Exercise 0.1 (1)**.** Let R be a Noetherian ring and A be a finitely generated R-algebra. Show that if $B \subset A$ is a subalgebra such that A is a finitely generated B-module, then B is also a finitely generated R-algebra.

Proof. We want to show that B is finitely generated as an R-algebra.

Suppose $\{y_1, \ldots, y_n\} \subset A$ generate A as an R-algebra, so $A = R[y_1, \ldots y_n]$. Since A is also finitely generated as a R-module, there exist $a_1, \ldots, a_m \in A$ such that $A = Ba_1 + \ldots + Ba_m$. Now using the module expression for A, write

$$y_i = \sum_j b_{ij} a_j$$

and similarly, since $a_i a_j \in A$,

$$a_i a_j = \sum_k b_{ijk} a_k.$$

Then given arbitrary $u, v \in A$, we can write

$$u = \sum_{i,j} \alpha_i b_{ij} a_j$$

and

$$v = \sum_{i,j} \beta_i b_{ij} a_j$$

We have then seen that

$$uv = \sum_{i,j,k,l} \alpha_i b_{ij} a_j \beta_k b_{kl} a_l = \sum_{i,j,k,l} (\alpha_i \beta_k)(b_{ij} b_{kl}) \sum_r b_{jlr} a_r = \sum_{i,,k,l,r} (\alpha_i b_k)(b_{ij} b_{kl} b_{jlr}) a_r.$$

This shows that A is generated by $a_1, \ldots, a_n$ as an $D = R[b_{ij}, b_{jlr} \mid j, l = 1, \ldots, n \quad i, r = 1, \ldots, m]$ algebra. In particular, D is Noetherian by corollary 6.15, so A is a Noetherian D-module by applying by theorem 6.11. Hence since B is a natural D-submodule of A it is finitely generated as a D-module, so $B = Db_1 + \ldots + Db_n$. However, this in particular expresses B as the R-algebra $R[b_{ij}, b_{ijk}, b_1, \ldots, b_n \mid i, j, k = 1, \ldots, n]$.

□

Exercise 0.2 (2)**.** Let K be a field and let A be a finitely generated K-algebra. Show that if A is a field, then A is finite-dimensional as a K-vector space. In particular, note that for every maximal ideal $\mathfrak{R} \subset A$, $A/\mathfrak{R}$ is a finite dimensional K-vector space.

Proof. If A is a field extension of K such that A is finite type over K, then by Zariski's lemma, we directly find that A is finite over K - i.e. that it is finitely generated as a K-module, and since K is a field, this is saying that A is finitely generated as a K-vector space. The latter part is corollary 11.7.

□