CURVES AND SURFACES - SCHLICHTKRULL

1. Parametrized curves and surfaces and trace

Definition 1.1 (parametrized continuous curve)**.** A **parametrized continuous curve** in $\mathbb{R}^n$ is a continuous map $\gamma\colon I \to \mathbb{R}^n$, for $n \geq 2$, where $I \subset \mathbb{R}$ is an open interval.
The image set $\mathcal{C} = \gamma(I) \subset \mathbb{R}^n$ si called the **trace** of the curve.
A point in the trace, which corresponds to more than one parameter value t, is called a **self-intersection** of the curve.

Example 1.2. circle The curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ given by $t \mapsto (r\cos t, r\sin t)$ parametrizes a curve and all points in the trace are self-intersections because values $t + 2\pi k$ correspond to the same point for all $k \in \mathbb{Z}$.

Definition 1.3 (smooth curve)**.** A parametrized continuous curve for which the map $\gamma\colon I \to \mathbb{R}^n$ is differentiable up to all orders is called a parametrized smooth curve; i.e., if each of the component functions of γ are infinitely differentiable.

Note. In these notes, we will only be concerned with smooth curves, and therefore we adopt the convention that from now on a parametrized curve is smooth, unless otherwise mentioned.

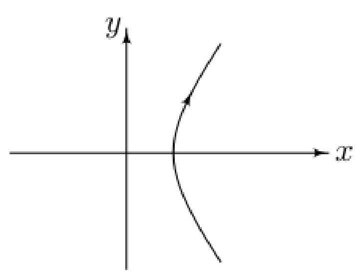
Example 1.4. Ellipse A map $\gamma(t) = (a\cos t, b\sin t)$, where $a, b > 0$ are constants, parametrizes the **ellipse** $\mathcal{C} = \left\{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$.

Example 1.5 (hyperbola)**.** Let $\gamma(t) = (a\cosh t, b\sinh t)$ where $a, b > 0$ and

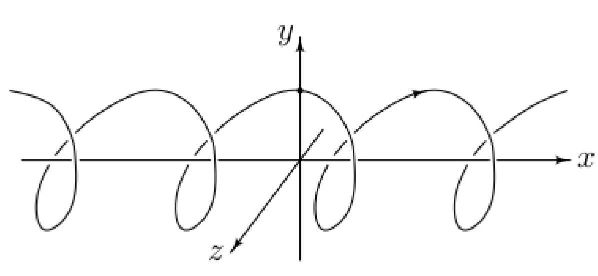
$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}.$$

Then

$$\mathcal{C} = \left\{ (\S, \dagger) \mid \frac{\S^{\in}}{\dashv^{\in}} - \frac{\dagger^{\in}}{\lfloor^{\in}} = \infty, \S > \prime \right\}.$$

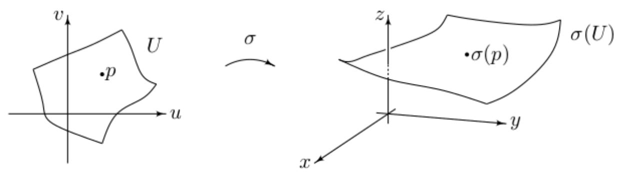


Example 1.6 (Helix)**.** The space curve $\gamma(t) = (\lambda t, r\cos(\omega t), r\sin(\omega t))$ where $r > 0$ and $\lambda, \omega \neq 0$ are constants, is called a **helix**.



1.1. **Surfaces.**

Definition 1.7 (Continuous surface)**.** A **parametrized continuous surface** in $\mathbb{R}^3$ is a continuous map $\sigma \colon U \to \mathbb{R}^3$ where $U \subset \mathbb{R}^2$ is an open, non-empty set.



Definition 1.8 (Smooth maps)**.** If U and V are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F \colon U \to V$ is said to be **smooth** (or C^∞, or **infinitely differentiable**) if each of its component functions has continuous partial derivatives of all orders.

Note. We adopt the convention that a *parametrized surface is smooth, unless otherwise mentioned.*

Example 1.9 (Plane)**.** Let $p, q_1, q_2 \in \mathbb{R}^3$ be fixed vectors and let

$$\sigma\,(u, v) = p + uq_1 + vq_2$$

for $(u, v) \in U = \mathbb{R}^2$. Then σ is a parametrized surface. If q_1, q_2 are linearly independent, the image $\sigma(U)$ is a plane. Otherwise, it is a line or a point.

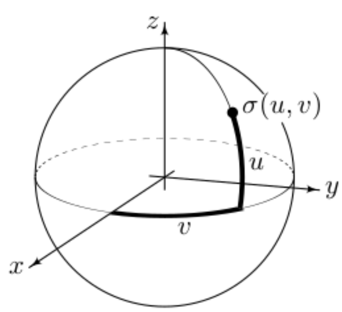
Example 1.10 (Sphere)**.** Let

$$\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$$

where $(u, v) \in \mathbb{R}^2$. This is the standard parametrization of the unit sphere

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\}.$$

The parameters u and v are called *latitude* and *longitude*, and together they are called **spherical coordinates**.



This parametrization covers the total sphere, but it is not injective. On the other hand, if say $u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $v \in (-\pi, \pi)$, then σ is injective, but it is not surjective since a half-circle on the back of the sphere is not in the image.

Example 1.11 (Cone)**.** Let $\lambda > 0$ and

$$\sigma\,(u, v) = (\lambda u \cos v, \lambda u \sin v, u)$$

where $(u, v) \in \mathbb{R}^2$. This gives the cone $\left\{(x, y, z) \mid x^2 + y^2 = \lambda^2 z^2\right\}$

1.2. **Graphs.**

Example 1.12 (Graph of affine linear maps)**.** Graph of $h(t) = at + b, \mathbb{R} \to \mathbb{R}, a, b \in \mathbb{R}$ is the line in $\mathbb{R}^2$ parametrized by $(t, at + b)$. All lines not perpendicular to the x-axis can be parametrized in this fashion.
Similarly, the graph of an affine linear map $h(t) = at+b, \mathbb{R} \to \mathbb{R}^2$ where $a, b \in \mathbb{R}^2$ is the line in $\mathbb{R}^3$ parametrized by $(t, a_1t + b_1, a_2t + b_2)$. All lines of direction not perpendicular to the x-axis can be parametrized in this fashion.

Example 1.13 (Surfaces that are graphs)**.** If $h\colon U \to \mathbb{R}$ is a smooth map defined on $U \subset \mathbb{R}^2$, then its graph is a surface

$$\{(u, v, h(u, v)) \mid (u, v) \in U\} \subset \mathbb{R}^3.$$

Equipped with the map

$$\sigma(u, v) = (u, v, h(u, v)), \quad (u, v) \in U,$$

the graph is clearly a parametrized smooth surface.

Example 1.14 (Planes not perp to xy-plane)**.** The graph of an affine linear function $\mathbb{R}^2 \to \mathbb{R}$ is a plane in $\mathbb{R}^3$. Say $h(u, v) = au + bv + c$ where $a, b, c \in \mathbb{R}$, then $\sigma(u, v) = (u, v, au + bv + c)$. All planes, except those which are perpendicular to the xy-plane, can be parametrized in this fashion.

Example 1.15 (Hemisphere)**.** The graph of the map $h(u, v) = \sqrt{1 - u^2 - v^2}$, defined on $B(0, 1) = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$ is a hemisphere.

1.3. **Level sets.** A plane curve is often described by means of an equation: e.g. a line represented by $ax + by = c, a, b, c \in \mathbb{R}$ and $(a, b) \neq (0, 0)$.
Similarly, a surface can be described by an equation. For example, a plane in $\mathbb{R}^3$ is the set of solutions to an equation $ax + by + cz = d$ where $(a, b, c) \neq (0, 0, 0)$.

Definition 1.16 (Level sets)**.** Let $\Omega \subset \mathbb{R}^n$ be open and let $f\colon \Omega \to \mathbb{R}$ be continuous. The **level sets** for f are the sets

$$\mathcal{C} = \{x \in \Omega \mid f(x) = c\}$$

of solutions in Ω to the equation $f(x) = c$, where $c \in \mathbb{R}$ is fixed.

Definition 1.17 (Critical points)**.** Let $f\colon \Omega \to \mathbb{R}$ be smooth, where $\Omega \subset \mathbb{R}^n$ is open. A point $p \in \Omega$ is called **critical** if

$$\frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = 0.$$

Example 1.18. The solutions to $ax + by = c$ comprise levels sets for $f(x, y) = ax + by$. If $(a, b) \neq (0, 0)$ then there are no critical points. In this case, the set of solutions form a line, hence can be parametrized as a curve. If $(a, b) = (0, 0)$, then f is the trivial function and all points are critical.

Example 1.19. Consider $f(x, y) = xy = 0$. Here $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$, and hence the origin is the only critical point. In fact, the level set given by $f(x, y) = 0$ is the union of two axes, which exactly fails to be a 'reasonable' curve at the origin.

1.4. **The implicit function theorem for two variables.** The implicit function theorem describes conditions under which a given equation in two variables can be solved to obtain one of the variables as a function of the other variables.

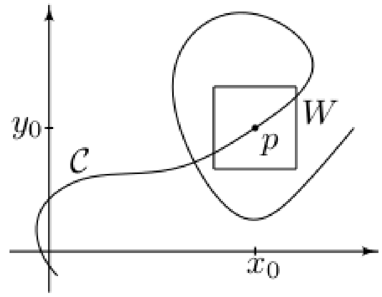
Theorem 1.20 (Implicit function theorem for two variables)**.** *Let $f\colon \Omega \to \mathbb{R}$ be a smooth function where $\Omega \subset \mathbb{R}^2$ is open. Let*

$$\mathcal{C} = \{(x, y) \in \Omega \mid f(x, y) = c\}$$

be the set of solutions to the equation $f(x, y) = c$. Let $p = (x_0, y_0) \in \mathcal{C}$ be given, and assume that $\frac{\partial f}{\partial y} \neq 0$ at p.
Then there exist open intervals I and J around x_0 and y_0, respectively, such that $I \times J \subset \Omega$ and there exists a smooth map $h\colon I \to J$ such that

$$\mathcal{C} \cap I \times J = \{(x, h(x)) \mid x \in I\}, \qquad (\omega)$$

that is, in the neighborhood $I \times J$ of p, $\mathcal{C}$ is the graph of h.



Proof. Assume for simplicity that $c = 0$ and $\frac{\partial f}{\partial x} > 0$ at p. These properties can be arranged by a simple replacement of f which does not affect the set $\mathcal{C}$. By continuity of $\frac{\partial f}{\partial y}$ and openness of Ω, we can choose $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \times [y_0 - \delta, y_0 + \delta] \subset \Omega$ and such that $\frac{\partial f}{\partial y} > a$ for some $a > 0$ on this neighborhood. Then $y \mapsto f(x, y)$ is strictly increasing on $[y_0 - \delta, y_0 + \delta]$ for each fixed x with $|x - x_0| < \delta$.
In particular, since $p \in \mathcal{C}$, we have $f(p) = f(x_0, y_0) = 0$ and hence

$$f(x_0, y_0 - \delta) < 0 \quad \text{and} \quad f(x_0, y_0 + \delta) > 0.$$

By continuity in x_0 of each of the maps $x \mapsto f(x, y_0 \pm \delta)$, there exists a positive number $\eta \leq \delta$ such that $f(x, y_0 - \delta) < 0$ and $f(x, y_0 + \delta) > 0$ for all x with $|x - x_0| < \eta$.
Let $I = (x_0 - \eta, x_0 + \eta)$ and let $x \in I$. We wish to define a map $h\colon I \to J$.

Since $y \mapsto f(x, y)$ is strictly increasing and continuous and $f(x, y_0 - \delta) < 0$ and $f(x, y_0 + \delta) \geq$, there exists a unique y between $y_0 - \delta$ and $y_0 + \delta$ with $f(x, y) = 0$. This value of y is denoted $h(x)$. Then h maps I into $J = (y_0 - \delta, y_0 + \delta)$ and satisfies $f(x, h(x)) = 0$.
The identity of the sets in (ω) follows from the uniqueness of y.

It only remains to show that h is smooth. We first show that h is continuous. Fix $x \in I$ and let $y = h(x)$. Then $f(x, y) = 0$. Choose $\delta_x \in \mathbb{R}$ sufficiently small that $x + \delta_x \in I$. Define δ_y such that $y + \delta_y = h(x + \delta_x)$, then also $f(x + \delta_x, y + \delta_y) = 0$.
The asserted continuity amounts to showing that $\delta_y \to 0$ as $\delta_x \to 0$. The map

$$t \mapsto \varphi(t) = f(x + t\delta_x, y + t\delta_y)$$

is zero both for $t = 0$ and $t = 1$. By the mean value theorem (Rolle's theorem), there exists a number $\theta \in (0, 1)$ depending on δ_x such that

$$\varphi'(\theta) = 0.$$

Differentiating φ by means of the chain rule, we thus obtain

$$\frac{\partial f}{\partial x}(x + \theta\delta_x, y + \theta\delta_y)\,\delta_x + \frac{\partial f}{\partial y}(x + \theta\delta_x, y + \theta\delta_y)\,\delta_y = 0$$

Hence

$$\delta_y = -\frac{\frac{\partial f}{\partial x}(x + \theta\delta_x, y + \theta\delta_y)}{\frac{\partial f}{\partial y}(x + \theta\delta_x, y + \theta\delta_y)}\delta_x$$

and since $\left|\frac{\partial f}{\partial x}\right|$ is bounded, and $\frac{\partial f}{\partial y} \geq a > 0$, it follows that $\delta_y \to 0$ when $\delta_x \to 0$, as claimed.
Next we prove that h is differentiable, which with the notation from above amounts to the convergence of $\frac{\delta_y}{\delta_x}$ as $\delta_x \to 0$. In fact, since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous, this follows immediately from the equation above. Moreover, the limit is given by

$$\lim_{\delta_x \to 0} \frac{\delta_y}{\delta_x} = -\frac{\frac{\partial f}{\partial x}(x, y)}{\frac{\partial f}{\partial y}(x, y)}.$$

Hence h is differentiable and satisfies

$$h'(x) = -\frac{\frac{\partial f}{\partial x}(x, h(x))}{\frac{\partial f}{\partial y}(x, h(x))}.$$

Finally, we prove by induction that h is smooth. Assuming that h is r times differentiable for some $r \in \mathbb{Z}_+$, we see from the equation above that so is h'. Hence h is $r+1$ times differentiable. $\square$

Corollary 1.21. *Let $f\colon \Omega \to \mathbb{R}$ be a smooth function, where $\Omega \subset \mathbb{R}^2$ is open. Let*

$$\mathcal{C} = \{(x, y) \in \Omega \mid f(x, y) = c\}$$

and let $p = (x_0, y_0) \in \mathcal{C}$. Assume that p is not a critical point.
Then there exists an open rectangle $W \subset \Omega$ around p, such that $\mathcal{C} \cap W$ is the graph of a smooth function h, considered either as $y = h(x)$ or as $x = h(y)$.
In particular, it follows that the level set can be parametrized as a smooth curve in a neighborhood of each non-critical point.

Proof. Either $\frac{\partial f}{\partial y} \neq 0$ or $\frac{\partial f}{\partial x} \neq 0$. In the first case, $y = h(x)$ by theorem 1.20. In the other case, interchange x and y in the theorem, to get the case $x = h(y)$. $\square$