Assignment 1

Jonas Trepiakas - 3039733855 - jtrepiakas@berkeley.edu

5: Let $F\colon X \times I \to X$ be a deformation retraction of X onto x_0 such that $F(x, 0) = \mathbb{1}$ and $F(X, 1) = x_0$ and $F(x_0, t) = x_0$ for all t. Let $U \subset X$ be a neighborhood of x_0 in X. Since F is continuous, $F^{-1}(U)$ is open and nonempty as $x_0 \in F^{-1}(U)$, and since U is open and I is open in I, we have by definition of the product topology that $U \times I$ is open in $X \times I$ and thus $F^{-1}(U) \cap U \times I$ is open and nonempty as $\{x_0\} \times I \subset F^{-1}(U) \cap U \times I$. We can thus write $F^{-1}(U) \cap U \times I$ as a union of basis elements $\bigcup_\alpha V_\alpha \cup W_\alpha$ where all V_α are open in X and W_α open in I. Since $\{x_0\} \times I \subset F^{-1}(U)$, we have $\bigcup_\alpha W_\alpha = I$, and since I is compact, there exist $\alpha_1, \ldots, \alpha_n$ such that $W_{\alpha_1} \cup \ldots \cup W_{\alpha_n} = I$. Then $V = V_{\alpha_1} \cap \ldots \cap V_{\alpha_n}$ is open as the finite intersection of open sets and nonempty as all contain x_0, and we find that $F(V \times I) \subset U$ since for any $(x, t) \in V \times I$, there exists a $1 \leq i \leq n$ such that $(x, t) \in V_{\alpha_i} \times W_{\alpha_i}$ and $F\left(V_{\alpha_1} \times W_{\alpha_1}\right) \subset F\left(F^{-1}(U) \cap U \times I\right) \subset U$. Now $V \times I$ is open and satisfies that the restriction $F|_{V\times I}\colon V \times I \to U$ is well-defined and a homotopy between the inclusion map $\mathbb{1}_V\colon V \to U$ and the constant function sending V to x_0 since $F(x, 0) = \mathbb{1}_V$ and $F(x, 1) = x_0$, and it is continuous since for any open set W of the subspace U, we have $F|_{V\times I}^{-1}(W) = F^{-1}(W) \cap V \times I$ which is the intersection of open sets and thus open.
Hence the inclusion map $\mathbb{1}_V\colon V \to U$ is nullhomotopic.

15: We have $S^\infty = \bigcup_n S^n$ and each S^n can be built inductively from S^{n-1} by attaching two n-cells (by the procedure on page 7, Hatcher). If now A is a subcomplex of S^∞ that contains an n-cell, it must also contain the boundary of this n-cell since a subcomplex is a closed subspace. The boundary of an n-cell, ∂e^n, is an $(n-1)$-cell which in the n-skeleton S^n would be S^{n-1} in this cell structure. So if the subcomplex A has finite dimension n, then it is of the form $e_1^n \bigcup_{i<n} e_1^i \cup e_2^i$ or $e_1^n \cup e_2^n \bigcup_{i<n} e_1^i \cup e_2^i$. If A is infinite dimensional, it must be S^∞ since it is a union of cells and assume it does not contain a cell e_z^l. Since it is not finite dimensional, it must contain a cell e^j with $j > l$ and hence by the finite case, it must contain $\bigcup_{i<j} e_1^i \cup e_2^i$ which contains e_z^j.