Representable functors

P. 52
Example (iv): Is the forgetful functor $U\colon$ Ring $\to$ Set representable?

Solution: We want to find an element R in Ring such that the forgetful functor is naturally isomorphic to the functor represented by R.
What are some natural considerations? $\mathbb{Z}$ again?
Suppose we map each $f \in C(\mathbb{Z}, R)$ to the unique element $f(1) \in U(R)$. This induces a well defined map $\alpha_R\colon C(\mathbb{Z} \to R) \to U(R)$ which is bijective and hence an isomorphism. We have the following natural isomorphism

$$\begin{tikzcd} C(\mathbb{Z} \to R) \arrow[r, "\alpha_R"] \arrow[d, "\phi_*"'] & U(R) \arrow[d, "U(\phi)"] \\ C(\mathbb{Z} \to S) \arrow[r, "\alpha_S"'] & U(S) \end{tikzcd}$$

Naturality follows as for an $f\colon \mathbb{Z} \to R$ a ring homomorphism, $\alpha_S \circ \varphi_* f$ maps to the element $\varphi\left(f(1)\right) \in U(S)$ and $U(\varphi) \circ \alpha_R(f) = U(\varphi)(f(1)) = \varphi\left(f(1)\right) \in U(S)$, so it is indeed a natural isomorphism.
Example (v): The functor $U(-)^n\colon$ Group $\to$ Set that sends a group G to the set of n-tuples of elements of G is represented by the free group F_n on n generators where α_G sends the map which sends the generator $x_i \to g_i$ to the set $(g_1, \ldots, g_n) \in \operatorname{Im} U(-)^n$.

$$\begin{tikzcd} C(F_n, G) \arrow[r, "\alpha_G"] \arrow[d, "\varphi_*"] & U(G)^n \arrow[d, "U(\varphi)^n"] \\ C(F_n, H) \arrow[r, "\alpha_H"] & U(H)^n \end{tikzcd}$$

Now, since for a map $f \in C(F_n, G)$ which maps $x_i \to g_i$, we have $\alpha_H \circ \varphi_*(f) = \alpha_H\left(\varphi \circ f\right) = (\varphi(g_1), \ldots, \varphi(g_n))$ and $U\left(\varphi\right)^n \circ \alpha_G\left(f\right) = U\left(\varphi\right)^n\left(g_1, \ldots, g_n\right) = (\varphi(g_1), \ldots, \varphi(g_n))$, the square is indeed commutative, so as α_G is a bijection, we indeed have a natural isomorphism.

Example (vi) Given any group presentation such as

$$S_3 := \langle s, t \mid s^2 = t^2 = 1, sts = tst \rangle$$

defines a functor $F\colon$ Group $\to$ Set that carries a group G to the set

$$\left\{(g_1, g_2) \in G^2 \mid g_1^2 = g_2^2 = e, g_1 g_2 g_1 = g_2 g_1 g_2\right\}.$$

The functor is represented by the group admitting the given presentation, in this case by S_3.
The presentation tells us that homomorphisms $S_3 \to G$ are classified by pairs of elements $g_1, g_2 \in G$ satisfying the listed relations.

(vii) The functor $(-)^\times\colon$ Ring $\to$ Set that sends a unital ring to its set of units is represented by the ring $\mathbb{Z}\left[x, x^{-1}\right]$ of Laurent polynomials in one variable.

$$\begin{tikzcd} C(\mathbb{Z}[x, x^{-1}], G) \arrow[r, dashed, "(x \to u) \to u"] \arrow[d, "\varphi_*"] & G^\times \arrow[d, "\varphi^\times"] \\ C(\mathbb{Z}[x, x^{-1}], H) \arrow[r, dashed, "(x \to v) \to v"] & H^\times \end{tikzcd}$$

Let f be the map sending $\varphi \in C\left(\mathbb{Z}[x, x^{-1}], G\right)$ to $\varphi(u)$ and g similarly but with H instead of G.
Then for $k \in C\left(\mathbb{Z}[x, x^{-1}], G\right)$, $g \circ \varphi_*(k) = g\left(\varphi \circ k\right) = \varphi\left(k(x)\right)$ and $\varphi^\times \circ f(k) = \varphi^\times\left(k(x)\right)$ and since $k(x)$ is a unit in G, this last part is well-defined and equals $\varphi\left(k(x)\right)$, hence we have naturality if f, g are bijections.

It suffices, by symmetry, to show it for f.
If $\varphi\colon \mathbb{Z}[x, x^{-1}] \to G$ is a homomorphism, then $1 = \varphi(1) = \varphi(x)\varphi(x^{-1})$, so $\varphi(x)$ is a unit; so f is indeed well-defined.
Injectivity follows since if $f(\varphi) = f(\psi)$, then $\varphi(x) = \psi(x)$, so given any $p \in \mathbb{Z}\left[x, x^{-1}\right]$, we have $p = \sum_{k\neq 0} a_k x^k$, so $\varphi(p) = \sum_{k\neq 0} a_k \varphi(x)^k = \sum_{k\neq 0} a_k \psi(x)^k = \psi(p)$, so $\varphi = \psi$.
For surjectivity, simply note that letting $\varphi(x) = u$ for some $u \in G^{\times}$, determines a well-defined homomorphism, so f is surjective.
This gives the result.

Thus this representation gives that any ring homomorphism $\mathbb{Z}\left[x, x^{-1}\right] \to R$ may be defined by sending x to any unit of R and is completely determined by the assignment, moreover, there are no ring homomorphisms that carry x to a non-unit.

Example (viii) The forgetful functor $U\colon$ Top $\to$ Set is represented by the singleton space: there is a natural bijection between elements of a topological space and continuous functions from the one-point space:

$$\begin{tikzcd}
C(\{x\}, X) \arrow[r, phantom, "(x\to x')\to\{x'\}"] \arrow[d, "f_*"] & U(X) \arrow[d, "U(f)"] \\
C(\{x\}, Y) \arrow[r, phantom, "(x\to y')\to\{y'\}"] & U(Y)
\end{tikzcd}$$

First, we check commutativity. Let f denote the map sending a $\varphi \in C(\{x\}, X)$ to $\{\varphi(x)\} \in U(X)$ and g likewise for Y.
Now for a continuous map $h\colon \{x\} \to X$, we have $g \circ f_*(h) = g\,(f \circ h) = f(h(x))$ and $U(f) \circ f(h) = U(f) \circ h(x) = f(h(x))$.

Bijectivity is clear, so we have a natural isomorphism.
Example xiii: The forgetful funcotr $U\colon$ Set_* $\to$ Set is represented by the two-element based set:

$$\begin{tikzcd}
C(\{*, y\} \to U_*) \arrow[r, "((*,y)\to(*,u))\to u"] \arrow[d, "f_*"] & U \arrow[d, "f"] \\
C(\{*, y\} \to V_*) \arrow[r, "((*,y)\to(*,v))\to v"'] & V
\end{tikzcd}$$

Example (xiv) The functor Path: Top $\to$ Set that carries a topological space to its set of paths and the functor Loop: Top_* $\to$ Set that carries a based space to its set of based loops are each representable by definition, by the unit interval I and the based circle S^1, respectively.

Free categories

Example (ix), (x) and (xi):
The functors ob, mor, iso: Cat $\to$ Set that take a locally small category to, respectively, its set of objects, its set of morphisms and its set of isomorphisms (point in a specified direction) are represented, respectively, by the terminal category $\mathbb{1}$, the category $\mathbb{2}$ and the category $\mathbb{I}$. In this sense, $\mathbb{2}$ is the **free** or **walking arrow**, and the category $\mathbb{I}$ is the **free** or **walking isomorphism**.

The adjective "free" is reserved for universal properties expressed by covariant represented functors. It could be applied to any of the objects listed above: e.g. $\mathbb{2}$ is the free category with an arrow, S^1 is the free space containing a loop. The dual term "cofree" for universal properties expressed by contravariant represented functors is less commonly used.

The following contravariant functors are representable:

Example (i) The contravariant power set functor $P\colon \mathrm{Set}^{op} \to \mathrm{Set}$ is represented by the set $Omega = \{\perp, \top\}$ with two elements.
The natural isomorphism $\mathrm{Set}(A, \Omega) \cong PA$ is defined by the bijection that associates a function $A \to \Omega$ with the subset that is the preimage of $\perp$; reversing perspectives, a subset $A' \subset A$ is identified iwth its **classifying function** $\chi_{A'}\colon A \to \Omega$ which sends exactly the elements of A' to the element $\perp$. The naturality condition stipulates that for any function $f\colon A \to B$, the diagram

$$\begin{tikzcd}
\mathrm{Set}(B,\Omega) \arrow[r, "\cong"] \arrow[d, "f^*"] & PB \arrow[d, "f^{-1}"] \\
\mathrm{Set}(A,\Omega) \arrow[r, "\cong"'] & PA
\end{tikzcd}$$

commutes. That is, naturality asserts that given a function $\chi_{B'}\colon B \to \Omega$ classifying the subset $B' \subset B$, the composite function $A \xrightarrow{f} B \xrightarrow{\chi_{B'}} \Omega$ classifies the subset $f^{-1}(B') \subset A$. That is, that $f^{-1}(B')$ is the same as $P \circ f^*(\chi_{B'}) = P(\chi_{B'} \circ f)$ which is presicely $(\chi_{B'} \circ f)^{-1}(\perp) = f^{-1}\left(\chi_{B'}^{-1}(\perp)\right) = f^{-1}(B')$.

Example (ii): The functor $\mathcal{O}\colon \mathrm{Top}^{op} \to \mathrm{Set}$ that sends a space to its set of open subsets is represented by the **Sierpinski space** S, the topological space with two points, one closed and one open. The natural bijection $\mathrm{Top}(X, S) \cong \mathcal{O}(X)$ associates a continuous function $X \to S$ to the preimage of the open points.

$$\begin{tikzcd}
Top(X,S) \arrow[r, "\cong"] & O(X) \\
Top(Y,S) \arrow[r, "\cong"'] \arrow[u, "f^*"] & O(Y) \arrow[u, "f^{-1}"']
\end{tikzcd}$$

Let $S = \{0, 1\}$ where 0 is open. Then naturality asserts that for a map $g\colon Y \to S$, the set $f^{-1}(g^{-1}(0))$ is the same as $(f^*(g))^{-1}(0) = (g \circ f)^{-1}(0) = f^{-1}\left(g^{-1}(0)\right)$.

Example (iii): The Sierpinski space also represents the functor $\mathcal{C}\colon \mathrm{Top}^{op} \to \mathrm{Set}$ that sends a space to its set of closed subsets. Composing the natural isomorphism $\mathcal{O} \cong \mathrm{Top}(-, S) \cong \mathcal{C}$, we see that the closed set and open set functors are naturally isomorphic. The composite natural isomorphism carries an open subset to its complement, which is closed. This recovers the natural isomorphism described in Example 1.4.3(v).

Example (iv) The functor $\mathrm{Hom}(- \times A, B)\colon \mathrm{Set}^{op} \to \mathrm{Set}$ that sends a set X to the set of functions $X \times A \to B$ is represented by the set B^A of functions from A to B. That is, there is a natural bijection between functions $X \times A \to B$ and functions $X \to B^A$. This natural isomorphism is referred to as **currying** in computer science; by fixing a variable in a two-variable function, one obtains a family of functions in a single variable.

$$\begin{tikzcd}
C(X,B^A) \arrow[r, "\cong"] \arrow[d, "f_*"] & \mathrm{Hom}(X \times A, B) \arrow[d, "(f,id)_*"] \\
C(Y,B^A) \arrow[r, "\cong"'] & \mathrm{Hom}(Y \times A, B)
\end{tikzcd}$$

Suppose $g\colon C(X, B^A) \to \mathrm{Hom}(X \times A, B)$ sends $k \in C(X, B^A)$ to the map $X \times A \to B$ defined by $(x, a) \to k(x)(a)$. This gives for any map in $C(X, B^A)$ a map in $\mathrm{Hom}(X \times A, B)$.
Define $h\colon C(Y, B^A) \to \mathrm{Hom}(Y \times A, B)$ similarly.

Conversely, for any $\varphi \in \mathrm{Hom}(X \times A, B)$, we can define $k \in C(X, B^A)$ by sending $x \to \varphi(x, -)\colon A \to B$. We thus have an injection both ways, so by Cantor-Bernstein, the sets are in bijections.
Naturality then asserts that for a $k \in C(X, B^A)$, $h \circ f_*(k) = h(k \circ f)$ which sends (y, a) to $k \circ f(y)(a)$ is the same as $(f, id)_* g(k)$ which sends (y, a) to $(f(y), a)$ and then to $k(f(y))(a) = k \circ f(y)(a)$.

Example (v) The functor $U(-)^*\colon \mathrm{Vect}_k^{op} \to \mathrm{Set}$ that sends a vector space to the set of vectors in its dual space is represented by the vector space k, i.e., linear maps $V \to k$ are, by definition, precisely the vectors in the dual space V^*.

$$\begin{tikzcd} C(V,k) \arrow[r] & U(V^*) \\ C(W,k) \arrow[r] \arrow[u, "f_*"] & U(W^*) \arrow[u, "U(f_*)"'] \end{tikzcd}$$

2.2.2: Suppose we have a G-set $X\colon BG \to \mathrm{Set}$. Suppose now we have a natural transformation $\varphi\colon G \implies X$, which is a G-equivariant map $\varphi\colon G \to X$. Now $\varphi(g) = g \cdot \varphi(e)$. Is the functor G representable?
We have

$$\begin{tikzcd} C(e,g) \arrow[r, "\alpha \to G(\alpha(e))"] \arrow[d, "f_*"] & G(g) \arrow[d, "G(f)"] \\ C(e,h) \arrow[r] & G(h) \end{tikzcd}$$

Exercise 1.4.iv: Prove that distinct parallel morphisms $f, g\colon c \to d$ define distinct natural transformations

$$f_*, g_*\colon C(-,c) \implies C(-,d) \quad \text{and} \quad f^*, g^*\colon C(d,-) \implies C(c,-)$$

Solution:

$$\begin{tikzcd} C(x,c) \arrow[r, "f_*"] \arrow[d, "h^*"] & C(x,d) \arrow[d, "h^*"] \\ C(y,c) \arrow[r, "f_*"] & C(y,d) \end{tikzcd}$$

Two natural transformations are distinct if the components are distinct. But all components for f_* are f_* and similarly for g_*.
So in particular, $\alpha_c = f_*$, so if the components were the same, then $f = f \circ 1_c = f_*(1_c) = g_*(1_c) = g \circ 1_c = g$, contradiction.

Proof of Cayley's theorem using the Yoneda embedding:
Regard a group G as a one-object category BG. Since there is a single object in BG whose endomorphisms define the group G, there is a unique contravariant represented functor $BG^{op} \to \mathrm{Set}$ which by example 1.3.9 corresponds to a right G-set, with G acting by right multiplication.
This is the image of the covariant Yoneda embedding $BG \to \mathrm{Set}^{BG^{op}}$. Corollary 2.2.8 then tells us that the only G-equivariant endomorphisms of the right G-set G are those maps defined by left multiplication by a fixed element of G, i.e. it is of the form g_* with $g \in G$.
In particular, any G-equivariant endomorphism of G must be an automorphism, a fact that is not otherwise obvious.
The Yoneda embedding thus defines an isomorphism between G and the automorphism group of the right G-set G, an object in $\mathrm{Set}^{BG^{op}}$. Composing with the faithful forgetful functor $\mathrm{Set}^{BG^{op}} \to \mathrm{Set}$, we obtain an isomorphism between G and a subgroup of the automorphism group $\mathrm{Sym}(G)$ of the set G.

Exercise 2.2.iv: Prove that the following are equivalent.
(i) $f\colon x \to y$ is an iso in C.
(ii) $f_*\colon C(-,x) \implies C(-,y)$ is a natural iso.
(iii) $f^*\colon C(y,-) \implies C(x,-)$ is a natural iso.

Solution: $(i) \implies (ii)$: for a morphism $h\colon c \to d$ in C,

$$\begin{tikzcd} C(d,x) \arrow[r, "f_*"] \arrow[d, "h^*"] & C(d,y) \arrow[d, "h^*"] \\ C(c,x) \arrow[r, "f_*"] & C(c,y) \end{tikzcd}$$

and $(f^{-1})_* f_* = \mathbb{1}$, so f_* is an iso. Hence f_* is a natural iso with clear commutativity.

$(i) \implies (iii)$ similarly.

$(iii) \implies (i)$: the Yoneda embedding gives a full and faithful embedding sending f to f^*. Since f^* i a natural iso, there exists a morphism g' in $\mathrm{Hom}\left(C(-,d), C(-,c)\right) \cong \mathrm{Hom}(d,c)$, with corresponding g in $\mathrm{Hom}(d,c)$ such that $g_* = g$. Thus $1 = g_* f_* = (gf)_*$ and similarly $1 = (fg)_*$. Hence f is an iso. Similarly for $(ii) \implies (i)$.

Universal properties and universal elements: this shows that isomorphic objects $x \cong y$ are **representably isomorphic**, meaning that $C(-,x) \cong C(-,y)$ and $C(x,-) \cong C(y,-)$ (exercise 2.2.iv). The Yoneda lemma supplies the converse:
Proposition 2.3.1: Consider a pair of objects x and y in a locally small category C.
(i) If either the co- or contravariant functors represented by x and y are naturally isomorphic, then x and y are isomorphic.
(ii) In particular, if x and y represent the same functor, then x and y are isomorphic.

Proof: The full and faithful Yoneda embeddings $C \to \mathrm{Set}^{C^{op}}$ and $C^{op} \to \mathrm{Set}^{C}$ create isomorphism since fully faithful functors create isomorphisms (exercise 1.5.iv), so an (natural) isomorphism between represented functors is induced by a unique isomorphism between their representing objects, which in particular must be isomorphic, proving (i) (uniqueness by the fact that the Yoneda embeddings are faithful).
Now (ii) is an immediate consequence since given a functor represented by both x and y, the representing natural isomorphisms compose to demonstate that x and y are represntably isomorphic.

There may be many isomorphisms between the objects x and y appearing in the proof of proposition 2.3.1, but there is a unique natural isomorphism commuting with the chosen representations. On account of this, one typically refers to *the* representing object of a representable functor. Category theorists often use the definite article "the" in contexts where the object in question is well-defined up to canonical isomorphism.

Corollary 2.3.2: The full subcategory of C spanned by its terminal objects is either empty or is a contractible groupoid. In particular, any two terminal objects in C are uniquely isomorphic.
Def: A **contractible groupoid** is a category that is equivalent to the terminal category $\mathbb{1}$. Explicitly, a contractible groupoid is a category with a unique morphism in each hom-set.

Proof: For any terminal objects t, t', $C(t,t')$ is a singleton by definition. Proposition 2.3.1 then implies that this morphism is an isomorphism: the objects t and t' are terminal iff they represent the functor $*\colon C^{op} \to \mathrm{Set}$ that is constant at the singleton set.

Def 2.3.3: A **universal property** of an object $c \in C$ is expressed by a representable functor F together with a **universal element** $x \in Fc$ that defines a natural isomorphism $C(c,-) \cong F$ or $C(-,c) \cong F$, as appropriate, via the Yoneda lemma.

Limits and colimits

Def 3.1.1: For any object $c \in C$ and any category J, the **constant functor** $c\colon J \to C$ sneds every object of J to c and every morphism in J to the identity morphism 1_c. The constant functors define an embedding $\Delta\colon C \to C^J$ (recall an embedding is a faithful functor that is injective on objects) that sends an object c to the constant functor at c and a morphism $f\colon c \to c'$ to the **constant natural transformation**, in which each component is defined to be the morphism f; i.e. $\Delta(f)\colon \text{constant } c \implies \text{constant } c'$ has components $f = \Delta(f)_j\colon \text{constant } c(j) \to \text{constant } c'(j) = c \to c'$.

Def 3.1.2: A **cone over** a diagram $F\colon J \to C$ with **summit** or **apex** $c \in C$ is a natural transformation

$\lambda\colon c \implies F$ whose domain is the constant functor at c. The components $(\lambda_j : c \to Fj)_{j\in J}$ of the natural transformation are called the **legs** of the cone.
Explicitly:

- The data of a cone over $F\colon J \to C$ with summit c is a collection of morphisms $\lambda_j : C \to Fj$, indexed by the objects $j \in J$.
- A family of morphisms $(\lambda_j : c \to Fj)_{j\in J}$ defines a cone over F if and only if, for each morphism $f\colon j \to k$ in J, the following triangle commutes in C :

$$\begin{tikzcd} & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ Fj \arrow[rr, "Ff"] & & Fk \end{tikzcd}$$

Dually, a **cone under** F with **nadir** c is a natural transformation $\lambda F \implies c$, whose **legs** are the components $(\lambda_j : F_j \to c)_{j\in J}$. The naturality condition asserts that, for each morphism $f\colon j \to k$ of J, the triangle

$$\begin{tikzcd} Fj \arrow[rr, "Ff"] \arrow[dr, "\lambda_j"'] & & Fk \arrow[dl, "\lambda_k"] \\ & c & \end{tikzcd}$$

commutes in C.
Cones under a diagram are also called **cocones** - a cone under $F\colon J \to C$ is precisely a cone over $F\colon J^{op} \to C^{op}$.

Questions

1. On page 75, it says: "By the Yoneda lemma, a limit consists of an object $\mathrm{lim}F \in C$ together with a universal cone $\lambda\colon \mathrm{lim}F \implies F$, called the limit cone, . . ." What does "universal" specifically mean here?

2.

Note: I believe the proof of proposition 2.3.1 shows that an isomorphism between represented functors is induced by a *unique* isomorphism between their representing objects, which in particular must be isomorphic.