1. Let $\sigma_{1,1}\colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be given by

$$[x_1 : x_2] \times [y_1 : y_2] \mapsto [x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2]$$

(a) It is clear that since $(x_1y_1)(x_2y_2) - (x_1y_2)(x_2y_1) = 0$,

$$\sigma_{1,1}\left(\mathbb{P}^1 \times \mathbb{P}^1\right) \subset \mathbb{V}\left(z_1z_4 - z_2z_3\right) = \left\{[z_1 : z_2 : z_3 : z_4] : \operatorname{rank}\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \leq 1\right\}.$$

Now, for the converse, suppose $z = [z_1 : z_2 : z_3 : z_4] \in \mathbb{V}\left(z_1z_4 - z_2z_3\right)$.

Then if $z_1, z_2 = 0$, either $z_3 \neq 0$ or $z_4 \neq 0$ (we will omit this from now), so we have $\sigma_{1,1}\left([0:1] \times [z_3 : z_4]\right) = [0 : 0 : z_3 : z_4] = z$.

Suppose $z_1, z_3 = 0$, then $\sigma_{1,1}\left([z_2 : z_4] \times [0 : 1]\right) = [0 : z_2 : 0 : z_4] = [z_1 : z_2 : z_3 : z_4]$.

If $z_2, z_4 = 0$ then either $z_1 \neq 0$ or $z_3 \neq 0$ so $\sigma_{1,1}\left([z_1 : z_3] \times [1 : 0]\right) = z$.

If $z_3, z_4 = 0$ then either $z_1 \neq 0$ or $z_2 \neq 0$, so $\sigma_{1,1}\left([1 : 0] \times [z_1 : z_2]\right) = z$.

Suppose $z_1, z_4 = 0$, so either $z_2 \neq 0$ or $z_3 \neq 0$, and in particular, the other is 0. Suppose $z_2 = 0$, so $z_3 \neq 0$. Then $\sigma_{1,1}\left([0 : z_3] \times [1 : 0]\right) = [0 : 0 : z_3 : 0] = [z_1 : z_2 : z_3 : z_4]$. If instead $z_3 = 0$, then $z_2 \neq 0$, so $\sigma_{1,1}\left([z_2 : 0] \times [0 : 1]\right) = [0 : z_2 : 0 : 0] = [z_1 : z_2 : z_3 : z_4]$.
If $z_2, z_3 = 0$, then either z_1 or z_4 is nonzero and the other zero, so for $z_1 = 0$, $\sigma_{1,1}\left([0 : z_4] \times [0 : 1]\right)$ and for $z_4 = 0$, $\sigma_{1,1}\left([z_1 : 0] \times [1 : 0]\right) = z$.

Thus we have $\mathbb{V}\left(z_1z_4 - z_2z_3\right) \subset \sigma_{1,1}\left(\mathbb{P}^1 \times \mathbb{P}^1\right)$.
Therefore $\mathbb{V}\left(z_1z_4 - z_2z_3\right)$ is precisely the image of $\sigma_{1,1}$.

(b) Let $\sigma_{1,2}\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$ be the morphism given by

$$[x_1 : x_2] \times [y_1 : y_2 : y_3] \mapsto [x_1y_1 : x_1y_2 : x_1y_3 : x_2y_1 : x_2y_2 : x_2y_3].$$

Let M be the matrix

$$M = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \end{pmatrix}$$

We claim that

$$\sigma_{1,2}\left(\mathbb{P}^1 \times \mathbb{P}^2\right) = \{[z_1 : \ldots : z_6] \mid \operatorname{rank} M \leq 1\} = A$$

This is equivalent to showing that $\sigma_{1,2}\left(\mathbb{P}^1 \times \mathbb{P}^2\right)$ is the vanishing of the 2×2 minors of M.

Now, since $z_1z_5 - z_4z_2 = x_1y_1x_2y_2 - x_2y_1x_1y_2 = 0$ and $z_2z_6 - z_5z_3 = x_1y_2x_2y_3 - x_2y_2x_1y_3 = 0$, we have that $\sigma_{1,2}\left(\mathbb{P}^1 \times \mathbb{P}^2\right) \subset A$.

Now, suppose conversely, that $z = [z_1 : \ldots : z_6] \in A$, so $z_1z_5 - z_4z_2 = 0 = z_2z_6 - z_5z_3$.

If $z_1 = z_2 = z_3 = 0$ then $\sigma_{1,2}\left([0 : 1] \times [z_4 : z_5 : z_6]\right) = z$ (here z_4, z_5 and z_6 cannot all be 0 as $[0 : \ldots : 0] \notin \mathbb{P}^5$). If $z_i \neq 0$ for some $i \in \{1, 2, 3\}$, then $z_{i+3}z_1 = z_iz_4$, since for $i = 1$, we get $z_1z_4 = z_1z_4$, for $i = 2$ we get $z_1z_5 = z_2z_4$ which is true since $z \in A$ and $\begin{pmatrix} z_1 & z_2 \\ z_4 & z_5 \end{pmatrix}$ is a 2×2 minor of M; if $i = 3$, we get $z_1z_6 = z_3z_4$ which is true as $\begin{pmatrix} z_1 & z_3 \\ z_4 & z_6 \end{pmatrix}$ is a 2×2 minor of M.
Completely equivalently, one can show that $z_{i+3}z_2 = z_iz_5$ and $z_{i+3}z_3 = z_iz_6$ which follow from the 2×2 minors in M.
Thus

$$\begin{aligned}
\sigma_{1,2}\left([z_i : z_{i+3}] \times [z_1 : z_2 : z_3]\right) &= [z_iz_1 : z_iz_2 : z_iz_3 : z_{i+3}z_1 : z_{i+3}z_2 : z_{i+3}z_3] \\
&= [z_iz_1 : z_iz_2 : z_iz_3 : z_iz_4 : z_iz_5 : z_iz_6] \\
&= [z_1 : z_2 : z_3 : z_4 : z_5 : z_6] = z
\end{aligned}$$

(c) This generalizes directly to letting M be the matrix $M = (z_{ij})$ with $z_{ij} = x_i y_j$ and $k = 1$.

2:
(a) Let $\varphi\colon k\left(\mathbb{P}^1\right) \to k(x)$ be defined by $\frac{F}{G} \mapsto \frac{F(x,1)}{G(x,1)}$.
One-to-one: By definition of being fields, we have $\frac{F}{G}+\frac{F'}{G'} = \frac{FG'+F'G}{GG'}$, so $\varphi\left(\frac{F}{G}+\frac{F'}{G'}\right) = \frac{F(x,1)G'(x,1)+F'(x,1)G(x,1)}{G(x,1)G'(x,1)} = \frac{F(x,1)}{G(x,1)} + \frac{F'(x,1)}{G'(x,1)} = \varphi\left(\frac{F}{G}\right) + \varphi\left(\frac{F'}{G'}\right)$, so φ is a homomorphism.
It suffices to show that $\varphi\left(\frac{F}{G}\right) = 0 \implies \frac{F}{G} = 0$.
First, $\Gamma_h(\mathbb{P}^1) = \frac{k[x,y]}{\mathbb{I}(\mathbb{P}^1)} = k\left[x,y\right]$, so $F, G \in k\left[x,y\right]$ are forms of the same degree with $G \neq 0$.
Suppose $\varphi\left(\frac{F}{G}\right) = \frac{F(x,1)}{G(x,1)} = 0$. Then $F(x,y) \in (y-1)$. We claim $F = 0$:
Suppose $F(x,y) = (g_0 + g_1 + \ldots + g_m)(y-1)$ with $g_m \neq 0$. Then $F_{m+1} = g_m z \neq 0$, so all lower F_i vanish as F is homogeneous. So $0 = F_0 = -g_0$. Then $0 = F_1 = g_0 z - g_1 \implies g_1 = 0$. Assume $g_0, \ldots, g_j = 0$, then $0 = F_{j+1} = g_j z - g_{j+1} = -g_{j+1}$, so $g_{j+1} = 0$. Hence $g_0, \ldots, g_m = 0$, contradicting $g_m \neq 0$. Thus $g = g_0 + \ldots + g_m = 0$ implying $F = 0$.
Thus $G \in k - \{0\}$, so $\frac{F}{G} = 0$.
Onto: Now suppose $\frac{f}{g} \in k(x)$, so $g \neq 0$. Let $d = \max\{\deg f, \deg g\}$, and let $f' = H_d(f)$ and $g' = H_d(g)$ be the homogenizations of f and g of degree d in $k\left[x,y\right]$. Then $f'(x,1) = f$ and $g'(x,1) = g$ and f', g' are forms of degree d with $g' \neq 0$. Then $\varphi\left(\frac{f'}{g'}\right) = \frac{f'(x,1)}{g'(x,1)} = \frac{f(x)}{g(x)}$, so φ is onto.

(b) We have that $\varphi\colon X \to Y$ is dominant if $\mathbb{I}\left(\varphi(X)\right) = \mathbb{I}(Y)$ if and only if $\mathbb{V}\left(\mathbb{I}\left(\varphi(X)\right)\right) = \mathbb{V}\left(\mathbb{I}\left(Y\right)\right) = Y$. Now, $\varphi(X) \subset \mathbb{V}(\mathbb{I}(Y))$, and supposing W is a projective algebraic set containing $\varphi(X)$, we have $\mathbb{I}(Y) = \mathbb{I}\left(\varphi(X)\right) \supset \mathbb{I}(W)$, so $\mathbb{V}\left(\mathbb{I}\left(\varphi(X)\right)\right) = \mathbb{V}\left(\mathbb{I}(Y)\right) \subset \mathbb{V}\left(\mathbb{I}\left(W\right)\right) = W$, so $\mathbb{V}\left(\mathbb{I}\left(\varphi(X)\right)\right)$ is the smallest projective algebraic set containing $\varphi(X)$. Hence $\mathbb{V}\left(\mathbb{I}\left(\varphi(X)\right)\right) = \overline{\varphi(X)}$, so we see that the equivalence φ dominant if and only if $\overline{\varphi(X)} = Y$ is also true for the projective case. Now $\overline{\varphi(X)} - U = Y - U$ is closed, so $\varphi^{-1}\left(\overline{\varphi(X)} - U\right) = \varphi^{-1}\left(\overline{\varphi(X)}\right) - \varphi^{-1}(U) = X - \varphi^{-1}(U)$ is closed, so $\varphi^{-1}(U)$ is open.

3:
(a) By the lemma on lecture note 24, we have that since $\varphi\colon X \to Y$ is an isomorphism with $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$, the pullback $\varphi^*\colon k(Y) \to k(X)$ is an isomorphism.
Suppose $\psi\colon Y \to X$ is the inverse to φ. By the lemma on lecture note 24, we have that ψ^* is the inverse to φ^*.
It thus remains to show that φ^* takes $\mathcal{O}_Q(Y)$ into $\mathcal{O}_P(X)$, that ψ^* takes $\mathcal{O}_P(X)$ into $\mathcal{O}_Q(Y)$ and that $\psi^* \circ \varphi^*$ is the identity on $\mathcal{O}_Q(Y)$ and that $\varphi^* \circ \psi^*$ is the identity on $\mathcal{O}_P(X)$.

Suppose $(U, \alpha) \in \mathcal{O}_Q(Y)$. That is, $Q \in U$. Now, $P \in X$ and φ is a morphism, so we can find some open set W containing P such that $\varphi|_W$ agrees with some map $U \to \mathbb{P}^m$ with $A \mapsto [F_1(A) : \ldots : F_{m+1}(A)]$ for $F_1, \ldots, F_{m+1}$ homogeneous of the same degree. Thus $P \in \varphi^{-1}(Y) \cap W := U'$, and thus $\varphi^*\left(U, \alpha\right) = (U', \alpha \circ \varphi) \in \mathcal{O}_P(X)$ since if $\alpha = \frac{G}{H}$, then $\left(\alpha \circ \varphi\right)(P) = \frac{G(F_1(P),\ldots,F_{m+1}(P))}{H(F_1(P),\ldots,F_{m+1}(P))} = \frac{G(Q)}{H(Q)}$ is well defined.
Since $\psi(Q) = P$, we can repeat the above to find that ψ^* maps $\mathcal{O}_P(X)$ into $\mathcal{O}_Q(Y)$. Now, we further have that for any $(U, \alpha) \in \mathcal{O}_Q(Y)$, $\psi^* \circ \varphi^*\left(U, \alpha\right) = \psi^*\left(U', \alpha \circ \varphi\right) = \left(U'', \alpha \circ \varphi \circ \psi\right) = \left(U'', \alpha\right) = (U, \alpha)$, and similarly, for any $(U, \alpha) \in \mathcal{O}_P(X)$, $\varphi^* \circ \psi^*\left(U, \alpha\right) = \varphi^*\left(U', \alpha \circ \psi\right) = \left(U'', \alpha \circ \varphi \circ \varphi\right) = \left(U'', \alpha\right) = (U, \alpha)$, so we can thus conclude that φ^* restricts to an isomorphism $\varphi^*|_{\mathcal{O}_Q(Y)}\colon \mathcal{O}_Q(Y) \to \mathcal{O}_P(X)$.
It remains to show that this is a homomorphism of rings.
Suppose $(U, \alpha), (V, \beta) \in \mathcal{O}_Q(Y)$. Then indeed $(U, \alpha) + (V, \beta) = (U \cap V, \alpha|_{U \cap V} + \beta|_{U \cap V}) \in \mathcal{O}_Q(Y)$ and $(U, \alpha) \cdot (V, \beta) = \left(U \cap V\right), \alpha|_{U \cap V} \cdot \beta|_{U \cap V}) \in \mathcal{O}_Q(Y)$ since $Q \in U \cap V$.
Now

$$\begin{aligned}\varphi^*\left((U \cap V, \alpha|_{U \cap V} + \beta|_{U \cap V})\right) &= \left(W \cap \varphi^{-1}(U) \cap \varphi^{-1}(V), \alpha|_{U \cap V} \circ \varphi + \beta|_{U \cap V} \circ \varphi\right) \\ &= \left(W \cap \varphi^{-1}(U), \alpha|_{U \cap V} \circ \varphi\right) + \left(W \cap \varphi^{-1}\left(V\right), \beta|_{U \cap V} \circ \varphi\right) = \varphi^*\left(U, \alpha\right) + \varphi^*\left(V, \beta\right).\end{aligned}$$

And replacing the $+$ with $\cdot$, we get $\varphi^*\left((U, \alpha) \cdot (V, \alpha)\right) = \varphi^*\left(U, \alpha\right) \cdot \varphi^*\left(V, \beta\right)$ also.
Hence φ^* is a ring homomorphism.

We recall that

$$m_P(X) = \{f \in \mathcal{O}_P(X)\colon f(P) = 0\} = \{\text{non-units in } \mathcal{O}_P(X)\} = \{(U, \alpha)\colon P \in U, \alpha(P) = 0\}$$

Now, suppose $(U,\alpha) \in m_Q(Y)$. Then $\varphi^*(U,\alpha) = (U', \alpha\circ\varphi)$ where $P \in U'$ and since $\alpha\circ\varphi(P) = \alpha(Q) = 0$, we have $(U', \alpha\circ\varphi) \in m_P(X)$. Similarly, we get $\psi^*(U,\alpha) \in m_Q(Y)$ for any $(U,\alpha) \in m_P(X)$.
Showing that $\psi^*\circ\varphi^*(U,\alpha) = (U,\alpha)$ for any $(U,\alpha) \in m_Q(Y)$ and $\varphi^*\circ\psi^*(V,\beta) = (V,\beta)$ for any $(V,\beta) \in m_P(X)$ is done the same as the first part of the problem. Thus $\varphi^*|_{\mathcal{O}_Q(Y)}$ restrict to an isomorphism $\varphi^*|_{m_Q(Y)} \colon m_Q(Y) \to m_P(X)$. It remains to show that this isomorphism is, in fact, a homomorphism. It suffices to show this for φ^*.
Letting $(U,\alpha), (V,\beta) \in m_Q(Y)$, we have $Q \in U\cap V$ and $\alpha(Q) = 0 = \beta(Q)$. Thus since $(U,\alpha) + (V,\beta) = (U\cap V, \alpha+\beta)$, we get

$$\begin{aligned}\varphi^*((U,\alpha)+(V,\beta)) = \varphi^*(U\cap V, \alpha+\beta) &= \left(W\cap\varphi^{-1}(U)\cap\varphi^{-1}(V), \alpha\circ\varphi+\beta\circ\varphi\right)\\ &= (W\cap\varphi^{-1}(U), \alpha\circ\varphi) + (W\cap\varphi^{-1}(V), \beta\circ\varphi) = \varphi^*(U,\alpha)+\varphi^*(V,\beta),\end{aligned}$$

showing that φ^* is an isomorphism of abelian groups.

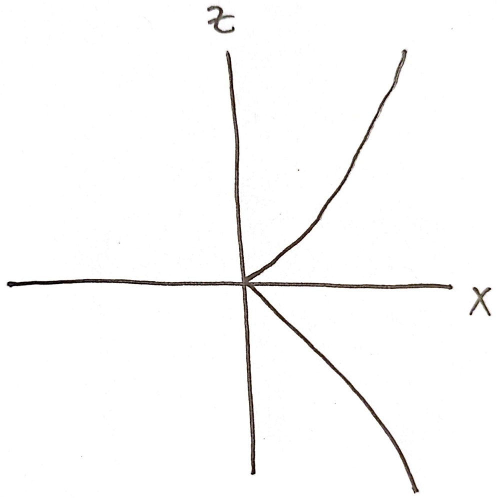
(b) Suppose $\varphi\colon X \to Y$ is an isomorphism. By (a), we have that φ^* restricts to an isomorphism of abelian groups $m_Q(Y) \to m_P(X)$.
We thus have that $\frac{m_P(X)}{m_P(X)^2}$ which is a quotient group is isomorphic to $\frac{m_Q(Y)}{m_Q(Y)^2}$, and hence, if X is smooth, then

$$\dim Y = \dim X \overset{\text{X smooth}}{=} \dim_k \frac{m_P(X)}{m_P(X)^2} = \dim_k \frac{m_Q(Y)}{m_Q(Y)^2}$$

so Y is smooth. If Y were smooth instead, interchange X and Y above, as well as P and Q, giving that X is smooth as well. So X is smooth if and only if Y is smooth.

(c) We have that $\Gamma(V(y)) = k[x,y]/I(V(y)) = k[x,y]/\sqrt{(y)} = k[x,y]/(y) = k[x]$ and $\Gamma\left(V(y-x^3)\right) = k[x,y]/I\left(V\left(y-x^3\right)\right) = k[x,y]/\sqrt{(y-x^3)} = k[x,y]/\left(y-x^3\right) = k[x]$, and $k[x] \cong k[x]$ by the trivial isomorphism (here we have concluded that the $\sqrt{(y)} = (y)$ and $\sqrt{(y-x^3)} = (y-x^3)$ since it is given in the problem that $V(y)$ and $V(y-x^3)$ are varieties. Thus $\Gamma(V(y)) \cong \Gamma\left(V(y-x^3)\right)$, so by the lemma on page 2 of lecture note 8, we have that $V(y)$ and $V(y-x^3)$ are isomorphic as affine algebraic varieties.

(d) The projective closure of a set $X \subset \mathbb{A}^n$ in $\mathbb{P}^n$ is $\mathbb{V}(H(I(X)))$ by a lemma on lecture note 19. Thus, the projective closure of $V\left(y-x^3\right)$ is $\mathbb{V}\left(H\left(I\left(V(y-x^3)\right)\right)\right) \overset{\text{variety}}{=} \mathbb{V}(H(y-x^3)) \overset{\text{principal ideal}}{=} \mathbb{V}\left(yz^2-x^3\right)$, since homogenizing a principal ideal is homoginizing the generator. Now, if $P \in \mathbb{V}(y)$, then $P = [a:0:b]$, so $P \notin U_2$. Suppose $P \in U_1$, then since $\mathbb{V}(y)\cap U_1 = V(y)$ and $\frac{d}{dy}y = 1 \neq 0$, we have that P is not a singular point of $\mathbb{V}(y)$. Similarly, if $P \in U_3$, then $\mathbb{V}(y)\cap U_3 = V(y)$ and $\frac{d}{dy}y = 1 \neq 0$ again, so P is not a singular point of $\mathbb{V}(y)$. Hence P is smooth by definition.
By (b), if $\mathbb{V}(y)$ and $\mathbb{V}(yz^2-x^3)$ were isomorphic, then $\mathbb{V}(yz^2-x^3)$ would be smooth as well. However, we have $[0:1:0] \in \mathbb{V}(yz^2-x^3), U_2$, so $\mathbb{V}\left(yz^2-x^3\right)\cap U_2 = V(z^2-x^3)$, and letting $f = z^2-x^3$, we have $f_x = -3x^2$ and $f_z = 2z$, so evaluating at $(0,0)$, we have $f_x(0,0) = 0 = f_z(0,0)$, so $\mathbb{V}(yz^2-x^3)$ is singular at $[0:1:0]$. Thus the projective closures of $V(y)$ and $V(y-x^3)$ are not isomorphic.
Geometrically, we see that $\mathbb{V}(yz^2-x^3)\cap U_2 = V(z^2-x^3)$ has a cusp at $(0,0)$ making it not smooth here while $\mathbb{V}(y)\cap U_2$ is the single point.



4:
Let $F \in k[x, y, z]$ be a homogeneous polynomial of degree n.
(a) We can write

$$F = \sum_{i+j+k=n, i,j,k \geq 0} \alpha_{i,j,k} x^i y^j z^k.$$

Then

$$F_x = \sum_{i+j+k=n, i \geq 1} \alpha_{i,j,k} \cdot i x^{i-1} y^j z^k$$
$$F_y = \sum_{i+j+k=n, j \geq 1} \alpha_{i,j,k} \cdot j x^i y^{j-1} z^k$$
$$F_z = \sum_{i+j+k=n, k \geq 1} \alpha_{i,j,k} \cdot k x^i y^j z^{k-1}$$

so

$$xF_x = \sum_{i+j+k=n, i \geq 1} \alpha_{i,j,k} \cdot i x^i y^j z^k$$
$$yF_y = \sum_{i+j+k=n, j \geq 1} \alpha_{i,j,k} \cdot j x^i y^j z^k$$
$$yF_z = \sum_{i+j+k=n, k \geq 1} \alpha_{i,j,k} \cdot k x^i y^j z^k.$$

giving the sum

$$xF_x + yF_y + zF_z = \sum_{i+j+k=n, i,j,k \geq 0} a_{i,j,k}(i+j+k) x^i y^j z^k = nF.$$

(b) *Claim:* $f_x = D(F_x)$.
Proof: Suppose $F = \sum_{i+j+k=n} \alpha_{i,j,k} x^i y^j z^k$. Then $F_x = \sum_{i+j+k=n, i \geq 1} \alpha_{i,j,k} i x^{i-1} y^j z^k$. So $D(F_x) = \sum_{i+j+k=n, i \geq 1} \alpha_{i,j,k} i x^{i-1} y^j$.
Now $f = \sum_{i+j+k=n} \alpha_{i,j,k} x^i y^j$, so $f_x = \sum_{i+j+k=n, i \geq 1} \alpha_{i,j,k} i x^{i-1} y^j$, proving the claim.

Suppose that P is a singular point of $\mathbb{V}(F)$ with $P \in U_3$. Thus P is a singular point of $V(F(x, y, 1)) = V(f)$ which means that $f_x(P) = 0 = f_y(P)$ and $f(P) = 0$. But if we denote the dehomoginization operator by D, we get that $f_x = D(F_x)$, so $D(F_x)(P) = 0 = D(F_y)(P)$. Further, $P \in V(F(x, y, 1))$ gives that $F(P) = 0$. Now, $z(P)F_z(P) = nF(P) = 0$ so since $P \in U_3, F_z(P) = 0$. Similarly for if $P \in U_2$ or U_1.
Conversely, if $F(P) = F_x(P) = F_y(P) = F_z(P) = 0$ then suppose $P \in U_3$ wlog, with $P = [a, b, 1]$. Note

that F_i is homogenous for $i = x, y, z$. Then $f_x(P) = D(F_x)(a, b, 1) = F_x(a, b, 1) = 0$. And similarly $f_y(P) = 0$. And $f(P) = D(F)(a, b, 1) = F(a, b, 1) = 0$ since F is homogeneous. Thus P is a singular point of $\mathbb{V}(F)$.

(c)

5:

(a) Let $F = x^2y^3 + x^2z^3 + y^2z^3$. Let $X = \mathbb{V}(F)$. Then by problem 4.(b), we have that P is a singular point if and only if $F(P) = F_x(P) = F_y(P) = F_z(P) = 0$. Now $F_x = 2xy^3 + 2xz^3$, $F_y = 3x^2y^2 + 2yz^3$ and $F_z = 3x^2z^2 + 3y^2z^2 = 3z^2\left(x^2 + y^2\right)$. If $z = 0$ then $x^2y^2 = 0$ so either $x = 0$ or $y = 0$.

If $x = 0$ then $yz^3 = 0$, so either $y = 0$ or $z = 0$. If $y = 0$ then $x = 0$ or $z = 0$.
If none of them are 0, then $x^2 = -y^2$ and $y^3 = -z^3$ and $-2z^3 = 3x^2y$. However, from the first two, we get $3x^2y = 3z^3$, so $-2z^3 = 3z^3$, hence $5z^3 = 0$ so $z = 0$, contradiction.
So the only singular points are $\{[1:0:0], [0:1:0], [0:0:1]\}$.

Let $f = F(x, y, 1) = x^2y^3 + x^2 + y^2$, so the multiplicity at $(0, 0)$ is 2, and the tangent cone is $V(x^2 + y^2))$.

Let $f = F(x, 1, z) = x^2 + x^2z^3 + z^3$, so the multiplicity at $(0, 0)$ is 2, and the tangent cone at $(0, 0)$ is $V(x^2)$.

Let $f = F(1, x, z) = y^3 + z^3 + y^2z^3$, so multiplicity at $(0, 0)$ is 3 and the tangent cone is $V\left(y^3 + z^3\right)$.

(b) Let $F = y^2z - x(x - z)(x - \lambda z), \lambda \in k$. Then $F_x = -(x - z)(x - \lambda z) - x(x - \lambda z) - x(x - z)$, and $F_y = 2yz$ and $F_z = y^2 + x(x - \lambda z) + \lambda x(x - z)$. If $y = 0$ then $x\left(x - \lambda z + \lambda x - \lambda z\right)$. If $x = 0$ then $z \neq 0$ and $\lambda = 0$. If $x \neq 0$ then $x(1 + \lambda) = 2\lambda z$. Also $x = z$ or $x = \lambda z$, so $\frac{2\lambda}{1+\lambda} \in \{1, \lambda\}$, so either $\lambda = 1$ or $\lambda^2 = \lambda \implies \lambda \in \{0, 1\}$. Or $\lambda = -1$ in which case $z = 0$ we get $[1:0:0]$, in either case $x = z$, but inserting into F_x, we get $\lambda = 1$.
Now, if $z = 0$ then $x = 0$ so $y \neq 0$.
So all the singular points are

$$\{[0:0:1], [1:0:1], [0:1:0]\}$$

Let $f = y^2 - x^2(x - z)$, then the multiplicity of $(0, 0)$ is 2 and tangent cone $V(y^2)$.
For $f = y^2 - x(x - 1)^2$ for $(0, 1)$, we insert $x' + 1$, so $y^2 - (x' + 1)x'^2$ which has multiplicity 2 and cone $V\left(y^2 - x'^2\right)$ which maps to $V\left(y^2 - (x - 1)^2\right)$.

Let $f = z - x(x - z)(x - \lambda z)$ which has multiplicity 1 and cone $V(z)$.

6:

(a) Firstly, $v_{2,2}(P) = \left[x_0^2 : x_0y_0 : x_0z_0 : y_0^2 : y_0z_0 : z_0^2\right]$, so $v_{2,2}(P)^* = \mathbb{V}\left(x_0^2w_1 + x_0y_0w_2 + \ldots y_0z_0w_5 + z_0^2w_6\right)$.

If $P \in C$ with $[C] = [a:b:c:d:e:f]$, then

$$ax_0^2 + bx_0y_0 + cx_0z_0 + dy_0^2 + ey_0z_0 + fz_0^2 = 0$$

so $[C] \in v_{2,2}(P)^*$. Conversely, for any $[C] \in v_{2,2}(P)^*$, we have $P \in C$ by the nature of $v_{2,2}$.

(b) We have that $P_1, \ldots, P_5 \in C$ if and only if $v_{2,2}(P_1)^* \cap \ldots \cap v_{2,2}(P_5)^*$ is nonempty, which is true by problem 4 on homework 10.

(c) Suppose $P = [0:0:1]$. Then

Let f be the homogenization of F with respect to z, so

$$f = \sum_{i+j+k=n, i,j,k \geq 0} \alpha_{i,j,k} x^i y^j$$

Then $f_x = \sum_{i+j+k=n, i \geq 1} \alpha_{i,j,k} i x^{i-1} y^j$ and $f_y = \sum_{i+j+k=n, j \geq 1} \alpha_{i,j,k} j x^i y^{j-1}$ vanish at P.

(c)