curves* if

(1) a_i bounds a one-holed Klein bottle for $i = 1, \dots, n-1$,
(2) $i\,(a_i, a_{i+1}) = 2$ for $i = 1, \dots, n-2$,
(3) $i\,(a_i, a_j) = 0$ for $|i - j| > 1$.

Here a_i bounding a one-holed Klein bottle means that if we collapse a_i to a point, we obtain a sphere with two crosscaps which is equivalent to the Klein bottle
Let K_i be the one-holed Klein bottle bounded by a_i. Then $K_i \cap K_{i+1}$ will be a Möbius strip for $i = 1, \dots, n-2$, and we denote its core curve by μ_{i+1}. Let μ_1 and μ_n be the core curves of $K_1 - K_2$ and $K_{n-1} - K_{n-2}$, respectively. Fix an orientation of a regular neighborhood of the union of the a_i. Let T_{a_i} be the right-handed Dehn twist about $a_i$and let u_i be the *crosscap transpostion* supported in K_i, swapping μ_i

and μ_{i+1} such that $u_i^2 = T_{a_i}$ (essentially a half-Dehn twist but for crosscaps instead of punctures).

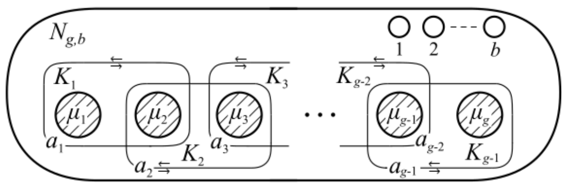


FIGURE 16. A chain of separating curves in $N_{g,b}$. [3, Figure 2]

Lemma 4.8. *The mapping* $\theta_C : B_n \to \mathrm{PMod}(N)$ *by* $\theta_C(\sigma_i) = u_i$ *for* $i = 1, \ldots, n-1$*, defines a homomorphisms called crosscap transposition representation.*

Remark. This is simply the geometric representation arising from the Yang-Baxter element associated to the Möbius band in $\mathcal{M}_1$ which is the half-twist on crosscaps.

4.0.3. *Transvection.*

Definition 4.9 (Transvection)**.** Given a homomorphism $\rho: B_n \to \mathrm{PMod}(S)$ and an element $\tau \in \mathrm{PMod}(S)$ such that τ commutes with $\rho(\sigma_i)$ for $1 \leqslant i \leqslant n-1$, we define a homomorphism $\rho^\tau : B_n \to \mathrm{Mod}(S)$, called a *transvection* of ρ, by

$$\rho^\tau(\sigma_i) = \tau\rho(\sigma_i), \quad i = 1, \ldots, n-1.$$

A homomorphism $\rho: B_n \to \mathrm{PMod}(S)$ is called *cyclic* if $\rho(B_n)$ is a cyclic group.

Note. Note that transvection defines an equivalence relation on the set of representations $B_n \to \mathrm{PMod}(S)$.

4.1. **The main theorems.** Following work by Castel and generalizations by Chen and Mukherjea on classifications of geometric representations of the braid group on orientable surfaces in certain ranges, Stukow and Szepietowski proved the following two theorems in [3] which classify all geometric representations on non-orientable surfaces in a certain range.

Theorem 4.10. *Let* $n \geqslant 14$ *and let* $N = N_{g,b}$ *with* $g \leqslant 2\left\lfloor \frac{n}{2} \right\rfloor + 1$ *and* $b \geqslant 0$*. Then any homomorphism* $\rho: B_n \to \mathrm{PMod}(N)$ *is either cyclic, or is a transvection of a standard twist representation, or is a transvection of a crosscap transposition representation.*

Theorem 4.11. *Theorem 4.10 still holds when* $\mathrm{PMod}(N)$ *is replaced by* $\mathrm{Mod}(N, \partial N)$*.*

Note that $\mathrm{Mod}(N, \partial N) \nleqslant \mathrm{PMod}(N)$ as, for example, the Dehn twist about a boundary curve is non-trivial in $\mathrm{Mod}(N, \partial N)$, but becomes trivial in $\mathrm{PMod}(N)$.

We have shown that the standard twist representation for odd genus and the crosscap transposition naturally arise as Yang-Baxter operators on appropriate objects in an appropriate category of surfaces.
By Theorem 4.10, these are in fact all the possible explicit geometric representations which we can look at on non-orientable surfaces up to transvection given in [3].

References

[1] Søren Galatius, Alexander Kupers, and Oscar Randal-Williams. "E_2-Cells and Mapping Class Groups". In: *Publ. Math. Inst. Hautes Études Sci. 130 (2019), 1–61.* 130 (2018). DOI: 10.48550/arXiv.1805.07187.

[2] Oscar Harr, Max Vistrup, and Nathalie Wahl. *Disordered arcs and Harer stability*. 2022. DOI: 10.48550/arXiv.2211.03858.

[3] Michał Stukow and Błażej Szepietowski. *Geometric representations of the braid group on a nonorientable surface*. 2024. DOI: 10.48550/arXiv.2408.04707.

[4] Nathalie Wahl and Oscar Randal-Williams. "Homological stability for automorphism groups". In: *Adv. Math.* 318 (2017). DOI: 10.1016/j.aim.2017.07.022. URL: https://doi.org/10.1016/j.aim.2017.07.022.

[5] Jeffrey R. Weeks. *The shape of space*. Second. Vol. 249. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 2002. ISBN: 0-8247-0709-5.