ASSIGNMENT 3

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Problem 0.1 (1 (Quotient manifolds))**.** Let M be a smooth manifold, and let $\tau\colon M \to M$ be a fixed-point free involution on M, i.e., a map $\tau\colon M \to M$ such that $\tau \circ \tau = \mathrm{id}$ and $\tau(x) \neq x$ for all $x \in M$.

(1) Show that τ is a diffeomorphism, and that the quotient M/τ is a topological manifold.
(2) Show that M/τ admits a unique smooth structure so that the quotient map $q\colon M \to M/\tau$ is a local diffeomorphism.
(3) Give an example of a fixed-point free involution on a smooth manifold, and describe its quotient.

Proof. (1) We assume τ is meant to be smooth here. Then we immediately find that τ is a diffeomorphism since τ has itself as an inverse as $\tau \circ \tau = \mathrm{id}$. To see that M/τ is a topological manifold, we can simply using Hausdorffness of M find τ-invariant neighborhoods U and V of $\tau^{-1}(\overline{x})$ and $\tau^{-1}(\overline{y})$ for arbitrary distinct points $\overline{x}, \overline{y} \in M/\tau$. Since these open neighborhoods are saturated with respect to τ, their images form open disjoint neighborhoods of $\overline{x}$ and $\overline{y}$, so M/τ is Hausdorff. Second countability follows if we can show that τ is an open map. But for an open subset U in M and a point $x \in U$, we can find a neighborhood U_x contained in U around x such that $\tau(U_x) \cap U_x = \varnothing$ simply by taking two disjoint neighborhoods of x and $\tau(x)$ (the one for x contained in U) and intersecting the image under τ of the neighborhood of x with the neighborhood of τ and then transferring it back to x also. Thus the image of the neighborhood of x is open in M/τ. But this shows that an arbitrary open set is the union of sets whose image is open under τ, so τ is an open map. Lastly, M/τ can be seen to be locally Euclidean as follows. Restricting τ to an open set on which it is injective gives an embedding, so choosing a point $\overline{x} \in M/\tau$ and a point $x \in \tau^{-1}(\overline{x})$ and a neighborhood as constructed above such that τ is indeed injective on this neighborhoods, we can intersect this neighborhood with a chart around x and compose τ^{-1} with the chart to get a chart around $\overline{x}$.

(2) We take the same charts as we constructed above. We need to check smoothness. Suppose $\left(U, \varphi \circ \tau^{-1}\right)$ and $\left(V, \psi \circ \tau^{-1}\right)$ are charts in M/τ with $U \cap V \neq \varnothing$. Then we get the transition map to be

$$\psi \circ \tau^{-1} \circ \left(\varphi \circ \tau^{-1}\right)^{-1} = \psi \circ \tau^{-1} \circ \tau \circ \varphi^{-1} = \psi \circ \varphi^{-1}$$

on some open subset of $\mathbb{R}^n$ which is assumed to be smooth by assumption. Hence this atlas indeed induces a smooth structure. In this structure, choose a chart (U, φ) about $x \in M$ such that $q|_U$ is a homeomorphism onto its image where $q\colon M \to M/\tau$ is the quotient map. Then q on this open set has a coordinate representation given by

$$\left(\varphi \circ q^{-1}\right) \circ q \circ \varphi^{-1} = \mathrm{id}$$

which is a diffeomorphism on any subset of $\mathbb{R}^n$, so q is a diffeomorphism on U, and as x was arbitrary, q is a local diffeomorphism.

(3) Here is a nice one: let $\iota\colon S_{g,1} \to S_{g,1}$ be the hyperelliptic involution of the surface with genus g and one boundary component. Then $S_{g,1}/\iota \cong D^2$.

But for the sake of the problem, we could take S^n with the antipodal action $\mathbb{Z}/2$ on S^n giving $\mathbb{RP}^n$ as the quotient. It is a fixed-point free involution, and we have checked that S^n is a smooth manifold and that its quotient $\mathbb{RP}^n$ is also smooth.

□

Problem 0.2 (2 (Covering spaces and smooth liftings))**.** Let M and N be smooth manifolds, and $f, g\colon M \to N$ two smooth maps. We say f and g are smoothly homotopic if there exists a smooth map $H\colon M \times \mathbb{R} \to N$ such that $H(x,t) = f(x)$ for all $t \in (-\infty, 0]$ and $H(x,t) = g(x)$ for all $t \in [1, \infty)$.

(1) If M is a smooth, compact connected manifold, and $p\colon M' \to M$ is a covering map of topological spaces, show that M' is a topological manifold which admits a unique smooth structure so that $p\colon M' \to M$ is a local diffeomorphism.
(2) Fix M a smooth, compact connected manifold and $p\colon M' \to M$ a covering. Let $f\colon N \to M$ be a smooth map, where we assume N is a connected smooth manifold. Suppose we have points $x \in N$ and $y \in p^{-1}(f(x))$ such that the induced maps
$$f_*\colon \pi_1(N, x) \to \pi_1(M, f(x))$$
and
$$p_*\colon \pi_1(M', y) \to \pi_1(M, f(x))$$
is such that $\operatorname{im} f_* \subset \operatorname{im} p_*$. Show that there exists a unique smooth map $f'\colon N \to M'$ such that $f'(x) = y$ and $p \circ f' = f$.
(3) Show that any smooth map $S^2 \to S^1 \times S^1$ admits a smooth nullhomotopy.

Proof. (1) **I will add the assumption that M' is connected as otherwise second-countability can fail**

Suppose $\dim M = m$. First, we show that M' is a topological manifold.
Locally Euclidean: Let $x \in M'$. Then there exists a chart (U, φ) around $p(x)$ in M which we can assume to be connected. Now, since p is a covering map, $p(x)$ has an evenly covered neighborhood V, so in particular, letting W be the component of $p^{-1}(V)$ which contains p, $p|_{W \cap p^{-1}(U)}\colon W \cap p^{-1}(U) \to U \cap V$ is a homeomorphism of open sets. Composing this with the chart homeomorphism, we get a homeomorphism of $W \cap p^{-1}(U)$ to an open subset of $\mathbb{R}^m$.

Hausdorffness: Let $x, y \in M'$. There are two cases to consider: $p(x) = p(y)$ and $p(x) \neq p(y)$. Suppose first that $p(x) \neq p(y)$. Then take two disjoint neighborhoods of $p(x)$ and $p(y)$. Then taking preimages, we get two disjoint open sets in M' containing x and y, respectively.

Second-countability: Since M' is connected and locally-Euclidean hence locally path-connected, M' is path-connected. By theorem 2.3.9 in AlgTop1, the monodromy action $\pi_1(M)$ on any fiber is transitive, so the fiber of the covering has cardinality $\leq |\pi_1(M)|$, and since the fundamental group of a manifold is countable, we obtain that the covering space has countably many sheets. Since M is second-countable, it has a countable basis for its topology. Now pulling back each of these open sets and taking components, we obtain a countable union of a countable collection of open sets which is thus countable. Lastly, this collection is a basis for M' since for any point $x \in M'$ and any open neighborhood U of x, we can find some local neighborhood V of x homeomorphic to an open neighborhood of $p(x)$ downstairs. Taking a basis element W contained in $p\,(U \cap V)$ containing $p(x)$, $p^{-1}(W)$ will be an open neighborhood of x contained in $U \cap V \subset U$.

Smooth structure Now we must show that M' admits a unique smooth structure such that $p\colon M' \to M$ is a local diffeomorphism. For this, it suffices to show that $p\colon M' \to M$ forces M' to have a smooth atlas compatible with its smooth structure. In particular, suppose $x \in M'$ and choose some local neighborhood U of x using that p is a covering map such that p is a homeomorphism of U onto its image $p(U) \subset M$. Now take a smooth chart (V, φ) in M around $p(x)$. We may assume that $V \subset \varphi(U)$ as otherwise we can just intersect V with $\varphi(U)$ and take φ to be the restriction onto this subspace. Then we let the composite $U \cap p^{-1}(V) \xrightarrow{p} V \xrightarrow{\varphi} \varphi(V)$ be a smooth chart for x in M'. To see that these are compatible, suppose $(V, \varphi \circ p)$ and $(W, \psi \circ p)$ are two charts with $V \cap W \neq \varnothing$. Then recall that $p\colon V \cap W \to p\,(V \cap W)$ is a homeomorphism, wo $\varphi \circ \psi^{-1} = \varphi \circ p \circ p^{-1} \circ \psi^{-1} = (\varphi \circ p) \circ (\psi \circ p)^{-1} : \psi \circ p\,(V \cap W) \to \varphi \circ p\,(V \cap W)$ is smooth by assumption of φ and ψ being smooth chart maps for M. Thus this collection gives a smooth atlas for M' hence a unique smooth structure by taking the maximal compatible smooth atlas. Furthermore, smooth charts are diffeomorphisms onto their images, so for a chart $(V, \varphi \circ p)$, we have that since $\varphi \circ p$ is a diffeomorphism on V and φ is a diffeomorphism on $p(V)$, the composite $V \xrightarrow{\varphi \circ p} \varphi \circ p(V) \xrightarrow{\varphi^{-1}} p(V)$ is a diffeomorphism on V. Hence p is a diffeomorphism on V. As these charts for an atlas, we conclude that p is a local diffeomorphism in this smooth structure on M'.

To see that this structure is the unique one making p a local diffeomorphism, suppose $\mathcal{A} = \{(W_\alpha, \psi_\alpha)\}$ is another smooth structure on M' making p a local diffeomorphism. Let $(V, \varphi \circ p)$ be a smooth chart from the smooth structure constructed previously.
Without loss of generality, let W_α be a smooth chart such that $p\colon W_\alpha \to p\,(W_\alpha)$ is a diffeomorphism. Then $\psi_\alpha \circ p^{-1}\colon p\,(W_\alpha) \to \mathbb{R}^m$ is a diffeomorphism of the open set $p\,(W_\alpha) \subset M$ onto an open subset of $\mathbb{R}^m$. Suppose $W_\alpha \cap V \neq \varnothing$ where $(V, \varphi \circ p)$ is a smooth chart from the smooth structure constructed before. Then $(\varphi \circ p) \circ p^{-1}\colon p\,(W_\alpha \cap V) \to \mathbb{R}^m$ is by construction a diffeomorphism as φ is a smooth chart map. Hence these two charts on M are compatible, so $\varphi \circ p \circ \psi_\alpha^{-1} = (\varphi \circ p) \circ p^{-1} \circ p \circ \psi_\alpha^{-1}\colon \psi_\alpha \circ p^{-1}\,(W_\alpha \cap V) \to (\varphi \circ p) \circ p^{-1}\,(W_\alpha \cap V)$ is a diffeomorphism, thus making the two charts compatible. Hence the two structures coincide.

□

(2) Since N is a connected manifold, it is locally path-connected hence path-connected. Theorem 2.7.2 now gives that because $\operatorname{im} f_* \subset \operatorname{im} p_*$, a unique topological

lift $f' \colon N \to M'$ exists making the following diagram commute

$$\begin{tikzcd} & (M', y) \arrow[d, "p"] \\ (N, x) \arrow[ur, "\exists! f'"] \arrow[r, "f"'] & (M, f(x)). \end{tikzcd}$$

Hence we must show that this map is smooth when we equip M' with the unique smooth structure from the previous exercise making p a local diffeomorphism.
To see this, take a chart $(V, \varphi \circ p)$ such that p is a diffeomorphism of V onto its image. Take also a chart (U, ψ) in N. Then the coordinate representation of f' becomes $\varphi \circ p \circ f' \circ \psi^{-1} = \varphi \circ f \circ \psi^{-1}$ which is a coordinate representation of f. Since f is assumed to be smooth, this coordinate representation is smooth, hence $p \circ f'$ is smooth since the charts chosen in the atlases were arbitrary.

(3) We have that $S^1 \times S^1 \cong T^2$ is the torus. Now, this is a manifold being the product of two manifolds and it is compact being the finite product of two compact spaces. Furthermore, it is connected being the product of two connected spaces. We can apply the first two subproblems to the usual topological covering space $p \colon \mathbb{R}^2 \to T^2$ which obtains T^2 as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ under the action of $\mathbb{Z}^2$ on the plane by translations. Since S^2 is simply connected, we trivially have that for any map $f \colon S^2 \to S^1 \times S^1$, $\operatorname{im} f_* = \{0\} \subset \operatorname{im} p_* \subset \pi_1(M, f(x))$. Thus a smooth lift $f' \colon S^2 \to S^1 \times S^1$ exists such that

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ S^2 \arrow[ur, "f'"] \arrow[r, "f"'] & S^1 \times S^1 \end{tikzcd}$$

commutes. But $\mathbb{R}^2$ is contractible, and this contraction can be done smoothly: namely, let $\varphi \colon \mathbb{R}^2 \times I \to \mathbb{R}^2$ be given by $\varphi(x,t) = x(1-t)$. This is clearly smooth and thus a smooth homotopy from id to c_0, the constant map at 0. This, in return, gives us a smooth homotopy $H(x,t) = p \circ \varphi(f'(x), t)$ from $p \circ f' = f$ at $t = 0$ to $c_{p(0)}$ at $t = 1$.

□

Problem 0.3 (3 (Mapping class groups of some manifolds))**.** For M a smooth manifold, we define the *mapping class group* of M by

$$\pi_0 \mathrm{Diff}(M) := \mathrm{Diff}(M)/\sim$$

as the group of self-diffeomorphisms up to isotopy, where an isotopy of two diffeomorphisms f, g is a map $H \colon M \times I \to M$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ and $H(-,t)$ is a self-diffeomorphism of M for all $t \in I$.

1. Show that $\pi_0 \mathrm{Diff}(\mathbb{R}) \cong \mathbb{Z}/2$.
2. Compute $\pi_0 \mathrm{Diff}\left(S^1\right)$.

Proof. (1) I will start by showing that any two orientation-preserving diffeomorphisms of $\mathbb{R}$ are isotopic. For this, it suffices to show that any orientation-preserving diffeomorphism is isotopic to the identity. Since $\mathbb{R} \cong (-1,1)$, any diffeomorphism f of $\mathbb{R}$ can be considered as a diffeomorphism of $(-1,1)$. Now, we claim that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. As f is orientation-preserving, we

have $\frac{df}{dx}(0) > 0$. In particular, there exists a neighborhood around 0 in which $\frac{df}{dx}$ is positive. By the mean value theorem, on this open neighborhood, f is strictly increasing. Since we can successively cover $\mathbb{R}$ by overlapping intervals where the orientations must agree since we choose it to be a continuous section of $\Lambda^1 T\mathbb{R}$, we must have that f is strictly increasing everywhere. Hence $\lim_{x\to\pm\infty} f(x) = \pm\infty$. Now the map $\varphi\colon \mathbb{R}\times I \to \mathbb{R}$ given by $\varphi(x,t) = x(1-t) + f(x)t$ is in fact a diffeomorphism at each time t since f is strictly increasing.
Now to see that $\pi_0\mathrm{Diff}\,(\mathbb{R}) \cong \mathbb{Z}/2$, if f and g are orientation reversing, then $f^{-1}g$ is orientation-preserving, so $f^{-1}g \simeq \mathrm{id}$ implies $g \simeq f$.

(2)
Let $p\colon \mathbb{R} \to S^1$ be the usual parametrization $t \mapsto e^{2\pi i t}$. Let $f\colon S^1 \to S^1$ be a diffeomorphism. By composing f with the isotopy $H(x,t) = xe^{-i\arg f((1,0))t}$, we may assume that f fixes $(1,0)$. Then by problem 2.2 above, there exists a unique smooth path $\tilde{f}\colon I \to \mathbb{R}$ sending 0 to 0 such that

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[d, "p"] \\
I \arrow[urr, "\tilde{f}"] \arrow[r, "p|_I"'] & S^1 \arrow[r, "f"'] & S^1
\end{tikzcd}$$

commutes. But since f is a diffeomorphism, it induces an isomorphism on fundamental groups, so f sends the generating circle $1 \in \mathbb{Z} \cong \pi_1(S^1)$ to either 1 or -1 (the two generators for $\mathbb{Z}$). But then by uniqueness of endpoints of lifts, $\tilde{f}(1) \in \{1,-1\}$. Now suppose firstly that $\tilde{f}(1) = 1$. Since $\tilde{f}$ is smooth and $f = p \circ \tilde{f}$ on I is a diffeomorphism on the interior of I, it must be a strictly increasing function, and since f is smooth, its limits in S^1 about $(1,0)$ agree from both sides, so we conclude that $\lim_{t\to 0+} \frac{d^n}{dt^n}\tilde{f}(t) = \lim_{t\to 1-} \frac{d^n}{dt^n}\tilde{f}(t)$ for all n. Now let $\varphi\colon \mathbb{R}\times I \to \mathbb{R}$ be the isotopy which is the identity on $\mathbb{R} - (0,1)$ and on $(0,1)$, it is given by

$$\varphi(x,t) = xt + f(x)(1-t)$$

Then clearly φ is smooth and $\varphi(x,1) = x$ and $\varphi(x,0) = f(x)$. Furthermore, φ is a diffeomorphism of $(0,1)$ at each time t and of $\mathbb{R} - (0,1)$. We must check that the limits of the derivatives of φ coincide for the two endpoints of the interval at each time t. Indeed,

$$\lim_{x\to 0+} \frac{d}{dx}\varphi(x,t) = \lim_{x\to 0+} t + (1-t)f'(x) = \lim_{x\to 1-} t + (1-t)f'(x) = \lim_{x\to 1-} \frac{d}{dx}\varphi(x,t)$$

since the limits for $f'(x)$ agree. Higher limits can be checked likewise. Hence $\varphi(-,t)$ is a diffeomorphism for all t. The composite $p \circ \varphi(\tilde{f}(x), t)$ gives an isotopy from f to the identity. For the case where $\tilde{f}(1) = -1$, simply note that the reflection map also induces this isomorphism, so if g is the reflection map, then $\widetilde{f \circ g^{-1}}(1) = 1$, so $f \circ g^{-1}$ is isotopic to the identity, hence f is isotopic to g.

$\square$

Problem 0.4 (4 (Local properties of homeomorphisms of $\mathbb{R}^d$))**.** Let $d \in \mathbb{N}$ and denote by D^d the unit disc in $\mathbb{R}^d$. Show that if $f, g \in \mathrm{Homeo}\left(\mathbb{R}^d\right)$ are two homeomorphisms such that there exists an open subset $U \subset \mathbb{R}^d$ where $f|_U = g|_U$, then f and g are isotopic.

Proof. Since isotopic forms an equivalence relation, it suffices to show the situation for $g = \mathrm{id}$. Now, a homeomorphism f of $\mathbb{R}^d$ extends to a homeomorphism of its one-point compactification S^d. Now remove an interior disc $D^2 \hookrightarrow S^d$ contained in the open subset $U \subset S^d$ on which f is the identity. The resulting space is a closed disc, and by assumption, f is the identity on the boundary. Using the Alexander trick for the punctured disc (the puncture being the point at infinity), we find an isotopy of f on this disc to the identity. Now glue back the disc we removed. Since the isotopy fixes the boundary pointwise at all times, the isotopy extends to an isotopy of S^d. Now transfer this isotopy back to $\mathbb{R}^d$ using the stereographic projection. This gives an isotopy of f with the identity on $\mathbb{R}^d$. $\square$