Problem 0.1. Let Y be a simply-connected CW-complex. Assume there exists a finite wedge of spheres $\bigvee_i S^{n_i}$ together with maps $i\colon Y \to \bigvee_i S^{n_i}, r\colon \bigvee_i S^{n_i} \to Y$ such that $r \circ i$ is homotopic to id_Y. Prove that Y is homotopy equivalent to some finite wedge of spheres $\bigvee_j S^{m_j}$.

Proof. Since $r \circ \iota \simeq \mathrm{id}_Y$, we have that the induced map $r_*\colon H_n\left(\bigvee_i S^{n_i}\right) \to H_n(Y)$ is surjective for all n. We want to make use of the Corollary 4.33 in Hatcher which says:

Corollary 0.2 (4.33 Hatcher)**.** *A map $f\colon X \to Y$ between simply-connected CW complexes is a homotopy equivalence if $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism for each n.*

By assumption Y is a simply-connected CW complex, and $\bigvee_i S^{n_i}$ is also a simply-connected CW complex. Attaching cells of dimension ≥ 2 to $\bigvee_i S^{n_i}$ does not change either of these properties - i.e., the space remains a simply-connected CW complex. We would like to modify $\bigvee_i S^{n_i}$ such that r_* becomes injective also on each homology group. Let $[f] \in H_n\left(\bigvee_i S^{n_i}\right)$ be in the kernel of r_*. For the sake

Since $r \circ i \simeq \mathrm{id}_Y$, we have $r_* \circ i_* = \mathrm{id}_{\pi_n(Y)}$ for any $n \geq 0$, so $r_* \colon \pi_n \left(\bigvee_i S^{n_i}\right) \to \pi_n(Y)$ is surjective for each n. We will extend r inductively to obtain a weak homotopy equivalence $\bigvee_j S^{m_j} \to Y$. By Whitehead's theorem, it will then follow that $\bigvee_j S^{m_j} \simeq Y$.
Now, r_* is surjective on π_0, and since both $\bigvee_i S^{n_i}$ and Y are path-connected, r_* is, in fact, bijective on π_0.

We will now inductively attach cells onto $\bigvee_i S^{n_i}$ so as to make r_* a weak homotopy equivalence. In particular, we will inductively attach cells by the constant attaching map $e^k_\alpha \to \bigvee_i S^{n_i}$ to the base point of $\bigvee_i S^{n_i}$. But this will simply be a wedge of spheres again, so if we let Z_k denote the space obtained after the k th inductive step (i.e., after having attached n-cells for $n = 1, \dots, k$), then Z_k will again have the form $Z_k = \bigvee_j S^{n_j}$.
Suppose we have already made r_* an isomorphism on π_n for $n = 0, \dots, k-1$.
Next, choose maps $\varphi_\alpha \colon \left(S^k, s_0\right) \to \left(\bigvee_i S^{n_i}, *\right)$ representing all nontrivial elements of the kernel of $r_* \colon \pi_k(\bigvee_i S^{n_i}) \to \pi_k\left(Y\right)$. By the Cellular Approximation Theorem, if we let S^k have the CW structure with s_0 being the single 0-cell with a k-cell attached, then φ_α may be assumed to be cellular. Next we can attach e^{k+1}_α cells to $Z_k = \bigvee_i S^{n_i}$ via the maps φ_α which produces a new CW complex Z_{k+1}, which still is of the form $\bigvee_i S^{n_i}$, so we shall continue to denote $Z_k = \bigvee_i S^{n_i}$. Since $r \circ \varphi_\alpha$ is based nullhomotopic by construction, r extends over the new cells, so r extends to a map $Z_{k+1} \to Y$. Note that by assumption, r_* was an isomorphism on π_n for $n \leq k-1$, and attaching $k+1$-cells to Z_k has not changed this (the same maps from before still work for surjectivity). However, we now claim that r_* is injective on π_k also. Suppose $h \colon \left(S^k, s_0\right) \to \left(Z_{k+1}, *\right)$ represents an element of the kernel of $r_* \colon \pi_k\left(Z_{k+1}\right) \to \pi_k\left(Y\right)$. By the Cellular Approximation Theorem again (using the standard CW structure on S^k as above), we may assume that h is cellular, so in particular, the image of h lies in Z_k, and thus h is in the kernel of $r_* \colon \pi_k(Z_k) \to \pi_k(Y)$ and is thus based homotopic to some φ_α, which is based nullhomotopic in Z_{k+1}, so h represents zero in $\pi_k\left(Z_{k+1}\right)$. Thus the kernel of $r_* \colon \pi_k\left(Z_{k+1}\right) \to \pi_k(Y)$ is trivial, so r induces isomorphisms on π_n for $n \leq k$ now.
Note now that since $\bigvee_i S^{n_i} = Z_0$ was assumed to be a finite wedge of spheres, so there exists largest n_M in the wedge. In particular, then for any $k > n_M$ and any map $\left(S^k, s_0\right) \to \left(Z_{n_M}, *\right)$ is based homotopic to a cellular map by the Cellular Approximation Theorem, and hence maps all of S^k □

Problem 0.3. Let X be a path-connected CW complex such that $H_1\left(X; \mathbb{Z}\right) = 0$. The goal of this problem is to construct a simply connected space Z and a map $X \to Z$ inducing an isomorphism in homology.

(1) Give an example of such X such that $\pi_1\left(X\right) \neq 1$.
(2) Consider a set of generators for $\pi_1(X)$, construct another CW complex Y by attaching cells to X, so that
    - Y is simply connected.
    - The inclusion $X \subset Y$ induces an isomorphism on homology in degrees ≥ 3.
(3) Show that $H_2\left(Y; \mathbb{Z}\right)$ is a sum of $H_2\left(X; \mathbb{Z}\right)$ together with a free abelian group. Let A be a set of generators for this free summand.

See Prop 4.40 in Hatcher

Proof. (1) Since H_1 is just the abelianization for π_1 for path-connected spaces, this is equivalent to finding a path-connected CW complex X whose fundamental group is nontrivial, but becomes trivial when abelianized. By corollary 1.28 in Hatcher, for any group G, we can construct a 2-dimensional CW complex X_G such that $\pi_1(X_G) \cong G$. So it suffices to find a nontrivial group whose abelianization is trivial. Such a group is called a perfect group, and we have many examples of such groups. For example, any non-abelian simple group is perfect, so for example A_5 is perfect. The construction of X_{A_5} can now be carried out as follows: A_5 is generated by (123) and (12345) which do not commute, so we can express (as with any other group) A_5 as

$$A_5 = \langle g_\alpha \mid r_\beta \rangle$$

So in this case, the number of generators is simply 2. Then we can construct X_{A_5} from $\bigvee_\alpha S^1$ by attaching 2-cells e^2_β by the loops specified by the words r_β. By Proposition 1.26 in Hatcher, $\pi_1(X_{A_5}) \cong A_5$, and $H_1(X_{A_5}) \cong \mathrm{ab}(A_5) \cong 1$.

(2) We want to attach cells to X to obtain a CW-complex Y which is simply connected and induce an isomorphism on homology in degrees ≥ 3 under the inclusion. To do this, suppose $f\colon (S^1, s_0) \to (X, x_0)$ is in $\pi_1(X, x_0)$. We can assume by the Cellular Approximation Theorem that f is cellular. Then we can attach a 2-cell along f which renders f based nullhomotopic. Attaching 2-cells for each nontrivial element in $\pi_1(X)$ like this simultaneously, we let Y be the resulting space. Then we claim that $\pi_1(Y) \cong 0$. To see this, suppose $g\colon (S^1, s_0) \to (Y, x_0)$ is some map. By giving S^1 the standard CW stucture of a single 0-cell which is s_0 and a single 1-cell attached, we get by cellular approximation, that g is based homotopic to a map $\tilde{g}\colon (S^1, s_0) \to (Y, x_0)$ which has image in X. Thus $\tilde{g}$ represents an element of $\pi_1(X, x_0)$, but by construction of Y, $\tilde{g}$ is then based nullhomotopic. Composing these homotopies, we find that g is based nullhomotopic, so $\pi_1(Y) \cong 0$.

It remains to show that the inclusion induces isomorphisms in homology in degrees ≥ 3. Let I be an indexing set for the attaching maps of the 2-cells $\{e^2_\alpha\}_{\alpha \in I}$ that we attached to obtain Y from X. Let also A_n be an indexing set for the n-cells in the CW structure of X (we can also view A_n as an indexing set for the n-simplices in the Δ-complex structure obtained from X using its CW structure). In either case, we obtain a chain complex from this CW/Δ-complex structure along with a chain map induced by the inclusion $X \hookrightarrow Y$ which is the identity in all degrees except degree 2 :

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{A_4} \mathbb{Z} \arrow[r, "\partial_4^X"] \arrow[d, equal] & \bigoplus_{A_3} \mathbb{Z} \arrow[r, "\partial_3^X"] \arrow[d, equal] & \bigoplus_{A_2} \mathbb{Z} \arrow[r, "\partial_2^X"] \arrow[d, hook] & \bigoplus_{A_1} \mathbb{Z} \arrow[r, "\partial_1^X"] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \bigoplus_{A_4} \mathbb{Z} \arrow[r, "\partial_4^Y"] & \bigoplus_{A_3} \mathbb{Z} \arrow[r, "\partial_3^Y"] & \bigoplus_{A_2 \sqcup I} \mathbb{Z} \arrow[r, "\partial_2^Y"] & \bigoplus_{A_1} \mathbb{Z} \arrow[r, "\partial_1^Y"] & \cdots
\end{tikzcd}$$

Now, recalling that the induced map $\iota_*\colon H_n(X) \to H_n(Y)$ is given by $[c] \mapsto [\iota \circ c]$, the maps on homology in degrees ≥ 3 will simply be the identity since for any $n \geq 3$, $\partial_n^Y = \partial_n^X$, so

$$H_n(Y) = \ker \partial_n^Y / \operatorname{im} \partial_{n+1}^Y = \ker \partial_n^X / \operatorname{im} \partial_{n+1}^X = H_n(X).$$

(3) Using the LES of the pair (Y,X), we find that

$$H_3\left(Y,X\right) \stackrel{\partial_*}{\to} H_2\left(X\right) \stackrel{i_*}{\to} H_2(Y) \stackrel{j_*}{\to} H_2(Y,X) \stackrel{\partial_*}{\to} H_1\left(X\right)$$

is exact. Now, note that since X is a CW subcomplex, it is, in particular, closed and the inclusion $X \hookrightarrow Y$ is a cofibration, so the quotienting map $(Y,X) \to (Y/X,*)$ induces an isomorphism $H_*\left(Y,X\right) \cong H_*\left(Y/X,*\right) \cong \tilde{H}_*(Y/X)$ (Corollary 1.7 together with Corollary 1.4, Chapter VII in Bredon's Topology and Geometry). Now, Y/X is a wedge of 2-spheres, so $\tilde{H}_3(Y/X) \cong 0$ by considering its chain in cellular or simplicial homology. As for $H_1(X)$, this vanishes by assumption on the space X, so we finally obtain that

$$0 \to H_2\left(X\right) \stackrel{i_*}{\to} H_2(Y) \stackrel{j_*}{\to} H_2(Y,X) \to 0$$

is a SES. Now, using the exact same argument as above, $H_2\left(Y,X\right) \cong \tilde{H}_2(Y/X)$ and Y/X is a wedge of 2-spheres indexed by I, so $\tilde{H}_2\left(Y/X\right) \cong \bigoplus_I \mathbb{Z}$. In particular, this is a free abelian group, and we can let A be a set of generators for this free summand. Since any free group is projective, this SES splits, so we find that

$$H_2(Y) \cong H_2(X) \oplus H_2(Y,X) \cong H_2(X) \oplus \bigoplus_I \mathbb{Z}$$

(4) Since Y is simply-connected, the Hurewicz theorem gives us an isomor

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