Proposition 0.1 (13.19)**.** *If $f\colon X \to Y$ is an identification map and K is a locally compact Hausdorff space then $f \times 1\colon X \times K \to Y \times K$ is an identification map.*

Proof. Let $\pi = (f \times 1)$. Let $A \subset Y \times K$ such that $\pi^{-1}(A)$ is open. Let $(x, y) \in \pi^{-1}(A)$ and choose open sets U_1, V such that $(x, y) \in U_1 \times V$ and $U_1 \times \overline{V} \subset \pi^{-1}(A)$. Now suppose $u \in f^{-1}(f(U_1))$. Then $f(u) \times \overline{V} \subset A$, so $u \times \overline{V} \subset \pi^{-1}(A)$ which is open in $U_1 \times \overline{V}$. By the tube lemma, we can find an open set U_u around u in X such that $U_u \times \overline{V} \subset \pi^{-1}(A)$. Let $U_2 = \bigcup_{u \in f^{-1}(f(U_1))} U_u$. This is open and $U_2 \times \overline{V} \subset \pi^{-1}(A)$. Continuing for U_i with $i > 2$ in the same way, we let $U = \bigcup_{i \in \mathbb{N}} U_i$. Then clearly $U \times \overline{V} \subset \pi^{-1}(A)$.
Now suppose $u' \in f^{-1}(f(U))$, so there exists $u \in U$ such that $f(u') = f(u)$. But then there exists $i \in \mathbb{N}$ such that $u \in U_i$ and hence $u' \in U_{i+1}$ by construction, so $u' \in U$. Hence $f^{-1}(f(U)) = U$, so U is saturated. Therefore $\pi(U \times V)$ is an open set contained in A which contains $(f(x), y)$. As f is surjective, we can for any $(y', y) \in A$ find $(x', y) \in X$ and repeat the above to obtain some open set whose image is contained in A and contains (y', y). Therefore A is open. □

Exercise 0.2 (13.2)**.** If X, Y are normal, $A \subset X$ is closed, and $f\colon A \to Y$ is a map, show that $Y \cup_f X$ is normal.

Proof.

$$Y \cup_f X = Y + X/\sim$$

0.1. **Homotopy.** Let C denote the constant homotopy, whichever one makes sense in the current context, i.e., $C\colon X \times I \to Y$ is $\operatorname{rel} X$.

Proposition 0.3 (14.13)**.** *We have $F * C \simeq F \operatorname{rel} X \times \partial I$, and, similarly, $C * F \simeq F \operatorname{rel} X \times \partial I$.*

Proof. We have a map $\varphi_1\colon (I, \partial I) \to (I, \partial I)$ given by

$$\varphi_1(t) = \begin{cases} 2t, & t \in \left[0, \frac{1}{2}\right] \\ 1, & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Then φ_1 is continuous, and we have

$$F * C \simeq F * C\,(x, \varphi_1(t)) =$$

□

□

Lemma 0.4. *$\tilde{H}_0(X)$ is the kernel of the map $H_0(X) \to H_0(*)$ where $\{*\}$ is a one-point space.*

Proof. We have $\tilde{H}_0(X) = \ker \varepsilon / \operatorname{Im} \partial$ where $\varepsilon\colon \Delta_0(X) \to \mathbb{Z}$ maps $\varepsilon\left(\sum n_x x\right) = \sum n_x$. Now, take any $\sum n_x x \in H_0(X)$. Since the differential maps to 0, $\Delta_0(X)/\operatorname{Im}(\partial\colon \Delta_1(X) \to \Delta_0(X)) = H_0(X)$. Now take the map $f\colon X \to *$. Then for any $\sum n_x x \in H_0(X)$, $f^*\left(\sum n_x x\right) = \sum n_x f(x) = \sum n_x *$. Hence $f^*\left(\sum n_x x\right) = 0$ if and only if $\sum n_x = 0$ if and only if $\sum n_x x \in \ker \varepsilon$. □