# 1. Double and Total Complexes

**Definition 1.1** (Double complex). A double complex (or bicomplex) in an abelian category  $\mathcal{A}$  is a family  $\{C_{p,q}\}$  of objects of  $\mathcal{A}$ , together with maps

$$d^h: C_{p,q} \to C_{p-1,q}$$
 and  $d^v: C_{p,q} \to C_{p,q-1}$ 

such that  $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$ .

It is useful to picture the double complex as a lattice in which the maps  $d^h$  go horizontally, the maps  $d^v$  go vertically, and each square anticommutes.

Each row  $C_{*q}$  and each columns  $C_{p*}$  is a chain complex.

We say that the double complex C is bounded if C has only finitely many nonzero terms along each diagonal line p + q = n. For example, if C is concentrated in the first quadrant of the plane (a first quadrant double complex).

1.0.1. Sign Trick. Are the maps  $d^v$  and  $d^h$  maps in Ch?

Because of anticommutativity, the chain map conditions fail, but we can construct chain maps  $f_{*q}$  from  $C_{*,q}$  to  $C_{*,q-1}$  by introducing signs:

$$f_{p,q} = (-1)^p d_{p,q}^v \colon C_{p,q} \to C_{p,q-1}.$$

Using this sign trick, we can identify the category of double complexes with the category Ch (Ch).

1.0.2. Total Complexes. To see why the anticommutativity condition  $d^v d^h + d^h d^v = 0$  is useful, we define the total complexes  $\text{Tot}(C) = Tot^{\prod}(C)$  and  $\text{Tot}^{\oplus}(C)$  as follows:

**Definition 1.2** (Total complexes). We define

$$\operatorname{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q} \text{ and } \operatorname{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

The formula  $d = d^h + d^v$  define maps

$$d \colon \operatorname{Tot}^{\prod}(C)_n = \prod_{p+q=n} C_{p,q} \text{ and } d \colon \operatorname{Tot}^{\oplus}(C)_n \to \operatorname{Tot}^{\oplus}(C)_{n-1}$$

such that  $d \circ d = 0$ , making  $\text{Tot}^{\Pi}(C)$  and  $\text{Tot}^{\oplus}(C)$  into chain complexes.

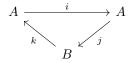
**Exercise 1.3.** Check that  $d = d^h + d^v$  define maps as claimed.

Solution. Let  $(\alpha_{p,q}) \in \text{Tot}^{\prod}(C)_n$ , so p+q=n. Then  $d((\alpha_{p,q}))=d^h((\alpha_{p,q}))+d^v((\alpha_{p,q}))=(\alpha_{p-1,q})+(\alpha_{p,q-1})\in \prod_{p+q=n-1}C_{p,q}$ . Clearly, this also works for direct products since the number of non-zero terms under d just multiplies by 2, hence is still finite. We also want to show that  $d \circ d = 0$ . For this, note that

$$\begin{split} d\circ d\left(\alpha\right) &= d\left(d^h(\alpha) + d^v(\alpha)\right) = d^h\left(d^h(\alpha) + d^v(\alpha)\right) + d^v\left(d^h(\alpha) + d^v(\alpha)\right) \\ &= d^hd^h(\alpha) + d^hd^v(\alpha) + d^vd^h(\alpha) + d^vd^v(\alpha) \\ &= 0. \end{split}$$

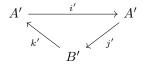
### 1.1. Exact Couples.

**Definition 1.4** (Exact Couple). An *exact couple* is an exact sequence of abelian groups of the form



where i, j and k are group homomorphisms. Define  $d: B \to B$  by  $d = j \circ k$ . Then  $d^2 = j(kj)k = 0$ , so  $H(B) := \ker d/\operatorname{im} d$  is defined - in particular, since A and B are abelian, the quotient H(B) is well-defined and a group.

**Definition 1.5** (Derived Couple). Out of a given exact couple, we can construct a new exact couple, called the *derived couple*:



where we define

- (1) A' = i(A) and B' = H(B).
- (2) i' is the induced map  $i' := i|_{A'} : A' \to A'$  by i'(ia) = i(ia)
- (3) We define j' by j'a' = [ja] where a' = ia for some a in A.
- (4) k' is defined by  $k'[b] = kb \in i(A)$ .

With these definitions, the derived couple is an exact couple.

**Exercise 1.6.** Check that the maps are well-defined and that the derived sequence is exact.

*Proof.* We must check that j' and k' are well-defined maps.

Suppose  $a'=ia=i\tilde{a}$ . Then  $a-\tilde{a}\in\ker i=\operatorname{im} k$  so  $a-\tilde{a}=k[b]$ . Hence Then  $ja-j\tilde{a}=jk[b]=d[b]\in\operatorname{im} d$ , so  $[ja]=[j\tilde{a}]$ .

Next, suppose  $[b] = [\tilde{b}]$ , so  $b - \tilde{b} \in \text{im } d$ , i.e.,  $b - \tilde{b} = jk(\bar{b})$ . Then  $kb - k\tilde{b} = kjk(\bar{b}) = 0$ , so  $k'[b] = k'[\tilde{b}]$ .

Lastly, exactness at B': suppose k'[b] = 0. Then kb = 0, so by exactness of the original exact couple, there exists some  $a \in A$  such that j(a) = b. Then let a' = i(a), so j'(a') = [j(a)] = [b], hence  $\ker k' \subset \operatorname{im} j'$ .

Conversely, k'j'(a') = k'[ja] = kja = 0, by exactness at B of the original couple.

# 1.2. The Spectral Sequence of a Filtered Complex.

**Definition 1.7** (Differential Complex). A differential complex K with differential operator D is an abelian group K together with a group homomorphism  $D: K \to K$  such that  $D^2 = 0$ .

Let K be a differential complex with differential operator D. Usually K comes with a grading  $K = \bigoplus_{k \in \mathbb{Z}} C^k$  and  $D \colon C^k \to C^{k+1}$  increases the degree by 1, but the grading is not absolutely necessary.

**Definition 1.8** (Subcomplex). A subcomplex K' of K is a graded subgroup such that  $DK' \subset K'$ .

**Definition 1.9** (Filtration, Associated Graded Complex). A sequence of subcomplexes

$$K = K_0 \supset K_1 \supset K_2 \supset K_3 \supset \dots$$

is called a filtration on K. This makes K into a filtered complex, with associated graded complex

$$GK = \bigoplus_{p=0}^{\infty} K_p / K_{p+1}.$$

For notational reasons, we usually extend the filtration to negative indices by defining  $K_p = K$  for p < 0.

**Example 1.10.** If  $K = \bigoplus K^{p,q}$  is a double complex with horizontal operator  $\delta$  and vertical operator d (which we assume to commute), we can form a single complex out of it by setting  $C^k = \bigoplus_{p+q=k} K^{p,q}$  and then letting  $K = \bigoplus C^k$  and the differential operator  $D: C^k \to C^{k+1}$  to be  $D = \delta + (-1)^p d$ . Then letting

$$K_p = \bigoplus_{i \ge p} \bigoplus_{q \ge 0} K^{i,q}$$

we obtain a filtration on K.

Suppose now that we have a general filtered complex  $K = K_0 \supset K_1 \supset ...$ , and let A be the group defined by

$$A = \bigoplus_{p \in \mathbb{Z}} K_p$$

 $A=\bigoplus_{p\in\mathbb{Z}}K_p.$  Then A is again a differential complex with operator D. Let  $i\colon A\to A$  be the inclusion  $K_{p+1} \hookrightarrow K_p$  on each p. Let B be the cokernel of  $i: A \to A$ . Then  $B = GK = \bigoplus_{p=0}^{\infty} K_p/K_{p+1}$ , and we have an exact sequence

$$0 \to A \stackrel{i}{\to} A \stackrel{j}{\to} GK \to 0.$$

#### 2. Introduction to Spectral Sequences

Consider the problem of computing the homology of the total chain complex  $T_* = \text{Tot}(E_{**})$  where  $E_{**}$  is a first quadrant double complex.

Firstly, it is convenient to forget the horizontal differentials and add a superscript zero, retaining only the vertical differentials  $d^v$  along the columns  $E_{p*}^0$ .

Let  $E_{pq}^1$  be the vertical homology  $H_q\left(E_{p*}^0\right)$  at the (p,q) spot.

# 3. Filtrations

**Definition 3.1** (Filtered *R*-module). A filtered *R*-module is an *R*-module *A* with an increasing sequence of submodules  $\{F_p\}_{p\in\mathbb{Z}}$  such that  $F_pA \subset F_{p+1}A$  for all p and such that  $\bigcup_p F_pA = A$  and  $\bigcap_p F_pA = \{0\}$ .

A filtration is said to be bounded if  $F_pA = \{0\}$  for p sufficiently small and  $F_pA = A$  for p sufficiently larger.

**Definition 3.2** (Associated graded module). The associated graded module is defined by  $G_pA = F_pA/F_{p-1}A$ .

**Definition 3.3** (Filtered chain complex). A filtered chain complex is a chain complex  $(C_*, \partial)$  together with a filtration  $\{F_pC_i\}_{p\in\mathbb{Z}}$  of each  $C_i$  such that the differential preserves the filtration, i.e., s.t.  $\partial (F_pC_i) \subset F_pC_{i-1}$ .

Note that we, in particular, obtain an induced differential  $\partial: G_pC_i \to G_pC_{i-1}$  by the universal property of cokernels

$$F_{p}C_{i} \xrightarrow{\partial} F_{p}C_{i-1}$$

$$\downarrow \qquad \qquad \downarrow$$

$$F_{p-1}C_{i} \xrightarrow{\partial} F_{p-1}C_{i-1}$$

$$\downarrow^{\text{coker}} \qquad \downarrow^{\text{coker}}$$

$$G_{p}C_{i} \xrightarrow{\cdots} G_{p}C_{i-1}$$

so we obtain an associated graded chain complex  $G_pC_*$ .

The filtration on  $C_*$  also induces a filtration on the homology of  $C_*$  by

$$F_p H_i(C_*) = \{ \alpha \in H_i(C_*) \mid (\exists x \in F_p C_i) : \alpha = [x] \}.$$

This filtration has associated graded pieces  $G_pH_i(C_*)$  which, in favorable cases, determine  $H_i(C_*)$ .

3.1. **Example.** Suppose we have a chain complex  $C_*$  and a filtration consisting of a single  $F_0C_*$ , so  $F_nC_*=0$  if n<0 and  $F_nC_*=F_0C_*$  if  $n\geq 0$ . Then  $G_nC_*=0$  for  $n\neq 0$  and  $G_0C_*=F_0C_*$  and