**1.1.1:** If $d_E$ and $d_T$ induce the same topology, then any function is continuous with respect to one if and only if it is continuous with respect to the other.
We show that the topologies are each finer than the other:

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$. Let $z_k = (x_1, \ldots, x_k, y_{k+1}, y_{k+2}, \ldots y_n)$. Then by repeated application of the triangle inequality, we get

$$\begin{aligned}
\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = d_E(x,y) &\leq d_E\left(x, (x_1, y_2, \ldots, y_n)\right) + d_E\left((x_1, y_2, \ldots, y_n), y\right) \\
&= d_E(x, z_1) + d_E(z_1, y) \\
&\leq d(x, z_2) + d(z_2, z_1) + d(z_1, y) \\
&\vdots \\
&\leq d(x, z_{n-1}) + d(z_{n-1}, z_{n-2}) + \ldots + d(z_2, z_1) + d(z_1, y) \\
&= \sum_{i}^{n} |x_i - y_i|
\end{aligned}$$

By Cauchy-Schwarz, we have

$$\sum_{i=1}^{n} |x_i - y_i| \leq \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\sqrt{n},$$

so we have $B_{d_T}(x, \varepsilon) \subset B_{d_E}(x, \varepsilon)$ and $B_{d_E}(x, \frac{\varepsilon}{\sqrt{n}}) \subset B_{d_T}(x, \varepsilon)$ . Since $x$ was arbitrary, we thus have $\tau_{d_T} \subset \tau_{d_E} \subset \tau_{d_T}$ by lemma 13.3 in Munkres, so the topologies are equal.

**1.1.2:** Assume first it is continuous in the metric sense. Let $f\colon X \to Y$ be continuous. Let $V \subset Y$ be open. Then it is the union of basis elements: $V = \bigcup_{i \in I} B_i(x_i, \varepsilon_i)$. Let $U = f^{-1}(V) = \bigcup_{i \in I} f^{-1}(B_i(x_i, \varepsilon_i))$. Let $x \in U$. Then $f(x) \in B_i(x_i, \varepsilon_i)$ for some $i \in I$. Since $B_i(x_i, \varepsilon_i)$ is open, we can choose a ball $B(f(x), \varepsilon_x) \subset B_i(x_i, \varepsilon_i)$. By definition of our continuity, we now can find a $\delta_x$ such that $B(x, \delta_x) \subset f^{-1}(B(f(x), \varepsilon_x)) \subset f^{-1}(V) = U$. Now since $x$ was arbitrary, we can find such a $\delta_x$ for any $x \in U$. Then

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} B(x, \delta_x) \subset U$$

so $\bigcup_{x \in U} B(x, \delta_x) = U$ and thus $U$ is a union of open sets and hence open.
Hence $f$ is continuous the topological sense also.

Now assume $f$ is continuous in the topological sense and we show it is continuous in the metric sense.

Let $x \in X$ and $\varepsilon > 0$. Then $B(f(x), \varepsilon)$ is an open set and thus $U = f^{-1}(B(f(x), \varepsilon))$ is open. Thus we can write it as a union of open balls $U = \bigcup_{i \in I} B(x_i, \delta_i)$. Then there exists some $i \in I$ such that $x \in B(x_i, \delta_i)$. Now since $B(x_i, \delta_i)$ is open, we can choose a basis element $B(x, \delta) \subset B(x_i, \delta_i)$. Now $f(B(x, \delta)) \subset f(B(x_i, \delta_i)) \subset f(U) \subset B(f(x), \varepsilon)$, which satisfies the requirement.

**1.2.1:** Firstly, since $\varnothing, X \in \tau_i$ for all $i \in I$, we have $\varnothing, X \in \bigcap_{i \in I} \tau_i$.

Now let $\{U_\alpha\}_{\alpha \in J}$ be a collection of sets in $\bigcap_{i \in I} \tau_i$. Then for all $\alpha \in J$, we have $U_\alpha \in \tau_i$ for all $i$, and hence $\bigcup_{\alpha \in J} U_\alpha \in \tau_i$ for all $i$ since each $\tau_i$ is a topology and hence closed under arbitrary unions. But then $\bigcup_{\alpha \in J} U_\alpha \in \bigcap_{i \in I} \tau_i$. So $\bigcap_{i \in I} \tau_i$ is closed under arbitrary unions of open sets.

Now let $U_1, \ldots, U_n$ be a finite collection of open sets in $\bigcap_{i \in I} \tau_i$. Then for all $i \in I$, we have $U_k \in \tau_i$ for all $k = 1, \ldots, n$, and therefore $\bigcap_{k=1}^{n} U_k \in \tau_i$ for all $i$ since each $\tau_i$ is a topology and thus closed under finite intersections. Then $\bigcap_{k=1}^{n} U_k \in \bigcap_{i \in I} \tau_i$, so $\bigcap_{i \in I} \tau_i$ is closed under finite intersections.
So $\bigcap_{i \in I} \tau_i$ is a topology on $X$.