Problem 0.1. Write down explicit 2×2 matrices that are generators for a Fuchsian group uniformizing the Wiman surfaces of Type I and Type II from the exercises from week 3.

Proof. We will construct the transformation as a composition of complex conjugation, followed by rotation and then inversion.
Firstly, we wish to find the point of inversion: $\alpha = ae^{i\theta}$. We have

$$\frac{L}{\sin\left(\frac{\pi}{2n}\right)} = \frac{a}{\sin\left(\frac{(n+1)\pi}{2n}\right)}, \quad a^2 = 1 + L^2$$

$$\implies a = \sqrt{\frac{\sin^2\left(\frac{(n+1)\pi}{2n}\right)}{\sin^2\left(\frac{(n+1)\pi}{2n}\right) - \sin^2\left(\frac{\pi}{2n}\right)}} \tag{Ω}$$

And clearly $\theta = \frac{\pi}{2n}$. Now the inversion at α is given by

$$\rho(z) = \frac{\alpha\bar{z} - 1}{\bar{z} - \bar{\alpha}}$$

so all together we get the hyperbolic translation to be

$$T(z) = \rho\left(e^{-\frac{(n-1)\pi i}{n}}\bar{z}\right) = \frac{\alpha e^{\frac{(n-1)\pi i}{n}} z - 1}{e^{\frac{(n-1)\pi i}{n}} z - \bar{\alpha}} = \frac{ae^{\frac{\pi i}{2n}} e^{\frac{(n-1)\pi i}{n}} z - 1}{e^{\frac{(n-1)\pi i}{n}} z - ae^{-\frac{i\pi}{2n}}} = \frac{-az - e^{\frac{i\pi}{2n}}}{-e^{\frac{-i\pi}{2n}} z - a}$$

Hence

$$M(T) = \begin{pmatrix} a & e^{\frac{i\pi}{2n}} \\ e^{-\frac{i\pi}{2n}} & a \end{pmatrix}$$

Since $T_j = e^{\frac{(j-1)i\pi}{n}} T\left(e^{-\frac{(j-1)i\pi}{n}}\right)$ we have

$$\begin{aligned} M(T_j) &= \begin{pmatrix} e^{\frac{(j-1)i\pi}{n}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & e^{\frac{i\pi}{2n}} \\ e^{-\frac{i\pi}{2n}} & a \end{pmatrix} \begin{pmatrix} e^{-\frac{(j-1)i\pi}{n}} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & e^{\frac{(2j-1)i\pi}{2n}} \\ e^{-\frac{(2j-1)i\pi}{2n}} & a \end{pmatrix} \end{aligned}$$

So $\Gamma_n = \langle T_1, \ldots, T_n \rangle$ is a Fuchsian group, and we have $W_{2n} \approx \mathbb{D}/\Gamma_n$. □

Let $T(z) = \frac{az+b}{cz+d}$.
Let $f\colon H \to D$ be the map $z \mapsto \frac{z-i}{z+i}$ and let $f^{-1}(z) = -\frac{i(1+z)}{z-1}$ be the inverse. It is indeed easy to check that f^{-1} takes ∂D to $\mathbb{R}\cup\{\infty\}$. Now $T(z) = z$ if and only if $cz^2+(d-a)z-b = 0$. We want the fixed points to be $f^{-1}\left(e^{\frac{i\pi}{2n}}\right) = -\frac{i\left(1+e^{\frac{i\pi}{2n}}\right)}{e^{\frac{i\pi}{2n}}-1}$ and $f^{-1}\left(e^{\frac{(2n+1)i\pi}{2n}}\right) = -\frac{i\left(1+e^{\frac{(2n+1)i\pi}{2n}}\right)}{e^{\frac{(2n+1)i\pi}{2n}}-1}$ (which lie in $\mathbb{R}$), so we have the equation

$$
\begin{aligned}
0 &= \left(z + \frac{i\left(1+e^{\frac{i\pi}{2n}}\right)}{e^{\frac{i\pi}{2n}}-1}\right)\left(z + \frac{i\left(1+e^{\frac{i\pi}{2n}}\right)}{e^{\frac{i\pi}{2n}}-1}\right) \\
&= z^2 + i\left[\frac{\left(1+e^{\frac{i\pi}{2n}}\right)}{e^{\frac{i\pi}{2n}}-1} + \frac{\left(1+e^{\frac{(2n+1)i\pi}{2n}}\right)}{e^{\frac{(2n+1)i\pi}{2n}}-1}\right]z - \frac{\left(1+e^{\frac{i\pi}{2n}}\right)}{e^{\frac{i\pi}{2n}}-1}\frac{\left(1+e^{\frac{(2n+1)i\pi}{2n}}\right)}{e^{\frac{(2n+1)i\pi}{2n}}-1} \\
&= z^2 + i\left[\frac{\left(1+e^{\frac{i\pi}{2n}}\right)}{e^{\frac{i\pi}{2n}}-1} + \frac{\left(1+e^{\frac{(2n+1)i\pi}{2n}}\right)}{e^{\frac{(2n+1)i\pi}{2n}}-1}\right]z - 1
\end{aligned}
$$

Now, we must choose a, b, c, d suitably so that $ad - bc = 1$. Now we have the system of equations

$$
\begin{aligned}
ad &= 1 + b^2 \\
d - a &= bi\left[\frac{\left(1+e^{\frac{i\pi}{2n}}\right)}{e^{\frac{i\pi}{2n}}-1} + \frac{\left(1+e^{\frac{(2n+1)i\pi}{2n}}\right)}{e^{\frac{(2n+1)i\pi}{2n}}-1}\right]
\end{aligned}
$$

Denote the right hand side of the last equation by bC.

$$
a^2C = 2 \implies a = \sqrt{\frac{2}{C}}
$$
$$
d = C + \sqrt{\frac{2}{C}}
$$

Hence

$$
a = \sqrt{\frac{2}{C}}, \quad d = C + \sqrt{\frac{2}{C}}
$$

Hence we have

$$
T(z) = \frac{\sqrt{\frac{2}{C}}z + 1}{z + C \pm \sqrt{\frac{2}{C}}}, \text{where} \quad C = i\left[\frac{\left(1+e^{\frac{i\pi}{2n}}\right)}{e^{\frac{i\pi}{2n}}-1} + \frac{\left(1+e^{\frac{(2n+1)i\pi}{2n}}\right)}{e^{\frac{(2n+1)i\pi}{2n}}-1}\right]
$$

Put in a different way, if we let $r_1 = f^{-1}\left(e^{\frac{i\pi}{2n}}\right) = -\frac{i\left(1+e^{\frac{i\pi}{2n}}\right)}{e^{\frac{i\pi}{2n}}-1}$ and $r_2 = f^{-1}\left(e^{\frac{(2n+1)i\pi}{2n}}\right) = -\frac{i\left(1+e^{\frac{(2n+1)i\pi}{2n}}\right)}{e^{\frac{(2n+1)i\pi}{2n}}-1}$, then $r_1 + r_2 = -C$, so

$$
T(z) = \frac{\sqrt{-\frac{2}{r_1+r_2}}z + 1}{z\sqrt{-\frac{2}{r_1+r_2}} - r_1 - r_2}, \quad G = \begin{pmatrix} \sqrt{-\frac{2}{r_1+r_2}} & 1 \\ 1 & \sqrt{-\frac{2}{r_1+r_2}} - r_1 - r_2 \end{pmatrix}
$$

To convert it to a transformation of the unit disk instead, we have

$$
S(z) = f \circ T \circ f^{-1}(z)
$$

so since $M(T) = G$, $M(f) = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $M\left(f^{-1}\right) = \begin{pmatrix} -i & -i \\ 1 & -1 \end{pmatrix}$, we have

$$\begin{aligned} M(S) &= M(f)M(T)M(f^{-1}) \\ &= \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \sqrt{-\frac{2}{r_1+r_2}} & 1 \\ 1 & \sqrt{-\frac{2}{r_1+r_2}} - r_1 - r_2 \end{pmatrix} \begin{pmatrix} -i & -i \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -i\left(2\sqrt{-\frac{2}{r_1+r_2}} - r_1 - r_2\right) & -i(r_1+r_2) - 2 \\ -i(r_1+r_2) + 2 & -i\left(2\sqrt{-\frac{2}{r_1+r_2}} - r_1 - r_2\right) \end{pmatrix} \end{aligned}$$

Now all of the other side pairings $S_2, \ldots, S_n$ are obtained by conjugation with rotations:

$$\begin{aligned} M(S_i) &= \begin{pmatrix} e^{\frac{(i-1)\pi i}{n}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -i\left(2\sqrt{-\frac{2}{r_1+r_2}} - r_1 - r_2\right) & -i(r_1+r_2) - 2 \\ -i(r_1+r_2) + 2 & -i\left(2\sqrt{-\frac{2}{r_1+r_2}} - r_1 - r_2\right) \end{pmatrix} \begin{pmatrix} e^{-\frac{(i-1)\pi i}{n}} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -i\left(2\sqrt{-\frac{2}{r_1+r_2}} - r_1 - r_2\right) & e^{\frac{(i-1)\pi i}{n}}\left[-i(r_1+r_2) - 2\right] \\ e^{-\frac{(i-1)\pi i}{n}}\left[-i(r_1+r_2) + 2\right] & -i\left(2\sqrt{-\frac{2}{r_1+r_2}} - r_1 - r_2\right) \end{pmatrix} \end{aligned}$$

Hence the other side pairing translations are given by $T_2, T_3, \dots, T_n$ where

$$T_i(z) = f^{-1}\left(e^{\frac{(i-1)\pi i}{n}} T\left(e^{-\frac{(i-1)\pi i}{n}} f(z)\right)\right)$$

and since the association $GL(2,\mathbb{C}) \to Homeo(\mathbb{C}^*)$ is a group homomorphism, it suffices to look at the associated matrices. Now $M(T) = G$, $M(f) = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ and $M\left(f^{-1}\right) = \begin{pmatrix} -i & -i \\ 1 & -1 \end{pmatrix}$, so

$$M(T_i) = \begin{pmatrix} -i & -i \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{-\frac{2}{r_1+r_2}} & e^{\frac{(i-1)\pi i}{n}} \\ 1 & e^{\frac{(i-1)\pi i}{n}}\left[\sqrt{-\frac{2}{r_1+r_2}} - r_1 - r_2\right] \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

where again $r_1 = f^{-1}\left(e^{\frac{i\pi}{2n}}\right) = -\frac{i\left(1+e^{\frac{i\pi}{2n}}\right)}{e^{\frac{i\pi}{2n}}-1}$ and $r_2 = f^{-1}\left(e^{\frac{(2n+1)i\pi}{2n}}\right) = -\frac{i\left(1+e^{\frac{(2n+1)i\pi}{2n}}\right)}{e^{\frac{(2n+1)i\pi}{2n}}-1}$. Then $\Gamma_n = \langle T_1, \dots, T_n \rangle$ is a Fuchsian group such that $\mathbb{H}/\Gamma_n \approx W_{2n}$.