MANDATORY ASSIGNMENT

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Exercise 0.1 (1.11)**.** Let $x, y, z \in V$ be linearly independent in a vector space over a field of characteristic $\neq 2$. Prove $x+y, y+z, z+x$ are linearly independent. Show by an example that the conclusion can fail in characteristic 2.

Proof. Suppose $x+y, y+z, z+x$ are not linearly independent. Then by lemma 1.10, there exist $a, b, c \in F$ where $\operatorname{char} F \neq 2$, such that not all a, b, c are 0 and $a\,(x+y)+b(y+z)+c(z+x)=0$. Thus $(a+c)x+(a+b)y+(b+c)z=0$, so by linear independence, we get $a+c=a+b=b+c=0$. Hence $a=b=c$ and so $2a=2b=2c=0$, giving $a=b=c=0$ as $\operatorname{char} F \neq 2$, giving a contradiction. Thus $x+y, y+z, z+x$ are linearly independent.

To show that this fails in characteristic 2, take for example $V=(\mathbb{Z}/2\mathbb{Z})^3$. The field $\mathbb{Z}/2\mathbb{Z}$ has characteristic 2, and we can let $x=e_1=(1,0,0), y=e_2=(0,1,0)$ and $z=e_3=(0,0,1)$. These are linearly independent since $(0,0,0)=ax+by+cz=(a,b,c)$ implies $a,b,c=0$ in $\mathbb{Z}/2\mathbb{Z}$. However,

$$(x+y)+(y+z)+(z+x)=2x+2y+2z=0+0+0=0$$

while the coefficients of each $(x+y),(y+z)$ and $(z+x)$ is $1 \neq 0$ in $\mathbb{Z}/2\mathbb{Z}$, so by lemma 1.10, $x+y, y+z, z+x$ are not linearly independent over $\mathbb{Z}/2\mathbb{Z}$. $\square$

Exercise 0.2 (2.5)**.** Let $A \in \operatorname{Hom}(U,V)$ and $x_1, \ldots, x_k \in U$. Assume $Ax_1, \ldots, Ax_k$ are distinct and linearly independent. Show that $L=\{x_1, \ldots, x_k\}$ is linearly independent and that $N(A)+\operatorname{span}(L)$ is a direct sum.

Proof. Suppose for contradiction that L is not linearly dependent. By lemma 1.10 there thus exist coefficients $\alpha_i, i=1,\ldots,n$, not all 0 such that $\alpha_1 x_1+\ldots+\alpha_k x_k=0$. By linearity, $0=A(0)=A\left(\alpha_1 x_1+\ldots+\alpha_k x_k\right)=\sum \alpha_i A(x_i)$. By linear independence and distinctness of $Ax_1,\ldots,Ax_k$, we get using lemma 1.10 that $\alpha_i=0$ for all i. Contradiction. Thus L is linearly independent.
Now, by the first line of page 14 and lemma 1.8.(a), $N(A)$ and $\operatorname{span}(L)$ are subspaces of U. To show that $N(A)+\operatorname{span}(L)$ is direct, it suffices by lemma 2.12 to show that $N(A)\cap\operatorname{span}(L)=\{0\}$. Suppose $x \in N(A) \in \operatorname{span}(L)$. By lemma 1.10.(2), there exists a unique linear combination $x=\sum \alpha_i x_i$. Now $0=A(x)=\sum \alpha_i A(x_i)$, so again, by linear independence and distinctness of $Ax_1,\ldots,Ax_k$, we get $\alpha_i=0$ for all i. Hence $x=0$. So $N(A)\cap\operatorname{span}(L)\subset\{0\}$, and clearly $A(0)=0$ and $0=\sum_i 0\cdot x_i$, so $\{0\}\subset N(A)\cap\operatorname{span}(L)$. Thus $N(A)\cap\operatorname{span}(L)=\{0\}$, and hence $N(A)+\operatorname{span}(L)$ is a direct sum. $\square$

Exercise 0.3 (5.3)**.** The graph of a map $A\colon X\to Y$ is defined as the set of all pairs (x, Ax) in $X\times Y$. Assuming X and Y are vector spaces, prove that A is linear if and only if its graph is a subspace of $X\oplus Y$. Prove then that a subspace $G\subset X\oplus Y$ is the graph of a linear map if and only if $X\oplus Y=G\oplus Y$.

Proof. Let ΓA denote the graph $\Gamma A = \{(x, Ax) \mid x \in X\} \subset X \times Y$. We must show that this forms a vector space as a subspace of $X \oplus Y$.
Firstly, since $A(0) = 0$ for any linear map, we have $(0, 0) \in \Gamma A$. Now, if $(x, Ax), (y, Ay) \in \Gamma A$, then $Ax+Ay = A(x+y)$ by linearity of A, so $(x, Ax)+(y, Ay) = (x + y, Ax + Ay) \in \Gamma A$, hence ΓA is closed under the additive operation inherited from $X \oplus Y$. Suppose now $\alpha \in F$ where F is the common field over which X and Y are vector spaces. Then $\alpha Ax = A(\alpha x)$ by linearity of A, so $\alpha (x, Ax) = (\alpha x, \alpha Ax) \in \Gamma A$, so ΓA is also closed under the scalar multiplication inherited from $X \oplus Y$. By definition 1.4, ΓA is a subspace of $X \oplus Y$.

Now suppose conversely that ΓA is a subspace of $X \oplus Y$. We must check definition 2.1. Suppose $x, y \in X$ and $\alpha, \beta \in F$. Then

$$\begin{aligned} A(\alpha x + \beta y) &= \alpha A(x) + \beta A(y) \\ &\iff (\alpha x + \beta y, \alpha A(x) + \beta A(y)) \in \Gamma A \\ &\iff \alpha (x, A(x)) + \beta (y, A(y)) \in \Gamma A \end{aligned}$$

and the last line is true by assumption of ΓA being a vector space under the operations inherited from $X \oplus Y$. We thus see that if ΓA is a linear subspace of $X \oplus Y$ then A is a linear map.

Now, to show that $X \oplus Y = G \oplus Y$, we interpret this as Y being identified with the subspace $\{0\} \times Y \subset X \oplus Y$ and then $G \oplus Y$ as the direct sum of subspaces.
Suppose G is the graph of some linear map $A\colon X \to Y$, $G = \Gamma A$. Let $(x, y) \in X \oplus Y$. Then we can write (x, y) as the sum $(x, Ax) + (0, y - Ax) \in G + Y$. So $G + Y = X \oplus Y$. To show that the sum is direct, suppose $(x, y) \in G \cap Y$. Since $(x, y) \in Y = \{0\} \times Y$, we have $x = 0$. Now since $(x, y) \in G$, we have $y = Ax = A(0) = 0$ by linearity of A, so indeed $(x, y) = (0, 0)$. Hence $G \cap Y = \{(0, 0)\}$, so $X \oplus Y = G \oplus Y$.

Conversely, suppose $X \oplus Y = G \oplus Y$ where again Y is identified with $\{0\} \times Y$. Then we must show that there exists a linear map $A\colon X \to Y$ with $\Gamma A = G$. To construct this as a map, we must simply show that there do not exist elements $(x, y), (x, y') \in G$ with $y \neq y'$ and that for each $x \in X$, there exists $y \in Y$ with $(x, y) \in G$. This by definition will mean that G is the graph of a function.
Firstly, suppose $(x, y), (x, y') \in G$ with $y \neq y'$. Then since G is a subspace, $(0, y - y') = (x, y) - (x, y') \in G$, so since $G \oplus Y = X \oplus Y$ is direct, and $(0, y - y') \in \{0\} \times Y$, we have $(0, y - y') \in G \cap Y = \{(0, 0)\}$, so $y = y'$.
Now, for existence, choose any $y \in Y$. Then $(x, y) \in X \oplus Y = G \oplus Y$, so by directness, there exist unique vectors $(a, b), (0, c)$ with $(a, b) \in G$ and $(0, c) \in Y$ such that $(x, y) = (a, b) + (0, c)$. Hence $x = a$, so indeed $(x, b) \in G$, and by the above, this b is the unique element in Y for which $(x, b) \in G$. Denote now for an arbitrary $x \in X$ by $A(x)$ the element $b \in Y$ for which $(x, b) \in G$. Thus $\Gamma A = \{(x, A(x)) \mid x \in X\} \subset X \times Y$ defines a function $X \to Y$ by sending $x \mapsto A(x)$.

But by the first part of this exercise, A is linear if and only if ΓA is a subspace of $X \oplus Y$. Since $\Gamma A = G$ by construction and G is a subspace of $X \oplus Y$, we can thus conclude that A is linear. □

Exercise 0.4 (6.25)**.** Assume $A^3 = A$ and char $F \neq 2$. Show that A is diagonable.

Proof. Since $A^3 = A$, we have $A(A-1)(A+1) = A\left(A^2-1\right) = 0$, so letting $q(x) = x(x-1)(x+1)$, we have $q(A) = 0$, hence by lemma 6.10, if $p(x)$ is the minimal polynomial, we have $p(x)|q(x)$, so since q splits without repetition of linear factors (here is where $\operatorname{char} F \neq 2$ comes into play to allow us to conclude that $x-1 \neq x+1$), p must also split without repetition, and by theorem 6.17.(i), this implies that A is diagonable. □

Exercise 0.5 (7.15)**.** Let $\dim V < \infty, F = \mathbb{C}$, and let $A \in \operatorname{End}(V)$ be normal. Prove that if B commutes with A, then it commutes with A^* as well.

Proof. Since A is normal, we have $AA^* = A^*A$, so also A^* is normal. By theorem 7.24, A and A^* being normal means that they are orthogonally diagonable. By theorem 6.19.(7), we now have that $BA^* = A^*B$ if and only if $BE_{\overline{\lambda}} = E_{\overline{\lambda}}B$ for all $\overline{\lambda} \in \sigma\left(A^*\right)$ where $E_{\overline{\lambda}}$ is the projection to $V_{\overline{\lambda}}$ along the other eigenspace in the decomposition

$$V = \oplus_{\overline{\lambda} \in \sigma(A^*)} V_{\overline{\lambda}}$$

But by theorem 7.21.(iii), $V_{\lambda,A} = V_{A^*,\overline{\lambda}}$, so

$$V = \oplus_{\lambda \in \sigma(A)} V_{\lambda}$$

and $E_{\overline{\lambda}} = E_{\lambda}$. But then we indeed get that $BE_{\overline{\lambda}} = E_{\overline{\lambda}}B$ if and only if $BE_\lambda = E_\lambda B$, and again using theorem 6.19.(7), this is true if and only if $AB = BA$ which we assumed to be true by assumption. □

Exercise 0.6 (8.6)**.** Let $A \in \operatorname{End}(V)$ be nilpotent, and $U \subset V$ invariant. Show that the quotient map $\overline{A} \in \operatorname{End}(V/U)$ is nilpotent.

Proof. Suppose $A^k = 0$ for some $k > 0$. We claim that $\overline{A}^k = 0$ for the same k. We recall by lemma 2.16 that $\overline{A} \in \operatorname{End}(V/U)$ is the unique endomorphism making $\overline{A} \circ \pi = \pi \circ A$ commute where $\pi \colon V \to V/U$ is the quotient map. It thus immediately follows that $\overline{A^k} = 0$ since this satisfies the commutative criterion. Now, we claim that suppose that for N we have shown $\overline{A}^N \circ \pi = \pi \circ A^N$. Then we get

$$\pi \circ A^{N+1} = (\pi \circ A) \circ A^N = \overline{A} \circ \pi \circ A^N = \overline{A}^{N+1} \circ \pi$$

so since the case for $N = 1$ was shown, we get by induction that $\overline{A}^k \circ \pi = \pi \circ A^k = 0$. Now, π is surjective by lemma 2.9, so given some $\overline{x} \in V/U$, let $x \in V$ be such that $\pi(x) = \overline{x}$. Then $\overline{A}^k \overline{x} = \overline{A}^k(\pi(x)) = \pi \circ A^K(x) = \pi(0) = \overline{0}$. So indeed $\overline{A}^k$ is equal to the zero endomorphism in $\operatorname{End}(V/U)$. Thus $\overline{A}$ is nilpotent.

□

Exercise 0.7 (10.11)**.** Show $\chi_{A^{-1}}(\lambda) = (-\lambda)^n \det(A)^{-1} \chi_A\left(\lambda^{-1}\right)$ for $A \in \mathrm{GL}(V)$, $\lambda \neq 0$ and $n = \dim V$.

Proof. We have

$$\begin{aligned}
\det\left(A^{-1} - \lambda I\right) &= \det\left(A^{-1}\left(I - \lambda A\right)\right) \\
&= \det\left(-A^{-1}\lambda\left(A - \lambda^{-1} I\right)\right) \\
&= \det(A^{-1})\det(-\lambda I)\det\left(A - \lambda^{-1} I\right) && \text{(Thm 10.1.(ii))} \\
&= \det(A)^{-1}(-\lambda)^n \chi_A\left(\lambda^{-1}\right)
\end{aligned}$$

where the last step follows since $\det\left(A^{-1}\right) = \det(A)^{-1}$ by theorem 10.3, $\det(-\lambda I) = (-\lambda)^{\dim V} = (-\lambda)^n$ by theorem 10.1.(i), and $\det\left(A - \lambda^{-1} I\right) = \chi_A(\lambda^{-1})$ by definition 10.19, (10.2) and that $\chi_A(x) := \chi_{[A]}(x)$. $\square$