HOMOTOPY THEORY

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For these notes, we will follow [2], [1] and [3].

1. COFIBRATIONS

For this section, we will follow chapter VII.1 in [1].

One of the fundamental questions in topology is the "extension problem". Namely, given a map $g\colon A \to Y$ defined on a subspace A of X, when can we extend this map to all of X.

This cannot always be done - for example, as is the case with $A = Y = S^n$ and $X = D^{n+1}$ choosing the map to be any degree -1 map.

Question 1.1. Is the extension problem a *homotopy-theoretic* problem? That is, does the answer depend only on the homotopy class of g?

The answer is: generally not. For example, we can take $X = [0,1]$, $A = \{0\} \cup \left\{\frac{1}{n} \mid n = 1, 2, \ldots\right\}$ and $Y = CA$, the cone on A. Choosing g to be the inclusion of A into Y, this cannot be extended to X as the extension would be discontinuous at $\{0\}$. However, $g \simeq g'$ with g' being the constant map of A to the vertex of the cone, and g' easily extends to X by the constant map.

It turns out, however, that under some very mild conditions on the spaces, the problem becomes homotopy theoretic. We will now discuss this.

Definition 1.2 (Homotopy extension property)**.** Let (X, A) and Y be given spaces. Then (X, A) is said to have the *homotopy extension property* with respect to Y if the following diagram can always be completed to be commutative.

$$
\begin{array}{ccc}
A \times I \cup X \times \{0\} & \longrightarrow & Y \\
\downarrow & \nearrow & \\
X \times I & &
\end{array}
$$

One can also depict this by the following diagram:

$$
\begin{array}{ccc}
A \times \{0\} & \longrightarrow & A \times I \\
\downarrow & \searrow\; Y \;\nwarrow & \downarrow \\
X \times \{0\} & \longrightarrow & X \times I
\end{array}
$$

If (X, A) has the homotopy extension property with respect to Y, then the extensibility of maps $g\colon A \to Y$ depends only on the homotopy class of g. For suppose $H\colon g \simeq g'$ and g' can be extended to $\tilde{g}'\colon X \to Y$, then define the map $A \times I \cup X \times \{0\}$ by $\tilde{g}' \times \{0\}$ on $X \times \{0\}$ and H on $A \times I$. The homotopy extension property for the pair (X, A) then guarantees the existence of a map $G\colon X \times I \to Y$ which equals g on $A \times \{1\}$, so $H(-, 1)\colon X \to Y$ extends g.

Definition 1.3 (Cofibration)**.** Let $f\colon A \to X$ be a map. Then f is called a *cofibration* if one can always fill in the following commutative diagram given the solid arrows:

$$
\begin{array}{ccc}
A \times \{0\} & \longrightarrow & A \times I \\
\downarrow{\scriptstyle f\times\mathrm{id}} & Y & \downarrow{\scriptstyle f\times\mathrm{id}} \\
X \times \{0\} & \longrightarrow & X \times I
\end{array}
$$

for *any* space Y.

Note. If f is an inclusion, the this is the same as the homotopy extension property for all Y. That attribute is sometimes referred to as the *absolute homotopy extension property*.

Theorem 1.4. *For an inclusion $A \subset X$, the following are equivalent:*
  (1) *The inclusion map $A \hookrightarrow X$ is a cofibration.*
  (2) *$A \times I \cup X \times \{0\}$ is a retract of $X \times I$.*

Proof. If the inclusion is a cofibration, then choosing $Y = A \times I \cup X \times \{0\}$ with all arrows being inclusions in the diagram of a cofibration, we obtain a map $X \times I \to A \times I \cup X \times \{0\}$ which is the identity on $A \times I \cup X \times \{0\}$.
Conversely, if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$, then we can always complete the diagram by mapping $X \times I \to A \times I \cup X \times \{0\} \to Y$ where the second map takes the maps $A \times I \to Y$ and $X \times \{0\} \to Y$ from the diagram. $\qquad\square$

Corollary 1.5. *If A is a subcomplex of a CW-complex X, then the inclusion $A \hookrightarrow X$ is a cofibration.*

Proof. We want to construct a retraction $X \times I \to A \times I \cup X \times \{0\}$. We will do so by constructing a retraction $\left(\left(A \cup X^{(r)}\right) \times I\right) \cup (X \times \{0\}) \to (A \times I) \cup (X \times \{0\})$ by induction on r. If it has been defined on the $(r-1)$-skeleton, then extending it over an r-cell is simply a matter of extending a map on $S^{r-1} \times I \cup D^r \times \{0\}$ over $D^r \times I$ which can be done since the pair $\left(D^r \times I, S^{r-1} \times I \cup D^r \times \{0\}\right)$ is homeomorphic to $(D^r \times I, D^r \times \{0\})$. See Figure 1
These maps for each cell fit together to give a map on the r-skeleton because of the weak topology on $X \times I$. The union of these maps for all r gives a map on $X \times I$, again because of the weak topology on $X \times I$. $\qquad\square$

Theorem 1.6. *Assume that $A \subset X$ is closed and that there exists a neighborhood U of A and a map $\varphi\colon X \to I$ such that*
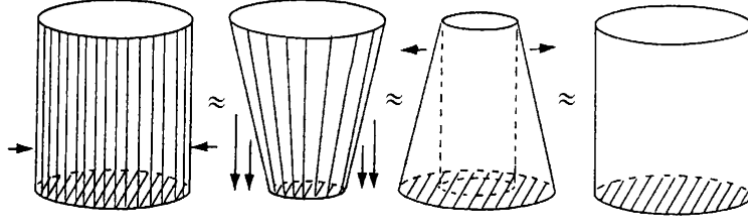 (1) *$A = \varphi^{-1}(0)$.*
  (2) *$\varphi(X - U) = \{1\}$.*

FIGURE 1. A homeomorphism of pairs.

(3) *U deforms to A through X with A fixed. That is, there is a map $H \colon U \times I \to X$ such that $H(a,t) = a$ for all $a \in A$, $H(u,0) = 0$, and $H(u,1) \in A$ for all $u \in U$.*

Then the inclusion $A \hookrightarrow X$ is a cofibration. The converse also holds.

Proof. We may assume that $\varphi = 1$ on a neighborhood of $X - U$ by replacing φ with $\min(2\varphi, 1)$. It suffices to show that there exists a retract $\Phi \colon U \times I \to X \times \{0\} \cup A \times I$ since then the map

$$r(x,t) = \begin{cases} \Phi(x, t(1 - \varphi(x))), & x \in U \\ (x, 0), & x \notin U \end{cases}$$

gives a retraction $X \times I \to A \times I \cup X \times \{0\}$.
We define Φ by

$$\Phi(u,t) = \begin{cases} H\left(u, \frac{t}{\varphi(u)}\right) \times \{0\}, & \varphi(u) > t \\ H(u, 1) \times \{t - \varphi(u)\}, & \varphi(u) \le t. \end{cases}$$

The only thing that needs checking here is that Φ is continuous at points $(u, 0)$ such that $\varphi(u) = 0$, i.e., points $(a, 0)$ for $a \in A$ - indeed here the expression for $\varphi(u) > t$ is not defined.
Recall that a map $f \colon X \to Y$ is continuous if for every point $x \in X$ and any neighborhood U of $f(x)$, there exists a neighborhood V of x such that $f(V) \subset U$. So let W be a neighborhood of $a = H(a, t)$. Then there exists a neighborhood $V \subset W$ containing a such that $H(V \times I) \subset W$, by assumption of H being continuous. So for $t < \varepsilon$ for some ε and $u \in V$, we have $\Phi(u, t) \in W \times [0, \varepsilon]$. Hence Φ is continuous.

To prove the converse, suppose that the inclusion $A \hookrightarrow X$ is a cofibration. Equivalently, $A \times I \cup X \times \{0\}$ is a retract of $X \times I$. Let $r \colon X \times I \to A \times I \cup X \times \{0\}$ be this retraction. Let $s(x) = r(x, 1)$ and set $U = s^{-1}(A \times (0, 1])$. Let p_X, p_I be the projections of $X \times I$ to its factors. Then put $H = p_X \circ r|_{U \times I} \colon U \times I \to X$. Now, $H(a, t) = p_X \circ r|_{U \times I}(a, t) = p_X(a, t) = a$ for all $a \in A$ and $t \in I$; $H(u, 0) = p_X \circ r|_{U \times I}(u, 0) = p_X(u, 0) = u$, and $H(u, 1) = p_X \circ r|_{U \times I}(u, 1) = u$ forces $(u, 1) \in A \times I$, hence $u \in A$. Thus, H satisfies condition (3).
For (1) and (2), let $\varphi(x) = \max_{t \in I} |t - p_I r(x, t)|$ which is possible since I is compact. Then $x \in \varphi^{-1}(0)$ implies that $\max_{t \in I} |t - p_I r(x, t)| = 0$, so for all $t \in I$, we have $|t - p_I r(x, t)| = 0$, so $r(x, t) \in A \times \{t\}$ for all $t \in (0, 1]$. Then $r(x, 0) = \lim_{n \to \infty} r\left(x, \frac{1}{n}\right) \in A \times I$ since $A \times I$ is closed. But $(x, 0) = r(x, 0)$, so $x \in A$. Conversely, for any $x \in A$, clearly, $\varphi(x) = 0$ since $r(x, t) = (x, t)$ for all

$t \in I$. This shows that φ satisfies (1). For (2), we have that for $x \in X - U$, with $U = s^{-1}(A \times (0,1])$, we have $r(x,1) = s(x) \notin A \times (0,1]$, so $r(x,1) \in X \times \{0\}$. Hence $\varphi(x) = \max_{t \in I} |t - p_I r(x,t)| = 1$, giving (2).

It remains to show that φ is continuous. Let $f(x,t) = |t - p_I r(x,t)|$ and $f_t = (x,t)$ all of which are continuous. Then

$$\varphi^{-1}((-\infty, b]) = \{x \mid f(x,t) \le b \text{ for all } t\} = \bigcap_{i \in I} f_t^{-1}((-\infty, b]).$$

is an intersection of closed sets and so is closed. Similarly,

$$\varphi^{-1}([a, \infty)) = \{x \mid f(x,t) \ge a \text{ for some } t\} = p_X \left(f^{-1}([a, \infty)) \right)$$

which si also closed since p_X is closed as a projection and I is compact. Since the complements of the intervals of the form $[a, \infty)$ and $(-\infty, b]$ give a subbase for the topology of $\mathbb{R}$, this shows that φ is continuous.

$\square$

Next, we recall that for a map $f \colon X \to Y$, the mapping cylinder M_f is defined as

$$M_f = ((X \times I) \sqcup Y) / ((x,0) \sim f(x)).$$

Consider the inclusion $\iota \colon X \hookrightarrow M_f$ where we include X as $X \times \{1\}$. Consider the map $\varphi \colon Mf \to I$ given by $\varphi(x,t) = 1 - 2t$ for $t \ge \frac{1}{2}$ and $\varphi(x,t) = 1$ on the rest of M_f. Choosing $U = X \times (\frac{1}{3}, 1]$, U clearly deformations retracts to $X \times \{1\}$ and satisfies the conditions of Theorem 1.6, hence the inclusion $X \hookrightarrow M_f$ is a cofibration. Also, the retraction $r \colon M_f \to Y$ is a homotopy equivalence with the homotopy inverse being the inclusion $Y \hookrightarrow M_f$. The diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \iota\ } & M_f \\
 & f \searrow \quad \simeq \swarrow r & \\
 & Y &
\end{array}
$$

commutes. Thus any map f is a cofibration up to a homotopy equivalence of spaces. Recall also that the mapping cone of a map $f \colon X \to Y$ is defined as

$$C_f := M_f / X \times \{1\} \cong M_f \cup CX.$$

In the case of an inclusion $\iota \colon A \hookrightarrow X$, we have $C_\iota = X \cup CA$.

There is a map $C_\iota \xrightarrow{h} X/A$, defined as the composite of the quotient map $X \cup CA \to X \cup CA / CA$ composed with the inverse of the homeomorphism $X/A \to X \cup CA / CA$.

Question 1.7. Is h a homotopy equivalence?

Theorem 1.8. *If $A \subset X$ is closed and the inclusion $\iota \colon A \to X$ is a cofibration, then $h \colon C_\iota \to X/A$ is a homotopy equivalence. In fact, it is a homotopy equivalence of pairs*

$$(X/A, *) \simeq (C_\iota, CA) \simeq (C_\iota, v),$$

where v is the vertex of the cone.

Proof. The mapping cone $C_\iota = X \cup CA$ consists of three different types of points: the vertex $v = \{A \times \{1\}\}$, the rest of the cone $\{(a,t) \mid 0 \le t < 1\}$ where $(a,0) = a \in A \subset X$, and points in X itself, which we identify with $X \times \{0\}$.

Define $f \colon A \times I \cup X \times \{0\} \to C_\iota$ as the collapsing map and extend f to $\overline{f} \colon X \times I \to C_\iota$

using that f is a cofibration. Then $\overline{f}(a,1) = v, \overline{f}(a,t) = (a,t)$ and $\overline{f}(x,0) = x$.
Let $\overline{f}_t = \overline{f}|_{X \times \{t\}}$. Since $\overline{f}_1(A) = \{v\}$, we can factorize $\overline{f}_1 \colon X \to C_\iota$ as $g \circ j$ where
$j \colon X \to X/A$ is the quotient map and $g \colon X/A \to C_\iota$ is the induced map

$$
\begin{array}{ccc}
X & & \\
\big\downarrow{\scriptstyle j} & \searrow^{\overline{f}_1} & \\
X/A & \dashrightarrow[g] & C_\iota.
\end{array}
$$

where g is induced and continuous by definition of the quotient topology.
We claim that g is a homotopy equivalence with homotopy inverse h. First, we
prove that $hg \simeq \mathrm{id}_{X/A}$.
Note that taking the composite $h\overline{f}_t \colon X \to X/A$ gives a homotopy between $h\overline{f}_0$ and
$h\overline{f}_1$. For all t, this homotopy takes A to the point $\{A\}$. Thus, it factors to give a
homotopy

$$hgj = h\overline{f}_1 \simeq h\overline{f}_0 = j$$

Let $H \colon X \times I \to X/A$ be the homotopy between hgj and j, so $H(x,0) = hgj(x)$ and
$H(x,1) = j(x)$. Then the map $\overline{H} \colon X/A \times I \to X/A$ defined by $\overline{H}([x],t) = H(x,t)$
defines a homotopy between hg and $\mathrm{id}_{X/A}$, so $hg \simeq \mathrm{id}_{X/A}$.
Next, we will show that $gh \simeq \mathrm{id}_{C_\iota}$. Consider $W = (X \times I)/(A \times \{1\})$ and the
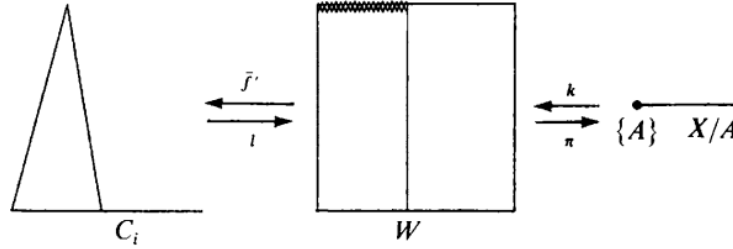maps illustrated in Figure 2.

FIGURE 2. DWIJJXNXJNXJUI.png

The map $\overline{f}'$ is induced by $\overline{f}$. The map k is the "top face" map. From this, we see
that

$$
\begin{aligned}
\overline{f}' \circ l &= \mathrm{id} \\
\pi \circ k &= \mathrm{id} \\
k \circ \pi &\simeq \mathrm{id} \\
\overline{f}' \circ k &= g \\
\pi \circ l &= l.
\end{aligned}
$$

Hence $gh = \overline{f}'k\pi l \simeq \overline{f}'l = \mathrm{id}$. $\qquad\square$

Example 1.9 (A non example)**.** An example of when the result of Theorem 1.6
does not hold is with $A = \{0\} \cup \{\frac{1}{n} \mid n = 1, 2, \ldots\}$ and $X = [0,1]$. In this case,
C_ι is not homotopy equivalent to X/A which is a one-point union of a countably
infinite sequence of circles with radii going to zero.

C_i has homeomorphs of circles joined along edges. However, the circles do not tend to a point ,so any prospective homotopy equivalence $X/A \to C_\iota$ would be discontinuous at the image of $\{0\}$ in X/A.

Corollary 1.10. *If $A \subset X$ is closed and the inclusion $A \hookrightarrow X$ is a cofibration, then the map $j\colon (X, A) \to (X/A, *)$ induces isomorphisms*

$$H_*(X, A) \xrightarrow{\cong} H_* (X/A, *) \cong \tilde{H}_* (X/A)$$

and

$$\tilde{H}^*(X/A) \cong H^*(X/A, *) \xrightarrow{\cong} H^*(X, A).$$

Proof. We have $H_* (X/A, *) \cong H_* (C_\iota, CA)$ by Theorem 1.8. And since $C_\iota = X \cup A \times \left[0, \frac{1}{2}\right]$ and $CA = A \times \left[0, \frac{1}{2}\right]$, where we collapse $A \times \left\{\frac{1}{2}\right\}$ in both, and attach $A \times \left[0, \frac{1}{2}\right]$ along $A \times \{0\}$ in $X \cup A \times \left[0, \frac{1}{2}\right]$, we obtain

$$H_*(C_\iota, CA) \cong H_* \left(X \cup A \times \left[0, \frac{1}{2}\right], A \times \left[0, \frac{1}{2}\right] \right) \cong H_*(X, A)$$

since $\left(X \cup A \times \left[0, \frac{1}{2}\right], A \times \left[0, \frac{1}{2}\right] \right) \simeq (X, A)$ by deformation retracting $A \times \left[0, \frac{1}{2}\right]$ down to $A \times \{0\} \subset X$. $\qquad\square$

1.0.1. *Interlude on pointed-spaces and operations on spaces.* We recall some important constructions:

Definition 1.11 (Unreduced Suspension)**.** For a space X, the *unreduced suspension* ΣX is the quotient obtained from $X \times I$ by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

Note. We have $\Sigma S^n = S^{n+1}$.

Definition 1.12 (Suspension of a map)**.** Given a map $f\colon X \to Y$, we can suspend f to $\Sigma f\colon \Sigma X \to \Sigma Y$ by letting Σf be the induced map on the quotients:

$$
\begin{array}{ccc}
X \times I & \xrightarrow{f \times \mathrm{id}} & Y \times I \\
\downarrow & & \downarrow \\
\Sigma X & \xrightarrow{\Sigma f} & \Sigma Y
\end{array}
$$

Exercise 1.13. For any homology theory, show that there is a natural isomorphism $\tilde{H}_I(X) \xrightarrow{\cong} \tilde{H}_{i+1} (\Sigma X)$. Here, natural means that for a map $f\colon X \to Y$, and its suspension $\Sigma f\colon \Sigma X \to \Sigma Y$, the following diagram commutes:

$$
\begin{array}{ccc}
\tilde{H}_i(X) & \xrightarrow{\cong} & \tilde{H}_{i+1} (\Sigma X) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle (\Sigma f)_*} \\
\tilde{H}_i(Y) & \xrightarrow{\cong} & \tilde{H}_{i+1} (\Sigma Y)
\end{array}
$$

Definition 1.14 (Wedge Sum/one-point union)**.** Given two pointed spaces $(X, x_0), (Y, y_0)$, we define the *wedge sum* $X \vee Y$ to be

$$X \vee Y = X \sqcup Y/ (x_0 \sim y_0),$$

i.e., the quotient of the disjoint union identifying x_0 and y_0 to a single point.

Definition 1.15 (Smash Product)**.** Inside the product $X \times Y$ of two pointed space $(X, x_0), (Y, y_0)$, we have natural copies of X and Y by $X \times \{y_0\}$ and $\{x_0\} \times Y$, respectively. These two copies intersect only at the point (x_0, y_0), so their union can be identified with the wedge sum $X \vee Y$. I.e., $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$. We define the *smash product* $X \wedge Y$ to be the quotient $X \times Y / X \vee Y$.

If $f \colon X \to Y$ is a pointed map, then the reduced mapping cylinder of f is defined as the quotient space M_f of $(X \times I) \cup Y$ modulo the relations identifying $(x, 0) \sim f(x)$ and the set $\{*\} \times I$ to the base point of M_f.
The reduced mapping cone is the quotient of the reduced mapping cylinder M_f obtained by identifying the image of $X \times \{1\}$ to a point, the base point.

The circle S^1 is defined as $I/\partial I$ with base point $\{\partial I\}$.
The reduced suspension of a pointed space X is $SX = X \wedge S^1$. It can also be considered as the quotient space $X \times I / (X \times \partial I \cup \{*\} \times I)$

Definition 1.16 (Well-pointed space)**.** A base point $x_0 \in X$ is said to be *nondegenerate* if the inclusion $\{x_0\} \hookrightarrow X$ is a cofibration. A pointed Hausdorff space X with nondegenerate base point is said to be *well-pointed*.

It is clear that any manifold or CW-complex satisfies Theorem 1.6 with A being any point of the space. Hence any manifold or CW-complex is well-pointed.

Example 1.17 (Pointed space that is not well-pointed)**.** Taking the pointed space $X = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$ with base point 0, this space is not well-pointed. This can for example be seen because it fails to satisfy Theorem 1.4 - any retraction would break continuity at $(0, 1)$.

Example 1.18. If $A \hookrightarrow X$ is a cofibration, then X/A with base point $\{A\}$ is well-pointed, as follows from Theorem 1.6.

Theorem 1.19. *If X is well-pointed, then so are the reduced cone CX and the reduced suspension SX. Moreover, the collapsing map $\Sigma X \to SX$, of the unreduced suspension to the reduced suspension, is a homotopy equivalence.*

Proof. Denote the base point of X by $*$. Consider the homeomorphism

$$h \colon (I \times I, I \times \{0\} \cup \partial I \times I) \overset{\cong}{\to} (I \times I, I \times \{0\})$$

which clearly exists. For example, take Figure 3

FIGURE 3.

Then the induced homeomorphism

$$\mathrm{id}_X \times h \colon X \times I \times I \overset{\cong}{\to} X \times I \times I$$

carries $X \times I \times \{0\} \cup X \times \partial I \times I$ to $X \times I \times \{0\}$. Hence it takes $A = X \times I \times \{0\} \cup X \times \partial I \times I \cup \{*\} \times I \times I$ to $X \times I \times \{0\} \cup \{*\} \times I \times I$. Therefore, the pair

$(X \times I \times I, A)$ is homeomorphic to the pair $I \times (X \times I, X \times \{0\} \cup \{*\} \times I)$. Now, X is well-pointed, so $X \times \{0\} \cup \{*\} \times I$ is a retract of $X \times I$ by Theorem 1.4 and the definition of well-pointed. It follows that A is a retract of $X \times I \times I$. By another application of 1.4, then the inclusion $X \times \partial I \cup \{*\} \times I \hookrightarrow X \times I$ is a cofibration. Hence the quotient by this, $SX = X \times I / (X \times \partial I \cup \{*\} \times I)$ is well-pointed, using the quotient of the above inclusion.

Next consider the homeomorphism $(I \times I, I \times \{0\} \cup \{1\} \times I) \stackrel{\cong}{\to} (I \times I, I \times \{0\})$ which can be seen similarly. The induced homeomorphism

$$1 \times h \colon X \times I \times I \stackrel{\cong}{\to} X \times I \times I$$

takes $A := X \times \{1\} \times I \cup \{*\} \times I \times I \cup X \times I \times \{0\}$ to $X \times I \times \{0\} \cup \{*\} \times I \times I$. Thus the pair $(X \times I \times I, A)$ is homeomorphic to $I \times (X \times I, X \times \{0\} \cup \{*\} \times I)$. Just as above, we have that $X \times \{0\} \cup \{*\} \times I$ is a retract of $X \times I$, so it follows that A is a retract of $X \times I \times I$. Thus the inclusion $X \times \{1\} \cup \{*\} \times I \hookrightarrow X \times I$ is a cofibration, which shows that $CX = X \times I / (X \times \{1\} \cup \{*\} \times I)$ is well-pointed.

The fact that $X \times \partial I \cup \{*\} \times I \hookrightarrow X \times I$ is a cofibration gives that there exists a neighborhood U of $X \times \partial I \cup \{*\} \times I$ and a map $\varphi \colon X \times I \to I$ that satisfy Theorem 1.6. We obtain an induced map $\overline{\varphi} \colon \Sigma X \to I$ which satisfies the same conditions, so $I \times \times \{*\} \times I \hookrightarrow X \times I / \{X \times \{0\}, X \times \{1\}\} = \Sigma X$ is a cofibration. Now Theorem 1.8 implies that $\Sigma X \cup CI = C_\iota \to \Sigma X / I$ is a homotopy equivalence. Hence we obtain that $\Sigma X \simeq \Sigma X \cup CI \simeq \Sigma X / I = SX$, via the collapsing map.

$\square$

Problem 1.20. Find $H_* \left(\mathbb{P}^2, \mathbb{P}^1 \right)$ using methods or results from this section.

Solution. Consider $\mathbb{P}^2$ as S^2 quotiented by the relation $x \simeq -x$. Then we can think of $\mathbb{P}^1$ as $S^1 \subset S^2$ under this relation. We want to show that the inclusion $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ is a cofibration. Using Theorem 1.6, it suffices to find a neighborhood U of $\mathbb{P}^1 \subset \mathbb{P}^2$ and a map $\overline{\varphi} \colon \mathbb{P}^2 \to I$ such that the conditions of the theorem are satisfied. We construct a preliminary map on S^2 towards this end. Define $\varphi \colon S^2 \to I$ to be $\varphi(x) = \min \{1, 2 |x_3|\}$, where x_3 is the last coordinate of x. Since $\varphi(x) = \varphi(-x)$, φ induces a map $\overline{\varphi} \colon \mathbb{P}^2 \to I$ such that the diagram

$$\begin{array}{ccc} S^2 & & \\ \downarrow & \searrow^{\varphi} & \\ \mathbb{P}^2 & \xrightarrow{\overline{\varphi}} & I \end{array}$$

commutes. Letting U be the image under the quotient map of $\left\{ x \in S^2 \mid |x_3| < \frac{1}{2} \right\}$, this becomes an open set in $\mathbb{P}^2$ since the above set is saturated with respect to the quotient map. It is also clear that U and $\overline{\varphi}$ satisfy the conditions of the theorem, hence the inclusion $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ is a cofibration. By Corollary 1.10, we obtain that $H_* \left(\mathbb{P}^2, \mathbb{P}^1 \right) \cong \tilde{H}_* \left(\mathbb{P}^2 / \mathbb{P}^1 \right)$. But $\mathbb{P}^2 / \mathbb{P}^1 \cong S^2$, so $H_* \left(\mathbb{P}^2, \mathbb{P}^1 \right) \cong \tilde{H}_* \left(S^2 \right)$. Now simply recall that

$$\tilde{H}_p \left(S^2 \right) \cong \begin{cases} \mathbb{Z}, & p = 2 \\ 0, & p \neq 2. \end{cases}$$

$\square$

Problem 1.21. Find $H_* \left(T^2, \{*\} \times S^1 \cup S^1 \times \{*\} \right)$ using methods from this section.

Solution. If we can show that the inclusion $A := \{*\} \times S^1 \cup S^1 \times \{*\} \hookrightarrow T^2$ is a cofibration, then we will again obtain that $H_*(T^2, A) \cong \tilde{H}_* \left(T^2/A\right) \cong \tilde{H}_* \left(S^2\right)$. But we have a CW-structure on the torus given by the square with identified sides. With this identificaiton, A simple becomes the 1-skeleton, hence it is a subcomplex, so by Corollary 1.5, the inclusion $A \hookrightarrow T^2$ is a cofibration. This finishes the solution. $\square$

Problem 1.22. For a space X, consider the pair (CX, X). What do the results of this section tell you about the homology of these, and related, spaces?

Solution. We can define a map $\varphi \colon CX \to I$ by $\varphi(x, t) = t$. Choosing $A = X = X \times \{0\} \subset CX$ and $U = CX - \{v\}$ where v is the vertex, this satisfies the conditions in Theorem 1.6 (H can be defined by $H((x, t_0), t) = (x, t_0)(1 - t) + (x, 0)t)$. Hence the inclusion $X \hookrightarrow CX$ is a cofibration, so we know that $H_* (CX, X) \cong \tilde{H}_* (CX/X)$. Similarly, one can show that the inclusion $X \hookrightarrow \Sigma X$ is a cofibration, so $H_* (\Sigma X, X) \cong \tilde{H}_* (\Sigma X/X) \cong \tilde{H}_* (\Sigma X \vee \Sigma X)$ and $H_* (SX, X) \cong \tilde{H}_* (SX \vee SX)$.

2. Homotopy Groups

2.1. **Homotopy.** We follow chapter 14 of [1] for this subsection.

To start of, we recall the basic definitions of homotopies.

Definition 2.1 (Homotopy). Two maps $f_0, f_1 \colon X \to Y$ are said to be *homotopic* if there exists a homotopy $F \colon X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$.

Definition 2.2 (Homotopy equivalence). A map $f \colon X \to Y$ is said to be a *homotopy equivalence* if it is an isomorphism in hTop.

Lemma 2.3 (Reparametrization Lemma). *Let φ_1, φ_2 be maps $(I, \partial I) \to (I, \partial I)$ which are equal on ∂I. Let $F \colon X \times I \to Y$ be a homotopy and let $G_i(x, t) = F(x, \varphi_i(t))$ for $i = 1, 2$. Then $G_1 \simeq G_2 \operatorname{rel} X \times \partial I$.*

We shall use c to denote the constant homotopy.

Proposition 2.4. $F * c \simeq F \operatorname{rel} X \times \partial I$ and $c * F \simeq F \operatorname{rel} X \times \partial I$.

Definition 2.5. If $F \colon X \times I \to Y$ is a homotopy, then we define $F^{-1} \colon X \times I \to Y$ by $F^{-1}(x, t) = F(x, 1 - t)$.

Note that F^{-1} is precisely the inverse to F in hTop.

Proposition 2.6. *For any homotopies F, G, H for which the concatenations are defined, we have*
$$(F * G) * H \simeq F * (G * H) \operatorname{rel} X \times \partial I.$$

Proposition 2.7. *For homotopies F_1, F_2, G_1, G_2, if $F_1 \simeq F_2 \operatorname{rel} X \times \partial I$ and $G_1 \simeq G_2 \operatorname{rel} X \times \partial I$, then $F_1 * G_1 \simeq F_2 * G_2 \operatorname{rel} X \times \partial I$.*

Note that all of the discussion of concatenation of homotopies goes through with no difficulties for the cases in which all homotopies are relative to some subspace $A \subset X$ or are homotopies of pairs $(X, A) \to (Y, B)$.

It follows that homotopy between maps of pairs $(X, A) \to (Y, B)$ is an equivalence relation. The set of homotopy classes of these maps is commonly denoted by $[X, A; Y, B]$ or just $[X; Y]$ if $A = \varnothing$.

Theorem 2.8. *If $f_0 \simeq f_1 \colon X \to Y$ then $M_{f_0} \simeq M_{f_1} \operatorname{rel} X + Y$ and $C_{f_0} \simeq C_{f_1} \operatorname{rel} Y + vertex$.*

To show this, one needs the following basic topological proposition:

Proposition 2.9. *If $f \colon X \to Y$ is a quotient map and K is locally compact Hausdorff, then $f \times 1 \colon X \times K \to Y \times K$ is a quotient map.*

Proof of Theorem 2.8. First, let $F \colon X \times I \to Y$ be the homotopy between f_0 and f_1. Now define $h \colon M_{f_0} \to M_{f_1}$ by $h(y) = y$ for $y \in Y$ and
$$h(x, t) = \begin{cases} F(x, 2t), & t \leq \frac{1}{2} \\ (x, 2t - 1), & \frac{1}{2} \leq t. \end{cases}$$

Define $k \colon M_{f_1} \to M_{f_0}$ likewise by the identity on Y nad
$$k(x, t) = \begin{cases} F^{-1}(x, 2t), & t \leq \frac{1}{2} \\ (x, 2t - 1), & \frac{1}{2} \leq t \end{cases}.$$

Then the composition $kh \colon M_{f_0} \to M_{f_1}$ is the identity on Y and $F * \left(F^{-1} * E\right)$ on the cylinder portion, where $E \colon X \times I \to M_{f_0}$ is induced by the identity on $X \times I \to X \times I$. This is homotopic to the identity rel $X \times \{1\} + Y$. Similarly for hk. In now remains to check the continuity of this homotopy. We have a homotopy $M_{f_0} \times I \to M_{f_0}$. We now claim that $M_{f_0} \times I \cong M_{f_0 \times I}$. Indeed then, using that $M_{f_0 \times I} = \frac{X \times I \times I \sqcup Y \times I}{((x,0,k) \sim (f_0(x),k))}$, it suffices to show continuity of the composition $X \times I \times I \sqcup Y \times I \to M_{f_0} \times I \to M_{f_0}$. For on $Y \times I$, it is the constant homotopy and on $X \times I \times I$ it is $F * \left(F^{-1} * E\right) \simeq E$ rel $X \times \partial I$. Now, that $M_{f_0} \times I \cong M_{f_0 \times I}$ follows from Proposition 2.9.

$\square$

Let $f \colon X \to Y$. If $\varphi \colon Y \to Y'$ is a map, then there is the induced map $F \colon M_f \to M_{\varphi \circ f}$ induced from φ on Y and the identity on $X \times I$.

Theorem 2.10. *If $\varphi \colon Y \to Y'$ is a homotopy equivalence then so is $F \colon (M_f, X) \to (M_{\varphi \circ f}, X)$ and hence so is $F \colon C_f \to C_{\varphi \circ f}$.*

Proof. Let $\psi \colon Y' \to Y$ be a homotopy inverse of φ and let $G \colon M_{\varphi \circ f} \to M_{\psi \circ \varphi \circ f}$ be the map induced by ψ on Y' and the identity on $X \times I$. The composition $GF \colon M_f \to M_{\psi \circ \varphi \circ f}$ is induced from $\psi \circ \varphi \colon Y \to Y$ and the identity on $X \times I$. Let $H \colon Y \times I \to Y$ be a homotopy from id to $\psi \circ \varphi$; i.e., $H(y,0) = y$ and $H(y,1) = \psi\varphi(y)$. By the proof of Theorem 2.8, there is a homotopy equivalence $h \colon M_f \to M_{\psi \circ \varphi \circ f}$ rel X given by $h(y) = y$ and

$$h(x,t) = \begin{cases} H\left(f(x), 2t\right), & t \le \frac{1}{2} \\ (x, 2t-1), & t \ge \frac{1}{2} \end{cases}.$$

We claim that $h \simeq GF$ rel X. Indeed, the homotopy H can be extended to $M_f \times I \to M_{\psi \circ \varphi \circ f}$ by putting

$$H\left((x,s),t\right) = \begin{cases} H\left(f(x), 2s+t\right), & 2s+t \le 1 \\ \left(x, \frac{2s+t-1}{t+1}\right), & 2s+t \ge 1 \end{cases}.$$

Then $H(-,0) = h$ and $H(-,1) = GF$, so since GF is a homotopy equvalence, so is h. Define $F' \colon M_{\psi \circ \varphi \circ f} \to M_{\varphi \circ \psi \circ \varphi \circ f}$ as the induced map on mapping cones with φ on Y and the identity on $X \times I$. Then similarly, $F'G$ is a homotopy equivalence. If k is a homotopy inverse of GF then $GFk \simeq$ id. If k' is a homotopy inverse of $F'G$ then $k'F'G \simeq$ id. Thus G has a right and left homotopy inverse: $R = Fk$ and $L = k'F'$. Then $R = \text{id} \circ R \simeq (LG)R = L(GR) \simeq L \circ \text{id} = L$, so $R \simeq L$. That is, G has a homotopy inverse. Therefore, G is a homotopy equivalence. Since G and GF are homotopy equivalences, so is F. $\square$

Problem 2.11. [1, Ex 14.1] Let $S^2 \cup A$ denote the union of the unit 2-sphere and the line segment joining the north and south poles. Show that $S^2 \vee S^1 \simeq S^2 \cup A$.

Proof. Define two maps $f_0, f_1 \colon \{0,1\} \to S^2$ where $f_0(t) = (\cos(2\pi t), \sin(2\pi t), 0)$ and f_1 is the constant map at $(1,0,0)$. Then $f_0 \simeq f_1$, so $C_{f_0} \simeq C_{f_1}$. Now, $C_{f_0} = S^2 \cup A$ while $C_{f_1} = S^2 \vee S^1$. $\square$

Problem 2.12. [1, Ex 14.2] Show that the union of a 2-sphere and a flat unit 2-cell through the origin is homotopically equivalent to the one-point union of two 2-spheres.

Proof. A 2-cell is contractible, an a 2-sphere with a 2-cell inside it is precisely the cone of the map $S^1 \sqcup S^1 \to S^1$ with the identity on both. By [1, Thm 14.19], this is homotopy equivalent to the cone on $S^1 \sqcup S^1 \to \{*\}$ which is $S^2 \vee S^2$. $\square$

Problem 2.13. Show that the union of a standard 2-torus with two disks, one spanning a latitudinal circle and the other spanning a longitudinal circle of the torus, is homotopically equivalent to a 2-sphere.

Proof. Using the identification of the torus as the quotient space of I^2 in the usual way, we can choose on spanning circle to be a 2-cell attached along $\{0\} \times I$ and the other to be a 2-cell attached along $I \times \{0\}$. These are contractible, and the quotient space becomes a 2-sphere. $\square$

2.2. **Homotopy Groups.** Recall that $[X, A; Y, B]$ denotes the set of homotopy classes of maps $X \to Y$ carrying A into B such that A goes into B during the entire homotopy.

To make a group then, we can select a point $y_0 \in Y$ and consider the set

$$[X \times I, X \times \partial I; Y, \{y_0\}]$$

In this case, the operation of concatenation of homotopies makes this set into a group. It is technically also better to choose a basepoint $x_0 \in X$ and consider

$$[X \times I, \{x_0\} \times I \cup X \times \partial I; Y, \{y_0\}] .$$

For the moment, let us set $A = \{x_0\} \times I \cup X \times \partial I$. Then maps $X \times I \to Y$ which carry A into $\{y_0\}$ are in bijective correspondence with maps $(X \times I)/A \to Y$ which take the point $\{A\}$ into $\{y_0\}$.

Definition 2.14 (Reduced Suspension)**.** We define the *reduced suspension* of X to be

$$SX = (X \times I)/A = (X \times I) / (\{x_0\} \times I \cup X \times \partial I)$$

The set of homotopy classes of pointed maps of a pointed space X to a pointed space Y with homotopies preserving the base points will be denoted by $[X; Y]_*$.
Thus $[SX; Y]_*$ is in canonical bijective correspondence with $[X \times I, A; Y, \{y_0\}]$.
Now, suppose we have pointed maps $f, g \colon SX \to Y$. Then they induce homotopies $f', g' \colon X \times I \to Y$ by precomposing with the quotient map $X \times I \to SX$. We can then define $f' * g' \colon X \times I \to Y$ as usual. The resulting pointed map $SX \to Y$ will be denoted $f * g$. Geometrically, $f * g$ is obtained by putting f on the bottom and g on the top of the one-point union $SX \vee SX$ and composing the resulting map $SX \vee SX \to Y$ with the map $SX \to SX \vee SX$ obtained by collapsing the middle parameter value $\frac{1}{2}$ copy of X in SX to the base point.
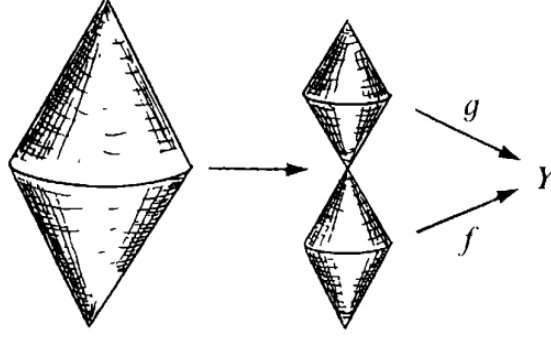For a map $f \colon (SX, \{A\}) \to (Y, \{y_0\})$, we denote its homotopy class in $[SX; Y]_*$ by $[f]$, and we define

$$[f] [g] = [f * g]$$

Under this operation, the set $[SX; Y]_*$ becomes a group.

Proposition 2.15. *The reduced suspension gives* $SS^{n-1} \cong S^n$.

Thus, we can define S^n as the n-fold reduced suspension of S^0. As a special case, the set $[S^n; Y]_*$ then becomes a group for $n > 0$.

FIGURE 4. The product of two map classes $SX \to Y$.

Definition 2.16 (n th homotopy group)**.** We define

$$\pi_n(Y, y_0) = [S^n; Y]_*$$

with this operation.

2.2.1. *A different way of defining* $\pi_n(Y, y_0)$. Note that reduced suspension supplies a parameter in $[0, 1]$ and the space S^n as constructed is the quotient space of I^n obtained by collapsing the boundary of the cube to a point. Pointed maps $S^n \to Y$ are in bijective correspondence with maps $I^n \to Y$ taking ∂I^n to the base point of Y. This is a more traditional way of defining $\pi_n(Y)$. This becomes the group of homotopy classes of maps $(I^n, \partial I^n) \to (Y, \{y_0\})$ with the operation being

$$f * g(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n), & t_1 \in \left[0, \frac{1}{2}\right] \\ g(2t_1 - 1, t_2, \ldots, t_n), & t_1 \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Proposition 2.17. *For* $n \geq 2$, $\pi_n(X, x_0)$ *is abelian.*

Proof. Consider the homotopy in Figure 5. We begin by shrinking the domains of f and g to smaller subcubes of I^n, where the region outside is mapped to the basepoint. This allows us to move the boxes around in a continuous manner. The rest is clear. $\square$
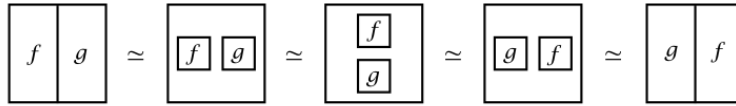

FIGURE 5. The homotopy in question

Next, we want to show that following:

Proposition 2.18. *If* X *is path-connected, then* $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ *for any two* $x_0, x_1 \in X$.

For this, we introduce an action of π_1 on π_n.

Definition 2.19 (The action of π_1 on π_n). Given a path $\gamma \colon I \to X$ from x_0 to x_1, we associate to a map $f \colon (I^n, \partial I^n) \to (X, x_1)$ the map $\gamma f \colon (I^n, \partial I^n) \to (X, x_0)$ by shrinking the domain of f to a smaller concentric cube in I^n, then inserting the path γ on each radial segment in the shell between this smaller cube and ∂I^n. See Figure 6
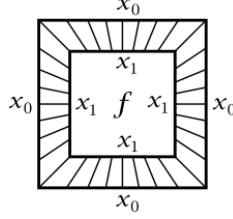

FIGURE 6. Depiction of γf.

Note. We have the following properties

(1) $\gamma (f + g) \simeq \gamma f + \gamma g$.
(2) $(\gamma \eta) f \simeq \gamma (\eta f)$.
(3) $\mathrm{id} f \simeq f$, where id denotes the constant path.

To see (1), first deform f and g to be constant on the right and left halves of I^n, respectively, producing maps which we may call $f + 0$ and $0 + g$, then we can excise a progressively wider symmetric middle slab of $\gamma(f + 0) + \gamma(0 + g)$ (which can be seen on the left in Figure 7) until it becomes $\gamma (f + g)$ (shown on the right).
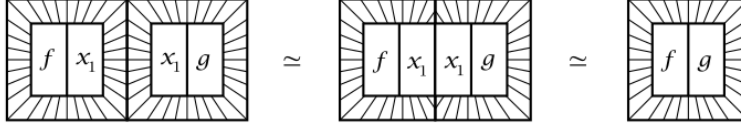

FIGURE 7.

Now if $\beta_\gamma \colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ is the change-of-basepoint transformation, $\beta_\gamma [f] = [\gamma f]$, then the above note shows that β_γ is a group isomorphism. This proves Proposition 2.18. If we restrict attention to loops γ at x_0, then since $\beta_{\gamma \eta} = \beta_\gamma \beta_\eta$, the map $[\gamma] \mapsto \beta_\gamma$ defines a homomorphism from $\pi_1 (X, x_0)$ to $\mathrm{Aut} (\pi_n (X, x_0))$ called the *action of π_1 on π_n* .

Note. For $n > 1$, this action makes $\pi_n(X, x_0)$ into a module over the group ring $\mathbb{Z} [\pi_1 (X, x_0)]$.

Definition 2.20 (Simple/abelian spaces). A space with trivial π_1 action on π_n is called 'n-simple', and 'simple' means ' n-simple for all n '. We call a space *abelian* if it has trivial action of π_1 on all homotopy groups π_n.

Proposition 2.21 (π_n is a functor). *A map $\varphi \colon (X, x_0) \to (Y, y_0)$ induces a map $\varphi_* \colon \pi_n (X, x_0) \to \pi_n (Y, y_0)$ defined by $\varphi_* [f] = [\varphi f]$. It is immediate from the definitions that φ_* is well-defined and a homomorphism for $n \geq 1$. The functorial properties are also clear.*

Corollary 2.22. *Homotopy equivalent spaces have isomorphic homotopy groups.*

Proposition 2.23. *A covering space projection $p\colon \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ induces isomorphisms $p_*\colon \pi_n\left(\tilde{X}, \tilde{x}_0\right) \to \pi_n(X, x_0)$ for all $n \geq 2$.*

Proof. Since S^n is path-connected and locally path-connected, and simply connected for $n \geq 2$, we find that any map $(S^n, s_0) \to (X, x_0)$ lifts to a map $(S^n, s_0) \to \left(\tilde{X}, \tilde{x}_0\right)$ when $n \geq 2$. This gives surjectivity of p_*. For injectivity, suppose $p_*[f] = [0]$ where $f\colon (S^n, s_0) \to \left(\tilde{X}, \tilde{x}_0\right)$. Let $c_{\tilde{x}_0}$ be the constant map at $\tilde{x}_0$. Then $p_*[\tilde{x}_0] = [0]$, so by uniqueness of the lifting theorem, $[f] = [c_{\tilde{x}_0}] = [0]$. $\qquad\square$

Definition 2.24 (Aspherical)**.** Spaces with $\pi_n = 0$ for all $n \geq 2$ are called *aspherical*.

Corollary 2.25. S^1, T^n *and K are aspherical since they have contractible covering spaces.*

Proposition 2.26.

$$\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha)$$

Next we define relative homotopy groups.

Definition 2.27 (Relative homotopy groups)**.** Regard I^{n-1} as a face of I^n with the last coordinate $s_n = 0$ and let J^{n-1} be the closure of $\partial I^n - I^{n-1}$. Then we define

$$\pi_n(X, A, x_0) := \left[I^n, \partial I^n, J^{n-1}; X, A, x_0\right]$$

We shall leave $\pi_0(X, A, x_0)$ undefined for now.

We can define a sum operation on $\pi_n(X, A, x_0)$ in the same way as for $\pi_n(X, x_0)$, except now the coordinate s_n now must remain free, so we must use one of the other coordinates. Thus we must have at least one other coordinate to define the same operation. So $\pi_n(X, A, x_0)$ is a group for $n \geq 2$, and it is abelian for $n \geq 3$. For $n = 1$, we have $I^1 = [0, 1]$, $I^0 = \{0\}$ and $J^0 = \{1\}$, so $\pi_1(X, A, x_0) = [I, \{0\}, \{1\}; X, A, x_0]$ is the set of homotopy classes of paths in X from a varying point in A to the fixed basepoint $x_0 \in A$. In general, this is not a group in any natural way.

Now, we saw before that $\pi_n(X, x_0)$ can be regarded as homotopy classes of maps $(S^n, x_0) \to (X, x_0)$. Similarly, collapsing J^{n-1} to a point, converts $\left(I^n, \partial I^n, J^{n-1}\right)$ to $\left(D^n, S^{n-1}, s_0\right)$. In this case, addition is done by the map $c\colon D^n \to D^n \vee D^n$ collapsing $D^{n-1} \subset D^n$ to a point.

Theorem 2.28 (Compression criterion)**.** *A map $f\colon \left(D^n, S^{n-1}, s_0\right) \to (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ if and only if it is homotopic rel S^{n-1} to a map with image contained in A.*

Proof. Suppose we have a homotopy rel S^{n-1} from f to a map g, so $[f] = [g]$ in $\pi_n(X, A, x_0)$. Viewing g as a map $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ whose image is contained in A, we can construct the homotopy $H \colon D^n \times I \to X$ by $H(x,t) = g((1-t)x + s_0 t)$ which is a homotopy from g to the constant map at x_0, hence $[g] = 0$ in $\pi_n(X, A, x_0)$.

Conversely, if $[f] = 0$ via a homotopy $F \colon D^n \times I \to X$ such that $F(x,0) = f(x)$ and $F(x,1) = x_0$ for all $x \in D^n$ and $F(x,t) \in A$ for all x with $|x| = 1$ as well as $F(s_0, t) = x_0$ for all t. We can construct a homotopy using F by restricting F to a family of n-disks in $D^n \times I$ starting with $D^n \times \{0\}$ and ending with the disk $D^n \times \{1\} \cup S^{n-1} \times I$, and where all the disks throughout the family have the same boundary. See Figure 8 for a depiction of this homotopy.
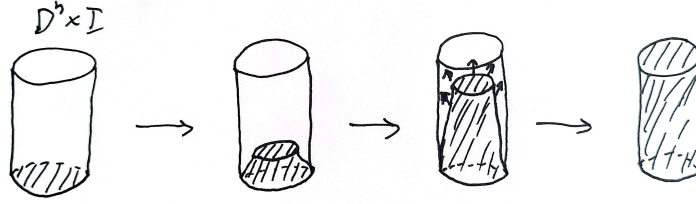

FIGURE 8.

This completes the proof.   □

Next, some things that carry over: a map $\varphi \colon (X, A, x_0) \to (Y, B, y_0)$ induces maps $\varphi_* \colon \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ which are homomorphisms when $n \geq 2$ and have properties analogous to those in the absolute case: $(\varphi\psi)_* = \varphi_*\psi_*, (\mathrm{id}_{(X,A,x_0)})_* = \mathrm{id}_{\pi_n(X,A,x_0)}$, and if $\varphi \simeq \psi$ through maps $(X, A, x_0) \to (Y, B, y_0)$, then $\varphi_* = \psi_*$.

2.2.2. *LES of relative homotopy groups.* Probably the most useful feature of relative homotopy groups $\pi_n(X, A, x_0)$ is that they fit into a long exact sequence

$$\ldots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \ldots \to \pi_0(X, x_0).$$

Here i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to I^{n-1} (the face of I^n with the last coordinate $s_n = 0$), or equivalently, by restricting maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ to S^{n-1}. The map ∂, called the *boundary map*, is a homomorphism when $n > 1$. In fact, we can show the following theorem

Theorem 2.29 (LES of relative homotopy groups)**.** *Given $x_0 \in B \subset A \subset X$, the sequence of relative homotopy groups*

$$\ldots \to \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \to \ldots \to \pi_1(X, A, x_0)$$

is exact and natural. In the case when $B = \{x_0\}$, we have that the LES

$$\ldots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \ldots \to \pi_0(X, x_0).$$

is exact and natural.

Proof. Exactness at $\pi_n(X, B, x_0)$: the composition $j_* i_*$ is zero because any map $(I^n, \partial I^n, J^{n-1}) \to (A, B, x_0)$ is zero in $\pi_n(X, A, x_0)$ by the compression criterion (Theorem 2.28). To see that $\ker j_* \subset \operatorname{im} i_*$, let $f: (I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ represent zero in $\pi_n(X, A, x_0)$. Using the compression criterion again, we then get that f is homotopic rel ∂I^n to a map with image in A, hence the class $[f] \in \pi_n(X, B, x_0)$ is indeed in the image of i_*. We conclude that $\ker j_* = \operatorname{im} i_*$, obtaining exactness at $\pi_n(X, B, x_0)$.

Exactness at $\pi_n(X, A, x_0)$: for a map $[f] \in \operatorname{im} j_*$, we have that j_* maps ∂I^n into B, hence in particular $I^{n-1} \subset \partial I^n$ into B, so $\partial j_*[f]$ represents a homotopy class in $\pi_{n-1}(A, B, x_0)$ with image in B, but then by the compression criterion, $\partial j_*[f] = 0$ in $\pi_{n-1}(A, B, x_0)$, so $\operatorname{im} j_* \subset \ker \partial$. Conversely, suppose $\partial[f] = 0$. By the compression criterion, representatives of $\partial[f]$ are homotopic rel ∂I^{n-1} to a map with image in B. In particular, $f|_{I^{n-1}}$ is homotopic to a map with image in B via a homotopy $F: I^{n-1} \times I \to A$ rel ∂I^{n-1}. We can tack F onto f to get a new map $(I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ which, as a map $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ is homotopic to f by the homotopy that tacks on increasingly longer initial segments of F. See Figure 9. Hence $[f] \in \operatorname{im} j_*$.
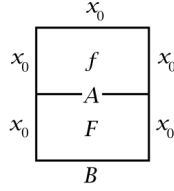

FIGURE 9.

Exactness at $\pi_n(A, B, x_0)$: First, $i_* \partial$ is zero since the restriction of a map $f: (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0)$ to I^n is homotopic rel ∂I^n to a constant map via f itself (a similar picture to Figure 8 works).

Conversely, if B is a point, then a nullhomotopy $f_t: (I^n, \partial I^n) \to (X, x_0)$ of $f_0: (I^n, \partial I^n) \to (A, x_0)$ gives a map $F: (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0)$ with $\partial([F]) = [f_0]$. So in this case, the proof is finished. For a general B, let F be a nullhomotopy of $f: (I^n, \partial I^n, J^{n-1}) \to (A, B, x_0)$ through maps $(I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ and let g be the restriction of F to I^{n-1} in $I^{n-1} \times I = I^n$ (see the first of the pictures in Figure 10). Next reparametrize the n th and $(n+1)$ st coordinates as in the second picture. Then we find that f with g tacked on is in the image of ∂. But as before, tacking g onto f gives the same element of $\pi_n(A, B, x_0)$
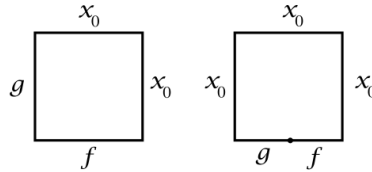

FIGURE 10.

□

Corollary 2.30. *Consider the inclusion $\iota\colon X = X \times \{0\} \hookrightarrow CX$. Then $\pi_n\left(CX, X, x_0\right) \cong \pi_{n-1}\left(X, x_0\right)$ for all $n \geq 1$. Taking $n = 2$, we can thus realize an group G, abelian or not, as a relative π_2 by choosing X to have $\pi_1(X) \cong G$.*

There are also change-of-basepoint isomorphisms β_γ for relative homotopy groups. One takes a path γ in $A \subset X$ from x_0 to x_1 which induces $\beta_\gamma\colon \pi_n(X, A, x_1) \to \pi_n(X, A, x_0)$ by setting $\beta_\gamma\left([f]\right) = [\gamma f]$, where γf is depicted in Figure 11.
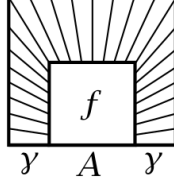

FIGURE 11.

Restricting to loops at the basepoint, the association $\gamma \mapsto \beta_\gamma$ defines an action of $\pi_1\left(A, x_0\right)$ on $\pi_n\left(X, A, x_0\right)$ analogous to the action of $\pi_1\left(X, x_0\right)$ on $\pi_n(X, x_0)$.

2.3. **Problem set 1.**

2.3.1. *Exercises.*

Exercise 2.31 (The action of the fundamental gorup, part 2)**.** Let X be a path-connected, semi-locally simply-connected space with basepoint x and $p\colon \tilde{X} \to X$ its universal cover. Show that for $n \geq 2$ and $\tilde{x} \in X$ with $p(\tilde{x}) = x$, the isomorphism $p_* = \pi_n(p)\colon \pi_n\left(\tilde{X}, \tilde{x}\right) \cong \pi_n(X, x)$ allows us to identify the action of $\pi_1(X, x)$ on $\pi_n(X, x)$ with the action of $\pi_1(X, x)$ on $\pi_n\left(\tilde{X}, \tilde{x}\right)$ induced by the group of deck transformations, i.e., the natural action of $\pi_1(X, x)$ on $\tilde{X}$. In particular, make the statement precise.

Proof. We want to show that for $[\gamma] \in \pi_1(X, x)$ and $[f] \in \pi_n(X, x)$, if $\tilde{g}$ is the lift for γ starting at $\tilde{x}_0$, and $\tilde{f}\colon (S^n, s_0) \to \left(\tilde{X}, \tilde{x}_0\right)$ is the lift of f, then $p_*\left(\tilde{\gamma}\tilde{f}\right) = \gamma f$. But this follows directly from how $\tilde{\gamma}\tilde{f}$ and γf we constructed. Namely, applying p to the square used in the definition, we see that we obtain γf from $\tilde{\gamma}\tilde{f}$ since $p \circ \tilde{\gamma} = \gamma$ and $p \circ \tilde{f} = f$. $\square$

Exercise 2.32. Let X and Y be pointed spaces and $n \geq 2$. Show that the inclusion $X \vee Y \hookrightarrow X \times Y$ induces a surjection $\pi_n(X \vee Y) \to \pi_n(X \times Y)$ for all n. Furthermore, this exhibits $\pi_n(X \times Y)$ as a retract of $\pi_n(X \vee Y)$ for all n. (Is this also true for $n = 1$?)

Proof. a $\square$

2.3.2. *Problems.*

Problem 2.33. Fix an isomorphism $H_n(S^n) \cong \mathbb{Z}$. We define the degree $\deg f$ of a map $f\colon S^n \to S^n$ to be the integer such that $f_*\colon H_n(S^n) \to H_n(S^n)$ sends 1 to $\deg f \in \mathbb{Z}$.

    (1) Show that taking the degree of a map $S^n \to S^n$ induces a well-defined map
$$\deg\colon \pi_n(S^n) \to \mathbb{Z}$$
    (2) Show that deg is a group homomorphism.
    (3) Show that the map deg is surjective.
    (4) Suppose that $n \geq 2$. Show that $\pi_n(S^n) \cong \mathbb{Z} \times A$ for some abelian group A.

Proof.   (1) Let $[f] \in \pi_n(S^n)$ and suppose f, f' are two representatives of this class. Then f and f' are homotopic by definition, so $f_* = (f')_*\colon \mathbb{Z} = H_n(S^n) \to H_n(S^n) = \mathbb{Z}$ are equal. In particular, $\deg f = f_*(1) = (f')_*(1) = \deg f'$. So the map is well-defined.
    (2) To show that degree is a group homomorphism, we must show that $\deg(f + g) = \deg f + \deg g$.

        For this, we will show a couple of results.

        Proposition 2.34. *Let $X = S_1^n \vee \ldots \vee S_k^n$ for $n > 0$. Then the homomorphism $H_n(S_1^n) \oplus \ldots \oplus H_n(S_k^n) \to H_n(X)$ induced by the inclusion maps is an isomorphism whose inverse is induced by the projections $X \to S_i^n$.*

        To prove this proposition, we must show the following lemma.

Lemma 2.35. *Let X be a Hausdorff space and let $x_0 \in X$ be a point having a closed neighborhood N in X of which $\{x_0\}$ is a strong deformation retract. Let Y be a Hausdorff space and let $y_0 \in Y$. Define $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$. Then the inclusion maps induce isomorphisms $\tilde{H}_i(X) \oplus \tilde{H}_i(Y) \cong \tilde{H}_i(X \vee Y)$ whose inverse is induced by the projections of $X \vee Y$ to X and Y.*

Proof of lemma. Consider $A = X$ and $U = X - N$ which is open, and $\overline{U} \subset A$. Then by excision, $H_*(X \vee Y, X) \cong H_*(N \cup Y, N) \cong \tilde{H}_*(Y)$

Consider the LES of the triple $(X \vee Y, \{x_0\} \times Y, \{x_0\} \times \{y_0\})$. We obtain

$$\ldots \to H_p(\{x_0\} \times Y, (x_0, y_0)) \xrightarrow{i_*} H_p(X \vee Y, (x_0, y_0)) \xrightarrow{j_*} H_p(X \vee Y, \{x_0\} \times Y) \to \ldots$$

Since $\pi_Y \circ i = \mathrm{id}_{\{x_0\} \times Y}$, i_* is injective.

Furthermore, we have

$$H_p(X \vee Y, (x_0, y_0)) \xrightarrow{(\pi_X)_*} H_p(\{x_0\} \times Y, (x_0, y_0)) \cong H_p(X \vee Y, \{x_0\} \times Y)$$

so $j_* = (\pi_X)_*$ under these identifications, so, in particular, j_* is surjective. Therefore, our exact sequence is a SES:

$$0 \to H_p(Y, pt) \xrightarrow{i_*} H_p(X \vee Y, pt) \xrightarrow{j_*} \underbrace{H_p(X \vee Y, Y)}_{\cong H_p(X, pt)} \to 0$$

It remains to show that this SES is split, but since $\pi_X \circ \iota_X = \mathrm{id}_{\{x_0\} \times X}$, we have that ι_{X*} provides a section.

$\square$

Proof of proposition. This follows by induction on the lemma. $\square$

Next, suppose that $E_1, \ldots, E_k$ are disjoint open subsets of S^n, each homeomorphic to $\mathbb{R}^n$ for $n > 0$. Let $f \colon S^n \to Y$ be a map which takes $S^n - \bigcup E_i$ to y_0. Then f factors through the quotient space $S^n / (S^n - \bigcup E_i) \cong S_1^n \vee \ldots \vee S_k^n$ where $S_i^n = S^n / (S^n - E_i)$:

$$f \colon S^n \xrightarrow{g} S_1^n \vee \ldots \vee S_k^n \xrightarrow{h} Y$$

Let $\iota_j \colon S_j^n \hookrightarrow S_1^n \vee \ldots \vee S_k^n$ be the j th inclusion and let $p_j \colon S_1^n \vee \ldots \vee S_k^n \to S_j^n$ be the j th projection. Then by the proposition, $\sum_j \iota_{j*} p_{j*} = \mathrm{id}_* \colon H_n(S_1^n \vee \ldots \vee S_k^n) \to H_n(S_1^n \vee \ldots \vee S_k^n)$. Let $g_j = p_j \circ g \colon S^n \to S_j^n$ and $h_j = h \circ \iota_j \colon S_j^n \to Y$ and let $f_j = h_j \circ g_j \colon S^n \to Y$. That is, f_j is the map which is f on E_j and maps the complement of E_j to the basepoint y_0.

Theorem 2.36. *In the above situation, $f_* = \sum_{j=1}^k f_{j*} \colon H_n(S^n) \to H_n(Y)$.*

Proof of theorem. We have $f_* = h_* \circ g_* = \sum_j h_* i_{j*} p_{j*} g_* = \sum_j h_{j*} g_{j*} = \sum_j f_{j*}$. $\square$

Now we get back to showing that $\deg(f + g) = \deg f + \deg g$.

Note that by way of defining $f + g$, this essentially maps I^n by f on the left half and g on the right half with the boundary mapping to the base point x_0. In particular, this factors through the quotient $I^n \to I^n / \partial I^n \cong S^n$, where now the two halves can be interpreted as, say, the upper and lower hemispheres. In particular, the equator is by assumption also mapped to x_0, so we can quotient further by $S^n \to S^n \vee S^n$ by "pinching" the equator

to a point. This is essentially what the proposition above describes. In particular, $f + g$ can be covered by the two open hemispheres and maps the equator to x_0, so by the theorem, we have $(f + g)_* = f_* + g_*$, i.e., $\deg(f + g) = (f + g)_* (1) = f_*(1) + g_*(1) = \deg f + \deg g$, as we wanted to show.

(3) Next we show that deg is surjective. First note that $\deg \mathrm{id} = \mathrm{id}_*(1) = 1$ by functoriality since $\mathrm{id}_* = \mathrm{id}_{H_n(S^n)}$. By functoriality, we thus hit all of $\mathbb{Z}$. More precisely, $\deg(*_n \mathrm{id}) = n$ for $n \in \mathbb{N}$ as deg is a homomorphism. Also $\deg(*_n(-\mathrm{id})) = -n$ for $n \in \mathbb{N}$ and $\deg(c_{x_0}) = 0$, so deg is surjective.

(4) Let $n \geq 2$. We have a SES

$$0 \to \ker \deg \to \pi_n (S^n) \overset{\deg}{\to} \mathbb{Z} \to 0.$$

Since $\mathbb{Z}$ is projective, this splits, so $\pi_n (S^n) \cong \mathbb{Z} \oplus \ker \deg$. But $\ker \deg$ is a subgroup of $\pi_n (S^n)$ which is abelian, hence is itself abelian.

$\square$

Problem 2.37. Fix $n \geq 1$. We say that a space X is *n-connected* if it is non-empty, path-connected, and $\pi_k(X, x) = 0$ for all $1 \leq k \leq n$ and $x \in X$. For (X, x_0) a pointed, path-connected space, show that the following are equivalent:

(1) X is *n*-connected.
(2) $\pi_k (X, x_0) = 0$ for all $1 \leq k \leq n$.
(3) Every map $S^k \to X$ can be extended to a map $D^{k+1} \to X$ for all $k \leq n$.
(4) Every map $S^k \to X$ is homotopic to a constant map for all $k \leq n$.

Proof. $(1 \implies 2)$: this follows since X being n-connected means that $\pi_k (X, x) = 0$ for all $x \in X$ and all $1 \leq k \leq n$, hence in particular for x_0.

$(2 \implies 3)$: Let $f \colon S^k \to X$ be a map. Then f represents some homotopy class $[f] \in \pi_k (X, x_0)$. But since $\pi_k(X, x_0) = 0$, f is homotopic to the constant map at x_0 rel s_0. Let $H \colon S^k \times I \to X$ be this homotopy. Define $\tilde{f} \colon D^{k+1} \to X$ by $\tilde{f}(x) = H(x, \|x\|)$. Then $\tilde{f}$ is continuous as a composite of continuous maps and $\tilde{f}|_{S^k}(-) = H(-, 1) = f(-)$, so $\tilde{f}$ indeed extends f.

$(3 \implies 4)$: Let $f \colon S^k \to X$ be a map. Extends f to a map $\tilde{f} \colon D^{k+1} \to X$. Define now a homotopy $H \colon S^k \times I \to X$ by $H(x, t) = \tilde{f}(xt)$. This is continuous and $H(x, 1) = \tilde{f}(x) = f(x)$ while $H(x, 0) = \tilde{f}(0) \in X$ is constant. Hence this gives a homotopy between f and $c_{\tilde{f}(0)}$.

$(4 \implies 3)$: Let $f \colon S^k \to X$ be a given map. By assumption, there exists a homotopy $H \colon S^k \times I \to X$ such that $H(-, 1) = f(-)$ and $H(-, 0) = c$ where c is some constant map at a point in X. But then H factors through the quotient

$$
\begin{array}{ccc}
S^k \times I & & \\
\downarrow & \searrow^{H} & \\
D^{k+1} & \xrightarrow{\tilde{H}} & X
\end{array}
$$

where we identify $S^k \times \{0\}$ to a point. But then $\tilde{H}|_{S^k}(-) = H(-, 1) = f(-)$, so $\tilde{H}$ extends f.

$(3 \implies 2)$: Let $[f] \in \pi_k(X, x_0)$ and f a representative. We want to show

that f is homotopic to the constant map at x_0 relative ∂I^k. Extend f to a map $\tilde{f} \colon D^{k+1} \to X$, and let $H \colon S^k \times I \to X$ be given by $H(x,t) = \tilde{f}(ts_0 + (1-t)x)$. This gives a homotopy between f and the constant map at x_0.

$(2 \implies 1)$: the only thing that requires showing is that given that $\pi_k(X, x_0) = 0$ for all k, we then have $\pi_k(X, x) = 0$ for all k and all $x \in X$. But this is precisely what the given hint says we are allowed to assume since X is path connected. So we are done. $\qquad\square$

Problem 2.38 (*n*-connected in the relative case)**.** The following four conditions are equivalent for $i > 0$:

(1) Every map $\left(D^i, \partial D^i\right) \to (X, A)$ is homotopic rel ∂D^i to a map $D^i \to A$.
(2) Every map $\left(D^i, \partial D^i\right) \to (X, A)$ is homotopic through such maps to a map $D^i \to A$.
(3) Every map $\left(D^i, \partial D^i\right) \to (X, A)$ is homotopic through such maps to a constant map $D^i \to A$.
(4) $\pi_i (X, A, x_0) = 0$ for all $x_0 \in A$.

When $i = 0$, we did not define the relative π_0, and (1)-(3) are each equivalent to saying that each path-component of X contains points in A since D^0 is a point and ∂D^0 is empty. The pair (X, A) is called *n-connected* if (1)-(4) hold for $0 < i \le n$ and (1)-(3) hold for $i = 0$.

2.4. **Whitehead's Theorem.**

Theorem 2.39 (Whitehead's Theorem)**.** *If a map $f \colon X \to Y$ between connected CW complexes induces isomorphisms $f_* \colon \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is stronger: X is a deformation retract of Y.*

The proof will require the following lemma:

Lemma 2.40 (Compression Lemma)**.** *Let (X, A) be a CW pair and let (Y, B) be any pair with $B \ne \varnothing$. For each n such that $X - A$ has cells of dimension n, assume that $\pi_n (Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f \colon (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$. When $n = 0$, the condition that $\pi_n (Y, B, y_0) = 0$ for all $y_0 \in B$ is to be regarded as saying that (Y, B) is 0-connected.*

Proof of lemma. Assume inductively that f has already been homotoped to take the skeleton X^{k-1} to B. Let Φ be the caracteristic (attaching) map of cell e^k of $X - A$. Then the composition $f\Phi \colon \left(D^k, \partial D^k\right) \to (Y, B)$ is in some class in $\pi_k (Y, B, y_0) = 0$, so it can be homotoped into B rel ∂D^k by the compression criterion when $k > 0$, or by (Y, B) being 0-connected for $k = 0$ (this is condition (3) in Problem 2.38). This homotopy of $f\Phi$ induces a homotopy rel X^{k-1} on the quotient space $X^{k-1} \cup e^k$ of $X^{k-1} \sqcup D^k$. Doing this for all k-cells of $X - A$ simultaneously, and taking the constant homotopy on A, we obtain a homotopy of $f|_{X^k \cup A}$ to a map into B. Since the inclusion of a subcomplex into a CW-complex is a cofibration, $f|_{X^k \cup A}$ extends to all of X (essentially the homotopy extension property). This completes the inductive step in the finite dimensional CW-complex case. In the general case, we perform the homotopy of the inductive step during the t-interval $\left[1 - \frac{1}{2^k}, 1 - \frac{1}{2^{k+1}}\right]$. Any finite skeleton X^k is eventually stationary under these homotopies, hence we have a well-defined homotopy $f_t, t \in [0, 1]$ with $f_1(X) \subset B$. $\square$

Proof of Whitehead's Theorem, 2.39. Let's tackle the case when f is the inclusion of a subcomplex first. Consider then the LES of the pair (Y, X). Since f by assumption induces isomorphisms on all homotopy groups, $f_* \colon \pi_*(X) \to \pi_*(Y)$, the relative homotopy groups $\pi_*(Y, X)$ are zero. Applying the lemma now to the identity map $(Y, X) \to (Y, X)$, we obtain a homotopy of the identity id $\colon Y \to Y$ to a map $Y \to X$ which is relative to X. That is, we obtain a deformation retract of Y onto X.

For the general case, recall that a map $f\colon X \to Y$, can be considered as the composition of the inclusion $X \hookrightarrow M_f$ and the retraction $M_f \to Y$. Since the retraction is a homotopy equivalence, it suffices to show that M_f deformation retracts onto X if f induces isomorphisms on homotopy groups, or equivalently, if the relative groups $\pi_n(M_f, X)$ are all zero (since $M_f \simeq Y$). If f is cellular - i.e., takes the n-skeleton of X to the n-skeleton of Y for all n - then (M_f, X) is a CW pair and we can apply the first paragraph of the proof.

If f is not cellular, we can either apply Theorem 4.8 in [2] which says that f is homotopic to a cellular map, or we can use the following argument.

First, using that $\pi_n(M_f, X) = 0$ for all n, apply the Compression Lemma to the inclusion $(X \cup Y, X) \hookrightarrow (M_f, X)$ to obtain a homotopy of the inclusion to a map into X rel X. The inclusion $X \cup Y \hookrightarrow M_f$ can be seen to be a cofibration using Theorem 1.6, so the pair $(M_f, X \cup Y)$ satisfies the homotopy extension property. So the homotopy in question extends to a homotopy from the identity of M_f to a map $g\colon M_f \to M_f$ taking $X \cup Y$ into X rel X. Applying the Compression lemma again to the composition

$$(X \times I \sqcup Y, X \times \partial I \sqcup Y) \to (M_f, X \cup Y) \xrightarrow{g} (M_f, X),$$

we get a deformation retraction of M_f onto X. $\qquad\square$

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