# HOMOTOPY THEORY

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For these notes, we will follow [2], [1] and [3].

## 1. The Compact-Open Topology

Recall that $Y^X$ denotes the *set* of *continuous* functions $X \to Y$.

**Definition 1.1.** The *compact-open topology* on $Y^X$ is the topology generated by the sets $M(K,U) = \left\{ f \in Y^X \mid f(K) \subset U \right\}$ where $K \subset X$ is compact and $U \subset Y$ is open.

Generated here means that these sets form a *subbasis* for the open sets.

**Lemma 1.2.** *Let $\mathcal{K}$ be a collection of compact subsets of $X$ containing a neighborhood base at each point of $X$. Let $\mathcal{B}$ be a subbasis for the open sets of $Y$. Then the collection*

$$\{M(K,U) \mid K \in \mathcal{K}, B \in \mathcal{B}\}$$

*forms a subbasis for the compact-open topology on $Y^X$.*

*Proof.* Recall first that a subbasis is a collection whose union is the whole space and such that the collection of finite intersections of elements of the subbasis form a basis.
In particular, noting that $M(K,U) \cap M(K,V) = M(K, U \cap V)$, this implies that it suffices to consider the case when $\mathcal{B}$ is a basis.
So to show that the collection in question is a subbasis, it suffices to show that given $f \in M(K,U)$, there exist $K_1, \ldots, K_n \in \mathcal{K}$ and $U_1, \ldots, U_n \in \mathcal{B}$ such that $f \in \bigcap_{i=1}^n M(K_i, U_i) \subset M(K,U)$.
For each $x \in K$, there is an open set $U_x \in \mathcal{B}$ with $f(x) \in U_x \subset U$ (since $\mathcal{B}$ was assumed to be a basis), and there exists a neighborhood $K_x \in \mathcal{K}$ of $x$ such that $f(K_x) \subset U_x$ (since $f$ is continuous and $\mathcal{K}$ was assumed to contain a neighborhood base at each point of $X$ ). Thus $f \in M(K_x, U_x)$. Now, covering $K$ with these sets $K \subset \bigcup_{x \in K} K_x$. By compactness of $K$, there exists a finite subcover $K \subset K_{x_1} \cup \ldots \cup K_{x_n}$. Then $f \in \bigcap_{i=1}^n M(K_{x_i}, U_{x_i}) \subset M(K,U)$. □

**Proposition 1.3.** *For $X$ locally compact Hausdorff, the "evaluation map" $e \colon Y^X \times X \to Y$, defined by $e(f,x) = f(x)$, is continuous.*

*Proof.* Let $(f,x) \in Y^X \times X$ and $U$ a neighborhood of $f(x) \in Y$. Now we make use of the following lemma:

**Lemma 1.4.** *If $X$ is a locally compact Hausdorff space, then each neighborhood of a point $x \in X$ contains a compact neighborhood of $X$. In particular, $X$ is completely regular.*

*Proof.* Let $C$ be a compact neighborhood of $x$ and $U$ an arbitrary neighborhood of $x$. Since $X$ is Hausdorff, $C$ is closed, so $(X-U) \cap C$ is a closed subspace of $C$, hence compact. Now, for each point $z \in (X-U) \cap C$, choose, by Hausdorffness, open neighborhoods $U_z', V_z'$ of $z$ and $x$, respectively, and consider $W' := \bigcup_{z \in (X-U) \cap C} U_z'$. Since this is open, $C - W'$ is closed hence compact. Furthermore, it is contained in $U$ and contains $x$.

*Alternative proof due to Bredon:* Let $C$ be a compact neighborhood of $x$ and $U$ an arbitrary neighborhood of $x$. Let $V \subset C \cap U$ be open with $x \in V$. Then $\overline{V} \subset C$ is compact Hausdorff, hence regular, so there exists a neighborhood $N \subset V$ of $x$ in $C$ which is closed in $\overline{V}$ and hence closed in $X$. Since $N$ is closed in the compact space

$C$, it is compact. Since $N$ is a neighborhood of $x$ in $\overline{V}$ and since $N = N \cap V$, $N$ is a neighborhood of $x$ in the open set $V$ and hence in $X$. $\square$

By Lemma 1.4, there exists a compact neighborhood $K$ of $x$ such that $f(K) \subset U$. Hence $f \in M(K,U)$, and $e(M(K,U) \times K) \subset U$. This finishes the proof. $\square$

**Theorem 1.5.** *Let $X$ be locally compact Hausdorff and $Y$ and $T$ arbitrary Hausdorff spaces. Given a function $f\colon X \times T \to Y$, define, for each $t \in T$, the function $f_t\colon X \to Y$ by $f_t(x) = f(x,t)$. Then $f$ is continuous if and only if both of the following conditions hold:*

1. *Each $f_t$ is continuous*
2. *The function $T \to Y^X$ taking $t \mapsto f_t$ is continuous.*

*Proof.* The "if" implication follows from the fact that $f$ is the composition

$$X \times T \overset{(x,t)\mapsto(f_t,x)}{\longrightarrow} Y^X \times X \overset{e}{\to} Y.$$

Now the evaluation map is continuous by Proposition 1.3 since $X$ is assumed to be locally compact Hausdorff and since $f_t$ is assumed to be continuous for all $t$ by condition (1); and $(x,t) \mapsto (f_t, x)$ is continuous since $t \mapsto f_t$ is assumed to be continuous by condition (2).
Conversely, for the "only if" implication, (1) follows from the fact that $f_t$ is the composition

$$X \overset{x\mapsto(x,t)}{\to} X \times T \overset{f}{\to} Y.$$

To prove (2), let $t \in T$ be given and $f_t \in M(K,U)$. It suffices to find a neighborhood $W$ of $t$ in $T$ such that $t' \in W$ implies that $f_{t'} \in M(K,U)$ (i.e., it suffices to prove conditions for continuity for a subbasis only). For $x \in K$, there are open neighborhoods $V_x \subset X$ of $x$ and $W_x \subset T$ of $t$ such that $f(V_x \times W_x) \subset U$. By compactness, $K \subset V_{x_1} \cup \ldots \cup V_{x_n} =: V$ for some $V_{x_i}$. Let $W = \bigcap_{i=1}^n W_{x_i}$. Then $f(K \times W) \subset f(V \times W) \subset U$. So $t' \in W$ implies that $f_{t'} \in M(K,U)$ as claimed. $\square$

*Note.* This theorem implies that a homotopy $X \times I \to Y$ with $X$ locally compact is the same thing as a path $I \to Y^X$ when we give $Y^X$ the compact-open topology.

*Note.* This is precisely the reason why, when we define $\mathrm{MCG}(X)$, we define it as $\pi_0 \operatorname{Homeo}^+(X, \partial X)$ where we equip $\operatorname{Homeo}^+(X, \partial X)$ with the subspace topology inherited from $X^X$ in the compact-open topology. By the above theorem, a path $I \to \operatorname{Homeo}^+(X, \partial X)$ given as $t \mapsto \gamma_t$ is continuous if and only if the associated function $\gamma\colon X \times I \to X$ given by $\gamma(x,t) = \gamma_t(x)$ is continuous. But since each $\gamma_t$ is a self-homeomorphism of $X$, this just tells us that $\gamma$ is an isotopy of $X$. So path components of $\operatorname{Homeo}^+(X, \partial X)$ correspond to isotopy classes of orientation-preserving self-homeomorphisms of $X$ fixing the boundary point-wise.

**Theorem 1.6** (The Exponential Law)**.** *Let $X$ and $T$ be locally compact Hausdorff spaces and let $Y$ be an arbitrary Hausdorff space. Then there is the homeomorphism*

$$Y^{X \times T} \overset{\cong}{\to} \left(Y^X\right)^T$$

*taking $f \mapsto f^*$ where $f^*(t)(x) = f(x,t) = f_t(x)$.*

*Proof.* By Theorem 1.5, the assignment $f \mapsto f^*$ is a bijection.
We must show it and its inverse to be continuous. Let $U \subset Y$ be open and $K \subset X, K' \subset T$ be compact. Then

$$\begin{aligned} f \in M(K \times K', U) &\iff (t \in K', x \in K \implies f_t(x) = f(x,t) \in U) \\ &\iff (t \in K' \implies f_t \in M(K,U)) \\ &\iff f^* \in M(K', M(K,U)). \end{aligned}$$

Now, the $K \times K'$ are compact subsets of $X \times T$, and the collection of all these over $X \times T$ contain a neighborhood basis at each point since $X$ and $T$ are both assumed to be locally compact. By Lemma 1.2, the collection

$$\{M(K \times K', U) \mid U \subset Y \text{ open}, K \subset X, K' \subset T \text{ both compact}\}$$

forms a subbasis for the compact-open topology on $Y^{X \times T}$. Also, the $M(K,U)$ give a subbasis for $Y^X$ and therefore the $M(K', M(K,U))$ form a subbasis for the topology on $\left(Y^X\right)^T$. Since we showed that these subbases correspond to one another under the exponential correspondence, the theorem is proved. □

**Proposition 1.7.** *If $X$ is locally compact Hausdorff and $Y$ and $W$ are Hausdorff, then there is the homeomorphism*

$$Y^X \times W^X \xrightarrow{\cong} (Y \times W)^X$$

*given by $(f,g) \mapsto f \times g$.*

*Proof.* It is clearly a bijection. If $K, K' \subset X$ are compact and $U \subset Y$ and $V \subset W$ are open, then

$$\begin{aligned} (f,g) \in M(K,U) \times M(K',V) &\iff (x \in K \implies f(x) \in U) \text{ and } (x \in K' \implies g(x) \in V) \\ &\iff ((x,y) \in K \times K' \implies f \times g(x,y) \in U \times V) \\ &\iff f \times g \in M(K, U \times W) \cap M(K', U \times W). \end{aligned}$$

so $(f,g) \mapsto f \times g$ is an open map.
Also $(f,g) \in M(K,U) \times M(K,V) \iff f \times g \in M(K, U \times V)$ which implies that the function is continuous. □

**Proposition 1.8.** *If $X$ and $T$ are locally compact Hausdorff spaces and $Y$ is an arbitrary Hausdorff space, then there is the homeomorphism*

$$Y^{X \sqcup T} \xrightarrow{\cong} Y^X \times Y^T$$

*taking $f \mapsto (f \circ \iota_X, f \circ \iota_T)$.*

*Proof.* The map is clearly well-defined and injective. Also, given $(f,g) \in Y^X \times Y^T$, we can define a function $f \cup g \colon X \sqcup T \to Y$ by $f$ on $X$ and $g$ on $T$, and clearly, $f \cup g \mapsto (f,g)$ under the correspondence, giving surjectivity. We must show that it is continuous and has continuous inverse.
Let $f \colon X \sqcup T \to Y$ and suppose $(f \circ \iota_X, f \circ \iota_T) \in M(K,U) \times M(K',V)$. Then $f \in M(K,U) \cap M(K',V)$ which is an open set that is mapped precisely to $M(K,U) \times M(K',V)$. Hence $f \mapsto (f \circ \iota_X, f \circ \iota_T)$ is continuous.

Conversely, note that under the correspondence, $M(C \sqcup C', U)$ is mapped to $M(C,U) \times M(C',U)$, so the map is also open. □

**Theorem 1.9.** *For $X$ locally compact and both $X$ and $Y$ Hausdorff, $Y^X$ is a covariant functor of $Y$ and a contravariant functor of $X$ from* Top *to* Top.

*Proof.* A map $\varphi\colon Y \to Z$ induces $\varphi^X\colon Y^X \to Z^X$ (put differently, $\varphi$ induces $\varphi_*\colon \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z)$.)) We must show that $\varphi^X$ is continuous. By Theorem 1.5, it suffices to show that the map $Y^X \times X \to Z$ given by $(f,x) \mapsto \varphi(f(x))$ is continuous, but this is the composition $\varphi \circ e$ which is thus continuous. Next, for the contravariant part, we must show that for $\psi\colon X \to T$, both spaces locally compact, we have that $Y^\psi\colon Y^T \to Y^X$ given by $\psi^*\colon f \mapsto f \circ \psi$ is continuous. By the same theorem as above, it suffices to show that $Y^T \times X \to Y$ taking $(f,x) \mapsto f(\psi(x))$ is continuous, but this is $e \circ (\mathrm{id} \times \psi)$, which is continuous. $\square$

**Corollary 1.10.** *For $A \subset X$ both locally compact and $X, Y$ Hausdorff, the restriction $Y^X \to Y^A$ is continuous.*

*Proof.* Apply the contravariant functor $\mathrm{Hom}(-,Y) = Y^-$ to the inclusion $\iota\colon A \hookrightarrow X$. $\square$

**Theorem 1.11.** *For $X, Y$ locally compact, and $X, Y, Z$ Hausdorff, the function*

$$Z^Y \times Y^X \to Z^X$$

*taking $(f,g) \mapsto f \circ g$ is continuous.*

*Proof.* Again, by Theorem 1.5, it suffices to show that the map $Z^Y \times Y^X \times X \to Z$ taking $(f,g,x) \mapsto (f \circ g)(x)$ is continuous, but this is simply $e \circ (\mathrm{id} \times e)$. $\square$

## 2. Methods of Calculation

### 2.1. Excision for Homotopy Groups.

**Theorem 2.1.** *Let $X$ be a CW complex decomposed as the union of subcomplexes $A$ and $B$ with nonempty connected intersection $C = A \cap B$. If $(A, C)$ is $m$-connected and $(B, C)$ is $n$-connected, $m, n \geq 0$, then the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.*

**Corollary 2.2** (Freudenthal Suspension Theorem)**.** *The unreduced suspension map $\pi_i(S^n) \to \pi_{i+1}\left(S^{n+1}\right)$, induced by the suspension map $S^n \to \Sigma S^n \cong S^{n+1}$, is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$. More generally, this holds for the suspension $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ whenever $X$ is an $(n-1)$-connected CW complex.*

*Proof of Corollary.* Decompose the unreduced suspension $\Sigma X = (X \times I) / (X \times \{0\}, X \times \{1\})$ as the union of two cones $C_+X$ and $C_-X$ intersecting in a copy of $X$. Recall that a map $f\colon X \to Y$ induces a suspended map $\Sigma f\colon \Sigma X \to \Sigma Y$. Now, if we consider $f$ to be any map $f\colon (S^n, s_0) \to (X, x_0)$, then we have a suspended map

$$\begin{array}{ccc} S^n \times I & \xrightarrow{f \times \mathrm{id}} & X \times I \\ \downarrow & & \downarrow \\ S^{n+1} \cong \Sigma S^n & \xrightarrow{\Sigma f} & \Sigma X \end{array}$$

So, in particular, $\Sigma f$ is some class in $\pi_{n+1}(\Sigma X)$. Define the suspension homomorphism $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ to be the map that sends $f$ to $\Sigma f$. This is a homomorphism (why?).

The unreduced suspension map is the same as the map

$$\pi_i(X) \cong \pi_{i+1}(C_+X, X) \to \pi_{i+1}(\Sigma X, C_-X) \cong \pi_{i+1}(\Sigma X).$$

(why?) where the two isomorphisms come from the LES of pairs and the middle map is induced by inclusion. The first map $\pi_i(X) \to \pi_{i+1}(C_+X, X)$ takes a map $\left(I^i, \partial I^n\right) \to (X, x_0)$ to the map $\left(I^{n+1}, \partial I^{n+1}, J^n\right) \to (C_+X, X, x_0)$ constructed by extending the given map radially to correspond with the height of $C_+X$. So one face of $I^{n+1}$ will be mapped to the vertex of $C_+X$.

Including this into $(\Sigma X, C_-X)$ gives the middle homomorphism, and then the map $\pi_{i+1}(\Sigma X, C_-X) \to \pi_{i+1}(\Sigma X)$ is simply the identity on our map.

From the LES of $(C_\pm X, X)$, we see that this pair is $n$-connected if $X$ is $(n-1)$-connected. Then Theorem 2.1 gives that the middle map is an isomorphism for $i + 1 < 2n$ and surjective for $i + 1 = 2n$.

$\square$

**Example 2.3** ($\pi_n\left(\bigvee_\alpha S^n_\alpha\right)$)**.** We want to show that $\pi_n\left(\bigvee_\alpha S^n_\alpha\right)$ for $n \geq 2$ is free abelian with basis the homotopy classes of the inclusions $S^n_\alpha \hookrightarrow \bigvee_\alpha S^n_\alpha$.

Suppose first that there are only *finitely many* summands $S^n_\alpha$. Then we can regard $\bigvee_\alpha S^n_\alpha$ as the $n$-skeleton of the product $\prod_\alpha S^n_\alpha$, where $S^n_\alpha$ is given the usual CW structure and $\prod_\alpha S^n_\alpha$ has the product CW structure. (See Hatcher appendix A).

By construction then $\prod_\alpha S^n_\alpha$ has cells only in dimensions a multiple of $n$, so the pair $(\prod_\alpha S^n_\alpha, \bigvee_\alpha S^n_\alpha)$ is $(2n-1)$-connected by Corollary **??**. So from the LES for the pair, we see that the inclusion $\bigvee_\alpha S^n_\alpha \hookrightarrow \prod_\alpha S^n_\alpha$ induces an isomorphism on homotopy groups in dimensions $\leq 2n-1$. Next we have $\pi_n\left(\prod_\alpha S^n_\alpha\right) \cong \bigoplus_\alpha \pi_n\left(S^n_\alpha\right) \cong \bigoplus_\alpha \mathbb{Z}$, a free abelian group with basis the inclusions $S^n_\alpha \hookrightarrow \prod_\alpha S^n_\alpha$, so pulling this back along the isomorphism $\pi_n\left(\bigvee_\alpha S^n_\alpha\right) \cong \pi_n\left(\prod_\alpha S^n_\alpha\right)$, the same is true for $\bigvee_\alpha S^n_\alpha$. This proves the claim when there are finitely many $S^n_\alpha$'s.

When there are infinitely many summands $S^n_\alpha$, consider the homomorphism $\Phi\colon \bigoplus_\alpha \pi_n\left(S^n_\alpha\right) \to \pi_n\left(\bigvee_\alpha S^n_\alpha\right)$ induced by the inclusions $S^n_\alpha \hookrightarrow \bigvee_\alpha S^n_\alpha$. Then $\Phi$ is surjective since any map $f\colon S^n \to \bigvee_\alpha S^n_\alpha$ has compact image contained in the wedge sum of finitely many $S^n_\alpha$'s, so by the finite case already proved, $[f]$ is in the image of $\Phi$.

Similarly, a nullhomotopy of $f$ has compact image contained in a finite wedge sum of $S^n_\alpha$'s, so the finite case also implies that $\Phi$ is injective.

**Proposition 2.4.** *If a CW pair $(X,A)$ is $r$-connected and $A$ is $s$-connected, with $r,s \geq 0$, then the map $\pi_i(X,A) \to \pi_i(X/A)$ induced by the quotient map $X \to X/A$ is an isomorphism for $i \leq r+s$ and a surjection for $i = r+s+1$.*

*Proof.* Consider $X \cup CA$. Since $A$ is closed and the inclusion $A \hookrightarrow X$ is a cofibration (since these are CW complexes), the map $h\colon C_\iota = X \cup CA \to X/A$ is a homotopy equivalence by Theorem **??**. So we have a commutative diagram

$$\begin{tikzcd}
\pi_i(X,A) \arrow[r] & \pi_i\left(X \cup CA, CA\right) \arrow[r] & \pi_i\left(X \cup CA/CA\right) = \pi_i(X/A) \\
& \pi_i(X \cup CA) \arrow[u, "\cong"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

where the vertical isomorphism comes from the LES of the pair $(X \cup CA, CA)$. Now, applying Theorem 2.1 to $(A,B) = (X, CA)$, since $(X,A)$ is $r$-connected and $(CA, A)$ is $(s+1)$-connected, we find that the homomorphism $\pi_i(X,A) \to \pi_i(X \cup CA, CA)$ induced by the inclusion is an isomorphism for $i < r+s+1$ and a surjection for $i = r+s+1$, which proves the result. $\square$

**Example 2.5** (Construction of spaces with a particular group as $\pi_n$)**.** Suppose $X$ is obtained from a wedge of spheres $\bigvee_\alpha S^n_\alpha$ by attaching cells $e^{n+1}_\beta$ via basepoint-preserving maps $\varphi_\beta\colon S^n \to \bigvee_\alpha S^n_\alpha, n \geq 2$. By cellular approximation, we know that $\pi_i(X) = 0$ for $i < n$, and we shall show that $\pi_n(X)$ is the quotient of the free abelian group $\pi_n\left(\bigvee_\alpha S^n_\alpha\right) \cong \bigoplus_\alpha \mathbb{Z}$ by the subgroup generated by the classes $[\varphi_\alpha]$. Any subgroup can be realized in this way, by choosing maps $\varphi_\beta$ to represent a set of generators for the subgroup. Let $X = \left(\bigvee_\alpha S^n_\alpha\right) \bigcup_\beta e^{n+1}_\beta$.

Then the LES of the pair $\left(X, \bigvee_\alpha S^n_\alpha\right)$ gives

$$\pi_{n+1}\left(X, \bigvee_\alpha S^n_\alpha\right) \xrightarrow{\partial} \pi_n\left(\bigvee_\alpha S^n_\alpha\right) \to \pi_n(X) \to 0.$$

so

$$\pi_n(X) \cong \pi_n\left(\bigvee_\alpha S^n_\alpha\right) / \operatorname{im} \partial$$

The quotient $X/\bigvee_\alpha S^n_\alpha$ is a wedge of spheres $S^{n+1}_\beta$, so by Proposition 2.4 and Example 2.3, the map $\pi_{n+1}\left(X, \bigvee_\alpha S^n_\alpha\right) \to \pi_{n+1}\left(X/\bigvee_\alpha S^n_\alpha\right) \cong \pi_{n+1}\left(\bigvee_\beta S^{n+1}_\beta\right)$ is an isomorphism, so $\pi_{n+1}\left(X, \bigvee_\alpha S^n_\alpha\right)$ is free with basis the caracteristic maps $\varphi_\beta$ of the cells $e^{n+1}_\beta$. The boundary map $\partial$ takes these to the classes $[\varphi_\beta]$, so the result follows.

2.1.1. *Eilenberg-MacLane Spaces.*

**Definition 2.6** (Eilenberg-MacLane space, $K(G, n)$)**.** A space $X$ having just one nontrivial homotopy group $\pi_n(X) \cong G$ is called an *Eilenberg-MacLane space* $K(G, n)$.

*Construction of Eilenberg-MacLane Spaces:*
Given arbitrary $G$ and $n$, and assuming $G$ is abelian if $n > 1$, we can construct a CW complex $K(G, n)$. To begin, construct the CW complex $X$ from Example 2.5. Then $X$ is an $(n-1)$- conneted CW complex of dimension $n+1$ such that $\pi_n(X) \cong G$ by construction. Alternatively, given the existence of Moore spaces $M(G, n)$ for any $G$ and $n$, we can take a Moore space $M(G, n)$ and use the Hurewicz isomorphism to conclude that $\pi_n(X) \cong H_n(X)$. Hence we just need to fix all homotopy groups of dimension greater than $n$. By Example **??**, we can construct a CW complex $X_n$ containing $X$ as a subcomplex such that $\pi_n(X_n) \cong \pi_n(X) \cong G$ while $\pi_k(X_n) \cong 0$ for all $k \neq n$.

**Example 2.7** (Constructing spaces with arbitrary (abelian) homotopy groups)**.** Recall that

$$\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n\left(X_\alpha\right),$$

so if we have a sequence of abelian groups $\{G_{n_i}\}_{i \in I}$, and let $X_{n_i}$ denote that $K(G_{n_i}, n_i)$ space, then we find that

$$\pi_k(\prod_{i\in I} X_{n_i}) \cong \prod_{i \in I} \pi_k\left(X_{n_i}\right) \cong \begin{cases} G_{n_i}, & k = n_i \text{ for some } i \in I \\ 0, & \text{else} \end{cases}$$

Having covered the existence of Eilenberg-MacLane spaces, we now find the following for uniqueness of these spaces:

**Proposition 2.8** (Uniqueness of Eilenberg-MacLane spaces)**.** *The homotopy type of a CW complex $K(G, n)$ is uniquely determined by $G$ and $n$.*

The proof is based on the following lemma giving a condition for when homomorphisms between homotopy groups are induced by some map:

**Lemma 2.9.** *Let $X$ be a CW complex of the form $(\bigvee_\alpha S^n_\alpha) \bigcup_\beta e^{n+1}_\beta$ for some $n \geq 1$. Then for every homomorphism $\psi\colon \pi_n(X) \to \pi_n(Y)$ with $Y$ path-connected there exists a map $f\colon X \to Y$ with $f_* = \psi$.*

*Proof.* The construction of $f$ is as one would expect: first let $f$ send the natural basepoint of $\bigvee_\alpha S^n_\alpha$ to a chosen basepoint $y_0 \in Y$. Now for every sphere $S^n_\alpha$ in $X$, we extend $f$ over the sphere via a map representing $\psi\left([i_\alpha]\right)$ where $i_\alpha$ is the inclusion $S^n_\alpha \hookrightarrow X$. This defines $f$ on the $n$-skeleton of $X$ : $f\colon X^n \to Y$. Since now $f_*\left[i_\alpha\right] = \psi\left[i_\alpha\right]$ for all $\alpha$ and the $[i_\alpha]$ generate $\pi_n\left(X^n\right)$, this defines $f_*$ on all of $\pi_n(X^n)$.
To extend $f$ over the $(n+1)$-cells, it will suffice to show that $f \circ \varphi_\beta$ is nullhomotopic, where $\varphi_\beta\colon S^n \to X^n$ is the attaching map for the $(n+1)$-cell $e^{n+1}_\beta$. But $f \circ \varphi_\beta$ is a representative of $f_*\left[\varphi_\beta\right] = \psi\left[\varphi_\beta\right]$. Thus we have transformed $f_*\left[\varphi_\beta\right]$ into an element in the image of $\psi\colon \pi_n(X) \to \pi_n(Y)$, and for this, we can use the extra structure of $X$, not just $X^n$. In $X$, $[\varphi_\beta]$ is trivial via the characteristic map of the cell $e^{n+1}_\beta$, so $\psi\left[\varphi_\beta\right] = \psi(0) = 0$, thus indeed $f \circ \varphi_\beta$ is nullhomotopic. Thus we obtain the desired extension $f\colon X \to Y$. To see that $f_* = \psi$, simply note that by

cellular approximation, any element of $\pi_n(X)$ can be represented as an element in $\pi_n(X^n)$, and on $\pi_n(X^n)$, $f_*$ agrees with $\psi$ by construction. $\square$

*Proof of Proposition 2.8.* Let $K'$ be any $K(G,n)$ CW complex, and let $K$ be the specific $K(G,n)$ CW complex constructed in Example 2.5. In particular, $K$ is of the form of Lemma 2.9. Since $\pi_n(K) = \pi_n(Y)$, we can apply Lemma 2.9 to obtain a map $f\colon K \to K'$ inducing the identity on $\pi_n$. Since all other homotopy groups of $K$ and $K'$ are trivial, Whitehead's theorem now gives that $f$ is a homotopy equivalence. Since homotopy equivalence is an equivalence relation, this finishes the proof. $\square$

### 2.2. **The Hurewicz Theorem.**

**Theorem 2.10.** *If a space $X$ is $(n-1)$-connected, $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$. If a pair $(X, A)$ is $(n-1)$-connected, $n \geq 2$, with $A$ simply connected and nonempty, then $H_i(X, A) = 0$ for $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$.*

*Remark.* This result is, in a sense, the best that we can expect. For example, $S^n$ has trivial homology groups above dimension $n$ but many nontrivial homotopy groups in this range when $n \geq 2$; and conversely, Eilenberg-MacLane spaces such as $\mathbb{CP}^\infty$ have trivial higher homotopy groups but many nontrivial homology groups.

**Corollary 2.11.** *A map $f\colon X \to Y$ between simply-connected CW complexes is a homotopy equivalence if $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism for each $n$.*

*Proof.* By replacing $Y$ with the mapping cylinder $M_f$, we may assume $f$ is the inclusion $X \hookrightarrow Y$. Since $X$ and $Y$ are simply-connected, $\pi_1(Y, X) = 0$. The relative Hurewicz theorem says that the first nonzero $\pi_n(Y, X)$ is isomorphic to the first nonzero $H_n(Y, X)$, but by the LES of the pair $(Y, X)$ in homology, $H_n(Y, X) \cong 0$ for all $n \geq 0$, so also $\pi_n(Y, X) \cong 0$ for all $n \geq 0$, so $f$ induces isomorphisms $\pi_n(X) \to \pi_n(Y)$ for all $n$. By Whitehead's theorem, $f$ is a homotopy equivalence. $\square$

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