Chapter 1

Exercise 0.1 (4)**.** Let M be a differentiable manifold and $\tau\colon M \to M$ be a fixed point free involution, i.e., a diffeomorphism with $\tau \circ \tau = \mathrm{id}_M$ and $\tau(x) \neq x$ for all x. Show that the quotient space M/τ is a topological manifold possessing a unique differentiable structure making the projection $M \to M/\tau$ locally diffeomorphic.

Proof. Hausdorff: let $x, y \in M/\tau$ be distinct. The preimage contains points x_1, x_2 and y_1, y_2. Choose open disjoint subsets U_1, U_2 containing x_1 and x_2 respectively which is disjoint from y_1, y_2. We can modify U_1 to be $U_1 \cap \tau(U_2)$, and then let $U_2 = \tau(U_1)$.
Let V_1, V_2 be disjoint open neighborhoods of y_1, y_2 respectively, disjoint from U_1, U_2. Replace V_1 by $V_1 \cap \tau(V_2)$ and let $V_2 = \tau(V_1)$. Then $U := U_1 \cup U_2$ is saturated with respect to τ and $V := V_1 \cup V_2$ is also, hence they descend to open sets in M/τ which are again disjoint. Indeed, suppose $\pi\colon M \to M/\tau$ is the quotient map and $\overline{z} \in \pi U \cap \pi V$. Then there exist $z_1 \in U$ and $z_2 \in V$ such that $\tau(z_1) = z_2$. But this contradicts $U \cap V = \varnothing$.

Second-countable: Take a countable open cover $\mathcal{B}$ of M. Now take the countable cover $\mathcal{B}' := \{U \cup \tau(U) \colon U \in \mathcal{B}\}$. This descends to a countable cover on M/τ.

Locally-homeomorphic to $\mathbb{R}^n$ *:*
Let $\overline{x} \in M/\tau$. Choose a chart (φ_x, U_x) around $x \in M$. If necessary, intersect U_x with U_1 constructed from before. This then implies that $\tau(U_x) \cap U_x = \varnothing$. Then $U_x \cup \tau(U_x)$ is a saturated open neighborhood descending to an open neighborhood $\overline{U_x}$ of $\overline{x}$. Now $\pi|_{U_x} \colon U_x \to \overline{U_x}$ is a homeomorphism, so we can define a chart for $\overline{x}$ as $\left(\varphi_x \circ (\pi|_{U_x})^{-1}, \overline{U_x}\right)$. This constitutes an atlas for M/τ.
For good measure, we prove the lemma

Lemma 0.2. $\pi|_{U_x} \colon U_x \to \overline{U_x}$ *is a homeomorphism.*

Proof. Suppose $\pi(y) = \pi(z)$ for $y, z \in U_x$ distinct. That forces $y = \tau(z)$. However, then $\tau(U_x) \cap U_x \neq \varnothing$, which is a contradiction by construction. Thus π is injective on U_x.
Now, an injective quotient map is a homeomorphism, giving the desired result. □

Lastly, we must prove that *there exists a structure making the quotient* $\pi\colon M \to M/\tau$ *a local diffeomorphism and it is the unique such structure:* by construction of the charts on M/τ, we can choose transition maps giving the identity as a coordinate representation. Hence the structure we constructed indeed gives a local diffeomorphism $M \cong M/\tau$.
Any other structure making it a local diffeomorphism would necessarily give a local diffeomorphism of M/τ in the two structures, thus forcing all charts to be compatible in the two structures, and by maximality, it forces the structures to be the same. □

Exercise 0.3 (5)**.** Show that $\mathbb{RP}^1 \cong S^1$.

Proof. This now follows by applying the previous exercise to S^1 with $\tau\colon S^1 \to S^1$ being the antipodal map $\tau(x) = -x$. Indeed $S^1/\tau \cong \mathbb{RP}^1$ is the usual quotient construction for $\mathbb{RP}^1$. □

Exercise 0.4 (9)**.** Prove that if M is a non-empty, n-dimensional smooth manifold and $k \leq n$, then there is an embedding $\mathbb{R}^k \to M$.

Proof. Let $x \in M$ and (U, φ) be a chart centered around x. Take some open ball $B(0, r) \subset \varphi(U)$. Then we define $f\colon (-r, r)^k \to M$ as φ^{-1}. Since $(-r, r)^k$ is diffeomorphic to $\mathbb{R}^k$, we get a diffeomorphism $\mathbb{R}^k \to f\left((-r, r)^k\right)$ if and only if f is a diffeomorphism onto its image which it trivially is as the restriction of a diffeomorphism to a submanifold. Indeed, it is a k-submanifold as, if we let $U' = \varphi^{-1}(B(0, r))$, we get $(\varphi|_{U'}, U')$ as a chart in the atlas, and letting $N = \operatorname{im} f$, we have $\varphi(N \cap U) = \varphi(N) = (-r, r)^k = B(0, r) \cap \mathbb{R}^k$. □