# MAPPING CLASS GROUPS, BRAID GROUPS AND GEOMETRIC REPRESENTATIONS

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## 1. Introduction

In this thesis, we develop the theory of mapping class groups of surfaces and its connection to braid groups. The mapping class group of a surface $S$ is the group $\mathrm{Mod}(S)$ of isotopy classes of homeomorphisms of the surface where isotopies are continuous deformations which are embeddings at all times. The braid group on $n$ strands, $B_n$, is the group of isotopy classes of a collection of strands in $\mathbb{C} \times [0,1]$ which we call braids. The goal of this thesis is to study connections between these groups. In particular, we study geometric representations which are homomorphisms from braid groups into the mapping class group of some surface.
Our goal is to place these representations in the more general categorical framework of monoidal categories - primarily following Harr, Vistrup and Wahl [4] and Wahl and Randal-Williams [12] where many representations are induced by so called Yang-Baxter operators on certain monoidal categories of surfaces.
Using a classification of geometric representations of braid groups on non-orientable surfaces by Stukow and Szepietowski [11], we show that each such geometric representation is obtained by an appropriate choice of monoidal category of surfaces and choice of Yang-Baxter operator - leaving certain cases out.

1.1. **Thesis Summary.** For the first 4 chapters, we will closely be following the book 'A Primer on Mapping Class Groups' by Farb and Margalit [2] together with [14].
We start by developing the necessary basics of surface topology which we will need with an emphasis on curves since studying the action of the mapping class group on simple closed curves on the surface will prove a strong tool. We develop essential techniques such as intersection numbers, the bigon criterion, the change of coordinates principle.

Afterwards, we give an account of the basics of mapping class groups for surfaces and compute basic examples such as the mapping class group of the torus being isomorphic to $\mathrm{SL}(2,\mathbb{Z})$.

Subsequently, we introduce a particular isotopy class of homeomorphisms called Dehn twists which will be important in the connection between braid groups and mapping class groups. We show that Dehn twists satisfy the relations given in the presentation for the braid group allowing us to consider homomorphisms from the braid group into mapping class groups of surfaces - this is called a geometric representation of the braid group.

We prove the Birman-Hilden theorem which precisely says that the geometric representation corresponding to sending generators to Dehn twists for a chain of curves is, in fact, an embedding of the braid group.

Afterwards, we show that general monoidal categories that come equipped with a so called Yang-Baxter element induce a geometric representation of the braid group.

We construct certain categories of surfaces with Yang-Baxter elements which induce different geometric representations of the braid group, amongst which we recover the Birman-Hilden embedding.

We then turn our attention to geometric representations of the braid group on non-orientable surfaces and once again recover previously constructed representations in a new light.

## 2. Curves, Surfaces

Just as linear transformations are determined by their actions on vectors, we shall see that in many cases, we can understand the mapping class groups of a surface by its action on simple loops on the surface.

Another important tool is the algebraic and geometric intersection number associated to a pair of simple curves on the surface which functions as the analogue of an inner product on a vector space.

Lastly, we shall introduce the change of coordinates principle which is a strong application of the classification of surfaces used to reduce situations with complicated curves on a surface to simpler cases - it plays a similar role to changes of bases for matrices.

### 2.1. Simple closed curves.

**Definition 2.1** (Primitive and multiple elements)**.** An element $g$ of a group $G$ is *primitive* if there does not exist any $h \in G$ so that $g = h^k$ for $|k| > 1$. The property of being a primitive is a conjugacy class invariant. In particular, it makes sense to say that a closed curve in a surface is primitive.
A closed curve in $S$ is a multiple if it is a map $S^1 \to S$ that factors through the map $S^1 \xrightarrow{\times n} S^1$ for $n > 1$, i.e., there exists a map $\tilde{\alpha}\colon S^1 \to S$ such that the following diagram commutes:

$$\begin{tikzcd} S^1 \arrow[r, "\times n"] \arrow[rr, bend left, dashed, "\tilde{\alpha}"] & S^1 \arrow[r, "\alpha"] & S \end{tikzcd}$$

**Definition 2.2** (Lifts)**.** We make a distinction between lifts: let $p\colon \tilde{S} \to S$ be a covering space. By a *lift* of a closed curve $\alpha$ to $\tilde{S}$ we will always mean the image of a lift $\mathbb{R} \to \tilde{S}$ of the map $\alpha \circ \pi$ where $\pi\colon \mathbb{R} \to S^1$ is the usual covering map. I.e., a lift of $\alpha\colon S^1 \to S$ is a map $\tilde{\alpha}\colon \mathbb{R} \to \tilde{S}$ such that the following diagram commutes

$$\begin{tikzcd} & & \tilde{S} \arrow[d, "p"] \\ \mathbb{R} \arrow[r, "\pi"'] \arrow[urr, "\tilde{\alpha}"] & S^1 \arrow[r, "\alpha"'] & S \end{tikzcd}$$

A lift is different from a *path lift* which is a proper subset of a lift. Namely, it would be the restriction of $\tilde{\alpha}$ to some interval of $\mathbb{R}$ of length $2\pi$ if the covering map $\pi$ is of the form $t \mapsto e^{it}$.

**Proposition 2.3.** *Suppose $p\colon \tilde{S} \to S$ is the universal cover and $\alpha$ is a simple closed curve in $S$ that is not a multiple of another closed curve. In this case, there is a* *bijective correspondence between cosets in $\pi_1(S)$ of the infinite cyclic subgroup $\langle\alpha\rangle$ and the lifts of $\alpha$.*

*Proof.* This can be seen as follows: first choose a basepoint $\alpha(1) = x_0 \in S$ and some $\tilde{x_0} \in p^{-1}(x_0)$. There exists a unique lift $\tilde{\alpha}$ of $\alpha$ such that

$$\begin{tikzcd} & & \tilde{S} \arrow[d, "p"] \\ \mathbb{R} \arrow[r] \arrow[urr, "\tilde{\alpha}"] & S^1 \arrow[r, "\alpha"] & S \end{tikzcd}$$

commutes and such that $\tilde{\alpha}(0) = \tilde{x} \in p^{-1}(\alpha \circ \pi(0))$ for some specific $\tilde{x}$ [1, Cor. 4.2]. But the set $p^{-1}(\alpha \circ \pi(0))$ is in bijective correspondence with the loops in $\pi_1(S)$ by the path lifting lemma. Now, under which path lifts are the lifts the same? The lifts of $\alpha$ to two points $\tilde{x}, \tilde{y} \in p^{-1}(\alpha \circ \pi(0))$ will be the same if $\alpha^k \cdot \tilde{x} = \tilde{y}$ where $\cdot$ denotes the monodromy action of $\pi_1(S)$ on the fiber. Now, there exist $\gamma_x$ and $\gamma_y$ in $\pi_1(S)$ such that $\gamma_x \cdot \tilde{x_0} = \tilde{x}$ and $\gamma_y \cdot \tilde{x_0} = \tilde{y}$, so $\alpha^k \gamma_x = \gamma_y$. Hence the lifts corresponding to $\gamma_x$ and $\gamma_y$ are the same if and only if $\alpha^k \gamma_x = \gamma_y$ for some $k$, i.e. if and only if $\gamma_x = \gamma_y$ in $\pi_1(S)/\langle\alpha\rangle$.

□

As usual, the group $\pi_1(S)$ acts on the set of lifts of $\alpha$ by deck transformations, and this action agrees with the usual left action of $\pi_1(S)$ on the cosets of $\langle\alpha\rangle$. The stabilizer of the lift corresponding to the coset $\gamma\langle\alpha\rangle$ is the cyclic group $\langle\gamma\alpha\gamma^{-1}\rangle$. See figure 1.

**Proposition 2.4.** *[2, Proposition 1.3] Let $S$ be a hyperbolic surface. If $\alpha$ is a closed curve in $S$ that is not homotopic into a neighborhood of a puncture, then $\alpha$ is homotopic to a unique geodesic closed curve $\gamma$.*

**Definition 2.5** (Simple curves)**.** A closed curve in $S$ is *simple* if it is topologically embedded, i.e., if the map $S^1 \to S$ is injective.

By [1, Thm 11.8], any closed curve $\alpha$ can be approximated (arbitrarily close) by a smooth closed curve which is homotopic to $\alpha$. Moreover, if $\alpha$ is simple, then the smooth approximation can be chosen to be simple. Smooth curves are advantageous because we can make use of notions such as transversality.

Simple closed curves are also natural to study because they represent primitive elements of $\pi_1(S)$.

**Proposition 2.6.** *[2, Proposition 1.4] Let $\alpha$ be a simple closed curve in a surface $S$. If $\alpha$ is not null homotopic, then each element of the corresponding conjugacy class in $\pi_1(S)$ is primitive.*

2.1.1. *Example: simple closed curves on the torus.*

**Proposition 2.7.** *The nontrivial homotopy classes of oriented simple closed curves in $T^2$ are in bijective correspondence with the set of primitive elements of $\pi_1(T^2) \approx \mathbb{Z}^2$ which is the set of elements $(p,q) \in \mathbb{Z}^2$ such that either $(p,q) = (0,\pm1)$ or $(p,q) = (\pm1, 0)$ or $\gcd(p,q) = 1$.*

*Proof.* Firstly, primitive elements of $\pi_1 T^2 \approx \mathbb{Z}^2$ are those $(a,b)$ such that there does not exist $(c,d)$ and $k \in \mathbb{Z}$ such that $|k| > 1$ and $(kc, kd) = (a,b)$. But this is

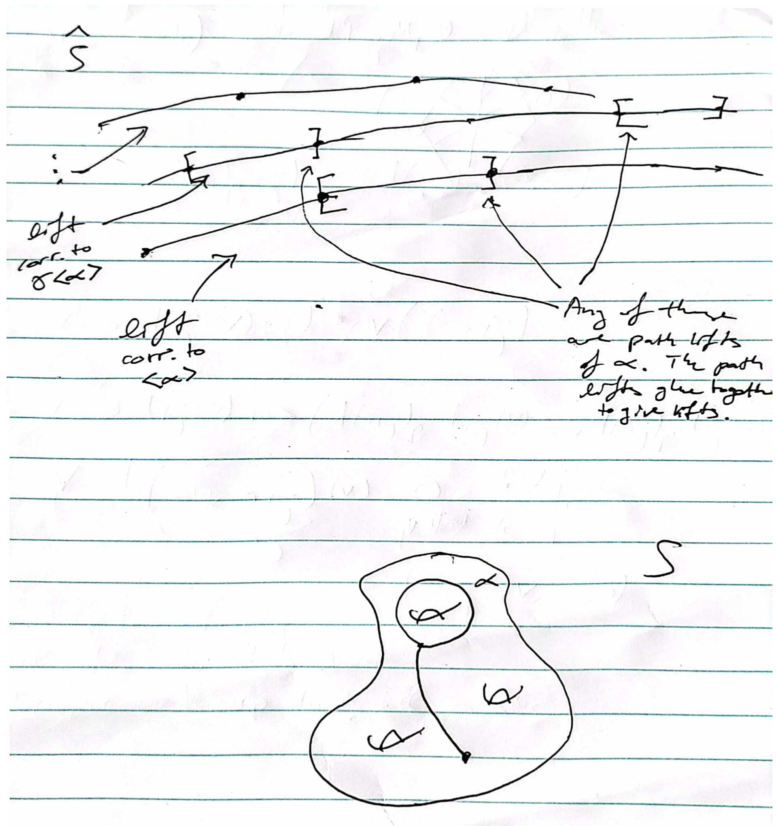



FIGURE 1

equivalent to saying that either $\gcd(a,b) = 1$ or $(a,b) \in \{(\pm 1, 0), (0, \pm 1)\}$.

We now claim that for a nontrivial closed curve $\alpha$ in $T^2$, $\alpha$ is homotopic to a simple closed curve if and only if $\alpha$ represents a primitive element of $\pi_1 T^2 \approx \mathbb{Z}^2$. Suppose $(p,q) = \alpha$ represents a primitive element of $\pi_1 T^2 \approx \mathbb{Z}^2$. Then if we choose the origin as basepoint, we can lift $\alpha$ to the straight-line path starting at the origin and ending at $(p,q) \in \mathbb{Z}^2$. The projected loop is simple since otherwise, we would have for $t, t' \in t, t' \in [0,1)$ with $t \neq t'$ that $(tp, tq) = (t'p, t'q) + (m,n)$ for $m, n \in \mathbb{Z}$. Then $p(t - t') = m$ and $q(t - t') = n$, so $t - t' \in \mathbb{Q}$; let $t - t' = \frac{a}{b}$ and assume $\gcd(a,b) = 1$. Then $pa = bm$ and $qa = bn$, so $b \mid p, q$ and hence $b \mid \gcd(p,q) = 1$, so $b = 1$. But then $\frac{p}{q} \in \mathbb{Z}$ contradicting $\gcd(p,q) = 1$ except when $q = \pm 1$. But then $\pm t = \pm t' + n$ necessarily gives $n = 0$ and $t = t'$, contradiction.

Conversely, suppose $\alpha$ is homotopic to a nontrivial simple closed curve which we will also denote $\alpha$. A lift of $\alpha$ will consist of a collection of biinfinite disjoint topological lines. We can homotopy this collection into a collection of disjoint straight biinfinite lines, where the integer points on the lines are fixed during the homotopy. Since these collections and the homotopy are equivariant with respect to deck transformations, this descends to a homotopy of $\alpha$. If the descended loop from the straight lines were not simple, we would have intersections in the universal cover - i.e. non-parallel lines which we do not have. This is the information of Lemma 2.8 below. As the straight-line respresentative is simple, we must therefore have that if $\alpha$ is represented by $(p, q)$, we have that $\gcd(p, q) = 1$ or $(p, q) \in \{(\pm 1, 0), (0, \pm 1)\}$.

**Lemma 2.8.** *Let $X$ be a topological space with a universal covering space $\tilde{X}$. A closed curve $\beta$ in $X$ is simple if and only if the following properties hold:*

1. *Each lift of $\beta$ to $\tilde{X}$ is simple.*
2. *No two lifts of $\beta$ intersect.*
3. *$\beta$ is not a nontrivial multiple of another closed curve.*

□

2.2. **Intersection numbers.** It is often useful to put an inner product on a vector space to check if two vectors are linearly independent. We can pursue something similar for surfaces.

**Definition 2.9** (Transversality for curves)**.** If $\alpha \cap \beta$ is finite and, at every intersection, each curve locally separates the other curve, then we say that $\alpha$ and $\beta$ are *transverse.*

**Definition 2.10** (Algebraic intersection number)**.** Let $\alpha$ and $\beta$ be a pair of transverse, oriented, simple closed curves in $S$. Their *algebraic intersection number* $\hat{i}(\alpha, \beta)$ is defined as the sum of the indices of the intersection points of $\alpha$ and $\beta$, where the intersection point is of index $+1$ when the orientation of intersection agrees with the orientation of $S$ and is $-1$ otherwise. See Figure 2.

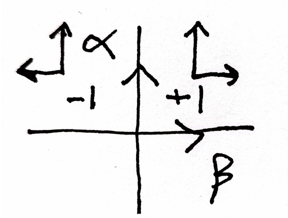



FIGURE 2. Curves $\alpha$ and $\beta$ intersecting. If the orientation of intersection created here is compatible with the orientation of the surface as depicted on the right side, the index of intersection for $\hat{i}(\alpha, \beta)$ is $+1$, otherwise, if the orientation is as on the left, the index is $-1$.

*Remark.* The algebraic intersection number only depends on the homology classes of the curves and defines a symplectic form on homology.

**Definition 2.11** (Geometric intersection number)**.** Let $\alpha, \beta$ be closed curves on a surface $S$. Their *geometric intersection number* is

$$i(\alpha, \beta) = \min_{\alpha' \simeq \alpha, \beta' \simeq \beta} \#(\alpha' \cap \beta')$$

Intersection numbers are a useful general tool, and we will encounter them not only in applications of the Bigon criterion which we describe in a moment, but also in direct computations. One such computation will be the mapping class group of the torus $T^2$. For this, we will need an explicit expression for the intersection number of curves on the torus.

**Example 2.12** (Intersection numbers on the torus)**.** By proposition 2.7, the non-trivial homotopy classes of oriented simple closed curves in $T^2$ are in bijective correspondence with the set of primitive elements of $\mathbb{Z}^2$. For two such homotopy classes $(p, q)$ and $p', q'$, we claim that

$$i\left((p,q),\left(p',q'\right)\right) = \left|pq' - p'q\right|.$$

and

$$\hat{i}\left((p,q),\left(p',q'\right)\right) = pq' - p'q.$$

Suppose first that $(p, q) = (1, 0)$. Through a homotopy, we can assume that $(p', q')$ is represented by a loop which first winds around the torus $p'$ times horizontally without intersecting $(1, 0)$ and then $q'$ times vertically. It then is clear that $i((p,q),(p',q')) = |q'| = |pq' - p'q|$. For the algebraic case, we choose $(1, 0)$ to be along the orientation direction. Then $\hat{i}((p,q),(p',q')) = q' = pq' - p'q$.
For the general case, suppose $\gcd(p, q) = 1$. Then by Bezout's lemma, there exist $a, b \in \mathbb{Z}$ such that $ap + bq = 1$. The system of equations

$$\begin{aligned} qd + pc &= 0 \\ ad - bc &= 1 \end{aligned}$$

has $(c, d) = (-q, p)$ as a solution. So letting

$$A = \begin{pmatrix} a & b \\ -q & p \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

we get $A\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since $A$ is a linear, orientation-preserving homeomorphism of $\mathbb{R}^2$ preserving $\mathbb{Z}^2$, it induces an orientation-preserving homeomorphism on the quotient $\mathbb{R}^2/\mathbb{Z}^2 \approx T^2$ whose action on the fundamental group $\pi_1\left(T^2\right) \approx \mathbb{Z}^2$ is given by $A$. Now, homeomorphisms preserve algebraic and geometric intersection numbers. Then

$$\begin{aligned} i\left((p,q),\left(p',q'\right)\right) = i\left(A(p,q), A\left(p',q'\right)\right) &= i\left((1,0),\left(ap'+bq', -qp'+pq'\right)\right) \\ &= \left|pq' - qp'\right| \end{aligned}$$

and likewise $\hat{i}((p,q),(p',q')) = pq' - qp'$.
The other primitive cases are checked easily.

**Definition 2.13** (Minimal position)**.** Two curves $\alpha$ and $\beta$ are in *minimal position* if $\#(\alpha \cap \beta) = i(\alpha, \beta)$.

2.3. **Bigons.** We want a procedure to put curves into minimal position so we can compute intersection numbers.
For this, we need the notion of a *bigon*:

**Definition 2.14** (Bigon)**.** Two transverse simple closed curves $\alpha$ and $\beta$ in a surface $S$ form a *bigon* if there is a topologically embedded disk in $S$ (the bigon) whose boundary is the union of an arc of $\alpha$ and an arc of $\beta$ intersecting in exactly two points.

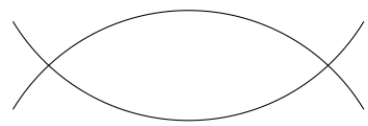


FIGURE 3. Local picture of a bigon [2, Figure 1.2]

**Proposition 2.15** (The bigon criterion). *[2, Proposition 1.7] Two transverse simple closed curves in a surface $S$ are in minimal position if and only if they do not form a bigon.*

**Corollary 2.16.** *Any two transverse simple closed curves that intersect exactly once are in minimal position.*

### 2.4. Homotopy versus isotopy for simple closed curves.

**Definition 2.17** (Isotopy). Two simple closed curves $\alpha$ and $\beta$ are *isotopic* if there is a homotopy

$$H\colon S^1 \times [0,1] \to S$$

from $\alpha$ to $\beta$ with the property that the closed curve $H\left(S^1 \times \{t\}\right)$ is simple for each $t \in [0,1]$.

**Proposition 2.18** (Baer). *[2, Proposition 1.10] Let $\alpha$ and $\beta$ be two essential simple closed curves in a surface $S$. Then $\alpha$ is isotopic to $\beta$ if and only if $\alpha$ is homotopoic to $\beta$.*

*Proof.* If $\alpha$ is isotopic to $\beta$ then they are clearly also homotopic.

Suppose $\alpha$ and $\beta$ are homotopic. Taking a tubular neighborhood around $\alpha$, we can find a disjoint simple loop $\tilde{\alpha}$ which is homotopic to $\alpha$ but disjoint from it. Then $\beta$ is homotopic to $\tilde{\alpha}$, and hence $i(\alpha, \beta) = i(\alpha, \tilde{\alpha}) = 0$. Performing an isotopy of $\alpha$, we may assume that $\alpha$ is transverse to $\beta$. If $\alpha$ and $\beta$ are not disjoint, then by the bigon criterion, they form a bigon. A bigon prescribes an isotopy that reduces intersection, so we may remove bigons by isotopy until $\alpha$ and $\beta$ are disjoint.

Suppose $\chi(S) < 0$. Lift $\alpha$ and $\beta$ to $\tilde{a}$ and $\tilde{\beta}$ with the same endpoints in $\partial\mathbb{H}^2$. There is a hyperbolic isometry $\varphi$ that leaves $\tilde{\alpha}$ and $\tilde{\beta}$ invariant and acts by translation on the lifts. As $\tilde{\alpha}$ and $\tilde{\beta}$ are disjoint, let $R$ denote the region between them. We claim that the quotient surface $R' = R/\langle\varphi\rangle$ is an annulus. The fundamental group of $R'$ is isomorphic to the group of deck transformations $\langle\varphi\rangle$ and is hence infinite cyclic. Furthermore, $R'$ has two boundary components. By considering representative examples of surfaces with two boundaries in the classification of surfaces, we obtain that $R'$ must be homeomorphic to an annulus. □

### 2.5. Digression on isotopies and their extensions.

**Definition 2.19** (General isotopies). Let $V$ and $M$ be manifolds. An isotopy from $V$ to $M$ is a map $F\colon V \times I \to M$ such that for each $t \in I$, the map

$$F_t\colon V \to M, \quad x \mapsto F(x,t)$$

is an embedding.

**Definition 2.20** (Isotopic embeddings and ambient isotopies)**.** If $F\colon V \times I \to M$ is an isotopy, we call the two embeddings $F_0$ and $F_1$ isotopic. If $V$ is a submanifold of $M$ and $F_0$ is the inclusion, we call $F$ an isotopy of $V$ in $M$. When $V = M$ and each $F_t$ is a diffeomorphism, and $F_0 = \mathbb{1}_M$, then $F$ is called an ambient isotopy.

**Definition 2.21** (Support of isotopy)**.** The *support* $\operatorname{Supp} F \subset V$ of an isotopy $F\colon V \times I \to M$ is the closure of $\{x \in V : F(x,t) \neq F(x,0) \text{ for some } t \in I\}$.

The vital theorem is the following:

**Theorem 2.22** (Isotopy extension theorem)**.** *[7, Theorem 1.3, chapter 8] Let $V \subset M$ be a compact submanifold and $F\colon V \times I \to M$ an isotopy of $V$. If either $F(V \times I) \subset \partial M$ or $F(V \times I) \subset M - \partial M$, then $F$ extends to an ambient isotopy of $M$ having compact support.*

## 2.6. **Arcs.**

Assume $S$ is a compact surface, possibly with boundary and possibly with finitely many marked points in the interior. Denote the set of marked points by $\mathcal{P}$.

**Definition 2.23.** A *proper arc* in $S$ is a map $\alpha\colon [0,1] \to S$ such that $\alpha^{-1}(\mathcal{P} \cup \partial S) = \{0,1\}$.

**Definition 2.24.** The arc $\alpha$ is *simple* if it is an embedding on its interior.

*Remark.* The homotopy class of a proper arc is taken to be the homotopy class within the class of proper arcs. Thus points on $\partial S$ cannot move off the boundary during the homotopy.
A homotopy (or isotopy) of an arc is said to be *relative to the boundary* if its endpoints stay fixed throughout the homotopy. An arc in a surface $S$ is *essential* if it is neither homotopic into a boundary component of $S$ nor a marked point of $S$.

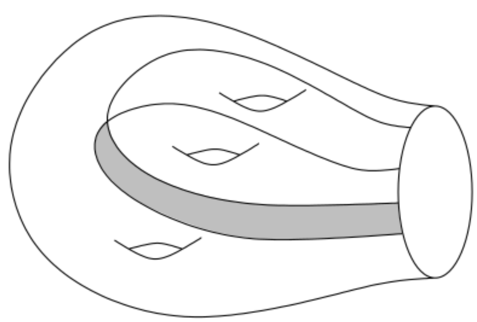


FIGURE 4. Bigon of arcs [2, Figure 1.4]

Note in this picture how if isotopies are considered relative to the boundary, then the two arcs are in minimal position, while if we consider general isotopies, then the half-bigon shows that they are not in minimal position as we can pull the top strand down under the bottom one along the boundary.

- The bigon criterion holds for arcs.
- Prop 1.10 (homotopy versus isotopy for curves) and theorem 1.13 (extension of isotopies) also work for arcs.

## 2.7. **Change of coordinates principle.**

### 2.7.1. *Classification of simple closed curves.*

**Definition 2.25.** Given a simple closed curve or a simple proper arc $\alpha$ in a surface $S$, the surface obtain by cutting $S$ along $\alpha$ is a compact surface $S_\alpha$ equipped with an attaching map $h$ (i.e.

(1) $S_\alpha/(x \sim h(x)) \approx S$
(2) the image of the distinguished boundary components under this quotient map is $\alpha$.

**Definition 2.26.** We say that a simple closed curve $\alpha$ in the surface $S$ is *nonseparating* if the cut surface $S_\alpha$ is connected, and *separating* if $S_\alpha$ is not connected.

As an important consequence of the classification of surfaces, we obtain the following theorem:

**Theorem 2.27.** *If $\alpha$ and $\beta$ are any two nonseparating simple closed curves in a surface $S$, then there is a homeomorphism $\varphi\colon S \to S$ with $\varphi(\alpha) = \beta$.*

*Proof.* The cut surface $S_\alpha$ and $S_\beta$ have two boundary component corresponding to $\alpha$ and $\beta$, respectively. Now, suppose $S_\alpha$ has $n_\alpha$ vertices, $m_\alpha$ edges and $t_\alpha$ triangles in a triangulation. Then in obtaining $S$ from $S_\alpha$, we identify the vertices and edges, but no triangles are identified, so we get $n_S = n_\alpha - 3$ and $m_S = m_\alpha - 3$, but $t_S = t_\alpha$. Thus $\chi(S_\alpha) = \chi(S)$.
Since both $S_\alpha$ and $S_\beta$ have the same Euler characteristic, number of boundary components and number of punctures, it follows that $S_\alpha \approx S_\beta$. Choose a homeomorphism $\varphi\colon S_\alpha \to S_\beta$ such that if $h_\alpha$ is the attaching map for $S_\alpha$ and $h_\beta$ is the attaching map for $S_\beta$, then $\varphi$ takes $\{x, h_\alpha(x)\}$ to $\{y, h_\beta(y)\}$ - i.e., the identification are respected under the map. This homeomorphism gives the desired homeomorphism of $S$ taking $\alpha$ to $\beta$. If we want an orientation preserving homeomorphism, we can postcompose by an orientation-reversing homeomorphism fixing $\beta$ if necessary. $\square$

*Remark.* By the "classification of disconnected surfaces", there are finitely many separating simple closed curves in $S$ up to homeomorphism.

**Corollary 2.28.** *There is an orientation-preserving homeomorphism of a surface taking one simple closed curve to another if and only if the corresponding cut surfaces (which may be disconnected) are homeomorphic.*

**Question 2.29.** Suppose $\alpha$ is any nonseparating simple closed curve on a surface $S$.

(1) Is there a simple closed curve $\gamma$ in $S$ so that $\alpha$ and $\gamma$ *fill* $S$, i.e., such that $\alpha$ and $\gamma$ are in minimal position and the complement of $\alpha \cup \gamma$ is a union of topological disks.
(2) Is there a simple closed curve $\beta$ in $S$ with $i(\alpha, \beta) = 0$? $i(\alpha, \beta) = 1$? $i(\alpha, \beta) = k$?

Figure 5 shows two filling simple closed curves on the genus 2 surface. By the classification of simple closed curves on a surface, there is a homeomorphism $\varphi\colon S_2 \to S_2$ such that $\varphi(\beta) = \alpha$. Then the image of $\gamma$ under $\varphi$ fills $S_2$ with $\alpha$ since filling is a topological property.

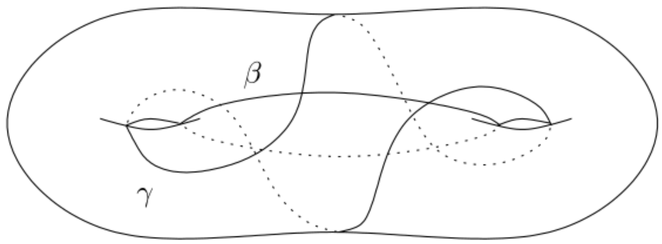



FIGURE 5. Two filling curves answering question (1) [2, Figure 1.7]

2.7.2. *Example of the change of coordinates principle.*

(1) *Pairs of simple closed curves that intersect once are all homeomorphic as in Corollary 2.28.* Suppose $\alpha_1$ and $\beta_1$ form such a pair on a surface $S$. Then $\beta_1$ must be an arc connecting the two boundary components in $S_{\alpha_1}$. But the boundary component is homeomorphic to $S^1$, so removing a point leaves it connected. Thus removing $\beta_1$ leaves $(S_{\alpha_1})_{\beta_1}$ path-connected. Similarly, $(S_{\alpha_2})_{\beta_2}$ is path-connected for any other pair $\alpha_2$ and $\beta_2$ that constitute a pair of simple closed curves that intersect once in $S$. By the classification of surfaces with boundary, $(S_{\alpha_1})_{\beta_1}$ is homeomorphic to $(S_{\alpha_2})_{\beta_2}$ which preserves equivalence classes on the boundary, and as we can construct this homeomorphism first for the $\beta$'s and then for the $\alpha$'s, this homeomorphism descends to a self-homeomorphism of $S$ taking the pair $\{\alpha_1, \beta_1\}$ to $\{\alpha_2, \beta_2\}$.

2.8. **Three facts about homeomorphisms.** Suppose $f\colon D \to D$ is an orientation-reversing map. Then $f$ restricts to a map on $S^1 \to S^1$ of degree $-1$. But thus $f$ is not isotopic to the identity as the identity has degree 1 and the isotopy would have to restrict to a homotopy on the boundary, but degree is a homotopy invariant for maps $S^n \to S^n$.
However, the straight-line homotopy does give a homotopy between $f$ and the identity.

As shown in the following theorem, this is one of the only two situations in which this can happen.

**Theorem 2.30.** *[2, Theorem 2.34] Let $S$ be any compact surface and let $f$ and $g$ be homotopic homeomorphisms of $S$. Then $f$ and $g$ are isotopic unless they are one of the two examples described above (on $S = D^2$ and $S = A$ ). In particular, if $f$ and $g$ are orientation-preserving, then they are isotopic.*

**Theorem 2.31.** *[2, Theorem 2.35] Let $S$ be a compact surface. Then every homeomorphism of $S$ is isotopic to a diffeomorphism of $S$.*

**Theorem 2.32** (Hamstrom)**.** *[2, Theorem 2.36] Let $S$ be a compact surface, possibly minus a finite number of points from the interior. Assume that $S$ is not homeomorphic to $S^2, \mathbb{R}^2, D^2, T^2$, the closed annulus, the once-punctured disk, or the once-punctured plane. Then the space* $\mathrm{Homeo}_0(S)$ *is contractible.*

## 3. Mapping class group basics

We will be defining the mapping class group as $\pi_0$ of a space of homeomorphisms, and as such, we must give this space a topology. For this, it turns out that we need the compact-open topology which we now describe.

### 3.1. The compact-open topology.

**Definition 3.1.** The *weak* or *compact-open $C^r$* topology on $C^r(M,N)$, where $M$ and $N$ are $C^r$ manifolds, is generated by sets defined as follows: let $f \in C^r(M,N)$. Let $(U,\varphi),(V,\psi)$ be charts on $M$ and $N$ ; let $K \subset U$ be compact such that $f(K) \subset V$ and let $0 < \varepsilon \leqslant \infty$. Then a *weak subbasic neighborhood*

$$\mathcal{N}^r\left(f;(U,\varphi),(V,\psi),K,\varepsilon\right) \tag{$\zeta$}$$

is the set of $C^r$ maps $g\colon M \to N$ such that $g(K) \subset V$ and

$$\|D^k\left(\psi f \varphi^{-1}\right)(x) - D^k\left(\psi g \varphi^{-1}\right)(x)\| < \varepsilon$$

for all $x \in \varphi(K)$, for $k = 0,\ldots,r$. The *compact-open $C^r$ topology* on $C^r(M,N)$ is generated by the set of weak subbasic neighborhoods, and defines the topological space $C^r_W(M,N)$. A neighborhood of $f$ is then any set containing the intersection of a finite number of sets of the type ($\zeta$).

We are interested in the subspace $\mathrm{Homeo}(S) \subset C^0_W(S,S)$, inheriting the subspace topology.

The compact-open topology might seem a bit confusing, but we have the following lemma [5, Prop A.14]:

**Lemma 3.2.** *Let $X,Y,Z$ be Hausdorff topological spaces. Suppose $Y$ is locally compact. Then a map $f\colon X \to C^0_W(Y,Z)$ is continuous if and only if the associated map $F\colon X \times Y \to Z$ defined by*

$$F(x,y) := f(x)(y)$$

*is continuous.*

### 3.2. Definitions and first examples.

**Definition 3.3.** Let $S$ be a surface which is the connected sum of $g \geqslant 0$ tori with $b \geqslant 0$ disjoint open disks removed and $n \geqslant 0$ points removed from the interior. Let $\mathrm{Homeo}^+(S,\partial S)$ denote the group of orientation-preserving self-homeomorphisms of $S$ that restrict to the identity on $\partial S$. We endow this group with the compact-open topology. The *mapping class group* of $S$, denoted $\mathrm{Mod}(S)$, is the group

$$\mathrm{Mod}(S) = \pi_0\left(\mathrm{Homeo}^+(S,\partial S)\right)$$

*Remark.* From Lemma 3.2, we see that a path $\gamma\colon I \to \mathrm{Homeo}^+(S,\partial S)$ is precisely equivalent to an isotopy $F\colon I \times S \to S$ from $\gamma(0)$ to $\gamma(1)$ (isotopy because at each time $t$, $\gamma(t)\colon S \to S$ is indeed a topological embedding as it is a homeomorphism). In fact, it's an isotopy of $S$. Here isotopies are required to fix boundaries.

If $\mathrm{Homeo}_0(S,\partial S)$ denotes the connected component of the identity in $\mathrm{Homeo}^+(S,\partial S)$, then we can equivalently write

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S,\partial S)/\mathrm{Homeo}_0(S,\partial S).$$

**Proposition 3.4.**

$$\begin{aligned}\mathrm{Mod}(S) &= \pi_0\left(\mathrm{Homeo}^+(S,\partial S)\right)\\ &\approx \mathrm{Homeo}^+(S,\partial S)/\textit{homotopy}\\ &\approx \pi_0\left(\mathrm{Diff}^+(S,\partial S)\right)\\ &\approx \mathit{Diff}^+(S,\partial S)/\sim\end{aligned}$$

*where* $\mathrm{Diff}^+(S,\partial S)$ *is the group of orientation preserving diffeomorphisms of* $S$ *that are the identity on the boundary and* $\sim$ *can be taken to be either smooth homotopy relative to the boundary or smooth isotopy relative to the boundary.*

3.2.1. *The Alexander Lemma.* Here we will describe some of the simplest examples of mapping class groups following [2, Chapter 2.1]

**Lemma 3.5** (Alexander lemma)**.** *The group* $\mathrm{Mod}\left(D^2\right)$ *is trivial.*

*Proof.* Let $\varphi\colon D^2 \to D^2$ be a homeomorphism with $\varphi|_{\partial D^2} = \mathrm{id}_{\partial D^2}$. Define

$$F(x,t) = \begin{cases}(1-t)\varphi\left(\frac{x}{1-t}\right), & 0 \leqslant |x| < 1-t\\ x, & 1-t \leqslant |x| \leqslant 1\end{cases}$$

for $0 \leqslant t < 1$, and let $F(x,1) = \mathrm{id}_{D^2}$. Then $F$ is an isotopy from $\varphi$ to the identity. The reason it is an isotopy is because at time $t$, $F(-,t)$ is a homeomorphism on the disk of radius $1-t$ where it is $\varphi$ and outside this disk, it is the identity. On the boundary of this disk, both $\varphi$ and the identity are the identity, so $F(-,t)$ is continuous by the pasting lemma, and thus a homeomorphism for each $t$.

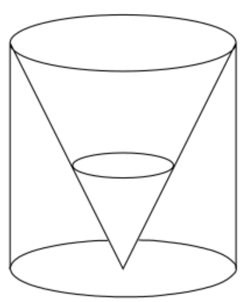


FIGURE 6. The Alexander trick. We envision $F$ as performing the homeomorphism on the disk $D^2$ and embedding it at time $t$ as a disk of radius $1-t$ as shown in the picture. Thus $F$ will have support the cone as shown. [2, Figure 2.4]

□

*Remark.* Also $0 \approx \mathrm{Mod}\left(D-\{0\}\right) \approx \mathrm{Mod}\left(S_{0,1}\right) \approx \mathrm{Mod}\left(S^2\right)$ using the Alexander trick again.

3.2.2. *The mapping class group of the thrice-punctured sphere,* $\mathrm{Mod}(S_{0,3})$. We will now present a couple other computations of some simple mapping class groups.

**Proposition 3.6.** *Any two essential simple proper arcs in $S_{0,3}$ with the same endpoints are isotopic. Any two essential arcs that both start and end at the same marked point of $S_{0,3}$ are isotopic.*

*Proof.* Let $\alpha$ and $\beta$ be two simple proper arcs in $S_{0,3}$ connecting the same two distinct marked points. By isotopy, we may modify $\alpha$ so that it intersects transversally with $\beta$. Letting the last marked point become the point at infinity, we can consider $\alpha$ and $\beta$ as being arcs in $\mathbb{R}^2 - \{p, q\}$ for the two marked points $p, q$. Now, suppose the arcs are disjoint. Then, choosing an intersection point, we can follow the path to the other intersection point and obtain either a bigon, in which case we can remove it by isotopy, or a bigon with path segments inside. Now, suppose the there is some point of $\alpha$ inside the bigon. Then since this is part of the arc $\alpha$, we can find a simple path connecting this point to two points of $\beta$. By transversality of the intersections along the compact arc, there cannot be infinitely many such intersections - if we assume the smooth case, this can also be seen by noting that we get 0-dimensional submanifolds which are closed and discrete. Hence we can choose the innermost such path of $\alpha$. By isotopy, we can remove the bigon formed by this alpha. Continuing a finite amount of times, we remove the original bigon. After a finite amount of reiterations, we can therefore remove all bigons, and we get disjoint $\alpha$ and $\beta$.
Now suppose we remove $\alpha \cup \beta$. Then we get a disjoint union of a disk and a punctured disk (by the classification of surfaces). Thus the embedded disk in $S_{0,3}$ gives an isotopy of $\alpha$ to $\beta$. $\square$

**Proposition 3.7.** *The natural map*

$$\mathrm{Mod}(S_{0,3}) \to \Sigma_3$$

*given by the action of* $\mathrm{Mod}(S_{0,3})$ *on the set of marked points of $S_{0,3}$ is an isomorphism.*

*Proof.* The map is a surjective homomorphism since, for any choice of punctures $S_{0,3}$, we can, using the classification of surfaces, modify where the punctures are so that the desired permutation can easily be obtain from a rotation.

It thus suffices to show injectivity. So suppose $\varphi$ is a homomorphism fixing the three marked points, call them $p, q$, and $r$. Choose an arc $\alpha$ in $S_{0,3}$ with distinct endpoints, say $p$ and $q$. Since $\varphi$ fixes the marked points, proposition 3.6 gives that $\varphi \circ \alpha$ is isotopic to $\alpha$, or equivalently, that $\alpha$ is isotopic to $\varphi^{-1} \circ \alpha$, say through an isotopy $F \colon I \times I \to S_{0,3}$. By theorem 2.22, we can extend $F$ to an ambient isotopy, and by composing with $\varphi$, we get an isotopy from $\varphi$ to a homeomorphism which fixes $\alpha$ pointwise.

Now cut $S_{0,3}$ along $\alpha$ so as to obtain a disk with one marked point. Since $\varphi$ preserves the orientations of $S_{0,3}$ and of $\alpha$, it follows that $\varphi$ induces a homemomorphism $\overline{\varphi}$ of this disk which is the identity on the boundary.
But $\mathrm{Mod}(S_{0,1}) \approx 0$, so $\overline{\varphi}$ is homotopic to the identity. And this homotopy induces a homotopy from $\varphi$ to the identity. $\square$

**Exercise 3.8.** Show similarly that $\mathrm{Mod}(S_{0,2}) \approx \mathbb{Z}/2\mathbb{Z}$.

*Solution.* Let $\alpha, \beta$ be arcs with the same distinct marked endpoints. Equivalently to before, we can reduce bigons by isotopy until $\alpha$ and $\beta$ are disjoint. Then removing $\alpha \cup \beta$ we would get two disjoint disks (firstly, $\alpha \cup \beta$ make up a closed simple curve which is trivial since $H_1\left(S^1\right) = \{0\}$ and thus separating. Therefore we get a disconnected space with as many vertices as edges whose Euler characteristic must add to $2 = \chi\left(S^2\right)$, so it must precisely have 1 face each, i.e., they are disks) which will descend to give the desired isotopy in $S_{0,2}$.
So assume no intersection. Let $\varphi$ be an orientation preserving homeomorphism fixing the marked points. Then $\varphi(\alpha)$ is isotopic to $\alpha$, so $\varphi$ is isotopic to a homeomorphism which fixes $\alpha$ pointwise, call it $\psi$. This induces a homeomorphism on $S^2 - \alpha$ which is a disk that is the identity on the boundary, and hence isotopic to the identity homeomorphism on the disk since $\mathrm{Mod}\left(D^2\right) \approx \{0\}$. This isotopy gives an isotopy of $\psi$ to the identity. The composition of all these isotopies gives an isotopy of $\varphi$ with the identity. Hence the map is injective.

**Theorem 3.9.** *The homomorphism*

$$\sigma \colon \mathrm{Mod}\left(T^2\right) \to \mathrm{SL}\left(2, \mathbb{Z}\right)$$

*given by the action on $H_1\left(T; \mathbb{Z}\right) \approx \mathbb{Z}^2$ is an isomorphism.*

*Proof.* Any homeomorphism $\varphi$ of $T^2$ induces an isomorphism on homology: $\varphi_* \colon \mathbb{Z}^2 \approx H_1(T^2) \to H_1\left(T^2\right) \approx \mathbb{Z}^2$, and since homotopic maps induce the same map on homology, we get a map $\sigma \colon \mathrm{Mod}\left(T^2\right) \to \mathrm{Aut}\left(\mathbb{Z}^2\right) \approx \mathrm{GL}\left(2, \mathbb{Z}\right)$. However, we must necessarily have that $\sigma(f) \in \mathrm{SL}(2, \mathbb{Z})$ since orientation-preserving homeomorphisms preserve algebraic intersection numbers which correspond to determinants for $T^2$ by Example 2.12.
It remains to prove that the map is bijective.

For surjectivity, any element $M$ of $\mathrm{SL}\left(2, \mathbb{Z}\right)$ induces an equivariant orientation-preserving linear homeomorphism of $\mathbb{R}^2$ which thus descends to a linear homeomorphism $\varphi_M$ of the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Using our identification of primitive elements in $\mathbb{Z}^2$ with homotopy classes of orientated simple closed curves in $T^2$, we have $\sigma\left([\varphi_M]\right) = M$.

For injectivity, note that $T^2$ is a $K\left(G, 1\right)$-space, so by Proposition 1B.9 in [5], we have a correspondence between homotopy classes of based maps $T^2 \to T^2$ and homomorphisms $\mathbb{Z}^2 \to \mathbb{Z}^2$.

Furthermore, any element $f \in \mathrm{Mod}\left(T^2\right)$ has a representative $\varphi$ that fixes a basepoint for $T^2$: suppose $\psi$ is a representative of $f$ and $x$ is the basepoint. Then we can take a regular neighborhood of the path from $x$ to $\psi(x)$ with support a disk. For any two points on a disk, we can find a homeomorphism taking one point to the other, so let $g$ be a homeomorphism taking $\psi(x)$ to $x$. Since $\mathrm{Mod}\left(D^2\right) = 0$ by the Alexander trick, we get an isotopy from the $g$ to the identity, and postcomposing $\psi$ with this isotopy gives an isotopy from a homeomorphism fixing $x$ to $\psi$. So if $f \in \ker\left(\sigma\right)$, then $\varphi$ is homotopic as a based map to the identity, so $\sigma$ is injective. □

3.2.3. *The Alexander method.* We state the following important proposition without proof.

**Proposition 3.10** (Alexander method)**.** *[2, Proposition 2.8] Let $S$ be a compact surface, possibly with marked points, and let $\varphi \in \mathrm{Homeo}^{+}(S, \partial S)$. Let $\gamma_1, \dots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in $S$ with the following properties.*

*(1) The $\gamma_i$ are pairwise in minimal position.*
*(2) The $\gamma_i$ are pairwise nonisotopic.*
*(3) For distinct $i, j, k$, at least one of $\gamma_i \cap \gamma_j, \gamma_i \cap \gamma_k$, or $\gamma_j \cap \gamma_k$ is empty.*

*(i) If there is a permutation $\sigma$ of $\{1, \dots, n\}$ so that $\varphi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to $\partial S$ for each $i$, then $\varphi(\cup\gamma_i)$ is isotopic to $\cup\gamma_i$ relative to $\partial S$.*
*If we regard $\cup\gamma_i$ as a (possibily disconnected) graph $\Gamma$ in $S$, with vertices at the intersection points and at the endpoints of arcs, then the composition of $\varphi$ with this isotopy gives an automorphism $\varphi_*$ of $\Gamma$.*
*(ii) Suppose now that $\{\gamma_i\}$ fills $S$. If $\varphi_*$ fixes each vertex and each edge of $\Gamma$ with orientations, then $\varphi$ is isotopic to the identity. Otherwise, $\varphi$ has a nontrivial power that is isotopic to the identity.*

## 4. Dehn Twists

Dehn twists will play an important part as generators for the mapping class group. In particular, we will show that certain relations hold for Dehn twists of curves when they form a chain which we will later use as the defining relations for the braid group. This will allow us to look at homomorphisms of the braid group into mapping class groups as long as the relations are preserved.

For an annulus $A = S^1 \times [0, 1]$, we orient $A$ via the induced orientation from the standard orientation of the plane under the embedding $(\theta, t) \mapsto (\theta, t + 1)$ where the plane is taken in $(\theta, r)$-coordinates.

Define the left twist map of $A$ as $T\colon A \to A$ given by $T(\theta, t) = (\theta + 2\pi t, t)$.

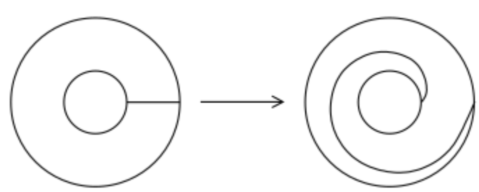


Figure 7. The effect of the right twist on a curve on the annulus. [2, Figure 3.1]

Let $S$ be an oriented surface and let $\alpha$ be a simple closed curve in $S$. Let $N$ be a tubular neighborhood of $\alpha$ and choose an orientation preserving homeomorphsim $\varphi\colon A \to N$. We then obtain a homeomorphism $T_\alpha\colon S \to S$, called a *Dehn twist about* $\alpha$, as follows:

$$T_\alpha(x) = \begin{cases} \varphi \circ T \circ \varphi^{-1} & \text{if } x \in N \\ x & \text{if } x \in S - N \end{cases}.$$

By the uniqueness of regular neighborhoods, the isotopy class of $T_\alpha$ does not depend on the choice of $N$ or the choice of homeomorphism $\varphi$ [6, Theorem 1.8]; nor does $T_\alpha$ depend on the choice of simple closed curve $\alpha$ within its isotopy class.

*Dehn twists on the torus.* Via the isomorphism $\mathrm{Mod}\left(T^2\right) \to \mathrm{SL}\left(2, \mathbb{Z}\right)$ from 3.9, the Dehn twists about the $(1,0)$-curve and the $(0,1)$-curve in $\mathrm{Mod}\left(T^2\right)$ correspond to the matrices

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

4.0.1. *Dehn twists via cutting and gluing and surgery.* One useful visual picture to have in mind for Dehn twists is to cut $S$ along $\alpha$, twist a neighborhood of one boundary component through an angle of $2\pi$ and then reglue.

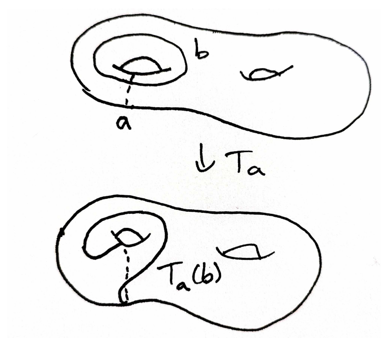



FIGURE 8. Dehn twist of $b$ along $a$ via a cutting along $a$

The global picture is nice, but it is often easier to think about things locally. Here surgery can be quite useful. Suppose we have a loop $b$ with $i\left(a,b\right) \neq 0$. Then the isotopy class $T_a(b)$ is determined by the following procedure: taking representatives $\alpha$ and $\beta$ of $a$ and $b$, respectively, each segment of $\beta$ crossing $\alpha$ is replaced with a segment that turns left before crossing, follows $\alpha$ all the way around parallel, and then turns right and continues to intersect.

This is of course a very nice way to visualize what happens to individual curves, however, when $i\left(a,b\right)$ is larger, it can be difficult to draw the global picture of $T_a(b)$ using this turn left-turn right procedure.

This is where surging the curves comes in handy. Suppose we start with one curve $\beta$ in the class $b$ and $i\left(a,b\right)$ parallel curves $\alpha_i$, each in the class $a$, each in minimal position with $\beta$. At each intersection point between $\beta$ and some $\alpha_i$, we do surgery as demonstrated in figure 9

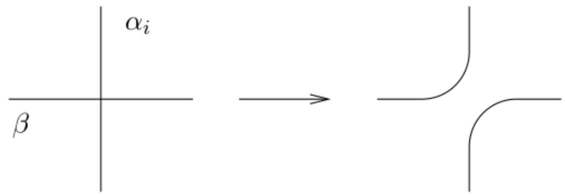



FIGURE 9. Dehn twist via surgery [2, Figure 3.3]

We then resolve the intersection in the unique way so that if we follow an arc of $\beta$ toward the intersection, the surgered arc turns left at the intersection.

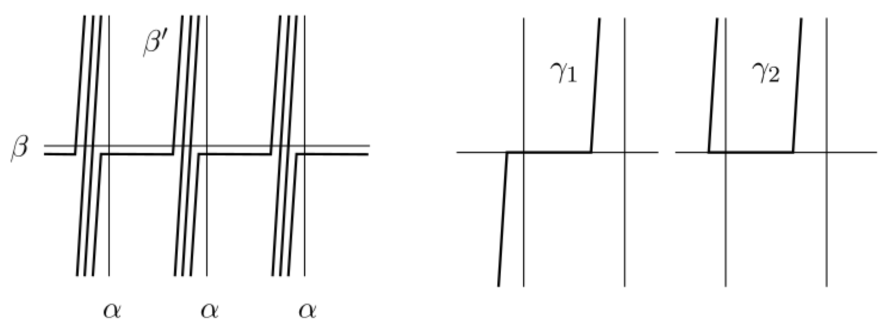
4.0.2. *Dehn twist facts.*

**Proposition 4.1.** *Let $a$ be the isotopy class of a simple closed curve $\alpha$ in a surface $S$. If $\alpha$ is not homotopic to a point or a puncture of $S$, then the Dehn twist $T_a$ is a nontrivial element of* $\operatorname{Mod}(S)$*.*

**Proposition 4.2.** *Let $a$ and $b$ be arbitrary isotopy classes of essential simple closed curves in a surface and let $k$ be an arbitrary integer. We have*

$$i\left(T_a^k(b), b\right) = |k|\, i\,(a,b)^2 .$$

*Proof.* Choose representative simple closed curves $\alpha$ and $\beta$ in minimal position and form a simple closed curve $\beta'$ in the class of $T_a(b)$ using the following surgical principle. We place $ki\,(a,b)$ parallel copies of $\alpha$ to one side of $\alpha$ and then we surger the as described in section 4.0.1(the reason is that for each twist, we essentially run along $\alpha$ which intersects $\beta$ $i(a,b)$ times, hence if we twist $k$ times, we will end up intersecting $\beta$ $ki\,(a,b)$ times).




(A) The setup

(B) The two possibilities based on side of intersection.

FIGURE 10. One Dehn twist about $\alpha$. [2, Figure 3.5]

Note that in the figure $\beta$ intersects $\alpha$ from the same side each time, but we could also have the case where it intersects $\alpha$ from the other side.
Now, by counting, we see that $|\beta \cap \beta'| = |k|\, i\,(a,b)^2$, so it suffices to show that $\alpha$ and $\beta$ are in minimal position. This is equivalent to showing that they form no

bigons by the bigon criterion.

First we cut $\beta$ and $\beta'$ at the points of intersection and obtain arcs $\{\beta_i\}$ and $\{\beta_i'\}$. Then for the potential bigons that can be formed from one arc of $\beta_i$ and one $\beta_j'$, we either have that the orientations at the intersection points are compatible or not compatible as for $\gamma_1$ and $\gamma_2$ in figure 10b. In a bigon, the orientations will be different, so we must be in the situation of $\gamma_2$. However, the vertical line segments of $\beta'$ are parallel to $\alpha$, so if $\gamma_2$ forms a bigon, then $\alpha$ also forms a bigon with $\beta$, contradicting them being in minimal position. □

**Corollary 4.3.** *Dehn twists about essential simple closed curves in a surface are nontrivial infinite order elements of the mapping class group.*

4.0.3. *Basic facts about Dehn twists.* Throughout this section, $a$ and $b$ denote arbitrary (unoriented) isotopy classes of simple closed curves.

**Lemma 4.4.** $T_a = T_b \iff a = b$.

*Proof.* The only if part simply says that Dehn twists are well-defined on isotopy classes.
For the if part, suppose $a \neq b$. We want to find an isotopy class $c$ of simple closed curves such that $i(a, c) = 0$ and $i(b, c) \neq 0$.
If $i(a, b) \neq 0$, then choosing $c = a$ works.
If $i(a, b) = 0$, then choosing a representative $\alpha$ of $a$ and $\beta$ of $b$ such that $\alpha$ and $\beta$ are disjoint, we havethat $\alpha$ is contained in the interior of some component of the cut surface $S_\beta$. This surface has at least one boundary component. If it has genus at least 1, then by the classification of surfaces and nontriviality of $\alpha$, it is easy to find some $\gamma$ such that $\gamma$ is contained in the interior and has geometric intersection number 1 with $\alpha$. If the genus is 0, we are looking at a component which is a sphere with at least one disk removed and potentially some punctures. Since $\alpha$ and $\beta$ are not isotopic, we cannot have a disk, once or twice punctured disk or annulus. For the remaining cases, one can also easily find a choice of $\gamma$. □

**Lemma 4.5.** *For any $f \in \mathrm{Mod}(S)$ and any isotopy class $a$ of simple closed curves in $S$, we have*

$$T_{f(a)} = f T_a f^{-1}.$$

*Proof.* Letting $\varphi$ be a representative of $f$, $\alpha$ a representative of $a$, and $\psi_\alpha$ a representative of $T_a$ whose support is an annulus, we get that $\varphi^{-1}$ takes a regular neighborhood of $\varphi(\alpha)$ to a regular neighborhood of $\alpha$ preserving the orientation. Then $\psi_\alpha$ twists the neighborhood of $\alpha$ and $\varphi$ takes this neighborhood of $\alpha$ back to a neighborhood of $\varphi(\alpha)$. Hence we get a Dehn twist about $\varphi(\alpha)$. □

**Corollary 4.6.** *For any $f \in \mathrm{Mod}(S)$ and any isotopy class $a$ of simple closed curves in $S$, we have*

$$f \text{ commutes with } T_a \iff f(a) = a.$$

**Corollary 4.7.** *If $a$ and $b$ are nonseparating simple closed curves in $S$, then $T_a$ and $T_b$ are conjugate in $\mathrm{Mod}(S)$.*

**Lemma 4.8.** *For any two isotopy classes $a$ and $b$ of simple closed curves in a surface $S$, we have*

$$i(a, b) = 0 \iff T_a(b) = b \iff T_a T_b = T_b T_a.$$

*Proof.* For the first if and only if, if $i(a,b)=0$, then choosing a regular neighborhood of $a$ such that $b$ is not contained in it, we see that the Dehn twist using this regular neighborhood leaves $b$ fixed, so $T_a(b)=b$. Conversely, if $T_a(b)=b$, then $i(T_a(b),b)=i(b,b)=0$, so by proposition 4.2, we have $i(a,b)^2=i(T_a(b),b)=0$, hence $i(a,b)=0$.
The second if and only if follows from Corollary 4.6. □

**Proposition 4.9.** *The above results have the following analogues for powers of Dehn twists: for $f\in \mathrm{Mod}(S)$, we have*

$$fT_a^j f^{-1}=T_{f(a)}^j$$

*and so $f$ commutes with $T_a^j$ if and only if $f(a)=a$. Also, for nontrivial Dehn twists $T_a,T_b$ and nonzero integers $j,k$, we have*

$$T_a^j=T_b^k \iff a=b \text{ and } j=k$$
$$T_a^jT_b^k=T_b^kT_a^j \iff i(a,b)=0.$$

4.0.4. *Relations between two Dehn twists.*

**Proposition 4.10** (Braid relation)**.** *If $a$ and $b$ are isotopy classes of simple closed curves with $i(a,b)=1$, then*

$$T_aT_bT_a=T_bT_aT_b.$$

*Equivalently, this reads $T_aT_b(a)=b$.*

*Proof.* We have

$$T_aT_bT_a=T_bT_aT_b \iff (T_aT_b)\,T_a\,(T_aT_b)^{-1}=T_b \iff T_{T_aT_b(a)}=T_b \iff T_aT_b(a)=b$$

By a change of coordinates principle, it suffices to check the last formula for any two isotopy classes $a$ and $b$ with $i(a,b)=1$. This is shown in Figure 11 with $\alpha$ and $\beta$ representatives of $a$ and $b$, respectively.

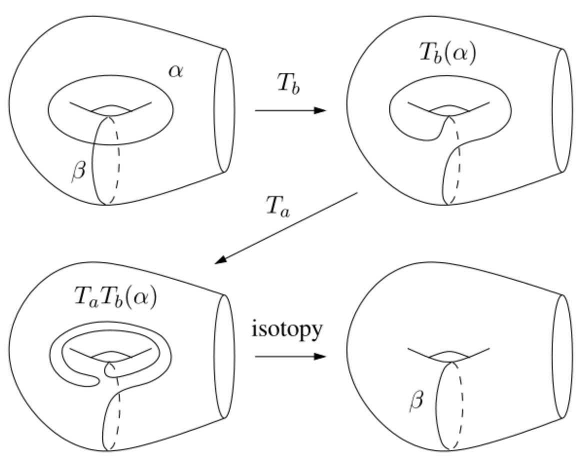



FIGURE 11. Checking $T_aT_b(\alpha)=\beta$. [2, Figure 3.9]

□

**Question 4.11.** Does the converse also work? I.e., if two Dehn twists satisfy the braid relation algebraically, do the corresponding curves necessarily have intersection number equal to 1?

**Proposition 4.12.** *[2, Proposition 3.13] If $a$ and $b$ are distinct isotopy classes of simple closed curves and the Dehn twists $T_a$ and $T_b$ satisfy $T_aT_bT_a = T_bT_aT_b$, then $i(a,b) = 1$.*

*Groups generated by two Dehn twists.*

**Theorem 4.13.** *Let $a$ and $b$ be two isotopy classes of simple closed curves in a surface $S$. If $i(a,b) \geqslant 2$, then $\langle T_a, T_b\rangle \approx F_2$, where $F_2$ is the free group of rank 2.*

*Proof.* The slick proof makes use of this basic but nice lemma from geometric group theory:

**Lemma 4.14** (Ping pong lemma)**.** *Let $G$ be a group acting on a set $X$. Let $g_1, \ldots, g_n$ be elements of $G$. Suppose that there are nonempty, disjoint subsets $X_1, \ldots, X_n$ of $X$ with the property that, for each $i$ and each $j \neq i$, we have $g_i^k(X_j) \subset X_i$ for every nonzero integer $k$. Then the group generated by the $g_i$ is a free group of rank $n$.*

**Exercise 4.15.** Complete the proof.

□

## 4.1. **Capping and including.**

4.1.1. *Including.* When $S$ is a closed subsurface of a surface $S'$, we can define a natural homomorphism $\eta\colon \mathrm{Mod}(S) \to \mathrm{Mod}(S')$ as follows. For $f \in \mathrm{Mod}(S)$, we represent it by some $\varphi \in \mathrm{Homeo}^+(S, \partial S)$. Then, if $\hat\varphi \in \mathrm{Homeo}^+(S', \partial S')$ denotes the element that agrees with $\varphi$ on $S$ and is the identity outside of $S$, we define $\eta(f)$ to be the class of $\hat\varphi$. The map $\eta$ is well defined because any homotopy between two elements of $\varphi \in \mathrm{Homeo}^+(S, \partial S)$ gives a homotopy between the corresponding elements of $\mathrm{Homeo}^+(S', \partial S')$ (the homotopy is simply relative to $S' - S$).

The goal is to find $\ker\eta$.

**Lemma 4.16.** *Let $\alpha_1, \ldots, \alpha_n$ be a collection of homotopically distinct simple closed curves in a surface $S$, each not homotopic to a point in $S$. Let $\beta$ and $\beta'$ be simple closed curves in $S$ that are both disjoint from $\cup\alpha_i$ and are homotopically distinct from each $\alpha_i$. If $\beta$ and $\beta'$ are isotopic in $S$, then they are isotopic in $S - \cup\alpha_i$.*

**Theorem 4.17** (The kernel of the inclusion homomorphism)**.** *Let $S$ be a closed subsurface of a surface $S'$. Assume that $S$ is not homeomorphic to a closed annulus and that no component of $S' - S$ is an open disk. Let $\eta\colon \mathrm{Mod}(S) \to \mathrm{Mod}(S')$ be the induced map. Let $\alpha_1, \ldots, \alpha_m$ denote the boundary components of $S$ that bound once-punctured disks in $S' - S$ and let $\{\beta_1, \gamma_1\}, \ldots, \{\beta_n, \gamma_n\}$ denote the pairs of boundary components of $S$ that bound annuli in $S' - S$. Then the kernel of $\eta$ is the free abelian group*

$$\ker\eta = \left\langle T_{\alpha_1}, \ldots, T_{\alpha_m}, \ldots, T_{\beta_1}T_{\gamma_1}^{-1}, \ldots, T_{\beta_n}T_{\gamma_n}^{-1}\right\rangle.$$

*In particular, i no connected component of $S' - S$ is an open annulus, an open disk, or an open once-marked disk, then $\eta$ is injective.*

4.1.2. *The capping homomorphism.* One useful special case of theorem 4.17 is the case where $S' - S$ is a once-punctured disk. We say that $S'$ is the surface obtained from $S$ by *capping* one boundary component. In this case, we have

**Proposition 4.18** (The capping homomorphism). *Let $S'$ be the surface obtained from a surface $S$ by capping the boundary component $\beta$ with a once-marked disk; call the marked point in this disk $p_0$. Denote by $\operatorname{Mod}(S, \{p_1, \ldots, p_k\})$ the subgroup of $\operatorname{Mod}(S)$ consisting of elements that fix the punctures $p_1, \ldots, p_k$, where $k \geqslant 0$. Let $\operatorname{Mod}(S', \{p_0, \ldots, p_k\})$ denote the subgroup of $\operatorname{Mod}(S')$ consisting of elements that fix the marked points $p_0, \ldots, p_k$ and then let $\mathcal{C}ap\colon \operatorname{Mod}(S, \{p_1, \ldots, p_k\}) \to \operatorname{Mod}(S', \{p_0, \ldots, p_k\})$ be the induced homomorphism. Then the following sequence is exact:*

$$1 \to \langle T_\beta \rangle \to \operatorname{Mod}(S, \{p_1, \ldots, p_k\}) \xrightarrow{\mathcal{C}ap} \operatorname{Mod}\left(S', \{p_0, \ldots, p_k\}\right) \to 1$$

## 5. Braid Groups

We finally introduce braid group. These are must more concrete objects to work with and allow for visualization. They hold not only connections to mapping class groups which we shall explore, but are natural in the study of knots and links as well. That is to say that they are naturally interesting objects to explore. In particular, we will show a explore one of these connections between braid groups and symmetric mapping class groups of punctured disks through the Birman-Hilden theorem. This will be the goal of this section.

**Definition 5.1** (Braids). Let $p_1, \ldots, p_n$ be distinguished points in $\mathbb{C}$. A *braid* is a collection of $n$ paths $f_i\colon [0,1] \to \mathbb{C} \times [0,1], 1 \leqslant i \leqslant n$, called *strands*, and a permutation $\overline{f} \in \Sigma_n$ such that the following hold:

- the strands $f_i([0,1])$ are disjoint
- $f_i(0) = p_i$
- $f_i(1) = p_{\overline{f}(i)}$
- $f_i(t) \in \mathbb{C} \times \{t\}$.

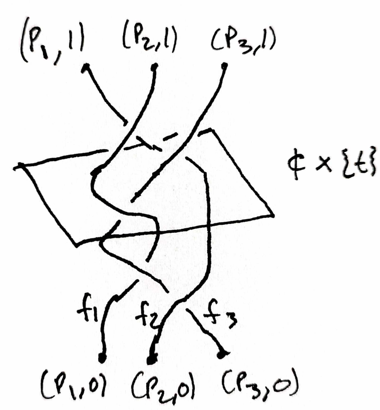



Figure 12. Depiction of a braid

Usually we picture this by its *braid diagram* which is the projection of the images of the strands to the plane $\mathbb{R} \times [0,1]$ (with indications as to which strands pass over and under which others).

**Definition 5.2.** The *braid group on $n$ strands*, denoted $B_n$, is the group of isotopy classes of braids.

*Remark.* Here an isotopy of the braid is a collection of isotopies $(h_1(x,t), \ldots, h_n(x,t))$ where $h_i$ is an isotopy of $f_i$ and such that $(h_1(-,t), \ldots, h_n(-,t))$ is a braid for each $t \in [0,1]$.

**Definition 5.3.** The product of the braid $(f_i(t))$ and the braid $(g_i(t))$ is the braid $(h_i(t))$, where

$$h_i(t) = \begin{cases} f_i(2t), & t \in \left[0, \frac{1}{2}\right] \\ g_{\overline{f}(i)}(2t-1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

For $1 \leqslant i \leqslant n-1$, let $\sigma_i \in B_n$ denote the braid whose only crossing is the $(i+1)$ st strand passing in front of the $i$ th strand.
We claim that the group $B_n$ is generated by the elements $\sigma_1, \ldots, \sigma_{n-1}$. This claim follows from the fact that any braid $\beta$ can be isotoped so that its finitely many corssings occur at different horizontal levels.

### 5.1. Fundamental groups of configuration spaces.

**Definition 5.4.** Let $S$ be a surface and let $C^{ord}(S,n)$ denote the configuration space of $n$ distinct, ordered points in $S$, given by $C^{ord}(S,n) = S^{\times n} - \mathrm{BigDiag}(S^{\times n})$ where $\mathrm{BigDiag}\,(S^{\times n}) = \{(x_1, \ldots, x_n) \in S^{\times n} \mid \exists 1 \leqslant i < j \leqslant n \colon x_i = x_j\}$.

Now, the symmetric group $\Sigma_n$ acts on $S^{\times n}$ by permuting the coordinates. This action preserves $\mathrm{BigDiag}\,(S^{\times n})$ and thus induces an action of $\Sigma_n$ by homeomorphisms on $C^{ord}\,(S,n)$. Since the action of $\Sigma_n$ permutes the $n$ coordinates and since these coordinates are always distinct for points in $C^{ord}\,(S,n)$, we see that this action is free. The quotient space

$$C\,(S,n) = C^{ord}\,(S,n)\,/\Sigma_n$$

is just the configuration space of $n$ distinct, *unordered* points in $S$.
Now we note the following lemma [9, Cor 12.27] and proposition [9, Prop 12.22]

**Lemma 5.5.** *Let $M$ be a connected $n$-manifold on which a discrete group $\Gamma$ acts continuously, freely, and properly. Then $M/\Gamma$ is an $n$-manifold.*

**Proposition 5.6.** *Every continuous action of a compact topological group on a Hausdorff space is proper*

Since $C\,(S,n)$ is the quotient of a manifold by a continuous free action of a finite group (hence compact), it follows that $C\,(S,n)$ is a manifold.
Since each strand of a braid is a map $f_i \colon I \to \mathbb{C} \times I$ with $f_i(t) \in \mathbb{C} \times \{t\}$, we can think of each $f_i$ as a map $I \to \mathbb{C}$, so that a braid essetially becomes a path $I \to \mathbb{C}^n$ with equal end points, i.e., a loop, where each $t$ is mapped to the point whose $i$th coordinate is $I \xrightarrow{f_i} \mathbb{C} \times \{t\} \xrightarrow{\text{proj}} \mathbb{C}$. By assumption, the strands are disjoint, so each $t$ is mapped to a point in $C^{\text{ord}}\,(\mathbb{C},n)$, and essentially, forgetting the strand index, we can regard this as a map $I \to C\,(\mathbb{C},n)$. Said in a different way, if we consider a slice $\mathbb{C} \times \{t\}$ and intersect it with any braid, then we get a point in $C\,(\mathbb{C},n)$, so the

whole braid, which can be seen as a path in $C(\mathbb{C}, n)$ between these intersections, gives an element of $\pi_1(C(\mathbb{C}, n))$, and this gives the isomorphism

$$B_n \approx \pi_1(C(\mathbb{C}, n))$$

In this case, the generator $\sigma_i$ of $B_n$ corresponds to the element of $\pi_1(C(\mathbb{C}, n))$ given by the loop of $n$-point configurations in $\mathbb{C}$ where the $i$ th and $(i+1)$ st points switch places by moving in a clockwise fashion, and the other $n-2$ points remain fixed.

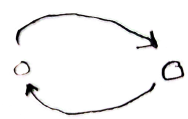


FIGURE 13

5.2. **Mapping Class Group of a punctured disk.**

**Question 5.7.** Is there any relationship between braid groups and mapping class groups?

On relationship is the following: let $D_n$ be a closed disk $D^2$ with $n$ marked points, then

$$B_n \approx \operatorname{Mod}(D_n) = \pi_0\left(\operatorname{Homeo}^+(D_n, \partial D_n)\right).$$

To see this, let $\varphi$ be a representative of some element in $\operatorname{Mod}(D_n)$ which leaves the set of marked points invariant under $\varphi$ - i.e., if $\{x_0, \ldots, x_n\} \subset D_n$ are the $n$ marked points, then $\varphi(\{x_0, \ldots, x_n\}) = \{x_0, \ldots, x_n\}$. Note that we do not regard these marked points as punctures because we will want to consider isotopies which "move" these marked points around which is not allowed when the marked points represent punctures. We see that $\varphi$ is simply a homeomorphism of $D^2$ fixing $\partial D^2$ pointwise, so by the Alexander lemma, $\varphi$ is isotopic to the identity. Now, throughout such an isotopy, the marked points must again be send to themselves, albeit they might move around through in the interior of $D^2$ (which we identify with $\mathbb{C}$ ) throughout the isotopy. Thus this isotopy produces a loop of these marked points, i.e., it produces a loop in $C(\mathbb{C}, n)$. So we have produced a braid.

**Question 5.8.** Does this association give a well-defined homomorphism $B_n \to \operatorname{Mod}(D_n)$ and would this be an isomorphism?

The answer is yes, and since $\sigma_i$ generate $B_n$, the images generate $\operatorname{Mod}(D_n)$. Now, the image of $\sigma_i$ will be the isotopy class of a homeomorphism of $D_n$ that has support a twice-punctured disk and is described on this support by figure 14 - we will call such a homeomorphism a half-twist.
The way to think about this is that while fixing the boundary pointwise, we take the two punctures and swap their places by moving them in semicircles to the other puncture's position and the movement of the remaining points can be thought of as if this were carried out on a surface made of rubber.

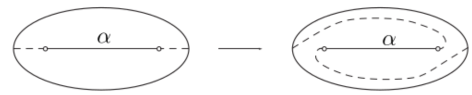



FIGURE 14. A half-twist [2, Figure 9.5]

We denote such a half-twist as $H_\alpha$, and we can think of $\alpha$ as either a simple closed curve with two punctures in its interior or a simple proper arc connecting two punctures.

5.3. **The Birman Exact Sequence.** Let $S$ be any surface, possibly with punctures (but no marked points) and let $(S, x)$ denote the surface obtained from $S$ by marking a point $x$ in the interior of $S$. There is a natural homomorphism

$$\mathcal{F}orget\colon \mathrm{Mod}\,(S, x) \to \mathrm{Mod}(S)$$

called the forgetful map which is realized by forgetting that the point $x$ is marked. This map is surjective as any homeomorphism of $S$ can be modified by isotopy to fix $x$ using the same argument as for the torus in Theorem 3.9 - namely, we pick some arc connecting $x$ to the image of $x$ and then create a local isotopy supported in a regular neighborhood of the arc moving the image of $x$ to $x$. The group $\mathrm{Mod}(S, x)$ is isomorphic to the subgroup $G$ of $\mathrm{Mod}\,(S - x)$ consisting of homeomorphisms preserving the puncture coming from $x$.
The forgetful map can then be interpreted as the map $G \to \mathrm{Mod}(S)$ obtained by "filling in" the puncture $x$. I.e., $\mathcal{F}orget$ is the map induced by the inclusion $S - x \hookrightarrow S$.

5.3.1. *Analyzing the kernel of $\mathcal{F}orget$.* Let $f \in \mathrm{Mod}(S, x)$ be an element of the kernel of $\mathcal{F}orget$ and let $\varphi$ be a homeomorphism representing $f$. We can think of $\varphi$ as a homeomorphism $\overline{\varphi}$ of $S$.
Since $\mathcal{F}orget(f) = 1$, there is an isotopy from $\overline{\varphi}$ to $\mathbb{1}_S$. During this isotopy, the image of the point $x$ traces out a loop $\alpha$ in $S$ based at $x$. Now we will introduce the $\mathcal{P}ush$ map: given a loop $\alpha$ in $S$ based at $x$, we can consider $\alpha\colon [0, 1] \to S$ as an isotopy $h\colon \{x\} \times I \to S$ with $h(x, 0) = h(x, 1) = x$ and extend this to the whole surface using 2.22 (here, $V = \{x\}$ ), denote this by $h\colon S \times I \to S$ also. Let $\varphi(x) = h(x, 1)$ be the homeomorphism of $S$ obtained at the end of the isotopy. Taking its isotopy class in $\mathrm{Mod}\,(S, x)$, we get $[\varphi] =: \mathcal{P}ush(\alpha) \in \mathrm{Mod}\,(S, x)$. Think of $\mathcal{P}ush(\alpha)$ as placing your finger on $x$ and pushing $x$ along $\alpha$, draggin the rest of the surface along as you go (indeed, locally, this is what must happen).
The question of whether this mapping class is independent of the choice of isotopy extension as well as the choice of $\alpha$ within its homotopy class. I.e., whether $\mathcal{P}ush$ defines a well-defined map

$$\mathcal{P}ush\colon \pi_1\,(S, x) \to \mathrm{Mod}\,(S, x)\,.$$

We will show this to prove the Birman exact sequence.

**Theorem 5.9** (Birman exact sequence)**.** *Let $S$ be a surface with $\chi(S) < 0$, possibily with punctures and/or boundary. Let $(S, x)$ be the surface obtained from $S$ by marking a point $x$ in the interior of $S$. Then the following sequence is exact:*

$$1 \to \pi_1\,(S, x) \overset{\mathcal{P}ush}{\to} \mathrm{Mod}\,(S, x) \overset{\mathcal{F}orget}{\to} \mathrm{Mod}(S) \to 1.$$

**Lemma 5.10.** *$\mathcal{P}ush\colon \pi_1(S,x) \to \mathrm{Mod}(S,x)$ is injective when $\chi(S) < 0$.*

*Proof.* Let $\varphi$ represent $\mathcal{P}ush(\alpha) \in \mathrm{Mod}(S,x)$. Then $\varphi$ is a map $(S,x) \to (S,x)$, so in particular, we have an induced homomorphism $\varphi_*\colon \pi_1(S,x) \to \pi_1(S,x)$. This homomorphism is the inner automorphism $I_\alpha(\gamma) = \alpha\gamma\alpha^{-1}$ as can be checked visually. Now, for $\chi(X) < 0$, the center of $\pi_1(S)$ is trivial [2, p. 22], so $I_\alpha$ is nontrivial whenever $\alpha \neq c_x$. If $\varphi$ had been homotopic to the identity, their induced homomorphisms would be equal, so we conclude that $\mathcal{P}ush(\alpha)$ is nontrivial whenever $\alpha$ is nontrivial. □

5.3.2. *Push maps along loops in terms of Dehn twists.* Let $\alpha \in \pi_1(S,x)$ be simple. Let $S^1 \times [0,2]$ be an annulus about $\alpha$ with $S^1 \times \{1\}$ identified with $\alpha$ with $(0,1)$ identified with the marked point $x$. We orient $S^1 \times [0,2]$ with the standard orientations on $S^1$ and $[0,2]$.
Let $F\colon A \times I \to A$ be the isotopy given by

$$F((\theta,r),t) = \begin{cases} (\theta + 2\pi r t, r), & 0 \leqslant r \leqslant 1 \\ (\theta + 2\pi(2-r)t, r), & 1 \leqslant r \leqslant 2. \end{cases}$$

By Theorem 2.22, we can extend $F$ to an ambient isotopy of $S$. Restricting $F$ to $\{x\} \times [0,1]$, we get

$$F((0,1),t) = (2\pi t, 1),$$

so $F$ pushes $x$ around the core of the annulus. Now, the homeomorphism (representing $\mathcal{P}ush(\alpha)$ ) $\varphi$ of $(S,x)$ given by $F((\theta,r),1)$ is a product of two Dehn twists, $T_a T_b^{-1}$ where $a$ is the closed curve identified with $S^1 \times \{0\}$ and $b$ is the closed curve identified with $S^1 \times \{2\}$.

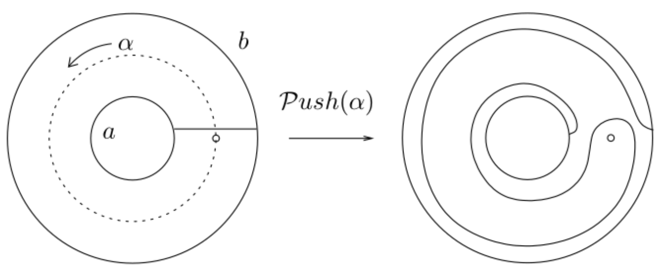



FIGURE 15. The point-pushing map $\mathcal{P}ush$ from the Birman exact sequence [2, Figure 4.4]

*Remark* (Naturality). For any $h \in \mathrm{Mod}(S,x)$ and any $\alpha \in \pi_1(S,x)$, we have

$$\mathcal{P}ush(h_*(\alpha)) = h\mathcal{P}ush(\alpha)h^{-1}.$$

*Proof of Birman exact sequence.* There is a natural fiber bundle

$$\mathrm{Homeo}^+(S,x) \to \mathrm{Homeo}^+(S) \xrightarrow{\mathcal{E}} S$$

where $\mathcal{E}$ is evaluation at $x$ and the fiber is the subgroup of $\mathrm{Homeo}^+(S)$ consisting of elements that fix the point $x$. We must explain why $\mathrm{Homeo}^+(S)$ is locally homeomorphic to a product of an open set $U$ of $S$ with $\mathrm{Homeo}^+(S,x)$ so that the restriction of $\mathcal{E}$ is projection to the first factor (see definition 10.5) Let $U$ be some open neighborhood of $x$ in $S$ that is homeomorphic to a disk. Given $u \in U$, we can choose $\varphi_u \in \mathrm{Homeo}^+(U)$ such that $\varphi_u(x) = u$ and such that $\varphi_u$ varies continuously as a function of $u$. We then get a homeomorphism $U \times \mathrm{Homeo}^+(S, x) \to \mathcal{E}^{-1}(U)$ given by

$$(u, \psi) \mapsto \varphi_u \circ \psi.$$

The inverse map is given by $\psi \mapsto \left(\psi(x), \varphi^{-1}_{\psi(x)} \circ \psi\right)$. For any other point $y \in S$, we can choose a homeomorphism $\xi$ taking $x$ to $y$, giving a homeomorphism $\mathcal{E}^{-1}(U) \to \mathcal{E}^1(\xi(U))$ by $\psi \mapsto \xi \circ \psi$, hence we have a fiber bundle.
By the long exact sequence of homotopy groups for the fiber bundle, we get

$$\ldots \to \underbrace{\pi_1\left(\mathrm{Homeo}^+(S)\right)}_{=1} \to \pi_1(S) \overset{\mathcal{P}ush}{\to} \pi_0\left(\mathrm{Homeo}^+(S,x)\right) \overset{\mathcal{F}orget}{\to} \pi_0\left(\mathrm{Homeo}^+(S)\right) \to \underbrace{\pi_0(S)}_{=1} \to \ldots$$

where $\pi_1\left(\mathrm{Homeo}^+(S)\right) = 1$ follows from Theorem 2.30. $\square$

5.4. **Generalized Birman Exact Sequence.** We now return to Question 5.8. We wish to show that $\pi_1(C(\mathbb{C}, n)) \approx \mathrm{Mod}(D_n)$.

**Definition 5.11** ($n$-stranded surface braid group of $S$)**.** For an arbitrary surface $S$, we call $\pi_1(C(S, n))$ the $n$-stranded surface braid group of $S$.

Let $S$ be a compact finite-type surface without marked points. Let $(S, \{x_1, \ldots, x_n\})$ denote $S$ with $n$ marked points $x_1, \ldots, x_n$ in the interior. Equivalently to the construction in the proof of the Birman exact sequence, there is a fiber bundle

$$\mathrm{Homeo}^+((S, \{x_1, \ldots, x_n\}), \partial S) \to \mathrm{Homeo}^+(S, \partial S) \to C(S^\circ, n)$$

where $S^\circ$ is the interior of $S$ and $\mathrm{Homeo}^+((S, \{x_1, \ldots, x_n\}), \partial S)$ is the group of orientation-preserving homeomorphisms of $S$ that preserve the set $\{x_1, \ldots, x_n\}$ (allowing permutations, however) and fix the boundary of $S$ pointwise.

**Theorem 5.12** (Birman exact sequence generalized)**.** *[2, Theorem 9.1] Let $S$ be a surface without marked points and with $\pi_1\left(\mathrm{Homeo}^+(S, \partial S)\right) = 1$. The following sequence is exact:*

$$1 \to \pi_1(C(S, n)) \overset{\mathcal{P}ush}{\to} \mathrm{Mod}(S, \{x_1, \ldots, x_n\}) \overset{\mathcal{F}orget}{\to} \mathrm{Mod}(S) \to 1.$$

**Corollary 5.13.** *When $S = D^2$, this gives the exact sequence*

$$1 \to \underbrace{\pi_1\left(C\left(D^2, n\right)\right)}_{B_n} \to \mathrm{Mod}(D_n) \to \underbrace{\mathrm{Mod}\left(D^2\right)}_{1} \to 1.$$

*Hence $B_n \approx \mathrm{Mod}(D_n)$.*

5.5. **Braid group and symmetric mapping class groups.**

5.5.1. *The construction of the homomorphism.* Let $S^1_g$ be a surface of genus $g$ with one boundary component. Define a homomorphism $\psi\colon B_n \to \mathrm{Mod}\left(S^1_g\right)$ for $n \leqslant 2g+1$ as follows. Choose a chain of simple closed curves $\{\alpha_i\}$ in $S^1_g$, that is, a collection of simple closed curves satisfying $i(\alpha_i, \alpha_{i+1}) = 1$ for all $i$ and $i(\alpha_i, \alpha_j) = 0$ otherwise, see Figure 16.
We then define $\psi$ via $\psi(\sigma_i) = T_{\alpha_i}$. This is well defined if and only if $T$ respects all the relations in $B_n$. Now, if $\sigma_i$ and $\sigma_j$ are given with $|i-j| \geqslant 2$, then $\sigma_i\sigma_j = \sigma_j\sigma_i$

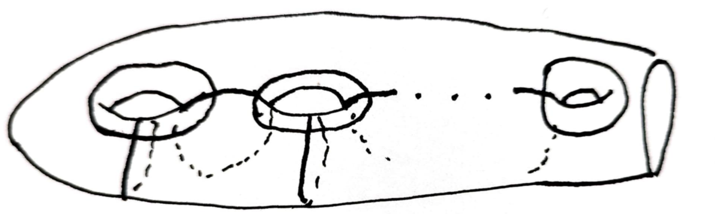


FIGURE 16. A chain of simple closed curves (Humphreys generators ).

and indeed also $i(\alpha_i, \alpha_j) = 0$ so by the disjointness relation for Dehn twists (fact 3.9), we get that $T$ respects this commutativity.
Similarly, we have that $\sigma_i\sigma_{i+1}\sigma_i$, for all $i$, is mapped to $T_{\alpha_i}T_{\alpha_{i+1}}T_{\alpha_i}$ which, by the braid relation on Dehn twists and the assumption that $i(\alpha_i, \alpha_{i+1}) = 1$ for all $i$, is equivalent to $T_{\alpha_{i+1}}T_{\alpha_i}T_{\alpha_{i+1}}$, so the braid relation in $B_n$ is respected under $\psi$.

**Question 5.14.** Can we say whether $\psi$ is injective?

5.6. **The Birman-Hilden Theorem.** Let $\iota$ be the order 2 element of $\mathrm{Homeo}^+\left(S_g^1\right)$ as shown in figure 17 and let $\mathrm{SHomeo}^+\left(S_g^1\right)$ be the centralizer in $\mathrm{Homeo}^+\left(S_g^1, \partial S_g^1\right)$ of $\iota$:

$$\mathrm{SHomeo}^+\left(S_g^1\right) = C_{\mathrm{Homeo}^+\left(S_g^1, \partial S_g^1\right)}\left(\iota\right).$$

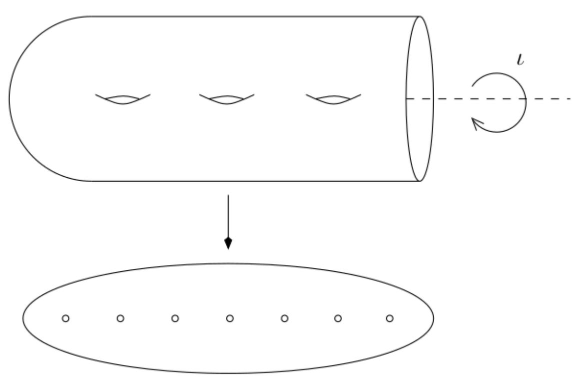



FIGURE 17. The Birman-Hilden double cover [2, Figre 9.13]

**Definition 5.15.** The group $\mathrm{SHomeo}^+\left(S_g^1\right)$ is called the group of orientation-preserving symmetric homeomorphisms of $S_g^1$. The symmetric mapping class group is the group

$$\mathrm{SMod}(S_g^1) = \mathrm{SHomeo}^+\left(S_g^1\right)/\text{isotopy},$$

i.e, the subgroup of $\mathrm{Mod}\left(S_g^1\right)$ that is the image of $\mathrm{SHomeo}^+\left(S_g^1\right)$. In particular, $\mathrm{SMod}\left(S_g^1\right)$ is not the same as $\pi_0\left(\mathrm{SHomeo}^+\left(S_g^1\right)\right)$ as isotopies are allowed to pass through all homeomorphisms in $\mathrm{SMod}\left(S_g^1\right)$, but not in the latter.

The homeomorphism $\iota$ has $2g+1$ fixed points in $S_g^1$. The quotient of $S_g^1$ by $\langle\iota\rangle$ is a topological disk $D_{2g+1}$ with $2g+1$ branch points of order 2, with each branch point coming from a fixed point of $\iota$. Since the elements of $\mathrm{SHomeo}^+\left(S_g^1\right)$ commute with $\iota$, they descend to homeomorphisms of the quotient disk, and by the commutativity, they must preserve the set of $2g+1$ fixed points of $\iota$, and so there is a homomorphism

$$\mathrm{SHomeo}^+\left(S_g^1\right) \to \mathrm{Homeo}^+\left(D_{2g+1}, \partial D_{2g+1}\right) \tag{$\Omega$}$$

by sending $\varphi$ to $\pi\circ\varphi$ where $\pi\colon S_g^1 \to D_{2g+1}$ is the quotient map. If $\varphi \mapsto \pi\circ\varphi = \mathbb{1}_{D_{2g+1}}$, then $\varphi \in \langle\iota\rangle$, and since $\iota \notin \mathrm{SHomeo}^+\left(S_g^1\right)$, we have $\varphi = 1$, so the homomorphism is injective. Furthermore, it is clearly surjective, hence an isomorphism.

**Definition 5.16.** We say two homeomorphisms $\varphi, \psi \in \mathrm{SHomeo}^+\left(S_g^1\right)$ are symmetrically isotopic whenever $[\varphi] = [\psi]$ in $\pi_0\left(\mathrm{SHomeo}^+\left(S_g^1\right)\right)$. Note that here, $\mathrm{SHomeo}^+\left(S_g^1\right)$ inherits the subspace topology from $\mathrm{Homeo}^+\left(S_g^1\right)$.

Recalling that ($\Omega$) is an isomorphism, we get

$$\begin{aligned}\mathrm{SHomeo}^+\left(S_g^1\right)/\text{symmetric isotopy} &= \pi_0\left(\mathrm{SHomeo}^+(S_g^1)\right)\\ &\approx \pi_0\left(\mathrm{Homeo}^+\left(D_{2g+1}, \partial D_{2g+1}\right)\right)\\ &= \mathrm{Mod}\left(D_{2g+1}\right)\\ &\approx B_{2g+1}.\end{aligned}$$

Since $\mathrm{SMod}\left(S_g^1\right) = \mathrm{SHomeo}^+\left(S_g^1\right)/\text{isotopy}$, if we can show that two symmetric homeomorphisms of $S_g^1$ which are isotopic must be symmetrically isotopic, then we will thus derive the Birman-Hilden theorem which we now state.

**Theorem 5.17** (Birman-Hilden)**.** $\mathrm{SMod}\left(S_g^1\right) \approx B_{2g+1}$.

**Example 5.18.** Taking $g=1$, we will find $\mathrm{Mod}\left(S_g^1\right)$. Note that by [2, Section 3.4] $\iota \in C\left(\mathrm{Homeo}^+\left(S_1^1\right)\right)$, so $\mathrm{Mod}\left(S_1^1\right) = \mathrm{SMod}\left(S_g^1\right)$, and now it is easy to see that

$$\mathrm{Mod}\left(S_1^1\right) = \mathrm{SMod}\left(S_g^1\right) \approx B_3 \approx \mathrm{Mod}\left(D_3\right)$$

*Remark.* The Birman-Hilden theorem also holds for surfaces with two symmetric boundary components that are interchanged by $\iota$, see figure 18. Hence

$$\mathrm{SMod}\left(S_g^2\right) \approx B_{2g+2}.$$

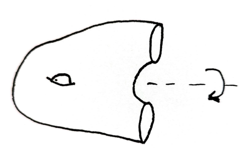


FIGURE 18. Symmetric boundary components interchanged by $\iota$.

### 5.7. **Proof of the Birman-Hilden Theorem.**

**Definition 5.19.** A closed curve $\alpha$ in $S_g$ is symmetric if $\iota(\alpha) = \alpha$ as sets.

**Lemma 5.20.** *Let $g \geqslant 2$ and let $\alpha$ and $\beta$ be two symmetric nonseparating simple closed curves in $S_g$. If $\alpha$ and $\beta$ are isotopic, then they are symmetrically isotopic.*

*Proof.* Let $\overline{\alpha}$ and $\overline{\beta}$ denote the images of $\alpha$ and $\beta$ in $S_{0,2g+2} \approx S_g / \langle \iota \rangle$. Now, $\overline{\alpha}$ and $\overline{\beta}$ must be simple proper arcs in $S_{0,2g+2}$ (look at it geometrically - in particular, remember $\alpha$ and $\beta$ are symmetric).
Let's look at an example to get a picture. If $\alpha$ and $\beta$ are symmetric choices of the usual generating loops in the homology of a torus, these will correspond to one loop being the "innermost" meridian loop while we choose the other loop to be a longitudinal loop whose plan of existence is normal to the axis of rotation. See figure 19.

FIGURE 19. Basis for homology groups of torus and double torus

Imagine in this figure now we cut the surface into a "lower" and "upper" half. Looking at the resulting paths of our loops, we indeed get arcs for the longitudinal loops and loop on the boundary for the meridian loops.
Now, an isotopy between these arcs will lift to a symmetric isotopy between $\alpha$ and $\beta$.
I.e., if $H$ is an isotopy between $\alpha$ and $\beta$, then there exist an induced map $\tilde{H}$ such that

$$\begin{tikzcd} & S_g \arrow[d] \\ I^2 \arrow[ur, "\tilde{H}"] \arrow[r, "H"'] & S_g / \langle \iota \rangle \end{tikzcd}$$

commutes since $I^2$ is simply connected, path-connected and locally path-connected [1, Cor 4.2].
We can modify $\alpha$ by a symmetric isotopy so that it is transverse to $\beta$. We claim that $\alpha$ cannot be disjoint from $\beta$. Indeed, then $\overline{\alpha}$ and $\overline{\beta}$ are disjoint including endpoints. But such arcs cannot correspond to isotopic curves in $S_g$: we can choose an arc $\overline{\gamma}$ that passes through an odd number of endpoints of $\overline{\alpha}$ and an even number of endpoints of $\overline{\beta}$. This will then lift to a loop $\gamma$ in $S_g$ with $i(\alpha, \gamma)$ odd and $i(\beta, \gamma)$ even, contradicting $\alpha$ isotopic to $\beta$.
Now, since $\alpha$ is isotopic to $\beta$ and $\alpha \cap \beta \neq \varnothing$, the bigon criterion gives that $\alpha$ and $\beta$ form a bigon $B$. Assume $B$ is the innermost bigon. As $\alpha$ and $\beta$ are fixed (they are symmetric) by $\iota$, we have that $\iota(B)$ is another innermost bigon in the graph $\alpha \cup \beta$.

But $\iota$ reverses the orientation of non-separating closed curves (while preserving orientations of separating closed curves), so since the bigon $B$ lies to one side of $\alpha$, $\iota(B)$ must lie on the other side of $\alpha$.

It follows that the image of $B$ in $S_{0,2g+2}$ is an innermost bigon $\overline{B}$ between $\overline{\alpha}$ and $\overline{\beta}$ as otherwise its preimage would have bigons inside it as well. Furthermore, since $\iota(B) \neq B$, there are no fixed points of $\iota$ in $B$ and hence no marked points of $S_{0,2g+2}$ in $\overline{B}$.
Now, considering the boundary of the bigon $\overline{B}$, it can have zero, one or two of its vertices on marked points of $S_{0,2g+2}$. In the first two cases, we can modify $\overline{\alpha}$ by isotopy in order to remove the bigon by pushing the vertex to the other, reducing the intersection number of $\overline{\alpha}$ and $\overline{\beta}$. In the last case, since $\overline{B}$ is innermost, we see that $\overline{\alpha} \cup \overline{\beta}$ is a simple loop bounding a disk, and we can push $\overline{\alpha}$ onto $\overline{\beta}$. Removing bigons inductively, we see that $\overline{\alpha}$ is isotopic to $\overline{\beta}$, and this isotopy lifts to a symmetric isotopy between $\alpha$ and $\beta$. □

**Proposition 5.21.** *Let $g \geqslant 2$ and let $\varphi, \psi \in \mathrm{SHomeo}^+(S_g)$. If $\varphi$ and $\psi$ are isotopic, then they are symmetrically isotopic.*

*Proof.* It suffices to treat the case where $\psi$ is the identity. Let $\overline{\varphi}$ denote the induced homeomorphism of $S_{0,2g+2}$.
Let $(\gamma_1, \ldots, \gamma_{2g+1})$ be a chain of nonseparating symmetric simple closed curves in $S_g$. By assumption, $\varphi$ is isotopic to the identity, so for each $i$, $\varphi(\gamma_i)$ is isotopic to $\gamma_i$. By Lemma 5.20, $\varphi(\gamma_i)$ is symmetrically isotopic to $\gamma_i$ for each $i$. Letting $\overline{\gamma_i}$ and $\overline{\varphi(\gamma_i)}$ be the images in $S_{0,2g+2}$, this implies that the arc $\overline{\varphi(\gamma_i)}$ is isotopic to $\overline{\gamma_i}$. By the 3.(ii) in 3.10 (the Alexander method), we deduce that $\overline{\varphi}$ is isotopic to the identity. This isotopy then lifts to a symmetric isotopy of $\varphi$ to the identity. □

*Proof of the Birman-Hilden theorem.* We have a commutative diagram

$$\begin{tikzcd}
\mathrm{SHomeo}^+(S_g) \arrow[r, two heads] \arrow[dr, "\text{mod isotopy}"'] & \mathrm{Homeo}^+(S_{0,2g+2}, \partial S_{0,2g+2}) \arrow[r, "\pi_0"] & \mathrm{Mod}(S_{0,2g+2}) \\
& \mathrm{SMod}(S_g) \arrow[ur, two heads] &
\end{tikzcd}$$

where the composite map factors through $\mathrm{SMod}(S_g) \to \mathrm{Mod}(S_{0,2g+2})$ by Proposition 5.21.
We wish to find the kernel of the map $\mathrm{SMod}(S_g) \to \mathrm{Mod}(S_{0,2g+2})$. Suppose $f \in \mathrm{SMod}(S_g)$ is mapped to [id]. Since the composition $\mathrm{SHomeo}^+(S_g) \to \mathrm{Homeo}^+(S_{0,2g+2}) \to \mathrm{SMod}(S_g)$ is a surjective homomorphism, we can choose a representative $\varphi \in \mathrm{SHomeo}^+(S_g)$. Let $\overline{\varphi}$ be the image in $\mathrm{Homeo}^+(S_{0,2g+2}, \partial S_{0,2g+2})$. By assumption, $\overline{\varphi}$ is isotopic to the identity, so this isotopy lifts to an isotopy of $\varphi$ to either the idenity or $\iota$. Therefore

$$\mathrm{SMod}(S_g) / \langle [\iota] \rangle \approx \mathrm{Mod}(S_{0,2g+2}).$$

□

*Remark.* We can generalize the proof of the Birman-Hilden theorem a bit to the case of $S_g^1$ quite simply: the quotient of $S_g^1$ by the hyperelliptic involution $\iota \colon S_g^1 \to S_g^1$

is a disk with $2g+1$ marked points. Since $\iota\colon S_g^1 \to S_g^1$ is not an element of $\mathrm{Homeo}^+\left(S_g^1, \partial S_g^1\right)$, it does not represent an element of $\mathrm{SMod}(S_g^1)$, and so we get

$$\mathrm{SMod}\left(S_g^1\right) \approx \mathrm{Mod}\left(D_{2g+1}\right) \approx B_{2g+1}.$$

## 6. Braided monoidal categories

The Birman-Hilden homomorphism $\psi\colon B_n \to \mathrm{Mod}\left(S_g^1\right)$ is an example of a geometric representation:

**Definition 6.1** (Geometric representation)**.** A geometric representation of a group $G$ is any homomorphism $G \to \mathrm{Mod}(S)$ for some surface $S$.

In particular, the Birman-Hilden theorem shows that this representation is an embedding. There are different geometric representations and embeddings of the braid group that one could look at, and one goal in this section will be to show that these naturally arise through Yang-Baxter operators in certain categories of surfaces.
We start by introducing the categorical language.

### 6.1. Monoidal categories.

We first introduce the notion of a monoidal category.

**Definition 6.2** (Monoidal category)**.** A monoidal category is a tuple $V = (V, \otimes, I, a, l, r)$ consisting of a category $V$, a functor $\otimes\colon V \times V \to V$ called the monoidal product, and object $I \in V$ called the unit, and natural isomorphisms

$$a\colon\ (-\otimes-)\otimes- \overset{\sim}{\Longrightarrow} -\otimes(-\otimes-)$$
$$l\colon I\otimes- \overset{\sim}{\Longrightarrow} -$$
$$r\colon\ -\otimes I \overset{\sim}{\Longrightarrow} -$$

called the associativity, left unit and right unit constraints, respectively. Additionally, we require that for all objects $A, B, C, D \in V$, the following two diagrams commute:

$$\begin{tikzcd}
 & (A\otimes B)\otimes(C\otimes D) \arrow[rd, "a"] & \\
((A\otimes B)\otimes C)\otimes D \arrow[ru, "a"] \arrow[d, "a\otimes D"'] & & A\otimes(B\otimes(C\otimes D)) \\
(A\otimes(B\otimes C))\otimes D \arrow[rr, "a"'] & & A\otimes((B\otimes C)\otimes D) \arrow[u, "A\otimes a"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
(A\otimes I)\otimes B \arrow[rr, "a"] \arrow[dr, "r\otimes B"'] & & A\otimes(I\otimes B) \arrow[dl, "A\otimes l"] \\
 & A\otimes B &
\end{tikzcd}$$

The monoidal category is called strict when all the natural isomorphisms are identity morphisms for all objects.

**Definition 6.3** (Monoidal functor)**.** If $\mathcal{V}$ and $\mathcal{W}$ are monoidal categories, then we define a *monoidal functor* to be a tuple $F = (F, \varphi_2, \varphi_0) : \mathcal{V} \to \mathcal{W}$ consisting of a functor $F\colon \mathcal{V} \to \mathcal{W}$, a family of natural isomorphisms

$$\varphi_{2,A,B}\colon FA\otimes FB \xrightarrow{\sim} F(A\otimes B),$$

and an isomorphism $\varphi_0\colon I \xrightarrow{\sim} FI$ such that the following three diagrams commute:

$$\begin{tikzcd}
(FA \otimes FB) \otimes FC \arrow[rr, "a"] \arrow[d, "\varphi_2 \otimes FC"'] & & FA \otimes (FB \otimes FC) \arrow[d, "FA \otimes \varphi_2"] \\
F(A \otimes B) \otimes FC \arrow[d, "\varphi_2"'] & & FA \otimes F(B \otimes C) \arrow[d, "\varphi_2"] \\
F((A \otimes B) \otimes C) \arrow[rr, "Fa"'] & & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
FA \otimes I \arrow[r, "r"] \arrow[d, "FA \otimes \varphi_0"'] & FA \\
FA \otimes FI \arrow[r, "\varphi_2"'] & F(A \otimes I) \arrow[u, "Fr"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes FA \arrow[r, "l"] \arrow[d, "\varphi_0 \otimes l"'] & FA \\
FI \otimes FA \arrow[r, "\varphi_2"'] & F(I \otimes A) \arrow[u, "Fl"']
\end{tikzcd}$$

A monoidal functor is called *strict* when each of the isomorphisms $\varphi_{2,A,B}$ and $\varphi_0$ are all identities.

*Remark.* A monoidal functor is also sometimes called a strong monoidal functor if one wants to distinguish it from a *lax monoidal functor* which is the definition of a monoidal functor without the requirements that $\varphi_0$ and $\varphi_{2,A,B}$ be isomorphisms.

**Definition 6.4.** A morphism $\theta\colon F \implies G$ of monoidal functors is a natural transformation $\theta\colon F \implies G$ such that the following diagrams commute:

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\varphi_2"] \arrow[d, "\theta_a \otimes \theta_b"'] & F(A \otimes B) \arrow[d, "\theta_{A \otimes B}"] \\
GA \otimes GB \arrow[r, "\varphi_2"'] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & FI \arrow[dd, "\theta_I"] \\
I \arrow[ur, "\varphi_0"] \arrow[dr, "\varphi_0"'] & \\
 & GI
\end{tikzcd}$$

**Definition 6.5** (Braiding)**.** A braiding for a monoidal category $V$ consists for a natural family of isomorphisms

$$c = c_{A,B}\colon A \otimes B \xrightarrow{\sim} B \otimes A$$

in $V$ such that the following diagrams commute

$$\begin{tikzcd}
 & (B \otimes A) \otimes C \arrow[r, "a"] & B \otimes (A \otimes C) \arrow[dr, "B \otimes c"] & \\
(A \otimes B) \otimes C \arrow[ur, "c \otimes C"] \arrow[dr, "a"'] & & & B \otimes (C \otimes A) \\
 & A \otimes (B \otimes C) \arrow[r, "c"'] & (B \otimes C) \otimes A \arrow[ur, "a"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & A \otimes (C \otimes B) \arrow[r, "a^{-1}"] & (A \otimes C) \otimes B \arrow[dr, "c \otimes B"] & \\
A \otimes (B \otimes C) \arrow[ur, "A \otimes c"] \arrow[dr, "a^{-1}"'] & & & (C \otimes A) \otimes B \\
 & (A \otimes B) \otimes C \arrow[r, "c"'] & C \otimes (A \otimes B) \arrow[ur, "a^{-1}"'] &
\end{tikzcd}$$

**Proposition 6.6.** *In a braided monoidal category, the following diagram commutes*

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "a"] \arrow[d, "c \otimes C"'] & A \otimes (B \otimes C) \arrow[r, "A \otimes c"] \arrow[ddd, dashed, no head, "c"] & A \otimes (C \otimes B) \arrow[r, "a^{-1}"] \arrow[ddd, dashed, no head, "c"] & (A \otimes C) \otimes B \arrow[d, "c \otimes B"] \\
(B \otimes A) \otimes C \arrow[d, "a"'] & & & (C \otimes A) \otimes B \arrow[d, "a"] \\
B \otimes (A \otimes C) \arrow[d, "B \otimes c"'] & & & C \otimes (A \otimes B) \arrow[d, "C \otimes c"] \\
B \otimes (C \otimes A) \arrow[r, "a^{-1}"'] & (B \otimes C) \otimes A \arrow[r, "c \otimes A"'] & (C \otimes B) \otimes A \arrow[r, "a"'] & C \otimes (B \otimes A)
\end{tikzcd}$$

*Proof.* [8, Proposition 2.1] □

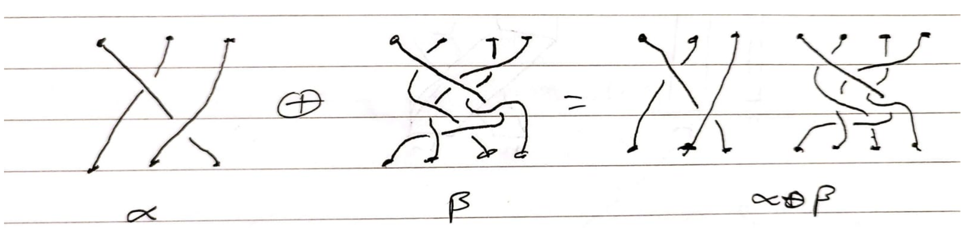
**Example 6.7** (Braids and labelled braids on strings)**.** We define the braid groupoid to be the category $\mathcal{B}$ whose objects are the natural numbers and whose morphisms are given by

$$B(m,n) = \begin{cases} B_n, & m = n \\ \varnothing, & m \neq n \end{cases}$$

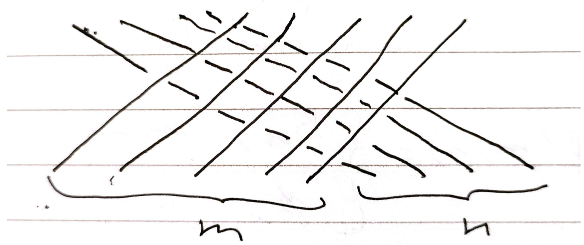
where composition of morphisms is defined to be the product of the braids (i.e., concatenation).
We can then equip $\mathcal{B}$ with a strinct monoidal structure by letting $\otimes \colon B_m \times B_n \to B_{m+n}$ be given by addition of braids which is described algebraically by

$$\sigma_i \otimes \sigma_j = \sigma_i \sigma_{m+j} \, (= \sigma_{m+j} \sigma_i)$$

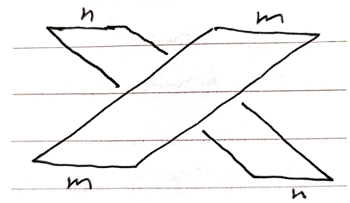
pictured as



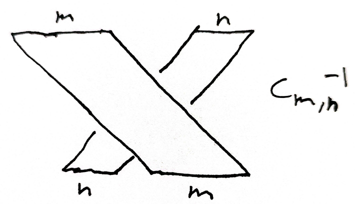

Furthermore, we can give $\mathcal{B}$ a braiding $c$ given by $c = c_{m,n} \colon \underbrace{m \otimes n}_{=m+n} \mapsto \underbrace{n \otimes m}_{=n+m}$ which, on morphisms, can be illustrated by concatenating from, say above, with the following braid




or, illustrated differently, as




Then this is clearly an isomorphism since it has the inverse




6.2. **Yang-Baxter operators.**

**Definition 6.8** (Yang-Baxter operator)**.** Let $T\colon \mathcal{A} \to \mathcal{V}$ be a functor from a category $\mathcal{A}$ to a monoidal category $\mathcal{V}$. A *Yang-Baxter operator on* $T$ is a natural family of isomorphisms

$$y = y_{A,B} \colon TA \otimes TB \xrightarrow{\sim} TB \otimes TA$$

such that the following diagram commutes.

$$\begin{tikzcd}
(TA \otimes TB) \otimes TC \arrow[r, "a"] \arrow[d, "y \otimes TC"'] & TA \otimes (TB \otimes TC) \arrow[r, "TA \otimes y"] & TA \otimes (TC \otimes TB) \arrow[r, "a^{-1}"] & (TA \otimes TC) \otimes TB \arrow[d, "y \otimes TB"] \\
(TB \otimes TA) \otimes TC \arrow[d, "a"'] & & & (TC \otimes TA) \otimes TB \arrow[d, "a"] \\
TB \otimes (TA \otimes TC) \arrow[d, "TB \otimes y"'] & & & TC \otimes (TA \otimes TB) \arrow[d, "TC \otimes y"] \\
TB \otimes (TC \otimes TA) \arrow[r, "a^{-1}"'] & (TB \otimes TC) \otimes TA \arrow[r, "y \otimes TA"'] & (TC \otimes TB) \otimes TA \arrow[r, "a"'] & TC \otimes (TB \otimes TA)
\end{tikzcd}$$

*Remark.* When $\mathcal{A} = \mathbb{1}$, we say that $y$ is a Yang-Baxter operator on $X = T(\mathcal{A}) \in \mathcal{V}$ if it is a Yang-Baxter operator on $T\colon \mathbb{1} = \mathcal{A} \to \mathcal{V}$.

Let $(\mathcal{X}, \otimes, I)$ be a monoidal category with $\tau \in \mathrm{Aut}_{\mathcal{X}}(X \otimes X)$ a Yang-Baxter operator in $\mathcal{X}$. Suppose $\mathcal{X}$ acts on a category $\mathcal{M}$ via a functor $\mathcal{M} \times \mathcal{X} \to \mathcal{M}$ which we also denote by $\otimes$. Then there is an action of the braid groupoid $\alpha_\tau\colon \mathcal{M} \times B \to \mathcal{M}$ given on objects by $\alpha_\tau(A, n) = A \otimes X^{\otimes n}$ and determined on morphisms by $\alpha_\tau(f, \sigma_i) = f \otimes \mathrm{id}_{X^{\otimes i-1}} \otimes \tau \otimes \mathrm{id}_{X^{\otimes n-i-1}}$.

**Example 6.9.** If $\mathcal{X} = (\mathcal{X}, \otimes, \mathcal{I})$ admits a braiding $b$, then $\tau = b_{X,X} \in \mathrm{Aut}_{\mathcal{X}}(X \otimes X)$ is a Yang-Baxter operator for any object $X$. The thing that needs verifying here is that the big Yang-Baxter diagram in definition 6.8 is satisfied, but this follows directly from proposition 6.6.
Likewise, for any functor $T\colon \mathcal{A} \to \mathcal{V}$ into a braided tensor category $\mathcal{V}$, we obtain a Yang-Baxter operator as

$$y_{A,B} = c_{TA,TB}\colon TA \otimes TB \xrightarrow{\sim} TB \otimes TA.$$

In particular, we obtain a Yang-Baxter operator on the inclusion functor $\iota\colon \mathbb{1} \to \mathcal{B}$ identifying $\mathbb{1}$ with the braid of a single string. We will denote this Yang-Baxter operator by $z$.

We want to show that the category of strong monoidal functors from the braid groupoid into $\mathcal{X}$ is equivalent to a naturally defined category of Yang-Baxter operators in $\mathcal{X}$.

**Proposition 6.10.** *For any strict monoidal category $\mathcal{V}$ and any Yang-Baxter $\tau$ on an element $X \in V$, there exists a unique strict monoidal functor $\Phi_{X,\tau}\colon \mathcal{B} \to \mathcal{V}$ such that $\Phi_{X,\tau} \circ z = y$.*

*Proof and construction.* Define $\Phi_{X,\tau}\colon \mathcal{B} \to \mathcal{V}$ on objects by $\Phi_{X,\tau}(n) = X^{\otimes n}$. For $0 \leqslant i < n$, define

$$y_i = X^{\otimes(i-1)} \otimes y \otimes X^{\otimes(n-i-1)}\colon X^{\otimes n} \to X^{\otimes n}.$$

These satisfy the braid group relations. Thus we obtain a monoid homomorphism $\Phi_{X,\tau,n}\colon \mathcal{B}_n \to \mathcal{V}(X^{\otimes n}, X^{\otimes n})$ taking $\sigma_i$ to $y_i$ for all $0 \leqslant i < n$. Clearly $\Phi_{X,\tau}$ is the unique strict monoidal functor with these properties. ☐

*Remark.* In particular, $\Phi_{X,\tau,n}\colon \mathcal{B}_n \to \mathrm{Aut}_{\mathcal{V}}(X^{\otimes n})$ for all $n$.

As we said in the beginning, our goal is to get geometric representations from Yang-Baxter operators, so given our construction of $\Phi_{X,\tau}$, we might hope to find a category $\mathcal{V}$ such that its objects are surfaces and $\mathrm{Aut}_{\mathcal{V}}(X^{\otimes n})$ might correspond to a mapping class group. Indeed, this is what we shall do now in two different cases: the category of decorated surfaces and the category of bidecorated surfaces.

### 6.3. Braided monoidal category of decorated and bidecorated surfaces.

**Definition 6.11** (Decorated surface)**.** A decorated surface is a pair $(S, I)$ where $S$ is a compact connected surface with at least one boundary component and $I\colon [-1, 1] \hookrightarrow \partial S$ is a parametrised interval in its boundary.

**Definition 6.12** ($\mathcal{M}_1$)**.** Let $\mathcal{M}_1$ denote the groupoid where the objects are decorated surfaces and morphisms are isotopy classes of diffeomorphisms/homeomorphisms restricting to the identity on a neighborhood of $I$.

*Remark.* In particular, $\mathrm{Aut}_{\mathcal{M}_1}(S) = \mathrm{Mod}(S)$.

We now construct a braided monoidal structure on $\mathcal{M}_1$ : given decorated surfaces $(S_1, I_1)$ and $(S_2, I_2)$, define $(S_1, I_1) \otimes (S_2, I_2) := (S_1 \natural S_2, I_1 \natural I_2)$ to be the surface obtained by gluing $S_1$ and $S_2$ along the right half-interval $I_1^+ \in \partial S_1$ and the left half-interval $I_2^- \in \partial S_2$, defining $I_1 \natural I_2 = I_1^- \cup I_2^+$.

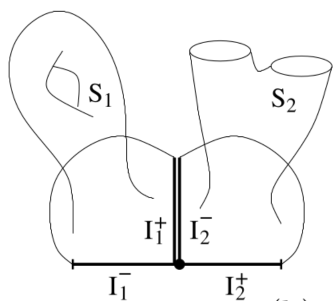
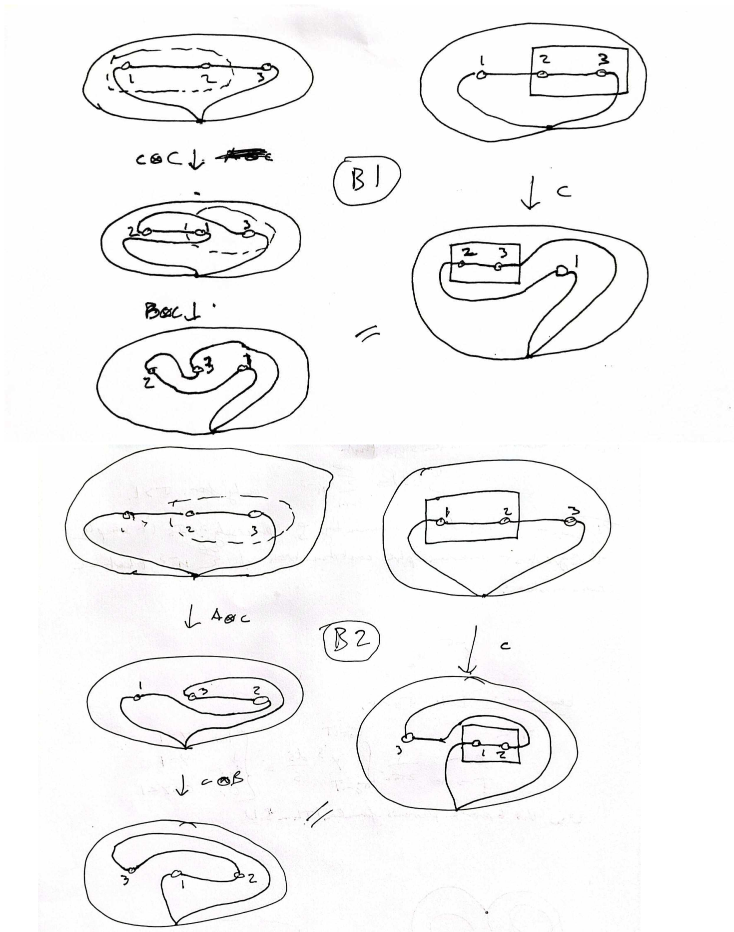



FIGURE 20. $S_1 \# S_2$ [12, Figure 2]

Furthermore, we define the unit object to be $I := \big(D^2, I\big)$. For it to be a strict unit, we define $\big(S_1 \natural D^2, I_1 \natural I\big) := (S_1, I_1)$ and $\big(D^2 \natural S_2, I \natural I_2\big) := (S_2, I_2)$.

Note that this category is not strict as associativity is not strict, but there is a way to construct an equivalent category which is strict [3, Section 3].

We define a braiding $c$ on $(S_1 \natural S_2, I_1 \natural I_2)$ as the half-Dehn twist which satisfies that $c\colon (S_1 \natural S_2, I_1 \natural I_2) \xrightarrow{\sim} (S_2 \natural S_1, I_2 \natural I_1)$ is a natural isomorphism because it has the opposite half-Dehn twist as the inverse. It is natural because the induced map will simply be the one induced by the naturality square.
The B1 and B2 diagrams can be verified pictorially as follows:




This is precisely the requirements needed in Proposition 6.10, so we obtain a monoidal functor $\Phi\colon \mathcal{B} \to \mathcal{M}_1$ such that $\Phi \circ z = y$ where $y \in \mathrm{Aut}_{\mathcal{M}_1}(S\natural S) = \mathrm{Mod}(S\natural S)$ is the induced Yang-Baxter operator on some decorated surface $S$ from the braiding. So again, we obtain a geometric representation $\Phi_n\colon \mathcal{B}_n \to \mathrm{Mod}\left(S^{\natural n}\right)$

We will also consider a different monoidal category of surfaces. Informally, a bidecorated surface is a surface with two intervals marked in its boundary.
To give a precise definition, we first define certain surfaces $X_i$ that will be convenient for the monoidal structure, we set $X_1 = D^2 \subset \mathbb{C}$ to be the unit disk, and then define embeddings $\iota_1^0, \iota_1^1\colon I \to X_1$ by

$$\iota_1^0(t) = e^{i\left(\frac{\pi}{4}+t\frac{\pi}{2}\right)} \quad \text{and} \quad \iota_1^1(t) = e^{i\left(5\frac{\pi}{4}+t\frac{\pi}{2}\right)}.$$

We denote by $\overline{\iota_1^i}\colon I \to X_1$ the reverse map $t \mapsto \iota_1^i(1-t)$ for $i = 0, 1$. Then we recursively define $X_{m+1}$ for $m \geqslant 1$ by

$$X_{m+1} := \frac{X_m \sqcup X_1}{\iota_m^i(t) \sim \overline{\iota_1^i}} \quad \text{for } t \in \left[\frac{1}{2}, 1\right]$$

and we define

$$\iota_{m+1}^{i}(t) = \begin{cases} \iota_{m}^{i}(t), & \text{if } t \leqslant \frac{1}{2} \\ \iota_{1}^{i}(t), & \text{else} \end{cases}.$$

In this process, the marked intervals will live in different boundary components every second time. The process for each of the two situations is illustrated below in figures 21 and 22.

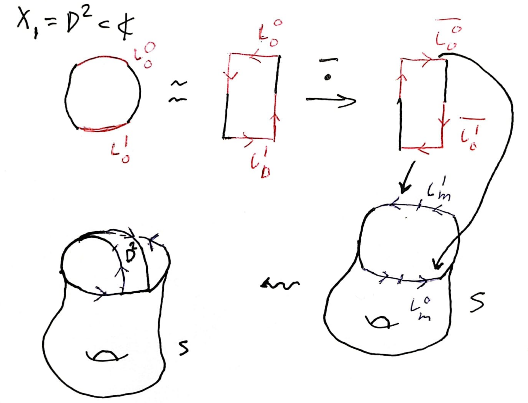



FIGURE 21. Marked intervals in single boundary components.

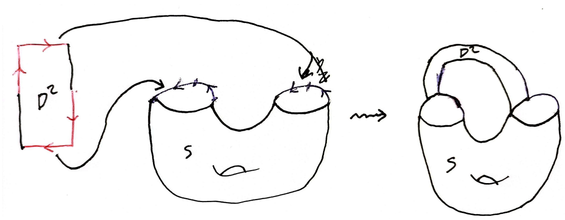



FIGURE 22. Marked intervals in different boundary components.

**Lemma 6.13.** *For $m \geqslant 1$, $X_m \approx S_{g,r}$ where*

$$(g, r) = \begin{cases} \left(\frac{m}{2} - 1, 2\right), & m \text{ even} \\ \left(\frac{m-1}{2}, 1\right), & m \text{ odd} \end{cases}.$$

*Proof.* Firstly, $X_m$ is clearly connected, and we have

$$\chi(X_m) = \chi(X_{m-1}) - 1$$

since, for example, the $\Delta$-structure on our surface $X_m$ can be chosen to be that for $X_{m-1}$ with the boundary subdivided into four 2-simplices with four vertices, adding an additional two 2-simplices and then a disk. By induction, we then get $\chi(X_m) = \chi(X_1) - (m-1) = 2 - m$.

Now, by the classification of surfaces with boundary and genus, we simply need to know how many boundary components $X_{m+1}$ has. But as can be seen from the figures, if $m$ is odd, we will have one boundary component, while if $m$ is even, we will have two boundary components. $\square$

**Definition 6.14** (Bidecorated surface)**.** A bidecorated surface is a tuple $(S, m, \varphi)$ where $S$ is a surface, $m \geqslant 1$ is an integer, and

$$\varphi\colon \partial X_m \sqcup \left(\sqcup_k S^1\right) \xrightarrow{\sim} \partial S$$

is a homeomorphism, giving a parametrization of the boundary of $S$. We think of $(S, m, \varphi)$ as a surface with two parametrized arcs

$$I_0 := \varphi \circ \iota_m^0 \quad \text{and} \quad I_1 := \varphi \circ \iota_m^1$$

in its boundary, and $k$ additional parametrized boundaries.

**Definition 6.15** ($\mathcal{M}_2$)**.** Let $\mathcal{M}_2$ denote the monoidal groupoid where objects are bidecorated surfaces together with a formal unit $U$. The Hom set between two bidecorated surfaces $(S, m, \varphi)$ and $(S', m', \varphi')$ is empty if $m \neq m'$ or $S$ and $S'$ are nonhomeomorphic. Otherwise, the Hom set consists of all mapping classes of homeomorphisms that preserve the boundary parametrizations:

$$\operatorname{Hom}_{\mathcal{M}_2}\left((S, m, \varphi), \left(S', m, \varphi'\right)\right) = \pi_0 \operatorname{Homeo}_\partial\left(S, S'\right) = \pi_0\left\{f \in \operatorname{Homeo}\left(S, S'\right) \mid f \circ \varphi = \varphi'\right\}$$

where $\operatorname{Homeo}(S, S')$ has the compact-open topology, and $\operatorname{Homeo}_\partial(S, S')$ the subspace topology.

*Remark.* We again obtain that $\operatorname{Aut}_{\mathcal{M}_2}((S, m, \varphi)) = \operatorname{Mod}(S)$.

The monoidal structure $\natural^2$ on $\mathcal{M}_2$ is defined as follows. The object $U$ is by definition a unit, and for the remaining objects, we define

$$(S, m, \varphi) \natural^2 \left(S', m', \varphi'\right) := \left(\frac{S \sqcup S'}{I_i(t) \sim \overline{I'_i}(t), t \in \left[\frac{1}{2}, 1\right]}, m + m', \varphi \natural^2 \varphi'\right)$$

for $i = 0, 1$, and where

$$\varphi \natural^2 \varphi'\colon \partial X_{m+m'} \sqcup \left(\sqcup_{k+k'} S^1\right) \hookrightarrow \partial\left(S \natural^2 S'\right)$$

is obtained using the canonical identification $\partial X_{m+m'} \approx \left(\partial X_{n - \iota_m\left(\frac{1}{2}, 1\right)}\right) \cup \left(\partial X_{m'} - \iota_{m'}\left(0, \frac{1}{2}\right)\right)$.
Now we will construct a Yang-Baxter element in $\mathcal{M}_2$ as follows. Let $D^{\natural^2 m} = D_1 \natural^2 \ldots \natural^2 \left(D_i \natural^2 D_{i+1}\right) \natural^2 \ldots \natural^2 D_m$, where subscripts are used to enumerate the disks. The underlying surface, by construction, will be $X_m$. Let $a_i$ denote the isotopy class of a curve in the interior $D_i \natural^2 D_{i+1} \approx S^1 \times I$ that is parallel to its boundary components, see figure 23.

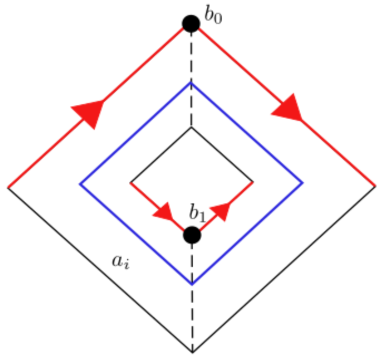



FIGURE 23. The curve $a_i$ in $D_i \natural D_{i+1}$ [4, Figure 4]

**Lemma 6.16.** *The curves $a_1, \ldots, a_{m-1}$ form a chain in $D^{\natural^2 m}$.*

*Proof.* The curve $a_i$ has image contained in $D_i \natural^2 D_{i+1}$, so it can only intersect $a_{i-1}$ and $a_{i+1}$ nontrivially. So it suffices to look at the subsurface of $D^{\natural^2 m}$ corresponding to $D_i \natural^2 D_{i+1} \natural^2 D_{i+2}$.

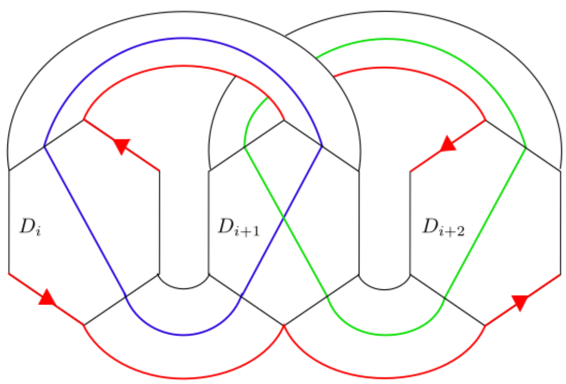



FIGURE 24. Intersection of $a_i$ and $a_{i+1}$ in $D_i \natural^2 D_{i+1} \natural^2 D_{i+2}$. [4, Figure 5]

□

Now by Lemma 4.8 and Proposition 4.10 (the braid relation), we get that the braid group relations hold for the Dehn twists $T_i \in \operatorname{Aut}_{\mathcal{M}_2}\left(D^{\natural^2 m}\right)$ where $T_i$ is the Dehn twist along the curve $a_i$ in $D^{\natural^2 m}$, i.e.,

$$\begin{aligned} T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} && \forall i \\ T_i T_j &= T_j T_i && \text{for } |i-j| > 1 \end{aligned}$$

Hence the same relations hold for the inverses $T_i^{-1}$.

If we add a disk to either side of $D^{\natural^2 m}$, we get

$$T_i \natural^2 \mathrm{id}_D = T_i \quad \text{and} \quad \mathrm{id}_D \natural^2 T_i = T_{i+1}$$

in $\operatorname{Aut}_{\mathcal{M}_2}\left(D^{\natural^2 m+1}\right)$. Hence this gives the relation

$$\left(T_1^{-1}\natural^2\mathrm{id}_D\right)\left(\mathrm{id}_D\natural^2T_1^{-1}\right)\left(T_1^{-1}\natural^2 id_D\right) = \left(\mathrm{id}_D\natural^2T_1^{-1}\right)\left(T_1^{-1}\natural^2\mathrm{id}_D\right)\left(\mathrm{id}_D\natural^2T_1^{-1}\right)$$

in $\operatorname{Aut}_{\mathcal{M}_2}\left(D^{\natural^2 3}\right)$, meaning that $T_1^{-1}$ is a Yang-Baxter element. This yields a monoidal functor

$$\Phi = \Phi_{D,T_1^{-1}}\colon (\mathcal{B},\otimes) \to \left(\mathcal{M}_2,\natural^2\right)$$

uniquely determined up to monoidal natural isomorphism by $\Phi(n) = D^{\natural^2 n}$ and $\Phi_{D,T_1^{-1},n}\left(\sigma_1\right) = D^{\natural^2 i-1}\natural^2T_1^{-1}\natural^2D^{\natural^2 n-i-1} = T_i^{-1} \in \operatorname{Aut}_{\mathcal{M}_2}\left(D^{\natural^2 m}\right) = \pi_0\mathrm{Homeo}_\partial\left(X_m\right)$. Seeing that this is exactly the setup for the Birman-Hilden theorem, we note that the homomorphisms

$$\Phi_m = \Phi_{D,T_1^{-1},m}\colon B_m \to \operatorname{Aut}_{\mathcal{M}_2}\left(D^{\natural^2 m}\right) \approx \operatorname{Aut}_{\mathcal{M}_2}\left(X_m\right) \approx \begin{cases}\mathrm{Mod}\left(S_{\frac{m}{2}-1,2}\right), & m \text{ even}\\ \mathrm{Mod}\left(S_{\frac{m-1}{2},1}\right), & m \text{ odd}\end{cases}.$$

recover the Birman-Hilden embeddings from section 5.5.1.

## 7. Geometric Representations of the Braid Group on non-orientable surfaces

We will now look at the classified geometric representations of braid groups on non-orientable surfaces as presented in [11]. In particular, we will explore how the representations fit in our categorical framework from the previous section.

We start by noting some facts about non-orientable surfaces.

A connected orientable (respectively nonorientable) surface of genus $g$ with $b$ boundary components will be denoted by $S_{g,b}$ (respectively $N_{g,b}$ ).
Now, recall that the Möbius band (which is also called a crosscap), is the mapping cylinder on the map $z \mapsto z^2$ (see figure 25) We will denote the Möbius band by $M$.

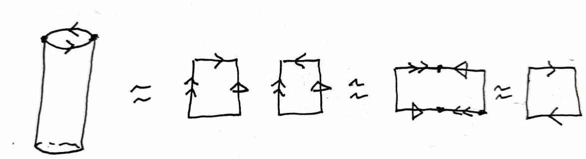


Figure 25. Möbius band

In this case, any of the two curves making up the half-circles which get identified can be cut along, rendering a connected surface. Thus gluing on a Möbius strip along the boundary, we obtain a surface of one higher genus.

We can then obtain $N_{g,b}$ from $S_{0,g+b}$ by gluing $g$ Möbius bands along $g$ distinct boundary components of $S_{0,g+b}$.

Note that by the classification of surfaces

$$N_{g,b} \approx \left(\mathbb{RP}^2\right)^{\#g} - \sqcup_b \mathring{D} \approx \left(\mathbb{RP}^2 - \mathring{D}\right)^{\natural g} - \sqcup_{b-1} \mathring{D}$$

where $\mathbb{RP}^2 - \mathring{D}$ is a Möbius band.
Note also that $\mathbb{RP}^2 \# \mathbb{RP}^2 \approx K$, the Klein bottle, see figure 26.

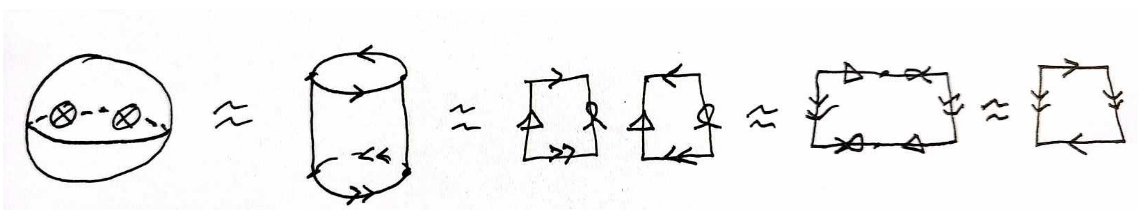


FIGURE 26. Klein-bottle as the sphere with two crosscaps

**Lemma 7.1.** *$N_{g,1} \approx S_{n,1} \natural M$ for $g = 2n+1$.*

*Proof.* For $g = 2n+1$ with $n \geqslant 0$, we have (see 27 for a visual argument for $T^2 \# M \approx K \# M$ - to see why the top-right surface is $K - D^2$, take the usual Klein-bottle, remove a disk such that it is embeddable in $\mathbb{R}^3$, and then enlarge the hole.)

$$\begin{aligned} N_{2n+1,1} \approx \left(\mathbb{RP}^2\right)^{\#2n+1} - \mathring{D} &\approx K^{\#n} \# M \approx K^{\#n-1} \# K \# M \\ &\approx K^{\#n-1} \# T^2 \# M \\ &\approx \dots \\ &\approx \left(T^2\right)^{\#n} \# M \\ &\approx S_{n,0} \# M \\ &\approx S_{n,1} \natural M \end{aligned}$$

$\square$

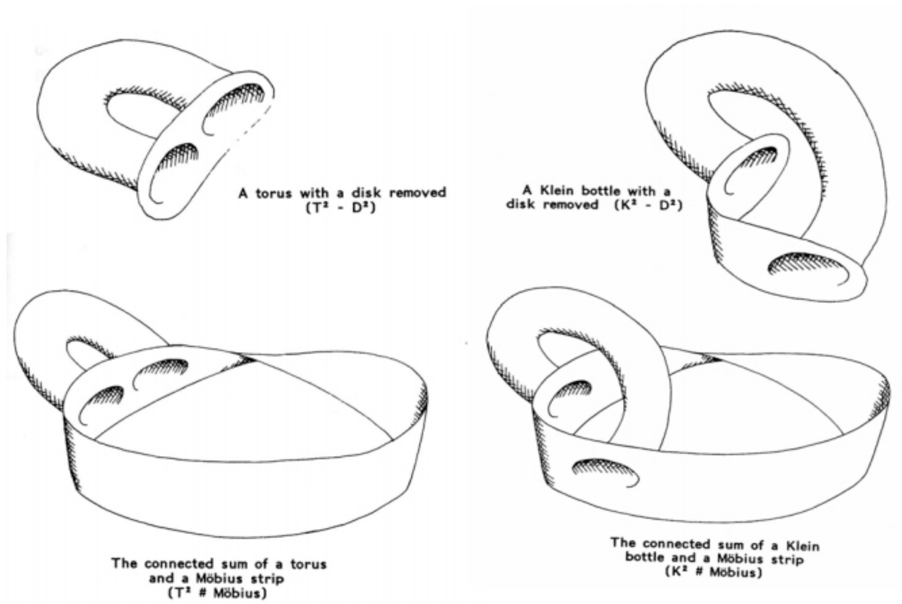



FIGURE 27. A hemeomorphism between $T^2\#M$ and $K\#M$ [13, Figure 5.7]

Hence for $g$ odd, we have an embedding $B_g \hookrightarrow N_{g,1}$ by the Birman-Hilden embedding into the $S_{n,1}$ summand. A similar thing can be done for the even case.

We will now introduce different types of representations of the braid group on non-orientable surfaces.

**Definition 7.2.** We call a curve two-sided (resp. one-sided) if its regular neighborhood is an annulus (resp a Möbius band).

7.0.1. *The standard twist representation.*

**Lemma 7.3.** *Take a chain of two-sided curves $C = (a_1, \ldots, a_{n-1})$. If we fix an orientation of a regular neighborhood of the union of the curves $a_i$, then $C$ determines the standard twist representation $\rho_C \colon B_n \to \mathrm{PMod}(S)$ defined by*

$$\rho_C(\sigma_i) = t_{a_i}, \quad i = 1, \ldots, n-1,$$

*where $t_{a_i}$ is the right-handed Dehn twist about $a_i$ with respect to the orientation.*

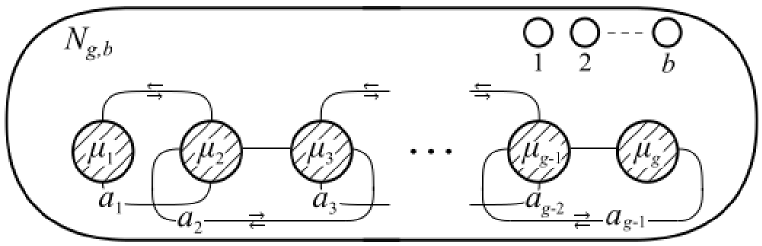



FIGURE 28. Standard chain of non-separating curves in $N_{g,b}$ [11, Figure 1]

**Question 7.4.** One might wonder whether the standard twist representation corresponds to an induced representation from a Yang-Baxter operator on some category of surfaces.

To answer this question, the intuitive monoidal category to look at given Figure 28 is $\mathcal{M}_1$ with a Yang-Baxter operator on the Möbius band with the parametrized interval as depicted on the left in Figure 29 and the Yang-Baxter operator given by the homeomorphism which is the Dehn twist about the curve in $M\natural M$ depicted on the right side in Figure 29.
Continuing in this manner, we find that $M^{\natural g}$ is as depicted in Figure 30 which is homeomorphic to $N_{g,1}$, and we see that the loops precisely coincide with those from the standard chain depicted in Figure 28.
Now, note that the Yang-Baxter equation in this case again is satisfied because of the braid relation for Dehn twists, Proposition 4.10. Thus we obtain a monoidal functor $\Phi_{M,std}\colon \mathcal{B} \to \mathcal{M}_1$ with $\Phi_{M,std,g}\colon \mathcal{B}_n \to \mathrm{Mod}\,(N_{g,1})$ the standard twist representation.

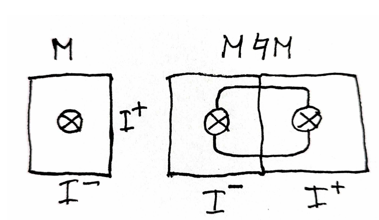



FIGURE 29. mobius-decorated.jpg

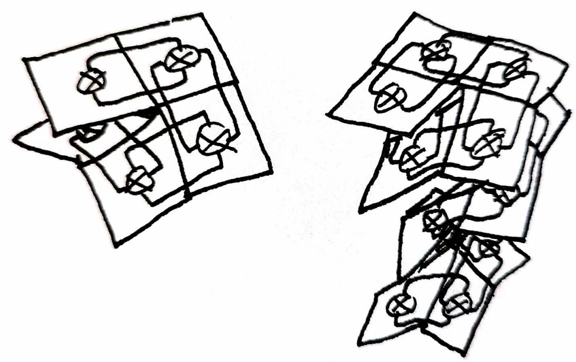


FIGURE 30. The standard chain in $M^{\natural g} \approx N_{g,1}$.

Recall that the Birman-Hilden embedding was obtained as an induced geometric representation from a Yang-Baxter operator in the category of bidecorated surfaces. The following proposition shows that the standard twist representation, in fact, is also the Birman-Hilden embedding now obtained from a Yang-Baxter operator in the category of decorated surfaces and in a seemingly different form.

**Proposition 7.5.** *For $b \geqslant 1$ and $g$ odd, the standard twist representation $\rho_C \colon B_g \to \mathrm{Mod}(N_{g,b})$ is the same as the Birman-Hilden embedding $B_g \hookrightarrow S_{\frac{g-1}{2},b-1} \# M$ into the orientable factor.*

*Proof.* To make use of the visualization from Figure 27, we need the steps to go from $K - D^2$ as a sphere with two crosscaps and a disk removed to the depiction in Figure 27.
Suppose we look at the standard chain of non-separating curves in $N_{3,b}$, so we have curves $a_1$ and $a_2$ as in Figure 28. Now, since $N_{3,1} = K \# M$, we can decompose the loops $a_1$ and $a_2$ and follow it through the homeomorphisms for $K - D$ and for $M - D$ as in in Figure 31 and Figure 32.
After the transformations, we reglue the surfaces and twist the tube from the Klein bottle around to obtain a torus as in Figure 33.
Now, noting that

$$N_{2g+1,b} \approx \left(\left(\mathbb{RP}^2 - \mathring{D}\right) \natural \left(\mathbb{RP}^2 - \mathring{D}\right)\right)^{\natural g} \natural M - \bigsqcup_{b-1} \mathring{D} = K^{\natural g} \natural M - \bigsqcup_{b-1} \mathring{D} = \left(K^{\natural g}\right)$$

we find that for $N_{2g+1,1}$ with $g > 1$, we get a similar picture as for $N_{3,1}$, depicted in Figure 34, where we can also twist the tube around.
In conclusion, the loops really correspond to a standard chain in $S_g$, so the standard twist representation $\rho_C \colon B_{2g+1} \to \mathrm{Mod}\left(N_{2g+1,1}\right) \approx \mathrm{Mod}\left(S_{g,1} \natural M\right)$ corresponds to the Birman-Hilden embedding into the $S_{g,1}$ factor.

□

*Note.* In the case where we do not have boundary components, we can simple remove a disk, perform the operations from Proposition 7.5 and then reglue. Thus we can actually extend the proposition to the case of $b = 0$ as well.

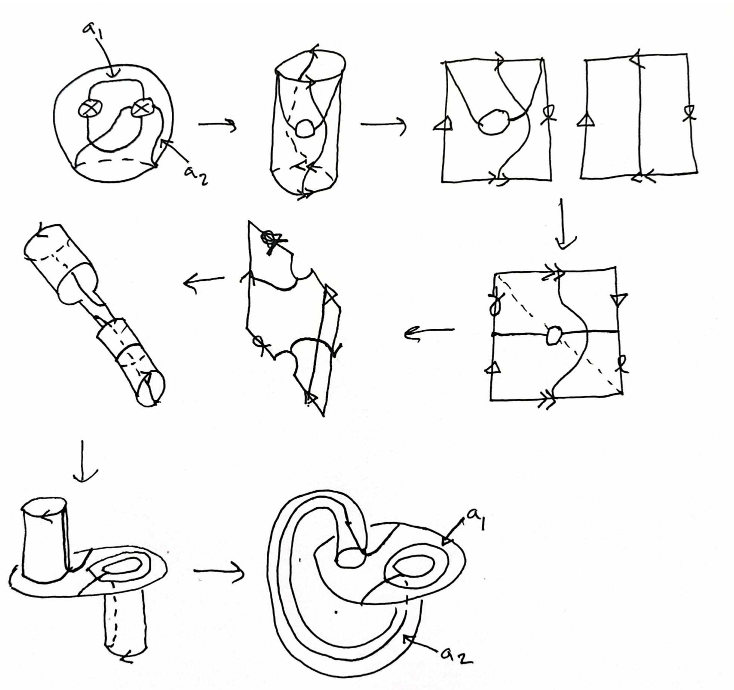



FIGURE 31. The loop $a_1$ and the arc $a_2$ throughout the homeomorphisms for $K - D^2$

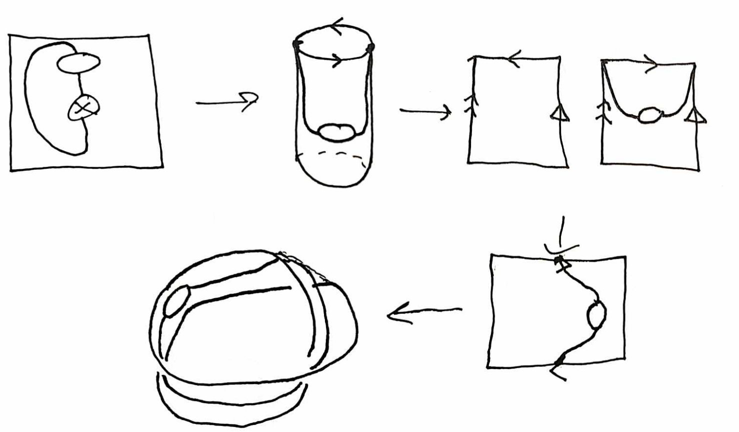


FIGURE 32. The arc from $a_2$ throughout the homeomorphisms for $M - D^2$

**Question 7.6.** What happens in the case where $g$ is even?

**Question 7.7.** If $g$ is odd and $g \geqslant 5$, the standard chain can be extended by adding a curve $a_g$ passing once through each of the first $g-1$ crosscaps. This extended

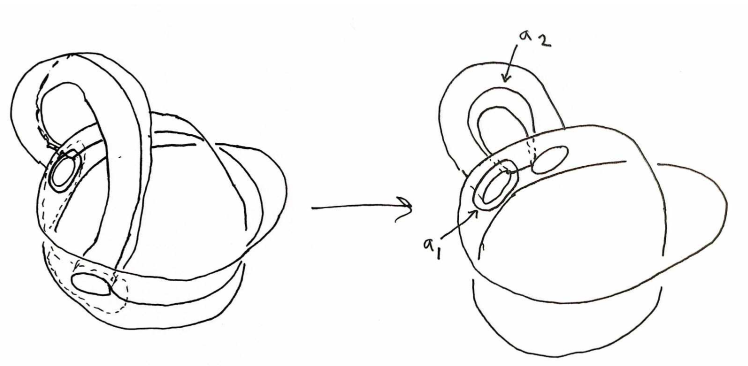



FIGURE 33. The connected sum of $K$ and $M$ by regluing.

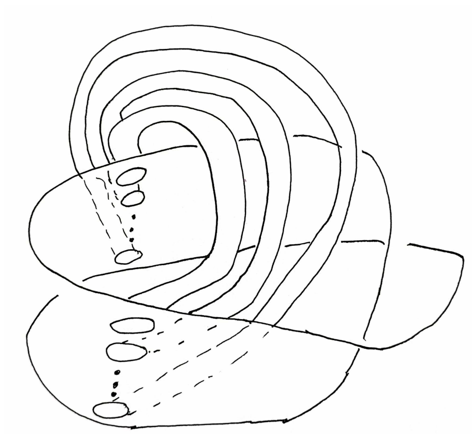


FIGURE 34. The case of $K^{\natural g}\natural M$

chain also determines a standard twist representation $\rho_{C'}\colon \mathcal{B}_{g+1} \to \mathrm{PMod}\,(N_{g,b})$. Does it correspond to something we know as well?

7.0.2. *The crosscap transposition representation.* Let $N = N_{g,b}$ be nonorientable. A sequence $C = (a_1, \ldots, a_{n-1})$ of separating curves in $N$ is called *a chain of separating curves* if

(1) $a_i$ bounds a one-holed Klein bottle for $i = 1, \ldots, n-1$,
(2) $i\,(a_i, a_{i+1}) = 2$ for $i = 1, \ldots, n-2$,
(3) $i\,(a_i, a_j) = 0$ for $|i - j| > 1$.

Here $a_i$ bounding a one-holed Klein bottle means that if we collapse $a_i$ to a point, we obtain a sphere with two crosscaps which is equivalent to the Klein bottle
Let $K_i$ be the one-holed Klein bottle bounded by $a_i$. Then $K_i \cap K_{i+1}$ will be a Möbius strip for $i = 1, \ldots, n-2$, and we denote its core curve by $\mu_{i+1}$. Let $\mu_1$ and $\mu_n$ be the core curves of $K_1 - K_2$ and $K_{n-1} - K_{n-2}$, respectively. Fix an orientation of a regular neighborhood of the union of the $a_i$. Let $T_{a_i}$ be the right-handed Dehn twist about $a_i$and let $u_i$ be the *crosscap transpostion* supported in $K_i$, swapping $\mu_i$

and $\mu_{i+1}$ such that $u_i^2 = T_{a_i}$ (essentially a half-Dehn twist but for crosscaps instead of punctures).

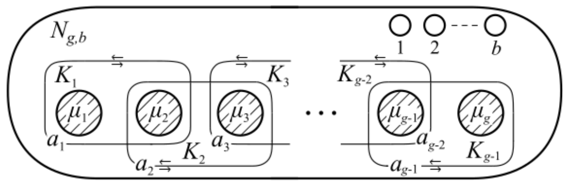



FIGURE 35. A chain of separating curves in $N_{g,b}$. [11, Figure 2]

**Lemma 7.8.** *The mapping $\theta_C \colon B_n \to \mathrm{PMod}(N)$ by $\theta_C(\sigma_i) = u_i$ for $i = 1, \ldots, n-1$, defines a homomorphisms called crosscap transposition representation.*

*Remark.* This is simply the geometric representation arising from the Yang-Baxter element associated to the Möbius band in $\mathcal{M}_1$ which is the half-twist on crosscaps.

7.0.3. *Transvection.*

**Definition 7.9** (Transvection)**.** Given a homomorphism $\rho \colon B_n \to \mathrm{PMod}(S)$ and an element $\tau \in \mathrm{PMod}(S)$ such that $\tau$ commutes with $\rho(\sigma_i)$ for $1 \leqslant i \leqslant n-1$, we define a homomorphism $\rho^\tau \colon B_n \to \mathrm{Mod}(S)$, called a *transvection* of $\rho$, by

$$\rho^\tau(\sigma_i) = \tau\rho(\sigma_i), \quad i = 1, \ldots, n-1.$$

A homomorphism $\rho \colon B_n \to \mathrm{PMod}(S)$ is called *cyclic* if $\rho(B_n)$ is a cyclic group.

*Note.* Note that transvection defines an equivalence relation on the set of representations $B_n \to \mathrm{PMod}(S)$.

7.1. **The main theorems.** Following work by Castel and generalizations by Chen and Mukherjea on classifications of geometric representations of the braid group on orientable surfaces in certain ranges, Stukow and Szepietowski proved the following two theorems in [11] which classify all geometric representations on non-orientable surfaces in a certain range.

**Theorem 7.10.** *Let $n \geqslant 14$ and let $N = N_{g,b}$ with $g \leqslant 2\left\lfloor \frac{n}{2} \right\rfloor + 1$ and $b \geqslant 0$. Then any homomorphism $\rho \colon B_n \to \mathrm{PMod}(N)$ is either cyclic, or is a transvection of a standard twist representation, or is a transvection of a crosscap transposition representation.*

**Theorem 7.11.** *Theorem 7.10 still holds when* $\mathrm{PMod}(N)$ *is replaced by* $\mathrm{Mod}(N, \partial N)$.

Note that $\mathrm{Mod}(N, \partial N) \nleqslant \mathrm{PMod}(N)$ as, for example, the Dehn twist about a boundary curve is non-trivial in $\mathrm{Mod}(N, \partial N)$, but becomes trivial in $\mathrm{PMod}(N)$.

We have shown that the standard twist representation for odd genus and the crosscap transposition naturally arise as Yang-Baxter operators on appropriate objects in an appropriate category of surfaces.
By Theorem 7.10, these are in fact all the possible explicit geometric representations which we can look at on non-orientable surfaces up to transvection given in [11].

## 8. Teichmüller Space

We assume $S$ to be a compact surface with finitely many points removed from the interior. For now, we assume $\chi(S) < 0$.

**Definition 8.1** (Hyperbolic metric)**.** A surface $S$ admits a hyperbolic metric if there exists a complete, finite-area Riemannian metric on $S$ of constant curvature $-1$ where the boundary of $S$ (if nonempty) is totally geodesic (meaning that the geodesics in $\partial S$ are geodesics in $S$ ).

**Definition 8.2** (Euclidean/flat metric)**.** We say that $S$ admits a Euclidean metric, or a flat metric, if there is a complete, finite-area Riemannian metric on $S$ with constant curvature 0 and totally geodesic boundary.

**Definition 8.3** (Hyperbolic structure)**.** A *hyperbolic structure* on $S$ is a diffeomorphism $\varphi\colon S \to X$ where $X$ is a surface with a complete, finite-area hyperbolic metric with totally geodesic boundary.
We record this as a pair $(X, \varphi)$ and call the diffeomorphism $\varphi$ a *marking* and either $X$ or $(X, \varphi)$ can be referred to as a *marked hyperbolic surface.*

**Definition 8.4** (Homotopic hyperbolic structures)**.** Two hyperbolic structures $\varphi_1 : S \to X_1$ and $\varphi_2 : S \to X_2$ on $S$ are *homotopic* if there is an isometry $I\colon X_1 \to X_2$ such that the markings $I \circ \varphi_1 : S \to X_2$ and $\varphi_2 : S \to X_2$ are homotopic, i.e, $\varphi_2 \simeq I \circ \varphi_1$. I.e., the following diagram commutes up to homotopy

$$\begin{tikzcd} & S \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ X_1 \arrow[rr, "I"] & & X_2 \end{tikzcd}$$

Note that the homotopy is allowed to move points in the boundary of $X_2$.

**Definition 8.5** (Teichmüller Space)**.** The Teichmüller space of $S$ is defined to be the set of homotopy classes of hyperbolic structures on $S$ :

$$\mathrm{Teich}(S) = \{\text{hyperbolic structures on } S\}/\text{homotopy}.$$

Or, alternatively,

$$\mathrm{Teich}(S) = \{(X, \varphi)\}/\sim$$

where the equivalence relation identifies marked hyperbolic surfaces if the hyperbolic structures they define are homotopic.

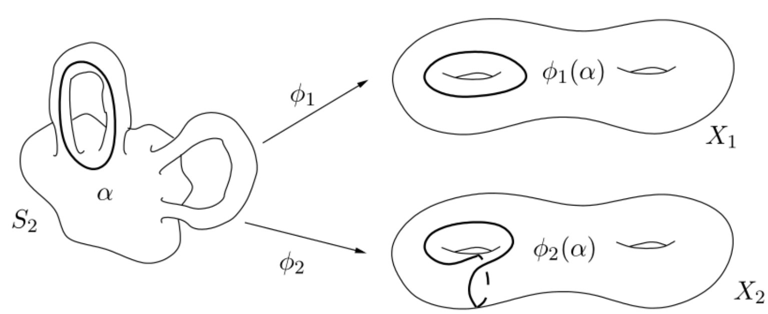



Figure 36. Example of endowing $S$ with two different hyperbolic structures which are not isometric since $\ell_{\mathcal{X}_1}(\alpha) \neq \ell_{\mathcal{X}_2}(\alpha)$.

8.1. **Teichmüller space as space of metrics.** A marking $\varphi\colon S \to X$ gives rise to a hyperbolic Riemannian metric on $S$ : namely, if $g_X$ is the metric on $X$, then $g_S(x,y) = g_X(\varphi(x),\varphi(y))$ is the pullback of the hyperbolic metric on $X$, so we have

$$\mathrm{Teich}(S) = \mathrm{HypMet}(S)/\mathrm{Diff}_0(S)$$

where the action of $\mathrm{Diff}_0(S)$ on the set of hyperbolic metrics $\mathrm{HypMet}(S)$ is by pullback as follows: if $g$ is a hyperbolic metric on $S$ and $[\varphi]$ is a diffeomorphism of $S$ isotopic to the identity, then $[\varphi](g) = g \circ \varphi$ where $g \circ \varphi(x,y) = g(\varphi(x),\varphi(y))$.

8.2. **Length functions.** Let $\mathcal{S}$ denote the set of isotopy classes of essential simple closed curves in $S$. The hyperbolic structure on $S$ corresponding to a point $\mathcal{X} \in \mathrm{Teich}(S)$ is defined only up to isotopy, so we have a well-defined length-function

$$\ell_{\mathcal{X}}\colon \mathcal{S} \to \mathbb{R}_+$$

described as follows: if $\mathcal{X} = [(X,\varphi)]$, and $c$ is an isotopy class of essential simple closed curves in $S$, then $\ell_{\mathcal{X}}(c)$ is the length of the unique geodesic in $X$ in the isotopy class $\varphi(c)$. Recall that by Proposition 2.4, every every isotopy class $c$ has a unique geodesic representative, so $\ell_{\mathcal{X}}(c)$ is well-defined.

*Remark.* We will prove later that we can essentially understand $\mathrm{Teich}(S)$ completely by its length function. More concretely, if $\mathbb{R}^{\mathcal{S}}$ is the set of real-valued functions on $\mathcal{S}$, the map $\ell\colon \mathrm{Teich}(S) \to \mathbb{R}^{\mathcal{S}}$ given by $\mathcal{X} \mapsto \ell_{\mathcal{X}}$ is injective. In fact, we will show that $\mathrm{Teich}(S)$ is determined by finitely many coordinates of the length function $\ell$.

## 9. Glossary

**Definition 9.1** (Equivariant maps)**.** Suppose a group $G$ acts on spaces $X$ and $Y$, and let $f\colon X \to Y$ be a map. Then $f$ is said to be equivariant if $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$ and all $g \in G$.

**Definition 9.2** (Closed surface)**.** A *closed surface* is a surface that is compact and without boundary.

**Definition 9.3** (Isotopy)**.** A topological isotopy is a homotopy $F\colon X \times I \to Y$ such that for each $t_0 \in I$, $F(x, t_0)\colon X \to Y$ is a topological embedding (homeomorphism onto some subspace of $Y$ ).
Two embeddings $f, g\colon X \to Y$ are said to be isotopic if there exists an isotopy $F\colon X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

**Definition 9.4** (Orientation)**.** A closed $n$-manifold $M$ is called orientable if $H_n(M; \mathbb{Z}) = \mathbb{Z}$. The choice of generator $[M]$ in $\mathbb{Z}$ is called an orientation, and the generator is called the fundamental class of $M$. A manifold together with a choice of orientation is called oriented. A compact $n$-manifold $M$ with boundary is called orientable if $H_n(M, \partial M; \mathbb{Z}) = \mathbb{Z}$. The choice of generator $[M, \partial M]$ in $\mathbb{Z}$ is called an orientation, and $[M, \partial M]$ is referred to as the fundamental class of $M$.
A smooth manifold $M$ is orientable if and only if the restriction of its tangent bundle to every smooth curve is trivial.

*Remark.* This makes sense since T. Radó showed that every surface is triangulable and it is clear then that the 2-cycles form a cyclic group. A choice of generator corresponds to choosing an orientation of each 2-simplex in the triangulation (compatibly).

**Definition 9.5** (Inner and outer automorphisms)**.** Let $G$ be any group and $\gamma \in G$. A conjugate automorphism

$$I_\gamma\colon g \mapsto \gamma g \gamma^{-1}$$

is called an *inner automorphism* of $G$. The group of inner automorphisms is denoted by $\mathrm{Inn}(G)$. It is isomorphic to $G/Z(G)$, and is a normal subgroup of $\mathrm{Aut}(G)$. The quotient

$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$$

is called the *outer automorphism group* of $G$.

## 10. Appendix

### 10.1. Fiber Bundles.

#### 10.1.1. *Hatcher.*

**Definition 10.1** (Homotopy lifting property)**.** A map $p\colon E \to B$ is said to have the homotopy lifting property with respect to a space $X$ if, given a homotopy $g\colon X \times I \to B$ and a map $\tilde{g}_0\colon X \times \{0\} \to E$ lifting $g(x, 0)$, so $p\tilde{g}_0(t, 0) = g(t, 0)$, then there exists a homotopy $\tilde{g}\colon X \times I \to E$ lifting $g_t$.

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{g}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{g}"] & B \end{tikzcd}$$

This is a special case of the *lift extension property for a pair* $(Z, A)$ (See [10])

**Definition 10.2** (Fibration)**.** A fibration is a map $p\colon E \to B$ having the homotopy lifting property with respect to all spaces $X$. For example, a projection $B \times F \to B$ is a fibration since we can choose lifts of the form $\tilde{g}(x,t) = (g(x,t), h(x))$ where $\tilde{g}(x,0) = (g(x,0), h(x))$.

**Definition 10.3** (Homotopy lifting property for a pair $(X,A)$)**.** The map $p\colon E \to B$ is said to have the *homotopy lifting property for a pair* $(X,A)$ if each homotopy $f\colon X \times I \to B$ lifts to a homotopy $\tilde{g}\colon X \times I \to E$ starting with a given lift $\tilde{g}_0\colon X \times \{0\} \to E$ and extending a given lift $\tilde{g}\colon A \times I \to E$.

**Theorem 10.4.** *Suppose $p\colon E \to B$ has the homotopy lifting property with respect to disks $D^k$ for all $k \geqslant 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_*\colon \pi_n(E,F,x_0) \to \pi_n(B,b_0)$ is an isomorphism for all $n \geqslant 1$. Hence if $B$ is path-connected, there is a long exact sequence*

$$\ldots \to \pi_n(F,x_0) \to \pi_n(E,x_0) \xrightarrow{p_*} \pi_n(B,b_0) \to \pi_{n-1}(F,x_0) \to \ldots \to \pi_0(E,x_0) \to 0$$

**Definition 10.5** (Fiber bundle)**.** A fiber bundle structure on a space $E$, with fiber $F$, consists of a projection map $p\colon E \to B$ such that each point of $B$ has a neighborhood $U$ for which there is a homeomorphism $h\colon p^{-1}(U) \to U \times F$ making the following diagram commute

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "proj"] \\
& U &
\end{tikzcd}$$

Thus $h\left(p^{-1}(b)\right) = \{b\} \times F$ by the projection map $p\colon E \to B$, but to indicate what the fiber is we sometimes write a fiber bundle as $F \to E \to B$, a "short exact sequence of spaces". The space $B$ is called the *base space* of the bundle, and $E$ is the *total space*.

*Remark.* This is just the definition of a covering map without the restriction of $F$ having the discrete topology.

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