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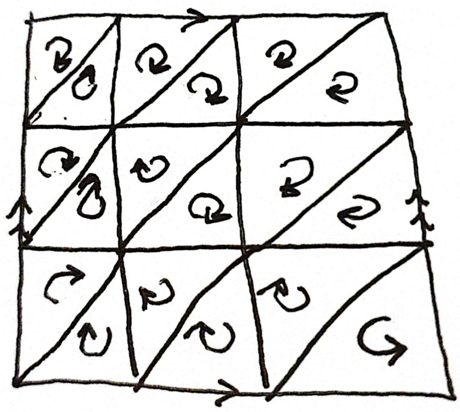
2: Show that the elementary 1-cycles, mentioned in section 8.1, generates $Z_1(K)$ for any complex K.

Solution: Suppose K is a simplicial complex. Let $\lambda = \sum \lambda_i(u_i, v_i) \in Z_1(K)$. We wish to show that there exist elementary 1-cycles $\sigma_1, \dots, \sigma_n$ such that $\lambda = \sum \sigma_i$.
We can assume without loss of generality that the λ_i are positive since otherwise we can replace $\lambda_i(u_i, v_i)$ by $(-\lambda_i)(v_i, u_i)$. Furthermore, suppose $(u_i, v_i) \neq (u_j, v_j)$ for all $i \neq j$.
Denote $\delta_i := (u_i, v_i)$. Then starting from some δ_{i_0}, since $0 = \partial\lambda = \sum \lambda_j(v_j - u_j)$, we have that there must exist some i_1 with $u_{i_1} = v_{i_0}$. Then append $\delta_{i_0}.\delta_{i_1}$ which we will let denote the path $u_{i_0}, v_{i_0}, v_{i_1}$. Now, inductively, we can continue to do so until we eventually get some δ_{i_k} such that $\delta_{i_k} = \delta_{i_s}$ for some $0 \leq s < k$. Then $\delta_{i_s}.\delta_{i_{s+1}}.\dots.\delta_{i_{k-1}}$ is a closed chain, and we have effectively removed the vertices in this chain from $\partial\lambda$ since $\partial \sum \delta_{i_h} = \sum(v_{i_h} - u_{i_h}) = v_{i_{k-1}} - u_{i_s}$, and since $(v_{i_{k-1}}, v_{i_k}) = (u_{i_k}, v_{i_k}) = \delta_{i_k} = \delta_{i_s} = (u_{i_s}, v_{i_s})$, we have $v_{i_{k-1}} = u_{i_s}$, so $\partial \sum \delta_{i_h} = 0$, and further we thus have that the curve $u_{i_s}, v_{i_s}, v_{i_{s+1}}, \dots, v_{i_{k-1}}$ is a closed oriented polygonal curve in K. Furthermore, if it were not simple, then for some $i_s \leq j < l \leq i_{k-1}$, we would have $\delta_j = \delta_l$, however, by construction, $\delta_{i_{k-1}}$ was the first repeated δ in the chain $\delta_{i_0}, \dots, \delta_{i_{k-1}}$, so this is not possible. Hence the curve is also simple. Taking a remaining $\delta \notin \delta_{i_s}.\dots.\delta_{i_{k-1}}$ and repeat the procedure, we receive another chain. Now, since there is only a finite number of δ in λ, this procedure must end at some point.
Denoting the chains in the end by $\sigma_1, \dots, \sigma_N$, we get $\lambda = \sum \sigma_{i=1}^N$ by construction, so λ is a sum of elementary 1-cycles.
Hence $Z_1(K)$ is generated by elementary cycles.

5: As for problem 4, but this time orient all the triangles compatibly, with the exception of one of them which is given the 'wrong' orientation.
Solution: We can orient the triangles as follows:



Here, all triangles are oriented compatibly except for the bottom-right one, which has the 'wrong' orientation.
Suppose we gave it the opposite orientation. Then the boundary of each edge of a triangle would cancel with the edge of another, so the boundary would be 0. Now, switching the orientation of the bottom-right triangle back, give it an orientation $\{v_0, v_1, v_2\}$ with $v_0 < v_1 < v_2$, which is the wrong orientation. Then we have that in the 'right' orientation, the edges $[v_2, v_1], [v_1, v_0]$ and $[v_0, v_2]$ with orientation, got cancelled, so there are neighboring edges $[v_1, v_2], [v_0, v_1]$ and $[v_2, v_0]$. Thus these do not get cancelled in the 'wrong' orientation either, so since all other edges still get cancelled, letting A denote the collection of oriented triangles in the compatible orientations, we end up with the sum of the oriented triangles being $\sum A + (v_2, v_1, v_0) - (v_0, v_1, v_2) = \sum A + 2(v_2, v_1, v_0)$, so the boundary becomes

$\partial \sum A + \partial\left(2(v_2, v_1, v_0)\right) = 2(v_2, v_1) + 2(v_1, v_0) + 2(v_0, v_2) = 2\partial\left[v_0, v_2, v_1\right]$, i.e. twice the boundary of the wrong triangle.