## 1. Preparation

### 1.1. Representability.

**Definition 1.1** (Initial and terminal objects)**.** An object $c$ in a category $C$ is initial if and only if the functor $C\,(c,-) : C \to \mathrm{Set}$ is naturally isomorphic to the constant functor $*\colon C \to \mathrm{Set}$ that sends every object to the singleton set. That is, for any $d, e \in C$ and any $f\colon d \to e$, we have

$$\begin{tikzcd} C(c,d) \arrow[r, "\approx"] \arrow[d, "f_*"] & \{*\} \arrow[d] \\ C(c,e) \arrow[r, "\approx"] & \{*\} \end{tikzcd}$$

Since these are isomorphisms in set, we can conclude that $C(c,d)$ and $C(c,e)$ are singletons, and as these is only one map between the singleton sets $\{*\} \to \{*\}$, we conclude that $\mathrm{Hom}\,(C(c,d), C(c,e))$ is also a singleton set.

**Theorem 1.2** (Yoneda lemma)**.** *For any functor $F\colon C \to \mathrm{Set}$, whose domain $C$ is locally small and any object $c \in C$, there is a bijection*

$$\mathrm{Hom}\,(C(c,-), F) \cong Fc$$

*that associates a natural transformation $\alpha\colon C\,(c,-) \implies F$ to the elements $\alpha_c\,(\mathbb{1}_c) \in Fc$. Moreover, this correspondence is natural in both $c$ and $F$.*

## 2. Limits and colimits

**Definition 2.1.** A diagram of shame $J$ in a category $C$ is a functor $F\colon J \to C$.

**Definition 2.2.** For any object $c \in C$ and any category $J$, the constant functor $c\colon J \to C$ sends every object of $J$ to $c$ and every morphism in $J$ to the identity morphism $\mathbb{1}_c$.

**Definition 2.3** (Embedding)**.** Recall that an embedding is a faithful functor that is injective on objects.
A fully faithful functor that is injective on objects defines a full embedding. The domain then defines a full subcategory of the codomain.

*Remark.* The constant functors define an embedding $\Delta\colon C \to \mathrm{Fun}(J, C)$ sending $c$ to the constant functor at $c$ and a morphism $f\colon c \to c'$ to the constant natural transformation, in which each component is defined to be the morphism $f$.

**Definition 2.4** (Cones)**.** A cone over a diagram $F\colon J \to C$ with summit or apex $c \in C$ is a natural transformation $\lambda\colon c \implies F$, so $\lambda \in \mathrm{Fun}(J, C)$, whose domain is the constant functor at $c$. The components $(\lambda_j\colon c \to Fj)_{j\in J}$ of the natural transformation are called the legs of the cone.

$$\begin{tikzcd} & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ F_j \arrow[rr, "Ff"'] & & F_k \end{tikzcd}$$

**Definition 2.5** (Cocones)**.** Dually, a cone under $F$ with nadir $c$ is a natural transformation $\lambda\colon F \implies c$ whose legs are the components $(\lambda_j\colon F_j \to c)_{j\in J}$.o
So naturality asserts that for each morphism $f\colon j \to k$ of $J$, the triangle

$$\begin{tikzcd} F_j \arrow[rr, "Ff"] \arrow[dr, "\lambda_j"'] & & F_k \arrow[dl, "\lambda_k"] \\ & c & \end{tikzcd}$$

commutes in $C$. Cones under a diagram are also called cocones.

**Definition 2.6** (Limits and colimits I)**.** For any diagram $F\colon J \to C$, there is a functor

$$\mathrm{Cone}(-, F)\colon C^{op} \to \mathrm{Set}$$

that sends $c \in C$ to the set of cones over $F$ with summit $c$.
A limit of $F$ is a representation for $\mathrm{Cone}\,(-, F)$.