ASSIGNMENT 5

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Exercise 0.1. Show that $\operatorname{Hom}(X \otimes Y, Z) \simeq \operatorname{Bil}(X, Y; Z)$ for all vector spaces. In particular, $(X \otimes Y)' \simeq \operatorname{Bil}(X, Y)$.

Solution. Suppose $B \in \operatorname{Bil}(X, Y; Z)$. By universality, there exists a unique linear map $L \colon X \otimes Y \to Z$ such that

$$\begin{tikzcd} X \times Y \arrow[r, "\otimes"] \arrow[dr, "M"'] & X \otimes Y \arrow[d, "L"] \\ & Z \end{tikzcd}$$

commutes, i.e., we can define a map $\operatorname{Bil}(X, Y; Z) \to \operatorname{Hom}(X \otimes Y, Z)$ by sending $M \mapsto L$.
Similarly, given some linear map $L \in \operatorname{Hom}(X \otimes Y, Z)$, we have

$$\begin{aligned} L \circ \otimes (a_1 x_1 + a_2 x_2, y) &= L\left((a_1 x_1 + a_2 x_2) \otimes y\right) = L\left(a_1 x_1 \otimes y + a_2 x_2 \otimes y\right) \\ &= a_1 L\left(x_1 \otimes y\right) + a_2 L\left(x_1 \otimes y\right) \end{aligned}$$

so $L \circ \otimes$ is linear in x, and by doing the same for y, $L \circ \otimes$ is seen to be linear in y as well. Hence $L \circ \otimes$ is bilinear, so each L induces a map M as well such that the diagram above commutes.
Furthermore, it is clear that if $L \mapsto M$, then $M \mapsto L$ and vice versa, so we, in fact, have a bijective correspondence between the classes $\operatorname{Hom}(X \otimes Y, Z)$ and $\operatorname{Bil}(X, Y; Z)$.

It remains to check that the maps $M \mapsto L$ and $L \mapsto M$ are linear.
Suppose we have a bilinear map $a_1 M_1 + a_2 M_2$. Let L_1 and L_2 be the linear maps corresponding to M_1 and M_2. Then

$$\begin{aligned} \left(a_1 M_1 + a_2 M_2\right)(x, y) &= a_1 M_1(x, y) + a_2 M_2(x, y) = a_1 L_1(x \otimes y) + a_2 L_2\left(x \otimes y\right) \\ &= \left(a_1 L_1 + a_2 L_2\right) \circ \otimes (x, y) \end{aligned}$$

so by uniqueness of the universal property, $a_1 M_1 + a_2 M_2$ maps to $a_1 L_1 + a_2 L_2$ which is indeed linear. Now, since the inverse map of a bijective linear map is also linear, we get that the other map is also linear. Since, as we remarked above, the composition of them is the identity, we find that $\operatorname{Hom}(X \otimes Y, Z) \simeq \operatorname{Bil}(X, Y; Z)$.

Choosing $Z = F$, we get $(X \otimes Y)' \simeq \operatorname{Bil}(X, Y)$.