ASSIGNMENT 5

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CONTENTS

I will start the assignment by definitions, lemmas and theorems used in the solutions. I have put the problems in a separate section after this.

1. Results and Theory

1.0.1. *Coordinate bundles and fibre bundles.*

Lemma 1.1. *[1, Lemma 2.8] Let $\mathcal{B}, \mathcal{B}'$ be coordinate bundles having the space base space, fibre and group. Then they are equivalent if and only if there exist continuous maps*

$$\overline{g}_{kj} \colon V_j \cap V_k' \to G$$

such that

$$\overline{g}_{ki}(x) = \overline{g}_{kj}(x) g_{ji}(x)$$
$$\overline{g}_{lj}(x) = g_{lk}'(x) \overline{g}_{kj}(x)$$

.

1.0.2. *Construction of a bundle from coordinate transformations.*

Definition 1.2. Let G be a topological group and X a space. By a *system of coordinate transformations in X with values in G* is meant an indexed covering $\{V_j\}$ of X by open sets and a collection of continuous maps

$$g_{ji} \colon V_i \cap V_j \to G$$

such that

$$g_{kj}(x) g_{ji}(x) = g_{ki}(x).$$

Remark. We have so far seen that any bundle over X with group G determines such a set of coordinate transformations. We now state a converse.

Theorem 1.3 (Existence)**.** *[1, Thm 3.2] If G is a topological transformation group of Y, and $\{V_j\}, \{g_{ij}\}$ is a system of coordinate transformations in the space X, then there exists a bundle $\mathcal{B}$ with base space X, fibre Y, group G and coordinate transformations $\{g_{ij}\}$. Furthermore, any such bundles are equivalent.*

1.0.3. *The Principal Bundle and the Principal Map.*

Definition 1.4 (Principal G-bundle)**.** A bundle $\mathcal{B} = \{B, p, X, Y, G\}$ is called a principal bundle if $Y = G$ and G operates on Y by left translations.

Definition 1.5 (Associated prinicipal bundle)**.** Let $\mathcal{B} = \{B, p, X, Y, G\}$ be an arbitrary bundle. The *associated principal bundle* $\tilde{B}$ of $\mathcal{B}$ is the bundle given by the construction/existence theorem using the same base space, the same $\{V_j\}$, the same $\{g_{ji}\}$ and the same group G as for $\mathcal{B}$, but replacing Y by G and allowing G to operate on itself by left translations.

Theorem 1.6 (Equivalence theorem)**.** *[1, Thm 10.3] Two bundles having the same base space, fibre and group are equivalent if and only if their associated principal bundles are equivalent.*

Proof. By Lemma 1.1, equivalence of bundles is purely a property of the coordinate transformations. □

Definition 1.7 (Manifold bundle)**.** Let M be a smooth manifold. A manifold bundle over M with structure group G is a fiber bundle $W \to E \to M$ with structure group G such that E is a manifold and $E \to M$ is continuous.
We say a manifold bundle over M is a smooth manifold bundle if it is a smooth fiber bundle as well as a manifold bundle and G acts by diffeomorphisms on M.

Definition 1.8 (Associated bundles)**.** Let M be a smooth manifold, and fix a manifold bundle $E \xrightarrow{\xi} M$ with fibre a smooth manifold W and structure group $G \leq \mathrm{Homeo}(W)$. Given another smooth manifold W' such that there exists an injective group homomorphism $\iota\colon G \hookrightarrow \mathrm{Homeo}(W')$, the associated W'-manifold bundle of ξ is defined as follows. Let $\{U_\alpha, \varphi_\alpha\}_\alpha$ be a cover of M by open neighborhoods together with trivializations φ_α of ξ. Transition maps $\varphi_\alpha \varphi_\beta^{-1}$ give rise to transition function $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G \leq \mathrm{Homeo}(W)$ satisfying the cocycle condition. We define the associated W'-manifold by gluing trivializations $U_\alpha \times W'$ along transition maps

$$\iota \circ g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G \xrightarrow{\iota} \mathrm{Homeo}(W').$$

Definition 1.9 (Structure group reduction)**.** Fix a manifold bundle $\xi\colon E \to M$ over a smooth manifold M, with fibre a smooth manifold W and structure group G. Given a subgroup $H \leq G$, ξ is said to admit a structure group reduction to H if it is isomorphic to a bundle so that all transition maps $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ take values in H.

1.0.4. *The Induced Bundle.*

Definition 1.10 (Induced Bundle)**.** Suppose $\mathcal{B}', X$ and η are as before. Form the product space $X \times B'$ and let $p\colon X \times B' \to X, h\colon X \times B' \to B'$ be the natural projections. Define $B = X \times_{X'} B' := \{(x, b') \in X \times B' \mid \eta(x) = p'(b')\}$ to be the fibered product.

We want to give $[p\colon B \to X]$ a fibre bundle structure (by giving it a coordinate bundle structure). Define $V_j = \eta^{-1}(V'_j)$ and set

$$\varphi_j(x, y) = \left(x, \varphi'_j\left(\eta(x), y\right)\right).$$

Let's give these maps some motivation. For these to be trivializations, we want φ_j to be homeomorphisms $p^{-1}(V_j) \cap B = p|_B^{-1}(V_j) \cong V_j \times Y$. Now, φ_j simply maps x to x in the first coordinate, but φ'_j by assumption maps $V'_j \times Y$ homeomorphically onto $p'^{-1}(V'_j)$. Hence in particular, $\varphi'_j(\eta(x), y) \in p'^{-1}(V'_j) \subset B'$. So $\left(x, \varphi'_j(\eta(x), y)\right) \in B$ if and only if $\eta(x) = p'\left(\varphi'_j(\eta(x), y)\right)$, but this is true by assumption. Furthermore, $\left(x, \varphi'_j(\eta(x), y)\right) \in X \times B'$, so applying p, we get $p\left(x, \varphi'_j(\eta(x), y)\right) = x$ which is in V_j when $x \in V_j$. Hence putting things together, φ_j maps $V_j \times Y$ to $p^{-1}(V_j) \cap B$. We, in fact, want to show that φ_j is a homeomorphism of these spaces. For this, simply note that the map $(u, v) \mapsto \left(u, \pi_2 \circ \varphi_j'^{-1}(v)\right)$ is an inverse.
Lastly, let for $x \in V_i \cap V_j$, $g_{ij}(x) = \varphi_{i,x}^{-1}\varphi_{j,x} = p_i\varphi_{j,x}$
Note then that

$$\begin{aligned}
g_{ij}(x)y &= p_i\varphi_{j,x}(y) \\
&= p_i\left(x, \varphi'_j(\eta(x), y)\right) \\
&= p'_i\varphi'_j(\eta(x), y) \\
&= g'_{ij}(\eta(x))\, y
\end{aligned}$$

So the clutching functions are simply $g'_{ij} \circ \eta$ which are indeed continuous.

Theorem 1.11 (Equivalence Theorem/pullbacks of fibre bundles with the same fibre and group exist)**.** *Let $\mathcal{B}, \mathcal{B}'$ be two bundles having the same fibre and group and $h\colon \mathcal{B} \to \mathcal{B}'$ a bundle map. Let $\eta\colon X \to X'$ be the induced map of base spaces. Then the induced bundle $\eta^*\mathcal{B}'$ is equivalent to $\mathcal{B}$, and there is an equivalence $h_0\colon \mathcal{B} \to \eta^*\mathcal{B}'$ such that h is the composite $h = h^* \circ h_0$ where $h^*\colon \eta^*\mathcal{B}' \to \mathcal{B}'$ is the induced map:*

$$\begin{tikzcd}
B \arrow[rrd, bend left] \arrow[rd, dashed, "\exists"] \arrow[rdd, bend right] & & \\
& X \times_{X'} B' \cong \eta^* B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B' \arrow[d] \\
& X \arrow[r, "\eta"] & X'
\end{tikzcd}$$

Note. A "Bundle Theory" is also called a Cartesian Fibration over Sm.

Definition 1.12 (Bundle Theory)**.** A bundle theory is a functor from some arbitrary category $\mathcal{B}$ to Sm subject to the following conditions.
Given a map $f\colon M \to N$ between smooth manifolds in Sm, there exists a map $f^*\colon \mathcal{B}(N) \to \mathcal{B}(M)$.
The solid arrows in the diagram below, the dashed lifts are in bijection and the

diagram commutes.

$$\begin{tikzcd}
B' \arrow[r, dashed, "\psi"] \arrow[d] \arrow[rr, bend left] & f^*B \arrow[r] \arrow[d] \arrow[dr, phantom, "\exists"] & B \arrow[d] \\
N \arrow[r, dashed, "\varphi"] \arrow[rr, bend right] & P \arrow[r, "f"] & M
\end{tikzcd}$$

In the sense that given φ, there exists a ψ, everything commutes and composite map above is mapped under the functor to the composite map below.
Furthermore, it is required to satisfy gluing (the cocycle condition). I describe this in the solution of the problem below.

A bundle $B \to M$ is called locally trivial if for each point $x \in M$, there exists a neighborhood $x \in U \overset{i}{\hookrightarrow} M$ and there exists a bundle $B' \to *$ and a pullback along $\pi\colon U \to *$ for B' such that there exists an isomorphism $i^*B \cong \pi^*B'$.

2. Problems

2.1. **Principal G-bundles.** Let G be a discrete group. Consider the category Sm^G where objects are smooth manifolds equipped with a free, fixed point free action by G which is properly discontinuous: the exists a cover $\{U_\alpha\}_{\alpha\in A}$ of M so that $\{g \cdot U_\alpha\}$ are pairwise disjoint for all $\alpha \in A$ and $g \in G$. Furthermore, morphisms are smooth maps which are G-equivariant: $f\colon M \to N$ is such that $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in M$.

Problem 2.1.
1. (2pts) Show that for $M \in \mathrm{Sm}^G$, the quotient M/G admits a structure of a smooth manifold so that the map $M \to M/G$ is a local diffeomorphism.
2. (5pts) Check that the association $M \mapsto M/G$ defines a functor $\mathrm{Sm}^G \to \mathrm{Sm}$, and show that this defines a locally trivial bundle theory on smooth manifolds.

Proof. (1) (2 pts) (I will assume that G acts by homeomorphisms on M) Using the covering space quotient theorem (theorem 12.14 in Lee's book on Topological Manifolds), we find that $M \to M/G$ is a covering space. To construct a smooth structure on M/G, let $p \in M/G$ and U an evenly covered open neighborhood of p. Then U splits into homeomorphic copies $\sqcup U_\alpha$ in M with $\pi|_{U_\alpha}\colon U_\alpha \cong U$ homeomorphisms. For $\tilde{p} \in U_\alpha$, choose a smooth chart $(V_{\tilde{p}}, \varphi_{\tilde{p}})$ contained in U_α. Since $\tilde{p} = g \cdot p$ for some g, we may as well denote these charts as $(V_{g,p}, \psi_{g,p})$. Now consider the charts $\left(\pi|_g(V_{g,p}), \psi_{g,p} \circ (\pi|_g)^{-1}\right)$. On an overlap the transition functions have the form

$$\psi_{g,p} \circ (\pi|_g)^{-1} \left(\psi_{g',p'} \circ (\pi|_{g'})^{-1}\right)^{-1} = \psi_{g,p} \circ (\pi|_g)^{-1}\, \pi|_{g'} \circ \psi^{-1}_{g',p'} = \psi_{g,p} \circ \psi^{-1}_{g',p'}$$

on the overlap, which is smooth by assumption. Hence we indeed obtain a smooth structure on M/G. In particular, the map $\pi\colon M \to M/G$ has coordinate form

$$\left(\psi_{g,p} \circ \pi|_g^{-1}\right) \pi \circ \psi_{g,p}^{-1} = \mathrm{id}$$

which is a diffeomorphism. So π is a local diffeomorphism when we equip M/G with this smooth structure.

(2) (5 pts) Define the functor $F\colon \mathrm{Sm}^G \to \mathrm{Sm}$ sending $M \mapsto M/G$ with the smooth structure defined in the first part of the exercise. Here, since maps $f\colon M \to N$ in Sm^G are G-equivariant, they, in particular, descend to smooth maps $\overline{f}\colon M/G \to N/G$, and we let $F(f) = \overline{f}$. Then indeed $F(\mathrm{id}_M) = \overline{\mathrm{id}_M} = \mathrm{id}_{M/G}$ and if $f\colon M \to N$ and $g\colon N \to P$, then $F\left(g \circ f\right) = \overline{g \circ f}$. But by pasting the two squares

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[r, "g"] \arrow[d] & P \arrow[d] \\ M/G \arrow[r, "\overline{f}"] & N/G \arrow[r, "\overline{g}"] & P/G \end{tikzcd}$$

we find that $\overline{g \circ f} = \overline{g} \circ \overline{f}$. So $F\left(g \circ f\right) = F(g) \circ F(f)$.
This shows that F is indeed a functor.
We want to show that this defines a bundle theory on Sm. So suppose we have some $N \in \mathrm{Sm}^G$ and $f\colon M \to N/G$ in Sm. Now, the quotient map $N \to N/G$ is a submersion (show this), so the pullback along f exists in Sm, giving

$$\begin{tikzcd} f^*N \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[d] \\ M \arrow[r] & N/G \end{tikzcd}$$

Lastly, we must then show that f^*N is in Sm^G. For this, note that the induced bundle f^*N is precisely the pullback which is equivalent as a fibre bundle to $M \times_{N/G} N$. But this inherits a natural action of G given by $g \cdot (m,n) = (m, g \cdot n)$. Choosing the same cover $\{U_\alpha\}$ for N as given in the condition of it being in Sm^G, i.e., $\{g \cdot U_\alpha\}$ being disjoint for all g and α, the neighborhoods $M \times U_\alpha \cap f^*N$ then satisfy the same conditions under this action of G. Lastly, the map $f^*N \cong M \times_{N/G} N \to N$ given by the projection to the N component which is the top map in the pullback diagram is naturally G-equivariant. This shows that the above diagram indeed can be made.
Now suppose we have some $P \in \mathrm{Sm}^G$ and a bundle map $P \to N$ giving the solid part of the diagram

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & M \times_{N/G} N \arrow[r] \arrow[d] & N \arrow[d] \\ P/G \arrow[r] & M \arrow[r] & N/G \end{tikzcd}$$

where the map $P \to N$ descends to the composite map $P/G \to M \to N/G$ on the bottom.

We then want to show that the dashed map exists. Let $p\colon P \to P/G$ and $q\colon f^*N \cong M \times_{N/G} N \to M$ be the projection. Let $k\colon P \to N$ be the map on the top. Let $f\colon P/G \to M$ be the map on the bottom. Define a map $h\colon P \to M \times_{N/G} N$ by $h(x) = \left(f(p(x)), k(p)\right)$. Then if $l\colon M \to N/G$ denotes the map on the bottom, $l \circ f\left(p(x)\right) = \pi\left(k(p)\right)$ where $\pi\colon N \to N/G$. By definition then $h(x) \in M \times_{N/G} N$.

Furthermore,

$$h(g\cdot x) = (f(p(g\cdot x)), k(g\cdot x)) = (f(p(x)), g\cdot k(x)) = g\cdot(f(p(x)), k(x)) = g\cdot h(x),$$

so h is G-equivariant.

Next we must check that the bundle theory is locally trivial. That is, we must check that for any $M \in \mathrm{Sm}^G$ and any point $x \in M/G$, there exists an open neighborhood U about x such that if we let $\pi\colon U \to *$ be the unique map and $i\colon U \to M/G$ the open embedding, there exists a manifold $N \in \mathrm{Sm}^G$ such that $N/G \cong *$, and such that the pullbacks are isomorphic: $i^*M \cong \pi^*N$.
Note that these pullbacks are really

$$\begin{tikzcd} U \times_{M/G} M \cong i^*M \arrow[r] \arrow[d] & M \arrow[d, "p"] \\ U \arrow[r] & M/G \end{tikzcd}$$

But clearly if $(u,m) \in U \times_{M/G} M$, then essentially $\overline{m} = u$, so $U \times_{M/G} M \cong p^{-1}(U)$, and

$$\begin{tikzcd} U \times N \cong U \times_* N \arrow[r] \arrow[d] & N \arrow[d] \\ U \arrow[r] & * \end{tikzcd}$$

So we find that the condition is indeed equivalent to the usual one: the existence of a neighborhood U about x and a homeomorphism $p^{-1}(U) \cong U \times N$. In this case, suppose $x \in M/G$ and simply choose one of the U_α such that $x \in p(U_\alpha)$. Note that this is open in M/G since the $g \cdot U_\alpha$ are pairwise disjoint and g acts by homeomorphisms (G is discrete and each g has g^{-1} as inverse). Choosing $U = p(U_\alpha)$, we get $p^{-1}(U) = \sqcup_{g\in G} U_\alpha \cong U_\alpha \times G \cong U \times G$ where $G \in \mathrm{Sm}^G$ is precisely G considered as a smooth manifold with the trivial charts $g \mapsto *$, at each $g \in G$. Indeed then $G/G \cong *$, so this satisfies the condition above. I.e., the functor $\mathrm{Sm}^G \to \mathrm{Sm}$ is locally trivial.

Lastly, we must check gluing. Namely that for $M \in \mathrm{Sm}^G$ and some open coordinate neighborhoods $U_i, U_j, U_k \subset M/G$, with coordinate maps $g_{ij}\colon U_i \cap U_j \to G$, $g_{jk}\colon U_j \cap U_k \to G$ and $g_{ki}\colon U_k \cap U_i \to G$, the maps satisfy $g_{ik}(x) = g_{ij}(x)g_{jk}(x)$ for $x \in U_i \cap U_j \cap U_k$. As we saw above, $p^{-1}(U_i) = U_i \times G$, and we shall call this coordinate function $\varphi_i\colon U_i \times G \to p^{-1}(U_i)$. Let $g_{ij}(x) = \varphi_{i,x}^{-1}\varphi_{j,x}$ where $\varphi_{i,x}(y) = \varphi_i(x,y)$ is the function considered only as a function of y. But then the condition $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$ follows trivially.

This completes the proof that the functor we constructed $\mathrm{Sm}^G \to \mathrm{Sm}$ is indeed a bundle theory over Sm. □

2.2. **Change of fibres of bundles.**

Problem 2.2 (Change of fibres of bundles)**.** (3pts) Let W_0 and W_1 be two smooth manifolds, and let G be a group which we assume as a simultaneous subgroup of both $\mathrm{Homeo}(W_0)$ and $\mathrm{Homeo}(W_1)$, i.e., we have injective group homomorphisms $\iota_0\colon G \hookrightarrow \mathrm{Homeo}(W_0)$ and $\iota_1\colon G \hookrightarrow (W_1)$. Given a fixed smooth manifold M, construct a bijection $\mathrm{Bun}_G^{W_0}(M) \to \mathrm{Bun}_G^{W_1}(M)$, where $\mathrm{Bun}_G^{W_i}(M)$ denotes the set of isomorphism classes of manifold bundles with fibre W_i and structure group G over the base space M.

Proof. (3pts) Let $\mathcal{B} = \{B, p, M, W_0, G\} \in \mathrm{Bun}_G^{W_0}$. By Theorem 1.6, the bundle $\mathcal{B}$ is equivalent to its associated principal bundle $\tilde{\mathcal{B}} = \{B, p, M, G, G\}$ which thus represents the same isomorphism class. But by assumption, G embeds into $\mathrm{Homeo}(W_1)$, so by Theorem 1.3, also $\tilde{B}$ is equivalent to $\{B, p, M, W_1, G\} =: \mathcal{B}'$ which has the same coordinate transformations. Thus $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{B}}'$ are equivalent. Now, seeing as equivalence of bundles is purely determined by their base space, fibre, structure group and coordinate transformations by Lemma 1.1, this gives an injective map $\mathrm{Bun}_G^{W_0} \to \mathrm{Bun}_G^{W_1}$. We can simply use the existence theorem directly. Seeing as we can do the exact same thing to obtain an injective map $\mathrm{Bun}_G^{W_1} \to \mathrm{Bun}_G^{W_0}$, we obtain a bijection by Schröder-Bernstein. □

2.3. **Associated frame bundles and structure group reductions.** I couldn't figure this one out in time.

Problem 2.3 (Associated frame bundles and structure group reductions)**.** For a rank d vector bundle $\xi\colon E \to M$ over a smooth manifold, we define the associated frame bundle $\mathrm{Fr}(\xi)$ as the associated $\mathrm{GL}_d(\mathbb{R})$-bundle.

(1) (1 pt) For M a smooth d-dimensional manifold, we define its frame bundle $\mathrm{Fr}(M)$ as the associated frame bundle of its tangent bundle TM. Show that $\mathrm{Fr}(M) \to M$ is a principal $\mathrm{GL}_d(\mathbb{R})$-bundle.

2.4. **Invertible Cobordisms and Boundaries of Compact Manifolds.**

Problem 2.4 (Invertible cobordisms and boundaries of compact manifolds)**.** Let $W_0\colon M_0 \rightsquigarrow \varnothing$ and $W_1\colon M_1 \rightsquigarrow \varnothing$ be two compact d-dimensional smooth cobordisms from compact $(d-1)$-dimensional smooth manifolds M_0 and M_1 to the empty manifold, viewed as a $(d-1)$-manifold. In other words, we have a smooth embedding $M_i \times \mathbb{R} \hookrightarrow W_i$ satisfying that $M_i \times (-\infty, 0]$ is closed, and such that their complement $W_i - (M_i \times \mathbb{R})$ is compact. We define $\mathrm{Int}(W_i)$ to be the complement of the image of $M_i \times (-\infty, t]$ for some $t \in \mathbb{R}$ (and hence any $t \in \mathbb{R}$), and observe that $\mathrm{Int}(W_i)$ is again a smooth manifold, being an open subset of W_i.

(1) (4pts) Assume that in the situation of the above, $\mathrm{Int}(W_0)$ is diffeomorphic to $\mathrm{Int}(W_1)$. Show that M_0 and M_1 are invertibly cobordant, i.e., there exists a cobordism $M_0 \rightsquigarrow M_1$ which is invertible in the category Cob_d.
(2) (6pts) Let W be a smooth, open (i.e., non-compact) d-manifold. We define a compact closure of W to be a compact cobordism $W'\colon M \rightsquigarrow \varnothing$ such that W is diffeomorphic to $\mathrm{Int}(W')$. Assume that W admits a compact closure $W'\colon M \rightsquigarrow \varnothing$. Show that the set of compact closures of W up to isomorphism of their interiors is in bijection with the set of invertible cobordisms over M.

Proof. (1) (4 pts)
Saying that $M_0 \rightsquigarrow M_1$ is invertible in Cob_d is precisely saying that there exists a cobordism $M_1 \rightsquigarrow M_0$ such that the composite cobordism $M_0 \rightsquigarrow M_1 \rightsquigarrow M_0$ is equivalent to the trivial cobordism $M_0 \rightsquigarrow M_0$. We will do this using the usual definition of cobordisms with boundaries. Then the problem is equivalently to show that we can find coborisms $M_0 \rightsquigarrow M_1 \rightsquigarrow M_0$ such that the composite is a

product cobordism. In this case, we are dealing with closed compact manifolds W_0, W_1 such that $\partial W_0 \cong M_0$ and $\partial W_1 \cong M_1$. Furthermore, the boundaries have closed collar neighborhoods $\partial W_i \times I$, and removing some open/usual collar neighborhoods of these boundaries $\partial W_i \times [0,1)$ leaves us with compact spaces which are, by assumption, diffeomorphic. Now, take the cobordism W_0 and choose a collar neighborhood of ∂W_0: $M_0 \times [0,1]$, where M_0 is identified with $M_0 \times 0$ in W_0. By assumption, there is a diffeomorphism $W_0 - (M_0 \times [0,1]) \cong W_1 - (M_1 \times [0,1])$. Now, the diffeomorphism extends to the closure of the interiors which is also M_i since the collar is a cylinder, so we obtain a diffeomorphism $h\colon M_0 \times 1 \cong M_1 \times 1$. Without loss of generality, we can reparametrize, to get the diffeomorphism $h\colon M_0 \times 1 \to M_1 \times 0$ since the boundaries of the interiors must map to each other. Now we can glue the collars by gluing the cobordisms they represent using theorem 1.4 in Milnor's book on h-cobordisms to get a cobordism c_h which is the manifold $M_0 \times [0,1] \cup_h M_1 \times [0,1]$. This indeed now gives a cobordism $M_0 \rightsquigarrow M_1$. We can likewise obtain the cobordism $M_1 \rightsquigarrow M_0$ which is also obtained by gluing $M_1 \times [0,1]$ with $M_0 \times [0,1]$ along $M_1 \times 1$ and $M_0 \times 0$. Denote this cobordism by $c_{h'}$. We claim that $c_h c_{h'} = \mathrm{id}_{M_0}$. That is, that $c_h c_{h'}$ is a product cobordism/trivial cobordism of M_0. One way to see this is by using theorem 1.6 in Milnor's book on h-cobordisms which says that $c_h c_{h'} = c_{h'h} = c_{\mathrm{id}_{M_0}}$ which indeed is the trivial cobordism. Alternatively, each collar neighborhood has no critical values, so c_h and $c_{h'}$ both have Morse number 0, and then corollary 3.8 in Milnor's book on h-cobordisms gives that $c_h c_{h'}$ also has Morse number 0, hence is trivial by theorem 3.4 in the same book. □

References

[1] Norman Steenrod. *The topology of fibre bundles.* Princeton Landmarks in Mathematics. Reprint of the 1957 edition, Princeton Paperbacks. Princeton University Press, Princeton, NJ, 1999, **pages** viii+229. ISBN: 0-691-00548-6.