Assignment 1

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Homework 1: Decide which of the following $\mathcal{B}$ are a basis for some topology on the respective given sets X.

(i) $X = \mathbb{N}, \mathcal{B} = \{U \subseteq \mathbb{N} \mid 10 \leq \#(\mathbb{N}\backslash U) < \infty\}$;

(ii) $X = \mathbb{N}, \mathcal{B} = \{U \subseteq \mathbb{N} \mid \#(\mathbb{N}\backslash U) \leq 10\}$;

(iii) $X = \mathbb{R}^2, \mathcal{B} = \{B_r(0) \mid r > 0\}$ where $B_r(0)$ denotes the open disk of radius r around 0 in $\mathbb{R}^2$ with respect to the Euclidean metric.

Solution: We will freely use that if $A \subseteq B$, the inclusion function is injective and thus $\#A \leq \#B$.
(i) We claim $\mathcal{B}$ is a basis for a topology on X. We check the requirements in theorem 3.6:
We have that any $U \in \mathcal{B}$ is by definition a subset of X. Now,
(a) Let $U_x = \mathbb{N}\backslash\{x, x+1, x+2, \ldots, x+9\}$ for $x \in \mathbb{N}$. Then $U_x \subseteq \mathbb{N}$ and $\#(\mathbb{N}\backslash U_x) = \#\{x, x+1, x+2, \ldots, x+9\} = 10$, so $U_x \in \mathcal{B}$, and

$$X = \bigcup_{x\in X} \{x\} \subseteq \bigcup_{x\in X} U_x \subseteq \bigcup_{U\in\mathcal{B}} U \subseteq X.$$

Hence $\bigcup_{U\in\mathcal{B}} U = \bigcup_{x\in X} U_x = X$.
(b) Let $U, V \in \mathcal{B}$ and let $x \in U \cap V$ (if $U \cap V = \varnothing$, the requirement is satisfied trivially). We have

$$\#\mathbb{N}\backslash(U \cap V) \overset{\text{De Morgan}}{=} \#\mathbb{N}\backslash U \cup \mathbb{N}\backslash V \geq \#\mathbb{N}\backslash U \geq 10$$

and

$$\#\mathbb{N}\backslash(U \cap V) \overset{\text{De Morgan}}{=} \#\mathbb{N}\backslash U \cup \mathbb{N}\backslash V \overset{\text{PIE}}{\leq} \#\mathbb{N}\backslash U + \#\mathbb{N}\backslash V < \infty,$$

where PIE stands for the principle of inclusion-exclusion (sætning 469, dismat). Thus $U \cap V \in \mathcal{B}$ and the condition is satisfies with $W = U \cap V$ since $x \in U \cap V \subseteq U \cap V$.
Now the claim follows by theorem 3.6.

(ii) We claim $\mathcal{B}$ is not a basis for a topology on $\mathbb{N}$.
Let $U = \{x \in \mathbb{N} \mid x \geq 11\}, V = \{x \in \mathbb{N} \mid x \geq 21 \vee x \leq 10\}$. Then

$$\#\mathbb{N}\backslash U = \#\{1, 2, \ldots, 10\} = 10, \qquad \#\mathbb{N}\backslash V = \#\{11, 12, \ldots, 20\} = 10.$$

Thus $U, V \in \mathcal{B}$. Now,

$$U \cap V = \{x \in \mathbb{N} \mid x \geq 11 \wedge (x \geq 21 \vee x \leq 10)\} = \{x \in \mathbb{N} \mid x \geq 21\} ..$$

Then $21 \in U \cap V$, and assume $\exists W \in \mathcal{B}$ such that $21 \in W \subseteq U \cap V = \{x \in \mathbb{N} \mid x \geq 21\}$. Then $W \cap (U \cap V)^c = \varnothing$, so

$$\{1, 2, \ldots, 20\} \subseteq \mathbb{N}\backslash W.$$

Hence $20 = \#\{1, \ldots, 20\} \leq \#\mathbb{N}\backslash W$, so $W \notin \mathcal{B}$ in contradiction with the assumption.
Now the claim follows by theorem 3.6.

(iii) We claim $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^2 = X$.
Any element in $\mathcal{B}$ is by definition a subset of X. Now, we claim $\bigcup_{r\in\mathbb{N}} B_r(0) = \mathbb{R}^2$: since $B_r(0) \subseteq \mathbb{R}^2$, $\bigcup_{r\in\mathbb{N}} B_r(0) \subseteq \mathbb{R}^2$ trivially.
Now assume $x \in \mathbb{R}^2$. Then $|x| \in \mathbb{R}$, so there exists $N \in \mathbb{N}$ such that $|x| < N$ (take e.g. $N = \lceil |x| \rceil + 1$). Then $x \in B_N(0)$ by definition and thus $x \in \bigcup_{r\in\mathbb{N}} B_r(0)$. Hence $\mathbb{R}^2 \subseteq \bigcup_{r\in\mathbb{N}} B_r(0)$. Thus the claim follows.
Now we have

$$\mathbb{R}^2 = \bigcup_{r\in\mathbb{N}} B_r(0) \subseteq \bigcup_{U\in\mathcal{B}} U \subseteq \mathbb{R}^2.$$

Thus $\bigcup_{U\in\mathcal{B}} U = \mathbb{R}^2$, so (a) in theorem 3.6 is satisfied.

Now let $B_r(0), B_s(0) \in \mathcal{B}$. We claim $B_r(0) \cap B_s(0) = B_{\min\{r,s\}}(0)$:
Let $x \in B_r(0) \cap B_s(0)$, then $x \in B_r(0)$ and $x \in B_s(0)$ so by definition $|x| < r$ and $|x| < s$, hence, since $\min\{r, s\} \in \{r, s\}$, $|x| < \min\{r, s\}$, so $x \in B_{\min\{r,s\}}(0)$.
Conversely, if $x \in B_{\min\{r,s\}}(0)$ then $|x| < \min\{r, s\} \leq r, s$, so by transitivity (or simply equality if $r = s$

), $|x| < r$ and $|x| < s$, hence $x \in B_r(0)$ and $x \in B_s(0)$ so $x \in B_r(0) \cap B_s(0)$.
Now condition (b) in theorem 3.6 is satisfied by choosing $W = B_{\min\{r,s\}}(0)$ (since $r, s > 0$ also $\min\{r, s\} > 0$ so it is in $\mathcal{B}$) since for any $x \in B_r(0) \cap B_s(0), x \in B_{\min\{r,s\}}(0) \subseteq B_r(0) \cap B_s(0)$.
Now the claim follows by theorem 3.6.

Homework 2. Let X be a non-empty set and fix a point $x_0 \in X$. Define

$$\mathcal{B} := \{\{x_0\}\} \cup \{\{x_0, x\} \mid x \in X \backslash \{x_0\}\}.$$

(i) Prove that $\mathcal{B}$ is the basis for some topology on X which we will denote by $\mathcal{T}$.

Solution: We check the conditions in theorem 3.6:
All sets in $\mathcal{B}$ are subsets of X.

Assume first $X\backslash\{x_0\} \neq \varnothing$.
Now, let $U_x = \{x_0, x\}$ for $x \in X\backslash\{x_0\}$, then $x \in U_x \in \mathcal{B}$, so

$$X = \bigcup_{x\in X} \{x\} \subseteq \bigcup_{x\in X\backslash\{x_0\}} U_x \subseteq \bigcup_{U\in\mathcal{B}} U \subseteq X$$

where the first inclusion follows by the above and because $x_0 \in U_x$ for any $x \in X\backslash\{x_0\}$. Thus $\bigcup_{U\in\mathcal{B}} U = X$, so (a) is satisfied.
Let $U, V \in \mathcal{B}$. If either U or V are $\{x_0\}$, then $U\cap V = \{x_0\}$. If $U = \{x_0, x\}, V = \{x_0, y\}$ for $x, y \in X\backslash\{x_0\}$ with $x \neq y$ then also $U \cap V = \{x_0\}$. In these cases, setting $W = \{x_0\}$, $W \in \mathcal{B}$ and satisfies the condition (b) of theorem 3.6: $x_0 \in W \subseteq U \cap V$.
If $x = y$ above, then $U \cap V = U \in \mathcal{B}$, hence for all $x \in U \cap V = U$, $x \in U \subseteq U \cap V$, so $W = U$ satisfies (b).
This exhausts all cases, hence (b) is checked and it follows that $\mathcal{B}$ is a basis for some topology on X by theorem 3.6.

If $X\backslash\{x_0\} = \varnothing$, then $\mathcal{B} = \{X\}$ which clearly is a basis for the trivial topology on X since $\mathcal{B} \subseteq \{\varnothing, X\}$ and for any $x \in X$ and every neighborhood U of x, U must be X and thus the condition in definition 3.1 is satisfied with $U' = X \in \mathcal{B}$ since $x \in U' \subseteq U$.

(ii) Find all subsets of X which are simultaneously open and closed with respect to $\mathcal{T}$.

Solution: If $A \subseteq X$ is simultaneously open and closed with respect to $\mathcal{T}$, then $A \in \mathcal{T}$ and $A^c = X\backslash A \in \mathcal{T}$.
Since $X\backslash\varnothing = X \in \mathcal{T}, X\backslash X = \varnothing \in \mathcal{T}$, we have that $X, \varnothing$ are simultaneously open and closed.

Assume now A is neither X nor $\varnothing$ - in particular we are dealing with the case $X\backslash\{x_0\} \neq \varnothing$ from (i).
Thus by proposition 3.2 and theorem 3.6.(a), $A = \bigcup_{U\in S} U, A^c = \bigcup_{U\in T} U$ where $S, T \subsetneq \mathcal{B}$ and $S, T \neq \varnothing$.
Then

$$\varnothing = A \cap A^c = \bigcup_{U\in S} U \cap \bigcup_{V\in T} V \overset{(7.52) \text{ dismat}}{=} \bigcup_{U\in S, V\in T} \underbrace{U \cap V}_{\text{contains } x_0}.$$

The last equality follows by using the distribute law for sets ((7.52) in the dismat book twice).
We saw in (i) that any intersection of basis elements contains x_0, hence the union above contains x_0 and hence cannot equal the empty set. Thus, no such A can exist. So the only subsets of X that are simultaneously open and closed with respect to $\mathcal{T}$ are X and $\varnothing$.

(iii) Let $Y = X\backslash\{x_0\}$. Prove that the subspace topology on Y obtained from $(X, \mathcal{T})$ equals the discrete topology on Y.

Solution: We note that any topology on Y is coarser than the discrete topology on Y (Remark 2.11), hence we must only show that the discrete topology on Y is coarser than the subspace topology on Y obtained from $(X, \mathcal{T})$.
Let A be any open set in the discrete topology on Y; let $y \in A$. Since $A \subseteq X\backslash\{x_0\}$, $A \cap \{x_0\} = \varnothing$, so $y \in \{x_0, y\} \cap Y \subseteq A$, and since $\{x_0, y\}$ is a basis element in X, it is open in the topology generated by

the basis by definition 3.7, hence $\{x_0, y\} \cap Y$ is open in the subspace topology on Y by definition 4.1. Letting $A_y = \{x_0, y\} \cap Y$, we thus get

$$A = \bigcup_{y \in A} \{y\} \subseteq \bigcup_{y \in A} A_y \subseteq A.$$

Hence $A = \bigcup_{y \in A} A_y$ which is open as it is the union of open sets. So A is open in the subspace topology.

Homework 3. Let $(X, <)$ be a totally ordered set. Assume that X has a maximal element with respect to $<$ which we call x_{max}. Let

$$\mathcal{S} = \{(x, x_{max}] \mid x \in X \backslash \{x_{max}\}\}.$$

Under what condition does $\mathcal{S}$ define a subbasis for some topology on X? In the case that it defines a subbasis, compare, if possible, the topology generated by $\mathcal{S}$ with the order topology on X.

Solution: By lemma 3.11, $\mathcal{S}$ is a subbasis for some topology on X if it satisfies (a) of definition 3.10.
Assume X has no minimal element. Then let $x \in X$. We then have $x \in (y, x_{max}] \in \mathcal{S}$ where $y \in X$ with $y < x$ (such an element exists since otherwise x would be a minimal element). Since $x \in X$ was arbitrary, we can for any $x \in X$ find $S_x \in \mathcal{S}$ such that $x \in S_x$. Hence

$$X = \bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} S_x \subseteq \bigcup_{S \in \mathcal{S}} S \subseteq X.$$

So (a) of definition 3.10 is satisfied.

Assume that X has a minimal element, call it x_{min}. Then $x_{min} \in X$ but there does not exist $x \in X \backslash \{x_{max}\}$ such that $x_{min} \in (x, x_{max}]$ since if such an x existed, $x < x_{min}$ by definition of intervals, and this contradicts the minimality of x_{min}. Thus $x \notin \bigcup_{S \in \mathcal{S}} S$, so $\bigcup_{S \in \mathcal{S}} S \neq X$ and thus (a) of definition 3.10 is not satisfied; so $\mathcal{S}$ is not a subbasis for some topology $\mathcal{T}$ on X.
Hence $\mathcal{S}$ defines a subbasis for some topology on X if and only if X does not have a minimal element.

Assume now that $\mathcal{S}$ does define a subbasis for some topology on X.
Then X has no minimal element, so the order topology on X is generated by the basis containing all intervals of the form (x, y) with $x, y \in X, x < y$ and all intervals of the for $(x, x_{max}], x \in X \backslash \{x_{max}\}$ (basis by proposition 4.13). Hence $\mathcal{S}$ is contained in this basis, and thus $\mathcal{S}$ is contained in the order topology on X. Hence, since topologies are closed under finite intersections, the basis generated by the subbasis $\mathcal{S}$ is also contained in the order topology, and again, since topologies are closed under finite intersections and arbitrary unions, the topology generated by $\mathcal{S}$ (the topology generated by the basis generated by $\mathcal{S}$) is contained in the order topology.

Thus, the order topology on X is finer than the topology generated by $\mathcal{S}$ on X. We claim it is strictly finer: since X has no minimal element and is totally ordered, we can choose $x \in X$ such that $x < x_{max}$. Since x is not minimal, we can choose $y \in X$ such that $y < x < x_{max}$. Again, since y is not minimal, we can choose $z \in X$ with $z < y < x < x_{max}$. Now $y \in (z, x)$ which is open in the order topology.

We claim that $\mathcal{S}$ is, in fact, even a basis for the topology it generates. For this it suffices to show that the basis it generates is equal to $\mathcal{S}$. Let $(x_1, x_{max}], \ldots, (x_n, x_{max}]$ be a finite collection of elements from $\mathcal{S}$, then since X is totally ordered, we can order $x_1, \ldots, x_n$, and assume without loss of generality that $x_1 \leq x_2 \leq \ldots \leq x_n$. Then

$$\bigcap_{i=1}^{n} (x_i, x_{max}] = (x_n, x_{max}] \in \mathcal{S}.$$

Hence $\mathcal{S}$ is closed under finite intersections and is thus equal to the basis it generates.
Thus if the topologies were equal, there would exist a basis element $(a, x_{max}]$ of the topology generated by $\mathcal{S}$ such that $y \in (a, x_{max}] \subseteq (z, x)$ (by definition 3.1); however then $x_{max} < x < x_{max}$ which is a contradiction. Hence the order topology is strictly finer than the topology generated by $\mathcal{S}$.