0.0.1. *Exercises.*

Exercise 0.1. Let $p\colon E\to B$ be a Serre (resp. Hurewicz) fibration. Given any map of spaces $f\colon B'\to B$, show that the projection $f^*E\to B'$ is a Serre (resp. Hurewicz) fibration, where

$$f^*(E)=B'\times_B E=\{(b',e)\mid f(b')=p(e)\}$$

is the pullback along f.

Proof. Consider the solid part of the diagram

$$\begin{tikzcd}
X\times\{0\} \arrow[r] \arrow[d] \arrow[rr, bend left] & f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
X\times I \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] \arrow[rr, bend right] & B' \arrow[r] & B
\end{tikzcd}$$

In the case where p is a Hurewicz fibration, X can be any space, while when p is a Serre fibration, it represents any disk D^n.
We then obtain the first dashed arrow $X\times I\to E$ because $E\xrightarrow{p}B$ is a Hurewicz/Serre fibration. But then we have maps $X\times I\to B'$ and $X\times I\to E$, so by the universal property of the pullback, this induces a unique map $X\times I\to f^*E$ in Top. $\square$

Exercise 0.2. Let G be a topological group and H a subgroup, and let G/H have the quotient topology from the projection $p\colon G\to G/H$ (here G/H is the space of cosets, not the space obtained by collapsing H to a point). Assume that there exists a nonempty open set $U\subset G/H$ such that $p\colon p^{-1}(U)\to U$ admits a section $s\colon U\to p^{-1}(U)$. Prove that $G\to G/H$ is a fiber bundle. Deduce that it is a fibration.

Proof. By assumption, $p\circ s=\mathrm{id}_U$. Now, picking some $x_0\in p^{-1}(U)$, the set $V:=x_0^{-1}\cdot p^{-1}(U)$ is a neighborhood of the identity $e\in G$. Let $y\in G/H$, and pick a $y_0\in p^{-1}(y)$. Then $y_0\cdot V$ is a neighborhood of y_0, hence $p(y_0\cdot V)$ is a neighborhood of y (it is open since V was saturated with respect to p by construciton and multiplication by y_0 is a homeomorphism of G, so saturated sets remain saturated). Defining $s'\colon p(y_0\cdot V)\to y_0\cdot V$ by $s'(\overline{x})=s\circ p\left(y_0^{-1}\cdot p^{-1}(\overline{x})\right)$. If $\overline{x}=\overline{z}$, then $z^{-1}\cdot x\in H$, so $\left(y_0^{-1}\cdot z\right)^{-1}\cdot\left(y_0^{-1}\cdot x\right)\in H$, hence $s'\left(\overline{x}\right)=s'\left(\overline{z}\right)$. We claim that s' is then also a section of $p|_{y_0V}\colon y_0\cdot V\to p\left(y_0\cdot V\right)$. To see this, we have

$$p\circ s'=p\circ s\circ p\circ\left(y_0^{-1}\cdot -\right)\circ p^{-1}=p\circ\left(y_0^{-1}\cdot -\right)\circ p^{-1}$$

Now if $\overline{x}=\overline{z}$ then again $z^{-1}\cdot x\in H$, so $\left(y_0^{-1}\cdot z\right)^{-1}\cdot\left(y_0^{-1}\cdot x\right)\in H$, from which the claim follows.

We claim that $p^{-1}(U)$ admits a trivialization $H\times U\cong p^{-1}(U)$ via the map $k\colon (h,u)\mapsto h\cdot s(u)$. Firstly, this is in $p^{-1}(U)$ since $p\left(h\cdot s(u)\right)=p\left(s(u)\right)=u\in U$. It is also continuous and injective as the composition $H\times U\overset{\mathrm{id}\times s}{\to}H\times p^{-1}(U)\overset{\mathrm{prod}}{\to}p^{-1}(U)$.
Furthermore, if $h\cdot v=h'\cdot v$ for some $v\in p^{-1}(U)$, then $h=h\cdot v\cdot v^{-1}=h'\cdot v\cdot v^{-1}=h'$, so the action of H on U is free.

Suppose $h \cdot s(U) \cap h' \cdot s(U) \neq \varnothing$, so for some $u, u' \in U$, $h \cdot s(u) = h' \cdot s(u')$. But then $u = p\left(h \cdot s(u)\right) = p\left(h' \cdot s(u')\right) = u'$, and so $h = h \cdot s(u) \cdot s(u)^{-1} = h' \cdot s(u) \cdot s(u)^{-1} = h'$. Hence $p^{-1}(U) = \sqcup_{h \in H} h \cdot s(U)$. Now define $r \colon p^{-1}(U) \to H \times U$ by $r(u) = \left(\sum_{h \in H} h \cdot \delta_{u \in h \cdot s(U)}, p(u)\right)$. Then

$$r \circ k(h, u) = r\left(h \cdot s(u)\right) = (h, u)$$

and

$$k \circ r\,(x) = k\,(h, u) = x$$

since (h, u) are by definition such that $p(x) = u$ and $x \in h \cdot s(U)$, so we must have $x = h \cdot s(u)$. Thus $k(h, u) = h \cdot s(u) = x$. So r is an inverse function to k. It remains to show that it is continuous. The coordinate $p(u)$ is continuous, so we must show that $r_1 \colon u \mapsto \sum_{h \in H} h \cdot \delta_{u \in h \cdot s(U)}$ is continuous. For this, note that for an open set $W \subset H$, $r_1^{-1}(W) = \bigcup_{h \in W} h \cdot s(U)$. Now, since $p\left(h \cdot s(u)\right) = p(s(u)) = u$, so $r_1^{-1}(W) \subset p^{-1}(U)$, and conversely, for any $x \in p^{-1}(U)$, there is an $h \in H$ such that $x \in h \cdot s(U)$, so $p^{-1}(U) \subset \bigcup_{h \in W} h \cdot s(U)$. Hence $r_1^{-1}(W) = p^{-1}(U)$ which is open.
This completes the proof that $G \to G/H$ is a fiber bundle.
Since any fiber bundle is a fibration, the last part follows directly.

□

Exercise 0.3. Recall that $S^3 \subset \mathbb{R}^4 \cong \mathbb{H}$ is a topological group, with $S^1 \subset \mathbb{C} \subset \mathbb{H}$ as a topological subgroup. Recall that

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}.$$

so $S^3 \subset \mathbb{H}$ here is considered as the group whose elements are elements in $\mathbb{H}$ with norm 1, and $S^1 \subset S^3$ as the subgroup

$$\left\{a + bi \mid a^2 + b^2 = 1\right\}.$$

(1) Prove (using the previous exercise) that $S^3 \to S^3/S^1$ is a fiber bundle with fiber S^1, and therefore a fiber bundle.
(2) Prove that $S^3/S^1 \cong S^2$. The fiber sequence $S^1 \to S^3 \to S^2$ is called the *Hopf* fibration.
(3) Use the LES associated to this fibration to compute $\pi_3(S^2)$.
(4) Show that $S^3 \times K(\mathbb{Z}, 2)$ and S^2 have isomorphic homotopy groups. Are they homotopy equivalent?

Proof. (1) Let S^3_+ denote the open upper hemisphere. Then if $p \colon S^3 \to S^3/S^1$ is the quotient map, S^3/S^1 looks □

0.0.2. *Problems.*

Problem 0.4. Suppose $p\colon E \to B$ is a Serre fibration and $f\colon X \to B$ is n-connected. Prove that the projection $E \times_B X \to E$ is also n-connected.

Proof. We are given the following commutative diagram

$$\begin{tikzcd} E \times_B X \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

Firstly, by Exercise 1 on Problem set 4, the map $E \times_B X \to X$ is also a Serre fibration.
Secondly, by assumption in the conventions section for problem set 4, all spaces are assumed to be locally path-connected and connected, hence all spaces are path-connected. In particular, both X and B are assumed to be path-connected, so by Theorem 4.41 in Hatcher, we have a LES

$$\ldots \to \pi_k(F', y_0) \to \pi_k(E \times_B X, y_0) \overset{(\pi_X)_*}{\to} \pi_k(X, x_0) \to \pi_{k-1}(F', y_0) \to \ldots \to \pi_0\left(E \times_B X, y_0\right) \to 0$$

where $F' = (\pi_X)^{-1}(x_0)$ for some $x_0 \in X$ and $y_0 \in F'$. Now,

$$F' = (\pi_X)^{-1}(x_0) = \{(e, x_0) \mid f(x_0) = p(e)\} \xrightarrow[\cong]{\pi_E} p^{-1}\left(f(x_0)\right) =: F$$

where we choose F to be the fiber of $p\colon E \to B$ (when repeating Theorem 4.41 for this fibration), and we choose $e_0 \in F$ to be $\pi_E(y_0)$. With these choices of fibers and basepoints, we obtain that the map $\pi_E|_{F'}\colon F' \to F$ is a homeomorphism (it has the inverse $e \mapsto (e, x_0)$) by construction, so the following diagram commutes:

$$\begin{tikzcd} (F', y_0) \arrow[r, "\cong"] \arrow[d, hook] & (F, e_0) \arrow[d, hook] \\ (E \times_B X, y_0) \arrow[r, "\pi_E"] \arrow[d, "\pi_X"] & (E, e_0) \arrow[d, "p"] \\ (X, x_0) \arrow[r, "f"] & (B, f(x_0)) \end{tikzcd} \tag{Ω}$$

With these choices of basepoints, Theorem 4.41 gives the following long exact sequences (the solid part of the diagram)

$$\begin{tikzcd} \pi_{k+2}(X, x_0) \arrow[r] \arrow[d, dashed] & \pi_{k+1}(F', y_0) \arrow[r] \arrow[d, dashed] & \pi_{k+1}(E \times_B X, y_0) \arrow[r, "(\pi_X)_*"] \arrow[d, dashed] & \pi_{k+1}(X, x_0) \arrow[r] \arrow[d, dashed] & \pi_k(F', y_0) \arrow[d, dashed] \\ \pi_{k+2}(B, f(x_0) \arrow[r] & \pi_{k+1}(F, e_0) \arrow[r] & \pi_{k+1}(E, e_0) \arrow[r, "p_*"] & \pi_{k+1}(B, f(x_0)) \arrow[r] & \pi_k(F, e_0) \end{tikzcd} \tag{Γ}$$

Now, applying π_{k+1} to (Ω), i.e., using functoriality of π_{k+1} on pointed topological spaces, we find that for $k+1 \geq 1$, we have

$$\begin{tikzcd}
\pi_{k+1}(F', y_0) \arrow[r, "\cong"] \arrow[d] & \pi_{k+1}(F, e_0) \arrow[d] \\
\pi_{k+1}(E \times_B X, y_0) \arrow[r, "(\pi_E)_*"] \arrow[d, "(\pi_X)_*"] & \pi_{k+1}(E, e_0) \arrow[d, "p_*"] \\
\pi_{k+1}(X, x_0) \arrow[r, "f_*", "\cong"'] & \pi_{k+1}(B, f(x_0))
\end{tikzcd} \tag{ζ}$$

commutes (since functoriality of π_{k+1} implies that compositions are preserved) - where also f_* is an isomorphism for $k < n-1$ and surjective for $k = n-1$.
We now claim that

$$\begin{tikzcd}
\pi_{k+1}(X, x_0) \arrow[r] \arrow[d, "f_*"] & \pi_k(F', y_0) \arrow[d, "(\pi_E|_{F'})_*", "\cong"'] \\
\pi_{k+1}(B, f(x_0)) \arrow[r] & \pi_k(F, e_0)
\end{tikzcd}$$

commutes. Consider the following diagram:

$$\begin{tikzcd}
\pi_{k+1}(E \times_B X, F') \arrow[r, "(\pi_X)_*", "\cong"'] \arrow[d, "(\pi_E)_*"] \arrow[rr, bend left, "\partial"] & \pi_{k+1}(X, x_0) \arrow[r] \arrow[d, "f_*"] & \pi_k(F', y_0) \arrow[d, "(\pi_E|_{F'})_*"] \\
\pi_{k+1}(E, F) \arrow[r, "p_*", "\cong"'] \arrow[rr, bend right, "\partial"'] & \pi_{k+1}(B, f(x_0)) \arrow[r] & \pi_k(F, e_0)
\end{tikzcd}$$

The outer triangle commutes by construction: since for $g\colon \left(D^{n+1}, S^n, s_0\right) \to (E \times_B X, F', y_0)$, we get

$$\partial \circ (\pi_E)_* ([g]) = \partial \left[\pi_E \circ g\right] = \left[(\pi_E \circ g)\,|_{S^n}\right] = \left[\pi_E|_{F'} \circ g|_{S^n}\right] = (\pi_E|_{F'})_* \left([g|_{S^n}]\right) = (\pi_E|_{F'})_* \circ \partial\left([g]\right).$$

Also, the left hand square commutes for $k+1 \geq 1$, since this is what we obtained from (ζ). From this, we can conclude that the right hand square also commutes for $k+1 \geq 1$, i.e., for $k \geq 0$. Explicitly, if we let k be the map $\pi_{k+1}(X, x_0) \to \pi_k(F', y_0)$ and l the map $\pi_{k+1}(B, f(x_0)) \to \pi_k(F, e_0)$, then we get

$$\begin{aligned}
(\pi_E|_{F'})_* \circ j &= (\pi_E|_{F'})_* \circ \partial \circ (\pi_X)_*^{-1} \\
&= \partial \circ (\pi_E)_* \circ (\pi_X)_*^{-1} \\
&= \partial \circ p_*^{-1} \circ f_* \\
&= l \circ f_*
\end{aligned}$$

giving commutativity.

Therefore, we can fill in the dashed arrows in diagram (Γ), giving that the following diagram commutes for $0 \leq k < n-1$.

$$\begin{tikzcd}
\pi_{k+2}(X,x_0) \arrow[r] \arrow[d, "f_*"] & \pi_{k+1}(F',y_0) \arrow[r] \arrow[d, "(\pi_E|_{F'})_*"', "\cong"] & \pi_{k+1}(E\times_B X,y_0) \arrow[r, "(\pi_X)_*"] \arrow[d, "(\pi_E)_*"] & \pi_{k+1}(X,x_0) \arrow[r] \arrow[d, "f_*"', "\cong"] & \pi_k(F',y_0) \arrow[d, "\cong"] \\
\pi_{k+2}(B,f(x_0)) \arrow[r] & \pi_{k+1}(F,e_0) \arrow[r] & \pi_{k+1}(E,e_0) \arrow[r, "p_*"] & \pi_{k+1}(B,f(x_0)) \arrow[r] & \pi_k(F,e_0)
\end{tikzcd}$$

By the 5-lemma, we obtain that $(\pi_E)_* : \pi_{k+1}(E\times_B X, y_0) \to \pi_{k+1}(E,e_0)$ is an isomorphism for $1 \le k+1 \le n-1$. Note that this also works for $1 = k+1$ despite π_0 not being a group (one can simply trace through the arguments in the proof of the 5-lemma and see that it still works with exactly the same arguments).
It remains to show that it is an isomorphism on π_0 and surjective on π_n. Surjectivity on π_n immediately follows by applying the 4-lemma to the following diagram:

$$\begin{tikzcd}
\pi_n(F',y_0) \arrow[r] \arrow[d, "(\pi_E|_{F'})_*"', "\cong"] & \pi_n(E\times_B X,y_0) \arrow[r, "(\pi_X)_*"] \arrow[d, "(\pi_E)_*"] & \pi_n(X,x_0) \arrow[r] \arrow[d, "f_*"', "\cong"] & \pi_{n-1}(F',y_0) \arrow[d, "\cong"] \\
\pi_n(F,e_0) \arrow[r] & \pi_n(E,e_0) \arrow[r, "p_*"] & \pi_n(B,f(x_0)) \arrow[r] & \pi_{n-1}(F,e_0)
\end{tikzcd}$$

For the isomorphism on π_0, note that we have assumed that E is path-connected, so it suffices to show that the induced map $(\pi_E)_* \colon \pi_0\left(E\times_B X, y_0\right) \to \pi_0(E,e_0) = 0$ is injective. Since the diagram (footnote: [1])

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r] \arrow[d, "\cong"', "f_*"] & \pi_0(F',y_0) \arrow[r] \arrow[d, "\cong"'] & \pi_0(E\times_B X,y_0) \arrow[r] \arrow[d, "(\pi_E)_*"] & 0 \\
\pi_1(B,f(x_0)) \arrow[r] & \pi_0(F,e_0) \arrow[r] & \underbrace{\pi_0(E,e_0)}_{0} \arrow[r] & 0
\end{tikzcd}$$

commutes with exact rows, we obtain by the 4-lemma that $(\pi_E)_* \colon \pi_0\left(E\times_B X, y_0\right) \to \pi_0(E,e_0) = 0$ is injective, so in particular $\pi_0\left(E\times_B X, y_0\right) = 0$. Note that the part of the 4-lemma which we used (the one concluding monomorphism) does not require the objects to be groups or, for that matter, the diagram to be in an abelian category - we just need the kernel to be concrete. (See footnote [2]). Thus $(\pi_E)_*$ is an isomorphism on π_0 also.

This completes the proof. $\square$

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[1] In this diagram, when we are talking about exactness of the rows, while π_0 of the spaces are not groups, we can still define exactness and exact sequences of pointed sets as one might expect: $(X,x) \xrightarrow{f} (Y,y) \xrightarrow{g} (Z,z)$ is exact if the composite gf is constant at z and $g(a) = z$ implies that there a b with $a = f(b)$

[2] For concreteness, what I am saying is that all the steps in the proof of the 4-lemma as laid out for example on Wikipedia (`https://en.wikipedia.org/wiki/Five_lemma`) all still hold in our situation ad verbum, hence I do not wish to repeat all the arguments since it's just a lengthy diagram chase. One thing to note, however, is that when we talk about 0 in π_0 in the arguments, this means to be in the same path-component as the basepoint. With this terminology, all things go through.

Problem 0.5. The (ordered) configuration space on n-points of a space X is the subspace

$$\mathrm{Conf}_n(X) = \{(x_1, \dots, x_n) \in X^n \mid \forall i \neq j \colon x_i \neq x_j\} \subset X^n$$

of those n-tuples of points that are pairwise distinct. In this problem, we consider the special case $X = \mathbb{R}^2$. You may assume that the map

$$\begin{aligned} \mathrm{Conf}_n(\mathbb{R}^2) &\to \mathrm{Conf}_{n-1}(\mathbb{R}^2) \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_{n-1}) \end{aligned}$$

is a fiber bundle. We define the base-point of $\mathrm{Conf}_n(\mathbb{R}^2)$ to be $x = ((1,0),(2,0),\dots,(n,0))$.

(1) Show that $\mathrm{Conf}_n(X)$ is an Eilenberg-MacLane space of type $(G,1)$ with $G = \pi_1 \mathrm{Conf}_n(\mathbb{R}^2)$.

For $1 \leq i < j \leq n$, define elements $\gamma_{i,j} \in \pi_1 \mathrm{Conf}_n(X)$ as follows:

$$\gamma_{i,j}(t) = ((1,0),\dots,(i-1,0),\rho(t),(i+1,0),\dots,(n,0))$$

where $\rho\colon [0,1] \to \mathbb{R}^2$ is a loop that starts and ends at $(i,0)$ and loops around $(j,0)$ and avoids the other points.

(2) Show that the $\gamma_{i,j}$ generate $\pi_1 \mathrm{Conf}_n(\mathbb{R}^2)$.

The unordered configuration space is defined as the set of subsets of $\mathbb{R}^2$ of cardinality n :

$$\mathrm{UConf}_n(\mathbb{R}^2) = \left\{ A \subset \mathbb{R}^2 \mid |A| = n \right\}.$$

This is topologised with the quotient topology coming from the map $\mathrm{Conf}_n(\mathbb{R}^2) \to \mathrm{UConf}_n(\mathbb{R}^n)$ sending $(x_1,\dots,x_n)$ to $\{x_1,\dots,x_n\}$.

(3) Show that $\mathrm{UConf}_n(X)$ is an Eilenberg-MacLane space of type $(G,1)$ with $G = \pi_1 \mathrm{UConf}_n(\mathbb{R}^2)$.

Proof. (1) The fiber above the base point $(x_1,\dots,x_{n-1}) \in \mathrm{Conf}_{n-1}(\mathbb{R}^2)$ under the map is $F = \{(x_1,\dots,x_n) \mid \forall i \colon x_n \neq x_i\} \cong \mathbb{R}^2 - \{x_1,\dots,x_{n-1}\}$. Since every fiber bundle is a Serre fibration, we find from the LES associated to the fibration that

$$\pi_k\left(\mathbb{R}^2 - \{x_1,\dots,x_{n-1}\}\right) \to \pi_k\left(\mathrm{Conf}_n(\mathbb{R}^2)\right) \to \pi_k\left(\mathrm{Conf}_{n-1}(\mathbb{R}^2)\right) \to \pi_{k-1}\left(\mathbb{R}^2 - \{x_1,\dots,x_{n-1}\}\right).$$

is exact for all $k \geq 1$.
By theorem 4.41, if $\mathrm{Conf}_{n-1}(\mathbb{R}^2)$ is path-connected, then

$$\underbrace{\pi_0\left(\mathbb{R}^2 - \{x_1,\dots,x_0\}\right)}_{0} \to \pi_0\left(\mathrm{Conf}_n(\mathbb{R}^2)\right) \to 0$$

is exact, so $\pi_0\left(\mathrm{Conf}_n(\mathbb{R}^2)\right) \cong 0$. Now, $\mathrm{Conf}_1(\mathbb{R}^2) = \mathbb{R}^2$ which is path-connected, so by induction, $\mathrm{Conf}_n(\mathbb{R}^2)$ is path-connected for all $n \geq 1$, hence $\pi_0\left(\mathrm{Conf}_n\left(\mathbb{R}^2\right)\right) = 0$.

Now, $\mathbb{R}^2 - \{x_1,\dots,x_{n-1}\} \simeq \bigvee_{n-1} S^1$. In particular, the universal cover of a graph is a tree hence contractible, so since $\pi_k\left(\bigvee_{n-1} S^1\right)$ coincides with π_k of the universal cover when $k \geq 2$, we find that $\pi_k\left(\mathbb{R}^2 - \{x_1, \ldots, x_{n-1}\}\right) \cong 0$ for $k \geq 2$. Thus $\pi_k\left(\operatorname{Conf}_n\left(\mathbb{R}^2\right)\right) \cong \pi_k\left(\operatorname{Conf}_{n-1}(\mathbb{R}^2)\right)$ for all $k \geq 2$. By repeated application, we thus get that $\pi_k\left(\operatorname{Conf}_n\left(\mathbb{R}^2\right)\right) \cong \pi_k\left(\operatorname{Conf}_1\left(\mathbb{R}^2\right)\right) \cong \pi_k\left(\mathbb{R}^2\right) \cong 0$ for $k \geq 2$ since $\operatorname{Conf}_1\left(\mathbb{R}^2\right) = \mathbb{R}^2$.

It remains to show that $\pi_1 \operatorname{Conf}_n(\mathbb{R}^2) \neq 0$. Noting that $\pi_0\left(\operatorname{Conf}_n(\mathbb{R}^2)\right) \cong 0$ and $\pi_2\left(\operatorname{Conf}_{n-1}(\mathbb{R}^2)\right)$, we obtain from the LES associated to the fibration $\operatorname{Conf}_n(\mathbb{R}^2) \to \operatorname{Conf}_{n-1}\left(\mathbb{R}^2\right)$ the following exact sequence:

$$0 \to \pi_1\left(\mathbb{R}^2 - \{x_1, \ldots, x_{n-1}\}\right) \to \pi_1 \operatorname{Conf}_n\left(\mathbb{R}^2\right) \to \pi_1 \operatorname{Conf}_{n-1}\left(\mathbb{R}^2\right) \to 0$$

Now, $\pi_1\left(\mathbb{R}^2 - \{x_1, \ldots, x_{n-1}\}\right) \cong \pi_1\left(\bigvee_{n-1} S^1\right) \cong F_{n-1} = \langle x_1, \ldots, x_{n-1}\rangle$, the free group on $n-1$ elements. Now, when $n = 1$, $\operatorname{Conf}_n\left(\mathbb{R}^2\right) = \mathbb{R}^2$, so $\pi_1 \operatorname{Conf}_1\left(\mathbb{R}^2\right) \cong 0$, and hence $\pi_1 \operatorname{Conf}_2\left(\mathbb{R}^2\right) \cong F_1 \cong \mathbb{Z}$. But free groups are projective, so by induction, the exact sequence

$$0 \to F_{n-1} \to \pi_1 \operatorname{Conf}_n\left(\mathbb{R}^2\right) \to \pi_1 \operatorname{Conf}_{n-1}(\mathbb{R}^2) \to 0 \qquad (\xi)$$

splits, hence each $\pi_1 \operatorname{Conf}_n(\mathbb{R}^2)$ is non-trivial as it contains an isomorphic copy of a free group. This shows that $\operatorname{Conf}_n(\mathbb{R}^2)$ is a $K(G,1)$ space for all $n \geq 1$.
We will even show below that the SES in (ξ) splits.

(2)

Note. Below, we will abuse terminology by identifying F_{n-1} with $\pi_1\left(\mathbb{R}^2 - \{x_1, \ldots, x_{n-1}\}\right)$ and ι_n having either as a domain.
Furthermore, these are always pointed maps.

To find generators, we first look at the actual maps in the LES for the fiber bundle. Let $p_n \colon \operatorname{Conf}_n(\mathbb{R}^2) \to \operatorname{Conf}_{n-1}\left(\mathbb{R}^2\right)$ be the map that forgets the last coordiante, and and let $\iota_n \colon \mathbb{R}^2 - \{x_1, \ldots, x_{n-1}\} \cong F \hookrightarrow \operatorname{Conf}_n\left(\mathbb{R}^2\right)$ be the inclusion of the fiber.

We now show that (ξ) splits. Define a map $s \colon \operatorname{Conf}_{n-1}\left(\mathbb{R}^2\right) \to \operatorname{Conf}_n\left(\mathbb{R}^2\right)$ by $s\left(x_1, \ldots, x_{n-1}\right) = (x_1, \ldots, x_{n-1}, x_n')$ where $x_n' = (1 + \max_{i \leq n-1} |x_i|, 0)$. This map is continuous since the norm is continuous, and we also have $p \circ s = \mathrm{id}_{\operatorname{Conf}_{n-1}(\mathbb{R}^2)}$. Since both p and s preserve basepoints, we find by taking π_1 that $p_* \circ s_* = \mathrm{id}_{\pi_1 \operatorname{Conf}_{n-1}(\mathbb{R}^2)}$. That is, s_* is a section, so (ξ) splits.

Thus we have

$$F_{n-1} \oplus \pi_1 \operatorname{Conf}_{n-1}(\mathbb{R}^2) \cong \pi_1 \operatorname{Conf}_n(\mathbb{R}^2)$$

under the isomorphism $(x,y) \mapsto (\iota_n)_*(x) + s_*(y)$. Since $\pi_1 \operatorname{Conf}_2\left(\mathbb{R}^2\right) \cong \pi_1 S^1$ induced by the deformation retract $\mathbb{R}^2 - \{x_1\} \times I \to \mathbb{R}^2 - \{x_1\}$ sending $(x,t) \mapsto (1-t)(x - x_1) + t\frac{x - x_1}{\|x - x_1\|} + x_1$. Under this isomorphism, we see that $\gamma_{1,2}$ which winds around the puncture once corresponds to a generator for $\pi_1 S^1$ - namely a loop that goes once around. Thus $\gamma_{1,2}$ indeed generates $\pi_1 \operatorname{Conf}_2\left(\mathbb{R}^2\right)$.
Now, suppose we have shown for $n \leq N-1$ that $\pi_1 \operatorname{Conf}_n\left(\mathbb{R}^2\right)$ is generated by $\{\gamma_{i,j}\}_{1 \leq i < j \leq n}$. Then using the isomorphism

$$\pi_1(F) \oplus \pi_1 \operatorname{Conf}_{N-1}\left(\mathbb{R}^2\right) \overset{(x,y) \mapsto \iota_{N*}(x) + s_*(y)}{\to} \pi_1 \operatorname{Conf}_N\left(\mathbb{R}^2\right)$$

we see that the $\gamma_{i,j} \in \pi_1 \operatorname{Conf}_{N-1}\left(\mathbb{R}^2\right)$ are mapped under the isomorphism to $\gamma_{i,j} \in \pi_1 \operatorname{Conf}_{N-1}\left(\mathbb{R}^2\right)$ (we will furthermore assume that the given ρ which loop around $(j,0)$ from $(i,0)$ are sufficiently close to he the other points $(1,0),\ldots,(N,0)$ at all times so that $1+\max_{i\leq N-1}|x_i|$ will be equal to $(N,0)$):

$$\gamma_{i,j}(t) = \left((1,0),\ldots,(i-1,0),\rho(t),(i+1,0),\ldots,\underbrace{(1+\max_{i\leq N-1}|x_i|,0)}_{(N,0)}\right)$$

So to complete the induction, it suffices to show that the remaining loops $\gamma_{i,N}$ correspond to generators of F_{n-1}. The problem is that under the isomorphism above, a loop $\alpha_i \in \pi_1\left(\mathbb{R}^2 - \{x_1,\ldots,x_{N-1}\}\right)$, which winds once around x_i and then returns to the basepoint, corresponds to the loop in $\pi_1(F)$ whose $N-1$ first coordinates remain fixed at $x_1,\ldots,x_{N-1}$ and where the last coordinate traces out a loop where it winds around x_i once and then returns back to the basepoint x_N. Hence all of the loops in $\pi_1(F)$ are not of the form $\gamma_{i,N}$. However, we claim that the loop is homotopic to a $\gamma_{i,N}$.

To see this, we will regard $\pi_1 \operatorname{Conf}_N\left(\mathbb{R}^2\right)$ in a different way. The group $\pi_1 \operatorname{Conf}_N\left(\mathbb{R}^2\right)$ is commonly called the pure braid group PB_N. To visualize a loop $\alpha\colon (I,\{0,1\}) \to \pi_1\left(\operatorname{Conf}_N(\mathbb{R}^2),((1,0),\ldots,(N,0))\right)$ consider α as $\alpha(t) = (\alpha_1(t),\ldots,\alpha_N(t))$ where $\alpha_i(t)$ is the i th coordinate of α. This α_i traces out a loop in $\mathbb{R}^2$ in the common sense. Define now a new map $\beta_i\colon I \to \mathbb{R}^2\times I$ by $\beta_i(t) = (\alpha_i(t),t)$. Then $(\beta_1,\ldots,\beta_N)$ is a braid. See Figure 1 for a typical example in $\pi_1 \operatorname{Conf}_3\left(\mathbb{R}^2\right)$.

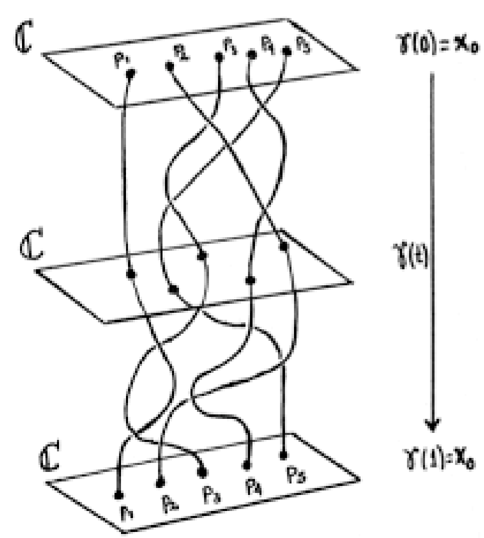

FIGURE 1. Here $\mathbb{C}$ can be replaced with $\mathbb{R}^2$.

So how do the loops in the image of $(\iota_N)_*$ look like? Well, since the first $N-1$ coordinates are constant, our β_i are of the form $\beta_i(t) = ((i,0),t)$ for $i \leq N-1$. And since α_N by construction traced out a loop that went around one of the punctures

$(i, 0)$ and not the others, β_N will be a strand which starts at $((N, 0), 0)$ then which wraps (in a continuous level-wise manner) around the i th braid once and the connects to $((N, 0), 1)$. See Figure 2 Now, what is a homotopy in $\pi_1 \operatorname{Conf}_n(\mathbb{R}^2)$?

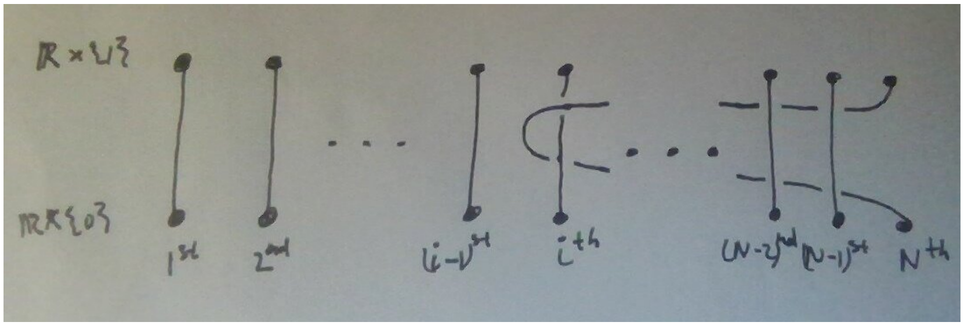

FIGURE 2.

It will have the form $(F_1(x,t), \ldots, F_N(x,t))$ where $F_i \colon I \times I \to \mathbb{R}^2$ depicts a loop. Furthermore, at each time $t_0 \in I$, $F_i(x, t_0) \neq F_j(x, t_0)$; i.e., the braids are disjoint/non-intersecting. Hence a homotopy in $\pi_1 \operatorname{Conf}_n(\mathbb{R}^2)$ is the same as an isotopy of the braid that the loop depicts.

With this setup in mind, we can now show that the loop in Figure 2 is homotopic to $\gamma_{i,N}$. This, in particular, is equivalent to saying that the braid in 2 is isotopic to the braid the $\gamma_{i,N}$ corresponds to. To see this, consider the isotopy depicted in Figure 3.

At the bottom, we obtain $\gamma_{i,N}$. Thus $\operatorname{im}(\iota_N)_*$ indeed corresponds to the remaining $\gamma_{i,j}$'s.

And this completes the inductive argument.

(3) As we equip $\operatorname{UConf}_n(\mathbb{R}^2)$ with the quotient topology obtained from the map $\operatorname{Conf}_n(\mathbb{R}^2) \to \operatorname{UConf}_n(\mathbb{R}^2)$, this map becomes continuous, and as $\operatorname{Conf}_n(\mathbb{R}^2)$ was path-connected and continuous images of path-connected spaces are path-connected, we find that $\pi_0 \operatorname{UConf}_n(\mathbb{R}^2) = 0$ for all n. Next, we claim that $\operatorname{Conf}_n(\mathbb{R}^2) \to \operatorname{UConf}_n(\mathbb{R}^2)$ is, in fact, a covering map. Let $\{x_1, \ldots, x_n\} \in \operatorname{UConf}_n(\mathbb{R}^2)$, where we have just given indices in some random fashion. We can define an action of Σ_n on $(x_1, \ldots, x_n)$ by $\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Since all x_i are pairwise distinct, so are all $\sigma \cdot (x_1, \ldots, x_n)$. Furthermore, distinct $\sigma \in \Sigma_n$ produce distinct tuples. Since this gives us finitely many distinct elements in $(\mathbb{R}^2)^{\times n} \cong \mathbb{R}^{2n}$ which is Hausdorff, one can show that Σ_n acts properly discontinuously on $\operatorname{Conf}_n(\mathbb{R}^2)$. Furthermore, the quotient map $q \colon \operatorname{Conf}_n(\mathbb{R}^2) \to \operatorname{Conf}_n(\mathbb{R}^2)/\Sigma_n$ agrees with the quotient map $\operatorname{Conf}_n(\mathbb{R}^2) \to \operatorname{UConf}_n(\mathbb{R}^2)$.

By proposition 2.4.3 in the AlgTop1 notes, the quotient map $q \colon \operatorname{Conf}_n(\mathbb{R}^2) \to \operatorname{Conf}_n(\mathbb{R}^2)/\Sigma_n$ is, in fact, a covering map. By proposition 4.1 in Hatcher, then $\pi_k \operatorname{Conf}_n(\mathbb{R}^2) \cong \pi_k \operatorname{UConf}_n$ for all $k \geq 2$ and all n. We have already shown that $\pi_k \operatorname{Conf}_n(\mathbb{R}^2) \cong 0$ for all $k \geq 2$, so $\pi_k \operatorname{UConf}_n(\mathbb{R}^2) \cong 0$ for all $k \geq 2$ also.

Furthermore, by Theorem 2.2.9 in the AlgTop1 notes, q_* induces an injection on fundamental groups, so since $\pi_1 \operatorname{Conf}_n(\mathbb{R}^2)$ was nontrivial, so $\pi_1 \operatorname{UConf}_n(\mathbb{R}^2)$ is nontrivial as it contains an isomorphic copy of $\pi_1 \operatorname{Conf}_n(\mathbb{R}^2)$.

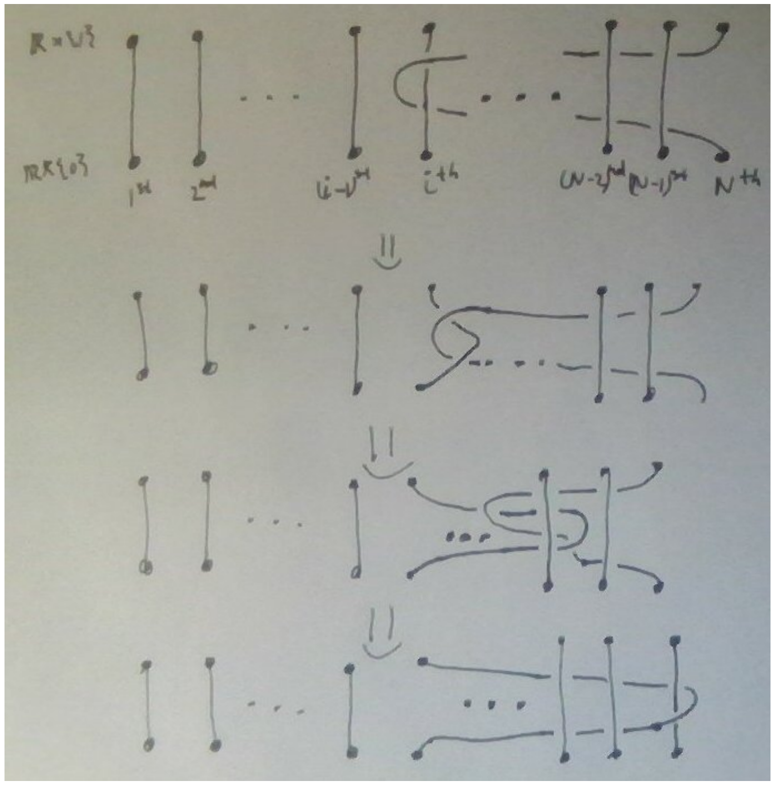

Figure 3.

This can also be seen very easily when we consider the visual representation of $\pi_1 \operatorname{UConf}_n\left(\mathbb{R}^2\right)$ as the standard braid group B_n. The pure braid group PB_n sits inside B_n as all the braids where each strand starts and ends at the same point - i.e., it starts at some $(p,0)$ and ends at $(p,1)$.

$\square$