Assignment 6

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p. 78:
16: Prove that $O(n)$ is homeomorphic to $\mathrm{SO}(n) \times Z_2$. Are these two isomorphic as topological groups?

Solution: We have that an orthogonal matrix A has property $AA^T = I$, so $\det A \det A^T = 1$ and $\det A^T = \det A$, so $\det A = \pm 1$. Now, for any $A \in O(n)$, either $A \in \mathrm{SO}(n)$ or $XA \in SO(n)$ where X has -1 as its $1,1$ coordinate, 1 as its i,i coordinate for $2 \leq i \leq n$ and 0 as its i,j coordinate for $i \neq j$. Thus we can define a map $f\colon \mathrm{SO}(n) \times Z_2 \to O(n)$ by $f(A,0) = A$ and $f(A,1) = XA$. By the above, this is a surjective function. Since $O(n)$ is compact and $\mathrm{SO}(n) \times Z_2$ is Hausdorff, we will have that f is a homeomorphism if and only if f is continuous. Now, given an open set $U \subset O(n)$, we have that $O(n) = \det^{-1}(-1) \cup \det^{-1}(1)$, so $\det^{-1}(-1)$ and $\det^{-1}(1)$ are open in $O(n)$ and separate $O(n)$, so $U \cap \det^{-1}(-1)$ and $U \cap \det^{-1}(1)$ separate U as disjoint open sets. Let $V = \{-x \mid x \in U \cap \det^{-1}(-1)\}$. Then $f^{-1}(U) = (U \cap \det^{-1}(1), 0) \cup (V, 1)$ which are open, so f is continuous. Hence f is a homeomorphism.

Now, we claim that f is an isomorphism of topological groups if n is odd, and that if n is even, then $O(n)$ is not isomorphic to $\mathrm{SO}(n) \times Z_2$ as topological groups.

We first consider the case where n is odd:
Define the map $g\colon \mathrm{SO}(n) \times Z_2 \to O(n)$ by $g(A,0) = A$ and $g(A,1) = -A$. Since n is odd, we have $\det(-A) = -\det(A)$, we again this is surjective, and continuity is checked similarly, so we find that g is a homeomorphism.
Furthermore, we have

$$g\left((A,t) * (B,s)\right) = g\left((AB, t+s)\right) = (-1)^{t+s} AB = (-1)^t A (-1)^s B = f(A,t) f(B,s).$$

Thus $O(n) \cong \mathrm{SO}(n) \times Z_2$ as topological groups.

Now suppose n is even. In this case we do not have $\det(-A) = -\det(A)$, so we can't make use of the map g which gives us the niceness of commutativity in matrix multiplication to make the group homomorphism work.
Indeed, in this case, suppose $O(n) \cong \mathrm{SO}(n) \times Z_2$ with isomorphism $\varphi\colon \mathrm{SO}(n) \times Z_2 \to O(n)$.
Now, suppose $X \in O(n)$ is in the center. Thus $XA = AX$ for all $A \in O(n)$. Then $X = AXA^T$, so

$$x_{ij} = \sum_{k=1}^{n} (AX)_{ik} a_{jk} = \sum_{k=1}^{n} \sum_{r=1}^{n} a_{ir} x_{rk} a_{jk}$$

Now, taking A to be the matrix with $a_{i1} = a_{1i} = 1$ and all other entries equal 0, we get

$$x_{ii} = a_{i1} x_{11} a_{i1}$$

so the diagonal entries for X are all equal, and furthermore, for $j \neq i$

$$x_{ij} = x_{ji} = 0.$$

Furthermore, X has determinant ± 1, so the product of the diagonal entries is ± 1, so $x_{ii} = x_{ii}^n = \pm 1$. I.e., the center of $O(n)$ is precisely $\{I, -I\}$.
Now, clearly $I, -I$ are in the center of $\mathrm{SO}(n)$ as well, however, we thus get the four elements $(I,0), (I,1), (-I,0), (-I,1)$ in the center of $\mathrm{SO}(n) \times Z_2$. Now, if we have groups G, H and $g \in G$ is in the center of G then for $\psi\colon G \to H$ and isomorphism, we have $\psi(g)\psi(x) = \psi(gx) = \psi(xg) = \psi(x)\psi(g)$ for all $x \in G$, so $\psi(g)$ is in the center of H.
However, as φ is injective, this thus gives 4 distinct elements in the center of $O(n)$ as the images of $(\pm I, 0)$ and $(\pm I, 1)$, contradicting the cardinality of the center being 2.
So no such isomorphism exists.

85:
27: Find an action of Z_2 on the torus with orbit space the cylinder.

Solution: Consider the torus identified with $S^1 \times S^1 = T$ and the action of Z_2 on $S^1 \times S^1$ given by $g(e^{i\theta}, e^{i\alpha}) = \left(e^{i\theta}, e^{-i\alpha}\right)$ where g is a generator for Z_2.
We check the conditions of definition 4.14:

since for $h \in Z_2$, $h\left(e^{i\theta}, e^{i\alpha}\right) = \left(e^{i\theta}, e^{(-1)^h i\alpha}\right)$, we have $(h+g)\left(x, e^{i\alpha}\right) = (x, e^{(-1)^{h+g} i\alpha}) = \left(x, e^{(-1)^h (-1)^h i\alpha}\right) = h\left((x, e^{(-1)^g i\alpha}\right) = h\left(g\left(x, e^{i\alpha}\right)\right)$.

(b) $0\left(x, e^{i\alpha}\right) = g^2\left(x, e^{i\alpha}\right) = g\left(g\left(x, e^{i\alpha}\right)\right) = g\left(x, e^{-i\alpha}\right) = \left(x, e^{i\alpha}\right)$.

(c) Let $Z_2 \times T \to T$ be given by $(h, x) \xrightarrow{f} h(x)$. Then for an open set $U \subset T$, we have $f^{-1}(U)$ is $(0, U) \cup (1, V)$ where $V = \left\{\left(x, e^{-i\alpha}\right) \mid \left(x, e^{i\alpha}\right) \in U\right\}$.
Now, define the map $\varphi \colon T \to T$ by $\varphi\left(x, e^{i\alpha}\right)$. Now, as the component functions are continuous (identity and conjugation), φ is continuous and $V = \varphi^{-1}(U)$, so $(0, U) \cup (1, V)$ is open. Hence f is continuous.

The orbits are precisely

$$\begin{aligned}
&\left\{(x, e^{i\alpha}), (x, e^{-i\alpha}\right\}, \alpha \in (0, \pi), x \in S^1 \\
&\{(x, 1)\}, x \in S^1 \\
&\{(x, -1)\}, x \in S^1.
\end{aligned}$$

Now define a map $g \colon S^1 \times S^1 \to S^1 \times I$ by $g\left(e^{i\theta}, e^{i\alpha}\right) = \left(e^{i\theta}, \frac{|\alpha - \pi|}{\pi}\right)$. Now, consider $g_2 = \pi_2 g$ which maps $e^{i\alpha} \to \frac{|\alpha - \pi|}{\pi}$. Then $g_2^{-1}(J)$ for any closed interval $J \subset I$ is the union of two closed arcs on S^1 which is closed, so g_2 is continuous and hence the components of g are continuous, so g is continuous.
Now, given any $(x, t) \in S^1 \times I$ we further have that $\pi - \pi t \in [0, \pi]$, so $g\left(x, e^{i(\pi - \pi t)}\right) = \left(x, \frac{|\pi - \pi t - \pi|}{\pi}\right) = (x, t)$, so g is surjective. As $S^1 \times S^1$ is compact as the product of compact sets, and $S^1 \times I$ is Hausdorff as the product of Hausdorff spaces, we have by corollary 4.4 that g is an identification map.

Now, the induced identification space of g on $S^1 \times S^1$ is precisely the orbits of f with the identification topology, so by theorem 4.2.(a), we have that the orbit space of f is homeomorphic to $S^1 \times I$ which is the cylinder.

31: The stabilizer of a point $x \in X$ consists of those elements $g \in G$ for which $g(x) = x$. Show that the stabilizer of any point is a closed subgroup of G when X is Hausdorff, and that points in the same orbit have conjugate stabilizers for any X.

Solution: Let G_x denote the set of stabilizers of $x \in X$. Firstly, $G_x \leq G$ algebraically and can be checked as $e \in G_x$ and if $g, h \in G_x$ then since $h^{-1}(x) = h^{-1}(h(x)) \overset{4.14.(a)}{=} (h^{-1} * h)(x) = e(x) = x$, we have $h^{-1} \in G_x$, so $gh^{-1} \in G_x$, so $G_x \leq G$.
Suppose X is Hausdorff, and let $m \colon G \times X \to X$ by $m(g, x) = g(x)$. Since X is Hausdorff and Hausdorff implies T_1, singletons are closed, so $X - \{x\}$ is open. Now let $g \in G - G_x$. Then $(g, x) \in m^{-1}\left(X - \{x\}\right)$ which is open, so since π_1 is an open map, we have that $g \in \pi_1\left(m^{-1}\left(X - \{x\}\right)\right)$ which is open in G. Furthermore, if $h \in \pi_1\left(m^{-1}\left(X - \{x\}\right)\right)$, then $\{h\} \times X \cap m^{-1}\left(X - \{x\}\right) \neq \varnothing$, so there exists $x' \in X$ such that $m(h, x') = h(x') \in X - \{x\}$, so $h \notin G_x$. Thus $\pi_1\left(m^{-1}\left(X - \{x\}\right)\right) \cap G_x = \varnothing$, so $G - G_x \subset \pi_1\left(m^{-1}\left(X - \{x\}\right)\right) \subset G - G_x$, so $G - G_x = \pi_1\left(m^{-1}\left(X - \{x\}\right)\right)$ is open, and hence $G_x = G - (G - G_x)$ is closed.

Now, suppose x, y are in the same orbit. Hence there exists $g \in G$ such that $g(x) = y$. Now, let $h \in G_x$. Then $(ghg^{-1})(y) = (gh)(g^{-1}(y)) = (gh)(x) = g(h(x)) = g(x) = y$, so $ghg^{-1} \in G_y$, hence $gG_xg^{-1} \subset G_y$. Conversely, if $h \in G_y$, then $\left(g^{-1} h \left(g^{-1}\right)^{-1}\right)(x) = \left(g^{-1} h g\right)(x) = \left(g^{-1} h\right)(g(x)) = \left(g^{-1} h\right)(y) = g^{-1}\left(h(y)\right) = g^{-1}(y) = x$, so $g^{-1} G_y g \subset G_x$ and thus $G_y \subset gG_xg^{-1}$, so $gG_xg^{-1} = G_y$, so points in the same orbit have conjugate stabilizers for any X.