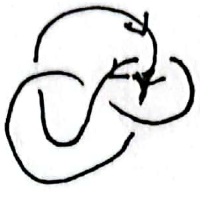
1. Let $K = 4_1$ as shown below with the given orientation

K

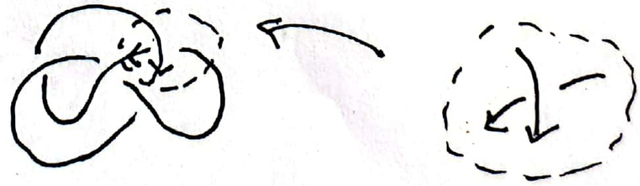


We can use the skein relations to compute the Alexander polynomial Δ_K of K with the relations

- $\Delta_{\text{trivial knot}} = 1$
- $\Delta_{L_+} - \Delta_{L_-} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_{L_0} = 0$, where L_+, L_- and L_0 are three links which differ only at one crossing shown as follows

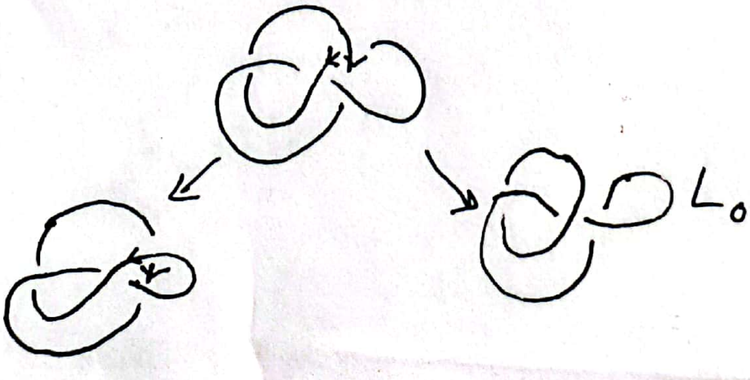
L_+ $\quad$ L_- $\quad$ L_0

We look at the circled crossing in K:



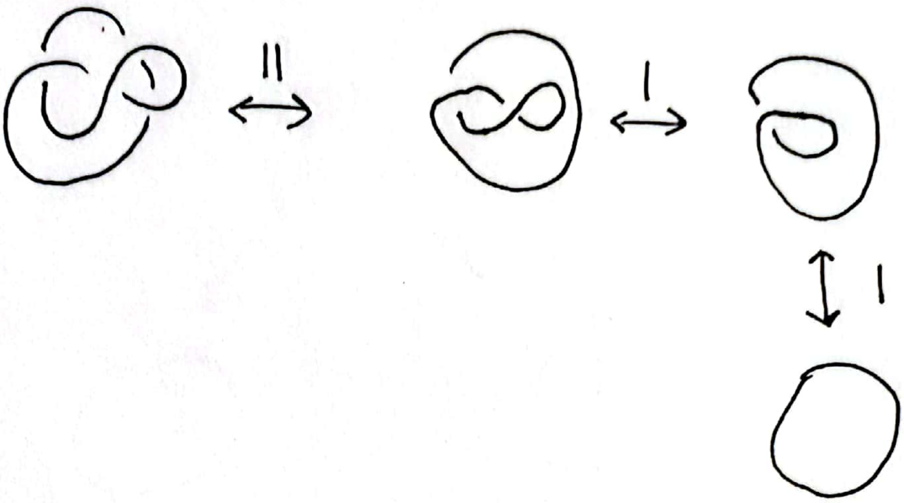
This crossing is of the L_- form, so we have

$K = L_-$



L_+ $\quad$ L_0

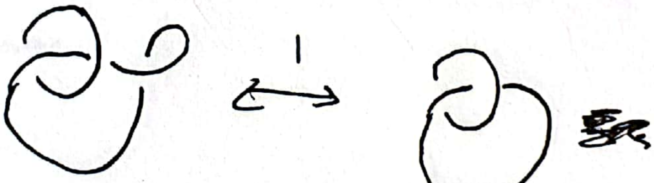
For L_+, we have by the following Reidemeister moves, that L_+ is the unknot:



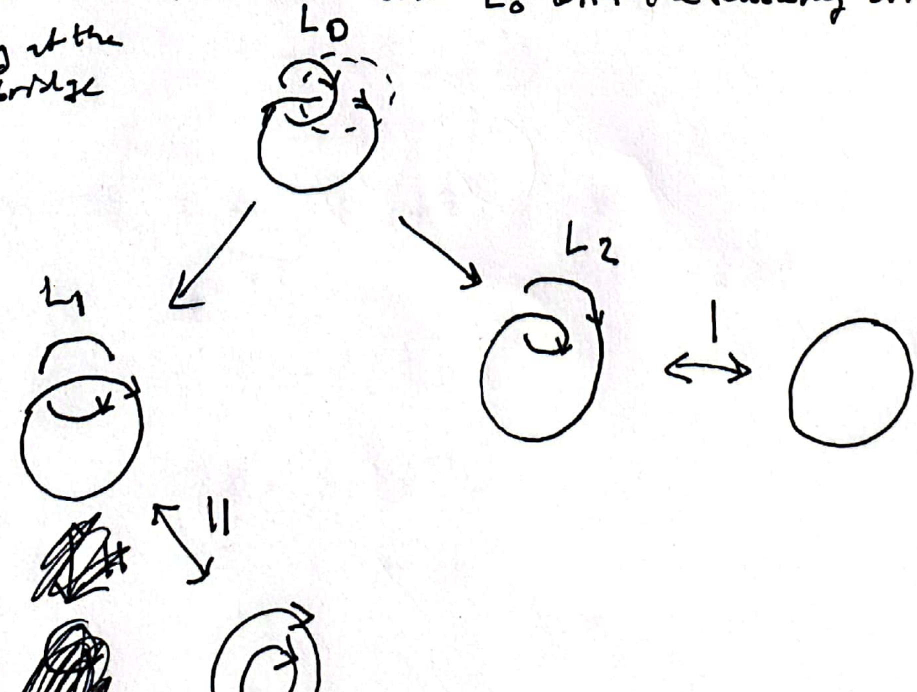
Unknot

So $\Delta_{L_+} = \Delta_{unknot} = 1$.

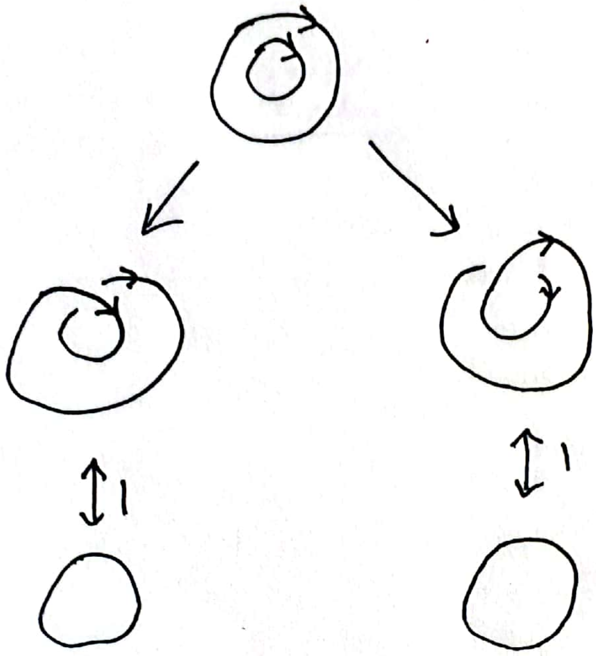
Now, L_0 by Reidemeister I is the Hopf link:



By skein relations, we have that L_0 with the following orientation is a negative crossing at the circled bridge



Now L_1 is the unlink of two components and since



We have $(t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta_{L_1} = \Delta_{unknot} - \Delta_{unknot} = 1 - 1 = 0 \Rightarrow \Delta_{L_1} = 0$

by the skein relations.

We thus have

$$\Delta_{L_0} = \underbrace{\Delta_{L_1}}_{=0} - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\underbrace{\Delta_{L_2}}_{=1}$$

$$= -(t^{\frac{1}{2}} - t^{-\frac{1}{2}})$$

So $\Delta_K = \Delta_{L_-} = \underbrace{\Delta_{L_+}}_{=1} + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\underbrace{\Delta_{L_0}}_{-(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}$

$$= 1 - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2$$

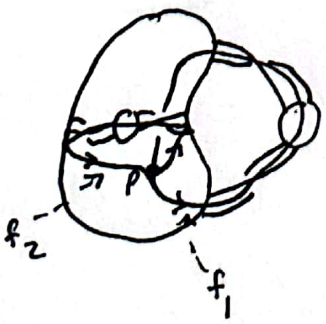
$$= 1 - t - t^{-1} + 2$$

$$= 3 - t - t^{-1}.$$

2. To calculate the Alexander polynomial using the Seifert form, we first find a Seifert surface for K using the Seifert algorithm, giving:

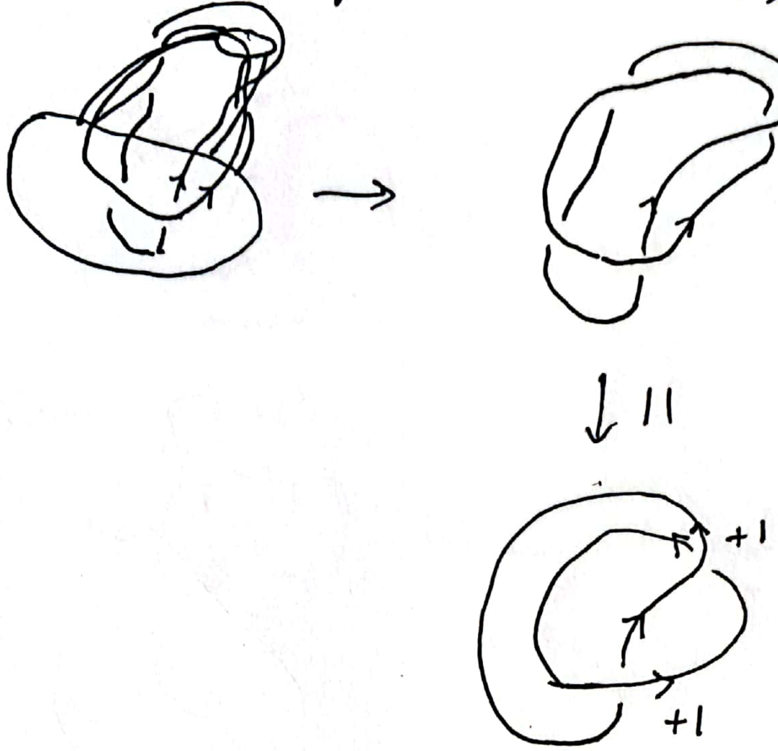
Call the surface F. We give F an orientation by considering the shaded in smaller discs to have the "upward-facing" side facing us, and the uncoloured disc to be facing down, i.e. its "upward direction" is into the paper.

To find generators for $H_1(F)$, the first homology group, we take the following two generators f_1 and f_2:



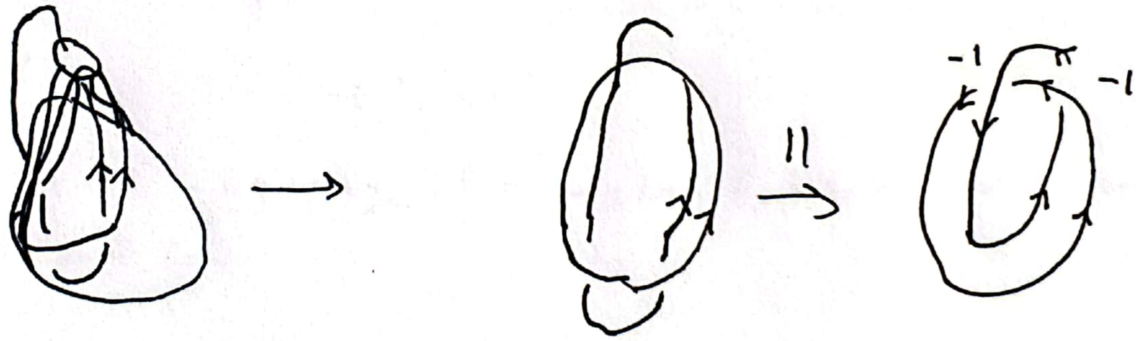
The Seifert matrix A is given by $A_{ij} = lk(f_i, f_j^+)$, where f_j^+ is the pushoff of f_j into $S^3 - F$ which runs parallel to f_j but slightly "upward w.r.t. orientation" to f_j.

To compute the linking number $lk(f_1, f_1^+)$, we look at



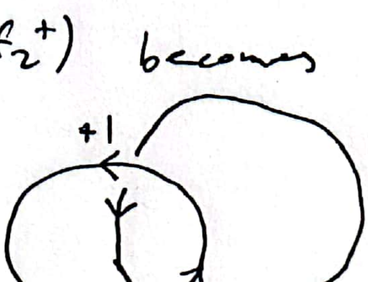
so $lk(f_1, f_1^+) = 1$

Now, $lk(f_2, f_2^+)$ is similarly



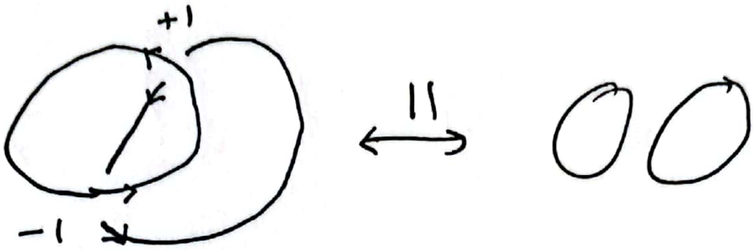
So $lk(f_2, f_2^+) = -1$

Now, $lk(f_1, f_2^+)$ becomes



so $lk(f_1, f_2^+) = 1$.

And lastly $lk(f_2, f_1^+)$ is



so $lk(f_2, f_1^+) = 0$

Thus $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Hence $tA - A^T = \begin{pmatrix} t & t \\ 0 & -t \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

$$= \begin{pmatrix} t-1 & t \\ -1 & 1-t \end{pmatrix}$$

which has determinant $(t-1)(1-t) + t = -(t-1)^2 + t$

$$= -t^2 + 2t - 1 + t$$
$$= -t^2 + 3t - 1$$

which by multiplying by t^{-1} which is a unit in $\mathbb{Z}[t, t^{-1}]$ is associated to $3 - t - t^{-1}$, so

$\Delta_K = 3 - t - t^{-1}$ by page 55 in Lickorish, which is also what we found in part (a).