0.1. **Cohomology in terms of Homological Algebra.** Recall the Universal Coefficient Theorem for Cohomology:

Theorem 0.1 (Universal Coefficient Theorme for Cohomology)**.** *Let R be a ring and A an $R-$ module. Let C_* be a complex of projective R-modules such that the subcomplex of boundaries B_* is also a complex of projective modules.*

(1) *For all n, there is a SES*

$$0 \to \mathrm{Ext}^1_R\left(H_{n-1}(C_*), A\right) \overset{\lambda_n}{\to} H^n\left(\mathrm{Hom}_R\left(C_*, A\right)\right) \overset{\mu_n}{\to} \mathrm{Hom}_R\left(H_n\left(C_*\right), A\right) \to 0$$

where both λ_n and μ_n are natural in C_* and A.

(2) *If R is a PID, then the SES in (1) is split, but it is not always naturally split.*

Also recall the basic properties:

Lemma 0.2. *For a finitely generated H, we have*

- $\mathrm{Ext}\left(H \oplus H', G\right) \cong \mathrm{Ext}\left(H, G\right) \oplus \mathrm{Ext}(H', G)$
- $\mathrm{Ext}(H, G) = 0$ *if H is free.*
- $\mathrm{Ext}(\mathbb{Z}/n, G) \cong G/nG$.

Corollary 0.3. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_{n-1}$, then $H^n(\mathrm{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z})) \cong (H_n/T_n) \oplus T_{n-1}$.*

Proof. By the Universal Coefficient theorem for cohomology, we have that

$$H^n\left(\mathrm{Hom}_{\mathbb{Z}}\left(C_*, \mathbb{Z}\right)\right) \cong \mathrm{Ext}^1_{\mathbb{Z}}\left(H_{n-1}(C_*), \mathbb{Z}\right) \oplus \mathrm{Hom}_{\mathbb{Z}}\left(H_n\left(C_*\right), \mathbb{Z}\right)$$

Now, $\mathrm{Ext}^1_{\mathbb{Z}}\left(H_{n-1}(C_*), \mathbb{Z}\right) \cong T_{n-1}$ and $\mathrm{Hom}_{\mathbb{Z}}\left(H_n\left(C_*\right), \mathbb{Z}\right) \cong H_n/T_n$. □

Proposition 0.4. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G.*

Proof. Suppose $\alpha\colon C_* \to C'_*$ is the chain map such that $\alpha_*\colon H_n(C_*) \to H_n(C'_*)$ is an isomorphism for all n. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}\left(H_{n-1}(C), G\right) \arrow[r] & H^n(C; G) \arrow[r, "h"] & \mathrm{Hom}(H_n(C), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(C'), G) \arrow[r] \arrow[u, "(\alpha_*)^*", "\cong"'] & H^n(C'; G) \arrow[r, "h"] \arrow[u, "\alpha^*"] & \mathrm{Hom}(H_n(C'), G) \arrow[r] \arrow[u, "(\alpha_*)^*", "\cong"'] & 0
\end{tikzcd}$$

which follows form naturality of the Universal Coefficient theorem. Then by the 5-lemma, we obtain that α^* is an isomorphism also.

□

0.2. **Cohomology of Spaces.** Define $S^{-n}(X; G) := \mathrm{Hom}_{\mathbb{Z}}\left(S_n(X), A\right)$, so $S^*(X; A)$ is a chain complex. We define $H^n(X; A) := H_{-n}\left(S^*(X; A)\right)$, called *singular cohomology of X with coefficients in A.*

Thus an n-cochain $\varphi \in S^{-n}(X; G)$ assigns to each n-simplex $\sigma\colon \Delta^n \to X$ a value $\varphi\left(\sigma\right) \in G$. Since the n-simplicies form a basis for $S_n(X)$, these values can be chosen arbitrarily, hence n-cochains are exactly equivalent to functions from singular n-simplices to G.

The *coboundary map* $\delta\colon S^{-n}(X;G) \to S^{-(n+1)}(X;G)$ is the dual ∂^*, so for a cochain $\varphi \in S^{-n}(X;G)$, its coboundary $\delta\varphi$ is the composition $\delta\varphi = \partial^*\varphi = \varphi \circ \partial$, i.e., the composition $C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\varphi} G$.
Hence for a singular $(n+1)$-simplex $\sigma\colon \Delta^{n+1} \to X$, we have

$$\delta\varphi(\sigma) = \sum_i (-1)^i \varphi\left(\sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_{n+1}]}\right).$$

Since δ^2 is the dual of $\partial^2 = 0$, we have $\delta^2 = 0$ also, so $H^n(X;G)$ can be defined as above.

Note. For a cochain $\varphi \in S^{-n}(X;G)$ to be a cocyle means that $\delta\varphi = \varphi\partial = 0$, i.e., it means that φ vanishes on boundaries.