Conventions for this assignment: We assume all topological spaces to be nice enough for covering theory (we can even assume locally contractible). Basepoints are assumed to be good basepoints, namely the inclusion $\{x\} \subset X$ is assumed to have the homotopy extension property. If X is a space, then ΩX denotes its loop space and there is a fiber sequence

$$\Omega X \to PX \to X$$

where PX is a contractible space.

Problem 0.1. Show that the homology of $\Omega\left(S^2 \vee S^3\right)$ is

$$H_*\left(\Omega\left(S^2 \vee S^3\right); \mathbb{Z}\right) \cong \mathbb{Z}^{F_n}$$

where F_n is the n th Fibbonaci number (We set $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$).

Proof. Using the fiber sequence

$$\Omega\left(S^2 \vee S^3\right) \to P\left(S^2 \vee S^3\right) \to S^2 \vee S^3$$

we obtain the following quadrant double complex:

$$\begin{tikzcd}
\vdots \arrow[d, dash] & \vdots & \vdots & \\
H_3(\Omega(S^2 \vee S^3)) \arrow[d, dash] & H_3(\Omega(S^2 \vee S^3)) & H_3(\Omega(S^2 \vee S^3)) & \\
H_2(\Omega(S^2 \vee S^3)) \arrow[d, dash] \arrow[dr] \arrow[ddrr] & H_2(\Omega(S^2 \vee S^3)) & H_2(\Omega(S^2 \vee S^3)) & \\
H_1(\Omega(S^2 \vee S^3)) \arrow[d, dash] \arrow[dr, "\cong"] & H_1(\Omega(S^2 \vee S^3)) & H_1(\Omega(S^2 \vee S^3)) & \\
\mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \cdots
\end{tikzcd}$$

In this complex, representing the E^2 page, we obtain that the d_2 emminating from $H_1\left(\Omega\left(S^2 \vee S^3\right)\right)$ to $\mathbb{Z}$ must be an isomorphism, since in E^∞ which represents the homology of the total space $P\left(S^2 \vee S^3\right)$ which is contractible, we have that the $\mathbb{Z}$ at $(2,1)$ vanishes, hence it must vanish in E^2 as d_2 is the only nontrivial map emminating or terminating at $(2,1)$. This gives surjectivity of this map, and the same argument on $H_1\left(\Omega\left(S^2 \vee S^3\right)\right)$, which must also vanish, gives that d_2 must be injective as well.
Next, we come to the inductive part of the diagram. Note that in E^∞, all $H_n\left(\Omega\left(S^2 \vee S^3\right)\right)$ must vanish for $n \geq 1$. Furthermore, any map in E^k for $k \geq 4$ has horizontal length greater than the greatest horizontal distance between nontrivial objects of the double complex, hence all maps in E^k, for $k \geq 4$, must be trivial, so $E^4 = E^\infty$. Hence all the homologies of $\Omega\left(S^2 \vee S^3\right)$ must vanish because of the maps d_2 and d_3. Firstly, note that d_2 maps $d_2\colon H_i\left(\Omega\left(S^2 \vee S^3\right)\right) \to H_{i-1}\left(\Omega\left(S^2 \vee S^3\right)\right)$, and in particular, this map must be surjective since it is the only nontrivial map terminating at the homologies in the second column which all must vanish. Next note that we similarly can see that the maps d_3 must be surjective (killing off the

terms in the third column) and injective as they must kill of the objects in the 0 th column.
Hence we find that we obtain a SES

$$0 \to H_{i-1}\left(\Omega\left(S^2 \vee S^3\right)\right) \to H_{i+1}\left(\Omega\left(S^2 \vee S^3\right)\right) \to H_i\left(\Omega\left(S^2 \vee S^3\right)\right) \to 0$$

Inserting $\mathbb{Z}$ for H_0 and H_1 when $i = 1$, we obtain, since $\mathbb{Z}$ is projective, that the sequence splits and $H_2\left(\Omega\left(S^2 \vee S^3\right)\right) \cong \mathbb{Z}^2$. Assume that $H_k\left(\Omega\left(S^2 \vee S^3\right)\right) \cong \mathbb{Z}^{F_k}$ for $k \leq N-1$. Then again

$$0 \to \mathbb{Z}^{F_{N-2}} \to H_N\left(\Omega\left(S^2 \vee S^3\right)\right) \to \mathbb{Z}^{F_{N-1}} \to 0$$

Again $\mathbb{Z}^{F_{N-1}}$ is projective, so the SES splits, so

$$H_N\left(\Omega\left(S^2 \vee S^3\right)\right) \cong \mathbb{Z}^{F_{N-2}} \oplus \mathbb{Z}^{F_{N-1}} \cong \mathbb{Z}^{F_{N-2}+F_{N-1}} \cong \mathbb{Z}^{F_N}.$$

Induction now finishes the proof.

$\square$