ASSIGNMENT 4

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Definition 1.1 (Fiber Bundle)**.** Let K be a topological group acting on a Hausdorff space F as a group of homeomorphisms. Let X and B be Hausdorff spaces. By a *fiber bundle* over a base space B with total space X, fiber F and structure group K, we mean a bundle map $p\colon X \to B$ together with a maximal chart atlas Φ over B. Explicitly, Φ is a collection of trivializations $\varphi\colon U \times F \to p^{-1}(U)$ such that

(1) each point of B has a neighborhood over which there is a chart in Φ
(2) if $\varphi\colon U \times F \to p^{-1}(U)$ is in Φ and $V \subset U$, then the restriction $\varphi|_{V \times F}$ is also in Φ.
(3) If $\varphi, \psi \in \Phi$ are charts over U then there exists a map $\theta\colon U \to K$ such that $\psi(u, y) = \varphi(u, \theta(u)(y))$
(4) the set Φ is maximal among the collections satisfying the (1),(2) and (3)

The fiber bundle is called smooth if all the spaces are smooth manifolds and all maps involved are smooth.

Definition 1.2 (Manifold bundle)**.** Let M be a smooth manifold. A manifold bundle over M with structure group G is a fiber bundle $W \to E \to M$ with structure group G such that E is a manifold and $E \to M$ is continuous.
We say a manifold bundle over M is a smooth manifold bundle if it is a smooth fiber bundle as well as a manifold bundle and G acts by diffeomorphisms on M.

Problem 1.3 (Manifold bundles over S^1)**.** We fix a smooth manifold M. The aim of this exercise is to study smooth manifold bundles over S^1 with fiber M.

(1) Let $f \in \mathrm{Diff}(M)$, and consider the mapping torus
$$T(f) := (M \times [0,1]) / \sim$$
where $\sim$ identifies $(x, 0)$ with $(f(x), 1)$ for all $x \in M$. Show that the projection map to the second factor yields a smooth manifold bundle
$$M \to T(f) \to S^1.$$
(2) Show that if f and g are isotopic diffeomorphisms, the bundles $T(f) \to S^1$ and $T(g) \to S^1$ are isomorphic bundles.
(3) Show that the map
$$\pi_0 \,\mathrm{Diff}(M) \to \mathrm{Bun}_M\left(S^1\right)$$
by
$$[f] \mapsto [T(f)]$$
from the mapping class group of M to the set of isomorphism classes of M-manifold bundles over S^1 is bijective.

Problem 1.4 (2)**.** Show that the following spaces admit the structure of smooth manifolds.

(1) $O(n)$, the set of orthogonal matrices of degree $n \times n$, topologized as a subspace of $\mathbb{R}^{n^2}$.
(2) $SO(n)$, the set of orthogonal matrices of degree $n \times n$ with determinant 1.
(3) $\mathrm{SL}_n(\mathbb{R})$, the set of $(n \times n)$-matrices with determinant 1.

Solution. (1) (2pts) The orthogonal group is the zero set $\mathbb{V}(I)$ of the ideal $I = (\{f_{ij}\})$ where

$$f_{i,j} = \sum_{k=1}^{n} x_{ki}x_{kj} \quad \text{for } i \neq j \quad \text{and} \quad f_{ii} = \sum_{k=1}^{n} x_{ki}^2 - 1$$

So defining a function $\varphi\colon \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ by $\varphi\left((x_{ij})\right) = ((f_{ij}))$, then since (f_{ij}) is symmetric, we may modify this map so that $\varphi\left(x_{ij}\right) = \left((f_{ij})_{i \geq j}\right)$ so $\varphi\colon \mathbb{R}^{n^2} \to \mathbb{R}^{\frac{n(n+1)}{2}}$.
We can also write this map as $\varphi(A) = A^t A - I$. Then we find that

$$\varphi'(A) = \frac{d}{dt}|_{t=0}\varphi(A+tX) = \frac{d}{dt}|_{t=0}\left(A+tX\right)\left(A+tX\right)^t - I = \frac{d}{dt}|_{t=0}AX^t t + A^t X t = AX^t + XA^t$$

Now if $A \in \varphi^{-1}(0)$ and $B \in \mathbb{R}^{\frac{n(n+1)}{2}}$ represents a symmetric matrix, then

$$\varphi'(\frac{1}{2}BA) = \frac{1}{2}\left(AA^tB^t + BAA^t\right) = \frac{1}{2}(B+B) = B$$

so φ' is surjective, hence has full rank. Therefore, by the rank lemma (Lemma 5.9 in JB) $O(n) = \varphi^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^{n^2}$ of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

(2) (1pt) The determinant function defines a continuous function on $M_n(\mathbb{R})$, hence also on $O(n)$. On $O(n)$, it takes values in $\{1, -1\}$, so in particular, $O(n) = \det|_{O(n)}^{-1}(-\infty, 0) \sqcup \det|_{O(n)}^{-1}(0, \infty)$ has two components, each of which is a manifold by a previous problem sheet's problem. But $\det|_{O(n)}^{-1}(0, \infty) = SO(n)$, so $SO(n)$ is also a smooth manifold, and in particular also of the same dimension as $O(n)$.

(3) (2pts) The determinant function is smooth and for example I has determinant 1. Now let I_t be I plus the matrix with a t in the $1,1$ entry and 0 elsewhere. Then $\det(I + tI) = (1+t)^n$, so $\frac{d}{dt}|_{t=0} \det\left(I + tI\right) = n(1+t)^{n-1}|_{t=0} = n \neq 0$, hence det has full rank at I, so 1 is a regular value, so $\det^{-1}(1)$ is a smooth submanifold of $M_n(\mathbb{R})$ by lemma 5.9 in BJ.

Problem 1.5 (3)**.** Fix a manifold M and consider the set $\mathrm{Vect}(M)$ of all isomorphism classes of finite dimensional real vector bundles over M.

(1) For $E, E' \in \mathrm{Vect}(M)$, construct a vector bundle $E \oplus E'$ over M which fiberwise is obtained by applying the direct sum $V \oplus V'$. Formulate a universal property of $E \oplus E'$.
(2) For $E, E' \in \mathrm{Vect}(M)$, construct a vector bundle $E \otimes E'$ over M which fiberwise is obtained by applying the tensor product $V \otimes V'$.

(3) Let $E \in \mathrm{Vect}(M)$ and fix $E' \subset E$ a subbundle of E, that is a vector bundle together with a map of bundles

$$\begin{tikzcd} V' \arrow[r, hook] \arrow[d] & V \arrow[d] \\ E \arrow[r] \arrow[dr] & E' \arrow[d] \\ & M \end{tikzcd}$$

that induces linear injective maps on fibres. Construct a vector bundle E/E' which fiberwise is given by taking the quotient vector space V/V'.

(4) Let E and E' be vector bundles over M, and assume we are given a bundle morphism $f\colon E \to E'$ such that the map on the fibres $V_p \to V'_p$ has constant rank for all $p \in M$. Construct a vector bundle $\ker f$ over M which fiberwise is obtained by taking $\ker\left(V_p \to V'_p\right)$.

Solution. We will use the approach of Bröcker and Jänich by constructing pre-vector bundles with the desired properties.

(1) (3pts) We define $E \oplus E' = \bigcup_{p\in M} E_p \oplus E'_p$ where E_p and E'_p are the fibers at p. Now take $\pi\colon E \oplus E' \to M$ to be the projection $(e_p, e'_p) \mapsto p$. The vector space structure on $(E \oplus E')_p = \pi^{-1}(p) = E_p \oplus E'_p$ is the precisely the direct sum of the vector space structures of E_p and E'_p.

For the pre-bundle atlas $\mathcal{B}$, let $\mathcal{B}_E, \mathcal{B}_{E'}$ be bundle atlases for E and E', respectively. Then for $(f_\alpha, U_\alpha) \in \mathcal{B}_E$ and $(g_\beta, V_\beta) \in \mathcal{B}_{E'}$, let $(f_\alpha \oplus g_\beta, U_\alpha \cap V_\beta) \in \mathcal{B}$ where

$$f_\alpha \oplus g_\beta\colon \pi^{-1}\left(U_\alpha \cap V_\beta\right) \to U_\alpha \cap V_\beta \times \mathbb{R}^n \times \mathbb{R}^m$$

sending $\left(e_p, e'_p\right) \mapsto \left(p, \pi_{\mathbb{R}^n} \circ f_\alpha(e_p), \pi_{\mathbb{R}^m} \circ g_\beta(e'_p)\right)$ is a bijective map which sends each fiber $(E \oplus E')_p$ linearly and isomorphically onto $\{p\} \times \mathbb{R}^n \oplus \mathbb{R}^m \cong \mathbb{R}^n \oplus \mathbb{R}^m$. Furthermore, the transition functions are of the form

$$\begin{aligned} f_\alpha \oplus g_\beta \circ (f_{\alpha'} \oplus g_{\beta'})^{-1}(p, \pi_{\mathbb{R}^n} \circ f_{\alpha'}(e_p), \pi_{\mathbb{R}^m} \circ g_{\beta'}(e'_p)) &= \left(p, \pi_{\mathbb{R}^n} \circ f_\alpha(e_p), \pi_{\mathbb{R}^m} \circ g_\beta(e'_p)\right) \\ &= \left(p, \tau\left(\pi_{\mathbb{R}^n} \circ f_{\alpha'}(e_p), \pi_{\mathbb{R}^m} \circ g_{\beta'}(e'_p)\right)\right) \end{aligned}$$

where $\tau = \tau_1 \oplus \tau_2$ where $\tau_1\colon U_\alpha \cap U_{\alpha'} \to \mathrm{GL}_n(\mathbb{R})$ and $\tau_2\colon V_\beta \cap V_{\beta'} \to \mathrm{GL}_m(\mathbb{R})$ are the transition functions for the trivializations $(f_\alpha, f_{\alpha'})$ and $(g_\beta, g_{\beta'})$, respecitively. Since each τ_1 and τ_2 is assumed to be continuous, τ is also. Hence $E \oplus E'$ is a vector bundle.

As for the universal property, $E \oplus E'$ is the product of E and E' in $\mathrm{Vect}(M)$, so the usual universal property of products applies.

(2)(3pts) Define $E \otimes E' := \bigcup_{p\in M} E_p \otimes E'_p$ and π the standard projection. Let $\mathcal{B}_E, \mathcal{B}_{E'}$ be bundle atlases for E and E' respectively. Then, recalling that $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{nm}$ and using this identification, we get for $(f_\alpha, U_\alpha) \in \mathcal{B}_E$ and $(g_\beta, V_\beta) \in \mathcal{B}_{E'}$, the map $f_\alpha \otimes g_\beta\colon \pi^{-1}\left(U_\alpha \cap V_\beta\right) \to U_\alpha \cap V_\beta \times \mathbb{R}^{nm}$ given by

$$f_\alpha \otimes g_\beta\left(e_p \otimes e'_p\right) = \left(p, f_\alpha(e_p) \otimes g_\beta\left(e'_p\right)\right)$$

on simple tensors, and we extend this linearly over the fiber.

The linearity then becomes automatic. To see that this is an isomorphism, suppose

$$(p, 0) = f_\alpha \otimes g_\beta\left(e_p \otimes e'_p\right) = (p, f_\alpha(e_p) \otimes g_\beta(e'_p))$$

so either $f_\alpha(e_p) = 0$ or $g_\beta(e'_p) = 0$. But then since f_α and g_β are isomorphisms on $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(V_\beta)$, respectively, this implies that either $e_p = 0$ or $e'_p = 0$, so $e_p \otimes e'_p = 0$. Surjectivity is inherited from that of f_α and g_β.
The transition maps then take on the form id and $\tau_1 \otimes \tau_2$ which is continuous.

(3) (3pts)

Definition 1.6 (Subbundle)**.** A bundle (E', p', B') is a subbundle of (E, p, B) provided E' is a subspace of E, B' is a subspace of B and $p' = p|_{E'} : E' \to B'$.

Definition 1.7 (Vector subbundles)**.** A vector subbundle (E', p', B') of (E, p, B) is a subbundle together with a map $E' \to E$ which also induces linear injective maps on fibres $V' \hookrightarrow V$:

$$\begin{tikzcd}
V' \arrow[rr, hook] \arrow[d] & & V \arrow[d] \\
E' \arrow[rr] \arrow[dr] & & E \arrow[dl] \\
& M &
\end{tikzcd}$$

With this in mind, we let $E/E' = \bigcup_{p \in M} E_p/E'_p$ and π the standard projection. Here E_p/E'_p is well-defined since E'_p is a subspace of E_p for all p by assumption. Suppose $\mathcal{B}_E, \mathcal{B}_{E'}$ are bundle atlases for E and E', respectively. Let $(f_\alpha, U_\alpha) \in \mathcal{B}_E$ and $(g_\beta, V_\beta) \in \mathcal{B}_{E'}$. Let $\varphi \colon E' \to E$ be the map in the definition of a sub-vector bundle. Then φ induces an injective linear map $\mathrm{id} \times \tilde{\varphi}$. Following through fiberwise, we find that we get an induced isomorphism $\psi_{\alpha\beta}$ such that the following diagram commutes

$$\begin{tikzcd}
p'^{-1}(U_\alpha \cap V_\beta) \arrow[r, "\varphi"] \arrow[d, "f_\alpha"', "\cong"] & p^{-1}(U_\alpha \cap V_\beta) \arrow[r, "\text{coker}"] \arrow[d, "\cong"', "g_\beta"] & \pi^{-1}(U_\alpha \cap V_\beta) \arrow[ddl, bend left, "\cong"', "\psi_{\alpha\beta}"] \\
U_\alpha \cap V_\beta \times \mathbb{R}^n \arrow[r, hook, dashed, "\mathrm{id}\times\tilde{\varphi}"'] \arrow[dr, "\mathrm{id}\times 0"'] & U_\alpha \cap V_\beta \times \mathbb{R}^m \arrow[d, "\mathrm{id}\times\text{coker}\,\tilde{\varphi}"] & \\
& U_\alpha \cap V_\beta \times \mathbb{R}^{m-n} &
\end{tikzcd}$$

In particular, since $\tilde{\varphi}$ is linear, we get a global frame on $\mathbb{R}^{m-n}$ which transfers back to a local frame on $\pi^{-1}(U_\alpha \cap V_\beta)$. The transition functions then become linear isomorphisms at each point of $U_\alpha \cap V_\beta$, which are smooth. The collection $\{(U_\alpha \cap V_\beta, \psi_{\alpha\beta})\}$ then gives a pre-bundle atlas for E/E'. So we have a pre-vector bundle.

(4) (3 pts) (I will write the proof here, but I am not sure whether I should receive credit for it since I only understood how to solve it after reading the proof in Lee's book, and I am not sure I can find a much more different approach. The approach that Lee uses is given in Theorem 10.34 in his book. In any case, here it is:)
Suppose f has constant rank r.
Let $\ker f = \bigcup_{p \in M} \ker\left(V_p \to V'_p\right)$ and π the natural projection onto M. Let $\mathcal{B}_E$ and $\mathcal{B}_{E'}$ be bundle atlases for E and E', respectively. Let $p \in M$ and choose some smooth local frame (σ_i) for E over some open neighborhood U of p. Then since f

has constant rank r, by rearranging, we may assume that $f \circ \sigma_1(p), \dots, f \circ \sigma_r(p)$ form a basis for $(\operatorname{im} f)|_{E_p}$. By continuity of the determinant, they remain linearly independent in some neighborhood V of p. Hence $f \circ \sigma_1, \dots, f \circ \sigma_r$ form a local frame over V. By the local frame condition, this implies that $\operatorname{im} f$ is a smooth vector bundle. Let now $V' = \operatorname{span}(\sigma_1, \dots, \sigma_r)$. Since $f|_{V'}$ is bijective, it is a smooth vector bundle isomorphism (prop 10.26 in Lee). So let $f|_V'^{-1} \colon \operatorname{im} f|_{V'} \to V'$ be the inverse. Let $\psi \colon V \to V$ be the map $\psi(v) = v - f_{V'}^{-1} \circ f_V(v)$. This is a smooth bundle morphism, and we claim that $\ker \psi = V'$. If $v \in V$ then $\psi(v) = v$, and if $\psi(v) = 0$ then $f|_V'^{-1} \circ f(v) = 0$, so $v \in V'$.
Now $\ker f|_V$ and V' span E_V, so since $\ker f|_V \subset \operatorname{im} \psi$, and $V' \subset \ker \psi$, we must have $\operatorname{im} \psi = \ker f|_V$, and $\psi|_{kerf|_V} = \mathrm{id}$, hence $\ker f|_V$ is a smooth vector bundle over V. Since we have smooth local frames at each point, these glue together using the local frame criterion for subbundles to give a vector bundle $\ker f$.

Problem 1.8 (4)**.** Let $(M, \cdot)$ be a commutative monoid. The *group completion* of M is defined as

$$M^{grp} := F(M)/\left(xy - (x+y)\right)$$

where $(F(M), +)$ is the free abelian group on the underlying set of M.

(1) Show that $(-)^{grp}$ defines a functor from the category of commutative monoids to the category of abelian groups, such that for any abelian group A, any monoid morphism $M \to A$ factors through M^{grp}.
(2) Let $\mathrm{Vect}_d(S^n)$ denote the set of isomorphism classes of vector bundles of rank d over S^n. Show that the clutching map defines a bijection

$$\left[S^{n-1}, \mathrm{GL}_d(\mathbb{R})\right] \to \mathrm{Vect}_d(S^n)$$

where $\left[S^{n-1}, \mathrm{GL}_d(\mathbb{R})\right]$ denotes the set of homotopy classes of continuous maps.

Proof. (1) (2 pts)
Firstly, we must show that M^{grp} is abelian. It suffices to show that the generating elements of $F(M)$ commute in M^{grp}. Let $x, y \in M$ be such generators. Then $x + y = xy$ since this is the relation being modded out by. Now M was assumed to be a *commutative* monoid, so $xy = yx$ in M. Hence since also $yx = y + x$, we have $x + y = xy = yx = y + x$, so M^{grp} has commuting generating elements, hence it is abelian. Now if $f \colon M \to N$ is a monoid morphism, then $f^{grp} \colon M^{grp} \to N^{grp}$ is defined on generators by $x \mapsto f(x)$, and then we extend it linearly on $F(M)$ and lastly take the quotient. We must check that this is a group homomorphism. By the definition of f^{grp}, we have that it distributes over sums of generating elements, so $f^{grp}(\sum x_i) = \overline{\sum f(x_i)} = \sum \overline{f(x_i)} = \sum f^{grp}(x_i)$. Hence since any $x, y \in M^{grp}$ can be written as such, we have $f^{grp}\left(\sum_i x_i + \sum_j y_j\right) = \sum_i f^{grp}(x_i) + \sum_j f^{grp}(y_j) = f^{grp}\left(\sum_i x_i\right) + f^{grp}\left(\sum_j y_j\right)$. Furthermore, if $\mathrm{id} \colon M \to M$ is the identity, then f^{grp} simply takes the form $f^{grp}(x) = x$ on all generating elements and extends linearly, hence f^{grp} is also the identity on M^{grp}. Now if $f \colon M \to N$ and $g \colon N \to S$, then $(g \circ f)^{grp}$ sends a generating element x to $g \circ f(x)$ which uniquely determines it. Conversely, $g^{grp} \circ f^{grp}$ sends x to $g^{grp}\left(f^{grp}(x)\right) = g^{grp}\left(f(x)\right) = g \circ f(x)$ and extends linearly onto all of M^{grp}. Hence the functions agree. So $(-)^{grp}$ is indeed a functor. Let now $\varphi \colon M \to A$ be a monoid morphism with A an abelian group. Then since $\varphi(xy) = \varphi(x)\varphi(y) = \varphi(y)\varphi(x) = \varphi(yx)$, we define a map $\tilde{\varphi} \colon M^{grp} \to A$ by

$\tilde{\varphi}(x) = \varphi(x)$ and extend linearly. This is well-defined on generators since $\tilde{\varphi}(x+y) = \varphi(x)\varphi(y) = \varphi(y)\varphi(x) = \tilde{\varphi}(y+x)$. Now we have $\varphi(xy) = \varphi(x)\varphi(y) = \tilde{\varphi}(x+y) = \tilde{\varphi}(\pi(x) + \pi(y)) = \tilde{\varphi}(\pi(xy))$ for $\pi\colon M \to M^{grp}$ the quotient map. Thus $\varphi = \tilde{\varphi} \circ \pi$. $\square$