PROBLEM SHEET

1. Geometric Representations

Problem 1.1 (Easy)**.** True or false: the crosscap transposition representation is induced by a Yang-Baxter operator on some category of surfaces.

Problem 1.2 (Easy)**.** True or false: the standard twist representations is induced by a Yang-Baxter operator on some category of surfaces.

Problem 1.3 (There is an easy and a harder way to do it)**.** Show that for g odd, the standard twist representation $\rho_C \colon \mathcal{B}_g \to \mathrm{MCG}\left(N_{g,b}\right)$ is the same as the Birman-Hilden embedding $\mathcal{B}_g \hookrightarrow S_{\frac{g-1}{2},b-1} \# M$ into the orientable factor.

Problem 1.4. Can you extend the previous problem to the case when g is even?

Problem 1.5. Experiment yourself by constructing your own geometric representations from Yang-Baxter operators on the category of decorated surfaces.

Problem 1.6 (Medium)**.** Read up on the category of bidecorated surfaces. See if you can recover the Birman-Hilden embedding in this category from a Yang-Baxter operator.

2. Mapping Class Group Examples

Problem 2.1 (Easy (might require algtop))**.** Give an example of a surface S of finite type and a self-diffeomorphism φ of S such that φ is homotopic to id_S but not isotopic to id_S.

Proposition 2.2. *Any two essential simple proper arcs in $S_{0,3}$ with the same endpoints are isotopic. Any two essential arcs that both start and end at the same marked point of $S_{0,3}$ are also isotopic.*

Problem 2.3. Given the previous proposition, find an isomorphism $\mathrm{MCG}\left(S_{0,3}\right) \cong \Sigma_3$

Problem 2.4. Show similarly that $\mathrm{MCG}\left(S_{0,3}\right) \cong \mathbb{Z}/2$.

3. Birman-Hilden

Problem 3.1. Prove the Birman exact sequence

$$1 \to \pi_1\left(S,x\right) \stackrel{\mathcal{P}ush}{\to} \mathrm{MCG}(S,x) \stackrel{\mathcal{F}orget}{\to} \mathrm{MCG}(S) \to 1$$

This generalizes to an exact sequence (you don't have to prove this)

$$1 \to \pi_1\left(C\left(S,n\right)\right) \stackrel{\mathcal{P}ush}{\to} \mathrm{MCG}\left(S,\{x_1,\ldots,x_n\}\right) \stackrel{\mathcal{F}orget}{\to} \mathrm{MCG}(S) \to 1$$

What do you obtain when we let $S = D^2$?
Let ι be the hyperelliptic involution of $S_{g,1}$. Let

$$\mathrm{SHomeo}^+\left(S_{g,1}\right) = C_{\mathrm{Homeo}^+\left(S_{g,1},\partial S_{g,1}\right)}\left(\iota\right)$$

Define

$$\mathrm{SMCG}\left(S_{g,1}\right) = \mathrm{SHomeo}^{+}\left(S_{g,1}\right)/\mathrm{isotopy}$$

Suppose we are given that two symmetric homeomorphisms are isotopic if and only if they are symmetrically isotopy, i.e., isotopic in $\mathrm{SHomeo}^{+}\left(S_{g,1}\right)$. Derive the Birman-Hilden theorem:

$$\mathrm{SMCG}\left(S_{g,1}\right) \cong B_{2g+1}.$$