

HOMOTOPY THEORY

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For these notes, we will follow [2], [1] and [3].

1. COFIBRATIONS

For this section, we will follow chapter VII.1 in [1].

One of the fundamental questions in topology is the "extension problem". Namely, given a map $g: A \rightarrow Y$ defined on a subspace A of X , when can we extend this map to all of X .

This cannot always be done - for example, as is the case with $A = Y = S^n$ and $X = D^{n+1}$ choosing the map to be any degree -1 map.

Question 1.1. Is the extension problem a *homotopy-theoretic* problem? That is, does the answer depend only on the homotopy class of g ?

The answer is: generally not. For example, we can take $X = [0, 1]$, $A = \{0\} \cup \{\frac{1}{n} \mid n = 1, 2, \dots\}$ and $Y = CA$, the cone on A . Choosing g to be the inclusion of A into Y , this cannot be extended to X as the extension would be discontinuous at $\{0\}$. However, $g \simeq g'$ with g' being the constant map of A to the vertex of the cone, and g' easily extends to X by the constant map.

It turns out, however, that under some very mild conditions on the spaces, the problem becomes homotopy theoretic. We will now discuss this.

Definition 1.2 (Homotopy extension property). Let (X, A) and Y be given spaces. Then (X, A) is said to have the *homotopy extension property* with respect to Y if the following diagram can always be completed to be commutative.

$$\begin{array}{ccc} A \times I \cup X \times \{0\} & \longrightarrow & Y \\ \downarrow & \nearrow \text{dashed} & \\ X \times I & & \end{array}$$

One can also depict this by the following diagram:

$$\begin{array}{ccccc} A \times \{0\} & \hookrightarrow & A \times I & & \\ \downarrow & & \swarrow & \searrow & \\ & & Y & & \\ \downarrow & \nearrow & \nwarrow & \searrow & \\ X \times \{0\} & \longrightarrow & X \times I & & \end{array}$$

If (X, A) has the homotopy extension property with respect to Y , then the extensibility of maps $g: A \rightarrow Y$ depends only on the homotopy class of g . For suppose $H: g \simeq g'$ and g' can be extended to $\tilde{g}': X \rightarrow Y$, then define the map $A \times I \cup X \times \{0\}$ by $\tilde{g}' \times \{0\}$ on $X \times \{0\}$ and H on $A \times I$. The homotopy extension property for the pair (X, A) then guarantees the existence of a map $G: X \times I \rightarrow Y$ which equals g on $A \times \{1\}$, so $H(-, 1): X \rightarrow Y$ extends g .

Definition 1.3 (Cofibration). Let $f: A \rightarrow X$ be a map. Then f is called a *cofibration* if one can always fill in the following commutative diagram given the solid arrows:

$$\begin{array}{ccc}
 A \times \{0\} & \xrightarrow{\quad} & A \times I \\
 \downarrow f \times \text{id} & \nearrow & \downarrow f \times \text{id} \\
 & Y & \\
 X \times \{0\} & \xrightarrow{\quad} & X \times I
 \end{array}$$

(Note: In the original image, there is a dashed arrow from $X \times \{0\}$ to Y and a solid arrow from $A \times I$ to Y . The diagram above represents the solid arrows and the dashed arrow.)

for any space Y .

Note. If f is an inclusion, then this is the same as the homotopy extension property for all Y . That attribute is sometimes referred to as the *absolute homotopy extension property*.

Theorem 1.4. For an inclusion $A \subset X$, the following are equivalent:

- (1) The inclusion map $A \hookrightarrow X$ is a cofibration.
- (2) $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.

Proof. If the inclusion is a cofibration, then choosing $Y = A \times I \cup X \times \{0\}$ with all arrows being inclusions in the diagram of a cofibration, we obtain a map $X \times I \rightarrow A \times I \cup X \times \{0\}$ which is the identity on $A \times I \cup X \times \{0\}$.

Conversely, if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$, then we can always complete the diagram by mapping $X \times I \rightarrow A \times I \cup X \times \{0\} \rightarrow Y$ where the second map takes the maps $A \times I \rightarrow Y$ and $X \times \{0\} \rightarrow Y$ from the diagram. \square

Corollary 1.5. If A is a subcomplex of a CW-complex X , then the inclusion $A \hookrightarrow X$ is a cofibration.

Proof. We want to construct a retraction $X \times I \rightarrow A \times I \cup X \times \{0\}$. We will do so by constructing a retraction $((A \cup X^{(r)}) \times I) \cup (X \times \{0\}) \rightarrow (A \times I) \cup (X \times \{0\})$ by induction on r . If it has been defined on the $(r-1)$ -skeleton, then extending it over an r -cell is simply a matter of extending a map on $S^{r-1} \times I \cup D^r \times \{0\}$ over $D^r \times I$ which can be done since the pair $(D^r \times I, S^{r-1} \times I \cup D^r \times \{0\})$ is homeomorphic to $(D^r \times I, D^r \times \{0\})$. See Figure 1

These maps for each cell fit together to give a map on the r -skeleton because of the weak topology on $X \times I$. The union of these maps for all r gives a map on $X \times I$, again because of the weak topology on $X \times I$. \square

Theorem 1.6. Assume that $A \subset X$ is closed and that there exists a neighborhood U of A and a map $\varphi: X \rightarrow I$ such that

- (1) $A = \varphi^{-1}(0)$.
- (2) $\varphi(X - U) = \{1\}$.

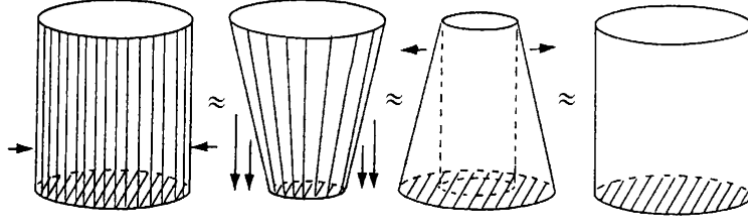


FIGURE 1. A homeomorphism of pairs.

- (3) U deforms to A through X with A fixed. That is, there is a map $H: U \times I \rightarrow X$ such that $H(a, t) = a$ for all $a \in A$, $H(u, 0) = 0$, and $H(u, 1) \in A$ for all $u \in U$.

Then the inclusion $A \hookrightarrow X$ is a cofibration. The converse also holds.

Proof. We may assume that $\varphi = 1$ on a neighborhood of $X - U$ by replacing φ with $\min(2\varphi, 1)$. It suffices to show that there exists a retract $\Phi: U \times I \rightarrow X \times \{0\} \cup A \times I$ since then the map

$$r(x, t) = \begin{cases} \Phi(x, t(1 - \varphi(x))), & x \in U \\ (x, 0), & x \notin U \end{cases}$$

gives a retraction $X \times I \rightarrow A \times I \cup X \times \{0\}$.

We define Φ by

$$\Phi(u, t) = \begin{cases} H\left(u, \frac{t}{\varphi(u)}\right) \times \{0\}, & \varphi(u) > t \\ H(u, 1) \times \{t - \varphi(u)\}, & \varphi(u) \leq t. \end{cases}$$

The only thing that needs checking here is that Φ is continuous at points $(u, 0)$ such that $\varphi(u) = 0$, i.e., points $(a, 0)$ for $a \in A$ - indeed here the expression for $\varphi(u) > t$ is not defined.

Recall that a map $f: X \rightarrow Y$ is continuous if for every point $x \in X$ and any neighborhood U of $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$. So let W be a neighborhood of $a = H(a, t)$. Then there exists a neighborhood $V \subset W$ containing a such that $H(V \times I) \subset W$, by assumption of H being continuous. So for $t < \varepsilon$ for some ε and $u \in V$, we have $\Phi(u, t) \in W \times [0, \varepsilon]$. Hence Φ is continuous.

To prove the converse, suppose that the inclusion $A \hookrightarrow X$ is a cofibration. Equivalently, $A \times I \cup X \times \{0\}$ is a retract of $X \times I$. Let $r: X \times I \rightarrow A \times I \cup X \times \{0\}$ be this retraction. Let $s(x) = r(x, 1)$ and set $U = s^{-1}(A \times (0, 1])$. Let p_X, p_I be the projections of $X \times I$ to its factors. Then put $H = p_X \circ r|_{U \times I}: U \times I \rightarrow X$. Now, $H(a, t) = p_X \circ r|_{U \times I}(a, t) = p_X(a, t) = a$ for all $a \in A$ and $t \in I$; $H(u, 0) = p_X \circ r|_{U \times I}(u, 0) = p_X(u, 0) = u$, and $H(u, 1) = p_X \circ r|_{U \times I}(u, 1) = u$ forces $(u, 1) \in A \times I$, hence $u \in A$. Thus, H satisfies condition (3).

For (1) and (2), let $\varphi(x) = \max_{t \in I} |t - p_I r(x, t)|$ which is possible since I is compact. Then $x \in \varphi^{-1}(0)$ implies that $\max_{t \in I} |t - p_I r(x, t)| = 0$, so for all $t \in I$, we have $|t - p_I r(x, t)| = 0$, so $r(x, t) \in A \times \{t\}$ for all $t \in (0, 1]$. Then $r(x, 0) = \lim_{n \rightarrow \infty} r(x, \frac{1}{n}) \in A \times I$ since $A \times I$ is closed. But $(x, 0) = r(x, 0)$, so $x \in A$. Conversely, for any $x \in A$, clearly, $\varphi(x) = 0$ since $r(x, t) = (x, t)$ for all

$t \in I$. This shows that φ satisfies (1). For (2), we have that for $x \in X - U$, with $U = s^{-1}(A \times (0, 1])$, we have $r(x, 1) = s(x) \notin A \times (0, 1]$, so $r(x, 1) \in X \times \{0\}$. Hence $\varphi(x) = \max_{t \in I} |t - p_I r(x, t)| = 1$, giving (2).

It remains to show that φ is continuous. Let $f(x, t) = |t - p_I r(x, t)|$ and $f_t = (x, t)$ all of which are continuous. Then

$$\varphi^{-1}((-\infty, b]) = \{x \mid f(x, t) \leq b \text{ for all } t\} = \bigcap_{i \in I} f_i^{-1}((-\infty, b]).$$

is an intersection of closed sets and so is closed. Similarly,

$$\varphi^{-1}([a, \infty)) = \{x \mid f(x, t) \geq a \text{ for some } t\} = p_X(f^{-1}([a, \infty)))$$

which is also closed since p_X is closed as a projection and I is compact. Since the complements of the intervals of the form $[a, \infty)$ and $(-\infty, b]$ give a subbase for the topology of \mathbb{R} , this shows that φ is continuous. \square

Next, we recall that for a map $f: X \rightarrow Y$, the mapping cylinder M_f is defined as

$$M_f = ((X \times I) \sqcup Y) / ((x, 0) \sim f(x)).$$

Consider the inclusion $\iota: X \hookrightarrow M_f$ where we include X as $X \times \{1\}$. Consider the map $\varphi: M_f \rightarrow I$ given by $\varphi(x, t) = 1 - 2t$ for $t \geq \frac{1}{2}$ and $\varphi(x, t) = 1$ on the rest of M_f . Choosing $U = X \times (\frac{1}{3}, 1]$, U clearly deformation retracts to $X \times \{1\}$ and satisfies the conditions of Theorem 1.6, hence the inclusion $X \hookrightarrow M_f$ is a cofibration. Also, the retraction $r: M_f \rightarrow Y$ is a homotopy equivalence with the homotopy inverse being the inclusion $Y \hookrightarrow M_f$. The diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota} & M_f \\ & \searrow f & \swarrow \simeq \\ & & Y \end{array}$$

commutes. Thus any map f is a cofibration up to a homotopy equivalence of spaces. Recall also that the mapping cone of a map $f: X \rightarrow Y$ is defined as

$$C_f := M_f / X \times \{1\} \cong M_f \cup CX.$$

In the case of an inclusion $\iota: A \hookrightarrow X$, we have $C_\iota = X \cup CA$.

There is a map $C_\iota \xrightarrow{h} X/A$, defined as the composite of the quotient map $X \cup CA \rightarrow X \cup CA / CA$ composed with the inverse of the homeomorphism $X/A \rightarrow X \cup CA / CA$.

Question 1.7. Is h a homotopy equivalence?

Theorem 1.8. *If $A \subset X$ is closed and the inclusion $\iota: A \rightarrow X$ is a cofibration, then $h: C_\iota \rightarrow X/A$ is a homotopy equivalence. In fact, it is a homotopy equivalence of pairs*

$$(X/A, *) \simeq (C_\iota, CA) \simeq (C_\iota, v),$$

where v is the vertex of the cone.

Proof. The mapping cone $C_\iota = X \cup CA$ consists of three different types of points: the vertex $v = \{A \times \{1\}\}$, the rest of the cone $\{(a, t) \mid 0 \leq t < 1\}$ where $(a, 0) = a \in A \subset X$, and points in X itself, which we identify with $X \times \{0\}$.

Define $f: A \times I \cup X \times \{0\} \rightarrow C_\iota$ as the collapsing map and extend f to $\bar{f}: X \times I \rightarrow C_\iota$

using that f is a cofibration. Then $\bar{f}(a, 1) = v$, $\bar{f}(a, t) = (a, t)$ and $\bar{f}(x, 0) = x$. Let $\bar{f}_t = \bar{f}|_{X \times \{t\}}$. Since $\bar{f}_1(A) = \{v\}$, we can factorize $\bar{f}_1: X \rightarrow C_i$ as $g \circ j$ where $j: X \rightarrow X/A$ is the quotient map and $g: X/A \rightarrow C_i$ is the induced map

$$\begin{array}{ccc} X & & \\ \downarrow j & \searrow \bar{f}_1 & \\ X/A & \xrightarrow{g} & C_i. \end{array}$$

where g is induced and continuous by definition of the quotient topology.

We claim that g is a homotopy equivalence with homotopy inverse h . First, we prove that $hg \simeq \text{id}_{X/A}$.

Note that taking the composite $h\bar{f}_t: X \rightarrow X/A$ gives a homotopy between $h\bar{f}_0$ and $h\bar{f}_1$. For all t , this homotopy takes A to the point $\{A\}$. Thus, it factors to give a homotopy

$$hgj = h\bar{f}_1 \simeq h\bar{f}_0 = j$$

Let $H: X \times I \rightarrow X/A$ be the homotopy between hgj and j , so $H(x, 0) = hgj(x)$ and $H(x, 1) = j(x)$. Then the map $\bar{H}: X/A \times I \rightarrow X/A$ defined by $\bar{H}([x], t) = H(x, t)$ defines a homotopy between hg and $\text{id}_{X/A}$, so $hg \simeq \text{id}_{X/A}$.

Next, we will show that $gh \simeq \text{id}_{C_i}$. Consider $W = (X \times I) / (A \times \{1\})$ and the maps illustrated in Figure 2.

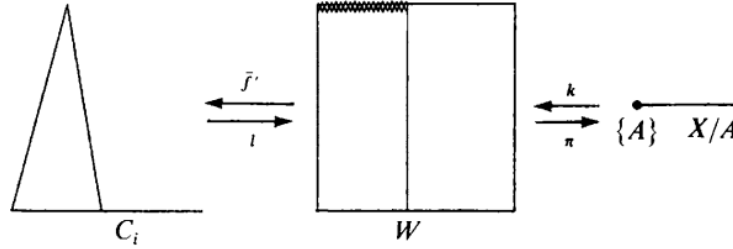


FIGURE 2. DWIJJXNXJNXJUI.png

The map \bar{f}' is induced by \bar{f} . The map k is the "top face" map. From this, we see that

$$\begin{aligned} \bar{f}' \circ l &= \text{id} \\ \pi \circ k &= \text{id} \\ k \circ \pi &\simeq \text{id} \\ \bar{f}' \circ k &= g \\ \pi \circ l &= l. \end{aligned}$$

Hence $gh = \bar{f}'k\pi l \simeq \bar{f}'l = \text{id}$. \square

Example 1.9 (A non example). An example of when the result of Theorem 1.6 does not hold is with $A = \{0\} \cup \{\frac{1}{n} \mid n = 1, 2, \dots\}$ and $X = [0, 1]$. In this case, C_i is not homotopy equivalent to X/A which is a one-point union of a countably infinite sequence of circles with radii going to zero.

C_i has homeomorphs of circles joined along edges. However, the circles do not tend to a point, so any prospective homotopy equivalence $X/A \rightarrow C_i$ would be discontinuous at the image of $\{0\}$ in X/A .

Corollary 1.10. *If $A \subset X$ is closed and the inclusion $A \hookrightarrow X$ is a cofibration, then the map $j: (X, A) \rightarrow (X/A, *)$ induces isomorphisms*

$$H_*(X, A) \xrightarrow{\cong} H_*(X/A, *) \cong \tilde{H}_*(X/A)$$

and

$$\tilde{H}^*(X/A) \cong H^*(X/A, *) \xrightarrow{\cong} H^*(X, A).$$

Proof. We have $H_*(X/A, *) \cong H_*(C_i, CA)$ by Theorem 1.8. And since $C_i = X \cup A \times [0, \frac{1}{2}]$ and $CA = A \times [0, \frac{1}{2}]$, where we collapse $A \times \{\frac{1}{2}\}$ in both, and attach $A \times [0, \frac{1}{2}]$ along $A \times \{0\}$ in $X \cup A \times [0, \frac{1}{2}]$, we obtain

$$H_*(C_i, CA) \cong H_*\left(X \cup A \times \left[0, \frac{1}{2}\right], A \times \left[0, \frac{1}{2}\right]\right) \cong H_*(X, A)$$

since $(X \cup A \times [0, \frac{1}{2}], A \times [0, \frac{1}{2}]) \simeq (X, A)$ by deformation retracting $A \times [0, \frac{1}{2}]$ down to $A \times \{0\} \subset X$. \square

1.0.1. *Interlude on pointed-spaces and operations on spaces.* We recall some important constructions:

Definition 1.11 (Unreduced Suspension). For a space X , the *unreduced suspension* ΣX is the quotient obtained from $X \times I$ by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

Note. We have $\Sigma S^n = S^{n+1}$.

Definition 1.12 (Suspension of a map). Given a map $f: X \rightarrow Y$, we can suspend f to $\Sigma f: \Sigma X \rightarrow \Sigma Y$ by letting Σf be the induced map on the quotients:

$$\begin{array}{ccc} X \times I & \xrightarrow{f \times \text{id}} & Y \times I \\ \downarrow & & \downarrow \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \end{array}$$

Exercise 1.13. For any homology theory, show that there is a natural isomorphism $\tilde{H}_I(X) \xrightarrow{\cong} \tilde{H}_{i+1}(\Sigma X)$. Here, natural means that for a map $f: X \rightarrow Y$, and its suspension $\Sigma f: \Sigma X \rightarrow \Sigma Y$, the following diagram commutes:

$$\begin{array}{ccc} \tilde{H}_i(X) & \xrightarrow{\cong} & \tilde{H}_{i+1}(\Sigma X) \\ \downarrow f_* & & \downarrow (\Sigma f)_* \\ \tilde{H}_i(Y) & \xrightarrow{\cong} & \tilde{H}_{i+1}(\Sigma Y) \end{array}$$

Definition 1.14 (Wedge Sum/one-point union). Given two pointed spaces $(X, x_0), (Y, y_0)$, we define the *wedge sum* $X \vee Y$ to be

$$X \vee Y = X \sqcup Y / (x_0 \sim y_0),$$

i.e., the quotient of the disjoint union identifying x_0 and y_0 to a single point.

Definition 1.15 (Smash Product). Inside the product $X \times Y$ of two pointed space $(X, x_0), (Y, y_0)$, we have natural copies of X and Y by $X \times \{y_0\}$ and $\{x_0\} \times Y$, respectively. These two copies intersect only at the point (x_0, y_0) , so their union can be identified with the wedge sum $X \vee Y$. I.e., $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$. We define the *smash product* $X \wedge Y$ to be the quotient $X \times Y / X \vee Y$.

If $f: X \rightarrow Y$ is a pointed map, then the reduced mapping cylinder of f is defined as the quotient space M_f of $(X \times I) \cup Y$ modulo the relations identifying $(x, 0) \sim f(x)$ and the set $\{*\} \times I$ to the base point of M_f .

The reduced mapping cone is the quotient of the reduced mapping cylinder M_f obtained by identifying the image of $X \times \{1\}$ to a point, the base point.

The circle S^1 is defined as $I/\partial I$ with base point $\{\partial I\}$.

The reduced suspension of a pointed space X is $SX = X \wedge S^1$. It can also be considered as the quotient space $X \times I / (X \times \partial I \cup \{*\} \times I)$

Definition 1.16 (Well-pointed space). A base point $x_0 \in X$ is said to be *nondegenerate* if the inclusion $\{x_0\} \hookrightarrow X$ is a cofibration. A pointed Hausdorff space X with nondegenerate base point is said to be *well-pointed*.

It is clear that any manifold or CW-complex satisfies Theorem 1.6 with A being any point of the space. Hence any manifold or CW-complex is well-pointed.

Example 1.17 (Pointed space that is not well-pointed). Taking the pointed space $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ with base point 0, this space is not well-pointed. This can for example be seen because it fails to satisfy Theorem 1.4 - any retraction would break continuity at $(0, 1)$.

Example 1.18. If $A \hookrightarrow X$ is a cofibration, then X/A with base point $\{A\}$ is well-pointed, as follows from Theorem 1.6.

Theorem 1.19. If X is well-pointed, then so are the reduced cone CX and the reduced suspension SX . Moreover, the collapsing map $\Sigma X \rightarrow SX$, of the unreduced suspension to the reduced suspension, is a homotopy equivalence.

Proof. Denote the base point of X by $*$. Consider the homeomorphism

$$h: (I \times I, I \times \{0\} \cup \partial I \times I) \xrightarrow{\cong} (I \times I, I \times \{0\})$$

which clearly exists. For example, take Figure 3



FIGURE 3.

Then the induced homeomorphism

$$\text{id}_X \times h: X \times I \times I \xrightarrow{\cong} X \times I \times I$$

carries $X \times I \times \{0\} \cup X \times \partial I \times I$ to $X \times I \times \{0\}$. Hence it takes $A = X \times I \times \{0\} \cup X \times \partial I \times I \cup \{*\} \times I \times I$ to $X \times I \times \{0\} \cup \{*\} \times I \times I$. Therefore, the pair

$(X \times I \times I, A)$ is homeomorphic to the pair $I \times (X \times I, X \times \{0\} \cup \{*\} \times I)$. Now, X is well-pointed, so $X \times \{0\} \cup \{*\} \times I$ is a retract of $X \times I$ by Theorem 1.4 and the definition of well-pointed. It follows that A is a retract of $X \times I \times I$. By another application of 1.4, then the inclusion $X \times \partial I \cup \{*\} \times I \hookrightarrow X \times I$ is a cofibration. Hence the quotient by this, $SX = X \times I / (X \times \partial I \cup \{*\} \times I)$ is well-pointed, using the quotient of the above inclusion.

Next consider the homeomorphism $(I \times I, I \times \{0\} \cup \{1\} \times I) \xrightarrow{\cong} (I \times I, I \times \{0\})$ which can be seen similarly. The induced homeomorphism

$$1 \times h: X \times I \times I \xrightarrow{\cong} X \times I \times I$$

takes $A := X \times \{1\} \times I \cup \{*\} \times I \times I \cup X \times I \times \{0\}$ to $X \times I \times \{0\} \cup \{*\} \times I \times I$. Thus the pair $(X \times I \times I, A)$ is homeomorphic to $I \times (X \times I, X \times \{0\} \cup \{*\} \times I)$. Just as above, we have that $X \times \{0\} \cup \{*\} \times I$ is a retract of $X \times I$, so it follows that A is a retract of $X \times I \times I$. Thus the inclusion $X \times \{1\} \cup \{*\} \times I \hookrightarrow X \times I$ is a cofibration, which shows that $CX = X \times I / (X \times \{1\} \cup \{*\} \times I)$ is well-pointed.

The fact that $X \times \partial I \cup \{*\} \times I \hookrightarrow X \times I$ is a cofibration gives that there exists a neighborhood U of $X \times \partial I \cup \{*\} \times I$ and a map $\varphi: X \times I \rightarrow I$ that satisfy Theorem 1.6. We obtain an induced map $\bar{\varphi}: \Sigma X \rightarrow I$ which satisfies the same conditions, so $I \times \times \{*\} \times I \hookrightarrow X \times I / \{X \times \{0\}, X \times \{1\}\} = \Sigma X$ is a cofibration. Now Theorem 1.8 implies that $\Sigma X \cup CI = C_i \rightarrow \Sigma X / I$ is a homotopy equivalence. Hence we obtain that $\Sigma X \simeq \Sigma X \cup CI \simeq \Sigma X / I = SX$, via the collapsing map. \square

Problem 1.20. Find $H_*(\mathbb{P}^2, \mathbb{P}^1)$ using methods or results from this section.

Solution. Consider \mathbb{P}^2 as S^2 quotiented by the relation $x \simeq -x$. Then we can think of \mathbb{P}^1 as $S^1 \subset S^2$ under this relation. We want to show that the inclusion $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ is a cofibration. Using Theorem 1.6, it suffices to find a neighborhood U of $\mathbb{P}^1 \subset \mathbb{P}^2$ and a map $\bar{\varphi}: \mathbb{P}^2 \rightarrow I$ such that the conditions of the theorem are satisfied. We construct a preliminary map on S^2 towards this end. Define $\varphi: S^2 \rightarrow I$ to be $\varphi(x) = \min\{1, 2|x_3|\}$, where x_3 is the last coordinate of x . Since $\varphi(x) = \varphi(-x)$, φ induces a map $\bar{\varphi}: \mathbb{P}^2 \rightarrow I$ such that the diagram

$$\begin{array}{ccc} S^2 & & \\ \downarrow & \searrow \varphi & \\ \mathbb{P}^2 & \xrightarrow{\bar{\varphi}} & I \end{array}$$

commutes. Letting U be the image under the quotient map of $\{x \in S^2 \mid |x_3| < \frac{1}{2}\}$, this becomes an open set in \mathbb{P}^2 since the above set is saturated with respect to the quotient map. It is also clear that U and $\bar{\varphi}$ satisfy the conditions of the theorem, hence the inclusion $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ is a cofibration. By Corollary 1.10, we obtain that $H_*(\mathbb{P}^2, \mathbb{P}^1) \cong \tilde{H}_*(\mathbb{P}^2/\mathbb{P}^1)$. But $\mathbb{P}^2/\mathbb{P}^1 \cong S^2$, so $H_*(\mathbb{P}^2, \mathbb{P}^1) \cong \tilde{H}_*(S^2)$. Now simply recall that

$$\tilde{H}_p(S^2) \cong \begin{cases} \mathbb{Z}, & p = 2 \\ 0, & p \neq 2. \end{cases}$$

\square

Problem 1.21. Find $H_*(T^2, \{*\} \times S^1 \cup S^1 \times \{*\})$ using methods from this section.

Solution. If we can show that the inclusion $A := \{*\} \times S^1 \cup S^1 \times \{*\} \hookrightarrow T^2$ is a cofibration, then we will again obtain that $H_*(T^2, A) \cong \tilde{H}_*(T^2/A) \cong \tilde{H}_*(S^2)$. But we have a CW-structure on the torus given by the square with identified sides. With this identification, A simply becomes the 1-skeleton, hence it is a subcomplex, so by Corollary 1.5, the inclusion $A \hookrightarrow T^2$ is a cofibration. This finishes the solution. \square

Problem 1.22. For a space X , consider the pair (CX, X) . What do the results of this section tell you about the homology of these, and related, spaces?

Solution. We can define a map $\varphi: CX \rightarrow I$ by $\varphi(x, t) = t$. Choosing $A = X = X \times \{0\} \subset CX$ and $U = CX - \{v\}$ where v is the vertex, this satisfies the conditions in Theorem 1.6 (H can be defined by $H((x, t_0), t) = (x, t_0)(1 - t) + (x, 0)t$). Hence the inclusion $X \hookrightarrow CX$ is a cofibration, so we know that $H_*(CX, X) \cong \tilde{H}_*(CX/X)$. Similarly, one can show that the inclusion $X \hookrightarrow \Sigma X$ is a cofibration, so $H_*(\Sigma X, X) \cong \tilde{H}_*(\Sigma X/X) \cong \tilde{H}_*(\Sigma X \vee \Sigma X)$ and $H_*(SX, X) \cong \tilde{H}_*(SX \vee SX)$.

2. HOMOTOPY GROUPS

2.1. Homotopy. We follow chapter 14 of [1] for this subsection.

To start of, we recall the basic definitions of homotopies.

Definition 2.1 (Homotopy). Two maps $f_0, f_1: X \rightarrow Y$ are said to be *homotopic* if there exists a homotopy $F: X \times I \rightarrow Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$.

Definition 2.2 (Homotopy equivalence). A map $f: X \rightarrow Y$ is said to be a *homotopy equivalence* if it is an isomorphism in \mathbf{hTop} .

Lemma 2.3 (Reparametrization Lemma). Let φ_1, φ_2 be maps $(I, \partial I) \rightarrow (I, \partial I)$ which are equal on ∂I . Let $F: X \times I \rightarrow Y$ be a homotopy and let $G_i(x, t) = F(x, \varphi_i(t))$ for $i = 1, 2$. Then $G_1 \simeq G_2 \text{ rel } X \times \partial I$.

We shall use c to denote the constant homotopy.

Proposition 2.4. $F * c \simeq F \text{ rel } X \times \partial I$ and $c * F \simeq F \text{ rel } X \times \partial I$.

Definition 2.5. If $F: X \times I \rightarrow Y$ is a homotopy, then we define $F^{-1}: X \times I \rightarrow Y$ by $F^{-1}(x, t) = F(x, 1 - t)$.

Note that F^{-1} is precisely the inverse to F in \mathbf{hTop} .

Proposition 2.6. For any homotopies F, G, H for which the concatenations are defined, we have

$$(F * G) * H \simeq F * (G * H) \text{ rel } X \times \partial I.$$

Proposition 2.7. For homotopies F_1, F_2, G_1, G_2 , if $F_1 \simeq F_2 \text{ rel } X \times \partial I$ and $G_1 \simeq G_2 \text{ rel } X \times \partial I$, then $F_1 * G_1 \simeq F_2 * G_2 \text{ rel } X \times \partial I$.

Note that all of the discussion of concatenation of homotopies goes through with no difficulties for the cases in which all homotopies are relative to some subspace $A \subset X$ or are homotopies of pairs $(X, A) \rightarrow (Y, B)$.

It follows that homotopy between maps of pairs $(X, A) \rightarrow (Y, B)$ is an equivalence relation. The set of homotopy classes of these maps is commonly denoted by $[X, A; Y, B]$ or just $[X; Y]$ if $A = \emptyset$.

Theorem 2.8. If $f_0 \simeq f_1: X \rightarrow Y$ then $M_{f_0} \simeq M_{f_1} \text{ rel } X + Y$ and $C_{f_0} \simeq C_{f_1} \text{ rel } Y + \text{vertex}$.

To show this, one needs the following basic topological proposition:

Proposition 2.9. If $f: X \rightarrow Y$ is a quotient map and K is locally compact Hausdorff, then $f \times 1: X \times K \rightarrow Y \times K$ is a quotient map.

Proof of Theorem 2.8. First, let $F: X \times I \rightarrow Y$ be the homotopy between f_0 and f_1 . Now define $h: M_{f_0} \rightarrow M_{f_1}$ by $h(y) = y$ for $y \in Y$ and

$$h(x, t) = \begin{cases} F(x, 2t), & t \leq \frac{1}{2} \\ (x, 2t - 1), & \frac{1}{2} \leq t. \end{cases}$$

Define $k: M_{f_1} \rightarrow M_{f_0}$ likewise by the identity on Y and

$$k(x, t) = \begin{cases} F^{-1}(x, 2t), & t \leq \frac{1}{2} \\ (x, 2t - 1), & \frac{1}{2} \leq t. \end{cases}$$

Then the composition $kh: M_{f_0} \rightarrow M_{f_1}$ is the identity on Y and $F * (F^{-1} * E)$ on the cylinder portion, where $E: X \times I \rightarrow M_{f_0}$ is induced by the identity on $X \times I \rightarrow X \times I$. This is homotopic to the identity $\text{rel } X \times \{1\} + Y$. Similarly for hk . It now remains to check the continuity of this homotopy. We have a homotopy $M_{f_0} \times I \rightarrow M_{f_0}$. We now claim that $M_{f_0} \times I \cong M_{f_0 \times I}$. Indeed then, using that $M_{f_0 \times I} = \frac{X \times I \times I \sqcup Y \times I}{((x,0,k) \sim (f_0(x),k))}$, it suffices to show continuity of the composition $X \times I \times I \sqcup Y \times I \rightarrow M_{f_0} \times I \rightarrow M_{f_0}$. For on $Y \times I$, it is the constant homotopy and on $X \times I \times I$ it is $F * (F^{-1} * E) \simeq E \text{ rel } X \times \partial I$. Now, that $M_{f_0} \times I \cong M_{f_0 \times I}$ follows from Proposition 2.9. \square

Let $f: X \rightarrow Y$. If $\varphi: Y \rightarrow Y'$ is a map, then there is the induced map $F: M_f \rightarrow M_{\varphi \circ f}$ induced from φ on Y and the identity on $X \times I$.

Theorem 2.10. *If $\varphi: Y \rightarrow Y'$ is a homotopy equivalence then so is $F: (M_f, X) \rightarrow (M_{\varphi \circ f}, X)$ and hence so is $F: C_f \rightarrow C_{\varphi \circ f}$.*

Proof. Let $\psi: Y' \rightarrow Y$ be a homotopy inverse of φ and let $G: M_{\varphi \circ f} \rightarrow M_{\psi \circ \varphi \circ f}$ be the map induced by ψ on Y' and the identity on $X \times I$. The composition $GF: M_f \rightarrow M_{\psi \circ \varphi \circ f}$ is induced from $\psi \circ \varphi: Y \rightarrow Y$ and the identity on $X \times I$. Let $H: Y \times I \rightarrow Y$ be a homotopy from id to $\psi \circ \varphi$; i.e., $H(y, 0) = y$ and $H(y, 1) = \psi(\varphi(y))$. By the proof of Theorem 2.8, there is a homotopy equivalence $h: M_f \rightarrow M_{\psi \circ \varphi \circ f} \text{ rel } X$ given by $h(y) = y$ and

$$h(x, t) = \begin{cases} H(f(x), 2t), & t \leq \frac{1}{2} \\ (x, 2t - 1), & t \geq \frac{1}{2} \end{cases}.$$

We claim that $h \simeq GF \text{ rel } X$. Indeed, the homotopy H can be extended to $M_f \times I \rightarrow M_{\psi \circ \varphi \circ f}$ by putting

$$H((x, s), t) = \begin{cases} H(f(x), 2s + t), & 2s + t \leq 1 \\ \left(x, \frac{2s+t-1}{t+1}\right), & 2s + t \geq 1 \end{cases}.$$

Then $H(-, 0) = h$ and $H(-, 1) = GF$, so since GF is a homotopy equivalence, so is h . Define $F': M_{\psi \circ \varphi \circ f} \rightarrow M_{\varphi \circ \psi \circ \varphi \circ f}$ as the induced map on mapping cones with φ on Y and the identity on $X \times I$. Then similarly, $F'G$ is a homotopy equivalence. If k is a homotopy inverse of GF then $GFk \simeq \text{id}$. If k' is a homotopy inverse of $F'G$ then $k'F'G \simeq \text{id}$. Thus G has a right and left homotopy inverse: $R = Fk$ and $L = k'F'$. Then $R = \text{id} \circ R \simeq (LG)R = L(GR) \simeq L \circ \text{id} = L$, so $R \simeq L$. That is, G has a homotopy inverse. Therefore, G is a homotopy equivalence. Since G and GF are homotopy equivalences, so is F . \square

Problem 2.11. [1, Ex 14.1] Let $S^2 \cup A$ denote the union of the unit 2-sphere and the line segment joining the north and south poles. Show that $S^2 \vee S^1 \simeq S^2 \cup A$.

Proof. Define two maps $f_0, f_1: \{0, 1\} \rightarrow S^2$ where $f_0(t) = (\cos(2\pi t), \sin(2\pi t), 0)$ and f_1 is the constant map at $(1, 0, 0)$. Then $f_0 \simeq f_1$, so $C_{f_0} \simeq C_{f_1}$. Now, $C_{f_0} = S^2 \cup A$ while $C_{f_1} = S^2 \vee S^1$. \square

Problem 2.12. [1, Ex 14.2] Show that the union of a 2-sphere and a flat unit 2-cell through the origin is homotopically equivalent to the one-point union of two 2-spheres.

Proof. A 2-cell is contractible, an a 2-sphere with a 2-cell inside it is precisely the cone of the map $S^1 \sqcup S^1 \rightarrow S^1$ with the identity on both. By [1, Thm 14.19], this is homotopy equivalent to the cone on $S^1 \sqcup S^1 \rightarrow \{*\}$ which is $S^2 \vee S^2$. \square

Problem 2.13. Show that the union of a standard 2-torus with two disks, one spanning a latitudinal circle and the other spanning a longitudinal circle of the torus, is homotopically equivalent to a 2-sphere.

Proof. Using the identification of the torus as the quotient space of I^2 in the usual way, we can choose on spanning circle to be a 2-cell attached along $\{0\} \times I$ and the other to be a 2-cell attached along $I \times \{0\}$. These are contractible, and the quotient space becomes a 2-sphere. \square

2.2. Homotopy Groups. Recall that $[X, A; Y, B]$ denotes the set of homotopy classes of maps $X \rightarrow Y$ carrying A into B such that A goes into B during the entire homotopy.

To make a group then, we can select a point $y_0 \in Y$ and consider the set

$$[X \times I, X \times \partial I; Y, \{y_0\}]$$

In this case, the operation of concatenation of homotopies makes this set into a group. It is technically also better to choose a basepoint $x_0 \in X$ and consider

$$[X \times I, \{x_0\} \times I \cup X \times \partial I; Y, \{y_0\}].$$

For the moment, let us set $A = \{x_0\} \times I \cup X \times \partial I$. Then maps $X \times I \rightarrow Y$ which carry A into $\{y_0\}$ are in bijective correspondence with maps $(X \times I)/A \rightarrow Y$ which take the point $\{A\}$ into $\{y_0\}$.

Definition 2.14 (Reduced Suspension). We define the *reduced suspension* of X to be

$$SX = (X \times I)/A = (X \times I) / (\{x_0\} \times I \cup X \times \partial I)$$

The set of homotopy classes of pointed maps of a pointed space X to a pointed space Y with homotopies preserving the base points will be denoted by $[X; Y]_*$.

Thus $[SX; Y]_*$ is in canonical bijective correspondence with $[X \times I, A; Y, \{y_0\}]$.

Now, suppose we have pointed maps $f, g: SX \rightarrow Y$. Then they induce homotopies $f', g': X \times I \rightarrow Y$ by precomposing with the quotient map $X \times I \rightarrow SX$. We can then define $f' * g': X \times I \rightarrow Y$ as usual. The resulting pointed map $SX \rightarrow Y$ will be denoted $f * g$. Geometrically, $f * g$ is obtained by putting f on the bottom and g on the top of the one-point union $SX \vee SX$ and composing the resulting map $SX \vee SX \rightarrow Y$ with the map $SX \rightarrow SX \vee SX$ obtained by collapsing the middle parameter value $\frac{1}{2}$ copy of X in SX to the base point.

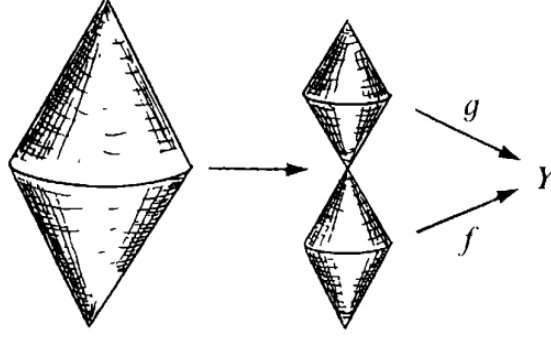
For a map $f: (SX, \{A\}) \rightarrow (Y, \{y_0\})$, we denote its homotopy class in $[SX; Y]_*$ by $[f]$, and we define

$$[f][g] = [f * g]$$

Under this operation, the set $[SX; Y]_*$ becomes a group.

Proposition 2.15. *The reduced suspension gives $SS^{n-1} \cong S^n$.*

Thus, we can define S^n as the n -fold reduced suspension of S^0 . As a special case, the set $[S^n; Y]_*$ then becomes a group for $n > 0$.

FIGURE 4. The product of two map classes $SX \rightarrow Y$.

Definition 2.16 (n th homotopy group). We define

$$\pi_n(Y, y_0) = [S^n; Y]_*$$

with this operation.

2.2.1. *A different way of defining $\pi_n(Y, y_0)$.* Note that reduced suspension supplies a parameter in $[0, 1]$ and the space S^n as constructed is the quotient space of I^n obtained by collapsing the boundary of the cube to a point. Pointed maps $S^n \rightarrow Y$ are in bijective correspondence with maps $I^n \rightarrow Y$ taking ∂I^n to the base point of Y . This is a more traditional way of defining $\pi_n(Y)$. This becomes the group of homotopy classes of maps $(I^n, \partial I^n) \rightarrow (Y, \{y_0\})$ with the operation being

$$f * g(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n), & t_1 \in [0, \frac{1}{2}] \\ g(2t_1 - 1, t_2, \dots, t_n), & t_1 \in [\frac{1}{2}, 1] \end{cases}.$$

Proposition 2.17. For $n \geq 2$, $\pi_n(X, x_0)$ is abelian.

Proof. Consider the homotopy in Figure 5. We begin by shrinking the domains of f and g to smaller subcubes of I^n , where the region outside is mapped to the basepoint. This allows us to move the boxes around in a continuous manner. The rest is clear. \square

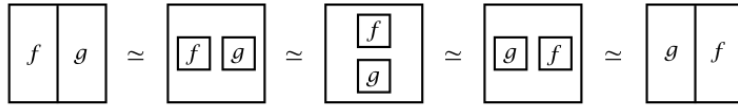


FIGURE 5. The homotopy in question

Next, we want to show that following:

Proposition 2.18. If X is path-connected, then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for any two $x_0, x_1 \in X$.

For this, we introduce an action of π_1 on π_n .

Definition 2.19 (The action of π_1 on π_n). Given a path $\gamma: I \rightarrow X$ from x_0 to x_1 , we associate to a map $f: (I^n, \partial I^n) \rightarrow (X, x_1)$ the map $\gamma f: (I^n, \partial I^n) \rightarrow (X, x_0)$ by shrinking the domain of f to a smaller concentric cube in I^n , then inserting the path γ on each radial segment in the shell between this smaller cube and ∂I^n . See Figure 6

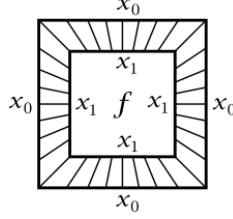


FIGURE 6. Depiction of γf .

Note. We have the following properties

- (1) $\gamma(f + g) \simeq \gamma f + \gamma g$.
- (2) $(\gamma\eta)f \simeq \gamma(\eta f)$.
- (3) $\text{id}f \simeq f$, where id denotes the constant path.

To see (1), first deform f and g to be constant on the right and left halves of I^n , respectively, producing maps which we may call $f + 0$ and $0 + g$, then we can excise a progressively wider symmetric middle slab of $\gamma(f + 0) + \gamma(0 + g)$ (which can be seen on the left in Figure 7) until it becomes $\gamma(f + g)$ (shown on the right).

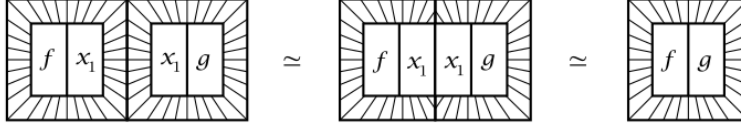


FIGURE 7.

Now if $\beta_\gamma: \pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$ is the change-of-basepoint transformation, $\beta_\gamma[f] = [\gamma f]$, then the above note shows that β_γ is a group isomorphism. This proves Proposition 2.18. If we restrict attention to loops γ at x_0 , then since $\beta_{\gamma\eta} = \beta_\gamma\beta_\eta$, the map $[\gamma] \mapsto \beta_\gamma$ defines a homomorphism from $\pi_1(X, x_0)$ to $\text{Aut}(\pi_n(X, x_0))$ called the *action of π_1 on π_n* .

Note. For $n > 1$, this action makes $\pi_n(X, x_0)$ into a module over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$.

Definition 2.20 (Simple/abelian spaces). A space with trivial π_1 action on π_n is called 'n-simple', and 'simple' means 'n-simple for all n'. We call a space *abelian* if it has trivial action of π_1 on all homotopy groups π_n .

Proposition 2.21 (π_n is a functor). A map $\varphi: (X, x_0) \rightarrow (Y, y_0)$ induces a map $\varphi_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0)$ defined by $\varphi_*[f] = [\varphi f]$. It is immediate from the definitions that φ_* is well-defined and a homomorphism for $n \geq 1$. The functorial properties are also clear.

Corollary 2.22. *Homotopy equivalent spaces have isomorphic homotopy groups.*

Proposition 2.23. *A covering space projection $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ induces isomorphisms $p_*: \pi_n(\tilde{X}, \tilde{x}_0) \rightarrow \pi_n(X, x_0)$ for all $n \geq 2$.*

Proof. Since S^n is path-connected and locally path-connected, and simply connected for $n \geq 2$, we find that any map $(S^n, s_0) \rightarrow (X, x_0)$ lifts to a map $(S^n, s_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ when $n \geq 2$. This gives surjectivity of p_* . For injectivity, suppose $p_*[f] = [0]$ where $f: (S^n, s_0) \rightarrow (\tilde{X}, \tilde{x}_0)$. Let $c_{\tilde{x}_0}$ be the constant map at \tilde{x}_0 . Then $p_*[c_{\tilde{x}_0}] = [0]$, so by uniqueness of the lifting theorem, $[f] = [c_{\tilde{x}_0}] = [0]$. \square

Definition 2.24 (Aspherical). Spaces with $\pi_n = 0$ for all $n \geq 2$ are called *aspherical*.

Corollary 2.25. S^1, T^n and K are aspherical since they have contractible covering spaces.

Proposition 2.26.

$$\pi_n\left(\prod_{\alpha} X_{\alpha}\right) \cong \prod_{\alpha} \pi_n(X_{\alpha})$$

Next we define relative homotopy groups.

Definition 2.27 (Relative homotopy groups). Regard I^{n-1} as a face of I^n with the last coordinate $s_n = 0$ and let J^{n-1} be the closure of $\partial I^n - I^{n-1}$. Then we define

$$\pi_n(X, A, x_0) := [I^n, \partial I^n, J^{n-1}; X, A, x_0]$$

We shall leave $\pi_0(X, A, x_0)$ undefined for now.

We can define a sum operation on $\pi_n(X, A, x_0)$ in the same way as for $\pi_n(X, x_0)$, except now the coordinate s_n now must remain free, so we must use one of the other coordinates. Thus we must have at least one other coordinate to define the same operation. So $\pi_n(X, A, x_0)$ is a group for $n \geq 2$, and it is abelian for $n \geq 3$. For $n = 1$, we have $I^1 = [0, 1]$, $I^0 = \{0\}$ and $J^0 = \{1\}$, so $\pi_1(X, A, x_0) = [I, \{0\}, \{1\}; X, A, x_0]$ is the set of homotopy classes of paths in X from a varying point in A to the fixed basepoint $x_0 \in A$. In general, this is not a group in any natural way.

Now, we saw before that $\pi_n(X, x_0)$ can be regarded as homotopy classes of maps $(S^n, x_0) \rightarrow (X, x_0)$. Similarly, collapsing J^{n-1} to a point, converts $(I^n, \partial I^n, J^{n-1})$ to (D^n, S^{n-1}, s_0) . In this case, addition is done by the map $c: D^n \rightarrow D^n \vee D^n$ collapsing $D^{n-1} \subset D^n$ to a point.

Theorem 2.28 (Compression criterion). *A map $f: (D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ if and only if it is homotopic rel S^{n-1} to a map with image contained in A .*

Proof. Suppose we have a homotopy $\text{rel } S^{n-1}$ from f to a map g , so $[f] = [g]$ in $\pi_n(X, A, x_0)$. Viewing g as a map $(D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$ whose image is contained in A , we can construct the homotopy $H: D^n \times I \rightarrow X$ by $H(x, t) = g((1-t)x + s_0 t)$ which is a homotopy from g to the constant map at x_0 , hence $[g] = 0$ in $\pi_n(X, A, x_0)$.

Conversely, if $[f] = 0$ via a homotopy $F: D^n \times I \rightarrow X$ such that $F(x, 0) = f(x)$ and $F(x, 1) = x_0$ for all $x \in D^n$ and $F(x, t) \in A$ for all x with $|x| = 1$ as well as $F(s_0, t) = x_0$ for all t . We can construct a homotopy using F by restricting F to a family of n -disks in $D^n \times I$ starting with $D^n \times \{0\}$ and ending with the disk $D^n \times \{1\} \cup S^{n-1} \times I$, and where all the disks throughout the family have the same boundary. See Figure 8 for a depiction of this homotopy.

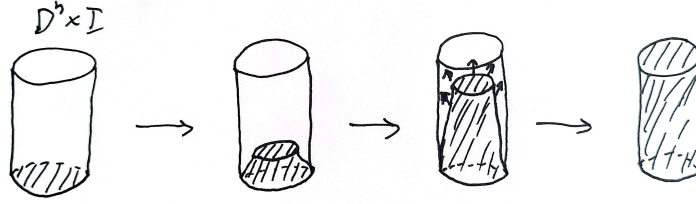


FIGURE 8.

This completes the proof. \square

Next, some things that carry over: a map $\varphi: (X, A, x_0) \rightarrow (Y, B, y_0)$ induces maps $\varphi_*: \pi_n(X, A, x_0) \rightarrow \pi_n(Y, B, y_0)$ which are homomorphisms when $n \geq 2$ and have properties analogous to those in the absolute case: $(\varphi\psi)_* = \varphi_*\psi_*$, $(\text{id}_{(X,A,x_0)})_* = \text{id}_{\pi_n(X,A,x_0)}$, and if $\varphi \simeq \psi$ through maps $(X, A, x_0) \rightarrow (Y, B, y_0)$, then $\varphi_* = \psi_*$.

2.2.2. LES of relative homotopy groups. Probably the most useful feature of relative homotopy groups $\pi_n(X, A, x_0)$ is that they fit into a long exact sequence

$$\dots \rightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \rightarrow \dots \rightarrow \pi_0(X, x_0).$$

Here i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ to I^{n-1} (the face of I^n with the last coordinate $s_n = 0$), or equivalently, by restricting maps $(D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$ to S^{n-1} . The map ∂ , called the *boundary map*, is a homomorphism when $n > 1$. In fact, we can show the following theorem

Theorem 2.29 (LES of relative homotopy groups). *Given $x_0 \in B \subset A \subset X$, the sequence of relative homotopy groups*

$$\dots \rightarrow \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \rightarrow \dots \rightarrow \pi_1(X, A, x_0)$$

is exact and natural. In the case when $B = \{x_0\}$, we have that the LES

$$\dots \rightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \rightarrow \dots \rightarrow \pi_0(X, x_0).$$

is exact and natural.

Proof. Exactness at $\pi_n(X, B, x_0)$: the composition $j_* i_*$ is zero because any map $(I^n, \partial I^n, J^{n-1}) \rightarrow (A, B, x_0)$ is zero in $\pi_n(X, A, x_0)$ by the compression criterion (Theorem 2.28). To see that $\ker j_* \subset \operatorname{im} i_*$, let $f: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, B, x_0)$ represent zero in $\pi_n(X, A, x_0)$. Using the compression criterion again, we then get that f is homotopic rel ∂I^n to a map with image in A , hence the class $[f] \in \pi_n(X, B, x_0)$ is indeed in the image of i_* . We conclude that $\ker j_* = \operatorname{im} i_*$, obtaining exactness at $\pi_n(X, B, x_0)$.

Exactness at $\pi_n(X, A, x_0)$: for a map $[f] \in \operatorname{im} j_*$, we have that j_* maps ∂I^n into B , hence in particular $I^{n-1} \subset \partial I^n$ into B , so $\partial j_* [f]$ represents a homotopy class in $\pi_{n-1}(A, B, x_0)$ with image in B , but then by the compression criterion, $\partial j_* [f] = 0$ in $\pi_{n-1}(A, B, x_0)$, so $\operatorname{im} j_* \subset \ker \partial$. Conversely, suppose $\partial [f] = 0$. By the compression criterion, representatives of $\partial [f]$ are homotopic rel ∂I^{n-1} to a map with image in B . In particular, $f|_{I^{n-1}}$ is homotopic to a map with image in B via a homotopy $F: I^{n-1} \times I \rightarrow A$ rel ∂I^{n-1} . We can tack F onto f to get a new map $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, B, x_0)$ which, as a map $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ is homotopic to f by the homotopy that tacks on increasingly longer initial segments of F . See Figure 9. Hence $[f] \in \operatorname{im} j_*$.

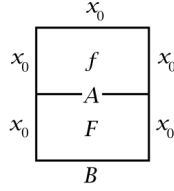


FIGURE 9.

Exactness at $\pi_n(A, B, x_0)$: First, $i_* \partial$ is zero since the restriction of a map $f: (I^{n+1}, \partial I^{n+1}, J^n) \rightarrow (X, A, x_0)$ to I^n is homotopic rel ∂I^n to a constant map via f itself (a similar picture to Figure 8 works).

Conversely, if B is a point, then a nullhomotopy $f_t: (I^n, \partial I^n) \rightarrow (X, x_0)$ of $f_0: (I^n, \partial I^n) \rightarrow (A, x_0)$ gives a map $F: (I^{n+1}, \partial I^{n+1}, J^n) \rightarrow (X, A, x_0)$ with $\partial([F]) = [f_0]$. So in this case, the proof is finished. For a general B , let F be a nullhomotopy of $f: (I^n, \partial I^n, J^{n-1}) \rightarrow (A, B, x_0)$ through maps $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, B, x_0)$ and let g be the restriction of F to I^{n-1} in $I^{n-1} \times I = I^n$ (see the first of the pictures in Figure 10). Next reparametrize the n th and $(n+1)$ st coordinates as in the second picture. Then we find that f with g tacked on is in the image of ∂ . But as before, tacking g onto f gives the same element of $\pi_n(A, B, x_0)$

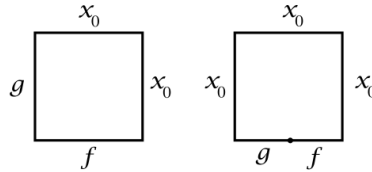


FIGURE 10.

□

Corollary 2.30. *Consider the inclusion $\iota: X = X \times \{0\} \hookrightarrow CX$. Then $\pi_n(CX, X, x_0) \cong \pi_{n-1}(X, x_0)$ for all $n \geq 1$. Taking $n = 2$, we can thus realize an group G , abelian or not, as a relative π_2 by choosing X to have $\pi_1(X) \cong G$.*

There are also change-of-basepoint isomorphisms β_γ for relative homotopy groups. One takes a path γ in $A \subset X$ from x_0 to x_1 which induces $\beta_\gamma: \pi_n(X, A, x_1) \rightarrow \pi_n(X, A, x_0)$ by setting $\beta_\gamma([f]) = [\gamma f]$, where γf is depicted in Figure 11.

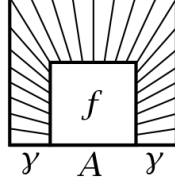


FIGURE 11.

Restricting to loops at the basepoint, the association $\gamma \mapsto \beta_\gamma$ defines an action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$ analogous to the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

2.3. Problem set 1.

2.3.1. Exercises.

Exercise 2.31 (The action of the fundamental group, part 2). Let X be a path-connected, semi-locally simply-connected space with basepoint x and $p: \tilde{X} \rightarrow X$ its universal cover. Show that for $n \geq 2$ and $\tilde{x} \in \tilde{X}$ with $p(\tilde{x}) = x$, the isomorphism $p_* = \pi_n(p): \pi_n(\tilde{X}, \tilde{x}) \cong \pi_n(X, x)$ allows us to identify the action of $\pi_1(X, x)$ on $\pi_n(X, x)$ with the action of $\pi_1(X, x)$ on $\pi_n(\tilde{X}, \tilde{x})$ induced by the group of deck transformations, i.e., the natural action of $\pi_1(X, x)$ on \tilde{X} . In particular, make the statement precise.

Proof. We want to show that for $[\gamma] \in \pi_1(X, x)$ and $[f] \in \pi_n(X, x)$, if \tilde{g} is the lift for γ starting at \tilde{x}_0 , and $\tilde{f}: (S^n, s_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ is the lift of f , then $p_* (\tilde{\gamma}\tilde{f}) = \gamma f$. But this follows directly from how $\tilde{\gamma}\tilde{f}$ and γf we constructed. Namely, applying p to the square used in the definition, we see that we obtain γf from $\tilde{\gamma}\tilde{f}$ since $p \circ \tilde{\gamma} = \gamma$ and $p \circ \tilde{f} = f$. □

Exercise 2.32. Let X and Y be pointed spaces and $n \geq 2$. Show that the inclusion $X \vee Y \hookrightarrow X \times Y$ induces a surjection $\pi_n(X \vee Y) \rightarrow \pi_n(X \times Y)$ for all n . Furthermore, this exhibits $\pi_n(X \times Y)$ as a retract of $\pi_n(X \vee Y)$ for all n . (Is this also true for $n = 1$?)

Proof. a □

2.3.2. Problems.

Problem 2.33. Fix an isomorphism $H_n(S^n) \cong \mathbb{Z}$. We define the degree $\deg f$ of a map $f: S^n \rightarrow S^n$ to be the integer such that $f_*: H_n(S^n) \rightarrow H_n(S^n)$ sends 1 to $\deg f \in \mathbb{Z}$.

- (1) Show that taking the degree of a map $S^n \rightarrow S^n$ induces a well-defined map

$$\deg: \pi_n(S^n) \rightarrow \mathbb{Z}$$

- (2) Show that \deg is a group homomorphism.
 (3) Show that the map \deg is surjective.
 (4) Suppose that $n \geq 2$. Show that $\pi_n(S^n) \cong \mathbb{Z} \times A$ for some abelian group A .

Proof. (1) Let $[f] \in \pi_n(S^n)$ and suppose f, f' are two representatives of this class. Then f and f' are homotopic by definition, so $f_* = (f')_*: \mathbb{Z} = H_n(S^n) \rightarrow H_n(S^n) = \mathbb{Z}$ are equal. In particular, $\deg f = f_*(1) = (f')_*(1) = \deg f'$. So the map is well-defined.
 (2) To show that degree is a group homomorphism, we must show that $\deg(f + g) = \deg f + \deg g$.

For this, we will show a couple of results.

Proposition 2.34. Let $X = S_1^n \vee \dots \vee S_k^n$ for $n > 0$. Then the homomorphism $H_n(S_1^n) \oplus \dots \oplus H_n(S_k^n) \rightarrow H_n(X)$ induced by the inclusion maps is an isomorphism whose inverse is induced by the projections $X \rightarrow S_i^n$.

To prove this proposition, we must show the following lemma.

Lemma 2.35. *Let X be a Hausdorff space and let $x_0 \in X$ be a point having a closed neighborhood N in X of which $\{x_0\}$ is a strong deformation retract. Let Y be a Hausdorff space and let $y_0 \in Y$. Define $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$. Then the inclusion maps induce isomorphisms $\tilde{H}_i(X) \oplus \tilde{H}_i(Y) \cong \tilde{H}_i(X \vee Y)$ whose inverse is induced by the projections of $X \vee Y$ to X and Y .*

Proof of lemma. Consider $A = X$ and $U = X - N$ which is open, and $\bar{U} \subset A$. Then by excision, $H_*(X \vee Y, X) \cong H_*(N \cup Y, N) \cong \tilde{H}_*(Y)$

Consider the LES of the triple $(X \vee Y, \{x_0\} \times Y, \{x_0\} \times \{y_0\})$. We obtain

$$\dots \rightarrow H_p(\{x_0\} \times Y, (x_0, y_0)) \xrightarrow{i_*} H_p(X \vee Y, (x_0, y_0)) \xrightarrow{j_*} H_p(X \vee Y, \{x_0\} \times Y) \rightarrow \dots$$

Since $\pi_Y \circ i = \text{id}_{\{x_0\} \times Y}$, i_* is injective.

Furthermore, we have

$$H_p(X \vee Y, (x_0, y_0)) \xrightarrow{(\pi_X)^*} H_p(\{x_0\} \times Y, (x_0, y_0)) \cong H_p(X \vee Y, \{x_0\} \times Y)$$

so $j_* = (\pi_X)_*$ under these identifications, so, in particular, j_* is surjective.

Therefore, our exact sequence is a SES:

$$0 \rightarrow H_p(Y, pt) \xrightarrow{i_*} H_p(X \vee Y, pt) \xrightarrow{j_*} \underbrace{H_p(X \vee Y, Y)}_{\cong H_p(X, pt)} \rightarrow 0$$

It remains to show that this SES is split, but since $\pi_X \circ \iota_X = \text{id}_{\{x_0\} \times X}$, we have that ι_{X*} provides a section. □

Proof of proposition. This follows by induction on the lemma. □

Next, suppose that E_1, \dots, E_k are disjoint open subsets of S^n , each homeomorphic to \mathbb{R}^n for $n > 0$. Let $f: S^n \rightarrow Y$ be a map which takes $S^n - \bigcup E_i$ to y_0 . Then f factors through the quotient space $S^n / (S^n - \bigcup E_i) \cong S_1^n \vee \dots \vee S_k^n$ where $S_i^n = S^n / (S^n - E_i)$:

$$f: S^n \xrightarrow{g} S_1^n \vee \dots \vee S_k^n \xrightarrow{h} Y$$

Let $\iota_j: S_j^n \hookrightarrow S_1^n \vee \dots \vee S_k^n$ be the j th inclusion and let $p_j: S_1^n \vee \dots \vee S_k^n \rightarrow S_j^n$ be the j th projection. Then by the proposition, $\sum_j \iota_{j*} p_{j*} = \text{id}_*: H_n(S_1^n \vee \dots \vee S_k^n) \rightarrow H_n(S_1^n \vee \dots \vee S_k^n)$. Let $g_j = p_j \circ g: S^n \rightarrow S_j^n$ and $h_j = h \circ \iota_j: S_j^n \rightarrow Y$ and let $f_j = h_j \circ g_j: S^n \rightarrow Y$. That is, f_j is the map which is f on E_j and maps the complement of E_j to the basepoint y_0 .

Theorem 2.36. *In the above situation, $f_* = \sum_{j=1}^k f_{j*}: H_n(S^n) \rightarrow H_n(Y)$.*

Proof of theorem. We have $f_* = h_* \circ g_* = \sum_j h_* \iota_{j*} p_{j*} g_* = \sum_j h_{j*} g_{j*} = \sum_j f_{j*}$. □

Now we get back to showing that $\deg(f + g) = \deg f + \deg g$.

Note that by way of defining $f + g$, this essentially maps I^n by f on the left half and g on the right half with the boundary mapping to the base point x_0 . In particular, this factors through the quotient $I^n \rightarrow I^n / \partial I^n \cong S^n$, where now the two halves can be interpreted as, say, the upper and lower hemispheres. In particular, the equator is by assumption also mapped to x_0 , so we can quotient further by $S^n \rightarrow S^n \vee S^n$ by "pinching" the equator

to a point. This is essentially what the proposition above describes. In particular, $f + g$ can be covered by the two open hemispheres and maps the equator to x_0 , so by the theorem, we have $(f + g)_* = f_* + g_*$, i.e., $\deg(f + g) = (f + g)_*(1) = f_*(1) + g_*(1) = \deg f + \deg g$, as we wanted to show.

- (3) Next we show that \deg is surjective. First note that $\deg \text{id} = \text{id}_*(1) = 1$ by functoriality since $\text{id}_* = \text{id}_{H_n(S^n)}$. By functoriality, we thus hit all of \mathbb{Z} . More precisely, $\deg(*_n \text{id}) = n$ for $n \in \mathbb{N}$ as \deg is a homomorphism. Also $\deg(*_n(-\text{id})) = -n$ for $n \in \mathbb{N}$ and $\deg(c_{x_0}) = 0$, so \deg is surjective.
- (4) Let $n \geq 2$. We have a SES

$$0 \rightarrow \ker \deg \rightarrow \pi_n(S^n) \xrightarrow{\deg} \mathbb{Z} \rightarrow 0.$$

Since \mathbb{Z} is projective, this splits, so $\pi_n(S^n) \cong \mathbb{Z} \oplus \ker \deg$. But $\ker \deg$ is a subgroup of $\pi_n(S^n)$ which is abelian, hence is itself abelian. \square

Problem 2.37. Fix $n \geq 1$. We say that a space X is n -connected if it is non-empty, path-connected, and $\pi_k(X, x) = 0$ for all $1 \leq k \leq n$ and $x \in X$. For (X, x_0) a pointed, path-connected space, show that the following are equivalent:

- (1) X is n -connected.
- (2) $\pi_k(X, x_0) = 0$ for all $1 \leq k \leq n$.
- (3) Every map $S^k \rightarrow X$ can be extended to a map $D^{k+1} \rightarrow X$ for all $k \leq n$.
- (4) Every map $S^k \rightarrow X$ is homotopic to a constant map for all $k \leq n$.

Proof. (1 \implies 2): this follows since X being n -connected means that $\pi_k(X, x) = 0$ for all $x \in X$ and all $1 \leq k \leq n$, hence in particular for x_0 .

(2 \implies 3): Let $f: S^k \rightarrow X$ be a map. Then f represents some homotopy class $[f] \in \pi_k(X, x_0)$. But since $\pi_k(X, x_0) = 0$, f is homotopic to the constant map at $x_0 \text{ rel } s_0$. Let $H: S^k \times I \rightarrow X$ be this homotopy. Define $\tilde{f}: D^{k+1} \rightarrow X$ by $\tilde{f}(x) = H(x, \|x\|)$. Then \tilde{f} is continuous as a composite of continuous maps and $\tilde{f}|_{S^k}(-) = H(-, 1) = f(-)$, so \tilde{f} indeed extends f .

(3 \implies 4): Let $f: S^k \rightarrow X$ be a map. Extends f to a map $\tilde{f}: D^{k+1} \rightarrow X$. Define now a homotopy $H: S^k \times I \rightarrow X$ by $H(x, t) = \tilde{f}(xt)$. This is continuous and $H(x, 1) = \tilde{f}(x) = f(x)$ while $H(x, 0) = \tilde{f}(0) \in X$ is constant. Hence this gives a homotopy between f and $c_{\tilde{f}(0)}$.

(4 \implies 3): Let $f: S^k \rightarrow X$ be a given map. By assumption, there exists a homotopy $H: S^k \times I \rightarrow X$ such that $H(-, 1) = f(-)$ and $H(-, 0) = c$ where c is some constant map at a point in X . But then H factors through the quotient

$$\begin{array}{ccc} S^k \times I & & \\ \downarrow & \searrow H & \\ D^{k+1} & \xrightarrow{\tilde{H}} & X \end{array}$$

where we identify $S^k \times \{0\}$ to a point. But then $\tilde{H}|_{S^k}(-) = H(-, 1) = f(-)$, so \tilde{H} extends f .

(3 \implies 2): Let $[f] \in \pi_k(X, x_0)$ and f a representative. We want to show

that f is homotopic to the constant map at x_0 relative ∂I^k . Extend f to a map $\tilde{f}: D^{k+1} \rightarrow X$, and let $H: S^k \times I \rightarrow X$ be given by $H(x, t) = \tilde{f}(ts_0 + (1-t)x)$. This gives a homotopy between f and the constant map at x_0 .

(2 \implies 1) : the only thing that requires showing is that given that $\pi_k(X, x_0) = 0$ for all k , we then have $\pi_k(X, x) = 0$ for all k and all $x \in X$. But this is precisely what the given hint says we are allowed to assume since X is path connected. So we are done. \square

Problem 2.38 (n -connected in the relative case). The following four conditions are equivalent for $i > 0$:

- (1) Every map $(D^i, \partial D^i) \rightarrow (X, A)$ is homotopic rel ∂D^i to a map $D^i \rightarrow A$.
- (2) Every map $(D^i, \partial D^i) \rightarrow (X, A)$ is homotopic through such maps to a map $D^i \rightarrow A$.
- (3) Every map $(D^i, \partial D^i) \rightarrow (X, A)$ is homotopic through such maps to a constant map $D^i \rightarrow A$.
- (4) $\pi_i(X, A, x_0) = 0$ for all $x_0 \in A$.

When $i = 0$, we did not define the relative π_0 , and (1)-(3) are each equivalent to saying that each path-component of X contains points in A since D^0 is a point and ∂D^0 is empty. The pair (X, A) is called n -connected if (1)-(4) hold for $0 < i \leq n$ and (1)-(3) hold for $i = 0$.

2.4. Whitehead's Theorem.

Theorem 2.39 (Whitehead's Theorem). *If a map $f: X \rightarrow Y$ between connected CW complexes induces isomorphisms $f_*: \pi_n(X) \rightarrow \pi_n(Y)$ for all n , then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is stronger: X is a deformation retract of Y .*

The proof will require the following lemma:

Lemma 2.40 (Compression Lemma). *Let (X, A) be a CW pair and let (Y, B) be any pair with $B \neq \emptyset$. For each n such that $X - A$ has cells of dimension n , assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f: (X, A) \rightarrow (Y, B)$ is homotopic rel A to a map $X \rightarrow B$. When $n = 0$, the condition that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$ is to be regarded as saying that (Y, B) is 0-connected.*

Proof of lemma. Assume inductively that f has already been homotoped to take the skeleton X^{k-1} to B . Let Φ be the characteristic (attaching) map of cell e^k of $X - A$. Then the composition $f\Phi: (D^k, \partial D^k) \rightarrow (Y, B)$ is in some class in $\pi_k(Y, B, y_0) = 0$, so it can be homotoped into B rel ∂D^k by the compression criterion when $k > 0$, or by (Y, B) being 0-connected for $k = 0$ (this is condition (3) in Problem 2.38). This homotopy of $f\Phi$ induces a homotopy rel X^{k-1} on the quotient space $X^{k-1} \cup e^k$ of $X^{k-1} \sqcup D^k$. Doing this for all k -cells of $X - A$ simultaneously, and taking the constant homotopy on A , we obtain a homotopy of $f|_{X^k \cup A}$ to a map into B . Since the inclusion of a subcomplex into a CW-complex is a cofibration, $f|_{X^k \cup A}$ extends to all of X (essentially the homotopy extension property). This completes the inductive step in the finite dimensional CW-complex case. In the general case, we perform the homotopy of the inductive step during the t -interval $[1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+1}}]$. Any finite skeleton X^k is eventually stationary under these homotopies, hence we have a well-defined homotopy $f_t, t \in [0, 1]$ with $f_1(X) \subset B$. \square

Proof of Whitehead's Theorem, 2.39. Let's tackle the case when f is the inclusion of a subcomplex first. Consider then the LES of the pair (Y, X) . Since f by assumption induces isomorphisms on all homotopy groups, $f_*: \pi_*(X) \rightarrow \pi_*(Y)$, the relative homotopy groups $\pi_*(Y, X)$ are zero. Applying the lemma now to the identity map $(Y, X) \rightarrow (Y, X)$, we obtain a homotopy of the identity $\text{id}: Y \rightarrow Y$ to a map $Y \rightarrow X$ which is relative to X . That is, we obtain a deformation retract of Y onto X .

For the general case, recall that a map $f: X \rightarrow Y$, can be considered as the composition of the inclusion $X \hookrightarrow M_f$ and the retraction $M_f \rightarrow Y$. Since the retraction is a homotopy equivalence, it suffices to show that M_f deformation retracts onto X if f induces isomorphisms on homotopy groups, or equivalently, if the relative groups $\pi_n(M_f, X)$ are all zero (since $M_f \simeq Y$). If f is cellular - i.e., takes the n -skeleton of X to the n -skeleton of Y for all n - then (M_f, X) is a CW pair and we can apply the first paragraph of the proof.

If f is not cellular, we can either apply Theorem 4.8 in [2] which says that f is homotopic to a cellular map, or we can use the following argument.

First, using that $\pi_n(M_f, X) = 0$ for all n , apply the Compression Lemma to the inclusion $(X \cup Y, X) \hookrightarrow (M_f, X)$ to obtain a homotopy of the inclusion to a map into $X \text{ rel } X$. The inclusion $X \cup Y \hookrightarrow M_f$ can be seen to be a cofibration using Theorem 1.6, so the pair $(M_f, X \cup Y)$ satisfies the homotopy extension property. So the homotopy in question extends to a homotopy from the identity of M_f to a map $g: M_f \rightarrow M_f$ taking $X \cup Y$ into $X \text{ rel } X$. However, we first of all do not know that this homotopy is $\text{rel } X$ nor that g maps all of M_f into X .

So we apply the Compression lemma again to the composition

$$(X \times I \sqcup Y, X \times \partial I \sqcup Y) \rightarrow (M_f, X \cup Y) \xrightarrow{g} (M_f, X),$$

to get a homotopy $\text{rel } X \times \partial I \sqcup Y$ of g to a map $X \times I \sqcup Y \rightarrow X$. In particular, this homotopy passes through the quotient $X \times I \sqcup Y \rightarrow M_f$, so we get a homotopy of $g \text{ rel } X \times \partial I \sqcup Y$ to a map $M_f \rightarrow X$.

Composing the homotopy from the identity of M_f to g with this homotopy, we get a deformation retraction of M_f onto X . \square

Note. Whitehead's theorem requires a map $f: X \rightarrow Y$ which induces isomorphisms on homotopy groups. Thus it does not apply simply to any two CW complexes X and Y with isomorphic homotopy groups since there might not exist such a map. For examples where this is the case, see [2, p. 348].

Corollary 2.41. *If X is a CW complex with $\pi_n(X) = 0$ for all $n \geq 0$, then $X \simeq \{0\}$.*

Proof. The inclusion of a 0-cell into the complex induces an isomorphism on homotopy groups, so by Whitehead's theorem, the complex deformation retracts to the 0-cell. \square

Lemma 2.42 (Extension Lemma). *Given a CW pair (X, A) and a map $f: A \rightarrow Y$ with Y -path connected, then f can be extended to a map $X \rightarrow Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X - A$ has cells of dimension n .*

Proof. Suppose that f has been extended over the $(n-1)$ -skeleton. Then an extension over an n -cell exists if and only if the composition of the cell's attaching map $S^{n-1} \rightarrow X^{n-1}$ with $f: X^{n-1} \rightarrow Y$ is nullhomotopic, which it is if $\pi_{n-1}(Y) = 0$. \square

2.5. Cellular Approximation.

Definition 2.43 (Cellular maps). A map $f: X \rightarrow Y$ between CW complexes, satisfying $f(X^n) \subset Y^n$ for all n , is called a *cellular map*.

Theorem 2.44 (Cellular Approximation Theorem). *Every map $f: X \rightarrow Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, then homotopy map be taken to be stationary on A .*

Recall the following about simplicial maps and simplicial approximations:

Definition 2.45 (Simplicial map). Let K and L be simplicial complexes. A function $s: |K| \rightarrow |L|$ is called *simplicial* if it takes simplexes of K linearly onto simplexes of L .

Definition 2.46 (Carrier of $f(x)$). Given a map $f: |K| \rightarrow |L|$ between polyhedra and a point $x \in |K|$, the point $f(x)$ lies in the interior of a unique simplex of L . Call this simplex the *carrier* of $f(x)$.

Definition 2.47 (Simplicial Approximation). A simplicial map $s: |K| \rightarrow |L|$ is a simplicial approximation of $f: |K| \rightarrow |L|$ if $s(x)$ lies in the carrier of $f(x)$ for each $x \in |K|$.

Theorem 2.48 (Simplicial approximation theorem). *Let $f: |K| \rightarrow |L|$ be a map between polyhedra. If m is chosen large enough, there is a simplicial approximation $s: |K^m| \rightarrow |L|$ to $f: |K^m| \rightarrow |L|$.*

Thus we may view cellular approximation as a CW analog of simplicial approximation since simplicial maps are cellular. Simplicial maps are much more rigid than cellular maps, however, and the core proof of cellular approximation will be a weaker form of simplicial approximation.

But first, a nice corollary:

Corollary 2.49. $\pi_n(S^k)$ for $n < k$.

Proof. If S^n and S^k are given their usual CW structure of a single 0-cell and then an n - or k -cell, respectively, then by the Cellular Approximation Theorem, any pointed map $S^n \rightarrow S^k$ is based homotopic to a cellular map, and hence maps the n -skeleton of S^n into the n -skeleton of S^k . But the n -skeleton of S^k is just the 0-cell. That is, any map $S^n \rightarrow S^k$ is based nullhomotopic, so $\pi_n(S^k) = 0$. \square

Proof of Cellular Approximation Theorem. To do \square

3. METHODS OF CALCULATION

3.1. The Hurewicz Theorem.

3.2. Problem Set 2.

Problem 3.1. Let $T = S^1 \times S^1$ be the torus and $i: D^2 \hookrightarrow T$ an embedding of the unit disk that is disjoint from $S^1 \times \{s_0\}$. Define $A := (S^1 \times \{s_0\}) \cup i(S^1) \subset T$. Let $x_0 = (s_0, s_0)$ and $x_1 \in i(S^1)$.

- (1) Draw a picture of (X, A) and the two points x_0 and x_1 .
- (2) Construct an explicit bijection of sets $\pi_1(T, A, x_1) \cong \mathbb{Z}^2 \sqcup \mathbb{Z}$.
- (3) Compute the relative homotopy groups $\pi_2(T, A, x_0)$ and $\pi_2(T, A, x_1)$.

Solution. (1)

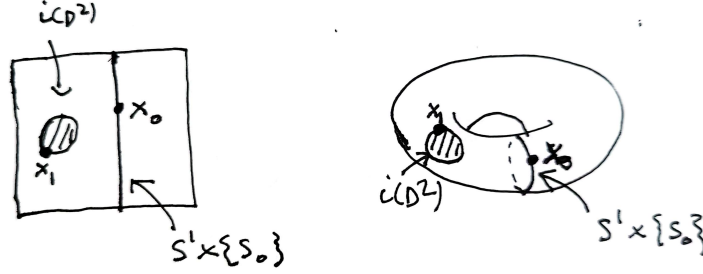


FIGURE 12. Note that in this figure, A are the parts drawn without the interior of the disk $i(D^2)$.

(2) Recall that

$$\pi_n(T, A, x_1) = [I^n, \partial I^n, J^{n-1}; T, A, x_1].$$

Thus $\pi_1(T, A, x_1)$ becomes the set of homotopy classes of maps $(I, \{0, 1\}, \{1\}) \rightarrow (T, A, x_1)$. That is, the set of paths in T starting at a point in A and ending at x_1 up to homotopy through such paths.

For any map $f: (I, \{0\}, \{1\}) \rightarrow (T, A, \{x_1\})$, can lift this to the universal cover since I is simply connected. Let $\tilde{f}: (I, \{0\}, \{1\}) \rightarrow (\mathbb{R}^2, p^{-1}(A), p^{-1}(\{x_1\}))$. Now, $p^{-1}(A)$ can be visualized as tiling \mathbb{R}^2 by tiles as the left picture in Figure 12, each tile of course contains precisely one element of the fiber $p^{-1}(\{x_1\})$. For the lift \tilde{f} , we choose a base point \tilde{x}_1 in $p^{-1}(\{x_1\})$. By the lifting theorem, there now exists a unique lift, call it $\tilde{f}: (I, \{1\}) \rightarrow (\mathbb{R}^2, \{\tilde{x}_1\})$, such that $f = p \circ \tilde{f}$. Now, $f(0) \in A$ is the only condition, so $\tilde{f}(0)$ lies in $p^{-1}(A)$. Homotopies through maps which start in A for f correspond in the universal cover to letting $\tilde{f}(0)$ run freely through its path component in $p^{-1}(A)$. We can construct a bijection $\pi_0(p^{-1}(A)) \cong \mathbb{Z}^2 \sqcup \mathbb{Z}$ by identifying the component of $p^{-1}(i(S^1)) \cap [n, n+1] \times [m, m+1]$ with $(n, m) \in \mathbb{Z}^2$ and identifying the vertical line in $p^{-1}(S^1 \times \{s_0\}) \cap [n, n+1]$ with $n \in \mathbb{Z} \subset \mathbb{Z}^2 \sqcup \mathbb{Z}$. This is obviously bijective. We can always homotopy f to be a straight-line in the universal cover, so the only thing that determines the equivalence class of f , given that \tilde{f} ends at \tilde{x}_1 , is which path component in $\mathbb{Z}^2 \sqcup \mathbb{Z}$ it start in. This gives an injective map $\varphi: \pi_1(T, A, x_1) \rightarrow \mathbb{Z}^2 \sqcup \mathbb{Z}$. To see that it is surjective, it is clear that choosing \tilde{x}_1 as above and choosing any point in the path component corresponding to an element $x \in \mathbb{Z}^2 \sqcup \mathbb{Z}$, taking the straight line between these two points gives a path $\tilde{f}: I \rightarrow \mathbb{R}^2$ such that $f := p \circ \tilde{f}$ gives a path $[f] \in \pi_1(T, A, x_1)$, and, by construction, $\varphi([f]) = x$.

Thus $\pi_1(T, A, x_1) \cong \mathbb{Z}^2 \sqcup \mathbb{Z}$.

(3) Let $\iota: A \rightarrow T$ be the inclusion. Then using the LES of relative homotopy groups, we have that

$$\pi_2(T, x_i) \rightarrow \pi_2(T, A, x_i) \rightarrow \pi_1(A, x_i) \xrightarrow{\iota_*} \pi_1(T, x_i)$$

is exact for $i = 0, 1$. For $i = 0, 1$, $\pi_1(A, x_i) \cong \mathbb{Z}$ and $\pi_1(T, x_i) \cong \mathbb{Z}^2$, while $\pi_2(T, x_i) \cong \pi_2(S^1) \times \pi_2(S^1) \cong 1$ for both $i = 0, 1$. Hence $\pi_2(T, A, x_i) \cong \ker \iota_*$. First, suppose $i = 0$. Then ι induces the map $\mathbb{Z} \cong \pi_1(A, x_0) \rightarrow \pi_1(T, x_0) \cong \mathbb{Z}^2$ given by $n \mapsto (0, n)$, so $\ker \iota_*$ is trivial in this case, so $\pi_2(T, A, x_0) \cong 0$. Suppose now that $i = 1$. Then any loop in the image of ι_* is clearly based nullhomotopic by contracting $i(D^2)$ to the point x_1 . Thus $\ker \iota_* = \pi_1(A, x_1) \cong \pi_1(S^1) \cong \mathbb{Z}$. So $\pi_2(T, A, x_1) \cong \mathbb{Z}$.

Problem 3.2. (1) Compute $\pi_1(S^1 \vee S^2)$ and describe the universal cover of $S^1 \vee S^2$.

(2) Show that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\bigoplus_{\mathbb{Z}} \mathbb{Z}$.

(3) Explicitly describe the action of $\pi_1(S^1 \vee S^2)$ on $\bigoplus_{\mathbb{Z}} \mathbb{Z} \cong \pi_2(S^1 \vee S^2)$.

Solution. (1) The universal cover of $S^1 \vee S^2 =: X$, which we will denote \tilde{X} , is clearly \mathbb{R} with a copy of S^2 attached to each integer of \mathbb{R} .

Let A_1 be the S^1 part together with a small open neighborhood of the base point in S^2 , and likewise, A_2 be S^2 together with a small open neighborhood of the base point in S^1 - here the base points are the points that get identified in the construction of $S^2 \vee S^1$. Applying van Kampen, we find that $\pi_1(S^2 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^2) / N$ where N is generated by all elements of the form $i_{12}(w)i_{21}(w)^{-1}$ for $w \in \pi_1(A_1 \cap A_2)$. But $A_1 \cap A_2$ is contractible, so $N \cong 0$. Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(S^2) \cong 0$, we conclude that $\pi_1(S^2 \vee S^1) \cong \mathbb{Z}$.

(2) To compute $\pi_2(S^1 \vee S^2)$, it suffices to compute π_2 of its universal cover since these are isomorphic. The universal cover is \mathbb{R} with S^2 attached at each integer. Since $\mathbb{R} \simeq \{*\}$, the universal cover is homotopy equivalent to $\bigvee_{\mathbb{Z}} S^2$ for example by using proposition 0.16 and 0.17 in Hatcher.

Since homotopy groups are invariant under based homotopy equivalences, it suffices to compute $\pi_2(\bigvee_{\mathbb{Z}} S^2)$.

But $\bigvee_{\mathbb{Z}} S^2$ is 1-connected, so if $\tilde{H}_2(\bigvee_{\mathbb{Z}} S^2) \cong H_2(\bigvee_{\mathbb{Z}} S^2)$ is nonzero, then by the Hurewicz theorem, we will obtain that $\pi_2(\bigvee_{\mathbb{Z}} S^2) \cong H_2(\bigvee_{\mathbb{Z}} S^2)$. Now, we can give $\bigvee_{\mathbb{Z}} S^2$ a Δ -complex (or cellular) structure with a single 0-simplex and a 2-simplex for each S^2 in $\bigvee_{\mathbb{Z}} S^2$. The associated simplicial chain complex then becomes

$$\dots \rightarrow 0 \rightarrow \bigoplus_{\mathbb{Z}} \mathbb{Z} \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow \dots$$

with 0 everywhere else. In particular then $H_2(\bigvee_{\mathbb{Z}} S^2) \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}$ since there cannot be any cancellation from the maps. Since the Hurewicz isomorphism takes $f \in \pi_2(\bigvee_{\mathbb{Z}} S^2)$, to $f_*[1] \in H_2(\bigvee_{\mathbb{Z}} S^2)$ for $[1] \in H_2(S^2)$ a generator, we find that through our proof using the Δ -complex of $\bigvee_{\mathbb{Z}} S^2$, we found that the inclusions $S^2 \hookrightarrow \bigvee_{\mathbb{Z}} S^2$ in fact induce generators on homology: i.e., the images of the different inclusions $\iota_i: H_2(S_i^2) \hookrightarrow H_2(\bigvee_{i \in \mathbb{Z}} S_i^2) \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}$ generate $\bigoplus_{\mathbb{Z}} \mathbb{Z}$, and hence also $\pi_2(\bigvee_{\mathbb{Z}} S^2)$ under the Hurewicz isomorphism.

(3) Recall that the action of $\pi_1(S^1 \vee S^2)$ on $\pi_n(S^1 \vee S^2)$ makes $\pi_n(S^1 \vee S^2)$ into a $\mathbb{Z}[\pi_1(S^1 \vee S^2)]$ -module. We saw that $\pi_1(S^1 \vee S^2) \cong \mathbb{Z}$. Let γ be a loop that goes once around the S^1 factor. This generates $\pi_1(S^1 \vee S^2) \cong \mathbb{Z}$, so it suffices to describe the action of γ on $\pi_2(S^1 \vee S^2)$ since π_2 now becomes a $\mathbb{Z}[\gamma]$ -module under this action. Since also $\pi_2(S^1 \vee S^2) \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}$, it suffices to describe the action of γ on an arbitrary basis element of $\bigoplus_{\mathbb{Z}} \mathbb{Z}$, say, corresponding to the image under p_* of some inclusion of some $S^2 \hookrightarrow \tilde{X}$. Suppose we choose the inclusion α into the S^2 attached to $1_n \in \mathbb{Z}_n$.

Then $p_*\alpha = [\eta_n \xi] = 1_n \in \mathbb{Z}_n$ where η_n is the loop that winds around the S^1 factor n times and ξ is the inclusion $S^2 \hookrightarrow S^1 \vee S^2$.

In particular then $\gamma p_*\alpha = [\eta_{n+1} \xi] = 1_{n+1} \in \mathbb{Z}_{n+1}$.

This completes the description, but I will also give an alternative description just for completeness where I expound on some details between the homomorphisms $H_2(\bigvee_{\mathbb{Z}} S^2) \cong \pi_2(\bigvee_{\mathbb{Z}} S^2) \cong \pi_2(S^1 \vee S^2)$ that underlies the above explanation. We will use the correspondence between the π_1 action on $\pi_n(X)$ and its action on $\pi_n(\tilde{X})$ where \tilde{X} was the universal covering space.

To this end, we have previously shown the following:

Lemma 3.3. *Let $p: \tilde{X} \rightarrow X$ be the universal cover of a path-connected space X . Under the isomorphism $\pi_n(X) \cong \pi_n(\tilde{X})$, for $n \geq 2$, the action of $\pi_1(X)$ on $\pi_n(X)$ corresponds to the action of $\pi_1(X)$ on $\pi_n(\tilde{X})$ induced by the action of $\pi_1(X)$ on \tilde{X} as deck transformations. More precisely, for $\gamma \in \pi_1(X, x_0)$, $\alpha \in \pi_n(\tilde{X}, \tilde{x}_0)$, $\tilde{\gamma}$ the lift of γ , and γ_* the homomorphism induced by the action of γ on \tilde{X} , we have $\gamma p_*(\alpha) = p_*(\beta_{\tilde{\gamma}}(\gamma_*(\alpha)))$.*

Let $\alpha \in \pi_2(\tilde{X})$ be the element corresponding under the isomorphism $\pi_2(X) \cong \pi_2(\tilde{X})$ to the class of our chosen inclusion $S^2 \hookrightarrow \bigvee_{\mathbb{Z}} S^2$. That is, α is the inclusion of S^2 into one of the S^2 in the universal cover. To understand $\gamma p_*(\alpha)$, we can thus look at $p_*(\beta_{\tilde{\gamma}}(\gamma_*(\alpha)))$. Now, $\gamma_*(\alpha)$ will simply be the inclusion of S^2 to the S^2 "above" the previous one in the universal cover. So if we previously included our S^2 into the S^2 attached to $n \in \mathbb{R} \subset \tilde{X}$, then $\gamma_*(\alpha)$ corresponds to including S^2 into the S^2 attached to $n+1 \in \mathbb{R} \subset \tilde{X}$. Then $\beta_{\tilde{\gamma}}$ is simply the change-or-basepoint transformation depicted in the picture on page 341 in Hatcher. I.e., it essentially shrinks α and attaches it inside a larger square where we put $\tilde{\gamma}$ on each radial line in-between the squares. If we understand our isomorphism $\pi_2(\bigvee_{\mathbb{Z}} S^2) \cong \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_i$ as the generator for \mathbb{Z}_i corresponding under the Hurewicz isomorphism to the inclusion of S^2 into the sphere attached to $i \in \mathbb{Z}$, then we find that $\alpha \mapsto \beta_{\tilde{\gamma}}(\gamma_*(\alpha))$ precisely sends $\alpha = 1_n \in \mathbb{Z}_n$ to $1_{n+1} \in \mathbb{Z}_{n+1}$. Under p_* , this may be interpreted again as sending $1_n \mapsto 1_{n+1}$ when n corresponds to $[\eta \xi]$ where η is the loop that winds around the S^1 factor n times and ξ is the inclusion of $S^2 \hookrightarrow S^1 \vee S^2$.

Problem 3.4. Let (X, A, x_0) be a pointed pair such that the inclusion $i: A \hookrightarrow X$ is based nullhomotopic (the nullhomotopy preserves the basepoint). The goal is to show that for $n \geq 2$, there is an isomorphism of groups:

$$\pi_n(X, A, x_0) \cong \pi_n(X, x_0) \times \pi_{n-1}(A, x_0).$$

- (1) Show that there is an exact sequence of groups

$$1 \rightarrow \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \rightarrow 1.$$

- (2) Using a based nullhomotopy $H: A \times [0, 1] \rightarrow X$, construct a natural group morphism

$$r_*: \pi_n(X, A, x_0) \rightarrow \pi_n(X, x_0)$$

such that $r_* \circ j_* = 1$.

- (3) Show that for any short exact sequence of groups

$$1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$$

such that α admits a retraction, there is a group isomorphism

$$B \cong A \times C.$$

Conclude the desired isomorphism.

Proof. (1) From the LES for relative homotopy groups, we obtain that

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \pi_{n-1}(X, x_0)$$

is exact. For $n \geq 2$, all the sets in the exact sequence are groups and the maps are group homomorphisms. Since homotopic maps relative to the base point induce the same maps on homotopy groups, we find by assumption that $i_* = 0$. Therefore,

$$1 \xrightarrow{0} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_*} \pi_{n-1}(A, x_0) \xrightarrow{0} 1$$

is exact.

(2) Let $[f] \in \pi_n(X, A, x_0)$ and consider a representative $f: (D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$. We put f on the bottom of a cylinder $D^n \times \{0\} \subset D^n \times I$. Now $H(f(x), t)$ gives a homotopy $S^{n-1} \times I \rightarrow X$, so we can use this on $S^{n-1} \times I \subset D^n \times I$ of the cylinder. Now we use that $D^n \times \{0\} \cup S^{n-1} \times I \cong D^n$ (see Figure 13). Denote this homeomorphism by $\varphi: D^n \rightarrow D^n \times \{0\} \cup S^{n-1} \times I$.

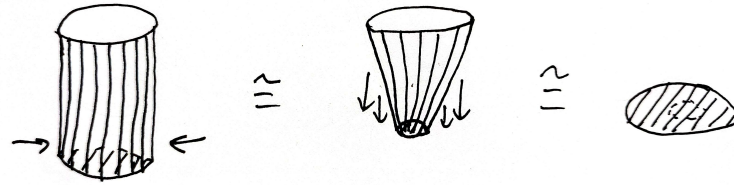


FIGURE 13.

Define $h: S^{n-1} \times I$ by $h(x, t) = H(f(x), t)$ and define $h \cup f: D^n \times \{0\} \cup S^{n-1} \times I$ by f on $D^n \times \{0\}$ and h on $S^{n-1} \times I$. Then define $h \cup f \circ \varphi: D^n \rightarrow X$. Now $h \cup f \circ \varphi$ maps ∂D^n to x_0 , so it factors through the quotient $D^n \rightarrow S^n$ and induces a map $\Gamma: (S^n, pt) \rightarrow (X, x_0)$, where pt is the point that the boundary collapses to. This is well-defined since if $f \simeq f' \text{ rel } s_0$ through a homotopy $F: D^n \times I \rightarrow X$, then $\tilde{h}(x, t, s) = H(F(x, s), t)$ gives a map $S^{n-1} \times I \times I$ - and this homotopy is constant

on the boundary $S^{n-1} \times \{1\}$. Then taking $\tilde{h} \cup F: (D^n \times \{0\} \cup S^{n-1} \times I) \times I \cong D^{n-1} \times \{0\} \times I \cup S^{n-1} \times I \times I \rightarrow X$, we obtain a homotopy $\tilde{h} \cup F \circ \varphi: D^n \times I \rightarrow X$ which is constant on the boundary throughout, hence induces the desired homotopy $S^n \times I \rightarrow X$ between Γ and $\Gamma' \text{ rel } \{pt\}$.

To see that it is a group morphism, see Figure 14. Here the top left picture depicts Γ obtained from $f + g \in \pi_n(X, A, x_0)$. The bottom left picture represents $\Gamma_f + \Gamma_g$, where Γ_f is obtained from f by the above procedure and Γ_g is obtained from g by the procedure. Hence $r_*([f] + [g]) = r_*([f]) + r_*([g])$.

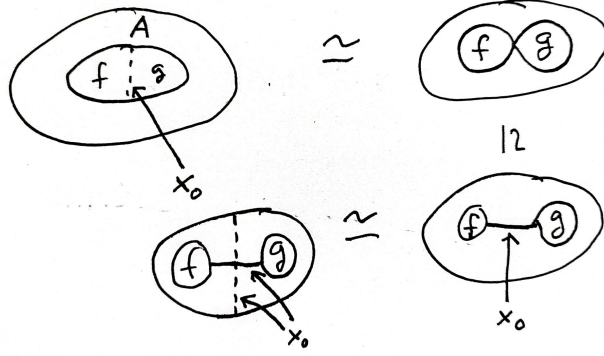


FIGURE 14.

Naturality amounts to showing that r_* defines a natural transformation from $\pi_n(-, -, -)$ to $\pi_n(-, -)$ on the category of based pairs (X, A) such that $A \hookrightarrow X$ is based nullhomotopic. That is, that given a map $f: (X, A, x_0) \rightarrow (Y, B, y_0)$, with both $A \hookrightarrow X$ and $B \hookrightarrow Y$ based nullhomotopic, the diagram

$$\begin{array}{ccc} \pi_n(X, A, x_0) & \xrightarrow{r_*} & \pi_n(X, x_0) \\ \downarrow f_* & & \downarrow f_* \\ \pi_n(Y, B, y_0) & \xrightarrow{r_*} & \pi_n(Y, y_0) \end{array}$$

commutes.

Now, if $H: A \times I \rightarrow X$ is the based nullhomotopy of $A \hookrightarrow X$ and $G: B \times I \rightarrow Y$ is the based nullhomotopy of $B \hookrightarrow Y$, then for $[f] \in \pi_n(X, A, x_0)$, we get the situation of Figure 15. In the central part, these maps agree - namely they are $f \circ g$. We are thus asking for a homotopy between $f \circ H(g(x), t)$ and $G(f \circ g(x), t)$. So we want a map $L: S^{n-1} \times I \times I \rightarrow X$. We may assume without loss of generality that H and G map $S^{n-1} \times \{0\}$ to x_0 and y_0 , respectively, instead of $S^{n-1} \times \{1\}$. Now we let L be given by

$$L(x, t, s) \begin{cases} f \circ H(g(x), (1 - 2s)t), & s \in [0, \frac{1}{2}] \\ G(f \circ g(x), 2s - 1), & s \in [\frac{1}{2}, 1]. \end{cases}$$

This gives naturality.

Now, for $[f] \in \pi_n(X, x_0)$, we have that the boundary is already mapped to x_0 , so $H(f(x), t)$ is constant on $S^{n-1} \times I$ since H is relative the basepoint. Hence

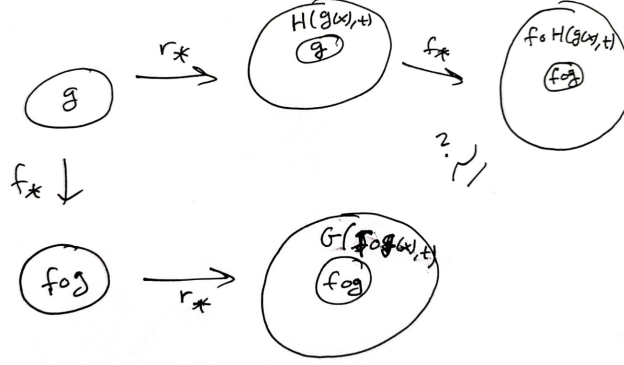


FIGURE 15.

$\Gamma \simeq f$ as depicted in Figure 16 where Γ is obtained from $j_*[f]$ which is simply $[f] \in \pi_n(X, A, x_0)$.

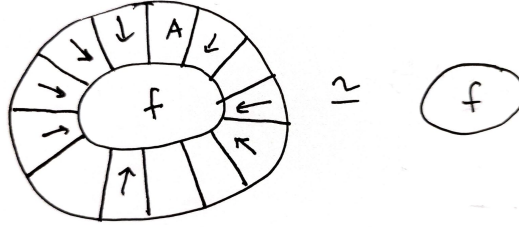


FIGURE 16.

This shows that $r_* \circ j_* = \text{id}$ which was what we wanted to show.

(3) Suppose

$$1 \rightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \rightarrow 1$$

is a short exact sequence and let $s: B \rightarrow A$ be a retraction - i.e., $s \circ \alpha = \text{id}$. We claim that $\varphi: B \rightarrow A \times C$ by $\varphi(b) = (s(b), \beta(b))$ is an isomorphism. Firstly, it is clearly a group homomorphism since s and β are assumed to be group homomorphisms. Next, for injectivity, if $\varphi(b) = 0$, then $s(b) = 0$ and $\beta(b) = 0$. But by exactness then there exists $a \in A$ such that $\alpha(a) = b$. Thus $a = \text{id}(a) = s \circ \alpha(a) = s(b) = 0$. But then since α is a group homomorphism, it takes 0 to 0, so $b = \alpha(a) = \alpha(0) = 0$. This gives injectivity.

For surjectivity, let $(a, c) \in A \times C$. Since β is surjective by exactness of the SES, there exists $b \in B$ such that $\beta(b) = c$. Then $s(\alpha(a) - \alpha \circ s(b) + b) = a - s(b) + s(b) = a$ while $\beta(\alpha(a) - \alpha \circ s(b) + b) = \beta(b) = c$ since $\beta \circ \alpha = 0$. Hence $\varphi(\alpha(a - s(b)) + b) = (a, c)$, so φ is also surjective.

To conclude the desired isomorphism of the problem, we simply note that by (1)

and (2), we precisely have an exact sequence where j_* admits a retraction, so by (3), we get an isomorphism

$$\pi_n(X, A, x_0) \cong \pi_n(X, x_0) \times \pi_{n-1}(A, x_0).$$

□

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