Exercise 0.1 (7.15)**.** Let $\dim V < \infty, F = \mathbb{C}$, and let $A \in \mathrm{End}(V)$ be normal. Prove that if B commutes with A, then it commutes with A^* as well.

Proof. □

Exercise 0.2 (8.6)**.** Let $A \in \mathrm{End}(V)$ be nilpotent, and $U \subset V$ invariant. Show that the quotient map $\overline{A} \in \mathrm{End}\,(V/U)$ is nilpotent.

Proof. Suppose $A^k = 0$ for some $k > 0$. We claim that $\overline{A}^k = 0$ for the same k. We recall by lemma 2.16 that $\overline{A} \in \mathrm{End}\,(V/U)$ is the unique endomorphism making $\overline{A} \circ \pi = \pi \circ A$ commute where $\pi \colon V \to V/U$ is the quotient map. It thus immediately follows that $\overline{A^k} = 0$ since this satisfies the commutative criterion. Now, we claim that suppose that for N we have shown $\overline{A}^N \circ \pi = \pi \circ A^N$. Then we get

$$\pi \circ A^{N+1} = (\pi \circ A) \circ A^N = \overline{A} \circ \pi \circ A^N = \overline{A}^{N+1} \circ \pi$$

so since the case for $N = 1$ was shown, we get by induction that $\overline{A}^k \circ \pi = \pi \circ A^k = 0$. Now, π is surjective by lemma 2.9, so given some $\overline{x} \in V/U$, let $x \in V$ be such that $\pi(x) = \overline{x}$. Then $\overline{A}^k \overline{x} = \overline{A}^k\,(\pi(x)) = \pi \circ A^K(x) = \pi(0) = \overline{0}$. So indeed $\overline{A}^k$ is equal to the zero endomorphism in $\mathrm{End}\,(V/U)$. Thus $\overline{A}$ is nilpotent.

□

Exercise 0.3 (10.11)**.** Show $\chi_{A^{-1}}(\lambda) = (-\lambda)^n \det(A)^{-1} \chi_A\left(\lambda^{-1}\right)$ for $A \in \mathrm{GL}(V)$, $\lambda \neq 0$ and $n = \dim V$.

Proof. We have

$$\begin{aligned}
\det\left(A^{-1} - \lambda I\right) &= \det\left(A^{-1}\left(I - \lambda A\right)\right) \\
&= \det\left(-A^{-1}\lambda\left(A - \lambda^{-1} I\right)\right) \\
&= \det(A^{-1}) \det(-\lambda I) \det\left(A - \lambda^{-1} I\right) && \text{(Thm 10.1.(ii))} \\
&= \det(A)^{-1} (-\lambda)^n \chi_A\left(\lambda^{-1}\right)
\end{aligned}$$

where the last step follows since $\det\left(A^{-1}\right) = \det(A)^{-1}$ by theorem 10.3, $\det(-\lambda I) = (-\lambda)^{\dim V} = (-\lambda)^n$ by theorem 10.1.(i), and $\det\left(A - \lambda^{-1} I\right) = \chi_A(\lambda^{-1})$ by definition 10.19, (10.2) and that $\chi_A(x) := \chi_{[A]}(x)$. □