1.
(a) Assume $i(a) = i(b)$, then $\frac{a}{1} = \frac{b}{1}$, so $a = a \cdot 1 = b \cdot 1 = b$.

(b) Let $\beta\left(\frac{a}{b}\right) = \varphi(a)\varphi(b)^{-1}$.

Clearly, for any $a \in R$, $\beta \circ i(a) = \beta\left(\frac{a}{1}\right) = \varphi(a)\varphi(1)^{-1} = \varphi(a)$.
To show that it is well-defined:
if $\frac{a}{b} = \frac{a'}{b'}$, then $ab' = ba'$, so $\varphi(a)\varphi(b') = \varphi(b)\varphi(a')$, hence $\frac{\varphi(a)}{\varphi(b)} = \frac{\varphi(a')}{\varphi(b')}$ where we can divide by $\varphi(b)\varphi(b')$ since neither is 0 as $\varphi(0) = 0$ and φ is injective.

2:
(a) Suppose φ is not the zero map.
Let $\varphi(1) = e$. Assume $\varphi(a) = 0$. Then $\varphi(1) = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1}) = 0$. However, then $\varphi(r) = \varphi(r) = \varphi(r)\varphi(1) = 0$ for all r, so φ is the zero map. Contradiction. So φ is injective.

(b) Assume such a φ exists. Firstly we show that there exist infinitely many irreducible polynomials in $k\,[y]$. Assume only finitely many exist, let these be $f_1, \ldots, f_n$. Then $f_1 \cdot \ldots \cdot f_n + 1$ is a polynomial of nonzero degree.
Since this is by assumption reducible, we must have that there exists f_i such that $f_i \mid f_1 \ldots f_n + 1$, but this means $f_i \mid 1$ which is a contradiction since k is an integral domain - so degrees don't match up.

Now if such a φ existed, then for any irreducible polynomial f in $k\,[x]$, we would have that there exists $f' \in k\,[x_1, \ldots, x_n]$ such that $\varphi(f') = \frac{1}{f}$. Choose f irreducible such that f does not divide any denominator of $\varphi(x_i)$ for any i. Then $\frac{1}{f} = k_1\varphi(x_1) + \ldots + k_n\varphi(x_n)$, but multiplying with the product of the denominators of each $\varphi(x_n)$, we get that the right hand side is in $k\,[y]$ while the left hand side is not. Contradiction.

3: We claim that $k\,[x]\,/(f) = \left\{k_0\overline{1} + k_1\overline{x} + \ldots + k_{n-1}\overline{x}^{n-1} \mid k_i \in k\right\}$.
$(\subset)$: Let g be any polynomial and assume there does not exist a representative in its equivalence class in $k\,[x]\,/(f)$ of the form on the right hand side. Then $\overline{g}$ is of degree $m \geq n$. Now let $f = a_0 + \ldots + a_n x^n$. Then letting $\overline{g} = b_m\overline{x}^m + \ldots + b_0\overline{1}$, we get $\overline{g} = b_m\overline{x}^m + \ldots + b_0\overline{1} - \frac{b_m}{a_n}\overline{f}$ which has degree $< m$, contradicitng the assumption on g.

$(\supset)$: This is clear.

(b) We claim that the dimension is $\frac{(d+1)(d+2)}{2}$.

We claim that $k\,[x, y]\,/I$ is generated by the basis

$$
\begin{aligned}
&1, x, \ldots, x^d \\
&y, yx, \ldots, yx^{d-1} \\
&\vdots \\
&y^{d-1}, y^{d-1}x \\
&y^d.
\end{aligned}
$$

For any $\overline{f} \in k\,[x, y]\,/I$, if there is a term of degree $\geq d$, then it is of the form $\sum_{i=0}^{m} a_i x^i y^{m-i}$, for $m \geq d$. But then this any term in this is of the form $a_i x^i y^{m-i}$ for $m \geq d$, but then if $i \leq d-1$, we have $x^i y^{m-i} = x^i y^{d-i} y^{m-d} = 0$, and if $i \geq d$ then $x^i y^{m-i} = x^d x^{i-d} y^{m-i} = 0$, so

$$\sum_{i=0}^{m} a_i x^i y^{m-i} = 0.$$

Conversely, any linear combination of the basis is clearly in $k\,[x, y]\,/I$. Thus the result follows since the basis is precisely $\frac{(d+1)(d+2)}{2}$ elements.

(c) The number of ways to put k objects into n bins is given by the ball-and-urn formula:

$$\binom{n+k-1}{n-1}.$$

We can think of the dimension of I as the number of ways to put d objects into $n+1$ bins which is

$$\binom{d+n}{n}$$

We see that for $n=2$ this corresponds, as we saw in (b), to $\frac{(d+2)(d+1)}{2}$.

4: For points P_1, P_2, we can define the polynomials $f_1(x) = \frac{x-P_2}{P_1-P_2}$ and $f_2(x) = \frac{x-P_1}{P_2-P_1}$. These satisfy the condition.
Fix some j. Now we thus can find for each i a function F_{ij} s.t. $F_{ij}(P_i) = 0$ and $F_{ij}(P_j) = 1$.
Then $\Pi_{i \neq j} F_{ij}$ vanishes on P_i for $i \neq j$ and is 1 at P_j.

5:
(a) There exist polynomials $T_1, \ldots, T_m \in k[x_1, \ldots, x_n]$ such that $\varphi(P) = (T_1(P), \ldots, T_m(P))$ for all $P \in X$. Similarly, there exist $S_1, \ldots, S_r \in k[x_1, \ldots, x_m]$ such that $\psi(P) = (S_1(P), \ldots, S_r(P))$ for all $P \in Y$. Hence $\psi \circ \varphi(P) = (S_1(T_1(P), \ldots, T_m(P)), \ldots, S_r(T_1(P), \ldots, T_m(P)))$ for all $P \in X$, and since compositions of polynomials is a polynomial, we find that $\psi \circ \varphi$ is a polynomial map.

(b) This is clear as $(\psi \circ \varphi)^*(f)(P) = f(\psi \circ \varphi(P)) = \psi^*(f(\varphi))(P) = \psi^*(\varphi^* f)(P) = \psi^* \circ \varphi^*(f)(P)$.