Assignment 1

Jonas Trepiakas - jtrepiakas@berkeley.edu - Student ID: 3039733855

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1.: Verify each of the following for arbitrary subsets A, B of a space X :

(a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(c) $\overline{\overline{A}} = \overline{A}$.
(d) $(A \cup B)^{\circ} \supseteq A^{\circ} \cup B^{\circ}$.

Solution:
(a) Let x be a limit point of $A \cup B$. If any neighborhood of x intersects $A - \{x\}$, then $x \in \overline{A} \subset \overline{A} \cup \overline{B}$, so assume there exists a neighborhood N_A of x that does not intersect $A - \{x\}$. Assume there exists a neighborhood N_B of x that does not intersect $B - \{x\}$. Then $x \in N_A \cap N_B$ as $x \in N_A$ and $x \in N_B$ and $N_A \cap N_B$ is a neighborhood of x by definition 1.3.(b).
Assume $y \in (N_A \cap N_B) \cap A \cup B - \{x\}$. Then $y \in N_A$ so since $N_A \cap A - \{x\} = \varnothing$, we have $y \notin A - \{x\}$. Similarly, since $y \in N_B$, we have $y \notin B - \{x\}$ as $N_B \cap B - \{x\} = \varnothing$.
Hence $y \notin A - \{x\} \cup B - \{x\} = A \cup B - \{x\}$ which contradicts $y \in (N_A \cap N_B) \cap A \cup B - \{x\}$. Thus $(N_A \cap N_B) \cap A \cup B - \{x\} = \varnothing$.
But then $N_A \cap N_B$ is a neighborhood of x that does not intersect $A \cup B - \{x\}$ contradicting $x \in \overline{A \cup B}$. Thus no such N_B can exist, so all neighborhood of x intersect $B - \{x\}$, hence $x \in \overline{B} \subset \overline{A} \cup \overline{B}$.

Conversely, if $x \in \overline{A} \cup \overline{B}$ then $x \in \overline{A}$ or $x \in \overline{B}$. Assume without loss of generality that $x \in \overline{A}$. Then any neighborhood of x intersects $A - \{x\} \subset A \cup B - \{x\}$, so $x \in \overline{A \cup B}$.

(c) By theorem 2.3, $\overline{A}$ is a closed set. Then by another use of theorem 2.3, $\overline{\overline{A}}$ is the smallest closed set containing $\overline{A}$ which is thus $\overline{A}$ as it is closed, so $\overline{\overline{A}} = \overline{A}$.
Alternatively, we can note: $x \in \overline{\overline{A}}$ if and only if x is a limit point of $\overline{A}$, and this happens if and only if $x \in \overline{A}$ since $\overline{A}$ is closed by theorem 2.3 and thus contains all its limit points by theorem 2.2.

(d) Let $x \in A^{\circ} \cup B^{\circ}$, then assume without loss of generality that $x \in A^{\circ}$. By definition, x is thus contained in the union of all open sets contained in A and hence is contained in some open set, say U, contained in A. Then $x \in U \subset A \subset A \cup B$, and by definition $U \subset (A \cup B)^{\circ}$, so $x \in (A \cup B)^{\circ}$.
To show that equality need not hold, consider the set $X = \{a, b\}$ with the indiscrete topology, i.e. $\tau = \{\varnothing, X\}$. Then let $A = \{a\}, B = \{b\}$. We have $(A \cup B)^{\circ} = X^{\circ} = X$, so $a \in (A \cup B)^{\circ}$. However, $A^{\circ} = \varnothing$ and $B^{\circ} = \varnothing$, so $a \notin A^{\circ} \cup B^{\circ}$.

3: Specify the interior, closure, and frontier of each of the following subsets of the plane:

(a) $A = \{(x, y) \mid 1 < x^2 + y^2 \leq 2\}$
(b) $B = \mathbb{E}^2$ with both axes removed
(c) $C = \mathbb{E}^2 - \{(x, \sin(\frac{1}{x})) \mid x > 0\}$.

Solution:
Lemma 1: Let $J = \{(a, b) \mid a, b \in \mathbb{E}\}$ with the convention that if $b \leq a$ then $(a, b) = \varnothing$. Then $J \times J$ is a basis for $\mathbb{E}^2$.

Proof: We first show that $J \times J$ determines a basis for a topology on $\mathbb{E}^2$ and then show that this topology coincides with the standard topology on $\mathbb{E}^2$.

$J \times J$ is clearly nonempty. Let $(a_1, b_1) \times (c_1, d_1), \ldots, (a_n, b_n) \times (c_n, d_n)$ be a finite number of members of $J \times J$. Then

$$(a_1, b_1) \times (c_1, d_1) \cap \ldots \cap (a_n, b_n) \times (c_n, d_n) = (\max\{a_i\}, \min\{b_i\}) \times (\max\{c_i\}, \min\{d_i\}) \in J$$

where we again remember the if say $\min\{b_i\} \leq \max\{a_i\}$ then the intersection becomes the empty set; similarly, for c_i and d_j.
Furthermore, for any $(x, y) \in \mathbb{E}^2$, we have $(x, y) \in (x - 1, x + 1) \times (y - 1, y + 1) \in J \times J$, so by theorem 2.5, we get that $J \times J$ determines a topology on $\mathbb{E}^2$.

Now let U be any open set in the standard topology on $\mathbb{E}^2$ and let $x \in U$.

Then by definition p. 28, we can find $\varepsilon_x > 0$ such that $B(x,\varepsilon_x) \subset U$. Then writing $x = (x_1, x_2)$, we have $V_x = \left(x_1 - \frac{\varepsilon_x}{\sqrt{2}}, x_1 + \frac{\varepsilon_x}{\sqrt{2}}\right) \times \left(x_2 - \frac{\varepsilon_x}{\sqrt{2}}, x_2 + \frac{\varepsilon_x}{\sqrt{2}}\right) \subset B(x, \varepsilon_x)$ since for any $(a, b) \in V$, we have $\|(x_1, x_2) - (a, b)\| < \sqrt{\frac{\varepsilon_x^2}{2} + \frac{\varepsilon_x^2}{2}} = \varepsilon_x$. Hence $x \in V_x \subset B(x, \varepsilon_x) \subset U$. As x was arbitrary, we can find such a V_x for any $x \in U$. Thus $U = \bigcup_{x \in U}\{x\} \subset \bigcup_{x\in U} V_x \subset U$, so $U = \bigcup_{x\in U} V_x$. Denoting the standard topology on $\mathbb{E}^2$ by τ and the topology induced by the basis $J \times J$ by τ_J, we thus have $\tau \subset \tau_J$.

Conversely, for any open set $U \in \tau_J$, we thus have $U = \bigcup_{i\in I}(a_i, b_i) \times (c_i, d_i)$. Let $x = (x_1, x_2) \in U$. Then there exists $i_x \in I$ such that $x \in (a_{i_x}, b_{i_x}) \times (c_{i_x}, d_{i_x})$.
Then for $\varepsilon_x = \min\{|b_{i_x} - x_1|, |a_{i_x} - x_1|, |d_{i_x} - x_2|, |c_{i_x} - x_2|\}$, we have $B(x, \varepsilon_x) \subset (a_{i_x}, b_{i_x}) \times (c_{i_x}, d_{i_x}) \subset U$. Since x was arbitrary, we can write $U = \bigcup_{x\in U}\{x\} \subset \bigcup_{x \in U} B(x, \varepsilon_x) \subset U$, so $U = \bigcup_{x\in X} B(x, \varepsilon_x)$. Thus $\tau_J \subset \tau$. Thus $J \times J$ is a basis for the standard topology on $\mathbb{E}^2$.

Lemma 2: For any $A \subset X$, we have $\partial A = \overline{A} - A^\circ$, where ∂A is the frontier of A : $\partial A = \overline{A} \cap \overline{X - A}$.

Proof: We have $x \in \overline{A}$ if and only if any neighborhood of x intersects $A - \{x\}$ if and only if for any neighborhood N of x, either $x \in N \subset A$ or $N \cap A - \{x\} \neq \varnothing \neq N \cap (X - A - \{x\})$. Now the last statement gives that if there exists N such that $x \in N \subset A$, then by definition of a neighborhood, there exists an open set U such that $x \in U \subset N \subset A$ and hence $x \in U \subset A^\circ$; if no such N exists, then for all neighborhoods N, we have $N \cap A - \{x\} \neq \varnothing \neq N \cap (X - A - \{x\})$ which is equivalent to $x \in \overline{A} \cap \overline{X - A} = \partial A$. Hence $x \in \overline{A}$ if and only if $x \in A^\circ \cup \partial A$ and furthermore, these two cases are disjoint: if $x \in A^\circ$ then $x \notin \overline{X - A} \supset \partial A$. And conversely by contraposition: $x \in \partial A$ implies $x \notin A^\circ$. Therefore $\partial A = \overline{A} - A^\circ$.

(a) We claim that $A^\circ = \{(x, y) \mid 1 < x^2 + y^2 < 2\}$. Let $\tilde{A}$ denote the right hand side.
For $(x, y) \in \tilde{A}$, we have that for $\varepsilon = \min\{\sqrt{2} - \|(x, y)\|, \|(x, y)\| - 1\}$, $B((x, y), \varepsilon) \subset \tilde{A} \subset A$. Hence $\tilde{A}$ is open, so $\tilde{A} \subset A^\circ$.
For $(x, y) \in A$ with $\|(x, y)\|^2 = 2$, we have that if $(x, y) \in A^\circ$ then there exists an open neighborhood U of (x, y) such that $(x, y) \in U \subset A$. However, since $J \times J$ is a basis for $\mathbb{E}^2$ there must thus exist $(a, b) \times (c, d)$ such that $(x, y) \in (a, b) \times (c, d) \subset U \subset A$, but then for $x' = \begin{cases} \frac{x+b}{2}, & x \geq 0 \\ \frac{x+a}{2}, & x < 0 \end{cases}$ and $y' = \begin{cases} \frac{y+d}{2}, & y \geq 0 \\ \frac{y-c}{2}, & y < 0 \end{cases}$, we have $\|(x', y')\| > 2$ but $(x', y') \in (a, b) \times (c, d) \subset U \subset A$, contradiction. Hence $(x, y) \notin A^\circ$, so $A^\circ \subset \tilde{A}$. Thus $A^\circ = \tilde{A}$.

We claim the closure of A is $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 2\}$. Let again $\tilde{A}$ denote the right hand side. By theorem 2.3, the closure of A is the smallest closed set containing A, hence $A \subset \overline{A}$. It remains to show that all (x, y) with $\|(x, y)\| = 1$ are limit points of A and that all points (x, y) with $\|(x, y)\| < 1$ or > 2 are not limit points of A.

Let (x, y) be such that $\|(x, y)\| = 1$ and let U be any open neighborhood of (x, y). Then we can find a $\varepsilon > 0$ such that $B((x, y), \varepsilon) \subset U$, and in this ball we can find some (x', y') with $1 < \|(x', y')\| \leq 2$. To be specific, there exists a basis element $(a, b) \times (c, d) \in J \times J$ with $(x, y) \in (a, b) \times (c, d) \subset U$, and again letting say $x' = \begin{cases} \frac{x+\min\{\frac{1}{4}, b\}}{2}, & 0 \leq x \\ \frac{x+\max\{-\frac{1}{4}, a\}}{2}, & 0 > x \end{cases}$ and $y' = \begin{cases} \frac{y+\min\{\frac{1}{4}, d\}}{2}, & 0 \leq y \\ \frac{y+\max\{-\frac{1}{4}, c\}}{2}, & 0 > y \end{cases}$ or anything that is simply "close" to (x, y) in $(a, b) \times (c, d)$ while having greater norm which does not exceed $\sqrt{2}$- which is possible to find as $(a, b) \times (c, d)$ is open -, we get $(x', y') \in (a, b) \times (c, d) \subset U$ while $1 < \|(x', y')\| \leq \sqrt{2}$ and hence U intersects A. Thus $\tilde{A} \subset \overline{A}$.

Now, for points (x, y) with $\|(x, y)\| < 1$, we have $(x, y) \in B((0, 0), 1)$ which is open and thus a neighborhood of (x, y) that is disjoint from A. Thus $\overline{A} \subset \mathbb{E}^2 - B((0, 0), 1)$. For points (x, y) with $\|(x, y)\| > \sqrt{2}$, we have that $(x, y) \in \mathbb{E}^2 - \overline{B((0, 0), \sqrt{2})}$ which is open as its compliment is $\overline{B((0, 0), \sqrt{2})}$ which is closed. But $A \subset \overline{B((0, 0), \sqrt{2})}$, and thus $\mathbb{E}^2 - \overline{B((0, 0)\sqrt{2})}$ is an open neighborhood that is disjoint from A, so (x, y) is not a limit point of A. Thus $\overline{A} \subset \overline{B((0, 0), \sqrt{2})} - B((0, 0), 1) = \tilde{A} \subset A$, hence

$\overline{A} = \tilde{A}$.

Now by lemma 2, $\partial A = \overline{A} - A^\circ = \{(x, y) \mid x^2 + y^2 \in \{1, 2\}\}$.

(b) We claim the interior is $B^\circ = B$. For any point (x, y), we have that $(x, y) \in B\left((x, y), \frac{\min\{|x|,|y|\}}{2}\right) \subset B$, thus B is open, so $B^\circ = B$.

The closure of B is all of $\mathbb{E}^2$: the closure contains B by theorem 2.3. Now let $(0, x) \in \mathbb{E}^2$ for any $x \in \mathbb{E}$. Then for any neighborhood N of $(0, x)$, we can find a basis element $(a, b) \times (c, d)$ such that $(0, x) \in (a, b) \times (c, d) \subset N$, so letting $y' = x$ if $x \neq 0$ and otherwise $y' = \frac{d}{2}$, and letting $x' = \frac{b}{2}$, we get $(x', y') \in (a, b) \times (c, d) \subset N$ and $(x', y') \in B$, hence any neighborhood of $(0, x)$ intersects B so $\{0\} \times \mathbb{E} \subset \overline{B}$. Equivalently, let $(x, 0) \in \mathbb{E}^2$ for any $x \in \mathbb{E}$. Then for any neighborhood N of $(x, 0)$, we can find a basis element $(a, b) \times (c, d)$ such that $(x, 0) \in (a, b) \times (c, d) \subset N$, so letting $x' = x$ if $x \neq 0$ and $x' = \frac{b}{2}$ otherwise, and $y' = \frac{d}{2}$, we get $(x', y') \in (a, b) \times (c, d) \subset N$ and $(x', y') \in B$, so any neighborhood of $(x, 0)$ intersects B hence $\mathbb{E} \times \{0\} \subset \overline{B}$, thus $\mathbb{E}^2 = B \cup \{0\} \times \mathbb{E} \cup \mathbb{E} \times \{0\} \subset \overline{B} \subset \mathbb{E}^2$ and hence $\overline{B} = \mathbb{E}^2$.

By lemma 2, we now get $\partial B = \overline{B} - B^\circ = \mathbb{E} \times \{0\} \cup \{0\} \times \mathbb{E}$.

(c) We claim that the interior of C is $C - \{(x, y) \mid x = 0, |y| \leq 1\}$.
Let $f\colon (0, \infty) \to \mathbb{E}$ be given by $f(x) = \sin(\frac{1}{x})$. This is continuous as the composition of continuous functions. Let $(x, y) \in C$ with $x > 0$. We can by continuity find δ such that $B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))$ for any $\varepsilon > 0$. Let $\varepsilon = \frac{|y - f(x)|}{2}$ (where $|y - f(x)| > 0$ since $(x, y) \notin \mathbb{E}^2 - C$). Then $B((x, y), \min\{\varepsilon, \delta\}) \subset C$ since if $(x', y') \in B((x, y), \min\{\varepsilon, \delta\}) \cap \mathbb{E}^2 - C$, then $|x - x'| \leq d((x, y), (x', y')) < \min\{\delta, \varepsilon\} \leq \delta$, and hence

$$d((x, y), (x', y')) \geq |y - y'| = |y - f(x')| \geq |y - f(x)| - |f(x) - f(x')| = \varepsilon,$$

which is a contradiction.

For $x = 0$ and $|y| > 1$, we can naturally choose $\varepsilon = |y| - 1$ and get $B((x, y), \varepsilon) \subset C$ since any point $(x', f(x')) \in \mathbb{E}^2 - C$ has $|f(x')| \leq 1$.

For $x < 0$, we can choose $\varepsilon = |x|$. Then $B((x, y), \varepsilon) \subset C$ since any $(x', f(x')) \in \mathbb{E}^2 - C$ has $x' > 0$.
It remains to show that the points $\{(0, y) \mid |y| \leq 1\}$ are limit points of $\mathbb{E}^2 - C$.
Take any neighborhood N of a fixed point $(0, y)$ with $|y| \leq 1$ and take a basis element $(a, b) \times (c, d) \subset N$ containing $(0, y)$. Then since sin is periodic with period 2π, we can find $N \in \mathbb{R}_+$ such that $\frac{1}{N} < b$ and $\sin\left(\frac{1}{N}\right) = y$. But then $\left(\frac{1}{N}, y\right) \in (a, b) \times (c, d) \cap \mathbb{E}^2 - C$, and thus any neighborhood of $(0, y)$ intersects $\mathbb{E}^2 - C$, so we get $C^\circ = C - \{(x, y) \mid x = 0, |y| \leq 1\}$.

Lemma 3: If $A \subset X$ then $\overline{X - A} = X - A^\circ$.
Proof: $x \in \overline{X - A}$ if and only if any neighborhood of x intersects $X - A$ if and only if there does not exist a neighborhood N of x such that $x \in N \subset A$ if and only if $x \in X - A^\circ$.

Thus $\overline{C} = \mathbb{E}^2 - \{(x, \sin(\frac{1}{x}) \mid x > 0\}^\circ$. For any point $(x', y') \in \{(x, \sin(\frac{1}{x}) \mid x > 0\}$, take a neighborhood N of it and let $(a, b) \times (c, d)$ be a basis element with $(x', y') \in (a, b) \times (c, d) \subset N$. Then choose any $y \in (c, d) - \{y'\}$. Now $(x', y) \in (a, b) \times (c, d) \subset N$, but $(x', y) \notin \{(x, \sin(\frac{1}{x})) \mid x > 0\}$, so $(x', y') \notin \{(x, \sin(\frac{1}{x})) \mid x > 0\}^\circ$, hence $\{(x, \sin(\frac{1}{x})) \mid x > 0\}^\circ = \varnothing$, so $\overline{C} = \mathbb{E}^2$.

Now by lemma 2,

$$\partial C = \overline{C} - C^\circ = (\mathbb{E}^2 - C) \cup \{(x, y) \mid x = 0, |y| \leq 1\} = \left\{\left(x, \sin(\frac{1}{x})\right) \mid x > 0\right\} \cup \{(x, y) \mid x = 0, |y| \leq 1\}$$

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12: Show that if X has a countable base for its topology, then X contains a countable dense subset.

Solution: Let $\{\beta_i\}_{i \in I}$ be a basis for the topology on X with I countable. Now for each $i \in I$, choose an

$x_i \in \beta_i$. We claim that the set $\{x_i\}_{i\in I}$ is a countable dense subset of X.

Assume $x \notin \overline{\{x_i\}_{i\in I}}$. Then there exists a neighborhood N of x such that N does not contain any x_i for $i \in I$. By definition of a neighborhood, there exists an open set U such that $x \in U \subset N$. Now since U is open and $\{\beta_i\}_{i\in I}$ is a basis for the topology on X, we can write $U = \bigcup_{j\in J} \beta_j$ for some $J \subset I$. But then for all $j \in J$, $x_j \in U \subset N$, so N contains a point x_i with $i \in I$, contradiction. Thus no such N exists, so $x \in \overline{\{x_i\}_{i\in I}}$ and as x was arbitrary, we get $X = \overline{\{x_i\}_{i\in I}}$.