

# 1. CURVES, SURFACES AND HYPERBOLIC GEOMETRY

1.1. **Simple closed curves.** There is a bijective correspondence

$$\left\{ \begin{array}{c} \text{Nontrivial} \\ \text{conjugacy classes} \\ \text{in } \pi_1(S) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Nontrivial free} \\ \text{homotopy classes of oriented} \\ \text{closed curves in } S \end{array} \right\}$$

**Definition 1.1** (Primitive and multiple elements). An element  $g$  of a group  $G$  is *primitive* if there does not exist any  $h \in G$  so that  $g = h^k$  for  $|k| > 1$ . The property of being a primitive is a conjugacy class invariant. In particular, it makes sense to say that a closed curve in a surface is primitive.

A closed curve in  $S$  is a multiple if it is a map  $S^1 \rightarrow S$  that factors through the map  $S^1 \xrightarrow{\times n} S^1$  for  $n > 1$ , i.e., there exists a map  $\tilde{\alpha}: S^1 \rightarrow S$  such that the following diagram commutes:

$$\begin{array}{ccccc} & & \tilde{\alpha} & & \\ & \swarrow & \text{---} & \searrow & \\ S^1 & \xrightarrow{\times n} & S^1 & \xrightarrow{\alpha} & S \end{array}$$

**Definition 1.2** (Lifts). We make a distinction between lifts: let  $p: \tilde{S} \rightarrow S$  be a covering space. By a *lift* of a closed curve  $\alpha$  to  $\tilde{S}$  we will always mean the image of a lift  $\mathbb{R} \rightarrow \tilde{S}$  of the map  $\alpha \circ \pi$  where  $\pi: \mathbb{R} \rightarrow S^1$  is the usual covering map. I.e., a lift of  $\alpha: S^1 \rightarrow S$  is a map  $\tilde{\alpha}: \mathbb{R} \rightarrow \tilde{S}$  such that the following diagram commutes

$$\begin{array}{ccccc} & & \tilde{S} & & \\ & \nearrow \tilde{\alpha} & \downarrow p & & \\ \mathbb{R} & \xrightarrow{\pi} & S^1 & \xrightarrow{\alpha} & S \end{array}$$

A lift is different from a *path lift* which is a proper subset of a lift. Namely, it would be the restriction of  $\tilde{\alpha}$  to some interval of  $\mathbb{R}$  of length  $2\pi$  if the covering map  $\pi$  is of the form  $t \mapsto e^{it}$ .

Now suppose  $p: \tilde{S} \rightarrow S$  is the universal cover and  $\alpha$  is a simple closed curve in  $S$  that is not a multiple of another closed curve. In this case, there is a bijective correspondence between cosets in  $\pi_1(S)$  of the infinite cyclic subgroup  $\langle \alpha \rangle$  and the lifts of  $\alpha$ . This can be seen as follows: first choose a basepoint  $\alpha(1) = x_0 \in S$  and some  $\tilde{x}_0 \in p^{-1}(x_0)$ . There exists a unique lift  $\tilde{\alpha}$  of  $\alpha$  such that

$$\begin{array}{ccccc} & & \tilde{S} & & \\ & \nearrow \tilde{\alpha} & \downarrow p & & \\ \mathbb{R} & \longrightarrow & S^1 & \xrightarrow{\alpha} & S \end{array}$$

commutes and such that  $\tilde{\alpha}(0) = \tilde{x} \in p^{-1}(\alpha \circ \pi(0))$  for some specific  $\tilde{x}$  [Bredon, Cor. 4.2]. But the set  $p^{-1}(\alpha \circ \pi(0))$  is in bijective correspondence with the loops in  $\pi_1(S)$  by the path lifting lemma. Now, under which path lifts are the lifts the same? The lifts of  $\alpha$  to two points  $\tilde{x}, \tilde{y} \in p^{-1}(\alpha \circ \pi(0))$  will be the same if  $\alpha^k \cdot \tilde{x} = \tilde{y}$  where  $\cdot$  denotes the monodromy action of  $\pi_1(S)$  on the fiber. Now, there exist  $\gamma_x$  and  $\gamma_y$  in  $\pi_1(S)$  such that  $\gamma_x \cdot \tilde{x}_0 = \tilde{x}$  and  $\gamma_y \cdot \tilde{x}_0 = \tilde{y}$ , so  $\alpha^k \gamma_x = \gamma_y$ . Hence the lifts corresponding to  $\gamma_x$  and  $\gamma_y$  are the same if and only if  $\alpha^k \gamma_x = \gamma_y$  for some  $k$ , i.e. if and only if  $\gamma_x = \gamma_y$  in  $\pi_1(S)/\langle \alpha \rangle$ .

As usual, the group  $\pi_1(S)$  acts on the set of lifts of  $\alpha$  by deck transformations, and this action agrees with the usual left action of  $\pi_1(S)$  on the cosets of  $\langle \alpha \rangle$ . The stabilizer of the lift corresponding to the coset  $\gamma \langle \alpha \rangle$  is the cyclic group  $\langle \gamma \alpha \gamma^{-1} \rangle$ . See figure 1.

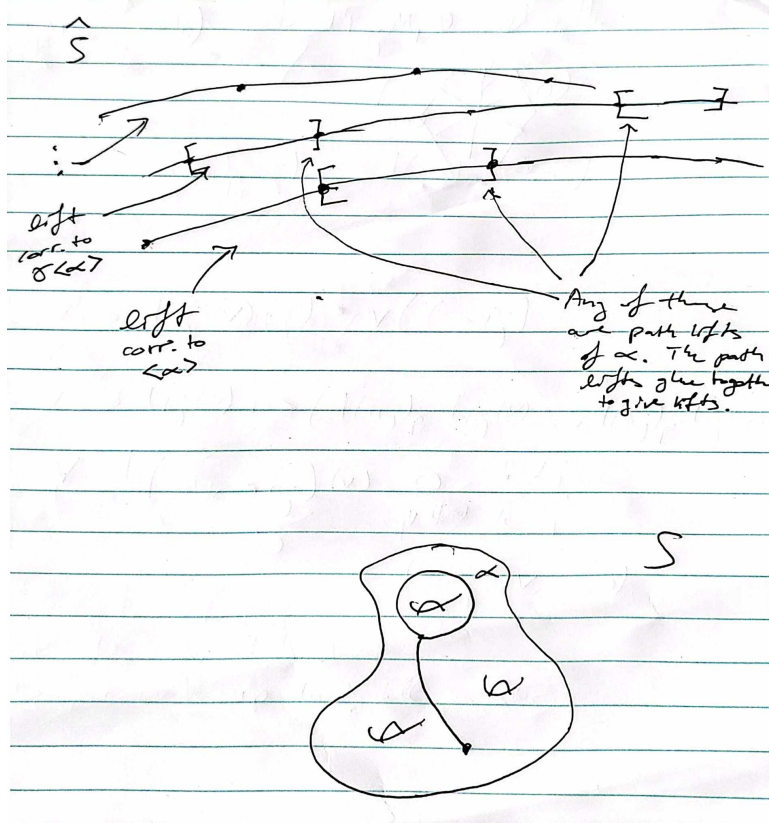


FIGURE 1.

**Theorem 1.3.** When  $S$  admits a hyperbolic metric and  $\alpha$  is a primitive element of  $\pi_1(S)$ , we have a bijective correspondence

$$\left\{ \begin{array}{c} \text{Elements of the conjugacy} \\ \text{class of } \alpha \text{ in } \pi_1(S) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Lifts to } \tilde{S} \text{ of the} \\ \text{closed curve } \alpha \end{array} \right\}$$

More precisely, we claim that the map which sends the lift given by the coset  $\gamma \langle \alpha \rangle$  to  $\gamma \alpha \gamma^{-1}$  is bijective and well-defined.

*Proof.* To show that it is well-defined, suppose  $\gamma \langle \alpha \rangle$  and  $\beta \langle \alpha \rangle$  give the same lift. Then  $\gamma = \beta \alpha^k$ . So in particular,

$$\gamma \alpha \gamma^{-1} = \beta \alpha^k \alpha \alpha^{-k} \beta^{-1} = \beta \alpha \beta^{-1}$$

so they do correspond to the same element of the conjugacy class  $[\alpha]$ . It is clear that this is a surjective map. Now suppose that  $\gamma \alpha \gamma^{-1} = \beta \alpha \beta^{-1}$ . Then

$\beta^{-1}\gamma\alpha(\beta^{-1}\gamma)^{-1} = \alpha$ , so in particular,  $\beta^{-1}\gamma \in C_{\pi_1(S)}(\alpha)$  which is a cyclic group generated by, say,  $\theta$ . But then  $\theta^l = \alpha$  since  $\alpha$  is trivially in the centralizer of  $\alpha$ ; however,  $\alpha$  is primitive, so  $l$  must be  $\pm 1$ , but then  $\alpha$  generates the centralizer of  $\alpha$ ,  $C_{\pi_1(S)}(\alpha) = \langle \alpha \rangle$ , and hence  $\gamma = \beta\alpha^l$ , so  $\gamma\langle \alpha \rangle = \beta\langle \alpha \rangle$ .  $\square$

*Remark.* If  $\alpha$  is any multiple, then we still have a bijective correspondence between elements of the conjugacy class of  $\alpha$  and the lifts of  $\alpha$ . However, if  $\alpha$  is not primitive and not a multiple, then there are more lifts of  $\alpha$  than there are conjugates. Indeed, if  $\alpha = \beta^k$ , where  $k > 1$ , then  $\beta\langle \alpha \rangle \neq \langle \alpha \rangle$  while  $\beta\alpha\beta^{-1} = \alpha$ .

**Example 1.4.** The above correspondence does not hold for the torus  $T^2$  because each closed curve has infinitely many lifts, while each element of  $\pi_1(T^2) \approx \mathbb{Z}^2$  is its own conjugacy class because  $\pi_1(T^2)$  is abelian.

*Geodesic representatives.*

**Proposition 1.5.** *Let  $S$  be a hyperbolic surface. If  $\alpha$  is a closed curve in  $S$  that is not homotopic into a neighborhood of a puncture, then  $\alpha$  is homotopic to a unique geodesic closed curve  $\gamma$ .*

**Corollary 1.6.** *For compact hyperbolic surfaces, there is a bijective correspondence:*

$$\left\{ \begin{array}{c} \text{Conjugacy classes} \\ \text{in } \pi_1(S) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Oriented geodesic} \\ \text{closed curves in } S \end{array} \right\}$$

**Simple closed curves.**

**Definition 1.7** (Simple curves). A closed curve in  $S$  is *simple* if it is topologically embedded, i.e., if the map  $S^1 \rightarrow S$  is injective.

By [Bredon, Thm 11.8], any closed curve  $\alpha$  can be approximated (arbitrarily close) by a smooth closed curve which is homotopic to  $\alpha$ . Moreover, if  $\alpha$  is simple, then the smooth approximation can be chosen to be simple. Smooth curves are advantageous because we can make use of notions such as transversality.

Simple closed curves are also natural to study because they represent primitive elements of  $\pi_1(S)$ .

**Proposition 1.8.** *Let  $\alpha$  be a simple closed curve in a surface  $S$ . If  $\alpha$  is not null homotopic, then each element of the corresponding conjugacy class in  $\pi_1(S)$  is primitive.*

**Example: simple closed curves on the torus.**

**Proposition 1.9.** *The nontrivial homotopy classes of oriented simple closed curves in  $T^2$  are in bijective correspondence with the set of primitive elements of  $\pi_1(T^2) \approx \mathbb{Z}^2$  which is the set of elements  $(p, q) \in \mathbb{Z}^2$  such that either  $(p, q) = (0, \pm 1)$  or  $(p, q) = (\pm 1, 0)$  or  $\gcd(p, q) = 1$ .*

**Closed geodesics.**

**Proposition 1.10.** *Let  $S$  be a hyperbolic surface. Let  $\alpha$  be a closed curve in  $S$  not homotopic into a neighborhood of a puncture. Let  $\gamma$  be the unique geodesic in the free homotopy class of  $\alpha$  guaranteed by proposition 1.5. If  $\alpha$  is simple, then  $\gamma$  is simple.*

*Proof.* Follows from the following lemma:

**Lemma 1.11.** *Let  $X$  be a topological space with a universal covering space  $\tilde{X}$ . A closed curve  $\beta$  in  $X$  is simple if and only if the following properties hold:*

- (1) *Each lift of  $\beta$  to  $\tilde{X}$  is simple.*
- (2) *No two lifts of  $\beta$  intersect.*
- (3)  *$\beta$  is not a nontrivial multiple of another closed curve.*

□

**Intersection numbers.** It is often useful to put an inner product on a vector space to check if two vectors are linearly independent. We can pursue something similar for surfaces.

**Definition 1.12** (Geometric intersection number). Let  $\alpha, \beta$  be closed curves on a surface  $S$ . Their *geometric intersection number* is

$$i(\alpha, \beta) = \min_{\alpha' \simeq \alpha, \beta' \simeq \beta} \#(\alpha' \cap \beta')$$

**Definition 1.13** (Preliminary definition for transversality). If  $\alpha \cap \beta$  is finite and, at every intersection, each curve locally separates the other curve, then we say that  $\alpha$  and  $\beta$  are *transverse*.

**Definition 1.14** (Minimal position). Two curves  $\alpha$  and  $\beta$  are in *minimal position* if  $\#(\alpha \cap \beta) = i(\alpha, \beta)$ .

**Bigons.** We want a procedure to put curves into minimal position so we can compute intersection numbers.

For this, we need the notion of a *bigon*:

**Definition 1.15** (Bigon). Two transverse simple closed curves  $\alpha$  and  $\beta$  in a surface  $S$  form a *bigon* if there is a topologically embedded disk in  $S$  (the bigon) whose boundary is the union of an arc of  $\alpha$  and an arc of  $\beta$  intersecting in exactly two points.

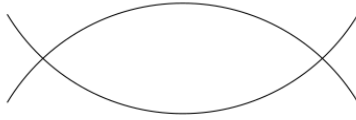


FIGURE 2. Local picture of a bigon

**Lemma 1.16.** *If transverse simple closed curves  $\alpha$  and  $\beta$  in a surface  $S$  do not form any bigons, then in the universal cover of  $S$ , any pair of lifts  $\tilde{\alpha}$  and  $\tilde{\beta}$  of  $\alpha$  and  $\beta$  intersect in at most one point.*

**Proposition 1.17** (The bigon criterion). *Two transverse simple closed curves in a surface  $S$  are in minimal position if and only if they do not form a bigon.*

**Corollary 1.18.** *Any two transverse simple closed curves that intersect exactly once are in minimal position.*

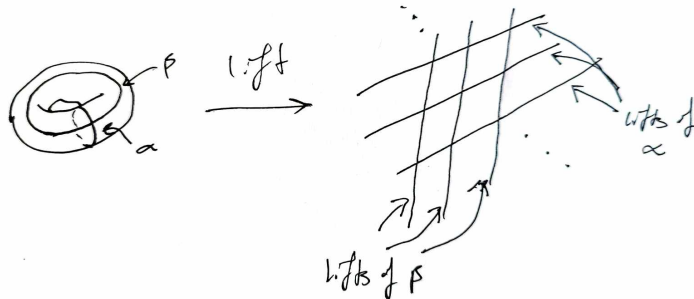


FIGURE 3. Lemma 1.16 illustrated

### Homotopy versus isotopy for simple closed curves.

**Definition 1.19** (Isotopy). Two simple closed curves  $\alpha$  and  $\beta$  are *isotopic* if there is a homotopy

$$H: S^1 \times [0, 1] \rightarrow S$$

from  $\alpha$  to  $\beta$  with the property that the closed curve  $H(S^1 \times \{t\})$  is simple for each  $t \in [0, 1]$ .

**Proposition 1.20** (Baer). *Let  $\alpha$  and  $\beta$  be two essential simple closed curves in a surface  $S$ . Then  $\alpha$  is isotopic to  $\beta$  if and only if  $\alpha$  is homotopic to  $\beta$ .*

*Proof.* If  $\alpha$  is isotopic to  $\beta$  then they are clearly also homotopic.

Suppose  $\alpha$  and  $\beta$  are homotopic. Taking a tubular neighborhood around  $\alpha$ , we can find a disjoint simple loop  $\tilde{\alpha}$  which is homotopic to  $\alpha$  but disjoint from it. Then  $\beta$  is homotopic to  $\tilde{\alpha}$ , and hence  $i(\alpha, \beta) = i(\alpha, \tilde{\alpha}) = 0$ . Performing an isotopy of  $\alpha$ , we may assume that  $\alpha$  is transverse to  $\beta$  (why?). If  $\alpha$  and  $\beta$  are not disjoint, then by the bigon criterion, they form a bigon. A bigon prescribes an isotopy that reduces intersection, so we may remove bigons by isotopy until  $\alpha$  and  $\beta$  are disjoint.

Suppose  $\chi(S) < 0$ . Lift  $\alpha$  and  $\beta$  to  $\tilde{\alpha}$  and  $\tilde{\beta}$  with the same endpoints in  $\partial\mathbb{H}^2$ . There is a hyperbolic isometry  $\varphi$  that leaves  $\tilde{\alpha}$  and  $\tilde{\beta}$  invariant and acts by translation on the lifts. As  $\tilde{\alpha}$  and  $\tilde{\beta}$  are disjoint, let  $R$  denote the region between them. We claim that the quotient surface  $R' = R/\langle\varphi\rangle$  is an annulus. The fundamental group of  $R'$  is isomorphic to the group of deck transformations  $\langle\varphi\rangle$  and is hence infinite cyclic. Furthermore,  $R'$  has two boundary components.  $\square$

*Extension of isotopies.*

**Definition 1.21.** An isotopy of a surface  $S$  is a homotopy  $H: S \times I \rightarrow S$  such that for each  $t \in [0, 1]$ , the map  $H(S, t): S \times \{t\} \rightarrow S$  is a homeomorphism. Given an isotopy between two simple closed curves in  $S$ , it will often be useful to promote this to an isotopy of  $S$  which we call an ambient isotopy of  $S$ .

**Proposition 1.22.** *Let  $S$  be any surface. If  $F: S^1 \times I \rightarrow S$  is a smooth isotopy of simple closed curves, then there is an isotopy  $H: S \times I \rightarrow S$  so that  $H|_{S \times 0}$  is the identity and  $H|_{F(S^1 \times 0) \times I} = F$ .*

*Proof.* [Hirsch, Ch 8, Thm 1.3]  $\square$

### 1.2. Small digression on Hirsch chapter 8.

**Definition 1.23** (Isotopy in general and their tracks). Let  $V$  and  $M$  be manifolds. An isotopy from  $V$  to  $M$  is a map  $F: V \times I \rightarrow M$  such that for each  $t \in I$ , the map

$$F_t: V \rightarrow M, \quad x \mapsto F(x, t)$$

is an embedding.

The *track* of the isotopy  $F$  is the embedding

$$\hat{F}: V \times I \rightarrow M \times I, \quad (x, t) \mapsto (F(x, t), t)$$

**Definition 1.24** (Isotopic embeddings and ambient isotopies). If  $F: V \times I \rightarrow M$  is an isotopy, we call the two embeddings  $F_0$  and  $F_1$  isotopic. If  $V$  is a submanifold of  $M$  and  $F_0$  is the inclusion, we call  $F$  an isotopy of  $V$  in  $M$ . When  $V = M$  and each  $F_t$  is a diffeomorphism, and  $F_0 = \mathbb{1}_M$ , then  $F$  is called an ambient isotopy.

**Definition 1.25** (Support of isotopy). The *support*  $\text{Supp } F \subset V$  of an isotopy  $F: V \times I \rightarrow M$  is the closure of  $\{x \in V: F(x, t) \neq F(x, 0) \text{ for some } t \in I\}$ .

**Theorem 1.26** (Isotopy extension theorem). *Let  $V \subset M$  be a compact submanifold and  $F: V \times I \rightarrow M$  an isotopy of  $V$ . If either  $F(V \times I) \subset \partial M$  or  $F(V \times I) \subset M - \partial M$ , then  $F$  extends to an ambient isotopy of  $M$  having compact support.*

#### Arcs.

Assume  $S$  is a compact surface, possibly with boundary and possibly with finitely many marked points in the interior. Denote the set of marked points by  $\mathcal{P}$ .

**Definition 1.27.** A *proper arc* in  $S$  is a map  $\alpha: [0, 1] \rightarrow S$  such that  $\alpha^{-1}(\mathcal{P} \cup \partial S) = \{0, 1\}$ .

**Definition 1.28.** The arc  $\alpha$  is *simple* if it is an embedding on its interior.

*Remark.* The homotopy class of a proper arc is taken to be the homotopy class within the class of proper arcs. Thus points on  $\partial S$  cannot move off the boundary during the homotopy.

A homotopy (or isotopy) of an arc is said to be *relative to the boundary* if its endpoints stay fixed throughout the homotopy. An arc in a surface  $S$  is *essential* if it is neither homotopic into a boundary component of  $S$  nor a marked point of  $S$ .

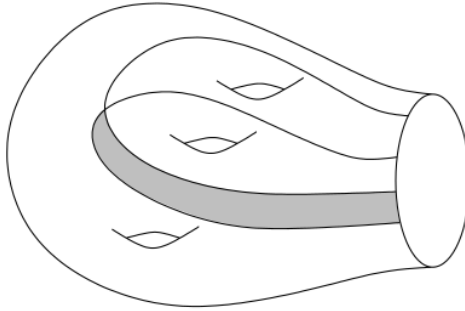


FIGURE 4. bigon-of-arcs.png

Note in this picture how if isotopies are considered relative to the boundary, then the two arcs are in minimal position, while if we consider general isotopies, then the half-bigon shows that they are not in minimal position as we can pull the top strand down under the bottom one along the boundary.

- The bigon criterion holds for arcs.
- Corollary 1.9 (geodesics are in minimal position) and prop 1.3 (existence and uniqueness of geodesic representatives) work for arcs in surfaces with punctures and/or boundary.
- Prop 1.10 (homotopy versus isotopy for curves) and theorem 1.13 (extension of isotopies) also work for arcs.

### Change of coordinates principle.

*Classification of simple closed curves.*

**Definition 1.29.** Given a simple closed curve or a simple proper arc  $\alpha$  in a surface  $S$ , the surface obtain by cutting  $S$  along  $\alpha$  is a compact surface  $S_\alpha$  equipped with an attaching map  $h$  (i.e.

- (1)  $S_\alpha / (x \sim h(x)) \approx S$
- (2) the image of the distinguished boundary components under this quotient map is  $\alpha$ .

**Definition 1.30.** We say that a simple closed curve  $\alpha$  in the surface  $S$  is *nonseparating* if the cut surface  $S_\alpha$  is connected, and *separating* if  $S_\alpha$  is not connected.

**Theorem 1.31.** If  $\alpha$  and  $\beta$  are any two nonseparating simple closed curves in a surface  $S$ , then there is a homeomorphism  $\varphi: S \rightarrow S$  with  $\varphi(\alpha) = \beta$ .

*Proof.* The cut surface  $S_\alpha$  and  $S_\beta$  have two boundary component corresponding to  $\alpha$  and  $\beta$ , respectively. Now, suppose  $S_\alpha$  has  $n_\alpha$  vertices,  $m_\alpha$  edges and  $t_\alpha$  triangles in a triangulation. Then in obtaining  $S$  from  $S_\alpha$ , we identify the vertices and edges, but no triangles are identified, so we get  $n_S = n_\alpha - 3$  and  $m_S = m_\alpha - 3$ , but  $t_S = t_\alpha$ . Thus  $\chi(S_\alpha) = \chi(S)$ .

Since both  $S_\alpha$  and  $S_\beta$  have the same Euler characteristic, number of boundary components and number of punctures, it follows that  $S_\alpha \approx S_\beta$ . Choose a homeomorphism  $\varphi: S_\alpha \rightarrow S_\beta$  such that if  $h_\alpha$  is the attaching map for  $S_\alpha$  and  $h_\beta$  is the attaching map for  $S_\beta$ , then  $\varphi$  takes  $\{x, h_\alpha(x)\}$  to  $\{y, h_\beta(y)\}$  - i.e., the identification are respected under the map. This homeomorphism gives the desired homeomorphism of  $S$  taking  $\alpha$  to  $\beta$ . If we want an orientation preserving homeomorphism, we can postcompose by an orientation-reversing homeomorphism fixing  $\beta$  if necessary.  $\square$

**Theorem 1.32.** When  $S$  is closed,  $\beta$  is separating if and only if it is the boundary of some subsurface of  $S$ . Which is equivalent to the vanishing of the homology class of  $\beta$  in  $H_1(S, \mathbb{Z})$ .

*Remark.* By the "classification of disconnected surfaces", there are finitely many separating simple closed curves in  $S$  up to homeomorphism.

**Corollary 1.33.** There is an orientation-preserving homeomorphism of a surface taking one simple closed curve to another if and only if the corresponding cut surfaces (which may be disconnected) are homeomorphic.

**Definition 1.34** (Topological type). The existence of a homeomorphism as in 1.33 is an equivalence relation. The equivalence class of a simple closed curve or a collection of simple closed curves is called its *topological type*.

A separating simple closed curve in the closed surface  $S_g$  divides  $S_g$  into two disjoint subsurfaces of genus  $k$  and  $g - k$ . The minimum of  $\{k, g - k\}$  is called the genus of the separating simple closed curve. There are  $\lfloor \frac{g}{2} \rfloor$  topological types of essential separating simple closed curves in a closed surface.

**Question 1.35.** Suppose  $\alpha$  is any nonseparating simple closed curve on a surface  $S$ .

- (1) Is there a simple closed curve  $\gamma$  in  $S$  so that  $\alpha$  and  $\gamma$  fill  $S$ , i.e., such that  $\alpha$  and  $\gamma$  are in minimal position and the complement of  $\alpha \cup \gamma$  is a union of topological disks.
- (2) Is there a simple closed curve  $\delta$  in  $S$  with  $i(\alpha, \delta) = 0$ ?  $i(\alpha, \delta) = 1$ ?  $i(\alpha, \delta) = k$ ?

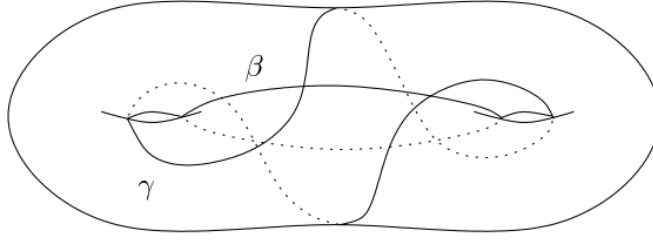


FIGURE 5. a

Figure ?? shows two filling simple closed curves on the genus 2 surface. By the classification of simple closed curves on a surface, there is a homeomorphism  $\varphi: S_2 \rightarrow S_2$  such that  $\varphi(\beta) = \alpha$ . Then the image of  $\gamma$  under  $\varphi$  fills  $S_2$  with  $\alpha$  since filling is a topological property (show this).

*Examples of change of coordinate principle.*

- (1) *Pairs of simple closed curves that intersect once are all of the same topological type.* Suppose  $\alpha_1$  and  $\beta_1$  form such a pair on a surface  $S$ . Then  $\beta_1$  must be an arc connecting the two boundary components in  $S_{\alpha_1}$ . But the boundary component is homeomorphic to  $S^1$ , so removing a point leaves it connected. Thus removing  $\beta_1$  leaves  $(S_{\alpha_1})_{\beta_1}$  path-connected. Similarly,  $(S_{\alpha_2})_{\beta_2}$  is path-connected for any other pair  $\alpha_2$  and  $\beta_2$  that constitute a pair of simple closed curves that intersect once in  $S$ . By the classification of surfaces with boundary,  $(S_{\alpha_1})_{\beta_1}$  is homeomorphic to  $(S_{\alpha_2})_{\beta_2}$  which preserves equivalence classes on the boundary, and as we can construct this homeomorphism first for the  $\beta$ 's and then for the  $\alpha$ 's, this homeomorphism descends to a self-homeomorphism of  $S$  taking the pair  $\{\alpha_1, \beta_1\}$  to  $\{\alpha_2, \beta_2\}$ .

**Three facts about homeomorphisms.** Suppose  $f: D \rightarrow D$  is an orientation-reversing map. Then  $f$  restricts to a map on  $S^1 \rightarrow S^1$ , and if  $f$  is smooth considered as such a map, then the reversal of orientation implies that since the fiber of any point is a single point, the degree of  $f$  must be  $-1$ . But thus  $f$  is not isotopic to



the identity as the identity has degree 1 and the isotopy would have to restrict to a homotopy on the boundary, but degree is a homotopy invariant for maps  $S^n \rightarrow S^n$ . However, the straight-line homotopy does give a homotopy between  $f$  and the identity.

On  $A = S^1 \times I$ , the orientation-reversing map that fixes the  $S^1$  factor and reflects the  $I$  factor is homotopic but not isotopic to the identity.

**Theorem 1.36.** *Let  $S$  be any compact surface and let  $f$  and  $g$  be homotopic homeomorphisms of  $S$ . Then  $f$  and  $g$  are isotopic unless they are one of the two examples described above (on  $S = D^2$  and  $S = A$ ). In particular, if  $f$  and  $g$  are orientation-preserving, then they are isotopic.*

**Theorem 1.37.** *Let  $S$  be a compact surface. Then every homeomorphism of  $S$  is isotopic to a diffeomorphism of  $S$ .*

**Theorem 1.38** (Hamstrom). *Let  $S$  be a compact surface, possibly minus a finite number of points from the interior. Assume that  $S$  is not homeomorphic to  $S^2, \mathbb{R}^2, D^2, T^2$ , the closed annulus, the once-punctured disk, or the once-punctured plane. Then the space  $\text{Homeo}_0(S)$  is contractible.*

## 2. MAPPING CLASS GROUP BASICS

**The compact-open topology.**

**Definition 2.1.** The *weak* or *compact-open*  $C^r$  topology on  $C^r(M, N)$ , where  $M$  and  $N$  are  $C^r$  manifolds, is generated by sets defined as follows: let  $f \in C^r(M, N)$ . Let  $(U, \varphi), (V, \psi)$  be charts on  $M$  and  $N$ ; let  $K \subset U$  be compact such that  $f(K) \subset V$  and let  $0 < \varepsilon \leq \infty$ . Then a *weak subbasic neighborhood*

$$\mathcal{N}^r(f; (U, \varphi), (V, \psi), K, \varepsilon) \quad (\zeta)$$

is the set of  $C^r$  maps  $g: M \rightarrow N$  such that  $g(K) \subset V$  and

$$\|D^k(\psi f \varphi^{-1})(x) - D^k(\psi g \varphi^{-1})(x)\| < \varepsilon$$

for all  $x \in \varphi(K)$ , for  $k = 0, \dots, r$ . The *compact-open*  $C^r$  topology on  $C^r(M, N)$  is generated by the set of weak subbasic neighborhoods, and defines the topological space  $C_W^r(M, N)$ . A neighborhood of  $f$  is then any set containing the intersection of a finite number of sets of the type  $(\zeta)$ .

We are interested in the subspace  $\text{Homeo}(S) \subset C_W^0(S, S)$ , inheriting the subspace topology.

The compact-open topology might seem a bit confusing, but we have the following lemma [Hatcher, Prop A.14]:

**Lemma 2.2.** *Let  $X, Y, Z$  be Hausdorff topological spaces. Suppose  $Y$  is locally compact. Then a map  $f: X \rightarrow C_W^0(Y, Z)$  is continuous if and only if the associated map  $F: X \times Y \rightarrow Z$  defined by*

$$F(x, y) := f(x)(y)$$

*is continuous.*

**2.1. Definitions and first examples.**

**Definition 2.3.** Let  $S$  be a surface which is the connected sum of  $g \geq 0$  tori with  $b \geq 0$  disjoint open disks removed and  $n \geq 0$  points removed from the interior. Let  $\text{Homeo}^+(S, \partial S)$  denote the group of orientation-preserving self-homeomorphisms of  $S$  that restrict to the identity on  $\partial S$ . We endow this group with the compact-open topology. The *mapping class group* of  $S$ , denoted  $\text{Mod}(S)$ , is the group

$$\text{Mod}(S) = \pi_0(\text{Homeo}^+(S, \partial S))$$

*Remark.* From Lemma 2.2, we see that a path  $\gamma: I \rightarrow \text{Homeo}^+(S, \partial S)$  is precisely equivalent to an isotopy  $F: I \times S \rightarrow S$  from  $\gamma(0)$  to  $\gamma(1)$  (isotopy because at each time  $t$ ,  $\gamma(t): S \rightarrow S$  is indeed a topological embedding as it is a homeomorphism). In fact, it's an isotopy of  $S$ . Here isotopies are required to fix boundaries.

If  $\text{Homeo}_0(S, \partial S)$  denotes the connected component of the identity in  $\text{Homeo}^+(S, \partial S)$ , then we can equivalently write

$$\text{Mod}(S) = \text{Homeo}^+(S, \partial S) / \text{Homeo}_0(S, \partial S).$$

**Proposition 2.4.**

$$\begin{aligned}
\text{Mod}(S) &= \pi_0 (\text{Homeo}^+ (S, \partial S)) \\
&\approx \text{Homeo}^+ (S, \partial S) / \text{homotopy} \\
&\approx \pi_0 (\text{Diff}^+ (S, \partial S)) \\
&\approx \text{Diff}^+ (S, \partial S) / \sim
\end{aligned}$$

where  $\text{Diff}^+ (S, \partial S)$  is the group of orientation preserving diffeomorphisms of  $S$  that are the identity on the boundary and  $\sim$  can be taken to be either smooth homotopy relative to the boundary or smooth isotopy relative to the boundary.

*The Alexander Lemma.*

**Lemma 2.5** (Alexander lemma). *The group  $\text{Mod} (D^2)$  is trivial.*

*Remark.* Also  $0 \approx \text{Mod} (D - \{0\}) \approx \text{Mod} (S_{0,1}) \approx \text{Mod} (S^2)$ .

*The mapping class group of the thrice-punctured sphere,  $\text{Mod} (S_{0,3})$ .*

**Proposition 2.6.** *Any two essential simple proper arcs in  $S_{0,3}$  with the same endpoints are isotopic. Any two essential arcs that both start and end at the same marked point of  $S_{0,3}$  are isotopic.*

*Proof.* Let  $\alpha$  and  $\beta$  be two simple proper arcs in  $S_{0,3}$  connecting the same two distinct marked points. By isotopy, we may modify  $\alpha$  so that it intersects transversally with  $\beta$ . Letting the last marked point become the point at infinity, we can consider  $\alpha$  and  $\beta$  as being arcs in  $\mathbb{R}^2 - \{p, q\}$  for the two marked points  $p, q$ . An example is illustrated below. Now, suppose the arcs are disjoint. Then, choosing an intersection point, we can follow the path to the other intersection point and obtain either a bigon, in which case we can remove it by isotopy, or a bigon with path segments inside. Now, suppose there is some point of  $\alpha$  inside the bigon. Then since this is part of the arc  $\alpha$ , we can find a simple path connecting this point to two points of  $\beta$ . There could, however, be infinitely many such paths inside the bigon, preventing us choosing the innermost (think concentric semicircles). However, by transversality, the preimages of the intersection points form a 0-dimensional submanifold of  $I$  which is closed (as the preimage of a closed path segment of  $\beta$ ) and discrete. But discrete subsets of compact spaces are finite. Hence we can choose the innermost such path of  $\alpha$ . By isotopy, we can remove the bigon formed by this  $\alpha$ . Continuing a finite amount of times, we remove the original bigon. After a finite amount of reiterations, we can therefore remove all bigons, and we get disjoint  $\alpha$  and  $\beta$ . Now suppose we remove  $\alpha \cup \beta$ . Then we get a disjoint union of a disk and a punctured disk (by the classification of surfaces - expound on this). Thus the embedded disk in  $S_{0,3}$  gives an isotopy of  $\alpha$  to  $\beta$ .  $\square$

**Proposition 2.7.** *The natural map*

$$\text{Mod} (S_{0,3}) \rightarrow \Sigma_3$$

*given by the action of  $\text{Mod} (S_{0,3})$  on the set of marked points of  $S_{0,3}$  is an isomorphism.*

*Proof.* This is just additional notes to the proof in the book. The reason it is surjective is that the previous proposition gives an isotopy between arcs which we can extend to an ambient isotopy relative to the boundary which is an element of the mapping class group. (can we be sure the last marked point stays fixed?)  $\square$

**Exercise 2.8.** Show similarly that  $\text{Mod}(S_{0,2}) \approx \mathbb{Z}/2\mathbb{Z}$ .

*Solution.* Let  $\alpha, \beta$  be arcs with the same distinct marked endpoints. Equivalently to before, we can reduce bigons by isotopy until  $\alpha$  and  $\beta$  are disjoint. Then removing  $\alpha \cup \beta$  we would get two disjoint disks (firstly,  $\alpha \cup \beta$  make up a closed simple curve which is trivial since  $H_1(S^1) = \{0\}$  and thus separating. Therefore we get a disconnected space with as many vertices as edges whose Euler characteristic must add to  $2 = \chi(S^2)$ , so it must precisely have 1 face each, i.e., they are disks) which will descend to give the desired isotopy in  $S_{0,2}$ .

So assume no intersection. Let  $\varphi$  be an orientation preserving homeomorphism fixing the marked points. Then  $\varphi(\alpha)$  is isotopic to  $\alpha$ , so  $\varphi$  is isotopic to a homeomorphism which fixes  $\alpha$  pointwise, call it  $\psi$ . This induces a homeomorphism on  $S^2 - \alpha$  which is a disk that is the identity on the boundary, and hence isotopic to the identity homeomorphism on the disk since  $\text{Mod}(D^2) \approx \{0\}$ . This isotopy gives an isotopy of  $\psi$  to the identity. The composition of all these isotopies gives an isotopy of  $\varphi$  with the identity. Hence the map is injective.

**Theorem 2.9.** *The homomorphism*

$$\sigma: \text{Mod}(T^2) \rightarrow \text{SL}(2, \mathbb{Z})$$

*given by the action on  $H_1(T; \mathbb{Z}) \approx \mathbb{Z}^2$  is an isomorphism.*

*Proof.* Additional notes on the proof: why can we for any element  $f \in \text{Mod}(T^2)$  choose a representative  $\varphi$  that fixes a basepoint for  $T^2$ ?  $\square$

**Corollary 2.10.** *Since  $H_1(S_{1,1}; \mathbb{Z}) \approx \mathbb{Z}^2$ , there is a homomorphism  $\sigma: \text{Mod}(S_{1,1}) \rightarrow \text{SL}(2, \mathbb{Z})$  which is determined which isomorphism the homomorphism induces in homology. This map is an isomorphism.*

**Exercise 2.11.** Prove this explicitly.

*The mapping class group of  $S_{0,4}$ .* Consider the torus  $T^2$  as  $I^2 / \sim$  under the usual identification. Then consider the linear map  $\iota: \mathbb{R} \rightarrow \mathbb{R}$  by  $\iota = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \in \text{SL}(2, \mathbb{Z})$  which rotates about the origin by  $\pi$  radians.

The map is equivariant with respect to the quotient map so it induces a map  $I^2 / \sim \rightarrow I^2 / \sim$  and we wish to take the quotient space that identifies fibers of this map. This is equivalent to taking the quotient space of  $\mathbb{R}^2$  induced by the following actions: for  $(a, b) \in \mathbb{R}^2$ ,

- (1) sending  $(a, b)$  to  $(a + 2k, b)$  for  $k \in \mathbb{Z}$ ,
- (2) sending  $(a, b)$  to  $(a, b + 2t)$  for  $t \in \mathbb{Z}$ ,
- (3) or sending  $(a, b)$  to  $(-a, -b)$ .

We claim the quotient of  $[0, 2] \times I$  under this action is a fundamental domain for the action. Clearly, the action is transitive. Now if  $(a, b), (c, d) \in (0, 2) \times I$  are in the same orbit, then

$$\begin{aligned} a &= (-1)^\alpha c + 2k \\ b &= (-1)^\alpha d + 2t \end{aligned}$$

for some  $k, t \in \mathbb{Z}$ . But then if  $b + d = 2t$ , we get  $b + d \in 2\mathbb{Z} \cap (0, 2) = \emptyset$ , so  $\alpha$  must be even, and  $b = d$ . But then  $a - c \in 2\mathbb{Z} \cap (-2, 2) = \{0\}$ , so  $a = c$  and  $b = d$ . The identifications on the boundary become as in figure 6 which becomes  $S^2$ .

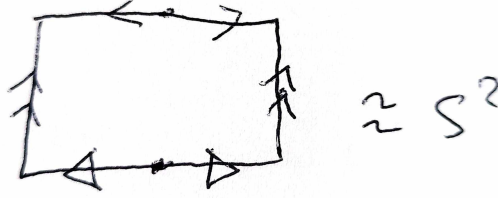


FIGURE 6.

We identify the quotient by  $S_{0,4}$  where the 4 marked points are the 4 fixed points under the involution, namely, the images of the center of  $I^2$ , the midpoints of the edges and corner vertices. This is clearly also a 2-fold cover of the sphere. Now, since for any  $A \in \text{SL}(2, \mathbb{Z})$ ,  $A(-I) = (-I)A$ , each element of  $\text{Mod}(T^2)$  induces an element of  $\text{Mod}(S_{0,4})$  by descending to the quotient.

**Proposition 2.12.** *The hyperelliptic involution induces a bijection between the set of homotopy classes of essential simple closed curves in  $T^2$  and the set of homotopy classes of essential simple closed curves in  $S_{0,4}$ .*

*Proof.* Notes:

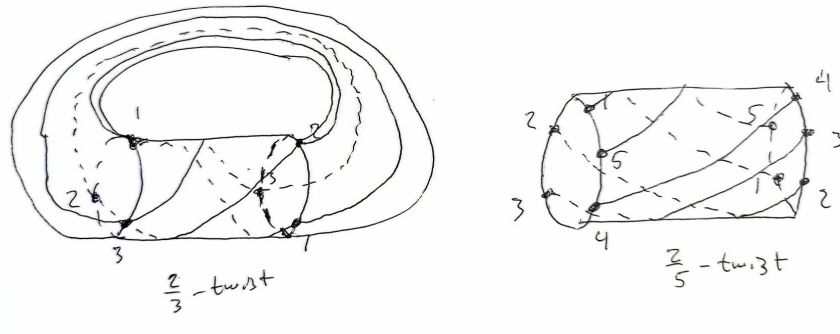


FIGURE 7. Twists along the meridian circle on the torus

Why is the preimage of a  $(p, q)$ -curve in  $S_{0,4}$  in  $T^2$  a  $(2p, 2q)$ -curve?

□

**Proposition 2.13.**  $\text{Mod}(S_{0,4}) \approx \text{PSL}(2, \mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ .

**Lemma 2.14.** *If the short exact sequence of groups*

$$1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$$

*has a right inverse for  $G \rightarrow H$ , then  $G$  is naturally isomorphic to  $N \ltimes H$ .*

*Proof.* Let  $f: N \rightarrow G$ ,  $g: G \rightarrow H$  and  $h: H \rightarrow G$  be the inverse. Then  $f$  and  $g$  are injective. Suppose  $z \in f(N) \cap h(H)$ . Then there exists a  $v \in N$  and  $u \in H$  such that  $f(v) = z = h(u)$ , so  $u = g(h(u)) = g(z) = g(f(v)) = 0$ , so  $z = 0$ . Since  $f(N)$  is the kernel of  $g$ , it is normal in  $G$ , so  $f(N)h(H)$  forms a subgroup of  $G$ . Now

suppose  $p \in G - f(N)h(H)$ . Then since  $g(p - h(g(p))) = 0$ , there exists  $n \in N$  such that  $p = f(n) + h(g(p)) \in f(N)h(H)$ , contradiction. So  $G = f(N)h(H)$ , giving  $G = f(N) \rtimes h(H) \approx N \rtimes H$ .  $\square$

*Proof.* To show 2.13, it thus suffices to find a homomorphism  $\text{Mod}(S_{0,4}) \rightarrow \text{PSL}(2, \mathbb{Z})$  with a right inverse, and show that the kernel is  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

Notes on the proof: for the involutions  $\iota_1, \iota_2$ , we can lift them to homeomorphisms of  $T^2$  by the lifting theorem [Bredon, Thm 4.1]. But why would these necessarily have to be homeomorphisms that rotate one of the factors of  $T^2 \approx S^1 \times S^1$  by  $\pi$ ?  $\square$

### 2.1.1. The Alexander method.

**Proposition 2.15** (Alexander method). *Let  $S$  be a compact surface, possibly with marked points, and let  $\varphi \in \text{Homeo}^+(S, \partial S)$ . Let  $\gamma_1, \dots, \gamma_n$  be a collection of essential simple closed curves and simple proper arcs in  $S$  with the following properties.*

- (1) *The  $\gamma_i$  are pairwise in minimal position.*
- (2) *The  $\gamma_i$  are pairwise nonisotopic.*
- (3) *For distinct  $i, j, k$ , at least one of  $\gamma_i \cap \gamma_j, \gamma_i \cap \gamma_k$ , or  $\gamma_j \cap \gamma_k$  is empty.*
- (i) *If there is a permutation  $\sigma$  of  $\{1, \dots, n\}$  so that  $\varphi(\gamma_i)$  is isotopic to  $\gamma_{\sigma(i)}$  relative to  $\partial S$  for each  $i$ , then  $\varphi(\cup \gamma_i)$  is isotopic to  $\cup \gamma_i$  relative to  $\partial S$ .*  
*If we regard  $\cup \gamma_i$  as a (possibly disconnected) graph  $\Gamma$  in  $S$ , with vertices at the intersection points and at the endpoints of arcs, then the composition of  $\varphi$  with this isotopy gives an automorphism  $\varphi_*$  of  $\Gamma$ .*
- (ii) *Suppose now that  $\{\gamma_i\}$  fills  $S$ . If  $\varphi_*$  fixes each vertex and each edge of  $\Gamma$  with orientations, then  $\varphi$  is isotopic to the identity. Otherwise,  $\varphi$  has a nontrivial power that is isotopic to the identity.*

**Lemma 2.16.** *Let  $S$  be a compact surface, possibly with marked points, and let  $\gamma_1, \dots, \gamma_n$  be a collection of essential simple closed curves and simple proper arcs in  $S$  that satisfy the three properties from proposition 2.15. If  $\gamma'_1, \dots, \gamma'_n$  is another such collection so that  $\gamma'_i$  is isotopic to  $\gamma_i$  relative to  $\partial S$  for each  $i$ , then there is an isotopy of  $S$  relative to  $\partial S$  that takes  $\gamma'_i$  to  $\gamma_i$  for all  $i$  simultaneously and hence takes  $\cup \gamma_i$  to  $\cup \gamma'_i$ .*

*Remark.* I put lemma 4.16 here since it's a slight generalization of (i) above and is used in the proof of Alexander's method too.

## 3. DEHN TWISTS

Let  $S$  be an oriented surface and let  $\alpha$  be a simple closed curve in  $S$ . Let  $N$  be a tubular neighborhood of  $\alpha$  and choose an orientation preserving homeomorphism  $\varphi: A \rightarrow N$ . We then obtain a homeomorphism  $T_\alpha: S \rightarrow S$ , called a *Dehn twist about  $\alpha$* , as follows:

$$T_\alpha(x) = \begin{cases} \varphi \circ T \circ \varphi^{-1} & \text{if } x \in N \\ x & \text{if } x \in S - N \end{cases}.$$

"By the uniqueness of regular neighborhoods, the isotopy class of  $T_\alpha$  does not depend on the choice of  $N$  or the choice of homeomorphism  $\varphi$ . Nor does  $T_\alpha$  depend on the choice of simple closed curve  $\alpha$  within its isotopy class." Huh, why???

*Dehn twists on the torus.* Via the isomorphism  $\text{Mod}(T^2) \rightarrow \text{SL}(2, \mathbb{Z})$  from 2.9, the Dehn twists about the  $(1, 0)$ -curve and the  $(0, 1)$ -curve in  $\text{Mod}(T^2)$  correspond to the matrices

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

3.0.1. *Dehn twist facts.*

**Proposition 3.1.** *Let  $a$  be the isotopy class of a simple closed curve  $\alpha$  in a surface  $S$ . If  $\alpha$  is not homotopic to a point or a puncture of  $S$ , then the Dehn twist  $T_a$  is a nontrivial element of  $\text{Mod}(S)$ .*

**Proposition 3.2.** *Let  $a$  and  $b$  be arbitrary isotopy classes of essential simple closed curves in a surface and let  $k$  be an arbitrary integer. We have*

$$i(T_a^k(b), b) = |k| i(a, b)^2.$$

**Proposition 3.3.** *Let  $\alpha$  and  $\beta$  be simple closed curves in a surface. Suppose that  $\alpha$  and  $\beta$  are in minimal position. Given a third simple closed curve  $\gamma$ , there exists a simple closed curve  $\gamma'$  that is homotopic to  $\gamma$  and that is in minimal position with respect to both  $\alpha$  and  $\beta$ .*

**Proposition 3.4.** *Let  $a_1, \dots, a_n$  be a collection of pairwise disjoint isotopy classes of simple closed curves in a surface  $S$  and let  $M = \prod_{i=1}^n T_{a_i}^{e_i}$ . Suppose that  $e_i > 0$  for all  $i$  or  $e_i < 0$  for all  $i$ . If  $b$  and  $c$  are arbitrary isotopy classes of simple closed curves in  $S$ , then*

$$\left| i(M(b), c) - \sum_{i=1}^n |e_i| i(a_i, b) i(a_i, c) \right| \leq i(b, c).$$

**Definition 3.5** (Pairs of filling curves). We say a pair of isotopy classes  $\{a, b\}$  of simple closed curves in a surface  $S$  *fill* if any pair of minimal position representatives fill (i.e., the complement of the representatives in the surface is a collection of disks and once-punctured disks). This is equivalent to saying that for every isotopy class  $c$  of essential simple closed curves in the surface, either  $i(a, c) > 0$  or  $i(b, c) > 0$ .

**Proposition 3.6.** *Let  $g, n \geq 0$  and assume  $\chi(S_{g,n}) < 0$ . Then there exists a pair of simple closed curves in  $S_{g,n}$  that fill  $S_{g,n}$ .*

The equivalence is intuitive since otherwise, it would be contained in one of the disks and hence not be essential, and the converse is similarly seen.

3.0.2. *Basic facts about Dehn twists.* Throughout this section,  $a$  and  $b$  denote arbitrary (unoriented) isotopy classes of simple closed curves.

**Lemma 3.7.**  $T_a = T_b \iff a = b$ .

**Lemma 3.8.** *For any  $f \in \text{Mod}(S)$  and any isotopy class  $a$  of simple closed curves in  $S$ , we have*

$$T_{f(a)} = f T_a f^{-1}.$$

**Corollary 3.9.** *For any  $f \in \text{Mod}(S)$  and any isotopy class  $a$  of simple closed curves in  $S$ , we have*

$$f \text{ commutes with } T_a \iff f(a) = a.$$

**Corollary 3.10.** *If  $a$  and  $b$  are nonseparating simple closed curves in  $S$ , then  $T_a$  and  $T_b$  are conjugate in  $\text{Mod}(S)$ .*

**Lemma 3.11.** *For any two isotopy classes  $a$  and  $b$  of simple closed curves in a surface  $S$ , we have*

$$i(a, b) = 0 \iff T_a(b) = b \iff T_a T_b = T_b T_a.$$

**Proposition 3.12.** *The above results have the following analogues for powers of Dehn twists: for  $f \in \text{Mod}(S)$ , we have*

$$f T_a^j f^{-1} = T_{f(a)}^j$$

*and so  $f$  commutes with  $T_a^j$  if and only if  $f(a) = a$ . Also, for nontrivial Dehn twists  $T_a, T_b$  and nonzero integers  $j, k$ , we have*

$$\begin{aligned} T_a^j &= T_b^k \iff a = b \text{ and } j = k \\ T_a^j T_b^k &= T_b^k T_a^j \iff i(a, b) = 0. \end{aligned}$$

3.0.3. *The center of the mapping class group.*

**Theorem 3.13.** *For  $g \geq 3$ ,  $Z(G)$ , for any finite index subgroup  $G \leq \text{Mod}(S_g)$ , is trivial. In particular,  $Z(\text{Mod}(S_g))$  is trivial for  $g \geq 3$ .*

3.0.4. *Relations between two Dehn twists.*

**Proposition 3.14** (Braid relation). *If  $a$  and  $b$  are isotopy classes of simple closed curves with  $i(a, b) = 1$ , then*

$$T_a T_b T_a = T_b T_a T_b.$$

*Equivalently, this read  $T_a T_b(a) = b$ .*

**Question 3.15.** Does the converse also work? I.e., if two Dehn twists satisfy the braid relation algebraically, do the corresponding curves necessarily have intersection number equal to 1?

**Proposition 3.16.** *If  $a$  and  $b$  are distinct isotopy classes of simple closed curves and the Dehn twists  $T_a$  and  $T_b$  satisfy  $T_a T_b T_a = T_b T_a T_b$ , then  $i(a, b) = 1$ .*

*Groups generated by two Dehn twists.*

**Theorem 3.17.** *Let  $a$  and  $b$  be two isotopy classes of simple closed curves in a surface  $S$ . If  $i(a, b) \geq 2$ , then  $\langle T_a, T_b \rangle \approx F_2$ , where  $F_2$  is the free group of rank 2.*

*Proof.* The slick proof makes use of this basic but nice lemma from geometric group theory:

**Lemma 3.18** (Ping pong lemma). *Let  $G$  be a group acting on a set  $X$ . Let  $g_1, \dots, g_n$  be elements of  $G$ . Suppose that there are nonempty, disjoint subsets  $X_1, \dots, X_n$  of  $X$  with the property that, for each  $i$  and each  $j \neq i$ , we have  $g_i^k(X_j) \subset X_i$  for every nonzero integer  $k$ . Then the group generated by the  $g_i$  is a free group of rank  $n$ .*

**Exercise 3.19.** Complete the proof.

□

3.1. **Cutting, capping and including.**



**3.1.1. Including.** When  $S$  is a closed subsurface of a surface  $S'$ , we can define a natural homomorphism  $\eta: \text{Mod}(S) \rightarrow \text{Mod}(S')$  as follows. For  $f \in \text{Mod}(S)$ , we represent it by some  $\varphi \in \text{Homeo}^+(S, \partial S)$ . Then, if  $\hat{\varphi} \in \text{Homeo}^+(S', \partial S')$  denotes the element that agrees with  $\varphi$  on  $S$  and is the identity outside of  $S$ , we define  $\eta(f)$  to be the class of  $\hat{\varphi}$ . The map  $\eta$  is well defined because any homotopy between two elements of  $\varphi \in \text{Homeo}^+(S, \partial S)$  gives a homotopy between the corresponding elements of  $\text{Homeo}^+(S', \partial S')$  (the homotopy is simply relative to  $S' - S$ ).

The goal is to find  $\ker \eta$ .

**Lemma 3.20.** *Let  $\alpha_1, \dots, \alpha_n$  be a collection of homotopically distinct simple closed curves in a surface  $S$ , each not homotopic to a point in  $S$ . Let  $\beta$  and  $\beta'$  be simple closed curves in  $S$  that are both disjoint from  $\cup \alpha_i$  and are homotopically distinct from each  $\alpha_i$ . If  $\beta$  and  $\beta'$  are isotopic in  $S$ , then they are isotopic in  $S - \cup \alpha_i$ .*

**Theorem 3.21** (The kernel of the inclusion homomorphism). *Let  $S$  be a closed subsurface of a surface  $S'$ . Assume that  $S$  is not homeomorphic to a closed annulus and that no component of  $S' - S$  is an open disk. Let  $\eta: \text{Mod}(S) \rightarrow \text{Mod}(S')$  be the induced map. Let  $\alpha_1, \dots, \alpha_m$  denote the boundary components of  $S$  that bound once-punctured disks in  $S' - S$  and let  $\{\beta_1, \gamma_1\}, \dots, \{\beta_n, \gamma_n\}$  denote the pairs of boundary components of  $S$  that bound annuli in  $S' - S$ . Then the kernel of  $\eta$  is the free abelian group*

$$\ker \eta = \langle T_{\alpha_1}, \dots, T_{\alpha_m}, \dots, T_{\beta_1} T_{\gamma_1}^{-1}, \dots, T_{\beta_n} T_{\gamma_n}^{-1} \rangle.$$

*In particular, if no connected component of  $S' - S$  is an open annulus, an open disk, or an open once-marked disk, then  $\eta$  is injective.*

**3.1.2. The capping homomorphism.** One useful special case of theorem 3.21 is the case where  $S' - S$  is a once-punctured disk. We say that  $S'$  is the surface obtained from  $S$  by *capping* one boundary component. In this case, we have

**Proposition 3.22** (The capping homomorphism). *Let  $S'$  be the surface obtained from a surface  $S$  by capping the boundary component  $\beta$  with a once-marked disk; call the marked point in this disk  $p_0$ . Denote by  $\text{Mod}(S, \{p_1, \dots, p_k\})$  the subgroup of  $\text{Mod}(S)$  consisting of elements that fix the punctures  $p_1, \dots, p_k$ , where  $k \geq 0$ . Let  $\text{Mod}(S', \{p_0, \dots, p_k\})$  denote the subgroup of  $\text{Mod}(S')$  consisting of elements that fix the marked points  $p_0, \dots, p_k$  and then let  $\text{Cap}: \text{Mod}(S, \{p_1, \dots, p_k\}) \rightarrow \text{Mod}(S', \{p_0, \dots, p_k\})$  be the induced homomorphism. Then the following sequence is exact:*

$$1 \rightarrow \langle T_\beta \rangle \rightarrow \text{Mod}(S, \{p_1, \dots, p_k\}) \xrightarrow{\text{Cap}} \text{Mod}(S', \{p_0, \dots, p_k\}) \rightarrow 1$$

*Remark.* In the case where  $S'$  is capped by an unmarked disk, the kernel is isomorphic to  $\pi_1(TS')$ , i.e., the fundamental group of the tangent bundle of  $S'$ .

#### 4. BRAID GROUPS

**Definition 4.1** (Braids). Let  $p_1, \dots, p_n$  be distinguished points in  $\mathbb{C}$ . A *braid* is a collection of  $n$  paths  $f_i: [0, 1] \rightarrow \mathbb{C} \times [0, 1]$ ,  $1 \leq i \leq n$ , called *strands*, and a permutation  $\bar{f} \in \Sigma_n$  such that the following hold:

- the strands  $f_i([0, 1])$  are disjoint
- $f_i(0) = p_i$

- $f_i(1) = p_{\bar{f}(i)}$
- $f_i(t) \in \mathbb{C} \times \{t\}$ .

Usually we picture this by its *braid diagram* which is the projection of the images of the strands to the plane  $\mathbb{R} \times [0, 1]$  (with indications as to which strands pass over and under which others).

**Definition 4.2.** The *braid group on  $n$  strands*, denoted  $B_n$ , is the group of isotopy classes of braids.

*Remark.* Here an isotopy of the braid is a collection of isotopies  $(h_1(x, t), \dots, h_n(x, t))$  where  $h_i$  is an isotopy of  $f_i$  and such that  $(h_1(-, t), \dots, h_n(-, t))$  is a braid for each  $t \in [0, 1]$ .

**Definition 4.3.** The product of the braid  $(f_i(t))$  and the braid  $(g_i(t))$  is the braid  $(h_i(t))$ , where

$$h_i(t) = \begin{cases} f_i(2t), & t \in [0, \frac{1}{2}] \\ g_{\bar{f}(i)}(2t - 1), & t \in [\frac{1}{2}, 1] \end{cases}.$$

For  $1 \leq i \leq n - 1$ , let  $\sigma_i \in B_n$  denote the braid whose only crossing is the  $(i + 1)$ st strand passing in front of the  $i$ th strand.

We claim that the group  $B_n$  is generated by the elements  $\sigma_1, \dots, \sigma_{n-1}$ . This claim follows from the fact that any braid  $\beta$  can be isotoped so that its finitely many crossings occur at different horizontal levels.

#### 4.1. Fundamental groups of configuration spaces.

**Definition 4.4.** Let  $S$  be a surface and let  $C^{ord}(S, n)$  denote the configuration space of  $n$  distinct, ordered points in  $S$ , given by  $C^{ord}(S, n) = \times_n S - \text{BigDiag}(\times_n S)$  where  $\text{BigDiag}(\times_n S) = \{(x_1, \dots, x_n) \in \times_n S \mid \exists 1 \leq i < j \leq n: x_i = x_j\}$ .

Now, the symmetric group  $\Sigma_n$  acts on  $\times_n S$  by permuting the coordinates. This action preserves  $\text{BigDiag}(\times_n S)$  and thus induces an action of  $\Sigma_n$  by homeomorphisms on  $C^{ord}(S, n)$ . Since the action of  $\Sigma_n$  permutes the  $n$  coordinates and since these coordinates are always distinct for points in  $C^{ord}(S, n)$ , we see that this action is free. The quotient space

$$C(S, n) = C^{ord}(S, n) / \Sigma_n$$

is just the configuration space of  $n$  distinct, *unordered* points in  $S$ .

Now we note the following lemma [LeeTM, Cor 12.27] and proposition [LeeTM, Prop 12.22]

**Lemma 4.5.** *Let  $M$  be a connected  $n$ -manifold on which a discrete group  $\Gamma$  acts continuously, freely, and properly. Then  $M/\Gamma$  is an  $n$ -manifold.*

**Proposition 4.6.** *Every continuous action of a compact topological group on a Hausdorff space is proper*

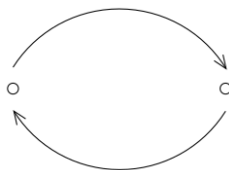
Since  $C(S, n)$  is the quotient of a manifold by a continuous free action of a finite group (hence compact), it follows that  $C(S, n)$  is a manifold.

Since each strand of a braid is a map  $f_i: I \rightarrow \mathbb{C} \times I$  with  $f_i(t) \in \mathbb{C} \times \{t\}$ , we can think of each  $f_i$  as a map  $I \rightarrow \mathbb{C}$ , so that a braid essentially becomes a path  $I \rightarrow \mathbb{C}^n$  with equal end points, i.e., a loop, where each  $t$  is mapped to the point whose  $i$ th coordinate is  $I \xrightarrow{f_i} \mathbb{C} \times \{t\} \xrightarrow{\text{proj}} \mathbb{C}$ . By assumption, the strands are disjoint, so each

$t$  is mapped to a point in  $C^{\text{ord}}(\mathbb{C}, n)$ , and essentially, forgetting the strand index, we can regard this as a map  $I \rightarrow C(\mathbb{C}, n)$ . Said in a different way, if we consider a slice  $\mathbb{C} \times \{t\}$  and intersect it with any braid, then we get a point in  $C(\mathbb{C}, n)$ , so the whole braid, which can be seen as a path in  $C(\mathbb{C}, n)$  between these intersections, gives an element of  $\pi_1(C(\mathbb{C}, n))$ , and this gives the isomorphism

$$B_n \approx \pi_1(C(\mathbb{C}, n))$$

In this case, the generator  $\sigma_i$  of  $B_n$  corresponds to the element of  $\pi_1(C(\mathbb{C}, n))$  given by the loop of  $n$ -point configurations in  $\mathbb{C}$  where the  $i$ th and  $(i+1)$ st points switch places by moving in a clockwise fashion, and the other  $n-2$  points remain fixed.



4.1.1. *The torsion of a braid group.* Since we just saw that  $B_n \approx \pi_1(C(\mathbb{C}, n))$ , and  $C(\mathbb{C}, n)$  is a finite-dimensional CW-complex that is  $K(B_n, 1)$ , we get that  $B_n$  is torsion-free [Hatcher, Prop 2.45]

4.2. **Mapping Class Group of a punctured disk.** Let  $D_n$  be a closed disk  $D^2$  with  $n$  marked points. Then

$$B_n \approx \text{Mod}(D_n) = \pi_0(\text{Homeo}^+(D_n, \partial D_n)).$$

To see this, let  $\varphi$  be a representative of some element in  $\text{Mod}(D_n)$  which leaves the set of marked points invariant under  $\varphi$  - i.e., if  $\{x_0, \dots, x_n\} \subset D_n$  are the  $n$  marked points, then  $\varphi(\{x_0, \dots, x_n\}) = \{x_0, \dots, x_n\}$ . Note that we do not regard these marked points as punctures because we will not want to consider isotopies which "move" these marked points around which is not allowed when the marked points represent punctures. We see that  $\varphi$  is simply a homeomorphism of  $D^2$  fixing  $\partial D^2$  pointwise, so by the Alexander lemma,  $\varphi$  is isotopic to the identity. Now, throughout such an isotopy, the marked points must again be sent to themselves, albeit they might move around through in the interior of  $D^2$  (which we identify with  $\mathbb{C}$ ) throughout the isotopy. Thus this isotopy produces a loop of these marked points, i.e., it produces a loop in  $C(\mathbb{C}, n)$ . So we have produced a braid.

**Question 4.7.** Does this association give a well-defined homomorphism  $B_n \rightarrow \text{Mod}(D_n)$  and would this be an isomorphism?

The answer is yes, and since  $\sigma_i$  generate  $B_n$ , the images generate  $\text{Mod}(D_n)$ . Now, the image of  $\sigma_i$  will be the homotopy class of a homeomorphism of  $D_n$  that has support a twice-punctured disk and is described on this support by figure 8.

We denote such a half-twist as  $H_\alpha$ , and we can think of  $\alpha$  as either a simple closed curve with two punctures in its interior or a simple proper arc connecting two punctures.

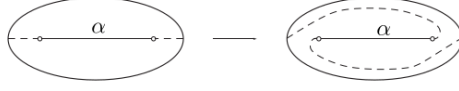


FIGURE 8. A half-twist

**4.3. The Birman Exact Sequence.** Let  $S$  be any surface, possibly with punctures (but no marked points) and let  $(S, x)$  denote the surface obtained from  $S$  by marking a point  $x$  in the interior of  $S$ . There is a natural homomorphism

$$\text{Forget}: \text{Mod}(S, x) \rightarrow \text{Mod}(S)$$

called the forgetful map which is realized by forgetting that the point  $x$  is marked. This map is surjective as any homeomorphism of  $S$  can be modified by isotopy to fix  $x$  (why?). The group  $\text{Mod}(S, x)$  is isomorphic to the subgroup  $G$  of  $\text{Mod}(S - x)$  consisting of homeomorphisms preserving the puncture coming from  $x$ . The forgetful map can then be interpreted as the map  $G \rightarrow \text{Mod}(S)$  obtained by "filling in" the puncture  $x$ . I.e., *Forget* is the map induced by the inclusion  $S - x \hookrightarrow S$ .

**4.3.1. Analyzing the kernel of *Forget*.** Let  $f \in \text{Mod}(S, x)$  be an element of the kernel of *Forget* and let  $\varphi$  be a homeomorphism representing  $f$ . We can think of  $\varphi$  as a homeomorphism  $\bar{\varphi}$  of  $S$ .

Since  $\text{Forget}(f) = 1$ , there is an isotopy from  $\bar{\varphi}$  to  $\mathbb{1}_S$ . During this isotopy, the image of the point  $x$  traces out a loop  $\alpha$  in  $S$  based at  $x$ . Now we will introduce the *Push* map: given a loop  $\alpha$  in  $S$  based at  $x$ , we can consider  $\alpha: [0, 1] \rightarrow S$  as an isotopy  $h: \{x\} \times I \rightarrow S$  with  $h(x, 0) = h(x, 1) = x$  and extend this to the whole surface using 1.26 (here,  $V = \{x\}$ ), denote this by  $h: S \times I \rightarrow S$  also. Let  $\varphi(x) = h(x, 1)$  be the homeomorphism of  $S$  obtained at the end of the isotopy. Taking its isotopy class in  $\text{Mod}(S, x)$ , we get  $[\varphi] =: \text{Push}(\alpha) \in \text{Mod}(S, x)$ . Think of  $\text{Push}(\alpha)$  as placing your finger on  $x$  and pushing  $x$  along  $\alpha$ , dragging the rest of the surface along as you go (indeed, locally, this is what must happen).

The question of whether this mapping class is independent of the choice of isotopy extension as well as the choice of  $\alpha$  within its homotopy class. I.e., whether *Push* defines a well-defined map

$$\text{Push}: \pi_1(S, x) \rightarrow \text{Mod}(S, x).$$

For that, we show look at the Birman exact sequence.

**Theorem 4.8** (Birman exact sequence). *Let  $S$  be a surface with  $\chi(S) < 0$ , possibly with punctures and/or boundary. Let  $(S, x)$  be the surface obtained from  $S$  by marking a point  $x$  in the interior of  $S$ . Then the following sequence is exact:*

$$1 \rightarrow \pi_1(S, x) \xrightarrow{\text{Push}} \text{Mod}(S, x) \xrightarrow{\text{Forget}} \text{Mod}(S) \rightarrow 1.$$

**Lemma 4.9.**  *$\text{Push}: \pi_1(S, x) \rightarrow \text{Mod}(S, x)$  is injective when  $\chi(S) < 0$ .*

*Proof.* Let  $\varphi$  represent  $\text{Push}(\alpha) \in \text{Mod}(S, x)$ . Then  $\varphi$  is a map  $(S, x) \rightarrow (S, x)$ , so in particular, we have an induced homomorphism  $\varphi_*: \pi_1(S, x) \rightarrow \pi_1(S, x)$ . This homomorphism is the inner automorphism  $I_\alpha(\gamma) = \alpha\gamma\alpha^{-1}$  (why?). Now, for  $\chi(S) < 0$ , the center of  $\pi_1(S)$  is trivial, so  $I_\alpha$  is nontrivial whenever  $\alpha \neq c_x$ . If  $\varphi$

had been homotopic to the identity, their induced homomorphisms would be equal, so we conclude that  $\mathcal{P}ush(\alpha)$  is nontrivial whenever  $\alpha$  is nontrivial.  $\square$

**4.3.2. Push maps along loops in terms of Dehn twists.** Let  $\alpha \in \pi_1(S, x)$  be simple. Let  $S^1 \times [0, 2]$  be an annulus about  $\alpha$  with  $S^1 \times \{1\}$  identified with  $\alpha$  with  $(0, 1)$  identified with the marked point  $x$ . We orient  $S^1 \times [0, 2]$  with the standard orientations on  $S^1$  and  $[0, 2]$ .

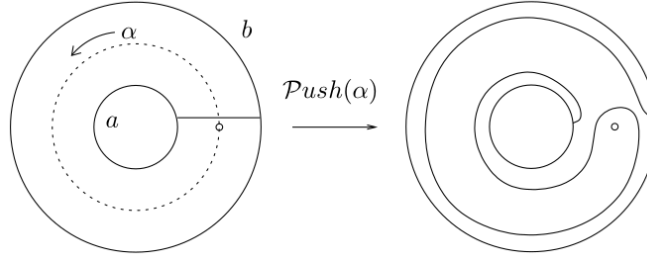
Let  $F: A \times I \rightarrow A$  be the isotopy given by

$$F((\theta, r), t) = \begin{cases} (\theta + 2\pi r t, r), & 0 \leq r \leq 1 \\ (\theta + 2\pi(2-r)t, r), & 1 \leq r \leq 2. \end{cases}$$

By 1.26, we can extend  $F$  to an ambient isotopy of  $S$ . Restricting  $F$  to  $\{x\} \times [0, 1]$ , we get

$$F((0, 1), t) = (2\pi t, 1),$$

so  $F$  pushes  $x$  around the core of the annulus. Now, the homeomorphism (representing  $\mathcal{P}ush(\alpha)$ )  $\varphi$  of  $(S, x)$  given by  $F((\theta, r), 1)$  is a product of two Dehn twists,  $T_a T_b^{-1}$  where  $a$  is the closed curve identified with  $S^1 \times \{0\}$  and  $b$  is the closed curve identified with  $S^1 \times \{2\}$ .



*Remark (Naturality).* For any  $h \in \text{Mod}(S, x)$  and any  $\alpha \in \pi_1(S, x)$ , we have

$$\mathcal{P}ush(h_*(\alpha)) = h\mathcal{P}ush(\alpha)h^{-1}.$$

*Proof.* Write a proof of the Birman exact sequence.  $\square$

**4.4. Generalized Birman Exact Sequence.** We now return to Question 4.7. We wish to show that  $\pi_1(C(\mathbb{C}, n)) \approx \text{Mod}(D_n)$ .

**Definition 4.10** ( $n$ -stranded surface braid group of  $S$ ). For an arbitrary surface  $S$ , we call  $\pi_1(C(S, n))$  the  $n$ -stranded surface braid group of  $S$ .

Let  $S$  be a compact finite-type surface without marked points. Let  $(S, \{x_1, \dots, x_n\})$  denote  $S$  with  $n$  marked points  $x_1, \dots, x_n$  in the interior. Equivalently to the construction in the proof of the Birman exact sequence, there is a fiber bundle

$$\text{Homeo}^+((S, \{x_1, \dots, x_n\}), \partial S) \rightarrow \text{Homeo}^+(S, \partial S) \rightarrow C(S^\circ, n)$$

where  $S^\circ$  is the interior of  $S$  and  $\text{Homeo}^+((S, \{x_1, \dots, x_n\}), \partial S)$  is the group of orientation-preserving homeomorphisms of  $S$  that preserve the set  $\{x_1, \dots, x_n\}$  (allowing permutations, however) and fix the boundary of  $S$  pointwise.

**Theorem 4.11** (Birman exact sequence generalized). *Let  $S$  be a surface without marked points and with  $\pi_1(\text{Homeo}^+(S, \partial S)) = 1$ . The following sequence is exact:*

$$1 \rightarrow \pi_1(C(S, n)) \xrightarrow{\mathcal{P}ush} \text{Mod}(S, \{x_1, \dots, x_n\}) \xrightarrow{\mathcal{F}orget} \text{Mod}(S) \rightarrow 1.$$

**Corollary 4.12.** *When  $S = D^2$ , this gives the exact sequence*

$$1 \rightarrow \underbrace{\pi_1(C(D^2, n))}_{B_n} \rightarrow \text{Mod}(D_n) \rightarrow \underbrace{\text{Mod}(D^2)}_1 \rightarrow 1.$$

Hence  $B_n \approx \text{Mod}(D_n)$ .

#### 4.5. Algebraic Structure of the Braid Group.

**4.6. The pure braid group.** Recall that by definition of a braid group, we are given a collection of  $n$  points  $p_1, \dots, p_n$ ,  $n$  paths  $f_i: I \rightarrow \mathbb{C} \times I$  and a permutation  $\bar{f} \in \Sigma_n$  such that  $f_i(0) = p_i$  and  $f_i(1) = p_{\bar{f}(i)}$ . We can thus define a homomorphism  $B_n \rightarrow \Sigma_n$  by sending  $f \mapsto \bar{f}$ .

**Definition 4.13.** We define the *pure braid group*  $PB_n$  as the kernel of the homomorphism  $B_n \rightarrow \Sigma_n$  given by  $f \mapsto \bar{f}$ .

We obtain the short exact sequence

$$1 \rightarrow PB_n \rightarrow B_n \rightarrow \Sigma_n \rightarrow 1.$$

So a pure braid is a braid where each strand begins and ends at the same point of  $\mathbb{C}$ .

**Exercise 4.14.** Show that

$$PB_n \approx \pi_1(C^{ord}(\mathbb{C}, n)) \approx \text{PMod}(D_n)$$

**4.7. Braid group and symmetric mapping class groups.** Let  $S_g^1$  be a surface of genus  $g$  with one boundary component. Define a homomorphism  $\psi: B_n \rightarrow \text{Mod}(S_g^1)$  for  $n \leq 2g + 1$  as follows. Choose a chain of simple closed curves  $\{\alpha_i\}$  in  $S_g^1$ , that is, a collection of simple closed curves satisfying  $i(\alpha_i, \alpha_{i+1}) = 1$  for all  $i$  and  $i(\alpha_i, \alpha_j) = 0$  otherwise. We then define  $\psi$  via  $\psi(\sigma_i) = T_{\alpha_i}$ . This is well defined if and only if  $T$  respects all the relations in  $B_n$ . Now, if  $\sigma_i$  and  $\sigma_j$  are given with  $|i - j| \geq 2$ , then  $\sigma_i \sigma_j = \sigma_j \sigma_i$  and indeed also  $i(\alpha_i, \alpha_j) = 0$  so by the disjointness relation for Dehn twists (fact 3.9), we get that  $T$  respects this commutativity. Similarly, we have that  $\sigma_i \sigma_{i+1} \sigma_i$ , for all  $i$ , is mapped to  $T_{\alpha_i} T_{\alpha_{i+1}} T_{\alpha_i}$  which, by the braid relation on Dehn twists and the assumption that  $i(\alpha_i, \alpha_{i+1}) = 1$  for all  $i$ , is equivalent to  $T_{\alpha_{i+1}} T_{\alpha_i} T_{\alpha_{i+1}}$ , so the braid relation in  $B_n$  is respected under  $\psi$ .

**Question 4.15.** Can we say whether  $\psi$  is injective?

**4.8. The Birman-Hilden Theorem.** Let  $\iota$  be the order 2 element of  $\text{Homeo}^+(S_g^1)$  as shown in figure 9 and let  $\text{SHomeo}^+(S_g^1)$  be the centralizer in  $\text{Homeo}^+(S_g^1, \partial S_g^1)$  of  $\iota$ :

$$\text{SHomeo}^+(S_g^1) = C_{\text{Homeo}^+(S_g^1, \partial S_g^1)}(\iota).$$

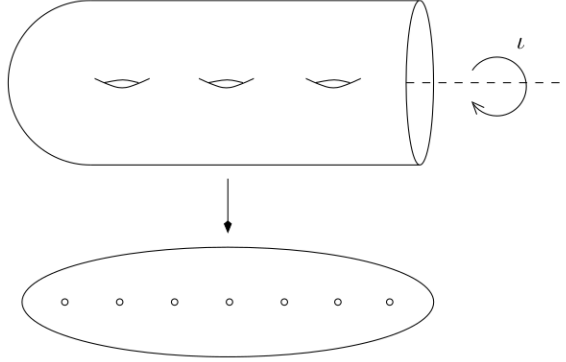


FIGURE 9. The Birman-Hilden double cover

**Definition 4.16.** The group  $\text{SHomeo}^+(S_g^1)$  is called the group of orientation-preserving symmetric homeomorphisms of  $S_g^1$ . The symmetric mapping class group is the group

$$\text{SMod}(S_g^1) = \text{SHomeo}^+(S_g^1) / \text{isotopy},$$

i.e., the subgroup of  $\text{Mod}(S_g^1)$  that is the image of  $\text{SHomeo}^+(S_g^1)$ .

The homeomorphism  $\iota$  has  $2g + 1$  fixed points in  $S_g^1$ . The quotient of  $S_g^1$  by  $\langle \iota \rangle$  is a topological disk  $D_{2g+1}$  with  $2g + 1$  cone points of order 2, with each cone point coming from a fixed point of  $\iota$ . Since the elements of  $\text{SHomeo}^+(S_g^1)$  commute with  $\iota$ , they descend to homeomorphisms of the quotient disk, and by the commutativity, they must preserve the set of  $2g + 1$  fixed points of  $\iota$ , and so there is a homomorphism

$$\text{SHomeo}^+(S_g^1) \rightarrow \text{Homeo}^+(D_{2g+1}, \partial D_{2g+1})$$

by sending  $\varphi$  to  $\pi \circ \varphi$  where  $\pi: S_g^1 \rightarrow D_{2g+1}$  is the quotient map. If  $\varphi \mapsto \pi \circ \varphi = \mathbb{1}_{D_{2g+1}}$ , then  $\varphi \in \langle \iota \rangle$ , and since  $\iota \notin \text{SHomeo}^+(S_g^1)$ , we have  $\varphi = 1$ .



**Question 4.17.** Why does any element of  $\text{Homeo}^+(D_{2g+1})$  lift to  $\text{SHomeo}^+(S_g^1)$ ?

**Definition 4.18.** We say two homeomorphisms  $\varphi, \psi \in \text{SHomeo}^+(S_g^1)$  are symmetrically isotopic whenever  $[\varphi] = [\psi]$  in  $\pi_0(\text{SHomeo}^+(S_g^1))$ . Note that here,  $\text{SHomeo}^+(S_g^1)$  inherits the subspace topology from  $\text{Homeo}^+(S_g^1)$  (actually, is this true?...)

If we have this, we get an isomorphism of the two, so

$$\begin{aligned} \text{SHomeo}^+(S_g^1) / \text{symmetric isotopy} &= \pi_0(\text{SHomeo}^+(S_g^1)) \\ &\approx \pi_0(\text{Homeo}^+(D_{2g+1}, \partial D_{2g+1})) \\ &= \text{Mod}(D_{2g+1}) \\ &\approx B_{2g+1}. \end{aligned}$$

We want to show that if two symmetric homeomorphisms of  $S_g^1$  are isotopic, then they must be symmetrically isotopic.

#### 4.9. The Birman-Hilden Theorem.

**Theorem 4.19** (Birman-Hilden).  $\text{SMod}(S_g^1) \approx B_{2g+1}$ .

**Example 4.20.** Taking  $g = 1$ , we will find  $\text{Mod}(S_1^1)$ . Note that by corollary 9.12,  $\iota \in C(\text{Homeo}^+(S_1^1))$ , so  $\text{Mod}(S_1^1) = \text{SMod}(S_1^1)$ , and now it is easy to see that

$$\text{Mod}(S_1^1) = \text{SMod}(S_1^1) \approx B_3 \approx \text{Mod}(D_3)$$

*Remark.* The Birman-Hilden theorem also holds for surfaces with two symmetric boundary components that are interchanged by  $\iota$ , see figure 10. Hence

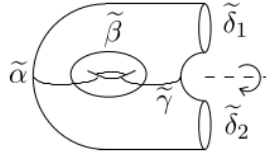


FIGURE 10. Symmetric boundary components interchanged by  $\iota$ .

$$\text{SMod}(S_g^2) \approx B_{2g+2}.$$

#### 4.10. Proof of the Birman-Hilden Theorem.

**Definition 4.21.** A closed curve  $\alpha$  in  $S_g$  is symmetric if  $\iota(\alpha) = \alpha$  as sets.

**Lemma 4.22.** Let  $g \geq 2$  and let  $\alpha$  and  $\beta$  be two symmetric nonseparating simple closed curves in  $S_g$ . If  $\alpha$  and  $\beta$  are isotopic, then they are symmetrically isotopic.

*Proof.* Let  $\bar{\alpha}$  and  $\bar{\beta}$  denote the images of  $\alpha$  and  $\beta$  in  $S_{0,2g+2} \approx S_g / \langle \iota \rangle$ . Now,  $\bar{\alpha}$  and  $\bar{\beta}$  must be simple proper arcs in  $S_{0,2g+2}$  (look at it geometrically - in particular, remember  $\alpha$  and  $\beta$  are symmetric).

Let's look at an example to get a picture. If  $\alpha$  and  $\beta$  are symmetric choices of the usual generating loops in the homology of a torus, these will correspond to one loop being the "innermost" meridian loop while we choose the other loop to be a longitudinal loop whose plan of existence is normal to the axis of rotation. See figure 11.

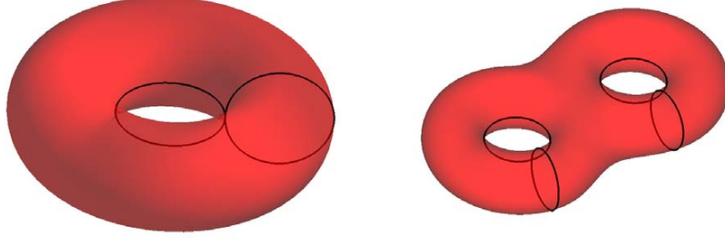


FIGURE 11. Basis for homologies of torus and double torus

Imagine in this figure now we cut the surface into a "lower" and "upper" half. Looking at the resulting paths of our loops, we indeed get arcs for the longitudinal loops and loop on the boundary for the meridian loops.

Now, an isotopy between these arcs will lift to a symmetric isotopy between  $\alpha$  and  $\beta$ .

I.e., if  $H$  is an isotopy between  $\alpha$  and  $\beta$ , then there exist an induced map  $\tilde{H}$  such that

$$\begin{array}{ccc} & & S_g \\ & \nearrow \tilde{H} & \downarrow \\ I^2 & \xrightarrow{H} & S_g / \langle \iota \rangle \end{array}$$

commutes since  $I^2$  is simply connected, path-connected and locally path-connected [Bredon, Cor 4.2].

We can modify  $\alpha$  by a symmetric isotopy so that it is transverse to  $\beta$ . We claim that  $\alpha$  cannot be disjoint from  $\beta$ . Indeed, for then  $\bar{\alpha}$  and  $\bar{\beta}$  are disjoint including endpoints. But such arcs cannot correspond to isotopic curves in  $S_g$ : we can choose an arc  $\bar{\gamma}$  that passes through an odd number of endpoints of  $\bar{\alpha}$  and an even number of endpoints of  $\bar{\beta}$ . This will then lift to a loop  $\gamma$  in  $S_g$  with  $i(\alpha, \gamma)$  odd and  $i(\beta, \gamma)$  even, contradicting  $\alpha$  isotopic to  $\beta$ .

Now, since  $\alpha$  is isotopic to  $\beta$  and  $\alpha \cap \beta \neq \emptyset$ , the bigon criterion gives that  $\alpha$  and  $\beta$  form a bigon  $B$ . Assume  $B$  is the innermost bigon. As  $\alpha$  and  $\beta$  are fixed (they are symmetric) by  $\iota$ , we have that  $\iota(B)$  is another innermost bigon in the graph  $\alpha \cup \beta$ . But  $\iota$  reverses the orientation of non-separating closed curves (while preserving orientations of separating closed curves), so since the bigon  $B$  lies to one side of  $\alpha$ ,  $\iota(B)$  must lie on the other side of  $\alpha$ .

It follows that the image of  $B$  in  $S_{0,2g+2}$  is an innermost bigon  $\bar{B}$  between  $\bar{\alpha}$  and  $\bar{\beta}$  (why?). Furthermore, since  $\iota(B) \neq B$ , there are no fixed points of  $\iota$  in  $B$  and hence no marked points of  $S_{0,2g+2}$  in  $\bar{B}$ .

Now, considering the boundary of the bigon  $\bar{B}$ , it can have zero, one or two of its vertices on marked points of  $S_{0,2g+2}$ . In the first two cases, we can modify  $\bar{\alpha}$  by isotopy in order to remove the bigon (why?), reducing the intersection number of  $\bar{\alpha}$  and  $\bar{\beta}$ . In the last case, since  $\bar{B}$  is innermost, we see that  $\bar{\alpha} \cup \bar{\beta}$  is a simple loop bounding a disk (why?), and we can push  $\bar{\alpha}$  onto  $\bar{\beta}$ . Removing bigons inductively,

we see that  $\bar{\alpha}$  is isotopic to  $\bar{\beta}$ , and this isotopy lifts to a symmetric isotopy between  $\alpha$  and  $\beta$ .  $\square$

**Proposition 4.23.** *Let  $g \geq 2$  and let  $\varphi, \psi \in \text{SHomeo}^+(S_g)$ . If  $\varphi$  and  $\psi$  are isotopic, then they are symmetrically isotopic.*

*Proof of the Birman-Hilden theorem.* We have a commutative diagram

$$\begin{array}{ccccc} \text{SHomeo}^+(S_g) & \longrightarrow & \text{Homeo}^+(S_{0,2g+2}, \partial S_{0,2g+2}) & \longrightarrow & \text{Mod}(S_{0,2g+2}) \\ & & \searrow & & \nearrow \\ & & \text{SMod}(S_g) & & \end{array}$$

We wish to find the kernel of the map  $\text{SMod}(S_g) \rightarrow \text{Mod}(S_{0,2g+2})$ . Suppose  $f \in \text{SMod}(S_g)$  is mapped to 0. Since the composition  $\text{SHomeo}^+(S_g) \rightarrow \text{Homeo}^+(S_{0,2g+2}) \rightarrow \text{Mod}(S_{0,2g+2})$  is a surjective homomorphism, we can choose a representative  $\varphi \in \text{SHomeo}^+(S_g)$ . Let  $\bar{\varphi}$  be the image in  $\text{Homeo}^+(S_{0,2g+2})$ . Then  $\bar{\varphi}$  is isotopic to the identity, say  $H: I^2 \rightarrow \text{Homeo}^+(S_{0,2g+2})$ . Then this lifts to a symmetric isotopy  $\tilde{H}: I^2 \rightarrow \text{SHomeo}^+(S_g)$  making the following diagram commute

$$\begin{array}{ccccc} & & I^2 & & \\ & \swarrow \tilde{H} & & \searrow H & \\ \text{SHomeo}^+(S_g) & \longrightarrow & \text{Homeo}^+(S_{0,2g+2}, \partial S_{0,2g+2}) & \longrightarrow & \text{Mod}(S_{0,2g+2}) \\ & & \searrow & & \nearrow \\ & & \text{SMod}(S_g) & & \end{array}$$

Now,  $\tilde{H}$  is an isotopy of  $\varphi$  to either the identity or  $\iota$ , so

$$\text{SMod}(S_g) / \langle [\iota] \rangle \approx \text{Mod}(S_{0,2g+2}).$$

$\square$

*Remark.* We can generalize the proof of the Birman-Hilden theorem a bit to the case of  $S_g^1$  quite simply: the quotient of  $S_g^1$  by the hyperelliptic involution  $\iota: S_g^1 \rightarrow S_g^1$  is a disk with  $2g + 1$  marked points. Since  $\iota: S_g^1 \rightarrow S_g^1$  is not an element of  $\text{Homeo}^+(S_g^1, \partial S_g^1)$ , it does not represent an element of  $\text{SMod}(S_g^1)$ , and so we get

$$\text{SMod}(S_g^1) \approx \text{Mod}(D_{2g+1}) \approx B_{2g+1}.$$

## 5. BRAIDED MONOIDAL CATEGORIES

**5.1. Monoidal categories.** We first introduce the notion of a monoidal category.

**Definition 5.1** (Monoidal category). A monoidal category is a tuple  $V = (V, \otimes, I, a, l, r)$  consisting of a category  $V$ , a functor  $\otimes: V \times V \rightarrow V$  called the tensor product, and object  $I \in V$  called the unit, and natural isomorphisms

$$a: (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$$

$$l: I \otimes - \xrightarrow{\sim} -$$

$$r: - \otimes I \xrightarrow{\sim} -$$

called the associativity, left unit and right unit constraints, respectively. Additionally, we require that for all objects  $A, B, C, D \in V$ , the following two diagrams commute:

$$\begin{array}{ccccc} & & (A \otimes B) \otimes (C \otimes D) & & \\ & \nearrow a & & \searrow a & \\ ((A \otimes B) \otimes C) \otimes D & & & & A \otimes (B \otimes (C \otimes D)) \\ & \searrow a \otimes D & & \nearrow A \otimes a & \\ & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{a} & A \otimes ((B \otimes C) \otimes D) & \end{array}$$

and

$$\begin{array}{ccc} (A \otimes I) \otimes B & \xrightarrow{a} & A \otimes (I \otimes B) \\ & \searrow r \otimes B \quad \swarrow A \otimes l & \\ & A \otimes B & \end{array}$$

The monoidal category is called strict when all the natural isomorphisms are identity morphisms for all objects.

**Definition 5.2** (Braiding). A braiding for a monoidal category  $V$  consists for a natural family of isomorphisms

$$c = c_{A,B}: A \otimes B \xrightarrow{\sim} B \otimes A$$

in  $V$  such that the following diagrams commute

$$\begin{array}{ccccc} & & (B \otimes A) \otimes C & \xrightarrow{a} & B \otimes (A \otimes C) \\ & \nearrow c \otimes C & & \searrow B \otimes c & \\ (A \otimes B) \otimes C & & & & B \otimes (C \otimes A) \\ & \searrow a & & \nearrow a & \\ & A \otimes (B \otimes C) & \xrightarrow{c} & (B \otimes C) \otimes A & \end{array}$$

and

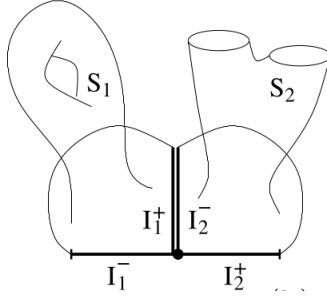
$$\begin{array}{ccccc}
& & A \otimes (C \otimes B) & \xrightarrow{a^{-1}} & (A \otimes C) \otimes B \\
& \nearrow^{A \otimes c} & & & \searrow^{c \otimes B} \\
A \otimes (B \otimes C) & & & & (C \otimes A) \otimes B \\
& \searrow_{a^{-1}} & & & \nearrow_{a^{-1}} \\
& & (A \otimes B) \otimes C & \xrightarrow{c} & C \otimes (A \otimes B)
\end{array}$$

## 5.2. Braided monoidal category of decorated surfaces.

**Definition 5.3** (Decorated surface). A decorated surface is a pair  $(S, I)$  where  $S$  is a compact connected surface with at least one boundary component and  $I: [-1, 1] \hookrightarrow \partial S$  is a parametrised interval in its boundary.

**Definition 5.4** ( $\mathcal{M}_2$ ). Let  $\mathcal{M}_2$  denote the groupoid where the objects are decorated surfaces and morphisms are isotopy classes of diffeomorphisms restricting to the identity on a neighborhood of  $I$ .

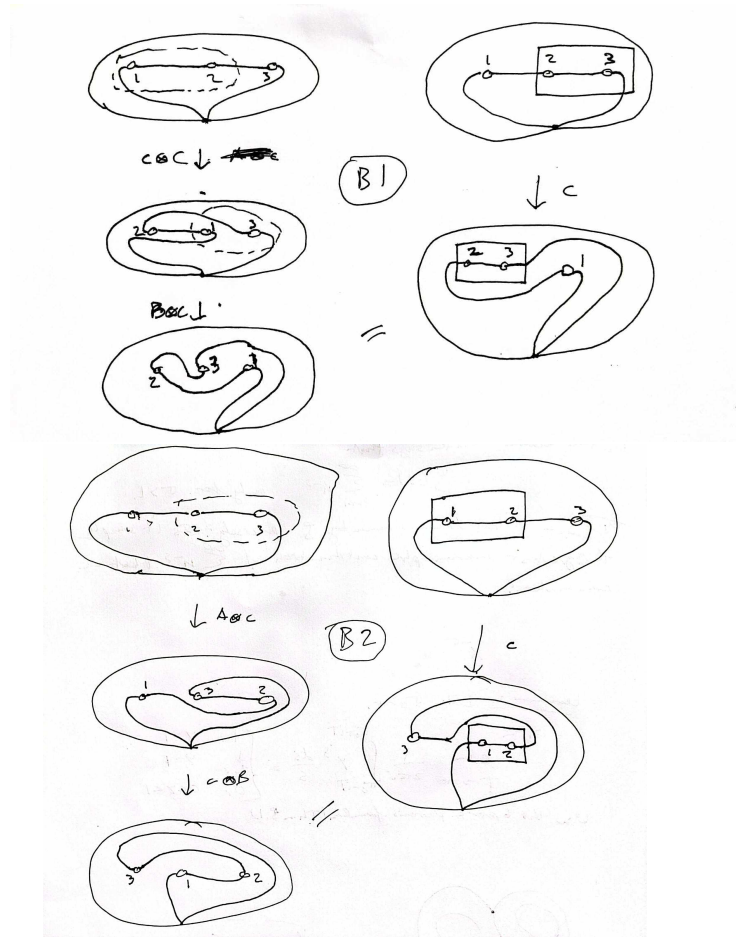
We now construct a braided monoidal structure on  $\mathcal{M}_2$  : given decorated surfaces  $(S_1, I_1)$  and  $(S_2, I_2)$ , define  $(S_1, I_1) \otimes (S_2, I_2) := (S_1 \# S_2, I_1 \# I_2)$  to be the surface obtained by gluing  $S_1$  and  $S_2$  along the right half-interval  $I_1^+ \in \partial S_1$  and the left half-interval  $I_2^- \in \partial S_2$ , defining  $I_1 \# I_2 = I_1^- \cup I_2^+$ .



Furthermore, we define the unit object to be  $I := (D^2, I)$ . In this case, the monoidal structure is clearly strict.

We define a braiding  $c$  on  $(S_1 \# S_2, I_1 \# I_2)$  as the half-Dehn twist which satisfies that  $c: (S_1 \# S_2, I_1 \# I_2) \xrightarrow{\sim} (S_2 \# S_1, I_2 \# I_1)$  is a natural isomorphism because we it has the opposite half-Dehn twist as the inverse (why would this be natural?...)

The B1 and B2 diagrams can be verified pictorially as follows:



## 6. GEOMETRIC REPRESENTATIONS OF THE BRAID GROUP ON NON-ORIENTABLE SURFACES

A connected orientable (respectively nonorientable) surface of genus  $g$  with  $b$  boundary components will be denoted by  $S_{g,b}$  (respectively  $N_{g,b}$ ).

Now, recall that the Möbius band, which are also called crosscaps), are the mapping cylinders on the map  $z \mapsto z^2$  (see figure 6)

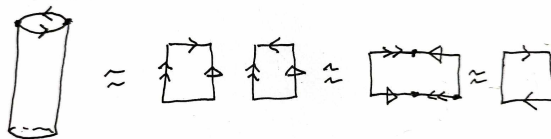


FIGURE 12. Möbius band

In this case, any of the two curves making up the half-circles which get identified can be cut along, rendering a connected surface. Thus gluing on a Möbius strip along the boundary, we obtain a surface of one higher genus.

We can then obtain  $N_{g,b}$  from  $S_{0,g+b}$  by gluing  $g$  Möbius bands along  $g$  distinct boundary components of  $S_{0,g+b}$ .

**Definition 6.1** (Geometric representation). A geometric representation of a group  $G$  is any homomorphism  $G \rightarrow \text{Mod}(S)$  for some surface  $S$ .

We will introduce three types of geometric representations of the braid group: the standard twist representation, the crosscap transposition representation, and lastly, transvection.

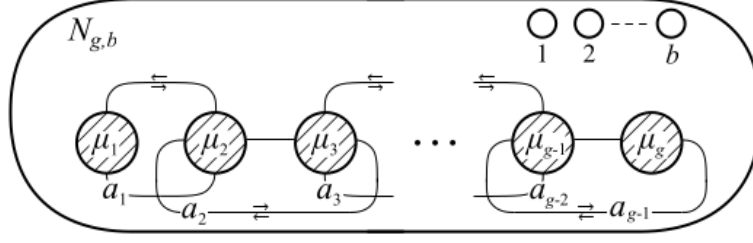
We call a curve two-sided (resp. one-sided) if its regular neighborhood is an annulus (resp a Möbius band). A sequence  $C(a_1, \dots, a_{n-1})$  of two-sided curves in  $S$  is called a chain of nonseparating curves if  $i(a_i, a_{i+1}) = 1$  for  $1 \leq i \leq n-2$  and  $i(a_i, a_j) = 0$  for  $|i - j| > 0$ .

### 6.0.1. The standard twist representation.

**Lemma 6.2.** *If we fix an orientation of a regular neighborhood of the union of the curves  $a_i$ , then  $C$  determines the standard twist representation  $\rho_C: B_n \rightarrow \text{PMod}(S)$  defined by*

$$\rho_C(\sigma_i) = t_{a_i}, \quad i = 1, \dots, n-1,$$

where  $t_{a_i}$  is the right-handed Dehn twist about  $a_i$  with respect to the orientation.



6.0.2. *The crosscap transposition representation.* Let  $N = N_{g,b}$  be nonorientable. A sequence  $C = (a_1, \dots, a_{n-1})$  of separating curves in  $N$  is called a *chain of separating curves* if

- (1)  $a_i$  bounds a one-holed Klein bottle for  $i = 1, \dots, n-1$ ,
- (2)  $i(a_i, a_{i+1}) = 2$  for  $i = 1, \dots, n-2$ ,
- (3)  $i(a_i, a_j) = 0$  for  $|i - j| > 1$ .

Here  $a_i$  bounding a one-holed Klein bottle means that if we collapse  $a_i$  to a point, we obtain a sphere with two crosscaps which is equivalent to the Klein bottle (see figure 6.0.2)

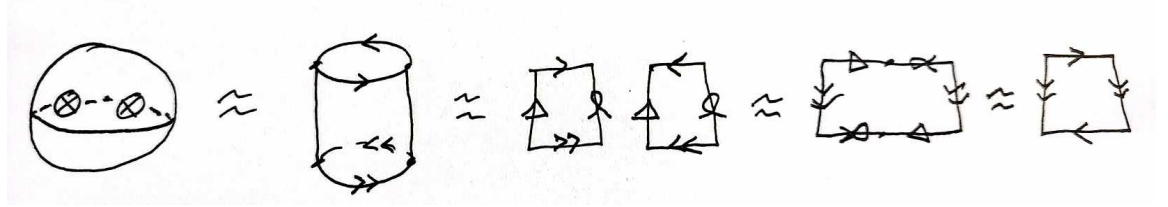


FIGURE 13. Klein-bottle as the sphere with two crosscaps

Let  $K_i$  be the one-holed Klein bottle bounded by  $a_i$ . Then  $K_i \cap K_{i+1}$  will be a Möbius strip for  $i = 1, \dots, n-2$ , and we denote its core curve by  $\mu_{i+1}$ . Let  $\mu_1$  and  $\mu_n$  be the core curves of  $K_1 - K_2$  and  $K_{n-1} - K_n$ , respectively. Fix an orientation of a regular neighborhood of the union of the  $a_i$ . Let  $T_{a_i}$  be the right-handed Dehn twist about  $a_i$  and let  $u_i$  be the *crosscap transposition* supported in  $K_i$ , swapping  $\mu_i$  and  $\mu_{i+1}$  such that  $u_i^2 = T_{a_i}$  (essentially a half-Dehn twist but for crosscaps instead of punctures).

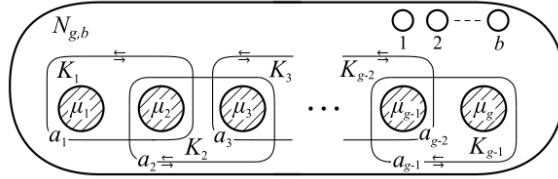


FIGURE 14. A chain of separating curves in  $N_{g,b}$ .



**Lemma 6.3.** *The mapping  $\theta_C: B_n \rightarrow \text{PMod}(N)$  by  $\theta_C(\sigma_i) = u_i$  for  $i = 1, \dots, n-1$ , defines a homomorphisms called *crosscap transposition representation*.*

6.0.3. *Transvection.*

**Definition 6.4** (Transvection). Given a homomorphism  $\rho: B_n \rightarrow \text{PMod}(S)$  and an element  $\tau \in \text{PMod}(S)$  such that  $\tau$  commutes with  $\rho(\sigma_i)$  for  $1 \leq i \leq n-1$ , we define a homomorphism  $\rho^\tau: B_n \rightarrow \text{Mod}(S)$ , called a *transvection* of  $\rho$ , by

$$\rho^\tau(\sigma_i) = \tau \rho(\sigma_i), \quad i = 1, \dots, n-1.$$

A homomorphism  $\rho: B_n \rightarrow \text{PMod}(S)$  is called *cyclic* if  $\rho(B_n)$  is a cyclic group.

### 6.1. The main theorems.

**Theorem 6.5.** *Let  $n \geq 14$  and let  $N = N_{g,b}$  with  $g \leq 2\lfloor \frac{n}{2} \rfloor + 1$  and  $b \geq 0$ . Then any homomorphism  $\rho: B_n \rightarrow \text{PMod}(N)$  is either cyclic, or is a transvection of a standard twist representation, or is a transvection of a crosscap transposition representation.*

**Theorem 6.6.** *Theorem 6.5 still holds when  $\text{PMod}(N)$  is replaced by  $\text{Mod}(N, \partial N)$ .*

Note that  $\text{Mod}(N, \partial N) \not\leq \text{PMod}(N)$  as, for example, the Dehn twist about a boundary curve is non-trivial in  $\text{Mod}(N, \partial N)$ , but becomes trivial in  $\text{PMod}(N)$ .

## 7. THE BURAU REPRESENTATIONS OF THE BRAID GROUP

Let  $\Lambda = \mathbb{Z}[t, t^{-1}]$  and  $n \geq 2$ . Over  $\Lambda$ , consider the  $n \times n$ -matrix

$$U_i = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

So letting  $U = \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$ ,  $U_i$  becomes a block matrix on identity matrices and  $U$ .

**Proposition 7.1.** *Each  $U_i$  is invertible, and the  $U_i$  for  $i \in \{1, \dots, n\}$ , for  $n \geq 2$ , satisfy the braid relations.*

*Proof.* By the Cayley-Hamilton theorem, any  $2 \times 2$ -matrix  $M$  over the ring  $\Lambda$  satisfies  $M^2 - \text{tr}(M)M + \det(M)I_2 = 0$ . For  $M = U$ , we get  $U^2 - (1-t)U - tI_2 = 0$ . Clearly, also  $U_n^2 - (1-t)U_n - tI_n = 0$  for all  $n$ , so we find that  $U_i^2 - (1-t)U_i - tI_n = 0$  for all  $i$ , giving  $U_i(U_i - (1-t)I_n) = tI_n$ , so  $U_i$  is invertible with inverse

$$U_i^{-1} = t^{-1}(U_i - (1-t)I_n) = \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t^{-1} & 1-t^{-1} & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix}.$$

Now, for all  $i, j$  with  $j - i \geq 2$ , we have

$$\begin{aligned} U_i U_j &= \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix} \begin{pmatrix} I_{j-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-j-1} \end{pmatrix} \\ &= \begin{pmatrix} I_{i-1} & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & I_{j-i-2} & \\ & & & 1-t & t \\ & & & 1 & 0 \\ & & & & I_{n-j-3} \end{pmatrix} \\ &= \begin{pmatrix} I_{j-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-j-1} \end{pmatrix} \begin{pmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{pmatrix} \end{aligned}$$

□

## 8. GLOSSARY

**Definition 8.1** (Equivariant maps). Suppose a group  $G$  acts on spaces  $X$  and  $Y$ , and let  $f: X \rightarrow Y$  be a map. Then  $f$  is said to be equivariant if  $f(g \cdot x) = g \cdot f(x)$  for all  $x \in X$  and all  $g \in G$ .

**Definition 8.2** (Closed surface). A *closed surface* is a surface that is compact and without boundary.

**Definition 8.3** (Isotopy). A topological isotopy is a homotopy  $F: X \times I \rightarrow Y$  such that for each  $t_0 \in I$ ,  $F(x, t_0): X \rightarrow Y$  is a topological embedding (homeomorphism onto some subspace of  $Y$ ).

Two embeddings  $f, g: X \rightarrow Y$  are said to be isotopic if there exists an isotopy  $F: X \times I \rightarrow Y$  such that  $F(x, 0) = f(x)$  and  $F(x, 1) = g(x)$ .

**Definition 8.4** (Orientation). A closed  $n$ -manifold  $M$  is called orientable if  $H_n(M; \mathbb{Z}) = \mathbb{Z}$ . The choice of generator  $[M]$  in  $\mathbb{Z}$  is called an orientation, and the generator is called the fundamental class of  $M$ . A manifold together with a choice of orientation is called oriented. A compact  $n$ -manifold  $M$  with boundary is called orientable if  $H_n(M, \partial M; \mathbb{Z}) = \mathbb{Z}$ . The choice of generator  $[M, \partial M]$  in  $\mathbb{Z}$  is called an orientation, and  $[M, \partial M]$  is referred to as the fundamental class of  $M$ .

A smooth manifold  $M$  is orientable if and only if the restriction of its tangent bundle to every smooth curve is trivial.

*Remark.* This makes sense since T. Radó showed that every surface is triangulable and it is clear then that the 2-cycles form a cyclic group. A choice of generator corresponds to choosing an orientation of each 2-simplex in the triangulation (compatibly).

**Definition 8.5** (Inner and outer automorphisms). Let  $G$  be any group and  $\gamma \in G$ . A conjugate automorphism

$$I_\gamma: g \mapsto \gamma g \gamma^{-1}$$

is called an *inner automorphism* of  $G$ . The group of inner automorphisms is denoted by  $\text{Inn}(G)$ . It is isomorphic to  $G/Z(G)$ , and is a normal subgroup of  $\text{Aut}(G)$ . The quotient

$$\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$$

is called the *outer automorphism group* of  $G$ .

## 9. APPENDIX

## 9.1. Fiber Bundles.

## 9.1.1. Bredon.

**Definition 9.1.** Let  $X, B$  and  $F$  be Hausdorff spaces and  $p: X \rightarrow B$  a map. Then  $p$  is called a bundle projection with fiber  $F$ , if each point of  $B$  has a neighborhood  $U$  such that there is a homeomorphism  $\varphi: U \times F \rightarrow p^{-1}(U)$  such that  $p(\varphi(b, y)) = b$  for all  $b \in U$  and  $y \in F$ . That is, on  $p^{-1}(U)$ ,  $p$  corresponds to projection  $U \times F \rightarrow U$ . Such a map  $\varphi$  is called a *trivialization* of the bundle over  $U$ .

**Definition 9.2.** An action of a group  $G$  on a space  $X$  is said to be *effective* if

$$(\forall x \in X: gx = x) \implies g = e.$$

**Definition 9.3.** Let  $K$  be a topological group acting effectively on the Hausdorff space  $F$  as a group of homeomorphisms. Let  $X$  and  $B$  be Hausdorff spaces. By a *fiber bundle* over the base space  $B$  with total space  $X$ , fiber  $F$ , and structure group  $K$ , we mean a bundle projection  $p: X \rightarrow B$  together with a collection  $\Phi$  of trivializations  $\varphi: U \times F \rightarrow p^{-1}(U)$ , of  $p$  over  $U$ , called *charts* over  $U$ , such that

- (1) each point of  $B$  has a neighborhood over which there is a chart in  $\Phi$  ;
- (2) if  $\varphi: U \times F \rightarrow p^{-1}(U)$  is in  $\Phi$  and  $V \subset U$  then the restriction of  $\varphi$  to  $V \times F$  is in  $\Phi$ ;
- (3) if  $\varphi, \psi \in \Phi$  are charts over  $U$ , then there is a map  $\theta: U \rightarrow K$  such that  $\psi \langle u, y \rangle = \varphi \langle u, \theta(u)(y) \rangle$  ;
- (4) the set  $\Phi$  is maximal among collections satisfying (1), (2) and (3).

The bundle is called *smooth* if all these spaces are manifolds and all maps involved are smooth.

**Definition 9.4.** A *vector bundle* is a fiber bundle in which the fiber is a Euclidean space and the structure group is the general linear group if this Euclidean space or some subgroup of that group.

#### 9.1.2. Hatcher.

**Definition 9.5** (Homotopy lifting property). A map  $p: E \rightarrow B$  is said to have the homotopy lifting property with respect to a space  $X$  if, given a homotopy  $g: X \times I \rightarrow B$  and a map  $\tilde{g}_0: X \times \{0\} \rightarrow E$  lifting  $g(x, 0)$ , so  $p\tilde{g}_0(t, 0) = g(t, 0)$ , then there exists a homotopy  $\tilde{g}: X \times I \rightarrow E$  lifting  $g_t$ .

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\tilde{g}_0} & E \\ \downarrow & \nearrow \tilde{g} & \downarrow p \\ X \times I & \xrightarrow{g} & B \end{array}$$

This is a special case of the *lift extension property for a pair*  $(Z, A)$  (See [Munkres])

**Definition 9.6** (Fibration). A fibration is a map  $p: E \rightarrow B$  having the homotopy lifting property with respect to all spaces  $X$ . For example, a projection  $B \times F \rightarrow B$  is a fibration since we can choose lifts of the form  $\tilde{g}(x, t) = (g(x, t), h(x))$  where  $\tilde{g}(x, 0) = (g(x, 0), h(x))$ .

**Definition 9.7** (Homotopy lifting property for a pair  $(X, A)$ ). The map  $p: E \rightarrow B$  is said to have the *homotopy lifting property for a pair*  $(X, A)$  if each homotopy  $f: X \times I \rightarrow B$  lifts to a homotopy  $\tilde{g}: X \times I \rightarrow E$  starting with a given lift  $\tilde{g}_0: X \times \{0\} \rightarrow E$  and extending a given lift  $\tilde{g}: A \times I \rightarrow E$ .

**Theorem 9.8.** Suppose  $p: E \rightarrow B$  has the homotopy lifting property with respect to disks  $D^k$  for all  $k \geq 0$ . Choose basepoints  $b_0 \in B$  and  $x_0 \in F = p^{-1}(b_0)$ . Then the map  $p_*: \pi_n(E, F, x_0) \rightarrow \pi_n(B, b_0)$  is an isomorphism for all  $n \geq 1$ . Hence if  $B$  is path-connected, there is a long exact sequence

$$\dots \rightarrow \pi_n(F, x_0) \rightarrow \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, x_0) \rightarrow \dots \rightarrow \pi_0(E, x_0) \rightarrow 0$$

**Definition 9.9** (Fiber bundle). A fiber bundle structure on a space  $E$ , with fiber  $F$ , consists of a projection map  $p: E \rightarrow B$  such that each point of  $B$  has a neighborhood

$U$  for which there is a homeomorphism  $h: p^{-1}(U) \rightarrow U \times F$  making the following diagram commute

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h} & U \times F \\ & \searrow p \quad \swarrow \text{proj} & \\ & U & \end{array}$$

Thus  $h(p^{-1}(b)) = \{b\} \times F$  by the projection map  $p: E \rightarrow B$ , but to indicate what the fiber is we sometimes write a fiber bundle as  $F \rightarrow E \rightarrow B$ , a "short exact sequence of spaces". The space  $B$  is called the *base space* of the bundle, and  $E$  is the *total space*.

*Remark.* This is just the definition of a covering map without the restriction of  $F$  having the discrete topology.

## 9.2. A couple of results on hyperelliptic involutions.

**Lemma 9.10.** *If  $S$  is a Riemann surface of genus  $g \geq 2$  admitting an involution  $J$  such that  $S/\langle J \rangle$  has genus 0, then  $S$  is a hyperelliptic Riemann surface with equation of the form  $y^2 = (x - a_1) \cdots (x - a_{2g+1})$*

**Proposition 9.11.** *The hyperelliptic involution  $J$  of a hyperelliptic Riemann surface  $S$  is the only automorphism of order 2 such that  $S/\langle J \rangle \approx \mathbb{P}^1$ .*

**Corollary 9.12.** *The hyperelliptic involution  $J$  of a hyperelliptic Riemann surface lies in the center of  $\text{Aut}(S)$ .*

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