

**1.3.6:** Let  $\tilde{X}$  be the covering space shows and consider any one of the basepoints  $p^{-1}(x_0)$  where  $x_0 = (0, 0)$  in  $X$ .

Consider this basepoint to be 0 and each subsequent basepoint in the fiber of 0 to be 1, 2, 3, etc. and preceding basepoints  $-1, -2, \dots$ . Now let the outermost circle of radius 1 in each copy of the Hawaiian earring unfold to the line segments between these integers. I.e. we place each copy of the earrings onto each integer and unfold the outermost circle to connect the components.

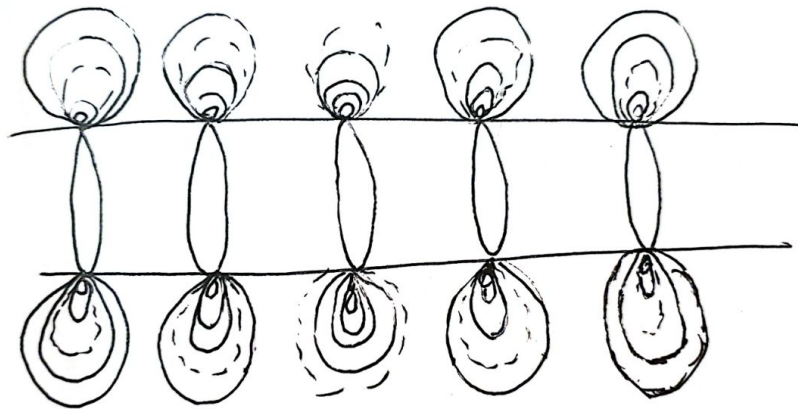
Let  $p$  be the covering map that collapses this covering space such that each of the circles  $C_n$  is mapped to the  $C_n$  in  $X$  and each segment  $[n, n+1]$  is mapped to  $C_1$ . This is a covering map since for any  $x$  that is not on  $C_1$ , we can find a neighborhood around this point that does not intersect  $C_1$ , and the preimage of this neighborhood will simply be the collection of identical neighborhoods each of which is homeomorphic to the neighborhood in  $X$  by the identity. For any point on  $C_1$  that is not the basepoint, its preimage will be of the form  $x' + \mathbb{Z}$ , and we can find a neighborhood that also only intersects  $C_1 - \{x_0\}$  and whose image will be the collection of some open intervals  $I + \mathbb{Z}$  where each  $I$  is clearly homeomorphic to the neighborhood on  $X$  by  $z \rightarrow e^{2\pi iz}$ .

For  $x_0$  whose fiber is  $\mathbb{Z}$ , we can choose a neighborhood that does not cover  $C_1$  whose preimage will then not contain some  $k + \mathbb{Z}$  for some  $k \in (0, 1)$ . The components will therefore be disconnected and each will be homeomorphic to the neighborhood by the identity. So this is a covering space.

Now let  $\tilde{\tilde{X}}$  denote the covering space obtained by first taking a copy of  $\tilde{X}$  and then connecting each component (i.e. a specific chosen  $[n, n+1]$  and  $\bigcup_{i \in \mathbb{N}} C_{n,i}$  where  $C_{n,i}$  is the circle of radius  $i$  centered at the integer  $n$ ) to its corresponding component in the copy by deforming the circles  $C_{k,k}$  into a line segment connecting the copies of integer the integer  $k$ . Let  $p'$  be the covering map that collapses  $\tilde{\tilde{X}}$  to  $\tilde{X}$  as one would expect (mapping the copy of  $\mathbb{R}$  to  $\mathbb{R}$  and each circle to each circle - the circles  $C_{k,k}$  also to its copy).

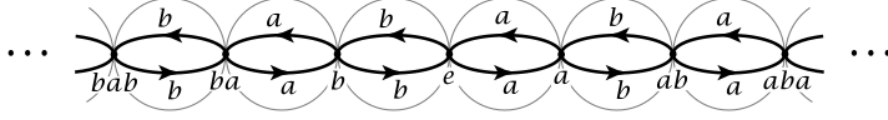
Checking that  $p'$  is a homeomorphism is done equivalently to  $p$ ; in this case, for any integer  $n$ , we choose a neighborhood that does not contain  $C_{n,n}$  nor any other integers, and hence its preimage will be two components, each homeomorphic to itself by the identity.

Now the composition of  $p \circ p'$  is not a covering map: take any neighborhood of  $x_0 = (0, 0)$ . This neighborhood will contain some  $C_k$  for  $k \in \mathbb{N}$ , and thus its preimage under  $p \circ p'$  will contain the linked copies of the earrings of  $k$  which is not a homeomorphism under  $p \circ p'$  as it is not injective on the disjoint components.



### 1.3.14:

From p. 78, we have that  $\mathbb{R}P^2 \vee \mathbb{R}P^2$  is the orbit space for the Cayley complex of  $G = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2, b^2 \rangle$ , depicted on p. 78 as an infinite string of 2-spheres linked at points as follows:



The action on this space is as follows: elements of any subgroup generated by  $ab$  act on  $\tilde{X}_G$  by translations by an even number of units, while each remaining element of  $\mathbb{Z}_2 * \mathbb{Z}_2$  acts by an antipodal map on one of the spheres and flips the chain end-for-end about the sphere.

This action satisfies (\*) on page 72 since it is clear that if we choose any point on the chain, then this point gets mapped to each copy of its point and each copy of its antipodal point by exactly one element of  $g$ ; hence any small neighborhood around the point restricted to that hemisphere or side of the equator works for (\*). Since it is a covering space action, we can use proposition 1.40 to deduce that that quotient map  $p : \tilde{X}_G \rightarrow X_G = \mathbb{R}P^2 \vee \mathbb{R}P^2$  is a normal covering space and by the remark on page 77, it is in fact a universal cover of  $\mathbb{R}P^2 \vee \mathbb{R}P^2$ , thus corresponding to  $\langle a, b \mid a^2, b^2 \rangle$ .

By playing around with the group we find that the subgroups are of the following forms:

1: The group itself  $G$ .

2: Subgroups of the form  $\langle (ab)^n \rangle$  or  $\langle (ba)^n \rangle$ .

3: Subgroups of the form  $\langle (ab)^n a \rangle$  or  $\langle (ba)^n b \rangle$ .

4: Subgroups of the form  $\langle (ba)^n b, (ba)^m \mid n < m \rangle$  and  $\langle (ab)^n a, (ba)^m \mid n < m \rangle$ .

By proposition 1.40.(c), each of these types of subgroups is isomorphic to the fundamental group the orbit space of  $\tilde{X}_G$  under the action induced by a generator of the subgroup.

For the trivial subgroup, we get the Cayley complex as shown above.

For subgroups of type two we get, by modding out,  $2n$  spheres and we link the end circles by a loop at the attaching points.

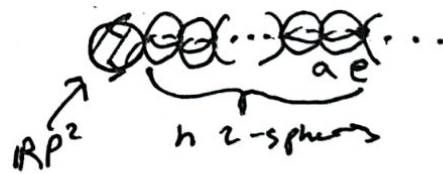
For subgroups of type three, we get an infinite chain of 2-spheres extending as shows in the picture from the basepoint  $e$ , onto which  $\mathbb{R}P^2$  is attached at one end. To get the other type three subgroup, we exchange each copy of  $a$  for  $b$  and  $b$  for  $a$ .

For subgroups of type three, we get a finite chain as shown where two copies of  $\mathbb{R}P^2$  have been attached at either end.

$$\langle (ab)^n \rangle:$$



$$\langle (ab)^n a \rangle; n \text{ even:}$$



$$\langle (ab)^n a \rangle; n \text{ odd:}$$



$$\langle (ab)^m, (ab)^n a \rangle :$$

$n < m,$   
 $m, n$  even

