Assignment 1

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1:
(a) 6.
(b) 9.

2: Any line $L \subset \mathbb{A}$ can be written as a graph $y = ax+b$ and is thus the hypersurface defined by $y-(ax+b)$. We thus have

$$L \cap C = V\left(y - (ax+b)\right) \cap V(f) = V\left(y - (ax+b), f\right)$$

So for any pair $(x_1, y_1) \in L \cap C$, we have $y_1 = ax_1 + b$ and thus $0 = f(x_1, y_1) = f(x_1, ax_1 + b)$, so $x_1 \in V\left(f(x, ax+b)\right)$. However, $f(x, ax+b)$ is a polynomial in $k[x]$ and as such has either $\mathbb{A}^1$ as its zero locus or finitely many zeros - in particular, since roots of $f(x, ax+b)$ correspond to linear factors and $k[x]$ is an integral domain, we have that $f(x, ax+b)$ has at most d roots in k (Dummit and Foote, proposition 17). Since $L \not\subset C$, we must have that $f(x, ax+b)$ has at most d roots, and since each y_i is uniquely determined by the corresponding x_i, there exists at most d roots of $f(x,y)$ in $L \cap C$.

3:
(a) $A = \{(t, \sin t) : t \in \mathbb{R}\} \subset \mathbb{A}^2_{\mathbb{R}}$ is not an algebraic set. Assume for contradiction that $A = V(S)$ for some $S \subset k[x,y]$. Let f be any polynomial in S. Let $L = V(y)$ - i.e. a line as in problem **2** with $a, b = 0$. Then by problem **2** $L \cap V(f)$ is finite, however,

$$\{(\pi k, 0) : k \in \mathbb{Z}\} = \{(t, \sin t) : t \in \mathbb{R}\} \cap \{(x, 0)\, x \in \mathbb{R}\} = L \cap A = L \cap \bigcap_{g \in S} V(g) \subset L \cap V(f)$$

and $\{(\pi k, 0) : k \in \mathbb{Z}\}$ is an infinite set hence $L \cap V(f)$ is an infinite set contradicting problem **2**.

(b) Similarly to (a), let $B = \left\{(x,y) \in \mathbb{A}^2_{\mathbb{C}} \colon x = 0, y \neq 0\right\}$ and $L = V(x)$. Then $L \not\subset B$ since $(0,0) \notin B$. Assume $B = V(S)$ for some $S \subset k[x,y]$ and let $f \in S$. Then

$$B = L \cap B = L \cap V(S) = L \cap \bigcap_{g \in S} V(g) \subset L \cap V(f).$$

Since B is an infinite set, so is $L \cap V(f)$, so by contraposition of problem **2**, we find B is not an algebraic set.

(c) Similarly to (a) and (b), let $C = \left\{(x,y) \in \mathbb{A}^2_{\mathbb{R}} \colon y = |x|\right\}$ and $L = V(y - x)$. Then $L \not\subset C$ since for example $(-1,-1) \notin C$ but is in L. Assume $C = V(S)$ where $S \subset k[x,y]$ and let $f \in S$. Then

$$\left\{(x,y) \in \mathbb{A}^2_{\mathbb{R}} \colon y = x, x \geq 0\right\} \subset L \cap C = L \cap V(S) = L \cap \bigcap_{g \in S} V(g) \subset L \cap V(f)$$

Since $\left\{(x,y) \in \mathbb{A}^2_{\mathbb{R}} \colon y = x, x \geq 0\right\}$ is an infinite set, so is $L \cap V(f)$, so by contraposition of problem **2**, C is not an algebraic set.

(d) $D = \left\{\left(t, t^2, t^3\right) \in \mathbb{A}^3_k \colon t \in k\right\}$ is an algebraic set since $D = V\left(y - x^2, z - x^3\right)$:

$(\subset)$: for any $(t, t^2, t^3) \in D$, we have $(t^2) - (t)^2 = 0$ and $(t^3) - (t)^3 = 0$.
$(\supset)$: For an arbitrary $(x,y,z) \in D$, we have $y = x^2$ and $z = x^3$ so $(x,y,z) = (x, x^2, x^3)$ giving the inclusion as x ranges over k.