1. Topological groups

Theorem 1.1. *Every topological group is completely regular.*

Proof. a ☐

2. Local Compactness

Proposition 2.1. *Every locally compact Hausdorff space is completely regular ($T_{3\frac{1}{2}}$).*

Proof. Idea: a locally compact Hausdorff space is homeomorphic to an open subset of its one-point compactification which is a compact Hausdorff space and thus of a normal space. ☐

3. Separation axioms

3.1. **Urysohn Lemma.**

Theorem 3.1 (Urysohn Lemma)**.** *Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map*

$$f : X \to [a, b]$$

such that $f(A) = \{a\}$ and $f(B) = \{b\}$.

3.2. **Strong Urysohn.**

Definition 3.2 (G_δ set)**.** A set $A \subset X$ is a G_δ set in X if A is the intersection of a countable collection of open sets of X.

Theorem 3.3. *Let X be normal. There exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) > 0$ for $x \notin A$, if and only if A is a closed G_δ set in X.*
A function satisfying the requirements of this theorem is said to ***vanish precisely on*** *A.*

Proof. The idea is to use the standard idea of the Urysohn lemma: namely, constructing sets U_p for all $p \in Q$ and defining a continuous function in $[0, 1]$ from these.
Suppose $A = \bigcap_{i\in\cap} V_i$ where V_i are open. If necessary, we can redefine $V_i' = \bigcap_{j=1}^{i} V_i$ so that $V_1' \supset V_2' \supset V_3' \supset \ldots$, so assume $V_1 \supset V_2 \supset \ldots$. Now let $U_1 = V_1$. Suppose $U_{\frac{1}{k}}$ is defined for $1 \le k \le n$. By normality, we can find an open set $U_{\frac{1}{n+1}}$ such that $A \subset U_{\frac{1}{n+1}} \subset \overline{U_{\frac{1}{n+1}}} \subset U_n \cap V_n$. In this way we define sets U_p for $p \in \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Suppose we have defined U_p for all $p \in P \subset \mathbb{Q} \cap (1, 0]$ where $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \subset P$. Choose $p \in (\mathbb{Q} \cap (1, 0]) - P$ and suppose $a < p < b$ where $a, b \in P$, the by normality we can find an open set U_p such that $\overline{U_a} \subset U_p \subset \overline{U_p} \subset U_b$. In this way we define U_p for all $p \in \mathbb{Q} \cap (0, 1]$. Now define $U_0 = A$. Define $U_p = \varnothing$ for $p < 0$ and $U_p = X$ for $p > 1$. Define $\mathbb{Q}(x) = \{p \mid x \in U_p\}$ and $f(x) = \inf \mathbb{Q}(x)$. Clearly if $x \in A$ then $f(x) = 0$. If $x \notin A$ then there exists N such that for all $i \ge N$, we have $x \notin V_i$. Hence for all $q \in [0, \frac{1}{N+1}) \cap \mathbb{Q}$, $x \notin U_q$, so $\inf \mathbb{Q}(x) \ge \frac{1}{N+1} > 0$, hence $f(x) > 0$ as desired. Now to prove continuity: let $x_0 \in X$ and $f(x_0) \in (c, d)$. Choose numbers p, q such that $c < p < f(x_0) < q < d$. Then $U_q - \overline{U_p}$ is an open neighborhood of x_0, and if $x \in U_q - \overline{U_p}$ then $x \in U_q \subset \overline{U_q}$ so $f(x) \le q$ and since $x \notin \overline{U_p}$, $x \notin U_p$ so $f(x) \ge p$. Thus $f(x) \in [p, q] \subset (c, d)$. Thus f is continuous. ☐

Theorem 3.4 (Strong Urysohn lemma)**.** *Let X be a normal space. There is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) = 1$ for $x \in B$, and $0 < f(x) < 1$ otherwise, if and only if A and B are disjoint closed G_δ sets in X.*

Proof. By the preceding theorem, we can find maps $f_1 : X \to [0, 1]$ such that $f_1(A) = \{0\}$ and $f_1(X - A) \subset (0, 1]$ and $g_2 : X \to [0, 1]$ such that $g_2(B) = \{0\}$ and $g_2(X - B) \subset (0, 1]$. Now defining $f_2(x) = 1 - g_2(x)$ we find that $f_2(B) = \{1\}$ and $f_2(X - B) \subset [0, 1)$.

$$h(x) = \frac{f_1(x)}{f_1(x) + g_2(x)}$$

satisfies the desired properties. ☐

3.3. **Tietze Extension Theorem.**

Theorem 3.5 (Tietze extension theorem)**.** *Let X be a normal space; let A be a closed subspace of X.*

(1) *Any continuous map $A \to [a,b] \subset \mathbb{R}$ may be extended to a continuous map $X \to [a,b]$.*
(2) *Any continuous map $A \to \mathbb{R}$ may be extended to a continuous map $X \to \mathbb{R}$.*

Remark. Suppose $A, B \subset X$ are closed disjoint subsets and X is normal. Then define a map $f\colon A \cup B \to [0,1]$ by $f(A) = \{0\}$ and $f(B) = \{1\}$ where $A \cup B$ is in the subspace topology. This is well-defined as A and B are disjoint, and it is continuous since if $C \subset [0,1]$ is a closed set, its preimage under f is either empty or A or B or $A \cup B$, all of which are closed.

Exercise 3.6 (Munkres, 35.3)**.** Let X be metrizable. Show that the following are equivalent:

(1) X is bounded under every metric that gives the topology of X.
(2) Every continuous function $\varphi\colon X \to \mathbb{R}$ is bounded.
(3) X is limit point compact.

Proof. $(i) \implies (ii)$: Suppose for any metric d giving the topology of X, $d(x,y) \leq M_d$ for all $x, y \in X$. Now let $\varphi\colon X \to \mathbb{R}$ be continuous. The map $X \to X \times \mathbb{R}$ by $x \mapsto x \times \varphi(x)$ is an embedding. Define a metric $\mu\colon X \times X \to \mathbb{R}$ by $\mu(x,y) = d(x,y) + |\varphi(x) - \varphi(y)|$. Then clearly $\mu \geq 0$ with $\mu(x,y) = 0$ if and only if $x = y$. Clearly, $\mu(x,y) = \mu(y,x)$. Further,

$$\mu(x,y) \leq d(x,z) + d(z,y) + |\varphi(x) - \varphi(z)| + |\varphi(z) - \varphi(y)| = \mu(x,z) + \mu(z,y).$$

Hence μ is indeed a metric. Now, suppose we take an ε-ball $B_d(x,\varepsilon)$ in X with the metric d, so $B_d(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$. Since

$$d(x,z) \leq d(x,z) + |\varphi(x) - \varphi(z)| = \mu(x,z)$$

we have $B_\mu(x,\varepsilon) \subset B_d(x,\varepsilon)$. Let $B = \varphi^{-1}\left(B(\varphi(x), \frac{\varepsilon}{2})\right)$. Since d induces the topology on X, we can choose $\delta' < \frac{\varepsilon}{2}$ such that $B_d(x,\delta') \subset B$. Then for $z \in B_d(x,\delta')$, we have

$$\mu(x,z) = d(x,z) + |\varphi(x) - \varphi(z)| < \delta' + \frac{\varepsilon}{2} < \varepsilon;$$

hence $B_d(x,\delta') \subset B_\mu(x,\varepsilon)$. Thus μ induces the same topology on X as d, so X is bounded under μ. But then there exists N such that $\mu(x,y) \leq N$ for all $x, y \in X$. Then since $d(x,y) \leq M_d$ for all $x, y \in X$, we have $|\varphi(x) - \varphi(y)| \leq N - M_d$ for all $x, y \in X$, so φ is bounded.

$(2) \implies (3)$: Now, suppose every continuous function $\varphi\colon X \to \mathbb{R}$ is bounded. Suppose A has an infinite set of points which has no limit points. Then $\overline{A} = A$, so A is closed. Take some sequence $(x_i)_{i \in \mathbb{Z}_+} \subset A$. Similarly, $\{x_i\}_{i \in \mathbb{Z}_+}$ is closed in X since it has no limit points in X. Define $f\colon \{x_i\}_{i \in \mathbb{Z}_+} \subset A \to \mathbb{R}$ by $f(x_i) = i$. Then if $(a,b) \subset \mathbb{R}$ with $(a,b) \cap \mathbb{Z}_+ = \{n, n+1, \ldots, n+k\}$, we have $f^{-1}((a,b)) = \{x_n, x_{n+1}, \ldots, x_{n+k}\}$ which is open since each $\{x_i\}$ is an open set: this is because since A is not limit point compact, for each x_i, there exists a neighborhood U_i such that $U_i \cap A = \{x_i\}$. Thus f is continuous and also f is a surjection onto $\mathbb{Z}_+$. Now, since X is a metric space and thus normal, and since A is closed and f is a map $\{x_i\}_{i \in \mathbb{Z}_+} \to \mathbb{R}$, the Tietze extension theorem says that f can be extended to a map $X \to \mathbb{R}$. However, also f is unbounded since it is surjective on $\mathbb{Z}_+$, yet all continuous maps $X \to \mathbb{R}$ are bounded by assumption. Thus A must contain some limit point, and hence X is limit point compact.

$(3) \implies (1)$: We show the contrapositive. Assume X is not bounded under a metric d inducing the topology on X. Let $x_0 \in X$. If there does not exist a point $y \in X$ such that $d(x_0, y) > N$ for some $N > 0$ then for any two points $x, y \in X$

$$d(x,y) \leq d(x, x_0) + d(x_0, y) \leq 2N,$$

so X will be bounded. Hence we can choose a point $x_1 \in X$ such that $d_1 := d(x_0, x_1) > \varepsilon$. Then to define d_n, choose x_n such that $d(x_0, x_n) > \varepsilon + \sum_{i=0}^{n-1} d(x_i, x_{i+1})$ and define $d_n := d(x_0, x_n)$. We claim that the set $\{x_i\}_{i \in Z_0^+}$ is not limit point compact. Suppose first x_n is a limit point. Then

$$d(x_n, x_m) > d(x_n, x_{m-1}) - d(x_m, x_{m-1}) > \ldots > d(x_n, x_0) - \sum_{r=0}^{m-1} d(x_{r+1}, x_r) > \varepsilon$$

hence $B_{x_i,\frac{\varepsilon}{2}} \cap \{x_j\}_{j\in\mathbb{Z}_0^+} = \{x_i\}$ for all $i \in \mathbb{Z}_0^+$.
Now suppose some $z \in X$ is a limit point. Then there exists x_N such that $x_N \in B\left(z, \frac{\varepsilon}{2}\right)$. By normality, we can choose some δ with $\frac{\varepsilon}{2} > \delta > 0$ such that $x_N \notin B(z,\delta)$, so $d(z,x_N) \geq \delta$. Then for any $M \neq N$

$$d(z,x_M) \geq d(x_M,x_N) - d(z,x_N) > \varepsilon - d(z,x_N) > \frac{\varepsilon}{2} > \delta$$

Hence $d(z,x_i) > \delta$ for all i, so z is not a limit point as $B(z,\delta) \cap \{x_j\}_{j\in\mathbb{Z}_0^+} = \varnothing$. Thus $\{x_j\}_{j\in\mathbb{Z}_0^+}$ is an infinite subset of X with no limit point in X, so X is not limit point compact. $\square$

Exercise 3.7 (Munkres 35.4)**.** Let Z be a topological space. If Y is a subspace of Z, we say that Y is a **retract** of Z if there is a continuous map $r\colon Z \to Y$ such that $r(y) = y$ for each $y \in Y$.

(1) Show that if Z is Hausdorff and Y is a retract of Z, then Y is closed in Z.
(2) Let A be a two-point set in $\mathbb{R}^2$. Show that A is not a retract of $\mathbb{R}^2$.
(3) Let S^1 be the unit circle in $\mathbb{R}^2$; show that S^1 is a retract of $\mathbb{R}^2 - \{0\}$.

Proof. (1) : Suppose z is a limit point of Y which is not in Y. Then all neighborhoods of z intersect Y. Since $z \neq r(z)$, we can take neighborhoods U, V of z and $r(z)$ respectively, such that $U \cap V = \varnothing$. Thus $U \not\subset \overline{V}$. Then $r^{-1}(V) - \overline{V}$ contains z and thus intersects Y. Let $y \in r^{-1}(V) - \overline{V}$. Then $y = r(y) \in V \subset \overline{V}$, contradiction. Hence no such z exists, so Y contains all its limit points. Hence Y is closed in Z.

(2) : Suppose $r\colon \mathbb{R}^2 \to A$ is a retract where $A = \{x, y\}$. Then since x, y are distinct and $\mathbb{R}^2$ is Hausdorff, A is discrete, so $r^{-1}(x), r^{-1}(y)$ is a separation of $\mathbb{R}^2$. But $\mathbb{R}^2$ is connected. Contradiction. Hence A is not a retract of $\mathbb{R}^2$. In fact, this reasoning shows that no finite subset of $\mathbb{R}^n$ with more than one point is a retraction of $\mathbb{R}^n$.

(3) : Define $r\colon \mathbb{R}^2 - \{0\} \to S^1$ by $r(x) = \frac{x}{\|x\|}$. This is clearly a retraction. $\square$

Exercise 3.8 (35.5)**.** A space Y is said to have the **universal extension property** if for each triple consisting of a normal space X, a closed subset A of X, and a continuous function $f\colon A \to Y$, there exists an extension of f to a continuous map of X into Y.

(1) Show that $\mathbb{R}^J$ has the universal extension property.
(2) Show that if Y is homeomorphic to a retract of $\mathbb{R}^J$, then Y has the universal extension property.

Proof. (1) : Firstly, the Tietze extension lemma thus says that $\mathbb{R}$ (as well as $[a,b]$) has the universal extension property. Now suppose we have a map $f\colon A \to \mathbb{R}^J$. Then each component function $f_\alpha\colon A \to \mathbb{R}$ is continuous and thus can be extended to a map $\tilde{f}_\alpha\colon X \to \mathbb{R}$. Hence the map $\tilde{f}\colon X \to \mathbb{R}^J$ defined by $\tilde{f}(x) = \left(\tilde{f}_1(x), \tilde{f}_2(x), \ldots\right)$ is continuous and an extension of $f\colon A \to \mathbb{R}^J$.

(2) : Suppose Y is a retract of $\mathbb{R}^J$ under $r\colon \mathbb{R}^J \to Y$. Then Y is closed in $\mathbb{R}^J$. Suppose $f\colon A \to Y$ is a continuous map where A is a closed subset of a normal space X.
Then there are finitely many components $f_{\alpha_1}\colon A \to Y_{\alpha_1}, \ldots, f_{\alpha_n}\colon A \to Y_{\alpha_n}$ which we must extend, so it suffices to show that if $r\colon \mathbb{R} \to Y$ is a retract then Y has the universal extension property. Since $\mathbb{R}$ is path-connected, so is Y. As it is closed as well, Y is an interval $[a,b]$. But $[a,b]$ has the universal extension property by the Tietze extension theorem, so r can be extended, and thus Y has the universal extension property. If $\tilde{Y}$ is homeomorphic to Y under a homeomorphism $\varphi\colon \tilde{Y} \to Y$ and we have a map $\tilde{f}\colon A \to \tilde{Y}$, then $\varphi \circ \tilde{f}\colon A \to Y$ is a continuous map and thus has an extension onto all of X, $\overline{\varphi \circ \tilde{f}}\colon X \to Y$, so in particular, $\varphi^{-1} \circ \overline{\varphi \circ \tilde{f}}\colon X \to \tilde{Y}$ is an extension of $\tilde{f}$ as it agrees with $\tilde{f}$ on A. $\square$

Exercise 3.9 (Munkres, 35.6)**.** Let Y be a normal space. Then Y is said to be an **absolute retract** if for every pair of spaces (Y_0, Z) such that Z is normal and Y_0 is a closed subspace of Z homeomorphic to Y, the space Y_0 is a retract of Z.

(1) Show that if Y has the universal extension property, then Y is an absolute retract.
(2) Show that if Y is an absolute retract and Y is compact, then Y has the universal extension property.

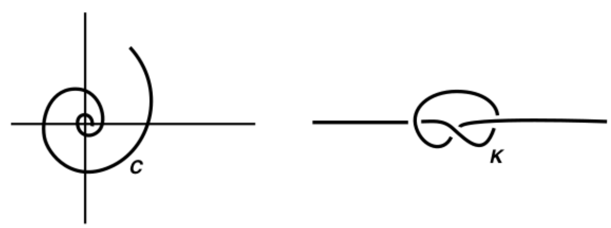
Proof. (a) : Suppose Y is normal and we have a normal space Z with a closed subspace Y_0 where Y_0 is homeomorphic to Y under $\varphi\colon Y_0 \to Y$. Suppose Y has the universal extension property. Since φ is continuous, it extends to a continuous map $\tilde{\varphi}\colon Z \to Y$ which agrees with φ on Y_0. Now $\varphi^{-1} \circ \tilde{\varphi}\colon Z \to Y_0$ is a continuous map and on Y_0, $\tilde{\varphi}$ equals φ, so $\varphi^{-1} \circ \tilde{\varphi}$ is the identity on Y_0. Hence $\varphi^{-1} \circ \tilde{\varphi}$ is a retract of Z onto Y_0.

(b) : Since Y is an absolute retract, it is normal and hence can be embedded into $[0,1]^J$. Now, by Tychonoff's theorem, $[0,1]^J$ is compact. Now, the embedding of Y is compact since Y is compact, and since compact subspaces of Hausdorff spaces are closed, the embedding of Y is a closed subspace. By the absolute retraction property, the embedding is a retract of $[0,1]^J$ and by problem 35.5, Y thus has the universal extension property. ☐

Exercise 3.10 (Munkres, 35.7). (1) Show that the logarithmic spiral

$$C = \{0 \times 0\} \cup \left\{ e^t \cos t \times e^t \sin t \mid t \in \mathbb{R} \right\}$$

is a retract of $\mathbb{R}^2$. Can you define a specific retraction $r \colon \mathbb{R}^2 \to C$?
(2) Show that the "knotted x-axis" K of the figure is a retract of $\mathbb{R}^3$.



Proof. (a) : The spiral is homeomorphic with $\mathbb{R}$ by the hemeomorphism:

$$C \to \mathbb{R}$$
$$(x,y) \mapsto \ln\left(\|(x,y)\|\right).$$

Now, $\mathbb{R}$ has the universal extension property by problem 35.5.(a), so by problem 35.6.(a), it is an absolute retract. Now, since $\mathbb{R}^2$ is normal and $C \subset \mathbb{R}^2$ is a closed subspace homeomorphic to $\mathbb{R}$, we find by the definition of an absolute retract that C is a retract of $\mathbb{R}^2$.

Define now a map $r \colon \mathbb{R}^2 \to C$ by

$$r(x,y) = e^{\ln\|(x,y)\|} \cos\left(\ln\|(x,y)\|\right) \times e^{\ln\|(x,y)\|} \sin\left(\ln\|(x,y)\|\right)$$

for $(x,y) \neq 0$ and $r(0,0) = (0,0)$. This is continuous as each component function is a composition of continuous functions which has limit 0 as $(x,y) \to 0$. And it is clearly a retraction.

(b) : This knotted x-axis is homeomorphic to the real line as well, so it has the universal extension property and is thus an absolute retract with the same reasoning as in part (a) of this problem. The knot is clearly a closed subset of $\mathbb{R}^3$ and $\mathbb{R}^3$ is normal, so again we find that it is a retract of $\mathbb{R}^3$. ☐

Theorem 3.11 (Munkres, exercise 35.8). *Let Y be a normal space. Then Y is an absolute retract if and only if Y has the universal extension property.*

Proof. One direction is exercise 35.6.(a).
We must show that if Y is an absolute retract, then Y has the universal extension property. We will make use of the following proposition:

Proposition 3.12 (Adjunction spaces of normal spaces are normal). *Suppose X and Y are disjoint normal spaces, A is closed in X and $f \colon A \to Y$ is a continuous map. Then $Y \cup_f X$ is normal.*

Proof. Suppose $B, C \subset Y \cup_f X$ are closed disjoint subsets. Then $\pi^{-1}(B)$ and $\pi^{-1}(C)$ are disjoint closed subsets of $X \cup Y$ which is normal.
Now, $\tilde{B} = \pi^{-1}(B) \cap Y$ and $\tilde{C} = \pi^{-1}(C) \cap Y$ are closed disjoint sets in Y, so as Y is normal, we can separate them by a continuous function $g \colon Y \to [0,1]$ such that $g\left(\pi^{-1}(B) \cap Y\right) = \{0\}$ and $g\left(\pi^{-1}(C) \cap Y\right) = \{1\}$.

Then $g \circ f \colon A \to [0,1]$ is a map such that $\pi^{-1}(B) \cap A \subset (g \circ f)^{-1}(0)$ and $\pi^{-1}(C) \cap A \subset (g \circ f)^{-1}(1)$. Define a map $h \colon A \cup Y \to [0,1]$ by

$$h(x) = \begin{cases} g \circ f(x), & x \in A \\ g(x), & x \in Y \end{cases}$$

Then h is continuous and separates $\pi^{-1}(B) \cap (A \cup Y)$ and $\pi^{-1}(C) \cap (A \cup Y)$. Now define $h\left(\pi^{-1}(B) \cap (X - A)\right) = \{0\}$ and $h\left(\pi^{-1}(C) \cap (X - A)\right) = \{1\}$. Then h is still continuous, and we can extend it to $X \cup Y$ by the Tietze extension lemma. Now $U = h^{-1}\left([0, \frac{1}{2})\right)$ and $V = h^{-1}\left((\frac{1}{2}, 1]\right)$ are open disjoint sets around $\pi^{-1}(B)$ and $\pi^{-1}(C)$ respectively. Suppose $[z] \in Y \cup_f X$. If $z' \in [z] \cap Y$, then $h(z') = g(z')$. Now, if $z'' \in [z] \cap A$, then $h(z'') = g\left(f(z'')\right) = g(z')$. Thus h factors through $Y \cup_f X$, so

$$\begin{tikzcd} Y \cup_f X \arrow[dr, "\tilde{h}"'] & X \cup Y \arrow[l, "\pi"'] \arrow[d, "h"] \\ & {[0,1]} \end{tikzcd}$$

commutes. Now by the universal property of quotient maps and since $h = \tilde{h} \circ \pi$ is continuous, we have that $\tilde{h}$ is continuous, and $\tilde{h}^{-1}\left([0, \frac{1}{2})\right)$ and $\tilde{h}^{-1}\left((\frac{1}{2}, 1]\right)$ are disjoint open sets around B and C, respectively. $\square$

Now suppose X is a normal space and $A \subset X$ is a closed subset and $f \colon A \to Y$ is continuous. Then we can form the adjunction space $Y \cup_f X$ which is normal by the above proposition. Since $Y \subset Y \cup_f X$ is homeomorphic with Y, and Y is an absolute retract, we have that $Y \subset Y \cup_f X$ is a retract of $Y \cup_f X$. I.e., there exists a map $r \colon Y \cup_f X \to Y$ such that $r([y]) = y$ for $[y] \in Y \subset Y \cup_f X$. Now define $f \colon X \to Y$ by

$$f(x) = i_Y^{-1} \circ r \circ i_X(x)$$

where $i_X \colon X \to Y \cup_f X$ and $i_Y \colon Y \to Y \cup_f X$ are the inclusions. Since i_X and i_Y are homeomorphisms and r is continuous, f is continuous and clearly agrees with the previous definition on A since $r \circ i_X(a) = r([a]) = r([f(a)]) = [f(a)]$ which maps to $f(a)$ by i_Y^{-1}. $\square$