1. Problems

Definition 1.1. Let M be a smooth manifold. A Morse function $f\colon M \to \mathbb{R}$ is a smooth map such that all its critical points are non-degenerate, with pairwise distinct critical values in $\mathbb{R}$.

Problem 1.2 (Reeb's Theorem)**.** (6 pts) Let M be a smooth, compact manifold of dimension d. Show that if M admits a Morse function with only two critical points, then M is homeomorphic to the sphere S^d. Indicate why the above proof fails in showing that M is diffeomorphic to the sphere S^d.

For the proof, we state a theorem that we will need:

Definition 1.3. For a smooth map $f\colon M \to \mathbb{R}$ on a smooth manifold M, let $M^a = f^{-1}(-\infty, a]$.

Theorem 1.4. *Let $f \in C^\infty(M)$ on a manifold M. Let $a < b$ and suppose that the set $f^{-1}[a,b]$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

Proof of Problem 1.2. Since M is compact, we have that $f(M) = [a,b] \subset \mathbb{R}$. Without loss of generality, assume that $f(M) = [0,1]$.

We shall need the following lemma from analysis:

Lemma 1.5 (Fermat's Theorem)**.** *Let $f\colon (a,b) \to \mathbb{R}$ be a function on an open interval $(a,b) \subset \mathbb{R}$. Suppose f has a local extremum at $x_0 \in (a,b)$. If f is differentiable at x_0, then $f'(x_0) = 0$.*

Now, we claim that the two critical points are precisely the preimages of 0 and 1. For suppose $x \in f^{-1}(0)$. Then x is a global minimum for f. Taking some chart centered around x, we have a local representation of f as a function $\mathbb{R}^d \to [0,1]$ with a global minimum at 0. Taking the partial derivatives of f and applying Fermat's theorem to each of them, we find that each partial derivative evaluated at 0 is 0: $\frac{\partial f}{\partial x^i}(0) = 0$. Hence we find that $Df(0) = 0$, so transfering back to the manifold, $Df(x) = 0$, so $x \in M$ is a critical point. The same argument applies to show that any $y \in f^{-1}(1)$ is a critical point. Since there are only two critical points, this immediately forces $f^{-1}(0)$ and $f^{-1}(1)$ to be singletons and thus global maximum and minimum of M. Suppose without loss of generality that $p \in M$ is the minimum and $q \in M$ is the maximum.
By Morse's Lemma, in some coordinate system about p, let's say in a neighborhood U, f takes the form

$$f(x_1, \ldots, x_n) = -x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2.$$

Now p is a global minimum, so in fact, we must have that $\lambda = 0$. That is

$$f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2$$

in this neighborhood. Since also $f(U)$ is open in the subspace topology and contains 0, we can find an open disk $\tilde{D}_1$ centered at 0 of radius ε_1 such that $\tilde{D}_1 \cap [0,1] \subset f(U)$, and let D_1 be the inverse of $\tilde{D}_1$ under this local diffeomorphism.
Similarly, in a neighborhood V of q, f takes the form

$$f(x_1, \ldots, x_n) = 1 - x_1^2 - x_2^2 - \ldots - x_n^2.$$

Again take some open disk $\tilde{D}_2$ centered at 1 of radius ε_2 such that $\tilde{D}_2 \cap [0,1] \subset f(V)$. Let D_2 be the inverse image under f of $\tilde{D}_2$.
We wish to show that there exists some $\varepsilon > 0$ such that $f^{-1}[0,\varepsilon]$ and $f^{-1}[1-\varepsilon,1]$ are homeomorphic to the closed n-disk D^n. There exist $\alpha, \beta \in (0,1)$ such that $f(M - D_1 \cup D_2) = [\alpha, \beta]$ since $M - D_1 \cup D_2$ is still compact. Now simply let $0 < \varepsilon < \min\left\{\alpha, 1-\beta, \varepsilon_1, 1-\varepsilon_2, 1-\varepsilon_1, \frac{1}{4}\right\}$. To see that this works, simply note that $f^{-1}[0,\varepsilon] \subset D_1 \cup D_2$. On D_1, f takes values in $[0,\varepsilon_1]$ and on D_2, f takes values in $[1-\varepsilon_2, 1]$. But $\varepsilon < \varepsilon_1$, so $[0,\varepsilon_1] \subset [0,\varepsilon]$, so $D_1 \subset f^{-1}[0,\varepsilon]$, while $\varepsilon < 1-\varepsilon_2$, so $[1-\varepsilon_2, 1] \not\subset f^{-1}[0,\varepsilon]$. Similarly, $1-\varepsilon > \varepsilon_1$, so $D_1 \subset [0,\varepsilon_1] \not\subset f^{-1}[1-\varepsilon,1]$ while $1-\varepsilon_2 > 1-\varepsilon$, so $D_2 \subset [1-\varepsilon_2, 1] \subset f^{-1}[1-\varepsilon,1]$.

Therefore, since $f^{-1}[0,\varepsilon] \subset D_1 \subset U$ and we know that on U, f takes the form

$$f(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2,$$

we know that $f^{-1}[0,\varepsilon]$ is precisely a closed disk about p. Likewise, $f^{-1}[1-\varepsilon,1]$ can be seen to be a closed disk about q.

But now by Theorem 1.4, since there are no critical points in $f^{-1}[\varepsilon, 1-\varepsilon]$ by assumption, M^{ε} is diffeomorphic to $M^{1-\varepsilon}$. Hence we find that $M^{1-\varepsilon}$ and $f^{-1}[1-\varepsilon,1]$ are both diffeomorphic to closed d-disks, and furthermore, M is obtained by gluing these d-disks along their boundary. We claim that this is sufficient to conclude that M is *homeomorphic* to S^d.

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