A common problem involves extending a map $f\colon X \to Z$ to a larger space Y ; the picture is

$$\begin{tikzcd} Y \arrow[dr, dashed, "g"] & \\ X \arrow[u, hook] \arrow[r, "f"] & Z \end{tikzcd}$$

Homotopy itself raises such a problem: if $f_0, f_1\colon X \to Z$, then $f_0 \sim f_1$ if we can extend $f_0 \cup f_1\colon X \times \{0\} \cup X \times \{1\} \to Z$ to all of $X \times I$.[1]

Theorem 1.6. Let $f\colon S^n \to Y$ be a continuous map into some space Y. The following conditions are equivalent:

1. f is nullhomotopic;
2. f can be extended to a continuous map $D^{n+1} \to Y$;
3. if $x_0 \in S^n$ and $k\colon S^n \to Y$ is the constant map at $f(x_0)$, then there is a homotopy $F\colon f \simeq k$ with $F(x_0, t) = f(x_0)$ for all $t \in I$.

Proof: ($2 \implies 1$): Let $\tilde{f}\colon D^{n+1} \to Y$ such that $\tilde{f} = f$ on S^n. Define $F\colon S^n \times I \to Y$ by $F(x,t) = \tilde{f}(tx)$. Then $F(x,0) = \tilde{f}(0)$ and $F(x,1) = \tilde{f}(x) = f(x)$.

($3 \implies 1$) is clear.

($1 \implies 3$) : Let $G\colon f \simeq c$ where c is constant. Thus $G(x_0,t)$ is a path from $f(x_0)$ to c. Let $F(x,t) = \begin{cases} G(x,2t) & t \in \left[0, \frac{1}{2}\right] \\ G(x_0, 2-2t) & t \in \left[\frac{1}{2}, 1\right] \end{cases}$ Then F is a homotopy between f and k.

1.3. Let $R\colon S^1 \to S^1$ be rotation by α radians. Prove that $R \simeq \mathbb{1}_S$ where $\mathbb{1}_S$ is the identity map of S^1. Conclude that every continuous map $f\colon S^1 \to S^1$ is homotopic to a continuous map $g\colon S^1 \to S^1$ with $g(1) = 1$.

Solution: $F\colon R \simeq \mathbb{1}_S$ by $F(x,t) = \frac{\mathbb{1}_S(x)t + (1-t)R(x)}{|\mathbb{1}_S(x)t + (1-t)R(x)|}$. Then $F(x,0) = \frac{R(x)}{|R(x)|} = R(x)$ and $F(x,1) = \mathbb{1}_S(x) = x$.
Thus for any map $f\colon S^1 \to S^1$, suppose $f(1) = e^{i\theta}$. Let R be the rotation of $-\theta$ radians. Then $f \simeq \mathbb{1}_S \circ f \simeq R \circ f$ and $R \circ f(1) = 1$ by definition of R.

1.5. Let $X = \{0\} \cup \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ and let Y be a countable discrete space. Show that X and Y do not have the same homotopy type.

Suppose $F\colon X \to Y$ and $G\colon Y \to X$ such that $FG \simeq \mathbb{1}_Y$ and $GF \simeq \mathbb{1}_X$. Then take a covering of X, $\mathcal{B}$, and since X is compact, let it be a finite covering. The $G^{-1}(\mathcal{B})$ is a finite covering of Y, but Y is a countable discrete space and thus not compact.

Example. Consider the space I and let A be the two-point subset $A = \{0,1\}$. Then show that $I/A \cong S^1$.

Solution: Instead of using more powerful techniques, let's just do it with basics. Let $\pi\colon I \to I/A$ be the quotient map. Define $f\colon I/A \to S^1$ by $f(x) = \begin{cases} e^{2\pi i x}, & x \in (0,1) \\ 1, & x = \{0,1\} \end{cases}$.
Let $g(x) = e^{2\pi i x}$. Then $g = f \circ \pi$. Now, suppose $U \subset S^1$ is open. Then since g is continuous, $g^{-1}(U)$ is open. That is, $(f \circ \pi)^{-1}(U) = \pi^{-1} \circ f^{-1}(U)$ is open. But π is a quotient map, so since $\pi_1^{-1}\left(f^{-1}(U)\right)$ is open, $f^{-1}(U)$ is open in I/A by definition of π. Hence f is continuous.

Now, note that g is a closed map. So if we define h to be the inverse of f, then for a closed set $U \subset I/A$, we have $h^{-1}(U) = g\left(\pi^{-1}(U)\right)$ which is closed. Hence h is continuous. So f is a homeomorphism.

[1] This expression can seem a bit weird, but remember that functions are sets.

Theorem 1.8 Let $f\colon X \to Y$ be a continuous surjection. Then f is an identification map (quotient map) if and only if $\forall Z, \forall g\colon Y \to Z$, we have that

$$\begin{tikzcd} X \arrow[rr, "gf"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

commutes.

Cor. 1.9 Let $f\colon X \to Y$ be an identification map. For some space Z, let $h\colon X \to Z$ be a continuous function that is constant on each fiber of f. Then $hf^{-1}\colon Y \to Z$ is continuous.

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "hf^{-1}"'] & \end{tikzcd}$$

Moreover, hf^{-1} is an open map (or a closed map) if and only if $h(U)$ is open (or closed) in Z whenever U is an open (or closed) set in X of the form $U = f^{-1}f(U)$.

Cor 1.10 Let X and Z be spaces and $h\colon X \to Z$ an identification map. Then $\varphi\colon X/\ker h \to Z$ defined by $[x] \mapsto h(x)$ is a homeomorphism.

1.10. Let $f\colon X \to Y$ be an identification, and let $g\colon Y \to Z$ be a continuous surjection. Then g is an identification if and only if gf is an identification.

Proof:

$$\begin{tikzcd} X \arrow[rr, "gf"] \arrow[dr, "f"] & & Z \\ & Y \arrow[ur, "g"] & \end{tikzcd}$$

If g is an identification, then if $(gf)^{-1}(U)$ is open, we have $f^{-1}\left(g^{-1}(U)\right)$ is open and since both f and g are identification, this means U is open. SO gf is an identification.

Conversely, if gf is an identification, then if $g^{-1}(U)$ is open, we have $f^{-1}(g^{-1}(U))$ is open, so $(gf)^{-1}(U)$ is open, and hence U is open.

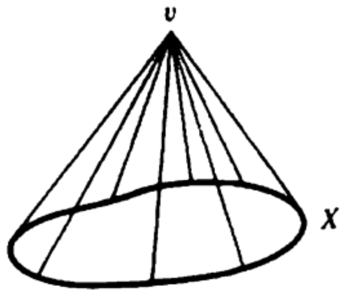
1.11 Let X and Y be spaces with equivalence relations $\sim$ and Δ respectively, and let $f\colon X \to Y$ be a continuous map preserving the relations (if $x \sim x'$, then $f(x)\Delta f(x')$) Prove that the induces map $\overline{f}\colon X/\sim \to Y/\Delta$ is continuous; moreover, if f is an identification, then so is $\overline{f}$.

Proof: Let $\pi\colon X \to X/\sim$ and $v\colon Y \to Y/\Delta$ be the identifications.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \arrow[d, "v"] \\ X/\sim \arrow[r, "\overline{f}"'] & Y/\Delta \end{tikzcd}$$

Suppose $\pi(x) = \pi(y)$. Then $x \sim y$, so $f(x)\Delta f(y)$ and hence $v(f(x)) = v(f(y))$, so $v \circ f$ is constant on each fiber of π. Corollary 1.9 gives that the map $(v \circ f) \circ \pi^{-1}$ is continuous. Now, since $\overline{f}([x]) = v(f(x)) = (v \circ f) \circ \pi^{-1}([x])$, we have that $\overline{f}$ is continuous.
If f is an identification, then it is surjective and then we get that $\overline{f}$ is an identification by 1.10 since compositions of identifications are identifications.

Def. If X is a space, define an equivalence relation on $X \times I$ by $(x,t) \sim (x',t')$ if $t = t' = 1$. Denote the equivalence class of (x,t) by $[x,t]$. The **cone** over X, denoted by CX, si the quotient space $X \times I/\sim$. We can usually view this as the following figure



This is fine when X is compact Hausdorff, but it may be misleading otherwise: the quotient topology may have more open sets than expected.
For example, consider X to be the set of positive integers regarded as points on the x-axis in $\mathbb{R}^2$. Let $C'X$ denote the subspace of $\mathbb{R}^2$ obtained by joining each $(n,0) \in X$ to $v = (0,1)$ with a line segment. There is a continuous bijection $CX \to C'X$, but CX is not homeomorphic to $C'X$.

Any homeomorphism would have to map the vertex to v. Now, suppose we take the open set

$$U = \bigcup_{i \in \mathbb{N}} \{i\} \times \left(1 - \frac{1}{i}, 1\right]$$

This is saturated and thus its image in the quotient topology is open, however, it is not open in $C'X$, for there is no open neighborhood around v contained in $\varphi(\pi(U))$.

Example 1.7 For spaces X and Y, every map $f\colon X \times I \to Y$ with $f(x,1) = y_0$, say, for all $x \in X$, induces a continuous map $\overline{f}\colon CX \to Y$, namely, $\overline{f}\colon [x,t] \mapsto f(x,t)$. In particular, let $f\colon S^n \times I \to D^{n+1}$ be the map $(u,t) \mapsto (1-t)u$; since $f(u,1) = 0$ for all $u \in S^n$, there is a map $\overline{f}\colon CS^n \to D^{n+1}$ with $[u,t] \mapsto (1-t)u$.
Since S^n is compact, $S^n \times I$ is compact and hence the quotient CS^n is compact. Since $\overline{f}$ is bijective, $\overline{f}$ is a homeomorphism as D^{n+1} is Hausdorff.
Thus D^{n+1} is the cone over S^n with vertex 0.

1.13. For fixed t with $0 \leqslant t < 1$, prove that $x \mapsto [x,t]$ defines a homeomorphism from a space X to a subspace of CX.

Proof: The map $x \mapsto [x,t]$, call it $f\colon X \to X_t$, with inverse $[x,t] \mapsto x$ is clearly bijective for $t \in [0,1)$ - here X_t is the subspace of CX defined as the image of f.

Since f is the restriction of a continuous map, it is continuous.
Now, suppose we take an open set $U \subset X$. Then $f(U) = \pi\left(U \times \{t\}\right)$. Since $f(U) = \pi\left(U \times (t-\varepsilon, t+\varepsilon)\right) \cap X_t$, $f(U)$ is open in CX, so it is a homeomorphism.

This, together with the obvious fact that for any space X, CX is contractible, shows that any space can be embedded in a contractible space.

Theorem 1.12 A space X has the same homotopy type as a point if and only if X is contractible.

Proof: Suppose $X \simeq p$ where p is a point. Then $\exists F\colon X \to p$ and $G\colon p \to X$ such that $FG \simeq \mathbb{1}_p$ and $GF \simeq \mathbb{1}_X$. But GF is a constant map, so X is contractible.

Conversely, if X is contractible, there exists some constant map $c\colon X \to X$ such that $\mathbb{1}_X \simeq c$. Let c denote the point in the image of c also. Define $F\colon X \to c$ as the map c restricting its image, and $G\colon p \to X$ as the inclusion. Then $FG\colon c \to c$ is the constant map, so $FG = \mathbb{1}_c$ and $GF\colon X \to X$ is the map which maps everything to c, so $\mathbb{1}_X \simeq c \simeq GF$, giving $\mathbb{1}_X \simeq GF$ by transitivity. Hence X has the homotopy type of a point.

Theorem 1.13 If Y is contractible, then any two maps $X \to Y$ are homotopic.

1.31. Let $a = (0, \dots, 0, 1)$ and $b = (0, \dots, 0, -1)$ be the north and south poles, respectively, of S^n. Show that the equator S^{n-1} is a deformation retract of $S^n - \{a, b\}$, hence S^{n-1} and $S^n - \{a, b\}$ have the same homotopy type.

Solution: Define the deformation retract

$$F(x,t) = \frac{x}{p(x)\|\frac{x}{p(x)}\|}t + (1-t)x$$

where p is the projection $S^n \to \mathbb{R}^n \subset \mathbb{R}^{n+1}$.

Def. Let $f\colon X \to Y$ be continuous and define

$$M_f = ((X \times I) \sqcup Y) / \sim$$

where $(x,t) \sim y$ if $y = f(x)$ and $t = 1$. Denote the class of (x,t) in M_f by $[x,t]$ and the class of y in M_f by $[y]$ (so that $[x,1] = [f(x)]$). The space M_f is called the **mapping cylinder** of f.

Def. Let $\{X_i \colon i \in I\}$ be a family of topological spaces indexed by I. Let

$$X = \sqcup_i X_i$$

be the disjoint union of the underlying sets. For each $i \in I$, let $\varphi_i \colon X_i \to X$ be the **canonical inection** (defined by $\varphi_i(x) = (x,i)$). The **disjoint union topology** on X is defined as the finest topology on X for which all the canonical injections φ_i are continuous.

Explicitly, the disjoint union topology can be described as follows: a subset $U \subset X$ is open in X if and only if $\varphi_i^{-1}(U)$ is open in X_i for each $i \in I$. Yet another formulation is that a subset $V \subset X$ is open relative to X if and only if its intersection with X_i is open relative to X_i for each i.

1.33 If Y is a one-point space, then $f\colon X \to Y$ must be constant. Prove that the mapping cylidner in this case i CX.

Proof: The mapping cylinder is

$$M_f = ((X \times I) \sqcup Y) / \sim = \{(x,t) \mid t \in [0,1]\} \cup \{y \mid y \in Y\} / \{(x,t) \sim y\} = \{(x,t) \mid t \in [0,1)\} \cup \{[y]\} = CX$$

1.34. (i) Define $i\colon X \to M_f$ by $i(x) = [x,0]$ and $j\colon Y \to M_f$ by $j(y)\,[y]$. Show that i and j are homeomorphisms to subspaces of M_f.

Solution: Let $\pi\colon ((X \times I) \sqcup Y) \to M_f$ be the quotient map. Then since i is simply $\pi \circ \iota_X$ where ι_X is the inclusion for $t = 0$ and $j = \pi \circ \iota_Y$ where ι_Y is the inclusion of Y, we have that i and j are continuous as the compositions of continuous maps. Since i and j are injective, they have inverses, call them $\bar{i}$ and $\bar{j}$. Now, let $j(Y)$ and $i(X)$ denote the images as subspaces of M_f. For any open set $U \subset j(Y)$ or of $i(X)$, we have that $j^{-1}(U) = \varphi_1^{-1}(U)$ where φ is the canonical injection, and $i^{-1}(U) = \varphi_0^{-1}(U)$.

(iii) Prove that Y is a deformation retract of M_f.

Proof: Let $F\colon M_f \times I \to M_f$ be defined by

$$\begin{aligned} F([x,t], s) &= [x, t(1-s)+s] \quad \text{if } x \in X, t, s, \in I \\ F([y], s) &= [y], \quad y \in Y, s \in I \end{aligned}$$

Then F is a deformation retract of M_f onto Y. It is continuous since for any $U \subset M_f$, we have $\pi_{M_f}^{-1}(U) \cap Y$ and $\pi_{M_f}^{-1}(U) \cap X \times I$ is open, and $F^{-1}(U) = F^{-1}(U \cap Y) \cup F^{-1}(U \cap X)$.

Simplexes

Def. A subset A of euclidean space is called **affine** if, for every pair of distinct points $x, x' \in A$, the line determined by x, x' is contained in A.

Note that, by default, $\varnothing$ and one-point subsets are affine.

Theorem. If $\{X_j : j \in J\}$ is a family of convex (or affine) subsets of $\mathbb{R}^n$, then $\bigcap X_j$ is also convex (or affine).

It thus makes sense to speak of the convex (or affine) set in $\mathbb{R}^n$ spanned by a subset X of $\mathbb{R}^n$ (called the **convex hull** of X), namely, the intersection of all convex (or affine) subsets of $\mathbb{R}^n$ containing X.
We denote the convex set spanned by X by $[X]$ (it exists since $\mathbb{R}^n$ itself is affine, hence convex).

These convex subsets of $\mathbb{R}^n$ are complicated and there are too many to easily describe: e.g., for every subset K of S^1, the set $D^2 - K$ is convex.
However, we can describe $[X]$ for finite X.

Def. An **affine combination** of points $p_0, p_1, \ldots, p_m$ in $\mathbb{R}^n$ is a point x with

$$x = t_0 p_0 + t_1 p_1 + \ldots + t_m p_m,$$

where $\sum_{i=0}^m t_i = 1$. A **convex combination** is an affine combination for which $t_i \geqslant 0$ for all i.

For example, a convex combination of x, x' has the form $tx + (1-t)x'$ for $t \in I$.

Theorem 2.2. If $p_0, p_1, \ldots, p_m \in \mathbb{R}^n$, then $[p_0, p_1, \ldots, p_m]$, the convex set spanned by these points, is the set of all convex combinations of $p_0, p_1, \ldots, p_m$.

Proof: Let S denote the set of all convex combinations of $p_0, \ldots, p_m$.

$$S = \left\{ \sum_{i=0}^m t_i p_i \mid \sum t_i = 1, t_i \geqslant 0 \forall i \right\}$$

We wish to show that S is a subset of any convex set containing $p_0, \ldots, p_m$.

Suppose A is a convex set containing $p_0, \ldots, p_m$. Then we proceed by induction, showing that $S \subset A$.
Firstly if $x \in S$ and if $t_i = 1$ then $x = p_i$ and $p_i \in A$.
Now, suppose that for any subset $B \subset \{p_0, \ldots, p_m\}$ such that $|B| \leqslant m$, we have that the convex combinations of B are in A. Now suppose $x = \sum_{i=0}^m t_i p_i$. Then if $t_0 = 0$, we have x is covered by the cases of B, so suppose $t_0 \neq 0$, then $\sum_{i=1}^m \frac{t_i}{1-t_0} p_i$ is covered by B, so

$$tp_0 + (1-t) \sum_{i=1}^m \frac{t_i}{1-t_0} p_i \in A$$

since A is convex, and $p_0, \sum_{i=1}^m \frac{t_i}{1-t_0} p_i \in A$.
Hence $S \subset A$, so as A was arbitrary, $S \subset [p_0, \ldots, p_m]$.

Conversely, we can show that $[p_0, \ldots, p_m] \subset S$ if we can show that S is a convex set.
Clearly, $\{p_0, \ldots, p_m\} \subset S$.
Now, suppose $\sum_{i=0}^m t_i p_i, \sum_{i=0}^m s_i p_i \in S$.
We claim

$$t \sum_{i=0}^m t_i p_i + (1-t) \sum_{i=0}^m s_i p_i \in S, \quad \forall t \in I.$$

This is clear since $t \sum_{i=0}^m t_i + (1-t) \sum_{i=0}^m s_i = t + 1 - t = 1$, and $tt_i, (1-t)s_i \geqslant 0$ for all i and $t \in I$ as $t \geqslant 0$.

Cor. 2.3. The affine set spanned by $\{p_0, p_1, \ldots, p_m\} \subset \mathbb{R}^n$ consists of all affine combinations of these points.

Def. An ordered set of points $\{p_0, p_1, \ldots, p_m\} \subset \mathbb{R}^n$ is **affine independent** if $\{p_1 - p_0, p_2 - p_0, \ldots, p_m - p_0\}$ is a linearly independent subset of the real vector space $\mathbb{R}^n$.

Theorem 2.4 The following conditions on an ordered set of points $\{p_0, \ldots, p_m\}$ in $\mathbb{R}^n$ are equivalent.

1. $\{p_0, \ldots, p_m\}$ is affine independent;
2. if $\{s_0, s_1, \ldots, s_m\} \subset \mathbb{R}$ satisfies $\sum_{i=0}^m s_i p_i = 0$ and $\sum_{i=0}^m s_i = 0$, then $s_0 = s_1 = \ldots = s_m = 0$
3. each $x \in A$, the affine set spanned by $\{p_0, \ldots, p_m\}$ has a unique expression as an affine combination:

$$x = \sum_{i=0}^m t_i p_i \quad \text{and} \quad \sum_{i=0}^m t_i = 1.$$

Cor. 2.5. Affine independence is a property of the set $\{p_0, \ldots, p_m\}$ that is independent of the given ordering.

Cor. 2.6. If A is the affine set in $\mathbb{R}^n$ spanned by an affine independent set $\{p_0, \ldots, p_m\}$, then A is a translate of an m-dimensional sub-vector-space V of $\mathbb{R}^n$, namely,

$$A = V + x_0$$

for some $x_0 \in \mathbb{R}^n$.

Def. A set of points $\{a_1, a_2, \ldots, a_k\}$ in $\mathbb{R}^n$ is in **general position** if every $n+1$ of its points forms an affine independent set.

Note. The property of being in general position depends on n. Thus, assume that $\{a_1, a_2, \ldots, _k\} \subset \mathbb{R}^n$ is in general position. If $n = 1$, we are saying that every pair $\{a_i, a_j\}$ is affine independent; that is, all the points are distinct. If $n = 2$, we are saying that no three points are collinear, and if $n = 3$, that no four points are coplanar.

Let $r_0, r_1, \ldots, r_m$ be real numbers. Recall that the $(m+1) \times (m+1)$ *Vandermonde matrix* V has its i th column $\left[1, r_i, r_i^2, \ldots, r_i^m\right]$; moreover, $\det V = \Pi_{j<i}(r_i - r_j)$, hence V is nonsingular if all the r_i are distinct. If one subtracts column 0 from each of the other columns of V, then the i th column (for $i > 0$) of the new matrix is

$$\left[0, r_i - r_0, r_i^2 - r_0^2, \ldots, r_i^m - r_0^m\right]$$

If V^* is the southeast $m \times m$ block of this new matrix, then $\det V^* = \det V$ (consider Laplace expansion across the first row).
We have

$$V^* = \begin{pmatrix} r_1 - r_0 & r_2 - r_0 & \ldots & r_m - r_0 \\ r_1^2 - r_0^2 & r_2^2 - r_0^2 & \ldots & r_m^2 - r_0^2 \\ \vdots & \vdots & \ldots & \vdots \\ r_1^m - r_0^m & r_2^m - r_0^m & \ldots & r_m^m - r_0^m \end{pmatrix}$$

Then

$$\det V^* = \sum_{i=1}^m (r_i - r_0)(-1)^{i+1} \det V^*[1, i]$$

$$V^*[1,i] = \begin{pmatrix} r_1^2 - r_0^2 & \ldots & r_{i-1}^2 - r_0^2 & r_{i+1}^2 - r_0^2 & \ldots & r_m^2 - r_0^2 & \\ \vdots & \vdots & \ldots & \ldots & \vdots & \vdots & \\ r_1^m - r_0^m & r_2^m - r_0^m & \ldots & r_{i-1}^m - r_0^m & r_{i+1}^m - r_0^m & \ldots & r_m^m - r_0^m \end{pmatrix}$$

Theorem 2.7. For every $k \geqslant 0$, euclidean space $\mathbb{R}^n$ contains k points in general position.

Proof: We may assume $k > n + 1$.
We thus want to find k points $p_0, p_1, \ldots, p_{k-1}$ such that for any $n+1$-subset, it forms an affine independent

set.
Suppose $q_0, \ldots, q_n$ is the $n+1$-subset. For these to be affine independent would mean that

$$\begin{pmatrix} q_{11} - q_{01} & \cdots & q_{n1} - q_{01} \\ q_{12} - q_{02} & \cdots & q_{n2} - q_{02} \\ \vdots & \cdots & \vdots \\ q_{1n} - q_{0n} & \cdots & q_{nn} - q_{0n} \end{pmatrix}$$

has nonzero determinant. Choosing $q_{ik} = r_i^k$ for some distinct $r_0, \ldots, r_n$, we get that the determinant is nonzero by the above discussion. Hence these $q_0, \ldots, q_n$ with

$$q_i = [q_{i1}, q_{i2}, \ldots, q_{in}]$$

form an affine independent subset.
So to form $p_0, p_1, \ldots, p_{k-1}$, we simply create p_i as we created the q 's above with different r_i. I.e., choose $r_0, \ldots, r_{k-1}$ and let

$$p_i = \left[r_i, r_i^2, \ldots, r_i^n\right]$$

2.1 Every affine subset $A \subset \mathbb{R}^n$ is spanned by a finite subset. Conclude that every nonempty affine subset of $\mathbb{R}^n$ is as described in Corollary 2.6.

Solution: Suppose $A \subset \mathbb{R}^n$ is affine. Any affine independent subset of A can consist of at most $n+1$ elements. Now, there exists a maximal N such that there exist $p_0, \ldots, p_N$ such that these vectors form an affine independent set and each $p_i \in A$. Now, since any affine combination of these is in A, we have that the span of these is in A.
Suppose that A is not equal to this span. Then there exists $a \in A$ such that $a \notin \operatorname{span}(p_0, \ldots, p_N)$. But then we claim $p_0, \ldots, p_N, a$ is affine independent. If $\sum_{i=0}^N c_i(p_i - a) = 0$, then $\sum_{i=0}^N c_i p_i = a \cdot \sum_{i=0}^N c_i$, so we must have $\sum_{i=0}^N c_i$ as otherwise $a \in \operatorname{span}(p_0, \ldots, p_N)$. But then $p_0 - a, \ldots, p_N - a$ are linearly independent.

Now, this gives that any affine subset is spanned by a finite subset, but then corollary 2.6 is true for any affine set in $\mathbb{R}^n$. Hence all affine sets in $\mathbb{R}^n$ are translates of m-dimensional sub-vector-spaces of $\mathbb{R}^n$, namely,

$$A = V + x_0$$

for some $x_0 \in \mathbb{R}^n$.

***2.2** Assume $n < k$ and that the vector space $\mathbb{R}^n$ is isomorphic to a subspace of $\mathbb{R}^k$ (not necessarily the subspace of all vectors whose last $k-n$ coordinates are 0). If X is a subset of $\mathbb{R}^n$, then the image under φ of the affine set spanned by X in $\mathbb{R}^n$ is the same as the affine set spanned by $\varphi(X)$ in $\mathbb{R}^k$.

Solution: Let $\varphi \colon \mathbb{R}^n \to \mathbb{R}^k$ be the isomorphism onto the subspace $\operatorname{Im}\varphi$. Let $A_m(X)$ denote the affine set spanned by X in $\mathbb{R}^m$. We wish to show that

$$\varphi\left(A_n(X)\right) = A_k(\varphi(X)).$$

By problem 2.1, there exist vectors $p_1, \ldots, p_m$ that span $A_n(X)$. Then $\varphi(p_1), \ldots, \varphi(p_m)$ span $\varphi\left(A_n(X)\right)$, and if

$$0 = \sum_{i=2}^m c_i \left(\varphi(p_i) - \varphi(p_1)\right)$$

then

$$0 = \varphi\left(\sum_{i=2}^m c_i \left(p_i - p_1\right)\right)$$

so since φ is an isomorphism, we find that $\varphi(p_1), \ldots, \varphi(p_m)$ are affine independent. So $\varphi\left(A_n(X)\right)$ is an affine set.
If we can show that $\varphi(X) \subset \varphi\left(A_n(X)\right)$, then we are done.

Let $\varphi(x) \in \varphi(X)$ with $x \in X$. Then since $x \in X \subset A_n(X)$, we have $\varphi(x) \in \varphi\left(A_n(X)\right)$ too.

2.3. Show that S^n contains an affine independent set with $n+2$ points.

Proof: By theorem 2.7, we can find $n+2$ points in $\mathbb{R}^{n+1}$ such that they are affine independent, say, $p_1, \ldots, p_{n+2}$. Then let $q_i := \frac{p_i}{\|p_i\|}$. We claim $q_1, \ldots, q_{n+2}$ are affine independent.

If not, then

$$0 = \sum_{i=2}^{n+2} c_i(q_i - q_1)$$

but then

$$0 = \sum_{i=2}^{n+2} c_i \Pi_{j \neq 1,i} |p_j| (p_i - p_1)$$

and since $|p_j| \neq 0$ for all j, we have that $c_2 = \ldots = c_{n+2} = 0$.

Def. Let $\{p_0, p_1, \ldots, p_m\}$ be an affine independent subset of $\mathbb{R}^n$, and let A be the affine set spanned by this subset. If $x \in A$, then theorem 2.4 gives a unique $(m+1)$-tuple $(t_0, t_1, \ldots, t_m)$ with $\sum t_i = 1$ and $x = \sum_{i=0}^{m} t_i p_i$. The entries of this $(m+1)$-tuple are called the **barycentric coordinates** of x (relative to the ordered set $\{p_0, p_1, \ldots, p_m\}$).

What does exercise 2.2 tell us then?
It says that the coordinates are invariant under such isomorphisms, so the barycentric coordinates of a point relative to $\{p_0, p_1, \ldots, p_m\} \subset \mathbb{R}^n$ do not depend on the ambient space $\mathbb{R}^n$.

Def. Let $\{p_0, p_1, \ldots, p_m\}$ be an affine independent subset of $\mathbb{R}^n$. The convex set spanned by this set, denoted by $[p_0, p_1, \ldots, p_m]$ is called the (affine) **m-simplex** with **vertices** $p_0, p_1, \ldots, p_m$.

Theorem 2.8 If $\{p_0, p_1, \ldots, p_m\}$ is affine independent, then each x in the m-simplex $[p_0, p_1, \ldots, p_m]$ has a unique expression of the form

$$x = \sum t_i p_i, \quad \text{where } \sum t_i = 1 \text{ and each } t_i \geqslant 0.$$

Def. If $\{p_0, \ldots, p_m\}$ is affine independent, the **barycenter** of $[p_0, \ldots, p_m]$ is $\frac{1}{m+1}(p_0 + p_1 + \ldots + p_m)$.

Example 2.5. Let e_i denote the standard point in $\mathbb{R}^{n+1}$ with a 1 on the i th coordinate and zero elsewhere. Then $\{e_0, e_1, \ldots, e_n\}$ is affine independent, and $[e_0, e_1, \ldots, e_n]$ consists of all convex combinations $x = \sum t_i e_i$. Here barycentric and cartesian coordinates coincide, and $[e_0, e_1, \ldots, e_n] = \Delta^n$, the standard n-simplex.

Def. Let $[p_0, \ldots, p_m]$ be an m-simplex. The **face opposite** p_i is

$$[p_0, \ldots, \hat{p_i}, \ldots, p_m] = \left\{ \sum t_j p_j : t_j \geqslant 0, \sum t_j = 1, \text{ and } t_i = 0 \right\}.$$

The **boundary** of $[p_0, \ldots, p_m]$ is the union of its faces.

Clearly, an m-simplex has $m+1$ faces. For an integer k with $0 \leqslant k \leqslant m-1$, one sometimes speaks of a **k-face** of $[p_0, p_1, \ldots, p_m]$, namely, a k-simplex spanned by $k+1$ of the vertices $\{p_0, \ldots, p_m\}$. In this terminology, the faces defined above are $(m-1)$-faces.

The following theorem will be needed when we discuss barycentric subdivision:
Theorem 2.9. Let S denote the n-simplex $[p_0, \ldots, p_n]$.

1. If $u, v \in S$, then $\|u - v\| \leqslant \sup_i \|u - p_i\|$.
2. $\operatorname{diam} S = \sup_{i,j} \|p_i - p_j\|$.
3. If b is the barycenter of S, then $\|b - p_i\| \leqslant \left(\frac{n}{n+1}\right) \operatorname{diam} S$.

Affine Maps

Def. Let $\{p_0, \ldots, p_m\} \subset \mathbb{R}^n$ be affine independent and let A denote the affine set it spans. An **affine map** $T\colon A \to \mathbb{R}^k$ for $k \geqslant 1$ is a function satisfying

$$T\left(\sum t_j p_j\right) = \sum t_j T(p_j)$$

whenever $\sum t_j = 1$. The restriction of T to $[p_0, \ldots, p_m]$ is also called an **affine map**.
Since the affine map $T\colon A \to \mathbb{R}^k$ is defined on $A = \operatorname{span}(p_0, \ldots, p_m)$, it is clear that the map is determined by its values on the affine independent subset; its restriction to a simplex is thus determined by its values on the vertices.

Moreover, uniqueness of barycentric coordinates relative to $\{p_0, \ldots, p_m\}$ shows that such an affine T exists, since the formula in the definition is well defined.

Theorem 2.10. If $[p_0, \ldots, p_m]$ is an m-simplex, $[q_0, \ldots, q_n]$ an n-simplex, and $f\colon \{p_0, \ldots, p_m\} \to [q_0, \ldots, q_n]$ any function, then there exists a unique affine map $T\colon [p_0, \ldots, p_m] \to [q_0, \ldots, q_n]$ with $T(p_i) = f(p_i)$ for $i = 0, 1, \ldots, m$.

***2.4** If $T\colon \mathbb{R}^n \to \mathbb{R}^k$ is affine, then $T(x) = \lambda(x) + y_0$, where $\lambda\colon \mathbb{R}^n \to \mathbb{R}^k$ is a linear transformation and $y_0 \in \mathbb{R}^k$ is fixed.

Solution: Since T is affine, there exist some $(p_0, \ldots, p_n) \subset \mathbb{R}^n$ such that

$$T\left(\sum t_j p_j\right) = \sum t_j T(p_j).$$

and such that $(p_1 - p_0, \ldots, p_n - p_0)$ is a basis for $\mathbb{R}^n$. Furthermore, $(p_0, \ldots, p_n)$ span $\mathbb{R}^n$. Clearly then $y_0 = T(0)$. Suppose $T(0) = \sum c_j T(p_j)$. Then

$$\lambda(x) = T(x) - T(0) = \sum (t_j - c_j) T(p_j)$$

Checking linearity: let $x = \sum t_j p_j, y = \sum s_j p_j$ with $\sum t_j = \sum s_j = 1$. Then $x + y = \sum (t_j + s_j - c_j)\, p_j$ and

$$\lambda(x+y) = T(x+y) - T(0) = \sum (t_j + s_j - c_j) T(p_j) - \sum c_j T(p_j) = \sum (t_j - c_j) T(p_j) + \sum (s_j - c_j) T(p_j) = \lambda(x) + \lambda(y).$$

For any scalar $k \in \mathbb{R}$, we have $kx = \sum k t_j p_j$, so $\sum k t_j = k$. Hence since $\sum c_j p_j = 0$, we have $\sum (k-1) c_j p_j = 0$, so $\sum k t_j - (k-1) c_j) p_j = x$ and $\sum k t_j - (k-1) c_j = k - (k-1) = 1$, so

$$\lambda(kx) = \sum \left(k t_j - (k-1)\, c_j\right) T(p_j) - \sum c_j T(p_j) = \sum \left(k t_j - k c_j\right) T(p_j) = k \sum (t_j - c_j) T(p_j) = k \lambda(x).$$

So λ is linear.

2.5 Every affine map is continuous.

Proof: Follows from 2.4 since linear maps into $\mathbb{R}^n$ are continuous.

Check that linear functionals are continuous when the domain is equipped with the natural topology (see theorem 2.27 in Roman). Then argue that the coordinate functions in the codomain of the linear map are also linear maps.

***2.6.** Prove that any two m-simplexes are homeomorphic via an affine map.

Proof: Suppose $[p_0, \ldots, p_m]$ and $[q_0, \ldots, q_m]$ are m-simplexes.
Define a map $T\colon [p_0, \ldots, p_m] \to [q_0, \ldots, q_m]$ by $T(q_i) = p_i$ and extend it affinely as follows:

$$T\left(\sum t_i p_i\right) = \sum t_i T(p_i), \quad \sum t_i = 1.$$

This is affine and hence continuous, and it has a clear inverse by sending $T(p_i) = q_i \mapsto p_i$ and then extending affinely again.

Alternatively, simply note that T is continuous and bijective, and since simplices are compact and Hausdorff, being subspaces of $\mathbb{R}^m$, we have that T is a homeomorphism.

***2.7.** Give an explicit formula for the affine map $\theta\colon \mathbb{R} \to \mathbb{R}$ carrying $[s_1, s_2] \to [t_1, t_2]$ with $\theta(s_i) = t_i, i = 1, 2$. In particular, give a formula for the affine map taking $[32, 212]$ onto $[0, 100]$.

Solution: The affine subset spanned by s_1, s_2 is $\{ts_1 + (1-t)s_2 \mid t \in \mathbb{R}\}$ If we want to find t for when $ts_1 + (1-t)s_2 = x$ then $t(s_1 - s_2) = x - s_2$, so $t = \frac{x-s_2}{s_1-s_2}$. Thus we defined

$$\theta(x) = \frac{x - s_2}{s_1 - s_2} t_1 + \left(1 - \frac{x - s_2}{s_1 - s_2}\right) t_2 = \frac{t_1(x - s_2) + (s_1 - x)\, t_2}{s_1 - s_2}$$

This is affine by construction.
Hence the $\theta\colon \mathbb{R} \to \mathbb{R}$ taking $[32, 212] \to [0, 100]$ is

$$\theta(x) = \frac{x - 212}{32 - 212} \cdot 0 + \left(1 - \frac{x - 212}{32 - 212}\right) 100 = \left(\frac{32 - x}{-180}\right) \cdot 100 = \frac{5 \cdot (x - 32)}{9}.$$

***2.8.** Let $A \subset \mathbb{R}^n$ be an affine set and let $T\colon A \to \mathbb{R}^k$ be an affine map. If $X \subset A$ is affine (or convex), then $T(x) \subset \mathbb{R}^k$ is affine (or convex). In particular, if a, b are distinct points in A and if l is the line segment with endpoints a, b, then $T(l)$ is the line segment with endpoints $T(a), T(b)$ if $T(a) \neq T(b)$, and $T(l)$ collapses to the points $T(a)$ if $T(a) = T(b)$.

Proof: Suppose $\{p_0, \dots, p_m\}$ are the affine independent set that spans A. Suppose $X \subset A$ is affine. Then let $T(x), T(x') \in T(X) \subset \mathbb{R}^k$. For any point $tT(x) + T(x')(1-t), t \in \mathbb{R}$ on the line going through $T(x)$ and $T(x')$, we have $T\left(tx + (1-t)x'\right) = \lambda\left(tx + (1-t)x'\right) + y_0 = t\lambda(x) + ty_0 + (1-t)\lambda(x') + (1-t)y_0 = tT(x) + T(x')(1-t)$, so the point lies on $T(X)$. Hence $T(X)$ is affine - convexity is shown analogously.
If a, b are distinct points in A and l the line segment between them, then $T(l)$ is a line segment between $T(a)$ and $T(b)$ by affineness of T if $T(a) \neq T(b)$.
If $T(a) = T(b)$, then for any point $x = ta + (1-t)b, t \in I$, we have $T(x) = tT(a) + (1-t)T(b) = T(a)$, so the whole line segment collapses to a point.

***2.10.** Show that, for $0 \leqslant i \leqslant m$, $[p_0, \dots, p_m]$ is homeomorphic to the cone $C\left[p_0, \dots, \hat{p_i}, \dots, p_m\right]$ with vertex p_i.

Solution:

$$\sum t_j p_j = (1-t) \sum_{j \neq i} c_j p_j + t p_i$$

so $t = t_i$, and hence

$$(1 - t_i) c_j = t_j \implies c_j = \frac{t_j}{1 - t_i}$$

So define the map

$$S\left(\sum t_j p_j\right) = \begin{cases} \left(\sum_{j \neq i} \frac{t_j}{1-t_i} p_j, t_i\right), & t_i \neq 1 \\ \pi\left(p_i, 1\right), & t_i = 1 \end{cases}$$

The fundamental group

The fundamental groupoid

Using our previous definition of homotopy, we have the following:
1.19.(ii). If X is contractible and Y is path connected, then any two continuous maps $X \to Y$ are homotopic.

Proof: X contractible means $\mathbb{1}_X \simeq c$ for some constant map $c\colon X \to X$. Now for any two maps $f, g\colon X \to Y$, we have $f \simeq f \circ \mathbb{1}_X \simeq f \circ c \simeq g \circ c \simeq g \circ \mathbb{1}_X \simeq g$, where $f \circ c \simeq g \circ c$ because Y is path connected: $f \circ c$ and $g \circ c$ are constant functions, so let x_0 and x_1 denote the images, respectively. Then since Y is path connected, there exists $h\colon I \to Y$ such that $h(0) = x_0$ and $h(1) = x_1$. Define a map $H(x,t) = h(t)$. Then $H(x,0) = f \circ c(x)$ and $H(x,1) = g \circ c(x)$, so H is a homotopy connecting $f \circ c$ and $g \circ c$.

Def. Let $A \subset X$ and let $f_0, f_1\colon X \to Y$ be continuous maps with $f_0|_A = f_1|_A$. We write

$$f_0 \simeq f_1 \operatorname{rel} A$$

if there is a continuous map $F\colon X \times I \to Y$ with $F\colon f_0 \simeq f_1$ and

$$F(a,t) = f_0(a) = f_1(a) \quad \forall a \in A, \forall t \in I.$$

The homotopy F is called a **relative homotopy** (more precisely, a homotopy rel A), in contract, the original definition is called a **free homotopy.**

Def. The equivalence class of path $f\colon I \to X \operatorname{rel} \{0,1\}$ is called the **path class** of f and is denoted by $[f]$.

***3.1.** Generalize theorem 3.1 as follows. Let $A \subset X$ and $B \subset Y$ be given. Assume that $f_0, f_1\colon X \to Y$ with $f_0|_A = f_1|_A$ and $f_i(A) \subset B$ for $i = 0,1$; assume $g_0, g_1\colon Y \to X$ with $g_0|_B = g_1|_B$. If $f_0 \simeq f_1 \operatorname{rel} A$ and $g_0 \simeq g_1 \operatorname{rel} B$, then $g_0 \circ f_0 \simeq g_1 \circ f_1 \operatorname{rel} A$.

Solution: Let $F\colon f_0 \simeq f_1 \operatorname{rel} A$ and $G\colon g_0 \simeq g_1 \operatorname{rel} B$. Then define

$$H(x,t) = G\left(F\left(x,t\right), t\right)$$

For any $a \in A$, we have $H(a,t) = G\left(F(a,t),t\right) = G(f_i(a),t) = g_i(f_i(a))$, for all t and $i = 0,1$.
Now since $H(x,0) = G(F(x,0),0) = G(f(x),0) = g_0(f_0(x))$ and $H(x,1) = G(F(x,1),1) = G(f_1(x),1) = g_1(f_1(x))$, so $H\colon g_0 \circ f_0 \simeq g_1 \circ f_1 \operatorname{rel} A$.

***3.2.** If $f\colon I \to X$ is a path with $f(0) = f(1) = x_0 \in X$, then there is a continuous $f'\colon S^1 \to X$ given by $f'\left(e^{2\pi i t}\right) = f(t)$. If $f, g\colon I \to X$ are paths with $f(0) = f(1) = x_0 = g(0) = g(1)$ and if $f \simeq g \operatorname{rel} \{0,1\}$ then $f' \simeq g' \operatorname{rel} \{1\}$.
Furthermore, if f and g are as above, then $f \simeq f_1 \operatorname{rel} \{0,1\}$ and $g \simeq g_1 \operatorname{rel} \{0,1\}$ implies that $f' * g' \simeq f_1' * g_1' \operatorname{rel} \{1\}$.

Solution: We have that $f'|_{\{1\}} = g'|_{\{1\}}$. Now, we have that there exist an $F\colon f \simeq g \operatorname{rel} \{0,1\}$, i.e., $F(x,0) = f$ and $F(x,1) = g$ and $F(a,t) = 1$ for $a \in \{0,1\}$ and $t \in I$.
Now, define $G\colon S^1 \times I \to X$ by

$$G(e^{2\pi i x}, t) = F(x,t), \quad x \in I$$

This is well-defined and continuous since for any open $U \subset X$, we have $G^{-1}(U) = (h \times \mathbb{1})F^{-1}(U)$ where $h\colon I \to S^1$ is given by $t \mapsto e^{2\pi i t}$. As $h \times \mathbb{1}$ is an open map and F is continuous, G is continuous.

Now, suppose $\alpha\colon f \simeq f_1 \operatorname{rel} \{0,1\}$ and $\beta\colon g \simeq g_1 \operatorname{rel} \{0,1\}$. Then α, β induce α', β' which are homotopies rel $\{1\}$. Here $f' \simeq f_1'$ and $g' \simeq g_1$ both rel $\{1\}$. The rest follows from 3.1 for example.

Def. If $f\colon I \to X$ is a path from x_0 to x_1, call x_0 the **origin** of f and write $x_0 = \alpha(f)$; call x_1 the **end** of f and write $x_1 = \omega(f)$. A path f in X is **closed** at x_0 if $\alpha(f) = x_0 = \omega(f)$.

If f, g are paths with $f \simeq g \operatorname{rel} \{0,1\}$ then $\alpha(f) = \alpha(g)$ and $\omega(f) = \omega(g)$; therefore we may speak of the **origin** and **end** of a path class and write $\alpha\,[f]$ and $\omega\,[f]$.

Def. If $p \in X$ then the constant function $i_p\colon I \to X$ with $i_p(t) = p$ for all $t \in I$ is called the **constant path** at p.

***3.4** Let $\sigma\colon \Delta^2 \to X$ be continuous, where $\Delta^2 = [e_0, e_1, e_2]$.

Define $\varepsilon_0\colon I \to \Delta^2$ as the affine map with $\varepsilon_0(0) = e_1$ and $\varepsilon_0(1) = e_2$; similarly, define ε_1 by $\varepsilon_1(0) = e_0$ and $\varepsilon_1(1) = e_2$ and ε_2 by $\varepsilon_2(0) = e_0$ and $\varepsilon_2(1) = e_1$. Finally, define $\sigma_i = \sigma \circ \varepsilon_i$ for $i = 0,1,2$.

(i) Prove that $\left(\sigma_0 * \sigma_1^{-1}\right) * \sigma_2$ is nullhomotopic rel $\{0,1\}$.

Solution: Define a map $h\colon \Delta^2 \times I \to \Delta^2$ by $h(x,t) = e_0 t + (1-t)x$. As Δ^2 is convex, this is a continuous homotopy. Now $\sigma \circ h\colon \sigma \simeq \sigma(e_0)$ is a homotopy $X \times I \to X$ contracting X.
Hence $\sigma \circ h$ contracts X to the point $\sigma(e_0)$ which is not moved under the homotopy. Thus $\left(\sigma_0 * \sigma_1^{-1}\right) * \sigma_2 \simeq \sigma(e_0)$ and (ii) is proven similarly.

(iii) Let $F\colon I \times I \to X$ be continuous, and define paths $\alpha, \beta, \gamma, \delta$ in X as indicated on page 41.
So $\alpha(t) = F(t,0), \beta(t) = F(t,1), \gamma(t) = F(0,t)$ and $\delta(t) = F(1,t)$. Prove that $\alpha \simeq \gamma * \beta * \delta^{-1} \operatorname{rel}\{0,1\}$.

Solution: We want a map $G\colon I \times I \to X$ such that $G(x,0) = F(x,0) = \alpha(x)$ and $G(x,1) = F(0,x) * F(x,1) * F(1,1-x) = \gamma * \beta * \delta^{-1}(x)$.

Define a map $\varphi\colon \partial I^2 \times I \to I^2$ by $\varphi((0,t),s) = s\left(\frac{1}{3}t, 0\right) + (1-s)(0,t), \varphi((t,1),s) = s\left(\frac{1+t}{3}, 0\right) + (1-s)(t,1)$ and $\varphi((1,t),s) = s\left(\frac{2+t}{3}, 0\right) + (1-s)(1,t)$, and $\varphi = \mathbb{1}$ on $I \times \{0\}$.

Consider $G(x,t) = F \circ \varphi((x,y),s)\colon \partial I^2 \times I \to X$. Then $F \circ \varphi((x,y),0) = F(x,y) = \begin{cases} \gamma(y), & x = 0 \\ \beta(x), & y = 1 \\ \delta(y), & x = 1 \\ \alpha(x), & y = 0 \end{cases}$ and

$F \circ \varphi\left((x,y),1\right) = \begin{cases} \alpha(\frac{y}{3}), & x = 0 \\ \alpha(\frac{1+x}{3}), & y = 1 \\ \alpha\left(\frac{2+y}{3}\right), & x = 1 \\ \alpha(x), & y = 0 \end{cases}$ Let $k\colon I \to \partial I^2$ be the map traversing $\{0\} \times I, I \times \{1\}$ and $\{1\} \times I$ during $\left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{3}, 1\right]$, respectively. Let $l\colon I \to \partial I^2$ be $l(t) = (t,0)$. Then $F \circ \varphi\left((k(x)), t\right) : I \times I \to X$ is a homotopy of k to $l \operatorname{rel}\{0,1\}$.

***3.6.**
(i) If $f \simeq g \operatorname{rel}\{0,1\}$, then $f^{-1} = g^{-1} \operatorname{rel}\{0,1\}$, where f, g are paths in X.

Solution: Let $F(x,t)$ be a continuous map such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ and $F(0,t) = x_0$ and $F(1,t) = x_1$ for all $t \in I$.

Then define $G(x,t) := F(1-x,t)$. We have $G(x,0) = F(1-x,0) = f(1-x) = f^{-1}(x)$ and $G(x,1) = F(1-x,1) = g(1-x) = g^{-1}(x)$. G is also continuous and $G(0,t) = F(1,t) = x_1$ and $G(1,t) = F(0,t) = x_0$ for all $t \in I$, so G is a homotopy between f^{-1} and g^{-1} rel $\{0,1\}$.

(ii) If f and g are paths in X with $\omega(f) = \alpha(g)$, then

$$(f * g)^{-1} = g^{-1} * f^{-1}.$$

Proof: We have $f * g = \begin{cases} f(2t), & t \in \left[0, \frac{1}{2}\right] \\ g(2t-1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$, so $f * g(1-t) = \begin{cases} f(2(1-t)), & (1-t) \in \left[0, \frac{1}{2}\right] \\ g\left(2(1-t) - 1\right), & (1-t) \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} g\left(1-2t\right), & t \in \left[0, \frac{1}{2}\right] \\ f(2-2t), & t \in \left[\frac{1}{2}, 1\right] \end{cases} = g^{-1} * f^{-1}$.

(iii) Give an example of a closed path f with $f * f^{-1} \neq f^{-1} * f$.

Solution: $f = e^{2\pi i t}$.

(iv) Show that if $\alpha(f) = p$ and f is not constant, then $i_p * f \neq f$.

Solution: Let $t_0 = \inf\{t\colon i_p \cdot f(t) \neq p\}$. Then $t_0 \geqslant \frac{1}{2}$. If $i_p * f = f$ then if $t_1 = \inf\{t\colon f(t) \neq p\}$, we have $t_0 = t_1$. Now $i_p * f(t_0) = f(2t_0 - 1)$, so since $t_0 = \inf\{t\colon f(2t-1) \neq p\}$, we have $2t_0 - 1 = t_1$, so $2t_0 - 1 = t_0$, hence $t_0 = 1$. But as f is continuous, $f(1) = \lim_{t \to 1-} f(t) = \lim_{t \to 1-} p = p$, contradiction.

Theorem 3.2. If X is a space, then the set of all path classes in X under the (not always defined) binary operation $[f]\,[g] = [f * g]$ forms a **groupid**.
I.e., it satisfies the following properties:

1. each path class $[f]$ has an origin $\alpha\,[f] = p \in X$ and an end $\omega\,[f] = q \in X$, and

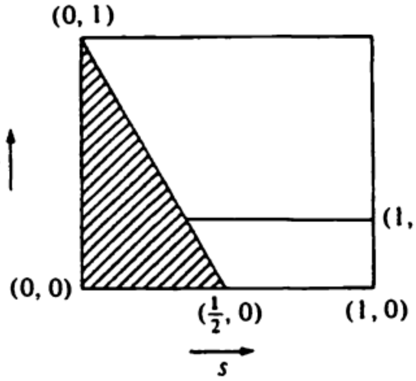
$$[i_p]\,[f] = [f] = [f]\,[i_q]\,.$$

2. associativity holds whenever possible;
3. if $p = \alpha\,[f]$ and $q = \omega\,[f]$, then

$$[f]\left[f^{-1}\right] = [i_p] \quad \text{and} \quad \left[f^{-1}\right][f] = [i_q]\,.$$

Here the objects of the category are the points of x and morphisms between objects are path classes with the source object as the starting point and the target object as the end point.

Proof: (i) We show that $i_p * f \simeq f \operatorname{rel}\{0,1\}$; the other half is similar.



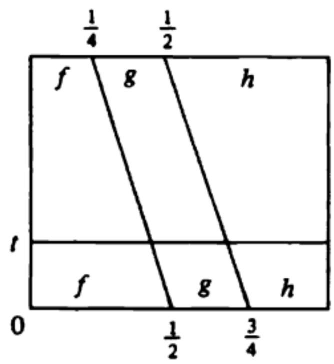
Draw the line in $I \times I$ joining $(0,1)$ to $(\frac{1}{2}, 0)$; its equation is $2s = 1-t$. For fixed t, define $\theta_t\colon \left[\frac{1-t}{2}, 1\right] \to [0,1]$ as the affine map matching the endpoints of these intervals. Then

$$\theta_t(s) = \frac{s - \frac{1-t}{2}}{1 - \frac{1-t}{2}}.$$

Define $H\colon I \times I \to X$ by

$$H(s,t) = \begin{cases} p, & 2s \leqslant 1-t \\ f\left(\theta_t(s)\right) = f\left(\frac{2s-1+t}{1+t}\right), & 2s \geqslant 1-t \end{cases}$$

(ii) For associativity, we use the picture



We define three maps: $\theta_{1,t}, \theta_{2,t}$ and $\theta_{3,t}$ as

$$\theta_{1,t}(s) = \frac{-\frac{1}{4}s}{-\frac{1}{2}}t + s(1-t) = \frac{st}{2} + s(1-t)$$

$$\theta_{2,t}(s)\left(s-\frac{1}{4}\right)t+(1-t)s$$

$$\theta_{3,t}(s)(2s-1)t+(1-t)s$$

Define

$$\theta_t(s)=\begin{cases}\theta_{1,t}(s), & s\in\left[0,\frac{1}{2}\right]\\ \theta_{2,t}(s), & s\in\left[\frac{1}{2},\frac{3}{4}\right]\\ \theta_{3,t}(s), & s\in\left[\frac{3}{4},1\right]\end{cases}.$$

Now define

$$H(s,t)=(f*g)*h\left(\theta_t(s)\right)$$

Then $H(s,0)=(f*g)*h$, and $H(s,1)=f*(g*h)$.

(iii) is shown similarly.

The groupoid here is not a group because multiplication is not always defined; we remedy this defect in the most naive possible way, namely, by restricting our attention to closed paths.

Def. Fix a point $x_0\in X$ and call it the **basepoint**. The **fundamental group** of X with basepoint x_0 is

$$\pi_1\left(X,x_0\right)=\{[f]:\ [f]\ \text{is a path class in } X \text{ with } \alpha\,[f]=x_0=\omega\,[f]\}$$

with binary operation

$$[f]\,[g]=[f*g]\,.$$

Theorem 3.3. $\pi_1\left(X,x_0\right)$ is a group for each $x_0\in X$.

Proof: Follows immediately from theorem 3.2.

The functor π_1

Theorem 3.4. $\pi_1\colon \mathrm{Top}_*\to\mathrm{Group}$ is a covariant functor. Moreover, if $h,k\colon(X,x_0)\to(Y,y_0)$ and $h\simeq k\,\mathrm{rel}\,\{x_0\}$, then $\pi_1(h)=\pi_1(k)$ (i.e., this is how π_1 works on morphisms).

Proof: We have that if $(X,x_0)\in\mathrm{Top}_*$ then $\pi_1(X,x_0)\in\mathrm{Group}$ by theorem 3.3.
We must show that if $f\colon(X,x_0)\to(Y,y_0)$ is a morphism in Top_* then $\pi_1(f)\colon\pi_1\left(X,x_0\right)\to\pi_1\left(Y,y_0\right)$ is a morphism in Group.

Now, define $\pi_1(f)\,[\alpha]=[f\circ\alpha]$. Then this criteria is true. We must have that if $(X,x_0)\xrightarrow{f}(Y,y_0)\xrightarrow{g}(Z,z_0)$ then $\pi_1\left(g\circ f\right)=\pi(g)\circ\pi(f)$. This is clear since $\pi_1\left(g\circ f\right)[\alpha]=\left[\left(g\circ f\right)\alpha\right]=\pi_1(g)\left[f\circ\alpha\right]=\pi_1(g)\left(\pi_1(f)\left[\alpha\right]\right)=\pi_1(g)\circ\pi_1(f)\left[\alpha\right]$.

Lastly, we must have that for $(X,x_0)\xrightarrow{\mathbb{1}_{(X,x_0)}}(X,x_0)$, we have $\pi_1\left(\mathbb{1}_{(X,x_0)}\right)=\mathbb{1}_{\pi_1(X,x_0)}$. This is clear by definition.

Suppose F is a homotopy of h and k rel $\{x_0\}$, both of which take x_0 to y_0. Thus $F(x_0,t)=y_0$ for all $t\in I$.
The claim is that $[h\circ\alpha]=\pi_1(h)\,[\alpha]=\pi_1(k)\,[\alpha]=[k\circ\alpha]$ for all $[\alpha]\in\pi_1\left(X,x_0\right)$. Now, $F\left(\alpha(x),t\right)$ is a continuous homotopy between $h\circ\alpha$ and $k\circ\alpha$. Furthermore, $\alpha(0)=\alpha(1)=x_0$, so $F\left(\alpha(x,t)\right)$ is a homotopy rel $\{0,1\}$; which means $[h\circ\alpha]=[k\circ\alpha]$, proving $\pi_1(h)=\pi_1(k)$.

Remark 1. One usually writes h_* instead of $\pi_1(h)$ and calls h_* the map **induced** by h.

Remark 2. There is a category appropriate to the fundamental group functor π_1. Define the **pointed homotopy category,** hTop_*, as the quotient category arising from the congruence of relative homotopy: if $f_0,f_1\colon(X,x_0)\to(Y,y_0)$, then $f_0\simeq f_1\,\mathrm{rel}\,\{x_0\}$. The objects of hTop_* are pointed spaces (X,x_0), morphisms $(X,x_0)\to(Y,y_0)$ are relative homotopy classes $[f]$, where $f\colon(X,x_0)\to(Y,y_0)$ is a pointed map, and composition is given by $[h][f] = [h \circ f]$ (when h, f can be composed in Top_*). By exercise 3.2, each closed path $f\colon (I, \{0,1\}) \to (Y, y_0)$ may be viewed as a pointed map $f'\colon \left(S^1, 1\right) \to (Y, y_0)$. If Hom sets in hTop_* are denoted by $[(X, x_0), (Y, y_0)]$, then $[f] \mapsto [f']$ is a bijection

$$\pi_1(Y, y_0) \xrightarrow{\sim} \left[\left(S^1, 1\right), (Y, y_0)\right].$$

Using exercise 3.2.(ii), one may introduce a multiplication in the Hom set, namely, $[f'][g'] = \left[(f * g)'\right]$, and the bijection is now an isomorphism.
Therefore, π_1 is an instance of a covariant Hom functor. Roughly speaking, the fundamental group of a space Y is just the set of morphisms $S^1 \to Y$. We shall elaborate on this theme when we introduce the higher homotopy group functors π_n (which, roughly speaking, are the morphisms of S^n into a space).

Back to fundamental groups.
Let x_0 be a basepoint of a space X, and let A be a subspace of X containing x_0 ; the inclusion $j\colon (A, x_0) \hookrightarrow (X, x_0)$ is a pointed map inducing a homomorphism $j_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$, namely, $[f] \mapsto [jf]$.
It is possible that f is not nullhomotopic in A, yet f is nullhomotopic in X, e.g., setting $X = CA$. The homomorphism j_* may thus have a kernel.

Theorem 3.5. Let $x_0 \in X$, and let X_0 be the path component of X containing x_0. Then

$$\pi_1(X_0, x_0) \simeq \pi_1(X, x_0).$$

Theorem 3.6. If X is path connected and $x_0, x_1 \in X$, then

$$\pi_1(X, x_0) \simeq \pi_1(X, x_1).$$

Notation: projections. $p\colon H \times K \to H$ and $q\colon H \times K \to K$ def by $p(h,k) = h$ and $q(h,k) = k$. If $\alpha\colon L \to H$ and $\beta\colon L \to K$ are functions from some set L, then there is a function $(\alpha, \beta) : L \to H \times K$ defined by $(\alpha, \beta)(x) = (\alpha(x), \beta(x))$.

$$\begin{tikzcd}
 & H \times K \arrow[dl, "p"'] \arrow[dr, "q"] & \\
H & & K \\
 & L \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \arrow[uu, dashed, "{(a,b)}"] &
\end{tikzcd}$$

Theorem 3.7. If (X, x_0) and (Y, y_0) are pointed spaces, then

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

3.7. If X is the $\sin\left(\frac{1}{x}\right)$ space, prove that $\pi_1(X, x_0) = \{1\}$ for every $x_0 \in X$.

Proof: By theorem 3.5, it suffices to show it for the two path components A and G separately.
A is clearly contractible, and hence $\pi_1(A, x_0) = \{1\}$ for any $x_0 \in A$.
For G, consider any point $x = (x_0, y_0) \in G$. Let $\gamma\colon I \to G$ be a loop.
Consider the map $f\colon G \times I \to G$ by $f((x, \sin\left(\frac{1}{x}\right)), t) = \left(x_0 t + (1-t)x, \sin\left(\frac{1}{x_0 t + (1-t)x}\right)\right)$. Then f contracts G, so $f \circ \gamma \simeq f(i_{x_0})$, and hence γ is nullhomotopic, so $\pi_1(G, x_0) \simeq \{1\}$. Thus $\pi_1(X, x_0) \simeq \{1\}$.

***3.8.** Give an example of a contractible space that is not locally path connected.

Solution: Consider $X = I \cap \mathbb{Q}$. THen X is not locally path connected and hence CX is not either. However, CX is contractible.

***3.9.** Let X be a space. Show that there is a category C with $\mathrm{ob}\, C = X$, with $\mathrm{Hom}(p,q) = \{\text{all path classes } [f] \text{ with } \alpha[f] = p \text{ and } \omega[f] = q\}$, and with composition $\mathrm{Hom}(p,q) \times \mathrm{Hom}(q,r) \to \mathrm{Hom}(p,r)$ defined by $([f],[g]) \mapsto [f * g]$. Show that every morphism in C is an equivalence.

Proof: The Hom sets are clearly disjoint.
Associativity follows since $f * (g * h) \simeq (f * g) * h$.
Identity is also clear with the constant paths.
Every morphism in C is an isomorphism since if for $f\colon I \to X$, we let $f^{-1}(t) = f(1-t)$ then $f * f^{-1} \simeq i_{f(0)}$ and $f^{-1} * f \simeq i_{f(1)}$.

3.10. If (X, x_0) is a pointed space, let the path component of X containing x_0 be the basepoint of $\pi_0(X)$; show that π_0 defines a functor $\text{Top}_* \to \text{Set}_*$.

Proof: Suppose π_0 takes morphisms to the underlying function between sets pointed sets.
We have that if $(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (Z, z_0)$ then $\pi_0(gf) = \pi_0(g) \circ \pi_0(f)$ by definition.
We also have that if $(X, x_0) \xrightarrow{\mathbb{1}_{(X,x_0)}} (X, x_0)$ then $\pi_0\left(\mathbb{1}_{(X,x_0)}\right) = \mathbb{1}_{\pi_0(X,x_0)}$ by definition. So π_0 defines a functor $\text{Top}_* \to \text{Set}_*$ in this way.

***3.11.** If $X = \{x_0\}$ is a one-point space, then $\pi_1(X, x_0) = \{1\}$.

Proof: Suppose $f\colon I \to X$. Then $f = i_{x_0}$, so $\pi_1(X, x_0) = \{[i_{x_0}]\} \cong \{1\}$.

Lemma 3.8. Assume that $F\colon \varphi_0 \simeq \varphi_1$ is a (free) homotopy, where $\varphi_i\colon X \to Y$ is continuous for $i = 0, 1$. Choose $x_0 \in X$ and let λ denote the path $F(x_0, -)$ in Y from $\varphi_0(x_0)$ to $\varphi_1(x_0)$. Then there is a commutative diagram

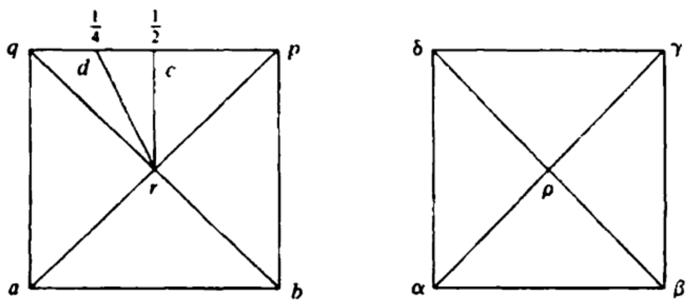
$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\varphi_{1*}"] \arrow[dr, "\varphi_{0*}"'] & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\psi"] \\
& \pi_1(Y, \varphi_0(x_0))
\end{tikzcd}$$

where ψ is the isomorphism $[g] \mapsto \left[\lambda * g * \lambda^{-1}\right]$.

Proof: To show that the diagram commutes, we must show that for any $[f] \in \pi_1(X, x_0)$, $[\varphi_0 \circ f] = \left[\lambda * (\varphi_1 \circ f) * \lambda^{-1}\right]$.
Define $G\colon I \times I \to Y$ bu

$$G(t, s) = F(f(t), s).$$

Note that $G\colon \varphi_0 \circ f \simeq \varphi_1 \circ f$.
Consider the two triangulations of the square $I \times I$ pictured below



Define a continuous map $H\colon I \times I \to I \times I$ by first defining it on each triangle and then invoking the gluing lemma. On each 2-simplex, H shall be an affine map; it thus suffices to evaluate H on each vertex. Define $H(a) = H(q) = \alpha, H(b) = H(p) = \beta; H(c) = \gamma; H(d) = \delta; H(r) = \rho$. By exercise 2.8, the vertical edge $[a, q]$ collapses to α, the vertical edge $[b, p]$ collapses to β. Also, $[q, d]$ goes to $[\alpha, \delta]$, $[d, c]$ goes to $[\delta, \gamma]$ and $[c, p]$ goes to $[\gamma, \beta]$. The map $J = G \circ H\colon I \times I \to Y$ is easily seen to be a relative homotopy

$$J\colon \varphi_0 \circ f \simeq (\lambda * (\varphi_1 \circ f)) * \lambda^{-1} \operatorname{rel}\{0, 1\}.$$

Therefore $\varphi_{0*}[f] = [\varphi_0 \circ f] = \left[\lambda * \varphi_1 \circ f * \lambda^{-1}\right]$ (using homotopy associativity). On the other hand, $\psi\varphi_{1*}[f] = \psi[\varphi_1 \circ f] = \left[\lambda * \varphi_1 \circ f * \lambda^{-1}\right]$.

This shows that freely homotopic maps φ_0 and φ_1 may not induce the same homomorphism between fundamental groups, because they differ by the isomorphism ψ.

Corollary 3.9. Assume that $\varphi_i\colon (X, x_0) \to (Y, y_0)$, for $i = 0, 1$, are freely homotopic.

1. φ_{0*} and φ_{1*} are conjugate; that is, there exists some $[\lambda] \in \pi_1(Y, y_0)$ with $\varphi_{0*}[f] = [\lambda]\,\varphi_{1*}([f])\,[\lambda]^{-1}$ for every $[f] \in \pi_1(X, x_0)$.
2. If $\pi_1(Y, y_0)$ is abelian, then $\varphi_{0*} = \varphi_{1*}$.

Theorem 3.10. If $\beta\colon X \to Y$ is a homotopy equivalence, then the induced homomorphism $\beta_*\colon \pi_1(X, x_0) \to \pi_1(Y, \beta(x_0))$ is an isomorphism for every $x_0 \in X$.

Proof: Choose a continuous map $\alpha\colon Y \to X$ with $\alpha \circ \beta \simeq \mathbb{1}_X$ and $\beta \circ \alpha \simeq \mathbb{1}_Y$. By the lemma, the lower triangle of the diagram below commutes.

$$\begin{tikzcd}
 & \pi_1(Y, \beta(x_0)) \arrow[dr, "\alpha_*"] & \\
\pi_1(X, x_0) \arrow[ur, "\beta_*"] \arrow[rr, "(\alpha\beta)_*"] \arrow[dr, "\mathbb{1}"'] & & \pi_1(X, \alpha\beta(x_0)) \arrow[dl, "\psi"] \\
 & \pi_1(X, x_0) &
\end{tikzcd}$$

Since ψ is an isomorphism, it follows that $(\alpha\beta)_*$ is an isomorphism. Now, the top triangle commutes because π_1 is a functor: $(\alpha\beta)_* = \alpha_*\beta_*$. It follows that β_* is injective and α_* is surjective. A similar diagram arising from $\beta\alpha \simeq \mathbb{1}_Y$ shows that β_* is surjective; thus β_* is an isomorphism.

Corollary 3.11. Let X and Y be path connected spaces having the same homotopy type. Then, for every $x_0 \in X$ and $y_0 \in Y$, we have

$$\pi_1(X, x_0) \cong \pi_1(Y, y_0).$$

Corollary 3.12. If X is a contractible space and $x_0 \in X$, then

$$\pi_1(X, x_0) = \{1\}.$$

Definition. A space X is called **simply connected** if it is path connected and $\pi_1(X, x_0) = \{1\}$ for every $x_0 \in X$.

Corollary 3.13. If $\beta\colon (X, x_0) \to (Y, y_0)$ is freely nullhomotopic, then the induced homomorphism $\beta_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is trivial.

$\pi_1(S^1)$

Lemma 3.14. Let X be a compact convex subset of some $\mathbb{R}^k$, let $f\colon (X, x_0) \to (S^1, 1)$ be continuous, let $t_0 \in \mathbb{Z}$ and let $\exp t$ denote $e^{2\pi i t}$. Then there exists a unique continuous $\tilde{f}\colon (X, x_0) \to (\mathbb{R}, t_0)$ with $\exp \tilde{f} = f$.

$$\begin{tikzcd}
 & (\mathbb{R}, t_0) \arrow[d, "\exp"] \\
(X, x_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & (S^1, 1)
\end{tikzcd}$$

Then $\tilde{f}$ is called a **lifting** of f.

Corollary 3.15. Let $f\colon (I, \{0, 1\}) \to (S^1, 1)$ be continuous.

1. There exists a unique continuous $\tilde{f}\colon I \to \mathbb{R}$ with $\exp \tilde{f} = f$ and $\tilde{f}(0) = 0$.

2. If $g\colon (I,\{0,1\}) \to (S^1,1)$ is continuous and $f \simeq g \operatorname{rel}\{0,1\}$, then $\tilde{f} \simeq \tilde{g} \operatorname{rel}\{0,1\}$ (where $\exp \tilde{g} = g$ and $\tilde{g}(0) = 0$); moreover, $\tilde{f}(1) = \tilde{g}(1)$.

Proof: (i) follows from the lemma because I is convex.

(ii): Suppose $F\colon I \times I \to (S^1,1)$ is the relative homotopy of f and $g \operatorname{rel}\{0,1\}$. Since $I \times I$ is convex compact, choose $(0,0)$ as a basepoint and lift F to $\tilde{F}\colon I \times I \to \mathbb{R}$. So $\exp \tilde{F} = F$ and $\tilde{F}(0,0) = 0$. We claim that $\tilde{F}\colon \tilde{f} \simeq \tilde{g} \operatorname{rel}\{0,1\}$.
We have $\exp \tilde{F}(x,0) = F(x,0) = f(x)$ and $\exp \tilde{F}(x,1) = g(x)$, so $\tilde{F}(x,0)$ and $\tilde{F}(x,1)$ are lifts for f and g respectively. Now, since $\tilde{F}(0,0) = 0$, we have by the previous lemma that $\tilde{F}(x,0) = \tilde{f}$ and $\tilde{F}(x,1) = \tilde{g}$ by uniqueness.
We now only need to show that $\tilde{F}(1,t)$ is constant for all t. But this is a lift of the constant function at 1, i_1, so it must be constant. Furthermore, $\tilde{F}(1,0) = \tilde{f}(1)$, so $\tilde{g}(1) = \tilde{f}(1)$.

Definition. If $f\colon (I,\{0,1\}) \to (S^1,1)$ is continuous, define the **degree** of f by

$$\deg f = \tilde{f}(1)$$

where $\tilde{f}$ is the unique lifting of f with $\tilde{f}(0) = 0$.

Theorem 3.16. The function $d\colon \pi_1(S^1,1) \to \mathbb{Z}$ given by $[f] \mapsto \deg f$ is an isomorphism. In particular, $\deg(f * g) = \deg f + \deg g$.

3.15. Let $f\colon (I,\{0,1\}) \to (S^1,a)$ be a closed path in S^1 at $a = \exp(\alpha)$. Define **degree** $f = \deg R \circ f$, where $R\colon S^1 \to S^1$ is rotation by $-2\pi\alpha$ radians. Prove that two closed paths f and g in S^1 (with $f(0) = a$ and $g(0) = b$) are homotopic (with closed paths at every time t of the homotopy) if and only if they have the same degree.

Proof:
Suppose $R_\alpha\colon S^1 \to S^1$ is rotation by $-2\pi\alpha$ radians. Then $R_\alpha \circ f\colon (I,\{0,1\}) \to (S^1,1)$ and $R_\beta \circ g\colon (I,\{0,1\}) \to (S^1,1)$. Clearly, $R_\alpha, R_\beta \simeq \mathbb{1}_{S^1}$. Suppose $F\colon R_\alpha \circ f \simeq R_\beta \circ g$. Then we can lift $R_\alpha \circ f$ to $\tilde{f}\colon I \to \mathbb{R}$ and $R_\beta \circ g$ to $\tilde{g}\colon I \to \mathbb{R}$, and $\tilde{f}(1) = \tilde{g}(1)$. But then $\deg R_\alpha \circ f = \deg R_\beta \circ g$, so $\deg f = \deg g$.

Conversely, if they have the same degree, then $\deg R_\alpha \circ f = \deg R_\beta \circ g$, so the lifts have the same endpoint. But then corollary 3.18 gives that $R_\alpha \circ f \simeq R_\beta \circ g$.

3.17. Prove that S^1 is not a retract of D^2.

Proof: Suppose $r\colon D^2 \to S^1$ is such that $r \circ i = \mathbb{1}_{S^1}$ for $i\colon S^1 \to D^2$ being the inclusion. Then

$$\begin{tikzcd} S^1 \arrow[r, "i"] \arrow[rr, bend right, "\mathbb{1}_{S^1}"'] & D^2 \arrow[r, "r"] & S^1 \end{tikzcd}$$

induces the following commutative diagram

$$\begin{tikzcd} \pi_1(S^1,1) \arrow[r, "i_*"] \arrow[rr, bend right, "\mathbb{1}_{\pi_1(S^1,1)}"'] & \pi_1(D^2,1) \arrow[r, "r_*"] & \pi_1(S^1,1) \end{tikzcd}$$

However, since $\pi_1(D^2,1) = 0$, we have that the image of $\pi_1(S^1,1)$ under $r_* \circ i_*$ is trivial, however, its image under $\mathbb{1}_{\pi_1(S^1,1)}$ is not trivial, contradiction.

3.18. Prove the Brouwer fixed point theorem for continuous maps $D^2 \to D^2$.

Proof: Suppose $f\colon D^2 \to D^2$ with $f(x) \neq x$ for all $x \in D^2$. Then define $r(x)\colon D^2 \to S^1$ to be the point of intersection of the ray going from $f(x)$ through x with the boundary $\partial D^2 = S^1$. We claim this

map is continuous.
We can write points on this ray as $h(x) + t(x - h(x))$ for $t \in \mathbb{R}_+$. Let now $t(x)$ be the t above such that

$$\|h(x) + t(x)\,(x - h(x))\,\| = 1$$

We must show that $t(x)$ is continuous. Taking the dot product of this point with itself, we get

$$\|h(x)\|^2 + 2t(x)(x - h(x)) \cdot h(x) + t(x)^2 \|(x - h(x))\|^2 = 1$$

We get for $h(x) \neq 0$ that

$$\begin{aligned} t(x) &= \frac{-2(x - h(x)) \cdot h(x) \pm \sqrt{4\|h(x)\|^2\|x - h(x)\|^2 - 4\|x - h(x)\|^2(\|h(x)\|^2 - 1)}}{2\|x - h(x)\|^2} \\ &= \frac{-2\,(x - h(x)) \cdot h(x) \pm \sqrt{4\|x - h(x)\|^2}}{2\|x - h(x)\|^2} \\ &= \frac{-\,(x - h(x)) \cdot h(x) \pm \|x - h(x)\|}{\|x - h(x)\|^2} \\ &= \frac{-\,(x - h(x)) \cdot h(x) \pm 1}{\|x - h(x)\|} \end{aligned}$$

This is continuous, giving that r is continuous.

But now we notice that if $x \in S^1 \subset D^2$, then $r(x) = x$, hence r is in fact a retraction of D^2 onto S^1. By exercise 3.17, this is a contradiction.

3.19. Let f be a closed path in S^1 at 1.

1. If f is not surjective , then $\deg f = 0$.
2. Give an example of a surjective f with $\deg f = 0$.

Solution:

(i) We claim that any path-connected proper subset of S^1 is simply connected.
Let $U \subset S^1$ be a path-connected proper subspace. Suppose $x_0 \in S^1 - U$. Let $\sigma\colon S^1 - \{x_0\} \to \mathbb{R}$ be the stereographic projection. This is an isomorphism so the image of a path-connected subspace is path-connected. But path-connected subspaces of $\mathbb{R}$ are intervals and thus simply connected. Since $\pi_1(U, x_1) \overset{\sigma_*}{\to} \{0\}$ is an isomorphism, we have $\pi_1(U, x_1) = \{0\}$. Now, $f\colon I \to S^1$ is continuous and I is path-connected, so $\operatorname{Im} f$ is path-connected. Hence $[f] = \left[i_{f(0)}\right] \in \pi_1(\operatorname{Im} f, f(0))$, so $\deg f = \deg i_{f(0)}$ by corollary 3.18, and $\deg i_{f(0)} = 0$ as the unique lift of $i_{f(0)}$ starting at 0 is precisely $i_0\colon R \to \mathbb{R}$.

(ii) Consider the map $F(x,t) = \begin{cases} e^{2xi\pi t}, & x \in \left[0, \frac{1}{2}\right] \\ e^{i\pi(2x-1)t}, & x \in \left[\frac{1}{2}, 1\right] \end{cases}$ and $G(x,t) = \begin{cases} e^{-2xi\pi t}, & x \in \left[0, \frac{1}{2}\right] \\ e^{-i\pi(2x-1)t}, & x \in \left[\frac{1}{2}, 1\right] \end{cases}$ Then $F * G(x,t) = f_t(x) * g_t(x)$ is a homotopy between $f_1(x) * g_1(x)$ and $i_1(x)$, and $f_1(x) * g_1(x)$ is surjective.

***3.20.** Let X be a space with basepoint x_0, and let $\{U_j : j \in J\}$ be an open cover of X by path connected subspaces such that:

1. $x_0 \in U_j$ for all j ;
2. $U_j \cap U_k$ is path connected for all j, k.

Prove that $\pi_1(X, x_0)$ is generated by the subgroups $\operatorname{Im} i_{j*}$ where $i_j\colon (U_j, x_0) \hookrightarrow (X, x_0)$ is the inclusion.

Proof: Suppose $[f] \in \pi_1(X, x_0)$. We claim there exists a decomposition $f = f_{j_1} * f_{j_2} * \ldots * f_{j_n}$ such that $\operatorname{Im} f_{j_i} \subset U_{k_i}$.
Now, $\left\{f^{-1}(U_j) : j \in J\right\}$ is an open cover of I which is compact metric, so there exists a Lebesgue number $\delta > 0$ for the cover. Subdivide I into $[0, i_1,] \cup [i_1, i_2] \cup \ldots \cup [i_{n-1}, i_n]$ where $i_n = 1$ and $|i_j - i_{j-1}| < \delta$ for all $j \in \{1, \ldots, n\}$.

Now, $f\left([i_{j-1}, i_j]\right) \subset U_{k_j}$. Let $\alpha_i\colon I \to X$ be the path going from x_0 to $f(t_i)$ in U_{k_i}. Let $f_i\colon I \to X$ be the path defined by $f_i(t) = f\left(t_i t + (1-t)t_{i-1}\right)$. Then

$$f \simeq f_1 * f_2 * \ldots * f_n \simeq f_1(x) * \alpha_1(x) * \alpha_1^{-1}(x) * f_2(x) * \alpha_2(x) * \alpha_2^{-1}(x) * f_3(x) * \ldots * \alpha_n(x) * f_n(x)$$

Now, $\operatorname{Im} f_1(x) * \alpha_1(x) \subset U_{k_1}$ and is a loop at x_0, so defining $g_1\colon I \to U_{k_1}$ as $g_1(x) = (f_1 * a_1)(x)$, we get $g_1 \in \pi_1\left(U_{k_1}, x_0\right)$. Similarly, $\operatorname{Im} \alpha_1^{-1} * f_2 * \alpha_2 \subset U_{k_2}$, so defining $g_2(x)\colon I \to U_{k_2}$ as $g_2(x) = \left(\alpha_1^{-1} * f_2 * \alpha_2\right)(x)$, we get $g_2 \in \pi_1\left(U_{k_2}, x_0\right)$. Define $i_j\colon \left(U_{k_j}, x_0\right) \to (X, x_0)$ as the inclusion of U_{k_j} into X, we get

$$f \simeq (i_1 \circ g_1) * (i_2 \circ g_2) * \ldots * (i_n \circ g_n)$$

so

$$[f] = i_{1*}\left([g_1]\right) i_{2*}\left([g_2]\right) \ldots i_{n*}\left([g_n]\right)$$

so indeed $\pi_1(X, x_0) = \langle \operatorname{Im} i_{1*}, \ldots, \operatorname{Im} i_{n*} \rangle$.

***3.21.** If $n \geqslant 2$, prove that S^n is simply connected.

Proof: Assume $n \geqslant 2$. Cover S^n by two hemispheres. Let

$$U = \{(x_1, \ldots, x_{n+1}) \mid x_{n+1} > -\varepsilon\}$$

and

$$V = \{(x_1, \ldots, x_{n+1}) \mid x_{n+1} < \varepsilon\}$$

where ε is small, say even $\varepsilon = \frac{1}{2}$ works.
Then $U \cap V$ becomes homotopy equivalent to S^{n-1} which is path-connected when $n \geqslant 2$. Furthermore, U and V form an open cover of S^n with $(1, 0, \ldots, 0, 0) \in U \cap V$. Thus the conditions in exercise 3.20 are fulfilled, so $\pi_1(S^n) = \langle \operatorname{Im} i_{1*}, \operatorname{Im} i_{2*} \rangle$, where $i_{1*}\colon \pi_1(U, (1, \ldots, 0)) \to \pi_1(S^n, (1, \ldots, 0))$ and $i_{2*}\colon \pi_1(V, (1, 0, \ldots, 0)) \to \pi_1(S^n, (1, 0, \ldots, 0))$ are the induced homomorphisms by the inclusions. Now, clearly, since U and V are simply connected, these are the trivial homomorphisms, so their images are trivial, and so $\pi_1(S^n, (1, \ldots, 0)) \cong \{1\}$. And as S^n is path connected, we can write $\pi_1(S^n) \cong \{1\}$.

Definition. A **topological group** is a group G whose underlying set is a topological space that is T_1 and such that

1. the multiplication map $\mu\colon G \times G \to G$, given by $(x, y) \mapsto xy$, where the product is the group product, is continuous if $G \times G$ has the product topology;

2. the inversion map $i\colon G \to G$, given by $x \mapsto x^{-1}$, is continuous.

Example. Both $\mathbb{R}^n$ under addition and S^1 under multiplication are topological groups.

***3.23.** Let G be a topological group and let H be a normal subgroup that is closed in G (alternatively, if T_1 is not required, H does not need to be closed). Prove that G/H is a topological group, where G/H is regarded as the quotient space of G by the kernel of the natural map.

Proof: Suppose $\overline{x} \in G/H$. Then

$$\pi^{-1}(\overline{x}) = \bigcup_{\pi(y) = \overline{x}} \{y\} = \bigcup_{yx^{-1} \in H} \{y\} = \left\{y\colon yx^{-1} \in H\right\}$$

So $\pi^{-1}(\overline{x}) = \left(\mu \circ (-, x^{-1})\right)^{-1}(H)$ which is closed as $\mu \circ \left(-, x^{-1}\right)$ is continuous. Hence $\{\overline{x}\}$ is closed, so G/H is T_1.
Now we must show that $\tilde{\mu}\colon G/H \times G/H \to G/H$ defined by $\tilde{\mu}(\overline{x}, \overline{y}) = \overline{xy}$ is continuous.
If $\overline{x} = \overline{x'}$ and $\overline{y} = \overline{y'}$ then $x'x^{-1}, y'y^{-1} \in H$, so $(x'y')(xy)^{-1} = x'y'y^{-1}x^{-1} = \underbrace{x'x^{-1}}_{\in H}\underbrace{x\,\underbrace{y'y^{-1}}_{\in H}\,x^{-1}}_{\in H}$, where $xy'y^{-1}x^{-1}$ is in H since H is normal and $y'y^{-1} \in H$.

Now, if $U \subset G/H$ is open, then $\tilde{\mu}^{-1}(U) = (\pi \times \pi)\, \mu^{-1}\left(\pi^{-1}(U)\right)$ which is open.
We must also show that $\tilde{i}\colon G/H \to G/H$ by $\tilde{i}(\overline{x}) = \overline{x^{-1}}$ is continuous.

Firstly, if $\overline{x} = \overline{x'}$ then $x'x^{-1} \in H$ so $x\left(x'\right)^{-1} \in H$, hence $\overline{x'^{-1}} = \overline{x^{-1}}$, so $\tilde{i}$ is well defined.
Suppose $U \subset G/H$ is open. Then $\tilde{i}^{-1}(U) = \pi\left(i^{-1}\pi^{-1}(U)\right)$ which is open.

***3.24.** Let G be a simply connected topological group and let H be a discrete closed normal subgroup. Prove that $\pi_1\left(G/H, 1\right) \cong H$.

Proof: We make use of the following theorem:

Theorem 10.2. Let G be a path connected topological group, and let H be a discrete normal subgroup of G. If $p\colon G \to G/H$ is the natural homomorphism, then (G, p) is a covering space of G/H.

Proof: Suppose $\overline{g} \in G/H$. Then $p^{-1}(\overline{g}) = \{x \colon p(x) = \overline{g}\} = \left\{x \colon xg^{-1} \in H\right\} = Hg$.
Firstly, we show p is an open map. Suppose $U \subset G$ is open. Then

$$p^{-1}p(U) = \bigcup_{x \in U} Hx = \bigcup_{h \in H} hU$$

Now, $g \mapsto hg$ is a homeomorphism of $G \to G$, so hU is open as U is open. Hence $p^{-1}p(U)$ is open, so $p(U)$ is open.

Since H is discrete, we have that for every $h \in H$, there exists an open set $W_h \subset G$ such that $W_h \cap H = \{h\}$. Now, since W_e is a neighborhood of e, there exists a symmetric neighborhood V of e with $VV^{-1} = V^2 \subset W_e$.
Let $U = p(V)$ which is an open neighborhood of e in G/H as p is open. Now

$$p^{-1}(U) = p^{-1}p(V) = \bigcup_{h \in H} hV,$$

where each hV is open in G. If $h \neq k$ and $hV \cap kV \neq \varnothing$, then there are $v, w \in V$ with $hv = kw$, so $vw^{-1} = k^{-1}h \in VV^{-1} \cap H \subset W \cap H = \{e\}$, contradiction. Hence $hV \cap kV \neq \varnothing$.

Finally, $p|_{hV}$ is a homeomorphism from hV to U : we already know that $p|_{hV}$ is open and continuous; $p|_{hV}$ is surjective, since $p(hV) = p(h)p(V) = p(V) = U$; it is injective since $p(hv) = p(hw)$ implies $p(v) = p(w)$ and $vw^{-1} \in VV^{-1} \cap H = \{e\}$.

It is now easy to see that if $\overline{x} \in G/H$, then $\overline{x}U$ is an open neighborhood of $\overline{x}$ in G/H that is evenly covered by p. Therefore (G, p) is a covering space of G/H. $\square$
Proof of 3.24. Since G is simply connected, it is path connected, so theorem 10.2 applies. Hence $p\colon G \to G/H$ is a covering space of G/H.
Now, suppose $\overline{f}\colon (I, \{0,1\}) \to (G/H, 1)$ is a loop at $\overline{1}$.
Then we can lift this to a unique path $f\colon (I, 0) \to (G, 1)$ with $p \circ f = \overline{f}$. Thus $f(1)$ is uniquely determined, and since $p \circ f(1) = \overline{f}(1) = \overline{1} \in H$, we have $f(1) \in H$, so define $d\colon \pi_1\left(G/H, 1\right) \to H$ by $d\left(\overline{f}\right) = f(1)$.

Surjective: suppose $h \in H$ and choose a path from 1 to h in G, say $f\colon I \to G$. Then $\overline{f} = p \circ f$ is a loop at $\overline{1}$ with $d(\overline{f}) = f(1) = h$.

Injective: suppose $d(\overline{f}) = f(1) = e$; then since G is simply connected, we have $f \simeq i_1\colon I \to G$, so $\overline{f} = p \circ f \simeq p \circ i_1 = i_{\overline{1}}$.
Injectivity follows if we can show that d is a homomorphism.

Homomorphism: Suppose $\overline{f}, \overline{g} \in \pi_1\left(G/H, 1\right)$. We wish to show that $d\left(\overline{f}\overline{g}\right) = d(\overline{f})d\left(\overline{g}\right)$. Now, $\overline{g}$ has a unique lift starting at 1, call it $g\colon I \to G$. It ends at $g(1) \in H$. Let $f\colon I \to G$ be the unique lift of $\overline{f}$ starting at 1. Then $d(\overline{f})d\left(\overline{g}\right) = f(1)g(1)$. Now, $g(1) \in H$, so there is a lift of $\overline{f}$ starting at $g(1)$, call it $\tilde{f}\colon I \to G$. Then $\tilde{f} \cdot g\colon I \to G$ is well defined, and starts at 1 and $p \circ \tilde{f} \cdot g = \overline{f}\overline{g}$, so $d\left(\overline{f}\overline{g}\right) = \tilde{f}(1)$. We claim $\tilde{f}(t) = f(t)g(1)$. Both at lifts of $\overline{f}$ and both start at $g(1)$, so uniqueness gives $\tilde{f}(t) = f(t)g\left(1\right)$. Now $d\left(\overline{f}\overline{g}\right) = \tilde{f}(1) = f(1)g(1) = d(\overline{f})d(\overline{g})$. $\square$

3.25. Let $GL(n, \mathbb{R})$ denote the multiplicative group of all $n \times n$ nonsingular real matrices. Regard $GL(n, \mathbb{R})$ as a subspace of $\mathbb{R}^{n^2}$, and show that it and its subgroups are topological groups.

Proof: Regarding $A, B \in GL(n, \mathbb{R})$ as elements of $\mathbb{R}^{n^2}$, multiplication of these two has entry $\sum_{r=1}^{n} a_{ir} b_{rj}$ on the i, j coordinate which is a polynomial on $\mathbb{R}^{n^2} \times \mathbb{R}^{n^2}$, so in particular, it is continuous. As each coordinate function is continuous, μ is continuous.

For $i\colon \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$, we have that $i(A) = \det(A)^{-1} adj A$. Now, $\det(A) \neq 0$ as $A \in GL(n, \mathbb{R})$, and it is a continuous map as det is continuous since it is a polynomial on the entries of A. Similarly, as the entries of $adj A$ are polynomials in the entries of A also, it is continuous. Hence the product is continuous, so i is continuous.

3.26. A discrete normal subgroup H of a connected topological group G is contained in the center of G, hence is abelian. Conclude that $\pi_1(G/H, 1)$ is abelian when G is simply connected and H is a discrete closed normal subgroup.

Proof: Fix $h \in H$ and let $\varphi\colon G \to H$ be defined by $\varphi(x) = xhx^{-1}h^{-1}$. Then $\varphi^{-1}(\{e\}) = C_G(H) = \{x \in G \mid \forall h \in H\colon xh = hx\}$ and as $\{e\}$ is open, $C_G(H)$ is open.
As H is discrete, $H - \{e\}$ is open, so $\varphi^{-1}(H - \{e\}) = \{x \in G \mid \exists h \in H\colon xh \neq hx\}$. Thus $\varphi^{-1}(\{e\}) \cup \varphi^{-1}(G - \{e\}) = G$, and as G is connected, and $e \in \varphi^{-1}(\{e\})$, we have $\varphi^{-1}(H - \{e\}) = \varnothing$, so $G = C_G(H)$ and so H lies in the center of G.
If G is simply connected and H is a discrete closed normal subgroup, then $\pi_1(G/H, 1) \cong H$ which is abelian as $H \subset G \subset C_G(H)$, so $\pi_1(G/H, 1)$ is abelian.

The next result is a vast generalization of the conclusion of the last exercise.

Definition. A pointed space (X, x_0) is called an **H-space** (after H. Hopf) if there is a pointed map $m\colon (X \times X, (x_0, x_0)) \to (X, x_0)$ such that each of the (necessarily pointed) maps $m(x_0, -)$ and $m(-, x_0)$ on (X, x_0) is homotopic to $\mathbb{1}_X \operatorname{rel} \{x_0\}$. One calls x_0 a **homotopy identity**.

What this is saying is that a space is an H-space if there exists a map m taking (x_0, x_0) to x_0 for some x_0 in the space, such that $m(x_0, -)$ and $m(-, x_0)$ are homotopic to $\mathbb{1}_X \operatorname{rel} \{x_0\}$.

Suppose $X = G$ is a topological group, $x_0 = e$ and $m = \mu$ (multiplication). Then clearly $m\colon (G \times G, ()) \to (G, e)$ is a pointed map and $m(e, -) = m(-, e) = \mathbb{1}_G$, so every topological group is an H-space with identity and multiplication (one even has equality instead of relative homotopy).

How about the induced map $m(x_0, -)_*: \pi_1(X, x_0) \to \pi_1(X, x_0)$?
Let us restate the definition of an H-space in terms of maps: if $k\colon X \to X$ is a constant map at x_0 and $(k, \mathbb{1}_X): X \to X \times X$ is the map $x \mapsto (x_0, x)$, then $m(x_0, -)$ is the composite $m \circ (k, \mathbb{1}_X)$. Similarly, $m(-, x_0)$ is the composite $m \circ (\mathbb{1}_X, k)$. In an H-space, therefore, each of these composites is homotopic to $\mathbb{1}_X \operatorname{rel} \{x_0\}$.

Recall an elementary property of direct products of groups: if $x \in G$ and $y \in H$, then in $G \times H$,

$$(x, 1)(1', y) = (x, y) = \left(1', y\right)(x, 1),$$

where 1 denotes the identity element in H and $1'$ denotes the identity element of G.

Theorem 3.20. If (X, x_0) is an H-space, then $\pi_1(X, x_0)$ is abelian.

Proof: In theorem 3.7, we showed $\theta\colon \pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X \times X, (x_0, x_0))$ defined by $([f], [g]) \mapsto [(f, g)]$ is an isomorphism. Choose $[f], [g] \in \pi_1(X, x_0)$. Then

$$\begin{aligned}
[g] &= (m \circ (k, \mathbb{1}_X))_* [g] && \text{(Definition of H-space)} \\
&= m_* (k, \mathbb{1}_X)_* [g] && (\pi_1 \text{ is a functor}) \\
&= m_* [(k, \mathbb{1}_X) \circ g] && \text{(definition of induced map)} \\
&= m_* [(kg, g)] && \\
&= m_* \theta ([kg], [g]) && \text{(definition of } \theta) \\
&= m_* \theta (e, [g]),
\end{aligned}$$

where $e = [k]$ is the identity element of $\pi_1(X, x_0)$. Similarly,

$$[f] = m_*\theta([f], e),$$

because $m \circ (\mathbb{1}_X, k) \simeq \mathbb{1}_X \operatorname{rel}\{x_0\}$. Since $m_*\theta\colon \pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)$ is a homomorphism, we have

$$\begin{aligned} m_*\theta([f], [g]) &= m_*\theta((e, [g])([f], e)) \\ &= m_*\theta((e, [g]))\, m_*\theta(([f], e)) = [g][f]. \end{aligned}$$

If instead one factors $([f], [g]) = ([f], e)(e, [g])$, one obtains $m_*\theta([f], [g]) = [f][g]$. We conclude that $[g][f] = [f][g]$, hence $\pi_1(X, x_0)$ is abelian. □

Corollary 3.21. If G is a topological group, then $\pi_1(G, e)$ is abelian.

The contrapositive of this last corollary is also interesting. If X is a space with $\pi_1(X, x_0)$ not abelian, then there is no way to define a multiplication on X making it a topological group. Indeed, one cannot even equip such an X with the structure of an H-space.

Munkres

§53 Covering spaces

Def. Let $p\colon E \to B$ be a continuous surjective map. The open set U of B is said to be **evenly covered** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into **slices**.

Def. Let $p\colon E \to B$ be continuous and surjective. If every point b of B has a neighborhood U that is evenly covered by p, then p is called a **covering map,** and E is said to be a **covering space** of B.

§54 The Fundamental Group of the Circle

Lemma 54.2. Let $p\colon E \to B$ be a covering map; let $p(e_0) = b_0$. Let the map $F\colon I \times I \to B$ be continuous, with $F(0,0) = b_0$. There is a unique lifting of F to a continuous map

$$\tilde{F}\colon I \times I \to E$$

such that $\tilde{F}(0,0) = e_0$. If F is a path homotopy, then $\tilde{F}$ is a path homotopy.

Bredon

Def. A topological space X is said to be **irreducible** if whenever $X = F \cup G$ with F and G closed, then either $X = F$ or $X = G$. A subspace is irreducible if it is so in the subspace topology.

1.3.6. A **Zariski space** is a topological space with the property that every descending chain $F_1 \supset F_2 \supset F_3 \supset \dots$ of closed sets is eventually constant. Show that every Zariski space can be expressed as a finite union $X = Y_1 \cup Y_2 \cup \dots \cup Y_n$ where the Y_i are closed and irreducible and $Y_i \not\subset Y_j$ for $i \neq j$. Also show that this decomposition is unique up to order.

Proof: Let X be a Zariski space. If X is irreducible, we are done. Hence, suppose $X = Y_1 \cup Y_2$ with Y_1 and Y_2 closed and $Y_1 \not\subset Y_2$.
Now, we can choose either Y_1 or Y_2 such that we can decompose them infinitely as a union of closed sets not containing each other as otherwise the result follows. Suppose we can choose $X \supset Y_1 \supset Y_{11} \supset \dots$ where $Y_1 = Y_{11} \cup Y_{12}$ and so forth with $Y_{11} = Y_{111} \cup Y_{112}$. Then we get an infinite descending proper chain, which is impossible. Thus we can write $X = Y_1 \cup Y_2 \cup \dots \cup Y_n$ where the Y_i are closed and irredubile and $Y_i \not\subset Y_j$ for $i \neq j$.
Suppose this decomposition were not unique. Then let

$$X = Y_1 \cup Y_2 \cup \dots \cup Y_n = Z_1 \cup \dots \cup Z_m$$

Then $Z_1 \subset Y_1 \cup \dots \cup Y_n$, so $Z_1 = \bigcup_{i=1}^n (Y_i \cap Z_1)$. But each $Y_i \cap Z_1$ is closed, so Z_1 is a union of closed sets, and since it is irreducible, we must have that there exists some $i \in \{1, \dots, n\}$ such that $Z_1 = Y_i \cap Z_1$. Hence $Y_i = Z_1$. We can let $\sigma(1) = i$. Do the same for $Z_2, \dots, Z_n$ Then we see that $Z_j = \varnothing$ for $j > n$, so $m = n$. Hence the decomposition differs by a permutation of the indices, given by σ.

1.3.7. Let X be the real line with the topology for which the open sets are $\varnothing$ together with the complements of finite subsets. Show that X is an irreducible Zariski space.

Proof: The closed sets are finite sets and the whole space.
It is thus clear that any descending chain of closed sets is eventually constant. Furthermore, if $X = F \cup G$ with F and G closed, then X is a union of finite sets, which is impossible, so X is irreducible.

Def 2.3. If X is a topological space and $x \in X$ then a set N is called a **neighborhood** of x in X if there is an open set $U \subset N$ with $x \in U$.

Def. 2.4. If X is a topological space and $x \in X$ then a collection $\{B_x\}$ of subsets of X containing x is called a **neighborhood basis** at x in X if each neighborhood of x in X contains some element of $\{B_x\}$ and each element of $\{B_x\}$ is a neighborhood of x.

5. Separation Axioms

Prop. 5.2. A Hausdorff space is regular iff the closed neighborhoods of any point form a neighborhood basis of the point.

Corollary. A subspace of a regular space is regular.

1.5.6. Show that a Hausdorff space is normal iff for any sets U open and C closed with $C \subset U$ there is an open set V with $C \subset V \subset \overline{V} \subset U$.

Proof: Suppose X is Hausdorff normal. Then $X - U$ is closed, so there exist W, V such that $C \subset V$ and $X - U \subset W$, and $V \cap W = \varnothing$. Then $V \subset X - W$ which is closed, and $X - W \subset U$. Hence $C \subset V \subset \overline{V} \subset X - W \subset U$.

Conversely, suppose for any such U and C there exists such a V.
Now, let $F, G \subset X$ be closed disjoint subsets. Then $F \subset X - G$ which is open. Hence there exists open V with $F \subset V \subset \overline{V} \subset X - G$ and there exists open W with $G \subset W \subset \overline{W} \subset X - F$. Then $V' = V - \overline{W} \neq \varnothing$

is open and $W' = W - \overline{V} \neq \varnothing$ is open. These are nonempty, since $F \subset V$ and $F \not\subset \overline{W} \subset X - F$. Similarly for G. They are open as taking a closed set away from an open set leaves an open set. And

$$V' \cap W' = \left(V - \overline{W}\right) \cap \left(W - \overline{V}\right) \subset \left(V - \overline{V}\right) = \varnothing$$

This V' and W' satisfy the conditions. □

1.5.8. Show that if a Zariski space is Hausdorff then it is finite.

Proof: Suppose X is a Zariski Hausdorff space that is infinite. Then $X = Y_1 \cup Y_2 \cup \ldots \cup Y_n$ where the Y_i are closed irreducible subsets with $Y_i \not\subset Y_j$ if $i \neq j$. Thus there must exist some Y_i which is infinite. Let $x_0, x_1 \in Y_i$. There exist U, V open neighborhoods of x_0 and x_1 respectively with $U \cap V = \varnothing$. Hence $x_1 \in Y_i \cap (Y_i - U)$ and $x_0 \in Y_i \cap (Y_i - V)$ which are both closed sets with union Y_i, so one of them is infinite. So one of them is the whole space Y_i as Y_i is irreducible. But then the other is empty, contradiction.

6. Nets (Moore-Smith Convergence)

The generalization of a sequence is called a net.

Def. 6.1. A **directed set** D is a partially ordered set such that, for any two elements $\alpha, \beta \in D$, there is a $\tau \in D$ with $\tau \geqslant \alpha$ and $\tau \geqslant \beta$.

Def. 6.2. A **net** in a topological space X is a directed set D together with a function $\varphi\colon D \to X$.

So a sequence is simply a net based on the natural numbers as indexing set.

Def. 6.3. If $\varphi\colon D \to X$ is a net in the topological space X and $A \subset X$, then we say that φ is **frequently** in A if for any $\alpha \in D$ there is a $\beta \geqslant \alpha$ such that $\varphi(\beta) \in A$. It is said to be **eventually** in A if there is an $\alpha \in D$ such that $\varphi(\beta) \in A$ for all $\beta \geqslant \alpha$.

Def. A net $\varphi\colon D \to X$ in a topological space is said to converge to $x \in X$ if, for every neighborhood $U \subset X$ of x, φ is eventually in U.

Prop 6.5 A topological space X is Hausdorff iff any two limits of any convergent net are equal.

Proof: Suppose X is Hausdorff and $\varphi\colon D \to X$ converges to $x_0, x_1 \in X$. Then take disjoint neighborhoods U, V of x_0, x_1, resp. By assumption, there exists $\alpha, \beta \in D$ such that for all $\gamma \geqslant \alpha, \beta$, we have $\varphi(\gamma) \in U \cap V = \varnothing$; contradiction (such γ exist by assumption of a directed set).
Conversely, suppose $x_0, x_1 \in X$ with $x_0 \neq x_1$. Let $\mathcal{A}$ be the set of open sets of x_0 and $\mathcal{B}$ the set of open sets of x_1. Define a directed set D as follows: let $D = \{(U, V) \mid U \in \mathcal{A}, V \in \mathcal{B}\}$ with partial order $(U, V) \geqslant (U', V')$ if and only if $U \subset U'$ and $V \subset V'$.

Define a net $\varphi\colon D \to X$ as $\varphi(U, V) = x_{U,V}$ where $x_{U,V} \in U \cap V$.

Then φ converges to x_0 : let W be an open set of x_0 and choose V as any open set in $\mathcal{B}$. Then for all $(U', V') \geqslant (W, V)$ we have $U' \subset W, V' \subset W$ so $x_{U',V'} \in U' \cap V' \subset W \cap V \subset W$. Similarly, we can show that φ converges to x_1.

But as any two limits of any convergent net are equal, we conclude $x_0 = x_1$ in contradiction with the assumption. □

Prop 6.6. A function $f\colon X \to Y$ between two topological spaces is continuous iff for every net φ in X converging to $x \in X$, the net $f \circ \varphi$ in Y converges to $f(x)$.

Proof: Suppose $f\colon X \to Y$ is continuous. Let $\varphi\colon D \to X$ be a net converging to $x \in X$. Let U be an open set around $f(x)$. Then $f^{-1}(U)$ is an open set around x. Thus there exists some $\alpha \in D$ such that for all $\beta \geqslant \alpha$, $\varphi(\beta) \in f^{-1}(U)$. Hence $f(\varphi(\beta)) \in U$. So $f \circ \varphi$ is eventually in U, so $f \circ \varphi$ converges to $f(x)$.

Conversely, suppose that for every net φ in X converging to $x \in X$, the net $f \circ \varphi$ in Y converges to $f(x)$. Suppose f is not continuous; then there exists some open set $U \subset Y$ such that $f^{-1}(U)$ is not open. Then for some $x \in f^{-1}(U)$, the image of every neighborhood intersects U^c.
For any neighborhood V of x, let $w_V \in V - f^{-1}(U)$. Define $\varphi(V) = w_V$ for V any neighborhood V of x with the ordering $V \geqslant W$ if $V \subset W$. Then suppose N is a neighborhood of x, and let $V \geqslant N$. Thus $V \subset N$ and $\varphi(V) = w_V \in V - f^{-1}(U) \subset N$, so φ is eventually in N. So φ converges to x. But then $f \circ \varphi$ converges to $f(x)$. But $f \circ \varphi$ is in $Y - U$ always, so it cannot converge to $f(x) \in U$. ☐

Given a particular net $\varphi\colon D \to X$, let $x_\alpha = \varphi(\alpha)$ for $\alpha \in D$. Then it is common to speak of $\{x_\alpha\}$ as being the net in question. Hence proposition 6.6 takes on the familiar form

$$f(\lim x_\alpha) = \lim (f(x_\alpha)).$$

Prop 6.7. If $A \subset X$ then $\overline{A}$ coincides with the set of limits of nets in A which converge in X.

Proof: Suppose $\varphi\colon D \to X$ is a net in A with limit in X. Hence for all $d \in D, \varphi(d) \in A$. If $\lim x_\alpha$ denotes the limit of φ, then for all open neighborhoods U of $\lim x_\alpha$, there exists some $\tilde{d} \in D$ such that for all $d \geqslant \tilde{d}$ we have $\varphi(d) \in U$. Hence $U \cap A \neq \varnothing$. So $x \in \overline{A}$.

Conversely, if $x \in \overline{A}$, then letting $\{U\}$ be the set of open sets at x with partial order $U \geqslant V \iff U \subset V$, we can let $\varphi(U) = a_U$ with $a_U \in A \cap U$. Then let V be any open set of x. Then for all $U \geqslant V$, we have $\varphi(U) \in A \cap U \subset A \cap V \subset V$, so φ is eventually in V. Thus φ converges to x, so x is a limit of a net in A which converges in X. ☐

In ordinary sequences, a subsequence can be thought of in two different ways: (1) by discarding elements of the sequence and renumbering, or (2) by composing the sequence, thought of as a function $\mathbb{Z}^+ \to X$, with a function $h\colon \mathbb{Z}^+ \to \mathbb{Z}^+$, such that $i > j \implies h(i) > h(j)$. The first of these turns out to be inadequate for nets in general spaces. For the second method, a little thought should convince the reader that the last condition of monotonicity of h is stronger than is necessary for the usual uses of subsequences. Modifying it leads to the more general notion of a "subset", which we now define.

Def. 6.8 If D and D' are directed sets and $h\colon D' \to D$ is a function, then h is called **final** if, $\forall \delta \in D, \exists \delta' \in D' \ni (\alpha' \geqslant \delta' \implies h(\alpha') \geqslant \delta)$.

Def. 6.9 A **subnet** of a net $\mu\colon D \to X$, is the composition $\mu \circ h$ of μ with a final function $h\colon D' \to D$.

Prop. 6.10. A net $\{x_\alpha\}$ is frequently in each neighborhood of a given point $x \in X \iff$ it has a subnet which converges in x.

Next we treat a powerful concept for nets which has no analogue for sequences.

Def. 6.11. A net in a set X is called **universal** if, for any $A \subset X$, the net is either eventually in A or eventually in $X - A$.

Prop. 6.12. The composition of a universal net in X with a function $f\colon X \to Y$ is a universal net in Y.

Proof: Let $A \subset Y$. Let $A' = f^{-1}(A)$. Then a net $\mu\colon D \to X$ is either eventually in A' or in $X - A'$. So there exists $\alpha \in D$ such that for all $\beta \geqslant \alpha$ either $\mu(\beta) \in A'$ or $\mu(\beta) \in X - A'$ which means either $f(\mu(\beta)) \in A$ or $f(\mu(\beta)) \in Y - A$. So $f \circ \mu$ is a universal net in Y.

Theorem 6.13. Every net has a universal subnet.

Proof: Let $\{x_\alpha \mid \alpha \in P\}$ be a net in X. Consider all collections C of subsets of X such that:

1. $A \in C \implies \{x_\alpha\}$ is frequently in A ; and
2. $A, B \in C \implies A \cap B \in C$.

For example, $C = \{X\}$ is such a collection. Order the family of all such collections C by inclusion. The union of any simply ordered set of such collections is clearly such a collection, i.e., satisfies (1) and (2). By the Maximality Principle, there is a maximal such collection C_0.
Let $P_0 = \{(A, \alpha) \in C_0 \times P \mid x_\alpha \in A\}$ and order P_0 by

$$(B, \beta) \geqslant (A, \alpha) \iff B \subset A \text{ and } \beta \geqslant \alpha.$$

This gives a partial order on P_0 making P_0 into a directed set. Map $P_0 \to P$ by taking (A, α) to α. This is clearly final and thus defines a subnet we shall denote by $\left\{x_{(A,\alpha)}\right\}$. We claim that this subnet is universal.
Suppose S is any subset of X such that $\left\{x_{(A,\alpha)}\right\}$ is frequently in S. Then for any $(A, \alpha) \in P_0$, there is a $(B, \beta) \geqslant (A, \alpha)$ in P_0 with $x_\beta = x_{(B,\beta)} \in S$. Then $B \subset A, \beta \geqslant \alpha$ and $x_\beta \in B$. Thus $x_\beta \in S \cap B \subset S \cap A$. We conclude that $\{x_a a\}$ is frequently in $S \cap A$ for any $A \in C_0$. But then we can throw S and all the sets $S \cap A$ for $A \in C_0$ into C_0 and conditions (1) and (2) will still hold. By maximality, we must have $S \in C_0$. If $\left\{x_{(A,\alpha)}\right\}$ we also frequently in $X - S$, then $X - S$ would be in C_0, and so $\varnothing = S \cap (X - S)$ would be in C_0, by (2), and this is contrary to (1). Thus we conclude that $\left\{x_{(A,\alpha)}\right\}$ is not frequently in $X - S$, and so is eventually in S.

We have shown that if $\left\{x_{(A,\alpha)}\right\}$ is frequently in a set S then, in fact, it is eventually in S. This implies that $\left\{x_{(A,\alpha)}\right\}$ is universal.

Prop 6.14. A subnet of a universal net is universal.

1.6.1. Show that a sequence is a universal net if and only if it is eventually constant.

Proof: If it is eventually constant, it is clear.

Conversely, suppose $x\colon \mathbb{N} \to X$ is a universal net, denoted by x_n. Suppose it is not eventually constant. Thus, for any $N \in \mathbb{N}$, there exists $M > N$ such that $x_N \neq x_M$. We can thus define a sequence $(x_{k_i})_{i\in\mathbb{N}}$ with $x_{k_i} \neq x_{k_{i+1}}$ for all $i \in \mathbb{N}$. By proposition 6.14, a subnet of a universal net is universal, so $\{x_{k_i}\}$ is universal. Let $A = \{x_{k_{2i}}\}_{i\in\mathbb{N}} \subset X$. Then $\{x_{k_i}\}$ is eventually in A or $X - A$. Suppose it is eventually in A, so there exists $N \in \mathbb{N}$ such that for $i \geqslant N, x_{k_i} \in A$. But then $x_{k_{2N+1}} \notin A$ by definition, contradiction. So $\{x_{k_i}\}$ is eventually in $X - A$, but then for some $N \in \mathbb{N}$, for all $i \geqslant N$ we have $x_{k_i} \in X - A$, but then $x_{k_{2N}} \in X - A$, contradiction. So x is not universal. ☐

7. Compactness

The following notion is mainly of use for locally compact spaces X, Y, but makes sense for all topological spaces.

Def. 7.12 A map $f\colon X \to Y$ between topological spaces is said to be **proper** if $f^{-1}(C)$ i s compact for each compact subset C of Y.

Theorem 7.13. If $f\colon X \to Y$ is a closed map and $f^{-1}(y)$ is compact for each $y \in Y$, then f is proper.

Proof: Let $C \subset Y$ be compact. Let $\{U_\alpha \mid \alpha \in A\}$ be a collection of open sets whose union contains $f^{-1}(C)$. For any $y \in C, f^{-1}(y)$ is compact, so there exists a finite subcollection $A_y \subset A$ such that $\{U_\alpha \mid \alpha \in A_y\}$ covers $f^{-1}(y)$. Let $U_y = \bigcup_{\alpha\in A_y} U_\alpha$. Let $V_y = Y - f(X - U_y)$ which is open and contains y. Thus $\bigcup_{y\in C} V_y$ contains C so there exists $y_1, y_2, \ldots, y_n$ which suffice to cover C. But then

$$\begin{aligned} f^{-1}(C) \subset f^{-1}(V_{y_1} \cup \ldots \cup V_{y_n}) &= f^{-1}(V_{y_1}) \cup \ldots \cup f^{-1}(V_{y_n}) \\ &\subset U_{y_1} \cup \ldots \cup U_{y_n} \end{aligned}$$

Hence $f^{-1}(C)$ is covered by a finite union of open sets from $\{U_\alpha \mid \alpha \in A\}$. So $f^{-1}(C)$ is compact.

Theorem 7.14. For a topological space X the following are equivalent:

1. X is compact.
2. Every collection of closed subsets of X with the finite intersection property has a nonempty intersection.
3. Every universal net in X converges.
4. Every net in X has a convergent subnet.

Proof: (1) $\implies$ (3) : let $x\colon D \to X$ be a universal cover where X is compact. Suppose $\{x_\alpha\}$ does not converge. So for any $x \in X$, we can find a neighborhood U such that $\{x_\alpha\}$ is not eventually in U. Let U_x denote this neighborhood. By compactness, X is covered by some $U_{x_1} \cup \ldots \cup U_{x_n}$. But by universality, $\{x_\alpha\}$ is in one of these open sets eventually; contradiction.
(3) $\implies$ (4) is clear as every net has a universal subnet.
(4) $\implies$ (2): Suppose $\{C_\alpha\}$ is a collection of closed subsets of X with the finite intersection property; i.e., for any $S \subset \{C_\alpha\}$ with S finite, $\bigcap_{A \in S} A \neq \varnothing$. Define a directed set as follows: let the elements be any finite intersection of elements of $\{C_\alpha\}$ and order by inclusion. Since the finite intersection of two finite intersections is finite, it is nonempty and thus this becomes a directed set; call it D.
Now define a net $\mu\colon D \to X$ as mapping C to some element of C, say x_C. This has a convergent subnet, so there is some x' and there exists a final function $h\colon D' \to D$ with $\mu \circ h$ converging to x'. Thus, for any neighborhood N of x', there exists some $\delta' \in D'$ such that for any $\alpha' \geqslant \delta'$, we have $\mu \circ h\,(\alpha') \in N$. But in particular, for any $C \in D$, we can find a $\delta' \in D'$ with $\alpha' \geqslant \delta' \implies h\,(\alpha') \geqslant C$, i.e., $h\,(\alpha') \subset C$. Thus $\{h\,(\alpha')\}_{\alpha' \in D'}$ eventually becomes a sequence in C which converges in X, so by proposition 6.7, $x' \in \overline{C} = C$ for any $C \in D$. Hence $x' \in \bigcap_{C \in D} C$, so $x' \in \bigcap_{C \in D} C \subset \bigcap C_\alpha$, so $\bigcap C_\alpha \neq \varnothing$.

1.7.2. Let X be a compact space and let $\{C_\alpha \mid \alpha \in A\}$ be a collection of closed sets, closed with respect to finite intersections. Let $C = \bigcap C_\alpha$ and suppose that $C \subset U$ with U open. Show that $C_\alpha \subset U$ for some α.

Proof: Firstly, $C \neq \varnothing$ by theorem 7.14.(2). The collection $D = \{C_\alpha \mid \alpha \in A\}$ is a directed set with partial order by inclusion. Suppose $C_\alpha \not\subset U$ for all α. Then define a net $\mu\colon D \to X$ by $\mu\,(C_\alpha) = x_{C_\alpha}$ where $x_{C_\alpha} \in C_\alpha - U$. This net has a convergent subnet, say $\mu \circ h$ with $h\colon D' \to D$ a final map. Suppose $\mu \circ h$ converges to x'. We claim $x' \in U$.
For any $C \in D$, we can find a $\delta' \in D'$ such that $\alpha' \geqslant \delta' \implies h\,(\alpha') \geqslant C$ which means $h\,(\alpha') \subset C$. This means that for any $C \in D$, $\mu \circ h\,(\alpha)$ eventually becomes a sequence in C, so its limit, x', is in $\overline{C} = C$. Thus $x' \in \bigcap_{C_\alpha \in D} C_\alpha = C \subset U$.
However, for any $C_\alpha \in D$, $\mu\,(C_\alpha) \in C_\alpha - U$, so μ is never in U and hence any subnet is also never in U; contradiction. □

Products

Prop 8.2. If X is compact, then the projection $\pi_Y\colon X \times Y \to Y$ is closed.

Proof: Let $C \subset X \times Y$ be closed. We wish to show that $Y - \pi_Y(C)$ is open. Let $y \notin \pi_Y(C)$, so $(x, y) \notin C$ for all $x \in X$. Then for any $x \in X$, there are open sets $U_x \subset X$ and $V_x \subset Y$ such that $x \in U_x, y \in V_y$ and $(U_x \times V_x) \cap C = \varnothing$.
Since X is compact, there are points $x_1, \ldots, x_n \in X$ such that $U_{x_1} \cup \ldots \cup U_{x_n} = X$. Let $V = V_{x_1} \cap \ldots \cap V_{x_n}$. Then

$$(X \times V) \cap C = (U_{x_1} \cup \ldots \cup U_{x_n}) \times (V_{x_1} \cap \ldots \cap V_{x_n}) \cap C = \varnothing.$$

Thus $y \in V \subset Y - \pi_Y(C)$ and V is open. Since y was arbitrary, it follows that $Y - \pi_Y(C)$ is open, so $\pi(Y)$ is closed. □

Corollary 8.3. If X is compact, then $\pi_Y\colon X \times Y \to Y$ is proper.

Proof: Follows from theorem 7.13 and proposition 8.2. □

Prop. 8.8 A net in a product space $X = \bigtimes X_\alpha$ converges to a point $(\ldots, x_\alpha, \ldots)$ if and only if its composition with each projection $\pi_\alpha\colon X \to X_\alpha$ converges to x_α.

Proof: Suppose $\mu\colon D \to X = \bigtimes X_\alpha$ is a net converging to $(\ldots, x_\alpha, \ldots)$.
Since π_α is continuous by proposition 8.1, we get by proposition 6.6, that $\pi_\alpha \circ \mu$ converges to x_α.

Conversely, suppose each $\mu\colon D \to X$ is a net and $\pi_\alpha \circ \mu$ converges to x_α for each α. Let $U \subset X$ be an open sets around $(\ldots, x_\alpha, \ldots)$ and take a basis open set, V, contained in U around this point too. Suppose the non-whole-space coordinates for this basis open set are $V_{\alpha_1}, \ldots, V_{\alpha_n}$. Then V_{α_i} is a neighborhood around x_{α_i}, so since $\pi_{\alpha_i} \circ \mu$ converges to x_{α_i}, there is some $\delta_i \in D$ such that for all $\delta \geqslant \delta_i$, we have $\pi_{\alpha_i} \circ \mu(\delta) \in V_{\alpha_i}$, we have that for $\delta \geqslant \delta_1, \ldots, \delta_n$, we have $\pi_{\alpha_j} \circ \mu(\delta) \in V_{\alpha_j}$ for $j = 1, \ldots, n$. And since for any other α, we have $\pi_\alpha \circ \mu \in X_\alpha$ which is the αth basis open set of V, we have $\mu(\delta) \in V$ for all $\delta \geqslant \delta_1, \ldots, \delta_n$. Thus μ is eventually in V, and as V was arbitrary, we have that μ converges to $(\ldots, x_\alpha, \ldots)$. $\square$

Theorem 8.9 (Tychonoff). The product of an arbitrary collection of compact spaces is compact.

Proof: Let $X = \bigtimes X_\alpha$ where X_α are compact. Let $f\colon D \to X$ be a universal net in X. Then the composition $\pi_\alpha \circ f$ is also a universal net by proposition 6.12. Therefore this composition converges, say to x_α, by theorem 7.14. But this means that the original net converges to the point whose α th coordinate is x_α by proposition 8.8, and so X is compact by theorem 7.14.

If X is a space and A is a set, the product of A copies of X is often denoted by X^A and can be thought of as the space of functions $f\colon A \to X$. In this context, proposition 8.8 takes the following form:
Proposition 8.10. A net $\{f_\alpha\}$ in X^A converges to $f \in X^A$ if and only if $\forall x \in X\colon f_\alpha(x) \to f(x)$. In particular, $\lim (f_\alpha(x)) = (\lim f_\alpha)(x)$.

When A also has a topology, the notation X^A is often used for the set of all *continuous* functions $f\colon A \to X$. In that context, a topology is often used on this set that differs from the product topology. There are several useful topologies in particular circumstances, and so the context must indicate what topology, if any, is meant by this notation.

Def. 8.11. If X and Y are spaces, then their **topological sum** or **disjoint union** $X + Y$ is the set $X \times \{0\} \cup Y \times \{y\}$ with the topology making $X \times \{0\}$ and $Y \times \{1\}$ clopen and the inclusions $x \mapsto (x, 0)$ of $X \to X + Y$ and $y \mapsto (y, 1)$ of $Y \to X + Y$ homeomorphisms to their images. More generally, if $\{X_\alpha \mid \alpha \in A\}$ is an indexed family of spaces then their **topological sum** $+_\alpha X_\alpha$ is $\bigcup \{X_\alpha \times \{\alpha\} \mid \alpha \in A\}$ given the topology making each $X_\alpha \times \{\alpha\}$ clopen and each inclusion $x \mapsto (x, \beta)$ of $X_\beta \to +_\alpha X_\alpha$ a homeomorphism to its image $X_\beta \times \{\beta\}$.

In ordinary parlance, if X and Y are disjoint spaces, one regards $X + Y$ as $X \cup Y$ with the topology making X and Y open subspaces.

1.8.6. Let $f, g\colon X \to Y$ be two maps. If Y is Hausdorff, then show that the subspace $A = \{x \in X \mid f(x) = g(x)\}$ is closed in X.

Proof: We have that $x \in A$ if and only if $(f(x), g(x)) \in \Delta \subset Y$ where $\Delta = \{(y, y) \mid y \in X\}$. Now, since Y is Hausdroff, Δ is closed, so since f and g are continuous, $x \mapsto (f(x), g(x))$ is continous, so $A = (f(x), g(x))^{-1}(\Delta)$ is closed in X. $\square$

Differentiable Manifolds

1. The implicit function theorem