Definition

Definition

Let S be a surface and $\mathrm{Homeo}^+(S, \partial S)$ denote the group of orientation-preserving self-homeomorphisms of S which fix the boundary pointwise.
Equipping $\mathrm{Homeo}^+(S, \partial S)$ with the compact-open topology inherited from $C^0(S, S)$, we define

$$\mathrm{Mod}(S) := \pi_0(\mathrm{Homeo}^+(S, \partial S)).$$

Interpretation

Lemma

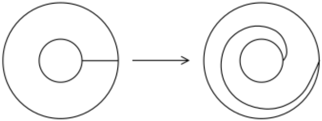
Let X, Y, Z be Hausdorff spaces with Y locally compact. Then a map $f\colon X \to C^0_W(Y, Z)$ is continuous if and only if $F\colon X \times Y \to Z$ defined by $F(x, y) = f(x)(y)$ is continuous.

Thus, a path $\gamma\colon I \to \mathrm{Mod}(S)$ is the same as a continuous map $I \times S \to S$ given by $(t, s) \mapsto \gamma(t)(s)$ which is an isotopy of S.

Definition of a Dehn Twist

Definition

Define the left twist map of the annulus $A = S^1 \times [1,2]$ as $T\colon A \to A$ given by $T(\theta, t) = (\theta + 2\pi t, t)$.



Definition

For an oriented surface S with a simple loop α in S, let N be a tubular neighborhood of α and choose an orientation-preserving homeomorphism $\varphi\colon A \to N$. We define a *Dehn twist about* α as

$$T_\alpha(x) = \begin{cases} \varphi \circ T \circ \varphi^{-1} & x \in N \\ x & x \in S - N \end{cases}$$

Braid Group Relations

Proposition

If a, b are isotopy classes of simple closed curves, then $i(a, b) = 0$ if and only if $T_a T_b = T_b T_a$. Furthermore, if $i(a, b) = 1$, then

$$T_a T_b T_a = T_b T_a T_b$$

.

Definition

The braid group on n strands, $\mathcal{B}_n$, is the free group on $n - 1$ generators σ_i with the above relations: $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for all i.

Geometric Representations

Definition

A geometric representation of a group G is a homomorphism $G \to \mathrm{Mod}(S)$ where S is a surface.

By the previous proposition on relations of Dehn-twists, we find that if we have a sequence of curves $\alpha_1, \ldots, \alpha_{n-1}$ such that $i(\alpha_i, \alpha_{i+1}) = 1$ for all i and $i(\alpha_i, \alpha_j) = 0$ whenever $|i-j| > 1$, then we obtain a well-defined homomorphism $\mathcal{B}_n \to \mathrm{Mod}(S)$ by sending $\sigma_i \mapsto T_{\alpha_i}$.

Yang-Baxter Operators

Definition

Let $T\colon \mathcal{A} \to \mathcal{V}$ be a functor from any category $\mathcal{A}$ to a monoidal category $\mathcal{V}$. A Yang-Baxter operator on T is a natural family of isomorphisms

$$y = y_{A,B}\colon TA \otimes TB \xrightarrow{\sim} TB \otimes TA$$

such that the following diagram commutes

$$\begin{tikzcd}
(TA \otimes TB) \otimes TC \arrow[r, "a"] \arrow[d, "y \otimes TC"'] & TA \otimes (TB \otimes TC) \arrow[r, "TA \otimes y"] & TA \otimes (TC \otimes TB) \arrow[r, "a^{-1}"] & (TA \otimes TC) \otimes TB \arrow[d, "y \otimes TB"] \\
(TB \otimes TA) \otimes TC \arrow[d, "a"'] & & & (TC \otimes TA) \otimes TB \arrow[d, "a"] \\
TB \otimes (TA \otimes TC) \arrow[d, "TB \otimes y"'] & & & TC \otimes (TA \otimes TB) \arrow[d, "TC \otimes y"] \\
TB \otimes (TC \otimes TA) \arrow[r, "a^{-1}"'] & (TB \otimes TC) \otimes TA \arrow[r, "y \otimes TA"'] & (TC \otimes TB) \otimes TA \arrow[r, "a"'] & TC \otimes (TB \otimes TA)
\end{tikzcd}$$

Yang-Baxter Elements

Remark

When $\mathcal{A} = \mathbb{1}$, we say that y is a Yang-Baxter operator on $X = T(\mathcal{A})$ if it is a Yang-Baxter operator on T. That is, $y \in \mathrm{Aut}_{\mathcal{V}}(X \otimes X)$.

If $T\colon \mathcal{A} \to \mathcal{V}$ is a functor with $\mathcal{V}$ braided monoidal, we obtain the Yang-Baxter operator

$$y_{A,B} = b_{TA,TB}\colon TA \otimes TB \xrightarrow{\sim} TB \otimes TA$$

We denote by z the Yang-Baxter operator obtained in this way from the inclusion functor $\iota\colon \mathbb{1} \to \mathcal{B}$, where $\mathcal{B}$ is the braid groupoid and the inclusion maps id to the single strand.

Obtaining representations from Yang-Baxter operators

Proposition

For any strict monoidal category $\mathcal{V}$ and any Yang-Baxter operator τ on an element $X \in \mathcal{V}$, there exists a unique strict monoidal functor $\Phi_{X,\tau} \colon \mathcal{B} \to \mathcal{V}$ such that $\Phi_{X,\tau} \circ z = y$.

Proof.

Since $\Phi_{X,\tau}$ is strict monoidal, this forces $\Phi_{X,\tau}(A) \otimes \Phi_{X,\tau}(B) = \Phi_{X,\tau}(A \otimes B)$ which forces $\Phi_{X,\tau}(n) = X^{\otimes n}$ and since $\sigma_i = 1 \otimes \ldots \otimes 1 \otimes \sigma_1 \otimes 1 \otimes \ldots \otimes 1$, it forces $\Phi_{X,\tau}(\sigma_i) = X \otimes \ldots \otimes X \otimes y \otimes X \otimes \ldots \otimes X =: y_i$. As y is a Yang-Baxter operator and $\mathcal{V}$ is strict, these satisfy the braid group relations. $\square$

Braided monoidal category of decorated surfaces

Definition

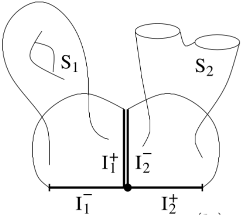
A decorated surface is a pair (S, I) where S is a compact connected surface with at least one boundary component, and $I\colon [-1, 1] \hookrightarrow \partial S$ is an interval in the boundary.

Let $\mathcal{M}_1$ be the groupoid where the objects are decorated surfaces and morphisms are isotopy classes of homeomorphisms restricting to the identity on a neighborhood of I and fixing the other boundaries.

Then $\mathrm{Aut}_{\mathcal{M}_1}(S) = \mathrm{Mod}(S)$.

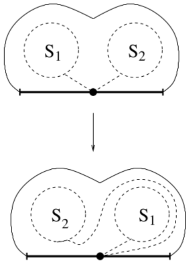
The Monoidal Structure

We equip it with the monoidal structure $(S_1, I_1) \otimes (S_2, I_2) := (S_1 \natural S_2, I_1 \natural I_2)$ where $I_1 \natural I_2 = I_1^- \cup I_2^+$ and $S_1 \natural S_2$ is obtained by gluing S_1 and S_2 along I_1^+ and I_2^-



The Braiding

The braiding is defined by a half twist of a pair-of-pants neighborhood of $\partial S_1 \cup \partial S_2$



By the proposition, we obtain a monoidal functor $\Phi\colon \mathcal{B} \to \mathcal{M}_1$ such that $\Phi \circ z = y$ where y is the Yang-Baxter operator on some decorated surface S which corresponds to the half-Dehn twist.

The main goal

Recall that a geometric representation of a group G is a group homomorphism $G \to \mathrm{Mod}(S)$ for some surface S.

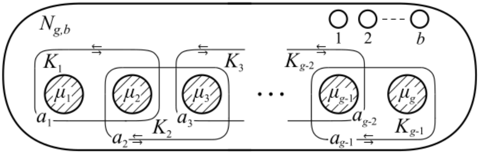
Theorem

Let $n \geq 14$ and $N = N_{g,b}$ with $g \leq 2\lfloor \frac{n}{2} \rfloor + 1$ and $b \geq 0$. Then any geometric representation $\mathcal{B}_n \to \mathrm{Mod}(N)$ is, up to transvection, either trivial, a standard twist representation or a crosscap transposition representation.

The goal is to find out whether these representations can be obtained from Yang-Baxter operators on an appropriately chosen category of surfaces.

The crosscap transposition representation

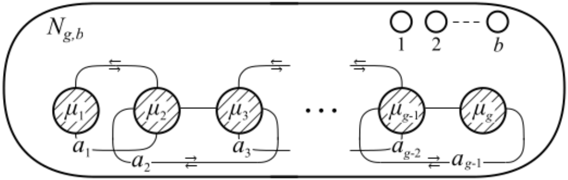
Given the setup in the figure with μ_i the core curve of the Möbius band $K_i - K_{i-1}$, we let u_i be the crosscap transposition supported in K_i which swaps μ_i and μ_{i+1} and $u_i^2 = T_{a_i}$ - i.e., a half-twist of the crosscaps.



The mapping $\theta_C \colon \mathcal{B}_n \to \mathrm{PMod}(N)$ sending $\sigma_i \mapsto u_i$ is called the crosscap transposition representation.
It is clear that this is the representation induced by the braiding from $\mathcal{M}_1$ on the Möbius band.

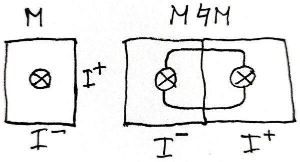
The standard twist representation

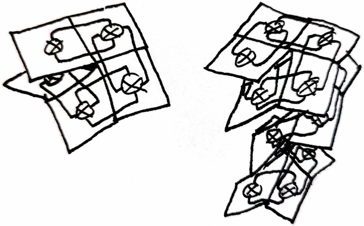
Take a chain of two-sided curves $C = (a_1, \ldots, a_{g-1})$ as depicted in the figure and fix an orientation of a regular neighborhood of their union. Then the map $\rho_C \colon \mathcal{B}_n \to \mathrm{PMod}(S)$ defined by $\rho_C(\sigma_i) = T_{a_i}$ is called the standard twist representation.



The Yang-Baxter Operator

We can obtain the standard twist representation as the Yang-Baxter element on the Möbius band M given by a Dehn twist about the curve denoted in the figure below for $M\natural M$.





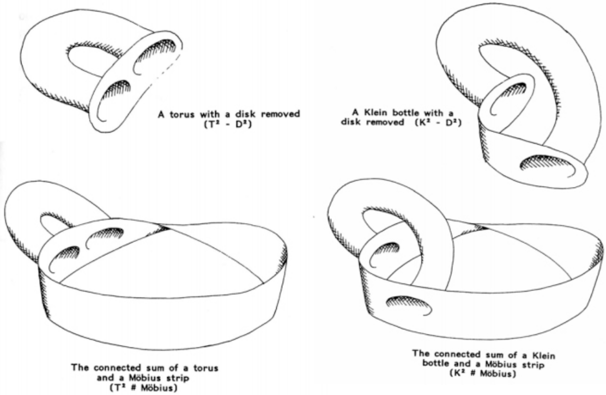
Recovering the Birman-Hilden embedding

Proposition

For $b \geq 1$ and g odd, the standard twist representation $\rho_C : \mathcal{B}_g \to \mathrm{Mod}(N_{g,b})$ is the same as the Birman-Hilden embedding $B_g \hookrightarrow S_{\frac{g-1}{2},b-1} \# M$ into the orientable factor.

Proof

We first note that $T^2 \# M \approx K \# M$

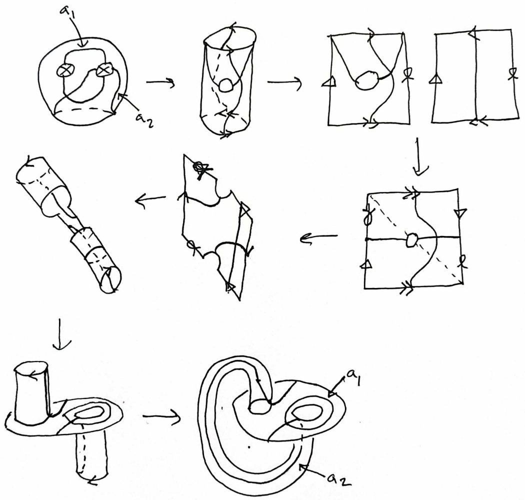


Proof

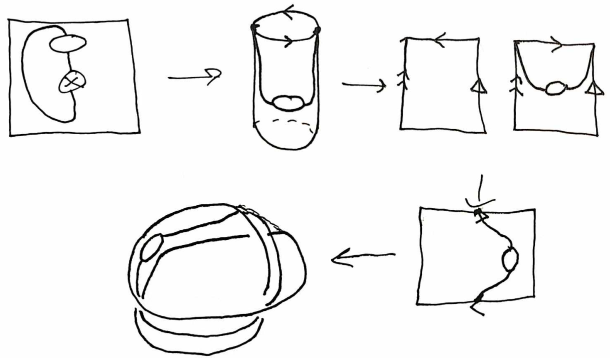
We will follow the loops through the following chain of homeomorphisms

$$\begin{aligned} N_{2n+1,1} \approx \left(\mathbb{RP}^2\right)^{\#2n+1} - \mathring{D} &\approx K^{\#n}\#M \approx K^{\#n-1}\#K\#M \\ &\approx K^{\#n-1}\#T^2\#M \\ &\vdots \\ &\approx \left(T^2\right)^{\#n}\#M \\ &\approx S_{n,0}\#M \\ &\approx S_{n,1}\natural M \end{aligned}$$

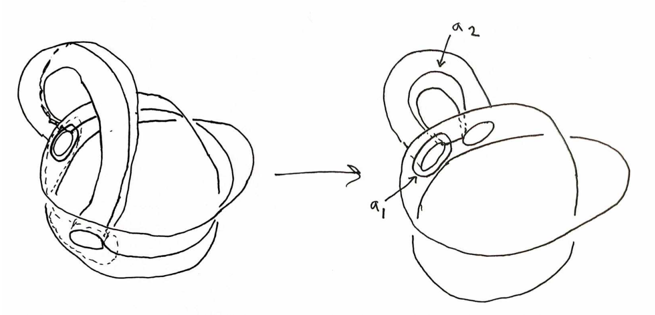
Proof



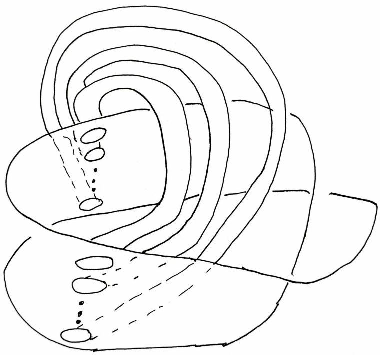
Proof



Proof



Proof



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