p. 91
1: Let C denote the unit circle in the plane. Suppose $f\colon C \to C$ is a map which is not homotopic to the identity. Prove that $f(x) = -x$ for some point x of C.

Solution: We prove the contrapositive.
Suppose $f(x) \neq -x$ for any point x of C. We must show that $f\colon C \to C$ is homotopic to the identity.

Since $f(x) \neq -x$ for all x, we can define the following map $F\colon C \times I \to C$ by

$$F(s,t) = \frac{s + t\,(f(s) - s)}{\|s + t\,(f(s) - s)\,\|}.$$

It is clear that $F(s,0) = s = \mathbb{1}(s)$ and $F(s,1) = f(s)$, so it remains to show that $F(s,t)$ is continuous and well-defined - i.e. that $\|s + t\,(f(s) - s)\,\| \neq 0$ for all $(s,t) \in C \times I$.
Now, if $s + t\,(f(s) - s) = s + tf(s) - ts = 0$ then $\frac{f(s)}{s} = \frac{t-1}{t}$, and since $f(s), s \in C$, $\|f(s)\| = 1 = \|s\|$, so $\|\frac{t-1}{t}\| = \|\frac{f(s)}{s}\| = \frac{\|f(s)\|}{\|s\|} = 1$. Thus since $t \in \mathbb{R}$, we have $\frac{t-1}{t} \in \{-1, 1\}$. Now $\frac{t-1}{t} = 1 \iff -1 = 0$, contradiction. Hence $\frac{t-1}{t} = -1$ which implies $2t - 1 = 0$ so $t = \frac{1}{2}$. Hence $0 = s + \frac{1}{2}\,(f(s) - s) = \frac{f(s)+s}{2}$ which implies $f(s) + s = 0$, so $f(s) = -s$ which we assumed not to be the case, contradiction. Hence $\|s + t\,(f(s) - s)\,\| \neq 0$ for all $(s,t) \in C \times I$.

Now, supposing $C \subset \mathbb{R}^2$, it suffices to note for continuity that we must simply check that the coordinate functions $\pi_i F \to \mathbb{R}, i = 1,2$ are continuous. But writing $s = (s_1, s_2)$ and $f(s) = (f_1(s), f_2(s))$, we find that the coordinate functions for $s + t\,(f(s) - s)$ is $s_i + t(f_i(s) - s_i)$ which is continuous as continuous functions are stable under sum, difference, product and composition. Since $s + t\,(f(s) - s) \neq 0$, we find that the quotient is continuous too, so F is continuous.

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13: Let G be a path-connected topological group. Given two loops α, β based at e in G, define a map $F\colon [0,1] \times [0,1] \to G$ by $F(s,t) = \alpha(s).\beta(t)$ where the dot denotes multiplication in G. Draw a diagram to show the effect of this map on the square, and prove that the fundamental group of G is abelian.

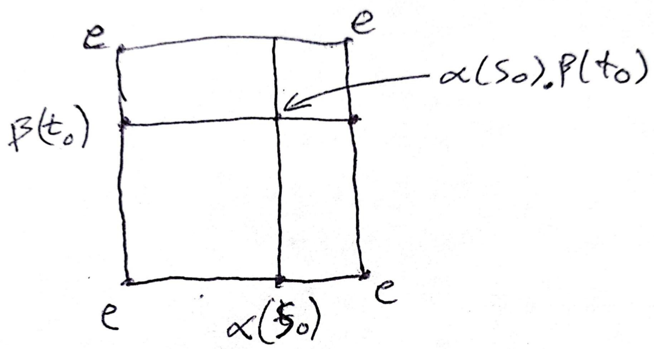
Solution: It suffices to show that $\pi(G,e)$ is abelian since G is path-connected.

Let $\langle\alpha\rangle, \langle\beta\rangle \in \pi(G,e)$. We want to show that $\langle\alpha\rangle * \langle\beta\rangle = \langle\beta\rangle * \langle\alpha\rangle$ where $*$ denotes the group operation of the fundamental group.

We must show that there is a homotopy $\alpha\beta \simeq \beta\alpha$, where $\alpha\beta$ and $\beta\alpha$ are the usual composition of the paths.
We have

$$\alpha\beta = (\alpha e).(e\beta) \simeq (e\alpha).(\beta e) = \beta\alpha$$

since $e\alpha \simeq \alpha e$ and $e\beta \simeq \beta e$. Thus the fundamental group of G is abelian.


e
e
α(s₀).β(t₀)
β(t₀)
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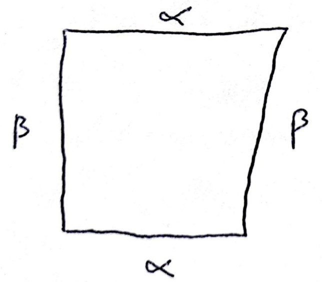
α
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α