

HOMOTOPY THEORY

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For these notes, we will follow [\[2\]](#), [\[1\]](#) and [\[3\]](#).

1. THE COMPACT-OPEN TOPOLOGY

Recall that Y^X denotes the *set of continuous functions* $X \rightarrow Y$.

Definition 1.1. The *compact-open topology* on Y^X is the topology generated by the sets $M(K, U) = \{f \in Y^X \mid f(K) \subset U\}$ where $K \subset X$ is compact and $U \subset Y$ is open.

Generated here means that these sets form a *subbasis* for the open sets.

Lemma 1.2. *Let \mathcal{K} be a collection of compact subsets of X containing a neighborhood base at each point of X . Let \mathcal{B} be a subbasis for the open sets of Y . Then the collection*

$$\{M(K, U) \mid K \in \mathcal{K}, U \in \mathcal{B}\}$$

forms a subbasis for the compact-open topology on Y^X .

Proof. Recall first that a subbasis is a collection whose union is the whole space and such that the collection of finite intersections of elements of the subbasis form a basis.

In particular, noting that $M(K, U) \cap M(K, V) = M(K, U \cap V)$, this implies that it suffices to consider the case when \mathcal{B} is a basis.

So to show that the collection in question is a subbasis, it suffices to show that given $f \in M(K, U)$, there exist $K_1, \dots, K_n \in \mathcal{K}$ and $U_1, \dots, U_n \in \mathcal{B}$ such that $f \in \bigcap_{i=1}^n M(K_i, U_i) \subset M(K, U)$.

For each $x \in K$, there is an open set $U_x \in \mathcal{B}$ with $f(x) \in U_x \subset U$ (since \mathcal{B} was assumed to be a basis), and there exists a neighborhood $K_x \in \mathcal{K}$ of x such that $f(K_x) \subset U_x$ (since f is continuous and \mathcal{K} was assumed to contain a neighborhood base at each point of X). Thus $f \in M(K_x, U_x)$. Now, covering K with these sets $K \subset \bigcup_{x \in K} K_x$. By compactness of K , there exists a finite subcover $K \subset K_{x_1} \cup \dots \cup K_{x_n}$. Then $f \in \bigcap_{i=1}^n M(K_{x_i}, U_{x_i}) \subset M(K, U)$. \square

Proposition 1.3. *For X locally compact Hausdorff, the "evaluation map" $e: Y^X \times X \rightarrow Y$, defined by $e(f, x) = f(x)$, is continuous.*

Proof. Let $(f, x) \in Y^X \times X$ and U a neighborhood of $f(x) \in Y$. Now we make use of the following lemma:

Lemma 1.4. *If X is a locally compact Hausdorff space, then each neighborhood of a point $x \in X$ contains a compact neighborhood of x . In particular, X is completely regular.*

Proof. Let C be a compact neighborhood of x and U an arbitrary neighborhood of x . Since X is Hausdorff, C is closed, so $(X - U) \cap C$ is a closed subspace of C , hence compact. Now, for each point $z \in (X - U) \cap C$, choose, by Hausdorffness, open neighborhoods U'_z, V'_z of z and x , respectively, and consider $W' := \bigcup_{z \in (X - U) \cap C} U'_z$. Since this is open, $C - W'$ is closed hence compact. Furthermore, it is contained in U and contains x .

Alternative proof due to Bredon: Let C be a compact neighborhood of x and U an arbitrary neighborhood of x . Let $V \subset C \cap U$ be open with $x \in V$. Then $\bar{V} \subset C$ is compact Hausdorff, hence regular, so there exists a neighborhood $N \subset V$ of x in C which is closed in \bar{V} and hence closed in X . Since N is closed in the compact space

C , it is compact. Since N is a neighborhood of x in \bar{V} and since $N = N \cap V$, N is a neighborhood of x in the open set V and hence in X . \square

By Lemma 1.4, there exists a compact neighborhood K of x such that $f(K) \subset U$. Hence $f \in M(K, U)$, and $e(M(K, U) \times K) \subset U$. This finishes the proof. \square

Theorem 1.5. *Let X be locally compact Hausdorff and Y and T arbitrary Hausdorff spaces. Given a function $f: X \times T \rightarrow Y$, define, for each $t \in T$, the function $f_t: X \rightarrow Y$ by $f_t(x) = f(x, t)$. Then f is continuous if and only if both of the following conditions hold:*

- (1) *Each f_t is continuous*
- (2) *The function $T \rightarrow Y^X$ taking $t \mapsto f_t$ is continuous.*

Proof. The "if" implication follows from the fact that f is the composition

$$X \times T \xrightarrow{(x,t) \mapsto (f_t, x)} Y^X \times X \xrightarrow{e} Y.$$

Now the evaluation map is continuous by Proposition 1.3 since X is assumed to be locally compact Hausdorff and since f_t is assumed to be continuous for all t by condition (1); and $(x, t) \mapsto (f_t, x)$ is continuous since $t \mapsto f_t$ is assumed to be continuous by condition (2).

Conversely, for the "only if" implication, (1) follows from the fact that f_t is the composition

$$X \xrightarrow{x \mapsto (x, t)} X \times T \xrightarrow{f} Y.$$

To prove (2), let $t \in T$ be given and $f_t \in M(K, U)$. It suffices to find a neighborhood W of t in T such that $t' \in W$ implies that $f_{t'} \in M(K, U)$ (i.e., it suffices to prove conditions for continuity for a subbasis only). For $x \in K$, there are open neighborhoods $V_x \subset X$ of x and $W_x \subset T$ of t such that $f(V_x \times W_x) \subset U$. By compactness, $K \subset V_{x_1} \cup \dots \cup V_{x_n} =: V$ for some V_{x_i} . Let $W = \bigcap_{i=1}^n W_{x_i}$. Then $f(K \times W) \subset f(V \times W) \subset U$. So $t' \in W$ implies that $f_{t'} \in M(K, U)$ as claimed. \square

Note. This theorem implies that a homotopy $X \times I \rightarrow Y$ with X locally compact is the same thing as a path $I \rightarrow Y^X$ when we give Y^X the compact-open topology.

Note. This is precisely the reason why, when we define $\text{MCG}(X)$, we define it as $\pi_0 \text{Homeo}^+(X, \partial X)$ where we equip $\text{Homeo}^+(X, \partial X)$ with the subspace topology inherited from X^X in the compact-open topology. By the above theorem, a path $I \rightarrow \text{Homeo}^+(X, \partial X)$ given as $t \mapsto \gamma_t$ is continuous if and only if the associated function $\gamma: X \times I \rightarrow X$ given by $\gamma(x, t) = \gamma_t(x)$ is continuous. But since each γ_t is a self-homeomorphism of X , this just tells us that γ is an isotopy of X . So path components of $\text{Homeo}^+(X, \partial X)$ correspond to isotopy classes of orientation-preserving self-homeomorphisms of X fixing the boundary point-wise.

Theorem 1.6 (The Exponential Law). *Let X and T be locally compact Hausdorff spaces and let Y be an arbitrary Hausdorff space. Then there is the homeomorphism*

$$Y^{X \times T} \cong (Y^X)^T$$

taking $f \mapsto f^$ where $f^*(t)(x) = f(x, t) = f_t(x)$.*

Proof. By Theorem 1.5, the assignment $f \mapsto f^*$ is a bijection.

We must show it and its inverse to be continuous. Let $U \subset Y$ be open and $K \subset X, K' \subset T$ be compact. Then

$$\begin{aligned} f \in M(K \times K', U) &\iff (t \in K', x \in K \implies f_t(x) = f(x, t) \in U) \\ &\iff (t \in K' \implies f_t \in M(K, U)) \\ &\iff f^* \in M(K', M(K, U)). \end{aligned}$$

Now, the $K \times K'$ are compact subsets of $X \times T$, and the collection of all these over $X \times T$ contain a neighborhood basis at each point since X and T are both assumed to be locally compact. By Lemma 1.2, the collection

$$\{M(K \times K', U) \mid U \subset Y \text{ open}, K \subset X, K' \subset T \text{ both compact}\}$$

forms a subbasis for the compact-open topology on $Y^{X \times T}$. Also, the $M(K, U)$ give a subbasis for Y^X and therefore the $M(K', M(K, U))$ form a subbasis for the topology on $(Y^X)^T$. Since we showed that these subbases correspond to one another under the exponential correspondence, the theorem is proved. \square

Proposition 1.7. *If X is locally compact Hausdorff and Y and W are Hausdorff, then there is the homeomorphism*

$$Y^X \times W^X \xrightarrow{\cong} (Y \times W)^X$$

given by $(f, g) \mapsto f \times g$.

Proof. It is clearly a bijection. If $K, K' \subset X$ are compact and $U \subset Y$ and $V \subset W$ are open, then

$$\begin{aligned} (f, g) \in M(K, U) \times M(K', V) &\iff (x \in K \implies f(x) \in U) \text{ and } (x \in K' \implies g(x) \in V) \\ &\iff ((x, y) \in K \times K' \implies f \times g(x, y) \in U \times V) \\ &\iff f \times g \in M(K, U \times W) \cap M(K', U \times W). \end{aligned}$$

so $(f, g) \mapsto f \times g$ is an open map.

Also $(f, g) \in M(K, U) \times M(K, V) \iff f \times g \in M(K, U \times V)$ which implies that the function is continuous. \square

Proposition 1.8. *If X and T are locally compact Hausdorff spaces and Y is an arbitrary Hausdorff space, then there is the homeomorphism*

$$Y^{X \sqcup T} \xrightarrow{\cong} Y^X \times Y^T$$

taking $f \mapsto (f \circ \iota_X, f \circ \iota_T)$.

Proof. The map is clearly well-defined and injective. Also, given $(f, g) \in Y^X \times Y^T$, we can define a function $f \cup g: X \sqcup T \rightarrow Y$ by f on X and g on T , and clearly, $f \cup g \mapsto (f, g)$ under the correspondence, giving surjectivity. We must show that it is continuous and has continuous inverse.

Let $f: X \sqcup T \rightarrow Y$ and suppose $(f \circ \iota_X, f \circ \iota_T) \in M(K, U) \times M(K', V)$. Then $f \in M(K, U) \cap M(K', V)$ which is an open set that is mapped precisely to $M(K, U) \times M(K', V)$. Hence $f \mapsto (f \circ \iota_X, f \circ \iota_T)$ is continuous.

Conversely, note that under the correspondence, $M(C \sqcup C', U)$ is mapped to $M(C, U) \times M(C', U)$, so the map is also open. \square

Theorem 1.9. *For X locally compact and both X and Y Hausdorff, Y^X is a covariant functor of Y and a contravariant functor of X from \mathbf{Top} to \mathbf{Top} .*

2. METHODS OF CALCULATION

2.1. Excision for Homotopy Groups.

Theorem 2.1. *Let X be a CW complex decomposed as the union of subcomplexes A and B with nonempty connected intersection $C = A \cap B$. If (A, C) is m -connected and (B, C) is n -connected, $m, n \geq 0$, then the map $\pi_i(A, C) \rightarrow \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.*

Corollary 2.2 (Freudenthal Suspension Theorem). *The unreduced suspension map $\pi_i(S^n) \rightarrow \pi_{i+1}(S^{n+1})$, induced by the suspension map $S^n \rightarrow \Sigma S^n \cong S^{n+1}$, is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$. More generally, this holds for the suspension $\pi_i(X) \rightarrow \pi_{i+1}(\Sigma X)$ whenever X is an $(n - 1)$ -connected CW complex.*

Proof of Corollary. Decompose the unreduced suspension $\Sigma X = (X \times I) / (X \times \{0\}, X \times \{1\})$ as the union of two cones C_+X and C_-X intersecting in a copy of X . Recall that a map $f: X \rightarrow Y$ induces a suspended map $\Sigma f: \Sigma X \rightarrow \Sigma Y$. Now, if we consider f to be any map $f: (S^n, s_0) \rightarrow (X, x_0)$, then we have a suspended map

$$\begin{array}{ccc} S^n \times I & \xrightarrow{f \times \text{id}} & X \times I \\ \downarrow & & \downarrow \\ S^{n+1} \cong \Sigma S^n & \xrightarrow{\Sigma f} & \Sigma X \end{array}$$

So, in particular, Σf is some class in $\pi_{n+1}(\Sigma X)$. Define the suspension homomorphism $\pi_i(X) \rightarrow \pi_{i+1}(\Sigma X)$ to be the map that sends f to Σf . This is a homomorphism (why?).

The unreduced suspension map is the same as the map

$$\pi_i(X) \cong \pi_{i+1}(C_+X, X) \rightarrow \pi_{i+1}(\Sigma X, C_-X) \cong \pi_{i+1}(\Sigma X).$$

(why?) where the two isomorphisms come from the LES of pairs and the middle map is induced by inclusion. The first map $\pi_i(X) \rightarrow \pi_{i+1}(C_+X, X)$ takes a map $(I^i, \partial I^i) \rightarrow (X, x_0)$ to the map $(I^{n+1}, \partial I^{n+1}, J^n) \rightarrow (C_+X, X, x_0)$ constructed by extending the given map radially to correspond with the height of C_+X . So one face of I^{n+1} will be mapped to the vertex of C_+X .

Including this into $(\Sigma X, C_-X)$ gives the middle homomorphism, and then the map $\pi_{i+1}(\Sigma X, C_-X) \rightarrow \pi_{i+1}(\Sigma X)$ is simply the identity on our map.

From the LES of $(C_\pm X, X)$, we see that this pair is n -connected if X is $(n - 1)$ -connected. Then Theorem 2.1 gives that the middle map is an isomorphism for $i + 1 < 2n$ and surjective for $i + 1 = 2n$. □

Example 2.3 ($\pi_n(\bigvee_\alpha S_\alpha^n)$). We want to show that $\pi_n(\bigvee_\alpha S_\alpha^n)$ for $n \geq 2$ is free abelian with basis the homotopy classes of the inclusions $S_\alpha^n \hookrightarrow \bigvee_\alpha S_\alpha^n$. Suppose first that there are only *finitely many* summands S_α^n . Then we can regard $\bigvee_\alpha S_\alpha^n$ as the n -skeleton of the product $\prod_\alpha S_\alpha^n$, where S_α^n is given the usual CW structure and $\prod_\alpha S_\alpha^n$ has the product CW structure. (See Hatcher appendix A). By construction then $\prod_\alpha S_\alpha^n$ has cells only in dimensions a multiple of n , so the pair $(\prod_\alpha S_\alpha^n, \bigvee_\alpha S_\alpha^n)$ is $(2n - 1)$ -connected by Corollary ???. So from the LES for the pair, we see that the inclusion $\bigvee_\alpha S_\alpha^n \hookrightarrow \prod_\alpha S_\alpha^n$ induces an isomorphism on homotopy

groups in dimensions $\leq 2n - 1$. Next we have $\pi_n(\prod_\alpha S_\alpha^n) \cong \bigoplus_\alpha \pi_n(S_\alpha^n) \cong \bigoplus_\alpha \mathbb{Z}$, a free abelian group with basis the inclusions $S_\alpha^n \hookrightarrow \prod_\alpha S_\alpha^n$, so pulling this back along the isomorphism $\pi_n(\bigvee_\alpha S_\alpha^n) \cong \pi_n(\prod_\alpha S_\alpha^n)$, the same is true for $\bigvee_\alpha S_\alpha^n$. This proves the claim when there are finitely many S_α^n 's.

When there are infinitely many summands S_α^n , consider the homomorphism $\Phi: \bigoplus_\alpha \pi_n(S_\alpha^n) \rightarrow \pi_n(\bigvee_\alpha S_\alpha^n)$ induced by the inclusions $S_\alpha^n \hookrightarrow \bigvee_\alpha S_\alpha^n$. Then Φ is surjective since any map $f: S^n \rightarrow \bigvee_\alpha S_\alpha^n$ has compact image contained in the wedge sum of finitely many S_α^n 's, so by the finite case already proved, $[f]$ is in the image of Φ .

Similarly, a nullhomotopy of f has compact image contained in a finite wedge sum of S_α^n 's, so the finite case also implies that Φ is injective.

Proposition 2.4. *If a CW pair (X, A) is r -connected and A is s -connected, with $r, s \geq 0$, then the map $\pi_i(X, A) \rightarrow \pi_i(X/A)$ induced by the quotient map $X \rightarrow X/A$ is an isomorphism for $i \leq r + s$ and a surjection for $i = r + s + 1$.*

Proof. Consider $X \cup CA$. Since A is closed and the inclusion $A \hookrightarrow X$ is a cofibration (since these are CW complexes), the map $h: C_\ell = X \cup CA \rightarrow X/A$ is a homotopy equivalence by Theorem ???. So we have a commutative diagram

$$\begin{array}{ccccc} \pi_i(X, A) & \longrightarrow & \pi_i(X \cup CA, CA) & \longrightarrow & \pi_i(X \cup CA/CA) = \pi_i(X/A) \\ & & \uparrow \cong & \nearrow \cong & \\ & & \pi_i(X \cup CA) & & \end{array}$$

where the vertical isomorphism comes from the LES of the pair $(X \cup CA, CA)$. Now, applying Theorem 2.1 to $(A, B) = (X, CA)$, since (X, A) is r -connected and (CA, A) is $(s + 1)$ -connected, we find that the homomorphism $\pi_i(X, A) \rightarrow \pi_i(X \cup CA, CA)$ induced by the inclusion is an isomorphism for $i < r + s + 1$ and a surjection for $i = r + s + 1$, which proves the result. \square

Example 2.5 (Construction of spaces with a particular group as π_n). Suppose X is obtained from a wedge of spheres $\bigvee_\alpha S_\alpha^n$ by attaching cells e_β^{n+1} via basepoint-preserving maps $\varphi_\beta: S^n \rightarrow \bigvee_\alpha S_\alpha^n, n \geq 2$. By cellular approximation, we know that $\pi_i(X) = 0$ for $i < n$, and we shall show that $\pi_n(X)$ is the quotient of the free abelian group $\pi_n(\bigvee_\alpha S_\alpha^n) \cong \bigoplus_\alpha \mathbb{Z}$ by the subgroup generated by the classes $[\varphi_\alpha]$. Any subgroup can be realized in this way, by choosing maps φ_β to represent a set of generators for the subgroup. Let $X = (\bigvee_\alpha S_\alpha^n) \cup_\beta e_\beta^{n+1}$.

Then the LES of the pair $(X, \bigvee_\alpha S_\alpha^n)$ gives

$$\pi_{n+1}\left(X, \bigvee_\alpha S_\alpha^n\right) \xrightarrow{\partial} \pi_n\left(\bigvee_\alpha S_\alpha^n\right) \rightarrow \pi_n(X) \rightarrow 0.$$

so

$$\pi_n(X) \cong \pi_n\left(\bigvee_\alpha S_\alpha^n\right) / \text{im } \partial$$

The quotient $X / \bigvee_\alpha S_\alpha^n$ is a wedge of spheres S_β^{n+1} , so by Proposition 2.4 and Example 2.3, the map $\pi_{n+1}(X, \bigvee_\alpha S_\alpha^n) \rightarrow \pi_{n+1}(X / \bigvee_\alpha S_\alpha^n) \cong \pi_{n+1}(\bigvee_\beta S_\beta^{n+1})$ is an isomorphism, so $\pi_{n+1}(X, \bigvee_\alpha S_\alpha^n)$ is free with basis the characteristic maps φ_β of the cells e_β^{n+1} . The boundary map ∂ takes these to the classes $[\varphi_\beta]$, so the result follows.

2.1.1. Eilenberg-MacLane Spaces.

Definition 2.6 (Eilenberg-MacLane space, $K(G, n)$). A space X having just one nontrivial homotopy group $\pi_n(X) \cong G$ is called an *Eilenberg-MacLane space* $K(G, n)$.

Construction of Eilenberg-MacLane Spaces:

Given arbitrary G and n , and assuming G is abelian if $n > 1$, we can construct a CW complex $K(G, n)$. To begin, construct the CW complex X from Example 2.5. Then X is an $(n-1)$ -connected CW complex of dimension $n+1$ such that $\pi_n(X) \cong G$ by construction. Alternatively, given the existence of Moore spaces $M(G, n)$ for any G and n , we can take a Moore space $M(G, n)$ and use the Hurewicz isomorphism to conclude that $\pi_n(X) \cong H_n(X)$. Hence we just need to fix all homotopy groups of dimension greater than n . By Example ??, we can construct a CW complex X_n containing X as a subcomplex such that $\pi_n(X_n) \cong \pi_n(X) \cong G$ while $\pi_k(X_n) \cong 0$ for all $k \neq n$.

Example 2.7 (Constructing spaces with arbitrary (abelian) homotopy groups). Recall that

$$\pi_n \left(\prod_{\alpha} X_{\alpha} \right) \cong \prod_{\alpha} \pi_n(X_{\alpha}),$$

so if we have a sequence of abelian groups $\{G_{n_i}\}_{i \in I}$, and let X_{n_i} denote that $K(G_{n_i}, n_i)$ space, then we find that

$$\pi_k \left(\prod_{i \in I} X_{n_i} \right) \cong \prod_{i \in I} \pi_k(X_{n_i}) \cong \begin{cases} G_{n_i}, & k = n_i \text{ for some } i \in I \\ 0, & \text{else} \end{cases}$$

Having covered the existence of Eilenberg-MacLane spaces, we now find the following for uniqueness of these spaces:

Proposition 2.8 (Uniqueness of Eilenberg-MacLane spaces). *The homotopy type of a CW complex $K(G, n)$ is uniquely determined by G and n .*

The proof is based on the following lemma giving a condition for when homomorphisms between homotopy groups are induced by some map:

Lemma 2.9. *Let X be a CW complex of the form $(\bigvee_{\alpha} S_{\alpha}^n) \bigcup_{\beta} e_{\beta}^{n+1}$ for some $n \geq 1$. Then for every homomorphism $\psi: \pi_n(X) \rightarrow \pi_n(Y)$ with Y path-connected there exists a map $f: X \rightarrow Y$ with $f_* = \psi$.*

Proof. The construction of f is as one would expect: first let f send the natural basepoint of $\bigvee_{\alpha} S_{\alpha}^n$ to a chosen basepoint $y_0 \in Y$. Now for every sphere S_{α}^n in X , we extend f over the sphere via a map representing $\psi([i_{\alpha}])$ where i_{α} is the inclusion $S_{\alpha}^n \hookrightarrow X$. This defines f on the n -skeleton of X : $f: X^n \rightarrow Y$. Since now $f_*[i_{\alpha}] = \psi[i_{\alpha}]$ for all α and the $[i_{\alpha}]$ generate $\pi_n(X^n)$, this defines f_* on all of $\pi_n(X^n)$.

To extend f over the $(n+1)$ -cells, it will suffice to show that $f \circ \varphi_{\beta}$ is nullhomotopic, where $\varphi_{\beta}: S^n \rightarrow X^n$ is the attaching map for the $(n+1)$ -cell e_{β}^{n+1} . But $f \circ \varphi_{\beta}$ is a representative of $f_*[\varphi_{\beta}] = \psi[\varphi_{\beta}]$. Thus we have transformed $f_*[\varphi_{\beta}]$ into an element in the image of $\psi: \pi_n(X) \rightarrow \pi_n(Y)$, and for this, we can use the extra structure of X , not just X^n . In X , $[\varphi_{\beta}]$ is trivial via the characteristic map of the cell e_{β}^{n+1} , so $\psi[\varphi_{\beta}] = \psi(0) = 0$, thus indeed $f \circ \varphi_{\beta}$ is nullhomotopic. Thus we obtain the desired extension $f: X \rightarrow Y$. To see that $f_* = \psi$, simply note that by

cellular approximation, any element of $\pi_n(X)$ can be represented as an element in $\pi_n(X^n)$, and on $\pi_n(X^n)$, f_* agrees with ψ by construction. \square

Proof of Proposition 2.8. Let K' be any $K(G, n)$ CW complex, and let K be the specific $K(G, n)$ CW complex constructed in Example 2.5. In particular, K is of the form of Lemma 2.9. Since $\pi_n(K) = \pi_n(Y)$, we can apply Lemma 2.9 to obtain a map $f: K \rightarrow K'$ inducing the identity on π_n . Since all other homotopy groups of K and K' are trivial, Whitehead's theorem now gives that f is a homotopy equivalence. Since homotopy equivalence is an equivalence relation, this finishes the proof. \square

2.2. The Hurewicz Theorem.

Theorem 2.10. *If a space X is $(n-1)$ -connected, $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$. If a pair (X, A) is $(n-1)$ -connected, $n \geq 2$, with A simply connected and nonempty, then $H_i(X, A) = 0$ for $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$.*

Remark. This result is, in a sense, the best that we can expect. For example, S^n has trivial homology groups above dimension n but many nontrivial homotopy groups in this range when $n \geq 2$; and conversely, Eilenberg-MacLane spaces such as $\mathbb{C}P^\infty$ have trivial higher homotopy groups but many nontrivial homology groups.

Corollary 2.11. *A map $f: X \rightarrow Y$ between simply-connected CW complexes is a homotopy equivalence if $f_*: H_n(X) \rightarrow H_n(Y)$ is an isomorphism for each n .*

Proof. By replacing Y with the mapping cylinder M_f , we may assume f is the inclusion $X \hookrightarrow Y$. Since X and Y are simply-connected, $\pi_1(Y, X) = 0$. The relative Hurewicz theorem says that the first nonzero $\pi_n(Y, X)$ is isomorphic to the first nonzero $H_n(Y, X)$, but by the LES of the pair (Y, X) in homology, $H_n(Y, X) \cong 0$ for all $n \geq 0$, so also $\pi_n(Y, X) \cong 0$ for all $n \geq 0$, so f induces isomorphisms $\pi_n(X) \rightarrow \pi_n(Y)$ for all n . By Whitehead's theorem, f is a homotopy equivalence. \square

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