

1.2.ii:

(i) If $f: x \rightarrow y$ is a split epimorphism then there exists $g: y \rightarrow x$ such that $fg = \mathbb{1}_y$.

Let $c \in C$ be arbitrary, then for any $h \in C(c, y)$, we have $h = \mathbb{1}_y h = fgh = f_*(gh)$ with $gh \in C(c, x)$ by composition. So f_* is surjective.

Conversely, if $f_*: C(c, x) \rightarrow C(c, y)$ is surjective for all $c \in C$, then letting $c = y$, we get a map $f_*: C(y, x) \rightarrow C(y, y)$ that is surjective. Since $\mathbb{1}_y \in C(y, y)$, let $g \in C(y, x)$ such that $fg = f_*(g) = \mathbb{1}_y$. Then f is a split epimorphism.

1.2.iii:

(i): if $f: x \rightarrow y$ and $g: y \rightarrow x$ are monomorphisms, then if for maps h, k we have $fgh = fgk$, then $gh = gk$ since f is monic and then $h = k$ since g is monic. Thus $gf: x \rightarrow x$ is monic.

(ii) Assume $f: x \rightarrow y$ and $g: y \rightarrow z$ are morphisms so that gf is monic. Now assume $fh = fk$ for some maps h, k with codomain x . Then composing with g on the left we get $gfh = gfk$ and since gf is monic, we get $h = k$. Therefore f is monic.

We thus have if for $f^{op}: y \rightarrow x$ and $g^{op}: z \rightarrow y$ monic, we have $(fg)^{op} = g^{op}f^{op}: z \rightarrow x$ is monic.

Taking the dual, noting that the dual of a monic function is an epic function, we get: if $f: x \rightarrow y$ and $g: y \rightarrow z$ are epic, then $fg: x \rightarrow z$ is epic too.

Similarly, we have from (ii): if $f^{op}: y \rightarrow x$ and $g^{op}: z \rightarrow y$ are morphisms so that $(gf)^{op} = f^{op}g^{op}$ is monic, then g^{op} is monic.

Taking the dual we get: if $f: x \rightarrow y$ and $g: y \rightarrow z$ are morphisms so that gf is epic then g is epic.

Restricting the morphisms of a category to only its monomorphisms or restricting to only its epimorphisms thus gives a subcategory since composition is well defined by (i) and (i') and the identities are trivial monomorphisms and epimorphisms; the rest of the requirements follow directly from the parent category.

1.3.i: Any functor between groups must send the single group object to the single group object - so there is one possible map of objects. By the functoriality axioms, we must have that the identity of the group is mapped to the identity of the other group.

The mapping of any morphism (any group element) must also fulfill $F(g_1g_2) = F(g_1)F(g_2)$ which simply says that F is a group homomorphism.

Conversely, any group homomorphism also gives a functor between groups: if $\varphi: G \rightarrow H$ is a group homomorphism. Then the corresponding functor, F , maps the single object G to the single object H and the morphism g_1 of G is mapped to $\varphi(g_1)$. By this, we have $F(e_G) = \varphi(e_G) = e_H$ and $F(g_1g_2) = \varphi(g_1g_2) = \varphi(g_1)\varphi(g_2) = F(g_1)F(g_2)$ so the functoriality axioms are satisfied. Hence F is a functor, so group homomorphisms correspond to functors between group categories.