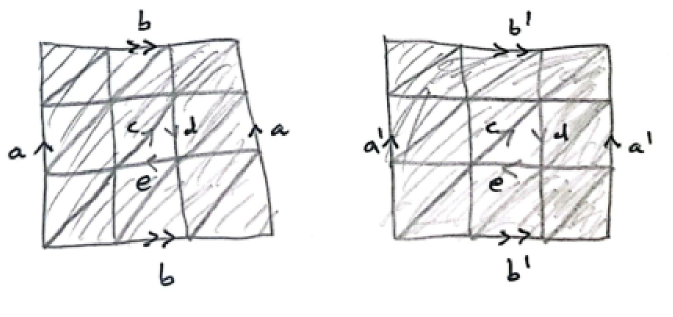
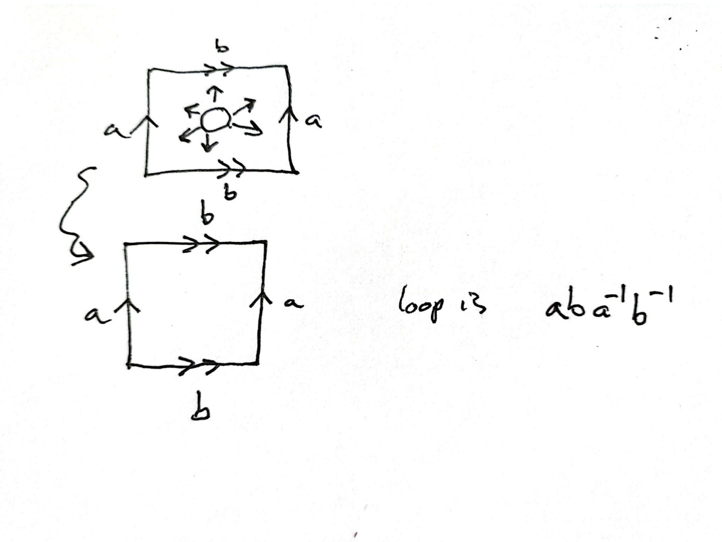
20: Use van Kampen's theorem to calculate the fundamental group of the double torus by dividing the surface into two halves, each of which is a punctured torus. Do the calculation again, this time splitting the surface into a disc and the closure of the complement of the disc.

Solution: Using the usual identification space of the double torus and the triangulation from problem 1 on page 124 from homework 10 shown below:



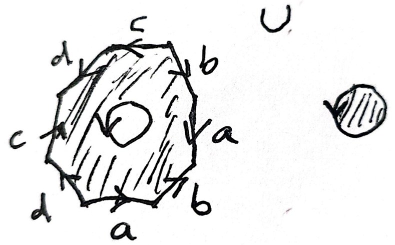
we can let the left punctured torus be denoted by by $|J|$ and the right punctured torus be denoted by $|K|$. Then $|J \cap K|$ is cde which is homeomorphic to a circle. Choose a basepoint p on the circle $|J \cap K|$. Now, by problem 25 on page 109 from homework 9, the punctured torus has fundamental group isomorphic to the fundamental group of the one-point union of two circles, $S^1 \vee S^1$, which, for example by example 1 on page 136, has fundamental group $\mathbb{Z} * \mathbb{Z}$, the free group on two generators. Now, the inclusion of the circle $|J \cap K|$ in either $|J|$ or $|K|$ (equivalent by symmetry) is the circle constituting the border of the puncture. Now, considering the deformation retraction given in problem 25 on homework 9, this circle corresponds the loop around the edges, given by $aba^{-1}b^{-1}$ as shown below



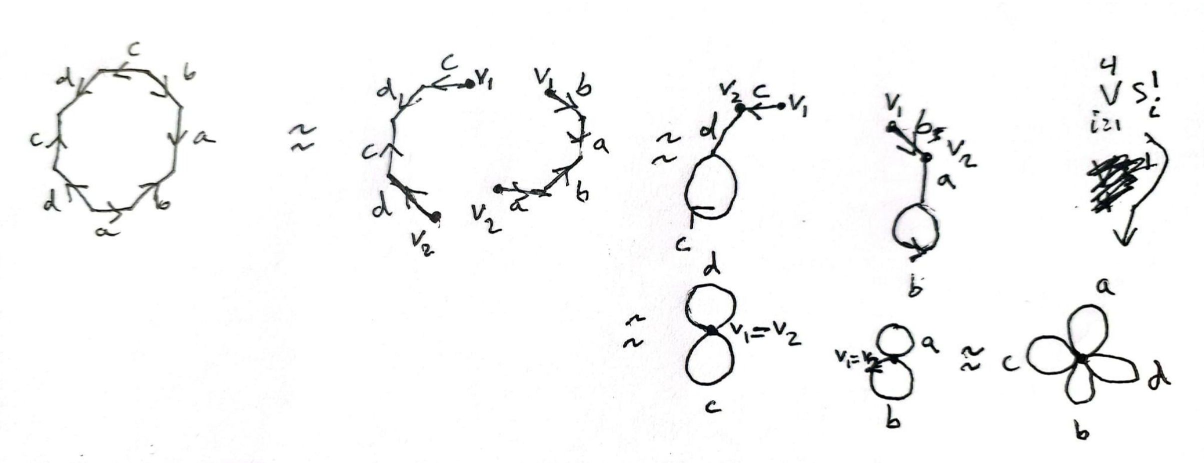
So, letting a, b be the generators for the fundamental group of the left punctured torus, and c, d the generators for the fundamental group of the right punctured torus, van Kampen gives that the fundamental group of the double holed torus is the free product $(\mathbb{Z} * \mathbb{Z}) * (\mathbb{Z} * \mathbb{Z}) \cong *_4\mathbb{Z}$ generated by a, b, c and d with the relation $aba^{-1}b^{-1} = cdc^{-1}d^{-1}$. So denoting the commutator of x and y by $[x, y] = xyx^{-1}y^{-1}$, we get that the fundamental group is

$$\langle a, b, c, d \mid [a, b] = [c, d] \rangle = \langle a, b, c, d \mid [a, b]\,[d, c] \rangle.$$

Suppose instead we divide the torus into a disc and the closure as shown below (the identification space can e.g. be found in Hatcher page 5)



Let $|J|$ denote the complement of the disc and $|K|$ the disc. Then $|J \cap K|$ is a circle again.
Let p denote a vertex on $|J \cap K|$ (interpreting it as a simplicial complex).
Now, $|J|$ is clearly deformation retractable to the outer loop $bab^{-1}a^{-1}dcd^{-1}c^{-1}$ by the projection from the center of the hole where the disc was removed. By the following figure, this outer loop is the identification space for the one-point union of 4 circle, $\bigvee_{i=1}^{4} S_i^1$:



Thus the fundamental group $\pi_1(|J|, p)$ is isomorphic to $\pi_1\left(\bigvee_{i=1}^{4} S_i^1\right)$ which is isomorphic to the free group on four generators $*_{i=1}^{4}\mathbb{Z}_i$ by example 1 on page 136.
Now, $|K|$ being a disc has trivial fundamental group as it is convex. The inclusions $|J| \to |J \cup K|$ and $|K| \to |J \cup K|$ thus identify the loop $bab^{-1}a^{-1}dcd^{-1}c^{-1} = [b, a][d, c]$ and the trivial loop. Thus we recover by van Kampen, that the fundamental group of the double torus is

$$\langle a, b, c, d \mid [b, a][d, c] \rangle$$

which was the same group that we got in the first part of the problem (b and a interchanged, however, this doesn't affect anything).

23: Let X be a path-connected triangulable space. How does attaching a disc to X affect the fundamental group of X?

Solution: We may suppose $X = |K|$ since the following proof is topologically invariant, and that $|K|$ has simplexes of dimension at most 2 by example 3 on page 136. Since $|K|$ has a finite number of simplexes, we can take a small ε-neighborhood of the image of the attaching map, call it $f : S^1 \to X$, which is path connected and deformation retracts onto a circle or a point - depending on whether f attaches along a loop, edge or a point. Now, by choosing a point p on the image of f, van Kampen gives that the fundamental group of $|K \cup_f D|$ with respect to p is $\pi_1(|K|, p) * \pi_1(D, p)/N$ where N is the normal subgroup formed by equating the inclusion of the circle $f\left(S^1\right)$ in each fundamental group. When included in $\pi_1(D, p)$, it is nulhomotopic as D is convex, and when included in $\pi_1(X, p)$, it is simply the loop $t \mapsto f\left(e^{i\theta + 2\pi i t}\right)$ with $f\left(e^{i\theta}\right) = p$. So since $\pi_1(D, p)$ is the trivial group, we get that attaching a disc to X simply makes the loop $t \mapsto f\left(e^{i\theta} + 2\pi i t\right)$ homotopic to the trivial loop at p in $\pi_1(X, p)$.

24: Let G be a finitely presented group. Construct a compact triasgulable space which has fundamental group G.

Solution: For G to be a finitely presented group means that G is of the form $\langle S \mid R \rangle$ such that S and R are finite, so we can write

$$G = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle.$$

Now, as in example 1 on page 136, suppose we take $\bigvee_{i=1}^{n} S_i^1$ which is triangulable as the one-point union of n triangles like in figure 6.13. Like in that example, we can take a maximal tree T which will consist of all edges connecting the central vertex/point to every other vertex, together with all the vertices. The fundamental group of this space is by the example isomorphic to $G(K, L)$ where K is the triangulation of $\bigvee_{i=1}^{n} S_i^1$ and L is the simplicial complex corresponding to the tree. Thus $G(K, L)$ consists of $g_{i,i+1}$ for the i such that v_i and v_{i+1} are the endpoints of an edge in $K - L$ with an enumeration as in figure 6.13. This is precisely $\langle g_{1,2}, g_{3,4}, \ldots, g_{2n-1,2n} \rangle$. Now, we have that r_i represents some loop in this interpretation of $G(K, L)$, namely, a relation of the form $g_{\alpha_1,\alpha_1+1}^{\varepsilon_1} \ldots x_{\alpha_k,\alpha_k+1}^{\varepsilon_k}$ corresponds the loop $E_{a_1} a_1 (a_1+1) E_{a_1+1}^{-1} E_{a_2} a_2 (a_2+1) E_{a_2+1}^{-1} \ldots E_{a_k} a_k (a_k+1) E_{a_k+1}^{-1}$ in $E(K, v)$.

Now, letting $\alpha_i \colon S^1 \to |K|$ correspond to the kiio r_i given above (we can let it map from S^1 without loss of generality) then since K is path-connected and triangulable, we get from problem 23 that attaching a disc along α_i to K will make the loop r_i nulhomotopic. Thus, we get that

$$G = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_m \rangle = \langle g_{1,2}, g_{3,4}, \ldots, g_{2n-1,2n} \rangle \cup_{\alpha_1} D \cup_{\alpha_2} D \cup_{\alpha_3} \ldots \cup_{\alpha_m} D$$