Exercise 0.1 (1)**.** Let

$$N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

be exact and consider

$$0 \to \operatorname{Hom}_R(N'', M) \xrightarrow{g^*} \operatorname{Hom}_R(N, M) \xrightarrow{f^*} \operatorname{Hom}_R(N', M).$$

(1) Show that the induced sequence is exact.
(2) Show that even though $N' \to N$ is injective, the map $\operatorname{Hom}_R(N, M) \to \operatorname{Hom}_R(N', M)$ may fail to be surjective.

Proof. (1) Suppose $g^*(k) = g^*(h)$, so $kg = hg$ as maps $N \to M$. For any $n'' \in N''$, there exists $n \in N$ such that $g(n) = n''$, and so $k(n'') = kg(n) = hg(n) = g(n'')$ for all $n'' \in N''$, hence $k = h$. Suppose $f^*(h) = 0$, so $hf = 0$. By injectivity of f, $\ker g = \operatorname{im} f \subset \ker h$. This means that h factors through g in the sense that there exists a map $l\colon N'' \to M$ such that $h = lg$. Thus $h \in \operatorname{im} g^*$.

(2) We have $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) = 0$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$, so while $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is exact,

$$0 \to 0 \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}) \to 0 \to 0$$

cannot be, since $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})$ contains $\mathrm{id} \neq 0$, and is thus not isomorphic to the trivial ring.

$\square$

Exercise 0.2 (3)**.** Let R be a commutative ring.

(1) Show that if $\{e_i \mid i \in I\}$ and $\left\{e'_j \mid j \in J\right\}$ are bases for E and E', respectively, where E and E' are both free R-modules, then $E \otimes_R E'$ is a free R-module with the simple tensors $e_i \otimes e'_j$ as a basis.
(2) Show that if E and E' are projective R-modules, then $E \otimes_R E'$ is a projective R-module.

Proof. (1) The simple tensors span $E \otimes_R E'$ since $\left(e_i, e'_j\right)$ are a basis for $E \times E'$ of which $E \otimes_R E'$ is a quotient.

Suppose

$$\sum c_{i,j} e_i \otimes e'_j = 0,$$

with $c_{i_0, j_0} \neq 0$. Then since $E \times E'$ is free on $\left\{\left(e_i, e'_j\right)\right\}$, we define a map $\tilde{\varphi}\colon E \times E' \to E \otimes_R E'$ by $\left(e_i, e'_j\right) = e_i \otimes e'_j$ when $(i, j) = (i_0, j_0)$ and 0 otherwise. Then the induced map $\varphi\colon E \otimes_R E' \to E \otimes_R E'$ is not identically 0. Then

$$0 = \sum c_{i,j} \varphi\left(e_i \otimes e'_j\right) = c_{i_0, j_0} e_{i_0} \otimes e'_{j_0}.$$

Now, suppose $e_{i_0} \otimes e'_{j_0} = 0$. But then let $f\colon E \times E' \to E \times E'$ be the identity. We then get a map $\overline{f}\colon E \otimes_R E' \to E \times E'$ sending $e_{i_0} \otimes e'_{j_0}$ to $\left(e_{i_0}, e'_{j_0}\right)$, but also $\overline{f}$ is a homomorphism of abelian groups, so it sends 0 to 0, giving $\left(e_{i_0}, e'_{j_0}\right) = (0, 0)$, contradicting it being a basis element. Hence the set $\left\{e_i \otimes e'_j \mid i \in I, j \in J\right\}$ is a basis for $E \otimes_R E'$.

(2) Suppose we have maps f, g as follows

$$\begin{tikzcd} E \times E' \arrow[r, "h"] \arrow[dr, dashed, "\overline{h}"] & E \otimes_R E' \arrow[d, "f"] \\ X \arrow[r, two heads, "g"] & Y \end{tikzcd}$$

Since E and E' are projective, restricting $\overline{h}$ to either coordinate induces a map into X. So for $e \in E$ fixed, there exists $k_e \colon E' \to X$ such that $g \circ k_e = \overline{h}(e, -)$. Define a map $k \colon E \times E' \to X$ by $k(e, e') = k_e(e')$. Then $g \circ k(e, e') = g \circ k_e(e') = \overline{h}(e, e') = f \circ h(e, e')$, hence the diagram

$$\begin{tikzcd} E \times E' \arrow[r, "h"] \arrow[d, "k"] \arrow[dr, dashed, "\overline{h}"] & E \otimes_R E' \arrow[d, "f"] \\ X \arrow[r, two heads, "g"] & Y \end{tikzcd}$$

commutes. By the universal property of tensor products, there exists a unique map $\overline{k} \colon E \otimes_R E' \to X$ such that

$$\begin{tikzcd} E \times E' \arrow[r, "h"] \arrow[d, "k"] & E \otimes_R E' \arrow[d, "f"] \arrow[dl, "\exists! \overline{k}"'] \\ X \arrow[r, two heads, "g"] & Y \end{tikzcd}$$

commutes. Thus $E \otimes_R E'$ is projective.

$\square$

Exercise 0.3 (4)**.** Let M be an R-module.

(1) Show that for any two-sided ideal $\mathfrak{a} \subset R$, the set

$$M[\mathfrak{a}] = \{x \in M \mid ax = 0 \text{ for each } a \in \mathfrak{a}\} \subset M.$$

is a submodule of M.

Proof. (1) A submodule is a subgroup which is closed under multiplication by elements of the ring.

Suppose $x, y \in M[\mathfrak{a}]$. Then for any $a \in \mathfrak{a}$, $a(x - y) = ax + a(-1)y = 0$ whenever $\mathfrak{a}$ is a right ideal.

Now, if $x \in M[\mathfrak{a}]$, then for any $r \in R$ and any $a \in \mathfrak{a}$, we have $ar \in \mathfrak{a}$, so $a \cdot (rx) = (ar) \cdot x = 0$ by assumption, so $rx \in M[\mathfrak{a}]$.

(2) If $\mathfrak{b} \subset \mathfrak{a}$, then for any $x \in M[\mathfrak{a}]$, we have that for all $b \in \mathfrak{b} \subset \mathfrak{a}$, $bx = 0$, so $x \in M[\mathfrak{b}]$. In particular, since $\mathfrak{a}^s \subset \mathfrak{a}^r$ for $r \leq s$, we have $M[\mathfrak{a}^r] \subset M[\mathfrak{a}^s]$.

(3) We define the scalar multiplication by $[r]_{R/\mathfrak{a}} \cdot [x]_{M[\mathfrak{a}^r]/M[\mathfrak{a}^{r-1}]} = [rx]_{M[\mathfrak{a}^r]/M[\mathfrak{a}^{r-1}]}$. We check that this is well-defined. Suppose $\overline{r} = \overline{r'}$ and $\overline{x} = \overline{x'}$. Then $r^{-1}r' \in \mathfrak{a}$ and $x' - x \in M\left[\mathfrak{a}^{r-1}\right]$. Then $r'x' - rx = r'(x' - x) + (r' - r)x \in M\left[\mathfrak{a}^{r-1}\right]$. The remaining module properties are inherited from M.

(4) Suppose $t \geq n$. Then for any $\overline{r} \in R/\mathfrak{m}^n$ and any $m \in \mathfrak{m}^t$, we have $rm \in \mathfrak{m}^n$, so $(R/\mathfrak{m}^n)[\mathfrak{m}^t] = R/\mathfrak{m}^n$. Suppose $t < n$. Then we claim $(R/\mathfrak{m}^n)[\mathfrak{m}^t] = (\mathfrak{m}^{n-t}R)/\mathfrak{m}^n$. $\mathfrak{m}^{n-t}R$ is generated by elements of the form $m_1 \cdots m_{n-t} r$, and $\mathfrak{m}^t$ is generated by elements of the form $m_{n-t+1} \cdots m_n$, so $rm_{n-t+1} \cdots m_n(r'm_1)m_2 \cdots m_{n-t}$ and $r'm_1 \in \mathfrak{m}$ as it is an ideal. Hence this is an element in $\mathfrak{m}^n$. So we have $(\mathfrak{m}^{n-t}R)/\mathfrak{m}^n \subset (R/\mathfrak{m}^n)[\mathfrak{m}^t]$. If

$n \in (R/\mathfrak{m}^n)[\mathfrak{m}^t]$, then for all $m \in \mathfrak{m}^t$, we have $mn = 0$, so $mn \in \mathfrak{m}^n$. Let's postpone this for later.

□

Exercise 0.4 (5)**.** Show that $\mathbb{Z}/3$ is a projective $\mathbb{Z}/6$-module.

Proof. $\mathbb{Z}/3$ is a projective $\mathbb{Z}/6$-module if and only if $\mathbb{Z}/3$ is a direct summand in a free $\mathbb{Z}/6$-module which it indeed is: $\mathbb{Z}/6 \approx \mathbb{Z}/3 \oplus \mathbb{Z}/2$. □