ASSIGNMENT 2

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Problem 0.1. Let $T = S^1 \times S^1$ be the torus and $i\colon D^2 \hookrightarrow T$ and embedding of the unit disk that is disjoint from $S^1 \times \{s_0\}$. Define $A := \left(S^1 \times \{s_0\}\right) \cup i\left(S^1\right) \subset T$. Let $x_0 = (s_0, s_0)$ and $x_1 \in i\left(S^1\right)$.

(1) Draw a picture of (X, A) and the two points x_0 and x_1.
(2) Construct an explicit bijection of sets $\pi_1(T, A, x_1) \cong \mathbb{Z}^2 \sqcup \mathbb{Z}$.
(3) Compute the relative homotopy groups $\pi_2(T, A, x_0)$ and $\pi_2(T, A, x_1)$.

Solution. (1)

FIGURE 1. Note that in this figure, A are the parts drawn without the interior of the disk $i(D^2)$.

(2) Recall that
$$\pi_n(T, A, x_1) = \left[I^n, \partial I^n, J^{n-1}; T, A, x_1\right].$$
Thus $\pi_1(T, A, x_1)$ becomes the set of homotopy classes of maps $(I, \{0,1\}, \{1\}) \to (T, A, x_1)$. That is, the set of paths in T starting at a point in A and ending at x_1 up to homotopy through such paths.
For any map $f\colon (I, \{0\}, \{1\}) \to (T, A, \{x_1\})$, can lift this to the universal cover since I is simply connected. Let $\tilde{f}\colon (I, \{0\}, \{1\}) \to \left(\mathbb{R}^2, p^{-1}(A), p^{-1}(\{x_1\})\right)$. Now, $p^{-1}(A)$ can be visualized as tiling $\mathbb{R}^2$ by tiles as the left picture in Figure 1, each tile of course contains precisely one element of the fiber $p^{-1}(\{x_1\})$. For the lift $\tilde{f}$, we choose a base point $\tilde{x_1}$ in $p^{-1}(\{x_1\})$. By the lifting theorem, there now exists a unique lift, call it $\tilde{f}\colon (I, \{1\}) \to \left(\mathbb{R}^2, \{\tilde{x_1}\}\right)$, such that $f = p \circ \tilde{f}$. Now, $f(0) \in A$ is the only condition, so $\tilde{f}(0)$ lies in $p^{-1}(A)$. Homotopies through maps which start in A for f correspond in the universal cover to letting $\tilde{f}(0)$ run freely through its path component in $p^{-1}(A)$. We can construct a bijection $\pi_0\left(p^{-1}(A)\right) \cong \mathbb{Z}^2 \sqcup \mathbb{Z}$ by identifying the component of $p^{-1}\left(i\left(S^1\right)\right) \cap [n, n+1] \times [m, m+1]$ with $(n, m) \in \mathbb{Z}^2$ and identifying the vertical line in $p^{-1}\left(S^1 \times \{s_0\}\right) \cap [n, n+1)$ with $n \in \mathbb{Z} \subset \mathbb{Z}^2 \sqcup \mathbb{Z}$.

This is obviously bijective. We can always homotopy f to be a straight-line in the universal cover, so the only thing that determines the equivalence class of f, given that $\tilde{f}$ ends at $\tilde{x_1}$, is which path component in $\mathbb{Z}^2 \sqcup \mathbb{Z}$ it start in. This gives an injective map $\varphi\colon \pi_1(T, A, x_1) \to \mathbb{Z}^2 \sqcup \mathbb{Z}$. To see that it is surjective, it is clear that choosing $\tilde{x_1}$ as above and choosing any point in the path component corresponding to an element $x \in \mathbb{Z}^2 \sqcup \mathbb{Z}$, taking the straight line between these two points gives a path $\tilde{f}\colon I \to \mathbb{R}^2$ such that $f := p \circ \tilde{f}$ gives a path $[f] \in \pi_1(T, A, x_1)$, and, by construction, $\varphi([f]) = x$.
Thus $\pi_1(T, A, x_1) \cong \mathbb{Z}^2 \sqcup \mathbb{Z}$.
(3) Let $\iota\colon A \to T$ be the inclusion. Then using the LES of relative homotopy groups, we have that

$$\pi_2(T, x_i) \to \pi_2(T, A, x_i) \to \pi_1(A, x_i) \xrightarrow{\iota_*} \pi_1(T, x_i)$$

is exact for $i = 0, 1$. For $i = 0, 1$, $\pi_1(A, x_i) \cong \mathbb{Z}$ and $\pi_1(T, x_i) \cong \mathbb{Z}^2$, while $\pi_2(T, x_i) \cong \pi_2(S_1) \times \pi_2(S^1) \cong 1$ for both $i = 0, 1$. Hence $\pi_2(T, A, x_i) \cong \ker \iota_*$.
First, suppose $i = 0$. Then ι induces the map $\mathbb{Z} \cong \pi_1(A, x_0) \to \pi_1(T, x_0) \cong \mathbb{Z}^2$ given by $n \mapsto (0, n)$, so $\ker \iota_*$ is trivial in this case, so $\pi_2(T, A, x_0) \cong 0$. Suppose now that $i = 1$. Then any loop in the image of ι_* is clearly based nullhomotopic by contracting $i(D^2)$ to the point x_1. Thus $\ker \iota_* = \pi_1(A, x_1) \cong \pi_1(S^1) \cong \mathbb{Z}$. So $\pi_2(T, A, x_1) \cong \mathbb{Z}$.

Problem 0.2.
1. Compute $\pi_1(S^1 \vee S^2)$ and describe the universal cover of $S^1 \vee S^2$.
2. Show that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\bigoplus_{\mathbb{Z}} \mathbb{Z}$.
3. Explicitly describe the action of $\pi_1(S^1 \vee S^2)$ on $\bigoplus_{\mathbb{Z}} \mathbb{Z} \cong \pi_2(S^1 \vee S^2)$.

Solution. (1) The universal cover of $S^1 \vee S^2 =: X$, which we will denote $\tilde{X}$, is clearly $\mathbb{R}$ with a copy of S^2 attached to each integer of $\mathbb{R}$.
Let A_1 be the S^1 part together with a small open neighborhood of the base point in S^2, and likewise, A_2 be S^2 together with a small open neighborhood of the base point in S^1 - here the base points are the points that get identified in the construction of $S^2 \vee S^1$. Applying van Kampen, we find that $\pi_1(S^2 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^2)/N$ where N is generated by all elements of the form $i_{12}(w)i_{21}(w)^{-1}$ for $w \in \pi_1(A_1 \cap A_2)$. But $A_1 \cap A_2$ is contractible, so $N \cong 0$. Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(S^2) \cong 0$, we conclude that $\pi_1(S^2 \vee S^1) \cong \mathbb{Z}$.

(2) To compute $\pi_2(S^1 \vee S^2)$, it suffices to compute π_2 of its universal cover since these are isomorphic. The universal cover is $\mathbb{R}$ with S^2 attached at each integer. Since $\mathbb{R} \simeq \{*\}$, the universal cover is homotopy equivalent to $\vee_{\mathbb{Z}} S^2$ for example by using proposition 0.16 and 0.17 in Hatcher.
Since homotopy groups are invariant under based homotopy equivalences, it suffices to compute $\pi_2(\bigvee_{\mathbb{Z}} S^2)$.
But $\bigvee_{\mathbb{Z}} S^2$ is 1-connected, so if $\tilde{H}_2(\bigvee_{\mathbb{Z}} S^2) \cong H_2(\bigvee_{\mathbb{Z}} S^2)$ is nonzero, then by the Hurewicz theorem, we will obtain that $\pi_2(\bigvee_{\mathbb{Z}} S^2) \cong H_2(\bigvee_{\mathbb{Z}} S^2)$. Now, we can give $\bigvee_{\mathbb{Z}} S^2$ a Δ-complex (or cellular) structure with a single 0-simplex and a 2-simplex for each S^2 in $\bigvee_{\mathbb{Z}} S^2$. The associated simplicial chain complex then becomes

$$\ldots \to 0 \to \bigoplus_{\mathbb{Z}} \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \ldots$$

with 0 everywhere else. In particular then $H_2\left(\bigvee_{\mathbb{Z}} S^2\right) \cong \oplus_{\mathbb{Z}}\mathbb{Z}$ since there cannot be any cancellation from the maps. Since the Hurewicz isomorphism takes $f \in \pi_2\left(\bigvee_{\mathbb{Z}} S^2\right)$, to $f_*[1] \in H_2\left(\bigvee_{\mathbb{Z}} S^2\right)$ for $[1] \in H_2\left(S^2\right)$ a generator, we find that through our proof using the Δ-complex of $\bigvee_{\mathbb{Z}} S^2$, we found that the inclusions $S^2 \hookrightarrow \bigvee_{\mathbb{Z}} S^2$ in fact induce generators on homology: i.e., the images of the different inclusions $\iota_i \colon H_2\left(S_i^2\right) \hookrightarrow H_2\left(\bigvee_{i\in\mathbb{Z}} S_i^2\right) \cong \bigoplus_{\mathbb{Z}}\mathbb{Z}$ generate $\bigoplus_{\mathbb{Z}}\mathbb{Z}$, and hence also $\pi_2\left(\bigvee_{\mathbb{Z}} S^2\right)$ under the Hurewicz isomorphism.

(3) Recall that the action of $\pi_1\left(S^1 \vee S^2\right)$ on $\pi_n\left(S^1 \vee S^2\right)$ makes $\pi_n\left(S^1 \vee S^2\right)$ into a $\mathbb{Z}\left[\pi_1\left(S^1 \vee S^2\right)\right]$-module. We saw that $\pi_1\left(S^1 \vee S^2\right) \cong \mathbb{Z}$. Let γ be a loop that goes once around the S^1 factor. This generates $\pi_1\left(S^1 \vee S^2\right) \cong \mathbb{Z}$, so it suffices to describe the action of γ on $\pi_2\left(S^1 \vee S^2\right)$ since π_2 now becomes a $\mathbb{Z}[\gamma]$-module under this action. Since also $\pi_2\left(S^1 \vee S^2\right) \cong \bigoplus_{\mathbb{Z}}\mathbb{Z}$, it suffices to describe the action of γ on an arbitrary basis element of $\bigoplus_{\mathbb{Z}}\mathbb{Z}$, say, corresponding to the image under p_* of some an inclusion of some $S^2 \hookrightarrow \tilde{X}$. Suppose we choose the inclusion α into the S^2 attached to $1_n \in \mathbb{Z}_n$.
Then $p_*\alpha = [\eta_n\xi] = 1_n \in \mathbb{Z}_n$ where η_n is the loop that winds around the S^1 factor n times and ξ is the inclusion $S^2 \hookrightarrow S^1 \vee S^2$.
In particular then $\gamma p_*\alpha = [\eta_{n+1}\xi] = 1_{n+1} \in \mathbb{Z}_{n+1}$.

This completes the description, but I will also give an alternative description just for completeness where I expound on some details between the homomorphisms $H_2\left(\bigvee_{\mathbb{Z}} S^2\right) \cong \pi_2\left(\bigvee_{\mathbb{Z}} S^2\right) \cong \pi_2\left(S^1 \vee S^2\right)$ that underlies the above explanation. We will use the correspondence between the π_1 action on $\pi_n(X)$ and its action on $\pi_n\left(\tilde{X}\right)$ where $\tilde{X}$ was the universal covering space.
To this end, we have previously shown the following:

Lemma 0.3. *Let $p \colon \tilde{X} \to X$ be the universal cover of a path-connected space X. Under the isomorphism $\pi_n(X) \cong \pi_n(\tilde{X})$, for $n \geq 2$, the action of $\pi_1(X)$ on $\pi_n(X)$ corresponds to the action of $\pi_1(X)$ on $\pi_n\left(\tilde{X}\right)$ induced by the action of $\pi_1(X)$ on $\tilde{X}$ as deck transformations. More precisely, for $\gamma \in \pi_1(X, x_0), \alpha \in \pi_n\left(\tilde{X}, \tilde{x}_0\right), \tilde{\gamma}$ the lift of γ, and γ_* the homomorphism induced by the action of γ on $\tilde{X}$, we have $\gamma p_*(\alpha) = p_*\left(\beta_{\tilde{\gamma}}\left(\gamma_*(\alpha)\right)\right)$.*

Let $\alpha \in \pi_2\left(\tilde{X}\right)$ be the element corresponding under the isomorphism $\pi_2(X) \cong \pi_2(\tilde{X})$ to the class of our chosen inclusion $S^2 \hookrightarrow \bigvee_{\mathbb{Z}} S^2$. That is, α is the inclusion of S^2 into one of the S^2 in the universal cover. To understand $\gamma p_*(\alpha)$, we can thus look at $p_*\left(\beta_{\tilde{\gamma}}\left(\gamma_*(\alpha)\right)\right)$. Now, $\gamma_*(\alpha)$ will simply be the inclusion of S^2 to the S^2 "above" the previous one in the universal cover. So if we previously included our S^2 into the S^2 attached to $n \in \mathbb{R} \subset \tilde{X}$, then $\gamma_*(\alpha)$ corresponds to including S^2 into the S^2 attached to $n+1 \in \mathbb{R} \subset \tilde{X}$. Then $\beta_{\tilde{\gamma}}$ is simply the change-or-basepoint transformation depicted in the picture on page 341 in Hatcher. I.e., it essentially shrinks α and attaches it inside a larger square where we put $\tilde{\gamma}$ on each radial line in-between the squares. If we understand our isomorphism $\pi_2\left(\bigvee_{\mathbb{Z}} S^2\right) \cong \bigoplus_{i\in\mathbb{Z}}\mathbb{Z}_i$ as the generator for $\mathbb{Z}_i$ corresponding under the Hurewicz isomorphism to the inclusion of S^2 into the sphere attached to $i \in \mathbb{Z}$, then we find that $\alpha \mapsto \beta_{\tilde{\gamma}}\left(\gamma_*(\alpha)\right)$ precisely sends $\alpha = 1_n \in \mathbb{Z}_n$ to $1_{n+1} \in \mathbb{Z}_{n+1}$. Under p_*, this may be interpreted again as sending $1_n \mapsto 1_{n+1}$ when n corresponds to $[\eta\xi]$ where η is the loop that winds around the S^1 factor n times and ξ is the inclusion of $S^2 \hookrightarrow S^1 \vee S^2$.

Problem 0.4. Let (X, A, x_0) be a pointed pair such that the inclusion $i\colon A \hookrightarrow X$ is based nullhomotopic (the nullhomotopy preserves the basepoint). The goal is to show that for $n \geq 2$, there is an isomorphism of groups:

$$\pi_n(X, A, x_0) \cong \pi_n(X, x_0) \times \pi_{n-1}(A, x_0).$$

(1) Show that there is an exact sequence of groups

$$1 \to \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \to 1.$$

(2) Using a based nullhomotopy $H\colon A \times [0,1] \to X$, construct a natural group morphism

$$r_*\colon \pi_n(X, A, x_0) \to \pi_n(X, x_0)$$

such that $r_* \circ j_* = 1$.

(3) Show that for any short exact sequence of groups

$$1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$$

such that α admits a retraction, there is a group isomorphism

$$B \cong A \times C.$$

Conclude the desired isomorphism.

Proof. (1) From the LES for relative homotopy groups, we obtain that

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \pi_{n-1}(X, x_0)$$

is exact. For $n \geq 2$, all the sets in the exact sequence are groups and the maps are group homomorphisms. Since homotopic maps relative to the base point induce the same maps on homotopy groups, we find by assumption that $i_* = 0$. Therefore,

$$1 \xrightarrow{0} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \xrightarrow{0} 1$$

is exact.

(2) Let $[f] \in \pi_n(X, A, x_0)$ and consider a representative $f\colon \left(D^n, S^{n-1}, s_0\right) \to (X, A, x_0)$. We put f on the bottom of a cylinder $D^n \times \{0\} \subset D^n \times I$. Now $H(f(x), t)$ gives a homotopy $S^{n-1} \times I \to X$, so we can use this on $S^{n-1} \times I \subset D^n \times I$ of the cylinder. Now we use that $D^n \times \{0\} \cup S^{n-1} \times I \cong D^n$ (see Figure 2). Denote this homeomorphism by $\varphi\colon D^n \to D^n \times \{0\} \cup S^{n-1} \times I$.
Define $h\colon S^{n-1} \times I$ by $h(x,t) = H(f(x), t)$ and define $h \cup f\colon D^n \times \{0\} \cup S^{n-1} \times I$ by f on $D^n \times \{0\}$ and h on $S^{n-1} \times I$. Then define $h \cup f \circ \varphi\colon D^n \to X$. Now $h \cup f \circ \varphi$ maps ∂D^n to x_0, so it factors through the quotient $D^n \to S^n$ and induces a map $\Gamma\colon (S^n, pt) \to (X, x_0)$, where pt is the point that the boundary collapses to. This is well-defined since if $f \simeq f' \operatorname{rel} s_0$ through a homotopy $F\colon D^n \times I \to X$, then $\tilde{h}(x,t,s) = H(F(x,s), t)$ gives a map $S^{n-1} \times I \times I$ - and this homotopy is constant on the boundary $S^{n-1} \times \{1\}$. Then taking $\tilde{h} \cup F\colon \left(D^n \times \{0\} \cup S^{n-1} \times I\right) \times I \cong D^{n-1} \times \{0\} \times I \cup S^{n-1} \times I \times I \to X$, we obtain a homotopy $\tilde{h} \cup F \circ \varphi\colon D^n \times I \to X$ which is constant on the boundary throughout, hence induces the desired homotopy $S^n \times I \to X$ between Γ and $\Gamma' \operatorname{rel}\{pt\}$.
To see that it is a group morphism, see Figure 3. Here the top left picture depicts

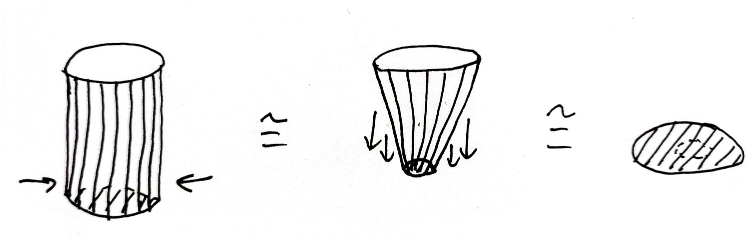

FIGURE 2.

Γ obtained from $f+g \in \pi_n(X, A, x_0)$. The bottom left picture represents $\Gamma_f + \Gamma_g$, where Γ_f is obtained from f by the above procedure and Γ_g is obtained from g by the procedure. Hence $r_*([f]+[g]) = r_*([f]) + r_*([g])$.

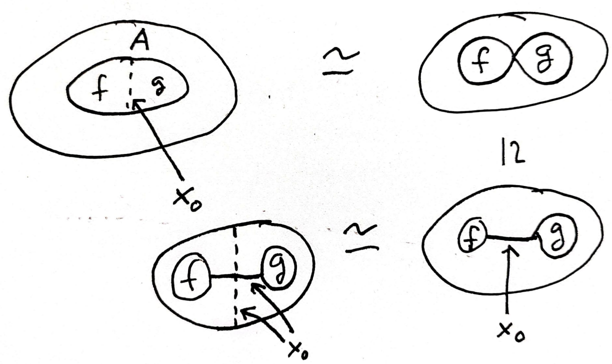

FIGURE 3.

Naturality amounts to showing that r_* defines a natural transformation from $\pi_n(-,-,-)$ to $\pi_n(-,-)$ on the category of based pairs (X, A) such that $A \hookrightarrow X$ is based nullhomotopic. That is, that given a map $f\colon (X, A, x_0) \to (Y, B, y_0)$, with both $A \hookrightarrow X$ and $B \hookrightarrow Y$ based nullhomotopic, the diagram

$$\begin{array}{ccc} \pi_n(X, A, x_0) & \xrightarrow{r_*} & \pi_n(X, x_0) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} \\ \pi_n(Y, B, y_0) & \xrightarrow{r_*} & \pi_n(Y, y_0) \end{array}$$

commutes.
Now, if $H\colon A \times I \to X$ is the based nullhomotopy of $A \hookrightarrow X$ and $G\colon B \times I \to Y$ is the based nullhomotopy of $B \hookrightarrow Y$, then for $[f] \in \pi_n(X, A, x_0)$, we get the situation of Figure 4. In the central part, these maps agree - namely they are $f \circ g$. We are thus asking for a homotopy between $f \circ H(g(x), t)$ and $G(f \circ g(x), t)$. So we want a map $L\colon S^{n-1} \times I \times I \to X$. We may assume without loss of generality that H and G map $S^{n-1} \times \{0\}$ to x_0 and y_0, respectively, instead of $S^{n-1} \times \{1\}$.

Now we let L be given by

$$L\,(x,t,s) \begin{cases} f\circ H\,(g(x),(1-2s)t)\,, & s\in\left[0,\frac{1}{2}\right] \\ G\,(f\circ g(x),2s-1)\,, & s\in\left[\frac{1}{2},1\right]. \end{cases}$$

This gives naturality.

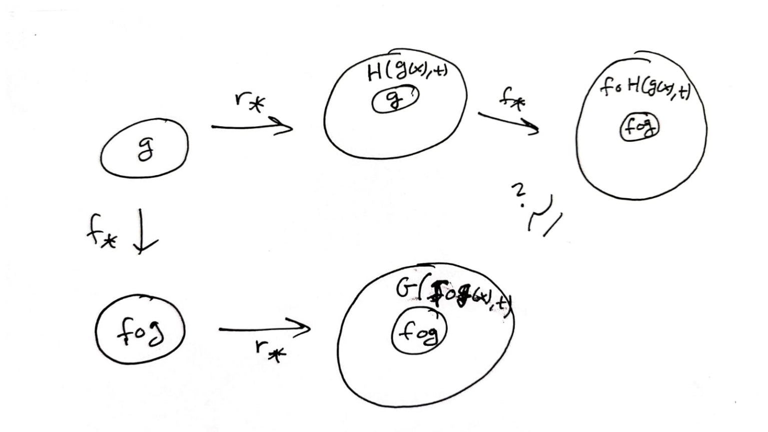

FIGURE 4.

Now, for $[f]\in\pi_n\,(X,x_0)$, we have that the boundary is already mapped to x_0, so $H\,(f(x),t)$ is constant on $S^{n-1}\times I$ since H is relative the basepoint. Hence $\Gamma\simeq f$ as depicted in Figure 5 where Γ is obtained from $j_*\,[f]$ which is simply $[f]\in\pi_n\,(X,A,x_0)$.

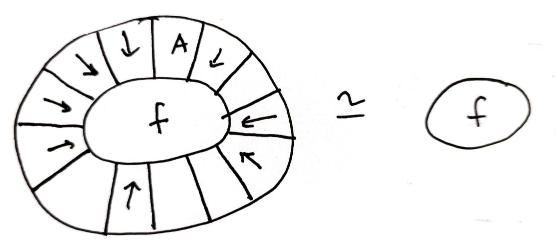

FIGURE 5.

This shows that $r_*\circ j_*=\mathrm{id}$ which was what we wanted to show.

(3) Suppose

$$1\to A\overset{\alpha}{\to}B\overset{\beta}{\to}C\to 1$$

is a short exact sequence and let $s\colon B\to A$ be a retraction - i.e., $s\circ\alpha=\mathrm{id}$. We claim that $\varphi\colon B\to A\times C$ by $\varphi(b)=(s(b),\beta(b))$ is an isomorphism. Firstly, it is clearly a group homomorphism since s and β are assumed to be group homomorphisms. Next, for injectivity, if $\varphi(b)=0$, then $s(b)=0$ and $\beta(b)=0$. But by exactness then there exists $a\in A$ such that $\alpha(a)=b$. Thus $a=\mathrm{id}(a)=s\circ\alpha(a)=s(b)=0$. But

then since α is a group homomorphism, it takes 0 to 0, so $b = \alpha(a) = \alpha(0) = 0$. This gives injectivity.
For surjectivity, let $(a, c) \in A \times C$. Since β is surjective by exactness of the SES, there exists $b \in B$ such that $\beta(b) = c$. Then $s(\alpha(a) - \alpha \circ s(b) + b) = a - s(b) + s(b) = a$ while $\beta\left(\alpha(a) - \alpha \circ s(b) + b\right) = \beta(b) = c$ since $\beta \circ \alpha = 0$. Hence $\varphi\left(\alpha(a - s(b)) + b\right) = (a, c)$, so φ is also surjective.
To conclude the desired isomorphism of the problem, we simply note that by (1) and (2), we precisely have an exact sequence where j_* admits a retraction, so by (3), we get an isomorphism

$$\pi_n\left(X, A, x_0\right) \cong \pi_n(X, x_0) \times \pi_{n-1}(A, x_0).$$

$\square$