1. Glossary for exam

- All subspaces have a complement (thm 2.14)
- $A, \tilde{A} \in \mathrm{Hom}(U, V)$ are equivalent if there exist $S \in \mathrm{GL}(U)$ and $T \in \mathrm{GL}(V)$ such that

$$\begin{tikzcd} U \arrow[r, "A"] \arrow[d, "S"] & V \arrow[d, "T"] \\ U \arrow[r, "\tilde{A}"] & V \end{tikzcd}$$

- $A, \tilde{A} \in \mathrm{End}(V)$ are called similar if there exists $T \in \mathrm{GL}(V)$ such that $TA = \tilde{A}T$.
- **Theorem 2.22:** Assume U, V fin.dim., then $A, \tilde{A}$ are equivalent iff they have the same rank.
- For $y \in V'$, the null-space of y has codimension one in V and only the non-zero scalar multiples of y have the same null-space as y. (lemma 3.15 + see thm 3.14)
- The dual map is the pullback: if $A \in \mathrm{Hom}(U, V)$, then $A' \in \mathrm{Hom}(V', U')$ is defined by $A'(y) = y \circ A$.

$$\begin{tikzcd} U \arrow[r, "A"] \arrow[dr, "A'(y)"'] & V \arrow[d, "y"] \\ & k \end{tikzcd}$$

- $\mathrm{Bil}(V) \cong M_{n,n}(F)$ when $\dim V = n$ through the isomorphism $B \overset{\cong}{\mapsto} (B(x_i, x_j))$. $\mathrm{Bil}(V)$ has basis $w_{i,j} = y_i y_j$ where (y_i) is a dual basis to a basis (x_i) for V and $(y_i y_j)(x, z) = y_i(x) y_j(z)$.
- $L^k(V) := \{\text{multilinear forms } V^{\times k} \to F\}$.
- Review chapter 4 again - bilinear, symmetric, skew-symmetric, alternating, quadratic forms.
- Polynomials $p(x) \in F[x]$ are uniquely determined by their associated functions $\lambda \mapsto p(\lambda)$ if and only if F is infinite.
- When A is diagonable, the projections $E_\lambda \colon V \to V_\lambda$ are called the spectral projections of A and the expansion of A as $A = \sum_{\lambda \in \sigma(A)} \lambda E_\lambda$ is called the spectral resolution of A.
- Using Gram-Schmidt, we find that every finite-dimensional inner product space has an orthonormal basis.
- Fitting's decomposition shows that all endomorphisms act on V by a nilpotent map on one subspace N and an invertible map on another subspace R, and these subspaces form a unique direct sum reduction $V = N \oplus R$.
- All transpositions are conjugate in S_n and every permutation is a product of transpositions.
- Bruhat decomposition shows that every isomorphism is the product STU of invertible upper triangular matrices S and U and a permutation matrix T.
- LU-decomposition is just a different form of Bruhat decomposition: for $A \in \mathrm{GL}(n, F)$, there exist matrices L, P, U such that $A = LPU$ with L lower triangular, U upper triangular, and P a permutation matrix.
- By the Jordan additive decomposition, for $A \in \mathrm{End}(V)$, it splits uniquely as $A = A_d + A_n$ where A_d is diagonable and A_n is nilpotent. Moreover,

$\sigma(A_d) = \sigma(A)$ and there exist $p_d, p_n \in F[x]$ such that $p_d(A) = A_d$ and $p_n(A) = A_n$.

Chapter 1

Chapter 4

Exercise 1.1 (1)**.** How does the matrix $[B]$ representing a bilinear form transform if the basis is changed by a transition matrix $[P]$?

Solution. Suppose $\mathcal{B} = \{x_1, \dots, x_n\}$ is the basis for V giving $[B]$ the representation. Write $[B]_{\mathcal{B}} = (B(x_i, x_j))$ for this representation. We can write

$$B = \sum B(x_i, x_j) w_{ij}.$$

Let P be a change of basis sending $x_j \mapsto \sum_i P_{ij} v_i$ and let $\mathcal{B}' = \{v_1, \dots, v_n\}$. Suppose (z_i) is the dual basis to $\mathcal{B}'$ and $\tilde{w}_{ij} = z_i z_j$. Then

$$B = \sum B\left(v_i, v_j\right) \tilde{w}_{ij}.$$

So $[B]_{\mathcal{B}'} = \left(B\left(v_i, v_j\right)\right)$. Now

$$B\left(x_i, x_j\right) = \sum B\left(v_s, v_t\right) \tilde{w}_{s,t}\left(P(x_i), P(x_j)\right) = \sum B\left(v_s, v_t\right) \tilde{w}_{s,t}\left(\sum P_{ki} v_k, \sum P_{rj} v_r\right)$$

$$\sum_{s,t} B(v_i, v_j) P_{si} P_{tj}$$

Exercise 1.2 (4.7)**.** Determine the symmetric bilinear form on $\mathbb{R}^2$ corresponding to the quadratic form $q(x) = 2x_1^2 + 6x_1x_2 - 3x_2^2$. Find a basis for which q has the form

$$ax_1^2 + bx_2^2.$$

Solution. The symmetric bilinear form $B((x_1, x_2), (y_1, y_2)) = 2x_1y_1 + 3x_1y_2 + 3x_2y_1 - 3x_2y_2$ works. The bilinear form is given the matrix $[B] = \begin{pmatrix} 2 & 3 \\ 3 & -3 \end{pmatrix}$ where $B(x, y) = x^t\,[B]\,y$.
Now setting up

$$\begin{pmatrix} 2 & 3 & x_1 \\ 3 & -3 & x_2 \end{pmatrix}$$

this is equivalent to the matrix

$$\begin{pmatrix} 1 & 0 & \frac{x_2}{5} + x_1 \\ 0 & 1 & \frac{3x_1 - 2x_2}{15} \end{pmatrix}$$

So choosing a basis as $\mathcal{B}' = \{y_1, y_2\}$ with $y_1 = \frac{x_2}{5} + x_1$ and $y_2 = \frac{3x_1 - 2x_2}{15}$, we get

$$q(y) = y^t\,[B]_{\mathcal{B}'}\,y = y_1^2 + y_2^2.$$

Chapter 5

Exercise 1.3 (5.4)**.** $X, Y \subset V$ subspaces, $A\colon V \to V/X \oplus V/Y$ by $v \mapsto (v + X, v + Y)$ and $B\colon V/X \oplus V/Y \to V/X + Y$ given by $(v_1 + X, v_2 + Y) \mapsto v_1 - v_2 + (X + Y)$. Give a codimension version of the Grassmann formula.

Solution. Firstly, $N(A) = X \cap Y$, and $R(B) = V/(X+Y)$ since for $v + (X+Y)$, we have $B(v + X, 0 + Y) = v + (X+Y)$. Now if $B(v_1 + X, v_2 + Y) = v_1 - v_2 + (X+Y) = 0 + (X+Y)$ then $v_1 - v_2 \in X + Y$ so $v_1 - v_2 = x + y$ and hence $v := v_1 - x = y + v_2$, so $A(v) = (v + X, v + Y) = (v_1 + X, v_2 + Y)$, hence $N(B) \subset R(A)$, but also, $R(A) \subset N(B)$ clearly. Hence $N(B) = R(A)$. Now by rank-nullity, we get

$$\begin{aligned}\operatorname{codim} X + \operatorname{codim} Y &= \operatorname{codim}(X+Y) + N(B) \\ &= \operatorname{codim}(X+Y) + \dim V - \dim(X \cap Y) \\ &= \operatorname{codim}(X+Y) + \operatorname{codim}(X \cap Y)\end{aligned}$$

□

Chapter 6

Chapter 7

Chapter 12

Exercise 1.4 (12.2)**.** Let U and V be normed spaces, and assume that V is complete. Show that then $B(U,V)$ is also complete with the operator norm.

Solution. Suppose B_n is a Cauchy sequence in $B(U,V)$, so

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \mid \|B_n - B_m\| < \varepsilon, \quad \forall m, n \geq N.$$

That is, for all $m, n \geq N$,

$$\sup_{\|x\|=1} \|B_n x - B_m x\| < \varepsilon$$

For a fixed n, this becomes a Cauchy sequence in V which thus converges, so we can define $Bx = \lim_{n\to\infty} B_n x$. We claim that B is a bounded operator too. It is clear that it is linear since each B_n is a continuous map. What remains is to show that B is bounded. It suffices to show that it is bounded on $S = \{x \mid \|x\| = 1\}$. Suppose it were not bounded and choose a sequence $(x_n) \subset S$ such that $\|Bx_n\| > n$. Choose $\varepsilon = \frac{1}{2}$ and let N be such that for $n, m \geq N$, we have

$$\|B_n - B_m\| < \varepsilon$$

Then for all k

$$\|B_n x_k - B_m x_k\| < \varepsilon$$

for all $n \geq N$, so in particular

$$\|Bx_k - B_m x_k\| = \lim_{n\to\infty} \|B_n x_k - B_m x_k\| < \varepsilon$$

But B_m is bounded, so let $\|B_m\| = R$. Choose M such that that for all $k \geq M$, we have $\|Bx_k\| \geq \|B_m x_k\|$, then

$$\|Bx_k\| - \|B_m x_k\| < \varepsilon$$

giving

$$\|Bx_k\| < \varepsilon + R$$

contradicting $\|Bx_k\| \to \infty$.

Exercise 1.5 (12.3)**.** Let $A \in B(V)$ for a complete normed space V. Prove

(1) If $\|A\| < 1$ then $\sum_{k=0}^{\infty} A^k$ converges in $B(V)$ to an inverse of $I - A$.
(2) If $B \in B(V)$ is invertible and $\|A\| < \frac{1}{\|B^{-1}\|}$ then $B - A$ is invertible.
(3) The set of invertible bounded operators is an open subset of $B(V)$.

Exercises

In these exercises all vector spaces denoted by V have finite dimension.

Exercise 6.1. Let $A \in \mathrm{End}(V)$. Prove $\alpha\lambda+\beta \in \sigma(\alpha A+\beta I)$ for all $\lambda \in \sigma(A)$ and all $\alpha, \beta \in \mathcal{F}$.

Exercise 6.2. Let $A \in \mathrm{End}(V)$ and assume $\mathrm{rank}(A) = 1$. Show that A is diagonable or $A^2 = 0$.

Exercise 6.3. Let $A \in \mathrm{End}(V)$ and $S \in \mathrm{GL}(V)$. Show that A and SAS^{-1} share their eigenvalues, and find the relation between their eigenspaces.

Exercise 6.4. Let $A \in \mathrm{End}(\mathbb{C}^n)$ be given by $Ax = (\xi_2, \dots, \xi_n, \xi_1)$ for each $x = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$. Show that $\sigma(A)$ is the set of all n-th roots of unity, and determine an eigenvector for each eigenvalue.

Exercise 6.5. Let $A \in \mathrm{End}(V)$ be an involution. Show $V = V_1 \oplus V_{-1}$ if $\mathrm{char}\, \mathcal{F} \neq 2$ (see Exercise 2.16).

Exercise 6.6. Is there a general relation between the eigenvectors and eigenvalues for A and those for A^{-1}, when A is invertible?

Exercise 6.7. Show all B-eigenspaces are A-invariant when $A, B \in \mathrm{End}(V)$ commute.

Exercise 6.8. Let $\mathcal{F} = \mathbb{C}$ and $A \in \mathrm{GL}(V)$. Show A is diagonable if A^2 is diagonable (use Exercises 6.5 and 6.7).

Exercise 6.9. On the vector space of all sequences $x = (\xi_1, \xi_2, \dots)$ with entries from a field $\mathcal{F}$, consider the left shift $S(x) = (\xi_2, \xi_3, \dots)$ and the right shift $T(x) = (0, \xi_1, \xi_2, \dots)$. What are their eigenvalues and eigenvectors?

Exercise 6.10. Let $A \in \mathrm{End}(V)$. Prove that $\sigma(A) = \sigma(A')$.

Exercise 6.11. Let $\mathcal{F} = \mathbb{R}$ and $W \subseteq \mathcal{F}[X]$ a finite-dimensional subspace. Prove that there exists $k \geq 0$ such that $(d/dt)^k f = 0$ for all $f \in W$.

Exercise 6.12. Let $V = \mathcal{F}[X]$ and let $y \in V'$ be a non-zero linear form which is multiplicative, that is, $y(pq) = y(p)y(q)$ for all $p, q \in V$. Show y is the evaluation $p(X) \mapsto p(\gamma)$ for some $\gamma \in \mathcal{F}$.

Exercise 6.13. Let $A \in \mathrm{End}(\mathcal{F}^3)$ be given by $A(\xi_1, \xi_2, \xi_3) = (\xi_2, \xi_1 - \xi_3, \xi_1)$. Find $p(A)$ for $p(X) = 1 - X + X^2$.

Exercise 6.14. Show $p(SAS^{-1}) = Sp(A)S^{-1}$ for $p \in \mathcal{F}[X]$ and $S \in \mathrm{GL}(V)$.

(Here invertible means there is a bounded inverse)

Solution. (1) geometric series.
(2) $B - A = B\left(1 - \frac{A}{B}\right)$. Now $\|\frac{A}{B}\| \leq \|A\|\|B^{-1}\| < 1$, so by (1), $1 - \frac{A}{B}$ has inverse $\sum_{k=0}^{\infty} \left(\frac{A}{B}\right)^k$. But then $B - A$ is a composition of invertible maps hence invertible since $\mathrm{GL}(V)$ is a group.

(3) Suppose $A \in B\left(B, \frac{1}{\|B^{-1}\|}\right)$, so $\|B-A\| < \frac{1}{\|B^{-1}\|}$. By (2), $B-(B-A)=A$ is then invertible. Hence $B\left(B, \frac{1}{\|B^{-1}\|}\right)$ is an open neighborhood of B in $B(V)$ consisting of invertible maps. Thus the set of invertible maps is open in $B(V)$.

Exercise 1.6 (12.6)**.** Let $S \in \operatorname{End}\left(\ell^2\right)$ denote the right shift taking the sequence $(x_1, x_2, \ldots)$ to $(0, x_1, x_2, \ldots)$. Show it is bounded and determine the operator norm $\|S\|$. Find also the adjoint S^*, and verify that $S^*S = I$ but $SS^* \neq I$.

Solution. Recall that we are dealing with the norm $\|\left(x_1, x_2, \ldots\right)\|^2 = \sum_{k=1}^{\infty} |x_k|^2$. But indeed then if $\|\left(x_1, \ldots\right)\| = 1$, then

$$\|S\left(x_1, \ldots\right)\|^2 = \|\left(0, x_1, x_2, \ldots\right)\|^2 = \sum_{k=1}^{\infty} |x_k|^2 = 1$$

so, in fact, S preserves the norm. But then since $\|Sx\| = \|x\|$ for all x by linearity, we have $\|S\| = 1$. Now, the inner product is $\langle x, y\rangle = \sum_k x_k \overline{y_k}$. Then

$$\langle Sx, y\rangle = \sum_{k=2}^{\infty} x_k y_{k-1} = \langle x, S^* y\rangle$$

if we define $S^*\left(y_1, y_2, \ldots\right) = \left(y_2, y_3, \ldots\right)$. We then indeed get $S^*S = I$ clearly, but $SS^*\left(x_1, x_2, \ldots\right) = \left(0, x_2, x_3, \ldots\right)$.

Exercise 1.7 (12.8)**.** Show $\|Ax \pm ix\|^2 = \|Ax\|^2 + \|x\|^2$ for A Hermitian and $\dim V < \infty$. Then show $A \pm iI$ is invertible and $(A-iI)(A+iI)^{-1}$ unitary.

Solution.

$$\begin{aligned}
\langle Ax \pm ix, Ax \pm ix\rangle &= \|Ax\|^2 + \langle Ax, \pm ix\rangle + \langle \pm ix, Ax\rangle + \langle \pm ix, \pm ix\rangle \\
&= \|Ax\|^2 + \|x\|^2 \mp i\langle Ax, x\rangle \pm i\langle x, Ax\rangle \\
&= \|Ax\|^2 + \|x\|^2 \mp i\langle Ax, x\rangle \pm i\langle Ax, x\rangle \\
&= \|Ax\|^2 + \|x\|^2.
\end{aligned}$$

Now, if $A \pm iI$ were not invertible, it would not be injective, so for $x \neq 0$, we would get

$$0 = \|Ax \pm ix\|^2 = \|Ax\|^2 + \|x\|^2$$

but $\|x\|^2 > 0$ and $\|Ax\|^2 \geq 0$, so this gives a contradiction.
Lastly, what is the adjoint of $(A-iI)(A+iI)^{-1}$? Well, $(A-iI)^* = A+iI$ by the rules on page 70. Hence the expression is of the form X^*X^{-1} which has adjoint $\left(X^{-1}\right)^* X$. Then $\left(X^{-1}\right)^* XX^*X^{-1}$.
Now, since A is self-adjoint, it is in particular normal, so $A+iI$ is normal and hence orthogonally diagonable. Writing $A+iI = \sum \lambda E_\lambda$, we get $(A+iI)^* = \sum \overline{\lambda} E_\lambda$, so $A+iI$ and $A-iI$ commute. Hence we get $XX^* = X^*X$, and the expression above becomes the identity.

Exercise 1.8 (12.4)**.** Give a simple proof of the Hahn-Banach theorem for a continuous linear form on a closed subspace of a Hilbert space.

Solution. Let V be a Hilbert space and let $U \subset V$ be a closed subspace. Then $V = U \oplus U^\perp$. Let $\mathcal{B}$ be a basis for U and extend it to a basis $\mathcal{A}$ for V. Take the duals $\mathcal{B}^{'}$ and $\mathcal{A}^{'}$. For $z \in U^*$ we can write $z = \sum_{y_i' \in \mathcal{B}'} a_i y_i$. Then z can also be considered a linear form on V by letting the coefficient for $y_i \in \mathcal{A}'$ be 0 if $y_i \notin \mathcal{B}'$

and a_i otherwise. The restrictions are clearly the same. By the Riesz-Fréchet representation theorem, since U is a closed subspace of a Hilbert space, it is also a Hilbert space, so by continuity of z, there exists $u \in U$ such that $z(x) = \langle x, u \rangle$ for all $x \in U$ and such that $\|z\| = \|u\|$. But since $z|_{U^\perp} = 0$, we also have $z(x) = \langle x, u \rangle$ for all $x \in V$, so by the Riesz-Fréchet theorem, $\|z\| = \|u\|$ over V as well.

Exercise 1.9 (13.1)**.** Prove $\rho(A+B) \leq \rho(A) + \rho(B)$ if A and B are normal. Prove it for general $A, B \in \mathrm{End}(V)$, now assuming they commute. Show the inequality can fail in general.

Solution. If A and B are normal, then they are orthogonally diagonable with respect to the associated inner product, hence $\rho(A) = \|A\|$ and $\rho(B) = \|B\|$. Now, in general, we have $\rho(X) \leq \|X\|$, we we get

$$\rho(A+B) \leq \|A+B\| \leq \|A\| + \|B\| = \rho(A) + \rho(B).$$

If A and B commute, then

$$\begin{aligned}
\rho(A+B) = \lim_{k\to\infty} \| (A+B)^k \|^{\frac{1}{k}} &= \lim_{k\to\infty} \| \sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i} \|^{\frac{1}{k}} \\
&\leq \lim_{k\to\infty} \left| \sum_{i=0}^{k} \binom{k}{i} \|A\|^i \|B\|^{k-i} \right|^{\frac{1}{k}} \\
&= \lim_{k\to\infty} \|A\| + \|B\| \\
&= \|A\| + \|B\| \\
&= \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}} + \lim_{k\to\infty} \|B^k\|^{\frac{1}{k}}
\end{aligned}$$

where the last equality follows from $\|X^k\| = \|X\|^k$ when X is diagonable (by the proof of lemma 13.4).

To show that it can fail in general, note that for $[A] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $[B] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have that the eigenvalues of both are precisely 1, hence $\rho(A) + \rho(B) = 2$, while $A + B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ which has characteristic polynomial $(x-3)(x-1)$ and thus 3 as an eigenvalue.

Exercise 1.10 (13.2)**.** Let $F = \mathbb{C}$. Find a counterexample to the statement: $\rho(p(A)) = p(\rho(A))$ for all polynomials p, where ρ is the spectral radius.

Solution. Consider $p(x) = ix - i$ and $A = -I$. So $p(A) = \begin{pmatrix} -2i & 0 \\ 0 & -2i \end{pmatrix}$ which has spectral radius 2. However, $-I$ has spectral radius 1 and $p(1) = 0$.

Exercise 1.11 (13.3)**.** Show $\rho(A^*A) = \|A^*A\| = \|A\|^2$ for the operator norm of an inner product.

Solution. The first equality holds when the matrix is orthogonally diagonable. But A^*A is self-adjoint, hence normal hence orthogonally diagonable.
The latter equality holds since

$$\|A^*A\| = \sup_{\|x\|=1} \langle A^*Ax, x\rangle = \sup_{\|x\|=1} \|Ax\|^2 = \|A\|^2$$