1:

(a) Suppose $J \subset k[x_1, \dots, x_{n+1}]$ is an ideal and that $(x_1, \dots, x_{n+1}) \subset \sqrt{J}$. Then by definition of radical ideals, for each $i \in \{1, \dots, n+1\}$, there exists $m_i \in \mathbb{N}$ such that $x_i^{m_i} \in J$. Then letting $m = \max\{m_1, \dots, m_{n+1}\}$, we have

$$(x_1, \dots, x_{n+1})^{(n+1)m} = \left\{x_1^{k_1} x_2^{k_2} \dots x_{n+1}^{k_{n+1}} \mid k_1 + \dots + k_{n+1} = (n+1)m, k_i \geq 0\right\}.$$

Let $x = x_1^{k_1} x_2^{k_2} \dots x_{n+1}^{k_{n+1}} \in (x_1, \dots, x_{n+1})^{(n+1)m}$ be an arbitrary element.
By the pigeonhole principle, there must exist a k_i such that $k_i \geq m \geq m_i$, so $x_i^{k_i} \in J$, and since J is a double-sided ideal, $x_1^{k_1} \dots x_i^{k_i} \dots x_{n+1}^{k_{n+1}} \in J$.
Since x was arbitrary, we get $(x_1, \dots, x_{n+1})^{(n+1)m} \subset J$.
Thus $(n+1)m = N$ works as an N.

(b) X is a projective algebraic set if there exists a set of homogeneous polynomials $S \subset k[x_1, \dots, x_{n+1}]$ such that $\mathbb{V}(S) = X$.

We have that $\mathbb{I}(X) = I(C(X))$ is prime if and only if $C(X)$ is irreducible. Now, suppose X is reducible as $X = X_1 \cup X_2$. So there exists sets of homogeneous polynomials $T, R \subset k[x_1, \dots, x_{n+1}]$ such that $X_1 = \mathbb{V}(T)$ and $X_2 = \mathbb{V}(R)$. We claim $C(X_1 \cup X_2) = C(X_1) \cup C(X_2)$.

We have $(0, \dots, 0) \neq (x_1, \dots, x_{n+1}) \in C(X_1 \cup X_2)$ if and only if $[x_1 : \dots : x_{n+1}] \in X = X_1 \cup X_2$ if and only if $[x_1 : \dots : x_{n+1}] \in X_1$ or $[x_1 : \dots : x_{n+1}] \in X_2$ if and only if $(x_1, \dots, x_{n+1}) \in C(X_1)$ or $(x_1, \dots, x_{n+1}) \in C(X_2)$.
If $(0, \dots, 0) = (x_1, \dots, x_{n+1})$, then $(x_1, \dots, x_{n+1)} \in C(X_1 \cup X_2), C(X_1), C(X_2)$ by definition.
Now, $C(X) = \{(x_1, \dots, x_{n+1})\colon [x_1 : \dots : x_{n+1}] \in X = \mathbb{V}(S) \vee (x_1, \dots, x_{n+1}) = (0, \dots, 0)\} = V(S)$ which is an algebraic set, and similarly $C(X_1) = V(T)$ and $C(X_2) = V(R)$ are algebraic sets.
Thus we get that $V(S) = C(X) = C(X_1 \cup X_2) = C(X_1) \cup C(X_2) = V(T) \cup V(R)$, and then if X is irreducible, then $V(S)$ is irreducible, so $V(S) = V(T)$ or $V(S) = V(R)$. But then $C(X) = V(S) = V(T) = C(X_1)$ or $C(X) = V(S) = V(R) = C(X_2)$, so $C(X)$ was not irreducible and hence $\mathbb{I}(X) = I(C(X))$ is not prime by the affine case.

Conversely, if $\mathbb{I}(X) = I(C(X))$ is prime then $C(X)$ is, as we showed above, an irreducible algebraic set. However, if $X = X_1 \cup X_2$ with X_1 and X_2 algebraic sets equaling $V(T)$ and $V(R)$ as above, respectively. Then $C(X) = C(X_1) \cup C(X_2)$, so either $V(T) = V(S)$ or $V(R) = V(S)$, and then we again get $C(X) = C(X_1)$ or $C(X) = C(X_2)$. Projectivizing this last bit gives $X = X_1$ or $X = X_2$, showing that X is irreducible.

2:

(a) Suppose X is closed in $\mathbb{P}^n$, so by definition it is a projective algebraic set, i.e. there exists a subset of homogeneous polynomials $S \subset k[x_1, \dots, x_{n+1}]$ such that $X = \mathbb{V}(S)$. Define

$$S_i = \{f \in k[x_1, \dots, x_n] \mid \exists g \in S\colon f(x_1, \dots, x_n) = g(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n), \forall (x_1, \dots, x_n) \in \mathbb{A}^n\}.$$

Then we claim $X \cap U_i = V(S_i) \subset \mathbb{A}^n$. Now, if $(a_1, \dots, a_n) \in V(S_i)$ then for all $g \in S$ we have $g(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = 0$, so since g is homogeneous, $[a_1 : \dots : a_{i-1} : 1 : a_{i+1} : \dots a_n] \in \mathbb{V}(g)$. Hence $[a_1 : \dots : 1 : \dots : a_n] \in \bigcap_{g \in S} \mathbb{V}(g) = \mathbb{V}(S) = X$, so $(a_1, \dots, a_n) \in \mathbb{V}(S) \cap U_i$.

Conversely, suppose $(a_1, \dots, a_n) \in X \cap U_i$. Then for all $g \in S$, $g(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots a_n) = 0$, so by definition of S_i, $(a_1, \dots, a_n) \in V(S_i)$.

Hence $X \cap U_i = V(S_i) \subset \mathbb{A}^n$ which is an algebraic set in $\mathbb{A}^n$ and thus closed in the Zariski topology on $\mathbb{A}^n \cong U_i$ for all i.

(b) If $W \subset U_i$ is open in the Zariski topology on $U_i \cong \mathbb{A}^n$, then $\mathbb{A}^n - (W \cap U_i)$ is closed, so there exist polynomials $S \subset k[x_1, \dots, x_n]$ such that $\mathbb{A}^n - (W \cap U_i) = V(S)$. Now define

$$S_i = \{f \text{ homogeneous} \in k[x_1, \dots, x_{n+1}] \mid \exists g \in k[x_1, \dots, x_n] \in S\colon f(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_{n+1}) = g(x_1, \dots, x_n)\}.$$

Since $W \subset U_i$, S_i is not empty, and hence $\mathbb{P}^n - W = \mathbb{V}(S_i)$, so W is open in $\mathbb{P}^n$.

(c) We show that $\mathbb{P}^n - X = \bigcup_i U_i - (U_i \cap X)$.

($\subset$): If $[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n - X = \bigcup_i (U_i) - X$, then there exists a U_i such that $[x_1 : \ldots : x_{n+1}] \in U_i$ and as it is not in X, $[x_1 : \ldots : x_{n+1}] \notin U_i \cap X$, so $[x_1 : \ldots : x_{n+1}] \in U_i - (U_i \cap X) \subset \bigcup_i U_i - (U_i \cap X)$.
Conversely, if $[x_1 : \ldots : x_{n+1}] \in \bigcup_i U_i - (U_i \cap X)$ then there exists i such that $[x_1 : \ldots : x_{n+1}] \in U_i - (U_i \cap X)$. Thus $[x_1 : \ldots : x_{n+1}] \notin X$ and $[x_1 : \ldots : x_{n+1}] \in U_i \subset \bigcup_i U_i = \mathbb{P}^n$, so $[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n - X$.

Now, if each $X \cap U_i$ is closed on each $U_i \cong \mathbb{A}^n$ then each $U_i - (U_i \cap X)$ is open in $\mathbb{P}^n$, so $\mathbb{P}^n - X = \bigcup_i U_i - (U_i \cap X)$ is open, so $X = \mathbb{P}^n - (\mathbb{P}^n - X)$ is closed.

(d) Suppose $X \subset \mathbb{P}^n$ is closed. Then by (a), $X \cap U_i$ is closed for each i in the Zariski topology on each $U_i \cong \mathbb{A}^n$.
Conversely, if each $X \cap U_i$ is closed in the Zariski topology on each $U_i \cong \mathbb{A}^n$, then by (c), X is closed in the Zariski topology on $\mathbb{P}^n$.
Now, if $X \subset \mathbb{P}^n$ is open, then $\mathbb{P}^n - X$ is closed, so by (a), $(\mathbb{P}^n - X) \cap U_i = (\mathbb{P}^n \cap U_i) - X = U_i - X$ is closed in U_i, so $U_i - (U_i - X) = U_i \cap (U_i \cap X^c)^c = U_i \cap (U_i^c \cup X) = U_i \cup X$ is open in U_i for each i.
Conversely, if $X \cap U_i$ is open for each U_i, then $U_i - (X \cap U_i) = U_i - X = U_i \cap X^c$ is closed for each U_i, so by (c), X^c is closed in $\mathbb{P}^n$, and thus $\mathbb{P}^n - X^c = X$ is open in $\mathbb{P}^n$.

3:
(b) Suppose $J \subset k[x_1, \ldots, x_{n+1}]$ is a radical homogeneous ideal. By (a), we have that J' is an ideal. Suppose now $f \in \sqrt{J'}$, so $f^n \in J'$.
Lemma: if $f, g \in k[x_1, \ldots, x_n]$ then $H(fg) = H(f)H(g)$.

Proof: Suppose $f = \sum_{\alpha_1 + \ldots + \alpha_n} a^{(\alpha)} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and $g = \sum_{\beta_1 + \ldots + \beta_n \leq M} b^{(\beta)} x_1^{\beta_1} \ldots x_n^{\beta_n}$.
Then dropping summation sign and using Einstein notation $H(f) = a^{(\alpha)} x_1^{\alpha_1} \ldots x_n^{\alpha_n} x_{n+1}^{N - \sum \alpha_i}$ and $H(g) = b^{(\beta)} x_1^{\beta_1} \ldots x_n^{\beta_n} x_{n+1}^{M - \sum \beta_i}$. Then $fg = a^{(\alpha)} b^{(\beta)} x_1^{\alpha_1 + \beta_1} \ldots x_n^{\alpha_n + \beta_n}$, so

$$H(fg) = a^{(\alpha)} b^{(\beta)} x_1^{\alpha_1 + \beta_1} \ldots x_n^{\alpha_n + \beta_n} x_{n+1}^{N + M - \sum_i (\alpha_i + \beta_i)} = \left[a^{(\alpha)} x_1^{\alpha_1} \ldots x_n^{\alpha_n} x_{n+1}^{N - \sum_i \alpha_i}\right] \left[b^{(\beta)} x_1^{\beta_1} \ldots x_n^{\beta_n} x_{n+1}^{M - \sum_i \beta_i}\right] = H(f)H(g)$$

We thus find that since $f^n \in J'$, $H(f)^n \overset{\text{lemma}}{=} H(f^n) \in J$, so since J is radical, $H(f) \in J$. Dehomoginizing, we find $f \in J'$, so $\sqrt{J'} \subset J' \subset \sqrt{J'}$, hence $\sqrt{J'} = J'$, so J' is a radical ideal.

(c) Suppose $I \subset k[x_1, \ldots, x_n]$ is radical. Suppose $F \in \sqrt{H(I)}$, so $F^n \in H(I)$. There exists an $f \in I$ such that $H(f) = F^n$.
Now, denote by $D(f)$ the dehomogenized polynomial of f - i.e. the polynomial f with $x_{n+1} = 1$. We have as a direct corollary from the definitions that $D(H(f)) = f$ and $H(D(f)) = f$, where $H(f)$ gives the smallest degree homogenization of f.
Lemma: For $f, g \in k[x_1, \ldots, x_{n+1}]$, $D(fg) = D(f)D(g)$.
Proof: Again, using Einstein summation, suppose $f = a^{(\alpha)} x_1^{\alpha_1} \ldots x_{n+1}^{\alpha_{n+1}}$ and $g = b^{(\beta)} x_1^{\beta_1} \ldots x_{n+1}^{\beta_{n+1}}$ then $fg = a^{(\alpha)} b^{(\beta)} x_1^{\alpha_1} x_1^{\beta_1} \ldots x_{n+1}^{\alpha_{n+1}} x_{n+1}^{\beta_{n+1}}$, so

$$D(fg) = a^{(\alpha)} b^{(\beta)} x_1^{\alpha_1} x_1^{\beta_1} \ldots x_n^{\alpha_n} x_n^{\beta_n} = D(f)D(g).$$

Hence dehomogenizing $H(f) = F^n$, we get $(D(F))^n = D(F^n) = D(H(f)) = f \in I$, so $D(F) \in I$ since I is radical, and hence $F = H(D(F)) \in H(I)$. Thus $\sqrt{H(I)} \subset H(I) \subset \sqrt{H(I)}$, so $\sqrt{H(I)} = H(I)$, so $H(I)$ is radical.

4:
Firstly, we give a trivial solution:
By theorem 3.26 in Linear Algebra Done Right by Axler, a homogeneous system of linear equations with more variables than equations has nonzero solutions.

Now we show the above:
Consider the matrix $(a_{i,j})$ where the i, j entry is $a_{i,j}$. The matrix is of dimension $m \times (n+1)$. It represents

the linear transformation $T\colon \mathbb{A}^{n+1} \to \mathbb{A}^m$ given by

$$T(x_1, \ldots, x_{n+1}) = \left(\sum_{k=1}^{n} a_{1,k} x_k, \ldots, \sum_{k=1}^{n} a_{m,k} x_k \right)$$

Since $m \leq n < n+1$, by the fundamental theorem of linear maps (3.22 in Linear Algebra done right by Axler), we have $n+1 = \dim \mathbb{A}^{n+1} = \dim \ker T + \dim \operatorname{range} T$, and as $\dim \operatorname{range} T \leq \dim \mathbb{A}^n \leq n$, we have $\dim \ker T \geq 1$, so T has nontrivial solutions.
Suppose $0 \neq (b_1, \ldots, b_{n+1}) \in \ker T$, so

$$\begin{aligned}(0, \ldots, 0) &= T(b_1, \ldots, b_{n+1}) \\ &= \left(\sum_{k=1}^{n} b_{1,k} x_k, \ldots, \sum_{k=1}^{n} b_{m,k} x_k \right)\end{aligned}$$

Hence $[b_1, \ldots, b_{n+1}] \in \mathbb{V}(a_{1,1}x_1 + \ldots + a_{1,n+1}x_{n+1}) \cap \ldots \cap \mathbb{V}\left(a_{m,1}x_1 + \ldots + a_{m,n+1}x_{n+1}\right) = \Lambda_1 \cap \ldots \cap \Lambda_m$, so $\Lambda_1 \cap \ldots \cap \Lambda_m \neq \varnothing$.

5:
(a) Suppose $F \in S_d$. Then there exists a homogeneous polynomial of degree d, $f \in k\left[x_1, \ldots, x_{n+1}\right]$ with $\overline{f} = F$. Now, we claim that the set of homogeneous polynomials of degree d in $k\left[x_1, \ldots, x_{n+1}\right]$ is precisely $A = (x_1, \ldots, x_{n+1})^d$. Any monomial of degree d is in A, and hence any homogeneous polynomial of degree d is in A. Conversely, any polynomial in A is by definition a sum of monomials of degree d and thus a homogeneous polynomial of degree d.
Since $A = \sum_{\alpha_1 + \ldots + \alpha_{n+1} = d} k x_1^{\alpha_1} \ldots x_{n+1}^{\alpha_{n+1}}$, it is finite-dimensional over k. Thus $f \in A$, so $F = \overline{f} \in \pi_I(A) = \sum_{\alpha_1 + \ldots + \alpha_{n+1} = d} k \overline{x}_1^{\alpha_1} \ldots \overline{x}_{n+1}^{\alpha_{n+1}}$. Now, if $f \in I$, then $F = 0$. Thus because π_I is surjective,

$$\pi_I(A) = S_d = \left\{ a \overline{x}^{\alpha_1} \ldots \overline{x}_{n+1}^{\alpha_{n+1}} \mid \alpha_1 + \ldots + \alpha_{n+1} = d, x_1^{\alpha_1} \ldots x_{n+1}^{\alpha_{n+1}} \notin I \right\}$$

which is generated by the basis

$$\left\{ \overline{x}_1^{\alpha_1} \ldots \overline{x}_{n+1}^{\alpha_{n+1}} \mid \alpha_1 + \ldots + a_{n+1} = d, x_1^{\alpha_1} \ldots x_{n+1}^{\alpha_{n+1}} \notin I \right\}$$

and is thus finite dimensional.
We check also the requirements for a vector space:
If $F, G \in S_d$, then there exist homogeneous polynomials of degree d $f, g \in k\left[x_1, \ldots, x_{n+1}\right]$ such that $\overline{f} = F$ and $\overline{g} = G$. So $F + G = \overline{f} + \overline{g} = \overline{f + g} \in \pi_I(A) = S_d$, since $f + g$ is homogeneous of degree d.
For scalar multiplication, let $F \in S_d$, then as above $\overline{f} = F$, so for $\lambda \in k$, $\lambda F = \lambda \overline{f} = \overline{\lambda f} \in \pi_I(A) = S_d$ since λf is homogeneous of degree d.
Furthermore, we can consider 0 as a form of any degree, and thus $0 \in S_d$ for any d, so we have an additive identity. The rest of the requirements are seen trivially as inherited from $k\left[x_1, \ldots, x_{n+1}\right]$.

(b) One upper bound is the upper bound in $k\left[x_1, \ldots, x_{n+1}\right]$. The subspace A is generated by the basis given in (a). For this, we use the balls-and-urns formula: we must partition d objects into $n+1$ urns, which we can do in $\binom{(n+1)+d-1}{(n+1)-1} = \binom{n+d}{n}$ ways. The image of the basis under π_I is a basis for $\pi_I(A) = S_d$. However, π_I may collapse some basis elements to 0 if they are contained in I, so the basis for S_d has an upper bound of $\binom{n+d}{n}$.