B (3.1.ii): For a fixed diagram $F \in C^J$, show that the cone functor $\mathrm{Cone}(-, F)$ is naturally isomorphic to $\mathrm{Hom}(\Delta(-), F)$, the restriction of the hom functor for the category C^J along the constant functor embedding defined in 3.1.1. Do you find this result surprising? Why or why not?

Solution: We want to define a natural isomorphism $\alpha\colon \mathrm{Cone}(-, F) \implies \mathrm{Hom}(\Delta(-), F)$.
When we look at the definition of a cone over a diagram F with apex c, we find it is a natural transformation $c \implies F$ which is in the set $\mathrm{Hom}(\Delta(c), F)$, so we suspect that the natural isomorphism will be quite obvious when we write things out.

Suppose we have a cone $\lambda = (\lambda_i\colon c \to Fi)_{i\in J} \in \mathrm{Cone}(c, F)$. This is locally, for a morphism $f\colon i \to j$ in J, is of the form

$$\begin{tikzcd} & c \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ Fi \arrow[rr, "Ff"] & & Fj \end{tikzcd}$$

Similarly, a natural transformation $\beta\colon \Delta(c) \implies F$ is a collection of morphisms $\beta_i\colon c \to Fi$ such that for any $f\colon i \to j$ in J, the following diagram commutes

$$\begin{tikzcd} c \arrow[r, "\beta_i"] \arrow[d, "\mathbb{1}_{\Delta c}"] & F(i) \arrow[d, "Ff"] \\ c \arrow[r, "\beta_j"] & F(j) \end{tikzcd}$$

We note that the data of the cone $\mathrm{Cone}(c, F)$ is a collection of morphisms $(\lambda_i\colon c \to Fi)_{i\in J}$ which by definition define a natural transformation $\Delta(c) \implies F$. Now, since J is small by definition of a diagram, we have that the collection J has only a set's worth of arrow. Since any object has an identity morphism, we thus deduce that J also only has a set's worth of objects, so we can conclude that $(\lambda_i\colon c \to F_i)_{i\in J}$ is of set size. Therefore, we can consider the collection of morphisms as a set $\{\lambda_i\colon c \to F_i\}_{i\in J} \in \mathrm{Set}$ which is an element of $\mathrm{Hom}(\Delta(c), F)$.
Now, define a collection of maps

$$\begin{aligned} \alpha_c\colon \mathrm{Cone}(c, F) &\to \mathrm{Hom}(\Delta(c), F) \\ \lambda = (\lambda_i\colon c \to Fi)_{i\in J} &\mapsto \{\lambda_i\colon c \to F_i : i \in J\} \end{aligned}$$

We claim that α is a natural isomorphism.

Consider a cone $\lambda = (\lambda_i\colon c \to F_i)_{i\in J} \in \mathrm{Cone}(c, F)$. Then naturality asserts that for a morphism $f\colon d \to c$ in C, we have $\mathrm{Hom}(\Delta(-), F)(f) \circ \alpha_c(\lambda) = \alpha_d \circ \mathrm{Cone}(-, F)(f)(\lambda)$.
Now

$$\begin{aligned} \mathrm{Hom}(\Delta(-), F)(f) \circ \alpha_c(\lambda) &= \mathrm{Hom}(\Delta(-), F)(f)\left(\{\lambda_i\colon c \to Fi\}_{i\in J}\right) \\ &= \{\lambda_i \circ f\colon d \to Fi\}_{i\in J}. \end{aligned}$$

And

$$\begin{aligned} \alpha_d \circ \mathrm{Cone}(-, F)(f)(\lambda) &= \alpha_d\,(\lambda_i \circ f\colon d \to F_i)_{i\in J} \\ &= \{\lambda_i \circ f\colon d \to F_i\}_{i\in J}. \end{aligned}$$

This gives naturality.

To see that α is an isomorphism, we must check that α_c is a bijection. We note that for any set $\{\lambda_i\colon c \to Fi : i \in J\} \in \mathrm{Hom}(\Delta(c), F)$, the collection $(\lambda_i\colon c \to Fi)_{i\in J}$ defines a cone over F with summit c since for any morphism $f\colon j \to k$ in J, the square

$$\begin{tikzcd} \Delta(c)(j) = c \arrow[r, "\lambda_j"] \arrow[d, "\Delta(c)(f)=\mathbb{1}_c"] & F(j) \arrow[d, "Ff"] \\ \Delta(c)(k) = c \arrow[r, "\lambda_k"] & F(k) \end{tikzcd}$$

commutes, and hence the following triangle commutes:

$$\begin{tikzcd} & c \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ Fi \arrow[rr, "Ff"] & & Fj \end{tikzcd}$$

which by the comment on page 74 means that the family of morphisms $(\lambda_i \colon c \to Fi)_{i \in J}$ defines a cone over F with summit c. So define a map $\beta_c \colon \mathrm{Hom}(\Delta(c), F) \to \mathrm{Cone}(c, F)$ which maps $\{\lambda_i \colon c \to Fi\}_{i \in J}$ to the cone λ. Then this is a two-sided inverse to α_c, so α_c is a bijection. Hence each α_c is invertible. So α is a natural isomorphism.

I don't find this result particularly surprising since cones over F with summit c is indeed just a natural transformation $c \implies F$, i.e. elements of $\mathrm{Mor}(c, F)$. The only thing to note is that this seems to boil down to the fact that we are dealing with a diagram which is defined on a small category, thus making our collection of morphisms into a set; however, this is also clear from the definition of a cone, so it's still not particularly surprising.

2.4.x: Fixing two objects A, B in a locally small category C, we define a functor

$$C(A, -) \times C(B, -) \colon C \to \mathrm{Set}$$

that carries an object X to the set $C(A, X) \times C(B, X)$ whose elements are pairs of maps $a \colon A \to X$ and $b \colon B \to X$ in C. What would it mean for this functor to be representable?

Solution: We will write $f \times g$ for (f, g).
Suppose $C(A, -) \times C(B, -)$ is representable by an element $c \in C$, so there is a natural isomorphism

$$C(c, -) \overset{\alpha}{\cong} C(A, -) \times C(B, -).$$

This means that for any morphism $f \colon X \to Y$ between two objects $X, Y \in C$, the following square commutes

$$\begin{tikzcd} C(c,X) \arrow[r, "\alpha_X"] \arrow[d, "f_*"] & C(A,X) \times C(B,X) \arrow[d, "f_* \times f_*"] \\ C(c,Y) \arrow[r, "\alpha_Y"] & C(A,Y) \times C(B,Y) \end{tikzcd}$$

Now, since C is locally small, Yoneda gives that any such natural isomorphism α is in bijection with some element in $C(A, c) \times C(B, c)$, namely $\alpha_c(\mathbb{1}_c)$. Thus there exists some morphisms $g \in C(A, c)$ and $h \in C(B, c)$ that uniquely represent and determine α, so $\alpha_c(\mathbb{1}_c) = g \times h$.
Now, suppose we have morphisms $p \colon A \to Y$ and $q \colon B \to Y$. Since α was an isomorphism, we can let $k = \alpha_Y^{-1}(p \times q) \in C(c, Y)$. Then letting $X = c$ we get that

$$\begin{tikzcd} C(c,c) \arrow[r, "\alpha_c"] \arrow[d, "k_*"] & C(A,c) \times C(B,c) \arrow[d, "k_* \times k_*"] \\ C(c,Y) \arrow[r, "\alpha_Y"] & C(A,Y) \times C(B,Y) \end{tikzcd}$$

commutes, so

$$k \circ g \times k \circ h = \alpha_Y(k) = p \times q$$

We can depict this as

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "p"'] & c \arrow[d, "\exists! k"] & B \arrow[l, "h"'] \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

commuting. Here k is unique since if k' also makes the above commute, then

$$k' = k'_* \mathbb{1}_c = \alpha_Y^{-1}(k_* \times k_*)\, \alpha_c \mathbb{1}_c = \alpha_Y^{-1}(k_* \times k_*)(g \times h) = \alpha_Y^{-1}(k \circ g \times k \circ h) = \alpha_Y^{-1}(\alpha_Y(k)) = k$$

This universal property precisely determines c together with the unique maps $g \colon A \to c$ and $h \colon B \to c$ to be the coproduct of A and B, i.e., $c = A \sqcup B$, where the underlying diagram is a discrete category of two elements.