GEOMETRIC TOPOLOGY

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Contents

1. Introduction

My primary two reference books for differential geometry for these notes will be [**KMS**] and [**LeeSM**].

2. Continuous maps

Definition 2.1. For a continuous map $f\colon M \to N$ between topological manifolds,

- f is called an immersion if locally at each point of M, it is of the form $\mathbb{R}^m \to \mathbb{R}^n$ sending $x \mapsto (x, 0)$.
- f is an embedding if it is an immersion, injective and induces a homeomorphism with its image.
- f is a submersion if it is locally of the form to $(x, y) \mapsto x$.

Definition 2.2 (Bundle as defined by Robert (is this supposed to be a fiber bundle?))**.** If $f\colon M \to N$ is a continuous map between topological manifolds, then f is called a bundle if it is locally on N of the form $X \times V \overset{\pi_2}{\to} V$. That is, there exist charts, in which f takes the form of a projection.

3. Smooth Manifolds

Proposition 3.1 (Manifolds are Locally Compact)**.** *Every topological manifold is locally compact.*

Definition 3.2. Let M be a topological space. A collection $\mathcal{X}$ of subsets of M is said to be *locally finite* if each point of M has a neighborhood that intersects at most finitely many of the sets in $\mathcal{X}$. Given a cover $\mathcal{U}$ of M, we say that another cover $\mathcal{V}$ is a *refinement of* $\mathcal{U}$ if for each $V \in \mathcal{V}$, there exists some $U \in \mathcal{U}$ such that $V \subset U$.

Definition 3.3 (Paracompactness)**.** We say that a topological space M is *paracompact* if every open cover of M admits an open, locally finite refinement.

Theorem 3.4 (Manifolds are Paracompact)**.** *Every topological manifold is paracompact. In fact, given a topological manifold M, an open cover $\mathcal{X}$ of M and any basis $\mathcal{B}$ for the topology of M, there exists a countable, locally finite open refinement of $\mathcal{X}$ consisting of elements of $\mathcal{B}$.*

Theorem 3.5. *The fundamental group of a topological manifold is countable.*

Definition 3.6. We say a set $B \subset M$ is a *regular coordinate ball* if there is a smooth coordinate ball $B' \supset \overline{B}$ and a smooth coordinate map $\varphi\colon B' \to \mathbb{R}^n$ such that for some positive real numbers $r < r'$,

$$\varphi(B) = B_r(0), \quad \varphi\left(\overline{B}\right) = \overline{B}_r(0), \quad \text{and} \quad \varphi(B') = B_{r'}(0).$$

Proposition 3.7. *Every smooth manifold has a countable basis of regular coordinate balls.*

Definition 3.8. A subset of a topological space X is said to be *precompact in* X if its closure in X is compact.

Exercise 3.9. For a Hausdorff space X, the following are equivalent

1. X is locally compact.
2. Each point of X has a precompact neighborhood.

(3) X has a basis of precompact open subsets.

Definition 3.10. A sequence $(K_i)_{i\in\mathbb{N}}$ of compact subsets of a topological space X is called an *exhaustion of X by compact sets* if $X = \bigcup_i K_i$ and $K_i \subset \operatorname{int} K_{i+1}$ for each i.

Proposition 3.11. *A second-countable, locally compact Hausdorff space admits an exhaustion by compact sets.*

Lemma 3.12. *[**GG**] Any smooth manifold is metrizable.*

4. Smooth Maps

4.1. A couple of nice formula.

Lemma 4.1 (Change of coordinates on tangent basis)**.** *Suppose $(U,\varphi),(V,\psi)$ are smooth charts on a smooth manifold M and $p \in U \cap V$. Let $\left(x^i\right),\left(\tilde{x}^i\right)$ be the coordinate functions for φ and ψ, respectively. We can write*

$$\psi \circ \varphi^{-1}(x) = \left(\tilde{x}^1(x), \dots, \tilde{x}^n(x)\right).$$

Now

$$\frac{\partial}{\partial x^i}|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(p)\frac{\partial}{\partial \tilde{x}^j}|_{p}$$

where $\tilde{p} = \varphi(p)$ and we are using Einstein summation.

There are very few strong things that we can at this point say about general smooth maps. This section will cover the big tools.

The most important construction on manifolds is that they posses partitions of unity.

Definition 4.2 (Partition of unity)**.** Suppose M is a topological space and let $\mathcal{X} = (X_\alpha)_{\alpha\in A}$ be an arbitrary open cover of M. A *partition of unity subordinate to $\mathcal{X}$* is an indexed family $(\psi_\alpha)_{\alpha\in A}$ of continuous functions $\psi_\alpha \colon M \to \mathbb{R}$ with the following properties:

(1) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and $x \in M$
(2) $\operatorname{supp}\psi_\alpha \subset X_\alpha$ for all $\alpha \in A$
(3) The family of supports $(\operatorname{supp}\psi_\alpha)_{\alpha\in A}$ is locally finite (or we say that the partition of unity or the space is locally finite), meaning that every point has a neighborhood that intersects $\operatorname{supp}\psi_\alpha$ for only finitely many values of α.
(4) $\sum_{\alpha\in A}\psi_\alpha(x) = 1$ for all $x \in M$.

When M is a smooth manifold, a *smooth partition of unity* is one for which each ψ_α is smooth.

Theorem 4.3 (Existence of Partitions of Unity)**.** *Suppose M is a smooth manifold with or without boundary, and $\mathcal{X} = (X_\alpha)_{\alpha\in A}$ is a cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.*

Theorem 4.4 (Existence of Partitions of Unity for Paracompact Spaces)**.** *If X is a paracompact space, then for every open cover, there exists a partition of unity subordinate to the covering.*

Definition 4.5 (Bump functions)**.** If M is a topological space, $A \subset M$ a closed subset, and $U \subset M$ an open subset containing A, a continuous function $\psi \colon M \to \mathbb{R}$ is called a *bump function for A supported in U* if $0 \le \psi \le 1$ on M, $\psi \equiv 1$ on A, and $\operatorname{supp} \psi \subset U$.

Proposition 4.6 (Existence of Smooth Bump Functions)**.** *Let M be a smooth manifold. For any closed subset $A \subset M$ and any open subset U containing A, there exists a smooth bump function for A supported in U.*

Note. The existence of bump functions give us a direct insight into just how different geometry is from complex analysis. In complex analysis, knowing a function in a small region determines it uniquely, while the very existence of smooth bump functions for smooth manifolds tells us that functions cannot be determined from local behavior. In fact, smooth partitions of unity give us a way to glue function patches together on the different parts of the manifold to give arbitrarily complicated functions. This makes the study of geometry very broad and flexible compared to complex analysis, for example.

Another strong property of general smooth maps is encapsulated in Sard's theorem:

Theorem 4.7 (Sard's theorem)**.** *The set of critical values of a smooth map between manifolds has Lebesgue measure zero.*

4.1.1. *Submersions, Immersions and Embeddings.*

Theorem 4.8. *[**GG**] Let X and Y be C^k-manifolds of dimensions n and m, respectively, with $n > m$. Let $\varphi \colon X \to Y$ be a C^k-map. Then*

(1) *If φ is a submersion, then φ is an open map.*
(2) *Let Z be a submanifold of Y. If φ is a submersion at each point in $\varphi^{-1}(Z)$, then $\varphi^{-1}(Z)$ is a C^k-submanifold of X with $\operatorname{codim} \varphi^{-1}(Z) = \operatorname{codim} Z$.*

Corollary 4.9 (Regular Value Theorem)**.** *If q is a regular value of a smooth map $f \colon M^{n+k} \to N^n$, then $f^{-1}(q)$ is a smooth submanifold of M of codimension n.*

Definition 4.10 (Locally Trivial Fibration/Bundle)**.** A locally trivial fibration (following [**JB**]) or a Bundle (following Robert), is a map $f \colon E \to M$ between smooth manifolds such that at each point $p \in M$, there exists a neighborhood U of p such that there exists a diffeomorphism $\varphi \colon U \times F \cong f^{-1}(U)$ for $F = f^{-1}(p)$, making the following diagram commute:

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & f^{-1}(U) \arrow[dl, "f|_{f^{-1}(U)}"] \\ & U & \end{tikzcd}$$

Theorem 4.11 (Fibration Theorem of Ehresmann)**.** *Let $f \colon E \to M$ be a proper submersion of smooth manifolds. Then f is a locally trivial fibration.*

5. Transversality and Function Spaces

For this section, we will closely be following [**GG**].

Definition 5.1. Given maps

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

we say that f is transverse to g, denoted $f \pitchfork g$ if for every $p \in X$ and every $q \in Y$ such that $f(p) = g(y)$, we have

$$(TX)_p \oplus (TY)_q \twoheadrightarrow (TZ)_{f(p)}$$

5.1. **Jet Bundles.**

Definition 5.2. Let X, Y be smooth manifolds and $p \in X$. Suppose $f, g\colon X \to Y$ are smooth with $f(p) = g(p) = q$.

(1) We that f has *first order contact with g at p* if $(df)_p = (dg)_p : T_pX \to T_qY$.
(2) We say that f *has k th order contact with g at p* if $(df) : TX \to TY$ has $(k-1)$ st order contact with (dg) at every point in T_pX. This is written as $f \sim_k g$ at p.
(3) Let $J^k(X,Y)_{p,q}$ denote the set of equivalence classes under " $\sim_k$ at p " of smooth maps $f\colon X \to Y$ where $f(p) = q$.
(4) Define $J^k(X,Y) := \bigcup_{(p,q)\in X\times Y} J^k(X,Y)_{p,q}$. An element $\sigma \in J^k(X,Y)$ is called a *k-jet of mappings (or just a k-jet) from X to Y.*
(5) Let σ be a k-jet. Then for some $(p,q) \in X \times Y$, $\sigma \in J^k(X,Y)_{p,q}$. Then p is called the source of σ and q is called the target of σ. The mapping $\alpha\colon J^k(X,Y) \to X$ given by $\sigma \mapsto$ source of σ is called the source map and the mapping $\beta\colon J^k(X,Y) \to Y$ given by $\sigma \mapsto$ target of σ is called the target map.

Definition 5.3 (k-jet or the k-prolongation of a map)**.** For a smooth map $f\colon X \to Y$, there is a canonically defined map $j^kf\colon X \to J^k(X,Y)$ called the k-jet of f defined by $j^kf(p) = [f,p]$, the equivalence class of f in $J^k(X,Y)_{p,f(p)}$, for every $p \in X$.

Lemma 5.4. *Let $U \subset \mathbb{R}^n$ be open and $p \in U$. Let $f, g\colon U \to \mathbb{R}^m$ be smooth. Then $f \sim_k g$ at p if and only if*

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(p) = \frac{\partial^{|\alpha|} g_i}{\partial x^\alpha}(p)$$

for every multi-index α with $|\alpha| \leq k$ and $1 \leq y \leq m$ where f_i and g_i are the coordinate functions determined by f and g, respectively, and $x_1, \ldots, x_n$ are coordinates on U.

Lemma 5.5. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open. Let $f_1, f_2\colon U \to V$ and $g_1, g_2\colon V \to \mathbb{R}^l$ be smooth. Let $p \in U$. If $f_1 \sim_k f_2$ at p and $g_1 \sim_k g_2$ at $q = f_1(p) = f_2(p)$, then $g_1 \circ f_1 \sim_k g_2 \circ f_2$ at p.*

Proof. We proceed by induction. First, we show the case when $k = 1$. In this case, the statement is precisely that

$$d\left(g_1 \circ f_1\right)_p = d\left(g_2 \circ f_2\right)_p$$

for all $p \in U$. But this is true by the chain rule (Lemma A.2):

$$d\left(g_1 \circ f_1\right)_p = \left(dg_1\right)_q \left(df_1\right)_p = \left(dg_2\right)_q \left(df_2\right)_p = d\left(g_2 \circ f_2\right)_p .$$

Suppose now the statement is true for $k-1$. Then since $(df_1) \sim_{k-1} (df_2)$ at p and $(dg_1) \sim_{k-1} (dg_2)$ at $q = f_1(p) = f_2(p)$, we have by induction that

$$(dg_1) \circ (df_1) \sim_{k-1} (dg_2) \circ (df_2) \quad \forall\, (p,v) \in \{p\} \times \mathbb{R}^n$$

which by the chain rule is precisely saying that

$$d\left(g_1 \circ f_1\right) \sim_{k-1} d\left(g_2 \circ f_2\right)$$

for all $(p,v) \in \{p\} \times \mathbb{R}^n$. But this is precisely the definition of $g_1 \circ f_1 \sim_k g_2 \circ f_2$ at p.

$\square$

Definition 5.6. Let X, Y, Z, W be smooth manifolds.

(1) Let $h\colon Y \to Z$ be smooth.a Then h induces a map $h_*\colon J^k(X,Y) \to J^k(X,Z)$ as follows: if $[f,p] \in J^k(X,Y)_{p,q}$, then $h_*[f,p] = [h \circ f, p] \in J^k(X,Z)_{p,h(q)}$.
(2) If $a\colon Z \to W$ is smooth, then $(a \circ h)_* = a_* \circ h_*$ and $(\mathrm{id}_Y)_* = \mathrm{id}_{J^k(X,Y)}$. So if h is a diffeomorphism, then h_* is a bijection.
(3) Let $g\colon Z \to X$ be a smooth diffeomorphism. Then g induces a map $g^*\colon J^k(X,Y) \to J^k(Z,Y)$ by $g^*[f,p] = \left[f \circ g, g^{-1}(p)\right] \in J^k(Z,Y)$.
(4) Let $a\colon W \to Z$ be a smooth diffeomorphism. Then $(g \circ a)^* = a^* g^*$ and $(\mathrm{id}_X)^* = \mathrm{id}_{J^k(X,Y)}$.

Next, let A_n^k be the vector space of polynomials in n variables of degree $\leq k$ which have constant term equal to 0. As coordinates for A_n^k, we can choose the coefficients of the polynomials. Let $B_{n,m}^k = \oplus_{i=1}^m A_n^k$. Both A_n^k and $B_{n,m}^k$ are smooth manifolds.

Let now $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ smooth. Define $T_k f\colon U \to A_n^k$ as $T_k f(x_0)$ being the kth order Taylor polynomial of f at x_0 without the constant term.
Let $V \subset \mathbb{R}^m$ be open. There is a canonical bijection $T_{U,V}\colon J^k(U,V) \to U \times V \times B_{n,m}^k$ given by

$$T_{U,V}\left([f, x_0]\right) = \left(x_0, f(x_0), T_k f_1(x_0), \ldots, T_k f_m(x_0)\right).$$

This map is well-defined and injective by Lemma 5.4.

Lemma 5.7. *Let $U, U' \subset \mathbb{R}^n$ be open and $V, V' \subset \mathbb{R}^m$ open. Suppose $h\colon V \to V'$ is smooth and $g\colon U \to U'$ a diffeomorphism. Then*

$$T_{U',V'}\left(g^{-1}\right)^* h_* T_{U,V}^{-1}\colon U \times V \times B_{n,m}^k \to U' \times V' \times B_{n,m}^k$$

is smooth.

Definition 5.8 (Smooth structure on $J^k(X,Y)$)**.** Let X, Y be smooth manifolds of dimension n and m, respectively. Let (U,φ) and (V,ψ) be smooth charts in X and Y, respectively. Let $U' = \varphi(U), V' = \psi(V)$. Then let $\tau_{U,V} := T_{U',V'} \circ \left(\varphi^{-1}\right)^* \psi_*\colon J^k(U,V) \to U' \times V' \times B_{n,m}^k$. We declare $\left(J^k(U,V), \tau_{U,V}\right)$ to be a chart for $J^k(X,Y)$. We equip $J^k(X,Y)$ with the smooth structure induced by these smooth charts.
We thus see that

$$\dim J^k(X,Y) = m + n + \dim\left(B_{n,m}^k\right)$$

Theorem 5.9. *Let X and Y be smooth manifolds with $n = \dim X$ and $m = \dim Y$. Then*

1. *$\alpha\colon J^k(X,Y) \to X, \beta\colon J^k(X,Y) \to Y$ and $\alpha \times \beta\colon J^k(X,Y) \to X \times Y$ are submersions.*
2. *If $h\colon Y \to Z$ is smooth, then $h_*\colon J^k(X,Y) \to J^k(X,Z)$ is smooth. If $g\colon X \to Y$ is a diffeomorphism, then $g^*\colon J^k(Y,Z) \to J^k(X,Z)$ is a diffeomorphism.*
3. *If $g\colon X \to Y$ is smooth, then $j^k g\colon X \to J^k(X,Y)$ is smooth.*

Proof. (3) Let $(U,\varphi), (V,\psi)$ be charts about x_0 and $g(x_0)$, respectively. Then in local coordinates,

$$\begin{aligned}\tau_{U,V} \circ j^k g \circ \varphi^{-1}(x) &= \tau_{U,V}\left[g, \varphi^{-1}(x)\right] T_{U',V'}\left[\psi \circ g \circ \varphi^{-1}, x\right] \\ &= \left(x, \psi \circ g \circ \varphi^{-1}(x), T_k\left(\psi_1 \circ g \circ \varphi^{-1}\right)(x), \ldots, T_k\left(\psi_m \circ g \circ \varphi^{-1}\right)(x)\right)\end{aligned}$$

Now, each $T_k\left(\psi_i \circ g \circ \varphi^{-1}\right)$ is smooth being a sum of partial derivatives of the $\psi_i \circ g \circ \varphi^{-1}$ which are smooth functions between Euclidean spaces. Since $j^k g$ is locally smooth everywhere, we find that it is smooth. □

Remark. $J^1(X,Y)$ is canonically isomorphic to $\mathrm{Hom}(TX,TY)$ where the isomorphism $\psi\colon J^1(X,Y) \to \mathrm{Hom}(TX,TY)$ is given as follows: let $\sigma = [f,p] \in J^1(X,Y)_{p,q}$. Then $\psi(\sigma) = (df)_p \in \mathrm{Hom}(T_p,X,T_qY)$. To see that this is well-defined and a diffeomorphism, note that if $[f,p] = [g,q]$, then $p = q$ firstly, and $(df)_p = (dg)_p$ by assumption. Hence $\psi([f,p]) = (df)_p = (dg)_p = \psi([g,q])$.
For the diffeomorphism part, we check that it is a local diffeomorphism and bijective. For bijectivity, if $\psi([f,p]) = \psi([g,q])$, then $p = q$ and $(df)_p = (dg)_q$ by assumption, so indeed $[f,p] = [g,q]$. Fur surjectivity, suppose $f \in \mathrm{Hom}(TX,TY) = \bigcup_{p\in X, q\in Y} \mathrm{Hom}(T_pX, T_qY)$, so there exists $p \in X$ such that $f\colon T_pX \to T_qY$. Then take a chart (U,φ) about $p \in X$ and (V,η) around q in Y with $\varphi(U) = \mathbb{R}^n$ and $\eta(V) = \mathbb{R}^m$, and identifying $T_pX \cong \varphi(U)$ and $T_qY \cong \mathbb{R}^m$. Now drawing f back on some closed set A to a map $A \subset U$ to V, we can use the extension lemma for smooth maps to get a global map $X \to Y$ which agrees with f on A. But the derivative of f is f itself as it is linear, so if $\tilde{f}\colon X \to Y$ is the global map, we get $\psi\left[\tilde{f}, p\right] = (df)_p$.
In local coordiantes, ψ sends

$$(p, f(p), T_1 f_1(p), \ldots, T_1 f_m(p)) \mapsto \begin{pmatrix} T_1 f_1(p) \\ \ldots \\ T_1 f_m(p) \end{pmatrix}$$

when we identify $A_n^1 \cong \mathbb{R}^n$, which is smooth.

Exercise 5.10. There is an obvious canonical projection $\pi_{k,l}\colon J^k(X,Y) \to J^l(X,Y)$ for $k > l$ defined by forgetting the jet information of order $> l$. Show that $J^k(X,Y)$ is a fiber bundle over $J^l(X,Y)$ with projection $\pi_{k,l}$ and identify the fiber.

Exercise 5.11. Let $J^1(X,\mathbb{R})_{X,0}$ be the set of all 1-jets whose target is 0.

1. Show that $J^1(X,\mathbb{R})_{X,0}$ is a vector bundle over X whose projection is the source mapping.
2. Show that $J^1(X,\mathbb{R})_{X,0}$ is canonically isomorphic (as vector bundles) with T^*X.

5.2. **The Whitney C^∞ topology (compact-open topology).**

Definition 5.12. For X, Y manifolds, $k \in \mathbb{Z}_{\geq 0}$ and $U \subset J^k(X,Y)$ open, let
$$M(U) := \left\{ f \in C^\infty(X,Y) \mid j^k f(X) \subset U \right\}.$$
The family of sets $\{M(U)\}$ for U an open subset of $J^k(X,Y)$ form a basis for a topology on $C^\infty(X,Y)$. This topology is called the *Whitney C^k topology.* Let W_k be the set of open subsets of $C^\infty(X,Y)$ in the Whitney C^k topology.
The Whitney C^∞ topology on $C^\infty(X,Y)$ the topology whose basis is $W = \bigcup_{k=0}^\infty W_k$.

How should we understand this topology?
We would like to describe a neighborhood basis of a function $f \in C^\infty(X,Y)$ in the Whitney C^k topology. It will turn out that the we can define δ-balls about f to be smooth maps whose first k partial derivatives are all δ-close to f in a metric on $J^k(X,Y)$ compatible with its topology.

First, choose a metric d on $J^k(X,Y)$ compatible with its topology using Lemma 3.12. Now for a continuous map $\delta\colon X \to \mathbb{R}_+$, define
$$B_\delta(f) := \left\{ g \in C^\infty(X,Y) \mid \forall x \in X\colon d\left(j^k f(x), j^k g(x)\right) < \delta(x) \right\}.$$
We claim now that $B_\delta(f)$ is an open set in $C^\infty(X,Y)$ for any such continuous function δ.
To see this, construct the map $\Delta\colon J^k(X,Y) \to \mathbb{R}$ defined by
$$\Delta(\sigma) = \delta\left(\alpha\left(\sigma\right)\right) - d\left(j^k f\left(\alpha(\sigma)\right), \sigma\right),$$
where, recall, α is the source map. We claim that this is continuous. Indeed, in local coordinates, α is simply a projection, and $j^k f$ is found to be smooth by Theorem 5.9. Since δ is continuous and d is also, Δ is found to be continuous. Hence $U = \Delta^{-1}\left(0, \infty\right)$ is open in $J^k(X,Y)$. Furthermore, we claim $B_\delta(f) = M(U)$.
To see this, we have $g \in M(U)$ if and only if $j^k g(X) \subset U = \Delta^{-1}(0,\infty)$ if and only if for all $x \in X$,
$$\delta\left(\alpha\left(j^k g(x)\right)\right) - d\left(j^k f\left(\alpha\left(j^k g(x)\right)\right), j^k g(x)\right) = \delta(x) - d\left(j^k f(x), j^k g(x)\right) > 0.$$
So $g \in M(U)$ if and only if $g \in B_\delta(f)$. Hence $B_\delta(f)$ is open. To see that this collection forms a basis, suppose W is some open neighborhood of $f \in C^\infty(X,Y)$. We wish to find a $\delta\colon X \to \mathbb{R}_+$ such that $B_\delta(f) \subset W$.
For this, let
$$m(x) = \inf\left\{ d\left(\sigma, j^k f(x)\right) \mid \sigma \in \alpha^{-1}(x) \cap \left(J^k(X,Y)\right) - V \right\},$$
where we let $m(x) = \infty$ if $\alpha^{-1}(x) \subset V$. Now, on any compact set $K \subset X$, m is bounded from below by some constant. So covering X by a countable collection $\{U_\alpha\}$ such that $K_\alpha \subset U_\alpha$ for each α is compact and the collection of compact sets $\{K_\alpha\}$ still covers X, and choosing a constant c_α for bounding m below on K_α, we construct a function $\delta\colon X \to \mathbb{R}_+$ such that $\delta(x) < m(x)$ for every $x \in X$ as follows: take a partition of unity (ψ_α) subordinate to $\{U_\alpha\}$. Then define $\delta(x) = \sum_\alpha c_\alpha \psi_\alpha(x)$.
With this, we find that if $g \in B_\delta(f)$, then $d\left(j^k f(x), j^k g(x)\right) < \delta(x) < m(x)$ for all $x \in X$, so $j^k g(x) \in V$. Hence $g \in M(V) \subset W$. So $B_\delta(f) \subset W$. Thus for any $f \in C^\infty(X,Y)$ and any open set in $C^\infty(X,Y)$ containing f, we can find a basis element $B_\delta(f) \subset W$ containing f. Lastly, we must just check that the intersection of two such basis sets is again a basis set. Let γ, δ be two continuous maps $X \to \mathbb{R}_+$.

Define $\eta(x) = \min\{\gamma(x), \delta(x)\}$. Then η is continuous and $B_\eta(f) = B_\delta(f) \cap B_\gamma(f)$. This finally shows that the collection $\{B_\delta(f)\}$ forms a neighborhood basis of f in the Whitney C^k topology on $C^\infty(X,Y)$.

On a compact manifold, we can define $B_n(f) = B_{\delta_n}(f)$ where $\delta_n(x) = \frac{1}{n}$ for all $x \in X$. Now if $\delta\colon X \to \mathbb{R}_+$ is continuous, then since X is compact, it is bounded from below by $\frac{1}{n}$ for some n. Hence $C^\infty(X,Y)$ is first-countable when X is compact. From the above, one can prove that a sequence of functions f_n in $C^\infty(X,Y)$ converges to f in the Whitney C^k topology if and only if $j^k f_n$ converges uniformly to $j^k f$.
So we see that on compact manifolds, the C^k Whitney topology takes on a nice form. Luckily for us, we will mostly care about compact manifolds, so we can use this interpretation. For the non-compact case, see the discussion in [**GG**].

Definition 5.13 (Residual, Baire space)**.** Let F be a topological space. Then

(1) A subset G of F is called *residual* if it is the countable intersection of open dense subsets of F.
(2) F is called a *Baire space* if every residual set is dense.

Proposition 5.14. *Let X and Y be smooth manifolds. Then $C^\infty(X,Y)$ is a Baire space in the Whitney C^∞ topology.*

5.3. **Transversality.**

Definition 5.15 (Transversality)**.** Let X and Y be smooth manifolds and $f\colon X \to Y$ a smooth map. Let W be a submanifold of Y and $x \in X$. Then f intersects W transversally at x, denoted by $f \pitchfork W$ at x, if either $f(x) \notin W$ or $f(x) \in W$ and $T_{f(x)}Y = T_{f(x)}W \oplus (df)_x\,(T_xX)$.

Proposition 5.16. *Let X and Y be smooth manifolds, $W \subset Y$ a submanifold. Suppose* $\dim W + \dim X < \dim Y$ *(i.e.,* $\dim X < \operatorname{codim} W$ *). Let $f\colon X \to Y$ be smooth and suppose $f \pitchfork W$. Then $f(X) \cap W = \varnothing$.*

Proof. Simple exercise. □

Lemma 5.17. *Let X, Y be smooth manifolds and $W \subset Y$ a submanifold, and $f\colon X \to Y$ smooth. Let $p \in X$ and $f(p) \in W$. Suppose there exists a neighborhood U of $f(p)$ in Y and a submersion $\varphi\colon U \to \mathbb{R}^k$, where $k = \operatorname{codim} W$, such that $W \cap U = \varphi^{-1}(0)$. Then $f \pitchfork W$ at p if and only if $\varphi \circ f$ is a submersion at p.*

Remark. Such a neighborhood U always exists. For there exists a chart neighborhood U of $f(p)$ and a chart $\alpha\colon U \to \mathbb{R}^m$ such that $W \cap U = \alpha^{-1}\left(0 \times \mathbb{R}^{m-k}\right)$ by the definition of W being a submanifold of dimension $m-k$. Letting $\pi\colon \mathbb{R}^m \to \mathbb{R}^k$ be the projection on the first factor, $\varphi = \pi \circ \alpha$ works.

Proof. We have that since $f(p) \in W$, $f \pitchfork W$ at p if and only if $T_{f(p)}Y = T_{f(p)}W \oplus (df)_p\,(T_pX)$. Since $\varphi\,(W \cap U) = 0$, $(d\varphi)_{f(p)}T_{f(p)}W = 0$, we have $\ker\,(d\varphi)_{f(p)} \supset T_{f(p)}W$ for all p. Recall also that $\dim T_{f(p)}U = \dim\ker\,(d\varphi)_{f(p)} + \dim\operatorname{im}\,(d\varphi)_{f(p)}$, so $\dim\ker\,(d\varphi)_{f(p)} = \dim W = \dim T_{f(p)}W$. Hence $\ker\,(d\varphi)_{f(p)} = T_{f(p)}W$.
Hence $f \pitchfork W$ at p if and only if

$$T_{f(p)}Y = \ker\,(d\varphi)_p \oplus (df)_p\,(T_pX)$$

Now

$$\dim \operatorname{im} (d\varphi \circ f)_p = \underbrace{\dim \operatorname{im} (d\varphi)_{f(p)}}_{=k} - \dim \left(\operatorname{im}(df)_p \cap \ker(d\varphi)_{f(p)}\right)$$

Furthermore,

$$\dim T_{f(p)}Y = \dim \ker (d\varphi)_{f(p)} + \dim (df)_p (T_pX) - \dim \Big(\ker (d\varphi)_{f(p)} \cap \operatorname{im}(df)_p\Big)$$

So we see that $\dim \operatorname{im} (d\varphi \circ f)_p = k$ if and only if $\dim \Big(\operatorname{im}(df)_p \cap \ker (d\varphi)_{f(p)}\Big) = 0$ if and only if

$$T_{f(p)}Y = \ker (d\varphi)_p \oplus \operatorname{im} (df)_p$$

$\square$

Theorem 5.18. *Let X and Y be smooth manifolds and W a submanifold of Y. Let $f\colon X \to Y$ be smooth and assume $f \pitchfork W$. Then $f^{-1}(W)$ is a submanifold of X. Also $\operatorname{codim} f^{-1}(W) = \operatorname{codim} W$. In particular, if $\dim X = \operatorname{codim} W$, then $f^{-1}(W)$ consists only of isolated points.*

Proof. It is sufficient to show that $f^{-1}(W)$ is locally a submanifold. Choose U and φ as in Lemma 5.17 and the remark following it. Now choose a neighborhood V of p such that $f(V) \subset U$. By the lemma, $\varphi \circ f$ is a submersion at p, so by contracting V if necessary, we may assume that $\varphi \circ f$ is a submersion on V (full rank is an open submanifold). Thus $f^{-1}(W) \cap V = (\varphi \circ f|_V)^{-1}(0)$ is a submanifold by the regular value theorem. $\square$

Proposition 5.19. *Let X and Y be smooth manifolds with W a submanifold of Y. Let $T_w = \{f \in C^\infty(X, Y) \mid f \pitchfork W\}$. Then T_W is an open subset of $C^\infty(X, Y)$ in the Whitney C^1, and hence C^∞, topology if W is a closed submanifold of Y.*

Theorem 5.20 (Thom Transversality Theorem)**.** *Let X and Y be smooth manifolds and W a submanifold of $J^k(X, Y)$. Let*

$$T_W = \left\{f \in C^\infty(X, Y) \mid j^k f \pitchfork W\right\}.$$

Then T_W is a residual subset of $C^\infty(X, Y)$ in the C^∞ topology.

5.3.1. *Multijet Spaces.*

Definition 5.21. Let X and Y be smooth manifolds. Define

$$X^s = X \times \ldots \times X$$
$$X^{(s)} = \{(x_1, \ldots, x_s) \in X^s \mid x_i \neq x_j, \quad 1 \leq i < j \leq s\}.$$

Let $\alpha\colon J^k(X, Y) \to X$ be the source map. Define $\alpha^s\colon J^k(X, Y)^s \to X^s$ by $(\sigma_1, \ldots, \sigma_s) \mapsto (\alpha\sigma_1, \ldots, \alpha\sigma_s)$. Then define $J^k_s(X, Y) = (\alpha^s)^{-1}\left(X^{(s)}\right)$, called the s-fold k-jet bundle.

A multijet bundle is some s-fold k-jet bundle, $X^{(s)}$ is a manifold since it is an open subset of X^s, so $J^k_s(X, Y)$ is an open subset of $J^k(X, Y)^s$, hence also a smooth manifold.

Let $f\colon X \to Y$ be smooth. Define $j^k_s f\colon X^{(s)} \to J^k_s(X, Y)$ by

$$j^k_s f\,(x_1, \ldots, x_s) = \left(j^k f(x_1), \ldots, j^k f(x_s)\right).$$

Theorem 5.22 (Multijet Transversality Theorem). *Let X and Y be smooth manifolds with W a submanifold of $J^k_s(X,Y)$. Let*

$$T_W = \left\{ f \in C^\infty(X,Y) \mid j^k_s f \pitchfork W \right\}.$$

Then T_W is a residual subset of $C^\infty(X,Y)$ in the C^∞ topology. Moreover, if W is compact, then T_W is open.

5.4. **The Whitney Embedding Theorem.**

Definition 5.23. Given smooth manifolds X, Y, let $\sigma = [f,p] \in J^1(X,Y)$. Then define $\operatorname{rank}\sigma = \operatorname{rank}(df)_p$ and $\operatorname{corank}\sigma = q - \operatorname{rank}\sigma$ where $q = \min\{\dim X, \dim Y\}$. Define

$$S_r = \left\{ \sigma \in J^1(X,Y) \mid \operatorname{corank}\sigma = r \right\}$$

Let's use these definitions to reformulate the definitions of critical points and degenerate critical points.

Firstly, for a map $f\colon X \to \mathbb{R}$, a point $p \in X$ is a critical point if $(df)_p = 0$. Thus $\operatorname{rank} j^1 f = \operatorname{rank}(df)_p = 0$, so $\operatorname{corank} j^1 f = 1$. Therefore if p is a critical point for f, then $[f,p] \in S_1$.
Conversely, if $[f,p] \in S_1$, then $\operatorname{corank}[f,p] = 1$, so $\operatorname{rank}(df)_p = 0$, but $(df)_p : T_pX \to \mathbb{R}$, so having rank 0 means that it must be the 0 map, so $(df)_p = 0$. Hence p is a critical point. So we find that $p \in X$ is a critical point for f if and only if $[f,p] \in S_1$.

Now we make use of the following proposition:

Proposition 5.24. *Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ smooth. Then a point $p \in U$ is a nondegenerate critical point for f if and only if p is a critical point and $j^1 f \pitchfork S_1$ at p.*

Proof. First recall that $J^1(U,\mathbb{R}) \cong U \times \mathbb{R} \times B^1_{n,1}$ by definition/construction. Now, $B^1_{n,1} \cong \operatorname{Hom}(\mathbb{R}^n,\mathbb{R})$. Since $T_p J^1(U,\mathbb{R}) \cong T_p\left(U \times \mathbb{R} \times \operatorname{Hom}(\mathbb{R}^n,\mathbb{R})\right) \cong T_{p_1}U \oplus T_{p_2}\mathbb{R} \oplus T_{p_3}\operatorname{Hom}(\mathbb{R}^n,\mathbb{R})$, we find that the projection $\pi\colon J^1(U,\mathbb{R}) \to \operatorname{Hom}(\mathbb{R}^n,\mathbb{R})$ under this identification on tangent spaces simply becomes the projection on the $T_{p_3}\operatorname{Hom}(\mathbb{R}^n,\mathbb{R})$ factor, hence π is a submersion. Furthermore, if $\pi(\sigma) = 0$, that means then in local coordinates, the first degree Taylor expansions without constant term of a smooth representative f for π at p vanish, so since these determine the equivalence class of $[f,p] = \sigma$, we have $(df)_p = 0$, that is, $\sigma \in S_1$. Hence $S_1 = \pi^{-1}(0)$. In particular, S_1 is a submanifold as the preimage of a regular value. Applying Lemma 5.17, $j^1 f \pitchfork S_1$ at p if and only if $\pi \circ j^1 f$ is a submersion at p. Now

$$\pi \circ j^1 f(x) = (df)_x = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right)$$

so $\pi \circ j^1 f$ is a submersion at p if and only if the map $\mathbb{R}^n \to \mathbb{R}^n$ given by

$$x \mapsto \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right)$$

is a submersion at p if and only if

$$\det H(f)_p = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right) \neq 0.$$

□

Problem 5.25 (Existence of Morse functions)**.** Show that any smooth manifold admits a Morse function.

Proof. The proof of this problem will consist of first showing that the set of Morse functions is an open dense subset of $C^\infty(M,\mathbb{R})$. We will thereafter intersect this set with another residual set in $C^\infty(M,\mathbb{R})$ which will force critical values to be distinct. Then we will finish the problem by making use of $C^\infty(M,\mathbb{R})$ being a Baire space in the Whitney C^∞ topology when M is a manifold.

Theorem 5.26. *Let M be a manifold. The set of Morse functions is an open dense subset of $C^\infty(M,\mathbb{R})$.*

Proof. Recall that S_1 is a submanifold of $J^1(M,\mathbb{R})$. Hence

$$T_{S_1} = \left\{ f \in C^\infty(M,\mathbb{R}) \mid j^1 f \pitchfork S_1 \right\}$$

is a residual subset of $C^\infty(X,Y)$ in the C^∞ topology.
By Theorem 5.24, $j^1 f \pitchfork S_1$ if and only if for all points $x \in X$, either $j_1 f(x) \notin S_1$ or $j_1 f(x) \in S_1$ and $j_1 f \pitchfork S_1$ at x. If $j_1 f(x) \notin S_1$, then x is not a critical value of f. If $j_1 f(x) \in S_1$, then x is a critical value. Then $j_1 f \pitchfork S_1$ at x precisely means that x is a nondegenerate critical point. Hence T_{S_1} precisely consists of all smooth maps $M \to \mathbb{R}$ which are Morse functions (not necessarily distinct critical values).
But by Proposition 5.14, $C^\infty(X,Y)$ is a Baire space in the Whitney C^∞ topology when X and Y are manifolds, so by definition, every residual set is dense. Hence T_{S_1} is dense in $C^\infty(M,\mathbb{R})$. Since 0 is an element, it is in particular nonempty. □

Theorem 5.27. *Let M be a smooth manifold. The set of Morse functions all of whose crtiical values are distinct form a residual set in $C^\infty(M,\mathbb{R})$*

Proof. Let $S = (S_1 \times S_1) \cap J_2^1(M,\mathbb{R}) \cap \left(\beta^2\right)^{-1}(\Delta\mathbb{R})$. We claim that S is a submanifold of the multijet bundle $J_2^1(M,\mathbb{R})$. It suffices to check that it is locally a submanifold. Let U be an open coordinate neighborhood in M diffeomorphic to $\mathbb{R}^n$. Recall that $J^1(U,\mathbb{R}) \cong U \times \mathbb{R} \times B^1_{n,1} \cong \mathbb{R} \times \mathbb{R} \times \mathrm{Hom}\left(\mathbb{R}^n, 1\right)$, so seeing as the coordinates on $J_1^2(X,Y)$ are inherited from the product smooth structure and that of an open subset of a smooth manifold, we find $J_1^2(U,\mathbb{R}) \cong (\mathbb{R}^n \times \mathbb{R}^n - \Delta\mathbb{R}^n) \times (\mathbb{R} \times \mathbb{R}) \times \mathrm{Hom}\left(\mathbb{R}^n,\mathbb{R}\right)^2$. Inserting this in the expression for S and noting that $\left(\beta^2\right)^{-1}(\Delta\mathbb{R})$ means that the codomain coordinates must be the same, so $(\mathbb{R} \times \mathbb{R})$ is replaced by $\Delta\mathbb{R}$, and intersecting with $(S_1 \times S_1)$ means that the coordinates for the partial derivatives all vanish, so $\mathrm{Hom}\left(\mathbb{R}^n,\mathbb{R}\right)^2$ reduces to $(0,0)$. So we get

$$S \cong (\mathbb{R}^n \times \mathbb{R}^n - \Delta\mathbb{R}^n) \times \Delta\mathbb{R} \times (0,0)$$

which indeed is a submanifold of

$$J_1^2\left(U,\mathbb{R}\right) \cong (\mathbb{R}^n \times \mathbb{R}^n - \Delta\mathbb{R}^n) \times (\mathbb{R} \times \mathbb{R}) \times \mathrm{Hom}\left(\mathbb{R}^n,\mathbb{R}\right)^2.$$

Since S is locally a submanifold of $J_2^1(M,\mathbb{R})$ at each point, it is a submanifold. Moreover, $\operatorname{codim} S = 2n+1$ where $n = \dim M$: since indeed $\dim J_1^2\left(U,\mathbb{R}\right) = 2n - 1 + 2 + 2n$ and $\dim S = 2n - 1 + 1$.
Now applying the Multijet Transversality Theorem (Theorem 5.22), we obtain that $T_S = \left\{ f \in C^\infty(M,\mathbb{R}) \mid j_2^1 f \pitchfork S \right\}$ is residual in $C^\infty(M,\mathbb{R})$ equipped with the C^∞ topology.

But by Proposition 5.14, $C^\infty(X,Y)$ is a Baire space in the Whitney C^∞ topology when X and Y are manifolds, so by definition, every residual set is dense. Hence T_S is dense in $C^\infty(M,\mathbb{R})$. Since 0 is an element, it is in particular nonempty.
Now, if $f\colon M \to \mathbb{R}$ is a smooth map. Then $j_2^1 f\colon M^{(s)} \to J_2^1(M,\mathbb{R})$. In particular, suppose that $j_2^1 f \pitchfork S$, then since $\operatorname{codim} S = 2n+1$, while $\dim M^{(2)} = \dim M \times M - \Delta M = 2n-1$, we obtain immediately from Proposition 5.16 that $j_2^1 f(M \times M - \Delta M) \cap S = \varnothing$.
So if p, q are critical points of f, the fact that $j_2^1 f(p,q) \notin S$ means that since $\left(j^1 f(p), j^1 f(q)\right) \in S_1 \times S_1 \cap J_2^1(M,\mathbb{R})$, it must be the failure of being in $\left(\beta^2\right)^{-1}(\Delta\mathbb{R})$ that prevents $j_2^1 f\,(M \times M - \Delta M)$ from intersecting S. I.e., the targets are not equal: $f(p) \neq f(q)$. Since p, q were arbitrary critical values, the critical values of any $f \in T_S$ are thus pairwise distinct.

Now taking the set T_S and T_{S_1} from Theorem 5.26, since T_{S_1} was shown to be an open dense subset of $C^\infty(M,\mathbb{R})$, and T_S was just shown to be residual in $C^\infty(M,\mathbb{R})$, i.e., the countable intersection of open dense subsets of $C^\infty(M,\mathbb{R})$, we find that $T_S \cap T_{S_1}$ is the countable intersection of open dense subsets of $C^\infty(M,\mathbb{R})$ also, hence residual in $C^\infty(M,\mathbb{R})$. From Proposition 5.14, we now obtain that $T_S \cap T_{S_1}$ is dense in $C^\infty(M,\mathbb{R})$, giving us the collection we wanted.

□

This completes the proof of Problem 5.25.

□

6. Bundles

For this section, we will closely be following [**Steenrod**] for the general fibre bundle theory, and [**LeeSM**] for the vector bundle theory.

6.1. Fibre Bundle Theory.

I will define things slightly differently.

Definition 6.1 (Bundle). A bundle is simply a triple (E, p, B) where $p\colon E \to B$ is a map.
The pullback

$$\begin{tikzcd} x^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ x \arrow[r] & B \end{tikzcd}$$

is called the fiber over x.

Definition 6.2 (Fiber bundle). A fiber bundle over B with standard fibre F is a bundle over B such that, given any $x\colon 1 \to B$, the pullback of E along x is isomorphic to F: $x^*E \cong F$.

Definition 6.3 (Locally trivial fibre bundle). If C is a site (???), then a locally trivial fibre bundle over B with typical fibre F is a bundle over B with a cover $(j_\alpha\colon U_\alpha \to B)_\alpha$ such that, for each index α, the pullback E_α of E along j_α is isomorphic in the slice category C/U_α to the trivial bundle $U_\alpha \times F$.

Definition 6.4 (Morphisms of bundles)**.** Let (E, p, B) and (E', p', B') be two bundles. A bundle morphism $(u, f) : (E, p, B) \to (E', p', B')$ is a pair of maps $u \colon E \to E'$ and $f \colon B \to B'$ such that

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

commutes.

Lemma 6.5. *Bundles together with bundle morphisms form a category, which we denote* Bun

Proof. Composition of two morphisms (u, f) and (u', f') is simply done componentwise: $(u', f') \circ (u, f) = (u' \circ u, f' \circ f)$. Now, clearly for a bundle (E, p, B), we have that $(\mathrm{id}_E, \mathrm{id}_B)$ forms an identity morphism, and associativity is inherited from associativity of morphism composition of the ambient category.

□

Definition 6.6 (Slice category)**.** For a category C and an object $c \in C$, we form the category c/C whose objects are morphisms $f \colon c \to x$ with domain c and in which a morphism from $f \colon c \to x$ to $g \colon c \to y$ is a map $h \colon x \to y$ such that

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

commutes. Likewise, there is a category C/c whose objects are morphisms $f \colon x \to c$ with codomain c, and where a morphism from $f \colon x \to c$ to $g \colon y \to c$ is a map $h \colon x \to y$ such that

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes.
The categories c/C and C/c are called the **slice categories** of C **under** and **over** c, respectively.

Proposition 6.7 (**[Riehl]**)**.** *If C is complete and cocomplete, then so are the slice categories c/C and C/c for any $c \in C$.*

So in particular, we have that since Top is complete and cocomplete, so is Top/X for any $X \in \mathrm{Top}$. So the product $E \times_X E'$ exists in Top/X for any $[E \to X], [E' \to X] \in \mathrm{Top}/X$.

Definition 6.8 ($\mathrm{Bun}(N)$)**.** For an object N in the category C, we let $\mathrm{Bun}(N)$ be the slice category C/N.

Definition 6.9 (Topological and smooth fiber bundles with structure group)**.** Let K be a topological group acting on a Hausdorff space F as a group of homeomorphisms. Let X and B be Hausdorff spaces. By a *fiber bundle* over a base space B with total space X, fiber F and structure group K, we mean a bundle map $p \colon X \to B$ together with a maximal chart atlas Φ over B. Explicitly, Φ is a collection of trivializations $\varphi \colon U \times F \to p^{-1}(U)$ such that

(1) each point of B has a neighborhood over which there is a chart in Φ
(2) if $\varphi\colon U \times F \to p^{-1}(U)$ is in Φ and $V \subset U$, then the restriction $\varphi|_{V\times F}$ is also in Φ.
(3) If $\varphi, \psi \in \Phi$ are charts over U then there exists a map $\theta\colon U \to K$ such that $\psi\,(u, y) = \varphi\,(u, \theta(u)(y))$
(4) the set Φ is maximal among the collections satisfying the (1),(2) and (3)

The fiber bundle is called smooth if all the spaces are smooth manifolds and all maps involved are smooth.

Example 6.10. The product bundle If we have a space $B = X \times Y$ and let $p\colon B \to X$ be the projection $p(x, y) = x$, then seeing as $p^{-1}(X) = X \times Y$, we automatically obtain an trivialization $\varphi\colon p^{-1}(X) \cong X \times Y$. The sections (aka cross sections) of B, i.e., continuous maps $X \to X \times Y$ is then just simply equivalent to graphs of maps $X \to Y$. The fibres are all homeomorphic. Since a single trivialization works for all of X, this exhibits $X \times Y$ as a fiber bundle over X with trivial structure group.

Example 6.11 (Möbius band)**.** Take the base space $X = S^1$ obtained from I by identifying ends. Let $Y = I$ be the fibre. We can obtain the Möbius bundle from $I\times I$ by matching the ends by a twist. This descends to a projection $p\colon B \to S^1 = X$ where B is the Möbius band. There are many cross-sections: any curve $I \to I \times I$ by $t \mapsto (t, \gamma(t))$ for some $\gamma\colon I \to I$ such that $\gamma(0) = 1 - \gamma(1)$ works. In particular, any two cross-sections agree on at least one point (see picture [**Steenrod**]. The structure group is $\mathbb{Z}/2$.

Example 6.12 (Klein Bottle)**.** The Klein bottle can be obtained similarly, choosing I as the fibre but S^1 as the base space and then quotienting the ends of $S^1 \times I$. Again, see [**Steenrod**]. See also Figure 1.

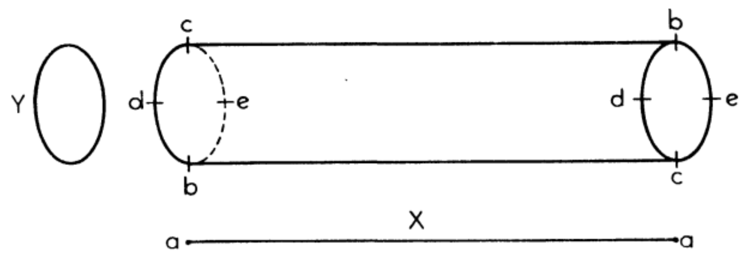


FIGURE 1. Construction of the Klein bottle.

Example 6.13 (Covering Spaces)**.** A covering space B of a space X is another example of a bundle. The projection $p\colon B \to X$ is the covering map. In particular, a covering space is a locally trivial fibre bundle where the fibre is a discrete space.

6.1.1. *Coordinate bundles and fibre bundles.*

Definition 6.14 (Transformation groups)**.** Recall that if G is a topological group and Y is a topological space, we say that G is a topological transformation group of Y relative to a map $\eta\colon G \times Y \to Y$ if

(1) η is continuous
(2) $\eta(e,-) = \mathrm{id}$
(3) $\eta(g_1 g_2, y) = \eta(g_1, \eta(g_2, y))$.

We shall often implicitly assume η as given and abbreviate $\eta(g, y)$ by $g \cdot y$, so that the above become that $\cdot$ is continuous, $e \cdot y = y$ for all y and $(g_1 g_2) \cdot y = g_1 \cdot (g_2 \cdot y)$.

Definition 6.15 (Effective action)**.** We say that G is effective if $g \cdot y = y$ for all y implies that $g = e$.

Definition 6.16 (Coordinate Bundle)**.** A coordinate bundle $\mathcal{B}$ is a collection as follows:

(1) A bundle space B
(2) a base space X
(3) a projection $p\colon B \to X$
(4) a space Y called the fibre
(5) an effective topological transformation group G acting on Y, called the (structure) group of the bundle
(6) A family $\{V_\alpha\}$ of open sets covering X called coordinate neighborhoods
(7) trivializations φ_α giving homeomorphisms

$$\varphi_a\colon V_a \times Y \to p^{-1}(V_a)$$

called coordinate functions.

restricted to the following requirements

(1)
$$\begin{tikzcd} V_\alpha \times Y \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi_1"'] & & p^{-1}(V_\alpha) \arrow[dl, "p"] \\ & V_\alpha & \end{tikzcd}$$

commutes.

(2) letting the map $\varphi_{j,x}\colon Y \to p^{-1}(x)$ be defined by

$$\varphi_{j,x}(y) = \varphi_j(x, y)$$

then for each $x \in V_\alpha \cap V_\beta$, $\varphi_{j,x}^{-1}\varphi_{i,x}(-)\colon Y \to Y$ is the same as $g\cdot(-)\colon Y \to Y$ for some $g \in G$.

(3) the map $g_{\alpha\beta}\colon V_\alpha \cap V_\beta \to G$ by $g_{\alpha\beta}(x) = \varphi_{\alpha,x}^{-1}\varphi_{\beta,x}$ is continuous.

And immediate consequence of the definition is that

$$g_{\gamma\beta}(x) g_{\beta\alpha}(x) = g_{\gamma\alpha}(x), \quad x \in V_\alpha \cap V_\beta \cap V_\gamma.$$

It is also convenient to introduce the map $p_\alpha\colon p^{-1}(V_\alpha) \to Y$ given by $p_\alpha(b) = \varphi_{\alpha,p(b)}^{-1}(b)$.

We obtain the identities

$$p_\alpha \varphi_\alpha(x, y) = y \tag{A_1}$$

$$\varphi_\alpha(p(b), p_\alpha(b)) = b \tag{A_2}$$

$$g_{\alpha\beta}(p(b)) \cdot p_\beta(b) = p_\alpha(b) \tag{A_3}$$

Definition 6.17 (Fibre bundle defined in terms of equivalences of coordinate bundles)**.** Two coordinate bundles $\mathcal{B}$ and $\mathcal{B}'$ are equivalent in the strict sense if they have the same bundle space, base space, projection, fibre and structure group and their coordinate functions satisfy that

$$\overline{g}_{kj}(x) = \varphi'^{-1}_{k,x}\varphi_{j,x}$$

coincide with the operation of an element of G and the map $\overline{g}_{kj}\colon V_j \cap V'_k \to G$ is continuous.
Then a fibre bundle is a maximal coordinate bundle with respect to this equivalence relation.

Definition 6.18 (Mappings of fibre bundles)**.** Let $\mathcal{B}$ and $\mathcal{B}'$ be two coordinate bundles having the same fibre and structure group. A bundle map $h\colon \mathcal{B} \to \mathcal{B}'$ is a tuple $\left(h, \overline{h}\right)$ with $h\colon B \to B'$ and $\overline{h}\colon X \to X'$ such that

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\overline{h}} & X' \end{array}$$

commutes and

$$\overline{g}_{\alpha\beta}(x) = \varphi'^{-1}_{\alpha,x'}h_x\varphi_{\beta,x} = p'_k h_x \varphi_{\beta,x}$$

coincides with the operation of some $g \in G$ on Y. Here $h_x\colon Y_x \to Y_{x'}$ is the map h restricted to the fibre above x, where $x' = \overline{h}(x)$. Note in particular, that this is well defined since by assumption, $p \circ \varphi_\beta = \pi_1$, so in particular, $\operatorname{im}\varphi_{\beta,x} \subset p^{-1}(x)$. Furthermore, the map

$$\overline{g}_{\alpha\beta}\colon V_\beta \cap \overline{h}^{-1}(V'_\alpha) \to G$$

is assumed to be continuous.

In particular, since $\overline{g}_{\alpha\beta}(x)$ acts by some $g \in G$ on Y which is through homeomorphisms, we obtain that since $\varphi'^{-1}_{\alpha,x'}$ and $\varphi_{\beta,x}$ are also homeomorphisms, h_x is a homeomorphism of the fibres.
The mapping transformations $\overline{g}_{\alpha\beta}$ satisfy

$$\overline{g}_{\alpha\beta}(x)g_{\beta\gamma}(x) = \overline{g}_{\alpha\gamma}(x) \tag{Ω_1}$$

$$g'_{\alpha\beta}\left(\overline{h}(x)\right)\overline{g}_{\beta\gamma}(x) = \overline{g}_{\alpha\gamma}(x) \tag{Ω_2}$$

Note. A quick note on terminology: we call the $\varphi_j\colon V_j \times Y \to p^{-1}(V_j)$ *coordinate functions*, the maps $g_{ji}\colon V_i \cap V_j \to G$ by $g_{ji}(x) = \varphi^{-1}_{j,x}\varphi_{i,x}$ *coordinate transformations*, and lastly, the maps $\overline{g}_{kj}\colon V_j \cap \overline{h}^{-1}(V'_k) \to G$ given by $\overline{g}_{kj}(x) = \varphi'^{-1}_{k,x'}h_x\varphi_{j,x}$ *mapping transformations*.

Lemma 6.19. *Let $\mathcal{B}, \mathcal{B}'$ be coordinate bundles having the same fibre Y and group G, and let $\overline{h}\colon X \to X'$ be a map of one base space into the other. Let $\overline{g}_{kj}\colon V_j \cap \overline{h}^{-1}(V'_k) \to G$ be a set of continuous maps satisfying (Ω_1) and (Ω_2). Then there exists a unique fibre bundle map $h\colon \mathcal{B} \to \mathcal{B}'$ inducing $\overline{h}$ and having $\left\{\overline{g}_{jk}\right\}$ as its mapping transformations.*

Proof. [**Steenrod**] We will define h on local patches and then glue these to obtain a global bundle map. Suppose we are given a $b \in B$ such that $p(b) = x \in V_j \cap \overline{h}^{-1}(V'_k)$. We want to end up having that $\overline{g}_{kj}(x) = \varphi'^{-1}_{k,x'}h_x\varphi_{j,x}$. Define

$$h_{kj}(b) = \varphi'_k\left(\overline{h}(x), \overline{g}_{kj}(x) \cdot p_j(b)\right)$$

As a composition of continuous maps, h_{kj} is then continuous as a function of b and $p'h_{kj}(b) = \overline{h}(x) = \overline{h}p(b)$. Now, we must check two things: (1) that h_{kj} and h_{li} agree on $V_i \cap V_j \cap \overline{h}^{-1}(V'_k \cap V'_l)$, and (2) that $\varphi'^{-1}_{\alpha,x'} h_x \varphi_{\beta,x}$ coincides with the operation of some $g \in G$ on Y.
(1) we have

$$\begin{aligned} h_{kj}(b) &= \varphi'_k\left(\overline{h}(x), \overline{g}_{kj}(x) \cdot p_j(b)\right) \\ &\overset{(\Omega_1)}{=} \varphi'_k\left(\overline{h}(x), \overline{g}_{ki}(x) g_{ij}(x) \cdot p_j(b)\right) \\ &\overset{(A_3)}{=} \varphi'_k\left(x', \overline{g}_{ki}(x) \cdot p_i(b)\right) = h_{ki}(b) \end{aligned}$$

Now, by construction, since $g'_{lk}(x') = \varphi'^{-1}_{l,x'}\varphi'_{k,x'}$, we have $\varphi'_l(x', g'_{lk}(x') \cdot y) = \varphi'_l\left(x', \varphi'^{-1}_{l,x'}\varphi'_{k,x'}(y)\right) = \varphi'_k(x', y)$. Hence

$$\begin{aligned} \varphi'_k\left(x', \overline{g}_{ki}(x) \cdot p_i(b)\right) &= \varphi'_l\left(x', g'_{lk}(x') \cdot \overline{g}_{ki}(x) \cdot p_i(b)\right) \\ &\overset{(\Omega_2)}{=} \varphi'_l\left(x', \overline{g}_{li}(x) \cdot p_i(b)\right) \\ &= h_{li}(b). \end{aligned}$$

Thus we can glue $\{h_{kj}\}$ together to form a global map on B.
(2) We have

$$\begin{aligned} \varphi'^{-1}_{k,x'} h_x \varphi_{j,x}(y) &= p'_k \varphi'_k\left(x', \overline{g}_{kj}(x) \cdot p_j\left(\varphi_{j,x}(y)\right)\right) \\ &= \overline{g}_{kj}(x) \cdot y \end{aligned}$$

□

Lemma 6.20. *Let $\mathcal{B}, \mathcal{B}'$ be coordinate bundles having the same fibre and group, and let $h\colon \mathcal{B} \to \mathcal{B}'$ be a bundle map such that the induced map $\overline{h}\colon X \to X'$ is a homeomorphism. Then h has a continuous inverse $h^{-1}\colon B' \to B$, and h^{-1} is a bundle map $\mathcal{B}' \to \mathcal{B}$.*

Proof. If $x_1, x_2 \in X$ lie in different fibers, then $h(x_1) \neq h(x_2)$ since h is fiber preserving. If x_1, x_2 lie in the same fiber, then $h(x_1) \neq h(x_2)$ since h is a linear isomorphism on this fiber.
Thus h is injective. Furthermore, surjectivity of $\overline{h}$ implies surjectivity of h, so h is a bijection. Now, for $x' \in V'_k \cap \overline{h}(V_j)$, let $x = \overline{h}^{-1}(x')$, and define

$$\overline{g}_{jk}(x') = \varphi^{-1}_{j,x} h^{-1}_x \varphi'_{k,x'}$$

Note that these $\overline{g}_{ij}$ satisfy (Ω_1) and (Ω_2), so there exists a unique bundle map with these charts in our setup, given by

$$b' \mapsto \varphi_j\left(\overline{h}^{-1}(x'), \overline{g}_{jk}(x') \cdot p'_k(b')\right).$$

Since h^{-1} induces $\overline{h}^{-1}$ and has $\overline{g}_{jk}$ as transformation maps, if we can show that h^{-1} is continuous, then it will be the unique bundle map. Firstly, $\overline{g}_{jk}(x') = \overline{g}_{kj}(x)^{-1}$. Since $g \mapsto g^{-1}$ is continuous in G and x is continuous in x', and $\overline{g}_{kj}(x)$ is continuous in x, it follows that $\overline{g}_{jk}(x')$ is continuous in x'. Now if $p'(b') = x'$ in $V'_k \cap h(V_j)$, then

$h^{-1}(b')$ will lie in the fibre above $\overline{h}^{-1}(x')$ and in the fibre, under the coordinate map φ_j, it will have coordinate $\overline{g}_{jk}(x') \cdot p'_k(b')$ by construction, so

$$h^{-1}(b') = \varphi_j\left(\overline{h}^{-1}(x'), \overline{g}_{jk}(x') \cdot p'_k(b')\right)$$

which shows that h^{-1} is continuous on $p'^{-1}\left(V'_k \cap \overline{h}(V_j)\right)$.

□

Definition 6.21. Two coordinate bundles $\mathcal{B}$ and $\mathcal{B}'$ with the same base space, fibre and group are said to be equivalent (or isomorphic) if there exists a fibre bundle map $\mathcal{B} \to \mathcal{B}'$ which induces the identity of the common base space.
Two fibre bundles having the same base space, fibre and group are said to be equivalent if they have representative coordinate bundles which are equivalent.

Lemma 6.22. *Let $\mathcal{B}, \mathcal{B}'$ be coordinate bundles having the space base space, fibre and group. Then they are equivalent if and only if there exist continuous maps*

$$\overline{g}_{kj} \colon V_j \cap V'_k \to G$$

such that

$$\overline{g}_{ki}(x) = \overline{g}_{kj}(x) g_{ji}(x)$$
$$\overline{g}_{lj}(x) = g'_{lk}(x) \overline{g}_{kj}(x)$$

.

Proof. Suppose $\mathcal{B}, \mathcal{B}'$ are equivalent through a bundle equivalence $h \colon \mathcal{B} \to \mathcal{B}'$. Define

$$\overline{g}_{kj} = \varphi'^{-1}_{k,x} h_x \varphi_{j,x}.$$

Then the relations which we know hold, (Ω_1), (Ω_2), become the desired relations in the lemma.
Conversely, suppose the $\overline{g}_{kj}$ are given. In the case of $\overline{h} = \mathrm{id}$, the relations in the lemma imply what we want, and the existence of such an h is guaranteed by Lemma 6.19. □

Before presenting the next lemma, we give some motivation and explanation of what "the same coordinate neighborhoods" is supposed to mean.
If $\mathcal{B}$ is a coordinate bundle with neighborhoods $\{V_j\}$ and $\{V'_k\}$ is a covering os X such that each V'_k is contained in some V_j, then we can construct a strictly equivalent coordinate bundle $\mathcal{B}'$ with neighborhoods $\{V'_k\}$ by restricting φ_j to $V'_k \times Y$ where j is such that $V'_k \subset V_j$. In this case, the coordinate functions $\overline{g}_{kj}$ are equal to the identity of G.
Now, suppose that we are given two coordinate bundles $\mathcal{B}, \mathcal{B}'$ with the same base space, fibre and group. The open sets $V_j \cap V'_k$ cover X and form a refinement of both $\{V_j\}$ and $\{V'_k\}$ as above. Thus we can form the refined bundles $\mathcal{B}_1$ and $\mathcal{B}'_1$ of $\mathcal{B}, \mathcal{B}'$ as above and by the above, $\mathcal{B}_1$ is strictly equivalent to $\mathcal{B}$, and $\mathcal{B}'_1$ is strictly equivalent to $\mathcal{B}'$. Now we can analyse $\mathcal{B}_1$ and $\mathcal{B}'_1$ through the following Lemma to conclude whether $\mathcal{B}$ and $\mathcal{B}'$ are equivalent.

Lemma 6.23. *Let $\mathcal{B}, \mathcal{B}'$ be two coordinate bundles with the same base space, fibre, group and coordinate neighborhoods. Let g_{ji}, g'_{ji} denote their coordinate transformations. Then $\mathcal{B}, \mathcal{B}'$ are equivalent if and only if there exist continuous functions $\lambda_j \colon V_j \to G$ such that*

$$g'_{ji}(x) = \lambda_j(x)^{-1} g_{ji}(x) \lambda_i(x).$$

Proof. If $\mathcal{B}, \mathcal{B}'$ are equivalent, then the maps $\overline{g}_{kj}$ from Lemma 6.22 can be used to define $\lambda_j(x) = \overline{g}_{jj}^{-1}(x)$. Then the relations in 6.22 give

$$g'_{ji}(x)\lambda_i^{-1}(x) = \overline{g}_{ji}(x) = \lambda_j^{-1}(x)g_{ji}(x)$$

so

$$g'_{ji}(x) = \lambda_j^{-1}(x)g_{ji}(x)\lambda_i(x)$$

Conversely, if we have

$$g'_{ji}(x) = \lambda_j(x)^{-1}g_{ji}(x)\lambda_i(x)$$

for some $\lambda_j \colon V_j \to G$, then define

$$\overline{g}_{ji}(x) = \lambda_j(x)^{-1}g_{ji}(x)$$

Then we have

$$\begin{aligned}
\overline{g}_{ki}(x) &= \lambda_k(x)^{-1}g_{ki}(x) \\
&= \overline{g}_{kj}(x)g_{kj}(x)^{-1}g_{ki}(x) \\
&= \overline{g}_{kj}(x)g_{ji}(x)
\end{aligned}$$

and

$$\begin{aligned}
g'_{lk}(x)\overline{g}_{kj}(x) &= \lambda_l(x)^{-1}g_{lk}(x)\lambda_k(x)\lambda_k(x)^{-1}g_{kj}(x) \\
&= \lambda_l(x)^{-1}g_{lj}(x) \\
&= \overline{g}_{lj}(x)
\end{aligned}$$

Thus we can apply Lemma 6.22 to conclude that $\mathcal{B}$ and $\mathcal{B}'$ are equivalent. $\square$

Lemma 6.24. *Let $\mathcal{B}, \mathcal{B}'$ be coordinate bundles having the same fibre and group, and let $h \colon \mathcal{B} \to \mathcal{B}'$ be a fibre bundle map. Corresponding to each section $f' \colon X' \to B'$, there exists a unique section $f \colon X \to B$ such that*

$$\begin{array}{ccc}
B & \xrightarrow{h} & B' \\
{\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\
X & \xrightarrow[\overline{h}]{} & X'
\end{array}$$

commutes. The section f is said to be induced by h and f' and will be denoted h^*f'.

Proof. Given $x \in X$, let $x' = \overline{h}(x)$. Since $f(x) \in Y_x$ and $h_x \colon Y_x \to Y_{x'}$ is supposed to be a linear isomorphism, it forces f to have the form $f(x) = h_x^{-1}f'(x')$, hence giving uniqueness. It remains to prove continuity. It suffices to show that f is continuous on local patches of the form $V_j \cap \overline{h}^{-1}(V'_k)$. Since $pf(x) = x$ is continuous, it remains to show that $p_jf(x)$ is continuous. Now, $\overline{g}_{kj}(x)$ is continuous and

$$\begin{aligned}
\overline{g}_{kj}(x) \cdot p_jf(x) &= \varphi'^{-1}_{k,x'}h_x\varphi_{j,x} \cdot p_jf(x) \\
&= \varphi'^{-1}_{k,x'}h_xf(x) \\
&= \varphi'^{-1}_{k,x'}f'\overline{h}(x) \\
&= p'_kf'\overline{h}(x)
\end{aligned}$$

so

$$p_jf(x) = \overline{g}_{kj}(x)^{-1}p'_kf'\overline{h}(x)$$

which is continuous. $\square$

6.1.2. *Construction of a bundle from coordinate transformations.*

Definition 6.25. Let G be a topological group and X a space. By a *system of coordinate transformations in X with values in G* is meant an indexed covering $\{V_j\}$ of X by open sets and a collection of continuous maps

$$g_{ji}\colon V_i \cap V_j \to G$$

satisfying the following condition (called the *cocycle condition*)

$$g_{kj}(x)g_{ji}(x) = g_{ki}(x).$$

Remark. We have so far seen that any bundle over X with group G determines such a set of coordinate transformations. We now state a converse.

Theorem 6.26 (Existence)**.** *If G is a topological transformation group of Y, and $\{V_j\}, \{g_{ij}\}$ is a system of coordinate transformations in the space X, then there exists a bundle $\mathcal{B}$ with base space X, fibre Y, group G and coordinate transformations $\{g_{ij}\}$. Furthermore, any such bundles are equivalent.*

Proof. Let us regard the indexing set J for the covering $\{V_j\}$ as a topological space with the discrete topology. Let $T \subset X \times Y \times J$ be the set of those triples (x, y, j) such that $x \in V_j$.
Define an equivalence relation on T :

$$(x, y, j) \sim (x', y', k) \quad \text{iff} \quad x = x', \text{ and } g_{kj}(x) \cdot y = y'$$

Let B be the set of equivalence classes of this relation in T. Define $q\colon T \to B$ by $(x, y, j) \mapsto [(x, y, j)]$ where $[-]$ denotes taking the equivalence class. We give B the quotient topology with respect to this map q.
Define $p\colon B \to X$ by

$$p\left([(x, y, j)]\right) = x$$

Now, if $W \subset X$ is open, then pulling it back along $pq\colon T \to X$, we have $(pq)^{-1}(W) = q^{-1}\left(p^{-1}(W)\right) T \cap (W \times Y \times J)$ which is open. Hence $p^{-1}(W)$ is open in B by definition of the quotient topology. Hence p is continuous. Now, define $\varphi_j\colon V_j \times Y \times j \to p^{-1}(V_j)$ by

$$\varphi_j\left(x, y\right) = q\left(x, y, j\right) = [(x, y, j)]$$

Then φ_j is continuous since q is continuous and furthermore, $pq\left(x, y, j\right) = x$, so $p\varphi_j(x, y) = x$. Therefore φ_j indeed maps $V_j \times Y$ into $p^{-1}(V_j)$.
We need to show that φ_j is also onto. Suppose $b = [(x, y, k)] \in p^{-1}(V_j)$, so $x \in V_j \cap V_k$. Then by the equivalence relation, $[(x, y, k)] = [(x, g_{jk} \cdot y, j)]$, so $b = \varphi_j\left(x, g_{jk}(x) \cdot y\right)$. Thus φ_j maps $V_j \times Y$ onto $p^{-1}(V_j)$.
We also need to show injectivity. Suppose $\varphi_j\left(x, y\right) = \varphi_j\left(x', y'\right)$. Then $[x, y, j] = [x', y', j]$, so $x = x'$ and $g_{jj}(x) \cdot y = y'$. But $g_{jj}(x) = e$, so $y = y'$. Thus φ_j is injective.

Lastly, we must show that $\varphi_j^{-1}\colon p^{-1}(V_j) \to V_j \times Y$ is continuous. But a set in $p^{-1}(V_j) \subset B$ is open if and only if its preimage under q is open in T by definition of the quotient topology on B. So suppose W is open in $V_j \times Y$. Then since T can be covered by the open sets $V_k \times Y \times k$, it is enough to show that $q^{-1}\varphi_j(W)$ intersects each $V_k \times Y \times k$ in an open set. Now

$$q^{-1}\varphi_j(W) \cap (V_k \times Y \times k) \subset (V_j \cap V_k) \times Y \times k$$

which is open in T. Restricting q to the set on the right can be factored as the composition

$$(V_j \cap V_k) \times Y \times k \xrightarrow{r} V_j \times Y \xrightarrow{\varphi_j} B$$

where $r\,(x, y, k) = (x, g_{jk}(x) \cdot y)$. Now r is continuous, so $r^{-1}(W) = q^{-1}\varphi_j(W) \cap (V_k \times Y \times k)$ is open. Hence $q^{-1}\varphi_j(W)$ is open, so $\varphi_j(W)$ is open.

Lastly, we must show that the coordinate transformations $\varphi_{j,x}^{-1}\varphi_{i,x}$ coincide with $g_{j,i}(x)$.
If $y' = \varphi_{j,x}^{-1}\varphi_{i,x}(y)$ then $\varphi_j(x, y') = \varphi_i(x, y)$ so $q\,(x, y', j) = q\,(x, y, i)$. Therefore, $y = g_{ji}(x) \cdot y$. Hence for each $y \in Y$,

$$\varphi_{j,x}^{-1}\varphi_{i,x}(y) = g_{ji}(x) \cdot y.$$

Hence we obtain a fibre bundle as desired.

□

Theorem 6.27. *The operation of assigning to each bundle with base space X, fibre Y and group G, the system of its coordinate transformations sets up a bijective correspondence between equivalence classes of bundles and equivalence classes of systems of coordinate transformations.*

Thus the classification of bundles reduces to the classification of coordinate transformation. In particular, in the latter problem, the fibre space Y plays no role.

6.1.3. *The Trivial/Product Bundle.*

Definition 6.28 (The Trivial/Product Bundle)**.** A coordinate bundle is called a *product bundle* (or a *trivial bundle*) if there is one coordinate neighborhood $V = X$ and the group G consists of the identity element e along.

Theorem 6.29. *If the group of a bundle consists of the identity element alone, then the bundle is equivalent to the product bundle.*

Proof. If $\mathcal{B}$ is a bundle with coordinate transformations $g_{ji} \colon V_j \cap V_i \to G = \{e\}$, then all the g_{ji} have the same value. So define $\mathcal{B}'$ as the bundle with the same base space, fibre and group and a singular coordinate transformation $g \colon X \to G = \{e\}$. This is continuous since X is open. Then define $\overline{g}_{ji} = e$. Then the conditions in Lemma 6.22 are satisfied. So $\mathcal{B}$ and $\mathcal{B}'$ are equivalent, and $\mathcal{B}'$ is a product bundle. □

6.1.4. *Enlarging the group of a bundle.*

Definition 6.30 (G-image of a bundle)**.** Let H be a closed subgroup of the topological group G. If $\mathcal{B}$ is a bundle with group H, the same coordinate neighborhoods, and the same coordinate transformations, altered only by regarding their values as belong to G, define a new bundle called the *G-image of $\mathcal{B}$*.

Note. If H operates on the fibre Y, it may or may not occur that G operates on Y or even that such operations can be defined. We instead view this bundle in the sense of its system of coordinate transformations as described by Theorem 6.27.

Now regarding equivalence of bundles only in terms of their base space, group and coordinate transformations as in Lemma 6.22, we find that if $\mathcal{B}$ and $\mathcal{B}'$ are equivalent in H, then their G-images are also equivalent.

Definition 6.31 (G-equivalence)**.** Let $H, K \leq G$ be closed subgroups, and let $\mathcal{B}, \mathcal{B}'$ be bundles having the same base space and structure groups H, K, respectively. We say that $\mathcal{B}, \mathcal{B}'$ are *equivalent in* G or G*-equivalent* if the G-images of $\mathcal{B}$ and $\mathcal{B}'$ are equivalent.

Theorem 6.32 (Equivalence Theorem)**.** *Let $\mathcal{B}$ be a bundle with group H and coordinate transformations $\{g_{ji}\}$. Let $H \leq G$. Then $\mathcal{B}$ is G-equivalent to the product bundle if and only if there exist maps $\lambda_j \colon V_j \to G$ such that*

$$g_{ji}(x) = \lambda_j(x)\lambda_i(x)^{-1}, \quad \text{for } x \in V_i \cap V_j.$$

Proof. Let $\mathcal{B}'$ be the bundle having the same coordinate neighborhoods as $\mathcal{B}$ but with the group consisting of the identity element. Then the result follows from Lemma 6.23. □

Definition 6.33 (Second definition of trivial/product bundle)**.** We shall allow ourselves to say that a bundle with group G is equivalent to the product bundle if it is G-equivalent to a product bundle.

Remark. This is also called a "simple bundle" in the literature.

Example 6.34 (Twisted Torus)**.** The twisted torus is obtained the same as for the Klein bottle except that reflection in the diameter de in Figure 1 is replaced by reflection in the center of the circle (rotation through π radians). Again, the group G is $\mathbb{Z}/2$ acting on $Y = S^1$ by the antipodal action. The base space X is also S^1. We define a coordinate system as follows: cover X by two open arcs V_1, V_2. Then $V_1 \cap V_2$ is the union of two disjoint open arcs. If we define $g_{12} \colon V_1 \cap V_2 \to \mathbb{Z}/2$ by $g_{12}(x) = \overline{1}$ if x is in one component, call it U, and $g_{12}(x) = e$ if x is in the other component, call it W. Then the existence theorem, Theorem 6.26, gives a vector bundle $B \to X$. Now, if this were $\mathbb{Z}/2$-equivalent to a product bundle, then by Theorem 6.32, there would exist continuous maps $\lambda_j \colon V_j \to \mathbb{Z}/2$, $j = 1, 2$, such that

$$g_{12}(x) = \lambda_1(x)\lambda_2(x)^{-1}$$

on $U \cup W$. Now, V_1, V_2 are connected, so λ_1, λ_2 take constant values on V_1 and V_2, respectively. But then, in particular, both λ_1 and λ_2^{-1} take the same value on U as they do on W, so g_{12} must be the same on U and W which is a contradiction to the construction. This shows that the twisted torus defined as above is not a product bundle.

Now consider this same situation but with G the full group of rotations of the circle, $SO(2)$, and $\mathbb{Z}/2 \leq G$. We can consider the G-image of the bundle. In this case, we again ask whether there exist continuous maps $\lambda_j \colon V_j \to SO(2)$ such that

$$g_{12}(x) = \lambda_1(x)\lambda_2(x)^{-1}$$

equals $\overline{1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on U and $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on W. This is indeed possible. Simply choose a path from $-I$ to I in $SO(2)$ parametrized by some interval in $S^1 - V_2$. Then extend this to all of V_1 by the constant functions on the ends. We can define λ_2 to be constantly e. This gives the desired conclusion.
Thus, the twisted torus is not a product bundle as a $\mathbb{Z}/2$-bundle, but it is equivalent to the product bundle in $SO(2)$.

6.2. **Quotient/factor spaces of groups.**

6.2.1. *Transitive groups.* Suppose B is a topological group and G is a closed subgroup of B.
We define a left translation action of B on B/G by

$$b \cdot x = p\left(b \cdot p^{-1}(x)\right), \quad \text{for } b \in B, x \in B/G.$$

Under this action, B becomes a group of homeomorphisms of B/G.
Define G_0 to be the intersection

$$G_0 := \bigcap_{b \in B} bGb^{-1}.$$

Arbitrary intersections of closed sets are closed, so G_0 is a closed invariant subgroup of B. One can check that B/G_0 is a topological transformation group of B/G.

Conversely, suppose B is a transitive topological transformation group of X and choose a basepoint $x_0 \in X$. Define $p' \colon B \to X$ by $p'(b) = b \cdot x_0$. Then p' is clearly continuous. Let G be the subgroup of B which stabilizes x_0. Then G is closed. This defines a bijective map $q \colon B/G \to X$ such that $qp(b) = p'(b)$ for all b. If $U \subset X$ is open, then $p'^{-1}(U)$ is open which equals $p^{-1}q^{-1}(U)$, so by construction of the quotient topology, $q^{-1}(U)$ is open. Hence q is continuous.
In general, q^{-1} is not continuous.
Suppose q^{-1} is continuous. Then for U open in B, $p'(U) = qp(U)$ is open in X, so p' is an open map. Conversely, if p' is open and $V \subset B/G$ is open, then $q(V) = p'p^{-1}(V)$ is open in X, so q^{-1} is continuous.

Theorem 6.35. *If B is compact, or if $p' \colon B \to X$ is open, then the natural map $q \colon B/G \to X$ is a homeomorphism, and the maps p' and $p \colon B \to B/G$ satisfy that*

$$\begin{tikzcd} B \arrow[d, "p"] \arrow[dr, "p'"] & \\ B/G \arrow[r, "q"] & X \end{tikzcd}$$

commutes.

6.2.2. *The bundle structure theorem.* We want to show that B is a bundle over B/G with respect to the projection p. Or, more generally, if $H \leq G$ is closed and $p \colon B/H \to B/G$ is the quotient, then B/H is a bundle over B/G with projection p.
It is an unsolved problem [**Steenrod**] whether this is always the case.

Definition 6.36 (Local section of G)**.** Let G be a closed subgroup of B. Then G is a point $x_0 \in B/G$. A *local section* of G in B is a function f mapping a neighborhood V of x_0 continuously into B and such that $pf(x) = x$ for each $x \in V$.

Theorem 6.37 (Bundle Structure Theorem)**.** *If the closed subgroup G of B admits a local section f, and if H is a closed subgroup of G and $p \colon B/H \to B/G$ is the map induced by the inclusion of cosets, then we can assign a bundle structure to B/H relative to p. The fibre of the bundle is G/H and the group is G/H_0 acting in G/H as left translations where H_0 is the largest subgroup of H invariant in G. Furthermore, any two sections lead to strictly equivalent bundles. Finally, the left translations of B/H by elements of B are bundle maps of this bundle onto itself.*

6.2.3. *Lie Groups.*

Definition 6.38 (Lie Group)**.** A *Lie group* B is a topological group and C^1 manifold where the multiplication and inversion operations are also C^1. It is a standard theorem of Lie theory that B is smoothly equivalent to an analytic manifold in which the two operations are analytic.

Remark. [**Steenrod**] A Lie group may have more than one connected component, but each component is open. Any closed subgroup G of B is itself a Lie group and the inclusion $G \hookrightarrow B$ is analytic and non-singular. Furthermore, an analytic structure is defined in the quotient space B/G in such a way that the projection $p\colon B \to B/G$ is analytic and of maximum rank at each point of B. A central step in this process is the construction of a local section of G in B. Hence the bundle structure theorem 6.37 applies to give

Corollary 6.39. *For any closed subgroup G of a Lie group B, the bundle structure theorem applies so B is a fibre bundle over B/G relative to the projection p with fibre G.*

6.2.4. *Orthogonal group.* Let $O(n)$ be the real orthogonal group of transformations in $\mathbb{R}^n$. It is a transitive group on S^{n-1}. If $x_0 \in S^{n-1}$, the subgroup leaving x_0 fixed is isomorphic to $O(n-1)$. Thus by the conclusions in Section 6.2.1, $S^{n-1} \cong O(n)/O(n-1)$, and $O(n)$ is a bundle over S^{n-1} with fibre and group $O(n-1)$.

6.2.5. *Stiefel Manifolds.* A k-frame, v^k, in $\mathbb{R}^n$ is an ordered set of k independent vectors in $\mathbb{R}^n$. Any fixed k-frame v_0^k can be transformed into any other v^k by an element of $\mathrm{GL}_n(\mathbb{R})$. Let $V'_{n,k}$ denote the set of all k-frames, and let $\mathrm{GL}_{v_0^k}(n,\mathbb{R})$ be the subgroup of $\mathrm{GL}(n,\mathbb{R})$ leaving fixed each vector of v_0^k. Then we may identity $V_{n,k} \cong \mathrm{GL}(n,\mathbb{R})\,/\,\mathrm{GL}_{v_0^k}(n,\mathbb{R})$. The quotient space is a manifold with a smooth structure. We assign this structure to $V_{n,k}$. With this structure, $V_{n,k}$ is called the *Stiefel manifold of k-frames in n-space.*

If we restrict attention to k-frames in which the vectors are orthonormal, the set of these, $\tilde{V}_{n,k}$, is a subspace of $V_{n,k}$. It is also a manifold. The group $O(n)$ maps $\tilde{V}_{n,k}$ onto itself and is transitive. The subgroup leaving fixed a v_0^k is the orthogonal group $O\left(n-k\right)$ operating in the space orthonormal to all the vectors of v_0^k. Thus

$$V_{n,k} \cong O(n)/O(n-k).$$

If we translate any v^k along its first vector to its end point on S^{n-1}, we obtain a $(k-1)$ frame of vectors tangent at a point of S^{n-1}. The process is clearly reversible. Thus we may interpret $V_{n,k}$ as the manifold of orthonormal $(k-1)$-frames tangent to S^{n-1}. In particular, $V_{n,2}$ is the manifold of unit tangent vectors on S^{n-1}.

Now let v_0^n be a fixed orthonormal n-frame in $\mathbb{R}^n$ and let v_0^k denote the first k vectors of v_0^n. Let $O(n-k)$ be the subgroup leaving v_0^k fixed. Then $O\left(n-k\right) \supset O(n-k-1)$. Passing to the coset space by these subgroups and introducing the natural projections (inclusion of cosets), we obtain a chain of Stiefel manifolds and projections

$$O(n) = V_{n,n} \to V_{n,n-1} \to \ldots \to V_{n,2} \to V_{n,1} \cong S^{n-1}$$

Each projection or any composition of them is a bundle map. By Theorem 6.37, the fibre of $V_{n,n-k+1} \to V_{n,n-k}$ is the coset space $O(k)/O(k-1) \cong S^{k-1}$, and the group of the bundle is $O(k)$.

Definition 6.40. A bundle in which the fibre is a k-sphere and the gorup is the orthogonal group is called a *k-sphere bundle.*

Thus the Stiefel manifolds provide a chain of sphere bundles connecting $O(n)$ and S^{n-1}.

6.2.6. *The Principal Bundle and the Principal Map.*

Definition 6.41 (Principal G-bundle)**.** A bundle $\mathcal{B} = \{B, p, X, Y, G\}$ is called a principal bundle if $Y = G$ and G operates on Y by left translations.

Definition 6.42 (Associated prinicipal bundle)**.** Let $\mathcal{B} = \{B, p, X, Y, G\}$ be an arbitrary bundle. The *associated principal bundle* $\tilde{B}$ of $\mathcal{B}$ is the bundle given by the construction/existence theorem using the same base space, the same $\{V_j\}$, the same $\{g_{ji}\}$ and the same group G as for $\mathcal{B}$, but replacing Y by G and allowing G to operate on itself by left translations.

Theorem 6.43 (Equivalence theorem for Associated Principal Bundles)**.** *Two bundles having the same base space, fibre and group are equivalent if and only if their associated principal bundles are equivalent.*

Proof. By Lemma 6.22, equivalence of bundles is purely a property of the coordinate transformations. □

Theorem 6.44 (Section Theorem)**.** *A principal bundle with group G is equivalent in G to the product bundle if and only if it admits a section.*

Proof. Suppose a section $f\colon X \to B$ is given. Define $\lambda_i(x) = p_i(f(x))$ for $x \in V_i$. Recall from (A_3) that

$$g_{ji}\left(p(b)\right) \cdot p_i(b) = p_j(b)$$

for $p(b) \in V_i \cap V_j$. Hence we obtain

$$g_{ji}(x) \cdot \lambda_i(x) = \lambda_j(x).$$

Now by Lemma 6.23, we see that the bundle is equivalent to a product bundle.
Conversely, suppose $\mathcal{B}$ is equivalent to a product bundle. By 6.23, there exist $\lambda_i\colon V_j \to G$ such that

$$g_{ji}(x) \cdot \lambda_i(x) = \lambda_j(x)$$

for $x \in V_i \cap V_j$. Now define

$$f_i(x)\varphi_i\left(x, \lambda_i(x)\right).$$

Note that $\lambda_i(x) \in G = Y$ since $\mathcal{B}$ is a principal bundle. Then f is continuous, and for $x \in V_i \cap V_j$, we have

$$f_i(x) = \varphi_i\left(x, \lambda_i(x)\right) = \varphi_i\left(x, g_{ij}(x)\lambda_j(x)\right) = \varphi_j(x, \lambda_j(x)) = f_j(x).$$

It follows that the f_i glue together to give a global section f which agrees with f_i on V_i for each i. □

Corollary 6.45. *A bundle with group G is equivalent in G to a product bundle if and only if the associated principal bundle admits a section.*

Definition 6.46 (Manifold bundle)**.** Let M be a smooth manifold. A manifold bundle over M with structure group G is a fiber bundle $W \to E \to M$ with structure group G such that E is a manifold and $E \to M$ is continuous.
We say a manifold bundle over M is a smooth manifold bundle if it is a smooth fiber bundle as well as a manifold bundle and G acts by diffeomorphisms on M.

Definition 6.47 (Associated bundles)**.** Let M be a smooth manifold, and fix a manifold bundle $E \xrightarrow{\xi} M$ with fibre a smooth manifold W and structure group $G \leq \mathrm{Homeo}(W)$. Given another smooth manifold W' such that there exists an injective group homomorphism $\iota \colon G \hookrightarrow \mathrm{Homeo}\,(W')$, the associated W'-manifold bundle of ξ is defined as follows. Let $\{U_\alpha, \varphi_\alpha\}_\alpha$ be a cover of M by open neighborhoods together with trivializations φ_α of ξ. Transition maps $\varphi_\alpha\varphi_\beta^{-1}$ give rise to transition function $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G \leq \mathrm{Homeo}(W)$ satisfying the cocycle condition. We define the associated W'-manifold by gluing trivializations $U_\alpha \times W'$ along transition maps

$$\iota \circ g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G \xrightarrow{\iota} \mathrm{Homeo}\,(W')\,.$$

Definition 6.48 (Structure group reduction)**.** Fix a manifold bundle $\xi \colon E \to M$ over a smooth manifold M, with fibre a smooth manifold W and structure group G. Given a subgroup $H \leq G$, ξ is said to admit a structure group reduction to H if it is isomorphic to a bundle so that all transition maps $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$ take values in H.

Problem 6.49 (Change of fibres of bundles)**.** Let W_0 and W_1 be two smooth manifolds, and let G be a group which we assume as a simultaneous subgroup of both $\mathrm{Homeo}(W_0)$ and $\mathrm{Homeo}(W_1)$, i.e., we have injective group homomorphisms $\iota_0 \colon G \hookrightarrow \mathrm{Homeo}(W_0)$ and $\iota_1 \colon G \hookrightarrow (W_1)$. Given a fixed smooth manifold M, construct a bijection $\mathrm{Bun}_G^{W_0}(M) \to \mathrm{Bun}_G^{W_1}(M)$, where $\mathrm{Bun}_G^{W_i}(M)$ denotes the set of isomorphism classes of manifold bundles with fibre W_i and structure group G.

Proof. Let $\mathcal{B} = \{B, p, X, W_0, G\} \in \mathrm{Bun}_G^{W_0}$. By Theorem 6.43, the bundle $\mathcal{B}$ is equivalent to its associated principal bundle $\tilde{\mathcal{B}} = \{B, p, X, G, G\}$. But by assumption, G embeds into $\mathrm{Homeo}(W_1)$, so by Theorem 6.26, also $\tilde{B}$ is equivalent to $\{B, p, X, W_1, G\} =: \mathcal{B}'$ which has the same coordinate transformations. Thus $\tilde{B}$ and $\tilde{B}'$ are equivalent. Now, seeing as equivalence of bundles is purely determined by their base space, fibre, structure group and coordinate transformations, this gives an injective map $\mathrm{Bun}_G^{W_0} \to \mathrm{Bun}_G^{W_1}$. Seeing as we can do the exact same thing to obtain an injective map $\mathrm{Bun}_G^{W_1} \to \mathrm{Bun}_G^{W_0}$, we obtain a bijection by Schröder-Bernstein. $\square$

6.2.7. *Associated bundles and relative bundles.*

Definition 6.50. Two bundles, having the same base space X and the same group G, are said to be *associated* if their associated principal bundles are equivalent.

Exercise 6.51. Check that the relation of being associated is reflexive, symmetric and transitive.

Definition 6.52 (Relative bundle)**.** Let $\mathcal{B} = \{B, p, X, Y, G\}$ be a bundle. Let $A \subset X$ be a closed subspace and $H \leq G$ a closed subgroup. If, for every i, j and every $x \in V_i \cap V_j \cap A$, the coordinate transformation $g_{ji}(x)$ is an element of H, then the portion of the bundle over A may be regarded as a bundle with group H. One simply restricts the coordinate neighborhoods and functions to A. Whenever this occurs, we say that $\mathcal{B}$ is a *relative* (G, H)*-bundle over the base space* (X, A).

Definition 6.53 ((G,H)-equivalence)**.** Let $\mathcal{B}$ be a (G,H)-bundle over (X,A) and let $\mathcal{B}'$ be an (H,H)-bundle over (X,A). A (G,H)-equivalence of $\mathcal{B}$ and $\mathcal{B}'$ is a map $h\colon \mathcal{B}\to\mathcal{B}'$ which is, first, a G-equivalence of the two absolute bundles over X, and, second, an H-equivalence when restricted to the portions of $\mathcal{B},\mathcal{B}'$ lying over A.

Slogan. The smaller the group of a bundle, the simpler the bundle.

6.2.8. *The canonical section of a relative bundle.* Let $\mathcal{B}$ be a (G,H)-bundle over (X,A). Let $\mathcal{B}'$ denote the associated bundle over X having G/H as fibre and G acting on the fibre by left translations. Let e_0 denote the coset of H treated as an element of G/H. We define a section over A of the bundle $\mathcal{B}'$ by

$$f_0(x)=\varphi_j'(x,e_0),\quad x\in V_j\cap A.$$

If $x\in V_i\cap V_j\cap A$, then

$$\varphi_j'(x,e_0)=\varphi_i'\left(x,g_{ij}(x)\cdot e_0\right)=\varphi_i'(x,e_0)$$

since $g_{ij}(x)\in H$. Thus f_0 defines a section over A. We call f_0 the *canonical section of the* (G,H)*-bundle.*

6.2.9. *Structure Group Reduction.*

Definition 6.54. For a bundle where the fibres are of the form G/H, if G operates effectively on G/H, we obtain an associated bundle; otherwise, a weakly associated bundle.

Theorem 6.55. *Let $H\le G$ be a closed subgroup which has a local section. A (G,H)-bundle over (X,A) is (G,H)-equivalent to an (H,H)-bundle over (X,A) if and only if the canonical section (defined only over A) can be extended to a full section of the weakly associated bundle with fibre G/H.*

Corollary 6.56. *If H has a local section in G, then a G-bundle over X is G-equivalent to an H-bundle if and only if the weakly associated bundle with fibre G/H has a section.*

Tomorrow, check out the link https://math.stackexchange.com/questions/2015174/structure-group-of-tangent-bundle-of-riemannian-manifold

6.2.10. *Associated frame bundles and structure group reductions.*

Problem 6.57. For a rank d vector bundle $\xi\colon E\to M$ over a smooth manifold, we define the associated frame bundle $\mathrm{Fr}\,(\xi)$ as the associated $\mathrm{GL}_d\,(\mathbb{R})$-bundle.

(1) For M a smooth d-dimensional manifold, we define its frame bundle $\mathrm{Fr}(M)$ as the associated frame bundle of its tangent bundle TM. Show that $\mathrm{Fr}(M)\to M$ is a principal $\mathrm{GL}_d\,(\mathbb{R})$-bundle.
(2) Show that a manifold is orientable if and only if its frame bundle $\mathrm{Fr}(M)$ admits a $\mathrm{GL}_d^+(\mathbb{R})$ reduction of its structure group, where $\mathrm{GL}_d^+(\mathbb{R})$ is the subgroup of the general linear group consisting of invertible matrices with positive determinant.
(3) Show that a structure bundle reduction of the frame bundle $\mathrm{Fr}(M)$ to the orthogonal group $O(n)\le\mathrm{GL}_d(\mathbb{R})$ corresponds to a choice of a bundle metric on the tangent bundle TM of M.

6.2.11. *The Induced Bundle.*

Definition 6.58 (First definition of the induced bundle)**.** Suppose we have a bundle $\mathcal{B}'$ over a base space X', fibre Y and group G which is uniquely determined up to isomorphism by a system of coordinate transformations $\{V'_\alpha\}$ and $\left\{g'_{\alpha\beta}\right\}$. Suppose now we have a map $\eta\colon X \to X'$. The *induced bundle* $\eta^*\mathcal{B}'$ having base space X, fibre Y and group G is defined by pulling back the system of coordinate transformations by letting $\{V_\alpha\}$ with $V_\alpha = \eta^{-1}(V'_\alpha)$ and $\{g_{\alpha\beta}\}$ with $g_{\alpha\beta}(x) = g'_{\alpha\beta} \circ \eta(x)$ be the system of coordinate transformations of $\eta^*\mathcal{B}'$ and then constructing a bundle using the Existence theorem (Theorem 6.26). We define a map $h\colon \eta^*\mathcal{B}' \to \mathcal{B}'$ (which, recall, is a map $B \to B'$) by

$$h(b) = \varphi'_j\left(\eta p(b), p_j(b)\right), \qquad p(b) \in V_j$$

Recall that $p_j\colon p^{-1}(V_j) \to Y$ is given by $p_j(b) = \varphi^{-1}_{j,p(b)}(b) \in Y$. Indeed then $\eta p(b) \in X'$, so $(\eta p(b), p_j(b)) \in X' \times Y$, and φ'_j is defined on some open subset of this space. To show that h is well-defined, we must show that it agrees on overlaps. If $p(b) \in V_i \cap V_j$, then

$$\begin{aligned}\varphi'_j\left(\eta(p(b)), p_j(b)\right) &= \varphi'_i\left(\eta(p(b)), g'_{ij}\left(\eta\left(p(b)\right) \cdot p_j(b)\right)\right)\\ &= \varphi'_i\left(\eta(p(b)), g_{ij}(x) \cdot p_j(b)\right) = \varphi'_i\left(\eta(p(b)), p_i(b)\right)\end{aligned}$$

Furthermore, all the maps in the definition of h are continuous, so h is continuous.

In particular, $p'h(b) = \eta(p(b))$, so indeed h induces η on $X \to X'$. I.e.,

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\eta} & X' \end{array}$$

commutes. Lastly, we want to show that h is a bundle map.
This means that we must show that for $x \in V_j \cap \eta^{-1}(V'_k)$, the map

$$\overline{g}_{kj}(x) = \varphi'^{-1}_{k,x'} h_x \varphi_{j,x} = p'_k h_x \varphi_{j,x}\colon Y \to Y$$

coincides with the operation of some $g \in G$ on Y. That is, that $\overline{g}_{kj}\colon V_j \cap V_k \to G$ is continuous for any k, j. But indeed

$$\begin{aligned}\overline{g}_{kj}(x) \cdot y &= \varphi'^{-1}_{k,x'} h_x \varphi_{j,x}(y)\\ &= \varphi'^{-1}_{k,x'} \varphi'_j\left(x', p_j\left(\varphi_{j,x}(y)\right)\right)\\ &= \varphi'^{-1}_{k,x'} \varphi'_j\left(x', y\right)\\ &= \varphi'^{-1}_{k,x'} \varphi'_{j,x'}(y)\\ &= g'_{k,j}(x') \cdot y\end{aligned}$$

so $\overline{g}_{kj} = g'_{kj} \circ \eta = g_{kj}$, and it is a continuous map of $V_k \cap V_j$ into G.

Definition 6.59 (Second definition of the induced bundle)**.** Suppose $\mathcal{B}', X$ and η are as before. Form the product space $X \times B'$ and let $p\colon X \times B' \to X, h\colon X \times B' \to B'$ be the natural projections. Define $B = X \times_{X'} B' := \{(x, b') \in X \times B' \mid \eta(x) = p'(b')\}$ to be the fibered product.

We want to give $[p\colon B \to X]$ a fibre bundle structure (by giving it a coordinate bundle structure). Define $V_j = \eta^{-1}(V_j')$ and set

$$\varphi_j(x,y) = \left(x, \varphi_j'\left(\eta(x), y\right)\right).$$

Let's give these maps some motivation. For these to be trivializations, we want φ_j to be homeomorphisms $p^{-1}(V_j) \cap B = p|_B^{-1}(V_j) \cong V_j \times Y$. Now, φ_j simply maps x to x in the first coordinate, but φ_j' by assumption maps $V_j' \times Y$ homeomorphically onto $p'^{-1}(V_j')$. Hence in particular, $\varphi_j'(\eta(x), y) \in p'^{-1}(V_j') \subset B'$. So $\left(x, \varphi_j'(\eta(x), y)\right) \in B$ if and only if $\eta(x) = p'\left(\varphi_j'(\eta(x), y)\right)$, but this is true by assumption. Furthermore, $\left(x, \varphi_j'(\eta(x), y)\right) \in X \times B'$, so applying p, we get $p\left(x, \varphi_j'(\eta(x), y)\right) = x$ which is in V_j when $x \in V_j$. Hence putting things together, φ_j maps $V_j \times Y$ to $p^{-1}(V_j) \cap B$. We, in fact, want to show that φ_j is a homeomorphism of these spaces. For this, simply note that the map $(u,v) \mapsto \left(u, \pi_2 \circ \varphi_j'^{-1}(v)\right)$ is an inverse.
Lastly, let for $x \in V_i \cap V_j$, $g_{ij}(x) = \varphi_{i,x}^{-1}\varphi_{j,x} = p_i \varphi_{j,x}$
Note then that

$$\begin{aligned} g_{ij}(x)y &= p_i \varphi_{j,x}(y) \\ &= p_i\left(x, \varphi_j'\left(\eta(x), y\right)\right) \\ &= p_i' \varphi_j'\left(\eta(x), y\right) \\ &= g_{ij}'\left(\eta(x)\right) y \end{aligned}$$

So the clutching functions are simply $g_{ij}' \circ \eta$ which are indeed continuous.

Theorem 6.60 (Equivalence Theorem/pullbacks of fibre bundles with the same fibre and group exist)**.** *Let $\mathcal{B}, \mathcal{B}'$ be two bundles having the same fibre and group and $h\colon \mathcal{B} \to \mathcal{B}'$ a bundle map. Let $\eta\colon X \to X'$ be the induced map of base spaces. Then the induced bundle $\eta^*\mathcal{B}'$ is equivalent to $\mathcal{B}$, and there is an equivalence $h_0\colon \mathcal{B} \to \eta^*\mathcal{B}'$ such that h is the composite $h = h^* \circ h_0$ where $h^*\colon \eta^*\mathcal{B}' \to \mathcal{B}'$ is the induced map:*

$$\begin{tikzcd} B \arrow[dr, dashed, "\exists"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_{X'} B' \cong \eta^* B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B' \arrow[d] \\ & X \arrow[r, "\eta"] & X' \end{tikzcd}$$

Definition 6.61 (Orientability)**.** A smooth manifold M is called *orientable* if for all smooth maps $S^1 \to M$, f^*TM is trivializable. That is, $[f^*TM \to S]$ is a trivial bundle.

6.3. **A Bundle Theory.**

Note. A "Bundle Theory" is also called a Cartesian Fibration over Sm.

Definition 6.62 (Essential fibers)**.** For a functor $F\colon \mathcal{B} \to \mathcal{C}$ and an object $M \in \mathcal{C}$, the (essentail) fiber above M is the fibered category $\mathcal{B} \times_{\mathcal{C}} \mathbb{1}$ making

$$\begin{tikzcd} \mathcal{B} \times_{\mathcal{C}} \mathbb{1} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d, "F"] \\ * \arrow[r, "* \mapsto M"] & \mathcal{C} \end{tikzcd}$$

commute.

Definition 6.63 (Bundle Theory)**.** A bundle theory is a functor from some arbitrary category $\mathcal{B}$ to Sm subject to the following conditions.
Given a map $f\colon M \to N$ between smooth manifolds in Sm, there exists a map $f^*\colon \mathcal{B}(N) \to \mathcal{B}(M)$.
The solid arrows in the diagram below, the dashed lifts are in bijection and the diagram commutes.

$$\begin{tikzcd} B' \arrow[r, dashed, "\psi"] \arrow[d] \arrow[rr, bend left] & f^*B \arrow[r] \arrow[d] \arrow[dr, phantom, "\exists"] & B \arrow[d] \\ N \arrow[r, dashed, "\varphi"] \arrow[rr, bend right] & P \arrow[r, "f"] & M \end{tikzcd}$$

In the sense that given φ, there exists a ψ, everything commutes and composite map above is mapped under the functor to the composite map below.
Furthermore, it is required to satisfy gluing (the cocycle condition): given $U_{ijk} \hookrightarrow U_{ij} \hookrightarrow U_i \hookrightarrow M$ and a bundle $B \in \mathcal{B}(M)$, we can consider the restricted bundles $B|_{U_i} = B_{U_i} = B_i \in \mathcal{B}(U_i)$ for each i, and likewise for B_{ij} and B_{ijk} for all combinations of i, j and k. For these, we have transition
A bundle $B \to M$ is called locally trivial if for each point $x \in M$, there exists a neighborhood $x \in U \overset{i}{\hookrightarrow} M$ and there exists a bundle $B' \to *$ and a pullback along $\pi\colon U \to *$ for B' such that there exists an isomorphism $i^*B \cong \pi^*B'$.

6.4. **Principal G-bundles.** Let G be a discrete group. Consider the category Sm^G where objects are smooth manifolds equipped with a free, fixed point free action by G which is properly discontinuous: the exists a cover $\{U_\alpha\}_{\alpha \in A}$ of M so that $\{g \cdot U_\alpha\}$ are pairwise disjoint for all $\alpha \in A$ and $g \in G$. Furthermore, morphisms are smooth maps which are G-equivariant: $f\colon M \to N$ is such that $f\,(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in M$.

Problem 6.64. (1) Show that for $M \in \mathrm{Sm}^G$, the quotient M/G admits a structure of a smooth manifold so that the map $M \to M/G$ is a local diffeomorphism.
(2) Check that the association $M \mapsto M/G$ defines a functor $\mathrm{Sm}^G \to \mathrm{Sm}$, and show that this defines a locally trivial bundle theory on smooth manifolds.

Proof. (1) (I will assume that G acts by homeomorphisms on M) Using the covering space quotient theorem (theorem 12.14 in Lee's book on Topological Manifolds), we find that $M \to M/G$ is a covering space. To construct a smooth structure on M/G, let $p \in M/G$ and U an evenly covered open neighborhood of p. Then U splits into homeomorphic copies $\sqcup U_\alpha$ in M with $\pi|_{U_\alpha}\colon U_\alpha \cong U$ homeomorphisms. For $\tilde{p} \in U_\alpha$, choose a smooth chart $(V_{\tilde{p}}, \varphi_{\tilde{p}})$ contained in U_α. Since $\tilde{p} = g \cdot p$ for some g, we may as well denote these charts as $(V_{g,p}, \psi_{g,p})$. Now consider the charts $\left(\pi|_g(V_{g,p}), \psi_{g,p} \circ (\pi|_g)^{-1}\right)$. On an overlap the transition functions have the form

$$\psi_{g,p} \circ (\pi|_g)^{-1} \left(\psi_{g',p'} \circ (\pi|_{g'})^{-1}\right)^{-1} = \psi_{g,p} \circ (\pi|_g)^{-1} \pi|_{g'} \circ \psi_{g',p'}^{-1} = \psi_{g,p} \circ \psi_{g',p'}^{-1}$$

on the overlap, which is smooth by assumption. Hence we indeed obtain a smooth structure on M/G. In particular, the map $\pi\colon M \to M/G$ has coordinate form

$$\left(\psi_{g,p} \circ \pi|_g^{-1}\right) \pi \circ \psi_{g,p}^{-1} = \mathrm{id}$$

which is a diffeomorphism. So π is a local diffeomorphism when we equip M/G with this smooth structure.

(2) Define the functor $F\colon \mathrm{Sm}^G \to \mathrm{Sm}$ sending $M \mapsto M/G$ with the smooth structure defined in the first part of the exercise. Here, since maps $f\colon M \to N$ in Sm^G are G-equivariant, they, in particular, descend to smooth maps $\overline{f}\colon M/G \to N/G$, and we let $F(f) = \overline{f}$. Then indeed $F(\mathrm{id}_M) = \overline{\mathrm{id}_M} = \mathrm{id}_{M/G}$ and if $f\colon M \to N$ and $g\colon N \to P$, then $F(g \circ f) = \overline{g \circ f}$. But by pasting the two squares

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[r, "g"] \arrow[d] & P \arrow[d] \\ M/G \arrow[r, "\overline{f}"] & N/G \arrow[r, "\overline{g}"] & P/G \end{tikzcd}$$

we find that $\overline{g \circ f} = \overline{g} \circ \overline{f}$. So $F(g \circ f) = F(g) \circ F(f)$.
This shows that F is indeed a functor.
We want to show that this defines a bundle theory on Sm. So suppose we have some $N \in \mathrm{Sm}^G$ and $f\colon M \to N/G$ in Sm. Now, the quotient map $N \to N/G$ is a submersion (show this), so the pullback along f exists in Sm, giving

$$\begin{tikzcd} f^*N \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & N \arrow[d] \\ M \arrow[r] & N/G \end{tikzcd}$$

Lastly, we must then show that f^*N is in Sm^G. For this, note that the induced bundle f^*N is precisely the pullback which is equivalent as a fibre bundle to $M \times_{N/G} N$. But this inherits a natural action of G given by $g \cdot (m, n) = (m, g \cdot n)$. Choosing the same cover $\{U_\alpha\}$ for N as given in the condition of it being in Sm^G, i.e., $\{g \cdot U_\alpha\}$ being disjoint for all g and α, the neighborhoods $M \times U_\alpha \cap f^*N$ then satisfy the same conditions under this action of G. Lastly, the map $f^*N \cong M \times_{N/G} N \to N$ given by the projection to the N component which is the top map in the pullback diagram is naturally G-equivariant. This shows that the above diagram indeed can be made.

Now suppose we have some $P \in \mathrm{Sm}^G$ and a bundle map $P \to N$ giving the solid part of the diagram

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & M \times_{N/G} N \arrow[r] \arrow[d] & N \arrow[d] \\ P/G \arrow[r] & M \arrow[r] & N/G \end{tikzcd}$$

where the map $P \to N$ descends to the composite map $P/G \to M \to N/G$ on the bottom.

We then want to show that the dashed map exists. Let $p\colon P \to P/G$ and $q\colon f^*N \cong M \times_{N/G} N \to M$ be the projection. Let $k\colon P \to N$ be the map on the top. Let $f\colon P/G \to M$ be the map on the bottom. Define a map $h\colon P \to M \times_{N/G} N$ by $h(x) = (f(p(x)), k(p))$. Then if $l\colon M \to N/G$ denotes the map on the bottom, $l \circ f\,(p(x)) = \pi\,(k(p))$ where $\pi\colon N \to N/G$. By definition then $h(x) \in M \times_{N/G} N$. Furthermore,

$$h\,(g \cdot x) = (f\,(p\,(g \cdot x))\,, k\,(g \cdot x)) = (f\,(p\,(x))\,, g \cdot k(x)) = g \cdot (f\,(p(x))\,, k(x)) = g \cdot h(x),$$

so h is G-equivariant.

Next we must check that the bundle theory is locally trivial. That is, we must check that for any $M \in \mathrm{Sm}^G$ and any point $x \in M/G$, there exists an open neighborhood U about x such that if we let $\pi\colon U \to *$ be the unique map and $i\colon U \to M/G$ the open embedding, there exists a manifold $N \in \mathrm{Sm}^G$ such that $N/G \cong *$, and such that the pullbacks are isomorphic: $i^*M \cong \pi^*N$.
Note that these pullbacks are really

$$\begin{tikzcd} U \times_{M/G} M \cong i^*M \arrow[r] \arrow[d] & M \arrow[d, "p"] \\ U \arrow[r] & M/G \end{tikzcd}$$

But clearly if $(u, m) \in U \times_{M/G} M$, then essentially $\overline{m} = u$, so $U \times_{M/G} M \cong p^{-1}(U)$, and

$$\begin{tikzcd} U \times N \cong U \times_* N \arrow[r] \arrow[d] & N \arrow[d] \\ U \arrow[r] & * \end{tikzcd}$$

So we find that the condition is indeed equivalent to the usual one: the existence of a neighborhood U about x and a homeomorphism $p^{-1}(U) \cong U \times N$. In this case, suppose $x \in M/G$ and simply choose one of the U_α such that $x \in p\,(U_\alpha)$. Note that this is open in M/G since the $g \cdot U_\alpha$ are pairwise disjoint and g acts by homeomorphisms (G is discrete and each g has g^{-1} as inverse). Choosing $U = p\,(U_\alpha)$, we get $p^{-1}(U) = \sqcup_{g \in G} U_\alpha \cong U_\alpha \times G \cong U \times G$ where $G \in \mathrm{Sm}^G$ is precisely G considered as a smooth manifold with the trivial charts $g \mapsto *$, at each $g \in G$. Indeed then $G/G \cong *$, so this satisfies the condition above. I.e., the functor $\mathrm{Sm}^G \to \mathrm{Sm}$ is locally trivial.

Lastly, we must check gluing. Namely that for $M \in \mathrm{Sm}^G$ and some open coordinate neighborhoods $U_i, U_j, U_k \subset M/G$, with coordinate maps $g_{ij} \colon U_i \cap U_j \to G, g_{jk} \colon U_j \cap U_k \to G$ and $g_{ki} \colon U_k \cap U_i \to G$, the maps satisfy $g_{ik}(x) = g_{ij}(x) g_{jk}(x)$ for $x \in U_i \cap U_j \cap U_k$. As we saw above, $p^{-1}(U_i) = U_i \times G$, and we shall call this coordinate function $\varphi_i \colon U_i \times G \to p^{-1}(U_i)$. Let $g_{ij}(x) = \varphi_{i,x}^{-1} \varphi_{j,x}$ where $\varphi_{i,x}(y) = \varphi_i(x,y)$ is the function considered only as a function of y. But then the condition $g_{ij}(x) g_{jk}(x) = g_{ik}(x)$ follows trivially.

This completes the proof that the functor we constructed $\mathrm{Sm}^G \to \mathrm{Sm}$ is indeed a bundle theory over Sm. $\square$

Note. The bundles constructed by sending an object $M \in \mathrm{Sm}^G$ to M/G above exhibits $M \to M/G$ as a principal G-manifold bundle.

Lemma 6.65. *For any locally trivial bundle theory $\mathcal{B} \to \mathrm{Sm}$, every $B \in \mathcal{B}(\mathbb{R})$ is trivial. (here $\mathcal{B}(\mathbb{R})$ denotes the fiber of $\mathbb{R}$ under the functor)*

Example 6.66 (Examples of pullback bundles)**.**
(1) The restriction of a vector bundle $p \colon B \to X$ over a subspace $Y \hookrightarrow X$ can be viewed as a pullback with respect to this inclusion map since the inclusion $p^{-1}(Y) \hookrightarrow E$ is certainly an isomorphism on each fiber.
(2) When $f \colon Y \to X$ has image a point, the pullback f^*B is the product $Y \times p^{-1}(b)$ where $f(Y) = b$.
(3) If B is trivial over some $U \subset X$, then the pullback along a map $f \colon Y \to X$, f^*B over $f^{-1}(U)$ is trivial since if we take a global frame over U, $s \colon U \to B$, then we can pull these back to give a frame over $f^{-1}(U)$: $s^* \colon f^{-1}(U) \to f^*B|_U$ by $s^*(x) = (x, s \circ f(x))$.

Note. Some key properties of pullbacks are:
(1) $(fg)^*B \cong g^*(f^*B)$.
(2) $\mathrm{id}^*B \cong B$.
(3) $f^*(B_1 \oplus B_2) \cong f^*B_1 \oplus f^*B_2$.
(4) $f^*(B_1 \otimes B_2) \cong f^*B_1 \otimes f^*B_2$.

Theorem 6.67. *Given a vector bundle $p \colon E \to B$ and homotopic maps $f_0, f_1 \colon A \to B$, then the induced bundles f_0^*E and f_1^*E are isomorphic if A is compact Hausdorff or, more generally, paracompact.*

Remark. This theorem holds for fiber bundles as well (with the same proof).

Corollary 6.68. *A homotopy equivalence $f \colon A \to B$ of paracompact spaces induced a bijection $f^* \colon \mathrm{Vect}^n(B) \to \mathrm{Vect}^n(A)$. In particular, every vector bundle over a contractible paracompact base is trivial.*

6.4.1. *Clutching functions and sphere bundles.* Write S^k as the union of its upper and lower hemispheres D_+^k and D_-^k with $D_+^k \cap D_-^k = S^{k-1}$. Given a map $f \colon S^{k-1} \to \mathrm{GL}(n, \mathbb{R})$, let E_f be the quotient of the disjoint union $D_+^k \times \mathbb{R}^n \sqcup D_-^k \times \mathbb{R}^n$ obtained by identifying $(x, v) \in \partial D_-^k \times \mathbb{R}^n$ with $(x, f(x)v) \in \partial D_+^k \times \mathbb{R}^n$. There is then a natural projection $E_f \to S^k$ and this is an n-dimensional vector bundle, as one can most easily see by taking an equivalent definition in which the two hemispheres of S^k are enlarged slightly to open balls and the identification occurs over their intersection, a product $S^{k-1} \times (-\varepsilon, \varepsilon)$, with the map f used in each slice $S^{k-1} \times \{t\}$. From this viewpoint, the construction of E_f follows from the existence theorem (Theorem 6.26). In particular, in this case, there are only two coordinate neighborhoods and thus only one clutching function on their intersection, so the cocycle condition is satisfied trivially.

Note. The map f is called the *clutching function* for E_f.

6.5. **Vector Bundles.** The theory of vector bundles is quite vast, so we will give several different perspectives on parts of the subject, primarily following [**LeeSM**], [**JB**], [**BT**], [**MS**] and [**Dieck**].

6.5.1. *Carrying things over.*

Definition 6.69 (Vector Bundle)**.** A *vector bundle* is a fiber bundle in which the fiber is $\mathbb{R}^n$ and the structure group is $\mathrm{GL}(n, \mathbb{R})$ or a subgroup thereof.

If $\mathcal{B}, \mathcal{B}'$ are two vector bundles over the same base space X, then locally, the map between total spaces looks like $g_{ji}(x)$ which is in $\mathrm{GL}(n, \mathbb{R})$, hence a smooth diffeomorphism, or homeomorphism if the vector bundles are not smooth. Since also a bundle map $\mathcal{B} \to \mathcal{B}'$ must be a bijection, we find that it is a bijective local diffeomorphism/homeomorphism, hence a diffeomorphism/homeomorphism. So we obtain:

Lemma 6.70. *If $\mathcal{B}, \mathcal{B}'$ are vector bundles over the same base space, then $\mathcal{B}$ is isomorphic to $\mathcal{B}'$ if and only if there exists a bundle map $\mathcal{B} \to \mathcal{B}'$ which induces a homeomorphism on total spaces.*

Definition 6.71. We can define $\mathbb{RP}^n$ as the set of all unordered $\{x, -x\}$ as x ranges over $S^n \subset \mathbb{R}^{n+1}$, topologized as a quotient space of S^n. Let $\gamma_n^1 = [\pi \colon B \to X]$ be the vector bundle where $B \subset \mathbb{RP}^n \times \mathbb{R}^{n+1}$ is the set consisting of all $(\{\pm x\}, v)$ such that $\exists \lambda \in \mathbb{R} \colon v = \lambda x$. Define $\pi \colon B \to \mathbb{RP}^n$ by $\pi(\{\pm x\}, v) = \{\pm x\}$. Thus each fiber $\pi^{-1}(\{\pm x\})$ can be identified with the line through x and $-x$ in $\mathbb{R}^{n+1}$. Each such line is given its usual vector space structure. The resulting bundle γ_n^1 will be called the *canonical line bundle over* $\mathbb{RP}^n$.

Lemma 6.72. *The canonical line bundle over $\mathbb{RP}^n$, γ_n^1, is no isomorphic to a trivial bundle.*

Proof. It suffices to show that no global frame exists - i.e., that a nowhere vanishing section does not exists.
Suppose there exists a section $s \colon \mathbb{RP}^n \to B = \big\{(\{\pm x\}, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid \exists \lambda \in \mathbb{R} \colon v = \lambda x\big\}$. Consider the composition $S^n \to \mathbb{RP}^n \to B$, $x \mapsto s(\{\pm x\})$. Since $\pi \circ s(\{\pm x\}) = \{\pm x\}$, we have $s(\{\pm x\}) = (\{\pm x\}, t(x)x)$ for some continuous map $t \colon S^n \to \mathbb{R}$. Then x and $-x$ are mapped to the same point under the composition $S^n \to \mathbb{RP}^n \to B$, so $t(x)x = t(-x)(-x)$, so $t(-x) = -t(x)$. Hence by the intermediate value theorem, there exists an $x \in S^n$ such that $t(x) = 0$. Then $s(\{\pm x\}) = (\{\pm x\}, 0)$. □

Theorem 6.73. *A vector bundle is trivial if and only if it admits a global frame.*

Definition 6.74 (Parallelizable)**.** If the tangent bundle of a manifold is trivial, then the manifold is said to be *parallelizable.*

Exercise 6.75. Open subsets of $\mathbb{R}^n$ are parallelizable.
S^1 is also parallizable as its tangent bundle admits a global frame as depicted in Figure 2.

6.5.2. *Random stuff.*

Definition 6.76 (Vector Bundle)**.** A vector bundle over $X \in \mathrm{Top}$ consists of the following data:

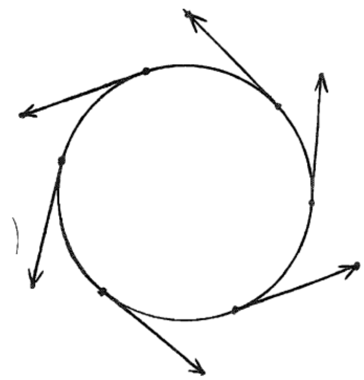


FIGURE 2. S1-parallelizable.png

- An object $\left[E \xrightarrow{\pi} X\right]$ in Top/X.
- An $\mathbb{R}$-vector space structure internal to Top/X :
  (1) a morphism $+\colon E \times_X E \to E$
  (2) a morphism $\cdot\colon \mathbb{R} \times E \to E$
  which satisfy the vector space axioms.

which are required to satisfy

- (local triviality) there exists an open cover $\{U_\alpha\}$ of X where if $U := \sqcup_{\alpha \in I} U_\alpha$ $[U \to X] \in \mathrm{Top}/X$ is such that there exists an isomorphism of vector space objects in Top/U
$$U \times_I \mathbb{R}^n \cong U \times_X E$$
  where $n\colon I \to \mathbb{N}$. Here $\mathbb{R}^n = \sqcup_{i \in I} \mathbb{R}_i^{n(i)}$

Definition 6.77 (Vect(X))**.** Topological vector bundles over X and bundle morphisms between them constitute a category denoted Vect(X).

Viewed in top, the last condition implies that there is a diagram of the form

$$\begin{tikzcd}
U \times k^n \arrow[r, "\cong"] \arrow[dr] & U \times_X E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "\pi"] \\
& U \arrow[r] & X
\end{tikzcd}$$

where the homeomorphism in the top left is fiber-wise linear.

All of this is fine so far, but we want to look at smooth manifolds, so we now reformulate our definitions a bit.

Remark. From now on, Bun will denote that subcategory consisting of topological manifolds. Then Bun(N) will denote Sm/N.

We would like to define vector bundles the same as before but replacing Top by Sm. However, the category Sm is not complete, so what is $+\colon E \times_X E \to E$ supposed to be?

Lemma 6.78 (Pullbacks along submersions exist)**.** *If $f\colon M \to N$ and $g\colon P \to N$ are morphisms in* Sm *and f is a submersion, then the pullback exists:*

$$\begin{tikzcd} \exists X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M \arrow[d, "f"] \\ P \arrow[r, "g"] & N \end{tikzcd}$$

Now, since $\pi\colon E \to X$ is a bundle, it is a submersion, so the pullback $E \times_X E$ exists. Then we can define $+\colon E \times_X E \to E$ in the same way as before.

Definition 6.79 (Vect)**.** Topological vector bundles form a category Vect whose morphisms are bundle maps

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}$$

such that

$$\begin{tikzcd} E \arrow[r] \arrow[dr] & E' \times_{X'} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E' \arrow[d, "\pi"] \\ & X \arrow[r] & X' \end{tikzcd}$$

commutes.

Definition 6.80 (Subvector bundle)**.** If E is an n-dimensional vector bundle over X and $E' \subset E$ is a subset, so that around every point in X, there is a bundle chart (f, U) with

$$f\left(\pi^{-1}(U) \cap E'\right) = U \times \mathbb{R}^k \subset U \times \mathbb{R}^n$$

then $(E', \pi|_{E'}, X)$ is in a natural manner a vector bundle over X and is called a k-dimensional subvector bundle of E.

6.6. **Frames.**

Definition 6.81. Let $E \to M$ be a vector bundle. If $U \subset M$ is an open subset, a k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ of E over U is said to be *linearly independent* if $(\sigma_1(p), \dots, \sigma_k(p))$ is linearly independent in E_p for each $p \in U$. Similarly, they are said to *span E* if their values span E_p for each $p \in U$. A *local frame for E over U* is an ordered k-tuple $(\sigma_1, \dots, \sigma_k)$ of linearly independent local sections over U which span E. It is called a *global frame* if $U = M$. If $E \to M$ is a smooth vector bundle, a frame is called *smooth* if each section σ_i is a smooth section.

Proposition 6.82 (Completion of Local Frames for Vector Bundles)**.** *Suppose $\pi\colon E \to M$ is a smooth vector bundle of rank k.*

(1) *If $(\sigma_1, \dots, \sigma_m)$ is a linearly independent smooth local section of E over an open subset $U \subset M$ with $1 \le m < k$, then for each $p \in U$, there exist smooth sections $\sigma_{m+1}, \dots, \sigma_k$ defined on some neighborhood V of p such that $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.*
(2) *If $(v_1, \dots, v_m)$ is a linearly independent m-tuple of elements of E_p for some $p \in M$ with $1 \le m \le k$, then there exists a smooth local frame (σ_i) for E over some neighborhood of p such that $\sigma_i(p) = v_i$ for $i = 1, \dots, m$.*

(3) *If $A \subset M$ is a closed subset and $(\tau_1, \ldots, \tau_k)$ is a linearly independent k-tuple of sections of $E|_A$ that are smooth, then there exists a smooth local frame $(\sigma_1, \ldots, \sigma_k)$ for E over some neighborhood of A such that $\sigma_i|_A = \tau_i$ for $i = 1, \ldots, k$.*

Example 6.83 (Global frame for a product bundle)**.** If $E = M \times \mathbb{R}^k \to M$ is a product bundle, the standard basis $(e_1, \ldots, e_k)$ for $\mathbb{R}^k$ yields a global frame $(\tilde{e}_i)$ for E defined by $\tilde{e}_i(p) = (p, e_i)$. If M is a smooth manifold with or without boundary, then this global frame is smooth.

Example 6.84 (The Local Frames Associated with Local Trivializations)**.** Suppose $\pi\colon E \to M$ is a smooth vector bundle. If $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a smooth local trivialization of E, we use the same idea as in the preceding example to construct a local frame for E over U. Define maps $\sigma_1, \ldots, \sigma_k \colon U \to E$ by $\sigma_i(p) = \Phi^{-1}(p, e_i) = \Phi^{-1} \circ \tilde{e}_i(p)$:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"'] \\
& U \arrow[ul, bend left, "\sigma_i"] \arrow[ur, bend right, "\tilde{e}_i"'] &
\end{tikzcd}$$

The σ_i are smooth since they have coordinate representation $\pi \circ \sigma_i(p) = \pi_1 \circ \Phi \circ \sigma_i(p) = \pi_1 (p, e_i) = p$ on U. To see that the σ_i form a basis, simply note that $\sigma_1(p), \ldots, \sigma_k(p)$ are the images of the standard basis under $\Phi^{-1}(p, -)$ which is assumed to be a linear isomorphism. Hence they form a basis. We say that this local frame (σ_i) is *associated with* Φ.

Proposition 6.85. *Every smooth local frame for a smooth vector bundle is associated with a smooth local trivialization as in Example 6.84.*

This proposition is what makes it possible for us to go back and forth between trivializations and frames when working with vector bundles.

Corollary 6.86. *Let $\pi\colon E \to M$ be a smooth vector bundle of rank k, let (V, φ) be a smooth chart on M with coordinate functions $\left(x^i\right)$, and suppose there exists a smooth local frame (σ_i) for E over V. Define $\tilde{\varphi}\colon \pi^{-1}(V) \to \varphi(V) \times \mathbb{R}^k$ by*

$$\tilde{\varphi}\left(v^i \sigma_i(p)\right) = \left(x^1(p), \ldots, x^m(p), v^1, \ldots, v^k\right).$$

Then $\left(\pi^{-1}(V), \tilde{\varphi}\right)$ is a smooth coordinate chart for E.

6.7. Gluing vector bundles.

Proposition 6.87 (Topological vector bundles reconstructed from transition functions (see ncatlab))**.** *Let $[\pi\colon E \to X]$ be a topological vector bundle, $\{U_i \subset X\}_{i \in I}$ an open cover of the X and $\left\{U_i \times k^n \xrightarrow{\varphi_i} E|_{U_i}\right\}_{i \in I}$ be local trivializations.*
Write

$$\left\{g_{ij} := \varphi_j^{-1} \circ \varphi_i \colon U_i \cap U_j \to \mathrm{GL}(n, k)\right\}_{i,j \in I}$$

for the corresponding transition functions. Then there is an isomorphism of vector bundles over X :

$$\left(\left(\sqcup_{i \in I} U_i\right) \times k^n\right) / \left(\left\{g_{ij}\right\}_{i,j \in I}\right) \overset{(\varphi_i)_{i \in I}}{\to} E$$

from the vector bundle glued from the transition functions to the original bundle E.

Definition 6.88 (Pre-vector bundles)**.** An *n-dimensional pre-vector bundle* is a quadruple $(E, \pi, X, \mathcal{B})$ consisting of

(1) A set E
(2) A topological space X
(3) A surjective map $\pi \colon E \to X$
(4) A vector space structure on every fibre $E_x := \pi^{-1}(x)$.
(5) A *pre-bundle atlas* $\mathcal{B}$ which is a set $\{(f_\alpha, U_\alpha)\}_{\alpha \in A}$ where $\{U_\alpha\}_{\alpha \in A}$ is an open covering of X and

$$f_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$$

is a bijective map which maps the fibre E_x linearly and isomorphically into $\{x\} \times \mathbb{R}^n$ for every $x \in U_\alpha$ in such a way that all transition function $U_\alpha \cap U_\beta \to \mathrm{GL}\,(n, \mathbb{R})$ of $\mathcal{B}$ are continuous.

Note. Given a pre-vector bundle as above, the Existence theorem for fibre bundles (Theorem 6.26) gives us a fibre bundle $[\pi \colon E \to X]$ with structure group $\mathrm{GL}\,(n, \mathbb{R})$ such that every fibre has the structure of a vector space and the trivializations restrict to vector space isomorphisms $f_x := f|_{E_x} \colon E_x \to \{x\} \times \mathbb{R}^n$ for each $x \in \pi^{-1}(U_\alpha)$. Furthermore, this vector bundle is unique up to equivalence.

6.8. **Riemannian metrics and Euclidean vector bundles.** Recall that a real valued function $\mu \colon V \to \mathbb{R}$ on a finite dimensional vector space V is called *quadratic* if there exist linear maps $l_i, l_i' \colon V \to \mathbb{R}$ such that

$$\mu(v) = \sum l_i(v) l_i'(v).$$

One can then show that the map $V \times V \to \mathbb{R}$ sending

$$(v, w) \mapsto v \cdot w = \frac{1}{2}\left(\mu(v + w) - \mu(v) - \mu(w)\right)$$

is a bilinear form.
Note that $v \cdot v = \mu(v)$. The quadratic function μ is called positive definite if $\mu(v) > 0$ whenever $v \neq 0$.

Definition 6.89 (Euclidean vector space)**.** A Euclidean vector space is a real vector space V together with a positive definite quadratic function

$$\mu \colon V \to \mathbb{R}.$$

The real number $v \cdot w$ is then called the inner product of v and w, and $v \cdot v = \mu(v) =: \|v\|^2$, where $\|v\|$ is called the norm of v.

Definition 6.90 (Euclidean vector bundle)**.** A *Euclidean vector bundle* is a real vector bundle $\mathcal{B}$ together with a continuous function

$$\mu \colon B \to \mathbb{R}$$

such that the restriction of μ to each fiber of $\mathcal{B}$ is positive definite and quadratic. The function μ itself will be called a *Euclidean metric* on the vector bundle $\mathcal{B}$.

Definition 6.91 (First definition of Riemannian metric and Riemannian manifold)**.** In the case of the tangent bundle TM of a smooth manifold, a Euclidean metric $\mu \colon TM \to \mathbb{R}$ is called a *Riemannian metric*, and M together with μ is called a *Riemannian manifold*. (One usually requires μ to be a smooth function also).

Note. The notation $\mu = ds^2$ is often used for a Riemannian metric.

Lemma 6.92. *Let ξ be a trivial vector bundle of dimension n over X and let μ be any Euclidean metric on ξ. Then there exist n sections $s_1, \dots, s_n$ of ξ which are normal and orthogonal in the sense that*

$$s_i(b) \cdot s_j(b) = \delta_{ij}$$

for each $b \in B$ (where $\cdot$ here denotes the inner product associated to μ).

If $[E \to X]$ is a vector bundle, then there exists a vector bundle $(E \otimes E)^*$. Given this, we have the following definition:

Definition 6.93 (Riemannian metric, scalar product, second definition)**.** If $[\pi\colon E \to X]$ is a vector bundle then, by a *scalar product* or a *Riemannian metric* for E, we mean a continuous section $s\colon X \to (E \otimes E)^*$ such that for every $x \in X$, the bilinear form determined by this is symmetric and positive definite. That is, such that for every $x \in X$, the bilinear form

$$\begin{aligned} E_x \times E_x &\to \mathbb{R} \\ (v, w) &\mapsto s\,(v \otimes w) =: \langle v, w \rangle_x \end{aligned}$$

is symmetric and positive definite.
The metric is said to be smooth if X is a smooth manifold and E and s are smooth.

Lemma 6.94. *If E is a vector bundle over a smooth manifold X, then we can equip E with a smooth Riemannian metric.*

Proof. For any bundle chart (φ, U) such that $\varphi\colon \pi^{-1}(U) \cong U \times \mathbb{R}^n$, choose the standard inner product on $\mathbb{R}^n$. Let $s\colon U \to (E|_U \otimes E|_U)^*$ given by $s(u) = \langle -, - \rangle_u$ be this Riemannian metric. We want to patch these local Riemannian metrics together to give a global smooth Riemannian metric on

□

6.9. **Examples of Vector Bundles.**

Lemma 6.95 (Orthogonal vector bundle)**.** *Given a vector bundle $\pi\colon E \to X$ equipped with a Riemannian metric and $F \subset E$ a subvector bundle, then*

$$F^\perp := \bigcup_{x \in X} F_x^\perp$$

is also a subvector bundle.

Proof. Let (φ, U) be a bundle chart of E such that

$$\varphi\left(\pi^{-1}(U) \cap F\right) = U \times \mathbb{R}^k \subset U \times \mathbb{R}^n$$

Let $\sigma_1, \dots, \sigma_n$ be the associated frame, so by construction, $\sigma_i = \Phi^{-1} \circ \tilde{e}_i$ where $\tilde{e}_i\colon U \to U \times \mathbb{R}^n$ is the map $\tilde{e}_i(p) = (p, e_i)$. Hence in particular, $\sigma_1, \dots, \sigma_k$ is a smooth local frame for F over U, and $\sigma_{k+1}, \dots, \sigma_n$ is a smooth local frame for $F^\perp$ over U. □

Lemma 6.96. *If E is equipped with a Riemannian metric and $F \subset E$ is a subbundle, then the composition*

$$F^\perp \hookrightarrow E \overset{proj}{\to} E/F$$

is a bundle isomorphism.

Definition 6.97 (Normal bundle)**.** If M is a smooth manifold and $X \subset M$ is a submanifold, then the normal bundle of X in M is defined to be

$$\perp X := (TM|_X)/TX.$$

Definition 6.98 (Riemannian manifold)**.** A manifold M, whose tangent bundle has a smooth Riemannian metric, is called a *Riemannian manifold*.

Lemma 6.99. *The bundle $\Lambda^k E$ of k-fold exterior powers is a vector bundle when (E, π, X) is an n-dimensional vector bundle with bundle atlas $\mathcal{U}$. We construct this by forming a pre-vector bundle. Define $\Lambda^k E := \bigsqcup_{x \in X} \Lambda^k E_x$. Each $\Lambda^k E_x$ has dimension $\binom{n}{k}$, hence $\varphi_x \colon \Lambda^k E_x \cong \mathbb{R}^{\frac{n!}{k!(n-k)!}}$. Let the projection be the canonical one. Let the atlas be given by $\{(f_\alpha, U_\alpha)\}$ where $f_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{\frac{n!}{k!(n-k)!}}$ is given by $\pi(-) \times \varphi_-$ and $(f, U_\alpha) \in \mathcal{U}$.*

Lemma 6.100 (Orientation cover)**.** *Using the exerior power bundle, we can construct the 1-dimensional bundle $\Lambda^n E$ for (E, π, X) an n-dimensional vector bundle. Define the equivalence relation in $\Lambda^n E - \{$zero section $\}$ by $x \sim y \iff y = \lambda x$ for some $\lambda > 0$. Give the equivalence classes $\tilde{X}(E)$ the quotient topology. Then we obtain a two sheeted cover of X by the canonical projection*

$$\tilde{X}(E) \xrightarrow{\tilde{\pi}} X$$

which is called the orientation cover of E.

7. Morse Theory

Lemma 7.1. *Let $k \colon \partial D_0^n \to \partial D_1^n$ be a diffeomorphism between the respective boundaries of two n-disks. Then we can extend k to a diffeomorphism $K \colon D_0 \to D_1$ if $n \leq 6$.*

Exercise 7.2. We obtain a closed surface from two disks D_1 and D_2 by pasting them together along their boundaries by a diffeomorphism $h \colon \partial D_1 \to \partial D_2$. This is called a *twisted* 2*-sphere*. Show that a twisted 2-sphere is diffeomorphic to the 2-sphere S^2 using Lemma 7.1.

Exercise 7.3. Show that any homeomorphism $h \colon S^1 \to S^1$ can be extended to a homeomorphism $H \colon D^2 \to D^2$.
Show the same thing for diffeomorphisms.

Theorem 7.4 (Gluing manifolds with boundary)**.** *Let M_1 and M_2 be manifolds with boundary, and let $\varphi \colon \partial M_1 \to \partial M_2$ be a diffeomorphism between the boundaries. Then we can construct a new manifold $W = M_1 \cup_\varphi M_2$ by gluing the boundaries of M and N using φ. The resulting manifold W is unique up to diffeomorphism. (It is allowed to glue only certain components of the boundary instead of the entire boundary).*

7.1. Definitions and Lemmas.

If $f \in C^\infty(M)$, let M^a denote the set of points $x \in M$ such that $f(x) \leq a$.

Lemma 7.5. *If a is not a critical value of f then M^a is a smooth codimension 0 manifold with boundary.*

Proof. **[LeeSM]** □

Definition 7.6 (Index and nullity of $f_{**} = H(f)$)**.** The *index* of a bilinear function H on a vector space V is defined to be the maximal dimension of a subspace of V on which H is negative definite; The nullity is the dimension of the null-space, i.e., the subspace of all $v \in V$ such that $H(v,w) = 0$ for all $w \in V$.
The index of f_{**} on TM_p will be referred to as the index of f at p.

Remark. A critical point p of f is non-degenerate if and only if f_{**} on TM_p has nullity equal to 0.

Lemma 7.7. *Let f be a smooth function in a neighborhood V of 0 in $\mathbb{R}^n$ with $f(0) = 0$. Then*

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i g_i \left(x_1, \ldots, x_n\right)$$

for some suitable smooth function g_i defined in V, with $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Lemma 7.8 (Morse Lemma)**.** *Let p be a non-degenerate critical point for f. Then there is a local coordinate system $\left(y^1, \ldots, y^n\right)$ in a neighborhood U of p with $y^i(p) = 0$ for all i and such that the identity*

$$f = f(p) - \left(y^1\right)^2 - \ldots - \left(y^\lambda\right)^2 + \left(y^{\lambda+1}\right)^2 + \ldots + \left(y^n\right)^2$$

holds throughout U, where λ is the index of f at p.

Remark. By Sylvester's law of inertia, λ does not depend on the way the Hessian is diagonalized.

Definition 7.9 (1-parameter group of diffeomorphisms)**.** A 1-*parameter group of diffeomorphisms* of a manifold M is a smooth map $\varphi \colon \mathbb{R} \times M \to M$ such that $\varphi_t = \varphi(t, -) \colon M \to M$ is a self-diffeomorphisms of M for each $t \in \mathbb{R}$, and for all $t, s \in \mathbb{R}$, we have $\varphi_{t+s} = \varphi_t \circ \varphi_s$.

Definition 7.10 (Vector field generating a 1-parameter group of diffeomorphisms)**.** Given a 1-parameter group φ of diffeomorphisms of M, we define a vector field X on M as follows: for every smooth real valued function f, let

$$X_q(f) = \lim_{h \to 0} \frac{f\left(\varphi_h(q)\right) - f(q)}{h}.$$

This vector field X is said to *generate* the group φ.

Lemma 7.11. *A smooth vector field on M which vanishes outside of a compact set $K \subset M$ generates a unique* 1*-parameter group of diffeomorphisms of M.*

7.2. **Homotopy Type in Terms of Critical Values.**

Theorem 7.12. *Let f be a smooth real-valued function on a manifold M. Let $a < b$ and suppose that the set $f^{-1}[a,b]$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

Proof. Choose a Riemannian metric on M, and let $\langle X, Y\rangle$ denote the inner product of two tangent vectors as determined by this metric.
The gradient, ∇f, of f on M is characterized by

$$\langle \xi, \nabla f\rangle = \xi(f)$$

for any vector field ξ on M, where $\xi(f)$ is the directional derivative of f along ξ. Let $\rho\colon M \to \mathbb{R}$ be a smooth function equal to $\frac{1}{\langle \nabla f, \nabla f\rangle}$ throughout the compact set $f^{-1}[a,b]$ and which vanishes outside of a compact neighborhood of this set - using bump functions. Then the vector field ξ defined by

$$\xi_q = \rho(q)\left(\nabla f\right)_q$$

satisfies the conditions of Lemma 7.11, hence generates a unique 1-parameter family of diffeomorphisms of M, $\varphi\colon \mathbb{R}\times M \to M$.

Consider now, for a fixed $q \in M$, the function $t \mapsto f\left(\varphi_t(q)\right)$. If $\varphi_t(q) \in f^{-1}[a,b]$, then

$$\frac{d\left(f \circ \varphi_t(q)\right)}{dt} = \left\langle (\nabla f)_q, \frac{d\varphi_t(q)}{dt}\right\rangle = \left\langle \left(\nabla f\right)_q, \xi_q\right\rangle = 1$$

Hence since the derivative is constant, the map

$$t \mapsto f\left(\varphi_t(q)\right)$$

is linear and with derivative 1 as long as $f \circ \varphi_t(q) \in [a,b]$. Therefore also $x \in M^a = f^{-1}(-\infty, a]$ if and only if $f(x) \in (-\infty, a]$. Now $a = f\left(\varphi_a(q)\right)$, so $f(x) = f\left(\varphi_c(q)\right)$ for $c = f(x) \leq a$. Thus $\varphi_{b-a}(x) = \varphi_{b-a+c}(q) \in f^{-1}(-\infty, b]$. It is also easy to see that it is bijective.
Now define a 1-parameter family of maps

$$r_t\colon M^b \to M^b$$

by

$$r_t(q) = \begin{cases} q, & f(q) \leq a \\ \varphi_{t(a-f(q))}(q), & a \leq f(q) \leq b \end{cases}$$

When $f(q) = a$, we have $\varphi_0(q)$ which should equal q and indeed, $\varphi_0 \circ \varphi_0(q) = \varphi_{0+0}(q) = \varphi_0(q)$ and since φ_t is a diffeomorphism for all t, we can take the inverse and find $\varphi_0(q) = q$. $\square$

Theorem 7.13. *Let $f\colon M \to \mathbb{R}$ be a smooth function, and let p be a non-degenerate critical point with index λ. Setting $f(p) = c$, suppose that $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact and contains no critical point of f other than p for some $\varepsilon > 0$. Then, for all sufficiently small ε, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ-cell attached.*

Proof. The idea of the proof of this theorem is indicated in 3 in the case of the height function on a torus.
Here is the idea: we will introduce a new function $F\colon M \to \mathbb{R}$ which coincides with the height function f except that $F < f$ in a small neighborhood of the critical point p. Thus the region $F^{-1}(-\infty, c-\varepsilon]$ will consist of $M^{c-\varepsilon}$ together with a region H near p which we will call a "handle". More on that later. We will choose a suitable cell $e^\lambda \subset H$ and then pushing H down onto e^λ will give a deformation retract from $M^{c-\varepsilon} \cup H$ to $M^{c-\varepsilon} \cup e^\lambda$. Then we will apply Theorem 7.12 to F and the region $F^{-1}[c-\varepsilon, c+\varepsilon]$ giving a deformation retract of $M^{c-\varepsilon} \cup H$ to $M^{c+\varepsilon}$.

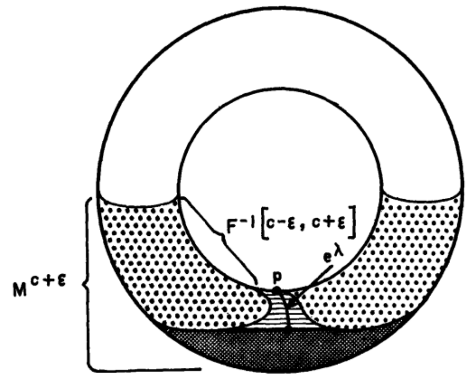


Figure 3. 32.png

Now, to the proof:
Choose a coordinate system $\left(u^1, \ldots, u^n\right)$ centered at p so that

$$f = c - \left(u^1\right)^2 - \ldots - \left(u^\lambda\right)^2 + \left(u^{\lambda+1}\right)^2 + \ldots + \left(u^n\right)^2$$

holds in U.
Next, choose $\varepsilon > 0$ such that

(1) $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact and contains no critical points other than p.
(2) The image of U under the diffeomorphism

$$\left(u^1, \ldots, u^n\right) : U \to \mathbb{R}^n$$

contains the closed ball

$$\left\{\left(u^1, \ldots, u^n\right) \mid \sum \left(u^i\right)^2 \leq 2\varepsilon\right\}.$$

Now, we finally define

$$e^\lambda = \left\{\left(u^1, \ldots, u^n\right) \in U \mid \sum_{i=1}^{\lambda} \left(u^i\right)^2 \leq \varepsilon \quad \text{and} \quad u^{\lambda+1} = \ldots = u^n = 0\right\}$$

See Figure 4.
The coordinate lines represent the planes $u^{\lambda+1} = \ldots = u^n = 0$ and $u^1 = \ldots = u^\lambda = 0$, respectively. The circle represents the boundary of the ball of radius $\sqrt{2\varepsilon}$, the hyperbolas represent the hypersurfaces $f^{-1}(c-\varepsilon)$ and $f^{-1}(c+\varepsilon)$. The region $M^{c-\varepsilon}$ is heavily shaded, the region $f^{-1}[c-\varepsilon, c]$ has big dots which are not so densely packed, while the region $f^{-1}[c, c+\varepsilon]$ has small dots which are tightly packed. The horizontal dark line through p represents the cell e^λ.
Note that $e^\lambda \cap M^{c-\varepsilon}$ is precisely the boundary ∂e^λ, so that e^λ is attached to $M^{c-\varepsilon}$ as required.

Now we will construct a new function $F\colon M \to \mathbb{R}$. Let $\mu\colon \mathbb{R} \to \mathbb{R}$ e a smooth function such that $\mu(0) > \varepsilon, \mu(r) = 0$ for $r \geq 2\varepsilon$ and $-1 < \mu'(r) \leq 0$ for all r. Now

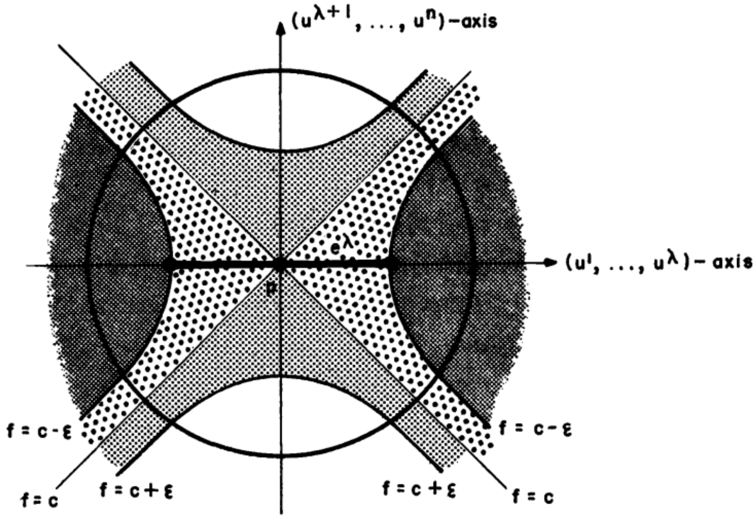


FIGURE 4. Setup

let $F \equiv f$ outside of U and on U,

$$F = f - \mu \left(\sum_{i=1}^{\lambda} \left(u^i\right)^2 + 2 \sum_{i=\lambda+1}^{n} \left(u^i\right)^2 \right).$$

F is clearly smooth.
For convenience, let

$$\xi, \eta \colon U \to [0, \infty)$$

be given by

$$\xi = \sum_{i=1}^{\lambda} \left(u^i\right)^2$$
$$\eta = \sum_{i=\lambda+1}^{n} \left(u^i\right)^2$$

so that $f = c - \xi + \eta$ and

$$F(q) = c - \xi(q) + \eta(q) - \mu\left(\xi(q) + 2\eta(q)\right)$$

for all $q \in U$.

Assertion. The region $F^{-1}(-\infty, c+\varepsilon]$ coincides with the region $M^{c+\varepsilon} = f^{-1}(-\infty, c+\varepsilon]$.

Proof. Smth smth smth
Verify the continuity in Case 2 later

$\square$

$\square$

Remark. A modification of the proof of Theorem 7.13 shows that the set M^c is also a deformation retract of $M^{c+\varepsilon}$. In fact, M^c is a deformation retract of $F^{-1}(-\infty, c]$

which is a deformation retract of $M^{c+\varepsilon}$. Combining this with Theorem 7.13, we see that $M^{c-\varepsilon} \cup e^{\lambda}$ is a deformation retract of M^c.

Theorem 7.14. *If f is a smooth function on a manifold M with no degenerate critical points, and if each M^a is compact, then M has the homotopy type of a CW-complex, with one cell of dimension λ for each critical point of index λ.*

Problem 7.15 (Reeb's Theorem)**.** Let M be a smooth, compact manifold of dimension d. Show that if M admits a Morse function with only two critical points, then M is homeomorphic to the sphere S^d. Indicate why the above proof fails in showing that M is diffeomorphic to the sphere S^d.

7.3. **The Cobordism Category.**

Definition 7.16 (Smooth manifold triad)**.** $(W; V_0, V_1)$ is a smooth manifold triad if W is a compact smooth n-manifold and ∂W is the disjoint union of two open and closed submanifolds V_0 and V_1.

Definition 7.17. If $(W; V_0, V_1)$ and $(W'; V_1', V_2')$ are two smooth manifold triads and $h\colon V_1 \to V_1'$ is a diffeomorphism, then we can form a third triad $(W \cup_h W'; V_0, V_2')$ where $W \cup_h W'$ is the space formed from W and W' by identifying points of V_1 and V_1' under h according to the following theorem.

Theorem 7.18. *There exists a smooth structure which is unique up to diffeomorphism fixing $V_0, h(V_1) = V_1'$ and V_2' on $W \cup_h W'$ such that the inclusion maps $W \hookrightarrow W \cup_h W', W' \hookrightarrow W \cup_h W'$ are diffeomorphisms onto their images.*

Definition 7.19 (Cobordism)**.** Given two closed smooth n-manifolds M_0 and M_1 (so M_0, M_1 compact and $\partial M_0 = \partial M_1 = \varnothing$), a *cobordism* from M_0 to M_1 is a 5-tuple $(W; V_0, V_1; h_0, h_1)$ where $(W; V_0, V_1)$ is a smooth manifold triad and $h_i\colon V_i \to M_i$ is a diffeomorphism for $i = 0, 1$.

Definition 7.20 (Equivalence)**.** Two cobordisms $(W; V_0, V_1; h_0, h_1)$ and $(W'; V_0', V_1'; h_0', h_1')$ from M_0 to M_1 are said to be *equivalent* if there exists a diffeomorphism $g\colon W \to W'$ carrying V_0 to V_0' and V_1 to V_1', such that for $i = 0, 1$, the following triangle commutes:

$$\begin{tikzcd} V_i \arrow[rr, "g|_{V_i}"] \arrow[dr, "h_i"'] & & V_i' \arrow[dl, "h_i'"] \\ & M_i & \end{tikzcd}$$

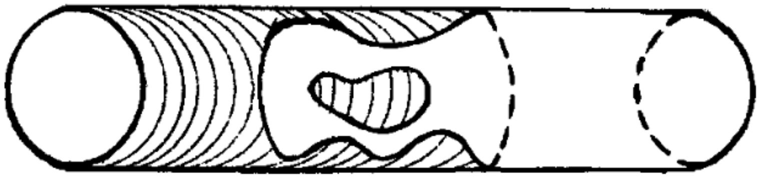
Definition 7.21 (Composition of cobordisms)**.** Given a cobordism equivalence class c from M_0 to M_1 and c' from M_1 to M_2, there is a well-defined class cc' from M_0 to M_2 formed using Theorem 7.18 as follows: let $(W; V_0, V_1; h_0, h_1)$ be the cobordism from M_0 to M_1 and $(W'; V_0', V_1'; h_0', h_1')$ from M_1 to M_2. Then the cobordism formed by $(W \cup_{\mathrm{id}} W'; V_0, V_1'; h_0, h_1')$ is a cobordism from M_0 to M_2, and furthermore, the inclusions $j_h\colon W \to W \cup_{\mathrm{id}} W'$ and $j_{h'}\colon W' \to W \cup_{\mathrm{id}} W'$ are diffeomorphisms onto their images.

This composition is associative.

Definition 7.22 (Identity cobordism)**.** For every closed manifold M, the identity cobordism class ι_M is the equivalence class of $(M \times I; M \times 0, M \times 1; p_0, p_1)$ where $p_i(x, i) = x$, for $x \in M$ and $i = 0, 1$. Hence $\iota_{M_1} c = c = c\iota_{M_2}$ when c is a cobordism class from M_1 to M_2.

Definition 7.23 (Trivial cobordism)**.** A cobordism $c = (W; V_0, V_1; h_0, h_1)$ is called a trivial cobordism if it is equivalent to an identity cobordism.

Note. Note also that there are non-trivial inverses: In particular, the manifolds in a cobordism are **not** assumed to be connected.



Definition 7.24. Consider cobordism classes from M to itself. These form a monoid H_M. The invertible coboridisms in H_M form a group G_M.

Definition 7.25 (c_h)**.** Given a diffeomorphism $h\colon M \to M'$, define c_h as the class of $(M \times I; M \times 0, M \times 1; j, h_1)$ where $j(x,0) = x$ and $h_1(x,1) = h(x)$ for $x \in M$.

So a diffeomorphism $M \to M'$ gives a cobordism c_h from M to M'.

Theorem 7.26. *$c_h c'_h = c_{h'h}$ for any two diffeomorphisms $h\colon M \to M'$ and $h'\colon M' \to M''$.*

Proof. Let $W = M \times I \cup_h M' \times I$. Let $c_h = (M \times I, M \times 0, M \times I, ; j_0, j_1)$ and $c_{h'} = (M' \times I, M' \times 0, M' \times 1, j'_0, j'_1)$. So recall that this is formed by taking a tube on M and a tube on M' and then gluing an end of the tube of M to an end of the tube of M' through a twist by the diffeomorphism h. Then W is still a smooth manifold. The resulting cobordism is $(W, M \times 0, M' \times 1, j_0, j'_1)$. We must show that this is the same, or more precisely, that this cobordism is *equivalent* to the cobordism $(M \times I, M \times 0, M \times 1, j, h_1)$ where $j(x,0) = x$ and $h_1(x,1) = h'h(x)$. So we must define a diffeomorphism $g\colon M \times I \to W$ carrying $M \times 0$ to $M \times 0$ and $M \times 1$ to $M' \times I$, such that for $i = 0,1$, the following triangle commutes

$$\begin{tikzcd}
M \times 1 \arrow[rr, "g|_{M\times 1}"] \arrow[dr, "h_1"'] & & M' \times 1 \arrow[dl, "j'_1"] \\
& M'' &
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \times 0 \arrow[rr, "g|_{M\times 0}"] \arrow[dr, "j"] & & M \times 0 \arrow[dl, "j_0"] \\
& M &
\end{tikzcd}$$

Define $g\colon M \times I \to W$ by

$$g(x,t) = \begin{cases} j_h(x, 2t), & t \in \left[0, \frac{1}{2}\right] \\ j_{h'}\left(h(x), 2t-1\right), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

where $j_h\colon M \times I \to W$ is the inclusion and $j_{h'}\colon M' \times I \to W$ is the other inclusion given in the construction of $c_h c_{h'}$. Then indeed $g|_{M\times 0}$ maps into $M \times 0$ and $g|_{M\times 1}$ maps into $M' \times 1$. Furthermore, $j_0 \circ g(x,0) = j_0 \circ j_h(x,0) = x$ and $j'_1 \circ g(x,1) = j'_1 \circ j_{h'}(h(x),1) = j'_1(h(x),1) = h'h(x) = h_1(x,1)$, so $j'_1 \circ g = h_1$. $\square$

7.3.1. *Isotopies and Pseudo-Isotopies.*

Definition 7.27. Two diffeomorphisms $h_0, h_1\colon M \to M'$ are (smoothly) isotopic if there exists a smooth map $f\colon M \times I \to M'$ such that $f_t = f(-,t)\colon M \to M'$ is a diffeomorphism for every t and $f_0 = h_0$ and $f_1 = h_1$.

Two diffeomorphisms $h_0, h_1 \colon M \to M'$ are *pseudo-isotopic* if there exists a diffeomorphism $g \colon M \times I \to M' \times I$ such that $g(x,0) = (h_0(x), 0)$ and $g(x,1) = (h_1(x), 1)$.

Lemma 7.28. *Isotopy and pseudo-isotopiy are equivalence relations.*

Theorem 7.29. $c_{h_0} = c_{h_1}$ *if and only if* h_0 *is pseudo-isotopic to* h_1*.*

Proof. Let $g \colon M \times I \to M' \times I$ be a pseudo-isotopy between h_0 and h_1. Define $h_0^{-1} \times \mathrm{id} \colon M' \times I \to M \times I$ by

$$\left(h_0^{-1} \times \mathrm{id}\right)(x,t) = \left(h_0^{-1}(x), t\right).$$

We claim that $\left(h_0^{-1} \times \mathrm{id}\right) \circ g$ is an equivalence between c_{h_1} and c_{h_0}. Firstly, $\left(h_0^{-1} \times \mathrm{id}\right) \circ g$ is indeed a map $M \times I \to M \times I$. If we write $c_{h_0} = (M \times I; M \times 0, M \times 1; j_0, k_0)$ and $c_{h_1} = (M \times I; M \times 0, M \times 1; j_0', k_0')$ where $j_0(x,0) = x$, $j_0'(x,0) = x$ and $k_0(x,1) = h_0(x)$ and $k_0'(x,1) = h_1(x)$, then firstly, $\left(h_0^{-1} \times \mathrm{id}\right) \circ g(x,0) = \left(h_0^{-1} \times \mathrm{id}\right)(h_0(x), 0) = (x,0)$ and $\left(h_0^{-1} \times \mathrm{id}\right) \circ g(x,1) = \left(h_0^{-1} \times \mathrm{id}\right)(h_1(x), 1) = \left(h_0^{-1} \circ h_1(x), 1\right) \in M \times 1$, and lastly,

$$k_0 \circ \left(h_0^{-1} \times \mathrm{id}\right) \circ g(x,1) = k_0 \left(h_0^{-1} \circ h_1(x), 1\right) = h_1(x) = k_0'(x,1)$$

and

$$j_0 \circ \left(h_0^{-1} \times \mathrm{id}\right) \circ g(x,0) = j_0(x,0) = x = j_0'(x,0)$$

so $\left(h_0^{-1} \times \mathrm{id}\right) \circ g$ defines an equivalence from c_{h_1} to c_{h_0}. □

7.3.2. *Interlude.* A different way to define a cobordism is as follows:

Definition 7.30. A smooth compact n-dimensional manifold is said to be a cobordism between two $(n-1)$-dimensional smooth manifolds M_L and M_R if there exist open embeddings $M_L \times \mathbb{R} \hookrightarrow M$ and $M_R \times \mathbb{R} \hookrightarrow M$ such that the images of $M_R \times [0, \infty)$ and $M_L \times (-\infty, 0]$ are closed. We denote this by $M_L \rightsquigarrow M_R$

Definition 7.31 (Gluing cobordisms/composition of cobordisms)**.** Given cobordisms $M_L \rightsquigarrow M_R = N_L \rightsquigarrow N_R$, we can form the composite cobordism $M \circ N$ as the pullback

$$\begin{tikzcd}
M_L \times \mathbb{R} \arrow[dr, hook] & & M_R \times \mathbb{R} = N_L \times R \arrow[dl, hook] \arrow[dr, hook] & & N_R \times \mathbb{R} \arrow[dl, hook] \\
& M \arrow[dr, hook] & & N \arrow[dl, hook] & \\
& & M \circ N \arrow[u, phantom, "\wedge"] & &
\end{tikzcd}$$

Definition 7.32 (Isomorphism/Equivalence of Cobordisms)**.** In this definition, two cobordisms $M_1 \colon M_L \rightsquigarrow M_R$ and $M_2 \colon M_L \rightsquigarrow M_R$ are isomorphic/equivalent when there exist maps making the following diagram commute:

$$\begin{tikzcd}
M_L \times R \arrow[r, hook] \arrow[d, dash, "\mathrm{id}"'] & M_1 \arrow[d, "f"', "\cong"] & M_R \times \mathbb{R} \arrow[l, hook'] \arrow[d, dash, "\mathrm{id}"] \\
M_L \times \mathbb{R} \arrow[r, hook] & M_2 & M_R \times \mathbb{R} \arrow[l, hook']
\end{tikzcd}$$

Definition 7.33 (Identity cobordism)**.** For a smooth compact manifold M, the identity cobordism of M is the cobordism from M to M given by $M \times \mathbb{R}$ where we embed $M \times \mathbb{R}_{<0} \hookrightarrow M \times \mathbb{R}$ and $M \times \mathbb{R}_{>0} \hookrightarrow M \times \mathbb{R}$ by the inclusions.

Definition 7.34 (Trivial cobordism)**.** A cobordism is trivial if it is equivalent to an identity cobordism.

7.4. **Elementary Cobordisms.**

Definition 7.35 (Gradient-like vector fields for Morse functions)**.** Let f be a Morse function for the triad $(W^n; V, V')$. A vector field ξ on W^n is a *gradient-like vector field for* f if

1. $\xi(f) > 0$ throughout the complement of the set of critical points of f
2. Given any critical point p of f, there are coordinates $(x, y) = (x_1, \dots, x_\lambda, x_{\lambda+1}, \dots, x_n)$ in a neighborhood U of p such that $f = f(p) - |x|^2 + |y|^2$ and ξ has coordinates $(-x_1, \dots, -x_\lambda, x_{\lambda+1}, \dots, x_n)$ throughout U.

Remark. The first condition essentially says that outside the critical points of f, ξ points in the direction into which f is increasing. If we think of f as a height function for the manifold, then ξ points "upward" along the manifold.

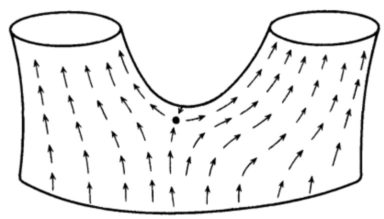


FIGURE 5. A gradient-like vector field

Theorem 7.36. *Let $f\colon M \to \mathbb{R}$ be a Morse function on a compact manifold M. Then there exists a gradient-like vector field ξ for f.*

Definition 7.37. If ξ is a vector field on M, an *integral curve of* ξ is a smooth curve $\gamma\colon J \to M$ such that

$$\gamma'(t) = \xi_{\gamma(t)}, \forall t \in J$$

Proposition 7.38. *[**LeeSM**] Let ξ be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exists $\varepsilon > 0$ and a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at p.*

Remark. We identify the triad $(W; V_0, V_1)$ with the cobordism $(W; V_0, V_1; i_0, i_1)$ where $i_0\colon V_0 \to V_0$ and $i_1\colon V_1 \to V_1$ are the identity maps.

Definition 7.39 (Product cobordism)**.** A triad $(W; V_0, V_1)$ is said to be a *product cobordism* if it is diffeomorphic to the trivial cobordism $(V_0 \times [0,1]; V_0 \times 0, V_0 \times 1)$.

Theorem 7.40 (Identifying product/trivial cobordisms)**.** *If the Morse number μ of a triad $(W; V_0, V_1)$ is zero, then $(W; V_0, V_1)$ is a product cobordism.*

Proof. Let $f\colon W \to \mathbb{R}$ be a Morse function with no critical points. Since W is compact, we have $f(W) = [a, b]$. Choose a gradient-like vector field ξ for f. As $\xi(f) > 0$ on all of W, we can define a new vector field ζ on W by

$$\zeta = \frac{1}{\xi(f)}\xi.$$

Consider the integral curve $c_p(t)$ of ζ starting at a point p of $f^{-1}(a)$. Then

$$\frac{d}{dt} f\left(c_p(t)\right) = c'(t) f = \zeta_{c(t)}(f) = \frac{1}{\xi(f)} \xi(f) = 1$$

Since it starts at the level set $f = a$ at time $t = 0$, it will reach the level set $f = b$ at time $t = b - a$. Define a map $h\colon f^{-1}(a) \times [0, b-a] \to W = W_{[a,b]}$ by

$$h(p,t) = c_{p(t)}.$$

The proof follows from h being a diffeomorphism as it depends smoothly on p and t and that two distinct integral curves do not meet. $\square$

Theorem 7.41 (Collar Neighborhood Theorem). *Let W be a compact smooth manifold with boundary. There exists a neighborhood of ∂W (called a collar neighborhood) diffeomorphic to $\partial W \times [0,1)$.*

Definition 7.42 (Two-sided). A connected, closed submanifold $M^{n-1} \subset W^n - \partial W^n$ is said to be *two-sided* if some neighborhood of M^{n-1} on W^n is cut into two components when M^{n-1} is deleted.

Theorem 7.43 (The Bicollaring Theorem). *Suppose that every component of a smooth submanifold M of W is compact and two-sided. Then there exists a "bicollar" neighborhood of M in W diffeomorphic to $M \times (-1,1)$ in such a way that M corresponds to $M \times 0$.*

7.4.1. *Handlebody decomposition/surgery.* First, the setup.
Suppose $(W; V, V')$ is a triad with Morse function $f\colon W \to \mathbb{R}$ and gradient-like vector field ξ for f. Suppose $p \in W$ is a critical point, and $V_0 = f^{-1}(c_0)$ and $V_1 = f^{-1}(c_1)$ are level sets such that $c_0 < f(p) < c_1$ and that $c = f(p)$ is the only critical value in the interval $[c_0, c_1]$.
Now, since ξ is a gradient-like vector field for f, there exists a neighborhood U of p in W and a coordinate diffeomorphism $g\colon B(0, 2\varepsilon) \to U$ such that $f \circ g(x,y) = c - \|x\|^2 + \|y\|^2$ and so that ξ has coordinates $(-x_1, \ldots, -x_\lambda, x_{\lambda+1}, \ldots, x_n)$ throughout U, for some $-1 \le \lambda \le n$ and some $\varepsilon > 0$.
Now let $V_{-\varepsilon} = f^{-1}\left(c - \varepsilon^2\right)$ and $V_\varepsilon = f^{-1}\left(c + \varepsilon^2\right)$. We may assume that $4\varepsilon^2 < \min\{|c - c_0|, |c - c_1|\}$ so that $V_{-\varepsilon}$ lies between V_0 and $f^{-1}(c)$ and V_ε lies between $f^{-1}(c)$ and V_1.

Definition 7.44 (Characteristic embedding). The characteristic embedding $\varphi_L\colon S^{\lambda-1} \times B^{n-\lambda} \to V_0$ is obtained as follows.
First, define an embedding $\varphi\colon S^{\lambda-1} \times B^{n-\lambda} \to V_{-\varepsilon}$ by $\varphi(u, \theta v) = g(\varepsilon u \cosh\theta, \varepsilon v \sinh\theta)$ for $u \in S^{\lambda-1}, v \in S^{n-\lambda-1}$ and $0 \le \theta < 1$. Then $f \circ \varphi(u, \theta v) = c - \|\varepsilon u \cosh\theta\|^2 + \|\varepsilon v \sinh\theta\|^2 = c - \varepsilon^2$, so indeed, $\varphi(u, \theta v) \in V_{-\varepsilon}$. Since φ is also an injective continuous map from a compact space to a Hausdorff space, it is an embedding.
Starting now at the point $\varphi(u, \theta v) \in V_{-\varepsilon}$, the integral curve of ξ (which, recall, goes "upward") is a non-singular (non-vanishing Jacobian) curve which leads from $\varphi(u, \theta v)$ back to some well-defined point $\varphi_L(u, \theta v) \in V_0$.
Define the *left-hand sphere* S_L of p in V_0 to be the image $\varphi_L\left(S^{\lambda-1} \times 0\right)$.

Definition 7.45 (Surgery). Given a manifold V of dimension $n-1$ and an embedding $\varphi\colon S^{\lambda-1} \times B^{n-\lambda} \to V$, let $\chi(V, \varphi)$ denote the quotient manifold obtained from the disjoint union

$$\left(V - \varphi\left(S^{\lambda-1} \times 0\right)\right) \sqcup \left(B^\lambda \times S^{n-\lambda-1}\right)$$

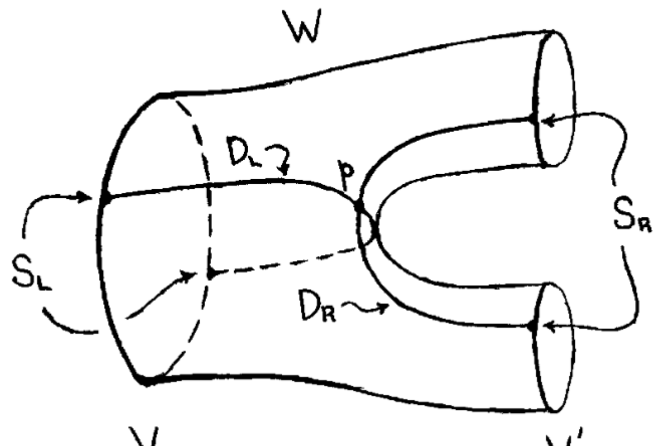


Figure 3.2

by identifying $\varphi(u, \theta v)$ with $(\theta u, v)$ for each $u \in S^{\lambda-1}, v \in S^{n-\lambda-1}$ and $0 < \theta < 1$. If V' denotes any manifold diffeomorphic to $\chi(V, \varphi)$ then we will say that V' can be obtained from V by *surgery* of type $(\lambda, n-\lambda)$.

So surgery on an $(n-1)$-manifold has the effect of removing an embedded sphere of dimension $\lambda - 1$ and replacing it by an embedded sphere of dimension $n - \lambda - 1$.

Definition 7.46. An *elementary cobordism* is a triad $(W; V, V')$ possessing a Morse function f with exactly one critical point p.

Theorem 7.47. *If $V' = \chi(V, \varphi)$ can be obtained from V by surgery of type $(\lambda, n-\lambda)$, then there exists an elementary cobordism $(W; V, V')$ and a Morse function $f \colon M \to \mathbb{R}$ with exactly one critical point of index λ.*

Proof. Let

$$L_\lambda = \left\{(x, y) \in \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda} \mid -1 \leq -\|x\|^2 + \|y\|^2 \leq 1, \|x\|\|y\| < \sinh 1 \cosh 1\right\},$$

which is a manifold with two boundaries: the "left" boundary $\left\{-\|x\|^2 + \|y\|^2 = -1\right\}$ is diffeomorphic to $S^{\lambda-1} \times B^{n-\lambda}$. Indeed, recall that

$$\cosh^2 x - \sinh^2 x = 1,$$

and consider the map

$$(u, \theta v) \mapsto (u \cosh\theta, v \sinh\theta)$$

Show that it is a diffeomorphism. Similarly for the "right" boundary.
Consider the orthogonal trajectories of the surfaces $-\|x\|^2 + \|y\|^2 = \text{constant}$.
The trajectories of the surface $-\|x\|^2 + \|y\|^2 = c$ can be parametrized by $t \mapsto \left(tx, t^{-1}y\right)$. To see this, pick a point (x, y) such that $-\|x\|^2 + \|y\|^2 = c$, that is $(x, y) = (cu \sinh\theta, cv \cosh\theta)$. Then the derivative with respect to θ is

$$(cu \cosh\theta, cv \sinh\theta)$$

and since

$$(cu \cosh\theta, cv \sinh\theta) \cdot (cu \sinh\theta, -cv \cosh\theta) = c^2 \left(\|u\|^2 \cosh\theta \sinh\theta - \|v\|^2 \cosh\theta \sinh\theta\right) = 0$$

since $\|u\| = \|v\| = 1$.

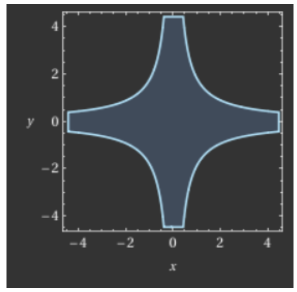


FIGURE 6. $|xy| < \sinh 1 \cosh 1$

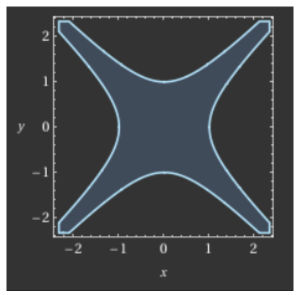


FIGURE 7. $-1 \leq -\|x\|^2 + \|y\|^2 \leq 1$

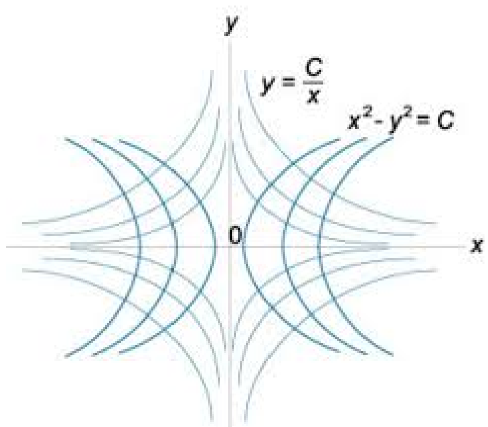


FIGURE 8. level-sets.jpeg

We now construct a manifold $W = \omega(V, \varphi)$ as follows. Start with the disjoint union

$$\left(V - \varphi\left(S^{\lambda-1} \times 0\right)\right) \times D^1 \sqcup L_\lambda.$$

Now for each $u \in S^{\lambda-1}, v \in S^{n-\lambda-1}, 0 < \theta < 1$ and $c \in D^1$, identify the point $(\varphi(u, \theta v), c)$ in the first summand with the point $(x, y) \in L_\lambda$ such that

(1) $-\|x\|^2 + \|y\|^2 = c$

(2) (x, y) lies on the orthogonal trajectory which passes through the point $(u\cosh\theta, v\sinh\theta)$.

This defines a diffeomorphism

$$\varphi\left(S^{\lambda-1} \times \left(B^{n-\lambda} - 0\right)\right) \times D^1 \cong L_\lambda \cap \left(\mathbb{R}^\lambda - 0\right) \times \left(\mathbb{R}^{n-\lambda} - 0\right)$$

(Finish the proof)

□

Theorem 7.48. *Let $(W; V, V')$ be an elementary cobordism with characteristic embedding $\varphi_L \colon S^{\lambda-1} \times B^{n-\lambda} \to V$. Then $(W; V, V')$ is diffeomorphic to the triad $(\omega(V, \varphi_L); V, \chi(V, \varphi_L))$.*

Theorem 7.49. *Let $(W; V, V')$ be an elementary cobordism possessing a Morse function with one critical point, of index λ. Let D_L be the left-hand disk associated to a fixed gradient-like vector field. Then $V \cup D_L$ is a deformation retract of W.*

Corollary 7.50.

$$H_n(W, V) \cong \begin{cases} \mathbb{Z}, & n = \lambda \\ 0, & n \neq \lambda \end{cases}.$$

A generator for $H_\lambda(W, V)$ is represented by D_L.

7.4.2. *Problems.*

Problem 7.51 (Invertible cobordisms and boundaries of compact manifolds)**.** Let $W_0\colon M_0 \rightsquigarrow \varnothing$ and $W_1\colon M_1 \rightsquigarrow \varnothing$ be two compact d-dimensional smooth cobordisms from compact $(d-1)$-dimensional smooth manifolds M_0 and M_1 to the empty manifold, viewed as a $(d-1)$-manifold. In other words, we have a smooth embedding $M_i \times \mathbb{R} \hookrightarrow W_i$ satisfying that $M_i \times (-\infty, 0]$ is closed, and such that their complement $W_i - (M_i \times \mathbb{R})$ is compact. We define $\operatorname{int}(W_i)$ to be the complement of the image of $M_i \times (-\infty, t]$ for some $t \in \mathbb{R}$ (and hence any $t \in \mathbb{R}$), and observe that $\operatorname{int}(W_i)$ is again a smooth manifold, being an open subset of W_i.

(1) Assume that in the situation of the above, $\operatorname{int}(W_0)$ is diffeomorphic to $\operatorname{int}(W_1)$. Show that M_0 and M_1 are invertibly cobordant, i.e., there exists a cobordism $M_0 \rightsquigarrow M_1$ which is invertible in the category Cob_d.
(2) Let W be a smooth, open (i.e., non-compact) d-manifold. We define a compact closure of W to be a compact cobordism $W'\colon M \rightsquigarrow \varnothing$ such that W is diffeomorphic to $\operatorname{int}(W')$. Assume that W admits a comapct closure $W'\colon M \rightsquigarrow \varnothing$. Show that the set of compact closures of W up to isomorphism of their interiors is in bijection with the set of invertible cobordisms over M.

Proof. (1)
Saying that $M_0 \rightsquigarrow M_1$ is invertible in Cob_d is precisely saying that there exists a cobordism $M_1 \rightsquigarrow M_0$ such that the composite cobordism $M_0 \rightsquigarrow M_1 \rightsquigarrow M_0$ is equivalent to the trivial cobordism $M_0 \rightsquigarrow M_0$. We will do this using the usual definition of cobordisms with boundaries. Then the problem is equivalently to show that we can find coborisms $M_0 \rightsquigarrow M_1 \rightsquigarrow M_0$ such that the composite is a product cobordism - i.e., has Morse number 0. In this case, we are dealing with closed compact manifolds W_0, W_1 such that $\partial W_0 \cong M_0$ and $\partial W_1 \cong M_1$. Furthermore, the boundaries have closed collar neighborhoods $\partial W_i \times I$, and removing some open/usual collar neighborhoods of these boundaries $\partial W_i \times [0,1)$ leaves us with compact spaces which are, by assumption, diffeomorphic. Now, take the cobordism W_0 and choose a collar neighborhood of ∂W_0: $M_0 \times [0,1]$, where M_0 is identified with $M_0 \times 0$ in W_0. By assumption, there is a diffeomorphism $W_0 - (M_0 \times [0,1]) \cong W_1 - (M_1 \times [0,1])$. Now, the diffeomorphism extends to the closure of the interiors which is also M_i since the collar is a cylinder, so we obtain a diffeomorphism $h\colon M_0 \times 1 \cong M_1 \times 1$. Without loss of generality, we can reparametrize, to get the diffeomorphism $h\colon M_0 \times 1 \to M_1 \times 0$ since the boundaries of the interiors must map to each other. Now we can glue the collars by gluing the cobordisms they represent using theorem 1.4 in Milnor's book on h-cobordisms to get a cobordism c_h which is the manifold $M_0 \times [0,1] \cup_h M_1 \times [0,1]$. This indeed now gives a cobordism $M_0 \rightsquigarrow M_1$. We can likewise obtain the cobordism $M_1 \rightsquigarrow M_0$ which is also obtained by gluing $M_1 \times [0,1]$ with $M_0 \times [0,1]$ along $M_1 \times 1$ and $M_0 \times 0$. Denote this cobordism by $c_{h'}$. We claim that $c_h c_{h'} = \mathrm{id}_{M_0}$. That is, that $c_h c_{h'}$ is a product cobordism/trivial cobordism of M_0. One way to see this is by using theorem 1.6 in Milnor's book on h-cobordisms which says that $c_h c_{h'} = c_{h'h} = c_{\mathrm{id}_{M_0}}$ which indeed is the trivial cobordism. Alternatively, each collar neighborhood has no critical values, so c_h and $c_{h'}$ both have Morse number 0, and then corollary 3.8 in Milnor's book on h-cobordisms gives that $c_h c_{h'}$ also has Morse number 0, hence is trivial by theorem 3.4 in the same book. ◻

7.5. **Morse Functions.** The goal is to be able to factor cobordisms into compositions of simpler cobordisms.

Definition 7.52 (Critical points and non-degenerate critical points)**.** Let W be a smooth manifold and $f\colon W \to \mathbb{R}$ a smooth function. A point $p \in W$ is a critical point of f if, in some coordinate system,

$$\frac{\partial f}{\partial x^1}|_p = \frac{\partial f}{\partial x^2}|_p = \ldots = \frac{\partial f}{\partial x^n}|_p = 0.$$

Such a point is called a non-degenerate critical point if $\det\left(H(f)_p\right) = \det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}|_p\right) \neq 0$

Lemma 7.53 (Morse Lemma)**.** *If p is a non-degenerate critical point of f, then in some coordiante system about p,*

$$f\left(x_1, \ldots, x_n\right) = c - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2$$

for λ between 0 and n and c some constant.

Definition 7.54 (Index of a critical point)**.** The λ from the Morse Lemma (Lemma 7.53) is called the index of the critical point p.

Definition 7.55 (Morse Function)**.** A *Morse function* on a smooth manifold triad $(W; V_0, V_1)$ is a smooth function $f\colon W \to [a, b]$ such that

(1) $f^{-1}(a) = V_0$ and $f^{-1}(b) = V_1$
(2) All the critical points of f are interior (lie in $W - \partial W$) and are non-degenerate.

Corollary 7.56. *A Morse function has only finitely many zeros.*

Proof. Suppose we have a Morse function $f\colon W \to [a, b]$ and suppose that p is a critical point. By definition, it is non-degenerate since f is a Morse function, so by the Morse Lemma, in some neighborhood of p, f takes the form

$$f\left(x_1, \ldots, x_n\right) = c - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2$$

so in particular, $\frac{\partial f}{\partial x^i}\left(x_1, \ldots, x_n\right) = -2x_i$ in this neighborhood for all i. Hence $(x_1, \ldots, x_n) = (0, \ldots, 0)$ in this neighborhood is the only critical point (in particular, in local coordinates, $p = (0, \ldots, 0)$). This shows that critical points of a Morse function are isolated. Since the manifold of a smooth manifold triad is, in particular, compact, there are only finitely many critical points since a collection of isolated points in a compact space is finite. $\square$

Definition 7.57 (Morse number μ)**.** The *Morse number* μ of $(W; V_0, V_1)$ is the minimum over all Morse functions f on $(W; V_0, V_1)$ of the number of critical points of f.

Theorem 7.58 (Existence of Morse functions)**.** *Every smooth manifold triad $(W; V_0, V_1)$ possesses a Morse function.*

Remark. We proved a stronger version of this theorem in Problem 5.25. We will also outline the proof idea from Milnor's book.

To prove the existence theorem of Morse functions, we need the following lemmas:

Lemma 7.59. *There exists a smooth function $f\colon W \to [0,1]$ with $f^{-1}(0) = V_0, f^{-1}(1) = V_1$, such that f has no critical points in a neighborhood of the boundary of W.*

Lemma 7.60 (M. Morse)**.** *If f is a C^2 mapping of an open subset $U \subset R^n$ to the real line, then, for almost all linear mappings $L\colon R^n \to R$, the function $f + L$ has only nondegenerate critical points.*

Proof. The idea of the proof is to consider the manifold $U \times \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$ the its submanifold $M = \{(x, L) \mid d(f(x) + L(x)) = 0\}$. Then $x \mapsto (x, -df(x))$ is a diffeomorphism $U \cong M$. Composing with a projection $\pi\colon M \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ sending $(x, L) \mapsto L$, which, under the identification, corresponds to $x \mapsto -df(x)$; one sees that π is critical at $x \approx (x, L) \in M \cong U$ if and only if $d\pi = -\frac{\partial^2 f}{\partial x_i \partial x_j}$ is singular. So x is a degenerate critical point of $f + L$ if and only if it is a critical point of π. By Sard's theorem, the set of critical values of π has measure zero. So if we can show that the image of π does not have measure zero, the result follows. For this, notice that π maps $x \mapsto -df(x)$ and we have a diffeomorphism $U \cong M$ by $x \mapsto (x, -df(x))$, so π is an open map from U into $\mathbb{R}^n$, hence in particular, the image is open and thus not measure zero. □

Lemma 7.61 (Lemma B)**.** *Let K be a compact subset of an open set $U \subset \mathbb{R}^n$. If $f\colon U \to \mathbb{R}$ is C^2 and has only nondegenerate critical points in K, then there is a number $\delta > 0$ such that if $g\colon U \to \mathbb{R}$ is C^2 and at all points of K, we have*

$$\left|\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i}\right| < \delta, \quad \left|\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 g}{\partial x_i \partial x_j}\right| < \delta$$

for $i, j = 1, \dots, n$, then g likewise has only nondegenerate critical points in K.

Proof. Just an extra note on the proof:

$$||df| - |dg||^2 \leq |df - dg|^2 = |d(f-g)|^2 = \sum_i \left|\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i}\right|^2$$

giving the possibility of bounding this by $\frac{\mu}{2}$. The other bound is done similarly. □

Lemma 7.62 (Lemma C)**.** *Suppose $h\colon U \to U'$ is a diffeomorphism of one open subset of $\mathbb{R}^n$ onto another and carries the compact set $K \subset U$ onto $K' \subset U'$. Given a number $\varepsilon > 0$, there is a number $\delta > 0$ such that if a smooth map $f\colon U' \to \mathbb{R}$ satisfies*

$$|f| < \delta, \quad \left|\frac{\partial f}{\partial x_i}\right| < \delta, \quad \left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right| < \delta, \quad i, j = 1, \dots, n$$

at all points of $K' \subset U'$, then $f \circ h$ satisfies

$$|f \circ h| < \varepsilon, \quad \left|\frac{\partial f \circ h}{\partial x_i}\right| < \varepsilon, \quad \left|\frac{\partial^2 f \circ h}{\partial x_i \partial x_j}\right| < \varepsilon, \quad i, j = 1, \dots, n$$

at all points of K.

Definition 7.63 (The compact-open C^2 topology)**.** **[Hirsch]** What Milnor calls the C^2 topology on $F(M, \mathbb{R})$ of smooth real-valued functions on a compact manifold, M with boundary is, I believe, simply the compact-open C^2 topology on $F(M, \mathbb{R}) = C^\infty(M)$.

The compact open topology on $F(M,\mathbb{R})$ is generated by sets defined as follows. Let $f \in F(M,\mathbb{R})$ and (φ, U) a chart on M. Let $K \subset U$ be compact. Define a weak subbasic neighborhood

$$\mathcal{N}^2\left(f;(\varphi,U),K,\varepsilon\right)$$

to be the set of all smooth maps $g\colon M \to \mathbb{R}$ such that

$$\left\|D^k\left(f\varphi^{-1}\right)(x) - D^k\left(g\varphi^{-1}\right)(x)\right\| < \varepsilon$$

for all $x \in \varphi(K)$, $k = 0,1,2$. The C^2 topology on $F(M,\mathbb{R})$ is generated by these sets.

Theorem 7.64. *If M is a compact manifold without boundary, the Morse functions form an open dense subset of $F(M,\mathbb{R})$ in the C^2 topology.*

Proof. Neat proof. Check it out in [**Milnor-h-cobordism**]. □

Proof of Theorem 7.58. The proof follows neatly from the previous theorem and lemmas. Again, check [**Milnor-h-cobordism**]. □

Remark. In the C^2 topology, the Morse functions also form an open dense subset of smooth maps $f\colon (W;V_0,V_1) \to ([0,1],0,1)$.

Lemma 7.65. *Let $f\colon W \to [0,1]$ be a Morse function for the triad $(W;V_0,V_1)$ with critical points $p_1,\ldots,p_k$. Then f can be approximated by a Morse function g with the same critical points such that $g(p_i) \neq g(p_j)$ whenever $i \neq j$.*

Now, the goal is to decompose cobordisms into simple cobordisms using Morse functions.

Lemma 7.66. *Let $f\colon (W;V_0,V_1) \to ([0,1],0,1)$ be a Morse function, and suppose that $0 < c < 1$ where c is not a critical value of f. Then both $f^{-1}[0,c]$ and $f^{-1}[c,1]$ are smooth manifolds with boundary.*

Corollary 7.67. *Any cobordism can be expressed as a composition of cobordisms with Morse number 1.*

7.6. **h-cobordism.**

Definition 7.68 (h-cobordism)**.** A compact cobordism $W\colon M_0 \rightsquigarrow M_1$ between closed manifolds M_0 and M_1 is called an h-cobordism if the inclusion $M_i \hookrightarrow W$ is a homotopy equivalence for $i \in \{0,1\}$.

Theorem 7.69 (h-cobordism theorem)**.** *Let $W\colon M_0 \rightsquigarrow M_1$ be a smooth, compact h-cobordism between closed, simply connected smooth manifolds M_0 and M_1, where we assume $\dim M_i \geq 5$. Then, there exists a diffeomorphism $W \cong M_0 \times I$ that restricts to the identity on the M_0 component of W.*

Corollary 7.70. *Two simply connected closed smooth manifolds of dimension ≥ 5 that are h-cobordant are diffeomorphic.*

Proof. The diffeomorphism of W to $V \times [0,1]$ sends the boundary $V \times \{1\}$ which is diffeomorphic to both V to $\partial_1 W$. □

Theorem 7.71 (Cerf's pseudo-isotopy theorem)**.** *Let M be a simply connected smooth manifold of dimension at least 5, and let $f,g \in \mathrm{Diff}(M)$ be two pseudo-isotopic diffeomorphisms of M. Then f and g are isotopic diffeomorphisms.*

Problem 7.72 (Connected sums and homology)**.** Let M, N be two connected smooth d-dimensional manifolds with empty boundary, and fix two embeddings of the d-disc into each; namely, fix an embedding $S^0 \times D^d \hookrightarrow M \sqcup N$ which is a bijection on path components. We define $M\#N$ to be the handle attachment of $D^1 \times D^{d-1}$ on $M \sqcup N$ via $S^0 \times D^d$; namely, $M\#N$ is the upper component of the boundary of the following manifold with boundary

$$\left(((M \sqcup N) \times I) - \left(S^0 \times D^d\right)\right) \cup_\partial D^1 \times D^d.$$

You may assume that connected sums are well-defined, i.e., independent of the choice of the embeddings $D^d \hookrightarrow M$ and $D^d \hookrightarrow N$.

(1) Given two Morse functions f_M and f_N on M and N, respectively, construct a Morse function on $M\#N$.
(2) Compute the homology of $M\#N$ in terms of the homology of M and N.
(3) Let $W_g^n := \#_g (S^n \times S^n)$, for $n \in \mathbb{N}$. Compute the homology of W_g^n.

Problem 7.73 (Poincaré conjecture)**.** Let M be a closed manifold of dimension at least 6. Assume that $M \simeq S^d$, where $\simeq$ denotes the equivalence relation of homotopy equivalence. Show that M is homeomorphic to the sphere, and indicate why the argument fails for showing that M is diffeomorphic to the sphere.

Proof. Firstly, we claim that if $M \simeq S^d$, then $M - D_1 \simeq S^d - D^d$ where D_1 is some disc in M. If $F\colon M \to S^d$ and $G\colon S^d \to M$ give a homotopy equivalence, then $G \circ F \simeq \mathrm{id}_M$ implies that $D_1 \simeq G(F(D_1))$

Removing two disks open discs D_1, D_2 from M, we get a compact cobordism from S^{d-1} to S^{d-1}. Now, since $d \geq 6$, $\pi_1 S^{d-1} = 1$. Furthermore, since $M \simeq S^d$, we have $M - (D_1 \cup D_2) \simeq S^d - \left(D^d \sqcup D^d\right) \simeq S^{d-1}$. Hence the inclusion becomes the inclusion into the equator for both D_1 and D_2. In particular, we get isomorphisms on both π_1 and $H_*(-;\mathbb{Z})$ since the spaces are homotopy equivalent. By Lemma B.1, the inclusions of the boundaries are thus homotopy equivalences. Therefore, $M - (D_1 \cup D_2)$ is an h-cobordism. Since S^{d-1} is a closed, simply connected smooth manifold for $d \geq 6$, the h-cobordism theorem tells us that there exists a diffeomorphism $M - (D_1 \cup D_2) \cong S^{d-1} \times I$ that restricts to the identity on the M_0 component of W. In particular, the restriction of the identity on one component, D_1 say, gives that regluing by the identity preserves the diffeomorphism, so we have $M - D_2 \cong D^d$. The other gluing might is completed under a diffeomorphism, so we find that M is diffeomorphic to a twisted sphere. From the last week's problem sheet, we know that twisted spheres are homeomorphic to normal spheres, but not necessarily diffeomorphic. This is where the diffeomorphism part fails. □

Problem 7.74 (Contractible manifolds with simply connected boundaries)**.** Let M be a compact manifold with non-empty boundary, of dimension d at least 6. Assume that ∂M is simply connected. Show that the following four statements are equivalent

(1) M is diffeomorphic to D^d.
(2) M is homeomorphic to D^d.
(3) M is contractible.

Proof. If M is diffeomorphic to D^d, then it is naturally also homeomorphic to D^d and indeed also contractible since we can just pull back the contraction.

Remove a disc $D^d \subset \operatorname{int} M$. We wish to apply the h-cobordism theorem to obtain a diffeomorphism $M - D^d \cong S^{d-1} \times I$, restricting to the identity on S^{d-1} in M, so that we can reglue D^d along the identity, thus obtaining a diffeomorphism $M \cong D^d$. Note that we have a smooth, compact cobordism between closed, simply connected smooth manifolds ∂M and $S^{d-1} = \partial D^d$. It remains to show that this is an h-cobordism, i.e., that the inclusions are homotopy equivalences. Since both spaces are simply connected, it suffices to show that the inclusions induce isomorphisms on π_1 and $H_*(-;\mathbb{Z})$. Consider $S^{d-1} \hookrightarrow M - D^d$. Let D_1 denote the disc in question and choose a disc D_2 containing D_1. Since $M - D_1 \cap D_2 \cong S^{d-1}$, Mayer-Vietoris gives us a LES

$$0 \to H_p(S^{d-1}) \to H_p\left(M - D_1\right) \oplus \underbrace{H_p(D_2)}_{=0} \to \underbrace{H_p(M)}_{=0} \to \dots$$

Furthermore, recall that in the proof of Mayer-Vietoris, we find that the map $H_p\left(S^{d-1}\right) \to H_p\left(M - D_1\right)$ in question is precisely the inclusion map. Hence the inclusion map $S^{d-1} \hookrightarrow M - D^d$ is an isomorphism which was what we wanted to show. Also, M is contractible, so $\pi_1(M) = 1$, so the inclusion also induces an isomorphism on fundamental groups. We therefore obtain using Lemma B.1 that the inclusion $S^{d-1} \hookrightarrow M - D^d$ is a homotopy equivalence. Next, we must show that the inclusion $\partial M \hookrightarrow M - D^d$ is also a homotopy equivalence. Firstly, we again have an isomorphism on fundamental groups for the same reason. Next, by Theorem B.2, we have that

$$H_*\left(M - D, \partial M\right) \cong H^*\left(M - D, \partial D\right)$$

Now $H_*\left(M - D, \partial D\right) \cong 0$, and we claim that this implies that $H^*\left(M - D, \partial D\right) \cong 0$.
To see this, note that the universal coefficient theorem for cohomology gives that

$$0 \to \operatorname{Ext}^1_R\left(H_{n-1}(M - D, \partial D), \mathbb{Z}\right) \to \underbrace{H^n(M - D, \partial D; \mathbb{Z})}_{\cong 0} \to \operatorname{Hom}_R\left(H_n(M - D, \partial D), \mathbb{Z}\right) \to 0$$

is exact, hence $\operatorname{Ext}^1_R\left(H_{n-1}(M - D, \partial D), \mathbb{Z}\right) \cong 0 \cong \operatorname{Hom}_R\left(H_n(M - D, \partial D), \mathbb{Z}\right)$. Now, since $M - D$ and ∂D are manifolds, they are in particular homotopy equivalent to finite CW-complexes and hence to finite Δ-complexes, hence so is $M - D/\partial D$. Now, we know from corollary 8.4.4 in the AlgTop1 notes that then $H_p\left(M - D, \partial D\right) \cong \tilde{H}_p\left(M - D/\partial D\right)$ is a finitely generated abelian group. But then vanishing of $\operatorname{Ext}^1(-, \mathbb{Z})$ means that the torsion part is trivial and the vanishing of $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ means that the torsionfree part is trivial. Hence $H_n\left(M - D, \partial D\right) \cong 0$ for all n.
By the LES associated to the pair $(M - D, \partial M)$, we thus obtain that the inclusion $\partial M \hookrightarrow M - D$ induces an isomorphism on integral homology.
This completes the proof. $\square$

Problem 7.75. Show that any diffeomorphism $S^1 \to S^1$ can be extended to a diffeomorphism $D^2 \to D^2$.

Proposition 7.76 (Characterization of the 5-disk)**.** *Suppose W^5 is a compact simply connected smooth manifold that has integral homology of a point. Let $V = \partial W$.*

(1) *If V is diffeomorphic to S^4, then W is diffeomorphic to D^5.*
(2) *If V is homeomorphic to S^4, then W is homeomorphic to D^5.*

Proof. Form a 5-manifold $M = W \sqcup_h D^5$ where h is the diffeomorphism $V \to S^4 = \partial D^5$. Then by Mayer-Vietoris, we find that

$$H_p\left(S^4\right) \to \underbrace{H_p(D^5) \oplus H_p(W)}_{0} \to H_p(M) \to H_{p-1}(S^4) \to 0$$

is exact for all p. Thus M is a simply connected manifold with the integral homology of S^5. By the above, M is diffeomorphic to S^5. Now we use the following theorem:

Theorem 7.77. *Any two smooth orientation-preserving imbeddings of an n-disk into a connected oriented n-manifold are ambient isotopic.*

Thus there is a diffeomorphism $g\colon M \to M$ that maps $D^5 \subset M$ into a disc D^5_1 such that $D^5_2 = M - \operatorname{int} D^5_1$ is also a disc. Then g maps $W \subset M$ diffeomorphically onto D^5_2.

For the homeomorphism part, consider the double $D(W)$ of W, $D(W) = W \sqcup_{\mathrm{id}} W$. The submanifold $V \subset D(W)$ has a bicollar neighborhood in $D(W)$, and $D(W)$ is homeomorphic to S^5. Now we use the following theorem:

Theorem 7.78. *If an $(n-1)$-sphere Σ, topologically embedded in S^n, has a bicollar neighborhood, then there exists a homeomorphisms $h\colon S^n \to S^n$ that maps Σ to $S^{n-1} \subset S^n$. Thus $S^n - \Sigma$ has two components and the closure of each is an n-disc with boundary Σ.*

It follows that W is homeomorphic to D^5. This completes the proof. $\square$

Proposition 7.79 (The smooth Schoenfliess theorem in dimension ≥ 5.)**.** *Suppose Σ is a smoothly embedded $(n-1)$-sphere in S^n. If $n \geq 5$, there is a smooth ambient isotopy that carries Σ into the equator $S^{n-1} \subset S^n$.*

7.7. **Handles.**

Definition 7.80. An *n-dimensional k-handle* is a contractible smooth manifold $D^k \times D^{n-k}$.

Definition 7.81. Let X, Y be topological spaces, and let $K \subset X$ and $L \subset Y$ be subspaces such that there exists a homeomorphism $\varphi\colon K \to L$. We obtain a new space, which call X glued to Y along φ by taking $X \sqcup Y/x \sim \varphi(x)$. We call φ the attaching map.

Definition 7.82. There are five subsets of a k-handle which will be of interest to us. They are:

- the *attaching region* defined to be $\partial D^k \times D^{n-k}$. Note that the attaching map of a k-handle is a homeomorphism of the attaching region onto a subset of the space being glued to.
- The *attaching sphere*, also called the *boundary of the core*, denoted $A^k = \partial D^k \times \{0\}$.
- The *core*, denoted $C^k = D^k \times \{0\}$.
- The *belt sphere*, also called the *transverse sphere*, denoted $B^k = \{0\} \times \partial D^{n-k}$.
- The *co-core*, denoted $K^k = \{0\} \times D^{n-k}$.

Envisioning D^k and D^{n-k} as products of the unit interval, we can envision the different parts of a handle as in Figure 9 below.

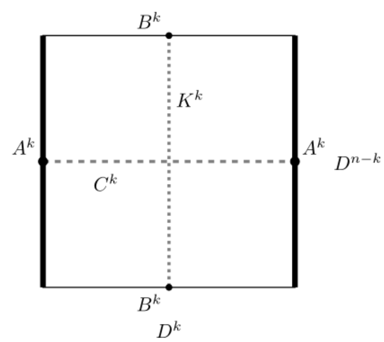


FIGURE 9. An n-dimensional k-handle

Convince yourself that Figure **??** is, in fact, a 2-dimensional 1-handle.

Exercise 7.83. Construct a handle decomposition of the 2-torus, T^2.

Exercise 7.84. Construct a handle decomposition of the 2-sphere, S^2.

Definition 7.85 (Handle attachment)**.** Consider an n-dimensional manifold M with boundary ∂M. Given an embedding $\varphi^q\colon S^{q-1} \times D^{n-q} \hookrightarrow \partial M$, we say that the manifold $M + (\varphi^q)$ defined by $M \sqcup_{\varphi^q} D^q \times D^{n-q}$ is obtained from M by attaching a handle of index q by φ^q.

Definition 7.86 (Handle decomposition and handlebody)**.** A *handle decomposition* of a compact manifold is a finite sequence of cobordisms $W_0 \rightsquigarrow W_1 \rightsquigarrow \ldots \rightsquigarrow W_l$ such that

(1) $W_0 = \varnothing$
(2) W_l is diffeomorphic to M.
(3) W_i is obtained from W_{i-1} by attaching a handle.

A *handlebody* is a compact manifold expressed as the union of handles.

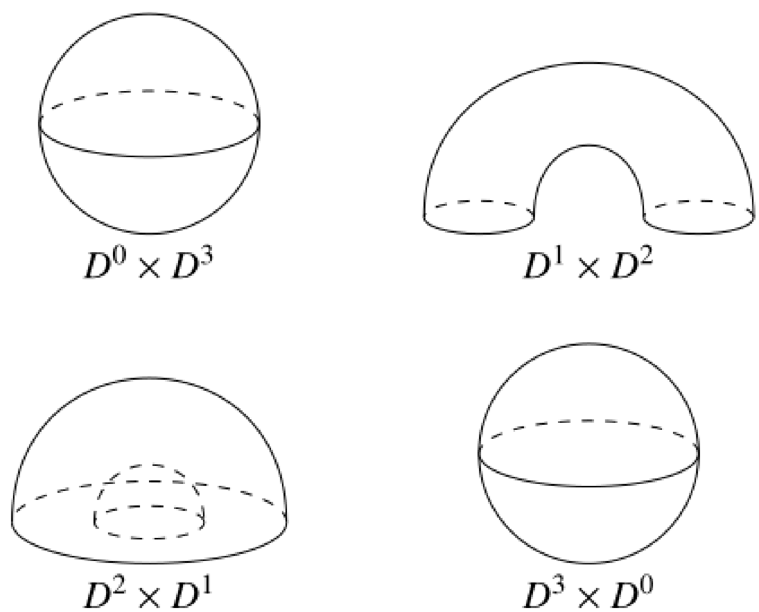


FIGURE 10. Different handlebodies.

In this way, $M + (\varphi^q)$ obviously carries the structure of a topological manifold. To get a smooth structure, one has to use the technique of straightening the angle to get rid of corners at the place where the handle is glued to M.
Let us now consider the boundary $\partial\,(M + (\varphi^q))$.
To describe this, delete from ∂M the interior of the image of φ^q. So we obtain a manifold with boundary together with a diffeomorphism from $S^{q-1} \times S^{n-q-1}$ to its boundary induced by $\varphi^q|_{S^{q-1}\times S^{n-q-1}}$. If we use this diffeomorphism to glue $D^q \times S^{n-q-1}$ to it, we obtain a closed manifold, namely, $\partial\,(M + (\varphi^q))$.

Now, let W be a cobordism $\partial_0 W \rightsquigarrow \partial_1 W$. Then we want to construct W from $\partial_0 W \times [0,1]$ by attaching handles as follows.
Note that the following construction will not change $\partial_0 W = \partial_0 W \times \{0\}$.
If $\varphi^q \colon S^{q-1} \times D^{n-q} \hookrightarrow \partial_0 W \times \{1\}$ is an embedding, we get by attaching a handle the compact manifold $W_1 = \partial_0 W \times [0,1] + (\varphi^q)$ that is given by $\partial_0 W\,[0,1] \sqcup_{\varphi^q} D^q \times D^{n-q}$.
In particular, its boundary is $\partial_0 W_1 \sqcup \partial 1 W_1$, where $\partial_0 W_1$ is the same as $\partial_0 W$.
Now, we can iterate this process where we attach a handle to W_1 at $\partial_1 W_1$. Thus we obtain a compact manifold with boundary

$$W = \partial_0 W \times [0,1] + (\varphi_1^{q_1}) + (\varphi_2^{q_2}) + \ldots + (\varphi_r^{q_r})$$

whose boundary is the disjoint union $\partial_0 W \sqcup \partial_1 W$ where $\partial_0 W$ is $\partial_0 W \times \{0\}$. We cal a description of W as above a *handlebody decomposition* of W relative $\partial_0 W$.

Lemma 7.87. *Let W be a compact manifold whose boundary ∂W is the disjoint sum $\partial_0 W \sqcup \partial_1 W$. Then W possesses a handlebody decomposition relative $\partial_0 W$, i.e., W is, up to diffeomorphism relative $\partial_0 W = \partial_0 W \times \{0\}$, of the form*

$$W = \partial_0 W \times [0,1] + (\varphi_1^{q_1}) + (\varphi_2^{q_2}) + \ldots + (\varphi_r^{q_r}).$$

7.8. **Dynamical Systems/Flows.**

Definition 7.88. Let M be a smooth manifold. A smooth map

$$\Phi\colon \mathbb{R} \times M \to M$$

is called a *flow* or *dynamical system* if, for all $x \in M$ and $t, s \in \mathbb{R}$, we have

- $\Phi(0, x) = x$
- $\Phi(t, \Phi(s, x)) = \Phi(t + s, x)$.

If we write $\Phi_t\colon M \to M$ for the map $x \mapsto \Phi(t, x)$, this reads, $\Phi_0 = \mathrm{id}_m$ and $\Phi_t \circ \Phi_s = \Phi_{t+s}$, so, in particular, $\Phi_{-t} = \Phi_t^{-1}$.

Definition 7.89 (Integral curve/flow line)**.** If $\Phi\colon \mathbb{R} \times M \to M$ is a flow, and $x \in M$, the curve

$$\alpha_x\colon \mathbb{R} \to M, \quad t \mapsto \Phi_t(x)$$

is called the *integral curve* or *flow line* of x. The image $\alpha_x(\mathbb{R})$ is called the orbit of x.

Remark. If a flow is given on a manifold, then exactly one orbit passes through each point $p \in M$.

Lemma 7.90. *There are three types of orbits. A flow line $\alpha_x\colon \mathbb{R} \to M$ of a flow i either an injective immersion, a periodic immersion or it is constant.*
We say α_x is a periodic immersion, if it is an immersion and there exists some $p > 0$ with $\alpha_x(t + p) = \alpha_x(t)$ for all t.
If α_x is constant, we call x a fixed point of the flow.

Proof. [**JB**] □

Definition 7.91. Let M be a smooth manifold. By a *local flow* Φ on M, we understand a smooth map

$$\Phi\colon A \to M$$

from an open subset $A \subset \mathbb{R} \times M$ containing $0 \times M$, such that for each $x \in M$, the intersection $A \cap (\mathbb{R} \times \{x\})$ is connected, and such that

- $\Phi_0 = \mathrm{id}_A$
- $\Phi_t \circ \Phi_s = \Phi_{t+s}$.

for all t, s, x for which both sides are defined.

Definition 7.92 (Velocity field)**.** If Φ is a local or global flow on M, then the vector field

$$\Phi\colon M \to TM, \quad x \mapsto \alpha_x'(0)$$

is called the *velocity field* of the flow.

Remark. For all flow lines and all $t \in (a_x, b_x)$, $\alpha_x'(t) = \Phi'(\alpha_x(t))$, where (a_x, b_x) is the domain of definition of the flow line α_x.

Proof. For $z = a_x(t)$, we have $\alpha_z(s) = \Phi_s(z) = \Phi_s(\Phi_t(x)) = \Phi_{s+t}(x) = \alpha_x(s + t)$ by definition. Hence $\Phi'(\alpha_x(t)) = \alpha_z'(0) = \alpha_x'(t)$. □

Theorem 7.93 (Integrability theorem for vector fields)**.** *Every vector field is the velocity field of a unique maximal local flow; on a compact manifold even of a global one.*

Appendix A. Analysis

For an m-tuple $I=(i_1,\dots,i_m)$ with $1\leq i_j\leq n$, we let $|I|=m$ denote the number of indices in I, and

$$\partial_I=\frac{\partial^m}{\partial x^{i_1}\cdots\partial x^{i_m}},$$
$$(x-a)^I=\left(x^{i_1}-a^{i_1}\right)\cdots\left(x^{i_m}-a^{i_m}\right)$$

Theorem A.1 (Taylor's Theorem)**.** *Let $U\subset\mathbb{R}^n$ be open and $a\in U$. Suppose $f\in C^{k+1}(U)$ for some $k\geq 0$. If W is any convex subset of U containing a, then for all $x\in W$,*

$$f(x)=T_k(x)+R_k(x)$$

where T_k is the k-th order Taylor polynomial of f at a, defined by

$$T_k(x)=f(a)+\sum_{m=1}^{k}\frac{1}{m!}\sum_{I\colon |I|=m}\partial_I f(a)(x-a)^I$$

and R_k is the kth remainder term, given by

$$R_k(x)=\frac{1}{k!}\sum_{I\colon |I|=k+1}(x-a)^I\int_0^1(1-t)^k\partial_I f\left(a+t(x-a)\right)dt.$$

Lemma A.2 (Chain Rule for Total Derivatives)**.** *Suppose V,W,X are finite-dimensional vector spaces, $U\subset V$ and $\tilde{U}\subset W$ oepn, and $f\colon U\to\tilde{U}$ and $g\colon\tilde{U}\to X$ are maps. If f is differentiable at $a\in U$ and g is differentiable at $f(a)\in\tilde{U}$, then $g\circ f$ is differentiable at a, and*

$$d\left(g\circ f\right)(a)=dg\left(f(a)\right)df(a).$$

Lemma A.3 (The Chain Rule for Partial Derivatives)**.** *Suppose $U\subset\mathbb{R}^n$ and $\tilde{U}\subset\mathbb{R}^m$ are open, and let $\left(x^1,\dots,x^n\right)$ denote the standard coordinates on U and $\left(y^1,\dots,y^m\right)$ those on $\tilde{U}$. Then if $F\colon U\to\tilde{U}$ and $G\colon\tilde{U}\to\mathbb{R}^p$ are of class C^1, then $G\circ F$ is C^1 and*

$$\frac{\partial\left(G^i\circ F\right)}{\partial x^j}(x)=\sum_{k=1}^{m}\frac{\partial G^i}{\partial y^k}(F(x))\frac{\partial F^k}{\partial x^j}(x).$$

Appendix B. Homotopy Theory

Lemma B.1. *Let $f\colon X\to Y$ and $\pi_1 X=1$. If f induces isomorphisms on π_1 and $H_*(-;\mathbb{Z})$, then f is a homotopy equivalence.*

Theorem B.2. *Suppose M is a compact R-orientable n-manifold whose boundary ∂M is decomposed as the union of two compact $(n-1)$-dimensional manifolds A and B with common boundary $\partial A=\partial B=A\cap B$. Then cap product with a fundamental class $[M]\in H_n\left(M,\partial M;R\right)$ gives isomorphisms $D_M\colon H^k\left(M,A;R\right)\to H_{n-k}\left(M,B;R\right)$ for all k. The possibility that A,B or $A\cap B$ is empty is not excluded.*

Appendix C. Random Stuff

1.

- Pinch map. $M\#N\to M\vee N$.
- Morse inequalities.
- $\#_g\left(S^n\times S^n\right)$

Exercise C.1 (?). M smooth, closed, $2n$-dim. If $M \simeq W_g^n$, then $M \cong W_g^n \# \Sigma$, Σ homotopy sphere.

Spherical modification.