Homework for Weeks 4,5

5.0.1. Problem. Exhibit an explicit homotopy equivalence of $\mathbb{R}^n$ and $\text{int}(B^n)$. That is, give two maps $f : \mathbb{R}^n \to \text{int}(B^n)$ and $g : \text{int}(B^n) \to \mathbb{R}^n$ and two homotopies such that the compositions $f \circ g \simeq id_{\text{int}(B^n)}$ and $g \circ f \simeq id_{\mathbb{R}^n}$ are homotopic (via your maps) to the identity on each respective domain.

5.0.2. Problem. Describe a CW structure on $\mathbb{R}P^n$ for each n. Draw these to the best of your ability for $n = 1, 2$. Hint[1]

[1] Recall that one definition of $\mathbb{R}P^n$ is the quotient of S^n by the antipodal map

Homework for Week 3

Let $Gr_k(\mathbb{R}^n)$ be the set of k dimensional subspaces of $\mathbb{R}^n$. We can topologize this space as follows.

Firstly, we define the set

$$V_k(\mathbb{R}^n) = \{k \text{ linearly independent vectors in } \mathbb{R}^n\} \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n \text{ (k times)}.$$

We can give $V_k(\mathbb{R}^n) \subset \mathbb{R}^{nk}$ the subspace topology. We have a quotient map $q : V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$ taking k linearly independent vectors to their span:

$$q : V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n) \tag{3.1}$$

$$(v_1, \cdots, v_k) \mapsto \text{span}\langle v_1, \cdots, v_k\rangle \tag{3.2}$$

Thus, we can give $\mathrm{Gr}_k(\mathbb{R}^n)$ the quotient topology with respect to this map q.

We call $\mathrm{Gr}_k(\mathbb{R}^n)$ the Grassmanian of k-planes in $\mathbb{R}^n$.

(Note we can also consider $\mathrm{Gr}_k(\mathbb{R}^n)$ equally well to be the quotient of the space $V_k^O(\mathbb{R}^n) := \{k \text{ orthonormal vectors in } \mathbb{R}^n\} \subset \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (k times). The two different quotient topologies agree and it may be useful to go back and forth in perspective for this problem).

3.0.3. Problem (Grassmannian is Hausdorff). We will show that the Grassmannian is Hausdorff

(i) Prove the following lemma:

3.0.1. Lemma: — *Let Y be a topological space. A map $f : \mathrm{Gr}_k(\mathbb{R}^n) \to Y$ is continuous if and only if the composition $f \circ q : V_k^O(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n) \to Y$ is continuous.*

(Hint: there is nothing special about the role of the grassmannian in this statement. We may as well replaced $q : V_k^O(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$ with any quotient map $q : A \to A/\sim$).

(ii) Prove the following lemma:

3.0.2. Lemma: — *Given a topological space X, if for any two points $x, y \in X$, there exists a continuous function (possibly depending on x and y) from $f : X \to \mathbb{R}$ satisfying $f(x) \neq f(y)$, then X is Hausdorff.*

(iii) Given a vector $w \in \mathbb{R}^n$, define a continuous function $\rho_w : \mathrm{Gr}_k(\mathbb{R}^n) \to \mathbb{R}$ that takes a k-plane P and spits out the euclidean distance from w to P.

(iv) Prove that $\mathrm{Gr}_k(\mathbb{R}^n)$ is Hausdorff. (Hint: use the previous lemma).

3.0.4. Problem (Grassmannian is locally Euclidean). We will show that the Grassmannian is locally Euclidean

(i) Given a plane X, consider the open neighborhood $U_X := \{P \in \mathrm{Gr}_k(\mathbb{R}^n) : P \cap X^\perp = \{0\}\}$. Play around with $\mathrm{Gr}_1(\mathbb{R}^2)$, the space of lines in the plane. What does the open set U_X look like?

(ii) Prove that $U_X \cong \mathbb{R}^{(k)(n-k)}$ as sets (don't worry about continuity). (An k plane in U_X can be thought of as a linear assignment from point on X to $X^\perp$). Thus we have a map $T : U_X \to \mathrm{Hom}(\mathbb{R}^k \to \mathbb{R}^{n-k})$.

(iii) Let the map $j : \mathbb{R}^n \to X \oplus X^\perp \to X$ be the projection of a vector to the subspace X. Fix an orthonormal basis $\{x_1, \cdots, x_n\}$ for the plane X. Given a plane $Y \in U_X$ show that there exists basis vectors $y_1, \cdots, y_k$ so that $j(y_i) = x_i$. Explain why the y_i vary continuously with respect to Y.

(iv) Prove that $T : U_X \to \mathbb{R}^{(k)(n-k)}$ is continuous. (Hint: write down a formula for $T(Y)$ in terms of $x_i's$ and $y_i's$. It may help to think about the $\mathrm{Gr}_1(\mathbb{R}^2)$ case).

If you are interested in the grassmannian, you can read more in Milnor and Stasheff's book on Characteristic Classes Chapter 5.

Homework for Week 2

Prove (one (or more) of $\{2.1.1, 2.1.2\}$) and also (optionally $\{2.1.3\}$).

2.1 Compactness

2.1.1. Problem (Product of two compact spaces is compact). Suppose (X, τ_X) and (Y, τ_Y) are two compact topological spaces. Show that $X \times Y$ equipped with the product topology is also compact.

2.1.1. Remark. A much more general fact about product of compact spaces is true, but requires some much more sophisticated machinery. (Namely, the product of an arbitrary collection of compact sets is compact).

2.1.2. Problem (The interval is compact). Make the following outline of the proof that the interval $I = [0, 1]$ is a compact space precise:

We would like to use the fact that for metric spaces, compactness is equivalent to sequential compactness. Thus, to prove I is compact, it is enough to construct for any sequence (x_n) in I, a subsequence of it that converges in I. A hint for the first step in constructing such a subsequence is as follows: divide the interval into two pieces: $[0, 1/2]$ and $[1/2, 1]$, one of these pieces has infinitely many terms of our original sequence. (Can we use/continue this process to somehow extract a subseqence that is convergent?). Hint 2: use the completeness of $\mathbb{R}$.

2.1.3. Problem. What happens if I take a mobius strip and perform a cut along the center (equatorial?) line. What happens to the resulting object(s) if I perform equatorial cut(s) along (it?) those. Feel free to take a piece of paper, make the appropriate twist (and tape) to form a mobius strip. And empirically find out the answer using a pair of scissors.

Homework for Weeks 0,1

EITHER submit problems 3 of $\{1.1.1, 1.1.2, 1.2.1, 1.2.2\}$ OR submit 1.3.1.

1.2 Metric Spaces

1.2.1. Problem. Recall the definitions of the taxicab metric d_T and the euclidean metric d_E on $\mathbb{R}^n$. We previously defined:

1. $d_E(x,y) = |x-y| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$
2. $d_T(x,y) = \sum_i |x_i - y_i|$ for $x = (x_1, \cdots, x_n), y = (y_1, \cdots y_n) \in \mathbb{R}^n$.

Prove that for any metric space (Z, d_Z), a function between the metric spaces $f : (\mathbb{R}^n, d_E) \to (Z, d_Z)$ is continuous if and only if it is continuous as a function from $(\mathbb{R}^n, d_T) \to (Z, d_Z)$.

Hint: Show that any set that is open with respect to d_T is open with respect to d_E.

1.2.2. Problem. Consider the following definition of continuity: a map between metric spaces $f : X \to Y$ is continuous if for every point $x \in X$ and every $\epsilon > 0$ then there is a ball of radius δ so that $f(B(x,\delta)) \subset B(f(x),\epsilon)$.

Prove that a map is continuous in this sense if and only if for every open set $V \subset Y$ we have that $f^{-1}(V)$ is open in X.

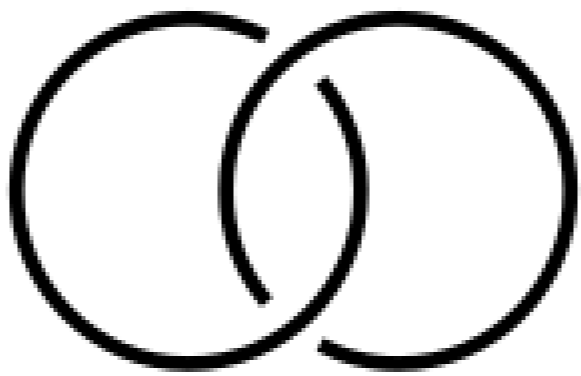
1.3 Topologies

1.3.1. Problem. Let $\{\tau_i\}_{i\in I}$ be a collection of topologies on X. Show that $\cap_{i\in I}\tau_i$ is a topology on X.

1.3.2. Problem. Let X be a set, and $\sigma \subset \mathcal{P}(X)$ be a basis. Show that the set of arbitrary unions of elements of σ form a topology on X.

1.4 Misc (Optional, but really interesting!)

1.4.1. Problem (The Hopf Link is Fibered). A link is like a knot but with many components. (Formally we say a link is a codimension 2 submanifold of S^3). The Hopf link is the most basic link. It looks like:



Prove that the hopf link is fibered.

** If you were gone last friday, or are interested in the question and need some help, feel free to ask the instructors.