Problem 0.1 (2.2. Algebra Structure on C_p^∞)**.** Define carefully addition, multiplication, and scalar multiplication on C_p^∞. Prove that addition in C_p^∞ is commutative.

Solution. Suppose $[(f,U)],[(g,V)] \in C_p^\infty$. Define $[(f,U)]+[(g,V)] = [(f+g,U\cap V)], [(f,U)]\cdot[(g,V)] = [(f\cdot g, U\cap V)]$ and $\lambda[(f,U)] = [(\lambda f, U)]$ for $\lambda \in \mathbb{R}$ as scalar.
We must show that these is well-defined. Firstly, since $p\in U,V$ we have $p \in U\cap V$. Now, since $f\colon U\to\mathbb{R}$ and $g\colon V\to\mathbb{R}$ are smooth, we have that both are smooth on $U\cap V$ as this is an open subspace of U and V. Hence $f+g, f\cdot g \in C^\infty(U\cap V)$ and $\lambda f \in C^\infty(U)$ as $C^\infty(W)$ is a ring for any open set W. Hence $(f+g,U\cap V)$ is in some equivalence class in C_p^∞.
Suppose now $[(f,U)] = \left[\left(\tilde{f},\tilde{U}\right)\right]$ and $[(g,V)] = \left[\left(\tilde{g},\tilde{V}\right)\right]$. Then $f=\tilde{f}$ on some $W\subset U\cap\tilde{U}$ and $g=\tilde{g}$ on some $S\subset V\cap\tilde{V}$. Thus $f+g=\tilde{f}+\tilde{g}$ and $f\cdot g = \tilde{f}\cdot\tilde{g}$ on $W\cap S\subset U\cap\tilde{U}\cap V\cap\tilde{V}$, and $W\cap S$ is open. By definition then $[(f+g,U\cap V)] = \left[\left(\tilde{f}+\tilde{g},\tilde{U}\cap\tilde{V}\right)\right]$ and $[(f\cdot g,U\cap V)] = \left[\left(\tilde{f}\cdot\tilde{g},\tilde{U}\cap\tilde{V}\right)\right]$, so addition and multiplication are well-defined.
And similarly since $f=\tilde{f}$ on some $W\subset U\cap\tilde{U}$, we have $\lambda f = \lambda\tilde{f}$ on W, so by definition $[(\lambda f,U)] = \left[\left(\lambda\tilde{f},\tilde{U}\right)\right]$.

Commutativity of addition (and even multiplication) follows from the commutativity of these in $\mathbb{R}$ and the above.

Problem 0.2 (Transformation rule for a wedge product of coverctors, 3.7.)**.** Cuppose two sets of covectors on a vector space V, $\beta^1,\dots,\beta^k$ and $\gamma^1,\dots,\gamma^k$ are related by

$$\beta^i = \sum_{j=1}^k a_j^i \gamma^j, \quad i=1,\dots,k,$$

for a $k\times k$ matrix $A = \left[a_j^i\right]$. Show that

$$\beta^1\wedge\ldots\wedge\beta^k = (\det A)\,\gamma^1\wedge\ldots\wedge\gamma^k.$$

Solution. By proposition 3.27, we have

$$\beta^1\wedge\ldots\wedge\beta^k(v_1,\dots,v_k) = \det\left[\beta^i(v_j)\right].$$

Now since $\gamma^1\wedge\ldots\wedge\gamma^k(v_1,\dots,v_k) = \det\left[\gamma^i(v_j)\right]$, we must show that $\left[\beta^i(v_j)\right] = \left[a_j^i\right]\left[\gamma^i(v_j)\right]$. I.e., we must show $\beta^i(v_j) = \sum_{r=1}^k \alpha_r^i\gamma^r(v_j)$, but this is precisely the definition of β^i. Hence $\det\left[\beta^i(v_j)\right] = \det\left(\left[a_j^i\right]\left[\gamma^i(v_j)\right]\right) = \det A\det\left[\gamma^i(v_j)\right] = (\det A)\,\gamma^1\wedge\ldots\wedge\gamma^k(v_1,\dots,v_k)$. This shows the desired equality.

Exercise 0.3 (4.4 (Wedge product of a 2-form with a 1-form))**.** Let ω be a 2-form and τ a 1-form on $\mathbb{R}^3$. If X,Y,Z are vector fields on M, find an explicit formula for $(\omega\wedge\tau)(X,Y,Z)$ in terms of the values of ω and τ on the vector fields X,Y,Z.

Solution. Let $X=x_1, Y=x_2, Z=x_3$. Then

$$\begin{aligned}(\omega\wedge\tau)(X,Y,Z) &= \frac{1}{2}\sum_{\sigma\in S_3}(\operatorname{sgn}\sigma)\,\omega(x_{\sigma1},x_{\sigma2})\,\tau(x_{\sigma3})\\ &= \omega(X,Y)\,\tau(Z)+\omega(Y,Z)\,\tau(X)+\omega(Z,X)\,\tau(Y)\end{aligned}$$

Exercise 0.4 (4.4 (Exterior calculus))**.** Suppose the standard coordinates on $\mathbb{R}^3$ are called ρ,φ and θ. If $x=\rho\sin\varphi\cos\theta, y=\rho\sin\varphi\sin\theta$ and $z=\rho\cos\varphi$, calculate dx,dy,dz and $dx\wedge dy\wedge dz$ in terms of $d\rho, d\varphi$ and $d\theta$.

Solution. We have

$$\begin{aligned}dx &= \frac{\partial x}{\partial\rho}d\rho+\frac{\partial x}{\partial\varphi}d\varphi+\frac{\partial x}{\partial\theta}d\theta = \sin\varphi\cos\theta d\rho+\rho\cos\varphi\cos\theta d\varphi-\rho\sin\varphi\sin\theta\\ dy &= \sin\varphi\sin\theta d\rho+\rho\cos\varphi\sin\theta d\varphi+\rho\sin\varphi\cos\theta d\theta\\ dz &= \cos\varphi d\rho-\rho\sin\varphi d\varphi\end{aligned}$$

Hence

$$\begin{aligned} dx \wedge dy \wedge dz &= \sin^3\varphi\cos^2\theta\rho^2 d\rho\wedge d\varphi\wedge d\theta + \rho^2\cos^2\varphi\cos^2\theta\sin\varphi d\rho\wedge d\varphi\wedge d\theta \\ &+ \rho^2\cos^2\varphi\sin\varphi\sin^2\theta d\rho\wedge d\varphi\wedge d\theta + \rho^2\sin^3\varphi\sin^2\theta d\rho\wedge d\varphi\wedge d\theta \\ &= \left(\rho^2\sin^3\varphi + \rho^2\cos^2\varphi\sin\varphi\right) d\rho\wedge d\varphi\wedge d\theta \\ &= \rho^2\sin\varphi d\rho\wedge d\varphi\wedge d\theta \end{aligned}$$

Exercise 0.5 (Wedge product and cross product)**.** The correspondence between differential forms and vector fields on an open subset of $\mathbb{R}^3$ in subsection 4.6 also makes sense pointwise. Let V be a vector space of dimension 3 with basis e_1, e_2, e_3 and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1-covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V, we associate vector $v_\alpha = \langle a_1, a_2, a_3\rangle \in \mathbb{R}^3$. To the 2-covector

$$\gamma = c_1\alpha^2\wedge\alpha^3 + c_2\alpha^3\alpha^1 + c_3\alpha^1\wedge\alpha^2$$

on V, we associate the vector $v_\gamma = \langle c_1, c_2, c_3\rangle \in \mathbb{R}^3$. Show that under this correspondence, the wedge product of 1-covectors corresponds to the cross product of vectors in $\mathbb{R}^3$: if $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then $v_{\alpha\wedge\beta} = v_\alpha \times v_\beta$.

Proof. We have

$$\alpha\wedge\beta = (a_1b_2 - a_2b_1)\,\alpha^1\wedge\alpha^2 + (a_3b_1 - a_1b_3)\,\alpha^3\wedge\alpha^1 + (a_2b_3 - b_2a_3)\,\alpha^2\wedge\alpha^3$$

which corresponds to

$$v_{\alpha\wedge\beta} = \begin{pmatrix} a_2b_3 - b_2a_3 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - b_1a_2 \end{pmatrix} = \det\begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = v_a \times v_b$$

□

Example 0.6 (Smoothness of a projection map)**.** Let M and N be manifolds and $\pi\colon M\times N \to M, \pi(p,q) = p$ the projection to the first factor. Prove that π is a C^∞ map.

Proof. Suppose $(p,q) \in M\times N$. Choose a chart (U,φ) around p in M and a chart (V,ψ) around q in N. Then $(U\times V, \varphi\times\psi)$ is a chart around (p,q) in $M\times N$. Now $\varphi\circ\pi\circ(\varphi\times\psi)^{-1} : \varphi(U)\times\psi(V) \to \mathbb{R}^m$ is the projection map onto the first coordinate on $\varphi(U)\times\psi(V)$ which is C^∞. Hence π is C^∞. □