Problem 0.1. Let F be the homotopy fibre of the map $S^n \to S^n$ of degree k, for $n \geq 2$.

(1) Show that $H^i(F) = 0$ for $0 < i < n$.
(2) Using the Serre spectral sequence, compute that

$$H^i(F) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/k, & i = 1 + m(n-1),\ m > 0 \\ 0, & \text{otherwise} \end{cases}.$$

(3) Show that for $x, y \in H^*(F)$, if $\deg(x), \deg(y) > 0$, then $x \smile y = 0$.

Proof. (1) Since $\pi_1 S^n = 0$, the Serre spectral sequence to the homotopy fiber sequence

$$F \to S^n \to S^n$$

gives the following double complex:

$$\begin{tikzcd}
H^{n-1}(F) \arrow[d, dash] \arrow[ddddr] & & \\
\vdots \arrow[d, dash] & & \\
H^2(F) \arrow[d, dash] & & \\
H^1(F) \arrow[d, dash] & & \\
\mathbb{Z} \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & \cdots \\
& n &
\end{tikzcd}$$

We apply the LSSS for cohomology and find that $H^i(S^n) = F_0^n$, and since $H^i(F)$ is the only nontrivial entry on the antidiagonal in degree i, and since there are no maps to kill off $H^i(F)$ for $0 < i < n-1$, we obtain that $H^i(F) = H^i(S^n) = 0$ for $0 < i < n-1$.

$\square$