ASSIGNMENT 5

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Problem 0.1. Let $F := \mathrm{hofib}\,(i\colon \mathbb{CP}^n \to \mathbb{CP}^\infty)$ be the homotopy fiber of the inclusion.

(1) Show that F is $2n$-conected.
There is a LES coming from the Puppe sequence

$$\ldots \to [X, \Omega\mathbb{CP}^\infty]_* \to [X, F]_* \to [X, \mathbb{CP}^n]_* \to [X, \mathbb{CP}^\infty]_*$$

Let X be a CW complex of dimension $\leq 2n$.

(2) Show that if $f\colon X \to \mathbb{CP}^n$ induces the 0-map on H^2, then f is nullhomotopic.
(3) Show that there is a counter-example when X is allowed to be $(2n+1)$-dimensional.

Solution. (1) The LES for the pair $(\mathbb{CP}^\infty, \mathbb{CP}^n)$ and the fibration $E_f \to B$ can be identified, so in particular, $\pi_k\,(F, \gamma_0) \cong \pi_{k+1}\,(\mathbb{CP}^\infty, \mathbb{CP}^n)$.

Recall now (p. 7, Hatcher) that $\mathbb{CP}^n$ has a cell structure of a single cell in every even degree up to $2n$: $\mathbb{CP}^n = e^0 \cup e^2 \cup \ldots \cup e^{2n}$. By definition, $\mathbb{CP}^\infty$ is defined as the sequential colimit over the sequence $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^{n+1}$: $\mathbb{CP}^\infty := \lim_{\leftarrow_n} \mathbb{CP}^n$. In paritcular (Hatcher, p.7), it also has a cell structure with one cell in each even dimension with $\mathbb{CP}^n$ constituting its $2n$-skeleton. From Corollary 4.12 in Hatcher, it now follows that $(\mathbb{CP}^\infty, \mathbb{CP}^n)$ is $(2n+1)$-connected, so F is $2n$-connected.

(2)
Firstly, F is $2n$-connected, so we can replace it using Proposition 4.15 in Hatcher by a CW complex with a single 0-cell and cells of dimension $> 2n$. Then by Cellular Approximation, $[X, F]_* = 0$, so $[X, \mathbb{CP}^n]_*$ injects into $[X, \mathbb{CP}^\infty]_*$. What is this injection?
Recall that the Puppe sequence given is obtained by applying $[X, -]_*$ to the Puppe sequence from p. 409 in Hatcher:

$$\ldots \to \Omega^2\mathbb{CP}^\infty \to \Omega F \to \Omega\mathbb{CP}^n \to \Omega\mathbb{CP}^\infty \to F \to \mathbb{CP}^n \xrightarrow{i} \mathbb{CP}^n$$

so the map $[X, \mathbb{CP}^n]_* \to [X, \mathbb{CP}^\infty]_*$ is given by postcomposing with the inclusion i, and we are told that this is injective. In particular, $[f]$ is zero if and only if $[i \circ f] \in [X, \mathbb{CP}^\infty]_*$ is zero.
But now since X is a CW complex, we have by Theorem 4.57 in Hatcher that $H^k(X;\mathbb{Z}) \cong [X, K(\mathbb{Z}, k)]$, so since $\mathbb{CP}^\infty$ is $K\,(\mathbb{Z}, 2)$, we have $H^2(X;\mathbb{Z}) \cong [X, \mathbb{CP}^\infty]$. Now, by theorem 4.57, the correspondence $[X, \mathbb{CP}^\infty]_* \xrightarrow{\cong} H^2(X;\mathbb{Z})$ takes $[g]$ to $g^*(\alpha)$ for some fundamental class $\alpha \in H^2\,(\mathbb{CP}^\infty;\mathbb{Z})$. But $(i \circ f)^*\,(\alpha) = f^*\,(i^*(\alpha)) = 0$ by assumption on f. But the 0 map in $[X, \mathbb{CP}^n]_*$ sending all of X to the basepoint clearly also induces the 0-map on H^2, so by the same reasoning, it is mapped to

0 in $H^2(X;\mathbb{Z})$ under this correspondence. Hence $[i\circ f]=[i\circ 0]$ by Theorem 4.57, and so by the injectivity from the Puppe sequence, $[f]=[0]$ in $[X,\mathbb{CP}^n]_*$, so f is based nullhomotopic, which was what we wanted to show.

(3) Consider the Hopf fibration $S^1\to S^{2n+1}\xrightarrow{f}\mathbb{CP}^n$. Since f induces a nontrivial isomorphism $\pi_k\left(S^{2n+1}\right)\xrightarrow{f_*}\pi_k\left(\mathbb{CP}^n\right)$ for $k>2$, f cannot be a nullhomotopic in $\left[S^{2n+1},\mathbb{CP}^n\right]_*$, however, it does induces the 0 map on H^2 since $H^2\left(S^{2n+1}\right)=0$.

Problem 0.2. Let Y be a locally compact CW complex. Use $[Y;\Omega Z]\cong\pi_1\operatorname{Map}(Y,Z)$ to define a group structure on $[Y,\Omega Z]$ and show that the part

$$\ldots\to[Y,\Omega F]_*\to[Y,\Omega E]_*\to[Y,\Omega B]_*$$

of the Puppe sequence can be made into a LES of groups. Using this and the previous exercise, show that $[X,\mathbb{CP}^n]\cong H^2(X;\mathbb{Z})$ for $X=\Sigma Y$ and Y an $(2n-1)$-dimensional CW complex.

Proof. I will assume that the formulation of the problem was meant for based spaces and based maps. In that case, given maps $[f],[g]\in[Y,\Omega Z]$, we can define $[f]+[g]$ as the $\varphi^{-1}\left(\varphi[f]\cdot\varphi[g]\right)$ where φ is the isomorphism $\varphi\colon\ [Y;\Omega Z]\cong\pi_1\operatorname{Map}(Y,Z)$, and $\cdot$ denotes concatenation of loops. This makes the sets into groups as they inherit the group structure from $\pi_1\operatorname{Map}(Y,Z)$. In general, $[A,B]$ is group with multiplication $(f\cdot g)(x)=f(x)\cdot g(x)$ whenever B is an H-group which loop spaces are (Bredon, Theorem 3.3, VII.(3)).
Then the sequence

$$[Y,\Omega F]_*\to[Y,\Omega E]_*\to[Y,\Omega B]_*$$

can be made into a LES of groups if we can show that the maps between these groups become group homomorphisms (exactness is already given). For this, note that the Puppe sequence in those degrees was already a LES of groups, and then taking $[Y,-]_*$ of this sequence just becomes postcomposition which distributes over the loop operation since it is simply concatenation, so the postcomposed function just distributes over the concatenation on $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$.

Now similarly to the previous problem, we then get $[X,\mathbb{CP}^n]\cong[\Sigma Y,\mathbb{CP}^n]\cong[Y,\Omega\mathbb{CP}^n]$, and also $[X,\mathbb{CP}^\infty]\cong H^2(X;\mathbb{Z})$, so if we can show that $[Y,F]_*=0$ and $[\Sigma Y,F]_*=0$, then we are done, as then $[X,\mathbb{CP}^n]\cong[X,\mathbb{CP}^\infty]\cong H^2(X;\mathbb{Z})$ by the LES from problem 1 (now as groups).
And $[Y,F]_*=0=[\Sigma Y,F]_*=0$ follows since Y is $(2n-1)$-dimensional hence ΣY is $(2n)$-dimensional, so by Cellular Approximation, and replacing F by a CW-complex with cells only of dimension greater than $2n$ along with the basepoint, we obtain that any map in $[Y,F]_*$ or $[\Sigma Y,F]_*$ is based nullhomotopic, giving the result. □