1:

(a,b) First we show φ is surjective: let $(z,w) \in Y$. Then $z^3 - w^2 = 0$ so $z^3 = w^2$. Then $\varphi(\sqrt{z}) = \left(z, z^{\frac{3}{2}}\right) = \left(z, \left(z^3\right)^{\frac{1}{2}}\right) = (z,w)$. Thus φ is surjective.

Assume $\varphi(z) = \varphi(w)$. Then $\left(z^2, z^3\right) = \left(w^2, w^3\right)$. Thus $z = \pm w$. If $z = -w$, then $w^3 = z^3 = (-w)^3 = -w^3$ and thus $w = 0$, but then $z = -w = 0$ so $(z,w) = (0,0)$. So φ is injective.

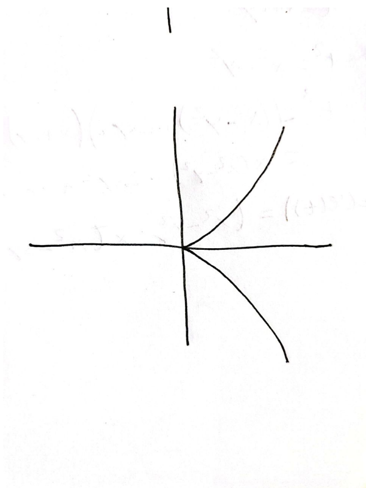
Now define $\psi \colon k[x,y] \to k[t]$ by $\psi(x) = t^2, \psi(y) = t^3$ with $k = \mathbb{C}$. We claim $\operatorname{Ker}\psi = \left(x^3 - y^2\right)$.
$(\supset)$: Since ψ is a homomorphism, we have

$$\psi(x^3 - y^2) = \left(t^2\right)^3 - \left(t^3\right)^2 = 0.$$

$(\subset)$: Let $f \in \operatorname{Ker}\psi$. Thus $f(t^2, t^3) = 0$.
Now since $f \in k\left[x,y\right] = k\left[y\right]\left[x\right]$, and $y^2 - x^3$ is monic in y, we can write $f(x,y) = (x^3 - y^2)g + r$ where $g, r \in k\left[x,y\right]$ and $r = r_0 + yr_1$ with $r_0, r_1 \in k\left[x\right]$. Now since $0 = f(t^2,t^3) = r_0(t^2) + t^3 r_1(t^2)$. Since the degree of t in $r_0(t^2)$ is even while it is odd in $t^3 r_1(t^2)$, we find $r_0(t^2) = 0$ and $r_1(t^2) = 0$ for all $t \in k$, so $r_0, r_1 = 0$ as they must be constant 0 by problem 1.8 in Fulton.
Therefore $x^3 - y^2 | f$ and so $\operatorname{Ker}\psi \subset \left(x^3 - y^2\right)$.

This furthermore gives us that $k\left[x,y\right] / \left(x^3 - y^2\right) \cong \operatorname{Im}\psi \subset k\left[x\right]$. Now, clearly $t \notin \operatorname{Im}\psi$, so in particular, $k\left[x,y\right]/(x^3 - y^2) \ncong k\left[x\right]$.
By the lemma on lecture 8, we thus have that Y is not isomorphic to $\mathbb{A}^1$.
This shows (b), and since if φ were an isomorphism, it would by the lemma induce and isomorphism of $k\left[x,y\right]/(x^3 - y^2)$ with $k\left[x\right]$, we also have that φ is not an isomorphism, completing (a).

(c)



We notice there is a kink at $(0,0)$, so it is not differentiable there.

(d) $\varphi^*(x)(t) = x \circ \varphi(t) = t^2$ and $\varphi^*(y)(t) = y \circ \varphi(t) = t^3$, so $\varphi^* \colon \Gamma(Y) = k\left[x,y\right]/(x^3 - y^2) \to k\left[t\right] = \Gamma\left(\mathbb{A}^1\right)$ is given by $x \to t^2$ and $y \to t^3$. Now for $f = 3x^2 + y + 5$, we have

$$\varphi^*\overline{f} = \varphi^*\overline{f}(t) = f \circ \varphi(t) = 3(t^2)^2 + t^3 + 5 = 3t^4 + t^3 + 5.$$

2:

(a) We have $\varphi^*(x)(t) = (x)\left(\varphi(t)\right) = x\left(t^2 - 1, t\left(t^2 - 1\right)\right) = t^2 - 1$ and $\varphi^*(y)(t) = (y)(\varphi(t)) = t\left(t^2 - 1\right)$. So for $\varphi^* \colon k\left[x,y\right] = \Gamma\left(\mathbb{A}^2\right) \to \Gamma\left(\mathbb{A}^1\right) = k\left[t\right]$, we have $\varphi^*(x) = t^2 - 1$ and $\varphi^*(y) = t\left(t^2 - 1\right)$.

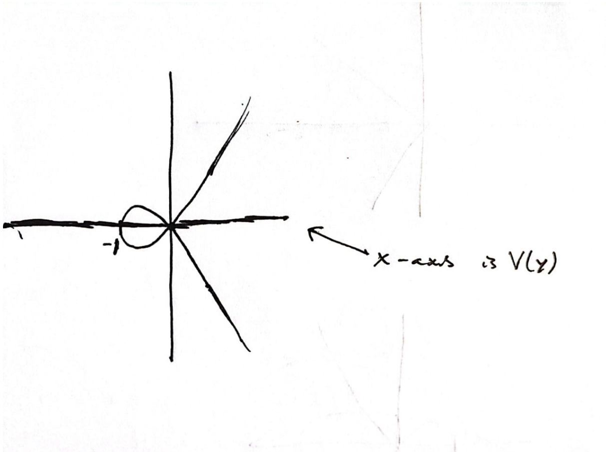
(b) $\varphi^{-1}\left(V(y)\right) = \varphi^{-1}\left(\{(x,0) \mid x \in \mathbb{C}\}\right) = \left\{t \in \mathbb{C} \mid t\left(t^2 - 1\right) = 0\right\} = \{0, 1, -1\}$.

(c) Let $Y = V\left(y^2 - x^2(x+1)\right) \subset \mathbb{A}^2$.

We first show that φ is surjective: let $(x,y) \in Y$. Thus $y^2 = x^2(x+1)$. Assume $x \neq 0$. Then $\varphi(\frac{y}{x}) = \left(\frac{y^2}{x^2} - 1, \frac{y}{x}\left(\frac{y^2}{x^2} - 1\right)\right) = \left((x+1) - 1, \frac{y}{x}(x+1-1)\right) = (x,y)$ where the second to last equality follows since $\frac{y^2}{x^2} = x+1$ in Y.
If $x = 0$, then since $y^2 = x^2(x+1) = 0$ in Y, we have $y = 0$. And so $\varphi(1) = (0,0) = (x,y)$.
Thus φ is surjective.

Now assume $\varphi(t) = \varphi(s)$. Then $\left(t^2 - 1, t\left(t^2-1\right)\right) = \left(s^2-1, s\left(s^2-1\right)\right)$, hence $s = \pm t$ from the first coordinate, and if $s = -t$, we get from the second coordinate that $t\left(t^2-1\right) = s\left(s^2-1\right) = (-t)\left((-t)^2-1\right) = -t\left(t^2-1\right)$, so $t\left(t^2-1\right) = 0$, hence $t = 0$ or $t = \pm 1$. Thus φ is only not injective in ± 1, i.e. $\varphi(1) = \varphi(-1)$.

(d)



We have $Y \cap V(y) = \{-1, 0\}$. Now $\varphi^{-1}(V(y))$ is precisely the points that map onto $Y \cap V(y) = \{-1, 0\}$, and we thus see that as t traverses $\mathbb{R}$, $\varphi(t)$ traverses the graph depicted, Y, starting from the bottom. The first time it attains the value 0 thus corresponds to $t = -1$, then it completes a half-circle and attains the value -1 at $t = 0$, whereafter it completes another half-circle and attains the value 0 again at $t = 1$. These are the only times $\varphi(t), t \in \mathbb{C}$ attains values on $Y \cap V(y) = \{(x,0) \colon x \in \mathbb{R}\}$.

3: Since $X \subset \mathbb{A}^n$ is an algebraic set, there exist polynomials $f_1, \dots, f_s \in k\left[x_1, \dots, x_n\right]$ such that $X = V\left(f_1, \dots, f_s\right)$ by Hilbert's basis theorem; similarly, there exist $g_1, \dots, g_r \in k\left[x_1, \dots, x_m\right]$ such that $Y = V\left(g_1, \dots, g_r\right)$ by Hilbert's basis theorem.

Now we can consider each f_i and g_i as a function on $k\left[x_1, \dots, x_{n+m}\right]$ by letting $\tilde{f}_i(x_1, \dots, x_{n+m}) = f_i(x_1, \dots, x_n)$ and $\tilde{g}_i\left(x_1, \dots, x_{n+m}\right) = g_i\left(x_{n+1}, \dots, x_{n+m}\right)$. We then claim that

$$X \times Y = V\left(\tilde{f}_1, \dots, \tilde{f}_s, \tilde{g}_1, \dots, \tilde{g}_r\right).$$

$(\subset)$: Let $(x,y) = (x_1, \dots, x_n, y_1, \dots, y_m) \in X \times Y$. Then for each $\tilde{f}_i$, we have $\tilde{f}_i(x,y) = f_i(x) = 0$ and for each g_i, we have $\tilde{g}_i(x,y) = g_i(y) = 0$.

$(\supset)$: Let $(x,y) = (x_1, \dots, x_n, y_1, \dots, y_m) \in V\left(\tilde{f}_1, \dots, \tilde{f}_s, \tilde{g}_1, \dots, \tilde{g}_r\right)$. Then for all i we have $0 = \tilde{f}_i(x,y) = f_i(x)$ and $0 = \tilde{g}_i(x,y) = g_i(y)$. Thus $x = (x_1, \dots, x_n) \in V\left(f_1, \dots, f_s\right) = X$ and $y = (y_1, \dots, y_m) \in V(g_1, \dots, g_r) = Y$.

(b) Let $T_i \colon k\left[x_1, \dots, x_{n+m}\right] \to k\left[x\right]$ by $T_i\left(x_1, \dots, x_{n+m}\right) = x_i$.
Then $T = \left(T_1, \dots, T_n\right) : k\left[x_1, \dots, x_{n+m}\right] \to k\left[x_1, \dots, x_n\right]$ and the projection $X \times Y \to X$ agree on all points of $X \times Y$: $T(x_1, \dots, x_{n+m}) = (x_1, \dots, x_n) = pr_X\left(x_1, \dots, x_{n+m}\right)$.

Similarly, $S = \left(T_{n+1}, \dots, T_{n+m}\right) : k\left[x_1, \dots, x_{n+m}\right] \to k\left[x_1, \dots, x_m\right]$ agrees with the projection $X \times Y \to Y$ since $S\left(x_1, \dots, x_{n+m}\right) = \left(x_{n+1}, \dots, x_{n+m}\right) = pr_Y\left(x_1, \dots, x_{n+m}\right)$ for all $\left(x_1, \dots, x_{n+m}\right) \in X \times Y$. By definition, the projection maps $X \times Y \to X$ and $X \times Y \to Y$ are thus morphisms.

(c) Consider the projections $pr_X \colon X \times Y \to X$ and $pr_Y \colon X \times Y \to Y$. Then by (b) these are morphisms. Now $X = pr_X\,(X \times Y)$, so $pr_X^{-1}(X) = X \times Y$ and since $X \times Y$ is irreducible, X is irreducible too by the lemma on page 3 on the notes of lecture 9.
Similarly, $Y = pr_Y(X \times Y)$, so $pr_Y^{-1}(Y) = X \times Y$, so since $X \times Y$ is irreducible, Y is irreducible by the same lemma.

(d) Suppose $X \times Y = A \cup B$. Let $X_A = \{p \in X \colon p \times Y \subset A\}$ and $X_B = \{p \in X \colon p \times Y \subset B\}$. Considering the sets in the Zariski topology, we see that

$$p \times Y = (p \times Y \cap A) \cup (p \times Y \cap B)$$

and since each $p \times Y$ is irreducible as it is isomorphic to Y, each $p \times Y$ is contained in A or B. Hence $X = A \cup B = X_A \cup X_B$. Now, for some $y \in Y$, the inclusion $X \to X \times Y$ by $x \to (x, y)$ is a morphism. Thus the preimages $\{x \colon (x, y) \in A\}$ are closed, and arbitrary intersections of closed sets are closed, so $X_A = \{x \colon x \times Y \subset A\} = \bigcap_{y \in Y} \{x \colon (x, y) \in A\}$ is closed and likewise for X_B. Thus they are algebraic subsets, so X is reducible, a contradiction.

4: We will show $(i) \implies (ii) \implies (iii) \implies (i)$.

$(i) \implies (ii)$:
If V is a point, say $V = \{(a_1, \ldots, a_n)\}$, then $I(V) = (x_1 - a_1, \ldots, x_n - a_n)$. Now this is the kernel of the evaluation function on $k\,[x_1, \ldots, x_n]$ at the point $(a_1, \ldots, a_n)$. That is, define $\varphi \colon k\,[x_1, \ldots, x_n] \to k$ by $\varphi(f) = f(a_1, \ldots, a_n)$. Since $k \subset k\,[x_1, \ldots, x_n]$, this is clearly surjective, and $\varphi(f) = 0$ if and only if $f(a_1, \ldots, a_n) = 0$. Writing $g(x_1, \ldots, x_n) = f(x_1 + a_1, \ldots, x_n + a_n)$, we thus find $g(0, \ldots, 0) = 0$ and as the composition of polynomial functions is a polynomial function by a previous homework assignment, we have that the constant term of the polynomial g is 0. Hence $g \in (x_1, \ldots, x_n)$ and thus $f \in (x_1 - a_1, \ldots, x_n - a_n)$. Thus

$$k\,[x_1, \ldots, x_n] \,/\, (x_1 - a_1, \ldots, x_n - a_n) \cong k$$

so since k is a field, $(x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal. In particular,

$$\Gamma(V) = k\,[x_1, \ldots, x_n]\,/I(V) = k\,[x_1, \ldots, x_n]\,/(x_1 - a_1, \ldots, x_n - a_n) = k$$

$(ii) \implies (iii)$: Clearly, if $\Gamma(V) = k$, then any point of $\Gamma(V) = k$ functions as a basis. Namely, for any $f \in \Gamma(V)$, it corresponds to some $k_1 \in k$, and since $k = \mathrm{span}(k_1)$ since k is a field, we have that f generates $\Gamma(V)$. By definition then $\dim_k \Gamma(V) = \dim_k k = 1 < \infty$.

$(iii) \implies (i)$: We may assume k is algebraically closed.
Since V is an affine variety, we have by Hilbert's basis theorem that $V = V(I)$ for some ideal I of $k\,[x_1, \ldots, x_n]$.
Now $\dim_k \Gamma(V) = \dim_k k\,[x_1, \ldots, x_n]\,/I(V) < \infty$ by assumption, so by corollary 4 in section 1.7 in Fulton, $V\,(I(V)) = V$ is a finite set of points. Since V is a variety, it must in particular be a single point, since if $V = \{p_1, \ldots, p_r\}$ where each p_i is a point, then $V = \{p_1\} \cup \ldots \cup \{p_r\}$ and each $\{p_i\}$ is a variety as it is the hypersurface of the evaluation function at p_i in $k\,[x_1, \ldots, x_n]$.

5: By problem 1.(b) in homework 3, we have that there is a natural bijection between radical ideals in $k\,[x_1, \ldots, x_n]\,/I(X) = \Gamma(X)$ and radical ideals in $k\,[x_1, \ldots, x_n]$ containing $I(X)$. Now by corollary 1 to Hilbert's Nullstellensatz in section 1.7 in Fulton, we have that there is a bijective correspondence between radical ideals and algebraic sets of $k\,[x_1, \ldots, x_n]$ and $\mathbb{A}^n$ given by $I(V(I)) = I$ and $V(I(V)) = V$.
Now any radical ideal R containing $I(X)$ corresponds to an algebraic set $V(R) \subset V(I(X)) = X$. And any algebraic set $V \subset X$ corresponds to a radical ideal $\sqrt{I(V)} \supset I(X)$ Hence we have a bijective correspondence between radical ideals of $k\,[x_1, \ldots, x_n]$ containing $I(X)$ and algebraic subsets of X.
Composing the bijective correspondences, we thus get a bijective correspondence between algebraic subsets of X and radical ideals in $\Gamma(X)$.