NOTES ON ISOSPECTRAL RIEMANN SURFACES

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We will follow the text 'Isospectral Riemann Surfaces' by Peter Buser.

1. Pasting

We will have two pasting constructions which will give us two different surfaces. They will both be made up of eight hyperbolic octagons glued together as shown in Figure **??**.

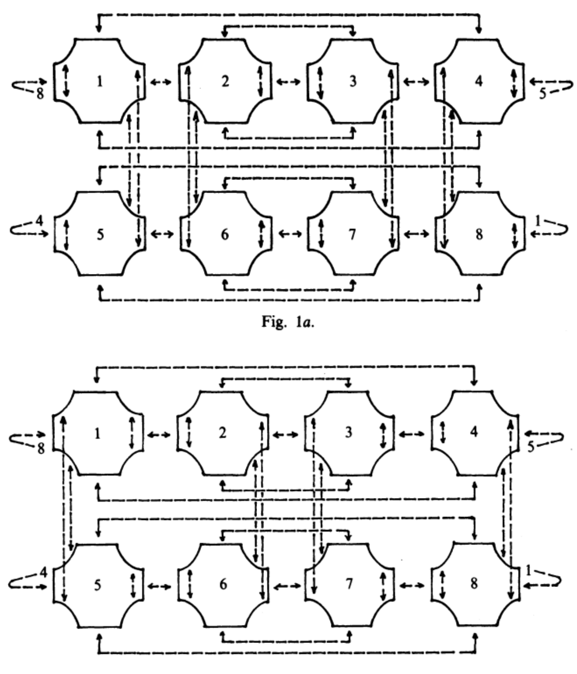

Figure 1. octagon-gluing.png

1.1. Genus five.

For a pentagon with sides r, p, q, q', p', the following trigonometric formulae hold

$$\cosh r = \coth p \coth p'$$
$$\cosh r = \sinh q \sinh q'.$$

Moreover, such pentagons exist for any $q', r \in \mathbb{R}_{>0}$.
Consequently, for any length of the sides b' and c' in Figure 2 we can construct such an octagon. By construction, $a' = a'' = a^* = a^{**}$ and $c' = c''$, so the pasting of section 1 is possible.

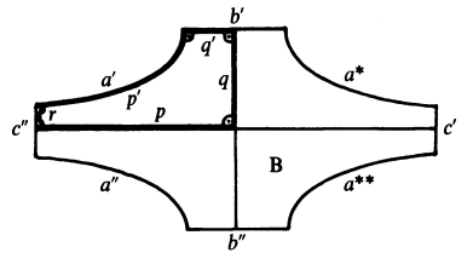

FIGURE 2. hyperbolic-octagon.png

Lemma 1.1. *Let $0 < c' < b' < 1$. Then any curve δ on B which connects two non adjacent sides of B has length $\ell(\delta) \geq c'$. Equality holds for $\delta = c'$ and $\delta = c''$.*

Proof. Recall that with negative curvature, the unique shortest connecting curve δ between two non intersecting geodesics is the common perpendicular between these geodesics. Now $\sinh q \sinh q' = \cosh r > 1$ and $q' = \frac{1}{2}b' < 1$ together imply $q > 1$. Similarly, $p', p > 1$. It follows from Figure 2 that if δ is not c' or c'' then it has length $\ell(\delta) > c' = c''$. □

Proposition 1.2. *Under the hypothesis of Lemma 1.1, S_1 and S_2 are non isometric.*

1.2. **The length spectrum.**

Definition 1.3 (Length Spectrum)**.** For a compact Riemannian manifold we define the *length spectrum* to be the function $\ell \mapsto n_M(\ell)$ which associates to each positive real number ℓ the cardinality $n_M(\ell)$ of the set of all closed geodesics of length ℓ on M.

Corollary 1.4. *Recall that for a compact Riemann surface of genus $g \geq 2$, each free homotopy class of an essential closed curve, there is a unique closed geodesic. Hence the number of closed geodesics of length $\leq \ell$ is finite for all ℓ. The length spectrum here may also be defined by giving the list of all possible lengths, arranged in increasing order.*

Theorem 1.5. *Two compact Riemann surfaces of genus $g \geq 2$ are isospectral with respect to the Laplacian if and only if they have the same length spectrum.*

Proposition 1.6. *The surfaces S_1 and S_2 obtained in section 1 have the same length spectrum.*