ASSIGNMENT 1

JONAS TREPIAKAS - KU-ID: HVN548

Exercise 0.1 (H1.1)**.** *Proof.*

$$f * e(n) = \sum_{d|n} f(d)e(\frac{n}{d}) = \sum_{d|n} f(d)\delta_{\frac{n}{d},1} = f(n)$$

and since the sets $\{d\colon d \mid n\}$ and $\left\{\frac{n}{d}\colon d \mid n\right\}$ are equal, we have

$$g * f = \sum_{d|n} g(d) f\left(\frac{n}{d}\right) = \sum_{d|n} g\left(\frac{n}{d}\right) f\left(\frac{n}{\frac{n}{d}}\right) = f * g(n)$$

□

Exercise 0.2 (H1.2)**.** *Proof.*

$$\mu * 1(n) = \sum_{d|n} \mu(d) 1\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d)$$

If $n = p$ is a prime, we trivially have $\{d\colon d \mid n\} = \{1, p\}$, so $\sum_{d|n} \mu(d) = 1 - 1 = 0 = e(p)$, so it is true for n a prime.
Now, if $n = p^{\alpha}$, then

$$\mu * 1(n) = \sum_{d|n} \mu(d) 1\left(\frac{n}{d}\right) = \mu(p) + \mu(1) = 0$$

since in all other terms, μ is evaluated at a non-squarefree integer.
Lastly, for $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, it reduces to the previous case because μ is only non-zero on squarefree integers, so

$$\mu * 1(n) = \sum_{d \mid \frac{n}{p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1}}} \mu(d) = 0$$

since the sets $\left\{d\colon d \mid \frac{n}{p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1}}\right\}$ and $\{d\colon d \mid p_1 \cdots p_k\}$ are equal.
Thus, indeed, $\mu * 1 = e$. □

Exercise 0.3 (H1.3)**.** We claim that the set of arithmetic functions with Dirichlet convolution as a binary operation is an abelian semigroup. For this, if $f, g\colon \mathbb{N} \to \mathbb{C}$, then clearly $f * g\colon \mathbb{N} \to \mathbb{C}$ too. Also, $f * g(n) = \sum_{ab=n} f(a)g(b) = \sum_{ba=n} g(b)f(a) = g * f(n)$ by commutativity of multiplication in $\mathbb{C}$. Lastly,

$$(f * g) * h(n) = \sum_{ab=n} f * g(a)h(b) = \sum_{ab=n} \sum_{cd=a} f(c)g(d)h(b) = \sum_{cdb=n} f(c)g(d)h(b)$$

and

$$f * (g * h)(n) = \sum_{ab=n} f(a) g * h(b) = \sum_{ab=n} \sum_{cd=b} f(a)g(c)h(d) = \sum_{acd=n} f(a)g(c)h(d)$$

(all of this is just Theorem 5.1.4 in the book for Introduction to Number Theory by Risager).
Now, if $f = 1 * g$ then $\mu * f = \mu * (1 * g) = (\mu * 1) * g = e * g = g * e = g$ by the above together with H1.1. Likewise, if $g = \mu * f$, then $1 * g = 1 * (\mu * f) = (1 * \mu) * f = (\mu * 1) * f = e * f = f * e = f$ again.

Exercise 0.4 (H1.4)**.** We have

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{f(n)}{n^s}\right| &\leq \sum_{n=1}^{\infty}\frac{Cn^k}{n^\sigma}\\ &\leq \sum_{n=1}^{\infty}\frac{C}{n^{\sigma-k}}\\ &< \infty\end{aligned}$$

as $\sigma - k > 1$. Thus the series converges absolutely.

Exercise 0.5 (H1.5)**.** *Proof.* We know that L_f converges absolutely for $\sigma > 1 + k_f$ and L_g converges absolutely for $\sigma > 1 + k_g$. Now,

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{\sum_{d|n} f(d)g(\frac{n}{d})}{n^s}\right| &\leq \sum_{n=1}^{\infty}\sum_{d|n}\frac{C_f C_g d^{k_f}\left(\frac{n}{d}\right)^{k_g}}{n^\sigma}\\ &= \sum_{n=1}^{\infty} C\frac{1}{n^{\sigma-k_g}}\end{aligned}$$

and by symmetric, likewise for a different constant C with k_f replacing k_g. Hence the sum defining $L_{f*g}(s)$ is absolutely convergent for $\sigma > \min\{1 + k_f, 1 + k_g\}$, and in this half-plane,

$$L_f(s)L_g(s) = \lim_{n\to\infty}\sum_{k=1}^{n}\sum_{t=1}^{n}\frac{f(k)}{k^s}\frac{g(t)}{t^s} = \lim_{n\to\infty}\sum_{k=1}^{n}\sum_{d|k}\frac{f(d)g(\frac{n}{d})}{k^s} = L_{f*g}(s)$$

$\square$

Exercise 0.6 (H1.6)**.** We have $L_1(s)L_\mu(s) = L_{1*\mu}(s) = L_e(s) = 1$, but $L_1(s) = \zeta(s)$ and $L_\mu(s) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$, so the result follows from H1.5 since $1(n) = O(1)$ and $\mu(n) = O(1)$, so $L_{1*\mu}$ is absolutely convergent for $\sigma > 1$ with $L_1(s)L_\mu(s) = L_{1*\mu}(s)$.

Exercise 0.7 (H 1.7)**.** *Proof.* For $f(n) = n^w$, we have $\sigma_w(n) = f * 1(n)$. The abscissa of convergence for 1 is 1 and for f it is $1+\Re(w)$. For $\sigma > \min\{1, 1 + \Re(w)\}$, we have $\sum_{n=1}^{\infty}\frac{\sigma_w(n)}{n^s} = L_{\sigma_w}(s) = L_f(s)L_1(s)$. Now $L_1(s) = \zeta(s)$, and

$$L_f(s) = \sum_{n=1}^{\infty}\frac{n^w}{n^s} = \sum_{n=1}^{\infty}\frac{1}{n^{s-w}} = \zeta(s-w).$$

Thus $\sum_{n=1}^{\infty}\frac{\sigma_w(n)}{n^s} = \zeta(s-w)\zeta(s)$ $\square$