Exercise 0.1. Let $U \subset V = \ell^2$ consist of all sequences with finite support (i.e., $x_k \neq 0$ only for finitely many $k \in \mathbb{N}$). Show that U has no orthocomplement and that the inclusion map $U \hookrightarrow V$ has no adjoint.

Solution. Suppose W is an orthocomplement to U. So $U \oplus W = \ell^2$. Then W in particular must be the subspace containing all sequences for which there are infinitely many non-zero entries. Now, choose some element $v \in U$, and let v_N be the highest index entry in v which is non-zero. Now let w be given by $\left(0, \ldots, 0, v_N, \frac{1}{N+1}, \frac{1}{N+2}, \ldots\right)$ where the entry at $M > N$ is $\frac{1}{M}$ and the entry at $m < N$ is 0. Then $\sum |w_k|^2 = |v_N|^2 + \sum_{k>n} \frac{1}{k^2} < \infty$, so $w \in \ell^2$ and since it does not have finite support, $w \in W$. In particular, then, if $U \oplus W$ is an orthogonal direct sum, we must have

$$|w_N|^2 = \sum_{k \in \mathbb{N}} v_k \overline{w_k} = \langle v, w \rangle = 0,$$

contradicting that w_N was a nonzero entry from F. Thus no orthocomplement can exist to U.

Let $A \in \operatorname{Hom}(U, \ell^2)$ be the inclusion map. Suppose A has an adjoint A^*. Suppose $y \perp \operatorname{Im} A$. Then assume y_N is some non-zero entry, and let $v \in U$ be the element for which $v_N = y_N$ and $v_n = 0$ when $n \neq N$. Then $\langle Av, y \rangle = \langle v, y \rangle = |y_N|^2 \neq 0$ which contradicts $y \perp \operatorname{Im} A$. Hence $y = 0$, so $(\operatorname{Im} A)^{\perp} = \{0\}$ and thus $N(A^*) = \{0\}$ by lemma 7.16, so A^* is injective. Now define the functional $z \in U'$ by $z(x) = \|x\| = \langle x, x \rangle = \langle Ax, x \rangle = \langle x, A^* x \rangle$. By uniqueness of theorem 7.13.(a), $A^* x = x$ for all $x \in U$, so in particular, $A^* A = \mathbb{1}_U$. Now suppose $y \in \ell^2 - U$ which is not empty since for example $\left(\frac{1}{n}\right) \in \ell^2 - U$. Then $A^* y = A^* A A^*(y)$, so $y = AA^*(y) \in \operatorname{Im} A$ by injectivity of A^*, so y has finite support, contradiction.