

**1.1.i:** (i) Let  $\mathcal{A}$  be a category and  $A, B \in \text{ob}(\mathcal{A})$ . Let  $f \in \mathcal{A}(A, B)$  and assume  $g, h \in \mathcal{B}, \mathcal{A}$  such that  $fg = \mathbb{1}_B = fh$  and  $gf = \mathbb{1}_A = hf$ . Then

$$h = h\mathbb{1}_B = h(fg) = (hf)g = \mathbb{1}_A g = g.$$

(ii): Let  $f: x \rightarrow y$  and  $g, h: y \rightarrow x$  such that  $gf = \mathbb{1}_x$  and  $fh = \mathbb{1}_y$ . Then

$$g = g\mathbb{1}_y = g(fh) = (gf)h = \mathbb{1}_x h = h$$

Thus we can denote  $g = f^{-1} = h$  and we get  $ff^{-1} = \mathbb{1}_y$  and  $f^{-1}f = \mathbb{1}_x$ , so  $f$  is an isomorphism by definition with  $f^{-1}$  as its inverse.

**1.1.iii:**

(i) Let  $\text{ob}(c/C)$  be all morphisms in  $C$  with domain  $c$ . Let  $c/C(f, g)$  be all maps in  $C$  from the codomain of  $f$  to the codomain of  $g$ . For any  $f, g, h \in \text{ob}(c/C)$  and for any  $\alpha \in c/C(f, g)$  and  $\beta \in c/C(g, h)$ , define the composition of  $\alpha$  with  $\beta$  as the map  $\beta \circ \alpha$  in  $C$  whose existence is guaranteed by  $C$  being a category. For the identity: since  $C$  is a category, we have that for each  $x \in \text{ob}(C)$ , there exists an identity on  $x$ , and thus since for any map  $f \in \text{ob}(c/C)$ , say  $f: c \rightarrow x$ , we have  $f = \mathbb{1}_x f$  in  $C$ , so  $\mathbb{1}_x \in c/C(f, f)$  where  $\mathbb{1}_x$  represents the morphism  $(f: c \rightarrow x) \rightarrow (f: c \rightarrow x)$ ; and since  $x$  was arbitrary, all objects in  $c/C$  have an identity.

For associativity: let  $\alpha \in c/C(f, g), \beta \in c/C(g, h), \gamma \in c/C(h, k)$ . Then the  $(\gamma\beta)\alpha = \gamma(\beta\alpha)$  follows from associativity of morphism composition in  $C$ .

Similarly, for the identity laws follow from the identity in  $C$ : for any  $\alpha \in c/C(f, g)$  where say  $c \xrightarrow{f} x_f$  and  $c \xrightarrow{g} x_g$ , we have  $\mathbb{1}_{x_f} \in c/C(f, f)$  and  $\mathbb{1}_{x_g} \in c/C(g, g)$  and  $\mathbb{1}_{x_g}\alpha = \alpha = \alpha\mathbb{1}_{x_f}$  since this is true in  $C$ .