Exercise 0.1. Let A and C be abelian groups. The goal of this exercise is to "classify" short exact sequences of the form $0 \to A \to (?) \to C \to 0$. To begin with, we construct some special short exact sequences.

(1) For a function $k\colon C \times C \to A$, define a binary operation "$+_k$ " on the set $C \times A$ by

$$(c, a) +_k (c', a') = (c + c', a + a' + k(c, c'))\,.$$

Prove that this operation makes $C \times A$ into an abelian group, provided k satisfies $k(x, 0) = 0 = k(0, x)$ for all $x \in C$ and that

$$k(x_1, x_2) - k(x_0 + x_1, x_2) + k(x_0, x_1 + x_2) - k(x_0, x_1) = 0$$

for all $x_0, x_1, x_2 \in C$. In the rest of this problem, we shall write $C \times_k A$ for this group, with underlying set $C \times A$ and group operation $+_k$.

Solution. (1):
We have that, choosing $x_0 = x_2 = c'$ and $x_1 = c$, we get

$$k(c, c') - k(c', c) = k\,(c' + c, c') - k\,(c', c + c')$$

hence $k(c, -c) = k(-c, c)$ for all c.

$$(c, a) +_k (c', a') = (c + c', a + a' + k(c, c')) = (c' + c, a' + a + k(c', c))\,.$$