1. Theory

Recall that

Definition 1.1 (Dirichlet Series)**.** Let f be an arithmetic function. Then the corresponding Dirichlet series is defined, for $s \in \mathbb{C}$, by

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Lemma 1.2.

$$0 \leq 3 + 4\cos\theta + \cos 2\theta = 2\left(1 + \cos\theta\right)^2$$

Lemma 1.3. *Let $\sigma > 1$. Then*

$$\Re\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it)\right) \geq 0$$

For the proof of the lemma, one shows that

$$\Re\left(\frac{1}{n^s}\right) = \frac{1}{n^\sigma}\cos\left(t\log n\right), \quad s = \sigma + it \tag{A_1}$$

Proof.

$$\Re\left(-\frac{\zeta'}{\zeta}(s)\right) = \Re\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}\cos\left(t\log n\right).$$

Hence

$$\Re\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma + it) - \frac{\zeta'}{\zeta}(\sigma + 2it)\right) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}\left[3 + 4\cos\left(t\log n\right) + \cos\left(2t\log n\right)\right] \overset{(1.2)}{\geq} 0$$

$\square$

2. Week 1

Exercise 2.1 (E1.1. Abel summation)**.** Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ and $f\colon [1,x] \to \mathbb{C}$ be C^1. Define $A(t) = \sum_{n\le t} a_n$. Then for $x > 1$, we have

$$\sum_{n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

3. Week 2

Let $\psi(x) := \sum_{n\le x} \Lambda(n)$.

Exercise 3.1 (E2.6)**.** Show that

$$\theta(x) := \sum_{p\le x} \log p = \psi(x) + O\left(x^{\frac{1}{2}} \log^2 x\right)$$

Exercise 3.2 (E2.7)**.** Show that

$$\pi(x) = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t\log^2 t}dt + O\left(x^{\frac{1}{2}} \log x\right).$$

Proof. By Abel summation, we first find that

$$\theta(x) := \sum_{p\le x} \log p = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt$$

and from the previous exercise, we now find that

$$\pi(x) = \frac{\psi(x)}{\log x} + \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt + O\left(x^{\frac{1}{2}} \log x\right)$$

The result follows if we can show that

$$\frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt = \int_2^x \frac{\psi(t)}{t\log^2 t}dt + O\left(x^{\frac{1}{2}} \log x\right).$$

Now $\psi(t) \le \pi(t)\log t$, so

$$\begin{aligned}\left|\int_2^x \frac{\psi(t)}{t\log^2 t} - \frac{\pi(t)}{t\log x}dt\right| &\le \left|\int_2^x \frac{\pi(t)}{t\log t} - \frac{\pi(t)}{t\log x}dt\right| \\ &= \left|\int_2^x \frac{\pi(t)}{t}\frac{\log\left(\frac{x}{t}\right)}{\log x\log t}dt\right|\end{aligned}$$

□

4. Week 3

Exercise 4.1 (E3.1)**.** Let $m \ge 0$. Show that

$$\sum_{n\le x} \log^m n = x\log^m x + O\left(x\log^{m-1} x\right).$$

Proof. Let $a_n = 1$ for all n. Then $A(x) = \lfloor x \rfloor$, so

$$\sum_{n \le x} \log^m n = \lfloor x \rfloor \log^m x - \int_1^x m \lfloor t \rfloor \frac{1}{t} \log^{m-1} t dt$$
$$= x \log^m x - (x - \lfloor x \rfloor) \log^m x - m \int_1^x \frac{\lfloor t \rfloor}{t} \log^{m-1}(t) dt$$

Thus we must show that

$$\left| (x - \lfloor x \rfloor) \log^m x + m \int_1^x \frac{\lfloor t \rfloor}{t} \log^{m-1}(t) dt \right| \le C x \log^{m-1} x$$

But $\frac{\lfloor t \rfloor}{t} \log^{m-1}(t) \le \log^{m-1}(x)$ giving that the right hand term is $O\left(x \log^{m-1} x\right)$. For the left hand term, it suffices to show that $(x - \lfloor x \rfloor) \log x \le x$, but this is clear since $x - \lfloor x \rfloor \le 1$ and $\log x \le x$. $\square$

Exercise 4.2 (E3.2)**.** Let $d(n) = \sum_{d|n} 1$. Show $d(n) \le 2\sqrt{n}$. If we consider the set $D \subset \mathbb{N}$ of positive divisors of n, then we can define a bijection $D \to D$ by $k \mapsto \frac{n}{k}$. Suppose now that $d(n) > 2\sqrt{n}$. Suppose $d \mid n$ and $d \ge \sqrt{n}$. Then since $\frac{d}{n} \cdot d = n$, we must have $\frac{d}{n} \le \sqrt{n}$. This implies that under this bijection, either the source or target lies in $\{1, \ldots, \lfloor \sqrt{n} \rfloor\}$. Hence $d(n) = |D| \le 2 \left| \{1, \ldots, \lfloor \sqrt{n} \rfloor\} \right| \le 2\sqrt{n}$.

Exercise 4.3 (E3.3)**.** Prove that for every $\varepsilon > 0$, there exists a constant C_ε such that $d(n) \le C_\varepsilon n^\varepsilon$.
Hint:

(1) Show that $d(n_1 n_2) = d(n_1) d(n_2)$ if $(n_1, n_2) = 1$.
(2) Show that
$$\frac{d(n)}{n^\varepsilon} = \prod_{p^\alpha || n} \frac{\alpha + 1}{p^{\alpha\varepsilon}}$$
where $p^\alpha \mid\mid n$ means that α is a positive integer, $p^\alpha \mid n$ and $p^{\alpha+1} \not| n$.
(3) Split the product in 2. Into the product over those primes $p < 2^{\frac{1}{\varepsilon}}$ and the product over the rest. Show that the second product is bounded by 1.
(4) Show that the factors in the first product are less than $1 + (\varepsilon \log 2)^{-1}$.

Proof. We follow the hint:
(1) Suppose $(n_1, n_2) = 1$. Let D be the set of divisors of $n_1 n_2$, D_1 the set of divisors of n_1 and D_2 the set of divisors of n_2. Suppose $d_1 \in D_1, d_2 \in D_2$. Then $d_1 a = n_1, d_2 b = n_2$, so $d_1 d_2 ab = n_1 n_2$, hence $d_1 d_2 \in D$. We thus obtain a map $D_1 \times D_2 \to D$ sending $(d_1, d_2) \mapsto d_1 d_2$. We claim this is a bijection. Suppose $d_1 d_2 = d_1' d_2'$. If $d_1 \mid d_2'$, then $d_1 = 1$, in which case, $d_1' = 1$, and thus $d_2 = d_2'$. Suppose thus that $d_1 \ne 1$, so $d_1 \not| d_2'$. Then since $(d_1', d_2') = 1$, we have $d_1 \mid d_1'$. Similarly, $d_1' \mid d_1$. So $d_1 = d_1'$. And again $d_2 = d_2'$. This gives injectivity. For surjectivity, if $d \mid n_1 n_2$, then consider $d_1 := \frac{d}{(n_2, d)}$ and $d_2 := \frac{d}{(n_1, d)}$. Then $d_1 d_2 = d$ and $d_1 \in D_1, d_2 \in D_2$.
(2) Clearly, $n^\varepsilon = \prod_{p^\alpha || n} p^{\alpha\varepsilon}$. It thus suffices to show that $\prod_{p^\alpha || n} (\alpha + 1) = d(n)$. But if we factorize n as $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, then it is clear that the divisors corresponds precisely to tuples $(a_1, \ldots, a_m)$ with $0 \le a_i \le \alpha_i$. There are precisely $\alpha_1 + 1$ choices for each a_i, giving $(\alpha_1 + 1) \cdots (\alpha_m + 1) = d(n)$ which indeed is what we wanted to show.

(3) We can split the product as

$$\frac{d(n)}{n^\varepsilon} = \underbrace{\prod_{\substack{p^\alpha || n \\ p < 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}}_{A} \cdot \underbrace{\prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}}_{B}$$

We claim that $B \leq 1$. Indeed

$$\prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}} \leq \prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \underbrace{\frac{\alpha+1}{2^\alpha}}_{\leq 1} \leq 1$$

(4) For the factors in the first product, we have $\alpha = \left\lfloor \frac{\log n}{\log p} \right\rfloor$ and $\log p < \frac{1}{\varepsilon} \log 2$, and $\alpha \leq \frac{\log n}{\log p}$, so $\frac{\log p}{\log n} \leq \frac{1}{\alpha}$

$$\varepsilon^2 \log p < \varepsilon \log 2$$

$$\frac{\alpha+1}{p^{\alpha\varepsilon}} \leq \frac{\log n + \log p}{p^{\alpha\varepsilon} \log p} \leq 1 + \frac{1}{\varepsilon \log 2} = \frac{\varepsilon \log 2 + 1}{\varepsilon \log 2}$$

What we want to bound is

$$\prod_{\substack{p^\alpha || n \\ p < 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}$$

Note here that p is bounded and as α increases, we should expect the denominator to take over. However, while α is small, we might have some large terms since p^ε might be large. All our terms are however bounded by p^ε by the looks of it? Then we would get that the product is the product is bounded by $\prod_{p < 2^{\frac{1}{\varepsilon}}} \frac{\log n}{\log p} \frac{1}{p^\varepsilon}$ □

Exercise 4.4 (E3.4)**.** Show that

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

is absolutely convergent in $\Re(s) > 1$.

Proof. Fix some $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. Then choosing an $\varepsilon > 0$ with $1 + \varepsilon < \sigma$, we have that $d(n) \leq C_\varepsilon n^\varepsilon$, so

$$\sum \left| \frac{d(n)}{n^s} \right| \leq \sum C_\varepsilon \frac{n^\varepsilon}{n^\sigma} \leq C_\varepsilon \sum \frac{1}{n^{\sigma - \varepsilon}} < \infty.$$

□

Exercise 4.5 (E3.5)**.** Show that the average order of $d(n)$ is $\log n$, i.e., that

$$\frac{1}{x} \sum_{n \leq x} d(n) = \log x + o(\log x).$$

Hint: Show that

$$\sum_{n \leq x} d(n) = \sum_{a \leq x} \left[\frac{x}{a} \right]$$

where $[b]$ is the integer part of b.

Proof. We follow the hint. For each $n \in \mathbb{N}$, let D_n denote the set of positive divisors of n. Then we want to find $\left|D_1 \cup \ldots \cup D_{[x]}\right|$. Now, $\left[\frac{x}{a}\right]$ is precisely the amount of multiples of a smaller than or equal to x, i.e., the amount of numbers in between 1 and x which have a as a divisor. Hence the right hand side indeed counts the number of divisors of the numbers less than or equal to x which is precisely the left hand side. Now, recall also the bound

$$\log x + \frac{1}{x} \leq \sum_{a \leq x} \frac{1}{a} \leq \log x + 1$$

so

$$1 + \frac{1}{x \log x} \leq \frac{1}{\log x} \sum_{a \leq x} \frac{1}{a} \leq 1 + \frac{1}{\log x}.$$

In particular, taking the limit as $x \mapsto \infty$, the outer functions tend to 1, so

$$\lim_{x \to \infty} \frac{1}{\log x} \sum_{a \leq x} \frac{1}{x} = 1.$$

In particular,

$$\frac{1}{x \log x} \sum_{n \leq x} d(n) \leq \frac{1}{\log x} \sum_{a \leq x} \frac{1}{a} \to 1, \quad x \to \infty.$$

For a lower bound, we have

$$\frac{1}{\log x} \sum_{a \leq x} \frac{1}{a} - \frac{1}{x \log x} \sum_{a \leq x} \frac{1}{a} = \frac{1}{\log x} \sum_{a \leq x} \frac{x-1}{ax} \leq \frac{1}{\log x} \sum_{a \leq x} \left[\frac{x}{a}\right]$$

But

$$\frac{1}{x} + \frac{1}{x^2 \log x} \leq \frac{1}{x \log x} \sum_{a \leq x} \frac{1}{a} \leq \frac{1}{x} + \frac{1}{x \log x}$$

so letting $x \to \infty$,

$$\lim_{x \to \infty} \frac{1}{x \log x} \sum_{a \leq x} \frac{1}{a} = 0$$

Hence also

$$1 \leq \lim_{x \to \infty} \frac{1}{x \log x} \sum_{n \leq x} d(n) \leq 1$$

giving the desired result. $\square$

Exercise 4.6 (E3.6)**.** Let

$$\chi_4(n) = \begin{cases} (-1)^{\frac{n-1}{2}}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}.$$

Show that χ_4 is a Dirichlet character modulo 4 and find $L(1, \chi_4)$. Use the value to give (yet another) proof- based on the irrationality of π - that there are infinitely many primes. Hint: Remember (or prove by playing around with arctan(1)) that

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$

Proof. We must check 3 criteria for χ_4 to be a Dirichlet character mod 4.
(i) It must be 4-periodic. Now if n is even, then $n+4$ is even, so then $\chi_4(n+4) = 0 = \chi_4(n)$.
If n is odd, then so is $n+4$, so

$$\chi_4(n+4) = (-1)^{\frac{n+4-1}{2}} = (-1)^{\frac{n-1}{2}+2} = (-1)^{\frac{n-1}{2}} = \chi_4(n).$$

So χ_4 is 4-periodic.
(ii) We must check that $\chi_4(n) = 0$ if and only if $(n,4) \neq 1$. Now, $\chi_4(n) = 0$ if and only if n is even if and only if $(n,4) \in \{2,4\}$ if and only if $(n,4) \neq 1$.
(iii) We must check that χ_4 is multiplicative. Indeed, if either n or m is even, then

$$\chi_4(nm) = 0 = \chi(n)\chi(m).$$

If both n, m are odd, then

$$\chi_4(nm) = (-1)^{\frac{nm-1}{2}} = \begin{cases} -1, & nm \equiv 3 \pmod 4 \\ 1, & nm \equiv 1 \pmod 4 \end{cases}$$

Now, if n and m are both equivalent to 3 mod 4, then their product is equivalent to 1 mod 4, which works out. If only one is equivalent to 3 mod 4, then nm is also, so it checks out, and similarly, if both are equivalent to 1 mod 4, then so is their product. Now, by definition,

$$L(1,\chi_4) := \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan(1) = \frac{\pi}{4}$$

Now, since $\chi_4 \neq \chi_0^4$, we know that $L(s,\chi_4)$ is convergent and analytic for $\Re(s) > 0$. In particular, it is continuous at $s = 1$. But for $\Re(s) > 1$, we know that $L(s,\chi_4) = \prod_p (1 - \chi_4(p)p^{-s})^{-1}$, so by continuity,

$$\frac{\pi}{4} = L(1,\chi_4) = \prod_p \left(1 - \chi_4(p)p^{-1}\right)^{-1}$$

Now, all the terms in the product are rational, so by irrationality of π, this forces there to be infinitely many primes. □

Exercise 4.7 (E3.7)**.** Let $\{a_n\}$ be a sequence of complex numbers satisfying that $\sum_{n \le x} a_n = O\left(x^\delta\right)$ for some $\delta > 0$. Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^\infty \sum_{n \le t} a_n \frac{1}{t^{s+1}} dt$$

for $\Re(s) > \delta$, and that the sum converges to an analytic function in this region.

Proof. Let $f(x) = x^s$. Then

$$\sum_{n \le x} \frac{a_n}{n^s} = \sum_{n \le x} a_n \frac{1}{x^s} + s \int_1^x \sum_{n \le t} a_n \frac{1}{t^{s-1}} dt$$

when $s \neq 1$. But $\left|\sum_{n \le x} a_n\right| \le Cx^\delta$, so

$$\left| \sum_{n \le x} a_n \frac{1}{x^s} \right| \le Cx^{\delta - \sigma} \to 0, \quad x \to \infty$$

as $\delta - \sigma < 0$. Thus

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{\infty} \sum_{n \le t} a_n \frac{1}{t^{s+1}} dt.$$

□

5. Week 4

Exercise 5.1 (E4.1)**.** Let $K \ge 0$. Prove that

$$\log(K|t| + 4) = O(\log(|t| + 4))$$

for $t \in \mathbb{R}$. Let $c_1, c_2, c_3 > 0$. Prove that there exists a constant c_4 such that for all $t \in \mathbb{R}$,

$$c_1 + c_2 \log(|t| + 4) + c_3 \log(|2t| + 4) \le c_4 \log(|t| + 4).$$

Proof. If $0 \le K \le 1$, then $\log(K|t| + 4) \le \log(|t| + 4)$ by monotonicity of log. So assume $K > 1$. Then $\log(K|t| + 4) = \log K + \log\left(|t| + \frac{4}{K}\right) \le \log K + \log(|t| + 4)$. Now $\log(|t| + 4) > 1$, so there exists some C such that $C \log(|t| + 4) \ge \log K$. Hence $\log(K|t| + 4) = O(\log(|t| + 4))$. Since $c_1 + c_2 \log(|t| + 4) + c_3 \log(|2t| + 4)$ is a sum of terms that are all $O(\log(|t| + 4))$, so is their sum, so the conclusion holds. □

Exercise 5.2 (E4.2)**.** Let $f(s)$ be a complex polynomial of degree n with complex zeroes $z_1, z_2, \ldots, z_n$. Show that

$$\frac{f'}{f}(z) = \sum_{i=1}^{n} \frac{1}{z - z_i}.$$

Consider how Lemma 6.3 is a generalization of this.

Proof. Firstly, f' is entire, so $\frac{f'}{f}$ is holomorphic on $\mathbb{C} - \{z_1, \ldots, z_n\}$. Now, by Theorem 6.1 in KomAn, there exist unique functions g_i holomorphic on $\mathbb{C} - \{z_1, \ldots, z_n\}$ such that $g_i(z_i) \ne 0$ and

$$f(z) = (z - z_i)^{n_i} g_i(z)$$

where n_i is the multiplicity of z_i. In particular, $f'(z) = n_i(z - z_i)^{n_i - 1} g_i(z) + (z - z_i)^{n_i} g_i'(z)$ which has z_i a zero of order $n_i - 1$. Hence $\frac{f'}{f}$ has z_i as a simple pole. Applying the partial fraction decomposition to $\frac{f'}{f}$ (theorem 6.12 in KomAn), we get that

$$\frac{f'}{f}(z) = \sum_{i=1}^{n} \frac{c_i}{z - z_i}$$

for certain constants c_i. Now $\lim_{z \to z_i} (z - z_i) \frac{f'}{f}(z) = n_i$. Now, f is of degree n with n distinct zeroes, so n_i must be 1 for each i.

Now let us recall Lemma 6.3:

Lemma 5.3 (6.3)**.** *Let $f \colon B \to \mathbb{C}$ be analytic, $B \subset \mathbb{C}$ open, and assume*

(1) $\{z \mid |z| \le 1\} \subset B$
(2) $|f(z)| \le M$ *when* $|z| \le 1$
(3) $f(0) \ne 0$.

Let $0 < r < R < 1$. Then for $|z| < r$,

$$\frac{f'}{f}(z) = \sum_{\substack{f(z_k)=0 \\ |z_k| \leq R}} \frac{1}{z - z_k} + O\left(\log \frac{M}{|f(0)|}\right)$$

Note here that f is not required to be a polynomial. However, since f is holomorphic in B, it has an analytic representation on B, so essentially, Lemma 6.3 generalizes the representation to analytic functions. □

Exercise 5.4 (E4.3)**.** Show that the Riemann zeta function $\zeta(s)$ has no zeroes for $\frac{1}{2} \leq s < 1$.

Proof. Recall that for $\sigma > 0$ and $s \neq 1$, we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty (u - [u])\, u^{-s-1} du.$$

For $s \in [\frac{1}{2}, 1)$, $\frac{s}{s-1} \leq -1$. So we wish to show that

$$s\int_1^\infty (u - [u])\, u^{-s-1} du > -1$$

But

$$s\int_1^\infty (u - [u])\, u^{-s-1} du$$

is positive since the inner function and s are both positive on $[1, \infty)$.

□

Exercise 5.5 (E4.4)**.** Let χ be a Dirichlet character modulo q. Find the Dirichlet series representation for $L'(s,\chi)/L(s,\chi)$. Let χ_0 be the trivial Dirichlet character modulo q. Prove that for $\sigma > 1, t \in \mathbb{R}$,

$$R := \Re\left(-3\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} - 4\frac{L'(\sigma + it,\chi)}{L(\sigma + it,\chi)} - \frac{L'(\sigma + i2t,\chi^2)}{L(\sigma + i2t,\chi^2)}\right) \geq 0.$$

Proof. We want to represent $\frac{L'(s,\chi)}{L(s,\chi)}$ as a Dirichlet series. We imitate the idea for $\frac{\zeta'}{\zeta}$.

$$\begin{aligned}
\frac{L'(s,\chi)}{L(s,\chi)} &= \frac{d}{ds}\log\left(L(s,\chi)\right) \\
&= -\sum_p \frac{d}{ds}\log\left(1-\frac{\chi(p)}{p^s}\right) \\
&= -\sum_p \frac{d}{ds}\sum_{k=1}^{\infty}(-1)^{k+1}\left(-\frac{\chi(p)}{p^s}\right)^k \\
&= \sum_p \sum_{k=1}^{\infty}\frac{d}{ds}\left(\frac{\chi(p)}{p^s}\right)^k \\
&= \sum_p \sum_{k=1}^{\infty}\chi(p)^k(-k\log p)p^{-sk} \\
&= -\sum_p \sum_{k=1}^{\infty}k\log p\left(\frac{\chi(p)}{p^s}\right)^k
\end{aligned}$$

Thus We want to find $\Re\left(\left(\frac{\chi(p)}{p^s}\right)^k\right)$. We have

$$\begin{aligned}
\Re\left(\left(\frac{\chi(p)}{p^s}\right)^k\right) &= \frac{1}{2}\left[\left(\frac{\chi(p)}{p^s}\right)^k + \left(\frac{\overline{\chi(p)}}{\overline{p^s}}\right)^k\right] \\
&=
\end{aligned}$$

$$\Re\left(-\frac{L'(s,\chi)}{L(s,\chi)}\right) = \sum_p \sum_{k=1}^{\infty} k\log p\cos\left(tk\log p\right).$$

So

□

Exercise 5.6 (E4.5)**.** Let $\zeta(s)$ be the Riemann zeta function. Let K be a compact subset of $\{s \in \mathbb{C} \mid \Re(s) > 0\}$. Assume that $1 \in K$ and that K does not contain any zeroes of ζ. Show that

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1)$$

for $s \in K - \{1\}$. Show that there exists a constant $c > 0$ such that for $0 < \delta < 1$,

$$-\frac{\zeta'}{\zeta}(1+\delta) < \frac{1}{\delta} + c.$$

Proof. Since 1 is a simple pole of $\frac{\zeta'}{\zeta}$ and K has no other zeroes of ζ and hence neither of ζ', we have that

$$-(s-1)\frac{\zeta'}{\zeta}(s)$$

is holomorphic on K, hence bounded as K is compact. Thus

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1)$$

for $s \in K - \{1\}$. Thus for small $0 < \delta < 1$ such that $1 + \delta \in K - \{1\}$,

$$-\frac{\zeta'}{\zeta}(1+\delta) < \frac{1}{\delta} + c$$

for some $c > 0$. ☐

Exercise 5.7 (E4.6)**.** Use partial summation (Abel summation) to show that for $\sigma > 1$,

$$-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}dx$$

where $\psi(x) = \sum_{n \le x} \Lambda(n)$, and Λ is the von Mangoldt function.

Proof. Recall that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

for $\sigma = \Re(s) > 1$.
Let $f(x) = \frac{1}{x^s}$ and $a_n = \Lambda(n)$. Partial summation gives

$$\sum_{n\le x} \frac{\Lambda(n)}{n^s} = \underbrace{\sum_{n\le x} \Lambda(n)}_{\psi(x)} \frac{1}{x^s} + s\int_1^x \underbrace{\sum_{n\le t} \Lambda(n)}_{\psi(t)} \frac{1}{t^{s+1}}dt$$

By the prime number theorem,

$$\psi(x) = x + O\left(\frac{x}{e^{c'\sqrt{\log x}}}\right)$$

so

$$\frac{\psi(x)}{x^s} \to 0, \quad x \to \infty$$

Thus

$$-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \frac{\psi(t)}{t^{s+1}}dt$$

for $\sigma > 1$. ☐

6. Week 5

Exercise 6.1 (E5.1)**.** Show that

$$x\exp\left(-c\sqrt{\log x}\right) = O_m\left(\frac{x}{\log^m x}\right)$$

for every m, and that

$$x^{1-\varepsilon} = O_\varepsilon\left(x\exp\left(-c\sqrt{\log x}\right)\right)$$

for every $\varepsilon > 0$. Discuss what this means for the quality of the error-term in the prime number theorem.

Proof.

$$\frac{\log^m x}{e^{c\sqrt{\log x}}} = \frac{\sqrt{\log x}^{2m}}{e^{c\sqrt{\log x}}}$$

Now

Lemma 6.2. *For any $a > 0$ and any $b > 1$,*

$$\frac{x^a}{b^x} \to 0, \quad x \to \infty.$$

Let $v = \sqrt{\log x}$. Then the above reads $\frac{v^{2m}}{e^{cv}}$. Assuming $c > 0$, we find that for $v \to \infty$, $\frac{v^{2m}}{e^{cv}} \to 0$. So in fact,

$$x \exp\left(-c\sqrt{\log x}\right) = o\left(\frac{x}{\log^m x}\right)$$

Now

$$x^{1-\varepsilon} = xx^{-\varepsilon} = xe^{-\log(x)\varepsilon} \leq xe^{-c\sqrt{\log x}}.$$

Recall that we proved the following version of the prime number theorem:

Theorem 6.3 (Prime number theorem)**.** *There exists a $c' > 0$ such that*

$$\psi(x) = x + O\left(x \exp\left(-c'\sqrt{\log x}\right)\right)$$

So by the above,

$$\psi(x) = x + O_m\left(\frac{x}{\log^m(x)}\right)$$

So essentially, the error term is smaller than $\frac{x}{\log^m(x)}$ for any x but still larger than $x^{1-\varepsilon}$ for any $\varepsilon > 0$. □

Exercise 6.4 (E5.2)**.** Prove that the following two statements are equivalent:

(1) There exists a $c > 0$ such that

$$\psi(x) = x + O\left(x \exp\left(-c\sqrt{\log x}\right)\right)$$

(2) There exists a $c > 0$ such that

$$\pi(x) = li(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right)$$

where $li(x) = \int_2^x \frac{1}{\log t} dt$.

Proof. Suppose (1) is true. Then

$$\begin{aligned}
\pi(x) &= \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t\log^2 t} dt + O\left(x^{\frac{1}{2}} \log x\right) \\
&= \frac{x}{\log x} + O\left(\frac{x}{\log x} \exp\left(-c\sqrt{\log x}\right)\right) + \int_2^x \frac{1}{\log^2 t} + O\left(\frac{1}{\log^2 t \exp\left(c\sqrt{\log t}\right)}\right) dt + O\left(x^{\frac{1}{2}} \log x\right)
\end{aligned}$$

Now

$$\int_2^x \frac{1}{\log^2 t} dt = -\frac{t}{\log t}\bigg|_2^x + li(x)$$

giving

$$\pi(x) = li(x) + \frac{2}{\log 2} + O\left(\frac{x}{\log x} e^{-c\sqrt{\log x}}\right) + \int_2^x O\left(\frac{e^{-c\sqrt{\log t}}}{\log^2 t}\right) dt + O\left(x^{\frac{1}{2}} \log x\right)$$

All the middle terms apart from the last two are clearly $O\left(xe^{-c\sqrt{\log x}}\right)$. To take care of the last term, we use the lemma:

Lemma 6.5. *For any $a > 0$,*

$$\frac{\log x}{x^a} \to 0, \quad x \to \infty$$

Hence $x^{\frac{1}{2}} \log x = O\left(x^{\frac{3}{4}}\right) = O\left(xe^{-c'\sqrt{\log x}}\right)$.

For the last part

$$\int_2^x O\left(\frac{e^{-c\sqrt{\log t}}}{\log^2 t}\right) dt \leq$$

Note that the derivative of $xe^{-c\sqrt{\log x}}$ is

$$e^{-c\sqrt{\log x}} - c\frac{d}{dx}\left[\sqrt{\log x}\right] e^{-c\sqrt{\log x}} = e^{-c\sqrt{\log x}} - c\frac{1}{2}\frac{1}{x}\frac{1}{\sqrt{\log x}}e^{-c\sqrt{\log x}}$$

But as $x \to \infty$, this grows faster than $\frac{e^{-c}\sqrt{\log x}}{\log^2 x}$, which is what we wanted.

Now we want to show that (2) implies (1). So assume there exists a $c > 0$ such that

$$\pi(x) = li(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right).$$

Then recall that

$$\psi(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt - O\left(x^{\frac{1}{2}} \log^2 x\right)$$

So

$$\psi(x) = li(x) \log x + \log x O\left(xe^{-c\sqrt{\log x}}\right) - \int_2^x \frac{li(t)}{t} dt - \int_2^x O\left(e^{-c\sqrt{\log t}}\right) dt - O\left(x^{\frac{1}{2}} \log^2 x\right)$$

Now, by repeated integration by parts, we get

$$\begin{aligned}
li(x) &= \frac{t}{\log t}\Big|_2^x + \int_2^x \frac{1}{\log^2 t} dt \\
&= \frac{t}{\log t}\Big|_2^x + \left[\frac{t}{\log^2 t}\Big|_2^x + 2\int_2^x \frac{1}{\log^3 t} dt\right] \\
&= \frac{t}{\log t} + \frac{t}{\log^2 t}\Big|_2^x + 2\left[\frac{t}{\log^3 t}\Big|_2^x + 3\int_2^x \frac{1}{\log^4 t} dt\right] \\
&= x\sum_{r=1}^{k-1} \frac{(r-1)!}{\log^r x} + (k-1)!\int_2^x \frac{1}{\log^k t} dt
\end{aligned}$$

□

Exercise 6.6 (E5.3)**.** Let f be a Schwartz function on the real line, and let $\hat{f}$ be its Fourier transform. Show that

$$\sum_{n\in\mathbb{Z}} f\left(\frac{v+n}{t}\right) = \sum_{n\in\mathbb{Z}} |t| \hat{f}(nt) e^{2\pi inv}.$$

Proof. a □