Definition 0.1 (Tangent)**.** We define the tangent function as $\tan\colon \mathbb{C} - \left\{\frac{\pi}{2} + \pi\mathbb{Z}\right\} \to \mathbb{C}$ by

$$\tan z = \frac{\sin z}{\cos z}$$

The function $\tan z$ is holomorphic on its whole domain $\mathbb{C} - \left\{\frac{\pi}{2} + \pi\mathbb{Z}\right\}$ by theorem 1.18 in. [Berg]

Problem 0.2 (Problem 3.5 in [Berg])**.** Use Goursat's lemma to show the following: Let G be a star-shaped open set around z_0. For every $z \in G$, let $[z_0, z]$ be the straight line from z_0 to z parametrized by $\gamma(t) = (1-t)\, z_0 + tz$. Show that if f is holomorphic on G, then $F\colon G \to \mathbb{C}$ defined by

$$F(z) = \int_{[z_0,z]} f$$

is an antiderivative to f which satisfies $F(z_0) = 0$.

Problem 0.3 (Inverse of tangent, Arctan, problem 3.8 in [Berg])**.** Consider the path-connected open set

$$G = \mathbb{C} - \{iy \mid y \in \mathbb{R}, |y| \geq 1\}.$$

Define the function $\operatorname{Arctan}\colon G \to \mathbb{C}$ by

$$\operatorname{Arctan} z = \int_0^1 \frac{z dt}{1 + t^2 z^2}.$$

Show that Arctan is holomorphic on G with derivative

$$\frac{d}{dz} \operatorname{Arctan} z = \frac{1}{1+z^2}.$$

Show that $\operatorname{Arctan}|_{\mathbb{R}}$ is the inverse function to $\tan\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ by showing that Arctan maps G bijectively onto the strip

$$\left\{z = x + iy \mid -\frac{\pi}{2} < x < \frac{\pi}{2}\right\},$$

and showing that it is inverse to tan.
Now show that

$$\operatorname{Arctan} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \dots \quad \text{for} \quad |z| < 1.$$

Remark. Since holomorphic functions are analytic, $\operatorname{Arctan}|_{\mathbb{R}}$ is in particular smooth, and since $\tan\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is smooth, we see that both are diffeomorphisms.

References

[Berg] Christian Berg. Kompleks Funktionsteori.