We will cover the cup product following Hatcher.

Definition 0.1 (Cup Product)**.** For a ring R, let $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$. Then the *cup product* $\varphi \smile \psi \in C^{k+l}(X;R)$ is the cochain whose value on $\sigma\colon \Delta^{k+l} \to X$ is given by

$$(\varphi \smile \psi)(\sigma) = \varphi\left(\sigma|_{[v_0,\dots,v_k]}\right)\psi\left(\sigma|_{[v_k,\dots,v_{k+l}]}\right)$$

where the right-hand side is the product in R.

To see that this induces a cup product on cohomology, we need the following lemma:

Lemma 0.2. *$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k\varphi \smile \delta\psi$ for $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$.*

Using the lemma, it is clear that the cup product of two cocycles is again a cocycles, and that the cup product of a cocycle and a coboundary, in either order, is a coboundary. It follows that there is an induced cup product

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X;R).$$

This is associative and distributive since at the level of cochains the cup product has these properties.
If R has an identity, then there is an identity elements for the cup product, the class $1 \in H^0(X;R)$ defined by the 0-cocycle taking the value 1 on each singular 0-simplex.

0.0.1. *Relative cup product.* The cup product formula $(\varphi \smile \psi)(\sigma) = \varphi\left(\sigma|_{[v_0,\dots,v_k]}\right)\psi\left(\sigma|_{[v_k,\dots,v_{k+l}]}\right)$ also gives relative cup products

$$H^k(X;R) \times H^l(X,A;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$
$$H^k(X,A;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$
$$H^k(X,A;R) \times H^l(X,A;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$

since if φ or ψ vanishes on chains in A, then so does $\varphi \smile \psi$.

We can also define an even more general relative cup product

$$H^k(X,A;R) \times H^l(X,B;R) \xrightarrow{\smile} H^{k+l}(X,A \cup B;R)$$

when A and B are open subsets of X or subcomplexes of the CW complex X.

Construction. The absolute cup product restricts to a cup product $C^k(X,A;R) \times C^l(X,B;R) \to C^{k+l}(X,A \sqcup B;R)$ where $C^n(X,A \sqcup B;R)$ is the subgroup of $C^n(X;R)$ consisting of cochains vanishing on sums of chains in A and chains in B. If A and B are open in X, then the inclusions $C^n(X,A \cup B;R) \hookrightarrow C^n(X,A \sqcup B;R)$ induces isomorphisms on cohomology:

Proposition 0.3. *For a map $f\colon X \to Y$, the induced map $f^*\colon H^n(Y;R) \to H^n(X;R)$ satisfies $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$, and similarly in the relative case.*

Theorem 0.4. *The identity $\alpha \smile \beta = (-1)^{kl}\beta \smile \alpha$ holds for all $\alpha \in H^k(X,A;R)$ and $\beta \in H^l(X,A;R)$, when R is commutative.*

1. The Cohomology Ring

Since the cup product is associative and distributive, it is natural to try to make it the multiplication in a ring structure on the cohomology groups of a space X. This is easy to do if we define $H^*(X;R) = \bigoplus_{k\in\mathbb{Z}} H^k(X;R)$. That is, if we define $H^*(X;R)$ as the direct sum of the cohomology groups of the space. Then elements of $H^*(X;R)$ are finite sums $\Sigma_i \alpha_i$ with $\alpha_i \in H^i(X;R)$ and the product of two such sums is defined to be $(\Sigma_i \alpha_i)(\Sigma_j \beta_j) = \Sigma_{i,j} \alpha_i \beta_j$.

Exercise 1.1. Show that this makes $H^*(X;R)$ into a ring,w with identity if R has an identity. Similarly for $H^*(X,A;R)$ with the relative cup product.
Taking scalar multiplication by elements of R into account, these rings can also be regarded as R-algebras.

Example 1.2. Recall that $H^k(\mathbb{RP}^2;\mathbb{Z}_2) \cong \mathbb{Z}_2$ for $k = 0, 1, 2$ and is 0 otherwise. Also by example 3.8 in Hatcher on Cohomology, for a generator $\alpha \in H^1(\mathbb{RP}^2;\mathbb{Z}_2)$, $\alpha^2 = \alpha \smile \alpha$ is a generator of $H^2(\mathbb{RP}^2;\mathbb{Z}_2)$, hence $H^*(\mathbb{RP}^2;\mathbb{Z}_2) \cong \mathbb{Z}_2\,[\alpha]\,/\,\left(\alpha^3\right)$.

Adding cohomology classes of different dimensions to form $H^*(X;R)$ is convenient, but it has little topological significance. One can always regard the cohomology ring as a *graded ring*:

Definition 1.3 (Graded Ring)**.** A ring A with a decomposition $\bigoplus_{k\geq 0} A_k$ into additive subgroups $A_k \leq A$ such that the multiplication takes $A_k \times A_l$ to A_{k+l} is called a *graded ring*.

To indicate that $\alpha \in A$ lies in A_k, we write $|a| = k$.

Definition 1.4 (Degree/dimension)**.** The number $|a|$ is called the *degree* or *dimension* of a.

Definition 1.5 (Commutative/anticommutative/graded commutative)**.** A graded ring satisfying the commutativity property that $ab = (-1)^{|a||b|}ba$ is usually called *commutative* or any of the following less ambiguous terms: *graded commutative*, *anticommutative*, or *skew commutative*.

Example 1.6 (Polynomial Rings)**.** An example of a graded ring is $R\,[\alpha]$ or the truncated version: $R\,[\alpha]\,/\,(\alpha^n)$.
We have seen that $H^*\left(\mathbb{RP}^2;\mathbb{Z}/2\right) \cong \mathbb{Z}/2\,[\alpha]\,/\,\left(\alpha^3\right)$. More generally, we can show that $H^*\left(\mathbb{RP}^n;\mathbb{Z}/2\right) \cong \mathbb{Z}/2\,[\alpha]\,/\,\left(\alpha^{n+1}\right)$ and $H^*\left(\mathbb{RP}^\infty;\mathbb{Z}/2\right) \cong \mathbb{Z}/2\,[\alpha]$, where, in these cases, $|\alpha| = 1$.

Example 1.7 (Exterior Algebras)**.** The *exterior algebra* $\Lambda_R\,[\alpha_1, \ldots, \alpha_n]$ over a commutative ring R with identity is the free R-module with basis the finite products $\alpha_{i_1} \cdots \alpha_{i_k}, i_1 < \ldots < i_k$, with associative, distributive multiplication defined by the rules $\alpha_i\alpha_j = -\alpha_j\alpha_i$ for $i \neq j$ and $\alpha_i^2 = 0$ for all i. The empty producto f α_i 's is the identity element 1 in $\Lambda_R\,[\alpha_1, \ldots, \alpha_n]$.
In view of $\alpha_i\alpha_j = -\alpha_j\alpha_i$, the exterior algebra becomes an anticommutative graded ring by specifying odd dimensions for the generators α.
By the Künneth formula, we have

$$H^*\left(S^{k_1} \times \ldots \times S^{k_n};\mathbb{Z}\right) \cong H^*(S^{k_1};\mathbb{Z}) \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} H^*(S^{k_n};\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}\,[\alpha_1, \ldots, \alpha_n]$$

when all the k_i are odd, since the first isomorphism is given by the cross product.

When some k_i 's are even, one obtains the tensor product of an exterior algebra for the odd-dimensional speheres and truncated polynomial rings $\mathbb{Z}[\alpha]/\left(\alpha^2\right)$ for the even dimensional spheres.

1.1. **The Cross Product.**

Definition 1.8 (First definition of cross product, external cup product)**.** We define the *cross product*, or *external cup product* as it is sometimes called, by the map

$$H^*(X;R) \times H^*(Y;R) \xrightarrow{\times} H^*(X \times Y;R)$$

given by $a \times b = p_1^*(a) \smile p_2^*(b)$ where p_1, p_2 are the projections of $X \times Y$ onto X and Y, respectively.

Definition 1.9 (Cross Product, second definition)**.** Since the cup product is distributive, the cross product is bilinear, hence it induces an R-module homomorphism

$$\begin{tikzcd}
H^*(X;R) \times H^*(Y;R) \arrow[d] \arrow[drr, "\times"] & & \\
H^*(X;R) \otimes_R H^*(Y;R) \arrow[rr, "\times"] & & H^*(X \times Y;R)
\end{tikzcd}$$

which we also call the cross product, given by $a \otimes b \mapsto a \times b$.

This module homomorphism becomes a ring homomorphism if we define the multiplication in a tensor product of graded rings by $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$ where $|x|$ denotes the dimension of x.
This can be seen as follows (note that $ac = a \smile c$ and $bd = b \smile d$):

$$\begin{aligned}
\mu\left((a \otimes b)(c \otimes d)\right) &= (-1)^{|b||c|}\mu\left(ac \otimes bd\right) \\
&= (-1)^{|b||c|}(a \smile c) \times (b \smile d) \\
&= (-1)^{|b||c|}p_1^*(a \smile c) \smile p_2^*\left(b \smile d\right) \\
&= (-1)^{|b||c|}p_1^*(a) \smile p_1^*(c) \smile p_2^*(b) \smile p_2^*(d) \\
&= p_1^*(a) \smile p_2^*(b) \smile p_1^*(c) \smile p_2^*(d) \\
&= (a \times b)(c \times d) = \mu(a \otimes b)\mu(c \otimes d)
\end{aligned}$$

Theorem 1.10. *The cross product $H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y;R)$ is a finitely generated free R-module for all k.*