1. Objectives

- Read up on transversality in Lee - potentially supplied with Hirsch and Guillemin and Pollack.
- Write notes for section 1.2.5 in Farb and Margalit.
- Read about oriented intersection theory in Guillemin and Pollack.
- Work on section 2 in Farb and Margalit.

2. Questions

- How to understand intersection numbers? Any references? Guillemin and Pollack?
- Exercise sheets.
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3. Curves, Surfaces and Hyperbolic Geometry

3.1. **Simple closed curves.** There is a bijective correspondence

$$\left\{ \begin{array}{c} \text{Nontrivial} \\ \text{conjugacy classes} \\ \text{in } \pi_1(S) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Nontrivial free} \\ \text{homotopy classes of oriented} \\ \text{closed curves in } S \end{array} \right\}$$

Definition 3.1 (Primitive and multiple elements)**.** An element g of a group G is *primitive* if there does not exist any $h \in G$ so that $g = h^k$ for $|k| > 1$. The property of being a primitive is a conjugacy class invariant. In particular, it makes sense to say that a closed curve in a surface is primitive.
A closed curve in S is a multiple if it is a map $S^1 \to S$ that factors through the map $S^1 \overset{\times n}{\to} S^1$ for $n > 1$, i.e., there exists a map $\tilde{\alpha} \colon S^1 \to S$ such that the following diagram commutes:

$$\begin{tikzcd} S^1 \arrow[r, "\times n"] \arrow[rr, bend left, dashed, "\tilde{\alpha}"] & S^1 \arrow[r, "\alpha"] & S \end{tikzcd}$$

Definition 3.2 (Lifts)**.** We make a distinction between lifts: let $p \colon \tilde{S} \to S$ be a covering space. By a *lift* of a closed curve α to $\tilde{S}$ we will always mean the image of a lift $\mathbb{R} \to \tilde{S}$ of the map $\alpha \circ \pi$ where $\pi \colon \mathbb{R} \to S^1$ is the usual covering map. I.e., a lift of $\alpha \colon S^1 \to S$ is a map $\tilde{\alpha} \colon \mathbb{R} \to \tilde{S}$ such that the following diagram commutes

$$\begin{tikzcd} & & \tilde{S} \arrow[d, "p"] \\ \mathbb{R} \arrow[urr, "\tilde{\alpha}"] \arrow[r, "\pi"'] & S^1 \arrow[r, "\alpha"'] & S \end{tikzcd}$$

A lift is different from a *path lift* which is a proper subset of a lift. Namely, it would be the restriction of $\tilde{\alpha}$ to some interval of $\mathbb{R}$ of length 2π if the covering map π is of the form $t \mapsto e^{it}$.

Now suppose $p \colon \tilde{S} \to S$ is the universal cover and α is a simple closed curve in S that is not a multiple of another closed curve. In this case, there is a bijective correspondence between cosets in $\pi_1(S)$ of the infinite cyclic subgroup $\langle \alpha \rangle$ and the lifts of α. This can be seen as follows: first choose a basepoint $\alpha(1) = x_0 \in S$ and some $\tilde{x_0} \in p^{-1}(x_0)$. There exists a unique lift $\tilde{\alpha}$ of α such that

$$\begin{tikzcd} & & \tilde{S} \arrow[d, "p"] \\ \mathbb{R} \arrow[urr, "\tilde{\alpha}"] \arrow[r] & S^1 \arrow[r, "\alpha"] & S \end{tikzcd}$$

commutes and such that $\tilde{\alpha}(0) = \tilde{x} \in p^{-1}(\alpha \circ \pi(0))$ for some specific $\tilde{x}$ [Bredon, Cor. 4.2]. But the set $p^{-1}(\alpha \circ \pi(0))$ is in bijective correspondence with the loops in $\pi_1(S)$ by the path lifting lemma. Now, under which path lifts are the lifts the same? The lifts of α to two points $\tilde{x}, \tilde{y} \in p^{-1}(\alpha \circ \pi(0))$ will be the same if $\alpha^k \cdot \tilde{x} = \tilde{y}$ where $\cdot$ denotes the monodromy action of $\pi_1(S)$ on the fiber. Now, there exist γ_x and γ_y in $\pi_1(S)$ such that $\gamma_x \cdot \tilde{x_0} = \tilde{x}$ and $\gamma_y \cdot \tilde{x_0} = \tilde{y}$, so $\alpha^k \gamma_x = \gamma_y$. Hence the lifts corresponding to γ_x and γ_y are the same if and only if $\alpha^k \gamma_x = \gamma_y$ for some k, i.e. if and only if $\gamma_x = \gamma_y$ in $\pi_1(S)/\langle \alpha \rangle$.

As usual, the group $\pi_1(S)$ acts on the set of lifts of α by deck transformations, and this action agrees with the usual left action of $\pi_1(S)$ on the cosets of $\langle\alpha\rangle$. The stabilizer of the lift corresponding to the coset $\gamma\langle\alpha\rangle$ is the cyclic group $\langle\gamma\alpha\gamma^{-1}\rangle$. See figure 1.

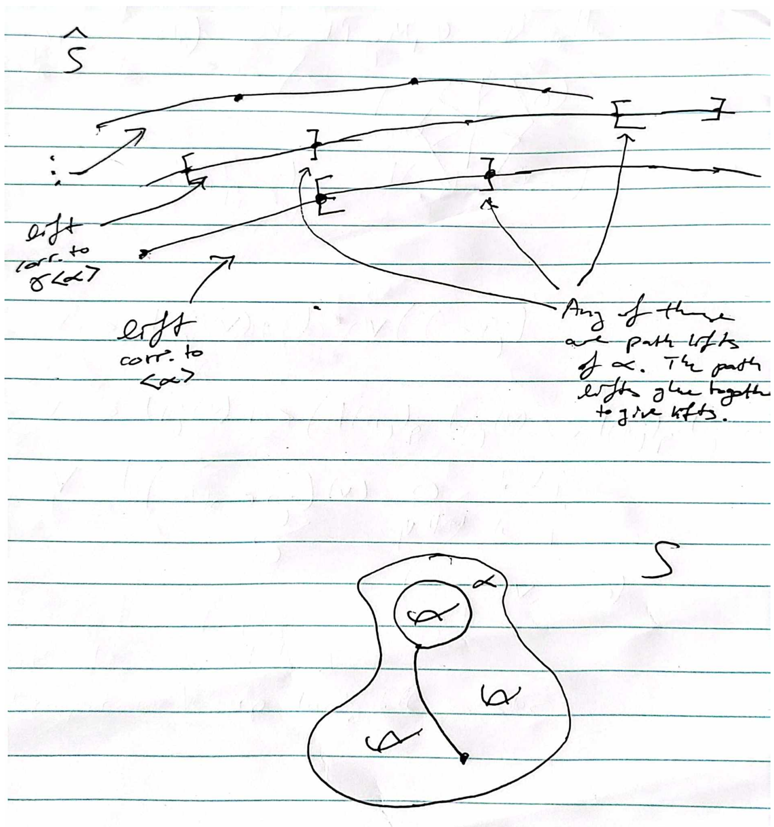

FIGURE 1.

Theorem 3.3. *When S admits a hyperbolic metric and α is a primitive element of $\pi_1(S)$, we have a bijective correspondence*

$$\left\{ \begin{array}{c} \text{Elements of the conjugacy} \\ \text{class of } \alpha \text{ in } \pi_1(S) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Lifts to } \tilde{S} \text{ of the} \\ \text{closed curve } \alpha \end{array} \right\}$$

More precisely, we claim that the map which sends the lift given by the coset $\gamma\langle\alpha\rangle$ to $\gamma\alpha\gamma^{-1}$ is bijective and well-defined.

Proof. To show that it is well-defined, suppose $\gamma\langle\alpha\rangle$ and $\beta\langle\alpha\rangle$ give the same lift. Then $\gamma = \beta\alpha^k$. So in particular,

$$\gamma\alpha\gamma^{-1} = \beta\alpha^k\alpha\alpha^{-k}\beta^{-1} = \beta\alpha\beta^{-1}$$

so they do correspond to the same element of the conjugacy class $[\alpha]$. It is clear that this is a surjective map. Now suppose that $\gamma\alpha\gamma^{-1} = \beta\alpha\beta^{-1}$. Then

$\beta^{-1}\gamma\alpha\left(\beta^{-1}\gamma\right)^{-1} = \alpha$, so in particular, $\beta^{-1}\gamma \in C_{\pi_1(S)}(\alpha)$ which is a cyclic group generated by, say, θ. But then $\theta^l = \alpha$ since α is trivially in the centralizer of α; however, α is primitive, so l must be ± 1, but then α generates the centralizer of α, $C_{\pi_1(S)}(\alpha) = \langle\alpha\rangle$, and hence $\gamma = \beta\alpha^l$, so $\gamma\langle\alpha\rangle = \beta\langle\alpha\rangle$. $\square$

Remark. If α is any multiple, then we still have a bijective correspondence between elements of the conjugacy class of α and the lifts of α. However, if α is not primitive and not a multiple, then there are more lifts of α than there are conjugates. Indeed, if $\alpha = \beta^k$, where $k > 1$, then $\beta\langle\alpha\rangle \neq \langle\alpha\rangle$ while $\beta\alpha\beta^{-1} = \alpha$.

Example 3.4. The above correspondence does not hold for the torus T^2 because each closed curve has infinitely many lifts, while each element of $\pi_1\left(T^2\right) \approx \mathbb{Z}^2$ is its own conjugacy class because $\pi_1\left(T^2\right)$ is abelian.

Geodesic representatives.

Proposition 3.5. *Let S be a hyperbolic surface. If α is a closed curve in S that is not homotopic into a neighborhood of a puncture, then α is homotopic to a unique geodesic closed curve γ.*

Corollary 3.6. *For compact hyperbolic surfaces, there is a bijective correspondence:*

$$\left\{\begin{array}{c}\text{Conjugacy classes}\\ \text{in } \pi_1(S)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Oriented geodesic}\\ \text{closed curves in } S\end{array}\right\}$$

Simple closed curves.

Definition 3.7 (Simple curves)**.** A closed curve in S is *simple* if it is topologically embedded, i.e., if the map $S^1 \to S$ is injective.

By [Bredon, Thm 11.8], any closed curve α can be approximated (arbitrarily close) by a smooth closed curve which is homotopic to α. Moreover, if α is simple, then the smooth approximation can be chosen to be simple. Smooth curves are advantageous because we can make use of notions such as transversality.
Simple closed curves are also natural to study because they represent primitive elements of $\pi_1(S)$.

Proposition 3.8. *Let α be a simple closed curve in a surface S. If α is not null homotopic, then each element of the corresponding conjugacy class in $\pi_1(S)$ is primitive.*

Example: simple closed curves on the torus.

Proposition 3.9. *The nontrivial homotopy classes of oriented simple closed curves in T^2 are in bijective correspondence with the set of primitive elements of $\pi_1\left(T^2\right) \approx \mathbb{Z}^2$ which is the set of elements $(p,q) \in \mathbb{Z}^2$ such that either $(p,q) = (0,\pm 1)$ or $(p,q) = (\pm 1, 0)$ or $\gcd(p,q) = 1$.*

Closed geodesics.

Proposition 3.10. *Let S be a hyperbolic surface. Let α be a closed curve in S not homotopic into a neighborhood of a puncture. Let γ be the unique geodesic in the free homotopy class of α guaranteed by proposition 3.5. If α is simple, then γ is simple.*

Proof. Follows from the following lemma:

Lemma 3.11. *Let X be a topological space with a universal covering space $\tilde{X}$. A closed curve β in X is simple if and only if the following properties hold:*

(1) Each lift of β to $\tilde{X}$ is simple.
(2) No two lifts of β intersect.
(3) β is not a nontrivial multiple of another closed curve.

□

Intersection numbers. It is often useful to put an inner product on a vector space to check if two vectors are linearly independent. We can pursue something similar for surfaces.

Definition 3.12 (Geometric intersection number)**.** Let α, β be closed curves on a surface S. Their *geometric intersection number* is

$$i(\alpha, \beta) = \min_{\alpha' \simeq \alpha, \beta' \simeq \beta} \#(\alpha' \cap \beta')$$

Definition 3.13 (Preliminary definition for transversality)**.** If $\alpha \cap \beta$ is finite and, at every intersection, each curve locally separates the other curve, then we say that α and β are *transverse*.

Definition 3.14 (Minimal position)**.** Two curves α and β are in *minimal position* if $\#(\alpha \cap \beta) = i(\alpha, \beta)$.

Bigons. We want a procedure to put curves into minimal position so we can compute intersection numbers.
For this, we need the notion of a *bigon*:

Definition 3.15 (Bigon)**.** Two transverse simple closed curves α and β in a surface S form a *bigon* if there is a topologically embedded disk in S (the bigon) whose boundary is the union of an arc of α and an arc of β intersecting in exactly two points.

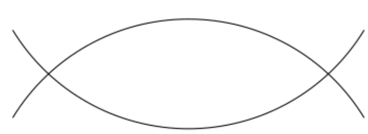

FIGURE 2. Local picture of a bigon

Lemma 3.16. *If transverse simple closed curves α and β in a surface S do not form any bigons, then in the universal cover of S, and pair of lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β intersect in at most one point.*

Proposition 3.17 (The bigon criterion)**.** *Two transverse simple closed curves in a surface S are in minimal position if and only if they do not form a bigon.*

Corollary 3.18. *Any two transverse simple closed curves that intersect exactly once are in minimal position.*

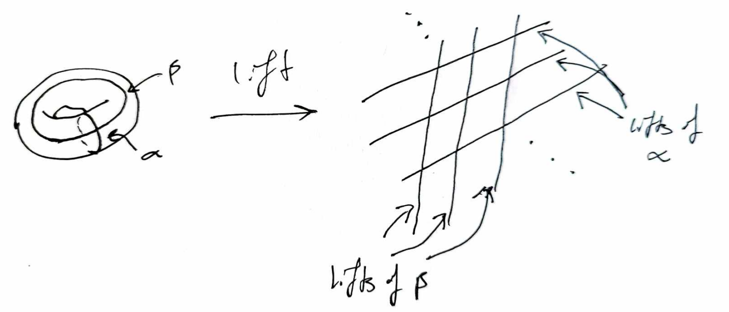

FIGURE 3. Lemma 3.16 illustrated

Homotopy versus isotopy for simple closed curves.

Definition 3.19 (Isotopy)**.** Two simple closed curves α and β are *isotopic* if there is a homotopy

$$H\colon S^1 \times [0,1] \to S$$

from α to β with the property that the closed curve $H\left(S^1 \times \{t\}\right)$ is simple for each $t \in [0,1]$.

Proposition 3.20 (Baer)**.** *Let α and β be two essential simple closed curves in a surface S. Then α is isotopic to β if and only if α is homotopoic to β.*

4. Mapping class group basics

4.1. Definitions and first examples.

Definition 4.1. Let S be a surface which is the connected sum of $g \geq 0$ tori with $b \geq 0$ disjoint open disks removed and $n \geq 0$ points removed from the interior. Let $\mathrm{Homeo}^+(S, \partial S)$ denote the group of orientation-preserving homeomorphisms of S that restrict to the identity on ∂S. We endow this group with the compact-open topology. The *mapping class group* of S, denoted $\mathrm{Mod}(S)$, is the group

$$\mathrm{Mod}(S) = \pi_0\left(\mathrm{Homeo}^+(S, \partial S)\right)$$

In other words, $\mathrm{Mod}(S)$ is the group of isotopy classes of elements of $\mathrm{Homeo}^+(S, \partial S)$, where isotopies are required to fix the boundary pointwise. If $\mathrm{Homeo}_0(S, \partial S)$ denotes the connected component of the identity in $\mathrm{Homeo}^+(S, \partial S)$, then we can equivalently write

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S, \partial S) / \mathrm{Homeo}_0(S, \partial S).$$

Proposition 4.2.

$$\begin{aligned}\mathrm{Mod}(S) &= \pi_0\left(\mathrm{Homeo}^+(S, \partial S)\right) \\ &\approx \mathrm{Homeo}^+(S, \partial S) / \textit{homotopy} \\ &\approx \pi_0\left(\mathrm{Diff}^+(S, \partial S)\right) \\ &\approx \textit{Diff}^+(S, \partial S) / \sim\end{aligned}$$

where $\mathrm{Diff}^+(S, \partial S)$ is the group of orientation preserving diffeomorphisms of S that are the identity on the boundary and $\sim$ can be taken to be either smooth homotopy relative to the boundary or smooth isotopy relative to the boundary.

The Alexander Lemma.

Lemma 4.3 (Alexander lemma)**.** *The group* $\operatorname{Mod}\left(D^2\right)$ *is trivial.*

Remark. Also $0 \approx \operatorname{Mod}(D - \{0\}) \approx \operatorname{Mod}(S_{0,1}) \approx \mod \left(S^2\right)$.

The mapping class group of the thrice-punctured sphere, $\operatorname{Mod}(S_{0,3})$.

Proposition 4.4. *Any two essential simple proper arcs in* $S_{0,3}$ *with the same endpoints are isotopic. Any two essential arcs that both start and end at the same marked point of* $S_{0,3}$ *are isotopic.*

Proposition 4.5. *The natural map*

$$\operatorname{Mod}(S_{0,3}) \to \Sigma_3$$

given by the action of $\operatorname{Mod}(S_{0,3})$ *on the set of marked points of* $S_{0,3}$ *is an isomorphism.*

Exercise 4.6. Show similarly that $\operatorname{Mod}(S_{0,2}) \approx \mathbb{Z}/2\mathbb{Z}$.

5. Exercises

Problem 5.1. Give an example of a surface S of finite type and self-diffeomorphism φ of S which is homotopic to id_S but not isotopic to id_S.

Proof. $\square$

6. Glossary

Definition 6.1 (Equivariant maps)**.** Suppose a group G acts on spaces X and Y, and let $f\colon X \to Y$ be a map. Then f is said to be equivariant if $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$ and all $g \in G$.

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