Exercise 0.1 (18)**.** Show that every free module is projective.

Proof. Suppose F is a free R-module on a set X. Let $f\colon F \to M$ be a map and $q\colon N \to M$ a surjective map. We have the diagram

$$\begin{tikzcd} & P \arrow[dl, "g"'] \arrow[d, "f"] & X \arrow[l] \arrow[dl, dashed, "\exists! \tilde{f}"] \\ N \arrow[r, "q"] & M & \end{tikzcd}$$

We define a map $X \to N$ by sending $x \in X$ to some $n \in N$ such that $q(n) = \tilde{f}(x)$ - this exists by surjectivity of q. Then this extends to a unique map $g\colon P \to N$ such that $q \circ g \circ \iota(x) = \tilde{f}(x)$ for all $x \in X$. But this extends to a unique map $\tilde{g}\colon P \to M$ such that $\tilde{g} \circ \iota = q \circ g \circ \iota = \tilde{f}$. Since $q \circ g$ is the unique map for the first equality, we obtain $\tilde{g} = q \circ g$. But for $\tilde{g} \circ \iota = \tilde{f}$, we obtain again by uniqueness that $\tilde{g} = f$. Hence $f = q \circ g$.

□

Exercise 0.2 (19)**.** Suppose R is a PID and let F be a projective R-module. Show that F is free.

Proof. Since F is projective, F is a direct summand in a free R-module, hence in particular, it is finitely generated. Hence $\bigoplus_{i \in I} R \approx F \oplus K$ and by the structure theorem, also $F \approx R^n \oplus \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{s_i} R/\left(p_i^{m_{ij}}\right)$. But $\bigoplus_{i \in I} R$ has no torsion, so F must also be torsion-free, hence $F \approx R^n$, i.e, F is free. □

Exercise 0.3 (20)**.** Let M be an R-module. We say that M is torsion-free if whenver $rm = 0$ for $r \in R$ and $m \in M$, then either $r = 0$ or $m = 0$.

(1) Show that R is torsion-free when considered as a module over itself if and only if R is an integral domain.
(2) Let R be an integral domain. Show that any projective R-module is torsion-free.

Proof. (1) Saying that R is torsion-free as a module over itself is the same as saying $rr' = 0$ for $r \in R$ and $r' \in R$ if and only if $r = 0$ or $r' = 0$ which is precisely the same as saying that R is an integral domain.
(2) If P is a projective R-module, then it is a direct summand in a free R-module. But any free R-module is torsion-free, so P must be as well.
(3) $\mathbb{Q}$ as a $\mathbb{Z}$-module is torsion-free, however $\mathbb{Q}$ is not projective over $\mathbb{Z}$ since the sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is exact, but
$$0 \to \underbrace{\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})}_{=0} \to \underbrace{\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q})}_{\approx \mathbb{Q}} \to \underbrace{\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})}_{=0} \to 0$$
is not exact.

□

Exercise 0.4 (21)**.** Let P be a projective R-module. Prove that there exists a free R-module F such that $P \oplus F$ is free.

Proof. P is a quotient of a free R-module, T, say $P = T/A$. Also, P is a direct summand of a free R-module $\overline{F}$, say $\overline{F} = P \oplus S$. Then

$$P \oplus \left(\oplus_{i \in \mathbb{N}} \overline{F}_i\right) \approx P \oplus \left(\oplus_{i \in \mathbb{N}} S_i \oplus P_i\right) \approx \oplus_{i \in \mathbb{N}} P \oplus S \approx \oplus_{i \in \mathbb{N}} \overline{F}.$$

□