Assignment 2

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Homework 4: Let X and Y be topological spaces and $f\colon X \to Y$ a continuous function. Let $Z := \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$ be equipped with the subspace topology coming from the product topology on $X \times Y$. Show that Z is homeomorphic to X.

Solution: We check that lemma 6.6 is satisfied:
Let $\varphi\colon X \to Z$ by $\varphi(x) = (x, f(x))$. This is clearly a well defined bijection. Let $A \subseteq Z$ be open. If $A = \varnothing$, then $\varphi^{-1}(A) = \varnothing$ which is open in X by definition of a topology, so assume A is nonempty and let $x \in \varphi^{-1}(A)$. Then $A = Z \cap (U \times V)$ where U is open in X and V is open in Y by definition 4.1 for the subspace topology and definition 4.6 for the product topology. Now, since f is continuous and $x \times f(x) \in Z \cap (U \times V)$ implies that $f(x) \in V$, we have that $f^{-1}(V)$ is a neighborhood of x, so since the intersection of a finite collection of open sets is open by definition of a topological space, $S = U \cap f^{-1}(V)$ is a neighborhood of x. Now, if $x' \in S$ then $f(x') \in V$ and $x' \in U$, so $\varphi(x') = x' \times f(x') \in Z \cap (U \times V) = A$, thus $x \in S \subseteq \varphi^{-1}(A)$.
Since x was arbitrary, we can for any $x \in A$, find an open neighborhood of x, S_x, as above. Then we have

$$A = \bigcup_{x\in A} \{x\} \subseteq \bigcup_{x \in A} S_x \overset{\forall x \in A\colon S_x \subseteq A}{\subseteq} A$$

Thus $A = \bigcup_{x\in A} S_x$ and by definition 2.1, A is open.

Now, we check the second part of Lemma 6.6: if $U \subseteq X$ is open, then $\varphi(U)$ is open in Z.

Let $U \subseteq X$ be open (if $U = \varnothing$, $(\varphi^{\circ -1})^{-1}(U) = \varnothing$ which is open, so assume U is nonempty) and let $x \times f(x) \in \varphi(U)$. Then $x \in U$ and hence $f(x) \in Y$, so $x \times f(x) \in (U \times Y) \cap Z$. We now claim $(U \times Y) \cap Z \subseteq \varphi(U)$. Let $x \times y \in (U \times Y) \cap Z$. Since $x \times y \in Z$, $f(x) = y$, and since $x \times y \in U \times Y$, we have $x \in U$, so $x \times y = x \times f(x) = \varphi(x) \in \varphi(U)$.
Now, since $(U \times Y) \cap Z$ was chosen for any $x \in U$ independent of x, we have

$$\varphi(U) = \bigcup_{x\in U} \{x \times f(x)\} \subseteq \bigcup_{x \in U} (U \times Y) \cap Z \subseteq \varphi(U)$$

hence $\varphi(U) = (U \times Y) \cap Z$. Since U is open in X and Y is open in Y by definition of a topology, we have that $(U \times Y) \cap Z$ is open in Z by definition of a subspace topology, hence $\varphi(U)$ is open in Z.
Hence φ is a homeomorphism by Lemma 6.6.

Homework 5: Let $X = \mathbb{R}$ be equipped with the topology

$$\mathcal{T} = \{\varnothing\} \cup \{U \subseteq X \mid \#(X \backslash U) < \infty\}.$$

(i) Prove or disprove that $(X, \mathcal{T})$ is Hausdorff.

(ii) Prove or disprove that $(X, \mathcal{T})$ is connected.

(iii) Find the closure of the sets $A = [0,1] = \{t \in \mathbb{R} \mid 0 \leq t \leq 1\}, B = (0,1) = \{t \in \mathbb{R} \mid 0 < t < 1\}, C = \{2,4,6,8\}$ in X with respect to $\mathcal{T}$.

(iv) Suppose that $f\colon X \to X$ is a bijection. Prove that f is a homeomorphism.

Solution:
(i) We claim that $(X, \mathcal{T})$ is not Hausdorff: We show that condition (b) in theorem 5.19 is not satisfied: Let x_n be the sequence $x_n = \frac{1}{n}$. Let x be any point of $\mathbb{R}$. Then for any neighborhood U of x, its complement is finite and can thus contain at most a finite subcollection of $\{x_n\}_{n\in\mathbb{N}}$. Let $N \in \mathbb{N}$ be the maximum of the indices in this subcollection, the for all $n > N$, we have that x_n is not in U^c, so $x_n \in (U^c)^c = U$ so since U was arbitrary, x_n converges to x by definition 5.14, so since x_n converges to every point in $\mathbb{R}$, we get contraposition of theorem 5.19.(b) that X is not Hausdorff.

(ii) If X could be separated as $X = U \cup V$ where U, V are open disjoint nonempty subsets of X, then $V = X \backslash U$, so $\#V = \#X \backslash U < \infty$ by definition of V being open, hence $\#X \backslash V = \#X - \#V = \infty$, where the second equality follows from theorem 468 in the dismat book, so V is not open. Thus no such separation exists, so X is connected by definition 8.1.

(iii) We saw in (i) that for any $x \in \mathbb{R}$, any neighborhood of x intersects $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \subseteq (0,1) \subseteq [0,1]$, thus by proposition 5.9, $\mathbb{R} \subseteq \overline{(0,1)}, \overline{[0,1]} \subseteq \mathbb{R}$ hence $\mathbb{R} = \overline{(0,1)}, \overline{[0,1]}$. Now, since $\#\mathbb{R}\backslash(\mathbb{R}\backslash\{2,4,6,8\}) = \#\{2,4,6,8\} = 4 < \infty$, we find that $\mathbb{R}\backslash\{2,4,6,8\}$ is open, hence its complement $\mathbb{R}\backslash(\mathbb{R}\backslash\{2,4,6,8\}) = \{2,4,6,8\}$ is closed by definition, so $\overline{\{2,4,6,8\}} = \{2,4,6,8\}$ by corollary 5.13 and proposition 5.12.

(iv) Since f is a bijection, $f^{\circ -1}$ is too, so we have that for any subset $U \subseteq X$, $\#\ f^{-1}(U) = \#U$. Now, for any nonempty open set $U \subseteq X$, we have $\mathbb{R}\backslash U$ is finite and

$$\mathbb{R}\backslash f^{-1}(U) = f^{-1}(X)\backslash f^{-1}(U) \overset{\text{week 1, 4.(v)}}{=} f^{-1}(X\backslash U).$$

And thus $\#\mathbb{R}\backslash f^{-1}(U) = \#f^{-1}(X\backslash U) = \#X\backslash U < \infty$, so $f^{-1}(U)$ is open. Now, if $U = \varnothing$ then $f^{-1}(U) = \varnothing$, which is open in X. Thus for any open set $U \subseteq X$, $f^{-1}(U)$ is open in X, so f is continuous.
Now, by replacing all f in the above by $f^{\circ -1}$ (the inverse of f), we find that $f^{\circ -1}$ is continuous too. Hence f is a homeomorphism.

Homework 6: Consider the following topological spaces $(X, \mathcal{T})$ and the given sequences $(a_n)_{n\in\mathbb{N}}$ in them. In each case either prove that the sequence does not converge, or find all the points the sequence converges to.

(i) $X = \mathbb{R}$, T generated by all the half-open intervals $[x,y), x,y \in \mathbb{R}, x < y$.
  (a) $a_n = \frac{1}{n}, n \in \mathbb{N}$.
  (b) $a_n = -\frac{1}{n}, n \in \mathbb{N}$.

(ii) $X = \{1,2,3,4,5\}$, $\mathcal{T} = \{\varnothing, X, \{1\}, \{2,3\}, \{1,2,3\}, \{2,3,4,5\}\}$.
  (a) $a_n = 4, n \in \mathbb{N}$.
  (b) $a_n = 2$ when $n \in \mathbb{N}$ is even, and $a_n = 3$ when $n \in \mathbb{N}$ is odd.

(iii) $X = \mathbb{R}$, $\mathcal{T} = \{\varnothing, \mathbb{R}\} \cup \{(t,\infty) \mid t \in \mathbb{R}\}$.
  (a) $a_n = \frac{1}{n}, n \in \mathbb{N}$.
  (b) $a_n = n, n \in \mathbb{N}$.

Solution:
(i):
(a): We claim $a_n \to 0$ as its only limit point. For any open set U containing 0, there exists a basis element $[x,y)$ such that $0 \in [x,y) \subseteq U$ by definition 3.1 and definition 3.7. Now we have $x \leq 0 < y$. Thus by the Archimedean property, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\frac{1}{y} < n$ hence since n is positive, $x \leq 0 < \frac{1}{n} < y$, so $\frac{1}{n} \in [x,y) \subseteq U$ and thus by definition 5.14, a_n converges to 0. Assume a_n also converges to $z \neq 0$. If $z < 0$, then $[z-1,0)$ contains z but does not contain any points of the sequence a_n, so z is not a limit point of a_n. If $z > 0$ then there exists $N \in \mathbb{N}$ such that $N < \frac{1}{z} \leq N+1$, and thus $z \in [\frac{1}{N+1}, \frac{1}{N})$ since inversion is continuous on $(0,\infty)$. Now, assume $\frac{1}{k} \in \left[\frac{1}{N+1}, \frac{1}{N}\right)$ for a $k \in \mathbb{N}$. Then $k = N+1$, and so for all $n > N+1$, $\frac{1}{n} \notin \left[\frac{1}{N+1}, \frac{1}{N}\right)$, so z is not a limit point of a_n. Thus 0 is the only limit point of the sequence.

(b) We claim that a_n has no limit points in $\mathbb{R}_l$ and thus does not converge.
Firstly, since for any $z \geq 0$, $[z, z+1)$ is a neighborhood of z that does not intersect the sequence since it contains only non-negative points of $\mathbb{R}$ while all points of the sequence are negative, the sequence does not converge to any $z \geq 0$. Assume $a_n \to z$ with $z < 0$.
There exists $N \in \mathbb{N}$ such that $-(N+1) < \frac{1}{z} \leq -N$, so $z \in \left[-\frac{1}{N}, -\frac{1}{N+1}\right)$ by continuity of inversion on $(-\infty, 0)$, which, as in (a), only contains $-\frac{1}{N}$ of the sequence, so for all $n > N$, $a_n = -\frac{1}{n} \notin \left[-\frac{1}{N}, -\frac{1}{N+1}\right)$, hence a_n does not converge to z.

(ii):
(a) We claim that $a_n = 4, n \in \mathbb{N}$ converges to 4 and 5. Let U be any neighborhood of either point. Then

since the only neighborhoods containing 4 or 5 are X and $\{2,3,4,5\}$ which both contain $4 = a_n, \forall n \in \mathbb{N}$, we have that a_n is in U for all $n \geq 1$, so by definition 5.14, a_n converges to 4 and 5.
For the points $1, 2, 3$, we have $1, 2, 3 \in \{1, 2, 3\}$ which does not contain $4 = a_n$ for all n, and thus the condition in definition 5.14 is not satisfied, so a_n does not converge to $1, 2$ or 3.

(b) We claim that a_n converges to the points $2, 3, 4$ and 5.
We note that for any neighborhood of any of the points $2, 3, 4$ and 5, $\{2, 3\}$ is contained in the neighborhood, and thus for all $n \geq 1$, a_n is in the neighborhood, so a_n converges to $2, 3, 4$ and 5 by definition 5.14.
Now, for the point 1, take the neighborhood $\{1\}$. If there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n \in \{1\}$ then $a_n = 1$ which is a contradiction. Hence a_n does not converge to 1.

(iii):
(a) We claim that a_n converges to all points in $\mathbb{R}_0^- = \{x \in \mathbb{R} \mid x \leq 0\}$.
Let $x \in \mathbb{R}_0^-$ and U be any neighborhood of x. Assume first that $U \neq X$. Then U is of the form (t, ∞) with $t < x \leq 0$. Now, since for all $n \geq 1$, $a_n > 0$, we have $a_n \in (t, \infty)$, so the condition in definition 5.14 is satisfied with $N = 1$. If $U = X$, a_n is similarly in X for any $n \geq 1$. Thus a_n converges to x.
We claim that a_n does not converge to any $x > 0$. Fix any $x > 0$, then $x \in (\frac{x}{2}, \infty)$. Now, by the Archimedean property, there exists $N \in \mathbb{N}$ such that $\frac{2}{x} < N$, and thus for all $n \geq N$, we have $a_n = \frac{1}{n} < \frac{x}{2}$, so for all $n \geq N$, $a_n \notin (\frac{x}{2}, \infty)$, and thus a_n does not satisfy the condition in definition 5.14, so it does not converge to any $x > 0$.

(b) We claim a_n converges to all points in $\mathbb{R}$.
Fix any $x \in \mathbb{R}$ and take any neighborhood U of x. Assume first $U \neq X$, then U is of the form (t, ∞). Then by the Archimedean property, there exists $N \in \mathbb{N}$ such that $x < N$, so we find that for all $n \geq N$, we have $t < x < N \leq n = a_n$, and thus for all $n \geq N$, $a_n \in (t, \infty)$. Now, if $U = X$, then a_n is trivially in U for all $n \geq 1$. Thus a_n converges to x by definition 5.14.