

1. ORIENTATIONS

We begin by attempting to give complete rigour and detail to the definitions of orientation and the many connected theorems.

For this section, we will follow [1] and [2]

Definition 1.1 (Local Homology Group). For $h_*(-)$ a homology theory and an n -manifold M , groups of the form $h_k(M, M - \{x\})$ are called *local homology groups*.

For a chart $\varphi: U \rightarrow \mathbb{R}^n$ on M centered at x , we get by excision that

$$h_k(M, M - \{x\}) \cong h_k(U, U - \{x\}) \xrightarrow{\varphi_*} h_k(\mathbb{R}^n, \mathbb{R}^n - \{0\}).$$

Hence for singular homology, we obtain $H_n(M, M - \{x\}; G) \cong G$.

Definition 1.2 (Local R -orientation). Let R be a commutative ring. A generator of $H_n(M, M - \{x\}; R) \cong R$ is called a *local R -orientation* of M about x .

Let $K \subset L \subset M$. The homomorphism $r_K^L: h_k(M, M - L) \rightarrow h_k(M, M - K)$ induced by inclusion is called restriction. We write r_x^L when $K = \{x\}$.

Proposition 1.3. *When A is a compact, convex set contained in some chart $\mathbb{R}^n \subset M$, then r_x^A is an isomorphism for each $x \in A$ and the groups are isomorphic to the coefficient group G .*

Proof. A is contained in the interior of some closed n -disk $D \subset \mathbb{R}^n \subset M$. Thus there is a commutative diagram

$$\begin{array}{ccc} h_n(M, M - A) & \longrightarrow & h_n(M, M - \{x\}) \\ \cong \uparrow & & \cong \uparrow \\ h_n(\mathbb{R}^n, \mathbb{R}^n - A) & \longrightarrow & h_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \\ \cong \uparrow & & \cong \uparrow \\ h_n(D, \partial D) & \xlongequal{\quad} & h_n(D, \partial D) \end{array}$$

□

Definition 1.4 (Orientation bundle). We construct a covering $\omega: h_k(M, M - \bullet) \rightarrow M$. Define

$$h_k(M, M - \bullet) = \bigsqcup_{x \in M} h_k(M, M - \{x\})$$

where $h_k(M, M - \{x\})$ is the fiber over x and is given the discrete topology.

Let U be an open neighborhood of x such that r_y^U is an isomorphism for each $y \in U$. Define bundle charts

$$\varphi_{x,U}: U \times G \rightarrow \omega^{-1}(U), \quad (y, a) \mapsto r_y^U (r_x^U)^{-1}(a).$$

We then give $h_k(M, M - \bullet)$ the topology that makes $\varphi_{x,U}$ in a homeomorphism onto an open subset. In particular, since $h_k(M, M - x)$ is given the discrete topology, this is equivalent to the map $\varphi_{x,U}(-, \alpha)$ being a homeomorphism onto an open subset for each $\alpha \in h_k(M, M - x)$. It then remains to show that the transition maps

$$\varphi_{y,V}^{-1} \varphi_{x,U}: (U \cap V) \times h_k(M, M - \{x\}) \rightarrow (U \cap V) \times h_k(M, M - \{y\})$$

are continuous.

Let $z \in U \cap V$, and choose W such that $z \in W \subset U \cap V$ and r_w^W is an isomorphism for each $w \in W$.

Consider the diagram

$$\begin{array}{ccccc}
 h_k(M, M - x) & \xleftarrow{r_x^U} & h_k(M, M - U) & \xrightarrow{r_w^U} & h_k(M, M - w) \\
 & & \downarrow r_W^U & \nearrow r_w^W & \uparrow r_w^V \\
 & & h_k(M, M - W) & \xleftarrow{r_W^V} & h_k(M, M - V) \\
 & & & & \downarrow r_y^V \\
 & & & & h_k(M, M - y)
 \end{array}$$

Let $\varphi_{x,U,p}: h_k(M, M - x) \rightarrow \omega^{-1}(p)$ be defined by

$$\varphi_{x,U,p}(y) = \varphi_{x,U}(p, y).$$

Then for $w \in U \cap V$, we have

$$\varphi_{x,U,w}^{-1} \varphi_{y,V,w} = r_y^V (r_W^V)^{-1} (r_w^W)^{-1} r_w^W r_W^U (r_x^U)^{-1} = r_y^V (r_W^V)^{-1} r_W^U r_x^U$$

Firstly, this coincides with the operation of an element of the coefficient group G since it is an isomorphism $G \rightarrow G$, and secondly, note that this does not depend on w , so the map

$$g_{x,U,y,V}: U \cap V \rightarrow G$$

defined by $g_{x,U,y,V}(p) = \varphi_{x,U,p}^{-1} \varphi_{y,V,p}$ is constant, hence continuous.

Thus ω is indeed a covering map.

But even moreso, the fibers are groups, so for $A \subset M$, denote by $\Gamma(A)$ the set of continuous sections over A of ω . If s and t are section, we can define $(s + t)(a) = s(a) + t(a)$. Then $s + t$ is again continuous, hence $\Gamma(A)$ is an abelian group.

Denote by $\Gamma_c(A) \subset \Gamma(A)$ the subgroup of sections with compact support, i.e., the sections which have values 0 in the fiber away from a compact set.

Proposition 1.5. *Let $z \in h_k(M, M - U)$. Then $y \mapsto r_y^U z \in h_k(M, M - y) \subset h_k(M, M - \bullet)$ is a continuous section of ω over U .*

Proof. The map $U \rightarrow U \times G$ by $y \mapsto (y, r_x^U z)$ is constant in the second coordinate, hence clearly continuous. Now composing with $\varphi_{x,U}$ gives us the section in question. \square

1.1. Homological Orientation. If we specify to singular homology with coefficient group R , and again let M be an n -manifold and $A \subset M$, then we can define an orientation along A as follows

Definition 1.6 (R -orientation of M along A). An R -orientation of M along A is a section $s \in \Gamma(A; R)$ of $\omega: H_n(M, M - \bullet; R) \rightarrow M$ such that $s(a) \in H_n(M, M - a; R) \cong R$ is a generator for each $a \in A$.

Thus s glues together the local orientations in a continuous manner.

When $A = M$, we call s an R -orientation of M .

Definition 1.7 (Orientation covering). Let $\text{Ori}(M) \subset H_n(M, M - \bullet; \mathbb{Z})$ be the subset of all generators of all fibers. Then the restriction $\text{Ori}(M) \rightarrow M$ of ω gives a 2-fold covering of M , called the *orientation covering* of M .

Proposition 1.8. *The following are equivalent:*

- (1) M is orientable
- (2) M is orientable along compact subsets.
- (3) The orientation covering is a trivial 2-fold covering map.
- (4) The covering $\omega: H_n(M, M - \bullet; \mathbb{Z}) \rightarrow M$ is a trivial covering map.

Proof. (1) \implies (2) is a subcase.

(2) \implies (3). The orientation covering is trivial if and only if the covering over each component is trivial, so we may assume that M is connected. Now, if a 2-fold covering $\tilde{M} \rightarrow M$ is trivial, then \tilde{M} splits as $M \times \{p, q\}$, and so \tilde{M} cannot be connected. Conversely, if \tilde{M} is not connected, then the covering restricted to each component must be a covering map, so the covering splits as a trivial covering.

Suppose then that $\text{Ori}(M) \rightarrow M$ is non-trivial. Since $\text{Ori}(M)$ is then connected, we can choose a path γ in $\text{Ori}(M)$ between two points of a given fiber. The image S of such a path is compact and connected, and the covering is non-trivial over S , so by assumption (2), the orientation covering has a section s over S , but then $\gamma(0) = s(\omega(\gamma(0))) = s(\omega(\gamma(1))) = \gamma(1)$, which gives a contradiction.

(3) \implies (4).

Let $s: M \rightarrow \text{Ori}(M) \cong M \times \{-1, 1\}$ be the section $m \mapsto (m, 1)$.

Now define a map $\varphi: M \times \mathbb{Z} \rightarrow H_n(M, M - \bullet; \mathbb{Z})$ by $\varphi(m, k) = ks(m)$. This is a bijective map by assumption on s being a section. It is furthermore continuous since s is continuous and since fiber-wise operations in $H_n(M, M - \bullet; \mathbb{Z})$ is continuous. Furthermore, it is also a morphism between coverings since it commutes with the projections: $\pi_M = \omega \circ \varphi$.

Lastly, one must show that it also has a continuous inverse. For this, we may take an open basis set in $M \times \mathbb{Z}$ - say $U \times \{k\}$, where \bar{U} is a convex subset of $\mathbb{R}^n \subset M$. Since φ is bijective, we obtain that $\varphi(U \times \{k\}) = ks(U) = U_\alpha$ if we choose α to be the element in $H_n(M, M - U) \cong \mathbb{Z}$ which maps to k under $r_{x,U}$ for $x \in U$. And by assumption, U_α is a basis open set for the topology on $H_n(M, M - \bullet; \mathbb{Z})$.

Hence φ is a homeomorphism, and even an isomorphism of covering spaces in the sense that $\pi_M = \omega \circ \varphi$.

Note. We could also say that it is trivial since every point is in the image of some section.

(4) \implies (1) : If ω is trivial, then it has a section with constant value in the set of generators.

□

1.2. Homology in the Dimension of the Manifold. Let M be an n -manifold and $A \subset M$ a closed subset. We will in this section use singular homology with coefficients in an abelian group G .

Proposition 1.9. *For each $\alpha \in H_n(M, M - A; G)$, the section*

$$J^A(\alpha): A \rightarrow H_n(M, M - \bullet; G), \quad x \mapsto r_x^A(\alpha)$$

of ω over A is continuous and has compact support.

Proof. Choose a representative $c \in \Delta_n(M; G)$ representing α . There exists a compact set K such that c is contained in K . Suppose $A - K$ is nonempty, and let $x \in A - K$. Then the image of c under

$$\Delta_n(K; G) \rightarrow \Delta_n(M; G) \rightarrow \Delta_n(M, K; G) \rightarrow \Delta_n(M, M - x; G)$$

is zero since $K \subset M - x$. Since this image represents r_x^A , the support of $J^A(\alpha)$ is contained in $A \cap K$ which is compact.

If $A - K$ is empty, K contains A , and then the support of $J^A(\alpha)$ is a closed subset of a compact space, hence compact.

The continuity follows from the more general case of Proposition 1.5. □

Thus we obtain a homomorphism

$$J^A: H_n(M, M - A; G) \rightarrow \Gamma_c(A; G), \quad \alpha \mapsto (x \mapsto r_x^A(\alpha)).$$

1.2.1. Direct Limits.

Definition 1.10. Let D be a directed set and G_α an abelian group defined for each $\alpha \in D$. Suppose we are given homomorphisms $f_{\beta, \alpha}: G_\alpha \rightarrow G_\beta$ for each $\beta > \alpha$ in D . Assume that for all $\gamma > \beta > \alpha$ in D , we have $f_{\gamma, \beta} f_{\beta, \alpha} = f_{\gamma, \alpha}$. Such a system is called a *direct system* of abelian groups. Then $G = \lim_{\rightarrow} G_\alpha$ is defined to be the quotient group of the direct sum $G = \bigoplus G_\alpha$ modulo the relations $f_{\beta, \alpha}(g) \sim g$ for all $g \in G_\alpha$ and all $\beta > \alpha$.

Note. Hence the direct limit is just the colimit of the direct system.

Proposition 1.11. Suppose we are given an abelian group A with homomorphisms $h_\alpha: G_\alpha \rightarrow A$ such that the cocone commutes. Since $\lim_{\rightarrow} G_\alpha$ is the colimit, we have a unique induced homomorphism $h: \lim_{\rightarrow} G_\alpha \rightarrow A$. Then

- (1) $\text{im } h = \{a \in A \mid a = h_\alpha(g) \text{ for some } g \text{ and } \alpha\} = \bigcup \text{im } h_\alpha$.
- (2) $\ker h = \{g \in \lim_{\rightarrow} G_\alpha \mid \exists \alpha \text{ and } g_\alpha \in G_\alpha: g = i_\alpha(g_\alpha) \text{ and } h_\alpha(g_\alpha) = 0\} = \bigcup i_\alpha(\ker h_\alpha)$.

Proof. Define $h(g_\alpha) = h_\alpha(g_\alpha)$. Then if $f_{\beta, \alpha}(g_\alpha) \sim g_\alpha$, we have $h(g_\alpha) = h_\alpha(g_\alpha) = h_\beta \circ f_{\beta, \alpha}(g_\alpha) = h(f_{\beta, \alpha}(g_\alpha))$, so h respects the equivalence relations, thus it is well-defined.

Now property (1) is clear by the way we defined h .

As for (2), note that if g represents the equivalence class of g_α and $h(g) = 0$, then $h_\alpha(g_\alpha) = 0$ which is what (2) is saying. □

Corollary 1.12. In the situation of Proposition 1.11, $h: \lim_{\rightarrow} G_\alpha \rightarrow A$ is an isomorphism if and only if the following two statements hold true:

- (1) $\forall a \in A, \exists \alpha \in D \text{ and } g_\alpha \in G_\alpha: h_\alpha(g_\alpha) = a$, and
- (2) if $h_\alpha(g_\alpha) = 0$ then $\exists \beta > \alpha: f_{\beta, \alpha}(g_\alpha) = 0$.

Theorem 1.13. The direct limit is an exact functor. So if we have direct systems $\{A'_\alpha\}, \{A_\alpha\}$ and $\{A''_\alpha\}$ based on the same directed set, and if we have an exact sequence $A'_\alpha \rightarrow A_\alpha \rightarrow A''_\alpha$ for each α , where the maps commute with the ones defining the direct systems, then the induced sequence

$$\lim_{\rightarrow} A'_\alpha \rightarrow \lim_{\rightarrow} A_\alpha \rightarrow \lim_{\rightarrow} A''_\alpha$$

is exact.

Proof. We have the following diagram, where all maps commute.

$$\begin{array}{ccccc} A'_\beta & \longrightarrow & A_\beta & \longrightarrow & A''_\beta \\ \downarrow & & \downarrow & & \downarrow \\ \lim_{\rightarrow} A'_\alpha & \longrightarrow & \lim_{\rightarrow} A_\alpha & \longrightarrow & \lim_{\rightarrow} A''_\alpha \end{array}$$

Suppose $a \in \lim_{\rightarrow} A_*$ is mapped to zero in $\lim_{\rightarrow} A''_*$. Then there exists $g \in \lim_{\rightarrow} A_\alpha$ such that there exists β and $g_\beta \in A_\beta$ such that $g = i_\beta(g_\beta)$ and $h_\beta(g_\beta) = 0$.

Recall here that h_β is a homomorphism $A_\beta \rightarrow \lim_{\rightarrow} A''_*$ and i_β is the inclusion $G_\beta \rightarrow \lim_{\rightarrow} G_\alpha$.

By commutativity of the diagram, there then exists $k_\beta \in A'_\beta$ such that $i_\beta(d_\beta(k_\beta)) = d_{\lim_{\rightarrow}} i'_\beta(k_\beta)$. Hence the kernel is contained in the image.

Now suppose let $\tilde{k} = d_{\lim_{\rightarrow}}(k) \in \lim_{\rightarrow} A_*$.

Then $\tilde{k} = i_\beta(d(\tilde{k})) = d_{\lim_{\rightarrow}} i'_\beta(\bar{k})$ for some $\bar{k} \in A'_\beta$.

But now

$$d_{\lim_{\rightarrow}}(\tilde{k}) = d_{\lim_{\rightarrow}} i_\beta(d(\tilde{k})) = i''_\beta d(d(\tilde{k})) = i''_\beta(0) = 0.$$

□

Theorem 1.14. Suppose we are given two directed sets D and E . Define an order on $D \times E$ by $(\alpha, \beta) \geq (\alpha', \beta')$ if and only if $\alpha \geq \alpha'$ and $\beta \geq \beta'$. Suppose $G_{\alpha, \beta}$ is a direct system based on $D \times E$. Then the maps $G_{\alpha, \beta} \rightarrow \lim_{\rightarrow, \beta} G_{\alpha, \beta} \rightarrow \lim_{\rightarrow, \alpha} (\lim_{\rightarrow, \beta} G_{\alpha, \beta})$ induce an isomorphism

$$\lim_{\rightarrow, \alpha, \beta} G_{\alpha, \beta} \xrightarrow{\cong} \lim_{\rightarrow, \alpha} \left(\lim_{\rightarrow, \beta} G_{\alpha, \beta} \right).$$

Proof.

□

Proposition 1.15. (1) For $A \supset B$ both closed, the following diagram commutes:

$$\begin{array}{ccc} H_n(M, M - A; G) & \longrightarrow & H_n(M, M - B; G) \\ \downarrow J^A & & \downarrow J^B \\ \Gamma_c(A, H_n(M, M - \bullet; G)) & \longrightarrow & \Gamma_c(B, H_n(M, M - \bullet; G)) \end{array}$$

(2) For $A, B \subset M$ both closed, the sequence

$$\begin{aligned} 0 \rightarrow \Gamma_c(A \cup B, H_n(M, M - \bullet; G)) &\xrightarrow{h} \Gamma_c(A, H_n(M, M - \bullet; G)) \oplus \Gamma_c(B, H_n(M, M - \bullet; G)) \\ &\xrightarrow{k} \Gamma_c(A \cap B, H_n(M, M - \bullet; G)) \end{aligned}$$

is exact, where h is the sum of restrictions and k is the difference of restrictions.

(3) If $A_1 \supset A_2 \supset A_3 \supset \dots$ are all compact and $A \cap A_i$, then the restriction homomorphisms $\Gamma(A_i, H_n(M, M - \bullet; G)) \rightarrow \Gamma(A, H_n(M, M - \bullet; G))$ induce an isomorphism

$$\lim_{\rightarrow} \Gamma(A_i, H_n(M, M - \bullet; G)) \xrightarrow{\cong} \Gamma(A, H_n(M, M - \bullet; G))$$

Proof. (1) Let $\alpha \in H_n(M, M - A; G)$, and denote by ι the inclusion $(M, M - A) \hookrightarrow (M, M - B)$. Then $\iota_* = r_B^A$, so $J^B(r_B^A(\alpha))(x) = r_x^B(r_B^A(\alpha))$. On the other hand, $J^A(\alpha)|_B(x) = J^A(\alpha)(x) = r_x^A(\alpha)$. Now, from the composition

$$(M, M - A) \hookrightarrow (M, M - B) \hookrightarrow (M, M - x)$$

we obtain by taking homology, that $r_x^A = r_x^B r_B^A$, which gives the result.

(2) Firstly, a section that is zero on both A and B is then also zero on $A \cup B$, which gives the injective part of h . Now, suppose $s - t$ is the zero section over $A \cap B$ for s a section over A and t a section over B . Then s and t agree on $A \cap B$, meaning that $s \cup t$ is well-defined and continuous, where $s \cup t$ is s on A and t on B , and $h(s \cup t) = (s, t)$. Likewise, if g is a section over $A \cup B$, then $k \circ h(g) = (g|_A)|_{A \cap B} - (g|_B)|_{A \cap B} = g|_{A \cap B} - g|_{A \cap B}$ is the zero section.

(3)

□

Theorem 1.16. *Let $A \subset M$ be closed. Then*

- (1) $H_i(M, M - A; G) = 0$ for $i > n$.
- (2) $J^A: H_n(M, M - A, G) \rightarrow \Gamma_c(A, H_n(M, M - \bullet; G))$ is an isomorphism.

Lemma 1.17 (The Bootstrap Lemma). *Let $P_M(A)$ be a statement about compact sets A in a given n -manifold M^n . If (i), (ii), (iii) hold, then $P_M(A)$ is true for all compact A in M^n .*

If M^n is separable metric, and $P_M(A)$ is defined for all closed sets A , and if (i), (ii), (iii), (iv) hold, then $P_M(A)$ is true for all closed sets A in M .

For general M^n , if $P_M(A)$ is defined for all closed sets A in M , for all M^n , and if all five statement (i) – (v) hold for all M^n , then $P_M(A)$ is true for all closed $A \subset M$ and all M^n .

Now note that for a given abelian group G and $g \in G$, the following maps are natural in $A \subset M$ (closed):

$$H_n(M, M - A) \cong H_n(M, M - A) \otimes \mathbb{Z} \rightarrow H_n(M, M - A) \otimes G \rightarrow H_n(M, M - A; G)$$

where the middle map is induced by the homomorphism $\mathbb{Z} \rightarrow G$ taking 1 to g .

In particular, this induces a map

$$H_n(M, M - \bullet) \rightarrow H_n(M, M - \bullet; G)$$

Lemma 1.18. *The sections $\Gamma(A; G)$ of ω over A correspond bijectively to continuous maps $\lambda: \text{Ori}(M)|_A \rightarrow G$ with the property $\lambda \circ t = -\lambda$, where t acts on G as multiplication by -1 .*

Proof. We may assume A is connected.

Let $s \in \Gamma(A; G)$ be a section of ω over A . That is, $w \circ s = \text{id}_A$, and s is a map $A \rightarrow H_n(M, M - \bullet; G)$. We can define an associated map $\lambda_s: \text{Ori}(M)|_A \rightarrow G$ by sending a generator in the fiber $x \in A$ to $s(x) \in H_n(M, M - \{x\}; G) \cong G$. If one chose the other generator, one would get the negative of the above map, so we have the relation $\lambda_s \circ t = -\lambda_s$. Subject to this relation, we obtain a well-defined map $\Gamma(A; G) \rightarrow S \subset \text{Hom}(\text{Ori}(M)|_A, G)$, where S is the subset for which $\lambda \circ t = -\lambda$ holds. This map is certainly injective, since the image tells us precisely the value

of s at any point in A .

It is furthermore surjective, since if $\text{Ori}(M)|_A$ is connected, then S can only consist of the zero section, and if it is not connected, it consists of a map on two components on which it is constant, and the relation $\lambda \circ t = -\lambda$ then determines that is must be the required values to constitute the induced map of a section. \square

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Theorem 1.19. *Suppose $A \subset M$ is a closed connected subset. Then*

- (1) $H_n(M, M - A; G) = 0$ if A is not compact.
- (2) $H_n(M, M - A; G) \cong G$ if M is R -orientable along A and A is compact. Moreover, $H_n(M, M - A; G) \rightarrow H_n(M, M - x; G)$ is an isomorphism for each $x \in A$.
- (3) $H_n(M, M - A; G) \cong {}_2G = \{g \in G \mid 2g = 0\}$ if M is not orientable along A and A is compact.

Proof. (1) By Lemma 5.1, a section in $\Gamma(A; G)$ is determined by its value at a single point. By the existence of the zero section, if a section is non-zero at any point, then it is non-zero at every point. Therefore, there do not exist non-zero sections with compact support over a non-compact A , so by Theorem 1.16, $H_n(M, M - A; G) \cong \Gamma_c(A; G) \cong 0$.

(2) Since A is compact, $H_n(M, M - A; G) \cong \Gamma_c(A; G) = \Gamma(A; G)$. A section is again determined by a single point. Recall now the commutative diagram

$$\begin{array}{ccc} H_n(M, M - A; G) & \xrightarrow{\cong} & \Gamma(A; G) \\ \downarrow r_x^A & & \downarrow b \\ H_n(M, M - x; G) & \xrightarrow{\cong} & \Gamma(\{x\}; G) \end{array}$$

from Proposition 1.15, the horizontal isomorphisms following from Theorem 1.16. If M is orientable along A , there by definition exists in $\Gamma(A; G)$ an element such that its value at x is a generator. Hence b is an isomorphism, and therefore also r_x^A is an isomorphism.

(3) By Lemma 1.18, a section in $\Gamma(A; G)$ corresponds to a continuous map $\lambda: \text{Ori}(M)|_A \rightarrow G$ with $\lambda t = -\lambda$. If M is not orientable along A , then $\text{Ori}(M)|_A$ is connected and therefore λ is constant as G has the discrete topology. The relation $\lambda t = -\lambda$ shows that λ is in ${}_2G$. Now by the commutative diagram from part (2), note that since λ must be constant, firstly $\Gamma(A; G) \cong {}_2G$, and secondly, b becomes injective, so $r_x^A: H_n(M, M - A; G) \rightarrow H_n(M, M - x; G) \cong G$ is injective and has image ${}_2G$, so the Hom term vanishes. \square

Proposition 1.20. *Let M be an n -manifold and $A \subset M$ be a closed connected subset. Then the torsion subgroup of $H_{n-1}(M, M - A; \mathbb{Z})$ is of order 2 if A is compact and M non-orientable along A , and is 0 otherwise.*

Proof. By UCT for homology,

$$\begin{aligned} \mathbb{Z}/2 \cong {}_2\mathbb{Z}/2 \cong H_n(M, M - A; \mathbb{Z}/2) &\cong H_n(M, M - A) \otimes \mathbb{Z}/2 \oplus \text{Tor}_1(H_{n-1}(M, M - A), \mathbb{Z}/2) \\ &\cong \text{Tor}_1(H_{n-1}(M, M - A), \mathbb{Z}/2) \\ &\cong \{g \in H_{n-1}(M, M - A) \mid 2g = 0\}. \end{aligned}$$

where $H_n(M, M - A) \cong {}_2\mathbb{Z} = 0$, and $H_n(M, M - A; \mathbb{Z}/2) \cong {}_2\mathbb{Z}/2 \cong \mathbb{Z}/2$ both follow from Theorem 1.19.

To see that this is the whole torsions subgroup, note that for odd k ,

$$\mathrm{Tor}_1(H_{n-1}(M, M - A), \mathbb{Z}/k) \cong H_n(M, M - A; \mathbb{Z}/k) \cong {}_2\mathbb{Z}/k \cong 0$$

When M is orientable along A and A is compact, we simply obtain

$$0 \rightarrow H_n(M, M - A) \otimes \mathbb{Z}/n \rightarrow H_n(M, M - A; \mathbb{Z}/n) \rightarrow \mathrm{Tor}_1(H_{n-1}(M, M - A), \mathbb{Z}/n) \rightarrow 0$$

and since $H_n(M, M - A) \cong \mathbb{Z}$ and $H_n(M, M - A; \mathbb{Z}/n) \cong \mathbb{Z}/n$ by Theorem 1.19, we find that Tor_1 vanishes for all n .

If A is non-compact, then Theorem 1.19 gives that Tor_1 trivially vanishes for all terms. □

1.3. Fundamental Class.

Theorem 1.21. *Let M be a compact connected n -manifold. Then one of the following assertions holds:*

- (1) *M is orientable, $H_n(M) \cong \mathbb{Z}$, and for each $x \in M$, the restriction $H_n(M) \rightarrow H_n(M, M - x)$ is an isomorphism.*
- (2) *M is non-orientable and $H_n(M) = 0$.*

Proof. Special case of Theorem 1.19. □

Under the hypothesis of Theorem 1.21, the orientations of M correspond to the generators of $H_n(M)$. Such a generator will be called a *fundamental class* or *homological class/orientation* of the orientable manifold.

Definition 1.22 (Degree). Let M and N be compact oriented n -manifolds. Let N be connected and suppose M has components M_1, \dots, M_r . Then we have fundamental classes $z(M_j)$ for each M_j and $z(M) \in H_n(M) \cong \bigoplus_j H_n(M_j)$ is the sum of the $z(M_j)$. Now, since $H_n(N) \cong \langle z(N) \rangle \cong \mathbb{Z}$, we obtain that there exists a *degree* $d(f) \in \mathbb{Z}$ such that $f_*z(M) = d(f)z(N)$.

Lemma 1.23 (Properties). (1) *The degree is a homotopy invariant.*

- (2) $d(f \circ g) = d(f)d(g)$.
- (3) *A homotopy equivalence has degree ± 1 .*
- (4) *If $M = M_1 \sqcup M_2$, then $d(f) = d(f|_{M_1}) + d(f|_{M_2})$.*
- (5) *If we pass in M or N to the opposite orientation, then the degree changes the sign.*

1.3.1. *Computations of degrees.* As usual, we can compute degrees in terms of local data of a map.

Let M and N be connected and set $K = f^{-1}(p)$. Let U be an open neighborhood of K in M . Then in particular $M - U = \overline{M - U} \subset \mathrm{int}(M - A) = M - A$, so excision gives the bottom left isomorphism in the following diagram, and the top right isomorphism follows from Theorem 1.21:

$$\begin{array}{ccccc}
z(M) \in & H_n(M) & \xrightarrow{f_*} & H_n(M) & \ni z(N) \\
\downarrow & \downarrow & & \downarrow \cong & \downarrow \\
z(U, K) \in & H_n(M, M-K) & \xrightarrow{f_*} & H_n(N, N-p) & \\
& \cong \uparrow i_* & & \uparrow = & \\
& H_n(U, U-K) & \xrightarrow{f_*^U} & H_n(N, N-p) & \ni z(N, p)
\end{array}$$

From the outer rectangle, we get $f_*^U z(U, K) = d(f)z(N, p)$, where $z(N, p)$ and $z(U, K)$ are the images of $z(N)$ and $z(M)$ under the indicated maps.

We want to show additivity of degree as in the case for spheres.

So suppose K if finite, and choose $U = \bigcup_{x \in K} U_x$ where the U_x are pair-wise disjoint open neighborhoods of x . Then

$$\bigoplus_{x \in K} H_n(U_x, U_x - x) \cong H_n(U, U - K), \quad H_n(U_x, U_x - x) \cong \mathbb{Z}.$$

The image $z(U_x, x)$ of $z(M)$ is a generator: it is the image under the following isomorphisms

$$H_n(M) \xrightarrow{\cong} H_n(M, M - x) \xrightarrow{\cong} H_n(U_x, U_x - x)$$

where the first follows from Theorem 1.21 and the second from excision. The local degree $d(f, x)$ is determined by $f_* z(U_x, x) = d(f, x)z(N, p)$, and by additivity above, we have $d(f) = \sum_{x \in K} d(f, x)$.

Proposition 1.24. *Let M be a connected, oriented, closed n -manifold. Then there exists for each $k \in \mathbb{Z}$ a map $f: M^n \rightarrow S^n$ of degree k .*

Remark. If f is C^1 in a neighborhood of x , then $d(f, x)$ is the sign of the determinant of $Dg(0)$ when $Dg(x)$ is regular, where g is f in local coordinates that preserve local orientations. By this we mean that for $\varphi: U_x \rightarrow \mathbb{R}^n$ centered at x , $\varphi_*: H_n(U_x, U_x - x) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - 0)$ sends $z(U_x, x)$ to the standard generator. Such charts are called *positive* with respect to the given orientations.

Proof. If $f: M \rightarrow S^n$ has degree a and $g: S^n \rightarrow S^n$ degree b , then gf has degree ab . Since the proposition is true for $M = S^n$, it suffices to find f having degree ± 1 . Let $\varphi: D^n \rightarrow M$ be an embedding. Then we have a map $f: M \rightarrow D^n/S^{n-1}$ which is the inverse of φ on $U = \varphi(\text{int } D^n)$ and sends $M - U$ to the basepoint. This map has degree ± 1 as can be seen by choosing any neighborhood of x in the interior of U and looking at the determinant of the differential locally. \square

1.4. Manifolds with Boundary.

Definition 1.25. For M an n -dimensional manifold with boundary, we call $z \in H_n(M, \partial M)$ a *fundamental class* if for each $x \in M - \partial M$, the restriction of z is a generator in $H_n(M, M - x)$.

Theorem 1.26. *Let M be a compact connected n -manifold with non-empty boundary. Then one of the following assertions hold:*

- (1) $H_n(M, \partial M) \cong \mathbb{Z}$, and a generator of this group is a fundamental class. The image of a fundamental class under $\partial: H_n(M, \partial M) \rightarrow H_{n-1}(\partial M)$ is a fundamental class. The interior $M - \partial M$ is orientable.
- (2) $H_n(M, \partial M) = 0$, and $M - \partial M$ is not orientable.

Proof. See [2, Thm 16.5.1]. We will follow that proof and only add a few extra words.

Let $\kappa: [0, \infty[\times \partial M \rightarrow U$ be a collar of M , i.e., a homeomorphism onto an open neighborhood U of ∂M such that $\kappa(0, x) = x$ for $x \in \partial M$ (See Milnor's h -cobordism book for existence). For simplicity of notation, we identify U with $[0, \infty) \times \partial M$ via κ ; similarly, for subsets of U . In this sense, $\partial M = 0 \times \partial M$. For $A = M - ([0, 1) \times \partial M) \subset M - \partial M$, we have isomorphisms

$$H_n(M, \partial M) \cong H_n(M, [0, 1) \times \partial M) \cong H_n(M - \partial M, (0, 1) \times \partial M) \cong \Gamma(A).$$

The first one by homotopy equivalence, the second by excision, and the third using Theorem 1.16.

Since A is connected, $\Gamma(A) \cong \mathbb{Z}$ or $\Gamma(A) \cong 0$. If $\Gamma(A) \cong \mathbb{Z}$, then $M - \partial M$ is orientable along A .

Let now $A_\varepsilon \cong A$ be the complement of $[0, \varepsilon) \times \partial M$. Since each compact subset of $M - \partial M$ is contained in some such M_ε for small enough ε , we see that $M - \partial M$ is orientable along all compact subsets, hence orientable by Proposition 1.8.

The isomorphism $H_n(M - \partial M, (0, 1) \times \partial M) \cong \Gamma(A)$ from above says that there exists some $z \in H_n(M - \partial M, (0, 1) \times \partial M)$ which restricts to a generator of $H_n(M - \partial M, M - \partial M - x)$ for each $x \in A$. For the corresponding element $z \in H_n(M, \partial M) \cong \mathbb{Z}$, the same assertion holds for any $x \in M - \partial M$ (simply shrink the collar to not contain x). Lastly, we must show that ∂z is a fundamental class. Let $x \in (0, 1) \times \partial M$. Consider the diagram:

$$\begin{array}{ccccc} H_{n-1}(\partial M) & \xrightarrow{\cong} & H_{n-1}(\partial M \cup A, A) & \xleftarrow{\cong} & H_{n-1}(\partial I \times \partial M, 1 \times \partial M) \\ \partial \uparrow & & \cong \uparrow \partial & & \cong \uparrow \partial \\ H_n(M, \partial M) & \longrightarrow & H_n(M, \partial M \cup A) & \xleftarrow{\cong} & H_n(I \times \partial M, \partial I \times \partial M) \\ & \searrow & \downarrow & & \\ & & H_n(M, M - x) & \xleftarrow{\cong} & H_n(I \times \partial M, I \times \partial M - x) \end{array}$$

Commutativity of the bottom left triangle tells us that the image of z under $H_n(M, \partial M) \rightarrow H_n(M, \partial M \cup A)$ gives an element whose restriction gives a generator in $H_n(M, M - x)$, but then by commutativity of the bottom right square, we get that the restriction of z transferred over by the isomorphism to $H_n(I \times \partial M, \partial I \times \partial M)$ is a generator of $H_n(I \times \partial M, I \times \partial M - x)$ at each point in $(0, 1) \times \partial M$. Hence z yields a fundamental class in $H_n(I \times \partial M, \partial I \times \partial M)$.

But since z is a generator in $H_n(I \times \partial M, \partial I \times \partial M)$, the upper part shows that z is a generator in $H_{n-1}(\partial M)$, thus a fundamental class of ∂M since this characterizes fundamental classes.

□

Example 1.27. Suppose that $B: M \rightarrow \emptyset$ is a cobordism. We have the fundamental classes $z(B) \in H_{n+1}(B, \partial B)$ and $z(M) = \partial z_B \in H_n(M)$ (here we crucially made use of our result in Theorem 1.26). This is already a lot of information. Indeed, suppose $f: M \rightarrow N$ is a map which has an extension to $B: F: B \rightarrow N$. Then the degree of f (if defined) is zero, $d(f) = 0$, for we have $f_* z(M) = f_* \partial z(B) = F_* i_* \partial z(B) = 0$, since $i_* \partial = 0$ by the exactness of the homology sequence for the pair (B, M) .

We call maps $f_\nu: M_\nu \rightarrow N$ *orientable bordant* if there exists a compact oriented cobordism $B: M_1 \rightarrow M_2$ with orientable boundary $\partial B = M_1 - M_2$ (meaning $\partial z(B) = z(M_1) - z(M_2)$) and an extension $F: B \rightarrow N$ of $f_1 \sqcup f_2: M_1 \sqcup M_2 \rightarrow N$. Under these assumptions, we have $d(f_1) = d(f_2)$. This fact is called the *bordism invariance* of the degree; it generalizes the homotopy invariance.

2. INTERSECTION THEORY

Definition 2.1 (*k*-disk bundle). A *k*-disk bundle is a vector bundle whose coordinate transformations are contained in $O(k) \subset GL(\mathbb{R}^k)$ and such that the local trivializations have the form $\pi^{-1}(U) \cong U \times D^k$.

Let N^n be a connected, oriented, closed n -manifold, and W^{k+n} an $(n+k)$ -manifold with boundary ∂W a $(k-1)$ -sphere bundle over N^n , and let $\pi: W^{n+k} \rightarrow N^n$ be a *k*-disk bundle over N .

Let us assume also that W is also oriented.

Definition 2.2. In the above situation, the *Thom class* of the disk bundle π is the class $\tau \in H^k(W, \partial W)$ given by

$$\tau = D_W(i_*[N])$$

where $D_W: H_{n-k}(W) \rightarrow H^k(W, \partial W)$ is the inverse of the Poincaré duality isomorphism. That is,

$$D(a) \cap [M] = a.$$

Thus

$$\tau \cap [W] = i_*[N].$$

We can deformation retract the punctured disk to its boundary, giving $H^k(W, W - N) \cong H^k(W, \partial W)$, so we will sometimes regard τ as being in $H^k(W, W - N)$.

Lemma 2.3. *In the above setup, suppose $A \subset N$ is closed. Let $\tilde{A} = \pi^{-1}(A) \subset W$ and $\partial \tilde{A} = \tilde{A} \cap \partial W$. Then $\check{H}^i(\tilde{A}, \partial \tilde{A}) = 0$ for $0 < i < k$.*

Proof. Suppose first that A is compact convex subset of a Euclidean neighborhood in N . It also suffices consider the case where A is connected, so $A \cong D^n$. Consider the pullback bundle of A :

$$\begin{array}{ccc} i^*(A) & \longrightarrow & W \\ \downarrow & & \downarrow \pi \\ A & \xhookrightarrow{i} & N \end{array}$$

Then $i^*(A) = A \times_N W \cong \pi^{-1}(A)$, so since any vector bundle over a contractible paracompact base space is trivial, we conclude that the bundle $\tilde{A} \rightarrow A$ is trivializable as $\tilde{A} \cong A \times D^k$ and $\partial \tilde{A} \cong A \times S^{k-1}$. Now the steps are as follows: calculate the homology of $A \times D^k$ and $A \times S^{k-1}$, then use UCT to obtain the cohomology, and then use the LES to find the cohomology of $(A \times D^k, A \times S^{k-1})$.

Now... But by the Künneth theorem,

$$H_m(A \times D^k) \cong H_m(A)$$

and

$$H_m(A \times S^{k-1}) \cong H_m(A) \oplus H_{m-k+1}(A).$$

□

Lemma 2.4. *The restriction $\tau_x \in \check{H}^k(\tilde{A}, \partial\tilde{A})$ of τ , when $A = \{x\}$, is a generator.*

Proof. Note that $(\tilde{A}, \partial\tilde{A}) \cong (D^k, S^{k-1})$.

Suppose first that $\tau_x = 0$ for some x .

Now, recall that

$$\tau_A = D_W(i_*[A]).$$

Then $\tau_x = 0$ if and only if $i_*[x] = 0$. But $i_*: H_*(N, N-x) \rightarrow H_*($

□

3. THOM-PONTRYAGIN THEORY

We start with an element $[f] \in \pi_{n+k}(S^n)$, so f is a pointed map $S^{n+k} \rightarrow S^n$.

Now insert a disk in place of the base point, and extend f to a map \bar{f} which is constant on the next disk, taking the disk to the basepoint of S^n , and is f elsewhere. There is a deformation retract of the sphere, collapsing this disk to a point, and composing with this retract gives f . Hence we may replace f by a pointed-homotopic map which is constant in a small neighborhood of the basepoint. Next, we can remove the base point of S^{n+k} and instead consider f as a map $\mathbb{R}^{n+k} \rightarrow S^n$ which is now constant to the base point outside some compact subset of \mathbb{R}^{n+k} .

Insert theorem

By the Smooth Approximation Theorem, we can also restrict attention to smooth maps $\mathbb{R}^{n+k} \rightarrow S^n$ and smooth homotopies.

We regard also S^n as the one-point compactification of \mathbb{R}^n , denoted $\mathbb{R}_+^n = \mathbb{R}^n \cup \{\infty\}$.

So suppose now we have a smooth map $f: \mathbb{R}^{n+k} \rightarrow \mathbb{R}_+^n$ as above.

If f is not null-homotopic, then it must be surjective, hence in particular the image does not have measure 0, so there exists a regular value $p \in \mathbb{R}^n \subset \mathbb{R}_+^n$. By following f by a translation in \mathbb{R}^n , we can assume that p is the origin $0 \in \mathbb{R}^n$ without changing the homotopy class of f .

Theorem 3.1 ([1], Thm 11.6). *Let $f: \mathbb{R}^n \rightarrow M^m$ be a smooth map. Assume that $p \in M^m$ is a regular value, let $K = f^{-1}\{p\}$, and assume that K is compact. Then there is an open neighborhood N of K inside a tubular neighborhood of K , with normal retraction $r: N \rightarrow K$, and an open neighborhood $E \cong \mathbb{R}^m$ of p in M^m such that the map $r \times f: N \rightarrow K \times E$ is a diffeomorphism.*

Using Theorem 3.1, we find that there is a disk E^n about 0 in \mathbb{R}^n and an embedding $M^k \times E^n \hookrightarrow N \subset \mathbb{R}^{n+k}$ onto an open neighborhood N of M^k whose inverse $N \rightarrow M^k \times E^n$ is $r \times f$, where $r: N \rightarrow M^k$ is the normal retraction.

Through another homotopy of f , we can assume that E^n is the open unit disk D^n .

We will refer to an embedding $g: M^k \times E^n \rightarrow \mathbb{R}^{n+k}$, with M^k compact, as a "fattened k -manifold".

4. TERMINOLOGY

Definition 4.1 (Neighborhood retract). If $A \subset X$ and A has a neighborhood in X of which it is a retract, then A is called a *neighborhood retract* (in X).

Note. Saying that $A \hookrightarrow X$ is a cofibration is stronger than saying that A is a neighborhood retract.

5. LEMMAS

Lemma 5.1. *Let $\pi: W \rightarrow N$ be a covering map and M a connected space. Suppose $f, g: M \rightarrow W$ are maps such that $\pi \circ f = \pi \circ g$ and that $f(x) = g(x)$ for some $x \in M$. Then $f = g$.*

Proof. Show that the set

$$Z = \{z \in M \mid f(z) = g(z)\}$$

is closed and open. □

REFERENCES

- [1] Glen E. Bredon. *Topology and geometry*. **volume** 139. Graduate Texts in Mathematics. Corrected third printing of the 1993 original. Springer-Verlag, New York, 1997.
- [2] Tammo tom Dieck. *Algebraic topology*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008, **pages** xii+567. ISBN: 978-3-03719-048-7.