

**Exercise 0.1.** Show that an  $R$ -module homomorphism  $f: M \rightarrow N$  is an isomorphism if and only if it is injective and surjective.

*Proof.* Suppose  $f$  is an isomorphism, so there exists a homomorphism  $f^{-1} \in \text{Hom}_R(N, M)$  such that  $f \circ f^{-1} = \mathbb{1}_N$  and  $f^{-1} \circ f = \mathbb{1}_M$ . Suppose  $f(v) = f(w)$ . Then  $v = f^{-1} \circ f(v) = f^{-1} \circ f(w) = w$ , so  $f$  is injective. Now for  $n \in N$ , we have  $n = f(f^{-1}(n))$ , so  $f$  is surjective.

Conversely, if  $f$  is injective, it has a left inverse as a function, and if it is surjective, it has a right inverse, and these are unique and equal (by the same general property for functions). Denote this function by  $f^{-1}: N \rightarrow M$ . We claim this is an  $R$ -module homomorphism. Indeed, as  $f$  is a group homomorphism, we know the inverse is as well. Now, since  $f \circ f^{-1}(rn) = rn$  and  $rn = rf \circ f^{-1}(n) = f(rf^{-1}(n))$ , we have by injectivity of  $f$  that  $f^{-1}(rn) = rf^{-1}(n)$ , so indeed  $f^{-1}$  is an  $R$ -module homomorphism.  $\square$

**Exercise 0.2.**  $\text{Hom}_R(M, N)$  has the structure of an abelian group with

$$\begin{aligned}(f + g)(x) &:= f(x) + g(x) \\ (-f)(x) &:= -f(x)\end{aligned}$$

*Proof.* We must check associativity, identity, inverse and commutativity when the set  $\text{Hom}_R(M, N)$  is equipped with the binary operator and inverse defined above. Associativity is inherited from associativity of functions. Now, let  $0_N$  denote the identity of the abelian group  $N$ . Define a map  $0$  by  $0(m) = 0_N$  for all  $m \in M$ . Then for any  $f \in \text{Hom}_R(M, N)$ , we have

$$(f + 0)(x) = f(x) + 0(x) = f(x) + 0_N = f(x)$$

and similarly,  $0 + f = f$ . Thus  $0$  is an identity for  $\text{Hom}_R(M, N)$ . Now

$$(f + (-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0_N$$

so  $f + (-f) = 0$ , and similarly,  $(-f) + f = 0$ .

Lastly, commutativity follows from commutativity of  $N$ .  $\square$

**Exercise 0.3.** For a homomorphism  $f \in \text{Hom}_R(M, N)$ , the kernel  $\ker(f) = \{x \mid f(x) = 0\}$  and the image  $\text{im}(f)$  are submodules.

*Proof.* For  $x, y \in \ker f$ , we have  $f(x - y) = f(x) - f(y) = \mathbb{1}_R$  by  $f$  being a group homomorphism, and  $f(rx) = rf(x) = r \cdot \mathbb{1}_R = \mathbb{1}_R$  by the definition of an  $R$ -module and an  $R$ -linear map, hence  $\ker f$  is also closed under multiplication by elements of  $R$ . The inclusion  $\ker f \rightarrow M$  is  $R$ -linear since if  $x, y \in \ker f$ , then  $\iota(rx + y) = rx + y = r\iota(x) + \iota(y)$ .

Likewise, if  $x, y \in \text{im } f$ , then let  $u, v \in M$  such that  $f(u) = x$  and  $f(v) = y$ . Then  $f(u - v) = f(u) - f(v) = x - y \in \text{im } f$  and  $f(ru) = rf(u) = rx \in \text{im } f$  for all  $r \in R$  and for all  $x, y \in \text{im } f$ . Furthermore, for  $x, y \in \text{im } f$ ,  $\iota(rx + y) = rx + y = r\iota(x) + \iota(y)$  so the inclusion  $\text{im } f \rightarrow N$  is  $R$ -linear.  $\square$

**Exercise 0.4.**  $R/I$  is a ring if  $I$  is a two-sided ideal of  $R$ .

*Proof.* If  $I$  is a two-sided ideal, then firstly,  $R/I$  is an abelian group under  $+$  since everything is abelian hence normal. All other requirements for a ring are inherited from  $R$ . We must only check that multiplication is well-defined. If  $r + I = r' + I$  and  $s + I = s' + I$  then  $s^{-1} \underbrace{r^{-1}r'}_{\in I} s' \in I$ , so indeed  $rs + I = r's' + I$ .  $\square$

**Exercise 0.5.** Show that a cokernel exists and is unique up to isomorphism.

*Proof.* For an  $R$ -linear map  $f: M \rightarrow N$ , define  $\text{coker } f := N/\text{im } f$ . Since  $\text{im } f$  is a submodule of  $N$ ,  $N/\text{im } f$  is an  $R$ -module. Furthermore, it satisfies the diagram

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{q} & \text{coker } f \\ & \searrow 0 & \downarrow g & \swarrow \exists! \bar{g} & \\ & & L & & \end{array}$$

since  $\bar{g}$  must be defined by  $\bar{g}(q(n)) = g(n)$ . We must check that this is well-defined. Suppose  $\bar{a} = \bar{b} \in \text{coker } f$ , so  $a - b \in \text{im } f$ , so  $f(m) = a - b$ . Then  $0 = gf(m) = \bar{g}(q(a) - q(b)) = \bar{g}(\bar{a} - \bar{b}) = \bar{g}(a) - \bar{g}(b)$ .

It is also  $R$ -linear because  $\bar{g}(\bar{x} + \bar{y}) = \bar{g}(\overline{x+y}) = g(x+y) = g(x) + g(y) = \bar{g}(\bar{x}) + \bar{g}(\bar{y})$ , and  $\bar{g}(r\bar{x}) = \bar{g}(\overline{rx}) = g(rx) = rg(x) = r\bar{g}(\bar{x})$ .

To check uniqueness, suppose  $(K, q')$  also satisfies the above diagram. Then letting  $g = q$  and  $L = \text{coker } f$ , we get

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{q'} & K \\ & \searrow 0 & \downarrow q & \swarrow \exists! \bar{q} & \\ & & \text{coker } f & & \end{array}$$

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{q} & \text{coker } f \\ & \searrow 0 & \downarrow q' & \swarrow \exists! \bar{q}' & \\ & & K & & \end{array}$$

Interchanging  $\text{coker } f$  and  $K$ , we also get a unique map  $\bar{q}': \text{coker } f \rightarrow K$ . These have the property that  $\bar{q}' \circ \bar{q} \circ q' = \bar{q}' \circ q = q'$  so by uniqueness of  $\mathbb{1}_K$  in the diagram

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{q'} & K \\ & \searrow 0 & \downarrow q' & \swarrow \mathbb{1}_K & \\ & & K & & \end{array}$$

we get  $\bar{q}' \circ \bar{q} = \mathbb{1}_K$ , and likewise,  $\bar{q} \circ \bar{q}' = \mathbb{1}_{\text{coker } f}$ . So  $\bar{q}: \text{coker } f \rightarrow K$  is an isomorphism.  $\square$

**Exercise 0.6.** Prove uniqueness of the direct product.

*Proof.* Suppose  $A$  and  $B$  are both direct products with maps  $\pi_{A,j}: A \rightarrow M_j$  and  $\pi_{B,j}: B \rightarrow M_j$  for all  $j$  such that the universal diagram is fulfilled. Then, since we have maps  $\pi_{B,j}: B \rightarrow M_j$  for all  $j$ , we have a unique map  $u: B \rightarrow A$  such that  $\pi_{A,j} \circ u = \pi_{B,j}$  for all  $j$ . And similarly, we have a map  $v: A \rightarrow B$  such that  $\pi_{B,j} \circ v = \pi_{A,j}$  for all  $j$ . But then  $\pi_{A,j} \circ u \circ v = \pi_{A,j}$  for all  $j$ , so since  $\pi_{A,j} \circ \mathbb{1}_A = \pi_{A,j}$  for all  $j$ , we get by uniqueness that  $u \circ v = \mathbb{1}_A$ , and similarly, interchanging  $A$  for  $B$  above, we get  $v \circ u = \mathbb{1}_B$ . Hence  $u: B \rightarrow A$  is an isomorphism.  $\square$

*Remark* (Direct product). Note that the direct product of a family  $(M_i)_{i \in I}$  is simply the universal cone over the diagram  $I \rightarrow \text{Mod}_R$  given by sending  $i \mapsto M_i$  where  $I$  is a discrete category.

*Remark* (Direct sum). The direct sum is the dual of the direct product. Given a diagram  $I \rightarrow \text{Mod}_R$  where  $I$  is discrete, we define the direct sum as the cone under  $I \rightarrow \text{Mod}_R$  and let its nadir be denoted  $\bigoplus_{i \in I} M_i$  together with  $R$ -linear maps  $\iota_j: M_j \rightarrow \bigoplus_{i \in I} M_i$ .

*Remark.* For each  $i \in I$ , there is a unique map  $f_j: M_j \rightarrow \prod_{j \in I} M_j$  with  $\pi_i f_j = 0$  if  $i \neq j$  and  $\pi_j f_j = \mathbb{1}_{M_j}$ . By the universal property, we thus get a unique map  $u: \bigoplus_{i \in I} M_i \rightarrow \prod_{i \in I} M_i$  given by  $u(x) = \sum_{i \in I} f_i(x(i))$ . This map is an isomorphism when  $I$  is finite, but not necessarily when  $I$  is infinite.

**Corollary 0.7.** *Summarizing, we have bijections*

$$\text{Hom}_R \left( N, \prod_{i \in I} M_i \right) \xrightarrow{\sim} \prod_{i \in I} \text{Hom}_R(N, M_i) \\ u \mapsto (\pi_i u)_{i \in I}$$

where we send  $(\pi_i u)_{i \in I}(j) = \pi_j u: N \xrightarrow{u} \prod_{i \in I} M_i \xrightarrow{\pi_j} M_j$  and

$$\text{Hom}_R \left( \bigoplus_{i \in I} M_i, N \right) \xrightarrow{\sim} \prod_{i \in I} \text{Hom}_R(M_i, N) \\ u \mapsto (u \iota_i)_{i \in I}$$

which are in fact isomorphisms of abelian groups.

**Exercise 0.8.** Show that the above bijections are isomorphisms of abelian groups.

*Proof.* Since they are bijections between abelian groups, we must simply show that they are homomorphisms.

Now,

$$u + v \mapsto (\pi_i(u + v))_{i \in I} = (\pi_i u + \pi_i v)_{i \in I}$$

and  $(\pi_i u + \pi_i v)(j) = (\pi_i u)(j) + (\pi_i v)(j)$  so  $u + v \mapsto (\pi_i u) + (\pi_i v)$  by the definition of addition in the direct product, hence this is indeed a homomorphism.

For the direct sum, we similarly have

$$u + v \mapsto ((u + v) \iota_i)_{i \in I} = (u \iota_i + v \iota_i)_{i \in I}$$

and  $(u \iota_i + v \iota_i)_{i \in I}(j) = (u \iota_i)_{i \in I}(j) + (v \iota_i)_{i \in I}(j)$ , so  $u + v \mapsto (u \iota_i)_{i \in I} + (v \iota_i)_{i \in I}$  which is indeed a homomorphism by the additive structure on the direct product which the direct sum inherits. □

**Exercise 0.9.** Show that cyclic left  $R$ -modules (of  $R$ ?) are precisely those of the form  $R/I$  for some left ideal  $I \subset R$ .

*Proof.* Suppose  $M$  is a cyclic left  $R$ -module. Then  $M = Rx$  for some  $x \in M$ . □

**Exercise 0.10.** Show that an  $R$ -module is free if and only if it has a basis.

*Proof.* Suppose  $M$  is a free  $R$ -module. Let  $X$  be a generating set and  $\mu: X \rightarrow M$  the map satisfying the universal property that for any  $R$ -module  $N$  and any map  $f: X \rightarrow N$ , there exists a unique  $R$ -linear map  $\tilde{f}: M \rightarrow N$  such that  $\tilde{f}\mu = f$ .

$$\begin{array}{ccc}
X & \xrightarrow{\mu} & M \\
& \searrow f & \downarrow \exists! \tilde{f} \\
& & N
\end{array}$$

We claim  $M$  has  $\mu(X)$  as a basis. Suppose there exists a finite linear combination  $\sum_{i=1}^n r_i x_i = 0$  with  $x_i \in \mu(X)$  and  $r_i \in R$  such that not all  $r_i$  are 0. Then let  $N = \bigoplus_X R$  and  $f: X \rightarrow \bigoplus_X R$  be the inclusion  $x \mapsto \iota_x(\mathbb{1}_R)$ . By uniqueness, we must have that  $0 = \tilde{f}(\mu(x_i)) = \iota_x(\mathbb{1}_R)$ , so  $\tilde{f}(\sum r_i x_i) = \sum r_i \tilde{f}(\mu(x_i)) = \sum r_i f(x_i) = \sum r_i \iota_{x_i}(\mathbb{1}_R)$ , but by definition then  $0 = (\sum r_i f(x_i))(j) = (\sum r_i \iota_{x_i}(\mathbb{1}_R))(j) = \sum r_i \iota_{x_i}(\mathbb{1}_R)(j) \sum r_i \mathbb{1}_R \delta_{i,j} = r_j$  for all  $j$ , hence we obtain linear independence. Conversely, suppose  $M$  has  $X \subset M$  as a basis. Let  $\mu: X \rightarrow M$  be the inclusion. We claim this is a free  $R$ -module.

Suppose  $N$  is any other  $R$ -module and we have any (not necessarily an  $R$ -module homomorphism) map  $f: X \rightarrow N$ . If  $\tilde{f}\mu = f$ , then we must have  $\tilde{f}(x) = f(x)$  for all  $x \in X$ . Now, for any linear combination  $\sum r_i x_i \in M$ , we have by linearity,  $\tilde{f}(\sum r_i x_i) = \sum r_i \tilde{f}(x_i) = \sum r_i f(x_i)$ , so  $\tilde{f}$  is indeed uniquely determined by  $f$ .  $\square$

**Exercise 0.11.** Complete the proof that every commutative ring  $R$  has invariant basis number.

*Proof.* By Zorn's lemma, every commutative ring  $R$  has a maximal ideal  $I \leq R$ . Then  $R/I$  is a field. Let now  $M$  be a free  $R$ -module with basis  $\{x_i\}_{i \in J}$ . We claim  $M/IM$  is an  $R/I$  module. Define  $\bar{r} \cdot \bar{x} = \overline{rx}$  where multiplication of  $rx$  is done in  $M$  over  $R$ , and define  $\bar{v} + \bar{w} = \overline{v+w}$ . Suppose  $\bar{r} = \bar{r}'$  and  $\bar{x} = \bar{x}'$ . So  $r' - r \in I$  and  $x' - x \in IM$ . Then  $r'x' - rx' \in IM$  and  $rx' - rx \in IM$ , so  $r'x' - rx = r'x' - rx' + rx' - rx \in IM$ , hence  $\overline{r'x'} = \overline{rx}$  is well defined. Similarly, if  $\bar{v} = \bar{v}'$  and  $\bar{w} = \bar{w}'$  then  $v - v', w - w' \in IM$ , so  $v + w - (v' + w') \in IM$ , hence  $\overline{v+w} = \overline{v'+w'}$ .

The properties for  $M/IM$  being an  $R/I$ -module are then inherited from  $M$  as an  $R$ -module.

But as  $R/I$  is a field,  $M/IM$  is a vector space over  $R/I$ , hence its dimension is well-defined. Now, suppose  $\sum \bar{r}_i \cdot \bar{x}_i = 0$  in  $M/IM$ . Then  $\sum r_i x_i = 0$ , hence  $\sum r_i x_i \in IM$ . But then there exists a linear combination  $\sum s_i x_i$  such that  $s_i \in I$  for all  $i$ , and such that  $\sum r_i x_i = \sum s_i x_i$ . Then  $\sum (r_i - s_i) x_i = 0$  and linear independence of  $\{x_i\}_{i \in J}$  gives  $r_i = s_i$  for all  $i$ , hence  $\bar{r}_i = \bar{0}$ . So  $\{\bar{x}_i\}_{i \in J}$  is linearly independent over  $R/I$  as well. Hence as it clearly also spans  $M/IM$ , we have that  $\dim_{R/I} M/IM = |J|$ . Thus an bases for  $M$  over  $R$  have the same cardinality, so  $R$  has invariant basis number.  $\square$

**Exercise 0.12.** Find a free non commutative ring  $R$  with bases of different cardinalities.