Exercise 0.1 (8.6)**.** Let $A \in \mathrm{End}(V)$ be nilpotent, and $U \subset V$ invariant. Show that the quotient map $\overline{A} \in \mathrm{End}\,(V/U)$ is nilpotent.

Proof. Suppose $A^k = 0$ for some $k > 0$. We claim that $\overline{A}^k = 0$ for the same k. We recall by lemma 2.16 that $\overline{A} \in \mathrm{End}\,(V/U)$ is the unique endomorphism making $\overline{A} \circ \pi = \pi \circ A$ commute where $\pi \colon V \to V/U$ is the quotient map. It thus immediately follows that $\overline{A^k} = 0$ since this satisfies the commutative criterion. Now, we claim that suppose that for N we have shown $\overline{A}^N \circ \pi = \pi \circ A^N$. Then we get

$$\pi \circ A^{N+1} = (\pi \circ A) \circ A^N = \overline{A} \circ \pi \circ A^N = \overline{A}^{N+1} \circ \pi$$

so since the case for $N = 1$ was shown, we get by induction that $\overline{A}^k \circ \pi = \pi \circ A^k = 0$. Now, π is surjective by lemma 2.9, so given some $\overline{x} \in V/U$, let $x \in V$ be such that $\pi(x) = \overline{x}$. Then $\overline{A}^k \overline{x} = \overline{A}^k (\pi(x)) = \pi \circ A^K(x) = \pi(0) = \overline{0}$. So indeed $\overline{A}^k$ is equal to the zero endomorphism in $\mathrm{End}\,(V/U)$. Thus $\overline{A}$ is nilpotent. $\square$