1. Hatcher

In this section, we will cover the cup product following Hatcher.

Definition 1.1 (Cup Product)**.** For a ring R, let $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$. Then the *cup product* $\varphi \smile \psi \in C^{k+l}(X;R)$ is the cochain whose value on $\sigma\colon \Delta^{k+l} \to X$ is given by

$$(\varphi \smile \psi)(\sigma) = \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right)\psi\left(\sigma|_{[v_k,\ldots,v_{k+l}]}\right)$$

where the right-hand side is the product in R.

To see that this induces a cup product on cohomology, we need the following lemma:

Lemma 1.2. *$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$ for $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$.*

Using the lemma, it is clear that the cup product of two cocycles is again a cocycles, and that the cup product of a cocycle and a coboundary, in either order, is a coboundary. It follows that there is an induced cup product

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X;R).$$

This is associative and distributive since at the level of cochains the cup product has these properties.
If R has an identity, then there is an identity elements for the cup product, the class $1 \in H^0(X;R)$ defined by the 0-cocycle taking the value 1 on each singular 0-simplex.

1.0.1. *Relative cup product.* The cup product formula $(\varphi \smile \psi)(\sigma) = \varphi\left(\sigma|_{[v_0,\ldots,v_k]}\right)\psi\left(\sigma|_{[v_k,\ldots,v_{k+l}]}\right)$ also gives relative cup products

$$H^k(X;R) \times H^l(X,A;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$
$$H^k(X,A;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$
$$H^k(X,A;R) \times H^l(X,A;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$

since if φ or ψ vanishes on chains in A, then so does $\varphi \smile \psi$.

We can also define an even more general relative cup product

$$H^k(X,A;R) \times H^l(X,B;R) \xrightarrow{\smile} H^{k+l}(X,A \cup B;R)$$

when A and B are open subsets of X or subcomplexes of the CW complex X.

Construction. The absolute cup product restricts to a cup product $C^k(X,A;R) \times C^l(X,B;R) \to C^{k+l}(X,A \sqcup B;R)$ where $C^n(X,A \sqcup B;R)$ is the subgroup of $C^n(X;R)$ consisting of cochains vanishing on sums of chains in A and chains in B. If A and B are open in X, then the inclusions $C^n(X,A \cup B;R) \hookrightarrow C^n(X,A \sqcup B;R)$ induces isomorphisms on cohomology:

Proposition 1.3. *For a map $f\colon X \to Y$, the induced map $f^*\colon H^n(Y;R) \to H^n(X;R)$ satisfies $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$, and similarly in the relative case.*

Theorem 1.4. *The identity $\alpha \smile \beta = (-1)^{kl}\beta \smile \alpha$ holds for all $\alpha \in H^k(X,A;R)$ and $\beta \in H^l(X,A;R)$, when R is commutative.*

2. The Cohomology Ring

Since the cup product is associative and distributive, it is natural to try to make it the multiplication in a ring structure on the cohomology groups of a space X. This is easy to do if we define $H^*(X;R) = \bigoplus_{k\in\mathbb{Z}} H^k(X;R)$. That is, if we define $H^*(X;R)$ as the direct sum of the cohomology groups of the space. Then elements of $H^*(X;R)$ are finite sums $\Sigma_i\alpha_i$ with $\alpha_i \in H^i(X;R)$ and the product of two such sums is defined to be $(\Sigma_i\alpha_i)(\Sigma_j\beta_j) = \Sigma_{i,j}\alpha_i\beta_j$.

Exercise 2.1. Show that this makes $H^*(X;R)$ into a ring,w with identity if R has an identity. Similarly for $H^*(X,A;R)$ with the relative cup product.
Taking scalar multiplication by elements of R into account, these rings can also be regarded as R-algebras.

Example 2.2. Recall that $H^k(\mathbb{RP}^2;\mathbb{Z}_2) \cong \mathbb{Z}_2$ for $k = 0, 1, 2$ and is 0 otherwise. Also by example 3.8 in Hatcher on Cohomology, for a generator $\alpha \in H^1(\mathbb{RP}^2;\mathbb{Z}_2)$, $\alpha^2 = \alpha \smile \alpha$ is a generator of $H^2(\mathbb{RP}^2;\mathbb{Z}_2)$, hence $H^*(\mathbb{RP}^2;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\left(\alpha^3\right)$.

Adding cohomology classes of different dimensions to form $H^*(X;R)$ is convenient, but it has little topological significance. One can always regard the cohomology ring as a *graded ring*:

Definition 2.3 (Graded Ring)**.** A ring A with a decomposition $\bigoplus_{k\geq 0} A_k$ into additive subgroups $A_k \leq A$ such that the multiplication takes $A_k \times A_l$ to A_{k+l} is called a *graded ring*.

To indicate that $\alpha \in A$ lies in A_k, we write $|a| = k$.

Definition 2.4 (Degree/dimension)**.** The number $|a|$ is called the *degree* or *dimension* of a.

Definition 2.5 (Commutative/anticommutative/graded commutative)**.** A graded ring satisfying the commutativity property that $ab = (-1)^{|a||b|}ba$ is usually called *commutative* or any of the following less ambiguous terms: *graded commutative*, *anticommutative*, or *skew commutative*.

Example 2.6 (Polynomial Rings)**.** An example of a graded ring is $R[\alpha]$ or the truncated version: $R[\alpha]/(\alpha^n)$.
We have seen that $H^*\left(\mathbb{RP}^2;\mathbb{Z}/2\right) \cong \mathbb{Z}/2[\alpha]/\left(\alpha^3\right)$. More generally, we can show that $H^*(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/\left(\alpha^{n+1}\right)$ and $H^*(\mathbb{RP}^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]$, where, in these cases, $|\alpha| = 1$.

Example 2.7 (Exterior Algebras)**.** The *exterior algebra* $\Lambda_R[\alpha_1,\ldots,\alpha_n]$ over a commutative ring R with identity is the free R-module with basis the finite products $\alpha_{i_1}\cdots\alpha_{i_k}, i_1 < \ldots < i_k$, with associative, distributive multiplication defined by the rules $\alpha_i\alpha_j = -\alpha_j\alpha_i$ for $i \neq j$ and $\alpha_i^2 = 0$ for all i. The empty producto f α_i 's is the identity element 1 in $\Lambda_R[\alpha_1,\ldots,\alpha_n]$.
In view of $\alpha_i\alpha_j = -\alpha_j\alpha_i$, the exterior algebra becomes an anticommutative graded ring by specifying odd dimensions for the generators α.
By the Künneth formula, we have

$$H^*\left(S^{k_1}\times\ldots\times S^{k_n};\mathbb{Z}\right) \cong H^*(S^{k_1};\mathbb{Z}) \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} H^*(S^{k_n};\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha_1,\ldots,\alpha_n]$$

when all the k_i are odd, since the first isomorphism is given by the cross product.

When some k_i 's are even, one obtains the tensor product of an exterior algebra for the odd-dimensional speheres and truncated polynomial rings $\mathbb{Z}[\alpha]/\left(\alpha^2\right)$ for the even dimensional spheres.

2.1. **The Cross Product.**

Definition 2.8 (First definition of cross product, external cup product)**.** We define the *cross product*, or *external cup product* as it is sometimes called, by the map

$$H^*(X;R) \times H^*(Y;R) \xrightarrow{\times} H^*\left(X \times Y;R\right)$$

given by $a \times b = p_1^*(a) \smile p_2^*(b)$ where p_1, p_2 are the projections of $X \times Y$ onto X and Y, respectively.

Definition 2.9 (Cross Product, second definition)**.** Since the cup product is distributive, the cross product is bilinear, hence it induces an R-module homomorphism

$$\begin{tikzcd}
H^*(X;R) \times H^*(Y;R) \arrow[d] \arrow[dr, "\times"] & \\
H^*(X;R) \otimes_R H^*(Y;R) \arrow[r, "\times"] & H^*(X \times Y;R)
\end{tikzcd}$$

which we also call the cross product, given by $a \otimes b \mapsto a \times b$.

This module homomorphism becomes a ring homomorphism if we define the multiplication in a tensor product of graded rings by $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$ where $|x|$ denotes the dimension of x.
This can be seen as follows (note that $ac = a \smile c$ and $bd = b \smile d$):

$$\begin{aligned}
\mu\left((a \otimes b)(c \otimes d)\right) &= (-1)^{|b||c|} \mu\left(ac \otimes bd\right) \\
&= (-1)^{|b||c|} (a \smile c) \times (b \smile d) \\
&= (-1)^{|b||c|} p_1^*(a \smile c) \smile p_2^*\left(b \smile d\right) \\
&= (-1)^{|b||c|} p_1^*(a) \smile p_1^*(c) \smile p_2^*(b) \smile p_2^*(d) \\
&= p_1^*(a) \smile p_2^*(b) \smile p_1^*(c) \smile p_2^*(d) \\
&= (a \times b)(c \times d) = \mu(a \otimes b)\mu(c \otimes d)
\end{aligned}$$

Theorem 2.10. *The cross product $H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y;R)$ is a finitely generated free R-module for all k.*

3. Bredon

We will need to define a homomorphism $D\colon \Delta_{n-1}(X) \to \Delta_n(X)$ with some desirable properties, so let us start with that. To this end, we construct it in the proof of the following theorem:

Theorem 3.1. *If X is contractible, then $H_i(X) = 0$ for all $i \neq 0$.*

Proof. Let $F\colon X \times I \to X$ be the homotopy with $F(x,0) = x$ and $F(x,1) = x_0$ for all $x \in X$, and some $x_0 \in X$. Define $D\sigma\colon \Delta_n \to X$ for each singular simplex $\sigma\colon \Delta_{n-1} \to X$ of X by

$$(D\sigma)\left(\sum_{i=0}^{n} \lambda_i e_i\right) = F\left(\sigma\left(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} e_{i-1}\right), \lambda_0\right) \tag{B}$$

where $\sum_{i=0}^{n}\lambda_i = 1$ and $\lambda = \sum_{i=1}^{n} = 1-\lambda_0$.
See Figure 1.

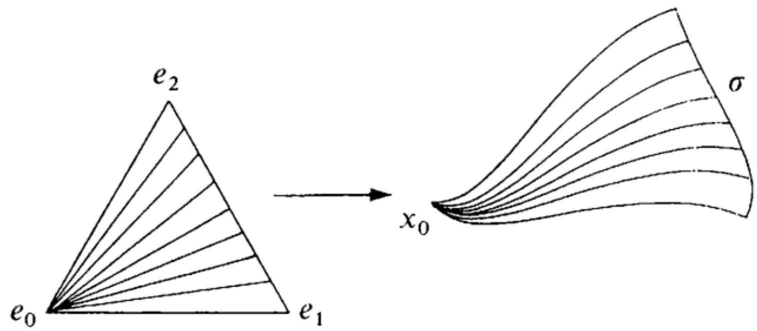

Figure 1. The Cone Construction

This D now extends to a homomorphism $D\colon \Delta_{n-1}(X) \to \Delta_n(X)$.

To compute the i th face of the singular simplex $D\sigma$ for $i > 0$, we put $\lambda_i = 0$ in (B), and we get $(D\sigma)^{(i)} = D\left(\sigma^{(i-1)}\right)$, and also $(D\sigma)^{(0)} = \sigma$.
But now we find that when $n > 1$,

$$\partial\left(D\sigma\right) = (D\sigma)^{(0)} - \sum_{i=1}^{n}(-1)^{i-1}(D\sigma)^{(i)} = (D\sigma)^{(0)} - \sum_{j=0}^{n-1}(-1)^{j}D(\sigma^{(j)}) = \sigma - D\left(\partial\sigma\right).$$

For $n = 1$ (so when σ is a 0-simplex), we have

$$\partial\left(D\sigma\right) = \sigma - \sigma_0, \quad \text{where } \sigma_0\colon \Delta_0 \to \{x_0\}\,.$$

and $D(\partial\sigma) = 0$ by definition.
Thus $\partial D + D\partial = \mathrm{id} - \varepsilon$, where $\varepsilon\colon \Delta_i(X) \to \Delta_i(X)$ is given by $\varepsilon = 0$ for $i \neq 0$ and $\varepsilon\left(\sum n_\sigma\sigma\right) = \left(\left(\sum n_\sigma\right)\sigma_0\right.$ for $i = 0$ and where σ_0 is the 0-simplex at x_0. Thus, in homology, $\mathrm{id} = \mathrm{id}_* = \varepsilon_*$ which is 0 in nonzero dimensions. □

3.1. **Cross Product.** We want to define a cross product

$$\times\colon \Delta_p(X) \times \Delta_q(Y) \to \Delta_{p+q}(X \times Y).$$

If $x \in X$, we denote also by x the singular 0-simplex sending e_0 to x.
Now for $\sigma\colon \Delta_q \to Y$, we let $x \times \sigma$ be the singular q-simplex of $X \times Y$ taking $w \mapsto (x, \sigma(w))$, and similarly for $y \in Y$. This defines $\times$ on $\Delta_0(X) \times \Delta_q(Y)$ and $\Delta_p(X) \times \Delta_0(Y)$. We also define $\times$ to be the zero map on elements (a, b) with either a or b being $0 \in \Delta_p(X) = \mathbb{Z}\,\mathrm{Map}(\Delta_p, X)$ or $0 \in \Delta_q(Y) = \mathbb{Z}\,\mathrm{Map}(\Delta_q, Y)$.

Theorem 3.2. *There exist bilinear maps* $\times\colon \Delta_p(X) \times \Delta_q(Y) \to \Delta_{p+q}(X \times Y)$ *such that:*

1. *for* $x \in X, y \in Y, \sigma\colon \Delta_q \to Y$ *and* $\tau\colon \Delta_p \to X$, $x \times \sigma$ *and* $\tau \times y$ *are as described above;*
2. *(naturality) if* $f\colon X \to X'$ *and* $g\colon Y \to Y'$, *and if* $f \times g\colon X \times Y \to X' \times Y'$ *denotes the product map, then*

$$(f \times g)_\Delta(a \times b) = f_\Delta(a) \times g_\Delta(b); \quad \textit{and}$$

(3) *(boundary formula)* $\partial(a \times b) = \partial a \times b + (-1)^{\deg a} a \times \partial b$.

Proof. Note that (3) holds when p or q is 0.

The method of proof here goes by the name of "acyclic models". It can be made general, but we will just carry it out in the specific situation of our theorem.

Let $\mathrm{id}_p \colon \Delta_p \to \Delta_p$ be the identity map, thought of as a singular p-simplex of the space Δ_p.

Let $p > 0$ and $q > 0$, and assume $\times$ has been defined for smaller $p+q$ satisfying the conditions above.
The idea is to first define $\mathrm{id}_p \times \mathrm{id}_q$ on the "models". To do this, we first use (3) to compute that $\partial\left(\mathrm{id}_p \times \mathrm{id}_q\right) = 0$. Since $\Delta_p \times \Delta_q$ is contractible, we saw that $H_*\left(\Delta_p \times \Delta_q\right) = 0$, so $\partial\left(\mathrm{id}_p \times \mathrm{id}_q\right)$ is a boundary of something. We let $\mathrm{id}_p \times \mathrm{id}_q$ denote this element. So $\partial\left(\mathrm{id}_p \times \mathrm{id}_q\right) = \partial \mathrm{id}_p \times \mathrm{id}_q + (-1)^p \mathrm{id}_p \times \partial \mathrm{id}_q$. Afterwards, we define $\sigma \times \tau$ in general by applying naturality (2) to maps $\sigma \colon \Delta_p \to X$ and $\tau \colon \Delta_q \to Y$.

To carry this out, we have

$$\partial\left(\mathrm{id}_p \times \mathrm{id}_q\right) = \partial \mathrm{id}_p \times \mathrm{id}_q + (-1)^p \mathrm{id}_p \times \partial \mathrm{id}_q \in \Delta_{p+q-1}\left(\Delta_p \times \Delta_q\right).$$

Now

$$\partial(rhs) = \partial\partial \mathrm{id}_p \times \mathrm{id}_q + (-1)^{p-1} \partial \mathrm{id}_p \times \partial \mathrm{id}_q + (-1)^p \partial \mathrm{id}_p \times \partial \mathrm{id}_q + \mathrm{id}_p \times \partial\partial \mathrm{id}_q = 0.$$

Hence the rhs is a $(p+q-1)$-cycle, so by the above, we can choose $\partial\left(\mathrm{id}_p \times \mathrm{id}_q\right) = \partial \mathrm{id}_p \times \mathrm{id}_q + (-1)^p \mathrm{id}_p \times \partial \mathrm{id}_q$.
Now if $\sigma \colon \Delta_p \to X$ and $\tau \colon \Delta_q \to Y$ are arbitrary singular simplices, then regarding them also as maps which then induce homomorphisms of chain groups, we have $\sigma = \sigma_\Delta(\mathrm{id}_p)$ and $\tau = \tau_\Delta(\mathrm{id}_p)$.
By (2), we must define $\sigma \times \tau = \sigma_\Delta(\mathrm{id}_p) \times \tau_\Delta(\mathrm{id}_q) = (\sigma \times \tau)_\Delta\left(\mathrm{id}_p \times \mathrm{id}_q\right)$. For $f \colon X \to X'$ and $g \colon Y \to Y'$, we then have

$$\begin{aligned}
(f \times g)_\Delta (\sigma \times \tau) &= (f \times g)_\Delta (\sigma \times \tau)_\Delta \left(\mathrm{id}_p \times \mathrm{id}_q\right) \\
&= (f \circ \sigma \times g \circ \tau)_\Delta \left(\mathrm{id}_p \times \mathrm{id}_q\right) \\
&= (f \circ \sigma)_\Delta \left(\mathrm{id}_p\right) \times (g \circ \tau)_\Delta \left(\mathrm{id}_q\right) \\
&= f_\Delta(\sigma) \times g_\Delta(\tau).
\end{aligned}$$

Hence naturality (2) holds in general in these dimensions.
For (3), we have

$$\begin{aligned}
\partial(\sigma \times \tau) &= \partial\left((\sigma \times \tau)_\Delta \left(\mathrm{id}_p \times \mathrm{id}_q\right)\right) \\
&= (\sigma \times \tau)_\Delta \left(\partial\left(\mathrm{id}_p \times \mathrm{id}_q\right)\right) && \text{(Ch map commutes w } \partial\text{)} \\
&= (\sigma \times \tau)_\Delta \left(\partial \mathrm{id}_p \times \mathrm{id}_q + (-1)^p \mathrm{id}_p \times \partial \mathrm{id}_q\right) \\
&= \partial \sigma_\Delta(\partial \mathrm{id}_p) \times \tau_\Delta(\mathrm{id}_q) + (-1)^p \sigma_\Delta(\mathrm{id}_p) \times \tau_\Delta\left(\partial \mathrm{id}_q\right) \\
&= \partial \sigma \times \tau + (-1)^p \sigma \times \partial \tau,
\end{aligned}$$

which extends to all chains by bilinearity. ∎

Definition 3.3. If (X, A) and (Y, B) are pairs of spaces, then $(X, A) \times (Y, B)$ denotes the pair $(X \times Y, X \times B \cup A \times Y)$.

Proposition 3.4. *The cross product $\Delta_p(X) \times \Delta_q(Y) \to \Delta_{p+q}(X \times Y)$ induces a bilinear map $\times \colon H_p(X, A) \times H_q(Y, B) \to H_{p+q}\left((X, A) \times (Y, B)\right)$ defined by $[a] \times [b] = [a \times b]$.*

Proof. If $a \in \Delta_p(X)$ with $\partial a \in \Delta_{p-1}(A)$ (i.e., represents a cycle of (X, A)) and $b \in \Delta_q(Y)$ with $\partial b \in \Delta_{q-1}(B)$, then

$$\partial\left(a \times b\right) = \partial a \times b + (-1)^p a \times \partial b \in \Delta_{p+q-1}\left((A \times Y) \cup (X \times B)\right)$$

so $a \times b$ is a cycle in $\Delta_{p+q}\left((X, A) \times (Y, B)\right)$.
To show that it does not depend on the choices of representatives of relative homology classes, we note that if we add chains in A or B, then these clearly vanish under ∂. Also,

$$\begin{aligned}(a + \partial a') \times (b + \partial b') &= a \times b + a \times \partial b' + \partial a' \times b + \partial a' \times \partial b' \\ &= a \times b \pm \partial(a \times b') + \partial(a' \times b) + \partial(a' \times \partial b') + (\text{ch in } A \times Y \cup X \times B)\end{aligned}$$

when a and b are relative cycles. $\square$

3.1.1. *Proving that the Homotopy Axiom holds for Singular Homology.* Let $X = I = [0, 1]$ and regard also I as the affine simplex $[\{0\}, \{1\}] : \Delta_1 \to I$ and let $\varepsilon_0, \varepsilon_1$ be the 0-similices $\varepsilon_0(e_0) = \{0\}$ and $\varepsilon_1(e_0) = \{1\}$ of I. Then $\partial I = \varepsilon_1 - \varepsilon_0$.

Next, given a chain $c \in \Delta_q(X)$, we have $I \times c \in \Delta_{q+1}(I \times X)$, and

$$\partial\left(I \times c\right) = \partial I \times c - I \times \partial c = \varepsilon_1 \times c - \varepsilon_0 \times c - I \times \partial c.$$

Define $D \colon \Delta_q(X) \to \Delta_{p+1}(I \times X)$ by $D(c) = I \times c$. Then

$$\left(\partial D + D\partial\right)(c) = \partial\left(D(c)\right) + D\left(\partial c\right) = \varepsilon_1 \times c - \varepsilon_0 \times c.$$

Let η_0 and η_1 be the maps $X \to I \times X$ given by $\eta_0(x) = (0, x)$ and $\eta_1(x) = (1, x)$. Then $\eta_{i\Delta}(c) = \varepsilon_i \times c$. Thus

$$\partial D + D\partial = \eta_{1\Delta} - \eta_{0\Delta}$$

Theorem 3.5. *If $f_0 \simeq f_1 \colon (X, A) \to (Y, B)$, then $f_{0\Delta} \simeq f_{1\Delta} \colon \Delta_*(X, A) \to \Delta_*(Y, B)$, and therefore $f_{0*} = f_{1*} \colon H_*(X, A) \to H_*(Y, B)$.*

Proof. Let $F \colon I \times (X, A) \to (Y, B)$ be a homotopy between f_0 and f_1, so $F \circ \eta_0 = f_0$ and $F \circ \eta_1 = f_1$, then it induces $F_\Delta \colon \Delta_*\left(I \times X, I \times A\right) \to \Delta_*(Y, B)$. If we compose this with $\partial D + D\partial = \eta_{1\Delta} - \eta_{0\Delta}$, we obtain

$$\partial\left(F_\Delta \circ D\right) + \left(F_\Delta \circ D\right)\partial = F_\Delta \circ \left(\partial D + D\partial\right) = F_\Delta \circ \eta_{1\Delta} - F_\Delta \eta_{0\Delta} = f_{1\Delta} - f_{0\Delta},$$

showing that $F_\Delta \circ D$ is the desired chain homotopy. From this the second statement follows immediately. $\square$

Corollary 3.6. *Singular homology satisfies the Homotopy Axiom.*

Definition 3.7 (Graded group)**.** A graded group is a collection of abelian groups C_i indexed by the integers.

Definition 3.8 (Degree of map of graded groups)**.** Suppose A_*, B_*, C_* and D_* are graded groups.
A map $f \colon A_* \to B_*$ is said to be of degree d if it takes A_i to B_{i+d} for all i.

We define

$$(A_* \otimes B_*)_n := \bigoplus_{i+j=n} A_i \otimes B_j.$$

If $f\colon A_* \to C_*$ and $g\colon B_* \to D_*$, we define $f \otimes g\colon A_* \otimes B_* \to C_* \otimes D_*$ by

$$(f \otimes g)(a \otimes b) = (-1)^{\deg(a)\deg(g)} f(a) \otimes g(b).$$

Then we obtain that

$$\begin{aligned}
(f \otimes g) \circ (h \otimes k)(a \otimes b) &= (-1)^{\deg k \deg a} (f \otimes g)(h(a) \otimes k(b)) \\
&= (-1)^{\deg k \deg a + \deg g(\deg h + \deg a)} (f \circ h)(a) \otimes (g \circ k)(b) \\
&= (-1)^{\deg k \deg a + \deg g(\deg h + \deg a) + (\deg g + \deg k)\deg a} (f \circ h) \otimes (g \circ k)(a \otimes b) \\
&= (-1)^{\deg g \deg h} (f \circ h) \otimes (g \circ k)(a \otimes b).
\end{aligned}$$

In particular, for the chain complexes $\Delta_*(X)$ and $\Delta_*(Y)$, we have the chain complex

$$(\Delta_*(X) \otimes \Delta_*(Y))_n = \bigoplus_{i+j=n} \Delta_i(X) \otimes \Delta_j(Y),$$

with boundary operator $\partial_\otimes = \partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial$, meaning

$$\partial_\otimes (a_p \otimes b_q) = \partial a \otimes b + (-1)^p a \otimes \partial b$$

since id has degree 0. The subscript on $\partial_\otimes$ will often be dropped.
So now we have a chain complex $(\Delta_*(X) \otimes \Delta_*(Y), \partial_\otimes)$.

Note. Note that if x and y are points, then $(\Delta_*(\{x\}) \otimes \Delta_*(\{y\}))_0 = \Delta_0(\{x\}) \otimes \Delta_0(\{y\}) \cong \mathbb{Z}$ generated by $x \otimes y$ and $\Delta_0(\{x\} \times \{y\}) \cong \mathbb{Z}$ generated by (x, y).

We ask whether this always works.
Recall the cross product defined on chain complexes:

$$\times\colon \Delta_*(X) \times \Delta_*(Y) \to \Delta_*(X \times Y).$$

Since it was bilinear, it induces a homomorphism

$$\times\colon \Delta_*(X) \otimes \Delta_*(Y) \to \Delta_*(X \times Y)$$

which, by definition, takes $a \otimes b$ to $a \times b$.
By Theorem 3.2, the product is natural in X and Y, and when X and Y are points, it is the canonical map $x \times y = (x, y)$, and also the boundary formula $\partial(a \times b) = \partial a \times b + (-1)^{\deg a} a \times \partial b$ holds.
Thus

$$\begin{aligned}
\partial(\times(a \otimes b)) = \partial(a \times b) &= \partial a \times b + (-1)^{\deg a} a \times \partial b \\
&= \times\left(\partial a \otimes b + (-1)^{\deg a} a \otimes \partial b\right) \\
&= \times(\partial_\otimes(a \otimes b)).
\end{aligned}$$

So $\times$ is a chain map $(\Delta_*(X) \otimes \Delta_*(Y), \partial_\otimes) \to (\Delta_*(X \times Y), \partial)$.

Lemma 3.9. *If X and Y are contractible, then there is a chain contraction of $\Delta_*(X) \otimes \Delta_*(Y)$. Consequently, $H_n(\Delta_*(X) \otimes \Delta_*(Y)) = 0$ for $n > 0$ and is $\mathbb{Z}$, generated by $[x_0 \otimes y_0]$, for $n = 0$.*

Proof. In Theorem 3.1, we constructed a chain contraction for X. I.e., we constructed a map $D\colon \Delta_p(X) \to \Delta_{p+1}(X)$ such that $\partial D + D\partial = \mathrm{id} - \varepsilon$ - where ε is the augmentation map.
Let $E = D \otimes \mathrm{id} + \varepsilon \otimes D$ on $\Delta_*(X) \otimes \Delta_*(Y)$. Then

$$\begin{aligned}
E\partial_{\otimes} + \partial_{\otimes} E &= (D \otimes \mathrm{id} + \varepsilon \otimes D)(\partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial) + (\partial \otimes \mathrm{id} + \mathrm{id} \otimes \partial)(D \otimes \mathrm{id} + \varepsilon \otimes D) \\
&= (-1)^{\deg \mathrm{id} \deg \partial} D\partial \otimes \mathrm{id} + (-1)^{\deg id \deg id} D \otimes \partial + (-1)^{\deg D \deg \partial} \varepsilon\partial \otimes D \\
&+ (-1)^{\deg D \deg \mathrm{id}} \varepsilon \otimes D\partial + (-1)^{\deg id \deg D} \partial D \otimes \mathrm{id} + (-1)^{\deg \mathrm{id} \deg \varepsilon} \partial\varepsilon \otimes D \\
&+ (-1)^{\deg \partial \deg D} D \otimes \partial + (-1)^{\deg \partial \deg \varepsilon} \varepsilon \otimes \partial D \\
&= D\partial \otimes \mathrm{id} + \varepsilon \otimes D\partial + \partial D \otimes \mathrm{id} + \varepsilon \otimes \partial D
\end{aligned}$$

Note that we here have used that ε is a chain map, but D might not be which is why the terms above survive. Now

$$\begin{aligned}
D\partial \otimes \mathrm{id} + \varepsilon \otimes D\partial + \partial D \otimes \mathrm{id} + \varepsilon \otimes \partial D &= (\mathrm{id} - \varepsilon - \partial D) \otimes \mathrm{id} + \varepsilon \otimes (\mathrm{id} - \varepsilon - \partial D) + \partial D \otimes \mathrm{id} + \varepsilon \otimes \partial D \\
&= \mathrm{id} \otimes \mathrm{id} - \varepsilon \otimes \varepsilon
\end{aligned}$$

□

Theorem 3.10. *There exists a natural (in X and Y) chain map*

$$\theta\colon \Delta_*(X \times Y) \to \Delta_*(X) \otimes \Delta_*(Y)$$

which is the canonical map $(x, y) \mapsto x \otimes y$ in degree 0.

Proof. The proof is simple and by acyclic models. See Bredon's book. □

Theorem 3.11. *Any two natural chain maps on $\Delta_*(X \times Y)$ to itself or on $\Delta_*(X) \otimes \Delta_*(Y)$ to itself or on one of these to the other which are the canonical isomorphisms in degree zero, with X and Y points, are naturally chain homotopic.*

Corollary 3.12 (The Eilenberg-Zilber Theorem)**.** *The chain maps*

$$\theta\colon \Delta_*(X \times Y) \to \Delta_*(X) \otimes \Delta_*(Y)$$

and

$$\times\colon \Delta_*(X) \otimes \Delta_*(Y) \to \Delta_*(X \times Y)$$

are natural homotopy equivalences which are naturally homotopy inverses of one another.

Corollary 3.13.

$$H_p(X \times Y; G) \cong H_P\left(\Delta_*(X) \otimes \Delta_*(Y) \otimes G\right),$$

and

$$H^p\left(X \times Y; G\right) \cong H^p\left(\mathrm{Hom}\left(\Delta_*(X) \otimes \Delta_*(Y)\right), G\right)$$

Theorem 3.14 (Algebraic Künneth Theorem)**.** *Let K_* and L_* be free chain complexes. Then there is a natural exact sequence*

$$0 \to \left(H_*(K_*) \otimes H_*(L_*)\right)_n \xrightarrow{\times} H_n(K_* \otimes L_*) \to \left(\mathrm{Tor}\left(H_*(K_*), H_*(L_*)\right)\right)_{n-1} \to 0$$

which splits non-naturally.

Note. Here

$$(\operatorname{Tor}(A_*, B_*))_n = \bigoplus_{i+j=n} \operatorname{Tor}(A_i, B_j)$$

Theorem 3.15 (Geometric Künneth Theorem)**.** *There is a natural exact sequence*

$$0 \to (H_*(X) \otimes H_n(Y))_n \xrightarrow{\times} H_n(X \times Y) \to (\operatorname{Tor}(H_*(X), H_*(Y)))_{n-1}$$

which splits non-naturally.