1. Problems

Definition 1.1. Let M be a smooth manifold. A Morse function $f\colon M \to \mathbb{R}$ is a smooth map such that all its critical points are non-degenerate, with pairwise distinct critical values in $\mathbb{R}$.

1.1. Reeb's Theorem.

Problem 1.2 (Reeb's Theorem)**.** (6 pts) Let M be a smooth, compact manifold of dimension d. Show that if M admits a Morse function with only two critical points, then M is homeomorphic to the sphere S^d. Indicate why the above proof fails in showing that M is diffeomorphic to the sphere S^d.

For the proof, we state a theorem that we will need:

Definition 1.3. For a smooth map $f\colon M \to \mathbb{R}$ on a smooth manifold M, let $M^a = f^{-1}(-\infty, a]$.

Theorem 1.4. *Let $f \in C^\infty(M)$ on a manifold M. Let $a < b$ and suppose that the set $f^{-1}[a,b]$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

Proof of Problem 1.2. Since M is compact, we have that $f(M) = [a,b] \subset \mathbb{R}$. Without loss of generality, assume that $f(M) = [0,1]$.

We shall need the following lemma from analysis:

Lemma 1.5 (Fermat's Theorem)**.** *Let $f\colon (a,b) \to \mathbb{R}$ be a function on an open interval $(a,b) \subset \mathbb{R}$. Suppose f has a local extremum at $x_0 \in (a,b)$. If f is differentiable at x_0, then $f'(x_0) = 0$.*

Now, we claim that the two critical points are precisely the preimages of 0 and 1. For suppose $x \in f^{-1}(0)$. Then x is a global minimum for f. Taking some chart centered around x, we have a local representation of f as a function $\mathbb{R}^d \to [0,1]$ with a global minimum at 0. Taking the partial derivatives of f and applying Fermat's theorem to each of them, we find that each partial derivative evaluated at 0 is 0: $\frac{\partial f}{\partial x^i}(0) = 0$. Hence we find that $Df(0) = 0$, so transfering back to the manifold, $Df(x) = 0$, so $x \in M$ is a critical point. The same argument applies to show that any $y \in f^{-1}(1)$ is a critical point. Since there are only two critical points, this immediately forces $f^{-1}(0)$ and $f^{-1}(1)$ to be singletons and thus global maximum and minimum of M. Suppose without loss of generality that $p \in M$ is the minimum and $q \in M$ is the maximum.
By Morse's Lemma, in some coordinate system about p, let's say in a neighborhood U, f takes the form

$$f(x_1, \ldots, x_n) = -x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2.$$

Now p is a global minimum, so in fact, we must have that $\lambda = 0$. That is

$$f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2$$

in this neighborhood. Since also $f(U)$ is open in the subspace topology and contains 0, we can find an open disk $\tilde{D}_1$ centered at 0 of radius ε_1 such that $\tilde{D}_1 \cap [0,1] \subset f(U)$,

and let D_1 be the inverse of $\tilde{D}_1$ under this local diffeomorphism.
Similarly, in a neighborhood V of q, f takes the form

$$f(x_1, \ldots, x_n) = 1 - x_1^2 - x_2^2 - \ldots - x_n^2.$$

Again take some open disk $\tilde{D}_2$ centered at 1 of radius ε_2 such that $\tilde{D}_2 \cap [0,1] \subset f(V)$.
Let D_2 be the inverse image under f of $\tilde{D}_2$.
We wish to show that there exists some $\varepsilon > 0$ such that $f^{-1}[0,\varepsilon]$ and $f^{-1}[1-\varepsilon, 1]$ are homeomorphic to the closed n-disk D^n. There exist $\alpha, \beta \in (0,1)$ such that $f(M - D_1 \cup D_2) = [\alpha, \beta]$ since $M - D_1 \cup D_2$ is still compact. Now simply let $0 < \varepsilon < \min\left\{\alpha, 1-\beta, \varepsilon_1, 1-\varepsilon_2, 1-\varepsilon_1, \frac{1}{4}\right\}$. To see that this works, simply note that $f^{-1}[0,\varepsilon] \subset D_1 \cup D_2$. On D_1, f takes values in $[0, \varepsilon_1]$ and on D_2, f takes values in $[1-\varepsilon_2, 1]$. But $\varepsilon < \varepsilon_1$, so $[0,\varepsilon_1] \subset [0,\varepsilon]$, so $D_1 \subset f^{-1}[0,\varepsilon]$, while $\varepsilon < 1-\varepsilon_2$, so $[1-\varepsilon_2, 1] \not\subset f^{-1}[0,\varepsilon]$. Similarly, $1-\varepsilon > \varepsilon_1$, so $D_1 \subset [0,\varepsilon_1] \not\subset f^{-1}[1-\varepsilon, 1]$ while $1-\varepsilon_2 > 1-\varepsilon$, so $D_2 \subset [1-\varepsilon_2, 1] \subset f^{-1}[1-\varepsilon, 1]$.

Therefore, since $f^{-1}[0,\varepsilon] \subset D_1 \subset U$ and we know that on U, f takes the form

$$f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2,$$

we know that $f^{-1}[0,\varepsilon]$ is precisely a closed disk about p. Likewise, $f^{-1}[1-\varepsilon, 1]$ can be seen to be a closed disk about q.

But now by Theorem 1.4, since there are no critical points in $f^{-1}[\varepsilon, 1-\varepsilon]$ by assumption, M^ε is diffeomorphic to $M^{1-\varepsilon}$. Hence we find that $M^{1-\varepsilon}$ and $f^{-1}[1-\varepsilon, 1]$ are both diffeomorphic to closed d-disks, and furthermore, M is obtained by gluing these d-disks along their boundary which is homeomorphic to S^{d-1}. We claim that this is sufficient to conclude that M is *homeomorphic* to S^d. The problem is that while we have individual diffeomorphisms $M^{1-\varepsilon} \cong D^n$ and $f^{-1}[1-\varepsilon, 1] \cong D^n$, the identifications of the boundaries might not be preserved under these diffeomorphisms, so we might not be able to reglue after. Let $\varphi_1 \colon M^{1-\varepsilon} \cong D^d$ and $\varphi_2 \colon f^{-1}[1-\varepsilon, 1] \cong D^d$ be the diffeomorphisms. Then $\varphi_1 \circ \varphi_2^{-1}$ is a diffeomorphism of S^{d-1}, and

$$M \cong D^d \sqcup_{\varphi_1 \circ \varphi_2^{-1}} D^d.$$

We construct a homeomorphism $\psi \colon D_1 \sqcup_{\mathrm{id}} D_2 \to D^d \sqcup_{\varphi_1 \circ \varphi_2^{-1}} D^d$ by

$$\psi(x) = \begin{cases} x & , x \in D_1 \\ 0 & , x \in D_2 \text{ and } x = 0 \\ \|x\| \varphi_1 \circ \varphi_2^{-1}\left(\frac{x}{\|x\|}\right), & x \in D_2 - \{0\} \end{cases}$$

As the sphere is compact and the twisted sphere Hausdorff, this map is a homeomorphism. The reason it might fail to be a diffeomorphism, is that on $D_2 - \{0\}$, as we let x approach 0, we might have non-agreeing derivatives from different directions.

A different way of obtaining a homeomorphism is as follows: since φ_1 and φ_2 can be chosen to both be orientation-preserving, for example by precomposing with an orientation reversing self-homeomorphism of the disk, we find that $\varphi_1 \circ \varphi_2^{-1}$ is isotopic through topological embeddings to the identity. Now applying an isotopy extension theorem, [1, Thm 1.3, p. 180], this isotopy extends to an ambient isotopy of S^d with compact support.

$\square$

1.2. **Existence of Morse functions.**

Problem 1.6 (Existence of Morse functions)**.** (6pts) Show that any smooth manifold M admits a Morse function.

Proof. Suppose M is of dimension k. By the Whitney embedding theorem, we can smoothly embed M in $\mathbb{R}^n$ for some $n \geq k$. Let $N \subset M \times \mathbb{R}^n$ be defined by

$$N = \left\{(q, v) : q \in M, v \in M_q^{\perp}\right\}$$

Lemma 1.7. *N is an n-dimensional manifold smoothly embedded in $\mathbb{R}^{2n}$.*

Define $E\colon N \to \mathbb{R}^n$ by $E(q, v) = q + v$.

Definition 1.8. A point $e \in \mathbb{R}^n$ is called a *focal point of (M, q) with multiplicity μ* if $e = q + v$ where $(q, v) \in N$ and the Jacobian of E at (q, v) has nullity $\mu > 0$. The point e will be called a *focal point* of M if e is a focal point of (M, q) for some $q \in M$.

Definition 1.9 (Critical point)**.** For our purposes, we will define a critical point of a smooth map f to be a point where the Jacobian is singular, i.e., $\det df = 0$. In particular, critical points in the usual definition where df vanishes at the point are included in this definition since if df vanishes at p, then $\det df(p) = 0$.

Now, since N is an n-manifold, note that $E\colon N \to \mathbb{R}^n$ is a map between two n-dimensional manifolds. In particular, dE is a map between two n-dimensional tangent spaces at each point. Therefore, by definition, a point $e \in \mathbb{R}^n$ is a focal point $e = q + v$ with $(q, v \in N)$ if and only if dE is not injective at (q, v) if and only if $\det dE_{(q,v)} = 0$ if and only if (q, v) is a critical point of E. But E is clearly smooth, so by Sard's theorem, the set of critical *values* of E which corresponds to the set of focal points has Lebesgue measure 0.

Let now $\left(U, \left(u^i\right) = \varphi\right)$ be a chart on M with $i = 1, \ldots, k$, and consider the inclusion $M \hookrightarrow \mathbb{R}^n$. Then we obtain natural coordinates in $\mathbb{R}^n$ given by $\mathbb{R}^k \overset{\varphi^{-1}}{\hookrightarrow} M \hookrightarrow \mathbb{R}^n$. We let $x_1\left(u_1, \ldots, u_k\right), \ldots, x_n\left(u_1, \ldots, u_k\right)$ be these maps and $x = \left(x_1, \ldots, x_n\right) : \mathbb{R}^k \to \mathbb{R}^n$.

Definition 1.10 (First and second fundamental forms)**.** Given the above setup, we call the following matrix the first fundamental form:

$$g_{ij} = \left(\tfrac{\partial x}{\partial u^i} \cdot \tfrac{\partial x}{\partial u^j}\right),$$

the dot signaling the usual dot product.
Similarly, we define a matrix (l_{ij}) called the second fundamental form where l_{ij} is the summand of the vector $\frac{\partial^2 x}{\partial u^i \partial u^j}$ which is normal to M.

$\square$

1.3. **On the Transversality Theorem.**

Problem 1.11 (On the transversality theorem)**.** Let M be a smooth manifold.

(1) Let $X \subset M$ be a smooth submanifold, and let $f\colon Y \to M$ be a smooth map, where Y is a smooth manifold. Show that f is smoothly homotopic to a map that intersects X transversally at every point.
(2) Show that in the above setting, if $f\colon Y \to M$ intersects X transversally, then $f^{-1}(X)$ is a smooth submanifold of Y such that $\dim Y + \dim f^{-1}(X) = \dim X$.

References

[1] Morris W. Hirsch. *Differential topology.* **volume** 33. Graduate Texts in Mathematics. Corrected reprint of the 1976 original. Springer-Verlag, New York, 1994, **pages** x+222. ISBN: 0-387-90148-5.