ASSIGNMENT 2

JONAS TREPIAKAS

Problem 0.1 (1)**.** Given a topological manifold M of dimension $d \in \mathbb{N}$, we define a smooth atlas on M as a collection of charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$, where $U_\alpha \subset M$ is open and $\varphi_\alpha \colon U_\alpha \xrightarrow{\cong} \mathbb{R}^d$ is a homeomorphism, such that the transition maps fit into diagrams

$$\begin{tikzcd} & U_\alpha \cap U_\beta \arrow[dl, "\varphi_\alpha"'] \arrow[dr, "\varphi_\beta"] & \\ \varphi_\alpha(U_\alpha \cap U_\beta) \arrow[rr, "\varphi_\beta\varphi_\alpha^{-1}"'] & & \varphi_\beta(U_\alpha \cap U_\beta) \end{tikzcd}$$

where the lower map $\varphi_\beta\varphi_\alpha^{-1}$ is a smooth map between open subsets of $\mathbb{R}^d$.

(1) (2.5 pts) Show that each smooth manifold (as defined in the lecture) admits a smooth atlas.
(2) (2.5 pts) Show that any topological manifold equipped with a smooth atlas admits the structure of a smooth manifold (as defined in the lecture)

Proof. We recall the definition given in the lecture:

Definition 0.2. For a topological space X, we let $C^0_K(X)$ denote the continuous functions on X with support contained in K.

Definition 0.3 (Smooth manifold)**.** A smooth n-manifold is a topological n-manifold M together with an $\mathbb{R}$-sub-algebra $C^\infty(M) \subset C^0(M)$ such that for every point $p \in M$, there exists a chart $i \colon \mathbb{R}^n \hookrightarrow M$ sending $0 \mapsto p$ which is an open topological embedding, such that for all compact subsets $K \subset \mathbb{R}^n$, $i^* \colon C^\infty_K(M) \cong C^\infty_K(\mathbb{R}^n)$ and $i^* \colon C^0_K(M) \to C^0_K(\mathbb{R}^n)$ are $\mathbb{R}$-algebra isomorphisms where $C^\infty_K(M) = C^\infty(M) \cap C^0_K(M)$ such that

$$\begin{tikzcd} C^\infty_K(M) \arrow[r, dashed, "\cong", "i^*"'] \arrow[d, hook] & C^\infty_K(\mathbb{R}^n) \arrow[d, hook] \\ C^0_K(M) \arrow[r, "\cong"] & C^0_K(\mathbb{R}^n) \end{tikzcd}$$

commutes and such that $C^\infty(M)$ admits countable locally finite sums.

(1) Suppose we are given a smooth n-manifold as defined in the definition above. Thus our data consists of a topological manifold M and an R-sub-algebra $C^\infty(M) \subset C^0(M)$.

Let $p \in M$ be a point. By assumption, there exists a topological embedding $i_p \colon \mathbb{R}^n \hookrightarrow M$ sending $0 \mapsto p$. For each $p \in M$, let $U_p := i_p(\mathbb{R}^n)$ and $\varphi_p = i_p^{-1}$. Then $\{(U_p, \varphi_p)\}_{p \in M}$ gives an atlas for M. Now take any two charts $(U_p, \varphi_p), (U_q, \varphi_q)$ such that $U_p \cap U_q \neq \varnothing$. We must show that $\varphi_q \circ \varphi_p^{-1} \colon \varphi_p(U_p \cap U_q) \to \varphi_q(U_p \cap U_q)$ is smooth as a function between open subsets of $\mathbb{R}^n$. Smoothness is a local property, so it suffices to check it locally at each point $x \in \varphi_p(U_p \cap U_q)$. Let x be such a point. Then we can find an open ball $B(x,\varepsilon) \subset \varphi_p(U_p \cap U_q)$, hence also the compact ball $\overline{B\left(x, \frac{\varepsilon}{2}\right)} \subset \varphi_p(U_p \cap U_q)$. Let $K = \overline{B\left(x, \frac{\varepsilon}{2}\right)}$. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function with support in $B\left(x, \frac{\varepsilon}{2}\right)$ and which has value 1 on some small open ball around x.

So $f \in C_K^\infty(\mathbb{R}^n)$. Then $(\varphi_q)^*(f) \in C_K^\infty(M)$, where by this expression, we really mean a function which agrees with $f \circ \varphi_q$ on $\varphi_q^{-1}(B(x,\varepsilon))$ and has value 0 outside of $\varphi_q^{-1}(B(x,\varepsilon))$. This ensures that $(\varphi_q)^*(f)$ really is defined on all of M.

Since $\varphi_p^{-1} = i$, we know that if $(\varphi_q)^*(f \cdot \pi_j) = (f \cdot \pi_j) \circ \varphi_q \in C_K^\infty(M)$ for all j, then $\varphi_q \circ \varphi_p^{-1}$ is smooth around x. Now, $\pi_j \cdot f$ is a product of two functions in $C^\infty(\mathbb{R}^n)$, and since f has support in K, the product is in $C_K^\infty(\mathbb{R}^n)$. Hence $(\varphi_q)^*(\pi_j \cdot f) \in C_K^\infty(M)$, and thus $i_p^*(\varphi_q)^*(\pi_j \cdot f) \in C_K^\infty(\mathbb{R}^n)$ and agrees with $\varphi_q \circ \varphi_p^{-1}$ in in a neighborhood of x. Therefore, $\varphi_q \circ \varphi_p^{-1}$ is smooth in a neighborhood of x. As x was arbitrary, this shows that $\varphi_q \circ \varphi_p^{-1}$ is smooth on all of $\varphi_p(U_p \cap U_q)$. Thus $\{(U_p, \varphi_p)\}_{p \in M}$ gives a smooth atlas which induces a smooth structure by taking the maximal atlas.

To see that $C^\infty(M)$ corresponds to the smooth functions on M in the smooth atlas structure, note that a function $f \in C^\infty(M)$ is locally smooth in the usual sense since for any compact set K, $C_K^\infty(M) \cong C_K^\infty(\mathbb{R}^n)$, where we can make use of this isomorphism by composing f with a bump function vanishing outside some small open set contained in K. Now using a partition of unity, we can write f as the countable sum of smooth functions each with support in an open set that is part of a locally finite collection covering M - I have included a more explicit description of this in part (2) of this problem at the end.

(2) Next, suppose we are given a topological manifold with a smooth atlas as given through charts $i_\alpha \colon \mathbb{R}^n \to M$. Smoothness of this chart amounts to $i_\beta^{-1} \circ i_\alpha$ being smooth on $i_\alpha^{-1}(i_\alpha(\mathbb{R}^n) \cap i_\beta(\mathbb{R}^n))$. Let $C^\infty(M)$ be the $\mathbb{R}$-algebra generated by all the smooth function using the smooth atlas definition. Now, precomposition is $\mathbb{R}$-linear. Furthermore, since i_α is continuous, i_α^* is indeed a linear map $C_K^0(M) \to C_K^0(\mathbb{R}^n)$. As i_α is a homeomorphism, it has a continuous inverse, so simply applying $(-)^*$ to both compositions, we see that i_α^* must be an isomorphism (note that here again, $(\varphi_q)^*(f)$ for $f \in C_K^0(\mathbb{R}^n)$ is again defined by taking a product with a bump function and extending it by 0 to the rest of M just as in the first part of this problem). Now, if we have a smooth map $f \colon M \to \mathbb{R}$ with support in K, then by definition, $f \circ i_\alpha \colon \mathbb{R}^n \to \mathbb{R}$ is smooth, so i_α^* indeed restricts to a map $C_K^\infty(M) \to C_K^\infty(\mathbb{R}^n)$. Now, given a map $f \in C_K^\infty(\mathbb{R}^n)$, the map $f \circ i_\alpha^{-1}$ which we can extend using a bump function to all of M is smooth since precomposing it with i_α gives f as a coordinate representation which is smooth by assumption.

So completely equivalently, $\left(i_\alpha^{-1}\right)^*$ gives a map $C_K^\infty(\mathbb{R}^n) \to C_K^\infty(M)$ where we again extend using a bump function, and these maps are inverses to each other.

It remains to show that $C^\infty(M)$ admits countable locally finite sums. But this follows from smooth manifolds in our smooth atlas definition admitting smooth partitions of unity. Since a partition of unity adds up to 1, any smooth function f in our smooth atlas definition is the countable sum of functions where we weigh f against each element of the partition of unity such that only finitely many terms are nonzero around each point. But around each points by construction f is weighted such that it has support in a compact set, hence this term is smooth in Robert's definition too. Now using that $C^\infty(M)$ admits countable locally finite sums (which this sum using a partition of unity is), we obtain that f is in $C^\infty(M)$. To be more precise, choose for each point $p \in M$ a coordinate ball (U_p, φ_p) such that $p \in U_p$ and $\varphi_p(U_p) = B(p, \varepsilon)$. Let $X_p = \varphi_p^{-1}\left(B\left(p, \frac{\varepsilon}{3}\right)\right)$ which is an open set contained in the compact set $K_p = \varphi_p^{-1}\left(\overline{B\left(p, \frac{\varepsilon}{2}\right)}\right)$. Then $\{X_p\}_{p \in M}$ is an open covering of M, so by second countability, we may take a countable subcovering which we will denote $\mathcal{X}$ indexed by a set A. Now take a smooth partition of unity subordinate to $\mathcal{X}$ (theorem 2.23 in Lee). This gives a sum

$$\sum_{\alpha \in A} \psi_\alpha(x) = 1$$

for all $x \in M$ such that the family of supports is locally finite, $\operatorname{supp} \psi_\alpha \subset X_\alpha$ and $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and $x \in M$. Now for a function f that is smooth on M in the smooth atlas sense, $\psi_\alpha \cdot f$ is smooth in Robert's sense, so f being the countable sum of functions which are nonzero only on finitely many X_α is thus also smooth, so $f \in C^\infty(M)$.

□

Problem 0.4 (2 (Some examples))**.** Find a smooth structure on the following topological manifolds

(1) $S^n, n \in \mathbb{N}$
(2) $\mathbb{RP}^n, n \in \mathbb{N}$
(3) $\mathbb{CP}^n, n \in \mathbb{N}$
(4) $\mathrm{GL}_n(\mathbb{R}), n \in \mathbb{N}$
(5) The Grassmannian $\mathrm{Gr}_d(\mathbb{R}^n), d \leq n$.

Proof. (1) We must find a smooth atlas (i.e., charts such that the transition functions are all smooth). Choose the standard topological atlas for S^n : $\left(U_i^\pm, \varphi_i^\pm\right)$ given by

$$U_i^\pm = \{(x_1, \ldots, x_{n+1}) \in S^n \mid \pm x_i > 0\}$$

and $\varphi_i^\pm : U_i^\pm \to \mathbb{R}^n$ is given by forgetting the i th coordinate:

$$(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, x_{n+1}).$$

Since each coordinate of $\varphi_i^\pm$ is a projection, it is continuous, and since it has the inverse

$$(x_1, \ldots, x_n) \mapsto \left(x_1, \ldots, x_{i-1}, \sqrt{1 - \sum_j x_j^2}, x_i, \ldots, x_n\right)$$

Essentially, this is just the homeomorphism of the hemispheres of S^n with D^n. Clearly, each point of S^n is in some open hemisphere, so this collection of charts is an atlas. We need to check that it's a smooth atlas. All the transition functions will be of the form

$$(x_1, \ldots, x_n) \mapsto \left(x_1, \ldots, x_{i-1}, \sqrt{1 - \sum_j x_j^2}, x_i, \ldots, \hat{x_j}, \ldots x_n\right)$$

which has smooth coordinate functions, hence is smooth.

(2) We check that the standard atlas on $\mathbb{RP}^n$ is smooth. That is, let $U_i \subset \mathbb{RP}^n$ be the subset

$$\{[x_1, \dots, x_{n+1}] : x_i \neq 0\}$$

and $\varphi_i \colon U_i \to \mathbb{R}^n$ given by

$$\varphi_i\left([x_1, \dots, x_{n+1}]\right) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right).$$

We have already previously checked that these are charts. Now $U_i \cap U_j$ consists of all points whose i th and j th coordinates are nonzero. The transition functions will then have the form

$$(x_1, \dots, x_n) \mapsto \left(\frac{x_1}{x_k}, \dots, \frac{x_{i-1}}{x_k}, \frac{1}{x_k}, \frac{x_i}{x_k}, \dots, \frac{x_n}{x_k} \dots\right)$$

where x_k will be x_j if $j < i$ and x_{j+1} if $i < j$. In either case, this is smooth since we are looking at an open domain where x_k is nonzero everywhere.

(3) We take the same maps and subsets as for $\mathbb{RP}^n$, now interpreted over $\mathbb{C}$. So we get

$$\varphi_j\left([a_1 + b_1 i, \dots, a_{n+1} + b_{n+1} i]\right) = \left(\frac{a_1 + ib_1}{a_j + ib_j}, \dots, \frac{a_n + ib_n}{a_j + ib_j}\right)$$

Now we also have the homeomorphism $\psi \colon \mathbb{C}^n \cong \mathbb{R}^{2n}$

$$(z_1, \dots, z_n) \mapsto \left(\Re z_1, \Im z_1, \dots, \Re z_n, \Im z_n\right).$$

The transition map will then be of the form

$$\begin{aligned}
(x_1, y_1, x_2, y_2, \dots, x_n, y_n) &\mapsto (x_1 + iy_1, \dots, x_n + iy_n) \\
&\mapsto [x_1 + iy_1, \dots, x_{j-1} + iy_{j-1}, 1, x_j + iy_j, \dots, x_n + iy_n] \\
&\mapsto \left(\frac{x_1 + iy_1}{x_k + iy_k}, \dots, \frac{1}{x_k + iy_k}, \dots, \frac{x_n + iy_n}{x_k + iy_k}\right) \\
&\mapsto \frac{1}{\|z_k\|^2}\left(\Re z_{1k}, \Im z_{1k}, \Re z_{2k}, \Im z_{2k}, \dots, \Re z_{nk}, \Im z_{nk}\right)
\end{aligned}$$

Where we let $z_k = x_k + iy_k$ and

$$z_{kj} = (x_k + iy_k)(x_j - iy_j) = x_k x_j + y_k y_j + i\left(y_k x_j - x_k y_j\right).$$

Then, clearly, each $\Re z_{jk}, \Im z_{jk}$ is smooth, so the transition map is smooth since $\|z_k\|$ is nonzero in the intersection as $z_k \neq 0$.

(4) For $\mathrm{GL}_n(\mathbb{R})$, we simply identify a matrix

$$(\alpha_{ij}) \mapsto \left(\alpha_{11}, \alpha_{12}, \dots, \alpha_{1n}, \alpha_{21}, \alpha_{22}, \dots, \alpha_{n(n-1)}, \alpha_{nn}\right) \in \mathbb{R}^{n^2}$$

Since the determinant is continuous, we take a chart about (α_{ij}) to be the preimage of some open neighborhood of its identified point in $\mathbb{R}^{n^2}$ such that the determinant is nonzero in this neighborhood. But since the same chart map is used for every chart domain, the transition maps will all be identity maps on some open subsets of $\mathbb{R}^{n^2}$ which are smooth, hence this atlas is trivially a smooth atlas.

(5) **I did not finish this in time** For the Grassmannian, we first define some charts. Recall that $\mathrm{Gr}_d(\mathbb{R}^n)$ inherits the quotient topology from $V_d(\mathbb{R}^n) \to$

$\mathrm{Gr}_d(\mathbb{R}^n)$ - or alternatively as the quotient $\tilde{V}_d(\mathbb{R}^n) \to \mathrm{Gr}_d(\mathbb{R}^n)$. Now let $X \in \mathrm{Gr}_d(\mathbb{R}^n)$ and define an open neighborhood of X given by

$$\mathcal{U}_X = \left\{P \in \mathrm{Gr}_d(\mathbb{R}^n) \colon P \cap X^\perp = \{0\}\right\}.$$

Given a d-dimensional subspace $P \in \mathcal{U}_X$, take a point $(x, y) \in P$ where $x \in X$ and $y \in X^\perp$. Then if there exists a $y' \in X^\perp$ such that also $(x, y') \in P$, then as P is a subspace, $(0, y - y') \in P$, but this contradicts trivial intersection with $X^\perp$ unless $y = y'$.

Choosing a linear isomorphism $P \cong \mathbb{R}^d$, we then obtain a well-defined map $T \colon \mathcal{U}_X \to \mathrm{Hom}\left(\mathbb{R}^d \to \mathbb{R}^{n-d}\right)$ where $P \in \mathcal{U}_X$ is sent to the map sending $x \in \mathbb{R}^d \mapsto y \in \mathbb{R}^{n-d}$. In particular, this shows that $\mathcal{U}_X \cong \mathbb{R}^{k(n-k)}$.

Let now $j \colon \mathbb{R}^n = X \oplus X^\perp \to X$ be the projection to the subspace X. Fix some orthonormal basis $\{x_1, \ldots, x_d\}$ for X. Given $Y \in \mathcal{U}_X$, we have that any $z \in Y$ can be written as

$$z = \underbrace{\sum_{i=1}^{k} \langle z, x_i \rangle x_i}_{:= x \in X} + \underbrace{\left(z - \sum_{i=1}^{k} \langle z, x_i \rangle x_i\right)}_{:= y \in X^\perp}$$

□

Problem 0.5 (3)**.**

(1) Let M and N be two smooth manifolds, and let $f \colon M \to N$ be a smooth embedding which is a homeomorphism onto its image. Show that f is actually a diffeomorphism onto its image.
(2) Let M and N be two smooth, connected compact manifolds of the same dimension. Assume that we have an embedding $f \colon M \to N$. Show that f is a diffeomorphism.

Proof. (1) To start with, by theorem 5.31 in Lee's book, there is a unique topological and smooth structure on $f(M)$ with respect to which f is a smooth embedding, namely the slice charts.
Let $f \colon M \to N$ be a smooth embedding which is a homeomorphism onto its image. A smooth embedding is an injective smooth immersion. So locally, f can be represented as a map $\mathbb{R}^m \to \mathbb{R}^n$ by $x \mapsto (x, 0)$. Choosing the restriction of these charts to the first m coordinates, we obtain smooth charts for $f(M)$ which are given by $(x, 0) \mapsto x$ on the restriction of the charts on M giving this local representation to $f(M)$. I.e., if (U, φ) is a chart for M such that f has a local representation with this chart as $x \mapsto (x, 0)$, then we let $V = U \cap f(M)$, $\hat{V} = \pi \circ \varphi(V)$ where $\pi \colon \mathbb{R}^n \to \mathbb{R}^m$ is the projection onto the first m coordinates, and let $\psi = \pi \circ \varphi|_V : V \to \hat{V}$. Then (V, ψ) will be the smooth charts for $f(M)$. Compatibility of these charts is inherited from N. This gives $f(M)$ the structure of an m-manifold. Furthermore, with respect to this structure, f becomes locally of the form $x \mapsto (x, 0) \mapsto x$, i.e., the identity. So $f \colon M \to f(M)$ is a local diffeomorphism. Note also that f was assumed to be injective, and since we restricted the codomain, it is now also surjective. So f is a bijective local diffeomorphism, which is a diffeomorphism.
(2)
Compact subsets of a Hausdorff space are closed, so since M is compact, $f(M) \subset N$ is compact, hence closed. However, f is also an embedding, hence a homeomorphism onto its image, so as M is open, $f(M)$ is open. As N is connected and $f(M) \neq \varnothing$, we must have $f(M) = N$. Now part (1) establishes the result. □

Problem 0.6 (4 (Bump Functions))**.** Let M be a smooth manifold. Fix K a compact subset of M, and U an open neighborhood of K. Show that there exists a bump function $b_{K,U} \colon M \to \mathbb{R}$, i.e., a smooth map such that $b_{K,U}$ is 1 on K, and 0 on the complement of U.

Proof. **I did not finish this one in time** We saw in class that in the situation where we have $K \subset U \subset \mathbb{R}^n$, then $\varphi_\varepsilon * \chi_U$ is a smooth function which is 1 on $K \subset U$ and 0 outside of U. We will denote this function $b_{K,U,\mathbb{R}^n}$.
Assume that the existence of bump functions has already been shown for $\mathbb{R}^n$ - i.e., the situation above where $M = \mathbb{R}^n$. Now suppose

□

Problem 0.7 (5)**.**
(1) Show that there is no smooth surjective map $f \colon \mathbb{R}^n \to \mathbb{R}^m$ whenever $n < m$.
(2) Let M be a connected compact manifold of dimension d, and fix a smooth map $f \colon M \to \mathbb{R}^{d+1}$. Show that there is a point $p \in \mathbb{R}^{d+1}$ and a line in $\mathbb{R}^{d+1}$ through p that meets $f(M)$ in finitely many points.

Proof. (1) Suppose $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is a smooth surjective map. As $n < m$, we have that Df has rank at most n at all points, so all points of $\mathbb{R}^n$ are critical values of f. By Sard's theorem then, $f(\mathbb{R}^n)$ has measure zero in $\mathbb{R}^m$. This in particular implies, that no open set can be contained in $f(\mathbb{R}^n)$ since any open subset of $\mathbb{R}^m$ has Lebesgue measure greater than 0. But then $f(\mathbb{R}^n)$ cannot be surjective, giving us a contradiction.

(2) As formulated, this problem is false: for example, we can take the map $f \colon (0,1) \to \mathbb{R}^2$ by $f(x) = e^{-\frac{1}{x^2}} e^{2\pi i \frac{1}{x}}$. This extends smoothly to $f(0) = 0$ and $f(1) = \lim_{x \to 1-} f(x)$. Then connecting $f(0)$ and $f(1)$ with a smooth arc, we obtain by gluing, a smooth map $S^1 \to \mathbb{R}^2$ which contradicts the problem statement. □