0.0.1. *Exercises.*

Exercise 0.1. Let $p\colon E \to B$ be a Serre (resp. Hurewicz) fibration. Given any map of spaces $f\colon B' \to B$, show that the projection $f^*E \to B'$ is a Serre (resp. Hurewicz) fibration, where

$$f^*(E) = B' \times_B E = \{(b', e) \mid f(b') = p(e)\}$$

is the pullback along f.

Proof. Consider the solid part of the diagram

$$\begin{tikzcd}
X \times \{0\} \arrow[r] \arrow[d] \arrow[rr, bend left] & f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
X \times I \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] \arrow[rr, bend right] & B' \arrow[r] & B
\end{tikzcd}$$

In the case where p is a Hurewicz fibration, X can be any space, while when p is a Serre fibration, it represents any disk D^n.
We then obtain the first dashed arrow $X \times I \to E$ because $E \xrightarrow{p} B$ is a Hurewicz/Serre fibration. But then we have maps $X \times I \to B'$ and $X \times I \to E$, so by the universal property of the pullback, this induces a unique map $X \times I \to f^*E$ in Top. $\square$

Exercise 0.2. Let G be a topological group and H a subgroup, and let G/H have the quotient topology from the projection $p\colon G \to G/H$ (here G/H is the space of cosets, not the space obtained by collapsing H to a point). Assume that there exists a nonempty open set $U \subset G/H$ such that $p\colon p^{-1}(U) \to U$ admits a section $s\colon U \to p^{-1}(U)$. Prove that $G \to G/H$ is a fiber bundle. Deduce that it is a fibration.

Proof. By assumption, $p \circ s = \mathrm{id}_U$. Now, picking some $x_0 \in p^{-1}(U)$, the set $V := x_0^{-1} \cdot p^{-1}(U)$ is a neighborhood of the identity $e \in G$. Let $y \in G/H$, and pick a $y_0 \in p^{-1}(y)$. Then $y_0 \cdot V$ is a neighborhood of y_0, hence $p(y_0 \cdot V)$ is a neighborhood of y (it is open since V was saturated with respect to p by construciton and multiplication by y_0 is a homeomorphism of G, so saturated sets remain saturated). Defining $s'\colon p(y_0 \cdot V) \to y_0 \cdot V$ by $s'(\overline{x}) = s \circ p\left(y_0^{-1} \cdot p^{-1}(\overline{x})\right)$. If $\overline{x} = \overline{z}$, then $z^{-1} \cdot x \in H$, so $\left(y_0^{-1} \cdot z\right)^{-1} \cdot \left(y_0^{-1} \cdot x\right) \in H$, hence $s'(\overline{x}) = s'(\overline{z})$. We claim that s' is then also a section of $p|_{y_0 V}\colon y_0 \cdot V \to p(y_0 \cdot V)$. To see this, we have

$$p \circ s' = p \circ s \circ p \circ \left(y_0^{-1} \cdot -\right) \circ p^{-1} = p \circ \left(y_0^{-1} \cdot -\right) \circ p^{-1}$$

Now if $\overline{x} = \overline{z}$ then again $z^{-1} \cdot x \in H$, so $\left(y_0^{-1} \cdot z\right)^{-1} \cdot \left(y_0^{-1} \cdot x\right) \in H$, from which the claim follows.

We claim that $p^{-1}(U)$ admits a trivialization $H \times U \cong p^{-1}(U)$ via the map $k\colon (h, u) \mapsto h \cdot s(u)$. Firstly, this is in $p^{-1}(U)$ since $p\left(h \cdot s(u)\right) = p\left(s(u)\right) = u \in U$. It is also continuous and injective as the composition $H \times U \overset{\mathrm{id} \times s}{\to} H \times p^{-1}(U) \overset{\mathrm{prod}}{\to} p^{-1}(U)$.
Furthermore, if $h \cdot v = h' \cdot v$ for some $v \in p^{-1}(U)$, then $h = h \cdot v \cdot v^{-1} = h' \cdot v \cdot v^{-1} = h'$, so the action of H on U is free.

Suppose $h \cdot s(U) \cap h' \cdot s(U) \neq \varnothing$, so for some $u, u' \in U$, $h \cdot s(u) = h' \cdot s(u')$. But then $u = p\left(h \cdot s(u)\right) = p\left(h' \cdot s(u')\right) = u'$, and so $h = h \cdot s(u) \cdot s(u)^{-1} = h' \cdot s(u) \cdot s(u)^{-1} = h'$. Hence $p^{-1}(U) = \sqcup_{h \in H} h \cdot s(U)$. Now define $r \colon p^{-1}(U) \to H \times U$ by $r(u) = \left(\sum_{h \in H} h \cdot \delta_{u \in h \cdot s(U)}, p(u)\right)$. Then

$$r \circ k(h,u) = r\left(h \cdot s(u)\right) = (h,u)$$

and

$$k \circ r\,(x) = k\,(h,u) = x$$

since (h,u) are by definition such that $p(x) = u$ and $x \in h \cdot s(U)$, so we must have $x = h \cdot s(u)$. Thus $k(h,u) = h \cdot s(u) = x$. So r is an inverse function to k. It remains to show that it is continuous. The coordinate $p(u)$ is continuous, so we must show that $r_1 \colon u \mapsto \sum_{h \in H} h \cdot \delta_{u \in h \cdot s(U)}$ is continuous. For this, note that for an open set $W \subset H$, $r_1^{-1}(W) = \bigcup_{h \in W} h \cdot s(U)$. Now, since $p\left(h \cdot s(u)\right) = p(s(u)) = u$, so $r_1^{-1}(W) \subset p^{-1}(U)$, and conversely, for any $x \in p^{-1}(U)$, there is an $h \in H$ such that $x \in h \cdot s(U)$, so $p^{-1}(U) \subset \bigcup_{h \in W} h \cdot s(U)$. Hence $r_1^{-1}(W) = p^{-1}(U)$ which is open.
This completes the proof that $G \to G/H$ is a fiber bundle.
Since any fiber bundle is a fibration, the last part follows directly.

□

Exercise 0.3. Recall that $S^3 \subset \mathbb{R}^4 \cong \mathbb{H}$ is a topological group, with $S^1 \subset \mathbb{C} \subset \mathbb{H}$ as a topological subgroup. Recall that

$$\mathbb{H} = \{a + bi + cj + dk \mid a,b,c,d \in \mathbb{R}\}.$$

so $S^3 \subset \mathbb{H}$ here is considered as the group whose elements are elements in $\mathbb{H}$ with norm 1, and $S^1 \subset S^3$ as the subgroup

$$\left\{a + bi \mid a^2 + b^2 = 1\right\}.$$

(1) Prove (using the previous exercise) that $S^3 \to S^3/S^1$ is a fiber bundle with fiber S^1, and therefore a fiber bundle.
(2) Prove that $S^3/S^1 \cong S^2$. The fiber sequence $S^1 \to S^3 \to S^2$ is called the *Hopf* fibration.
(3) Use the LES associated to this fibration to compute $\pi_3(S^2)$.
(4) Show that $S^3 \times K(\mathbb{Z},2)$ and S^2 have isomorphic homotopy groups. Are they homotopy equivalent?

Proof. (1) Let S^3_+ denote the open upper hemisphere. Then if $p \colon S^3 \to S^3/S^1$ is the quotient map, S^3/S^1 looks □

0.0.2. *Problems.*

Problem 0.4. Suppose $p\colon E \to B$ is a Serre fibration and $f\colon X \to B$ is n-connected. Prove that the projection $E \times_B X \to E$ is also n-connected.

Proof. We are given the following commutative diagram

$$\begin{tikzcd} E \times_B X \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

Firstly, by Exercise 1 on Problem set 4, the map $E \times_B X \to X$ is also a Serre fibration.
Secondly, by assumption in the conventions section for problem set 4, all spaces are assumed to be locally path-connected and connected, hence all spaces are path-connected. In particular, both X and B are assumed to be path-connected, so by Theorem 4.41 in Hatcher, we have a LES

$$\ldots \to \pi_k(F', y_0) \to \pi_k(E \times_B X, y_0) \overset{(\pi_X)_*}{\to} \pi_k(X, x_0) \to \pi_{k-1}(F', y_0) \to \ldots \to \pi_0\left(E \times_B X, y_0\right) \to 0$$

where $F' = (\pi_X)^{-1}(x_0)$ for some $x_0 \in X$ and $y_0 \in F'$. Now,

$$F' = (\pi_X)^{-1}(x_0) = \{(e, x_0) \mid f(x_0) = p(e)\} \xrightarrow[\cong]{\pi_E} p^{-1}(f(x_0)) =: F$$

where we choose F to be the fiber of $p\colon E \to B$ (when repeating Theorem 4.41 for this fibration), and we choose $e_0 \in F$ to be $\pi_E(y_0)$. With these choices of fibers and basepoints, we obtain that the map $\pi_E|_{F'}\colon F' \to F$ is a homeomorphism (it has the inverse $e \mapsto (e, x_0)$) by construction, so the following diagram commutes:

$$\begin{tikzcd} (F', y_0) \arrow[r, "\cong"] \arrow[d, hook] & (F, e_0) \arrow[d, hook] \\ (E \times_B X, y_0) \arrow[r, "\pi_E"] \arrow[d, "\pi_X"] & (E, e_0) \arrow[d, "p"] \\ (X, x_0) \arrow[r, "f"] & (B, f(x_0)) \end{tikzcd} \tag{Ω}$$

With these choices of basepoints, Theorem 4.41 gives the following long exact sequences (the solid part of the diagram)

$$\begin{tikzcd} \pi_{k+2}(X, x_0) \arrow[r] \arrow[d, dashed] & \pi_{k+1}(F', y_0) \arrow[r] \arrow[d, dashed] & \pi_{k+1}(E \times_B X, y_0) \arrow[r, "(\pi_X)_*"] \arrow[d, dashed] & \pi_{k+1}(X, x_0) \arrow[r] \arrow[d, dashed] & \pi_k(F', y_0) \arrow[d, dashed] \\ \pi_{k+2}(B, f(x_0)) \arrow[r] & \pi_{k+1}(F, e_0) \arrow[r] & \pi_{k+1}(E, e_0) \arrow[r, "p_*"] & \pi_{k+1}(B, f(x_0)) \arrow[r] & \pi_k(F, e_0) \end{tikzcd} \tag{Γ}$$

Now, applying π_{k+1} to (Ω), i.e., using functoriality of π_{k+1} on pointed topological spaces, we find that for $k+1 \geq 1$, we have

$$\begin{tikzcd}
\pi_{k+1}(F', y_0) \arrow[r, "\cong"] \arrow[d] & \pi_{k+1}(F, e_0) \arrow[d] \\
\pi_{k+1}(E \times_B X, y_0) \arrow[r, "(\pi_E)_*"] \arrow[d, "(\pi_X)_*"] & \pi_{k+1}(E, e_0) \arrow[d, "p_*"] \\
\pi_{k+1}(X, x_0) \arrow[r, "f_*", "\cong"'] & \pi_{k+1}(B, f(x_0))
\end{tikzcd} \tag{ζ}$$

commutes (since functoriality of π_{k+1} implies that compositions are preserved) - where also f_* is an isomorphism for $k < n-1$ and surjective for $k = n-1$.
We now claim that

$$\begin{tikzcd}
\pi_{k+1}(X, x_0) \arrow[r] \arrow[d, "f_*"] & \pi_k(F', y_0) \arrow[d, "(\pi_E|_{F'})_*", "\cong"'] \\
\pi_{k+1}(B, f(x_0)) \arrow[r] & \pi_k(F, e_0)
\end{tikzcd}$$

commutes. Consider the following diagram:

$$\begin{tikzcd}
\pi_{k+1}(E \times_B X, F') \arrow[r, "(\pi_X)_*", "\cong"'] \arrow[d, "(\pi_E)_*"] \arrow[rr, bend left, "\partial"] & \pi_{k+1}(X, x_0) \arrow[r] \arrow[d, "f_*"] & \pi_k(F', y_0) \arrow[d, "(\pi_E|_{F'})_*"] \\
\pi_{k+1}(E, F) \arrow[r, "p_*", "\cong"'] \arrow[rr, bend right, "\partial"'] & \pi_{k+1}(B, f(x_0)) \arrow[r] & \pi_k(F, e_0)
\end{tikzcd}$$

The outer triangle commutes by construction: since for $g\colon \left(D^{n+1}, S^n, s_0\right) \to (E \times_B X, F', y_0)$, we get

$$\partial\circ(\pi_E)_*([g]) = \partial\,[\pi_E \circ g] = [(\pi_E \circ g)\,|_{S^n}] = [\pi_E|_{F'} \circ g|_{S^n}] = (\pi_E|_{F'})_*\,([g|_{S^n}]) = (\pi_E|_{F'})_*\circ\partial\,([g])\,.$$

Also, the left hand square commutes for $k+1 \geq 1$, since this is what we obtained from (ζ). From this, we can conclude that the right hand square also commutes for $k+1 \geq 1$, i.e., for $k \geq 0$. Explicitly, if we let k be the map $\pi_{k+1}(X, x_0) \to \pi_k(F', y_0)$ and l the map $\pi_{k+1}(B, f(x_0)) \to \pi_k(F, e_0)$, then we get

$$\begin{aligned}
(\pi_E|_{F'})_* \circ j &= (\pi_E|_{F'})_* \circ \partial \circ (\pi_X)_*^{-1} \\
&= \partial \circ (\pi_E)_* \circ (\pi_X)_*^{-1} \\
&= \partial \circ p_*^{-1} \circ f_* \\
&= l \circ f_*
\end{aligned}$$

giving commutativity.

Therefore, we can fill in the dashed arrows in diagram (Γ), giving that the following diagram commutes for $0 \leq k < n-1$.

$$
\begin{tikzcd}
\pi_{k+2}(X,x_0) \arrow[r] \arrow[d, two heads, "f_*"] & \pi_{k+1}(F',y_0) \arrow[r] \arrow[d, "(\pi_E|_{F'})_*"', "\cong"] & \pi_{k+1}(E\times_B X,y_0) \arrow[r, "(\pi_X)_*"] \arrow[d, "(\pi_E)_*"] & \pi_{k+1}(X,x_0) \arrow[r] \arrow[d, "f_*"', "\cong"] & \pi_k(F',y_0) \arrow[d, "\cong"] \\
\pi_{k+2}(B,f(x_0)) \arrow[r] & \pi_{k+1}(F,e_0) \arrow[r] & \pi_{k+1}(E,e_0) \arrow[r, "p_*"] & \pi_{k+1}(B,f(x_0)) \arrow[r] & \pi_k(F,e_0)
\end{tikzcd}
$$

By the 5-lemma, we obtain that $(\pi_E)_*\colon \pi_{k+1}(E\times_B X,y_0)\to\pi_{k+1}(E,e_0)$ is an isomorphism for $1\le k+1\le n-1$. Note that this also works for $1=k+1$ despite π_0 not being a group (one can simply trace through the arguments in the proof of the 5-lemma and see that it still works).

It remains to show that it is an isomorphism on π_0 and surjective on π_n. Surjectivity on π_n immediately follows by applying the 4-lemma to the following diagram:

$$
\begin{tikzcd}
\pi_n(F',y_0) \arrow[r] \arrow[d, "(\pi_E|_{F'})_*"', "\cong"] & \pi_n(E\times_B X,y_0) \arrow[r, "(\pi_X)_*"] \arrow[d, "(\pi_E)_*"] & \pi_n(X,x_0) \arrow[r] \arrow[d, "f_*"', "\cong"] & \pi_{n-1}(F',y_0) \arrow[d, "\cong"] \\
\pi_n(F,e_0) \arrow[r] & \pi_n(E,e_0) \arrow[r, "p_*"] & \pi_n(B,f(x_0)) \arrow[r] & \pi_{n-1}(F,e_0)
\end{tikzcd}
$$

For the isomorphism on π_0, note that we have assumed that E is path-connected, so it suffices to show that $E\times_B X$ is also path-connected. Define a map $s\colon (E,e_0)\to(E\times_B X,y_0)$ by $s(e)=(e,x_0)$

$\square$