1.30:
(a) In $\mathbb{R}$, $x^2+y^2+1=0$ has no solutions, so $V\left(x^2+y^2+1\right)=\varnothing$ and hence $I\left(V\left(x^2+y^2+1\right)\right)=I\left(\varnothing\right)=k\left[x,y\right]=(1)$.

(b) Let V be an algebraic subset of $\mathbb{A}^2\left(\mathbb{R}\right)$. By corollary 2, we have that the algebraic sets are precisely $\mathbb{A}^2\left(\mathbb{R}\right),\varnothing$, points and irreducible plane curves $V(F)$ where F is an irreducible polynomial. Since $\mathbb{A}^2\left(\mathbb{R}\right)=V(0),\varnothing=V(1)$ and for any point $(a,b)\in\mathbb{A}^2(\mathbb{R})$, $(a,b)=V\left((x-a)^2+(y-b)^2\right)$. Thus for any collection of points $\{(a_1,b_1),\ldots,(a_n,b_n)\}$, it is the zero locus of

$$\left((x-a_1)^2+(y-b_1)^2\right)\cdot\left((x-a_2)^2+(y-b_2)^2\right)\cdots\left((x-a_n)^2+(y-b_n)^2\right)$$

2.14:
(b) Assume $V=V(F_1)$.
Now $V^T=V\left(I(V(F_1))^T\right)=V\left(F_1\circ T\right)=V\left(F_1(T_1,\ldots,T_n)\right)$. Write $F_1=\sum a_ix_i+a_0$. We can let $T_i=\frac{1}{a_i}x_i-\frac{a_0}{la_i}$ where l is the amount of a_i that are nonzero - if $a_i=0$, let $T_i=x_i$. Then $F_1\left(T_1,\ldots,T_n\right)=\sum_{a_i\neq 0}x_i$.

Practice for intersection multiplicities:

Consider the case $P=(0,0)$ and $f=(x^2+y^2)^3-4x^2y^2$ and $g=(x^2+y^2)^3+3x^2y-y^3$. Find $I_P(f,g)$.

Solution: We follow the algorithm:
(1) P is indeed $(0,0)$.
(2) f and g have no common factors which can be checked by computer.
(3) Indeed $(0,0)\in V(f)\cap V(g)$.
(4) We have

$$\begin{aligned}TC_PV(f)&=V(-4x^2y^2)=V(x)\cup V(y)\\TC_PV(g)&=V(3x^2y-y^3)=V(y)\cup V(\sqrt{3}x-y)\cup V\left(\sqrt{3}x+y\right).\end{aligned}$$

So they have the line $V(y)$ in common.
(5) Since the line is already $V(y)$, we find $f(x,0)=x^6$ and $g(x,0)=x^6$.

(6) As $r\neq 0$, we let $h=f-g=y^3-3x^2y-4x^2y^2=y\left(y^2-3x^2-4x^2y\right)$. Then $I_P(f,g)=I_P(g,h)=I_P(f,h)$. We will see that $I_P(f,h)$ is easier to compute.

We repeat from step 2:
(2) Again, h and g have no common divisors.
(3) P is still a common vanishing point.
(4) We now have

$$TC_PV(h)=V\left(y^3-3x^2y\right)=V(y)\cup V\left(y^2-3x^2\right)=V(y)\cup V(y-\sqrt{3}x)\cup V\left(y+\sqrt{3}x\right).$$

Again $V(y)$ is a common tangent cone line.
(5) We now have $h(x,0)=0$ and $g(x,0)=x^6$.
(6) We have

$$I_P(h,g)=I_P(y,g)+I_P(y^2-3x^2-4x^2y,g)$$

Now since $g=x^6+yB$, we find $I_P(y,g)=6$.
Now let $h_2=y^2-3x^2-4x^2y$.
Again (2) and (3) are satisfied, and

$$TC_PV(h_2)=V\left(y^2-3x^2\right)=V(y-\sqrt{3}x)\cup V(y+\sqrt{3}x).$$

Here we see the problem. These tangent cone lines are also tangent cone lines for g, however, they are not for f.
Now, does it matter that we chose f and not g?

No, we only care about the vanishings in each case which remain in both cases. (5) is still satisfied the same, so we find

$$I_P(h, g) = I_P(h, f) = 6 + I_P(y^2 - 3x^2 - 4x^2y, f) = 6 + mult_P(h_2)mult_P(f) = 6 + 2 \cdot 4 = 14$$

To show that single points are projective algebraic sets: We can write the point with last coordinate 1 if it does not live in $\mathbb{V}(z)$: $[a_1 : \ldots : a_n : 1] \in \mathbb{P}^n$.
The homogeneous ideal $(x_n - a_n x_{n+1}, x_{n-1} - a_{n-1}x_{n+1}, \ldots, x_1 - a_1 x_{n+1})$ vanishes at $[a_1 : \ldots : a_n : 1]$ precisely.
If last coordinate is 0, we can proceed similarly.

$$\begin{aligned} v_{2,2}^{-1}\left(\mathbb{V}\left(a_1x_1 + \ldots + a_6x_6\right)\right) &= \left\{[x : y : z] : a_1x^2 + a_2xy + \ldots + a_6 + z^2 = 0\right\} \\ &= \mathbb{V}\left(a_1x^2 + \ldots + a_6z^2\right) \end{aligned}$$