Problem 0.1. Let F be the homotopy fibre of the map $S^n \to S^n$ of degree k, for $n \ge 2$.

- (1) Show that $H^i(F) = 0$ for 0 < i < n.
- (2) Using the Serre spectral sequence, compute that

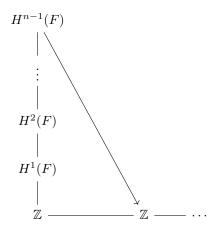
$$H^{i}(F) = \begin{cases} \mathbb{Z}, & i = 0\\ \mathbb{Z}/k, & i = 1 + m(n-1), m > 0. \\ 0, & \text{otherwise} \end{cases}$$

(3) Show that for $x, y \in H^*(F)$, if $\deg(x), \deg(y) > 0$, then $x \smile y = 0$.

Proof. (1) Since $\pi_1 S^n = 0$, the Serre spectral sequence to the homotopy fiber sequence

$$F \to S^n \to S^n$$

gives the following double complex:



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We apply the LSSS for cohomology and find that $H^i(S^n) = F_0^n$, and since $H^i(F)$ is the only nontrivial entry on the antidiagonal in degree i, and since there are no maps to kill off $H^i(F)$ for 0 < i < n-1, we obtain that $H^i(F) = H^i(S^n) = 0$ for 0 < i < n-1.

All that's missing is i = n - 1. For this, note that by the LES for the fibration, we get the following exact sequence:

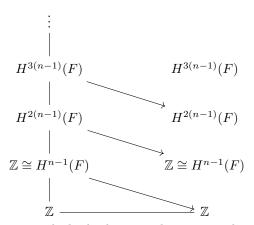
$$\underbrace{\mathbb{Z}}_{\pi_n(S^n)} \stackrel{\cdot k}{\to} \underbrace{\mathbb{Z}}_{\pi_n(S^n)} \to \pi_{n-1}(F) \to \underbrace{0}_{\pi_{n-1}(S^n)}$$

hence $\pi_{n-1}(F) \cong \operatorname{coker}\left(\mathbb{Z} \stackrel{\cdot k}{\to} \mathbb{Z}\right) \cong \mathbb{Z}/k$, and by the Hurewicz theorem, we get $H_{n-1}(F) \cong \pi_{n-1}(F) \cong \mathbb{Z}/k$. Now using the UCT, we obtain

$$0 \to \underbrace{\operatorname{Ext}(H_{n-2}(F), \mathbb{Z})}_{=0} \to H^{n-1}(F) \to \operatorname{Hom}\left(\underbrace{H_{n-1}(F)}_{=\mathbb{Z}/k}, \mathbb{Z}\right) \to 0$$

so $H^{n-1}(F) = 0$ as we wanted.

By the LSSS, the E^{∞} page has the form $E_{0,0}^{\infty} = E_{n,0}^{\infty} \cong \mathbb{Z}$, so in particular, on the E^k page, we get the following double complex:



This is the only page on which the horizontal maps can be nontrivial, so given the E^{∞} page, we conclude that the maps must be isomorphisms (including the trivial ones by just inductively shifting down by n-1 enough times). Hence we get periodicity, so

$$H^{i}(F) = \begin{cases} \mathbb{Z}, & i = 0\\ \mathbb{Z}/k, & i = 1 + m(n-1), m > 0, \\ 0, & \text{otherwise} \end{cases}$$

which was what we wanted to show.

(3) Suppose $\deg(x)+\deg(y)=2$ so both are of degree 1, then since $H^1(F)=0$, we have x=0=y so $x\smile y=0$. Suppose we have shown it for $\deg(x)+\deg(y)\le N-1$ now. If $\deg x+\deg y=N$, then firstly we can assume $x,y\ne 0$ since otherwise $x\smile y=0$. Hence $x\in H^{1+m(n-1)}(F)$ and $y\in H^{1+m'(n-1)}$, so $x\smile y\in H^{2+(m+m')(n-1)}(F)=0$, so directly, $x\smile y=0$.