ASSIGNMENT 1

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Problem 0.1 (1)**.** Let $\pi\colon \mathbb{R} \to \mathrm{GL}\left(\mathbb{R}^2\right)$ by $\pi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ with respect to the standard basis $\{e_1, e_2\}$ of $\mathbb{R}^2$. Consider the tensor product

$$\pi \otimes \pi \colon \mathbb{R} \to \mathrm{GL}\left(\mathbb{R}^2 \otimes \mathbb{R}^2\right).$$

Identify $\mathbb{R}^2 \otimes \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$ via $e_i \otimes e_j \mapsto E_{ij}$.

(1) Fix the following basis $\mathcal{B} = \{A_1, A_2, A_3, A_4\}$ of $M_{2\times 2}(\mathbb{R}) \cong \mathbb{R}^2 \otimes \mathbb{R}^2$:

$$A_1 = I_2, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Compute $(\pi \otimes \pi)_{\mathcal{B}} : \mathbb{R} \to \mathrm{GL}_4(\mathbb{R})$.

(2) Compute the decomposition of $M_{2\times 2}(\mathbb{R})$ into minimal invariant subspaces (with respect to the representation $\pi \otimes \pi$) and prove that the stated subspaces are indeed minimal.

Proof. (1) Let $\varphi_{\mathcal{B}}\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the change of basis given by $\varphi_{\mathcal{B}}(A_i) = e_i$.
By definition,

$$(\pi \otimes \pi)_{\mathcal{B}}(t) e_j = \varphi_{\mathcal{B}}\left(\pi \otimes \pi(t) A_j\right)$$

Now,

$$\begin{aligned} A_1 &= e_1 \otimes e_1 + e_2 \otimes e_2 \\ A_2 &= e_1 \otimes e_1 - e_2 \otimes e_2 \\ A_3 &= e_2 \otimes e_1 + e_1 \otimes e_2 \\ A_4 &= e_1 \otimes e_2 - e_2 \otimes e_1 \end{aligned}$$

so

$$
\begin{aligned}
\pi\otimes\pi(t)A_1 &= \pi(t)e_1\otimes\pi(t)e_1+\pi(t)e_2\otimes\pi(t)e_2\\
&= (\cos te_1-\sin te_2)\otimes(\cos te_1-\sin te_2)+(\sin te_1+\cos te_2)\otimes(\sin te_1+\cos te_2)\\
&= \cos^2 te_1\otimes e_1+\sin^2 te_2\otimes e_2-\sin t\cos t\,(e_1\otimes e_2+e_2\otimes e_1)\\
&+\sin^2 te_1\otimes e_1+\cos^2 te_2\otimes e_2+\cos t\sin t\,(e_1\otimes e_2+e_2\otimes e_1)\\
&= e_1\otimes e_1+e_2\otimes e_2\\
&= A_1\\
\pi\otimes\pi(t)A_2 &= \pi(t)e_1\otimes\pi(t)e_1-\pi(t)e_2\otimes\pi(t)e_2\\
&= (\cos te_1-\sin te_2)\otimes(\cos te_1-\sin te_2)-(\sin te_1+\cos te_2)\otimes(\sin te_1+\cos te_2)\\
&= \cos^2 te_1\otimes e_1+\sin^2 te_2\otimes e_2-\cos t\sin t\,(e_1\otimes e_2+e_2\otimes e_1)\\
&-\sin^2 te_1\otimes e_1-\cos^2 te_2\otimes e_2-\sin t\cos t\,(e_1\otimes e_2+e_2\otimes e_1)\\
&= \cos(2t)e_1\otimes e_1-\cos(2t)e_2\otimes e_2-\sin(2t)\,(e_1\otimes e_2+e_2\otimes e_1)\\
&= \cos(2t)A_2-\sin(2t)A_3\\
\pi\otimes\pi(t)A_3 &= (\sin te_1+\cos te_2)\otimes(\cos te_1-\sin te_2)+(\cos te_1-\sin te_2)\otimes(\sin te_1+\cos te_2)\\
&= \cos t\sin te_1\otimes e_1-\cos t\sin te_2\otimes e_2+\cos^2 te_2\otimes e_1-\sin^2 te_1\otimes e_2\\
&+\cos t\sin te_1\otimes e_1-\sin t\cos te_2\otimes e_2+\cos^2 te_1\otimes e_2-\sin^2 te_2\otimes e_1\\
&= \sin(2t)e_1\otimes e_1-\sin(2t)e_2\otimes e_2+\cos(2t)(e_2\otimes e_1+e_1\otimes e_2)\\
&= \sin(2t)A_2+\cos(2t)A_3\\
\pi\otimes\pi(t)A_4 &= (\cos te_1-\sin te_2)\otimes(\sin te_1+\cos te_2)-(\sin te_1+\cos te_2)\otimes(\cos te_1-\sin te_2)\\
&= \cos t\sin te_1\otimes e_1-\cos t\sin te_2\otimes e_2+\cos^2 te_1\otimes e_2-\sin^2 te_2\otimes e_1\\
&-\sin t\cos te_1\otimes e_1+\cos t\sin te_2\otimes e_2+\sin^2 te_1\otimes e_2-\cos^2 te_2\otimes e_1\\
&= e_1\otimes e_2-e_2\otimes e_1\\
&= A_4
\end{aligned}
$$

Thus the matrix for $(\pi\otimes\pi)_{\mathcal{B}}$ with respect to the standard basis becomes

$$
\begin{pmatrix}
1 & 0 & 0 & 0\\
0 & \cos(2t) & \sin(2t) & 0\\
0 & -\sin(2t) & \cos(2t) & 0\\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(2) It suffices to decompose $(\pi\otimes\pi)_{\mathcal{B}}$ into irreducible subspaces. Now since $(\pi\otimes\pi)_{\mathcal{B}}(e_i)=e_i$ for $i=1,4$, we have that $\mathrm{span}(e_1)$ and $\mathrm{span}(e_4)$ are $(\pi\otimes\pi)_{\mathcal{B}}$ invariant subspaces. Furthermore, since they are 1-dimensional, they are irreducible. Let $W=\mathrm{span}(e_2,e_3)$. On W, $(\pi\otimes\pi)_{\mathcal{B}}$ restricts to the representation

$$
\begin{pmatrix}
\cos(2t) & \sin(2t)\\
-\sin(2t) & \cos(2t)
\end{pmatrix}
$$

which is a rotation matrix. By example 1.20, this representation is irreducible since it is irreducible for all s when we insert $s=\frac{1}{2}t$, hence also for all t. Hence $\mathrm{span}(e_1),\mathrm{span}(e_2,e_3)$ and $\mathrm{span}(e_4)$ are the minimal invariant subspaces of $\mathbb{R}^4$ with respect to $(\pi\otimes\pi)_{\mathcal{B}}$. By changing back in bases, we find that $\mathrm{span}(A_1),\mathrm{span}(A_2,A_3)$ and $\mathrm{span}(A_4)$ are the minimal invariant subspaces of $\pi\otimes\pi$. $\square$

Problem 0.2 (2)**.** Let $\pi\colon \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ be given by $\pi(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and let $\pi_0\colon \mathbb{R} \to \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$ be given by $\pi_0(t) = 1$ be the one-dimensional trivial representation of $\mathbb{R}$. Find non-zero intertwining operators from π_0 to π and from π to π_0.

Solution. An intertwining operator from π_0 to π is a linear map $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ such that $\pi(g) \circ \varphi = \varphi \circ \pi_0(g)$ for all g. Thus we want to find φ such that for all t

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \circ \varphi = \varphi \circ I$$

For this, we can simply choose $\varphi(e_1) = e_1$, giving $[\varphi] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Now, an intertwining operator from π to π_0 is a linear map $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\pi_0(t) \circ \psi = \psi \circ \pi(t) = \psi \circ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

If we define $\psi(e_1) = 0$ and $\psi(e_2) = e_1$ so $[\psi] = \begin{pmatrix} 0 & 1 \end{pmatrix}$, we get $\pi_0(t) \circ \psi(e_1) = 0 = \psi \circ \pi(t)(e_1)$, while $\pi_0(t) \circ \psi(e_2) = e_1 = \psi \circ \pi(t) e_2$. So ψ is an intertwining operator from π to π_0. ☐

Problem 0.3 (3)**.** Suppose that π, σ_1, σ_2 are finite dimensional completely reducible representations of a group G. Show that if $\pi \oplus \sigma_1$ is equivalent to $\pi \oplus \sigma_2$, then σ_1 is equivalent to σ_2.

Solution. Suppose $\pi\colon G \to \mathrm{GL}(V)$, $\sigma_1\colon G \to \mathrm{GL}(U_1)$ and $\sigma_2\colon G \to \mathrm{GL}(U_2)$ are the representations.
Now, by assumption, there exists an isomorphism $\varphi\colon V \oplus U_1 \to V \oplus U_2$ such that $\pi \oplus \sigma_2 = \varphi \circ (\pi \oplus \sigma_1) \circ \varphi^{-1}$. Note that since π, σ_1, σ_2 are completely reducible, we can write $\pi \simeq \pi_1 \oplus \ldots \oplus \pi_n, \sigma_1 \simeq \sigma_{1,1} \oplus \ldots \oplus \sigma_{1,m}$ and $\sigma_2 \simeq \sigma_{2,1} \oplus \ldots \oplus \sigma_{2,k}$. Then

$$\pi \oplus \sigma_1 \simeq \bigoplus_{i=1}^{n} \pi_i \oplus \bigoplus_{j=1}^{m} \sigma_{1,j}$$

and

$$\pi \oplus \sigma_2 \simeq \bigoplus_{i=1}^{n} \pi_i \oplus \bigoplus_{j=1}^{k} \sigma_{2,j}$$

are also completely reducible by theorem 3.7 (as well as the direct sum of two irreducible representations again being irreducible). Furthermore, by corollary 3.10, $m = k$.

Now, φ must take an irreducible subspace of $V \oplus U_1$ to an irreducible subspace of $V \oplus U_2$. So φ induces a bijection using Corollary 3.10:
$\tilde{\varphi}\colon \{V_1, \ldots, V_n, U_{1,1}, \ldots, U_{1,m}\} \to \{V_1, \ldots, V_n, U_{2,1}, \ldots, U_{2,m}\}$ where V_i is the irreducible subspace of π_i, $U_{1,i}$ of $\sigma_{1,i}$ and $U_{2,i}$ of $\sigma_{2,i}$. Thus, for each i, $\varphi \circ \sigma_{1,i} \circ \varphi^{-1}$ must be one of the π_j or $\sigma_{2,j}$. Now, note that if it is π_j, then in the decomposition of $\pi \oplus \sigma_1$, we have 2 irreducible representations equivalent to $\sigma_{1,i}$. Hence in the decomposition of $\pi \oplus \sigma_2$, we must also have two, so there must be another irreducible factor also equivalent to $\sigma_{1,i}$. The same argument applies as above, and after a finite number of steps, we find that there must be some $\sigma_{2,j}$ equivalent to $\sigma_{1,i}$, since otherwise we would obtain at least $n+1$ irreducible factors equivalent to $\sigma_{1,i}$ in $\pi \oplus \sigma_1$, but only n in $\pi \oplus \sigma_2$, giving a contradiction. Let φ_{i,j_i} be the isomorphism

$U_{1,i} \to U_{2,j_i}$ such that $\varphi_{i,j_i} \circ \sigma_{1,i} \circ \varphi_{i,j_i}^{-1} = \sigma_{2,j_i}$. Applying this argument to each $\sigma_{1,i}$, we obtain isomorphisms $\varphi_{1,j_1}, \dots, \varphi_{m,j_m}$. But then pasting these together by letting $\varphi(x) = \varphi_{i,j_i}(x)$ when $x \in U_{1,i}$, we get an isomorphism $\varphi\colon U_1 \to U_2$ such that $\varphi \circ \sigma_{1,i} \circ \varphi^{-1} = \sigma_{2,j_i}$ for all i, so $\varphi \circ \sigma_1 \circ \varphi^{-1} = \sigma_2$. Thus $\sigma_1 \simeq \sigma_2$. $\square$

Problem 0.4 (4)**.** Let G be a finite group and π_1, π_2 two one-dimensional representations of G. Let $\pi = \pi_1 \oplus \pi_2$. Compute $\pi \otimes \pi$ and $\operatorname{Sym}^2 \pi$, that is, find a decomposition of those representations into irreducible ones.

Solution. Suppose $\pi_1, \pi_2\colon G \to \mathrm{GL}(V)$ where V is a one-dimensional vector space over a field k. Fix an isomorphism $V \cong k$, so that we can from now assume $V = k$. Then since $\mathrm{GL}(k) \cong k^\times = k - \{0\}$, we have that $\pi_1(g) = c_g$ and $\pi_2(g) = d_g$ where c_g and d_g are non-zero constants. Since π_1, π_2 are one-dimensional, π is two-dimensional, hence $\pi \otimes \pi$ is four-dimensional.

$$\pi \otimes \pi(g) = \pi(g) \otimes \pi(g) = (\pi_1(g), \pi_2(g)) \otimes (\pi_1(g), \pi_2(g)) = (c_g, d_g) \otimes (c_g, d_g)$$

Thus

$$\pi \otimes \pi(g)\left((x,y) \otimes (v,w)\right) = \pi(g)(x,y) \otimes \pi(g)(v,w) = (c_g x, d_g y) \otimes (c_g v, d_g w)$$

Hence, if we let $v_1 = e_1 \otimes e_1, v_2 = e_1 \otimes e_2, e_3 = e_2 \otimes e_1$ and $e_4 = e_2 \otimes e_2$, a matrix representation for $\pi \otimes \pi(g)$ in the basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ becomes

$$\begin{pmatrix} c_g^2 & 0 & 0 & 0 \\ 0 & c_g d_g & 0 & 0 \\ 0 & 0 & c_g d_g & 0 \\ 0 & 0 & 0 & d_g^2 \end{pmatrix}$$

Another way to see this is as follows: By example 3.13, $\pi \otimes \pi$ is equivalent to $\prod\colon G \to \mathrm{GL}_{2\times 2}(V)$ given by $\prod(g)A = (\pi)_{\mathcal{B}}\,(g) A\,(\pi)_{\mathcal{B}}\,(g)^t$. Choosing any basis $\mathcal{B}$, we find that $\pi(g) = (\pi_1(g), \pi_2(g))$ and each $\pi_i(g)$ is constant nonzero, so its matrix representation becomes

$$\pi(g) = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

Hence if $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$, we get

$$\pi(g) A \pi(g)^t = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \alpha_{11} c & \alpha_{12} d \\ \alpha_{21} c & \alpha_{22} d \end{pmatrix} = \begin{pmatrix} \alpha_{11} c^2 & \alpha_{12} cd \\ \alpha_{21} cd & \alpha_{22} d^2 \end{pmatrix}$$

Hence each subspace $V_i \subset M_{2\times 2}(F)$ where V_i is spanned by the matrix which has entry 1 in the i th entry and 0 elsewhere, where we just enumerate the entries using a lexicographic ordering, for example. So

$$V_1 = \operatorname{span} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$V_2 = \operatorname{span} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$V_3 = \operatorname{span} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$V_4 = \operatorname{span} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

These are all 1-dimensional hence irreducible. As shown above, they are also invariant under $\prod$. So

$$\pi \otimes \pi = (\pi \otimes \pi)_{V_1} \oplus \ldots \oplus (\pi \otimes \pi)_{V_4}$$

is a decomposition of $\pi \otimes \pi$ into irreducible components.

Now we wish to find $\operatorname{Sym}^2 \pi$. Recall that

$$\operatorname{Sym}^2 k^2 = k^2 \otimes k^2 / I$$

where I is generated by elements of the form

$$v \otimes w - w \otimes v.$$

Thus we get the following identification in $\operatorname{Sym}^2 k^2$:

$$(0,1) \otimes (1,0) = (1,0) \otimes (0,1)$$

So in fact,

$$\operatorname{Sym}^2 k^2 \cong \operatorname{span}((1,0) \otimes (1,0)) \oplus \operatorname{span}((0,1) \otimes (1,0)) \oplus \operatorname{span}((0,1) \otimes (0,1))$$

Now, since $\operatorname{Sym}^2 \pi$ is simply the descent of $\pi \otimes \pi$ to the quotient $k^2 \times k^2/I$, V_1 and V_4 remain invariant. Furthermore, $\operatorname{Sym}^2 \pi\,((1,0) \otimes (0,1))) = (1,0) \otimes (0,1)$, so $W_2 := \operatorname{span}((0,1) \otimes (1,0))$ is also invariant. Let $W_1 := V_1$ and $W_3 := V_4$. Then we have shown that W_1, W_2 and W_3 are invariant under $\operatorname{Sym}^2 \pi$. Furthermore, each is 1-dimensional, and $\operatorname{Sym}^2 k^2$ decomposes to a direct sum $\operatorname{Sym}^2 k^2 \cong W_1 \oplus W_2 \oplus W_3$. Since each $\operatorname{Sym}^2 \pi|_{W_i}$ is invariant, this gives a decomposition of $\operatorname{Sym}^2 \pi$ into three irreducible representations.