1. Hatcher

1.1. **Exact Couples.**

Definition 1.1 (Exact Couple)**.** An *exact couple* is an exact sequence of abelian groups of the form

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

where i, j and k are group homomorphisms. Define $d\colon B \to B$ by $d = j \circ k$. Then $d^2 = j(kj)k = 0$, so $H(B) := \ker d / \operatorname{im} d$ is defined - in particular, since A and B are abelian, the quotient $H(B)$ is well-defined and a group.

Definition 1.2 (Derived Couple)**.** Out of a given exact couple, we can construct a new exact couple, called the *derived couple*:

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd}$$

where we define

(1) $A' = i(A)$ and $B' = H(B)$.
(2) i' is the induced map $i' := i|_{A'}\colon A' \to A'$ by $i'(ia) = i(ia)$
(3) We define j' by $j'a' = [ja]$ where $a' = ia$ for some a in A.
(4) k' is defined by $k'[b] = kb \in i(A)$.

With these definitions, the derived couple is an exact couple.

Exercise 1.3. Check that the maps are well-defined and that the derived sequence is exact.

Proof. We must check that j' and k' are well-defined maps.
Suppose $a' = ia = i\tilde{a}$. Then $a - \tilde{a} \in \ker i = \operatorname{im} k$ so $a - \tilde{a} = k[b]$. Hence Then $ja - j\tilde{a} = jk[b] = d[b] \in \operatorname{im} d$, so $[ja] = [j\tilde{a}]$.
Next, suppose $[b] = \left[\tilde{b}\right]$, so $b - \tilde{b} \in \operatorname{im} d$, i.e., $b - \tilde{b} = jk(\bar{b})$. Then $kb - k\tilde{b} = kjk(\bar{b}) = 0$, so $k'[b] = k'\left[\tilde{b}\right]$.

Lastly, exactness at B': suppose $k'[b] = 0$. Then $kb = 0$, so by exactness of the original exact couple, there exists some $a \in A$ such that $j(a) = b$. Then let $a' = i(a)$, so $j'(a') = [j(a)] = [b]$, hence $\ker k' \subset \operatorname{im} j'$.
Conversely, $k'j'(a') = k'[ja] = kja = 0$, by exactness at B of the original couple. □

Definition 1.4 (Spectral Sequence)**.** A *spectral sequence* $(E_{*,*}, d)$ (in homological Serre grading), starting on page $r_0 \geq 1$, consists of:

(1) a bigraded group $\left(E^r_{p,q}\right)_{p,q\in\mathbb{Z}}$ for each $r \geq r_0$, called the r *th page* of the spectral sequence.
(2) For all $r \geq r_0$ and $p, q \in \mathbb{Z}$ a map of abelian groups

$$d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

called the$r$ *th differential* which squares to zero in the sense that

$$d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$$

holds for all p, q, r.

(3) For all $r \geq r_0$ and $p, q \in \mathbb{Z}$, isomorphisms of abelian groups

$$E^{r+1}_{p,q} \cong \frac{\ker\left(d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r-1}\right)}{\operatorname{im}\left(d^r_{p+q,q-r+1} \colon E^r_{p+r,q-r+1} \to E^r_{p,q}\right)}$$

Definition 1.5. We say that a spectral sequence $(E_{*,*}, d)$ *converges* to a graded abelian group H_* and write

$$E^2_{p,q} \implies H_{p+q}$$

if there is a filtration

$$0 \subset F^0_n \subset F^1_n \subset \ldots \subset F^{n-1}_n \subset F^n_n = H_n$$

and isomorphisms $E^\infty_{p,q} \cong F^p_{p+q}/F^{p-q}_{p+q}$.

Note that if $(E_{*,*}, d)$ is a first quadrant spectral sequence, then $E^{r+1}_{p,q} \cong E^r_{p,q}$ for $r > \max\{p, q+1\}$ because $d^r_{p,q}$ maps to the 0 group and $d_{p+r,q-r+1}$ comes from the 0 group.

Definition 1.6 (E^∞-page)**.** For a first quadrant spectral sequence, we define the *E^∞-page* as:

$$E^\infty_{p,q} := E^r_{p,q}, \quad \text{for } r \gg p, q$$

Lemma 1.7. *If*

$$1 \to A \to B \to C \to 1$$

is a SES of groups, then $B = A \rtimes C$.

Theorem 1.8 (Leray-Serre spectral sequence)**.** *For every abelian group G and every fiber sequence*

$$F \to E \to B$$

such that $\pi_1(B)$ acts trivially on $H_*(F; G)$, there is a natural, convergent Leray-Serre spectral sequence of signature

$$E^2_{p,q} = H_p\left(B; H_q(F; G)\right) \implies H_{p+q}(E; G)$$

meaning that the $E^2_{p,q}$ page is given by $E^2_{p,q} = H_p\left(B; H_p(F; G)\right)$ and there is a natural filtration

$$0 = F^n_{-1} \subset F^0_n \subset \ldots \subset F^n_n = H_n(E; G)$$

and natural SES:

$$0 \to F^{p+q}_{p-1} \hookrightarrow F^{p+q}_p \twoheadrightarrow E^\infty_{p,q} \to 0$$

Note. The SES in Theorem 1.8 splits as

$$\begin{aligned}
H_n(E; G) = F^n_n &\cong F^n_{n-1} \rtimes E^\infty_{n,0} \\
&\cong F^n_{n-2} \rtimes E^\infty_{n-1,1} \rtimes E^\infty_{n,0} \\
&\vdots \\
&\cong F^n_0 \rtimes E^\infty_{1,n-1} \rtimes \ldots \rtimes E^\infty_{n,0} \\
&\cong E^\infty_{0,n} \rtimes E^\infty_{1,n-1} \rtimes \ldots \rtimes E^\infty_{n,0}
\end{aligned}$$

Example 1.9. Suppose
$$F \to E \to B$$
is a fiber sequence and that $H_n(E;G) = 0$ for an abelian group G. Then $E^\infty_{p,n-p} = 0$ for all $0 \le p \le n$.
This can be seen because $F^n_n = H_n(E;G) = 0$, and $0 \subset F^0_n \subset \ldots \subset F^n_n = 0$, hence $E^\infty_{p,n-p} \cong F^p_n/F^{p-1}_n \cong 0$.

Example 1.10. Suppose that the $E^\infty_{p,q}$ are abelian groups. Then the semidirect products reduce to normal direct products, so that
$$H_n(E;G) \cong \bigoplus_{p=0}^{n} E^\infty_{p,n-p}$$
For example, if G is a field, then $H_n(E;G)$ is a G-vector space, hence abelian, so each F^p_n being subgroups of $H_n(E;G)$ is abelian, so each $E^\infty_{p,q} \cong F^p_n/F^{p-q}_n$ is abelian.

1.2. **Serre Classes.** Let $\mathcal{C}$ be one of the following classes of abelian groups:

(1) $\mathcal{FG}$, finitely generated abelian groups.
(2) $\mathcal{T}_p$, torsion abelian groups whose elements have orders divisible only by primes from a fixed set P of primes.
(3) $\mathcal{F}_p$, the finite groups in $\mathcal{T}_p$.

Note. P could be all primes and then $\mathcal{T}_p$ would be all torsion abelian groups and $\mathcal{F}_p$ would be all finite abelian groups.

Theorem 1.11. *If X is simply-connected, then $\pi_n(X) \in \mathcal{C}$ for all n if and only if $H_n(X;\mathbb{Z}) \in \mathcal{C}$ for all $n > 0$. This holds also if X is path-connected and abelian, that is, the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial for all $n \geq 1$.*

Theorem 1.12 (Hurewicz modulo $\mathcal{C}$)**.** *If a path-connected abelian space X has $\pi_i(X) \in \mathcal{C}$ for $i < n$, then the Hurewicz homomorphism $h\colon \pi_n(X) \to H_n(X)$ is an isomorphism $\mod \mathcal{C}$, meaning that the kernel and cokernel of h belong to $\mathcal{C}$.*

In order to prove this, we need a lemma:

Lemma 1.13. *Let $F \to X \to B$ be a fibration of path-connected spaces, with $\pi_1(B)$ acting trivially on $H_*(F)$. Then if two of F, X and B have $H_n \in \mathcal{C}$ for all $n > 0$, so does the third.*

Proof. We shall show the following:

(1) For a SES of abelian groups $0 \to A \to B \to C \to 0$, the group B is in $\mathcal{C}$ if and only if A and C are in $\mathcal{C}$.
(2) If A and B are in $\mathcal{C}$, then $A \otimes B$ and $\mathrm{Tor}(A,B)$ are in $\mathcal{C}$.

Case 1: Suppose $H_n(F), H_n(B) \in \mathcal{C}$ for all $n > 0$. In the Serre spectral sequence, we have
$$E^2_{p,q} = H_p\left(B; H_q(F)\right) \cong H_p(B) \otimes H_q(F) \bigoplus \mathrm{Tor}\left(H_{p-1}(B), H_q(F)\right) \in \mathcal{C}$$
for $(p,q) \neq (0,0)$. Here we use property (2) twice - once for $H_p(B) \otimes H_q(F) \in \mathcal{C}$ and once for $\mathrm{Tor}\left(H_{p-1}(B), H_q(F)\right) \in \mathcal{C}$.

We proceed by induction now - having shown the base case $r = 2$. Suppose $E^r_{p,q} \in \mathcal{C}$ for $(p,q) \neq (0,0)$. Then both $\ker d_r$ and $\operatorname{im} d_r$ are in $\mathcal{C}$ as can easily be checked in

each case. Hence the quotient $E^{r+1}_{p,q}$ is also in $\mathcal{C}$ as can also be checked in each case. Thus also $E^{\infty}_{p,q} \in \mathcal{C}$ for $(p,q) \neq (0,0)$.
Now, the groups $E^{\infty}_{p,n-p}$ are quotients in the filtration $0 \subset F_0H_n(X) \subset \ldots \subset F_nH_n(X) = H_n(X)$, so by induction on p, the subgroups $F_pH_n(X)$ are in $\mathcal{C}$ for $n > 0$, so in particular, $H_n(X) \in \mathcal{C}$. Here the induction starts with $F_0H_n(X) \in \mathcal{C}$ since $F_0H_n(X) \cong E^{\infty}_{0,n} \in \mathcal{C}$.

Case 2: Suppose $H_n(F), H_n(X) \in \mathcal{C}$ for all $n > 0$. Since $H_n(X) \in \mathcal{C}$, all the subgroups filtering $H_n(X)$ also lie in $\mathcal{C}$, hence also their quotients $E^{\infty}_{p,n-p} \in \mathcal{C}$. Now, $H_1(B) = E^2_{1,0} \cong E^{\infty}_{1,0} \in \mathcal{C}$ since any differentials out of and into $E^2_{1,0}$ must vanish. So suppose now inductively that $H_p(B) \in \mathcal{C}$ for $0 < p < k$.
But now

$$E^2_{p,q} = H_p\left(B, H_q(F)\right) \cong H_p(B) \otimes H_q(F) \bigoplus \operatorname{Tor}\left(H_{p-1}(B), H_q(F)\right)$$

Note that in this application, we use the Künneth theorem

$$H_n(C_* \otimes D_*) \cong \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \oplus \bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(C_*), H_j(D_*))$$

where, in our case, C_* is the usual singular chain complex for B and D_* in this case is the chain complex consisting of a single nontrivial element in degree p where it is $H_q(F)$. Thus $E^2_{p,q} \in \mathcal{C}$ for $p < k, (p,q) \neq (0,0)$. Since ker and im are subgroups, they inherit this property also as well as their quotient, so $E^r_{p,q} \in \mathcal{C}$ for $p < k, (p,q) \neq (0,0)$.
Next, since $E^{r+1}_{k,0} = \ker d_r \subset E^r_{k,0}$, we have a SES

$$0 \to E^{r+1}_{k,0} \to E^r_{k,0} \to \operatorname{im} d_r \to 0$$

with $\operatorname{im} d_r \subset E^r_{k-r,r-1}$, and so $\operatorname{im} d_r \in \mathcal{C}$ by induction since $E^r_{k-r,r-1} \in \mathcal{C}$. Then property (1) says that $E^{r+1}_{k,0} \in \mathcal{C}$ if and only if $E^r_{k,0} \in \mathcal{C}$. By downward induction on r, we obtain $E^r_{k,0} \in \mathcal{C}$ if and only if $E^2_{k,0} = H_k(B) \in \mathcal{C}$. But $E^{\infty}_{k,0} \in \mathcal{C}$, so we conclude that $H_k(B) \in \mathcal{C}$ for all k.

Case 3: Suppose $H_n(B), H_n(X) \in \mathcal{C}$ for all $n > 0$. This is similar to Case 2, so we will omit this.

□

Lemma 1.14. *If $\pi \in \mathcal{C}$, then $H_k\left(K\left(\pi, n\right)\right) \in \mathcal{C}$ for all $k, n > 0$.*

Proof. Using the path fibration $K\left(\pi, n-1\right) \to P \to K\left(\pi, n\right)$ and the previous lemma, it suffices to show the case $n = 1$. For the classes $\mathcal{FG}$ and $\mathcal{F}_P$, the group π is a product of cyclic groups in $\mathcal{C}$, and hence $K\left(G_1, 1\right) \times K\left(G_2, 1\right)$ is a $K\left(G_1 \times G_2, 1\right)$, so by the Künneth formula, it suffices to show the case when π is cyclic.
If $\pi = \mathbb{Z}$, we are in the case of $\mathcal{C} = \mathcal{FG}$, and S^1 is a $K(\mathbb{Z}, 1)$, and obviously $H_k\left(S^1\right) \in \mathcal{C}$. If $\pi = \mathbb{Z}/m$, we know that $H_k\left(K\left(\mathbb{Z}/m, 1\right)\right)$ is $\mathbb{Z}/m$ for odd k and 0 for even $k > 0$, since we can choose an infinite-dimensional lens space for $K\left(\mathbb{Z}/m, 1\right)$. Hence $H_k\left(K\left(\mathbb{Z}/m, 1\right)\right) \in \mathcal{C}$ for $k > 0$.

For the class $\mathcal{T}_p$, we can use the construction in section 1.B in Hatcher's Algebraic Topology of a $K\left(\pi, 1\right)$ CW complex $B\pi$ with the property that for any subgroup $G \subset \pi$, BG is a subcomplex of $B\pi$ (TODO). An element $x \in H_k\left(B\pi\right)$ with $k > 0$

is represented by a singular chain $\sum_i n_i\sigma_i$ with compact image contained in some finite subcomplex of $B\pi$. This finite subcomplex can involve generation by only finitely many elements of π, hence is contained in a subcomplex BG for some finitely generated subgroup $G \subset \pi$. Since $G \in \mathcal{F}_P$, by the first part of the proof, we know that the element of $H_k(BG)$ represented by $\Sigma_i n_i\sigma_i$ has finite order divisible only by primes in P, so the same is true for its image $x \in H_k(B\pi)$. □

Proof of Theorems 1.11 and 1.12. Assume first that X is simply-connected. Consider the Postnikov tower for X :

$$\ldots \to X_n \to X_{n-1} \to \ldots \to X_2 = K(\pi_2(X), 2)$$

where $X_n \to X_{n-1}$ is a fibration with fiber $F_n = K(\pi_n(X), n)$. If $\pi_i(X) \in \mathcal{C}$ for all i, then by Lemma 1.14, we have $H_k(X_2) \in \mathcal{C}$ for all k, and $H_k(F_n) = H_k(K(\pi_n(X), n)) \in \mathcal{C}$ for all k and n. Since $X_n \to X_{n-1}$ is a fibration with fiber F_n, we obtain by induction and Lemma 1.13 that $H_k(X_n) \in \mathcal{C}$ for all n and all k.
Now, by Cor. 4.12 in Hatcher, we know that the inclusion $X^n \hookrightarrow X$ induces an isomorphism on π_i for $i < n$ and a surjection for $i = n$, so by attaching cells of dimensions $\geq n+1$, we can obtain a space X' such that the composite inclusion $X^n \hookrightarrow X \hookrightarrow X'$ induces an isomorphism on all homotopy groups, hence is a homotopy equivalence.
Thus, up to homotopy equivalence, we can build X_n from X by attaching cells of dimension $\geq n+1$, so $H_i(X) \cong H_i(X_n)$ for $n \geq i$, and therefore $H_i(X) \in \mathcal{C}$ for all $i > 0$.
Next, the Hurewicz maps $\pi_n(X) \to H_n(X)$ and $\pi_n(X_n) \to H_n(X_n)$ are equivalent, and we can deal with the latter via the fibration $F_n \to X_n \to X_{n-1}$. Recall that $F_n = K(\pi_n(X), n)$, so by the Hurewicz theorem, $H_i(F_n) = 0$ for $0 < i < n$, hence the associated spectral sequence to the fibration has nothing between the 0 th and n th rows, so the first nontrivial differential is $d_{n+1}\colon H_{n+1}(X_{n-1}) \to H_n(F_n)$. Recalling that from the spectral sequence, we have

$$0 \to F_{n-1}H_n(X_n) \to H_n(X_n) \to E^\infty_{n,0} \to 0$$

we get that since $F_iH_n(X_n)/F_{i-1}H_n(X_n) \cong E^\infty_{i,n-i} \cong 0$ for $0 < i < n$, this implies that $F_{n-1}H_n(X_n) \cong F_0H_n(X_n) \cong E^\infty_{0,n}$, so we get a SES

$$0 \to E^\infty_{0,n} \to H_n(X_n) \to E^\infty_{n,0} \to 0.$$

Now, also since the only possible nontrivial differential terminating at $E^r_{0,n}$ for all r is $d_{n+1}\colon H_{n+1}(X_{n-1}) \to H_n(F_n)$, we find that $E^\infty_{0,n}$ must be the cokernel, so we get that

$$H_{n+1}(X_{n-1}) \overset{d_{n+1}}{\to} H_n(F_n) \to E^\infty_{0,n} \to 0$$

is exact.
Next, what is the map $H_n(F_n) \to E^\infty_{0,n} \to H_n(X_n)$? One can read off that the latter map $E^\infty_{0,n} \to H_n(X_n)$ is the inclusion, and the former is the quotienting map. □

1.3. **Supplements.**

1.3.1. *Naturality.* Suppose we are given two fibrations and a map between them, a commutative diagram as below:

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ F' & \longrightarrow & X' & \longrightarrow & B' \end{array}$$

Suppose that the hypotheses of the LSSS are satisfied for both fibrations. Then the naturality properties are:

(1) There are induced maps $f_*^r \colon E^r_{pq} \to E'^r_{pq}$ commuting with differentials, with f_*^{r+1} the map on homology induced by f_*^r.
(2) The map $\tilde{f}_* \colon H_*(X;G) \to H_*(X';G)$ preserves filtrations, inducing a map on successive quotient groups which is the map f_*^∞.
(3) Under the isomorphisms $E^2_{pq} \cong H_p(B; H_q(F;G))$ and $E'^2_{pq} \cong H_p(B'; H_q(F';G))$, the map f_*^2 corresponds to the map induced by the maps $B \to B'$ and $F \to F'$.

1.4. **Spectral Sequence Comparison.**

Proposition 1.15. *Suppose we have a map of fibrations as in the diagram:*

$$\begin{array}{ccccc} F & \longrightarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ F' & \longrightarrow & X' & \longrightarrow & B' \end{array}$$

and that both fibrations satisfy the hypothesis of trivial action for the Serre spectral sequence. Then if two of the three maps $F \to F', B \to B'$ and $X \to X'$ induce isomorphisms on $H_*(-;R)$ with R a PID, so does the third.

1.5. **Cohomology.**

Theorem 1.16. *For a fibration $F \to X \to B$ with B path-connected and $\pi_1(B)$ acting trivially on $H^*(F;G)$, there is a spectral sequence $\{E_r^{p,q}, d_r\}$ with:*

(1) $d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ *and* $E_{r+1}^{p,q} = \ker d_r / \operatorname{im} d_r$ *at* $E_r^{p,q}$.
(2) *Stable terms* $E_\infty^{p,n-p}$ *isomorphic to the successive quotients* F_p^n / F_{p+1}^n *in a filtration* $0 \subset F_n^n \subset \ldots \subset F_0^n = H^n(X;G)$ *of* $H^n(X;G)$.
(3) $E_2^{p,q} \cong H^p(B; H^q(F;G))$.

1.6. **Multiplicative structure.**

Definition 1.17 (Weibel, multiplicative structure)**.** Suppose that for $r = a$ we are given a bigraded product

$$E_r^{p_1 q_1} \times E_r^{p_2 q_2} \to E_r^{p_1+p_2, q_1+q_2}$$

such that the differential d_r satisfies the Leibnitz relation

$$d_r(x_1 x_2) = d_r(x_1) x_2 + (-1)^{p_1} x_1 d_r(x_2), \quad x_i \in E_r^{p_i q_i}.$$

Then the product of two cycles (boundaries) is again a cycle (boundary), and by induction, we have the above product for every $r \geq a$. We shall call this a *multiplicative structure* on the spectral sequence.

When considering cohomology with coefficients in a ring R, we can construct a multiplicative structure on a spectral sequence with $r = 1$ with the following properties:

(1) The product $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$ is $(-1)^{qs}$ times the standard cup product

$$H^p(B; H^q(F; R)) \times H^s(B; H^t(F; R)) \to H^{p+s}(B; H^{q+t}(F; R))$$

sending a pair of cocycles (φ, ψ) to $\varphi \smile \psi$ where coefficients are multiplied via the cup product $H^q(F; R) \times H^t(F; R) \to H^{q+t}(F; R)$.

(2) The cup product in $H^*(X; R)$ restricts to maps $F_p^m \times F_s^n \to F_{p+s}^{m+n}$. These induce quotient maps $F_p^m/F_{p+1}^m \times F_s^n/F_{s+1}^n \to F_{p+s}^{m+n}/F_{p+s+1}^{m+n}$ that coincide with the products $E_\infty^{p,m-p} \times E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$.

We shall obtain these products by thinking of the cup product as the composition

$$H^*(X; R) \times H^*(X; R) \xrightarrow{\times} H^*(X \times X; R) \xrightarrow{\Delta^*} H^*(X; R)$$

of cross product with the map induced by the diagonal map $\Delta \colon X \to X \times X$. This can be seen because

$$\begin{aligned} \Delta^*(- \times -)(a, b)(x) &= a \times b \circ \Delta(x) \\ &= p_1^*(a) \smile p_2^*(b) \circ \Delta(x) \\ &= p_1^*(a)(x, x) p_2^*(b)(x, x) \\ &= a(x)b(x) \\ &= (a \smile b)(x) \end{aligned}$$

so $\Delta^* \circ (- \times -) = (- \smile -)$.

1.7. **The Spectral Sequence of a Filtered Complex.**

Definition 1.18 (Differential Complex)**.** A differential complex K with differential operator D is an abelian group K together with a group homomorphism $D\colon K \to K$ such that $D^2 = 0$.

Let K be a differential complex with differential operator D. Usually K comes with a grading $K = \bigoplus_{k\in\mathbb{Z}} C^k$ and $D\colon C^k \to C^{k+1}$ increases the degree by 1, but the grading is not absolutely necessary.

Definition 1.19 (Subcomplex)**.** A *subcomplex* K' of K is a graded subgroup such that $DK' \subset K'$.

Definition 1.20 (Filtration, Associated Graded Complex)**.** A sequence of subcomplexes

$$K = K_0 \supset K_1 \supset K_2 \supset K_3 \supset \dots$$

is called a *filtration* on K. This makes K into a *filtered complex*, with *associated graded complex*

$$GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}.$$

For notational reasons, we usually extend the filtration to negative indices by defining $K_p = K$ for $p < 0$.

Example 1.21. If $K = \bigoplus K^{p,q}$ is a double complex with horizontal operator δ and vertical operator d (which we assume to commute), we can form a single complex out of it by setting $C^k = \bigoplus_{p+q=k} K^{p,q}$ and then letting $K = \bigoplus C^k$ and the differential operator $D\colon C^k \to C^{k+1}$ to be $D = \delta + (-1)^p d$. Then letting

$$K_p = \bigoplus_{i\geq p} \bigoplus_{q\geq 0} K^{i,q}$$

we obtain a filtration on K.

Suppose now that we have a general filtered complex $K = K_0 \supset K_1 \supset \dots$, and let A be the group defined by

$$A = \bigoplus_{p\in\mathbb{Z}} K_p.$$

Then A is again a differential complex with operator D. Let $i\colon A \to A$ be the inclusion $K_{p+1} \hookrightarrow K_p$ on each p. Let B be the cokernel of $i\colon A \to A$. Then $B = GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}$, and we have an exact sequence

$$0 \to A \xrightarrow{i} A \xrightarrow{j} GK \to 0.$$

2. Weibel

3. Double and Total Complexes

Definition 3.1 (Double complex)**.** A *double complex* (or *bicomplex*) in an abelian category $\mathcal{A}$ is a family $\{C_{p,q}\}$ of objects of $\mathcal{A}$, together with maps

$$d^h \colon C_{p,q} \to C_{p-1,q} \quad \text{and} \quad d^v \colon C_{p,q} \to C_{p,q-1}$$

such that $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$.

It is useful to picture the double complex as a lattice in which the maps d^h go horizontally, the maps d^v go vertically, and each square anticommutes.

Each row C_{*q} and each columns C_{p*} is a chain complex.
We say that the double complex C is *bounded* if C has only finitely many nonzero terms along each diagonal line $p+q=n$. For example, if C is concentrated in the first quadrant of the plane (a *first quadrant double complex*).

3.0.1. *Sign Trick.* Are the maps d^v and d^h maps in Ch?
Because of anticommutativity, the chain map conditions fail, but we can construct chain maps f_{*q} from $C_{*,q}$ to $C_{*,q-1}$ by introducing signs:

$$f_{p,q} = (-1)^p d^v_{p,q} \colon C_{p,q} \to C_{p,q-1}.$$

Using this sign trick, we can identify the category of double complexes with the category Ch (Ch).

3.0.2. *Total Complexes.* To see why the anticommutativity condition $d^v d^h + d^h d^v = 0$ is useful, we define the *total complexes* $\mathrm{Tot}(C) = Tot^{\prod}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ as follows:

Definition 3.2 (Total complexes)**.** We define

$$\mathrm{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q} \quad \text{and} \quad \mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

The formula $d = d^h + d^v$ define maps

$$d\colon \mathrm{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q} \quad \text{and} \quad d\colon \mathrm{Tot}^{\oplus}(C)_n \to \mathrm{Tot}^{\oplus}(C)_{n-1}$$

such that $d \circ d = 0$, making $\mathrm{Tot}^{\prod}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ into chain complexes.

Exercise 3.3. Check that $d = d^h + d^v$ define maps as claimed.

Solution. Let $(\alpha_{p,q}) \in \mathrm{Tot}^{\prod}(C)_n$, so $p+q=n$. Then $d\left((\alpha_{p,q})\right) = d^h\left((\alpha_{p,q})\right) + d^v\left((\alpha_{p,q})\right) = (\alpha_{p-1,q}) + (\alpha_{p,q-1}) \in \prod_{p+q=n-1} C_{p,q}$. Clearly, this also works for direct products since the number of non-zero terms under d just multiplies by 2, hence is still finite. We also want to show that $d \circ d = 0$. For this, note that

$$\begin{aligned} d \circ d\,(\alpha) = d\left(d^h(\alpha) + d^v(\alpha)\right) &= d^h\left(d^h(\alpha) + d^v(\alpha)\right) + d^v\left(d^h(\alpha) + d^v(\alpha)\right) \\ &= d^h d^h(\alpha) + d^h d^v(\alpha) + d^v d^h(\alpha) + d^v d^v(\alpha) \\ &= 0. \end{aligned}$$

3.1. **Terminology.**

Definition 3.4 (Homology spectral sequence)**.** A *homology spectral sequence (starting with* E^a *)* in an abelian category $\mathcal{A}$ consists of the following data:

(1) A family $\left\{E^r_{pq}\right\}$ of objects of $\mathcal{A}$ defined for all integers p, q and $r \geq a$.
(2) Maps $d^r_{pq} \colon E^r_{pq} \to E^r_{p-r,q+r-1}$ that are differentials in the sense that $d^r d^r = 0$, so that the "lines of slope $-(r+1)/r$ " in the lattice E^r_{**} form chain complexes.
(3) Isomorphisms between E^{r+1}_{pq} and the homology of E^r_{**} at the spot E^r_{pq} :

$$E^{r+1}_{pq} \cong \ker d^r_{pq} / \operatorname{im} d^r_{p+r,q-r+1}.$$

Note that E^{r+1}_{pq} is a subquotient of E^r_{pq}, and that each differential d^r_{pq} decreases the total degree by one.

Definition 3.5 (Total degree)**.** The *total degree* of the term E^r_{pq} is $n = p + q$.

Example 3.6. A *first quadrant (homology) spectral sequence* is one with $E^r_{pq} = 0$ unless $p \geq 0$ and $q \geq 0$.
If we fix p and q, then $E^r_{pq} = E^{r+1}_{pq}$ for all large enough r (for $r > \max\{p, q+1\}$), because d^r landing in the (p, q) spot comes from the fourth quadrant, and the d^r leaving E^r_{pq} lands in the second quadrant.
We write E^∞_{pq} for this stable value of E^r_{pq}.

Definition 3.7 (Dual Definition, Cohomology spectral sequence)**.** A *cohomology spectral sequence (starting with* E_a *)* in $\mathcal{A}$ is a family $\{E^{pq}_r\}$ of objects $(r \geq a)$, together with maps d^{pq}_r going "to the right":

$$d^{pq}_r \colon E^{pq}_r \to E^{p+r,q-r+1}_r$$

which are differentials in the sense that $d_r d_r = 0$.
So it is the same thing as a homology spectral sequence, reindexed via $E^{pq}_r = E^r_{-p,-q}$, so that d_r *increases* the total degree $p+q$ of E^r_{pq} by one.

Definition 3.8 (Bounded convergence)**.** A homology spectral sequence is said to be *bounded* if for each n, there are only finitely many nonzero terms of total degree n in E^a_{**}.

Exercise 3.9. Show that if E_{**} is a bounded homology spectral sequence, then for each p and q, there is an r_0 such that $E^r_{pq} = E^{r+1}_{pq}$ for all $r \geq r_0$.

Proof. If the spectral sequence has at most N non-vanishing terms of degree n on page r, say, then on the following pages, we have at most N non-vanishing terms of degree n again, since these are homologies of the terms of degree n on the previous pages.
Hence, for the bounded sequence, for each n, there exists $L(n) \in \mathbb{Z}$ such that $E^r_{p,n-p} = 0$ for all $p \leq L(n)$ and all r. Similarly, there is a $T(n) \in \mathbb{Z}$ such that $E^r_{n-q,q} = 0$ for all $q \leq T(n)$ and all r.
Now we claim that $E^r_{p,q} = E^\infty_{p,q}$ for

$$r > \max\{p - L(p+q-1), q+1-T(p+q+1)\}.$$

This is because we have

(1) $p - r < L(p+q-1)$, so $0 = E^r_{p-r,p+q-1-(p-r)} = E^r_{p-r,q+r-1}$, so $\ker d^r_{p,q} = E^r_{p,q}$, and

(2) $q-r+1 < T(p+q+1)$, so $0 = E_{(p+q+1)-(q-r+1),q-r+1} = E_{p+r,q-r+1}$, and hence $d_{p+r,q-r+1} \colon 0 = E^r_{p+r,q-r+1} \to E^r_{p,q}$ is 0.

Thus

$$\begin{aligned} E^{r+1}_{pq} &= \ker d^r_{pq} / \operatorname{im} d^r_{p+r,q-r+1} \\ &= E^r_{pq}/0 \\ &= E^r_{pq} \end{aligned}$$

□

We write E^∞_{pq} for this stable value of E^r_{pq}.

Next, we say that a bounded spectral sequence *converges* to H_* if we are given a family of objects H_n of $\mathcal{A}$, each having a *finite* filtration

$$0 = F_s H_n \subset \ldots \subset F_{p-1} H_n \subset F_p H_n \subset \ldots \subset F_t H_n = H_n,$$

and we are given isomorphisms $E^\infty_{pq} \cong F_p H_{p+q} / F_{p-1} H_{p+q}$.
The traditional symbolic way of describing such a bounded convergence is like this:

$$E^a_{pq} \implies H_{p+q}.$$

Similarly, a cohomology spectral sequence is called *bounded* if there are only finitely many nonzero terms in each total degree in E^{**}_a. In a bounded cohomology spectral sequence, we write E^{pq}_∞ for the stable value of the terms E^{pq}_r and say the (bounded) spectral sequence converges to H^* if there is a *finite* filtration

$$0 = F^t H^n \subset \ldots \subset F^{p+1} H^n \subset F^p H^n \subset \ldots \subset F^s H^n = H^n,$$

so that

$$E^{pq}_\infty \cong F^p H^{p+q} / F^{p+1} H^{p+q}.$$

Example 3.10. If a first quadrant homology spectral sequence converges to H_*, then each H_n has a finite filtration of length $n+1$:

$$0 = F_{-1} H_n \subset F_0 H_n \subset \ldots \subset F_{n-1} H_n \subset F_n H_n = H_n.$$

The bottom piece $F_0 H_n = E^\infty_{0n}$ of H_n is located on the y-axis, and the top quotient $H_n / F_{n-1} H_n \cong E^\infty_{n0}$ is located on the x-axis.
Note also that each arrow landing on the x-axis is zero, and each arrow leaving the y-axis is zero, hence E^a_{0n} is a quotient of E^∞_{0n}, and each E^∞_{n0} is a subobject of E^a_{n0}.

Definition 3.11 (Fiber and base terms, edge homomorphism)**.** The terms E^r_{0n} on the y-axis are called the *fiber* terms, and the terms E^r_{n0} on the x-axis are called the *base* terms. The resulting maps $E^a_{0n} \to E^\infty_{0n} \subset H_n$ and $H_n \to E^\infty_{n0} \subset E^a_{n0}$ are known as the *edge homomorphisms* of the spectral sequence.

Similarly, if a first quadrant cohomology spectral sequence converges to H^*, then H^n has a finite filtration:

$$0 = F^{n+1} H^n \subset F^n H^n \subset \ldots \subset F^1 H^n \subset F^0 H^n = H^n.$$

In this case, the bottom piece $F^n H^n \cong E^{n0}_\infty$ is located on the x-axis, and the top quotient $H^n / F^1 H^n \cong E^{0n}_\infty$ is located on the y-axis. In this case, the edge homomorphisms are the maps $E^{n0}_a \to E^{n0}_\infty \subset H^n$ and $H^n \to E^{0n}_\infty \subset E^{0n}_a$.

Definition 3.12 (Collapsing of spectral sequence)**.** A (homology) spectral sequence *collapses at* $E^r (r \geq 2)$ if there is exactly one nonzero row or column in the lattice $\{E^r_{pq}\}$. If a collapsing spectral sequence converges to H_*, we can read the H_n off: H_n is the unique nonzero E^r_{pq} with $p+q=n$. The overwhelming majority of all applications of spectral sequences involve spectral sequences that collapse at E^1 or E^2.

Exercise 3.13 (2 columns)**.** Suppose that a spectral sequence converging to H_* has $E^2_{pq} = 0$ unless $p = 0, 1$. Show that there are exact sequences

$$0 \to E^2_{0,n} \to H_n \to E^2_{1,n-1} \to 0$$

Proof. We have $E^\infty_{p,n-p} \cong 0$ if $p > 1$, so $F_pH_n/F_{p-1}H_n \cong 0$ whenever $p > 1$, so $F_pH_n \cong F_{p-1}H_n$ for $p > 1$. Hence $H_n = F_nH_n \cong F_1H_n$. Now, $E^\infty_{1,n-1} \cong H_n/F_0H_n$, and $E^\infty_{0n} \cong F_0H_n/F_{-1}H_n \cong F_0H_n$, so we have a SES

$$0 \to F_0H_n \hookrightarrow H_n \to H_n/F_0H_n \to 0$$

which thus becomes

$$0 \to E^\infty_{0n} \to H_n \to E^\infty_{1,n-1} \to 0$$

Furthermore, all differentials on pages E^r for $r \geq 2$ are 0, so $E^\infty_{0n} \cong E^2_{0n}$ and $E^\infty_{1,n-1} \cong E^2_{1,n-1}$. So we get a SES

$$0 \to E^2_{0n} \to H_n \to E^2_{1,n-1} \to 0.$$

$\square$

Example 3.14 (2 rows)**.** Suppose that a spectral sequence converging to H_* has $E^2_{pq} = 0$ unless $q = 0, 1$. Show that there is a LES

$$\ldots \to H_{p+1} \to E^2_{p+1,0} \xrightarrow{d} E^2_{p-1,1} \to H_p \to E^2_{p0} \xrightarrow{d} E^2_{p-2,1} \to H_{p-1} \to \ldots$$

Proof. The maps $H_p \to E^2_{p,0}$ and $E^2_{p-1,1} \to H_p$ are the edge homomorphisms given, respectively, by the map $H_p \to H_p/F_{p-1}H_p \cong E^\infty_{p,0} \hookrightarrow E^2_{p,0}$, where the last part is the inclusion since $E^\infty_{p,0}$ is the kernel of a map out of $E^2_{p,0}$, and the map $E^2_{p-1,1} \to E^2_{p-1,1}/\operatorname{im} d \cong E^\infty_{p-1,1} \cong F_{p-1}H_p \subset H_p$. Thus the kernel of d is indeed the image of the edge map $H_p \to E^2_{p,0}$, giving exactness at $E^2_{p,0}$, and the image of the edge map $E^2_{p-1,1} \to H_p$ is the subgroup $F_{p-1}H_p$. Now, the kernel of the edge map $H_p \to E^2_{p,0}$ is the subgroup $F_{p-1}H_p$, giving exactness at H_p. Also the image of the edge map $E^2_{p-1,1} \to H_p$ is $\operatorname{im} d$ giving exactness at $E^2_{p-1,1}$. This proves the claim. $\square$

3.2. **The category of homology spectral sequences.** A morphism $f\colon E' \to E$ in the category of homology spectral sequences is a family of maps $f^r_{pq}\colon E'^r_{pq} \to E^r_{pq}$ in $\mathcal{A}$ (for r suitably large) such that

$$\begin{tikzcd} E'^r_{pq} \arrow[r, "f^r"] \arrow[d, "d^r"] & E^r_{pq} \arrow[d, "d^r"] \\ E'^r_{p-r,q-r+1} \arrow[r, "f^r"] & E^r_{p-r,q-r+1} \end{tikzcd}$$

for all p, q, and such that f_{pq}^{r+1} is the map induced by f_{pq}^{r} on homology. That is, from the commutative diagram

$$\begin{tikzcd}
E'^{r}_{p+r,q+r-1} \arrow[r, "f^r"] \arrow[d, "d^r"] & E^{r}_{p+r,q+r-1} \arrow[d, "d^r"] \\
E'^{r}_{pq} \arrow[r, "f^r"] \arrow[d, "d^r"] & E^{r}_{pq} \arrow[d, "d^r"] \\
E'^{r}_{p-r,q-r+1} \arrow[r, "f^r"] & E^{r}_{p-r,q-r+1}
\end{tikzcd}$$

we obtain a map on homology since if $[a] \in E'^{r+1}_{pq} = HE'^{r}_{pq}$, then $d^r a = 0$, so since $f^r d^r = d^r f^r$, we get that $d^r\left(f^r[a]\right) = f^r\left(d^r[a]\right) = f^r\left[d^r a\right] = 0$, hence f^r takes cycles to cycles, and similarly, if $[b] \in \operatorname{im} d^r$, say $[b] = d^r\left[\tilde{b}\right]$, then $f^r[b] = d^r f^r\left[\tilde{b}\right]$, so f^r also maps boundaries to boundaries, hence f^r induces a map on homology.

Lemma 3.15 (Mapping Lemma)**.** *Let $f\colon \left\{E_{pq}^{r}\right\} \to \left\{E'^{r}_{pq}\right\}$ be a morphism of spectral sequences such that for some fixed r, $f^r\colon E_{pq}^{r} \cong E'^{r}_{pq}$ is an isomorphism for all p and q. Then $f^s\colon E_{pq}^{s} \cong E'^{s}_{pq}$ for all $s \geq r$ as well.*

Proof. Suppose f^r is an isomorphism. Then since $f^r d^r = d^r f^r$, we must have that f^r induces an isomorphism on cycles and boundaries, so let B'^{r}_{pq} and B_{pq}^{r} denote the boundaries at pq and Z'^{r}_{pq} and Z_{pq}^{r} the cycles, respectively. Then we have the SES

$$\begin{tikzcd}
0 \arrow[r] & B'^{r}_{pq} \arrow[r] \arrow[d, "f^r"] & Z'^{r}_{pq} \arrow[r] \arrow[d, "f^r"] & E'^{r+1}_{pq} \arrow[r] \arrow[d, "f^{r+1}"] & 0 \\
0 \arrow[r] & B^{r}_{pq} \arrow[r] & Z^{r}_{pq} \arrow[r] & E^{r+1}_{pq} \arrow[r] & 0
\end{tikzcd}$$

Applying the 5-lemma, we get that f^{r+1} is an isomorphism for all pq. By induction, we obtain the claim. $\square$

3.2.1. E^∞ *terms.* Given a homology spectral sequence, we know that each E_{pq}^{r+1} is a subquotient of the previous term E_{pq}^{r}. Letting Z_{pq}^{r} be the kernel of $E_{pq}^{r-1} \to E_{p-r,q+r-1}$ and B_{pq}^{r} the image of $E_{p+r,q-r+1}^{r-1} \to E_{pq}^{r-1}$, we get a nested family of subobjects of E_{pq}^{a} :

$$0 = B_{pq}^{a} \subset \ldots \subset B_{pq}^{r} \subset B_{pq}^{r+1} \subset \ldots \subset Z_{pq}^{r+1} \subset Z_{pq}^{r} \subset \ldots \subset Z_{pq}^{a} = E_{pq}^{a}$$

such that $E_{pq}^{r} \cong Z_{pq}^{r}/B_{pq}^{r}$. Let

$$B_{pq}^{\infty} = \bigcup_{r=a}^{\infty} B_{pq}^{r} \quad \text{and} \quad Z_{pq}^{\infty} = \bigcap_{r=a}^{\infty} Z_{pq}^{r}$$

and define $E_{pq}^{\infty} = Z_{pq}^{\infty}/B_{pq}^{\infty}$.

4. Bott Tu

4.1. **Introduction to Spectral Sequences.** Consider the problem of computing the homology of the total chain complex $T_* = \mathrm{Tot}(E_{**})$ where E_{**} is a first quadrant double complex.
Firstly, it is convenient to forget the horizontal differentials and add a superscript zero, retaining only the vertical differentials d^v along the columns E^0_{p*}.
Let E^1_{pq} be the vertical homology $H_q\left(E^0_{p*}\right)$ at the (p,q) spot.

4.2. **Filtrations.**

Definition 4.1 (Filtered R-module)**.** A *filtered R-module* is an R-module A with an increasing sequence of submodules $\{F_p\}_{p\in\mathbb{Z}}$ such that $F_pA \subset F_{p+1}A$ for all p and such that $\bigcup_p F_pA = A$ and $\bigcap_p F_pA = \{0\}$.
A filtration is said to be *bounded* if $F_pA = \{0\}$ for p sufficiently small and $F_pA = A$ for p sufficiently larger.

Definition 4.2 (Associated graded module)**.** The *associated graded module* is defined by $G_pA = F_pA/F_{p-1}A$.

Definition 4.3 (Filtered chain complex)**.** A *filtered chain complex* is a chain complex (C_*, ∂) together with a filtration $\{F_pC_i\}_{p\in\mathbb{Z}}$ of each C_i such that the differential preserves the filtration, i.e., s.t. $\partial\left(F_pC_i\right) \subset F_pC_{i-1}$.

Note that we, in particular, obtain an induced differential $\partial\colon G_pC_i \to G_pC_{i-1}$ by the universal property of cokernels

$$\begin{tikzcd}
F_pC_i \arrow[r, "\partial"] \arrow[d] & F_pC_{i-1} \arrow[d] \\
F_{p-1}C_i \arrow[r, "\partial"] \arrow[d, "\text{coker}"] & F_{p-1}C_{i-1} \arrow[d, "\text{coker}"] \\
G_pC_i \arrow[r, dashed, "\partial"] & G_pC_{i-1}
\end{tikzcd}$$

so we obtain an associated graded chain complex G_pC_*.

The filtration on C_* also induces a filtration on the homology of C_* by

$$F_pH_i(C_*) = \{\alpha \in H_i(C_*) \mid (\exists x \in F_pC_i) : \alpha = [x]\}\,.$$

This filtration has associated graded pieces $G_pH_i(C_*)$ which, in favorable cases, determine $H_i(C_*)$.

4.3. **Example.** Suppose we have a chain complex C_* and a filtration consisting of a single F_0C_*, so $F_nC_* = 0$ if $n < 0$ and $F_nC_* = F_0C_*$ if $n \geq 0$.
Then $G_nC_* = 0$ for $n \neq 0$ and $G_0C_* = F_0C_*$ and