ASSIGNMENT 3

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Problem 0.1. Let Y be a simply-connected CW-complex. Assume there exists a finite wedge of spheres $\bigvee_i S^{n_i}$ together with maps $i\colon Y \to \bigvee_i S^{n_i}, r\colon \bigvee_i S^{n_i} \to Y$ such that $r \circ i$ is homotopic to id_Y. Prove that Y is homotopy equivalent to some finite wedge of spheres $\bigvee_j S^{m_j}$.

Proof. We want to make use of the Corollary 4.33 in Hatcher which says:

Corollary 0.2 (4.33 Hatcher)**.** *A map $f\colon X \to Y$ between simply-connected CW complexes is a homotopy equivalence if $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism for each n.*

Since $r_* \circ \iota_* = \mathrm{id}_*$, $\iota_*\colon H_n(Y) \to H_n\left(\bigvee_i S^{n_i}\right)$ is injective for each n. Now $H_n\left(\bigvee_i S^{n_i}\right) \cong \bigoplus_{A_n} \mathbb{Z}$ where we let A_n be an indexing set for the n-cells $\{e^n_\alpha\}$ in the CW structure of $\bigvee_i S^{n_i}$. Let $\mathcal{A}_n$ be a set of representative basis elements for $H_n\left(\bigvee_i S^{n_i}\right) \cong \bigoplus_{A_n} \mathbb{Z}$ corresponding to the inclusion of a sphere into the wedge. Any subgroup of a free group is free, so $H_n(Y) \cong \bigoplus_{B_n} \mathbb{Z}$ for some indexing set B_n for each n. Now, starting with a single 0-cell $*$ and attaching $|B_1|$ 1-cells to $*$, $|B_2|$ 2-cells to $*$, etc., we obtain a space $Z = \bigvee_j S^{m_j}$ which satisfies $H_n(Z) = H_n(Y)$. Now let $\mathcal{C}_n$ denote the basis set for $H_n(Y) \cong \bigoplus_{B_n} \mathbb{Z}$. Since $r_*\colon H_n\left(\bigvee_i S^{n_i}\right) \cong \bigoplus_{A_n} \mathbb{Z} \to \bigoplus_{B_n} \mathbb{Z} \cong H_n(Y)$ is surjective, we can choose representatives $\tilde{\mathcal{A}}_n \subset \mathcal{A}_n$ such that $r_*\left(\tilde{\mathcal{A}}_n\right)$ gives a set of (by construction, linearly independent) basis elements which generate $H_n(Y)$. Now defining a map $f\colon \bigvee_j S^{m_j} \to \bigvee_i S^{n_i}$ by sending, for each n, all the n-spheres to distinct elements of $\tilde{\mathcal{A}}_n$ (this map is bijective by construction), we obtain a map f such that $r \circ f$ induces an isomorphism on homology on all n. □

Problem 0.3. Let X be a path-connected CW complex such that $H_1(X;\mathbb{Z}) = 0$. The goal of this problem is to construct a simply connected space Z and a map $X \to Z$ inducing an isomorphism in homology.

(1) Give an example of such X such that $\pi_1(X) \neq 1$.
(2) Consider a set of generators for $\pi_1(X)$, construct another CW complex Y by attaching cells to X, so that
   - Y is simply connected.
   - The inclusion $X \subset Y$ induces an isomorphism on homology in degrees ≥ 3.
(3) Show that $H_2(Y;\mathbb{Z})$ is a sum of $H_2(X;\mathbb{Z})$ together with a free abelian group. Let A be a set of generators for this free summand.

Proof. (1) Since H_1 is just the abelianization for π_1 for path-connected spaces, this is equivalent to finding a path-connected CW complex X whose fundamental group is nontrivial, but becomes trivial when abelianized. By corollary 1.28 in

Hatcher, for any group G, we can construct a 2-dimensional CW complex X_G such that $\pi_1(X_G) \cong G$. So it suffices to find a nontrivial group whose abelianization is trivial. Such a group is called a perfect group, and we have many examples of such groups. For example, any non-abelian simple group is perfect, so for example A_5 is perfect. The construction of X_{A_5} can now be carried out as follows: A_5 is generated by (123) and (12345) which do not commute, so we can express (as with any other group) A_5 as

$$A_5 = \langle g_\alpha \mid r_\beta \rangle$$

So in this case, the number of generators is simply 2. Then we can construct X_{A_5} from $\bigvee_\alpha S^1$ by attaching 2-cells e^2_β by the loops specified by the words r_β. By Proposition 1.26 in Hatcher, $\pi_1(X_{A_5}) \cong A_5$, and $H_1(X_{A_5}) \cong \mathrm{ab}(A_5) \cong 1$.

Another example is the example from Exercise 5 in problem set 2: namely, the Poincaré homology sphere. We showed that $H_1\left(S^3/2I;\mathbb{Z}\right) \cong 0$ while $\pi_1\left(S^3/2I\right) \cong 2I \ncong 1$. Furthermore, we showed that $S^3/2I$ is a manifold, hence admits a CW complex structure, and furthermore, as the quotient of a path-connected space, it is also path-connected, so $S^3/2I$ satisfies all the criterions of the problem.

(2) We want to attach cells to X to obtain a CW-complex Y which is simply connected and induce an isomorphism on homology in degrees ≥ 3 under the inclusion. To do this, suppose $f\colon \left(S^1, s_0\right) \to (X, x_0)$ is a nontrivial element in $\pi_1(X, x_0)$. We can assume by the Cellular Approximation Theorem that f is cellular. Then we can attach a 2-cell along f which renders f based nullhomotopic. Attaching 2-cells for each nontrivial element in $\pi_1(X)$ like this simultaneously, we let Y be the resulting space. Then we claim that $\pi_1(Y) \cong 0$. To see this, suppose $g\colon \left(S^1, s_0\right) \to (Y, x_0)$ is some map. By giving S^1 the standard CW stucture of a single 0-cell which is s_0 and a single 1-cell attached, we get by cellular approximation, that g is based homotopic to a map $\tilde{g}\colon \left(S^1, s_0\right) \to (Y, x_0)$ which has image in X. Thus $\tilde{g}$ represents an element of $\pi_1(X, x_0)$, but by construction of Y, $\tilde{g}$ is then based nullhomotopic. Composing these homotopies, we find that g is based nullhomotopic, so $\pi_1(Y) \cong 0$.

It remains to show that the inclusion induces isomorphisms in homology in degrees ≥ 3. Let I be an indexing set for the attaching maps of the 2-cells $\left\{e^2_\alpha\right\}_{\alpha \in I}$ that we attached to obtain Y from X. Let also A_n be an indexing set for the n-cells in the CW structure of X (we can also view A_n as an indexing set for the n-simplices in the Δ-complex structure obtained from X using its CW structure). In either case, we obtain a chain complex from this CW/Δ-complex structure along with a chain map induced by the inclusion $X \hookrightarrow Y$ which is the identity in all degrees except degree 2 :

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{A_4}\mathbb{Z} \arrow[r, "\partial_4^X"] \arrow[d, equal] & \bigoplus_{A_3}\mathbb{Z} \arrow[r, "\partial_3^X"] \arrow[d, equal] & \bigoplus_{A_2}\mathbb{Z} \arrow[r, "\partial_2^X"] \arrow[d, hook] & \bigoplus_{A_1}\mathbb{Z} \arrow[r, "\partial_1^X"] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \bigoplus_{A_4}\mathbb{Z} \arrow[r, "\partial_4^Y"] & \bigoplus_{A_3}\mathbb{Z} \arrow[r, "\partial_3^Y"] & \bigoplus_{A_2 \sqcup I}\mathbb{Z} \arrow[r, "\partial_2^Y"] & \bigoplus_{A_1}\mathbb{Z} \arrow[r, "\partial_1^Y"] & \cdots
\end{tikzcd}$$

Now, recalling that the induced map $\iota_*\colon H_n(X) \to H_n(Y)$ is given by $[c] \mapsto [\iota \circ c]$, the maps on homology in degrees ≥ 3 will simply be the identity since for any $n \geq 3$, $\partial_n^Y = \partial_n^X$, so

$$H_n(Y) = \ker \partial_n^Y / \operatorname{im} \partial_{n+1}^Y = \ker \partial_n^X / \operatorname{im} \partial_{n+1}^X = H_n(X).$$

(3) Using the LES of the pair (Y, X), we find that

$$H_3(Y, X) \xrightarrow{\partial_*} H_2(X) \xrightarrow{i_*} H_2(Y) \xrightarrow{j_*} H_2(Y, X) \xrightarrow{\partial_*} H_1(X)$$

is exact. Now, note that since X is a CW subcomplex, it is, in particular, closed and the inclusion $X \hookrightarrow Y$ is a cofibration, so the quotienting map $(Y, X) \to (Y/X, *)$ induces an isomorphism $H_*(Y, X) \cong H_*(Y/X, *) \cong \tilde{H}_*(Y/X)$ (Corollary 1.7 together with Corollary 1.4, Chapter VII in Bredon's Topology and Geometry). Now, Y/X is a wedge of 2-spheres, so $\tilde{H}_3(Y/X) \cong 0$ by considering its chain in cellular or simplicial homology. As for $H_1(X)$, this vanishes by assumption on the space X, so we finally obtain that

$$0 \to H_2(X) \xrightarrow{i_*} H_2(Y) \xrightarrow{j_*} H_2(Y, X) \to 0$$

is a SES. Now, using the exact same argument as above, $H_2(Y, X) \cong \tilde{H}_2(Y/X)$ and Y/X is a wedge of 2-spheres indexed by I, so $\tilde{H}_2(Y/X) \cong \bigoplus_I \mathbb{Z}$. In particular, this is a free abelian group, and we can let A be a set of generators for this free summand. Since any free group is projective, this SES splits, so we find that

$$H_2(Y) \cong H_2(X) \oplus H_2(Y, X) \cong H_2(X) \oplus \bigoplus_I \mathbb{Z}$$

(4) Since Y is simply-connected, the Hurewicz theorem gives us an isomorphism $h\colon \pi_2(Y) \to H_2(Y)$ given by sending $f\colon (S^2, s_0) \to (Y, x_0)$ to $h([f]) = f_*(\alpha)$ where α is a generator of $H_2(S^2)$. In particular, we can represent each basis element α in A by some map $\psi_\alpha\colon (S^2, s_0) \to (Y, x_0)$ by pulling α back along the Hurewicz isomorphism. By the Cellular Approximation Theorem, we may again assume that each ψ_α is cellular (giving S^2 its standard CW structure). Now we let Z be the space obtained by attaching 3-cells to Y along the maps $\{\psi_\alpha\}_{\alpha \in A}$.

(5)
The inclusions $X \hookrightarrow Y \hookrightarrow Z$ now give the following maps of chain complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{A_4} \mathbb{Z} \arrow[r, "\partial_4^X"] \arrow[d, equal] & \bigoplus_{A_3} \mathbb{Z} \arrow[r, "\partial_3^X"] \arrow[d, equal] & \bigoplus_{A_2} \mathbb{Z} \arrow[r, "\partial_2^X"] \arrow[d, hook] & \bigoplus_{A_1} \mathbb{Z} \arrow[r, "\partial_1^X"] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \bigoplus_{A_4} \mathbb{Z} \arrow[r, "\partial_4^Y"] \arrow[d, equal] & \bigoplus_{A_3} \mathbb{Z} \arrow[r, "\partial_3^Y"] \arrow[d, hook] & \bigoplus_{A_2 \sqcup I} \mathbb{Z} \arrow[r, "\partial_2^Y"] \arrow[d, equal] & \bigoplus_{A_1} \mathbb{Z} \arrow[r, "\partial_1^Y"] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \bigoplus_{A_4} \mathbb{Z} \arrow[r, "\partial_4^Z"] & \bigoplus_{A_3 \sqcup I} \mathbb{Z} \arrow[r, "\partial_3^Z"] & \bigoplus_{A_2 \sqcup I} \mathbb{Z} \arrow[r, "\partial_2^Z"] & \bigoplus_{A_1} \mathbb{Z} \arrow[r, "\partial_1^Z"] & \cdots
\end{tikzcd}$$

By the exact same reasoning as before, since $\partial_n^Z = \partial_n^Y = \partial_n^X$ for $n \geq 4$, it follows that $H_n(Z) = H_n(Y) = H_n(X)$ for $n \geq 4$ with the inclusions again, by the exact same reasoning as in (2), inducing the isomorphisms (in fact, equalities, and the inclusions simply become the identity). For $n = 1$, we have that $H_1(X) = H_1(Y) = 0$, and so since

$$H_1(Z) = \ker \partial_1^Z / \operatorname{im} \partial_2^Z = \ker \partial_1^Y / \operatorname{im} \partial_2^Y = H_1(Y)$$

we also find that $H_1(Z) = 0$, so the inclusion $X \hookrightarrow Y \hookrightarrow Z$ trivially induces an isomorphism $H_1(X) \to H_1(Z)$.

For $n = 2$, note that we have the following commutative diagram (which commutes by naturality of the Hurewicz isomorphism):

$$\begin{tikzcd}
\pi_2(X) \arrow[r, "i_*"] \arrow[d, "h", "\cong"'] & \pi_2(Y) \arrow[r, "j_*"] \arrow[d, "h", "\cong"'] & \pi_2(Z) \arrow[d, "h", "\cong"'] \\
H_2(X) \arrow[r, "i_*"] & H_2(Y) \cong H_2(X) \oplus \bigoplus_I \mathbb{Z} \arrow[r, "j_*"] & H_2(Z)
\end{tikzcd}$$

First, recall that the splitting

$$H_2(Y) \cong H_2(X) \oplus \bigoplus_I \mathbb{Z}$$

was given by $(\alpha, \beta) \mapsto i_*(\alpha) + s(\beta)$ where s is the section for $H_2(Y) \xrightarrow{j_*} H_2(Y, X)$, so in particular, the inclusion $X \hookrightarrow Y$, becomes the inclusion $H_2(X) \hookrightarrow H_2(X) \oplus \bigoplus_I \mathbb{Z}$ into the $H_2(X)$ factor under this isomorphism.

In this diagram, each element $\alpha \in A$ is mapped to some representative $\left(S^2, s_0\right) \to (Y, x_0)$ in $\pi_2(Y)$ which, by construction, is based nullhomotopic in Z, so we see that $j_* \circ h^{-1}(\alpha) = 0$, hence $j_*(\alpha) = h \circ j_* \circ h^{-1}(\alpha) = 0$. Meanwhile, any element in $H_2(X) \subset H_2(X) \oplus \bigoplus_I \mathbb{Z}$ is pulled back along h to a nontrivial element which has nonzero image under j_* (by construction, Z eliminates only the elements of A). Thus $j_* \colon H_2(Y) \cong H_2(X) \oplus \bigoplus_I \mathbb{Z} \to H_2(Z)$ is injective on the $H_2(X)$ factor.

hence $j_* i_* = (j \circ i)_* : H_2(X) \to H_2(Z)$ is injective. Now, for any nontrivial $\gamma \in H_2(Z)$, this pulls back to a nontrivial element in $\pi_2(Z)$ which is the image of some $\beta \in \pi_2(Y)$ under j_* since $j_* \colon \pi_2(Y) \to \pi_2(Z)$ is surjective. This maps down to some element $(x, y) \in H_2(X) \oplus \bigoplus_I \mathbb{Z}$, and since j_* is 0 on all factors in $\bigoplus_I \mathbb{Z}$, we find that $j_*(x, 0) = \gamma$. So $(j \circ i)_*(x) = j_*(x, 0) = \gamma$, which shows that $(j \circ i)_*$ is also surjective. Thus $(j \circ i)_* : H_2(X) \to H_2(Z)$ is an isomorphism.

Lastly, for $n = 3$, it suffices to show that the inclusion $Y \hookrightarrow Z$ induces an isomorphism $H_3(Y) \to H_3(Z)$ since we already showed in (2) that $H_3(X) \to H_3(Y)$ induced by the inclusion is an isomorphism can be seen as follows: for each basis element in the I part of the summand of the domain of $\partial_3^Z : \bigoplus_{A_3 \sqcup I} \mathbb{Z}$, this is by construction of Z mapped to an element in A under ∂_3^Z which is nontrivial in $\bigoplus_{A_2 \sqcup I} \mathbb{Z}$, hence $\ker \partial_3^Z = \ker \partial_3^Y$, so $H_3(Z) = H_3(Y)$ and hence the inclusion induces the identity which is an isomorphism.

This completes the proof. $\square$