Assignment 4

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31: Give the set of real numbers the finite-complement topology. What are the components of the resulting space? Answer the same question for the half-open interval topology.

Solution: Let $\mathbb{R}_{fc}$ denote the reals with the finite complement topology. We claim that $\mathbb{R}_{fc}$ is connected. Assume there exist disjoint open sets $U, V \subset \mathbb{R}_{fc}$ such that $\mathbb{R} = U \cup V$. Then by definition, $\mathbb{R} - U$ and $\mathbb{R} - V$ are finite, however, as $U \cap V = \varnothing$, we have $V \subset \mathbb{R} - U$, so V is finite, and hence $\mathbb{R} - V$ is infinite, contradicting V being a nonempty open set - which would have to be infinite.
By theorem 3.20 (c) equivalent to (a), we have that $\mathbb{R}_{fc}$ is connected.

34: A space X is locally connected if for each $x \in X$, and each neighborhood U of x, there is a connected neighborhood V of x which is contained in U. Show that any euclidean space, and therefore any space which is locally euclidean (like a surface), is locally connected. If $X = \{0\} \cup \left\{ \frac{1}{n} \mid n = 1, 2, \dots \right\}$ with the subspace topology from the real line, show that X is not locally connected.

Solution: We have the continuous map $f\colon (-1,1) \to \mathbb{R}$ defined by $f(t) = \frac{t}{1-|t|}$ defines a homeomorphism from $(-1,1)$ to $\mathbb{R}$ with inverse $f^{-1}(t) = \frac{t}{1+|t|}$, and $(-1,1)$ is homeomorphism to any open interval (a,b) by the map $t \to a + (b-a)\frac{t+1}{2}$ with inverse $t \to -1 + 2\frac{t-a}{b-a}$. Hence any interval (a,b) is homeomorphic to $\mathbb{R}$ which is connected and as connectedness is a topological property, (a,b) is also connected. By induction on theorem 3.26, we have that any generalized open cube

$$(a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n) \subset \mathbb{R}^n$$

is connected. Now, the product topology on $\mathbb{R}^n$ is generated by the base that consists of all products $U_1 \times \ldots \times U_n$, such that $U_i \subset \mathbb{R}$ is open.
Taking for each U_i a basis element $(a_i, b_i) \subset U_i$, we see that $(a_1, b_1) \times \ldots \times (a_n, b_n) \subset U_1 \times \ldots \times U_n$, so the topology on $\mathbb{R}^n$ generated by the basis of all products of intervals $(a_1, b_1) \times \ldots \times (a_n, b_n)$ is finer than the topology induced by all product of open sets in $\mathbb{R}$.
Conversely, since each (a_i, b_i) is already an open set in $\mathbb{R}$, the topology generated by the base consisting of products of open sets of $\mathbb{R}$ is finer than the topology generated by the base consisting of products of open intervals. Thus, as these topologies are each finer than the other, the topologies are the same.

Now we are ready to show the result: let $x \in \mathbb{R}^n$ and U be a neighborhood of x. Then there exists an open set $V \subset U$ containing x. As V is open, we can now find a basis element $I = (a_1, b_1) \times \ldots \times (a_n, b_n) \subset V$ with $x \in I$. As I was shown to be connected, we thus find that $\mathbb{R}^n$ is locally connected.

Now suppose $X = \{0\} \cup \left\{ \frac{1}{n} \mid n = 1, 2, \dots \right\}$ with the subspace topology from the real line.
Let U be a neighborhood of 0, and suppose there exists a connected neighborhood V of 0 which is contained in U. As V is a neighborhood, there exists an open neighborhood $W \subset V$ of 0 and thus we can find some interval (a,b) containing 0 such that $0 \in (a,b) \cap X \subset W \subset V$. Now, choose a $N \in \mathbb{N}$ such that $\frac{1}{N} \in (a,b)$. Then choosing any $q \in \left(\frac{1}{N+1}, \frac{1}{N} \right)$, we have that $((-\infty, q) \cap V) \cup ((q, \infty) \cap V) = V$, and $0 \in (\infty, q) \cap V$ and $\frac{1}{N} \in (q, \infty) \cap V$, so $(-\infty, q) \cap V$ and $(q, \infty) \cap V$ are nonempty disjoint open sets in X separating V, contradicting V being connected - using formulation (c) in theorem 3.20 for connectedness.