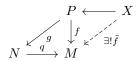
Exercise 0.1 (18). Show that every free module is projective.

*Proof.* Suppose F is a free R-module on a set X. Let  $f: F \to M$  be a map and  $q: N \to M$  a surjective map. We have the diagram



We define a map  $X \to N$  by sending  $x \in X$  to some  $n \in N$  such that  $q(n) = \tilde{f}(x)$  this exists by surjectivity of q. Then this extends to a unique map  $g \colon P \to N$  such that  $q \circ g \circ \iota(x) = \tilde{f}(x)$  for all  $x \in X$ . But this extends to a unique map  $\tilde{g} \colon P \to M$  such that  $\tilde{g} \circ \iota = q \circ g \circ \iota = \tilde{f}$ . Since  $q \circ g$  is the unique map for the first equality, we obtain  $\tilde{g} = q \circ g$ . But for  $\tilde{g} \circ \iota = \tilde{f}$ , we obtain again by uniqueness that  $\tilde{g} = f$ . Hence  $f = q \circ g$ .

**Exercise 0.2** (19). Suppose R is a PID and let F be a projective R-module. Show that F is free.

*Proof.* Since F is projective, F is a direct summand in a free R-module, hence in particular, it is finitely generated. Hence  $\bigoplus_{i\in I} R \approx F \oplus K$  and by the structure theorem, also  $F \approx R^n \oplus \bigoplus_{i=1}^r \bigoplus_{j=1}^{s_i} R/\left(p_i^{m_{ij}}\right)$ . But  $\bigoplus_{i\in I} R$  has no torsion, so F must also be torsion-free, hence  $F \approx R^n$ , i.e, F is free.

**Exercise 0.3** (20). Let M be an R-module. We say that M is torsion-free if whenver rm = 0 for  $r \in R$  and  $m \in M$ , then either r = 0 or m = 0.

- (1) Show that R is torsion-free when considered as a module over itself if and only if R is an integral domain.
- (2) Let R be an integral domain. Show that any projective R-module is torsion-free.

*Proof.* (1) Saying that R is torsion-free as a module over itself is the same as saying rr' = 0 for  $r \in R$  and  $r' \in R$  if and only if r = 0 or r' = 0 which is precisely the same as saying that R is an integral domain.

- (2) If P is a projective R-module, then it is a direct summand in a free R-module. But any free R-module is torsion-free, so P must be as well.
- (3)  $\mathbb{Q}$  as a  $\mathbb{Z}$ -module is torsion-free, however  $\mathbb{Q}$  is not projective over  $\mathbb{Z}$  since the sequence  $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$  is exact, but

$$0 \to \underbrace{\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \mathbb{Z}\right)}_{=0} \to \underbrace{\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \mathbb{Q}\right)}_{\approx \mathbb{Q}} \to \underbrace{\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}\right)}_{=0} \to 0$$

is not exact.

**Exercise 0.4** (21). Let P be a projective R-module. Prove that there exists a free R-module F such that  $P \oplus F$  is free.

*Proof.* P is a quotient of a free R-module, T, say P = T/A. Also, P is a direct summand of a free R-module  $\overline{F}$ , say  $\overline{F} = P \oplus S$ . Then

$$P \oplus \left( \oplus_{i \in \mathbb{N}} \overline{F}_i \right) \approx P \oplus \left( \oplus_{i \in \mathbb{N}} S_i \oplus P_i \right) \approx \oplus_{i \in \mathbb{N}} P \oplus S \approx \oplus_{i \in \mathbb{N}} \overline{F}.$$