0.1 Ch. 14-16

1. Let $(X, \mathcal{T}), (Y, \mathcal{T}_Y), (A, \mathcal{T}_A)$ be the topological spaces in question, where $\mathcal{T}_A$ is the topology inherited from Y.
Let $U \in \mathcal{T}_A$. Then $U = A \cap U'$, where $U' \in \mathcal{T}_Y$, and thus $U' = Y \cap U''$ where $U'' \in \mathcal{T}$. Hence $U = A \cap (Y \cap U'')$, and since $A \subset Y$, we have $U = A \cap U''$ which is in the topology inherited as a subspace from X by A.
Assume now that $A \cap U$ is in the topology inherited as a subspace from X, so $U \in \mathcal{T}$. Since $A \subset Y$, $A \cap U = A \cap (Y \cap U) \in \mathcal{T}_A$.
Hence we are done.

2. Let $\mathcal{T}_Y, \mathcal{T}'_Y$ be the corresponding topologies. Let $U \cap Y \in \mathcal{T}_Y$. Since $U \in T \subset T'$, we have $U \cap Y \in \mathcal{T}'_Y$. Since the additional open sets in $\mathcal{T}'$ may collapse under intersection with Y, this is not necessarily strictly finer. E.g. take T' to be the order topology on the real line, and T to be the order topology on $(-1, 1)$ and $Y = (-1, 1)$. Then $\mathcal{T}_Y = \mathcal{T}'_Y$.
Or e.g. on $X = \{a, b, c\}$, the topologies $T' = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$, $T = \{\varnothing, X, \{a\}, \{a, b\}\}$.
Here $T \subset T'$ strictly, but for $Y = \{a\}$, $\mathcal{T}_Y = \{\varnothing, Y\} = \mathcal{T}'_Y$.

3. $A = \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ which is open in both Y and $\mathbb{R}$.
$B = (-2, -\frac{1}{2}) \cup (\frac{1}{2}, 2)$ in Y, hence it is open. Since $1 \in B$ which is not maximal in $\mathbb{R}$, and B does not contain numbers greater than 1, B is not open in $\mathbb{R}$.
$C = (-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1)$ is not open in Y or $\mathbb{R}$ since any basis containing $\frac{1}{2}$ would contain numbers strictly less than $\frac{1}{2}$ in size.
D is not open for same reason as C.
Let $x \in E$. Then there exists $N \in \mathbb{N}$: $\frac{1}{N+1} < |x| < \frac{1}{N}$, so assuming wlog $x > 0$ and choosing $a \in (\frac{1}{N+1}, x), b \in (x, \frac{1}{N})$, we have $x \in (a, b)$ which is a basis element of $\mathbb{R}$ contained in $E \subset Y$. Thus E is an open set for both Y and $\mathbb{R}$.

4. Let $U \times V$ be an open set in the product topology of $X \times Y$. Since the collection of cartesian products of open sets from X and Y is a basis for the product topology, $U \times V$ can be written $U \times V = \bigcup_{i \in I} U_i \times V_i = \bigcup_{i \in I} U_i \times \bigcup_{i \in I} V_i$ where U_i, V_i are open in X and Y respectively. Hence

$$\pi_1 (U \times V) = \bigcup_{i \in I} U_i, \qquad \pi_2(U \times V) = \bigcup_{i \in I} V_i.$$

which are open in X and Y respectively.

Alternative: We want to show that for any open set $U \subset X \times Y$ and any $x \in U$, there exists an open neighbourhood around $\pi_1(x)$ such that it is contained in $\pi_1(U)$; similarly for π_2.
Let $x \in \pi_1(U)$. Then there exists $y \in \pi_2(U)$ such that $(x, y) \in U$. Since U is open, there exists a basis element U' in the topology of $X \times Y$ such that $(x, y) \in U' \subset U$ and $\pi_1(U'), \pi_2(U')$ are open. Hence since $x \in \pi_1(U') \subset \pi_1(U)$, we get what we wanted. Equivalent arguement for $\pi_2(U)$.

5. Let $(x, y) \in U \times V \subseteq X \times Y \subset X' \times Y'$, where $U \times V$ is open in $X \times Y$. Then U is an open subset of X and hence belongs to $\mathcal{T} \subset \mathcal{T}'$. Similarly, $V \in \mathcal{U}'$. Hence $U \times V$ is also an open set in the product topology on $X' \times Y'$, and since

these are the basis elements, we have by lemma 13.3 that $\mathcal{T}'$ is finer than $\mathcal{T}$.
Alternative: Let $U \subset X \times Y$ be open and $u \in U$. Then there exists a basis element U' in $X \times Y$ such that $x \in U' \subset U$, and since $\mathcal{T} \subset \mathcal{T}'$, U' is in $X' \times Y'$, hence $x \in U' \subset U$, and thus U is open in $X' \times Y'$ too.

For the converse: we have that X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$ respectively, which, apparently, means that $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and X' respectively, and $X = X'$ as sets. In this case, assume $X' \times Y'$ is finer than $X \times Y$. Let $U \in \mathcal{T}$. We want to show that $U \in \mathcal{T}'$. For any $V \in \mathcal{U}$, we have that $U \times V$ is open in $X \times Y$ and hence in $X' \times Y'$. Thus for any $x \in U$, choose $y \in V$ such that $(x, y) \in U \times V \subset X' \times Y'$. Then there exists a basis element $U' \times V'$ in $X' \times Y'$ such that $(x, y) \subset U' \times V' \subset U \times V$, since $U \times V$ is open in $X' \times Y'$. Since π_1 is an open mapping, we find $x \in U' \subset U$, so U is open in X'. Similarly for $\mathcal{U} \subset \mathcal{U}'$.

16.6: Let $(x, y) \in U \times V$ be an open set in $\mathbb{R}^2$ under the product topology. We have by 16.4, that $x \in U$ is open in $\mathbb{R}$, so there exists a basis element (a, b) such that $x \in (a, b) \subset U$ and similarly there exists (c, d) such that $y \in (c, d) \subset V$. We can choose a, b, c, d to be rational without loss of generality, since between any two real numbers, there are infinitely many rational and irrational numbers, so between, say, a and x we can find an alternative a that is rational such that $x \in (a, b)$. Now we thus have $(x, y) \in (a, b) \times (c, d) \subset U \times V$, and since $(a, b) \times (c, d)$ is open in $U \times V$, we get that it generates the collection generates the product topology on $\mathbb{R}^2$ by 13.2.

16.7: No, e.g. $X = \mathbb{Q}$ with the order topology. Then $Y = (\sqrt{2}, \pi) \cap \mathbb{Q} \subset X$ is convex in X, but since $\sqrt{2}, \pi \notin \mathbb{Q}$, $(\sqrt{2}, \pi) \cap \mathbb{Q}$ is not an interval or ray in X. To see where this went wrong, we can see that if $Y \subset X$ is convex, then by theorem 16.4, the order topology on $Y = (\sqrt{2}, \pi) \cap \mathbb{Q}$ is the same as the topology Y inherits as a subspace from $X = \mathbb{Q}$. Since open intervals and half-open intervals or rays are a basis of the order topology on Y, Y itself is a union of intervals. But we cannot guarantee that the union of intervals will be an interval in Y and hence an interval in X. For example in the example $Y = (\sqrt{2}, \pi) \cap \mathbb{Q}$, we can choose a sequence of intervals in $\mathbb{Q}$, $((q_i, p_i))_{i \in \mathbb{N}}$ such that $\bigcup_{i \in \mathbb{N}} (q_i, p_i) = (\sqrt{2}, \pi)$, however this is not a interval in $\mathbb{Q}$. So the incompleteness of $\mathbb{Q}$ is a problem.

16.8:

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Exercise 6:
(i) Assume $\mathcal{T}_2 \subset \mathcal{T}_1$. Let $x \in U \in S_2$. Since $U \in S_2$, it is particularly also open in the topology generated by the subbasis S_2, i.e. $U \in T_2 \subset T_1$. Hence there exist $\{V_i\}_{i \in I} \subset T_1$ where each $V_i = \bigcap_{j=1,\dots,m_i} S_{j,i}$ where $S_{j,i} \in S_1$, and

$$U = \bigcup_{i \in I} V_i.$$

Hence, there exists $i \in I$ such that $x \in V_i$, so

$$x \in \bigcap_{j=1,\ldots,m_i} S_{j,i} \subset U.$$

Conversely, since $\bigcap_{i=1,\ldots,n} V_i$ is open for $\{V_1, \ldots, V_n\} \subset S_1$, if $x \in U \in \mathcal{S}_2$, and letting $W_x = \bigcap_{i=1,\ldots,n} V_i$ where $x \in \bigcap_{i=1,\ldots,n} V_i \subset U$, we then get

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} W_x \subset U.$$

Hence $U \in T_1$, so $S_2 \subset T_1$, and thus the topology generated by S_2 is coarser than T_1, i.e. $T_2 \subset T_1$.

(ii) Since for all $x, y \in \mathbb{R}$ with $x < y$, $[x, y] \in \mathcal{S}$, we have for any $x \in \mathbb{R}$ that $\{x\} = [z, x] \cap [x, y] \in \mathcal{T}$, hence for any subset $U \subset \mathbb{R}$, we have

$$U = \bigcup_{x \in U} \{x\} \in \mathcal{T}..$$

Thus $\mathcal{T}$ is the discrete topology on $\mathbb{R}$.

Exercise 7:
(i) Let $x \in \pi_X(U)$. There then exists $y \in Y$ such that $x \times y \in U$.
Let $x \times y \in U \subset X \times Y$ be open. There exists by assumption a basis element $A \times B \subset X \times Y$ where A is open in X and B is open in Y such that $x \times y \in A \times B \subset U$. Then

$$x \in A \subset \pi_X(U).$$

Similarly for $y \in B \subset \pi_Y(U)$. Hence $\pi_X(U), \pi_Y(U)$ are open.

(ii) Let $X = Y = \mathbb{R}$ with standard topology. Let $Z = \{x \times x \in \mathbb{R}^2 \mid x > 0\}$. Z is clearly not open since any basis element $(a, b) \times (c, d)$ containing $x \times x$ contains elements outside of Z, but $\pi_X(Z) = \mathbb{R}_+$ and $\pi_Y(Z) = \mathbb{R}_+$ which are open, since $\mathbb{R}_+ = \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, n\right)$.

Exercise 8. Assume U is open in the subspace Z of X. Then $U = V \cap Z$ with V open in X. Hence

$$U = Z \cap V = Z \cap (Y \cap V).$$

Hence U is open in the subspace topology of Z in Y as a subspace of X.
Assume conversely that U is open in the subspace Z of Y as a subspace of X.
Then $U = Z \cap V$ where V is open in Y.
But then $V = Y \cap T$ where T is open in X, hence

$$U = Z \cap V = Z \cap (Y \cap T) = Z \cap T.$$

So U is open in the subspace Z of X.
The conclusion follows.

Exercise 9: We show that the topologies are not comparable:

$(\mathcal{T}_{lexi} \not\subset \mathcal{T}_{sub})$: We have $(0\times 0, 0\times 2)\cap I^2 = (0\times 0, 0\times 1]$ which is not contained in $\mathcal{T}_{lexi}$, since any neighborhood of 0×1 here contains an element $c\times d$ with $c>0$.
$(\mathcal{T}_{sub} \not\subset \mathcal{T}_{lexi}$): We have $0\times 1 \in \left(0\times \frac{1}{2}, \frac{1}{2}\times 0\right)$, however if some basis element $(a\times b, c\times d)$ of the subspace topology contained 0×1 and was contained in this set, then $a=0$ and $\frac{1}{2}\le b<1$ and either $c>0$ or $c=0$ and $d>1$. Assume $c>0$, then $0\times 2 \in (a\times b, c\times d)$ hence $(a\times b, c\times d)$ is not contained in the set. If $c=0$ and $d=1$, let $1<e<d$, then $0\times e\in (a\times b, c\times d)$, and thus it is not contained in the set.

Exercise 19:
(i) It is clear that it is a basis since $\bigcup_{n\in\mathbb{N}}(-n,n) = \mathbb{R}$ and $(a,b)\cap(c,d) = (\max\{a,c\},\min\{b,d\})$ and $(a,b)\backslash A\cap(c,d) = (\max\{a,c\},\min\{b,d\})\backslash A$.
Now, let $a,c\in X$ and assume wlog. $a<c$. Take any $b\in\mathbb{R}$ with $a<b<c$; then $a\in(-\infty,b)$ and $c\in(b,\infty)$, so X is Hausdorff.

(ii) Let $\rho\colon X\to X/A$ be the quotient map. Let $x\in X/A$ and x not be the point A collapsed to. Then $\rho^{-1}(\{x\}) = \{x\}$ which is closed, so $\{x\}$ is closed in X/A.
If b is the point A. Then $\rho^{-1}(b) = A$ which is also closed in $\mathbb{R}_K$, since $0\in(-1,1)-A$ is not a limit point of A in $\mathbb{R}_K$. Alternatively, A is the complement of the open set

$$\bigcup_{n\ge 2}\left[(-n,n)-A\right].$$

One way to see that X/A is not Hausdorff is to notice that if U is any neighborhood of the point A collapsed to and V is any neighborhood of 0, then $\rho^{-1}(V)$ must contain $(0,q)-A$ with $q>0\notin A$. Not let $0<\frac{1}{N}<q$. Then since $\frac{1}{N}\in\rho^{-1}(U)$, there exists a basis element containing $\frac{1}{N}$ contained in $\rho^{-1}(U)$. Any such basis element contains elements less than $\frac{1}{N}$ and thus must intersect $(0,q)$ since $\frac{1}{N}\in(0,q)$.

(iii) Assume it were a quotient map. Then

$$\begin{aligned}\rho^{-1}(\Delta) &= \{x\times y\in X\times X \mid \rho(x)=\rho(y)\}\\ &= \{x\times y\in X\times X \mid x=y \vee x,y\in A\}.\end{aligned}$$

Exercise 20:
Define the map $g\colon D_n\to S^n$ by $g(x) = A\left(\frac{x}{\|x\|}\right)$ where A is the matrix
Exercise 21:
26.8: Theorem: Let $f\colon X\to Y$; let Y be compact Hausdorff. Then f is continuous iff the graph of f,

$$G_f = \{x\times f(x)\mid x\in X\}$$

is closed in $X\times Y$.
Solution: Assume f is continuous and $x\times y\in X\times Y$ is a limit point of G_f that is not in G_f.
Since Y is Hausdorff, take disjoint neighborhoods U,V around $f(x)$ and y, respectively, which are not equal by assumption. Since f is continuous, $f^{-1}(U)$ is open and contains x, so $f^{-1}(U)\times V$ is open in $X\times Y$.

If $u \times v \in f^{-1}(U) \times V$ then $f(u) \in U$ and $v \in V$, so since disjoint, $f(u) \neq v$, so $f^{-1}(U) \times V \cap G_f = \varnothing$. So G_f is closed.
Conversely, assume G_f is closed. Let $B \subseteq Y$ be closed. Then $X \times B$ is closed in $X \times Y$, so $X \times B \cap G_f$ is closed, so by exercise 26.7, $C = \pi_X (X \times B \cap G_f)$ is closed. We claim $f^{-1}(B) = C$ which would prove continuity of f.
Take $x \in C$, then since the only element in G_f containing x in its first coordinate is $x \times f(x)$, we have $f(x) \in B$. Thus $x \in f^{-1}(B)$.
Now, if conversely $x \in f^{-1}(B) \subseteq X$, then $f(x) \in B$, so $x \times f(x) \in (X \times B) \cap G_f$, hence $x \in C$.

26.9: Generalize the tube lemma as follows:
Theorem: Let A and B be subspaces of X and Y, respectively; let N be an open set in $X \times Y$ containing $A \times B$. If A and B are compact, then there exist open sets U and V in X and Y, respectively, such that

$$A \times B \subset U \times V \subset N.$$

Solution: We have that $a \times B$ is homeomorphic to B for any $a \in A$ and thus compact. For any $a \times b \in a \times B$, we can find a basis element $U_a \times V_b$ containing $a \times b$ and contained in N. Then the union of these covers $a \times B$, so there is a finite subcovering

$$a \times B \subset \bigcup_{i \in S} U_i \times V_i \subset N$$

where S is finite. Now taking the union over $a \in A$, we get a covering of $A \times B$ in N, and since the product of compact spaces is compact, there exists a finite subcovering

$$A \times B \subset \bigcup_{i \in T} \bigcup_{j \in S_i} U_j \times V_j \subset N$$

where T and all S_i are finite. Since all V_j contain B, we can let V be the intersection of all V_j. Then

$$A \times B \subset \bigcup_{i \in T} \bigcup_{j \in S_i} U_j \times V = U \times V \subset N$$

10. (a) Prove the following partial converse to the uniform limit theorem: Theorem: Let $f_n\colon X \to \mathbb{R}$ be a sequence of continuous functions, with $f_n(x) \to f(x)$ for each $x \in X$. If f is continuous, and if the sequence f_n is monotone increasing, and if X is compact, then the convergence is uniform.
We say that f_n is monotone increasing if $f_n(x) \leq f_{n+1}(x)$ for all n and x.

Solution: Define the continuous non-negative monotone decreasing function $g_n(x) = f(x) - f_n(x)$. Assume there does not exist $N \in \mathbb{N}$ such that for all $n \geq N$, $g_n^{-1}((\varepsilon, \infty)) = \varnothing$. Since g_n is monotone decreasing, we have for all $n \in \mathbb{N}$ that $g_n^{-1}((\varepsilon, \infty)) \neq \varnothing$.

Now, take any finite open subcover of the open set $g_n^{-1}((\varepsilon, \infty))$; say $A_1, \ldots, A_{m_n}$. We have that any compact subspace of a metric space is bounded and closed, so in particular $g_n(X)$ is bounded and closed, and $g_n\left(\overline{A_i}\right)$ is bounded and closed for all i.
Then $C_i = \bigcap_{n\in\mathbb{Z}_+} g_n\left(\overline{A_i}\right)$ is bounded and closed for all i. Pick y to be maximal in C_i (if $y = 0$, we move on to another A_i until we have one where $y \neq 0$; such an A_i must exist since some element in $\bigcup_{i\leq n} A_i$ is contained in $\bigcap_{n\in\mathbb{Z}_+} g_n^{-1}((\varepsilon, \infty))$).
Let $D = \bigcap_{n\in\mathbb{Z}_+} g_n^{-1}(\{y\})$. It is closed and by assumption empty. But it is a nested sequence in a compact space having the finite intersection property. This contradicts that X is compact.

(b) Give an example to show that this theorem fails if you delete the requirement that X be compact, or if you delete the requirement that the sequence be monotone.
Solution: We showed in 21.9 that

$$f_n(x) = \frac{-1}{n^3[x - \left(\frac{1}{n}\right)]^2 + 1}$$

converges to 0, is monotonically increasing and the limit function is 0. However, the convergence is not uniform because the domain $\mathbb{R}$ is not compact.

26.11: Theorem: Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a collection of closed connected subsets of X that is simply ordered by proper inclusion. Then

$$Y = \bigcap_{A\in\mathcal{A}} A$$

is connected.

Solution: Assume $Y = C \cup D$ is a separation of Y. Then since C and D are open in Y, there exist open sets U and V in X such that $C = U \cap Y$ and $D = V \cap Y$. If the collection is finite, the intersection is one of the A and we are done. Assume the collection is infinite. Since all A are ordered by proper inclusion, $A - U \cup V$ is nonempty for all A and in particular closed. Now take any finite collection

$$\{A_1 - U \cup V, A_2 - U \cup V, \ldots, A_n - U \cup V\}.$$

These are all finite and by the total ordering, there exists an $1 \leq i \leq n$ such that

$$\bigcap_{k\leq n} A_k - U \cup V = A_i - U \cup V.$$

Thus the collection has the finite intersection property, but then

$$\varnothing = \bigcap_{A \in \mathcal{A}} A - U \cup V \neq \varnothing.$$

Contradiction.

29.4: Show that $[0,1]^\omega$ is not locally compact in the uniform topology.

Solution: Assume $0 \in U \subset C$ where U is open and C is compact. Now let $B_{\overline{\rho}}(0,\varepsilon) \subseteq U$. Let A be the set of sequences with which is 0 everywhere except at precisely one point where it is $\frac{\varepsilon}{2}$. Then $A \subset U \subseteq C$ and is clearly closed hence also compact. Furthermore, $[0,1]^\omega$ is a metric space, so A is sequentially compact, however clearly the sequence x_n with 0 everywhere except at the n'th coordinate has no convergent subsequence. Alternatively, any limit point of A is in A however, any point of A contains a $\frac{\varepsilon}{4}$- neighborhood around it and is thus isolated.

29.11: Prove the following:

(a) *Lemma:* If $p\colon X \to Y$ is a quotient map and if Z is locally compact Hausdorff, then the map

$$\pi = p \times i_Z \colon X \times Z \to Y \times Z$$

is a quotient map.

Solution: Let $A \subset Y \times Z$ be open. Let $x \times y \in \pi^{-1}(A)$. Choose any basis element $\pi(x \times y) \in U \times V \subset A$. Then $\pi^{-1}(U \times V) = p^{-1}(U) \times V$ which is open. So $x \times y \in \pi^{-1}(U \times V) \subset \pi^{-1}(A)$.
Now assume $x \times y \in \pi^{-1}(A) \subset X \times Z$ is open. $\pi_Z(A)$ is open in Z so choose a neighborhood V of y such that $\overline{V}$ is compact. Similarly, $\pi_X(A)$ is open, so $U_1 = p^{-1}(\pi_X(A))$ is open. Now, $x \times y \in U_1 \times V \subset U_1 \times \overline{V} \subset \pi^{-1}(A)$. Consider $p^{-1}(p(U_1))$. Since if $x \in p^{-1}(p(U_1))$, we have $p(x) \in p(U_1) = \pi_X(A)$, we have $x \times \overline{V} \subset \pi^{-1}(A)$, so $\pi^{-1}(A) \cap X \times \overline{V}$ is an open set in $X \times \overline{V}$ containing $x \times \overline{V}$, so by the tube lemma, we can find a neighborhood W_x of x such that $W_x \times \overline{V} \subset \pi^{-1}(A) \cap X \times \overline{V}$. Taking the union

$$U_2 \times \overline{V} = \bigcup_{x \in p^{-1}(p(U_1))} W_x \times \overline{V}.$$

We have $x \times y \in U_2 \times V \subset U_2 \times \overline{V} \subset \pi^{-1}(A)$. Continuing, we let $U = \bigcup_{i \in \mathbb{N}} U_i$. We claim $U \times V$ is saturated. Assume $\pi^{-1}(u \times v) \cap U \times V \neq \varnothing$. Then for some $x \in p^{-1}(u)$, we have $x \times v \in U \times V$. Now so for some $n \in \mathbb{N}$, $x \times v \in U_n \times V$. Then $p^{-1}(p(x)) = p^{-1}(u) \subset U_{n+1} \subset U$, so $p^{-1}(u) \times v \subset U \times V$, therefore $\pi^{-1}(u \times v) = p^{-1}(u) \times v \subset U \times V$; so $U \times V$ is saturated. Therefore So for all $x \times y \in \pi^{-1}(A)$, we can find $(U \times V)_{x \times y}$ saturated open containing $x \times y$. Then

$$A \subset \bigcup_{x \times y \in \pi^{-1}(A)} \pi\left((U \times V)_{x \times y}\right) \subset A$$

And since the union is open since $p(U)$ is open as U is saturated and V is open in Z, we find that A is open.

(b) Theorem. Let $p\colon A \to B$ and $q\colon C \to D$ be quotient maps. If B and C are locally compact Hausdorff spaces, then $p \times q\colon A \times C \to B \times D$ is a quotient map.

Solution: We have $p \times i_C\colon A \times C \to B \times C$ is a quotient map by (a) and $i_B \times q\colon B \times C \to B \times D$ is a quotient map by (a), hence the composite map $p \times q = (i_B \times q) \circ (p \times i_C)$ is a quotient map.

31.7: Let $p\colon X \to Y$ be a closed continuous surjective map such that $p^{-1}(\{y\})$ is compact for each $y \in Y$. (Such a map is called a perfect map).
a: Show that if X is Hausdorff, then so is Y.
Solution:

Week 7, exercise 29: Let Y be a Hausdorff space and $X \subset Y$ and open subspace. Assume X is locally compact. Let X^+ denote the one-point compactification of X, and let ∞ denote the point $X^+ - X$. Define $f\colon Y \to X^+$ by

$$f(y) \begin{cases} y & \text{if } y \in X \\ \infty & \text{else.} \end{cases}$$

Show that f is continuous.
Solution: Let U be an open set in X^+. If U is an open set in the subspace X of Y, then $f^{-1}(U) = U$ is open in X and since X is open in Y, we have that $f^{-1}(U)$ is open in Y.
If $\infty \in U$, then $U = X^+ - C$ where C is a compact subspace of X. Then $f^{-1}(U) = f^{-1}(X^+ - C) = Y - f^{-1}(C)$. Now, since C is compact in a Hausdorff space, it is closed, so $f^{-1}(C)$ is closed in Y, hence $Y - f^{-1}(C)$ is open in Y so $f^{-1}(U)$ is open. Thus f is continuous.

32.4: Every regular Lindelöf space is normal.

Solution: Let X be regular Lindelöf. Let $A, B \subset X$ be closed. For all $a \in A$ choose U_a, V_{B_a} open s.t. $a \in U_a, B \subset V_{B_a}$ and $U_a \cap V_{B_a} = \varnothing$. Then $\bigcup_{a \in A} U_a$ covers A. We can do the same for B and find a collection V_b such that $\bigcup_{b \in B} V_b$ covers B and each V_b is disjoint from A. Now, for each point outside of A, B choose a neighborhood disjoint from $A \cup B$ which is closed (using regularity). Now the full collection is an open covering of X, so since X is Lindelöf, it contains a countable subcover. Of this countable subcover, only element of the form U_a intersect A and only elements of the form V_b intersect B, so we must have that $\bigcup_{i \in \mathbb{N}} U_{a_i}$ contains A and $\bigcup_{i \in \mathbb{N}} V_{b_i}$ contains B. These might not be disjoint however, but let

$$U_n = \bigcup_{i \in \mathbb{N}} U_{a_i} - \bigcup_{i=1}^{n} \overline{V_{b_i}} \quad V_n = \bigcup_{i \in \mathbb{N}} V_{b_i} - \bigcup_{i=1}^{n} \overline{U_{a_i}}.$$

Then $U = \bigcup_{n \in \mathbb{N}} U_n$ and $V = \bigcup_{n \in \mathbb{N}} V_n$ are open disjoint sets that cover A and B, respectively. Hence X is normal.