1. Double and Total Complexes

Definition 1.1 (Double complex)**.** A *double complex* (or *bicomplex*) in an abelian category $\mathcal{A}$ is a family $\{C_{p,q}\}$ of objects of $\mathcal{A}$, together with maps

$$d^h\colon C_{p,q} \to C_{p-1,q} \quad \text{and} \quad d^v\colon C_{p,q} \to C_{p,q-1}$$

such that $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$.

It is useful to picture the double complex as a lattice in which the maps d^h go horizontally, the maps d^v go vertically, and each square anticommutes.

Each row C_{*q} and each columns C_{p*} is a chain complex.
We say that the double complex C is *bounded* if C has only finitely many nonzero terms along each diagonal line $p+q=n$. For example, if C is concentrated in the first quadrant of the plane (a *first quadrant double complex*).

1.0.1. *Sign Trick.* Are the maps d^v and d^h maps in Ch?
Because of anticommutativity, the chain map conditions fail, but we can construct chain maps f_{*q} from $C_{*,q}$ to $C_{*,q-1}$ by introducing signs:

$$f_{p,q} = (-1)^p d^v_{p,q}\colon C_{p,q} \to C_{p,q-1}.$$

Using this sign trick, we can identify the category of double complexes with the category $\mathrm{Ch}\,(\mathrm{Ch})$.

1.0.2. *Total Complexes.* To see why the anticommutativity condition $d^v d^h + d^h d^v = 0$ is useful, we define the *total complexes* $\mathrm{Tot}(C) = Tot^{\prod}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ as follows:

Definition 1.2 (Total complexes)**.** We define

$$\mathrm{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q} \quad \text{and} \quad \mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

The formula $d = d^h + d^v$ define maps

$$d\colon\ \mathrm{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q} \quad \text{and} \quad d\colon\ \mathrm{Tot}^{\oplus}(C)_n \to \mathrm{Tot}^{\oplus}(C)_{n-1}$$

such that $d \circ d = 0$, making $\mathrm{Tot}^{\prod}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ into chain complexes.

Exercise 1.3. Check that $d = d^h + d^v$ define maps as claimed.

Solution. Let $(\alpha_{p,q}) \in \mathrm{Tot}^{\prod}(C)_n$, so $p+q=n$. Then $d\left((\alpha_{p,q})\right) = d^h\left((\alpha_{p,q})\right) + d^v\left((\alpha_{p,q})\right) = (\alpha_{p-1,q}) + (\alpha_{p,q-1}) \in \prod_{p+q=n-1} C_{p,q}$. Clearly, this also works for direct products since the number of non-zero terms under d just multiplies by 2, hence is still finite. We also want to show that $d \circ d = 0$. For this, note that

$$\begin{aligned} d \circ d\,(\alpha) = d\left(d^h(\alpha) + d^v(\alpha)\right) &= d^h\left(d^h(\alpha) + d^v(\alpha)\right) + d^v\left(d^h(\alpha) + d^v(\alpha)\right) \\ &= d^h d^h(\alpha) + d^h d^v(\alpha) + d^v d^h(\alpha) + d^v d^v(\alpha) \\ &= 0. \end{aligned}$$

1.1. **Exact Couples.**

Definition 1.4 (Exact Couple)**.** An *exact couple* is an exact sequence of abelian groups of the form

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & B \arrow[ul, "k"] & \end{tikzcd}$$

where i, j and k are group homomorphisms. Define $d\colon B \to B$ by $d = j \circ k$. Then $d^2 = j(kj)k = 0$, so $H(B) := \ker d/\operatorname{im} d$ is defined - in particular, since A and B are abelian, the quotient $H(B)$ is well-defined and a group.

Definition 1.5 (Derived Couple)**.** Out of a given exact couple, we can construct a new exact couple, called the *derived couple*:

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & B' \arrow[ul, "k'"] & \end{tikzcd}$$

where we define

(1) $A' = i(A)$ and $B' = H(B)$.
(2) i' is the induced map $i' := i|_{A'}\colon A' \to A'$ by $i'(ia) = i(ia)$
(3) We define j' by $j'a' = [ja]$ where $a' = ia$ for some a in A.
(4) k' is defined by $k'[b] = kb \in i(A)$.

With these definitions, the derived couple is an exact couple.

Exercise 1.6. Check that the maps are well-defined and that the derived sequence is exact.

Proof. We must check that j' and k' are well-defined maps.
Suppose $a' = ia = i\tilde{a}$. Then $a - \tilde{a} \in \ker i = \operatorname{im} k$ so $a - \tilde{a} = k[b]$. Hence Then $ja - j\tilde{a} = jk[b] = d[b] \in \operatorname{im} d$, so $[ja] = [j\tilde{a}]$.
Next, suppose $[b] = \left[\tilde{b}\right]$, so $b - \tilde{b} \in \operatorname{im} d$, i.e., $b - \tilde{b} = jk(\bar{b})$. Then $kb - k\tilde{b} = kjk(\bar{b}) = 0$, so $k'[b] = k'\left[\tilde{b}\right]$.

Lastly, exactness at B': suppose $k'[b] = 0$. Then $kb = 0$, so by exactness of the original exact couple, there exists some $a \in A$ such that $j(a) = b$. Then let $a' = i(a)$, so $j'(a') = [j(a)] = [b]$, hence $\ker k' \subset \operatorname{im} j'$.
Conversely, $k'j'(a') = k'[ja] = kja = 0$, by exactness at B of the original couple. □

1.2. **The Spectral Sequence of a Filtered Complex.**

Definition 1.7 (Differential Complex)**.** A differential complex K with differential operator D is an abelian group K together with a group homomorphism $D\colon K \to K$ such that $D^2 = 0$.

Let K be a differential complex with differential operator D. Usually K comes with a grading $K = \bigoplus_{k\in\mathbb{Z}} C^k$ and $D\colon C^k \to C^{k+1}$ increases the degree by 1, but the grading is not absolutely necessary.

Definition 1.8 (Subcomplex)**.** A *subcomplex* K' of K is a graded subgroup such that $DK' \subset K'$.

Definition 1.9 (Filtration, Associated Graded Complex)**.** A sequence of subcomplexes

$$K = K_0 \supset K_1 \supset K_2 \supset K_3 \supset \dots$$

is called a *filtration* on K. This makes K into a *filtered complex*, with *associated graded complex*

$$GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}.$$

For notational reasons, we usually extend the filtration to negative indices by defining $K_p = K$ for $p < 0$.

Example 1.10. If $K = \bigoplus K^{p,q}$ is a double complex with horizontal operator δ and vertical operator d (which we assume to commute), we can form a single complex out of it by setting $C^k = \bigoplus_{p+q=k} K^{p,q}$ and then letting $K = \bigoplus C^k$ and the differential operator $D\colon C^k \to C^{k+1}$ to be $D = \delta + (-1)^p d$. Then letting

$$K_p = \bigoplus_{i \geq p} \bigoplus_{q \geq 0} K^{i,q}$$

we obtain a filtration on K.

Suppose now that we have a general filtered complex $K = K_0 \supset K_1 \supset \dots$, and let A be the group defined by

$$A = \bigoplus_{p \in \mathbb{Z}} K_p.$$

Then A is again a differential complex with operator D. Let $i\colon A \to A$ be the inclusion $K_{p+1} \hookrightarrow K_p$ on each p. Let B be the cokernel of $i\colon A \to A$. Then $B = GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}$, and we have an exact sequence

$$0 \to A \xrightarrow{i} A \xrightarrow{j} GK \to 0.$$

2. Introduction to Spectral Sequences

Consider the problem of computing the homology of the total chain complex $T_* = \mathrm{Tot}(E_{**})$ where E_{**} is a first quadrant double complex.
Firstly, it is convenient to forget the horizontal differentials and add a superscript zero, retaining only the vertical differentials d^v along the columns E^0_{p*}.
Let E^1_{pq} be the vertical homology $H_q\left(E^0_{p*}\right)$ at the (p,q) spot.

3. Filtrations

Definition 3.1 (Filtered R-module)**.** A *filtered R-module* is an R-module A with an increasing sequence of submodules $\{F_p\}_{p\in\mathbb{Z}}$ such that $F_pA \subset F_{p+1}A$ for all p and such that $\bigcup_p F_pA = A$ and $\bigcap_p F_pA = \{0\}$.
A filtration is said to be *bounded* if $F_pA = \{0\}$ for p sufficiently small and $F_pA = A$ for p sufficiently larger.

Definition 3.2 (Associated graded module)**.** The *associated graded module* is defined by $G_pA = F_pA/F_{p-1}A$.

Definition 3.3 (Filtered chain complex)**.** A *filtered chain complex* is a chain complex (C_*, ∂) together with a filtration $\{F_pC_i\}_{p\in\mathbb{Z}}$ of each C_i such that the differential preserves the filtration, i.e., s.t. $\partial\left(F_pC_i\right) \subset F_pC_{i-1}$.

Note that we, in particular, obtain an induced differential $\partial\colon G_pC_i \to G_pC_{i-1}$ by the universal property of cokernels

$$\begin{tikzcd}
F_pC_i \arrow[r, "\partial"] \arrow[d] & F_pC_{i-1} \arrow[d] \\
F_{p-1}C_i \arrow[r, "\partial"] \arrow[d, "\mathrm{coker}"] & F_{p-1}C_{i-1} \arrow[d, "\mathrm{coker}"] \\
G_pC_i \arrow[r, dashed, "\partial"] & G_pC_{i-1}
\end{tikzcd}$$

so we obtain an associated graded chain complex G_pC_*.

The filtration on C_* also induces a filtration on the homology of C_* by

$$F_pH_i(C_*) = \{\alpha \in H_i(C_*) \mid (\exists x \in F_pC_i) : \alpha = [x]\}\,.$$

This filtration has associated graded pieces $G_pH_i(C_*)$ which, in favorable cases, determine $H_i(C_*)$.

3.1. **Example.** Suppose we have a chain complex C_* and a filtration consisting of a single F_0C_*, so $F_nC_* = 0$ if $n < 0$ and $F_nC_* = F_0C_*$ if $n \geq 0$.
Then $G_nC_* = 0$ for $n \neq 0$ and $G_0C_* = F_0C_*$ and