1:

(a) We prove it by induction.

Suppose $f = f_0$. Since f is constant, if $f(\lambda a_1, \ldots, \lambda a_{n+1}) = 0$ for any $(a_1, \ldots, a_{n+1})$, then f is identically zero. Hence $f_0 = f = 0$.

Assume it is proved for all $d \leq N$.

Now assume $f(\lambda a_1, \ldots, \lambda a_{n+1}) = 0$ for all non-zero scalars $\lambda \in k$ and $f = f_0 + f_1 + \ldots + f_{N+1}$. Then we have for each f_i that

$$f_i(\lambda a_1, \ldots, \lambda a_{n+1}) = \lambda^i f_i(a_1, \ldots, a_{n+1}),$$

so

$$\begin{aligned}
c_0 + |\lambda| c_1 + \ldots + |\lambda|^N c_N &\geq \left| f_0(a_1, \ldots, a_{n+1}) + \lambda f_1(a_1, \ldots, a_{n+1}) + \ldots + \lambda^N f_N(a_1, \ldots, a_{n+1}) \right| \\
&= |f_0(\lambda a_1, \ldots, \lambda a_{n+1}) + \ldots + f_N(\lambda a_1, \ldots, \lambda a_{n+1})| \\
&= |f_{N+1}(\lambda a_1, \ldots, \lambda a_{n+1})| \\
&= |\lambda|^{N+1} |f_{N+1}(a_1, \ldots, a_{n+1})| = |\lambda|^{N+1} c_{N+1},
\end{aligned}$$

where $c_i = |f_i(a_1, \ldots, a_{n+1})|$.

So in particular, for positive λ, we have

$$c_0 + \lambda c_1 + \ldots + \lambda^N c_N \geq \lambda^{N+1} c_{N+1} \tag{ζ}$$

Now assuming $c_{N+1} \neq 0$, choose

$$\lambda = \frac{|c_0| + |c_1| + \ldots + |c_N|}{|c_{N+1}|} + 1.$$

Note that $\lambda \geq 1$, and thus $\lambda^j \leq \lambda^N$ for $j = 0, 1, \ldots, N$, so using the triangle inequality, we have

$$\begin{aligned}
c_0 + \ldots + c_N \lambda^N &= \left| c_0 + c_1 \lambda + \ldots + c_N \lambda^N \right| \\
&\leq (|c_0| + \ldots + |c_N|) \lambda^N \\
&< \left| c_{N+1} \lambda^{N+1} \right| = c_{N+1} \lambda^{N+1},
\end{aligned}$$

but this contradicts (ζ). Hence $0 = c_{N+1} = |f_{N+1}(a_1, \ldots, a_{n+1})|$, so $f_{N+1}(a_1, \ldots, a_{n+1}) = 0$. Hence for all $\lambda \in k$,

$$\begin{aligned}
0 = f(\lambda a_1, \ldots, \lambda a_{n+1}) &= f_0(\lambda a_1, \ldots, \lambda a_{n+1}) + \ldots + f_N(\lambda a_1, \ldots, \lambda a_{n+1}) + \underbrace{f_{N+1}(\lambda a_1, \ldots, \lambda a_{n+1})}_{=\lambda^{N+1} f(a_1, \ldots, a_{n+1}) = 0} \\
&= f_0(\lambda a_1, \ldots, \lambda a_{n+1}) + \ldots + f_N(\lambda a_1, \ldots, \lambda a_{n+1}).
\end{aligned}$$

By the inductive assumption, we now get $f_i(a_1, \ldots, a_{n+1})$ for all $i \in \{1, \ldots, N+1\}$, which gives the result by induction.

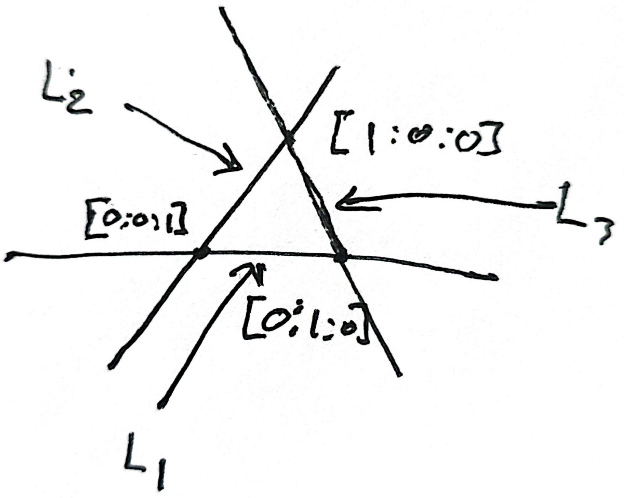
(b) Suppose $Y \subset \mathbb{A}^{n+1}$ is a cone. Then for any $(x_1, \ldots, x_{n+1}) \in Y$, and for any $\lambda \in k$, we have $(\lambda x_1, \ldots, \lambda x_{n+1}) \in Y$, so assume $f \in I(Y)$ and $f = f_0 + f_1 + \ldots + f_d$. Choose any point $(a_1, \ldots, a_{n+1}) \in Y$. Then for all non-zero scalars $\lambda \in k$, we have $(\lambda a_1, \ldots, \lambda a_{n+1}) \in Y$, so

$$f(\lambda a_1, \ldots, \lambda a_{n+1}), \quad \forall \lambda \in k,$$

so by (a), we have $f_i(a_1, \ldots, a_{n+1}) = 0$ for all $i = 0, \ldots, d$, and as $(a_1, \ldots, a_{n+1}) \in Y$ was arbitrary, we have that for any $i \in \{0, \ldots, d\}$, $f_i \in I(Y)$, so by past (2) of the last definition/proposition on lecture note 18, we have that $I(Y)$ is a homogeneous ideal.

2:

(a) As $U_1 = \left\{[x_1 \colon x_2 \colon x_3] \in \mathbb{P}^2 \colon x_1 \neq 0\right\}$, we have that the complement is $L_1 = \left\{[0 \colon x_2 \colon x_3] \in \mathbb{P}^2\right\}$, and similarly, $L_2 = \left\{[x_1 \colon 0 \colon x_3] \in \mathbb{P}^2\right\}$ and $L_3 = \left\{[x_1 \colon x_2 \colon 0] \in \mathbb{P}^2\right\}$. Hence $L_1 \cap L_2 = \left\{[x_1 \colon x_2 \colon x_3] \in \mathbb{P}^2 \colon x_1 = 0 = x_2\right\} = \left\{[0 \colon 0 \colon x_3] \in \mathbb{P}^2\right\} = \{[0 : 0 : 1]\}$. Similarly, $L_1 \cap L_3 = \{[0 \colon 1 \colon 0]\}$ and $L_2 \cap L_3 = \{[1 \colon 0 \colon 0]\}$.



(b) A point $[x_1 \colon x_2 \colon x_3] \in \mathbb{P}^2$ is in U_i if and only if $x_i \neq 0$, so it belongs to all three affine charts if and only if $x_1, x_2, x_3 \neq 0$, that is

$$U_1 \cap U_2 \cap U_3 = \left\{[x_1 \colon x_2 \colon x_3] \in \mathbb{P}^2 \colon x_1, x_2, x_3 \neq 0\right\}.$$

(c) A point $[x_1 \colon x_2 \colon x_3] \in \mathbb{P}^2$ belongs to only one affine chart, say U_i, if and only if $x_i \neq 0$ and the remaining coordinates are 0.
So the point belongs to only U_1 for example, if and only if $x_1 \neq 0$ and $x_2 = 0 = x_3$.

3:
(a) The corresponding projective algebraic set in $\mathbb{P}^2$ is $\left\{[x \colon y \colon 1] : x = y^2\right\} = \left\{[x \colon y \colon z] : zx = y^2\right\} = \mathbb{V}\left(zx - y^2\right)$ by homogenization.

(b) The intersection is

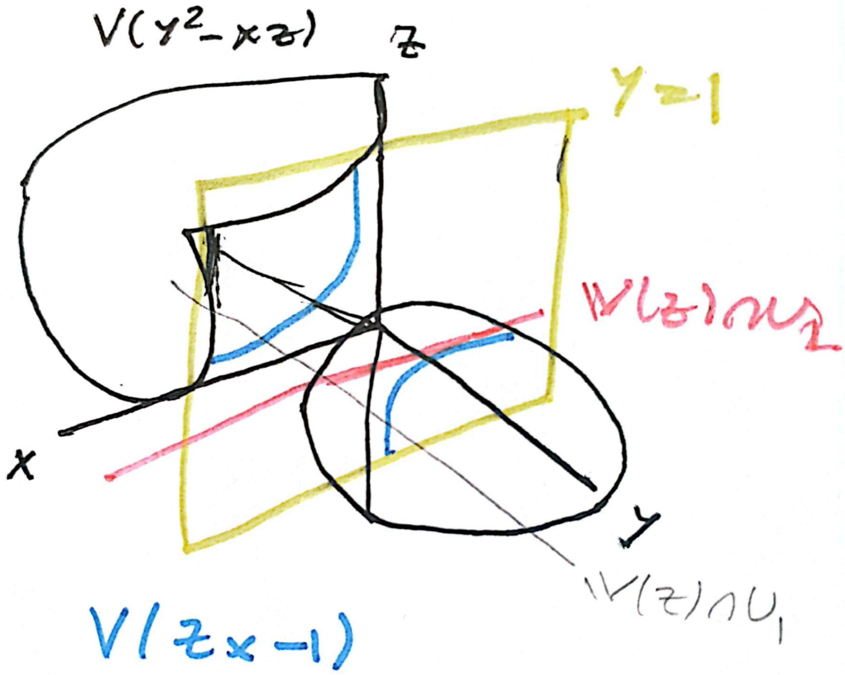
$$\mathbb{V}(zx - y^2) \cap \mathbb{V}(z) = \left\{[x \colon y \colon z] \in \mathbb{P}^2 \colon zx = y^2\right\} \cap \left\{[x \colon y \colon 0] \in \mathbb{P}^2\right\} = \left\{[x \colon y \colon 0] : 0 = y^2\right\} = \left\{[x \colon 0 \colon 0] \in \mathbb{P}^2\right\}.$$

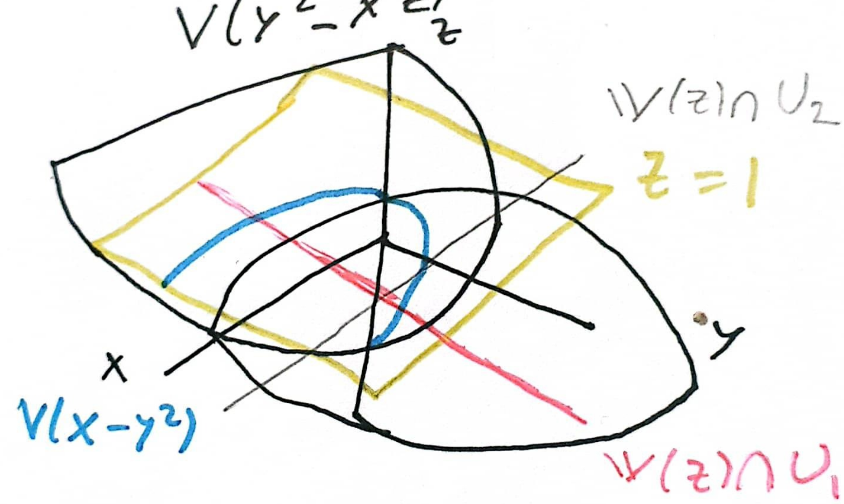
(c) We have

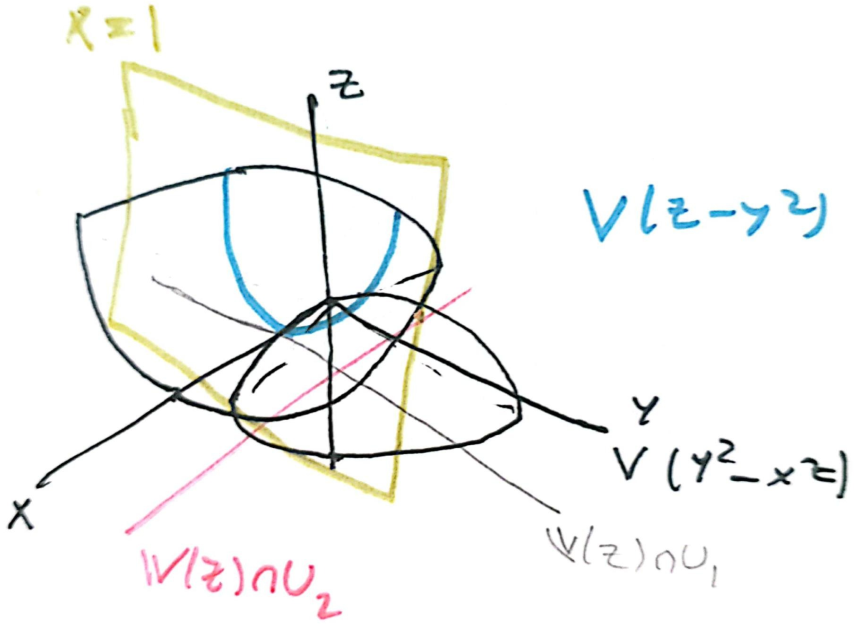
$$\mathbb{V}(zx - y^2) \cap U_1 = \left\{[x \colon y \colon z] : zx = y^2\right\} \cap \left\{[x \colon y \colon z] : x \neq 0\right\} = \left\{[x \colon y \colon z] : zx = y^2, x \neq 0\right\} = V\left(z - y^2\right)$$
$$\mathbb{V}(zx - y^2) \cap U_2 = \left\{[x \colon y \colon z] : zx = y^2\right\} \cap \left\{[x \colon y \colon z] : y \neq 0\right\} = \left\{[x \colon y \colon z] : zx = y^2, y \neq 0\right\} = V\left(zx - 1\right)$$
$$\mathbb{V}(zx - y^2) \cap U_3 = \left\{[x \colon y \colon z] : zx = y^2\right\} \cap \left\{[x \colon y \colon z] : z \neq 0\right\} = \left\{[x \colon y \colon z] : zx = y^2, z \neq 0\right\} = V\left(x - y^2\right)$$

Furthermore

$$\mathbb{V}(z) \cap U_1 = \{[x : y : 0]\} \cap \{[x : y : z] : x \neq 0\} = \{[x : y : 0] : x \neq 0\} = \{[1 : y : 0]\}$$
$$\mathbb{V}(z) \cap U_2 = \{[x : y : 0]\} \cap \{[x : y : z] : y \neq 0\} = \{[x : y : 0] : y \neq 0\} = \{[x : 1 : 0]\}$$
$$\mathbb{V}(z) \cap U_3 = \{[x : y : 0]\} \cap \{[x : y : z] : x \neq 0\} = \varnothing$$


V(y^2-xz)
z
y=1
W(z)∩U_2
x
y
W(z)∩U_1
V(zx-1)


V(y^2-xz)
z
W(z)∩U_2
z=1
x
y
V(x-y^2)
W(z)∩U_1



4:
(a) ($\implies$): Assume firstly that I is prime. Let $F, G \in k[x_1, \dots, x_{n+1}]$ be homogeneous. Assume $FG \in I$, then by assumption, $F \in I$ or $G \in I$.
($\Leftarrow$): Now assume conversely that for every pair of homogeneous polynomials F and G, if $FG \in I$, then $F \in I$ or $G \in I$.
Let now $f, g \in I$. Since I is a homogeneous ideal in $k[x_1, \dots, x_{n+1}]$, we can write $I = (\{F_i\})$ for homogeneous polynomials F_i. Thus we have $f = \sum F_{i_f}$ and $g = \sum F_{i_g}$. Hence

$$fg = \sum \underbrace{F_{i_f} F_{i_g}}_{\in I} \in I$$

as I is an ideal and thus closed under sums and multiplication.
Therefore I is a prime ideal.

(b) Suppose $f \in \sqrt{I}$. Then $f^n \in I$ for some n. Suppose $f = f_m + f_{m+1} + \ldots + f_d$ for some $0 \leq m \leq d$. Then the lowest degree term of f^n is of degree $n \cdot m$ and is precisely f_m^n. As I is a homogeneous polynomial, we have $f_m^n \in I$ by the last definition/proposition on lecture note 18, so in particular $f_m \in \sqrt{I}$. Now, as $\sqrt{I}$ is an ideal as was shown on HW2, we have $f - f_m \in \sqrt{I}$, which has lowest degree term of degree $m+1$. Repeating this process $d - m + 1$ times, we find that $f_i \in \sqrt{I}$ for all $i \in \{m, m+1, \dots, d\}$, so by the last definition/proposition on lecture note 18, we find that $\sqrt{I}$ is homogeneous as $f \in \sqrt{I}$ was arbitrary.

5: Suppose $V(f) \subset \mathbb{A}^2_{\mathbb{R}}$ is a circle.
We have that $V(f) \subset \mathbb{A}^2_{\mathbb{R}}$ is a circle if and only if $V(f) = V\left((x-a)^2 + (y-b)^2 - R^2\right)$ which has corresponding subset in $\mathbb{P}^2$ and homogenization

$$\left\{[x:y:1] : 0 = y^2 + a^2 - 2ax + b^2 - 2by - R^2 + x^2\right\} = \left\{[x:y:z] : 0 = y^2 + a^2z^2 - 2axz + b^2z^2 - 2byz - R^2z^2 + x^2\right\}$$

So $F = y^2 + a^2z^2 - 2axz + b^2z^2 - 2byz - R^2z^2 + x^2$. Now, letting $z = 0$, we find $y^2 + x^2 = 0$ which has solutions $[1:i]$ and $[1:-i]$ since $(1)^2 + (i)^2 = 1 - 1 = 0$ and $(1)^2 + (-i)^2 = 1 - 1 = 0$. Hence

$$\{[1:i:0], [1:-i:0]\} \subset \mathbb{V}(F) \subset \mathbb{P}^2_{\mathbb{C}}$$

so $\mathbb{V}(F)$ meets the line at infinite in $\{[1:i:0], [1:-i:0]\}$.
Furthermore, $V(f) \subset \mathbb{A}^2_{\mathbb{R}}$ is infinite since $(x-a)^2 + (y-b)^2 - \mathbb{R}^2$ has infinitely many roots over $\mathbb{R}^2$.

Conversely, suppose $\mathbb{V}(F) \subset \mathbb{P}^2_{\mathbb{C}}$ meets the line at infinity in $\{[1:i:0],[1:-i:0]\}$ and $V(f) \subset \mathbb{A}^2_{\mathbb{R}}$ is infinite.

Write $f = ax^2+by^2+cxy+dx+ey+g$. We find that $F = ax^2+by^2+cxy+dxz+eyz+gz^2 \in \mathbb{R}[x,y,z] \subset \mathbb{C}[x,y,z]$ is its homogenization of degree 2. As $\mathbb{V}(F)$ meets the line at infinite at $\{[1:i:0],[1:-i:0]\}$, we have $a-b+ic=0$ and $a-b-ic=0$, so $2ic=0$ gives $c=0$. Hence $f = ax^2+ay^2+dx+ey+g$. Now $f(x,y)=0$ if and only if $ax^2+ay^2+dx+ey+g=0$ if and only if (as f is of degree 2, a must be nonzero, so) $x^2+y^2+\frac{d}{a}x+\frac{e}{a}y+\frac{g}{a}=0$ if and only if

$$\left(\underbrace{x+\frac{d}{2a}}_{\in\mathbb{R}}\right)^2+\left(\underbrace{y+\frac{e}{2a}}_{\in\mathbb{R}}\right)^2=\frac{d^2}{4a^2}+\frac{e^2}{4a^2}-\frac{g}{a}.$$

Now, since $V(f) \subset \mathbb{A}^2_{\mathbb{R}}$ is infinite, we must have that $\frac{d^2}{4a^2}+\frac{e^2}{4a^2}-\frac{g}{a}>0$, and hence its square root exists, so we find that f is precisely a circle with center $\left(-\frac{d}{2a},-\frac{e}{2a}\right)$ and radius $\sqrt{\frac{d^2}{4a^2}+\frac{e^2}{4a^2}-\frac{g}{a}}$.