1. Rotman

1.1. Modules.

Definition 1.1 (Representations)**.** If M is an abelian group, then

$$\mathrm{End}_{\mathbb{Z}}(M) = \{\text{homomorphisms} f\colon M \to M\}$$

is a ring under pointwise addition and composition as multiplication. A representation of a ring R is a ring homomorphism $\varphi\colon R \to \mathrm{End}_{\mathbb{Z}}(M)$ for some abelian group M.

Definition 1.2 (Group ring)**.** Let G be a finite group and k be a commutative ring. The group ring is the set of all functions $\alpha\colon G \to k$ made into a ring with pointwise operations: for all $x \in G$,

$$(\alpha + \beta)(x) = \alpha(x) + \beta(x) \quad \text{and} \quad (\alpha\beta)(x) = \alpha(x)\beta(x).$$

Definition 1.3 (k-representation)**.** If G is a group and k is a commutative ring, then a k-representation of G is a function $\sigma\colon G \to \mathrm{Mat}_n(k)$ with

$$\sigma(xy) = \sigma(x)\sigma(y)$$
$$\sigma(1) = I$$

.

Lemma 1.4 ($\mathrm{Hom}_R(A,B)$ is an abelian group)**.** *For left (resp. right) R-modules, $\mathrm{Hom}_R(A,B)$ is an abelian group and if $p\colon A' \to A$ and $q\colon B \to B'$ are R-maps, then*

$$(f+g)\,p = fp + gp \quad \text{and} \quad q\,(f+g) = qf + qg.$$

Proposition 1.5. *Let R be a ring, A, B, B' be left R-modules. Then*

(1) $\mathrm{Hom}_R(A,-)$ *is an additive functor* ${}_R\mathrm{Mod} \to \mathrm{Ab}$ *sending* $B \to \mathrm{Hom}_R(A,B)$ *and a morphism* $q\colon B \to B'$ *to* $q_*\colon \mathrm{Hom}_R(A,B) \to \mathrm{Hom}_R(A,B')$ *by postcomposition.*

(2) *If A is a left R-module, then* $\mathrm{Hom}_R(A,B)$ *is a $Z(R)$-module, where $Z(R)$ is the center of R, if we define*

$$(rf)(a) = f(ra) = rf(a)$$

for all $r \in Z(R)$ and $f\colon A \to B$. Then $\mathrm{Hom}_R(A,-)$ *is a functor* ${}_R\mathrm{Mod} \to_{Z(R)} \mathrm{Mod}$

Proof. (1) Since $q\,(f+g) = qf + qg$, we have $q_*\,(f+g) = q_*(f) + q_*(g)$, so $\mathrm{Hom}_R\,(A,-)\,(q) = q_* \in \mathrm{Mor}\,(\mathrm{Hom}_R(A,B), \mathrm{Hom}_R(A,B'))$ in Ab.
Furthermore, for $q\colon A \to B$ and $p\colon B \to C$, we have

$$(pq)_*\,(a) = pqa = p_*\,(qa) = (p_*q_*)(a)$$

so composition is preserved. And for any $a\colon A \to B$, we have

$$(\mathbb{1}_B)_*\,(a) = \mathbb{1}_B \circ a = a$$

so $(\mathbb{1}_B)_* = \mathbb{1}_{\mathrm{Hom}_R(A,B)}$. □

Exercise 1.6 (Example of a quotient group which is not a quotient module)**.** We have that $\mathbb{Q}$ is a module over itself and $\mathbb{Q}/\mathbb{Z}$ is a quotient group, but since $\mathbb{Z}$ is not a submodule of $\mathbb{Q}$ - it is not closed under scalar multiplication from $\mathbb{Q}$ -, we are not guaranteed that $\mathbb{Q}/\mathbb{Z}$ is a quotient module. And in fact, it is not: $2\left(\frac{1}{2} + \mathbb{Z}\right) = \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$ but neither factor is zero, but $\mathbb{Q}$ is a field, so if $\mathbb{Q}/\mathbb{Z}$ were a quotient module (over $\mathbb{Q}$), it would have to be a vector space, but in a vector space, we have $av = 0$ iff $a = 0$ or $v = 0$.

1.2. **Isomorphism theorems.**

Theorem 1.7 (First isomorphism theorem)**.** *If $f\colon M \to N$ is an R-map of left R-modules, then there is an R-isomorphism*

$$\varphi\colon M/\ker f \to \operatorname{im} f$$

given by

$$\varphi\colon m + \ker f \mapsto f(m).$$

Theorem 1.8 (Second isomorphism)**.** *If S and T are submodules of a left R-module M, then there is an R-isomorphism*

$$S/\left(S \cap T\right) \to \left(S+T\right)/T$$

Theorem 1.9 (Third isomorphism theorem)**.** *If $T \subset S \subset M$ is a tower of submodules of a left R-module M, then the enlargement of cosets $e\colon M/T \to M/S$ induces an R-isomorphism*

$$\left(M/T\right)/\left(S/T\right) \to M/S$$

Theorem 1.10 (Fourth (Correspondence) isomorphism theorem)**.** *If T is a submodule of a left R-module M, then $\varphi\colon S \to S/T$ is a bijection:*

$$\varphi\colon \{\textit{intermediate submodules } T \subset S \subset M\} \to \{\textit{submodules of } M/T\}.$$

Moreover, $T \subset S \subset S'$ in M if and only if $S/T \subset S'/T$ in M/T.

Definition 1.11 (Simple/irreducible modules)**.** A left R-module M is simple (or irreducible) if $M \neq \{0\}$ and M has no proper nonzero submodules; i.e., $\{0\}$ and M are the only submodules of M.

Lemma 1.12. *A left R-module M is simple if and only if $M \approx R/I$, where I is a maximal left ideal.*

Theorem 1.13 (1st and 3rd isomorphism theorem rephrased)**.**
1. *If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence, then*
$$A \approx \operatorname{im} f \quad \textit{and} \quad B/\operatorname{im} f \approx C.$$
2. *If $T \subset S \subset M$ is a tower of submodules, then there is an exact sequence*
$$0 \to S/T \to M/S \to M/T \to 0.$$

Lemma 1.14. *Suppose M is an R-module. Then*

$$M = \bigcup_{\substack{M' \subset M \\ M' \textit{ fin. gen.}}} M'.$$

We claim that

1. $M \otimes_R A = \bigcup_{\substack{M' \subset M \\ M' \textit{ fin. gen.}}} M' \otimes_R A.$
2. *If $\mathrm{id}_{M'} \otimes f\colon M' \otimes_R A \to M' \otimes_R B$ is injective for all finitely generated $M' \subset M$, then so is*
$$\mathrm{id}_M \otimes f\colon \bigcup_{\substack{M' \subset M \\ M' \textit{ fin. gen.}}} M' \otimes_R A \to \bigcup_{\substack{M' \subset M \\ M' \textit{ fin. gen.}}} M' \otimes_R B$$

Proof. (1) Define a diagram $F\colon I \to {}_R\mathrm{Mod}$ which has maps to all finitely generated submodules of M and all inclusions between them. Then by the universal property, the $\operatorname{colim} F = \bigcup_{\substack{M' \subset M \\ M' \text{ fin. gen.}}} M' = M$ with maps $i_{M'}\colon M' \to \operatorname{colim} F$ satisfying the commutativity of the inclusions. Since $- \otimes A$ is a left adjoint, we have

$$M \otimes_R A = L\left(\operatorname{colim} F\right) = \operatorname{colim}\left(L \circ F\right) = \bigcup_{\substack{M' \subset M \\ M' \text{ fin. gen.}}} M' \otimes_R A$$

the last equality again following from the universal property.

(2)

$\square$