1. Week 1

Exercise 1.1 (E1.1. Abel summation). Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ and $f\colon [1,x] \to \mathbb{C}$ be C^1. Define $A(t) = \sum_{n\le t} a_n$. Then for $x > 1$, we have

$$\sum_{n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

2. Week 2

Let $\psi(x) := \sum_{n\le x} \Lambda(n)$.

Exercise 2.1 (E2.6). Show that

$$\theta(x) := \sum_{p\le x} \log p = \psi(x) + O\left(x^{\frac{1}{2}} \log^2 x\right)$$

Exercise 2.2 (E2.7). Show that

$$\pi(x) = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t\log^2 t}dt + O\left(x^{\frac{1}{2}} \log x\right).$$

Proof. By Abel summation, we first find that

$$\theta(x) := \sum_{p\le x} \log p = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt$$

and from the previous exercise, we now find that

$$\pi(x) = \frac{\psi(x)}{\log x} + \frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt + O\left(x^{\frac{1}{2}} \log x\right)$$

The result follows if we can show that

$$\frac{1}{\log x}\int_2^x \frac{\pi(t)}{t}dt = \int_2^x \frac{\psi(t)}{t\log^2 t}dt + O\left(x^{\frac{1}{2}} \log x\right).$$

Now $\psi(t) \le \pi(t)\log t$, so

$$\begin{aligned}\left|\int_2^x \frac{\psi(t)}{t\log^2 t} - \frac{\pi(t)}{t\log x}dt\right| &\le \left|\int_2^x \frac{\pi(t)}{t\log t} - \frac{\pi(t)}{t\log x}dt\right| \\ &= \left|\int_2^x \frac{\pi(t)}{t}\frac{\log\left(\frac{x}{t}\right)}{\log x\log t}dt\right|\end{aligned}$$

□

3. Week 3

Exercise 3.1 (E3.1). Let $m \ge 0$. Show that

$$\sum_{n\le x} \log^m n = x\log^m x + O\left(x\log^{m-1} x\right).$$

Proof. Let $a_n = 1$ for all n. Then $A(x) = \lfloor x \rfloor$, so

$$\sum_{n \le x} \log^m n = \lfloor x \rfloor \log^m x - \int_1^x m \lfloor t \rfloor \frac{1}{t} \log^{m-1} t dt$$
$$= x \log^m x - (x - \lfloor x \rfloor) \log^m x - m \int_1^x \frac{\lfloor t \rfloor}{t} \log^{m-1}(t) dt$$

Thus we must show that

$$\left| (x - \lfloor x \rfloor) \log^m x + m \int_1^x \frac{\lfloor t \rfloor}{t} \log^{m-1}(t) dt \right| \le C x \log^{m-1} x$$

But $\frac{\lfloor t \rfloor}{t} \log^{m-1}(t) \le \log^{m-1}(x)$ giving that the right hand term is $O\left(x \log^{m-1} x\right)$. For the left hand term, it suffices to show that $(x - \lfloor x \rfloor) \log x \le x$, but this is clear since $x - \lfloor x \rfloor \le 1$ and $\log x \le x$. $\square$

Exercise 3.2 (E3.2)**.** Let $d(n) = \sum_{d \mid n} 1$. Show $d(n) \le 2\sqrt{n}$. If we consider the set $D \subset \mathbb{N}$ of positive divisors of n, then we can define a bijection $D \to D$ by $k \mapsto \frac{n}{k}$. Suppose now that $d(n) > 2\sqrt{n}$. Suppose $d \mid n$ and $d \ge \sqrt{n}$. Then since $\frac{d}{n} \cdot d = n$, we must have $\frac{d}{n} \le \sqrt{n}$. This implies that under this bijection, either the source or target lies in $\{1, \dots, \lfloor \sqrt{n} \rfloor\}$. Hence $d(n) = |D| \le 2 \left|\{1, \dots, \lfloor \sqrt{n} \rfloor\}\right| \le 2\sqrt{n}$.

Exercise 3.3 (E3.3)**.** Prove that for every $\varepsilon > 0$, there exists a constant C_ε such that $d(n) \le C_\varepsilon n^\varepsilon$.
Hint:

(1) Show that $d(n_1 n_2) = d(n_1) d(n_2)$ if $(n_1, n_2) = 1$.
(2) Show that
$$\frac{d(n)}{n^\varepsilon} = \prod_{p^\alpha || n} \frac{\alpha + 1}{p^{\alpha\varepsilon}}$$
where $p^\alpha \mid\mid n$ means that α is a positive integer, $p^\alpha \mid n$ and $p^{\alpha+1} \nmid n$.
(3) Split the product in 2. Into the product over those primes $p < 2^{\frac{1}{\varepsilon}}$ and the product over the rest. Show that the second product is bounded by 1.
(4) Show that the factors in the first product are less than $1 + (\varepsilon \log 2)^{-1}$.

Proof. We follow the hint:
(1) Suppose $(n_1, n_2) = 1$. Let D be the set of divisors of $n_1 n_2$, D_1 the set of divisors of n_1 and D_2 the set of divisors of n_2. Suppose $d_1 \in D_1, d_2 \in D_2$. Then $d_1 a = n_1, d_2 b = n_2$, so $d_1 d_2 ab = n_1 n_2$, hence $d_1 d_2 \in D$. We thus obtain a map $D_1 \times D_2 \to D$ sending $(d_1, d_2) \mapsto d_1 d_2$. We claim this is a bijection. Suppose $d_1 d_2 = d_1' d_2'$. If $d_1 \mid d_2'$, then $d_1 = 1$, in which case, $d_1' = 1$, and thus $d_2 = d_2'$. Suppose thus that $d_1 \ne 1$, so $d_1 \nmid d_2'$. Then since $(d_1', d_2') = 1$, we have $d_1 \mid d_1'$. Similarly, $d_1' \mid d_1$. So $d_1 = d_1'$. And again $d_2 = d_2'$. This gives injectivity. For surjectivity, if $d \mid n_1 n_2$, then consider $d_1 := \frac{d}{(n_2, d)}$ and $d_2 := \frac{d}{(n_1, d)}$. Then $d_1 d_2 = d$ and $d_1 \in D_1, d_2 \in D_2$.
(2) Clearly, $n^\varepsilon = \prod_{p^\alpha || n} p^{\alpha\varepsilon}$. It thus suffices to show that $\prod_{p^\alpha || n} (\alpha + 1) = d(n)$. But if we factorize n as $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, then it is clear that the divisors corresponds precisely to tuples $(a_1, \dots, a_m)$ with $0 \le a_i \le \alpha_i$. There are precisely $\alpha_1 + 1$ choices for each a_i, giving $(\alpha_1 + 1) \cdots (\alpha_m + 1) = d(n)$ which indeed is what we wanted to show.

(3) We can split the product as

$$\frac{d(n)}{n^\varepsilon} = \underbrace{\prod_{\substack{p^\alpha || n \\ p < 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}}_{A} \cdot \underbrace{\prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}}_{B}$$

We claim that $B \leq 1$. Indeed

$$\prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}} \leq \prod_{\substack{p^\alpha || n \\ p \geq 2^{\frac{1}{\varepsilon}}}} \underbrace{\frac{\alpha+1}{2^\alpha}}_{\leq 1} \leq 1$$

(4) For the factors in the first product, we have $\alpha = \left\lfloor \frac{\log n}{\log p} \right\rfloor$ and $\log p < \frac{1}{\varepsilon} \log 2$, and $\alpha \leq \frac{\log n}{\log p}$, so $\frac{\log p}{\log n} \leq \frac{1}{\alpha}$

$$\varepsilon^2 \log p < \varepsilon \log 2$$

$$\frac{\alpha+1}{p^{\alpha\varepsilon}} \leq \frac{\log n + \log p}{p^{\alpha\varepsilon} \log p} \leq 1 + \frac{1}{\varepsilon \log 2} = \frac{\varepsilon \log 2 + 1}{\varepsilon \log 2}$$

What we want to bound is

$$\prod_{\substack{p^\alpha || n \\ p < 2^{\frac{1}{\varepsilon}}}} \frac{\alpha+1}{p^{\alpha\varepsilon}}$$

Note here that p is bounded and as α increases, we should expect the denominator to take over. However, while α is small, we might have some large terms since p^ε might be large. All our terms are however bounded by p^ε by the looks of it? Then we would get that the product is the product is bounded by $\prod_{p < 2^{\frac{1}{\varepsilon}}} \frac{\log n}{\log p} \frac{1}{p^\varepsilon}$

□

Exercise 3.4 (E3.4)**.** Show that

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s}$$

is absolutely convergent in $\Re(s) > 1$.

Proof. Fix some $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. Then choosing an $\varepsilon > 0$ with $1 + \varepsilon < \sigma$, we have that $d(n) \leq C_\varepsilon n^\varepsilon$, so

$$\sum \left| \frac{d(n)}{n^s} \right| \leq \sum C_\varepsilon \frac{n^\varepsilon}{n^\sigma} \leq C_\varepsilon \sum \frac{1}{n^{\sigma - \varepsilon}} < \infty.$$

□

Exercise 3.5 (E3.5)**.** Show that the average order of $d(n)$ is $\log n$, i.e., that

$$\frac{1}{x} \sum_{n \leq x} d(n) = \log x + o(\log x).$$

Hint: Show that

$$\sum_{n \leq x} d(n) = \sum_{a \leq x} \left[\frac{x}{a} \right]$$

where $[b]$ is the integer part of b.

Proof. We follow the hint. For each $n \in \mathbb{N}$, let D_n denote the set of positive divisors of n. Then we want to find $\left|D_1 \cup \ldots \cup D_{[x]}\right|$. Now, $\left[\frac{x}{a}\right]$ is precisely the amount of multiples of a smaller than or equal to x, i.e., the amount of numbers in between 1 and x which have a as a divisor. Hence the right hand side indeed counts the number of divisors of the numbers less than or equal to x which is precisely the left hand side. Then we find that

$$\left|\frac{1}{x}\sum_{n \leq x} d(n) - \log x\right| \leq \left|\sum_{a \leq x} \frac{1}{a} - \log x\right|$$

$\square$