Exercise 0.1. (1) Let $G\times X\to X$ be a continuous left properly discontinuous action. Prove that the action is free.

(2) Let $r\in\mathbb{R}$ be an irrational number, and let $G=\left(\mathbb{Z}^2,+\right)$ act on $X=\mathbb{R}$ as $(n,m).t=t+n+rm$. Prove that this action is free, but not properly discontinuous. Compare with the statement of exercise 3.

Solution. (i) Suppose it is not free, so there exists $g\in G-\{e\}$ such that $g.x=x$ for some $x\in X$. If $G\times X\to X$ is properly discontinuous, then there exists $V\subset X$ with $x\in V$ such that $V\cap gV=\varnothing$ for all $g\in G-\{e\}$. But then $x=g.x\in gV$ and $x\in V$, so $x\in V\cap gV=\varnothing$, contradiction. Hence $G\times X\to X$ can not be properly discontinuous. Taking the contraposition gives the desired result.
(ii)
Suppose $t=(n,m).t=t+n+rm$, so $n+rm=0$. But if $m\neq 0$, then $r=-\frac{n}{m}\in\mathbb{Q}$, contradicting irrationality of r, so $m=0$ and hence $n=0$, giving that the action is free.
Now, suppose there exists an open neighborhood V of 0 such that $(n,m)\,V\cap V\neq\varnothing$ implies $(n,m)=(0,0)$. Since V is open, we can find some basis open ball $B\,(0,\delta)\subset V$. But by Dirichlet's approximation theorem, we can find integers n,m with $m\geq 1$ such that $|n+rm|<\delta$, which implies that $n+rm\in(n,m)V\cap B\,(0,\delta)\subset(n,m)V\cap V$, giving $(0,0)=(n,m)$, but $m\geq 1$, contradiction.

Since this action is not properly discontinuous, it means that the quotient map $\mathbb{R}\to\mathbb{R}/G$ cannot be a covering map since 0, for example, has no evenly covered neighborhood(if it had, then this neighborhood would split into homeomorphic disjoint copies in $\mathbb{R}$ satisfying which would satisfy the requirement of the action being properly discontinuous at 0).

Exercise 0.2. Let X and B be Hausdorff path-connected spaces, and let $p\colon X\to B$ be a local homeomorphism, i.e., for all $x\in X$, there is a neighborhood $U\subset X$ of x such that $p(U)$ is open in B, and the restriction

$$p|_U\colon U\to p(U)$$

is a homeomorphism. Also, suppose that for all $b\in B$, $p^{-1}\left(\{b\}\right)$ is finite, of the same cardinality for all points. Show that p is a covering map.

Solution. Let $F=\left\{1,\ldots,n:=\left|p^{-1}\left(\{b\}\right)\right|\right\}\subset\mathbb{N}$. In particular, $p^{-1}\left(\{b\}\right)$ is non-empty for all $b\in B$ since it has the same cardinality for all $b\in B$ and so if it were empty, then $p^{-1}(B)=\varnothing$, contradicting p being a function. Now let $b\in B$ and let $p^{-1}(\{b\})=\{x_1,\ldots,x_n\}$. Then by Hausdorffness, we can choose open neighborhoods $U_{1,i}$ and U_i for x_1 and x_i for all $i=2,\ldots,n$ such that $U_{1,i}\cap U_i=\varnothing$. Let $V_1=\bigcap_{i=2}^n U_{1,i}$ which is a neighborhood of x_1. Now for each $i=2,\ldots,n$, define $V_i=p^{-1}\left(p(V_1)\right)\cap U_i$. Suppose $y\in V_i\cap V_j\neq\varnothing$. Then let $\gamma\colon I\to V_i$
So in particular, $p\left(U_i\right)$ is an open neighborhood of b. Let $V=\bigcap_{i=1}^n p\left(U_i\right)$. Then let $U_i'=U_i\cap p^{-1}(V)\subset U_i$ which is open for each i, and since a restriction of a homeomorphism to a smaller open set is still a homeomorphism onto its image, we get that $p|_{U_i'}\colon U_i'\to p\left(U_i'\right)=V$ is a homeomorphism for each i. Suppose $x\in U_i'\cap U_j'$ for distinct i and j. Then since the intersection of finitely many open path-connected sets with a common point is still path connected, we can choose a path $\gamma\colon I\to U_i'\cup U_j'$ from x_i to x_j. Thus, under

To show that the action is not properly discontinuous, we show something stronger:

Lemma 0.3. *If $G \neq \{e\}$ acts properly discontinuously on $\mathbb{R}$, then $G \approx \mathbb{Z}$.*

Proof. We have $\pi_1(\mathbb{R}/G) \approx G$. Let $\alpha \in \pi_1(\mathbb{R}/G, x_0) - \{e\}$. Then α lifts to a path $\tilde{\alpha}\colon I \to \mathbb{R}$ starting at some $\tilde{x_0} \in p^{-1}(x_0)$ where $p\colon \mathbb{R} \to \mathbb{R}/G$ is the covering map.

Firstly, we show that, for any two $y, z \in p^{-1}(x_0)$, $(y, z) \cap p^{-1}(x_0)$ is finite. For the map $\mathbb{R} \to \mathbb{R}$ by $x \mapsto g.x$ for some $g \in G$ is a homeomorphism, so letting V be an open neighborhood around $\tilde{x_0}$ for which $gV \cap V \neq \varnothing$ implies $g = e$, we would get that there exist infinitely many elements $g \in G$ for which $gV \subset (y, z)$ and for two distinct such g, g', $gV \cap g'V = \varnothing$ since otherwise $g^{-1}g'V \cap V \neq \varnothing$ and hence $g^{-1}g' = e$ so $g = g'$. Thus this would imply that there is an infinite disjoint union of open sets gV of the same measure (since $x \mapsto gx$ is a homeomorphism) contained in (y, z). Since (y, z) has finite measure, this implies that gV has measure 0, contradicting it being open.

This gives a total ordering on $p^{-1}(x_0)$ inherited from $\mathbb{R}$ with finitely many elements in $p^{-1}(x_0)$ between any two elements in $p^{-1}(x_0)$.
Let $\tilde{x_0} \in p^{-1}(x_0)$ and $\tilde{x_0}'$ be its successor in $p^{-1}(x_0)$. We claim that the image, γ, under p of the path $\tilde{\gamma}\colon I \to \mathbb{R}$ connecting $\tilde{x_0}$ to $\tilde{x_0}'$ linearly generates $\pi_1(\mathbb{R}/G)$ as a cyclic group. Suppose $\alpha \in \pi_1(\mathbb{R}/G)$ and lift it to a path $\tilde{\alpha}$ starting at $\tilde{x_0}$. Let $\tilde{x_n} = \tilde{\alpha}(1)$. By uniqueness of lifts, we may assume by homotopy that this is the straight line path between the points $\tilde{x_0}$ and $\tilde{x_n}$, and suppose $\tilde{\alpha}^{-1}\left(p^{-1}(x_0)\right) = \{0, t_1, \ldots, t_{n-1}, 1\}$.
Now, since any loop at x_0 lifts to a path between two points in $p^{-1}(x_0)$, we thus see that there are precisely two simple loops in $\pi_1(\mathbb{R}/G)$ corresponding to the projections of the path going from $\tilde{x_i}$ to $\tilde{x_{i+1}}$ and the path going from $\tilde{x_i}$ to $\tilde{x_{i-1}}$. In particular, since the inverse of one of them is again simple, these loops are each other's inverse.
Thus, letting $\gamma \in \pi_1(\mathbb{R}/G, x_0)$ denote one of these paths, we see that $p \circ \tilde{\alpha} = \gamma^{k_1} * \ldots * \gamma^{k_n}$ where $k_i = \pm 1$. So $\alpha \in \langle \gamma \rangle$. Thus $\mathbb{Z} \approx \langle \gamma \rangle = \pi_1(\mathbb{R}/G) \approx G$.

$\square$