Definition 0.1 (Coexact)**.** A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of pointed spaces (or pointed pairs) is called *coexact* if, for each pointed space (or pair) Y, the sequence of sets (pointed homotopy classes)

$$[C;Y] \xrightarrow{g^{\#}} [B;Y] \xrightarrow{f^{\#}} [A;Y]$$

is exact.

Theorem 0.2. *For any map $f\colon A \to X$ and for the inclusion $i\colon X \hookrightarrow C_f$, the sequence*

$$A \xrightarrow{f} X \xrightarrow{i} C_f$$

is coexact.

Proof. Clearly, $i \circ f \simeq *$, the constant map to the base point (by sliding A up along its cone to the vertex). Now suppose $\varphi \in \ker f^{\#} \in [X;Y]$. By assumption then $\varphi \circ f$ is nullhomotopic, say via a homotopy F. Then putting F on $A \times I$ and φ on X, we get a map $C_f \to Y$ extending φ - its image under $i^{\#}$ is given by restricting to $X \subset C_f$, which is φ. □

Corollary 0.3. *If $f\colon A \hookrightarrow X$ is a cofibration, where $A \subset X$ is closed, then*

$$A \to X \to X/A$$

is coexact.

Corollary 0.4. *Let $f\colon A \to X$ be any map. Then the sequence*

$$A \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{j} C_i \xrightarrow{k} C_j$$

is coexact, where j and k are the obvious inclusions.

We can replace C_i and C_j by simpler things. Note that $C_i = C_f \cup_X CX$, see Figure 1.

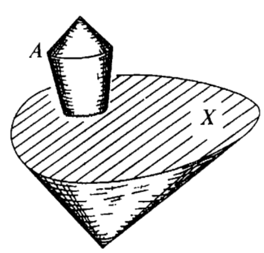

Figure 1.

In this figure, we can construct a deformation carrying CX to the base point through itself. Throughout this deformation, we can stretch the mapping cylinder of f to accommodate it.

Now using Theorem **??**, we obtain that the collapsing map $C_i \to C_i/CX = C_f/X = SA$ is a homotopy equivalence. Similarly, $C_j \simeq CX$. Under these homotopy equivalences, we will show that k becomes Sf. Consider Figure 2.

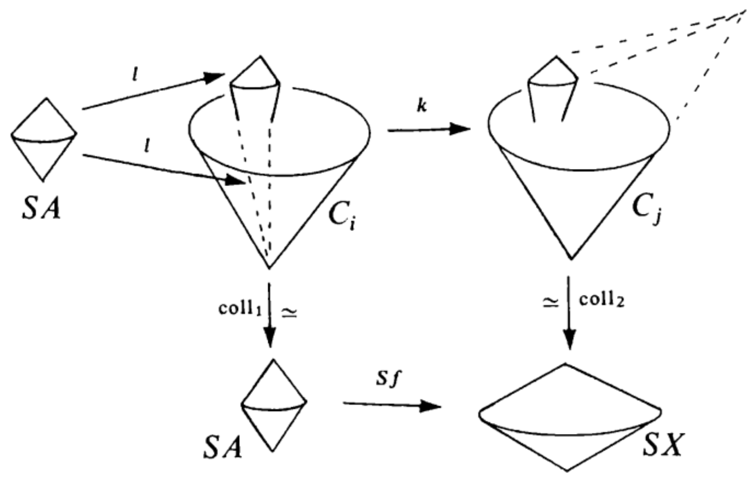

FIGURE 2.

The map l in the figure stretches the top cone of SA to the cylinder part of $C_f \subset C_i$ and is Cf on the bottom cone.
The map $coll_1$ is the collapse of the bottom of the picture and gives the homotopy equivalence $C_i \simeq SA$ obtained above. The map $coll_2$ is the collapse of the top of C_j in the picture (the dashed lines) and is the homotopy equivalence $C_j \simeq SX$.
Firstly, we clearly have $coll_1 \circ l \simeq \text{id}$, so l is a homotopy inverse of $coll_1$, i.e., $l \circ coll_1 \simeq \text{id}$ as well.
Also, $coll_2 \circ k \circ l = Sf \circ g \simeq Sf$, where g is the collapse of the top cone of SA.
Composing with $coll_1$ on the right gives $coll_2 \circ k \simeq coll_2 \circ k \circ l \circ coll_1 \simeq S_f \circ coll_1$, so the diagram

$$\begin{tikzcd} C_i \arrow[r, "k"] \arrow[d, "coll_1"] & C_j \arrow[d, "coll_2"] \\ SA \arrow[r, "Sf"] & SX \end{tikzcd}$$

is homotopy commutative.
Thus,

Corollary 0.5. *Give any map $f\colon A \to X$ of pointed spaces, the sequence*

$$A \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{g} SA \xrightarrow{Sf} SX$$

is coexact, where $g\colon C_f \to SA$ is the composition of the collapse $C_f \to C_f/X$ with the homotopy equivalence $SA \simeq C_f/X$ induced by the inclusion of $A \times I$ in $(A \times I) \sqcup X$ followed by the quotient map to C_f and then the collapsing of the subspace X of C_f.

Lemma 0.6. *Coexactness is preserved by suspension.*

Proof. Suppose $A \to B \to C$ is coexact. Then the sequence

$$[SC; Y] \to [SB; Y] \to [SA; Y]$$

is equivalent to the sequence

$$[C;\Omega Y] \to [B;\Omega Y] \to [A;\Omega Y]$$

which is exact by assumption. $\square$

Corollary 0.7 (Barratt-Puppe)**.** *If $f\colon A \to X$ is any map then the sequence*

$$A \xrightarrow{f} B \xrightarrow{i} C_f \xrightarrow{g} SA \xrightarrow{Sf} SX \xrightarrow{Si} SC_f \xrightarrow{Sg} S^2A \xrightarrow{S^2f}$$

is coexact. Furthermore, $SC_f \cong C_{Sf}$, etc. Similarly for maps of pairs of pointed spaces.