Definition 1.1 (Fiber Bundle)**.** Let K be a topological group acting on a Hausdorff space F as a group of homeomorphisms. Let X and B be Hausdorff spaces. By a *fiber bundle* over a base space B with total space X, fiber F and structure group K, we mean a bundle map $p\colon X \to B$ together with a maximal chart atlas Φ over B. Explicitly, Φ is a collection of trivializations $\varphi\colon U \times F \to p^{-1}(U)$ such that

(1) each point of B has a neighborhood over which there is a chart in Φ
(2) if $\varphi\colon U \times F \to p^{-1}(U)$ is in Φ and $V \subset U$, then the restriction $\varphi|_{V\times F}$ is also in Φ.
(3) If $\varphi, \psi \in \Phi$ are charts over U then there exists a map $\theta\colon U \to K$ such that $\psi\,(u, y) = \varphi\,(u, \theta(u)(y))$
(4) the set Φ is maximal among the collections satisfying the (1),(2) and (3)

The fiber bundle is called smooth if all the spaces are smooth manifolds and all maps involved are smooth.

Definition 1.2 (Manifold bundle)**.** Let M be a smooth manifold. A manifold bundle over M with structure group G is a fiber bundle $W \to E \to M$ with structure group G such that E is a manifold and $E \to M$ is continuous.
We say a manifold bundle over M is a smooth manifold bundle if it is a smooth fiber bundle as well as a manifold bundle and G acts by diffeomorphisms on M.

Problem 1.3 (Manifold bundles over S^1)**.** We fix a smooth manifold M. The aim of this exercise is to study smooth manifold bundles over S^1 with fiber M.

(1) Let $f \in \operatorname{Diff}(M)$, and consider the mapping torus
$$T(f) := (M \times [0,1]) \,/ \sim$$
where $\sim$ identifies $(x, 0)$ with $(f(x), 1)$ for all $x \in M$. Show that the projection map to the second factor yields a smooth manifold bundle
$$M \to T(f) \to S^1.$$
(2) Show that if f and g are isotopic diffeomorphisms, the bundles $T(f) \to S^1$ and $T(g) \to S^1$ are isomorphic bundles.
(3) Show that the map
$$\pi_0 \operatorname{Diff}(M) \to \operatorname{Bun}_M \left(S^1\right)$$
by
$$[f] \mapsto [T(f)]$$
from the mapping class group of M to the set of isomorphism classes of M-manifold bundles over S^1 is bijective.

Problem 1.4 (2)**.** Show that the following spaces admit the structure of smooth manifolds.

(1) $O(n)$, the set of orthogonal matrices of degree $n \times n$, topologized as a subspace of $\mathbb{R}^{n^2}$.
(2) $SO(n)$, the set of orthogonal matrices of degree $n \times n$ with determinant 1.
(3) $\mathrm{SL}_n(\mathbb{R})$, the set of $(n \times n)$-matrices with determinant 1.

Solution. (1) The orthogonal group is the zero set $\mathbb{V}(I)$ of the ideal $I = (\{f_{ij}\})$ where
$$f_{i,j} = \sum_{k=1}^{n} x_{ki}x_{kj} \quad \text{for } i \neq j \quad \text{and} \quad f_{ii} = \sum_{k=1}^{n} x_{ki}^2 - 1$$

So defining a function $\varphi\colon \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ by $\varphi\left((x_{ij})\right) = ((f_{ij}))$, then since (f_{ij}) is symmetric, we may modify this map so that $\varphi\left(x_{ij}\right) = \left(\left(f_{ij}\right)_{i \geq j}\right)$ so $\varphi\colon \mathbb{R}^{n^2} \to \mathbb{R}^{\frac{n(n+1)}{2}}$.

We can also write this map as $\varphi(A) = A^t A - I$. Then we find that

$$\varphi'(A) = \frac{d}{dt}|_{t=0}\varphi(A+tX) = \frac{d}{dt}|_{t=0}\left(A+tX\right)\left(A+tX\right)^t - I = \frac{d}{dt}|_{t=0} AX^t t + A^t X t = AX^t + XA^t$$

Now if $A \in \varphi^{-1}(0)$ and $B \in \mathbb{R}^{\frac{n(n+1)}{2}}$ represents a symmetric matrix, then

$$\varphi'(\frac{1}{2}BA) = \frac{1}{2}\left(AA^tB^t + BAA^t\right) = \frac{1}{2}(B+B) = B$$

so φ' is surjective, hence has full rank. Therefore, by the rank lemma (Lemma 5.9 in JB) $O(n) = \varphi^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^{n^2}$ of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Problem 1.5 (3)**.** Fix a manifold M and consider the set $\mathrm{Vect}(M)$ of all isomorphism classes of finite dimensional real vector bundles over M.

(1) For $E, E' \in \mathrm{Vect}(M)$, construct a vector bundle $E \oplus E'$ over M which fiberwise is obtained by applying the direct sum $V \oplus V'$. Formulate a universal property of $E \oplus E'$.

(2) For $E, E' \in \mathrm{Vect}(M)$, construct a vector bundle $E \otimes E'$ over M which fiberwise is obtained by applying the tensor product $V \otimes V'$.

(3) Let $E \in \mathrm{Vect}(M)$ and fix $E' \subset E$ a subbundle of E, that is a vector bundle together with a map of bundles

$$\begin{tikzcd} V' \arrow[r, hook] \arrow[d] & V \arrow[d] \\ E \arrow[r] \arrow[dr] & E' \arrow[d] \\ & M \end{tikzcd}$$

that induces linear injective maps on fibres. Construct a vector bundle E/E' which fiberwise is given by taking the quotient vector space V/V'.

Solution. We will use the approach of Bröcker and Jänich by constructing pre-vector bundles with the desired properties.

(1) (3pts) We define $E \oplus E' = \bigcup_{p \in M} E_p \oplus E'_p$ where E_p and E'_p are the fibers at p. Now take $\pi\colon E \oplus E' \to M$ to be the projection $(e_p, e'_p) \mapsto p$. The vector space structure on $\left(E \oplus E'\right)_p = \pi^{-1}(p) = E_p \oplus E'_p$ is the precisely the direct sum of the vector space structures of E_p and E'_p.

For the pre-bundle atlas $\mathcal{B}$, let $\mathcal{B}_E, \mathcal{B}_{E'}$ be bundle atlases for E and E', respectively. Then for $(f_\alpha, U_\alpha) \in \mathcal{B}_E$ and $(g_\beta, V_\beta) \in \mathcal{B}_{E'}$, let $(f_\alpha \oplus g_\beta, U_\alpha \cap V_\beta) \in \mathcal{B}$ where

$$f_\alpha \oplus g_\beta\colon \pi^{-1}\left(U_\alpha \cap V_\beta\right) \to U_\alpha \cap V_\beta \times \mathbb{R}^n \times \mathbb{R}^m$$

sending $\left(e_p, e'_p\right) \mapsto \left(p, f_\alpha(e_p), g_\beta(e'_p)\right)$ is a bijective map which sends each fiber $\left(E \oplus E'\right)_p$ linearly and isomorphically onto $\{p\} \times \mathbb{R}^n \oplus \mathbb{R}^m \cong \mathbb{R}^n \oplus \mathbb{R}^m$. Furthermore, the transition functions are of the form

$$f_\alpha \oplus g_\beta \circ (f_{\alpha'} \oplus g_{\beta'})^{-1}(p, f_{\alpha'}(e_p), g_{\beta'}(e'_p)) = \left(p, f_\alpha(e_p), g_\beta(e'_p)\right)$$

which is continuous since each coordinate function is of the form id, $f_\alpha \circ f_{\alpha'}^{-1}$ or $g_\beta \circ g_{\beta'}^{-1}$ which are assumed to be continuous. As for the universal property, $E \oplus E'$ is the product of E and E' in $\mathrm{Vect}(M)$, so the usual universal property of products applies.

(2)(3pts) Define $E \otimes E' := \bigcup_{p \in M} E_p \otimes E'_p$ and π the standard projection. Let $\mathcal{B}_E, \mathcal{B}_{E'}$ be bundle atlases for E and E' respectively. Then, recalling that $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{nm}$ and using this identification, we get for $(f_\alpha, U_\alpha) \in \mathcal{B}_E$ and $(g_\beta, V_\beta) \in \mathcal{B}_{E'}$, the map $f_\alpha \otimes g_\beta \colon \pi^{-1}(U_\alpha \cap V_\beta) \to U_\alpha \cap V_\beta \times \mathbb{R}^{nm}$ given by

$$f_\alpha \otimes g_\beta \left(e_p \otimes e'_p\right) = \left(p, f_\alpha(e_p) \otimes g_\beta\left(e'_p\right)\right)$$

on simple tensors, and we extend this linearly over the fiber.
The linearity then becomes automatic. To see that this is an isomorphism, suppose

$$(p, 0) = f_\alpha \otimes g_\beta \left(e_p \otimes e'_p\right) = (p, f_\alpha(e_p) \otimes g_\beta(e'_p))$$

so either $f_\alpha(e_p) = 0$ or $g_\beta(e'_p) = 0$. But then since f_α and g_β are isomorphisms on $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(V_\beta)$, respectively, this implies that either $e_p = 0$ or $e'_p = 0$, so $e_p \otimes e'_p = 0$.
The transition maps then take on the form id and $f_{\alpha'} \circ f_\alpha^{-1} \otimes g_{\beta'} \circ g_\beta^{-1}$ which are continuous.

(3) (3pts) Let $E/E' = \bigcup_{p \in M} E_p/E'_p$ and π the standard projection. Here E_p/E'_p is well-defined since E'_p is a subspace of E_p for all p by assumption. Suppose $\mathcal{B}_E, \mathcal{B}_{E'}$ are bundle atlases for E and E', respectively. Define for $(f_\alpha, U_\alpha) \in \mathcal{B}_E$ and $(g_\beta, V_\beta) \in \mathcal{B}_{E'}$, $\overline{f_{\alpha,\beta}} \colon \pi^{-1}(U_\alpha \cap V_\beta) \to U_\alpha \cap V_\beta \times \frac{\mathbb{R}^n}{\mathbb{R}^m} \cong U_\alpha \cap V_\beta \times \mathbb{R}^{n-m}$ by $x + E'_p \mapsto \left(p, \overline{f_\alpha(x)}\right) = (p, f_\alpha(x) + g_\beta(V_\beta))$. The transition maps are continuous as either the projection or the quotient of a continuous transition map.