1-5.

Problem 0.1. Let $\alpha\colon I \to \mathbb{R}^3$ be a regular parametrized curve (not necessarily arc length) and let $\beta\colon J \to \mathbb{R}^3$ be a reparametrization of $\alpha(I)$ by the arc length $s = s(t)$, measured from $t_0 \in I$. Let $t\colon J = s(I) \to I$ be the inverse of s.
The curvature of α is

$$k = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}$$

Show that the torsion of α is

$$\tau = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}.$$

Proof. We have

$$\ddot{\beta}(j) = (\alpha \ddot{\circ} t) = \alpha''(i)(\dot{t})^2(j) + \alpha''(i)\ddot{t}(j) = \frac{\alpha''}{|a'|^2} - \frac{\alpha' \langle \alpha', \alpha'' \rangle}{|\alpha'|^4} = k(i)n(i) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3} n(i)$$

hence

$$n \circ s = \frac{|\alpha'|}{|\alpha' \wedge \alpha''|}\alpha'' - \frac{\langle \alpha', \alpha'' \rangle}{|a'|}\alpha'$$

Now the binormal vector b is

$$b \circ s = \frac{\alpha'}{|\alpha'|} \wedge n = \frac{\alpha'}{|\alpha'|} \wedge \frac{|\alpha'|}{|\alpha' \wedge \alpha''|}\alpha'' = \frac{\alpha' \wedge \alpha''}{|\alpha' \wedge \alpha''|}$$

Hence by the chain rule

$$\dot{b}(j)s'(i) = (b \circ s)' = \frac{\alpha' \wedge \alpha'''}{|\alpha' \wedge \alpha''|} - \frac{\alpha' \wedge \alpha''}{|\alpha' \wedge \alpha''|^2}\frac{d}{dt}\left|\alpha' \wedge \alpha''\right|.$$

We know that $\dot{b} = \tau n$ as functions of j, so since $|n| = 1$, we have $\tau = \dot{b} \cdot n$ and this gives

$$\tau \circ s = \frac{|\alpha'|}{|\alpha'|\,|\alpha' \wedge \alpha''|^2}\left(\alpha' \wedge \alpha'''\right) \cdot \alpha'' = \frac{-(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}$$

$\square$