ASSIGNMENT 6

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Contents

1. Theory and Results

1.1. Jet Bundles.

Definition 1.1. Let X, Y be smooth manifolds and $p \in X$. Suppose $f, g \colon X \to Y$ are smooth with $f(p) = g(p) = q$.

(1) We that f has *first order contact with g at p* if $(df)_p = (dg)_p : T_pX \to T_qY$.
(2) We say that f *has k th order contact with g at p* if $(df) : TX \to TY$ has $(k-1)$ st order contact with (dg) at every point in T_pX. This is written as $f \sim_k g$ at p.
(3) Let $J^k(X,Y)_{p,q}$ denote the set of equivalence classes under " $\sim_k$ at p " of smooth maps $f \colon X \to Y$ where $f(p) = q$.
(4) Define $J^k(X,Y) := \bigcup_{(p,q)\in X\times Y} J^k(X,Y)_{p,q}$. An element $\sigma \in J^k(X,Y)$ is called a *k-jet of mappings (or just a k-jet) from X to Y*.
(5) Let σ be a k-jet. Then for some $(p,q) \in X \times Y$, $\sigma \in J^k(X,Y)_{p,q}$. Then p is called the source of σ and q is called the target of σ. The mapping $\alpha \colon J^k(X,Y) \to X$ given by $\sigma \mapsto$ source of σ is called the source map and the mapping $\beta \colon J^k(X,Y) \to Y$ given by $\sigma \mapsto$ target of σ is called the target map.

Definition 1.2 (k-jet or the k-prolongation of a map)**.** For a smooth map $f \colon X \to Y$, there is a canonically defined map $j^kf \colon X \to J^k(X,Y)$ called the k-jet of f defined by $j^kf(p) = [f,p]$, the equivalence class of f in $J^k(X,Y)_{p,f(p)}$, for every $p \in X$.

Lemma 1.3. *Let $U \subset \mathbb{R}^n$ be open and $p \in U$. Let $f, g \colon U \to \mathbb{R}^m$ be smooth. Then $f \sim_k g$ at p if and only if*

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(p) = \frac{\partial^{|\alpha|} g_i}{\partial x^\alpha}(p)$$

for every multi-index α with $|\alpha| \leq k$ and $1 \leq y \leq m$ where f_i and g_i are the coordinate functions determined by f and g, respectively, and $x_1, \dots, x_n$ are coordinates on U.

Lemma 1.4. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open. Let $f_1, f_2 \colon U \to V$ and $g_1, g_2 \colon V \to \mathbb{R}^l$ be smooth. Let $p \in U$. If $f_1 \sim_k f_2$ at p and $g_1 \sim_k g_2$ at $q = f_1(p) = f_2(p)$, then $g_1 \circ f_1 \sim_k g_2 \circ f_2$ at p.*

Proof. We proceed by induction. First, we show the case when $k = 1$. In this case, the statement is precisely that

$$d\left(g_1 \circ f_1\right)_p = d\left(g_2 \circ f_2\right)_p$$

for all $p \in U$. But this is true by the chain rule:

$$d\left(g_1 \circ f_1\right)_p = \left(dg_1\right)_q \left(df_1\right)_p = \left(dg_2\right)_q \left(df_2\right)_p = d\left(g_2 \circ f_2\right)_p .$$

Suppose now the statement is true for $k-1$. Then since $(df_1) \sim_{k-1} (df_2)$ at p and $(dg_1) \sim_{k-1} (dg_2)$ at $q = f_1(p) = f_2(p)$, we have by induction that

$$(dg_1) \circ (df_1) \sim_{k-1} (dg_2) \circ (df_2) \quad \forall\, (p, v) \in \{p\} \times \mathbb{R}^n$$

which by the chain rule is precisely saying that

$$d\left(g_1 \circ f_1\right) \sim_{k-1} d\left(g_2 \circ f_2\right)$$

for all $(p, v) \in \{p\} \times \mathbb{R}^n$. But this is precisely the definition of $g_1 \circ f_1 \sim_k g_2 \circ f_2$ at p.

□

Definition 1.5. Let X, Y, Z, W be smooth manifolds.

(1) Let $h \colon Y \to Z$ be smooth.a Then h induces a map $h_* \colon J^k(X, Y) \to J^k(X, Z)$ as follows: if $[f, p] \in J^k(X, Y)_{p,q}$, then $h_*\,[f, p] = [h \circ f, p] \in J^k\,(X, Z)_{p,h(q)}$.
(2) If $a \colon Z \to W$ is smooth, then $(a \circ h)_* = a_* \circ h_*$ and $(\mathrm{id}_Y)_* = \mathrm{id}_{J^k(X,Y)}$. So if h is a diffeomorphism, then h_* is a bijection.
(3) Let $g \colon Z \to X$ be a smooth diffeomorphism. Then g induces a map $g^* \colon J^k(X, Y) \to J^k(Z, Y)$ by $g^*\,[f, p] = \left[f \circ g, g^{-1}(p)\right] \in J^k(Z, Y)$.
(4) Let $a \colon W \to Z$ be a smooth diffeomorphism. Then $(g \circ a)^* = a^* g^*$ and $(\mathrm{id}_X)^* = \mathrm{id}_{J^k(X,Y)}$.

Next, let A_n^k be the vector space of polynomials in n variables of degree $\leq k$ which have constant term equal to 0. As coordinates for A_n^k, we can choose the coefficients of the polynomials. Let $B_{n,m}^k = \oplus_{i=1}^m A_n^k$. Both A_n^k and $B_{n,m}^k$ are smooth manifolds.

Let now $U \subset \mathbb{R}^n$ be open and $f \colon U \to \mathbb{R}$ smooth. Define $T_k f \colon U \to A_n^k$ as $T_k f(x_0)$ being the kth order Taylor polynomial of f at x_0 without the constant term.
Let $V \subset \mathbb{R}^m$ be open. There is a canonical bijection $T_{U,V} \colon J^k(U, V) \to U \times V \times B_{n,m}^k$ given by

$$T_{U,V}\left([f, x_0]\right) = \left(x_0, f(x_0), T_k f_1(x_0), \dots, T_k f_m(x_0)\right).$$

This map is well-defined and injective by Lemma 1.3.

Lemma 1.6. *Let $U, U' \subset \mathbb{R}^n$ be open and $V, V' \subset \mathbb{R}^m$ open. Suppose $h\colon V \to V'$ is smooth and $g\colon U \to U'$ a diffeomorphism. Then*

$$T_{U',V'}\left(g^{-1}\right)^* h_* T_{U,V}^{-1}\colon U \times V \times B_{n,m}^k \to U' \times V' \times B_{n,m}^k$$

is smooth.

Definition 1.7 (Smooth structure on $J^k(X,Y)$)**.** Let X, Y be smooth manifolds of dimension n and m, respectively. Let (U,φ) and (V,ψ) be smooth charts in X and Y, respectively. Let $U' = \varphi(U), V' = \psi(V)$. Then let $\tau_{U,V} := T_{U',V'} \circ \left(\varphi^{-1}\right)^* \psi_*\colon J^k(U,V) \to U' \times V' \times B_{n,m}^k$. We declare $\left(J^k(U,V), \tau_{U,V}\right)$ to be a chart for $J^k(X,Y)$. We equip $J^k(X,Y)$ with the smooth structure induced by these smooth charts.
We thus see that

$$\dim J^k(X,Y) = m + n + \dim\left(B_{n,m}^k\right)$$

Theorem 1.8. *Let X and Y be smooth manifolds with $n = \dim X$ and $m = \dim Y$. Then*

(1) *$\alpha\colon J^k(X,Y) \to X, \beta\colon J^k(X,Y) \to Y$ and $\alpha \times \beta\colon J^k(X,Y) \to X \times Y$ are submersions.*
(2) *If $h\colon Y \to Z$ is smooth, then $h_*\colon J^k(X,Y) \to J^k(X,Z)$ is smooth. If $g\colon X \to Y$ is a diffeomorphism, then $g^*\colon J^k(Y,Z) \to J^k(X,Z)$ is a diffeomorphism.*
(3) *If $g\colon X \to Y$ is smooth, then $j^k g\colon X \to J^k(X,Y)$ is smooth.*

Proof. (3) Let $(U,\varphi), (V,\psi)$ be charts about x_0 and $g(x_0)$, respectively. Then in local coordinates,

$$\begin{aligned}\tau_{U,V} \circ j^k g \circ \varphi^{-1}(x) &= \tau_{U,V}\left[g, \varphi^{-1}(x)\right] T_{U',V'}\left[\psi \circ g \circ \varphi^{-1}, x\right] \\ &= \left(x, \psi \circ g \circ \varphi^{-1}(x), T_k\left(\psi_1 \circ g \circ \varphi^{-1}\right)(x), \ldots, T_k\left(\psi_m \circ g \circ \varphi^{-1}\right)(x)\right)\end{aligned}$$

Now, each $T_k\left(\psi_i \circ g \circ \varphi^{-1}\right)$ is smooth being a sum of partial derivatives of the $\psi_i \circ g \circ \varphi^{-1}$ which are smooth functions between Euclidean spaces. Since $j^k g$ is locally smooth everywhere, we find that it is smooth. □

1.2. The Whitney C^∞ topology (compact-open topology).

Definition 1.9 (Residual, Baire space)**.** Let F be a topological space. Then

(1) A subset G of F is called *residual* if it is the countable intersection of open dense subsets of F.
(2) F is called a *Baire space* if every residual set is dense.

Proposition 1.10. *Let X and Y be smooth manifolds. Then $C^\infty(X,Y)$ is a Baire space in the Whitney C^∞ topology.*

1.3. Transversality.

Definition 1.11 (Transversality)**.** Let X and Y be smooth manifolds and $f\colon X \to Y$ a smooth map. Let W be a submanifold of Y and $x \in X$. Then f intersects W transversally at x, denoted by $f \pitchfork W$ at x, if either $f(x) \notin W$ or $f(x) \in W$ and $T_{f(x)}Y = T_{f(x)}W \oplus (df)_x\left(T_x X\right)$.

Proposition 1.12. *Let X and Y be smooth manifolds, $W \subset Y$ a submanifold. Suppose $\dim W + \dim X < \dim Y$ (i.e., $\dim X < \operatorname{codim} W$). Let $f\colon X \to Y$ be smooth and suppose $f \pitchfork W$. Then $f(X) \cap W = \varnothing$.*

Proof. Exercise. □

Lemma 1.13. *Let X, Y be smooth manifolds and $W \subset Y$ a submanifold, and $f\colon X \to Y$ smooth. Let $p \in X$ and $f(p) \in W$. Suppose there exists a neighborhood U of $f(p)$ in Y and a submersion $\varphi\colon U \to \mathbb{R}^k$, where $k = \operatorname{codim} W$, such that $W \cap U = \varphi^{-1}(0)$. Then $f \pitchfork W$ at p if and only if $\varphi \circ f$ is a submersion at p.*

Proof. We have that since $f(p) \in W$, $f \pitchfork W$ at p if and only if $T_{f(p)}Y = T_{f(p)}W \oplus (df)_p\,(T_pX)$. Since $\varphi\,(W \cap U) = 0$, $(d\varphi)_p T_p W = 0$, we have $\ker\,(d\varphi)_p = T_pW$ for all p. Hence $f \pitchfork W$ at p if and only if

$$T_{f(p)}Y = \ker\,(d\varphi)_p \oplus (df)_p\,(T_pX)$$

but $\dim\ker\,(d\varphi)_p = \dim T_{f(p)}U - \dim\operatorname{im}\,(d\varphi)_p = \dim T_{f(p)}Y - k$, so $f \pitchfork W$ at p if and only if $\dim\,(df)_p\,(T_pX) = k$, so in particular, since $(d\varphi)_{f(p)}$ is surjective, this happens if and only if $\dim\,(d\varphi \circ f)_p\,(T_pX) = k$, i.e., $\varphi \circ f$ is a submersion at p. □

Theorem 1.14 (Thom Transversality Theorem)**.** *Let X and Y be smooth manifolds and W a submanifold of $J^k(X, Y)$. Let*

$$T_W = \left\{ f \in C^\infty(X, Y) \mid j^k f \pitchfork W \right\}.$$

Then T_W is a residual subset of $C^\infty(X, Y)$ in the C^∞ topology.

1.3.1. *Multijet Spaces.*

Definition 1.15. Let X and Y be smooth manifolds. Define

$$\begin{aligned} X^s &= X \times \ldots \times X \\ X^{(s)} &= \{(x_1, \ldots, x_s) \in X^s \mid x_i \neq x_j, \quad 1 \leq i < j \leq s\}. \end{aligned}$$

Let $\alpha\colon J^k(X, Y) \to X$ be the source map. Define $\alpha^s\colon J^k(X, Y)^s \to X^s$ by $(\sigma_1, \ldots, \sigma_s) \mapsto (\alpha\sigma_1, \ldots, \alpha\sigma_s)$. Then define $J^k_s(X, Y) = (\alpha^s)^{-1}\left(X^{(s)}\right)$, called the s-fold k-jet bundle.

A multijet bundle is some s-fold k-jet bundle, $X^{(s)}$ is a manifold since it is an open subset of X^s, so $J^k_s(X, Y)$ is an open subset of $J^k(X, Y)^s$, hence also a smooth manifold.

Let $f\colon X \to Y$ be smooth. Define $j^k_s f\colon X^{(s)} \to J^k_s(X, Y)$ by

$$j^k_s f\,(x_1, \ldots, x_s) = \left(j^k f(x_1), \ldots, j^k f(x_s)\right).$$

Theorem 1.16 (Multijet Transversality Theorem)**.** *Let X and Y be smooth manifolds with W a submanifold of $J^k_s(X, Y)$. Let*

$$T_W = \left\{ f \in C^\infty(X, Y) \mid j^k_s f \pitchfork W \right\}.$$

Then T_W is a residual subset of $C^\infty(X, Y)$ in the C^∞ topology. Moreover, if W is compact, then T_W is open.

1.4. **Critical Values and Non-degenerate Critical Values.**

Definition 1.17. Given smooth manifolds X, Y, let $\sigma = [f, p] \in J^1(X, Y)$. Then define $\operatorname{rank}\sigma = \operatorname{rank}\,(df)_p$ and $\operatorname{corank}\sigma = q - \operatorname{rank}\sigma$ where $q = \min\{\dim X, \dim Y\}$. Define

$$S_r = \left\{ \sigma \in J^1(X, Y) \mid \operatorname{corank}\sigma = r \right\}$$

Let's use these definitions to reformulate the definitions of critical points and degenerate critical points.

Lemma 1.18. *$p \in X$ is a critical value for $f\colon X \to \mathbb{R}$ if and only if $[f,p] \in S_1$.*

Proof. Firstly, for a map $f\colon X \to \mathbb{R}$, a point $p \in X$ is a critical point if $(df)_p = 0$. Thus $\operatorname{rank} j^1 f = \operatorname{rank}(df)_p = 0$, so $\operatorname{corank} j^1 f = 1$. Therefore if p is a critical point for f, then $[f,p] \in S_1$.
Conversely, if $[f,p] \in S_1$, then $\operatorname{corank}[f,p] = 1$, so $\operatorname{rank}(df)_p = 0$, but $(df)_p : T_pX \to \mathbb{R}$, so having rank 0 means that it must be the 0 map, so $(df)_p = 0$. Hence p is a critical point. So we find that $p \in X$ is a critical point for f if and only if $[f,p] \in S_1$. $\square$

To relate non-degeneracy, we make use of the following proposition:

Proposition 1.19. *Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ smooth. Then a point $p \in U$ is a nondegenerate critical point for f if and only if p is a critical point and $j^1 f \pitchfork S_1$ at p.*

Proof. First recall that $J^1(U,\mathbb{R}) \cong U \times \mathbb{R} \times B^1_{n,1}$ by definition/construction. Now, $B^1_{n,1} \cong \operatorname{Hom}(\mathbb{R}^n, \mathbb{R})$. Since $T_pJ^1(U,\mathbb{R}) \cong T_p(U \times \mathbb{R} \times \operatorname{Hom}(\mathbb{R}^n,\mathbb{R})) \cong T_{p_1}U \oplus T_{p_2}\mathbb{R} \oplus T_{p_3}\operatorname{Hom}(\mathbb{R}^n,\mathbb{R})$, we find that the projection $\pi\colon J^1(U,\mathbb{R}) \to \operatorname{Hom}(\mathbb{R}^n,\mathbb{R})$ under this identification on tangent spaces simply becomes the projection on the $T_{p_3}\operatorname{Hom}(\mathbb{R}^n,\mathbb{R})$ factor, hence π is a submersion. Furthermore, if $\pi(\sigma) = 0$, that means then in local coordinates, the first degree Taylor expansions without constant term of a smooth representative f for π at p vanish, so since these determine the equivalence class of $[f,p] = \sigma$, we have $(df)_p = 0$, that is, $\sigma \in S_1$. Hence $S_1 = \pi^{-1}(0)$. In particular, S_1 is a submanifold as the preimage of a regular value. Applying Lemma 1.13, $j^1 f \pitchfork S_1$ at p if and only if $\pi \circ j^1 f$ is a submersion at p. Now

$$\pi \circ j^1 f(x) = (df)_x = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$$

so $\pi \circ j^1 f$ is a submersion at p if and only if the map $\mathbb{R}^n \to \mathbb{R}^n$ given by

$$x \mapsto \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right)$$

is a submersion at p if and only if

$$\det H(f)_p = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right) \neq 0.$$

$\square$

2. Problems

Definition 2.1. Let M be a smooth manifold. A Morse function $f\colon M \to \mathbb{R}$ is a smooth map such that all its critical points are non-degenerate, with pairwise distinct critical values in $\mathbb{R}$.

2.1. **Reeb's Theorem.**

Problem 2.2 (Reeb's Theorem)**.** (6 pts) Let M be a smooth, compact manifold of dimension d. Show that if M admits a Morse function with only two critical points, then M is homeomorphic to the sphere S^d. Indicate why the above proof fails in showing that M is diffeomorphic to the sphere S^d.

For the proof, we state a theorem that we will need:

Definition 2.3. For a smooth map $f\colon M \to \mathbb{R}$ on a smooth manifold M, let $M^a = f^{-1}(-\infty, a]$.

Theorem 2.4. *Let $f \in C^\infty(M)$ on a manifold M. Let $a < b$ and suppose that the set $f^{-1}[a, b]$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

Proof of Problem 2.2. Since M is compact, we have that $f(M) = [a, b] \subset \mathbb{R}$. Without loss of generality, assume that $f(M) = [0, 1]$.

We shall need the following lemma from analysis:

Lemma 2.5 (Fermat's Theorem)**.** *Let $f\colon (a, b) \to \mathbb{R}$ be a function on an open interval $(a, b) \subset \mathbb{R}$. Suppose f has a local extremum at $x_0 \in (a, b)$. If f is differentiable at x_0, then $f'(x_0) = 0$.*

Now, we claim that the two critical points are precisely the preimages of 0 and 1. For suppose $x \in f^{-1}(0)$. Then x is a global minimum for f. Taking some chart centered around x, we have a local representation of f as a function $\mathbb{R}^d \to [0, 1]$ with a global minimum at 0. Taking the partial derivatives of f and applying Fermat's theorem to each of them, we find that each partial derivative evaluated at 0 is 0: $\frac{\partial f}{\partial x^i}(0) = 0$. Hence we find that $Df(0) = 0$, so transfering back to the manifold, $Df(x) = 0$, so $x \in M$ is a critical point. The same argument applies to show that any $y \in f^{-1}(1)$ is a critical point. Since there are only two critical points, this immediately forces $f^{-1}(0)$ and $f^{-1}(1)$ to be singletons and thus global maximum and minimum of M. Suppose without loss of generality that $p \in M$ is the minimum and $q \in M$ is the maximum.
By Morse's Lemma, in some coordinate system about p, let's say in a neighborhood U, f takes the form

$$f(x_1, \ldots, x_n) = -x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2.$$

Now p is a global minimum, so in fact, we must have that $\lambda = 0$. That is

$$f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2$$

in this neighborhood. Since also $f(U)$ is open in the subspace topology and contains 0, we can find an open disk $\tilde{D}_1$ centered at 0 of radius ε_1 such that $\tilde{D}_1 \cap [0, 1] \subset f(U)$, and let D_1 be the inverse of $\tilde{D}_1$ under this local diffeomorphism.
Similarly, in a neighborhood V of q, f takes the form

$$f(x_1, \ldots, x_n) = 1 - x_1^2 - x_2^2 - \ldots - x_n^2.$$

Again take some open disk $\tilde{D}_2$ centered at 1 of radius ε_2 such that $\tilde{D}_2 \cap [0, 1] \subset f(V)$. Let D_2 be the inverse image under f of $\tilde{D}_2$.
We wish to show that there exists some $\varepsilon > 0$ such that $f^{-1}[0, \varepsilon]$ and $f^{-1}[1 - \varepsilon, 1]$

are homeomorphic to the closed n-disk D^n. There exist $\alpha, \beta \in (0,1)$ such that $f(M - D_1 \cup D_2) = [\alpha, \beta]$ since $M - D_1 \cup D_2$ is still compact. Now simply let $0 < \varepsilon < \min\left\{\alpha, 1-\beta, \varepsilon_1, 1-\varepsilon_2, 1-\varepsilon_1, \frac{1}{4}\right\}$. To see that this works, simply note that $f^{-1}[0,\varepsilon] \subset D_1 \cup D_2$. On D_1, f takes values in $[0,\varepsilon_1]$ and on D_2, f takes values in $[1-\varepsilon_2, 1]$. But $\varepsilon < \varepsilon_1$, so $[0,\varepsilon] \subset [0,\varepsilon_1]$, so $f^{-1}[0,\varepsilon] \subset D_1$, while $\varepsilon < 1-\varepsilon_2$, so $D_2 \cap f^{-1}[0,\varepsilon] = \varnothing$. Similarly, $1-\varepsilon > \varepsilon_1$, so $D_1 \cap f^{-1}[1-\varepsilon, 1] = \varnothing$ while $1-\varepsilon_2 > 1-\varepsilon$, so $D_2 \subset f^{-1}[1-\varepsilon, 1]$.

Therefore, since $f^{-1}[0,\varepsilon] \subset D_1 \subset U$ and we know that on U, f takes the form

$$f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2,$$

we know that $f^{-1}[0,\varepsilon]$ is precisely a closed disk about p. Likewise, $f^{-1}[1-\varepsilon, 1]$ can be seen to be a closed disk about q.

But now by Theorem 2.4, since there are no critical points in $f^{-1}[\varepsilon, 1-\varepsilon]$ by assumption, M^ε is diffeomorphic to $M^{1-\varepsilon}$. Hence we find that $M^{1-\varepsilon}$ and $f^{-1}[1-\varepsilon, 1]$ are both diffeomorphic to closed d-disks, and furthermore, M is obtained by gluing these d-disks along their boundary which is homeomorphic to S^{d-1}. We claim that this is sufficient to conclude that M is *homeomorphic* to S^d. The problem is that while we have individual diffeomorphisms $M^{1-\varepsilon} \cong D^n$ and $f^{-1}[1-\varepsilon, 1] \cong D^n$, the identifications of the boundaries might not be preserved under these diffeomorphisms, so we might not be able to reglue after. Let $\varphi_1 \colon M^{1-\varepsilon} \cong D^d$ and $\varphi_2 \colon f^{-1}[1-\varepsilon, 1] \cong D^d$ be the diffeomorphisms. Then $\varphi_1 \circ \varphi_2^{-1}$ is a diffeomorphism of S^{d-1}, and

$$M \cong D^d \sqcup_{\varphi_1 \circ \varphi_2^{-1}} D^d.$$

We construct a homeomorphism $\psi \colon D_1 \sqcup_{\mathrm{id}} D_2 \to D^d \sqcup_{\varphi_1 \circ \varphi_2^{-1}} D^d$ by

$$\psi(x) = \begin{cases} x & , x \in D_1 \\ 0 & , x \in D_2 \text{ and } x = 0 \\ \|x\| \varphi_1 \circ \varphi_2^{-1}\left(\frac{x}{\|x\|}\right), & x \in D_2 - \{0\} \end{cases}$$

As the sphere is compact and the twisted sphere Hausdorff, this map is a homeomorphism. The reason it might fail to be a diffeomorphism, is that on $D_2 - \{0\}$, as we let x approach 0, we might have non-agreeing derivatives from different directions.

□

2.2. **Existence of Morse functions.**

Problem 2.6 (Existence of Morse functions)**.** Show that any smooth manifold admits a Morse function.

Proof. The proof of this problem will consist of first showing that the set of Morse functions is an open dense subset of $C^\infty(M, \mathbb{R})$. We will thereafter intersect this set with another residual set in $C^\infty(M, \mathbb{R})$ which will force critical values to be distinct. Then we will finish the problem by making use of $C^\infty(M, \mathbb{R})$ being a Baire space in the Whitney C^∞ topology when M is a manifold.

Theorem 2.7. *Let M be a manifold. The set of Morse functions is an open dense subset of $C^\infty(M, \mathbb{R})$.*

Proof. Recall that S_1 is a submanifold of $J^1(M,\mathbb{R})$. Hence

$$T_{S_1} = \left\{ f \in C^\infty(M,\mathbb{R}) \mid j^1 f \pitchfork S_1 \right\}$$

is a residual subset of $C^\infty(X,Y)$ in the C^∞ topology.
By Theorem 1.19, $j^1 f \pitchfork S_1$ if and only if for all points $x \in X$, either $j_1 f(x) \notin S_1$ or $j_1 f(x) \in S_1$ and $j_1 f \pitchfork S_1$ at x. If $j_1 f(x) \notin S_1$, then x is not a critical value of f. If $j_1 f(x) \in S_1$, then x is a critical value. Then $j_1 f \pitchfork S_1$ at x precisely means that x is a nondegenerate critical point. Hence T_{S_1} precisely consists of all smooth maps $M \to \mathbb{R}$ which are Morse functions (not necessarily distinct critical values).
But by Proposition 1.10, $C^\infty(X,Y)$ is a Baire space in the Whitney C^∞ topology when X and Y are manifolds, so by definition, every residual set is dense. Hence T_{S_1} is dense in $C^\infty(M,\mathbb{R})$. Since 0 is an element, it is in particular nonempty. □

Theorem 2.8. *Let M be a smooth manifold. The set of Morse functions all of whose crtiical values are distinct form a residual set in $C^\infty(M,\mathbb{R})$*

Proof. Let $S = (S_1 \times S_1) \cap J^1_2(M,\mathbb{R}) \cap \left(\beta^2\right)^{-1}(\Delta\mathbb{R})$. We claim that S is a submanifold of the multijet bundle $J^1_2(M,\mathbb{R})$. It suffices to check that it is locally a submanifold. Let U be an open coordinate neighborhood in M diffeomorphic to $\mathbb{R}^n$. Recall that $J^1(U,\mathbb{R}) \cong U \times \mathbb{R} \times B^1_{n,1} \cong \mathbb{R} \times \mathbb{R} \times \operatorname{Hom}(\mathbb{R}^n, 1)$, so seeing as the coordinates on $J^2_1(X,Y)$ are inherited from the product smooth structure and that of an open subset of a smooth manifold, we find $J^2_1(U,\mathbb{R}) \cong (\mathbb{R}^n \times \mathbb{R}^n - \Delta\mathbb{R}^n) \times (\mathbb{R} \times \mathbb{R}) \times \operatorname{Hom}(\mathbb{R}^n,\mathbb{R})^2$. Inserting this in the expression for S and noting that $\left(\beta^2\right)^{-1}(\Delta\mathbb{R})$ means that the codomain coordinates must be the same, so $(\mathbb{R}\times\mathbb{R})$ is replaced by $\Delta\mathbb{R}$, and intersecting with $(S_1 \times S_1)$ means that the coordinates for the partial derivatives all vanish, so $\operatorname{Hom}(\mathbb{R}^n,\mathbb{R})^2$ reduces to $(0,0)$. So we get

$$S \cong (\mathbb{R}^n \times \mathbb{R}^n - \Delta\mathbb{R}^n) \times \Delta\mathbb{R} \times (0,0)$$

which indeed is a submanifold of

$$J^2_1(U,\mathbb{R}) \cong (\mathbb{R}^n \times \mathbb{R}^n - \Delta\mathbb{R}^n) \times (\mathbb{R}\times\mathbb{R}) \times \operatorname{Hom}(\mathbb{R}^n,\mathbb{R})^2 .$$

Since S is locally a submanifold of $J^1_2(M,\mathbb{R})$ at each point, it is a submanifold. Moreover, $\operatorname{codim} S = 2n+1$ where $n = \dim M$: since indeed $\dim J^2_1(U,\mathbb{R}) = 2n - 1 + 2 + 2n$ and $\dim S = 2n - 1 + 1$.
Now applying the Multijet Transversality Theorem (Theorem 1.16), we obtain that $T_S = \left\{ f \in C^\infty(M,\mathbb{R}) \mid j^1_2 f \pitchfork S \right\}$ is residual in $C^\infty(M,\mathbb{R})$ equipped with the C^∞ topology.

But by Proposition 1.10, $C^\infty(X,Y)$ is a Baire space in the Whitney C^∞ topology when X and Y are manifolds, so by definition, every residual set is dense. Hence T_S is dense in $C^\infty(M,\mathbb{R})$. Since 0 is an element, it is in particular nonempty.
Now, if $f\colon M \to \mathbb{R}$ is a smooth map. Then $j^1_2 f\colon M^{(s)} \to J^1_2(M,\mathbb{R})$. In particular, suppose that $j^1_2 f \pitchfork S$, then since $\operatorname{codim} S = 2n+1$, while $\dim M^{(2)} = \dim M \times M - \Delta M = 2n-1$, we obtain immediately from Proposition 1.12 that $j^1_2 f(M \times M - \Delta M) \cap S = \varnothing$.
So if p,q are critical points of f, the fact that $j^1_2 f(p,q) \notin S$ means that since $\left(j^1 f(p), j^1 f(q)\right) \in S_1 \times S_1 \cap J^1_2(M,\mathbb{R})$, it must be the failure of being in $\left(\beta^2\right)^{-1}(\Delta\mathbb{R})$ that prevents $j^1_2 f(M \times M - \Delta M)$ from intersecting S. I.e., the targets are not

equal: $f(p) \neq f(q)$. Since p, q were arbitrary critical values, the critical values of any $f \in T_S$ are thus pairwise distinct.

Now taking the set T_S and T_{S_1} from Theorem 2.7, since T_{S_1} was shown to be an open dense subset of $C^\infty(M, \mathbb{R})$, and T_S was just shown to be residual in $C^\infty(M, \mathbb{R})$, i.e., the countable intersection of open dense subsets of $C^\infty(M, \mathbb{R})$, we find that $T_S \cap T_{S_1}$ is the countable intersection of open dense subsets of $C^\infty(M, \mathbb{R})$ also, hence residual in $C^\infty(M, \mathbb{R})$. From Proposition 1.10, we now obtain that $T_S \cap T_{S_1}$ is dense in $C^\infty(M, \mathbb{R})$, giving us the collection we wanted.

□

This completes the proof.

□

2.3. **On the Transversality Theorem.**

Problem 2.9 (On the transversality theorem)**.** Let M be a smooth manifold.

(1) Let $X \subset M$ be a smooth submanifold, and let $f\colon Y \to M$ be a smooth map, where Y is a smooth manifold. Show that f is smoothly homotopic to a map that intersects X transversally at every point.
(2) Show that in the above setting, if $f\colon Y \to M$ intersects X transversally, then $f^{-1}(X)$ is a smooth submanifold of Y such that $\dim Y + \dim f^{-1}(X) = \dim X$.

Proof. (2) We were given the following lemma in class:

Lemma 2.10. *If $f\colon X \to Z$ and $g\colon Y \to Z$ are smooth maps between manifolds and $f \pitchfork g$, then the pullback exists:*

$$\begin{tikzcd} \exists W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

Saying that $f\colon Y \to M$ intersects X transversally then amounts to $f\colon Y \to M$ and $\iota\colon X \hookrightarrow M$ intersecting transversally, so the following pullback can be completed:

$$\begin{tikzcd} \exists W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\iota"] \\ Y \arrow[r, "f"'] & M \end{tikzcd}$$

In particular, W is the fiber product $X \times_M Y = \{(x, y) \mid \iota(x) = f(y)\} \cong f^{-1}(X)$. Thus $f^{-1}(X)$ is a smooth manifold.

I didn't get to the dimension part in time.

□