ASSIGNMENT 1

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Problem 0.1 (1)**.** Classify all topological 1-manifolds up to homeomorphism.

Proof. Let M be a 1-manifold. Since components of a manifold are themselves manifolds, we may assume M is connected. We also only consider manifolds without boundary. The claim is that M is homeomorphic to S^1 or $\mathbb{R}$.
Suppose (U, φ) is a maximal chart in the sense that it is not contained in any other chart. If U covers all of M, then $M \cong \varphi(U)$ is a connected open subset of $\mathbb{R}$ which is an interval, hence homeomorphic to $\mathbb{R}$. So suppose U does not cover all of M.
Since M is locally-compact (as it is locally Euclidean) and Hausdorff, M has a one-point compactification.
We must check that the one-point compactification of a 1-manifold is still a 1-manifold. This we will not show for now.

Suppose (U, φ) is a maximal chart for M which is now compact. Compact connected subsets of $\mathbb{R}$ are closed bounded intervals which are 1-manifolds with boundary, so U cannot cover all of M. Since U is open, it cannot be closed also since closed and open sets would give a contradiction to the connectedness assumption. Hence there exists a point $x \in M - U$ which is a limit point of U. Let (V, ψ) be a connected chart about x. Then $U \cap V$ has at least one component. Let $W \subset U \cap V$ be the component containing x. Then $\varphi(W), \psi(W)$ are open intervals in $\mathbb{R}$, hence homeomorphic. Let $f \colon \mathbb{R} \to \mathbb{R}$ be the homeomorphism sending $\psi(W)$ to $\varphi(W)$. Then φ and $f \circ \psi$ agree on W, so we can extend φ to a chart $(\varphi \cup f \circ \psi, U \cup B)$ where $B = B(x, \varepsilon) \subset V$ is some small open ball, defined by

$$\varphi \cup f \circ \psi(x) = \begin{cases} \varphi(x), & x \in U \\ f \circ \psi(x), & x \in B \end{cases}.$$

If $U \cup B$ yields a larger open set, this contradicts maximality of (φ, U), hence not such ball B can exist.
But then the intersection $V \cap U$ must have at least two components. If it have two components, then we can easily define a map $S^1 \to V \cup U = M$ such that the bottom hemisphere is send into U and the top is sent into V and some neighborhoods about $(1, 0)$ and $(-1, 0)$ are sent to the two components of $U \cap V$, giving a homeomorphism $M \cong S^1$. To see that there cannot be more than two components, we cite the graph argument that Milnor gives on page 56 in his Topology from a differentiable viewpoint.
Now as the non-compact connected manifolds have a one-point compactification by the above, we can obtain any non-compact one-manifold by deleting a point from S^1 which indeed gives a space homeomorphic to $S^1 - \{1, 0\} \cong \mathbb{R}$.

□

Problem 0.2 (2)**.** Show that the following spaces are topological manifolds

(1) $\mathbb{RP}^n, n \in \mathbb{N}$,
(2) $\mathbb{CP}^n, n \in \mathbb{N}$,
(3) Stiefel manifolds $V_d\left(\mathbb{R}^n\right)$: for $d \leq n$, let $V_d\left(\mathbb{R}^n\right)$ be the set of d-frames in $\mathbb{R}^n$, i.e., the collection of linearly independent vectors $v_1, \ldots, v_d \in \mathbb{R}^n$. This inherits a topology as a subspace of $\mathbb{R}^{nd}$.
(4) Stiefel manifolds: for $d \leq n$, let $\tilde{V}_d\left(\mathbb{R}^n\right)$ be the collection of orthonormal frames in $\mathbb{R}^n$ (with respect to the standard inner product on $\mathbb{R}^n$). Again, this inherits a topology as a subspace of $\mathbb{R}^{nd}$.
(5) $\mathrm{GL}_n\left(\mathbb{R}\right)/B_n\left(\mathbb{R}\right)$, where $B_n(\mathbb{R})$ consists of invertible upper triangular matrices.
(6) Show whether each of the above examples is compact or not.

Proof. (1) *Hausdorff:* let $\overline{x}, \overline{y} \in \mathbb{RP}^n$ be distinct. We have $\mathbb{RP}^n := S^n/\sim$ where $a, b \in S^n$ are identified if and only if $a = \pm b$. Hence $x \neq \pm y$. Take some neighborhood U_x of x which is disjoint from $-x, y, -y$ by Hausdorffness. Now let $U = U_y \cup -U_y$ where $-U_y = \{-u \mid u \in U_y\}$ which is homeomorphic to U since the antipodal map is a homeomorphism. Since $\mathbb{R}^{n+1}$ is a regular space, S^n is also regular, so there exist disjoint open sets V_y, W_y such that $\overline{U} \subset W_y$ and $y \in V_y$. Similarly, there exist open sets V_{-y}, W_{-y} such that $\overline{U} \subset W_{-y}$ and $-y \in V_{-y}$. Let $V' = V_y \cap -V_{-y}$, and then $V = V' \cup -V'$. Then $V \cap U \subset V' \cup -V' \cap (W_y \cap W_{-y}) = \varnothing$ since $V' \subset V_y$ and $-V' \subset V_{-y}$.
Furthermore, V and U are saturated with respect to the quotient map. I.e., $V = \pi^{-1}\left(\pi(V)\right)$ and $U = \pi^{-1}\left(\pi(U)\right)$, so $\pi(V), \pi(U)$ are open sets in $\mathbb{RP}^n$, and they form disjoint neighborhoods around $\overline{x}$ and $\overline{y}$.
Second-countable: Suppose we take the collection $\mathcal{B}$ consisting of open sets

$$\pi\left(B(x, \frac{1}{n}) \cup B\left(-x, \frac{1}{n}\right) \cap S^n\right)$$

with $n \in \mathbb{N}$ and $x \in \mathbb{Q}^{n+1} \cap S^n$. Then $\mathcal{B}$ is countable and consists of open sets in $\mathbb{RP}^n$. Now for any open set $U \subset \mathbb{RP}^n$ containing some $\overline{x} \in U$, we can take some $x \in \mathbb{Q}^{n+1} \cap S^n$ and some $n \in \mathbb{N}$ such that $\left(B\left(x, \frac{1}{n}\right) \cup B\left(-x, \frac{1}{n}\right)\right) \cap S^n \subset \pi^{-1}(U)$, hence U contains some open set $V \in \mathcal{B}$ such that $x \in V \subset U$. So $\mathcal{B}$ is a countable basis for $\mathbb{RP}^n$.
Different way of showing Hausdorff and second-countable if we are using a different definition of $\mathbb{RP}^n$: If we are using the definition of $\mathbb{RP}^n$ as the space of lines in $\mathbb{R}^{n+1}$, we can show Hausdorffness in a different way:

Proof. Let $[x], [y] \in \mathbb{RP}^n$ be distinct. Then $\frac{x}{\|x\|} \neq \frac{y}{\|y\|}$. Define $\rho \colon \mathbb{R}^{n+1} - \{0\} \to S^n$ by $\rho(x) = \frac{x}{\|x\|}$. Since S^n is Hausdorff, we can find open sets U, V around $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$, respectively. Since ρ is continuous, $\rho^{-1}(U)$ and $\rho^{-1}(V)$ are open and contain x and y. We claim that they are saturated with respect to $\pi \colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$. Suppose $w \in \pi^{-1}\left(\pi\left(\rho^{-1}(U)\right)\right)$. Then $[w] \in \pi\left(\rho^{-1}(U)\right)$. So there exists $u \in U$ such that $\pi\left(\rho^{-1}\left(u\right)\right) = [w]$. Then clearly, $u = \frac{w}{\|w\|}$, so $w = u\|w\| \in \rho^{-1}(U)$. Hence $\pi\left(\rho^{-1}(U)\right)$ and $\pi\left(\rho^{-1}(V)\right)$ are open sets around $[x]$ and $[y]$, respectively. Now, define an action on S^n by $\mathbb{Z}_2 \times S^n \to S^n$ by $(0, x) = x$ and $(1, x) = -x$. Then the orbit space is second countable since the quotient map is open, so as the orbit space is $\mathbb{RP}^n$, we are done.
Alternatively, for the Hausdorff condition, we can also say that $\mathbb{Z}_2$ acts properly on S^n and since if a group G acts properly on a Hausdorff space X then X/G is Hausdorff, we conclude that $\mathbb{RP}^n$ is Hausdorff. $\square$

Locally-Euclidean: Take a point $[x] = [x_1, \dots, x_n, x_{n+1}] \in \mathbb{RP}^n$. Then some x_i is non-zero. Let $U_i := \{[z_1, \dots, z_{n+1}] : z_i \neq 0\} \subset \mathbb{RP}^n$. So $[x] \in U_i$. Now define the map $\varphi_i \colon U_i \to \mathbb{R}^n$ by $\varphi([z_1, \dots, z_{n+1}]) = \left(\frac{z_1}{z_i}, \dots, \hat{z_i}, \dots, \frac{z_{n+1}}{z_i}\right)$ where $\hat{z_i}$ means that this coordinate is excluded. This is well-defined since $\frac{z_j}{z_i} = \frac{\lambda z_j}{\lambda z_i}$ for all $\lambda \neq 0$. In fact, this is a homeomorphism with inverse $\psi_i \colon \mathbb{R}^n \to U_i$ given by $\psi(y_1, \dots, y_n) = [y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n]$. Hence we choose (U_i, φ_i) to be one chart for $\mathbb{RP}^n$. The collection $U_1, \dots, U_{n+1}$ covers $\mathbb{RP}^n$, hence the collection of charts $\{U_i, \varphi_i\}$ for $i = 1, \dots, n+1$, gives an atlas for $\mathbb{RP}^n$.

(2) *Hausdorff:* The Hausdorff condition for $\mathbb{CP}^n$ is checked in the same way as for $\mathbb{RP}^n$, noting that $\mathbb{C}^{n+1}$ is a regular space.
Second-countable: This is also checked similarly as for $\mathbb{RP}^n$ where the collection $\mathcal{B}$ above is now taken for $x \in \mathbb{Q}[i]^{n+1} \cap S^n$.
Locally-Euclidean: Take the same open sets U_i as for $\mathbb{RP}^n$, now as subsets of $\mathbb{CP}^n$, and define φ_i and ψ_i in the same way. It is again clear that φ_i and ψ_i are inverses of each other, hence $\varphi_i \colon U_i \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$. So composing the charts φ_i with the isomorphism $\mathbb{C}^n \cong \mathbb{R}^{2n}$ gives the desired map - denote this also by φ_i. Then the collection $(U_i, \varphi_i)\,, i = 1, \dots, n+1$ will again define an atlas for $\mathbb{CP}^n$.

(3) *Hausdorff:* Let $(v_1, \dots, v_d)\,, (w_1, \dots, w_d) \in V_d(\mathbb{R}^n)$ be distinct d-frames. Then there exists i such that $v_i \neq w_i$, hence a coordinate j such that $v_{ij} \neq w_{ij}$. Take disjoint neighborhoods V, W containing v_{ij} and w_{ij}, respectively, in $\mathbb{R}$. Then $\mathbb{R}^{ij-1} \times V \times \mathbb{R}^{nd-ij} \cap V_d(\mathbb{R}^n)$ and $\mathbb{R}^{ij-1} \times W \times \mathbb{R}^{nd-ij} \cap V_d(\mathbb{R}^n)$ are disjoint neighborhoods of $(v_1, \dots, v_d)$ and $(w_1, \dots, w_d)$, respectively.

Second-countable: Every subspace of a second-countable space is second-countable, and as $\mathbb{R}^{nd}$ is second-countable, so is $V_d(\mathbb{R}^n)$.

Locally-Euclidean: Consider some d-frame $(v_1, \dots, v_d)$ as an $n \times d$ matrix. Since the columns are linearly independent, there exists some $d \times d$ submatrix with non-vanishing determinant. By continuity of of the determinant function, this $d \times d$ matrix has an open neighborhood in $M_d(\mathbb{R}^n)$ on which the determinant function is non-vanishing. Extend this neighborhood to one on $\mathbb{R}^{nd}$ by choosing $\mathbb{R}$ for the other coordinates and then taking the product of these sets to get an open neighborhood of $(v_1, \dots, v_d)$. On this neighborhood, the corresponding matrices have a $d \times d$ submatrix with non-vanishing determinant which means the d columns are linearly independent. Thus this open set is in fact contained in $V_d(\mathbb{R}^n)$ and hence also an open set in $V_d(\mathbb{R}^n)$ as we use the subspace topology. Naturally, the chart on this open set we choose to simply be the one sending $(w_1, \dots, w_d)$ to its coordinates in $\mathbb{R}^{nd}$. This is a bijective open map which is continuous since on any open set contained in the open set on $\mathbb{R}^{nd}$ which is the image of the chart, we still have the $d \times d$ submatrix on which the determinant function is non-vanishing, hence still linearly independent columns. Since a bijective, open continuous map is a homeomorphism, this gives a chart for $(v_1, \dots, v_d)$.

(d) *Hausdorff:* any orthonormal frame is also linearly independent, so we can use the same open sets intersecting with $\tilde{V}_d(\mathbb{R}^n)$ as for $V_d(\mathbb{R}^n)$ above.

Second-countable: Every subspace of a second-countable space is second-countable, and as $\mathbb{R}^{nd}$ is second-countable, so is $\tilde{V}_d(\mathbb{R}^n)$.

Locally-Euclidean: Suppose we are given an orthonormal frame $(v_1, \dots, v_d) \in \tilde{V}_d(\mathbb{R}^n)$. This information can be expressed by saying that $A^T A = I_d$ where A is the matrix whose ith column is v_i. But this is equivalent to saying that the entries of A satisfy $\sum_r \alpha_{ri}\alpha_{rj} = 0$ when $i \neq j$ and $\sum_r \alpha_{ri}^2 = 1$ which is the solution set of a map $\mathbb{R}^{nd} \to \mathbb{R}^d \times \mathbb{R}^{\frac{n(n-1)}{2}}$ given by $(x_1, \dots, x_{nd}) \mapsto \left(\sum_r x_{r1}^2 - 1, \sum_r x_{r2}^2 - 1, \dots, \sum_r x_{rd}^2 - 1, \sum_r x_{r1}x_{r2}, \dots, \sum_r x_{rn-1}\right.$

$$nd - \left(d + \frac{n(n-1)}{2}\right) =$$

(6) Since $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are quotients of S^n which is compact in $\mathbb{R}^{n+1}$ and $\mathbb{C}^{n+1}$, and quotient maps are continuous maps hence preserve compactness, we find that $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are compact.
For Stiefel manifolds, note that we showed that $V_d(\mathbb{R}^n)$ is an open subset of $\mathbb{R}^{nd}$ which is Hausdorff. Compact subsets of a Hausdorff space are closed, so $V_d(\mathbb{R}^n)$ would be closed and open, implying that $\mathbb{R}^{nd}$ is not connected, which is a contradiction. Thus $\mathbb{R}^{nd}$ cannot be compact.
For the Stiefel manifold of orthonormal frames, we expressed this as the zero set of a continuous map between Euclidean spaces, hence it is a closed subset of $\mathbb{R}^{nd}$. Furthermore, this set is also bounded since each entry of the $n \times d$ matrix is bounded by 1 as each column must have norm 1. Closed bounded subsets of $\mathbb{R}^{nd}$ are compact, hence $\tilde{V}_d(\mathbb{R}^n)$ is compact.

Suppose we take the flag $1 \subset E_1 := \operatorname{span}(e_1) \subset E_2 := \operatorname{span}(e_1, e_2) \subset \dots \subset \operatorname{span} E_{n-1} := (e_1, \dots, e_{n-1}) \subset \mathbb{R}^n := E_n$. Then any $A \in \mathrm{GL}_n(\mathbb{R})$ sends this flag to some other flag, and $B_n(\mathbb{R})$ is precisely the stabilizer of the flag under this action. Thus $\mathrm{GL}_n(\mathbb{R})/B_n(\mathbb{R})$ can be identified with all possible flags of length n in $\mathbb{R}^n$.
However, for any flag we can choose a basis for each space in the flag of orthonormal vectors. Now, putting these vectors in the columns of a matrix, we obtain a matrix in $O(n)$. Two such matrices represent the same flag if there is a matrix sending the i th column of one flag to the i th column of another flag. Thus $\mathrm{GL}_n(\mathbb{R})/B_n(\mathbb{R})$ can also be seen as a quotient of $O(n)$ which is compact as a closed bounded subset of $\mathbb{R}^{n^2}$, and so $\mathrm{GL}_n(\mathbb{R})/B_n(\mathbb{R})$ is also compact.

□

Problem 0.3 (3).
(1) Show that $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.
(2) Show that the dimension of a connected topological manifold is unique, i.e., such a manifold M cannot have dimensions m and n with $m \neq n$.

Proof. (1) If $m = n$, then $\mathbb{R}^m \cong \mathbb{R}^n$ by the identity.
Suppose $m \neq n$ and $\mathbb{R}^m \cong \mathbb{R}^n$. Then also the one point compactifications are homeomorphic, so $S^m \cong S^n$. Suppose $m > n \geq 0$. Then $\mathbb{Z} \cong H_m(S^m) \cong H_m(S^n) \cong 0$ which is a contradiction.

(2) Suppose a manifold M has dimensions m and n with $m \neq n$. Let $p \in M$ and choose two charts $(U, \varphi), (V, \psi)$ centered at 0 such that $\varphi(U) \subset \mathbb{R}^n$ and $\psi(V) \subset \mathbb{R}^m$, and assume $n > m$. We can take some $B(0, \varepsilon) \subset \varphi(U)$ and take its preimage under

φ to replace U with an open set whose image under φ is $B(0,\varepsilon)$, so assume without loss of generality that $\varphi(U) = B(0,\varepsilon)$. Then $\psi \circ \varphi^{-1} \colon \overline{B(0,\varepsilon)} \to \psi(U \cap V) \subset \mathbb{R}^m$ is a homeomorphism. Restricting to $S^{n-1} \cong \partial B\left(0, \frac{\varepsilon}{2}\right)$, we get an embedding $S^{n-1} \hookrightarrow \mathbb{R}^m$. But by the standard inclusion $S^m \hookrightarrow S^{n-1}$, we get an injective composite map $S^m \hookrightarrow S^{n-1} \hookrightarrow \mathbb{R}^m$, which is a contradiction by the Borsuk-Ulam theorem. □

Problem 0.4 (4)**.** (1) Let M and N be two topological manifolds. Show that $M \times N$ is again a topological manifold of dimension $\dim M + \dim N$.
(2) Let M and N be two topological manifolds of the same dimension. Show that $M \sqcup N$ is again a topological manifold.
(3) Show that the connected components of a manifold are again manifolds; in other words, every manifold is written as a disjoint union of a collection of connected manifolds. Can this collection be uncountable?

Proof. (1) *Hausdorff:* The product of two Hausdorff spaces is Hausdorff in the product topology, so as M, N are manifolds hence Hausdorff, so is $M \times N$. More explicitly, let $(m,n), (m',n') \in M \times N$ be distinct, so assume wlog. that $m \neq m'$. Then take U, V distinct open neighborhoods of m and m' respectively, and let W, W' be any neighborhoods of n and n', respectively. In the product topology, $U \times W$ and $V \times W'$ are neighborhoods of (m,n) and (m',n'), respectively, and $U \times W \cap V \times W' = (U \cap V) \times (W \cap W') = \varnothing$ as $U \cap V$ is empty.

Second-countable: Likewise, the finite product of second-countable spaces is countable, so as M, N are manifolds hence second-countable, so is $M \times N$.
We will also show it. Suppose $\mathcal{B}$ and $\mathcal{B}'$ are countable bases for M and N, respectively. We claim that $\mathcal{B} \times \mathcal{B}' = \{U \times V \mid U \in \mathcal{B}, V \in \mathcal{B}'\}$ is a countable basis for $M \times N$. Firstly, it is countable since the product of a finite collection of countable sets is countable. Now, suppose W is an open set in $M \times N$ and let $(m,n) \in W$. Because sets of the form $U \times V$ for U open in M and V open in N form a basis for the product topology, we can find such U and V such that $(m,n) \in U \times V \subset W$. But now we can find $U_{\mathcal{B}} \in \mathcal{B}$ and $V_{\mathcal{B}'} \in \mathcal{B}'$ such that $m \in U_{\mathcal{B}} \subset U$ and $n \in V_{\mathcal{B}'} \subset V$ by assumption of $\mathcal{B}$ and $\mathcal{B}'$ being bases. Then $(m,n) \in U_{\mathcal{B}} \times V_{\mathcal{B}'} \subset W$, showing that $\mathcal{B} \times \mathcal{B}'$ is a countable basis for the product topology on $M \times N$.

Locally-Euclidean:
Let $(p,q) \in M \times N$ be arbitrary with $m = \dim M$ and $n = \dim N$. Choose charts (U,φ) and (V,ψ) for p and q, respectively. Then define a chart $\varphi \times \psi \colon U \times V \to \mathbb{R}^{m+n}$ by sending $\varphi \times \psi\,(u,v) = (\varphi(u)_1, \ldots, \varphi(u)_m, \psi(v)_1, \ldots, \psi(v)_n)$ where $\varphi(u)_i$ is the i th coordinate of $\varphi(u)$ and $\psi(v)_j$ is the j th coordinate of $\psi(v)$. Taking the product topology on $M \times N$, $\varphi \times \psi$ becomes a homeomorphism of the open set $U \times V$ onto its image in $\mathbb{R}^{m+n}$ which is an open set. This is in particular because we have inverse maps $\varphi^{-1} \colon \varphi(U) \to U$ and $\psi^{-1} \colon \psi(V) \to V$, so by definition, $\left(\varphi^{-1} \times \psi^{-1}\right) \circ (\varphi \times \psi) = \left(\varphi^{-1} \circ \varphi\right) \times \left(\psi^{-1} \circ \psi\right) = \mathrm{id} \times \mathrm{id}$ and $(\varphi \times \psi) \circ \left(\varphi^{-1} \times \psi^{-1}\right) = \mathrm{id} \times \mathrm{id}$ likewise.

(2) We can define $M \sqcup N := M \times \{1\} \cup N \times \{0\}$. Suppose M and N are n-dimensional manifolds. Let $\tilde{p} \in M \sqcup N$, then either $\tilde{p} = (p, 0)$ with $p \in N$ or $\tilde{p} = (p, 1)$ with $p \in M$. Suppose without loss of generality that $p \in N$. Take some chart (U, φ) around p in N. Then $\tilde{U} := U \times [0, \frac{1}{2}) \cap M \sqcup N$ is an open set around $\tilde{p}$ in $M \sqcup N$. Define a chart $\tilde{\varphi} \colon \tilde{U} \to \mathbb{R}^n$ by $\tilde{\varphi}(p, 0) = \varphi(p)$. For some open set $V \subset \mathbb{R}^n$, we then have $\tilde{\varphi}^{-1}(V) = \varphi^{-1}(V) \times \{0\} = \varphi^{-1}(V) \times [0, \frac{1}{2}) \cap M \sqcup N$ which is clearly open. Likewise, take some open subset $W \times \{0\} \subset \tilde{U}$. Then $\tilde{\varphi}(W \times \{0\}) = \varphi(W)$ which is open since φ is a homeomorphism on U onto its open image. As φ is bijective, so is $\tilde{\varphi}$, so $\tilde{\varphi}$ is a homeomorphism onto its open image, so $\left(\tilde{U}, \tilde{\varphi}\right)$ is a chart around $\tilde{p}$ in $M \sqcup N$. As $\tilde{p}$ was arbitrary, we see that $M \sqcup N$ is a topological n manifold.

Hausdorff: The Hausdorff condition is clear: if $\tilde{p}, \tilde{q} \in M \sqcup N$ are distinct points which are both either in M or N, suppose without loss of generality both are in N, then $\tilde{p} = (p, 0), \tilde{q} = (q, 0)$, so we can take disjoint open neighborhoods U, V around p and q respectively using that N is a manifold, and then create open sets $\tilde{U} = U \times [0, \frac{1}{2}) \cap M \sqcup N, \tilde{V} = V \times [0, \frac{1}{2}) \cap M \sqcup N$ which are still disjoint and open in $M \sqcup N$ containing $\tilde{p}$ and $\tilde{q}$, respectively. If instead $\tilde{p}$ and $\tilde{q}$ are in, say, N and M, respectively, then $\tilde{p} = (p, 0)$ and $\tilde{q} = (q, 1)$, so take open neighborhoods U, V around p and q, respectively. Then $\tilde{U} = U \times [0, \frac{1}{2}) \cap M \sqcup N, \tilde{V} = V \times (\frac{1}{2}, 1] \cap M \sqcup N$ are disjoint open neighborhoods of $\tilde{p}$ and $\tilde{q}$, respectively.

Second-countable: Let $\mathcal{B}, \mathcal{B}'$ be countable bases for N and M, respectively. Then $\mathcal{B} \times \{0\} \cup \mathcal{B}' \times \{1\}$ gives a countable basis for $M \sqcup N$.

(3) Suppose M is a manifold and $N \subset M$ is a connected component. We will show that N is again a manifold in two ways: one will be to show that N is open in M, hence an open submanifold. Secondly, we will prove it from definitions.

Since M has a basis of coordinate balls, M is locally path-connected. Now for a general locally path-connected topological space, we have that path components are the same as components and that the components are are open in the topological space. (See Lee proposition A.43). Therefore, N is path-connected and open in M.

Lemma 0.5. *Open subsets U of a topological m-manifold M inherit the structure of a topological m-manifold.*

Proof. For Hausdorffness, let $p, q \in U$ and choose disjoint open sets V, W in M around p and q, respectively. Then $V \cap U$ and $W \cap U$ define open sets around p and q in U in the subspace topology.
For second-countability, let $\mathcal{B}$ be a countable basis for M. Then $\mathcal{B} \cap U = \{V \cap U \mid V \in \mathcal{B}\}$ is a countable basis for U.
Lastly, define the collection $\mathcal{A} = \{(V \cap U, \varphi|_U) : (V, \varphi) \text{ is a chart on } M\}$. For any $u \in U$, there exists some chart (V, φ) for u in M. Then $(V \cap U, \varphi|_U)$ is a chart in $\mathcal{A}$. Now since each $\varphi|_U$ is a restriction onto an open set, $\varphi|_U$ is still a homeomorphism onto its image which is open in $\mathbb{R}^m$, hence $\mathcal{A}$ defines an atlas for U. $\square$

This shows that $N \subset M$ is a manifold.

Now for the alternative direct checking of definitions:
Let $p \in N$. Since M is a manifold, there exists a chart (U, φ) around p. Then there exists a ball $B(\varphi(p), \varepsilon) \subset \varphi(U)$ since $\varphi(U)$ is open. Now $\varphi^{-1}(B(\varphi(p), \varepsilon))$ is a connected open subset of U containing p since homeomorphisms are in particular continuous hence preserve connectivity. Thus we in particular have that if we let $\tilde{U} := \varphi^{-1}(B(\varphi(p), \varepsilon))$, then $\left(\tilde{U}, \varphi|_{\tilde{U}}\right)$ is a chart in M hence also in N with N given the subspace topology. Thus every point $p \in N$ has a chart U_p contained in N. Hausdorffness can be checked as follows: if $p, q \in N$, then there exist disjoint neighborhoods V, W around p and q, respectively. Now $V \cap U_p$ and $W \cap U_q$ are open disjoint neighborhoods of p and q, respectively, in N. Alternatively, with N inheriting the subspace topology, Hausdorffness is inherited from M.
For second-countability, M has a countable basis $\mathcal{B}$, so since N has the subspace topology, $\mathcal{B} \cap N = \{U \cap N \mid U \in \mathcal{B}\}$ is a countable basis for N.
Hence N is a topological manifold of the same dimension as M.

To answer whether an uncountable collection of connected manifolds can be a manifold, we note that given an uncountable collection of connected manifolds, second-countability would imply that there exists a countable basis, suppose $\mathcal{B}$ is such a basis. Now let $\bigsqcup_{i \in I} M_i$ be the uncountable union of connected manifolds. For each i, choose an $x_i \in M_i$, and let (U_i, φ_i) be a chart around x_i contained in M_i (whose existence follows from the previous exercise). By assumption of $\mathcal{B}$ being a basis, there exists some $B_i \in \mathcal{B}$ such that $x_i \in B_i \subset U_i$. But this defines an injective map $I \to \mathcal{B}$ where I was assumed to be uncountable and $\mathcal{B}$ countable - this is a contradiction.

$\square$