Exercise 0.1 (12.2)**.** Let U and V be normed spaces, and assume that V is complete. Show that then $B(U,V)$ is also complete with the operator norm.

Solution. Suppose B_n is a Cauchy sequence in $B(U,V)$, so

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \mid \|B_n - B_m\| < \varepsilon, \quad \forall m,n \geq N.$$

That is, for all $m,n \geq N$,

$$\sup_{\|x\|=1} \|B_n x - B_m x\| < \varepsilon$$

For a fixed n, this becomes a Cauchy sequence in V which thus converges, so we can define $Bx = \lim_{n\to\infty} B_n x$. We claim that B is a bounded operator too. It is clear that it is linear since each B_n is a continuous map. What remains is to show that B is bounded. It suffices to show that it is bounded on $S = \{x \mid \|x\| = 1\}$. Suppose it were not bounded and choose a sequence $(x_n) \subset S$ such that $\|Bx_n\| > n$. Choose $\varepsilon = \frac{1}{2}$ and let N be such that for $n,m \geq N$, we have

$$\|B_n - B_m\| < \varepsilon$$

Then for all k

$$\|B_n x_k - B_m x_k\| < \varepsilon$$

for all $n \geq N$, so in particular

$$\|Bx_k - B_m x_k\| = \lim_{n\to\infty} \|B_n x_k - B_m x_k\| < \varepsilon$$

But B_m is bounded, so let $\|B_m\| = R$. Choose M such that that for all $k \geq M$, we have $\|Bx_k\| \geq \|B_m x_k\|$, then

$$\|Bx_k\| - \|B_m x_k\| < \varepsilon$$

giving

$$\|Bx_k\| < \varepsilon + R$$

contradicting $\|Bx_k\| \to \infty$.

Exercise 0.2 (12.3)**.** Let $A \in B(V)$ for a complete normed space V. Prove

(1) If $\|A\| < 1$ then $\sum_{k=0}^{\infty} A^k$ converges in $B(V)$ to an inverse of $I - A$.
(2) If $B \in B(V)$ is invertible and $\|A\| < \frac{1}{\|B^{-1}\|}$ then $B - A$ is invertible.
(3) The set of invertible bounded operators is an open subset of $B(V)$.

(Here invertible means there is a bounded inverse)

Solution. (1) geometric series.
(2) $B - A = B\left(1 - \frac{A}{B}\right)$. Now $\|\frac{A}{B}\| \leq \|A\|\|B^{-1}\| < 1$, so by (1), $1 - \frac{A}{B}$ has inverse $\sum_{k=0}^{\infty}\left(\frac{A}{B}\right)^k$. But then $B - A$ is a composition of invertible maps hence invertible since $\mathrm{GL}(V)$ is a group.
(3) Suppose $A \in B\left(B, \frac{1}{\|B^{-1}\|}\right)$, so $\|B - A\| < \frac{1}{\|B^{-1}\|}$. By (2), $B - (B - A) = A$ is then invertible. Hence $B\left(B, \frac{1}{\|B^{-1}\|}\right)$ is an open neighborhood of B in $B(V)$ consisting of invertible maps. Thus the set of invertible maps is open in $B(V)$.

Exercise 0.3 (12.6)**.** Let $S \in \mathrm{End}\left(\ell^2\right)$ denote the right shift taking the sequence $(x_1, x_2, \ldots)$ to $(0, x_1, x_2, \ldots)$. Show it is bounded and determine the operator norm $\|S\|$. Find also the adjoint S^*, and verify that $S^*S = I$ but $SS^* \neq I$.

Solution. Recall that we are dealing with the norm $\|(x_1, x_2, \ldots)\|^2 = \sum_{k=1}^{\infty} |x_k|^2$. But indeed then if $\|(x_1, \ldots)\| = 1$, then

$$\|S(x_1, \ldots)\|^2 = \|(0, x_1, x_2, \ldots)\|^2 = \sum_{k=1}^{\infty} |x_k|^2 = 1$$

so, in fact, S preserves the norm. But then since $\|Sx\| = \|x\|$ for all x by linearity, we have $\|S\| = 1$. Now, the inner product is $\langle x, y\rangle = \sum_k x_k \overline{y_k}$. Then

$$\langle Sx, y\rangle = \sum_{k=2}^{\infty} x_k y_{k-1} = \langle x, S^* y\rangle$$

if we define $S^*(y_1, y_2, \ldots) = (y_2, y_3, \ldots)$. We then indeed get $S^*S = I$ clearly, but $SS^*(x_1, x_2, \ldots) = (0, x_2, x_3, \ldots)$.

Exercise 0.4 (12.8)**.** Show $\|Ax \pm ix\|^2 = \|Ax\|^2 + \|x\|^2$ for A Hermitian and $\dim V < \infty$. Then show $A \pm iI$ is invertible and $(A - iI)(A + iI)^{-1}$ unitary.

Solution.

$$\begin{aligned}\langle Ax \pm ix, Ax \pm ix\rangle &= \|Ax\|^2 + \langle Ax, \pm ix\rangle + \langle \pm ix, Ax\rangle + \langle \pm ix, \pm ix\rangle \\ &= \|Ax\|^2 + \|x\|^2 \mp i\langle Ax, x\rangle \pm i\langle x, Ax\rangle \\ &= \|Ax\|^2 + \|x\|^2 \mp i\langle Ax, x\rangle \pm i\langle Ax, x\rangle \\ &= \|Ax\|^2 + \|x\|^2.\end{aligned}$$

Now, if $A \pm iI$ were not invertible, it would not be injective, so for $x \neq 0$, we would get

$$0 = \|Ax \pm ix\|^2 = \|Ax\|^2 + \|x\|^2$$

but $\|x\|^2 > 0$ and $\|Ax\|^2 \geq 0$, so this gives a contradiction.
Lastly, what is the adjoint of $(A - iI)(A + iI)^{-1}$? Well, $(A - iI)^* = A + iI$ by the rules on page 70. Hence the expression is of the form $X^* X^{-1}$ which has adjoint $\left(X^{-1}\right)^* X$. Then $\left(X^{-1}\right)^* X X^* X^{-1}$.
Now, since A is self-adjoint, it is in particular normal, so $A + iI$ is normal and hence orthogonally diagonable. Writing $A + iI = \sum \lambda E_\lambda$, we get $(A + iI)^* = \sum \overline{\lambda} E_\lambda$, so $A + iI$ and $A - iI$ commute. Hence we get $XX^* = X^*X$, and the expression above becomes the identity.

Exercise 0.5 (12.4)**.** Give a simple proof of the Hahn-Banach theorem for a continuous linear form on a closed subspace of a Hilbert space.

Solution. Let V be a Hilbert space and let $U \subset V$ be a closed subspace. Then $V = U \oplus U^\perp$. Let $\mathcal{B}$ be a basis for U and extend it to a basis $\mathcal{A}$ for V. Take the duals $\mathcal{B}^{'}$ and $\mathcal{A}^{'}$. For $z \in U^*$ we can write $z = \sum_{y_i' \in \mathcal{B}'} a_i y_i$. Then z can also be considered a linear form on V by letting the coefficient for $y_i \in \mathcal{A}'$ be 0 if $y_i \notin \mathcal{B}'$ and a_i otherwise. The restrictions are clearly the same. By the Riesz-Fréchet representation theorem, since U is a closed subspace of a Hilbert space, it is also a Hilbert space, so by continuity of z, there exists $u \in U$ such that $z(x) = \langle x, u\rangle$ for all $x \in U$ and such that $\|z\| = \|u\|$. But since $z|_{U^\perp} = 0$, we also have $z(x) = \langle x, u\rangle$ for all $x \in V$, so by the Riesz-Fréchet theorem, $\|z\| = \|u\|$ over V as well.

Exercise 0.6 (13.1)**.** Prove $\rho(A + B) \leq \rho(A) + \rho(B)$ if A and B are normal. Prove it for general $A, B \in \mathrm{End}(V)$, now assuming they commute. Show the inequality can fail in general.

Solution. If A and B are normal, then they are orthogonally diagonable with respect to the associated inner product, hence $\rho(A) = \|A\|$ and $\rho(B) = \|B\|$. Now, in general, we have $\rho(X) \le \|X\|$, we we get

$$\rho(A+B) \le \|A+B\| \le \|A\| + \|B\| = \rho(A) + \rho(B).$$

If A and B commute, then

$$\begin{aligned}
\rho(A+B) = \lim_{k\to\infty} \|\,(A+B)^k\,\|^{\frac{1}{k}} &= \lim_{k\to\infty} \|\sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i}\|^{\frac{1}{k}} \\
&\le \lim_{k\to\infty} \left|\sum_{i=0}^{k} \binom{k}{i} \|A\|^i \|B\|^{k-i}\right|^{\frac{1}{k}} \\
&= \lim_{k\to\infty} \|A\| + \|B\| \\
&= \|A\| + \|B\| \\
&= \lim_{k\to\infty} \|A^k\|^{\frac{1}{k}} + \lim_{k\to\infty} \|B^k\|^{\frac{1}{k}}
\end{aligned}$$

where the last equality follows from $\|X^k\| = \|X\|^k$ when X is diagonable (by the proof of lemma 13.4).

To show that it can fail in general, note that for $[A] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $[B] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have that the eigenvalues of both are precisely 1, hence $\rho(A) + \rho(B) = 2$, while $A + B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ which has characteristic polynomial $(x-3)(x-1)$ and thus 3 as an eigenvalue.

Exercise 0.7 (13.2)**.** Let $F = \mathbb{C}$. Find a counterexample to the statement: $\rho\left(p\left(A\right)\right) = p\left(\rho(A)\right)$ for all polynomials p, where ρ is the spectral radius.

Solution. Consider $p(x) = ix - i$ and $A = -I$. So $p(A) = \begin{pmatrix} -2i & 0 \\ 0 & -2i \end{pmatrix}$ which has spectral radius 2. However, $-I$ has spectral radius 1 and $p(1) = 0$.

Exercise 0.8 (13.3)**.** Show $\rho\left(A^*A\right) = \|A^*A\| = \|A\|^2$ for the operator norm of an inner product.

Solution. The first equality holds when the matrix is orthogonally diagonable. But A^*A is self-adjoint, hence normal hence orthogonally diagonable.
The latter equality holds since

$$\|A^*A\| = \sup_{\|x\|=1} \langle A^*Ax, x\rangle = \sup_{\|x\|=1} \|Ax\|^2 = \|A\|^2$$