

# 1. GLOSSARY FOR EXAM

- All subspaces have a complement (thm 2.14)
- $A, \tilde{A} \in \text{Hom}(U, V)$  are equivalent if there exist  $S \in \text{GL}(U)$  and  $T \in \text{GL}(V)$  such that

$$\begin{array}{ccc} U & \xrightarrow{A} & V \\ \downarrow S & & \downarrow T \\ U & \xrightarrow{\tilde{A}} & V \end{array}$$

- $A, \tilde{A} \in \text{End}(V)$  are called similar if there exists  $T \in \text{GL}(V)$  such that  $TA = \tilde{A}T$ .
- **Theorem 2.22:** Assume  $U, V$  fin.dim., then  $A, \tilde{A}$  are equivalent iff they have the same rank.
- For  $y \in V'$ , the null-space of  $y$  has codimension one in  $V$  and only the non-zero scalar multiples of  $y$  have the same null-space as  $y$ . (lemma 3.15 + see thm 3.14)
- The dual map is the pullback: if  $A \in \text{Hom}(U, V)$ , then  $A' \in \text{Hom}(V', U')$  is defined by  $A'(y) = y \circ A$ .

$$\begin{array}{ccc} U & \xrightarrow{A} & V \\ & \searrow A'(y) & \downarrow y \\ & & k \end{array}$$

## CHAPTER 1

**Exercise 1.1** (12.2). Let  $U$  and  $V$  be normed spaces, and assume that  $V$  is complete. Show that then  $B(U, V)$  is also complete with the operator norm.

*Solution.* Suppose  $B_n$  is a Cauchy sequence in  $B(U, V)$ , so

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \mid \|B_n - B_m\| < \varepsilon, \quad \forall m, n \geq N.$$

That is, for all  $m, n \geq N$ ,

$$\sup_{\|x\|=1} \|B_n x - B_m x\| < \varepsilon$$

For a fixed  $n$ , this becomes a Cauchy sequence in  $V$  which thus converges, so we can define  $Bx = \lim_{n \rightarrow \infty} B_n x$ . We claim that  $B$  is a bounded operator too. It is clear that it is linear since each  $B_n$  is a continuous map. What remains is to show that  $B$  is bounded. It suffices to show that it is bounded on  $S = \{x \mid \|x\| = 1\}$ . Suppose it were not bounded and choose a sequence  $(x_n) \subset S$  such that  $\|Bx_n\| > n$ . Choose  $\varepsilon = \frac{1}{2}$  and let  $N$  be such that for  $n, m \geq N$ , we have

$$\|B_n - B_m\| < \varepsilon$$

Then for all  $k$

$$\|B_n x_k - B_m x_k\| < \varepsilon$$

for all  $n \geq N$ , so in particular

$$\|Bx_k - B_m x_k\| = \lim_{n \rightarrow \infty} \|B_n x_k - B_m x_k\| < \varepsilon$$

But  $B_m$  is bounded, so let  $\|B_m\| = R$ . Choose  $M$  such that for all  $k \geq M$ , we have  $\|Bx_k\| \geq \|B_m x_k\|$ , then

$$\|Bx_k\| - \|B_m x_k\| < \varepsilon$$

giving

$$\|Bx_k\| < \varepsilon + R$$

contradicting  $\|Bx_k\| \rightarrow \infty$ .

**Exercise 1.2** (12.3). Let  $A \in B(V)$  for a complete normed space  $V$ . Prove

- (1) If  $\|A\| < 1$  then  $\sum_{k=0}^{\infty} A^k$  converges in  $B(V)$  to an inverse of  $I - A$ .
- (2) If  $B \in B(V)$  is invertible and  $\|A\| < \frac{1}{\|B^{-1}\|}$  then  $B - A$  is invertible.
- (3) The set of invertible bounded operators is an open subset of  $B(V)$ .

(Here invertible means there is a bounded inverse)

*Solution.* (1) geometric series.

(2)  $B - A = B(1 - \frac{A}{B})$ . Now  $\|\frac{A}{B}\| \leq \|A\|\|B^{-1}\| < 1$ , so by (1),  $1 - \frac{A}{B}$  has inverse  $\sum_{k=0}^{\infty} (\frac{A}{B})^k$ . But then  $B - A$  is a composition of invertible maps hence invertible since  $\text{GL}(V)$  is a group.

(3) Suppose  $A \in B(B, \frac{1}{\|B^{-1}\|})$ , so  $\|B - A\| < \frac{1}{\|B^{-1}\|}$ . By (2),  $B - (B - A) = A$  is then invertible. Hence  $B(B, \frac{1}{\|B^{-1}\|})$  is an open neighborhood of  $B$  in  $B(V)$  consisting of invertible maps. Thus the set of invertible maps is open in  $B(V)$ .

**Exercise 1.3** (12.6). Let  $S \in \text{End}(\ell^2)$  denote the right shift taking the sequence  $(x_1, x_2, \dots)$  to  $(0, x_1, x_2, \dots)$ . Show it is bounded and determine the operator norm  $\|S\|$ . Find also the adjoint  $S^*$ , and verify that  $S^*S = I$  but  $SS^* \neq I$ .

*Solution.* Recall that we are dealing with the norm  $\|(x_1, x_2, \dots)\|^2 = \sum_{k=1}^{\infty} |x_k|^2$ . But indeed then if  $\|(x_1, \dots)\| = 1$ , then

$$\|S(x_1, \dots)\|^2 = \|(0, x_1, x_2, \dots)\|^2 = \sum_{k=1}^{\infty} |x_k|^2 = 1$$

so, in fact,  $S$  preserves the norm. But then since  $\|Sx\| = \|x\|$  for all  $x$  by linearity, we have  $\|S\| = 1$ . Now, the inner product is  $\langle x, y \rangle = \sum_k x_k \overline{y_k}$ . Then

$$\langle Sx, y \rangle = \sum_{k=2}^{\infty} x_k y_{k-1} = \langle x, S^*y \rangle$$

if we define  $S^*(y_1, y_2, \dots) = (y_2, y_3, \dots)$ . We then indeed get  $S^*S = I$  clearly, but  $SS^*(x_1, x_2, \dots) = (0, x_2, x_3, \dots)$ .

**Exercise 1.4** (12.8). Show  $\|Ax \pm ix\|^2 = \|Ax\|^2 + \|x\|^2$  for  $A$  Hermitian and  $\dim V < \infty$ . Then show  $A \pm iI$  is invertible and  $(A - iI)(A + iI)^{-1}$  unitary.

*Solution.*

$$\begin{aligned} \langle Ax \pm ix, Ax \pm ix \rangle &= \|Ax\|^2 + \langle Ax, \pm ix \rangle + \langle \pm ix, Ax \rangle + \langle \pm ix, \pm ix \rangle \\ &= \|Ax\|^2 + \|x\|^2 \mp i \langle Ax, x \rangle \pm i \langle x, Ax \rangle \\ &= \|Ax\|^2 + \|x\|^2 \mp i \langle Ax, x \rangle \pm i \langle Ax, x \rangle \\ &= \|Ax\|^2 + \|x\|^2. \end{aligned}$$

Now, if  $A \pm iI$  were not invertible, it would not be injective, so for  $x \neq 0$ , we would get

$$0 = \|Ax \pm ix\|^2 = \|Ax\|^2 + \|x\|^2$$

but  $\|x\|^2 > 0$  and  $\|Ax\|^2 \geq 0$ , so this gives a contradiction.

Lastly, what is the adjoint of  $(A - iI)(A + iI)^{-1}$ ? Well,  $(A - iI)^* = A + iI$  by the rules on page 70. Hence the expression is of the form  $X^*X^{-1}$  which has adjoint  $(X^{-1})^*X$ . Then  $(X^{-1})^*XX^*X^{-1}$ .

Now, since  $A$  is self-adjoint, it is in particular normal, so  $A + iI$  is normal and hence orthogonally diagonalizable. Writing  $A + iI = \sum \lambda E_\lambda$ , we get  $(A + iI)^* = \sum \bar{\lambda} E_\lambda$ , so  $A + iI$  and  $A - iI$  commute. Hence we get  $XX^* = X^*X$ , and the expression above becomes the identity.

**Exercise 1.5** (12.4). Give a simple proof of the Hahn-Banach theorem for a continuous linear form on a closed subspace of a Hilbert space.

*Solution.* Let  $V$  be a Hilbert space and let  $U \subset V$  be a closed subspace. Then  $V = U \oplus U^\perp$ . Let  $\mathcal{B}$  be a basis for  $U$  and extend it to a basis  $\mathcal{A}$  for  $V$ . Take the duals  $\mathcal{B}'$  and  $\mathcal{A}'$ . For  $z \in U^*$  we can write  $z = \sum_{y_i' \in \mathcal{B}'} a_i y_i'$ . Then  $z$  can also be considered a linear form on  $V$  by letting the coefficient for  $y_i \in \mathcal{A}'$  be 0 if  $y_i \notin \mathcal{B}'$  and  $a_i$  otherwise. The restrictions are clearly the same. By the Riesz-Fréchet representation theorem, since  $U$  is a closed subspace of a Hilbert space, it is also a Hilbert space, so by continuity of  $z$ , there exists  $u \in U$  such that  $z(x) = \langle x, u \rangle$  for all  $x \in U$  and such that  $\|z\| = \|u\|$ . But since  $z|_{U^\perp} = 0$ , we also have  $z(x) = \langle x, u \rangle$  for all  $x \in V$ , so by the Riesz-Fréchet theorem,  $\|z\| = \|u\|$  over  $V$  as well.

**Exercise 1.6** (13.1). Prove  $\rho(A + B) \leq \rho(A) + \rho(B)$  if  $A$  and  $B$  are normal. Prove it for general  $A, B \in \text{End}(V)$ , now assuming they commute. Show the inequality can fail in general.

*Solution.* If  $A$  and  $B$  are normal, then they are orthogonally diagonalizable with respect to the associated inner product, hence  $\rho(A) = \|A\|$  and  $\rho(B) = \|B\|$ . Now, in general, we have  $\rho(X) \leq \|X\|$ , we get

$$\rho(A + B) \leq \|A + B\| \leq \|A\| + \|B\| = \rho(A) + \rho(B).$$

If  $A$  and  $B$  commute, then

$$\begin{aligned} \rho(A + B) &= \lim_{k \rightarrow \infty} \|(A + B)^k\|^{\frac{1}{k}} = \lim_{k \rightarrow \infty} \left\| \sum_{i=0}^k \binom{k}{i} A^i B^{k-i} \right\|^{\frac{1}{k}} \\ &\leq \lim_{k \rightarrow \infty} \left| \sum_{i=0}^k \binom{k}{i} \|A\|^i \|B\|^{k-i} \right|^{\frac{1}{k}} \\ &= \lim_{k \rightarrow \infty} (\|A\| + \|B\|) \\ &= \|A\| + \|B\| \\ &= \lim_{k \rightarrow \infty} \|A^k\|^{\frac{1}{k}} + \lim_{k \rightarrow \infty} \|B^k\|^{\frac{1}{k}} \end{aligned}$$

where the last equality follows from  $\|X^k\| = \|X\|^k$  when  $X$  is diagonalizable (by the proof of lemma 13.4).

To show that it can fail in general, note that for  $[A] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  and  $[B] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , we have that the eigenvalues of both are precisely 1, hence  $\rho(A) + \rho(B) = 2$ , while  $A + B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  which has characteristic polynomial  $(x - 3)(x - 1)$  and thus 3 as an eigenvalue.

**Exercise 1.7** (13.2). Let  $F = \mathbb{C}$ . Find a counterexample to the statement:  $\rho(p(A)) = p(\rho(A))$  for all polynomials  $p$ , where  $\rho$  is the spectral radius.

*Solution.* Consider  $p(x) = ix - i$  and  $A = -I$ . So  $p(A) = \begin{pmatrix} -2i & 0 \\ 0 & -2i \end{pmatrix}$  which has spectral radius 2. However,  $-I$  has spectral radius 1 and  $p(1) = 0$ .

**Exercise 1.8** (13.3). Show  $\rho(A^*A) = \|A^*A\| = \|A\|^2$  for the operator norm of an inner product.

*Solution.* The first equality holds when the matrix is orthogonally diagonalizable. But  $A^*A$  is self-adjoint, hence normal hence orthogonally diagonalizable.

The latter equality holds since

$$\|A^*A\| = \sup_{\|x\|=1} \langle A^*Ax, x \rangle = \sup_{\|x\|=1} \|Ax\|^2 = \|A\|^2$$