Exercise 0.1. Find $\dim\{A \in \operatorname{End}(V) \mid A(W) \subset W\}$ for a subspace $W \subset V$.

Solution.
Suppose V is finite-dimensional. Then W is also finite dimensional, so choose a basis $\{x_1, \ldots, x_k\}$ for W and extend it to a basis $\{x_1, \ldots, x_n\}$ for V. Then any $A \in U = \{A \in \operatorname{End}(V) \mid A(W) \subset W\}$, can by theorem 2.28 be written as

$$A = \sum_{i,j} \alpha_{ij} E_{ij}$$

where i, j run over $\{1, \ldots, n := \dim V\}$ and $E_{ij}(x_k) = \delta_{jk} x_i$. Since $A(W) \subset W$, we get by uniqueness of linear combinations (lemma 1.10), that $E_{ij}(x_i) \in \operatorname{span}(x_1, \ldots, x_k)$ for $i \in \{1, \ldots, k\}$ which is equivalent to $\alpha_{ij} = 0$ whenever $j \in \{1, \ldots, k\}$ and $i \in \{k+1, \ldots, n\}$. This is the only requirement for $A(W)$ to be contained in W, so any A of this form is also in U. Hence a basis for U is all E_{ij} such that $(i,j) \notin \{k+1, \ldots, n\} \times \{1, \ldots, k\}$ which has $n^2 - (n-k)k = n^2 - nk + k^2$ elements, so $\dim U = n^2 - (n-k)k$.

For V infinite dimensional, $\dim U = \infty$:

Let $\{v_\alpha\}_{\alpha \in I}$ be some basis for W and extend it to a basis $\{v_\alpha\}_{\alpha \in I \cup J}$ for V. If $|I| = \infty$, then define a map $\varphi_{\alpha,\beta} \colon V \to V$ by $\varphi_{\alpha,\beta}(v_\alpha) = v_\beta$ for $\alpha, \beta \in I$ distinct, and the zero map on the remaining basis elements. There are infinitely many such maps and clearly, each one is in U. Suppose $0 = \sum_{i,j} c_{ij} \varphi_{\alpha_i,\beta_j}$ is some linear combination. Then applying this on v_{β_j}, we get $0 = \sum_i c_{ij} v_{\alpha_i}$, so by linear independence, we get $c_{ij} = 0$ for all i. Since j was arbitrarily chosen in the linear combination, we get c_{ij} for all i and j occurring in the sum. Hence $\{\varphi_{\alpha,\beta}\}_{\alpha,\beta \in I}$ is a linearly independent subset of U, so $\dim U = \infty$.

If $|I| = n < \infty$ is finite, then J is infinite. We can do the above construction again to get maps $\varphi_{\alpha,\beta} \colon V \to V$ but now for $\alpha, \beta \in J$ distinct. In this case, they all map W to 0 which is indeed in W, so they are in U. Completely equivalently to the above, we see again that they are linearly independent, so $\dim U = \infty$.