1. Preparation

1.1. Representability.

Definition 1.1 (Initial and terminal objects). An object c in a category C is initial if and only if the functor $C(c, -): C \to \operatorname{Set}$ is naturally isomorphic to the constant functor $*: C \to \operatorname{Set}$ that sends every object to the singleton set. That is, for any $d, e \in C$ and any $f: d \to e$, we have

$$C(c,d) \xrightarrow{\approx} \{*\}$$

$$\downarrow^{f_*} \qquad \downarrow$$

$$C(c,e) \xrightarrow{\approx} \{*\}$$

Since these are isomorphisms in set, we can conclude that C(c,d) and C(c,e) are singletons, and as these is only one map between the singleton sets $\{*\} \to \{*\}$, we conclude that $\operatorname{Hom}(C(c,d),C(c,e))$ is also a singleton set.

Theorem 1.2 (Yoneda lemma). For any functor $F: C \to \text{Set}$, whose domain C is locally small and any object $c \in C$, there is a bijection

$$\operatorname{Hom}\left(C(c,-),F\right)\cong Fc$$

that associates a natural transformation $\alpha \colon C(c, -) \Longrightarrow F$ to the elements $\alpha_c(\mathbb{1}_c) \in Fc$. Moreover, this correspondence is natural in both c and F.

2. Limits and colimits

Definition 2.1. A diagram of shame J in a category C is a functor $F: J \to C$.

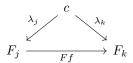
Definition 2.2. For any object $c \in C$ and any category J, the constant functor $c: J \to C$ sends every object of J to c and every morphism in J to the identity morphism $\mathbb{1}_c$.

Definition 2.3 (Embedding). Recall that an embedding is a faithful functor that is injective on objects.

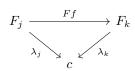
A fully faithful functor that is injective on objects defines a full embedding. The domain then defines a full subcategory of the codomain.

Remark. The constant functors define an embedding $\Delta: C \to \operatorname{Fun}(J, C)$ sending c to the constant functor at c and a morphism $f: c \to c'$ to the constant natural transformation, in which each component is defined to be the morphism f.

Definition 2.4 (Cones). A cone over a diagram $F: J \to C$ with summit or apex $c \in C$ is a natural transformation $\lambda \colon c \Longrightarrow F$, so $\lambda \in \operatorname{Fun}(J,C)$, whose domain is the constant functor at c. The components $(\lambda_j \colon c \to Fj)_{j \in J}$ of the natural transformation are called the legs of the cone.



Definition 2.5 (Cocones). Dually, a cone under F with nadir c is a natural transformation $\lambda \colon F \implies c$ whose legs are the components $(\lambda_j \colon F_j \to c)_{j \in J}$.o So naturality asserts that for each morphism $f \colon j \to k$ of J, the triangle



commutes in C. Cones under a diagram are also called cocones.

Definition 2.6 (Limits and colimits I). For any diagram $F \colon J \to C$, there is a functor

$$Cone(-,F): C^{op} \to Set$$

that sends $c \in C$ to the set of cones over F with summit c. A limit of F is a representation for Cone (-,F).