1. Curves, Surfaces and Hyperbolic Geometry

1.1. **Simple closed curves.** There is a bijective correspondence

$$\left\{\begin{array}{c}\text{Nontrivial}\\ \text{conjugacy classes}\\ \text{in } \pi_1(S)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Nontrivial free}\\ \text{homotopy classes of oriented}\\ \text{closed curves in } S\end{array}\right\}$$

Definition 1.1 (Primitive and multiple elements)**.** An element g of a group G is *primitive* if there does not exist any $h \in G$ so that $g = h^k$ for $|k| > 1$. The property of being a primitive is a conjugacy class invariant. In particular, it makes sense to say that a closed curve in a surface is primitive.
A closed curve in S is a multiple if it is a map $S^1 \to S$ that factors through the map $S^1 \stackrel{\times n}{\to} S^1$ for $n > 1$, i.e., there exists a map $\tilde{\alpha}\colon S^1 \to S$ such that the following diagram commutes:

$$\begin{tikzcd} S^1 \arrow[r, "\times n"] \arrow[rr, bend left, dashed, "\tilde{\alpha}"] & S^1 \arrow[r, "\alpha"] & S \end{tikzcd}$$

Definition 1.2 (Lifts)**.** We make a distinction between lifts: let $p\colon \tilde{S} \to S$ be a covering space. By a *lift* of a closed curve α to $\tilde{S}$ we will always mean the image of a lift $\mathbb{R} \to \tilde{S}$ of the map $\alpha \circ \pi$ where $\pi\colon \mathbb{R} \to S^1$ is the usual covering map. I.e., a lift of $\alpha\colon S^1 \to S$ is a map $\tilde{\alpha}\colon \mathbb{R} \to \tilde{S}$ such that the following diagram commutes

$$\begin{tikzcd} & & \tilde{S} \arrow[d, "p"] \\ \mathbb{R} \arrow[urr, "\tilde{\alpha}"] \arrow[r, "\pi"'] & S^1 \arrow[r, "\alpha"'] & S \end{tikzcd}$$

A lift is different from a *path lift* which is a proper subset of a lift. Namely, it would be the restriction of $\tilde{\alpha}$ to some interval of $\mathbb{R}$ of length 2π if the covering map π is of the form $t \mapsto e^{it}$.

Now suppose $p\colon \tilde{S} \to S$ is the universal cover and α is a simple closed curve in S that is not a multiple of another closed curve. In this case, there is a bijective correspondence between cosets in $\pi_1(S)$ of the infinite cyclic subgroup $\langle\alpha\rangle$ and the lifts of α. This can be seen as follows: first choose a basepoint $\alpha(1) = x_0 \in S$ and some $\tilde{x_0} \in p^{-1}(x_0)$. There exists a unique lift $\tilde{\alpha}$ of α such that

$$\begin{tikzcd} & & \tilde{S} \arrow[d, "p"] \\ \mathbb{R} \arrow[urr, "\tilde{\alpha}"] \arrow[r] & S^1 \arrow[r, "\alpha"] & S \end{tikzcd}$$

commutes and such that $\tilde{\alpha}(0) = \tilde{x} \in p^{-1}(\alpha \circ \pi(0))$ for some specific $\tilde{x}$ [Bredon, Cor. 4.2]. But the set $p^{-1}(\alpha \circ \pi(0))$ is in bijective correspondence with the loops in $\pi_1(S)$ by the path lifting lemma. Now, under which path lifts are the lifts the same? The lifts of α to two points $\tilde{x}, \tilde{y} \in p^{-1}(\alpha \circ \pi(0))$ will be the same if $\alpha^k \cdot \tilde{x} = \tilde{y}$ where $\cdot$ denotes the monodromy action of $\pi_1(S)$ on the fiber. Now, there exist γ_x and γ_y in $\pi_1(S)$ such that $\gamma_x \cdot \tilde{x_0} = \tilde{x}$ and $\gamma_y \cdot \tilde{x_0} = \tilde{y}$, so $\alpha^k\gamma_x = \gamma_y$. Hence the lifts corresponding to γ_x and γ_y are the same if and only if $\alpha^k\gamma_x = \gamma_y$ for some k, i.e. if and only if $\gamma_x = \gamma_y$ in $\pi_1(S)/\langle\alpha\rangle$.

As usual, the group $\pi_1(S)$ acts on the set of lifts of α by deck transformations, and this action agrees with the usual left action of $\pi_1(S)$ on the cosets of $\langle\alpha\rangle$. The stabilizer of the lift corresponding to the coset $\gamma\langle\alpha\rangle$ is the cyclic group $\langle\gamma\alpha\gamma^{-1}\rangle$. See figure 1.

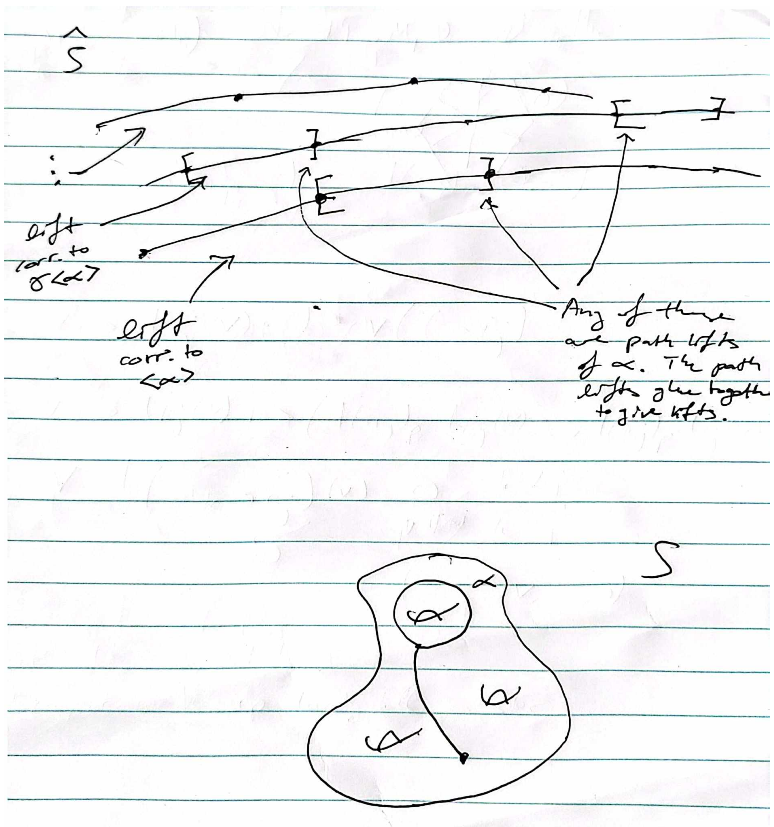

FIGURE 1.

Theorem 1.3. *When S admits a hyperbolic metric and α is a primitive element of $\pi_1(S)$, we have a bijective correspondence*

$$\left\{\begin{array}{c}\textit{Elements of the conjugacy}\\ \textit{class of } \alpha \textit{ in } \pi_1(S)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{Lifts to } \tilde{S} \textit{ of the}\\ \textit{closed curve } \alpha\end{array}\right\}$$

More precisely, we claim that the map which sends the lift given by the coset $\gamma\langle\alpha\rangle$ to $\gamma\alpha\gamma^{-1}$ is bijective and well-defined.

Proof. To show that it is well-defined, suppose $\gamma\langle\alpha\rangle$ and $\beta\langle\alpha\rangle$ give the same lift. Then $\gamma = \beta\alpha^k$. So in particular,

$$\gamma\alpha\gamma^{-1} = \beta\alpha^k\alpha\alpha^{-k}\beta^{-1} = \beta\alpha\beta^{-1}$$

so they do correspond to the same element of the conjugacy class $[\alpha]$. It is clear that this is a surjective map. Now suppose that $\gamma\alpha\gamma^{-1} = \beta\alpha\beta^{-1}$. Then

$\beta^{-1}\gamma\alpha\left(\beta^{-1}\gamma\right)^{-1} = \alpha$, so in particular, $\beta^{-1}\gamma \in C_{\pi_1(S)}(\alpha)$ which is a cyclic group generated by, say, θ. But then $\theta^l = \alpha$ since α is trivially in the centralizer of α; however, α is primitive, so l must be ± 1, but then α generates the centralizer of α, $C_{\pi_1(S)}(\alpha) = \langle\alpha\rangle$, and hence $\gamma = \beta\alpha^l$, so $\gamma\langle\alpha\rangle = \beta\langle\alpha\rangle$. $\square$

Remark. If α is any multiple, then we still have a bijective correspondence between elements of the conjugacy class of α and the lifts of α. However, if α is not primitive and not a multiple, then there are more lifts of α than there are conjugates. Indeed, if $\alpha = \beta^k$, where $k > 1$, then $\beta\langle\alpha\rangle \neq \langle\alpha\rangle$ while $\beta\alpha\beta^{-1} = \alpha$.

Example 1.4. The above correspondence does not hold for the torus T^2 because each closed curve has infinitely many lifts, while each element of $\pi_1\left(T^2\right) \approx \mathbb{Z}^2$ is its own conjugacy class because $\pi_1\left(T^2\right)$ is abelian.

Geodesic representatives.

Proposition 1.5. *Let S be a hyperbolic surface. If α is a closed curve in S that is not homotopic into a neighborhood of a puncture, then α is homotopic to a unique geodesic closed curve γ.*

References

[Bredon] Glen E. Bredon Geometry and Topology