HOMOTOPY THEORY

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For these notes, we will follow [2], [1] and [3].

1. Homotopy Groups

1.1. **Homotopy.** We follow chapter 14 of [1] for this subsection.

To start of, we recall the basic definitions of homotopies.

Definition 1.1 (Homotopy)**.** Two maps $f_0, f_1 \colon X \to Y$ are said to be *homotopic* if there exists a homotopy $F \colon X \times I \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$.

Definition 1.2 (Homotopy equivalence)**.** A map $f \colon X \to Y$ is said to be a *homotopy equivalence* if it is an isomorphism in hTop.

Lemma 1.3 (Reparametrization Lemma)**.** *Let φ_1, φ_2 be maps $(I, \partial I) \to (I, \partial I)$ which are equal on ∂I. Let $F \colon X \times I \to Y$ be a homotopy and let $G_i(x,t) = F(x, \varphi_i(t))$ for $i = 1, 2$. Then $G_1 \simeq G_2 \operatorname{rel} X \times \partial I$.*

We shall use c to denote the constant homotopy.

Proposition 1.4. *$F * c \simeq F \operatorname{rel} X \times \partial I$ and $c * F \simeq F \operatorname{rel} X \times \partial I$.*

Definition 1.5. If $F \colon X \times I \to Y$ is a homotopy, then we define $F^{-1} \colon X \times I \to Y$ by $F^{-1}(x,t) = F(x, 1-t)$.

Note that F^{-1} is precisely the inverse to F in hTop.

Proposition 1.6. *For any homotopies F, G, H for which the concatenations are defined, we have*

$$(F * G) * H \simeq F * (G * H) \operatorname{rel} X \times \partial I.$$

Proposition 1.7. *For homotopies F_1, F_2, G_1, G_2, if $F_1 \simeq F_2 \operatorname{rel} X \times \partial I$ and $G_1 \simeq G_2 \operatorname{rel} X \times \partial I$, then $F_1 * G_1 \simeq F_2 * G_2 \operatorname{rel} X \times \partial I$.*

Note that all of the discussion of concatenation of homotopies goes through with no difficulties for the cases in which all homotopies are relative to some subspace $A \subset X$ or are homotopies of pairs $(X, A) \to (Y, B)$.
It follows that homotopy between maps of pairs $(X, A) \to (Y, B)$ is an equivalence relation. The set of homotopy classes of these maps is commonly denoted by $[X, A; Y, B]$ or just $[X; Y]$ if $A = \varnothing$.

Theorem 1.8. *If $f_0 \simeq f_1 \colon X \to Y$ then $M_{f_0} \simeq M_{f_1} \operatorname{rel} X + Y$ and $C_{f_0} \simeq C_{f_1} \operatorname{rel} Y +$ vertex.*

To show this, one needs the following basic topological proposition:

Proposition 1.9. *If $f \colon X \to Y$ is a quotient map and K is locally compact Hausdorff, then $f \times 1 \colon X \times K \to Y \times K$ is a quotient map.*

Proof of Theorem 1.8. First, let $F \colon X \times I \to Y$ be the homotopy between f_0 and f_1. Now define $h \colon M_{f_0} \to M_{f_1}$ by $h(y) = y$ for $y \in Y$ and

$$h(x,t) = \begin{cases} F(x, 2t), & t \leq \frac{1}{2} \\ (x, 2t-1), & \frac{1}{2} \leq t. \end{cases}$$

Define $k \colon M_{f_1} \to M_{f_0}$ likewise by the identity on Y nad

$$k(x,t) = \begin{cases} F^{-1}(x, 2t), & t \leq \frac{1}{2} \\ (x, 2t-1), & \frac{1}{2} \leq t \end{cases}.$$

Then the composition $kh\colon M_{f_0} \to M_{f_1}$ is the identity on Y and $F * \left(F^{-1} * E\right)$ on the cylinder portion, where $E\colon X \times I \to M_{f_0}$ is induced by the identity on $X \times I \to X \times I$. This is homotopic to the identity $\operatorname{rel} X \times \{1\} + Y$. Similarly for hk. In now remains to check the continuity of this homotopy. We have a homotopy $M_{f_0} \times I \to M_{f_0}$. We now claim that $M_{f_0} \times I \cong M_{f_0 \times I}$. Indeed then, using that $M_{f_0 \times I} = \frac{X \times I \times I \sqcup Y \times I}{((x,0,k) \sim (f_0(x),k)}$, it suffices to show continuity of the composition $X \times I \times I \sqcup Y \times I \to M_{f_0} \times I \to M_{f_0}$. For on $Y \times I$, it is the constant homotopy and on $X \times I \times I$ it is $F * \left(F^{-1} * E\right) \simeq E \operatorname{rel} X \times \partial I$. Now, that $M_{f_0} \times I \cong M_{f_0 \times I}$ follows from Proposition 1.9.

□

Let $f\colon X \to Y$. If $\varphi\colon Y \to Y'$ is a map, then there is the induced map $F\colon M_f \to M_{\varphi \circ f}$ induced from φ on Y and the identity on $X \times I$.

Theorem 1.10. *If $\varphi\colon Y \to Y'$ is a homotopy equivalence then so is $F\colon (M_f, X) \to (M_{\varphi \circ f}, X)$ and hence so is $F\colon C_f \to C_{\varphi \circ f}$.*

Proof. Let $\psi\colon Y' \to Y$ be a homotopy inverse of φ and let $G\colon M_{\varphi \circ f} \to M_{\psi \circ \varphi \circ f}$ be the map induced by ψ on Y' and the identity on $X \times I$. The composition $GF\colon M_f \to M_{\psi \circ \varphi \circ f}$ is induced from $\psi \circ \varphi\colon Y \to Y$ and the identity on $X \times I$. Let $H\colon Y \times I \to Y$ be a homotopy from id to $\psi \circ \varphi$; i.e., $H(y, 0) = y$ and $H(y, 1) = \psi\varphi(y)$. By the proof of Theorem 1.8, there is a homotopy equivalence $h\colon M_f \to M_{\psi \circ \varphi \circ f} \operatorname{rel} X$ given by $h(y) = y$ and

$$h(x,t) = \begin{cases} H\left(f(x), 2t\right), & t \leq \frac{1}{2} \\ (x, 2t-1), & t \geq \frac{1}{2} \end{cases}.$$

We claim that $h \simeq GF \operatorname{rel} X$. Indeed, the homotopy H can be extended to $M_f \times I \to M_{\psi \circ \varphi \circ f}$ by putting

$$H\left((x,s),t\right) = \begin{cases} H\left(f(x), 2s+t\right), & 2s+t \leq 1 \\ \left(x, \frac{2s+t-1}{t+1}\right), & 2s+t \geq 1 \end{cases}.$$

Then $H(-, 0) = h$ and $H(-, 1) = GF$, so since GF is a homotopy equivalence, so is h. Define $F'\colon M_{\psi \circ \varphi \circ f} \to M_{\varphi \circ \psi \circ \varphi \circ f}$ as the induced map on mapping cones with φ on Y and the identity on $X \times I$. Then similarly, $F'G$ is a homotopy equivalence. If k is a homotopy inverse of GF then $GFk \simeq \mathrm{id}$. If k' is a homotopy inverse of $F'G$ then $k'F'G \simeq \mathrm{id}$. Thus G has a right and left homotopy inverse: $R = Fk$ and $L = k'F'$. Then $R = \mathrm{id} \circ R \simeq (LG)\, R = L\,(GR) \simeq L \circ \mathrm{id} = L$, so $R \simeq L$. That is, G has a homotopy inverse. Therefore, G is a homotopy equivalence. Since G and GF are homotopy equivalences, so is F. □

Problem 1.11. [1, Ex 14.1] Let $S^2 \cup A$ denote the union of the unit 2-sphere and the line segment joining the north and south poles. Show that $S^2 \vee S^1 \simeq S^2 \cup A$.

Proof. Define two maps $f_0, f_1\colon \{0,1\} \to S^2$ where $f_0(t) = (\cos(2\pi t), \sin(2\pi t), 0)$ and f_1 is the constant map at $(1,0,0)$. Then $f_0 \simeq f_1$, so $C_{f_0} \simeq C_{f_1}$. Now, $C_{f_0} = S^2 \cup A$ while $C_{f_1} = S^2 \vee S^1$. □

Problem 1.12. [1, Ex 14.2] Show that the union of a 2-sphere and a flat unit 2-cell through the origin is homotopically equivalent to the one-point union of two 2-spheres.

Proof. A 2-cell is contractible, an a 2-sphere with a 2-cell inside it is precisely the cone of the map $S^1 \sqcup S^1 \to S^1$ with the identity on both. By [1, Thm 14.19], this is homotopy equivalent to the cone on $S^1 \sqcup S^1 \to \{*\}$ which is $S^2 \vee S^2$. $\square$

Problem 1.13. Show that the union of a standard 2-torus with two disks, one spanning a latitudinal circle and the other spanning a longitudinal circle of the torus, is homotopically equivalent to a 2-sphere.

Proof. Using the identification of the torus as the quotient space of I^2 in the usual way, we can choose on spanning circle to be a 2-cell attached along $\{0\} \times I$ and the other to be a 2-cell attached along $I \times \{0\}$. These are contractible, and the quotient space becomes a 2-sphere. $\square$

1.2. **Homotopy Groups.** Recall that $[X, A; Y, B]$ denotes the set of homotopy classes of maps $X \to Y$ carrying A into B such that A goes into B during the entire homotopy.
To make a group then, we can select a point $y_0 \in Y$ and consider the set

$$[X \times I, X \times \partial I; Y, \{y_0\}]$$

In this case, the operation of concatenation of homotopies makes this set into a group. It is technically also better to choose a basepoint $x_0 \in X$ and consider

$$[X \times I, \{x_0\} \times I \cup X \times \partial I; Y, \{y_0\}] .$$

For the moment, let us set $A = \{x_0\} \times I \cup X \times \partial I$. Then maps $X \times I \to Y$ which carry A into $\{y_0\}$ are in bijective correspondence with maps $(X \times I)/A \to Y$ which take the point $\{A\}$ into $\{y_0\}$.

Definition 1.14 (Reduced Suspension)**.** We define the *reduced suspension* of X to be

$$SX = (X \times I)/A = (X \times I)/(\{x_0\} \times I \cup X \times \partial I)$$

The set of homotopy classes of pointed maps of a pointed space X to a pointed space Y with homotopies preserving the base points will be denoted by $[X; Y]_*$.
Thus $[SX; Y]_*$ is in canonical bijective correspondence with $[X \times I, A; Y, \{y_0\}]$.
Now, suppose we have pointed maps $f, g \colon SX \to Y$. Then they induce homotopies $f', g' \colon X \times I \to Y$ by precomposing with the quotient map $X \times I \to SX$. We can then define $f' * g' \colon X \times I \to Y$ as usual. The resulting pointed map $SX \to Y$ will be denoted $f * g$. Geometrically, $f * g$ is obtained by putting f on the bottom and g on the top of the one-point union $SX \vee SX$ and composing the resulting map $SX \vee SX \to Y$ with the map $SX \to SX \vee SX$ obtained by collapsing the middle parameter value $\frac{1}{2}$ copy of X in SX to the base point.
For a map $f \colon (SX, \{A\}) \to (Y, \{y_0\})$, we denote its homotopy class in $[SX; Y]_*$ by $[f]$, and we define

$$[f]\,[g] = [f * g]$$

Under this operation, the set $[SX; Y]_*$ becomes a group.

Proposition 1.15. *The reduced suspension gives $SS^{n-1} \cong S^n$.*

Thus, we can define S^n as the n-fold reduced suspension of S^0. As a special case, the set $[S^n; Y]_*$ then becomes a group for $n > 0$.

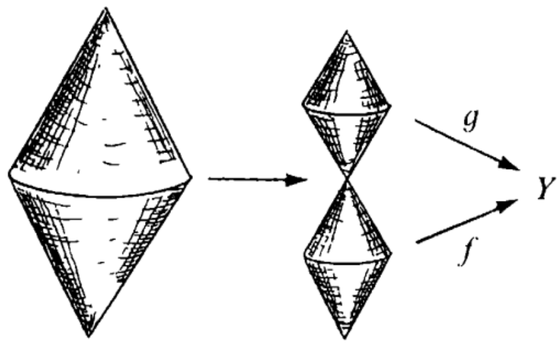

FIGURE 1. The product of two map classes $SX \to Y$.

Definition 1.16 (n th homotopy group)**.** We define

$$\pi_n(Y, y_0) = [S^n; Y]_*$$

with this operation.

1.3. **Homotopy Groups using H-Spaces/Groups/Cogroups.** From now on, unless otherwise indicated, we regard the n-sphere S^n as having the cogroup structure as the reduced suspension $S^n = SS^{n-1} = S^{n-1} \wedge S^1$ - i.e., the map γ in the definition of an H-cogroup will be $\gamma\colon S^n \to S^n \vee S^n$ given by

$$\gamma(t, x) = \begin{cases} (2t, x)_1, & t \leq \frac{1}{2} \\ (2t-1, x)_2, & t \geq \frac{1}{2}. \end{cases}$$

The 0-sphere S^0 is $\{0, 1\}$ with base point $\{0\}$.

For a based space X with base point x_0, we define the nth homotopy group

$$\pi_n(X, x_0) = [S^n, *; X, x_0].$$

This is a group with the product defined by Theorem **??**.(2).

Theorem 1.17. *If X is an H-space then the multiplication in $\pi_n(X, x_0)$ is induced by the H-space multiplication and is abelian for $n \geq 1$.*

Proof. This follows directly from Theorem **??**. □

Lemma 1.18. *In the pointed category, $[SX; Y] \cong [X; \Omega Y]$ as groups.*

Proof. Recall the characteristic correspondence for the compact-open topology (Theorem **??**):

$$f\colon X \times S^1 \to Y \leftrightarrow f'\colon X \to Y^{S^1}$$

given by $f'(x)(t) = f(x, t)$. Recall that $SX = X \wedge S^1$, so a map $g\colon X \wedge S^1 \to Y$ induces a map $f\colon X \times S^1 \to Y$ by the composition $X \times S^1 \to X \wedge S^1 \xrightarrow{g} Y$. Then f is continuous if and only if the map $f'\colon X \to Y^{S^1}$ is continuous, where $f'(x)(*) = f(x, *) = *$, so $f'(*) = * \in Y^{S^1}$, the basepoint of ΩY. So the correspondence induces a bijective correspondence between pointed maps $SX \to Y$ and pointed maps $X \to \Omega Y$, and pointed homotopies correspond as well.

It remains to show that the correspondence is a group homomorphism. Recall that

SX is an H-cogroup and ΩX is an H-group, so using Theorem **??**, we get that for $f, g\colon SX \to Y$, the product in $[SX; Y]$ is induced by

$$(f * g)(x, t) = \begin{cases} f(x, 2t), & t \leq \frac{1}{2} \\ g(x, 2t-1), & t \geq \frac{1}{2}, \end{cases}$$

which is equal to $(f * g)'(x)(t)$, while the multiplication in $[X; \Omega Y]$ is given by $(f' \cdot g')(x) = f'(x) * g'(x)$ where $*$ is loop concatenation. At time t, this is $f'(x)(2t)$ for $t \leq \frac{1}{2}$ and $g'(x)(2t-1)$ for $t \geq \frac{1}{2}$. Thus $(f * g)' = f' \cdot g'$. □

All of the above immediately carries over to pointed pairs (X, A) with a base point in A, so $[SX, SA; Y, B]$ is a group that is canonically isomorphic to $[X, A; \Omega Y, \Omega B]$. Note in particular that $D^n \cong SD^{n-1}$ for all $n \geq 2$, so $D^n = D^1 \wedge S^{n-1} \supset S^0 \wedge S^{n-1} = S^{n-1}$, so $\left(D^n, S^{n-1}\right) = S^{n-1}\left(D^1, S^0\right)$, the $(n-1)$-fold reduced suspension. Hence we can define the relative homotopy group by

$$\pi_n(Y, B, *) = \left[D^n, S^{n-1}; Y, B\right] = \left[S^{n-1}\left(D^1, S^0\right); Y, B\right].$$

Note that this is defined on pointed spaces and pointed maps, so this set is really $\left[D^n, S^{n-1}, s_0; Y, B, *\right]$. This becomes a group for $n \geq 2$.
Next note that we have a quotienting map

$$I^n = \underbrace{D^1}_{=I} \times I \times \ldots \times I \to D^1 \wedge S^1 \wedge \ldots \wedge S^1 = D^n.$$

Under this map, ∂I^n corresponds to S^{n-1} and the base point corresponds to $J^{n-1} = \left(I \times \partial I^{n-1}\right) \cup \left(\{0\} \times I^{n-1}\right)$. Thus under this quotienting map, we obtain a bijection

$$\pi_n(Y, B, *) = \left[D^n, S^{n-1}, s_0; Y, B, *\right] \cong \left[I^n, \partial I^n, J^{n-1}; Y, B, *\right].$$

Corollary 1.19. *$\pi_n(Y, *)$ is abelian for $n \geq 2$ and $\pi_n(Y, B, *)$ is abelian for $n \geq 3$. Moreover, the group structure is independent of the suspension coordinate used to define it.*

Proof. Recall from Lemma 1.18, that $[S^n; Y] = \left[S^1 \wedge \ldots \wedge S^1; Y\right] \cong \left[S^{n-1}; \Omega Y\right]$ as groups. Now the loop structure corresponds to the suspension in the last coordinate by definition, and by Theorem **??**, this is the same as the group $[S^n; Y]$ with the suspension operation on any of the coordinates, since the choice of coordinate is arbitrary, this shows that the group structure on $\pi_n(Y, *) = \left[SS^{n-1}; Y\right] = \left[S^1 \wedge \ldots \wedge S^1; Y\right]$ is independent of the suspension coordinate used to define it.
The product in $\left[S^{n-1}; \Omega Y\right]$ is furthermore abelian for $n - 1 \geq 1$ by Theorem 1.17 (using that ΩY is an H-space), and the relative case is similar. □

Corollary 1.20.

$$\pi_n(Y, *) \cong \pi_{n-1}(\Omega Y, *) \cong \ldots \cong \pi_1\left(\Omega^{n-1} Y, *\right) \cong \pi_0(\Omega^n Y, *)$$

and similarly in the relative case.

Theorem 1.21. *Let A be a closed subspace of X containing the base point $*$. Suppose that $F\colon X \times I \to X$ is a deformation of X contracting A to $*$; i.e.,*

$$\begin{aligned} F(A \times I) &\subset A \\ F(x, 0) &= x \\ F(A \times \{1\}) &= * \\ F(\{*\} \times I) &= * \end{aligned}$$

then the quotient map $X \to X/A$ *is a homotopy equivalence. Similarly for pairs* (X, X') *with* $A \subset X'$.

Proof. Let $\psi \colon X/A \to X$ be defined by the commutative diagram

$$\begin{tikzcd} X \arrow[d, "\varphi"] \arrow[dr, "F_{X\times\{1\}}"] & \\ X/A \arrow[r, "\psi"] & X \end{tikzcd}$$

We claim that. Let $\varphi \colon X \to X/A$ be the quotienting map. We claim that $\psi\varphi \simeq \mathrm{id}_X$ and $\varphi\psi \simeq \mathrm{id}_{X/A}$. Since $F(A \times I) \subset A$, F induces a homotopy $F' \colon X/A \times I \to X/A$, where $F_{X/A\times\{1\}} = \varphi\psi$, so since $F_{X/A\times\{0\}} = \mathrm{id}_{X/A}$, we get the result. $\square$

1.3.1. *A different way of defining $\pi_n(Y, y_0)$.* Note that reduced suspension supplies a parameter in $[0,1]$ and the space S^n as constructed is the quotient space of I^n obtained by collapsing the boundary of the cube to a point. Pointed maps $S^n \to Y$ are in bijective correspondence with maps $I^n \to Y$ taking ∂I^n to the base point of Y. This is a more traditional way of defining $\pi_n(Y)$. This becomes the group of homotopy classes of maps $(I^n, \partial I^n) \to (Y, \{y_0\})$ with the operation being

$$f * g\left(t_1, \ldots, t_n\right) = \begin{cases} f\left(2t_1, t_2, \ldots, t_n\right), & t_1 \in \left[0, \frac{1}{2}\right] \\ g\left(2t_1 - 1, t_2, \ldots, t_n\right), & t_1 \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Proposition 1.22. *For $n \geq 2$, $\pi_n(X, x_0)$ is abelian.*

Proof. Consider the homotopy in Figure 2. We begin by shrinking the domains of f and g to smaller subcubes of I^n, where the region outside is mapped to the basepoint. This allows us to move the boxes around in a continuous manner. The rest is clear. □

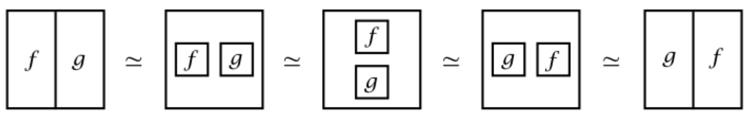

FIGURE 2. The homotopy in question

Next, we want to show that following:

Proposition 1.23. *If X is path-connected, then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for any two $x_0, x_1 \in X$.*

For this, we introduce an action of π_1 on π_n.

Definition 1.24 (The action of π_1 on π_n)**.** Given a path $\gamma \colon I \to X$ from x_0 to x_1, we associate to a map $f \colon (I^n, \partial I^n) \to (X, x_1)$ the map $\gamma f \colon (I^n, \partial I^n) \to (X, x_0)$ by shrinking the domain of f to a smaller concentric cube in I^n, then inserting the path γ on each radial segment in the shell between this smaller cube and ∂I^n. See Figure 3

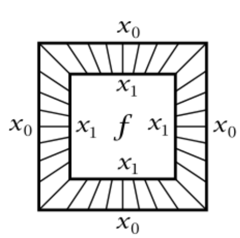

FIGURE 3. Depiction of γf.

Note. We have the following properties

(1) $\gamma(f + g) \simeq \gamma f + \gamma g$.
(2) $(\gamma\eta) f \simeq \gamma(\eta f)$.

(3) id$f \simeq f$, where id denotes the constant path.

To see (1), first deform f and g to be constant on the right and left halves of I^n, respectively, producing maps which we may call $f+0$ and $0+g$, then we can excise a progressively wider symmetric middle slab of $\gamma(f+0)+\gamma(0+g)$ (which can be seen on the left in Figure 4) until it becomes $\gamma(f+g)$ (shown on the right).

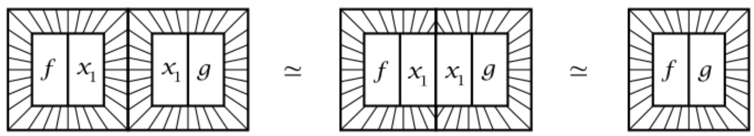

FIGURE 4.

Now if $\beta_\gamma\colon \pi_n(X,x_1)\to\pi_n(X,x_0)$ is the change-of-basepoint transformation, $\beta_\gamma[f]=[\gamma f]$, then the above note shows that β_γ is a group isomorphism. This proves Proposition 1.23. If we restrict attention to loops γ at x_0, then since $\beta_{\gamma\eta}=\beta_\gamma\beta_\eta$, the map $[\gamma]\mapsto\beta_\gamma$ defines a homomorphism from $\pi_1(X,x_0)$ to $\mathrm{Aut}(\pi_n(X,x_0))$ called the *action of* π_1 *on* π_n .

Note. For $n>1$, this action makes $\pi_n(X,x_0)$ into a module over the group ring $\mathbb{Z}[\pi_1(X,x_0)]$.

Definition 1.25 (Simple/abelian spaces)**.** A space with trivial π_1 action on π_n is called 'n-simple', and 'simple' means ' n-simple for all n '. We call a space *abelian* if it has trivial action of π_1 on all homotopy groups π_n.

Proposition 1.26 (π_n is a functor)**.** *A map* $\varphi\colon (X,x_0)\to(Y,y_0)$ *induces a map* $\varphi_*\colon \pi_n(X,x_0)\to\pi_n(Y,y_0)$ *defined by* $\varphi_*[f]=[\varphi f]$. *It is immediate from the definitions that* φ_* *is well-defined and a homomorphism for* $n\geq 1$. *The functorial properties are also clear.*

Corollary 1.27. *Homotopy equivalent spaces have isomorphic homotopy groups.*

Proposition 1.28. *A covering space projection* $p\colon \left(\tilde{X},\tilde{x}_0\right)\to(X,x_0)$ *induces isomorphisms* $p_*\colon \pi_n\left(\tilde{X},\tilde{x}_0\right)\to\pi_n(X,x_0)$ *for all* $n\geq 2$.

Proof. Since S^n is path-connected and locally path-connected, and simply connected for $n\geq 2$, we find that any map $(S^n,s_0)\to(X,x_0)$ lifts to a map $(S^n,s_0)\to\left(\tilde{X},\tilde{x}_0\right)$ when $n\geq 2$. This gives surjectivity of p_*. For injectivity, suppose $p_*[f]=[0]$ where $f\colon (S^n,s_0)\to\left(\tilde{X},\tilde{x}_0\right)$. Let $c_{\tilde{x}_0}$ be the constant map at $\tilde{x}_0$. Then $p_*[\tilde{x}_0]=[0]$, so by uniqueness of the lifting theorem, $[f]=[c_{\tilde{x}_0}]=[0]$. □

Definition 1.29 (Aspherical)**.** Spaces with $\pi_n=0$ for all $n\geq 2$ are called *aspherical.*

Corollary 1.30. S^1, T^n *and* K *are aspherical since they have contractible covering spaces.*

Proposition 1.31.

$$\pi_n\left(\prod_\alpha X_\alpha\right)\cong\prod_\alpha \pi_n(X_\alpha)$$

Next we define relative homotopy groups.

Definition 1.32 (Relative homotopy groups)**.** Regard I^{n-1} as a face of I^n with the last coordinate $s_n = 0$ and let J^{n-1} be the closure of $\partial I^n - I^{n-1}$. Then we define

$$\pi_n(X, A, x_0) := \left[I^n, \partial I^n, J^{n-1}; X, A, x_0\right]$$

We shall leave $\pi_0(X, A, x_0)$ undefined for now.

We can define a sum operation on $\pi_n(X, A, x_0)$ in the same way as for $\pi_n(X, x_0)$, except now the coordinate s_n now must remain free, so we must use one of the other coordinates. Thus we must have at least one other coordinate to define the same operation. So $\pi_n(X, A, x_0)$ is a group for $n \geq 2$, and it is abelian for $n \geq 3$. For $n = 1$, we have $I^1 = [0,1], I^0 = \{0\}$ and $J^0 = \{1\}$, so $\pi_1(X, A, x_0) = [I, \{0\}, \{1\}; X, A, x_0]$ is the set of homotopy classes of paths in X from a varying point in A to the fixed basepoint $x_0 \in A$. In general, this is not a group in any natural way.

Now, we saw before that $\pi_n(X, x_0)$ can be regarded as homotopy classes of maps $(S^n, x_0) \to (X, x_0)$. Similarly, collapsing J^{n-1} to a point, converts $\left(I^n, \partial I^n, J^{n-1}\right)$ to $\left(D^n, S^{n-1}, s_0\right)$. In this case, addition is done by the map $c\colon D^n \to D^n \vee D^n$ collapsing $D^{n-1} \subset D^n$ to a point.

Theorem 1.33 (Compression criterion)**.** *A map $f\colon \left(D^n, S^{n-1}, s_0\right) \to (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ if and only if it is homotopic* $\operatorname{rel} S^{n-1}$ *to a map with image contained in A.*

Proof. Suppose we have a homotopy $\operatorname{rel} S^{n-1}$ from f to a map g, so $[f] = [g]$ in $\pi_n(X, A, x_0)$. Viewing g as a map $\left(D^n, S^{n-1}, s_0\right) \to (X, A, x_0)$ whose image is contained in A, we can construct the homotopy $H\colon D^n \times I \to X$ by $H(x,t) = g((1-t)x + s_0 t)$ which is a homotopy from g to the constant map at x_0, hence $[g] = 0$ in $\pi_n(X, A, x_0)$.
Conversely, if $[f] = 0$ via a homotopy $F\colon D^n \times I \to X$ such that $F(x, 0) = f(x)$ and $F(x, 1) = x_0$ for all $x \in D^n$ and $F(x,t) \in A$ for all x with $|x| = 1$ as well as $F(s_0, t) = x_0$ for all t. We can construct a homotopy using F by restricting F to a family of n-disks in $D^n \times I$ starting with $D^n \times \{0\}$ and ending with the disk $D^n \times \{1\} \cup S^{n-1} \times I$, and where all the disks throughout the family have the same boundary. See Figure 5 for a depiction of this homotopy.
This completes the proof. □

Next, some things that carry over: a map $\varphi\colon (X, A, x_0) \to (Y, B, y_0)$ induces maps $\varphi_*\colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ which are homomorphisms when $n \geq 2$ and have properties analogous to those in the absolute case: $(\varphi\psi)_* = \varphi_*\psi_*, (\mathrm{id}_{(X,A,x_0)})_* = \mathrm{id}_{\pi_n(X,A,x_0)}$, and if $\varphi \simeq \psi$ through maps $(X, A, x_0) \to (Y, B, y_0)$, then $\varphi_* = \psi_*$.

1.3.2. *LES of relative homotopy groups.* Probably the most useful feature of relative homotopy groups $\pi_n(X, A, x_0)$ is that they fit into a long exact sequence

$$\ldots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \ldots \to \pi_0(X, x_0).$$

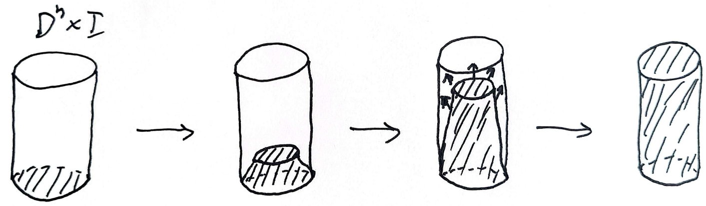

FIGURE 5.

Here i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $\left(I^n, \partial I^n, J^{n-1}\right) \to (X, A, x_0)$ to I^{n-1} (the face of I^n with the last coordinate $s_n = 0$), or equivalently, by restricting maps $\left(D^n, S^{n-1}, s_0\right) \to (X, A, x_0)$ to S^{n-1}. The map ∂, called the *boundary map*, is a homomorphism when $n > 1$. In fact, we can show the following theorem

Theorem 1.34 (LES of relative homotopy groups)**.** *Given $x_0 \in B \subset A \subset X$, the sequence of relative homotopy groups*

$$\ldots \to \pi_n\left(A, B, x_0\right) \xrightarrow{i_*} \pi_n\left(X, B, x_0\right) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}\left(A, B, x_0\right) \to \ldots \to \pi_1(X, A, x_0)$$

is exact and natural. In the case when $B = \{x_0\}$, we have that the LES

$$\ldots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \ldots \to \pi_0(X, x_0).$$

is exact and natural.

Proof. Exactness at $\pi_n\left(X, B, x_0\right)$: the composition $j_* i_*$ is zero because any map $\left(I^n, \partial I^n, J^{n-1}\right) \to (A, B, x_0)$ is zero in $\pi_n\left(X, A, x_0\right)$ by the compression criterion (Theorem 1.33). To see that $\ker j_* \subset \operatorname{im} i_*$, let $f\colon \left(I^n, \partial I^n, J^{n-1}\right) \to (X, B, x_0)$ represent zero in $\pi_n\left(X, A, x_0\right)$. Using the compression criterion again, we then get that f is homotopic $\operatorname{rel} \partial I^n$ to a map with image in A, hence the class $[f] \in \pi_n\left(X, B, x_0\right)$ is indeed in the image of i_*. We conclude that $\ker j_* = \operatorname{im} i_*$, obtaining exactness at $\pi_n\left(X, B, x_0\right)$.
Exactness at $\pi_n(X, A, x_0)$: for a map $[f] \in \operatorname{im} j_*$, we have that j_* maps ∂I^n into B, hence in particular $I^{n-1} \subset \partial I^n$ into B, so $\partial j_*[f]$ represents a homotopy class in $\pi_{n-1}\left(A, B, x_0\right)$ with image in B, but then by the compression criterion, $\partial j_*[f] = 0$ in $\pi_{n-1}\left(A, B, x_0\right)$, so $\operatorname{im} j_* \subset \ker \partial$. Conversely, suppose $\partial[f] = 0$. By the compression criterion, representatives of $\partial[f]$ are homotopic $\operatorname{rel} \partial I^{n-1}$ to a map with image in B. In particular, $f|_{I^{n-1}}$ is homotopic to a map with image in B via a homotopy $F\colon I^{n-1} \times I \to A \operatorname{rel} \partial I^{n-1}$. We can tack F onto f to get a new map $\left(I^n, \partial I^n, J^{n-1}\right) \to (X, B, x_0)$ which, as a map $\left(I^n, \partial I^n, J^{n-1}\right) \to (X, A, x_0)$ is homotopic to f by the homotopy that tacks on increasingly longer initial segments of F. See Figure 6. Hence $[f] \in \operatorname{im} j_*$.
Exactness at $\pi_n(A, B, x_0)$: First, $i_* \partial$ is zero since the restriction of a map $f\colon \left(I^{n+1}, \partial I^{n+1}, J^n\right) \to (X, A, x_0)$ to I^n is homotopic $\operatorname{rel} \partial I^n$ to a constant map via f itself (a similar picture to Figure 5 works).
Conversely, if B is a point, then a nullhomotopy $f_t\colon (I^n, \partial I^n) \to (X, x_0)$ of $f_0\colon (I^n, \partial I^n) \to (A, x_0)$ gives a map $F\colon \left(I^{n+1}, \partial I^{n+1}, J^n\right) \to (X, A, x_0)$ with $\partial([F]) = [f_0]$. So in

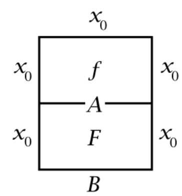

Figure 6.

this case, the proof is finished. For a general B, let F be a nullhomotopy of $f\colon \left(I^n, \partial I^n, J^{n-1}\right) \to (A, B, x_0)$ through maps $\left(I^n, \partial I^n, J^{n-1}\right) \to (X, B, x_0)$ and let g be the restriction of F to I^{n-1} in $I^{n-1} \times I = I^n$ (see the first of the pictures in Figure 7). Next reparametrize the n th and $(n+1)$ st coordinates as in the second picture. Then we find that f with g tacked on is in the image of ∂. But as before, tacking g onto f gives the same element of $\pi_n(A, B, x_0)$

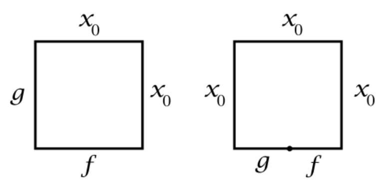

Figure 7.

□

Corollary 1.35. *Consider the inclusion $\iota\colon X = X \times \{0\} \hookrightarrow CX$. Then $\pi_n\left(CX, X, x_0\right) \cong \pi_{n-1}\left(X, x_0\right)$ for all $n \geq 1$. Taking $n = 2$, we can thus realize an group G, abelian or not, as a relative π_2 by choosing X to have $\pi_1(X) \cong G$.*

There are also change-of-basepoint isomorphisms β_γ for relative homotopy groups. One takes a path γ in $A \subset X$ from x_0 to x_1 which induces $\beta_\gamma\colon \pi_n(X, A, x_1) \to \pi_n(X, A, x_0)$ by setting $\beta_\gamma\left([f]\right) = [\gamma f]$, where γf is depicted in Figure 8.

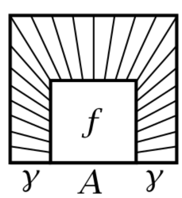

Figure 8.

Restricting to loops at the basepoint, the association $\gamma \mapsto \beta_\gamma$ defines an action of $\pi_1\left(A, x_0\right)$ on $\pi_n\left(X, A, x_0\right)$ analogous to the action of $\pi_1\left(X, x_0\right)$ on $\pi_n(X, x_0)$.

Problem 1.36 (n-connected in the relative case)**.** The following four conditions are equivalent for $i > 0$:

(1) Every map $\left(D^i, \partial D^i\right) \to (X, A)$ is homotopic rel ∂D^i to a map $D^i \to A$.
(2) Every map $\left(D^i, \partial D^i\right) \to (X, A)$ is homotopic through such maps to a map $D^i \to A$.
(3) Every map $\left(D^i, \partial D^i\right) \to (X, A)$ is homotopic through such maps to a constant map $D^i \to A$.
(4) $\pi_i(X, A, x_0) = 0$ for all $x_0 \in A$.

When $i = 0$, we did not define the relative π_0, and (1)-(3) are each equivalent to saying that each path-component of X contains points in A since D^0 is a point and ∂D^0 is empty. The pair (X, A) is called *n-connected* if (1)-(4) hold for $0 < i \leq n$ and (1)-(3) hold for $i = 0$.

1.4. **Whitehead's Theorem.**

Theorem 1.37 (Whitehead's Theorem)**.** *If a map $f\colon X \to Y$ between connected CW complexes induces isomorphisms $f_*\colon \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is stronger: X is a deformation retract of Y.*

The proof will require the following lemma:

Lemma 1.38 (Compression Lemma)**.** *Let (X, A) be a CW pair and let (Y, B) be any pair with $B \neq \varnothing$. For each n such that $X - A$ has cells of dimension n, assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f\colon (X, A) \to (Y, B)$ is homotopic* rel A *to a map $X \to B$. When $n = 0$, the condition that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$ is to be regarded as saying that (Y, B) is 0-connected.*

Proof of lemma. Assume inductively that f has already been homotoped to take the skeleton X^{k-1} to B. Let Φ be the caracteristic (attaching) map of cell e^k of $X - A$. Then the composition $f\Phi\colon \left(D^k, \partial D^k\right) \to (Y, B)$ is in some class in $\pi_k(Y, B, y_0) = 0$, so it can be homotoped into $B \operatorname{rel} \partial D^k$ by the compression criterion when $k > 0$, or by (Y, B) being 0-connected for $k = 0$ (this is condition (3) in Problem 1.36). This homotopy of $f\Phi$ induces a homotopy $\operatorname{rel} X^{k-1}$ on the quotient space $X^{k-1} \cup e^k$ of $X^{k-1} \sqcup D^k$. Doing this for all k-cells of $X - A$ simultaneously, and taking the constant homotopy on A, we obtain a homotopy of $f|_{X^k \cup A}$ to a map into B. Since the inclusion of a subcomplex into a CW-complex is a cofibration, $f|_{X^k \cup A}$ extends to all of X (essentially the homotopy extension property). This completes the inductive step in the finite dimensional CW-complex case. In the general case, we perform the homotopy of the inductive step during the t-interval $\left[1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+1}}\right]$. Any finite skeleton X^k is eventually stationary under these homotopies, hence we have a well-defined homotopy $f_t, t \in [0, 1]$ with $f_1(X) \subset B$. □

Proof of Whitehead's Theorem, 1.37. Let's tackle the case when f is the inclusion of a subcomplex first. Consider then the LES of the pair (Y, X). Since f by assumption induces isomorphisms on all homotopy groups, $f_*\colon \pi_*(X) \to \pi_*(Y)$, the relative homotopy groups $\pi_*(Y, X)$ are zero. Applying the lemma now to the identity map $(Y, X) \to (Y, X)$, we obtain a homotopy of the identity $\mathrm{id}\colon Y \to Y$ to a map $Y \to X$ which is relative to X. That is, we obtain a deformation retract of Y onto X.

For the general case, recall that a map $f\colon X \to Y$, can be considered as the composition of the inclusion $X \hookrightarrow M_f$ and the retraction $M_f \to Y$. Since the

retraction is a homotopy equivalence, it suffices to show that M_f deformation retracts onto X if f induces isomorphisms on homotopy groups, or equivalently, if the relative groups $\pi_n(M_f, X)$ are all zero (since $M_f \simeq Y$). If f is cellular - i.e., takes the n-skeleton of X to the n-skeleton of Y for all n - then (M_f, X) is a CW pair and we can apply the first paragraph of the proof.
If f is not cellular, we can either apply Theorem 4.8 in [2] which says that f is homotopic to a cellular map, or we can use the following argument.
First, using that $\pi_n(M_f, X) = 0$ for all n, apply the Compression Lemma to the inclusion $(X \cup Y, X) \hookrightarrow (M_f, X)$ to obtain a homotopy of the inclusion to a map into $X \operatorname{rel} X$. The inclusion $X \cup Y \hookrightarrow M_f$ can be seen to be a cofibration using Theorem **??**, so the pair $(M_f, X \cup Y)$ satisfies the homotopy extension property. So the homotopy in question extends to a homotopy from the identity of M_f to a map $g\colon M_f \to M_f$ taking $X \cup Y$ into $X \operatorname{rel} X$. However, we first of all do not know that this homotopy is $\operatorname{rel} X$ nor that g maps all of M_f into X.
So we apply the Compression lemma again to the composition

$$(X \times I \sqcup Y, X \times \partial I \sqcup Y) \to (M_f, X \cup Y) \xrightarrow{g} (M_f, X)\,,$$

to get a homotopy $\operatorname{rel} X \times \partial I \sqcup Y$ of g to a map $X \times I \sqcup Y \to X$. In particular, this homotopy passes through the quotient $X \times I \sqcup Y \to M_f$, so we get a homotopy of $g \operatorname{rel} X \times \partial I \cup Y$ to a map $M_f \to X$.
Composing the homotopy from the identity of M_f to g with this homotopy, we get a deformation retraction of M_f onto X. □

Note. Whitehead's theorem requires a map $f\colon X \to Y$ which induces isomorphisms on homotopy groups. Thus it does not apply simply to any two CW complexes X and Y with isomorphic homotopy groups since there might not exist such a map. For examples where this is the case, see [2, p. 348].

Corollary 1.39. *If X is a CW complex with $\pi_n(X) = 0$ for all $n \geq 0$, then $X \simeq \{0\}$.*

Proof. The inclusion of a 0-cell into the complex induces an isomorphism on homotopy groups, so by Whitehead's theorem, the complex deformation retracts to the 0-cell. □

Lemma 1.40 (Extension Lemma)**.** *Given a CW pair (X, A) and a map $f\colon A \to Y$ with Y-path connected, then f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X - A$ has cells of dimension n.*

Proof. Suppose that f has been extended over the $(n-1)$-skeleton. Then an extension over an n-cell exists if and only if the composition of the cell's attaching map $S^{n-1} \to X^{n-1}$ with $f\colon X^{n-1} \to Y$ is nullhomotopic, which it is if $\pi_{n-1}(Y) = 0$. □

1.5. **Cellular Approximation.**

Definition 1.41 (Cellular maps)**.** A map $f\colon X \to Y$ between CW complexes, satisfying $f(X^n) \subset Y^n$ for all n, is called a *cellular map.*

Theorem 1.42 (Cellular Approximation Theorem)**.** *Every map $f\colon X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, then homotopy map be taken to be stationary on A.*

Remark. Cellular approximation tells us that $\pi_n(X)$ only depends on the $(n+1)$-skeleton.

Recall the following about simplicial maps and simplicial approximations:

Definition 1.43 (Simplicial map)**.** Let K and L be simplicial complexes. A function $s\colon |K| \to |L|$ is called *simplicial* if it takes simplexes of K linearly onto simplexes of L.

Definition 1.44 (Carrier of $f(x)$)**.** Given a map $f\colon |K| \to |L|$ between polyhedra and a point $x \in |K|$, the point $f(x)$ lies in the interior of a unique simplex of L. Call this simplex the *carrier* of $f(x)$.

Definition 1.45 (Simplicial Approximation)**.** A simplicial map $s\colon |K| \to |L|$ is a simplicial approximation of $f\colon |K| \to |L|$ if $s(x)$ lies in the carrier of $f(x)$ for each $x \in |K|$.

Theorem 1.46 (Simplicial approximation theorem)**.** *Let $f\colon |K| \to |L|$ be a map between polyhedra. If m is chosen large enough, there is a simplicial approximation $s\colon |K^m| \to |L|$ to $f\colon |K^m| \to |L|$.*

Thus we may view cellular approximation as a CW analog of simplicial approximation since simplicial maps are cellular. Simplicial maps are much more rigid than cellular maps, however, and the core proof of cellular approximation will be a weaker form of simplciial approximation.

But first, a nice corollary:

Corollary 1.47. *$\pi_n\left(S^k\right)$ for $n < k$.*

Proof. If S^n and S^k are given their usual CW structure of a single 0-cell and then an n- or k-cell, respectively, then by the Cellular Approximation Theorem, any pointed map $S^n \to S^k$ is based homotopic to a cellular map, and hence maps the n-skeleton of S^n into the n-skeleton of S^k. But the n-skeleton of S^k is just the 0-cell. That is, any map $S^n \to S^k$ is based nullhomotopic, so $\pi_n\left(S^k\right) = 0$. □

Proof of Cellular Approximation Theorem. Long. To do □

Example 1.48 (Cellular Approximation for Pairs)**.** Every map $f\colon (X,A) \to (Y,B)$ of CW pairs can be deformed through maps $(X,A) \to (Y,B)$ to a cellular map. This follows from the theorem by first deforming the restriction $f\colon A \to B$ to be cellular, then extending this to a homotopy of f on all of X, then deforming the resulting map to be cellular staying fixed on A. As a further refinement, the homotopy of f can be taken to be stationary on any subcomplex of X where f is already cellular.

Corollary 1.49 (Geometric Version of n-connectedness)**.** *A CW pair (X,A) is n-connected if all the cells in $X - A$ have dimension greater than n. In particular, the pair (X, X^n) is n-connected, hence the inclusion $X^n \hookrightarrow X$ induces isomorphisms on π_i for $i < n$ and a surjection on π_n.*

Proof. Recall that (X,A) is n-connected if every map $\left(D^i, \partial D^i\right) \to (X,A)$ is homotopic through such maps to a map $D^i \to A$. Now let $f\left(D^i, \partial D^i\right) \to (X,A)$ be any map. Then by the Cellular Approximation theorem for Pairs, f is homotopic through maps $\left(D^i, \partial D^i\right) \to (X,A)$ to a cellular map, $\tilde{f}\colon \left(D^i, \partial D^i\right) \to (X,A)$. But by assumption, all cells in $X-A$ have dimension greater than $n \geq i$. Hence $\tilde{f}$ maps D^i into A. The last part of the statement now follows from the LES

$$\ldots \to \pi_n(X^n) \xrightarrow{\iota_*} \pi_n(X) \to \underbrace{\pi_n\left(X, X^n\right)}_{0} \to \pi_{n-1}(X^n) \xrightarrow{\iota_*} \pi_{n-1}(X) \to \underbrace{\pi_{n-1}\left(X, X^n\right)}_{0} \to \ldots$$

□

1.6. **CW Approximation.**

Definition 1.50 (Weak Homotopy Equivalence)**.** A map $f\colon X \to Y$ is called a *weak homotopy equivalence* if it induces isomorphisms $\pi_n\left(X, x_0\right) \to \pi_n\left(Y, f(x_0)\right)$ for all $n \geq 0$ and all choices of basepoint x_0.

Remark (Reformulation of Whitehead's Theorem)*.* Whitehead's Theorem thus says that a weak homotopy equivalence between CW complexes is, in fact, a homotopy equivalence.

Definition 1.51 (CW Approximation)**.** For a space X, a weak homotopy equivalence $f\colon Z \to X$, where Z is a CW complex, is called a *CW approximation* to X.

Remark. CW approximations to a given space X are unique up to homotopy equivalence since if $f\colon Z \to X$ and $f'\colon Z' \to X$ are CW approximations, then consider the composition $Z \to X \hookrightarrow M_{f'}$. Since $f'\colon Z' \to X$ is assumed to be a weak homotopy equivalence, we find by the relative LES that $\pi_n(M_f, Z') \cong \pi_n\left(X, Z'\right) = 0$ for all $n \geq 0$, so by the Compression Lemma (with A chosen to be the basepoint of Z), we find that the map $Z \to X \hookrightarrow M_{f'}$ is homotopic to a map $Z \to Z' \subset M_{f'}$ relative to the basepoint. But taking π_n of $Z \to X \to M_{f'} \to Z'$, we get $\pi_n(Z) \overset{\cong}{\to} \pi_n(X) \overset{\cong}{\to} \pi_n(M_{f'}) \overset{\cong}{\to} \pi_n\left(Z'\right)$ where $\pi_n(X) \overset{\cong}{\to} \pi_n(M_{f'})$ follows from $\iota\colon X \simeq M_{f'}$ being a homotopy equivalence; $\pi_n\left(M_{f'}\right) \overset{\cong}{\to} \pi_n(Z')$ follows from the homotopy that we got from the compression lemma, and the first isomorphism $f_*\colon \pi_n(Z) \overset{\cong}{\to} \pi_n(X)$ follows from f being a weak homotopy equivalence. Applying Whitehead's theorem, we find that this composition is a homotopy equivalence $Z \simeq Z'$.

Proposition 1.52. *Every space X has a CW approximation $f\colon Z \to X$. If X is path-connected, Z can be chosen to have a single* 0*-cell, with all other cells attached by basepoint-preserving maps. Thus every connected CW complex is homotopy equivalent to a CW complex with these additional properties.*

Proof. The construction of a CW approximation $f\colon Z \to X$ is inductive, so we first describe the induction step. Suppose we are given a CW complex A with a map $f\colon A \to X$ and suppose we have chosen a basepoint 0-cell a_γ in each component of A. Then for an integer $k \geq 0$, we will attach k-cells to A to form a CW complex B with a map $f\colon B \to X$ extending f such that

- For each basepoint a_γ, the induced map $f_*\colon \pi_i\left(B, a_\gamma\right) \to \pi_i\left(X, f\left(a_\gamma\right)\right)$ is injective for $i = k-1$ (when $k > 0$) and surjective for $i = k$.

We do this in two steps (the first step is omitted when $k = 0$):

(1) We have been given a CW complex A and a map $f\colon A \to X$ alongside basepoints a_γ. Now for each nontrivial element α of the kernel $\ker f_*$ ranging over all basepoints, choose a map $\varphi_\alpha\colon \left(S^{k-1}, s_0\right) \to \left(A, a_\gamma\right)$ representing α. We may assume that the φ_α are all cellular (by the Cellular Approximation

Theorem) where S^{k-1} is given its standard CW structure with s_0 as a 0-cell. Attaching cells e^k_α to A via the maps φ_α then produces a CW complex. Now, $f \circ \varphi_\alpha$ is nullhomotopic, so f extends over the cell e^k_α.

(2) Choose maps $f_\beta \colon S^k \to X$ representing all nontrivial elements of $\pi_k\left(X, f(a_\gamma)\right)$ for all the a_γ 's Then attach cells e^k_β to A via the constant maps at the appropriate basepoints a_γ and extend f over the resulting spheres S^k_β via f_β.

By the construction, then surjectivity of $f_* \colon \pi_i\left(B, a_\gamma\right) \to \pi_i\left(X, f(a_\gamma)\right)$ for $i = k$ follows. Now let α be in the kernel of $f_* \colon \pi_{k-1}(B, a_\gamma) \to \pi_{k-1}\left(X, f(a_\gamma)\right)$, and let $h \colon S^{k-1} \to B$ be a cellular map that represent α. Since h is cellular, its image is contained in the $(n-1)$-skeleton of B which is a subskeleton (could be all) of A. Since h has image in A, it is in the kernel of $f_* \colon \pi_{k-1}(A, a_\gamma) \to \pi_{k-1}\left(X, f(a_\gamma)\right)$ and thus it is homotopic to some φ_α and therefore nullhomotopic in B.

Note. In step (1), it suffices to attach cells for just the generators of the kernels when $k > 1$, and just for the generators of $\pi_k\left(X, f(a_\gamma)\right)$ in step (2) when $k > 0$.

Note. If the given map $f \colon A \to X$ happened to already be injective or surjective on π_i for some $i < k-1$ or $i < k$, respectively, then this remains true after attaching the k-cells. This is because attaching k-cells does not affect π_i if $i < k-1$, by cellular approximation, not does it affect surjectivity on any π_i, simply because the same maps still hold and work.

Now to construct a CW approximation $f \colon Z \to X$, one can start with A consisting of one point for each path-component of X, with $f \colon A \to X$ mapping each of these points to the corresponding path-component. This gives a bijection on π_0 by construction, hence it provides us with the inductive base case. Now we can attach 1-cells to A to create a surjection on π_1 for each path-component, then 2-cells to improve this to an isomorphism on π_1 and a surjection on π_2 and so forth for each successive π_i in turn. After all cells have been attached, on has a CW complex Z with a weak homotopy equivalence $f \colon Z \to X$. □

Example 1.53. One can apply this technique to produce a CW approximation to a pair (X, X_0) also. First one constructs a CW approximation $f_0 \colon Z_0 \to X_0$, then one starts with the composition $Z_0 \to X_0 \hookrightarrow X$ and attaches cells to Z_0 to create a weak homotopy equivalence $f \colon Z \to X$ extending f_0. Then we get

$$\begin{tikzcd}
\pi_n(Z_0) \arrow[r] \arrow[d, "\cong"] & \pi_n(Z) \arrow[r] \arrow[d, "\cong"] & \pi_n(Z, Z_0) \arrow[r] \arrow[d] & \pi_{n-1}(Z_0) \arrow[r] \arrow[d, "\cong"] & \pi_{n-1}(Z) \arrow[d, "\cong"] \\
\pi_n(X_0) \arrow[r] & \pi_n(X) \arrow[r] & \pi_n(X, X_0) \arrow[r] & \pi_{n-1}(X_0) \arrow[r] & \pi_{n-1}(X)
\end{tikzcd}$$

By the five-lemma, it follows that $\pi_n\left(Z, Z_0\right) \to \pi_n\left(X, X_0\right)$ is an isomorphism for each n.

Proposition 1.54. *If (X, A) is an n-connected CW pair, then there exists a CW pair $(Z, A) \simeq (X, A)\operatorname{rel} A$ such that all cells of $Z - A$ have dimension greater than n.*

Proof. Starting with the inclusion $A \hookrightarrow X$, attach cells of dimension $n+1$ and higher to A to produce a CW complex Z and a map $f \colon Z \to X$ using the procedure of Proposition 1.52. In particular then by the Proposition proof, f_* induces an injection of π_n and isomorphisms on all higher homotopy groups. Now, the induced map on π_n is also surjective since it is true for $A \hookrightarrow Z \xrightarrow{f} X$ as (X, A) is n-connected and hence $\pi_n(A) \xrightarrow{\cong} \pi_n(X)$ is an isomorphism. Since f is equal to this inclusion on the n-skeleton, this gives that f_* is also surjective. By cellular approximation $A \hookrightarrow Z$ induces an isomorphism on homotopy groups in dimensions below n, and likewise n-connectedness does the same for $A \hookrightarrow X$. But then since

$$\begin{tikzcd} \pi_n(A) \arrow[r, "\iota_*", "\cong"'] \arrow[rr, bend right, "\iota_*", "\cong"'] & \pi_n(Z) \arrow[r, "f_*"] & \pi_n(X) \end{tikzcd}$$

commutes, we find that f_* is also an isomorphism on all $n \geq 0$. Thus f is a weak homotopy equivalence, and hence a homotopy equivalence by Whitehead's theorem.

To see that f is a homotopy equivalence rel A, we could apply Proposition **??**, but here is an alternative argument. Let W be the quotient space of the mapping cylinder M_f obtained by collapsing each segment $\{a\} \times I$ to a point, for $a \in A$. Assuming f has been made cellular, W is a CW complex (why?) containing X and Z as subcomplexes, and W deformation retracts to X just as M_f does. Also, $\pi_i(W, Z) = 0$ for all i since f induces isomorphisms on all homotopy groups (by the LES), so W deformation retracts onto Z by Whitehead's Theorem (Theorem 1.37). The deformation retract of W onto X and the deformation retract of W onto Z are stationary on A, hence give a homotopy equivalence $X \simeq Z \operatorname{rel} A$. $\square$

Example 1.55 (Postnikov Towers)**.** For each connected CW complex X and each integer $n \geq 1$, we can construct a CW complex X_n containing X as a subcomplex such that

(1) $\pi_i(X_n) = 0$ for $i > n$.
(2) The inclusion $X \hookrightarrow X_n$ induces an isomorphism on π_i for $i \leq n$.

Idea. Take X and fill out any spheres of dimension $> n$ by filling them in.

Indeed, we attach $(n+2)$-cells to X using cellular maps $S^{n+1} \to X$ that generate $\pi_{n+1}(X)$ to form a space with π_{n+1} trivial. Then for this space, we attach $(n+3)$-cells to make π_{n+2} trivial, and so on. The result is a CW complex X_n with the desired properties. The inclusion $X \hookrightarrow X_n$ extends to a map $X_{n+1} \to X$ since X_{n+1} is obtained from X by attaching cells of dimension $n+3$ and greater, and $\pi_i(X_n) = 0$ for $i > n$, so we can apply the Extension Lemma (Lemma 1.40). Thus we get a commutative diagram as follows:

$$\begin{tikzcd} & \vdots \arrow[d] \\ & X_3 \arrow[d] \\ & X_2 \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & X_1 \end{tikzcd}$$

This is called a *Postnikov tower* for X. One can regard the spaces X_n as truncations of X which provides successively better approximations to X as n increases.

Next, let us try to give some names to the properties which the constructions that attaching cells to make a map more nearly a weak homotopy equivalence can achieve.
Suppose we are given a map $f\colon A \to B$. We can factor this as the map $A \hookrightarrow M_f \xrightarrow{r} B$ ([1, **page** 433]), so the setup is now that we have a pair (X, A) where $A \subset X$ is a nonempty CW complex.

Definition 1.56 (n-connected CW model)**.** Given a pair (X, A) where $A \subset X$ is a nonempty CW complex, we define an *n-connected CW model for* (X, A) to be an n-connected CW pair (Z, A) and a map $f\colon Z \to X$ with $f|_A = \mathrm{id}$ such that $f_*\colon \pi_i(Z) \to \pi_i(X)$ is an isomorphism for $i > n$ and an injection for $i = n$, for all choices of basepoint.

Since (Z, A) is n-connected, the map $\pi_i(A) \to \pi_i(Z)$ induced by the inclusion is an isomorphism for $i < n$ and a surjection for $i = n$.
So in the critical dimension $i = n$, the maps $A \hookrightarrow Z \xrightarrow{f} X$ induces a composition $\pi_n(A) \overset{\iota_*}{\twoheadrightarrow} \pi_n(Z) \overset{f_*}{\hookrightarrow} X$ of a surjection followed by an injection.

Remark. One can think of Z as a homotopy-theoretic hybrid of A and X. As n increases, the hybrid looks more and more like A and less and less like X.

Proposition 1.57. *For every pair (X, A) with A a nonempty CW complex, there exist n-connected CW models $f\colon (Z, A) \to (X, A)$ for all $n \geq 0$, and these models can be chosen to have the additional property that Z is obtained from A by attaching cells of dimension greater than n.*

Proof. This is what we showed above. □

Question 1.58. Are n-connected CW models unique up to homotopy equivalence?

Proposition 1.59. *Suppose we are given:*

(1) *an n-connected CW model $f\colon (Z, A) \to (X, A)$,*
(2) *an n'-connected CW model $f'\colon (Z', A') \to (X', A')$,*
(3) *a map $g\colon (X, A) \to (X', A')$.*

Then if $n \geq n'$, there is a map $h\colon Z \to Z'$ such that $h|_A = g$ and $gf \simeq f'h \operatorname{rel} A$, so the following diagram commutes up to homotopy rel A*. Furthermore, such a map h is unique up to homotopy* rel A.

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ \downarrow h & & \downarrow g \\ Z' & \xrightarrow{f'} & X' \end{array}$$

Proof. By Proposition 1.57, we may assume that the cells of $Z - A$ have dimension $> n$. Let W be the quotient space of a mapping cylinder of f' obtained by collapsing each line segment $\{a'\} \times I$ to a point for $a' \in A'$, say to $\{a'\} \times \{0\}$. Then W contains copies of Z' and X', with Z' being $Z' \times \{0\} \subset Z' \times I \cup_{f'} X'$ and X' being a deformation retract of W, contained as the X' copy in $Z' \times I \cup_{f'} X'$. The assumption that (Z', A') is an n'-connected CW model for (X', A') implies that the relative homotopy groups $\pi_i(W, Z*)$ are zero for $i > n'$ since the inclusion $\iota \colon Z' \hookrightarrow W$ is, up to homotopy, the same as $f' \colon Z' \to X'$ which induces isomorphisms for $i > n'$. More precisely, we have a commutative diagram as follows:

$$\begin{tikzcd} & W \arrow[dd, "r"] \\ Z' \arrow[ur, "\iota"] \arrow[dr, "f'"'] & \\ & X' \end{tikzcd}$$

where r is a retractions (the ending map of a deformation retraction which is thus homotopic to the identity on W), so r_* induces an isomorphism on homotopy groups. Hence $(f')_*$ is an isomorphism if and only if ι_* is.
Now using the inclusion $X' \hookrightarrow W$, we can view gf as a map

$$(Z, A) \xrightarrow{f} (X, A) \xrightarrow{g} (X', A') \hookrightarrow (W, Z')$$

Recall that we collapsed strands $\{a'\} \times I$ so A' gets included into W in a unique way and it is both contained in the X' and Z' copy (we write Z' above because we want to make use of (W, Z') having trivial homotopy groups in dimensions $> n$.) Using the compression lemma (Lemma 1.38) and the hypothesis that $n \geq n'$, we find that gf is homotopic $\operatorname{rel} A$ to a map h' with image in Z', considering gf as a map $(Z, A) \to (W, Z')$. More precisely, $i_{X'} gf \simeq h' \operatorname{rel} A$ as maps $Z \to W$ where $i_{X'} \colon X' \hookrightarrow W$ is the inclusion. Let $h \colon Z \to Z'$ be the map with restricted codomain. Then $h' = i_{Z'} h$, so $gf = r i_{X'} gf \simeq r i_{Z'} h = f' h \operatorname{rel} A$.
This gives the desired map $h \colon Z \to Z'$ making the diagram above commute.
For uniqueness, suppose h_0 and h_1 are two maps $Z \to Z'$ whose compositions with f' are homotopic to $gf \operatorname{rel} A$. Let $h'_i = i_{Z'} h_i$, for $i = 0, 1$. Then the homotopy for $h'_0 \simeq h'_1 \operatorname{rel} A$ gives a map $(Z \times I, Z \times \partial I \cup A \times I) \to (W, Z')$ and by the compression lemma (Lemma 1.38), this map can be deformed $\operatorname{rel} Z \times \partial I \cup A \times I$ to a map with image in Z', which gives the desired homotopy $h_0 \simeq h_1 \operatorname{rel} A$. □

Corollary 1.60. *An n-connected CW model for (X, A) is unique up to homotopy equivalence* $\operatorname{rel} A$.

Proof. Consider the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, shift left, "h"] & X \arrow[d, equal] \\ Z' \arrow[r] \arrow[u, shift left, "h'"] & X \end{tikzcd}$$

By uniqueness, $hh' \simeq \mathrm{id}$ and $h'h \simeq \mathrm{id} \operatorname{rel} A$. □

Corollary 1.61. *Given CW approximations $Z \to X$ and $Z' \to X'$ and a map $X \to X'$, there exists a unique map $Z \to Z'$ up to homotopy equivalence making the following diagram commute:*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Z' \arrow[r] & X' \end{tikzcd}$$

Example 1.62. Let $(Z_i, A)_{i \in \mathbb{N}_0}$ be a collection of i-connected CW models for (X, A). By the Proposition, we obtain the right hand part of the following tower:

$$\begin{tikzcd}
& \vdots \arrow[d] & \\
& Z_2 \arrow[d] \arrow[ddr] & \\
& Z_1 \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[uur] & Z_0 \arrow[r] & X
\end{tikzcd}$$

and the left hand side is obtained by the inclusions $A \hookrightarrow Z_i$. The left hand side commutes by the part of the proposition saying $h|_A = g$ and then noting hat g in our case is the identity throughout. The right hand side commutes up to homotopy by the proposition as well. We can also make the right hand side strictly commutative by replacing the maps $Z_i \to X$ by the composite $Z_i \to Z_{i-1} \to \ldots \to Z_0 \to X$.

Example 1.63 (Whitehead Towers)**.** Let X be an arbitrary CW complex with a subspace A that is a point. The resulting tower obtained from Example 1.62 gives a sequence of maps $\ldots \to Z_2 \to Z_1 \to Z_0 \to X$ where Z_n is now n-connected since (Z_n, A) is n-connected and $0 = \pi_k(Z_n, A, A) = \pi_k(Z_n, A)$ for $k \leq n$. Furthermore, the map $Z_n \to X$ induces an isomorphism on all homotopy groups π_i with $i > n$. The space Z_0 is path-connected and homotopy equivalent (by Whitehead's Theorem, Theorem 1.37) to the component of X containing A, so one may as well assume Z_0 equals this component.
Next, Z_1 is simply-connected, and the map $Z_1 \to X$ has the homotopy properties of the universal cover of the component Z_0 of X. By analogy, one can view $Z_n \to X$ as 'n-connected covers' of X for larger n.

Proposition 1.64. *A weak homotopy equivalence $f\colon X \to Y$ induces isomorphisms $f_*\colon H_n(X; G) \to H_n(Y; G)$ and $f^*\colon H^n(Y; G) \to H^n(X; G)$ for all n and all coefficient groups G.*

Proposition 1.65. *A weak homotopy equivalence $f\colon Y \to Z$ induces bijections $[X, Y] \to [X, Z]$ and $[X, Y]_* \to [X, Z]_*$ for all CW complexes X.*

1.7. **Moore-Postnikov factorizations and Whitehead covers.**

Definition 1.66. A map $f\colon X \to Y$ is called *n-truncated* if, for all $x \in X$, the map
$$f_*\colon \pi_k(X, x) \to \pi_k(Y, f(x))$$
is an isomorphism for $k > n$ and an injection for $k = n$.

Definition 1.67. A space X is called *n-truncated* if $\pi_k(X, x) = 0$ for all $k > n$ and all $x \in X$.

Example 1.68. $K(\pi, n)$ and $\tau_{\leq n} X$ are n-truncated for any π and any X.

Note. S^k is not n-truncated for any n when $k \geq 2$.

Exercise 1.69. Prove this.

Lemma 1.70 (Variation of extension lemma)**.** *Let (X, A) be a CW pair such that $X - A$ has only cells of dimension $\geq n+2$. For Y an n-truncated space, we have*

(1) *for all $f\colon A \to Y$, we can extend f to a map $\tilde{f}\colon X \to Y$.*
(2) *If $g, h\colon X \to Y$ are such that $g|_A \simeq h|_A$, then $g \simeq h$.*

Proof. The first part is just the extension lemma.
For the second part, we have a map $g \sqcup h\colon X \times \{0,1\} \to Y$ and a homotopy $H\colon A \times [0,1] \to Y$ which glue to give a map

$$X \times \{0,1\} \cup A \times [0,1] \overset{(g \sqcup h) \cup H}{\to} Y$$

Now, $X \times [0,1]$ can be obtained from $X \times \{0,1\} \cup A \times [0,1]$ by only attaching cells of dimension $\geq n+2$, so by part (1), we obtain that we can extend the map $(g \sqcup h) \cup H$ to a map $X \times [0,1] \to Y$. $\square$

Corollary 1.71. *If (X, A) is a CW pair with $X - A$ having cells of dimension $\geq n+2$ only, and if Y is n-truncated, then*

$$[X, Y] \overset{\cong}{\to} [A, Y]\,.$$

Remark. If $Y = K(\mathbb{Z}, k)$ for $k \leq n$, then we obtain $H^k(X, \mathbb{Z}) \cong [X, Y] \cong [A, Y] \cong H^k(A, \mathbb{Z})$, which we could also see since $A \hookrightarrow X$ is $(n+1)$-connected.

Definition 1.72 (Postnikov truncation)**.** An *n th Postnikov truncation* of a space A is a CW pair (X, A) s.t.

(1) $X - A$ contains only cells of dimension $\geq n+2$.
(2) X is n-truncated.

In this case, we define $\tau_{\geq n} A := X$.

Lemma 1.73. *The n th Postnikov truncation is unique up to homotopy equivalence.*

Proof. Suppose (X, A) and (Y, A) are both n th Postnikov truncations. Then Lemma 1.70 implies that the identity maps $A \hookrightarrow Y$ and $A \hookrightarrow X$ can be extended to maps $X \to Y$ and $Y \to X$ restricting to the identity on A. Part 2 of the lemma then implies that $fg \simeq \mathrm{id}$ and $gf \simeq \mathrm{id}$, which completes the proof. $\square$

1.7.1. *Postnikov Tower.* Recall the Postnikov tower

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \tau_{\leq 3} X \arrow[d, "p_3"] \\
 & \tau_{\leq 2} X \arrow[d, "p_2"] \\
 & \tau_{\leq 1} X \arrow[d, "p_1"] \\
X \arrow[r, "i_0"] \arrow[ur, "i_1"] \arrow[uur, "i_2"] \arrow[uuur, "i_3"] & \tau_{\leq 0} X
\end{tikzcd}$$

Remark. Since $(\tau_{\leq p}X, X)$ is p-connected, the inclusion induces an isomorphism on π_k for $k \leq p-1$ and a surjection on p, so we have

$$\begin{tikzcd} \pi_k X \arrow[r] \arrow[rr, bend right, "\cong"'] & \pi_k\left(\tau_{\leq n}X\right) \arrow[r] & \pi_k\left(\tau_{\leq n-1}X\right) \end{tikzcd}$$

which implies that the latter map is surjective, and that the former map is surjective. But if gf is an isomorphism with f surjective, then g must be injective. Hence the latter map is injective also, so $\pi_k\left(\tau_{\leq n}X\right) \to \pi_k\left(\tau_{\leq n-1}X\right)$ is an isomorphism for $k \leq n-1$. For $k > n$, π_k of both is 0, so the map is also an isomorphism.

Thus we can conclude that $p_n \colon \tau_{\leq n}X \to \tau_{\leq n-1}X$ is an isomorphism on π_k for $k \neq n$.

From this, we can conclude using the LES of the pair $(\tau_{\leq n}X, \tau_{\leq n-1}X)$that $\operatorname{hofib}(p_n)$ is an Eilenberg-MacLane space of type $K\left(\pi_n X, n\right)$:

$$\ldots \to 0 \to \underbrace{\pi_{n+1}\left(\tau_{\leq n-1}X\right)}_{\cong 0} \to \pi_n\left(\operatorname{hofib}(p_n)\right) \overset{\cong}{\to} \underbrace{\pi_n\left(\tau_{\leq n}X\right)}_{\cong \pi_n X} \to \underbrace{\pi_n\left(\tau_{\leq n-1}X\right)}_{\cong 0} \to \ldots$$

Example 1.74. For example, since $\pi_3\left(S^2\right) \cong \mathbb{Z}$ from the LES of the Hopf fibration, we get

$$\begin{tikzcd} K\left(\underbrace{\pi_3 S^2}_{\cong \mathbb{Z}}, 3\right) \arrow[r] & \tau_{\leq 3}S^2 \arrow[d] \\ K\left(\underbrace{\pi_2 S^2}_{\mathbb{Z}}, 2\right) \arrow[r] & \tau_{\leq 2}S^2 \arrow[d] \\ K\left(0,1\right) \arrow[r] & \tau_{\leq 1}S^2 \end{tikzcd}$$

So $\tau_{\leq 3}S^2$ is a space with $\pi_k\left(\tau_{\leq 3}S^2\right) \cong \begin{cases} \mathbb{Z}, & k = 2,3 \\ 0, & \text{else} \end{cases}$ by the LES of the homotopy fiber sequence $K\left(\mathbb{Z}, 3\right) \to \tau_{\leq 3}S^2 \to \tau_{\leq 2}S^2$.

Definition 1.75 (Whitehead cover)**.** The *n th Whitehead cover* of X is the homotopy fiber $\tau_{>n}X$ in

$$\tau_{>n}X \to X \to \tau_{\leq n}X$$

Note. Note that $\tau_{>n}X$ is the CW model for $(X, *)$ (by its LES) which we know is unique up to homotopy equivalence.

Example 1.76. If X is a connected CW complex with universal cover $\tilde{X} \to X$, then $\tilde{X} \simeq \tau_{>1}X$.

Example 1.77. The Hopf fibration $S^1 \to S^3 \to S^2$ is an isomorphism on $\pi_{>2}$ by the LES, and S^3 is 2-connected, so $S^3 \cong \tau_{>2}S^2$.

Remark. Recall that for a connected X, its Whitehead tower is

$$\ldots \to \tau_{>n+1}X \to \tau_{>n}X \to \ldots \to \tau_{>2}X \to \tau_{>1}X \to X.$$

In particular, by the LES associated to the fibration

$$\mathrm{hofib}\left(\tau_{>n}X \to \tau_{>n-1}X\right) \to \tau_{>n}X \to \tau_{>n-1}X$$

we find that

$$\mathrm{hofib}\left(\tau_{>n}X \to \tau_{>n-1}X\right) = K\left(\pi_n(X), n-1\right).$$

1.8. **Problem Set 1.**

1.8.1. *Exercises.*

Exercise 1.78 (The action of the fundamental gorup, part 2)**.** Let X be a path-connected, semi-locally simply-connected space with basepoint x and $p\colon \tilde{X} \to X$ its universal cover. Show that for $n \geq 2$ and $\tilde{x} \in X$ with $p(\tilde{x}) = x$, the isomorphism $p_* = \pi_n(p)\colon \pi_n\left(\tilde{X}, \tilde{x}\right) \cong \pi_n(X,x)$ allows us to identify the action of $\pi_1(X,x)$ on $\pi_n(X,x)$ with the action of $\pi_1(X,x)$ on $\pi_n\left(\tilde{X}, \tilde{x}\right)$ induced by the group of deck transformations, i.e., the natural action of $\pi_1(X,x)$ on $\tilde{X}$. In particular, make the statement precise.

Proof. We want to show that for $[\gamma] \in \pi_1(X,x)$ and $[f] \in \pi_n(X,x)$, if $\tilde{g}$ is the lift for γ starting at $\tilde{x}_0$, and $\tilde{f}\colon (S^n, s_0) \to \left(\tilde{X}, \tilde{x}_0\right)$ is the lift of f, then $p_*\left(\tilde{\gamma}\tilde{f}\right) = \gamma f$. But this follows directly from how $\tilde{\gamma}\tilde{f}$ and γf we constructed. Namely, applying p to the square used in the definition, we see that we obtain γf from $\tilde{\gamma}\tilde{f}$ since $p \circ \tilde{\gamma} = \gamma$ and $p \circ \tilde{f} = f$. ∎

Exercise 1.79. Let X and Y be pointed spaces and $n \geq 2$. Show that the inclusion $X \vee Y \hookrightarrow X \times Y$ induces a surjection $\pi_n(X \vee Y) \to \pi_n(X \times Y)$ for all n. Furthermore, this exhibits $\pi_n(X \times Y)$ as a retract of $\pi_n(X \vee Y)$ for all n. (Is this also true for $n = 1$?)

Proof. a ∎

1.8.2. *Problems.*

Problem 1.80. Fix an isomorphism $H_n(S^n) \cong \mathbb{Z}$. We define the degree $\deg f$ of a map $f\colon S^n \to S^n$ to be the integer such that $f_*\colon H_n(S^n) \to H_n(S^n)$ sends 1 to $\deg f \in \mathbb{Z}$.

(1) Show that taking the degree of a map $S^n \to S^n$ induces a well-defined map
$$\deg\colon \pi_n(S^n) \to \mathbb{Z}$$
(2) Show that deg is a group homomorphism.
(3) Show that the map deg is surjective.
(4) Suppose that $n \geq 2$. Show that $\pi_n(S^n) \cong \mathbb{Z} \times A$ for some abelian group A.

Proof. (1) Let $[f] \in \pi_n(S^n)$ and suppose f, f' are two representatives of this class. Then f and f' are homotopic by definition, so $f_* = (f')_*: \mathbb{Z} = H_n(S^n) \to H_n(S^n) = \mathbb{Z}$ are equal. In particular, $\deg f = f_*(1) = (f')_*(1) = \deg f'$. So the map is well-defined.

(2) To show that degree is a group homomorphism, we must show that $\deg(f+g) = \deg f + \deg g$.

For this, we will show a couple of results.

Proposition 1.81. *Let $X = S^n_1 \vee \ldots \vee S^n_k$ for $n > 0$. Then the homomorphism $H_n(S^n_1) \oplus \ldots \oplus H_n(S^n_k) \to H_n(X)$ induced by the inclusion maps is an isomorphism whose inverse is induced by the projections $X \to S^n_i$.*

To prove this proposition, we must show the following lemma.

Lemma 1.82. *Let X be a Hausdorff space and let $x_0 \in X$ be a point having a closed neighborhood N in X of which $\{x_0\}$ is a strong deformation retract. Let Y be a Hausdorff space and let $y_0 \in Y$. Define $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$. Then the inclusion maps induce isomorphisms $\tilde{H}_i(X) \oplus \tilde{H}_i(Y) \cong \tilde{H}_i(X \vee Y)$ whose inverse is induced by the projections of $X \vee Y$ to X and Y.*

Proof of lemma. Consider $A = X$ and $U = X - N$ which is open, and $\overline{U} \subset A$. Then by excision, $H_*(X \vee Y, X) \cong H_*(N \cup Y, N) \cong \tilde{H}_*(Y)$

Consider the LES of the triple $(X \vee Y, \{x_0\} \times Y, \{x_0\} \times \{y_0\})$. We obtain

$$\ldots \to H_p(\{x_0\} \times Y, (x_0, y_0)) \xrightarrow{i_*} H_p(X \vee Y, (x_0, y_0)) \xrightarrow{j_*} H_p(X \vee Y, \{x_0\} \times Y) \to \ldots$$

Since $\pi_Y \circ i = \mathrm{id}_{\{x_0\} \times Y}$, i_* is injective.

Furthermore, we have

$$H_p(X \vee Y, (x_0, y_0)) \xrightarrow{(\pi_X)_*} H_p(\{x_0\} \times Y, (x_0, y_0)) \cong H_p(X \vee Y, \{x_0\} \times Y)$$

so $j_* = (\pi_X)_*$ under these identifications, so, in particular, j_* is surjective. Therefore, our exact sequence is a SES:

$$0 \to H_p(Y, pt) \xrightarrow{i_*} H_p(X \vee Y, pt) \xrightarrow{j_*} \underbrace{H_p(X \vee Y, Y)}_{\cong H_p(X, pt)} \to 0$$

It remains to show that this SES is split, but since $\pi_X \circ \iota_X = \mathrm{id}_{\{x_0\} \times X}$, we have that ι_{X*} provides a section.

□

Proof of proposition. This follows by induction on the lemma. □

Next, suppose that $E_1, \ldots, E_k$ are disjoint open subsets of S^n, each homeomorphic to $\mathbb{R}^n$ for $n > 0$. Let $f\colon S^n \to Y$ be a map which takes $S^n - \bigcup E_i$ to y_0. Then f factors through the quotient space $S^n / (S^n - \bigcup E_i) \cong S_1^n \vee \ldots \vee S_k^n$ where $S_i^n = S^n / (S^n - E_i)$:

$$f\colon S^n \xrightarrow{g} S_1^n \vee \ldots \vee S_k^n \xrightarrow{h} Y$$

Let $\iota_j\colon S_j^n \hookrightarrow S_1^n \vee \ldots \vee S_k^n$ be the j th inclusion and let $p_j\colon S_1^n \vee \ldots \vee S_k^n \to S_j^n$ be the j th projection. Then by the proposition, $\sum_j \iota_{j*} p_{j*} = \mathrm{id}_*\colon H_n(S_1^n \vee \ldots \vee S_k^n) \to H_n(S_1^n \vee \ldots \vee S_k^n)$. Let $g_j = p_j \circ g\colon S^n \to S_j^n$ and $h_j = h \circ \iota_j\colon S_j^n \to Y$ and let $f_j = h_j \circ g_j\colon S^n \to Y$. That is, f_j is the map which is f on E_j and maps the complement of E_j to the basepoint y_0.

Theorem 1.83. *In the above situation, $f_* = \sum_{j=1}^k f_{j*}\colon H_n(S^n) \to H_n(Y)$.*

Proof of theorem. We have $f_* = h_* \circ g_* = \sum_j h_* i_{j*} p_{j*} g_* = \sum_j h_{j*} g_{j*} = \sum_j f_{j*}$. □

Now we get back to showing that $\deg(f+g) = \deg f + \deg g$.

Note that by way of defining $f+g$, this essentially maps I^n by f on the left half and g on the right half with the boundary mapping to the base point x_0. In particular, this factors through the quotient $I^n \to I^n/\partial I^n \cong S^n$, where now the two halves can be interpreted as, say, the upper and lower hemispheres. In particular, the equator is by assumption also mapped to x_0, so we can quotient further by $S^n \to S^n \vee S^n$ by "pinching" the equator to a point. This is essentially what the proposition above describes. In particular, $f+g$ can be covered by the two open hemispheres and maps the equator to x_0, so by the theorem, we have $(f+g)_* = f_* + g_*$, i.e., $\deg(f+g) = (f+g)_*(1) = f_*(1) + g_*(1) = \deg f + \deg g$, as we wanted to show.

(3) Next we show that deg is surjective. First note that $\deg \mathrm{id} = \mathrm{id}_*(1) = 1$ by functoriality since $\mathrm{id}_* = \mathrm{id}_{H_n(S^n)}$. By functoriality, we thus hit all of $\mathbb{Z}$. More precisely, $\deg(*_n \mathrm{id}) = n$ for $n \in \mathbb{N}$ as deg is a homomorphism. Also $\deg(*_n(-\mathrm{id})) = -n$ for $n \in \mathbb{N}$ and $\deg(c_{x_0}) = 0$, so deg is surjective.

(4) Let $n \geq 2$. We have a SES

$$0 \to \ker \deg \to \pi_n(S^n) \overset{\deg}{\to} \mathbb{Z} \to 0.$$

Since $\mathbb{Z}$ is projective, this splits, so $\pi_n(S^n) \cong \mathbb{Z} \oplus \ker \deg$. But $\ker \deg$ is a subgroup of $\pi_n(S^n)$ which is abelian, hence is itself abelian.

□

Problem 1.84. Fix $n \geq 1$. We say that a space X is *n-connected* if it is non-empty, path-connected, and $\pi_k(X, x) = 0$ for all $1 \leq k \leq n$ and $x \in X$. For (X, x_0) a pointed, path-connected space, show that the following are equivalent:

(1) X is n-connected.
(2) $\pi_k(X, x_0) = 0$ for all $1 \leq k \leq n$.
(3) Every map $S^k \to X$ can be extended to a map $D^{k+1} \to X$ for all $k \leq n$.
(4) Every map $S^k \to X$ is homotopic to a constant map for all $k \leq n$.

Proof. ($1 \implies 2$): this follows since X being n-connected means that $\pi_k(X, x) = 0$ for all $x \in X$ and all $1 \leq k \leq n$, hence in particular for x_0.
($2 \implies 3$) : Let $f\colon S^k \to X$ be a map. Then f represents some homotopy class $[f] \in \pi_k(X, x_0)$. But since $\pi_k(X, x_0) = 0$, f is homotopic to the constant map at $x_0 \operatorname{rel} s_0$. Let $H\colon S^k \times I \to X$ be this homotopy. Define $\tilde{f}\colon D^{k+1} \to X$ by $\tilde{f}(x) = H(x, \|x\|)$. Then $\tilde{f}$ is continuous as a composite of continuous maps and $\tilde{f}|_{S^k}(-) = H(-,1) = f(-)$, so $\tilde{f}$ indeed extends f.
($3 \implies 4$) : Let $f\colon S^k \to X$ be a map. Extends f to a map $\tilde{f}\colon D^{k+1} \to X$. Define now a homotopy $H\colon S^k \times I \to X$ by $H(x,t) = \tilde{f}(xt)$. This is continuous and $H(x,1) = \tilde{f}(x) = f(x)$ while $H(x,0) = \tilde{f}(0) \in X$ is constant. Hence this gives a homotopy between f and $c_{\tilde{f}(0)}$.

($4 \implies 3$) : Let $f\colon S^k \to X$ be a given map. By assumption, there exists a homotopy $H\colon S^k \times I \to X$ such that $H(-,1) = f(-)$ and $H(-,0) = c$ where c is some constant map at a point in X. But then H factors through the quotient

$$\begin{tikzcd}
S^k \times I \arrow[d] \arrow[dr, "H"] & \\
D^{k+1} \arrow[r, "\tilde{H}"'] & X
\end{tikzcd}$$

where we identify $S^k \times \{0\}$ to a point. But then $\tilde{H}|_{S^k}(-) = H(-,1) = f(-)$, so $\tilde{H}$ extends f.

($3 \implies 2$) : Let $[f] \in \pi_k(X, x_0)$ and f a representative. We want to show that f is homotopic to the constant map at x_0 relative ∂I^k. Extend f to a map $\tilde{f}\colon D^{k+1} \to X$, and let $H\colon S^k \times I \to X$ be given by $H(x,t) = \tilde{f}(ts_0 + (1-t)x)$. This gives a homotopy between f and the constant map at x_0.

$(2 \implies 1)$: the only thing that requires showing is that given that $\pi_k(X, x_0) = 0$ for all k, we then have $\pi_k(X, x) = 0$ for all k and all $x \in X$. But this is precisely what the given hint says we are allowed to assume since X is path connected. So we are done. $\square$

Definition 1.85 (n-connected maps)**.** A map $f\colon X \to Y$ is called *n-connected* if it induces isomorphisms on all homotopy groups in degree $< n$ and an epimorphism in degree n.

2. Methods of Calculation

2.1. Excision for Homotopy Groups.

Theorem 2.1. *Let X be a CW complex decomposed as the union of subcomplexes A and B with nonempty connected intersection $C = A \cap B$. If (A, C) is m-connected and (B, C) is n-connected, $m, n \geq 0$, then the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.*

Corollary 2.2 (Freudenthal Suspension Theorem)**.** *The unreduced suspension map $\pi_i(S^n) \to \pi_{i+1}\left(S^{n+1}\right)$, induced by the suspension map $S^n \to \Sigma S^n \cong S^{n+1}$, is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$. More generally, this holds for the suspension $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ whenever X is an $(n-1)$-connected CW complex.*

Proof of Corollary. Decompose the unreduced suspension $\Sigma X = (X \times I)/\left(X \times \{0\}, X \times \{1\}\right)$ as the union of two cones C_+X and C_-X intersecting in a copy of X. Recall that a map $f\colon X \to Y$ induces a suspended map $\Sigma f\colon \Sigma X \to \Sigma Y$. Now, if we consider f to be any map $f\colon (S^n, s_0) \to (X, x_0)$, then we have a suspended map

$$\begin{array}{ccc} S^n \times I & \xrightarrow{f \times \mathrm{id}} & X \times I \\ \downarrow & & \downarrow \\ S^{n+1} \cong \Sigma S^n & \xrightarrow{\Sigma f} & \Sigma X \end{array}$$

So, in particular, Σf is some class in $\pi_{n+1}(\Sigma X)$. Define the suspension homomorphism $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ to be the map that sends f to Σf. This is a homomorphism (why?).

The unreduced suspension map is the same as the map

$$\pi_i(X) \cong \pi_{i+1}\left(C_+X, X\right) \to \pi_{i+1}(\Sigma X, C_-X) \cong \pi_{i+1}\left(\Sigma X\right).$$

(why?) where the two isomorphisms come from the LES of pairs and the middle map is induced by inclusion. The first map $\pi_i(X) \to \pi_{i+1}(C_+X, X)$ takes a map $\left(I^i, \partial I^n\right) \to (X, x_0)$ to the map $\left(I^{n+1}, \partial I^{n+1}, J^n\right) \to (C_+X, X, x_0)$ constructed by extending the given map radially to correspond with the height of C_+X. So one face of I^{n+1} will be mapped to the vertex of C_+X.

Including this into $(\Sigma X, C_-X)$ gives the middle homomorphism, and then the map $\pi_{i+1}\left(\Sigma X, C_-X\right) \to \pi_{i+1}(\Sigma X)$ is simply the identity on our map.

From the LES of $(C_\pm X, X)$, we see that this pair is n-connected if X is $(n-1)$-connected. Then Theorem 2.1 gives that the middle map is an isomorphism for $i + 1 < 2n$ and surjective for $i + 1 = 2n$.

□

Corollary 2.3. *$\pi_n\left(S^n\right) \cong \mathbb{Z}$, generated by the identity map, for all all $n \geq 1$. In particular, the degree map $\deg\colon \pi_n\left(S^n\right) \to \mathbb{Z}$ is an isomorphism.*

Proof. From Corollary 2.2, we obtain a sequence of suspension homomorphisms

$$\pi_1\left(S^1\right) \to \pi_2\left(S^2\right) \to \pi_3\left(S^3\right) \to \dots$$

where the first map is surjective and all subsequent maps are isomorphisms. Since $\pi_1\left(S^1\right) \cong \mathbb{Z}$, this implies that $\pi_i\left(S^i\right)$ is either $\mathbb{Z}$ or a cyclic finite group for all $i \geq 2$. Note that there exist basepoint-preserving maps $S^n \to S^n$ for arbitrary degree and

since degree is a homotopy invariant, this implies that $\pi_n(S^n)$ must be infinite for all $n \geq 2$ since otherwise there would be maps $S^n \to S^n$ that are based homotopic but of different degrees which is a contradiction.
Lastly, we claim that the degree map $\pi_n(S^n) \to \mathbb{Z}$ is an isomorphism.
One way to see this is first to note that deg is a homomorphism by Problem 1.80, where we also showed it to be surjective.
A different way to show surjectivity is to note that

Proposition 2.4. $\deg \Sigma f = \deg f$, *where* $\Sigma f\colon S^{n+1} \to S^{n+1}$ *is the suspension of the map* $f\colon S^n \to S^n$.

Proof. Let $CS^n = (S^n \times I)\,/\,(S^n \times \{1\})$ be the cone on S^n and let $S^n = S^n \times \{0\} \subset CS^n$ be the base, so $CS^n/S^n \cong \Sigma S^n$. First the map f induces $Cf\colon (CS^n, S^n) \to (CS^n, S^n)$ with quotient Σf. The naturality of the boundary maps in the homology LES of the pair (CS^n, S^n) then gives commutativity of the diagram

$$\begin{tikzcd} \tilde{H}_{n+1}\left(S^{n+1}\right) \arrow[r, "\partial", "\cong"'] \arrow[d, "\Sigma f_*"] & \tilde{H}_n(S^n) \arrow[d, "f_*"] \\ \tilde{H}_{n+1}(S^{n+1}) \arrow[r, "\partial", "\cong"'] & \tilde{H}_n\left(S^n\right) \end{tikzcd}$$

So if f_* is multiplication by $\deg f$, then so is Σf_*. $\square$

Now since $\gamma_k\colon z \mapsto z^k$ is a degree k map of S^1, this shows that deg is surjective as a map $\pi_1\left(S^1\right) \to \mathbb{Z}$, and now by the above Proposition, the degree map on $\Sigma\gamma_k\colon S^{n+1} \to S^{n+1}$ has degree k also, so by repeated suspension, we have degree k maps in all $\pi_n(S^n)$, giving surjectivity of $\pi_n(S^n) \to \mathbb{Z}$.

$\square$

Example 2.5 ($\pi_n\left(\bigvee_\alpha S^n_\alpha\right)$)**.** We want to show that $\pi_n\left(\bigvee_\alpha S^n_\alpha\right)$ for $n \geq 2$ is free abelian with basis the homotopy classes of the inclusions $S^n_\alpha \hookrightarrow \bigvee_\alpha S^n_\alpha$.
Suppose first that there are only *finitely many* summands S^n_α. Then we can regard $\bigvee_\alpha S^n_\alpha$ as the n-skeleton of the product $\prod_\alpha S^n_\alpha$, where S^n_α is given the usual CW structure and $\prod_\alpha S^n_\alpha$ has the product CW structure. (See Hatcher appendix A).
By construction then $\prod_\alpha S^n_\alpha$ has cells only in dimensions a multiple of n, so the pair $\left(\prod_\alpha S^n_\alpha, \bigvee_\alpha S^n_\alpha\right)$ is $(2n-1)$-connected by Corollary 1.49. So from the LES for the pair, we see that the inclusion $\bigvee_\alpha S^n_\alpha \hookrightarrow \prod_\alpha S^n_\alpha$ induces an isomorphism on homotopy groups in dimensions $\leq 2n-1$. Next we have $\pi_n\left(\prod_\alpha S^n_\alpha\right) \cong \bigoplus_\alpha \pi_n\left(S^n_\alpha\right) \cong \bigoplus_\alpha \mathbb{Z}$, a free abelian group with basis the inclusions $S^n_\alpha \hookrightarrow \prod_\alpha S^n_\alpha$, so pulling this back along the isomorphism $\pi_n\left(\bigvee_\alpha S^n_\alpha\right) \cong \pi_n\left(\prod_\alpha S^n_\alpha\right)$, the same is true for $\bigvee_\alpha S^n_\alpha$.
This proves the claim when there are finitely many S^n_α's.
When there are infinitely many summands S^n_α, consider the homomorphism $\Phi\colon \bigoplus_\alpha \pi_n\left(S^n_\alpha\right) \to \pi_n\left(\bigvee_\alpha S^n_\alpha\right)$ induced by the inclusions $S^n_\alpha \hookrightarrow \bigvee_\alpha S^n_\alpha$. Then Φ is surjective since any map $f\colon S^n \to \bigvee_\alpha S^n_\alpha$ has compact image contained in the wedge sum of finitely many S^n_α's, so by the finite case already proved, $[f]$ is in the image of Φ.
Similarly, a nullhomotopy of f has compact image contained in a finite wedge sum of S^n_α's, so the finite case also implies that Φ is injective.

Proposition 2.6. *If a CW pair* (X, A) *is* r*-connected and* A *is* s*-connected, with* $r, s \geq 0$, *then the map* $\pi_i(X, A) \to \pi_i(X/A)$ *induced by the quotient map* $X \to X/A$ *is an isomorphism for* $i \leq r + s$ *and a surjection for* $i = r + s + 1$.

Proof. Consider $X \cup CA$. Since A is closed and the inclusion $A \hookrightarrow X$ is a cofibration (since these are CW complexes), the map $h\colon C_\iota = X \cup CA \to X/A$ is a homotopy equivalence by Theorem **??**. So we have a commutative diagram

$$\begin{tikzcd}
\pi_i(X,A) \arrow[r] & \pi_i\left(X \cup CA, CA\right) \arrow[r] & \pi_i\left(X \cup CA/CA\right) = \pi_i(X/A) \\
& \pi_i(X \cup CA) \arrow[u, "\cong"] \arrow[ur, "\cong"] &
\end{tikzcd}$$

where the vertical isomorphism comes from the LES of the pair $(X \cup CA, CA)$. Now, applying Theorem 2.1 to $(A,B) = (X, CA)$, since (X,A) is r-connected and (CA, A) is $(s+1)$-connected, we find that the homomorphism $\pi_i\left(X,A\right) \to \pi_i\left(X \cup CA, CA\right)$ induced by the inclusion is an isomorphism for $i < r+s+1$ and a surjection for $i = r+s+1$, which proves the result. $\square$

Example 2.7 (Construction of spaces with a particular group as π_n)**.** Suppose X is obtained from a wedge of spheres $\bigvee_\alpha S^n_\alpha$ by attaching cells e^{n+1}_β via basepoint-preserving maps $\varphi_\beta\colon S^n \to \bigvee_\alpha S^n_\alpha, n \geq 2$. By cellular approximation, we know that $\pi_i(X) = 0$ for $i < n$, and we shall show that $\pi_n(X)$ is the quotient of the free abelian group $\pi_n\left(\bigvee_\alpha S^n_\alpha\right) \cong \bigoplus_\alpha \mathbb{Z}$ by the subgroup generated by the classes $[\varphi_\alpha]$. Any subgroup can be realized in this way, by choosing maps φ_β to represent a set of generators for the subgroup. Let $X = \left(\bigvee_\alpha S^n_\alpha\right) \bigcup_\beta e^{n+1}_\beta$.
Then the LES of the pair $\left(X, \bigvee_\alpha S^n_\alpha\right)$ gives

$$\pi_{n+1}\left(X, \bigvee_\alpha S^n_\alpha\right) \xrightarrow{\partial} \pi_n\left(\bigvee_\alpha S^n_\alpha\right) \to \pi_n(X) \to 0.$$

so

$$\pi_n(X) \cong \pi_n\left(\bigvee_\alpha S^n_\alpha\right) / \operatorname{im}\partial$$

The quotient $X/\bigvee_\alpha S^n_\alpha$ is a wedge of spheres S^{n+1}_β, so by Proposition 2.6 and Example 2.5, the map $\pi_{n+1}\left(X, \bigvee_\alpha S^n_\alpha\right) \to \pi_{n+1}\left(X/\bigvee_\alpha S^n_\alpha\right) \cong \pi_{n+1}\left(\bigvee_\beta S^{n+1}_\beta\right)$ is an isomorphism, so $\pi_{n+1}\left(X, \bigvee_\alpha S^n_\alpha\right)$ is free with basis the caracteristic maps φ_β of the cells e^{n+1}_β. The boundary map ∂ takes these to the classes $[\varphi_\beta]$, so the result follows.

2.1.1. *Moore Spaces.*

Definition 2.8 (Moore Space)**.** Given an abelian group G and an integer $n \geq 1$, a CW complex X such that $H_n(X) \cong G$ and $\tilde{H}_i(X) \cong 0$ for $i \neq n$, then X is said to be a *Moore space*, commonly written $M(G, n)$ to indicate the dependence on G and n. It is also sensible to require that X be simply connected when $n > 1$.

Lemma 2.9 (Existence of Moore spaces)**.** *For any abelian group G and any integer $n \geq 1$, we can construct a $M(G, n)$-space.*

Proof. As a simple case, when $G = \mathbb{Z}/m$, we can take X to be S^n with a cell e^{n+1} attached by a map $S^n \to S^n$ of degree m. More generally, any finite generated G can be realized by taking wedges of examples of this type for finite cyclic summands of G together with copies of S^n for infinite cyclic summands of G.

For the general nonfinitely generated case, let $F \to G$ be a homomorphism of a free abelian group F onto G, sending a basis for F onto some set of generators of G. The kernel K of this homomorphism is a subgroup of a free abelian group, hence is itself free abelian. Choose bases $\{x_\alpha\}$ for F and $\{\gamma_\beta\}$ for K, and write $y_\beta = \sum_\alpha d_{\beta\alpha} x_\alpha$. Let $X^n = \bigvee_\alpha S^n_\alpha$, so $H_n(X^n) \cong F$. We will construct X from X^n by attaching cells e^{n+1}_β via maps $f_\beta \colon S^n \to X^n$ such that the composition of f_β with the projection onto the summand S^n_α has degree $d_{\beta\alpha}$. Then the cellular boundary map d_{n+1} will be the inclusion $K \hookrightarrow F$, hence X will have the desired homotopy groups.

We let the map f_β be the map which maps the complement of $\sum_\alpha |d_{\beta\alpha}|$ disjoint balls in S^n to the 0-cell of X^n while sending $|d_{\beta\alpha}|$ of the balls onto the summand S^n_α by maps of degree $+1$ if $d_{\beta\alpha} > 0$ or degree -1 if $d_{\beta\alpha} < 0$. By Theorem 1.83, we get that the composition of f_β with the projection onto the summand S^n_α has degree $d_{\beta\alpha}$.
This finishes the construction of a $M(G, n)$ space. □

Corollary 2.10. *For any abelian groups $\{G_n\}_{n\in\mathbb{N}}$, we can construct a space X such that $H_n(X) \cong G_n$ for all n.*

Proof. Take the wedge of the Moore spaces $M(G_n, n)$ for $n \in \mathbb{N}$. □

2.1.2. *Eilenberg-MacLane Spaces.*

Definition 2.11 (Eilenberg-MacLane space, $K(G, n)$)**.** A space X having just one nontrivial homotopy group $\pi_n(X) \cong G$ is called an *Eilenberg-MacLane space* $K(G, n)$.

Construction of Eilenberg-MacLane Spaces:
Given arbitrary G and n, and assuming G is abelian if $n > 1$, we can construct a CW complex $K(G, n)$. To begin, construct the CW complex X from Example 2.7. Then X is an $(n-1)$- conneted CW complex of dimension $n+1$ such that $\pi_n(X) \cong G$ by construction. Alternatively, given the existence of Moore spaces $M(G, n)$ for any G and n, we can take a Moore space $M(G, n)$ and use the Hurewicz isomorphism to conclude that $\pi_n(X) \cong H_n(X)$. Hence we just need to fix all homotopy groups of dimension greater than n. By Example 1.55, we can construct a CW complex X_n containing X as a subcomplex such that $\pi_n(X_n) \cong \pi_n(X) \cong G$ while $\pi_k(X_n) \cong 0$ for all $k \neq n$.

Example 2.12 (Constructing spaces with arbitrary (abelian) homotopy groups)**.** Recall that

$$\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n\left(X_\alpha\right),$$

so if we have a sequence of abelian groups $\{G_{n_i}\}_{i\in I}$, and let X_{n_i} denote that $K(G_{n_i}, n_i)$ space, then we find that

$$\pi_k(\prod_{i\in I} X_{n_i}) \cong \prod_{i\in I} \pi_k\left(X_{n_i}\right) \cong \begin{cases} G_{n_i}, & k = n_i \text{ for some } i \in I \\ 0, & \text{else} \end{cases}$$

Having covered the existence of Eilenberg-MacLane spaces, we now find the following for uniqueness of these spaces:

Proposition 2.13 (Uniqueness of Eilenberg-MacLane spaces)**.** *The homotopy type of a CW complex $K(G,n)$ is uniquely determined by G and n.*

The proof is based on the following lemma giving a condition for when homomorphisms between homotopy groups are induced by some map:

Lemma 2.14. *Let X be a CW complex of the form $(\bigvee_\alpha S^n_\alpha)\bigcup_\beta e^{n+1}_\beta$ for some $n \geq 1$. Then for every homomorphism $\psi\colon \pi_n(X) \to \pi_n(Y)$ with Y path-connected there exists a map $f\colon X \to Y$ with $f_* = \psi$.*

Proof. The construction of f is as one would expect: first let f send the natural basepoint of $\bigvee_\alpha S^n_\alpha$ to a chosen basepoint $y_0 \in Y$. Now for every sphere S^n_α in X, we extend f over the sphere via a map representing $\psi\left([i_\alpha]\right)$ where i_α is the inclusion $S^n_\alpha \hookrightarrow X$. This defines f on the n-skeleton of X : $f\colon X^n \to Y$. Since now $f_*\left[i_\alpha\right] = \psi\left[i_\alpha\right]$ for all α and the $[i_\alpha]$ generate $\pi_n\left(X^n\right)$, this defines f_* on all of $\pi_n(X^n)$.
To extend f over the $(n+1)$-cells, it will suffice to show that $f\circ\varphi_\beta$ is nullhomotopic, where $\varphi_\beta\colon S^n \to X^n$ is the attaching map for the $(n+1)$-cell e^{n+1}_β. But $f\circ\varphi_\beta$ is a representative of $f_*\left[\varphi_\beta\right] = \psi\left[\varphi_\beta\right]$. Thus we have transformed $f_*\left[\varphi_\beta\right]$ into an element in the image of $\psi\colon \pi_n(X) \to \pi_n(Y)$, and for this, we can use the extra structure of X, not just X^n. In X, $[\varphi_\beta]$ is trivial via the characteristic map of the cell e^{n+1}_β, so $\psi\left[\varphi_\beta\right] = \psi(0) = 0$, thus indeed $f\circ\varphi_\beta$ is nullhomotopic. Thus we obtain the desired extension $f\colon X \to Y$. To see that $f_* = \psi$, simply note that by cellular approximation, any element of $\pi_n(X)$ can be represented as an element in $\pi_n(X^n)$, and on $\pi_n(X^n)$, f_* agrees with ψ by construction. □

Proof of Proposition 2.13. Let K' be any $K\left(G,n\right)$ CW complex, and let K be the specific $K(G,n)$ CW complex constructed in Example 2.7. In particular, K is of the form of Lemma 2.14. Since $\pi_n(K) = \pi_n(Y)$, we can apply Lemma 2.14 to obtain a map $f\colon K \to K'$ inducing the identity on π_n. Since all other homotopy groups of K and K' are trivial, Whitehead's theorem now gives that f is a homotopy equivalence. Since homotopy equivalence is an equivalence relation, this finishes the proof. □

2.2. **The Hurewicz Theorem.**

Theorem 2.15 (The Little Hurewicz Theorem)**.** *If a space X is $(n-1)$-connected, $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$. If a pair (X,A) is $(n-1)$-connected, $n \geq 2$, with A simply connected and nonempty, then $H_i(X,A) = 0$ for $i < n$ and $\pi_n(X,A) \cong H_n(X,A)$.*

Remark. This result is, in a sense, the best that we can expect. For example, S^n has trivial homology groups above dimension n but many nontrivial homotopy groups in this range when $n \geq 2$; and conversely, Eilenberg-MacLane spaces such as $\mathbb{CP}^\infty$ have trivial higher homotopy groups but many nontrivial homology groups.

Proof. We may assume X is a CW complex and (X, A) is a CW pair by taking CW approximations to X and (X, A). For CW pairs, the relative case then reduces to the absolute case since $\pi_i(X, A) \cong \pi_i(X/A)$ for $i \leq n$ by Proposition 2.6, while $H_i(X, A) \cong \tilde{H}_I(X/A)$ as $A \hookrightarrow X$ is a cofibration of a closed subspace.
In the absolute case, using Proposition 1.54, we can replace X by a homotopy equivalence CW complex with $(n-1)$-skeleton a point, hence $\tilde{H}_i(X) = 0$ for $i < n$. To show that $\pi_n(X) = H_n(X)$, we can take the Postnikov truncation $X_{n+1} = \tau_{\leq n+1} X$, obtained by throwing away all cells of dimension $> n+1$ from X. Then X has the form $(\bigvee_\alpha S_\alpha^n) \cup_\beta e_\beta^{n+1}$. We may assume that the attaching maps φ_β of the cells e_β^{n+1} are basepoint-preserving (we can do this since we used Proposition 1.54 which used Proposition 1.52 and its proof, and in this proof, we could choose the cells to be basepoint-preserving.) But now by Example 2.7, we get that $\pi_n(X) = \operatorname{coker}(\pi_{n+1}(X, X^n) \to \pi_n(X^n))$ which is the cokernel of a map $\bigoplus_\beta \mathbb{Z} \to \bigoplus_\alpha \mathbb{Z}$. We claim that this is the same as the cellular boundary map $d \colon \tilde{H}_{n+1}\left(X^{n+1}/X^n\right) \cong H_{n+1}\left(X^{n+1}, X^n\right) \to H_n\left(X^n, X^{n-1}\right) \cong \tilde{H}_n\left(X^n/X^{n-1}\right)$. For a cell e_β^{n+1}, the coefficients of $d\left(e_\beta^{n+1}\right)$ are precisely the degrees of the compositions $q_\alpha \varphi_\beta$ where q_α collapses all n-cells except e_α^n to a point. Similarly, since the isomorphism $\pi_n(S^n) \cong \mathbb{Z}$ is obtained by the degree map, and the map $\pi_{n+1}(X/X^n) \cong \pi_{n+1}(X, X^n) \xrightarrow{\partial} \pi_n(X^n)$ sends the $(n+1)$-cell e_β^{n+1} to $\deg \varphi_\beta^n$ also (the degree of the attaching map), by construction. Hence the maps are the same, so $\pi_n(X) \cong \operatorname{coker}(\pi_{n+1}(X, X^n) \to \pi_n(X^n)) \cong \operatorname{coker} d = H_n(X)$ where the last equality holds since there are no $(n-1)$-cells.

□

Corollary 2.16 (Homology version of Whitehead's Theorem)**.** *A map $f \colon X \to Y$ between simply-connected CW complexes is a homotopy equivalence if $f_* \colon H_n(X) \to H_n(Y)$ is an isomorphism for each n.*

Proof. By replacing Y with the mapping cylinder M_f, we may assume f is the inclusion $X \hookrightarrow Y$. Since X and Y are simply-connected, $\pi_1(Y, X) = 0$. The relative Hurewicz theorem says that the first nonzero $\pi_n(Y, X)$ is isomorphic to the first nonzero $H_n(Y, X)$, but by the LES of the pair (Y, X) in homology, $H_n(Y, X) \cong 0$ for all $n \geq 0$, so also $\pi_n(Y, X) \cong 0$ for all $n \geq 0$, so f induces isomorphisms $\pi_n(X) \to \pi_n(Y)$ for all n. By Whitehead's theorem, f is a homotopy equivalence. □

Lemma 2.17 (Uniqueness of Moore Spaces)**.** *The homotopy type of a CW complex Moore space $M(G, n)$ is uniquely determined by G and n if $n > 1$, so $M(G, n)$ is simply-connected.*

Proof. Let Y be any $M(G, n)$ space and X be the Moore space constructed in Lemma 2.9. By Hurewicz, $G \cong H_n(X) \cong \pi_n(X)$ and similarly for Y.
Since X is a CW complex of the form $(\bigvee_\alpha S_\alpha^n) \cup_\beta e_\beta^{n+1}$, it follows from Lemma 2.14 that there exists a map $f \colon X \to Y$ inducing the identity on homotopy groups. We would like to be able to conclude that this map then also induces the identity on homology, or just an isomorphism, however, this requires the stronger Hurewicz

theorem. Instead, we can fix this problem differently: there is an induced map $\tilde{f}\colon X \to M_f$, and we have that this induces an isomorphism on homotopy groups in degrees $\leq n$, so (M_f, X) is n-connected. Since X is simply connected, the Little Hurewicz Theorem further gives us that $H_{n+1}(M_f, X) \cong \pi_{n+1}(M_f, X)$, so if (M_f, X) is furthermore $(n+1)$-connected, then we would find that f also also induces an isomorphism on H_n by the LES. So see this, we can first attach $(n+2)$-cells to Y so that $\pi_{n+1}(Y) = 0$. With this Y, (M_f, X) becomes $(n+1)$-connected, so for the enlarged Y, f induces an isomorphism on H_n, and now we can note that adding $(n+2)$-cells to Y did not affect the homology of Y in degree n, so f induced an isomorphism to begin with. $\square$

Definition 2.18 (Hurewicz map)**.** Thinking of $\pi_n(X, A, x_0)$ for $n > 0$ as $[D^n, \partial D^n, s_0; X, A, x_0]$, the *Hurewicz map* $h\colon \pi_n(X, A, x_0) \to H_n(X, A)$ is defined by $h([f]) = f_*(\alpha)$ where α is a fixed generator of $H_n(D^n, \partial D^n) \cong \mathbb{Z}$, and $f_*\colon H_n(D^n, \partial D^n) \to H_n(X, A)$ is induced by f.

If we have a homotopy $f \simeq g$ through maps $(D^n, \partial D^n, s_0) \to (X, A, x_0)$, through maps $(D^n, \partial D^n, s_0) \to (X, A, x_0)$ or even through maps $(D^n, \partial D^n) \to (X, A)$, we have $f_* = g_*$, so h is well-defined.

Proposition 2.19. *The Hurewicz maps $h\colon \pi_n(X, A, x_0) \to H_n(X, A)$ is a homomorphism, assuming $n > 1$ so that $\pi_n(X, A, x_0)$ is a group.*

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