1. Objectives

- Read up on transversality in Lee - potentially supplied with Hirsch and Guillemin and Pollack.
- Read up on classification of surfaces. Potentially through Munkres, or through the two papers in the folder "Classification of surfaces" under the Topology folder. One of them deals with surfaces with boundary.
- Read about oriented intersection theory in Guillemin and Pollack.
- Work on section 2 in Farb and Margalit.
- Read section on $K(G, 1)$-spaces.

2. Questions

- Grad school for algtop, geotop, alg?
- How does one check that γ and β fill the genus 2 surface in figure 1.7?
- How to find (or show existence of) orientation-preserving or orientation-reversing maps?

3. Curves, Surfaces and Hyperbolic Geometry

3.1. **Simple closed curves.** There is a bijective correspondence

$$\left\{\begin{array}{c}\text{Nontrivial}\\ \text{conjugacy classes}\\ \text{in } \pi_1(S)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Nontrivial free}\\ \text{homotopy classes of oriented}\\ \text{closed curves in } S\end{array}\right\}$$

Definition 3.1 (Primitive and multiple elements)**.** An element g of a group G is *primitive* if there does not exist any $h \in G$ so that $g = h^k$ for $|k| > 1$. The property of being a primitive is a conjugacy class invariant. In particular, it makes sense to say that a closed curve in a surface is primitive.

A closed curve in S is a multiple if it is a map $S^1 \to S$ that factors through the map $S^1 \overset{\times n}{\to} S^1$ for $n > 1$, i.e., there exists a map $\tilde{\alpha}\colon S^1 \to S$ such that the following diagram commutes:

$$\begin{tikzcd} S^1 \arrow[r, "\times n"] \arrow[rr, bend left, dashed, "\tilde{\alpha}"] & S^1 \arrow[r, "\alpha"] & S \end{tikzcd}$$

Definition 3.2 (Lifts)**.** We make a distinction between lifts: let $p\colon \tilde{S} \to S$ be a covering space. By a *lift* of a closed curve α to $\tilde{S}$ we will always mean the image of a lift $\mathbb{R} \to \tilde{S}$ of the map $\alpha \circ \pi$ where $\pi\colon \mathbb{R} \to S^1$ is the usual covering map. I.e., a lift of $\alpha\colon S^1 \to S$ is a map $\tilde{\alpha}\colon \mathbb{R} \to \tilde{S}$ such that the following diagram commutes

$$\begin{tikzcd} & & \tilde{S} \arrow[d, "p"] \\ \mathbb{R} \arrow[urr, "\tilde{\alpha}"] \arrow[r, "\pi"'] & S^1 \arrow[r, "\alpha"'] & S \end{tikzcd}$$

A lift is different from a *path lift* which is a proper subset of a lift. Namely, it would be the restriction of $\tilde{\alpha}$ to some interval of $\mathbb{R}$ of length 2π if the covering map π is of the form $t \mapsto e^{it}$.

Now suppose $p\colon \tilde{S} \to S$ is the universal cover and α is a simple closed curve in S that is not a multiple of another closed curve. In this case, there is a bijective correspondence between cosets in $\pi_1(S)$ of the infinite cyclic subgroup $\langle\alpha\rangle$ and the lifts of α. This can be seen as follows: first choose a basepoint $\alpha(1) = x_0 \in S$ and some $\tilde{x_0} \in p^{-1}(x_0)$. There exists a unique lift $\tilde{\alpha}$ of α such that

$$\begin{tikzcd} & & \tilde{S} \arrow[d, "p"] \\ \mathbb{R} \arrow[urr, "\tilde{\alpha}"] \arrow[r] & S^1 \arrow[r, "\alpha"] & S \end{tikzcd}$$

commutes and such that $\tilde{\alpha}(0) = \tilde{x} \in p^{-1}(\alpha \circ \pi(0))$ for some specific $\tilde{x}$ [Bredon, Cor. 4.2]. But the set $p^{-1}(\alpha \circ \pi(0))$ is in bijective correspondence with the loops in $\pi_1(S)$ by the path lifting lemma. Now, under which path lifts are the lifts the same? The lifts of α to two points $\tilde{x}, \tilde{y} \in p^{-1}(\alpha \circ \pi(0))$ will be the same if $\alpha^k \cdot \tilde{x} = \tilde{y}$ where $\cdot$ denotes the monodromy action of $\pi_1(S)$ on the fiber. Now, there exist γ_x and γ_y in $\pi_1(S)$ such that $\gamma_x \cdot \tilde{x_0} = \tilde{x}$ and $\gamma_y \cdot \tilde{x_0} = \tilde{y}$, so $\alpha^k\gamma_x = \gamma_y$. Hence the lifts corresponding to γ_x and γ_y are the same if and only if $\alpha^k\gamma_x = \gamma_y$ for some k, i.e. if and only if $\gamma_x = \gamma_y$ in $\pi_1(S)/\langle\alpha\rangle$.

As usual, the group $\pi_1(S)$ acts on the set of lifts of α by deck transformations, and this action agrees with the usual left action of $\pi_1(S)$ on the cosets of $\langle\alpha\rangle$. The stabilizer of the lift corresponding to the coset $\gamma\langle\alpha\rangle$ is the cyclic group $\langle\gamma\alpha\gamma^{-1}\rangle$. See figure 1.

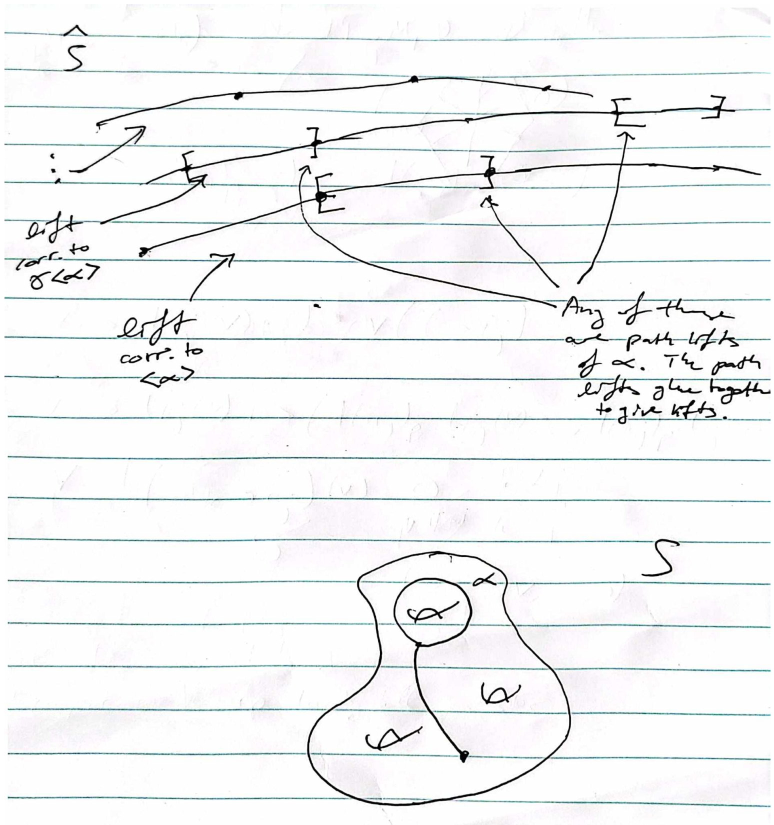


FIGURE 1.

Theorem 3.3. *When S admits a hyperbolic metric and α is a primitive element of $\pi_1(S)$, we have a bijective correspondence*

$$\left\{\begin{array}{c}\text{Elements of the conjugacy}\\ \text{class of } \alpha \text{ in } \pi_1(S)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Lifts to } \tilde{S} \text{ of the}\\ \text{closed curve } \alpha\end{array}\right\}$$

More precisely, we claim that the map which sends the lift given by the coset $\gamma\langle\alpha\rangle$ to $\gamma\alpha\gamma^{-1}$ is bijective and well-defined.

Proof. To show that it is well-defined, suppose $\gamma\langle\alpha\rangle$ and $\beta\langle\alpha\rangle$ give the same lift. Then $\gamma = \beta\alpha^k$. So in particular,

$$\gamma\alpha\gamma^{-1} = \beta\alpha^k\alpha\alpha^{-k}\beta^{-1} = \beta\alpha\beta^{-1}$$

so they do correspond to the same element of the conjugacy class $[\alpha]$. It is clear that this is a surjective map. Now suppose that $\gamma\alpha\gamma^{-1} = \beta\alpha\beta^{-1}$. Then

$\beta^{-1}\gamma\alpha\left(\beta^{-1}\gamma\right)^{-1} = \alpha$, so in particular, $\beta^{-1}\gamma \in C_{\pi_1(S)}(\alpha)$ which is a cyclic group generated by, say, θ. But then $\theta^l = \alpha$ since α is trivially in the centralizer of α; however, α is primitive, so l must be ± 1, but then α generates the centralizer of α, $C_{\pi_1(S)}(\alpha) = \langle\alpha\rangle$, and hence $\gamma = \beta\alpha^l$, so $\gamma\langle\alpha\rangle = \beta\langle\alpha\rangle$. $\square$

Remark. If α is any multiple, then we still have a bijective correspondence between elements of the conjugacy class of α and the lifts of α. However, if α is not primitive and not a multiple, then there are more lifts of α than there are conjugates. Indeed, if $\alpha = \beta^k$, where $k > 1$, then $\beta\langle\alpha\rangle \neq \langle\alpha\rangle$ while $\beta\alpha\beta^{-1} = \alpha$.

Example 3.4. The above correspondence does not hold for the torus T^2 because each closed curve has infinitely many lifts, while each element of $\pi_1\left(T^2\right) \approx \mathbb{Z}^2$ is its own conjugacy class because $\pi_1\left(T^2\right)$ is abelian.

Geodesic representatives.

Proposition 3.5. *Let S be a hyperbolic surface. If α is a closed curve in S that is not homotopic into a neighborhood of a puncture, then α is homotopic to a unique geodesic closed curve γ.*

Corollary 3.6. *For compact hyperbolic surfaces, there is a bijective correspondence:*

$$\left\{ \begin{array}{c} \textit{Conjugacy classes} \\ \textit{in } \pi_1(S) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Oriented geodesic} \\ \textit{closed curves in } S \end{array} \right\}$$

Simple closed curves.

Definition 3.7 (Simple curves)**.** A closed curve in S is *simple* if it is topologically embedded, i.e., if the map $S^1 \to S$ is injective.

By [Bredon, Thm 11.8], any closed curve α can be approximated (arbitrarily close) by a smooth closed curve which is homotopic to α. Moreover, if α is simple, then the smooth approximation can be chosen to be simple. Smooth curves are advantageous because we can make use of notions such as transversality.
Simple closed curves are also natural to study because they represent primitive elements of $\pi_1(S)$.

Proposition 3.8. *Let α be a simple closed curve in a surface S. If α is not null homotopic, then each element of the corresponding conjugacy class in $\pi_1(S)$ is primitive.*

Example: simple closed curves on the torus.

Proposition 3.9. *The nontrivial homotopy classes of oriented simple closed curves in T^2 are in bijective correspondence with the set of primitive elements of $\pi_1\left(T^2\right) \approx \mathbb{Z}^2$ which is the set of elements $(p,q) \in \mathbb{Z}^2$ such that either $(p,q) = (0,\pm 1)$ or $(p,q) = (\pm 1, 0)$ or $\gcd(p,q) = 1$.*

Closed geodesics.

Proposition 3.10. *Let S be a hyperbolic surface. Let α be a closed curve in S not homotopic into a neighborhood of a puncture. Let γ be the unique geodesic in the free homotopy class of α guaranteed by proposition 3.5. If α is simple, then γ is simple.*

Proof. Follows from the following lemma:

Lemma 3.11. *Let X be a topological space with a universal covering space $\tilde{X}$. A closed curve β in X is simple if and only if the following properties hold:*

(1) Each lift of β to $\tilde{X}$ is simple.
(2) No two lifts of β intersect.
(3) β is not a nontrivial multiple of another closed curve.

□

Intersection numbers. It is often useful to put an inner product on a vector space to check if two vectors are linearly independent. We can pursue something similar for surfaces.

Definition 3.12 (Geometric intersection number)**.** Let α, β be closed curves on a surface S. Their *geometric intersection number* is

$$i(\alpha, \beta) = \min_{\alpha' \simeq \alpha, \beta' \simeq \beta} \# \left(\alpha' \cap \beta'\right)$$

Definition 3.13 (Preliminary definition for transversality)**.** If $\alpha \cap \beta$ is finite and, at every intersection, each curve locally separates the other curve, then we say that α and β are *transverse.*

Definition 3.14 (Minimal position)**.** Two curves α and β are in *minimal position* if $\#(\alpha \cap \beta) = i(\alpha, \beta)$.

Bigons. We want a procedure to put curves into minimal position so we can compute intersection numbers.
For this, we need the notion of a *bigon*:

Definition 3.15 (Bigon)**.** Two transverse simple closed curves α and β in a surface S form a *bigon* if there is a topologically embedded disk in S (the bigon) whose boundary is the union of an arc of α and an arc of β intersecting in exactly two points.

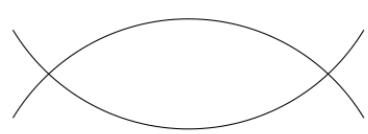


FIGURE 2. Local picture of a bigon

Lemma 3.16. *If transverse simple closed curves α and β in a surface S do not form any bigons, then in the universal cover of S, and pair of lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β intersect in at most one point.*

Proposition 3.17 (The bigon criterion)**.** *Two transverse simple closed curves in a surface S are in minimal position if and only if they do not form a bigon.*

Corollary 3.18. *Any two transverse simple closed curves that intersect exactly once are in minimal position.*

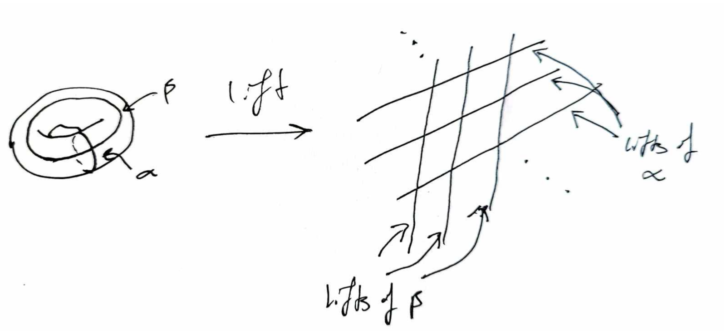


FIGURE 3. Lemma 3.16 illustrated

Homotopy versus isotopy for simple closed curves.

Definition 3.19 (Isotopy)**.** Two simple closed curves α and β are *isotopic* if there is a homotopy

$$H\colon S^1 \times [0,1] \to S$$

from α to β with the property that the closed curve $H\left(S^1 \times \{t\}\right)$ is simple for each $t \in [0,1]$.

Proposition 3.20 (Baer)**.** *Let α and β be two essential simple closed curves in a surface S. Then α is isotopic to β if and only if α is homotopoic to β.*

Proof. If α is isotopic to β then they are clearly also homotopic.

Suppose α and β are homotopic. Taking a tubular neighborhood around α, we can find a disjoint simple loop $\tilde{\alpha}$ which is homotopic to α but disjoint from it. Then β is homotopic to $\tilde{\alpha}$, and hence $i(\alpha,\beta) = i(\alpha,\tilde{\alpha}) = 0$. Performing an isotopy of α, we may assume that α is transverse to β (why?). If α and β are not disjoint, then by the bigon criterion, they form a bigon. A bigon prescribes an isotopy that reduces intersection, so we may remove bigons by isotopy until α and β are disjoint.

Suppose $\chi(S) < 0$. Lift α and β to $\tilde{a}$ and $\tilde{\beta}$ with the same endpoints in $\partial\mathbb{H}^2$. There is a hyperbolic isometry φ that leaves $\tilde{\alpha}$ and $\tilde{\beta}$ invariant and acts by translation on the lifts. As $\tilde{\alpha}$ and $\tilde{\beta}$ are disjoint, let R denote the region between them. We claim that the quotient surface $R' = R/\langle\varphi\rangle$ is an annulus. The fundamental group of R' is isomorphic to the group of deck transformations $\langle\varphi\rangle$ and is hence infinite cyclic. Furthermore, R' has two boundary components. □

Extension of isotopies.

Definition 3.21. An isotopy of a surface S is a homotopy $H\colon S \times I \to S$ such that for each $t \in [0,1]$, the map $H(S,t) : S \times \{t\} \to S$ is a homeomorphism. Given an isotopy between two simple closed curves in S, it will often be useful to promote this to an isotopy of S which we call an ambient isotopy of S.

Proposition 3.22. *Let S be any surface. If $F\colon S^1 \times I \to S$ is a smooth isotopy of simple closed curves, then there is an isotopy $H\colon S \times I \to S$ so that $H|_{S\times 0}$ is the identity and $H|_{F(S^1\times 0)\times I} = F$.*

Proof. [Hirsch, Ch 8, Thm 1.3] □

3.2. **Small digression on Hirsch chapter 8.**

Definition 3.23 (Isotopy in general and their tracks)**.** Let V and M be manifolds. An isotopy from V to M is a map $F\colon V \times I \to M$ such that for each $t \in I$, the map

$$F_t\colon V \to M, \quad x \mapsto F(x,t)$$

is an embedding.
The *track* of the isotopy F is the embedding

$$\hat{F}\colon V \times I \to M \times I, \quad (x,t) \mapsto (F(x,t),t)$$

Definition 3.24 (Isotopic embeddings and ambient isotopies)**.** If $F\colon V \times I \to M$ is an isotopy, we call the two embeddings F_0 and F_1 isotopic. If V is a submanifold of M and F_0 is the inclusion, we call F an isotopy of V in M. When $V = M$ and each F_t is a diffeomorphism, and $F_0 = \mathbb{1}_M$, then F is called an ambient isotopy.

Definition 3.25 (Support of isotopy)**.** The *support* $\operatorname{Supp} F \subset V$ of an isotopy $F\colon V \times I \to M$ is the closure of $\{x \in V : F(x,t) \neq F(x,0) \text{ for some } t \in I\}$.

Theorem 3.26 (Isotopy extension theorem)**.** *Let $V \subset M$ be a compact submanifold and $F\colon V \times I \to M$ an isotopy of V. If either $F(V \times I) \subset \partial M$ or $F(V \times I) \subset M - \partial M$, then F extends to an ambient isotopy of M having compact support.*

Arcs.

Assume S is a compact surface, possibly with boundary and possibly with finitely many marked points in the interior. Denote the set of marked points by $\mathcal{P}$.

Definition 3.27. A *proper arc* in S is a map $\alpha\colon [0,1] \to S$ such that $\alpha^{-1}(\mathcal{P} \cup \partial S) = \{0,1\}$.

Definition 3.28. The arc α is *simple* if it is an embedding on its interior.

Remark. The homotopy class of a proper arc is taken to be the homotopy class within the class of proper arcs. Thus points on ∂S cannot move off the boundary during the homotopy.
A homotopy (or isotopy) of an arc is said to be *relative to the boundary* if its endpoints stay fixed throughout the homotopy. An arc in a surface S is *essential* if it is neither homotopic into a boundary component of S nor a marked point of S.

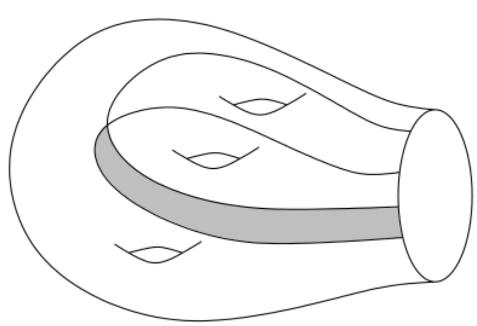


FIGURE 4. bigon-of-arcs.png

Note in this picture how if isotopies are considered relative to the boundary, then the two arcs are in minimal position, while if we consider general isotopies, then the half-bigon shows that they are not in minimal position as we can pull the top strand down under the bottom one along the boundary.

- The bigon criterion holds for arcs.
- Corollary 1.9 (geodesics are in minimal position) and prop 1.3 (existence and uniqueness of geodesic representatives) work for arcs in surfaces with punctures and/or boundary.
- Prop 1.10 (homotopy versus isotopy for curves) and theorem 1.13 (extension of isotopies) also work for arcs.

Change of coordinates principle.

Classification of simple closed curves.

Definition 3.29. Given a simple closed curve or a simple proper arc α in a surface S, the surface obtain by cutting S along α is a compact surface S_α equipped with an attaching map h (i.e.

1. $S_\alpha / (x \sim h(x)) \approx S$
2. the image of the distinguished boundary components under this quotient map is α.

Definition 3.30. We say that a simple closed curve α in the surface S is *nonseparating* if the cut surface S_α is connected, and *separating* if S_α is not connected.

Theorem 3.31. *If α and β are any two nonseparating simple closed curves in a surface S, then there is a homeomorphism $\varphi\colon S \to S$ with $\varphi(\alpha) = \beta$.*

Proof. The cut surface S_α and S_β have two boundary component corresponding to α and β, respectively. Now, suppose S_α has n_α vertices, m_α edges and t_α triangles in a triangulation. Then in obtaining S from S_α, we identify the vertices and edges, but no triangles are identified, so we get $n_S = n_\alpha - 3$ and $m_S = m_\alpha - 3$, but $t_S = t_\alpha$. Thus $\chi(S_\alpha) = \chi(S)$.
Since both S_α and S_β have the same Euler characteristic, number of boundary components and number of punctures, it follows that $S_\alpha \approx S_\beta$. Choose a homeomorphism $\varphi\colon S_\alpha \to S_\beta$ such that if h_α is the attaching map for S_α and h_β is the attaching map for S_β, then φ takes $\{x, h_\alpha(x)\}$ to $\{y, h_\beta(y)\}$ - i.e., the identification are respected under the map. This homeomorphism gives the desired homeomorphism of S taking α to β. If we want an orientation preserving homeomorphism, we can postcompose by an orientation-reversing homeomorphism fixing β if necessary. $\square$

Theorem 3.32. *When S is closed, β is separating if and only if it is the boundary of some subsurface of S. Which is equivalent to the vanishing of the homology class of β in $H_1(S, \mathbb{Z})$.*

Remark. By the "classification of disconnected surfaces", there are finitely many separating simple closed curves in S up to homeomorphism.

Corollary 3.33. *There is an orientation-preserving homeomorphism of a surface taking one simple closed curve to another if and only if the corresponding cut surfaces (which may be disconnected) are homeomorphic.*

Definition 3.34 (Topological type)**.** The existence of a homeomorphism as in 3.33 is an equivalence relation. The equivalence class of a simple closed curve or a collection of simple closed curves is called its *topological type*.

A sepraratіng simple closed curve in the closed surface S_g divides S_g into two disjoint subsurfaces of genus k and $g-k$. The minimum of $\{k, g-k\}$ is called the genus of the separating simple closed curve. There are $\lfloor \frac{g}{2} \rfloor$ topological types of essential separating simple closed curves in a closed surface.

Question 3.35. Suppose α is any nonseparating simple closed curve on a surface S.

(1) Is there a simple closed curve γ in S so that α and γ *fill* S, i.e., such that α and γ are in minimal position and the complement of $\alpha \cup \gamma$ is a union of topological disks.
(2) Is there a simple closed curve δ in S with $i(\alpha, \beta) = 0$? $i(\alpha, \beta) = 1$? $i(\alpha, \beta) = k$?

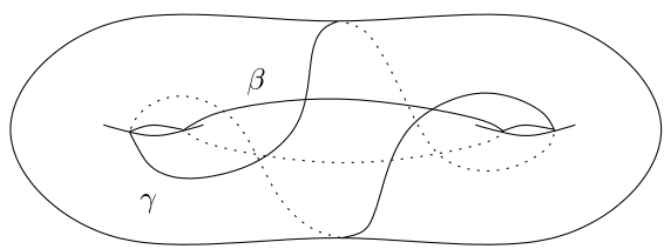


FIGURE 5. a

Figure **??** shows two filling simple closed curves on the genus 2 surface. By the classification of simple closed curves on a surface, there is a homeomorphism $\varphi\colon S_2 \to S_2$ such that $\varphi(\beta) = \alpha$. Then the image of γ under φ fills S_2 with α since filling is a topological property (show this).

Examples of change of coordinate principle.

(1) *Pairs of simple closed curves that intersect once are all of the same topological type.* Suppose α_1 and β_1 form such a pair on a surface S. Then β_1 must be an arc connecting the two boundary components in S_{α_1}. But the boundary component is homeomorphic to S^1, so removing a point leaves it connected. Thus removing β_1 leaves $(S_{\alpha_1})_{\beta_1}$ path-connected. Similarly, $(S_{\alpha_2})_{\beta_2}$ is path-connected for any other pair α_2 and β_2 that constitute a pair of simple closed curves that intersect once in S. By the classification of surfaces with boundary, $(S_{\alpha_1})_{\beta_1}$ is homeomorphic to $(S_{\alpha_2})_{\beta_2}$ which preserves equivalence classes on the boundary, and as we can construct this homeomorphism first for the β's and then for the α's, this homeomorphism descends to a self-homeomorphism of S taking the pair $\{\alpha_1, \beta_1\}$ to $\{\alpha_2, \beta_2\}$.

Three facts about homeomorphisms. Suppose $f\colon D \to D$ is an orientation-reversing map. Then f restricts to a map on $S^1 \to S^1$, and if f is smooth considered as such a map, then the reversal of orientation implies that since the fiber of any point is a single point, the degree of f must be -1. But thus f is not isotopic to the identity as the identity has degree 1 and the isotopy would have to restrict to a homotopy on the boundary, but degree is a homotopy invariant for maps $S^n \to S^n$. However, the straight-line homotopy does give a homotopy between f and the identity.

On $A = S^1 \times I$, the orientation-reversing map that fixes the S^1 factor and reflects the I factor is homotopic but not isotopic to the identity.

Theorem 3.36. *Let S be any compact surface and let f and g be homotopic homeomorphisms of S. Then f and g are isotopic unless they are one of the two examples described above (on $S = D^2$ and $S = A$). In particular, if f and g are orientation-preserving, then they are isotopic.*

Theorem 3.37. *Let S be a compact surface. Then every homeomorphism of S is isotopic to a diffeomorphism of S.*

Theorem 3.38 (Hamstrom)**.** *Let S be a compact surface, possibly minus a finite number of points from the interior. Assume that S is not homeomorphic to $S^2, \mathbb{R}^2, D^2, T^2$, the closed annulus, the once-punctured disk, or the once-punctured plane. Then the space* $\mathrm{Homeo}_0(S)$ *is contractible.*

4. Mapping class group basics

The compact-open topology.

Definition 4.1. The *weak* or *compact-open* C^r topology on $C^r(M,N)$, where M and N are C^r manifolds, is generated by sets defined as follows: let $f \in C^r(M,N)$. Let $(U,\varphi),(V,\psi)$ be charts on M and N ; let $K \subset U$ be compact such that $f(K) \subset V$ and let $0 < \varepsilon \leqslant \infty$. Then a *weak subbasic neighborhood*

$$\mathcal{N}^r\left(f;(U,\varphi),(V,\psi),K,\varepsilon\right) \tag{ζ}$$

is the set of C^r maps $g\colon M \to N$ such that $g(K) \subset V$ and

$$\left\|D^k\left(\psi f \varphi^{-1}\right)(x) - D^k\left(\psi g \varphi^{-1}\right)(x)\right\| < \varepsilon$$

for all $x \in \varphi(K)$, for $k = 0,\dots,r$. The *compact-open C^r topology* on $C^r(M,N)$ is generated by the set of weak subbasic neighborhoods, and defines the topological space $C^r_W(M,N)$. A neighborhood of f is then any set containing the intersection of a finite number of sets of the type (ζ).

We are interested in the subspace $\mathrm{Homeo}(S) \subset C^0_W(S,S)$, inheriting the subspace topology.

The compact-open topology might seem a bit confusing, but we have the following lemma [Hatcher, Prop A.14]:

Lemma 4.2. *Let X, Y, Z be Hausdorff topological spaces. Suppose Y is locally compact. Then a map $f\colon X \to C^0_W(Y,Z)$ is continuous if and only if the associated map $F\colon X \times Y \to Z$ defined by*

$$F(x,y) := f(x)(y)$$

is continuous.

4.1. **Definitions and first examples.**

Definition 4.3. Let S be a surface which is the connected sum of $g \geqslant 0$ tori with $b \geqslant 0$ disjoint open disks removed and $n \geqslant 0$ points removed from the interior. Let $\mathrm{Homeo}^+(S,\partial S)$ denote the group of orientation-preserving homeomorphisms of S that restrict to the identity on ∂S. We endow this group with the compact-open topology. The *mapping class group* of S, denoted $\mathrm{Mod}(S)$, is the group

$$\mathrm{Mod}(S) = \pi_0\left(\mathrm{Homeo}^+(S,\partial S)\right)$$

Remark. From Lemma 4.2, we see that a path $\gamma\colon I \to \mathrm{Homeo}^+(S,\partial S)$ is precisely equivalent to an isotopy $F\colon I \times S \to S$ from $\gamma(0)$ to $\gamma(1)$ (isotopy because at each time t, $\gamma(t)\colon S \to S$ is indeed a topological embedding as it is a homeomorphism). In fact, it's an isotopy of S. Here isotopies are required to fix boundaries.

If $\mathrm{Homeo}_0(S,\partial S)$ denotes the connected component of the identity in $\mathrm{Homeo}^+(S,\partial S)$, then we can equivalently write

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S,\partial S)/\mathrm{Homeo}_0(S,\partial S).$$

Proposition 4.4.

$$\begin{aligned}\mathrm{Mod}(S) &= \pi_0\left(\mathrm{Homeo}^+(S, \partial S)\right)\\ &\approx \mathrm{Homeo}^+(S, \partial S)/\textit{homotopy}\\ &\approx \pi_0\left(\mathrm{Diff}^+(S, \partial S)\right)\\ &\approx \mathit{Diff}^+(S, \partial S)/\sim\end{aligned}$$

where $\mathrm{Diff}^+(S, \partial S)$ *is the group of orientation preserving diffeomorphisms of* S *that are the identity on the boundary and* $\sim$ *can be taken to be either smooth homotopy relative to the boundary or smooth isotopy relative to the boundary.*

The Alexander Lemma.

Lemma 4.5 (Alexander lemma)**.** *The group* $\mathrm{Mod}\left(D^2\right)$ *is trivial.*

Remark. Also $0 \approx \mathrm{Mod}(D - \{0\}) \approx \mathrm{Mod}(S_{0,1}) \approx \mathrm{Mod}\left(S^2\right)$.

The mapping class group of the thrice-punctured sphere, $\mathrm{Mod}(S_{0,3})$*.*

Proposition 4.6. *Any two essential simple proper arcs in* $S_{0,3}$ *with the same endpoints are isotopic. Any two essential arcs that both start and end at the same marked point of* $S_{0,3}$ *are isotopic.*

Proof. Let α and β be two simple proper arcs in $S_{0,3}$ connecting the same two distinct marked points. By isotopy, we may modify α so that it intersects transversally with β. Letting the last marked point become the point at infinity, we can consider α and β as being arcs in $\mathbb{R}^2 - \{p, q\}$ for the two marked points p, q. An example is illustrated below. Now, suppose the arcs are disjoint. Then, choosing an intersection point, we can follow the path to the other intersection point and obtain either a bigon, in which case we can remove it by isotopy, or a bigon with path segments inside. Now, suppose the there is some point of α inside the bigon. Then since this is part of the arc α, we can find a simple path connecting this point to two points of β. There could, however, be infinitely many such paths inside the bigon, preventing us choosing the innermost (think concentric semicircles). However, by transversality, the preimages of the intersection points form a 0-dimensional submanifold of I which is closed (as the preimage of a closed path segment of β) and discrete. But discrete subsets of compact spaces are finite. Hence we can choose the innermost such path of α. By isotopy, we can remove the bigon formed by this alpha. Continuing a finite amount of times, we remove the original bigon. After a finite amount of reiterations, we can therefore remove all bigons, and we get disjoint α and β.
Now suppose we remove $\alpha \cup \beta$. Then we get a disjoint union of a disk and a punctured disk (by the classification of surfaces - expound on this). Thus the embedded disk in $S_{0,3}$ gives an isotopy of α to β. □

Proposition 4.7. *The natural map*

$$\mathrm{Mod}(S_{0,3}) \to \Sigma_3$$

given by the action of $\mathrm{Mod}(S_{0,3})$ *on the set of marked points of* $S_{0,3}$ *is an isomorphism.*

Proof. This is just additional notes to the proof in the book. The reason it is surjective is that the previous proposition gives an isotopy between arcs which we can extend to an ambient isotopy relative to the boundary which is an element of the mapping class group. (can we be sure the last marked point stays fixed?) □

Exercise 4.8. Show similarly that $\operatorname{Mod}(S_{0,2}) \approx \mathbb{Z}/2\mathbb{Z}$.

Solution. Let α, β be arcs with the same distinct marked endpoints. Equivalently to before, we can reduce bigons by isotopy until α and β are disjoint. Then removing $\alpha \cup \beta$ we would get two disjoint disks (firstly, $\alpha \cup \beta$ make up a closed simple curve which is trivial since $H_1\left(S^1\right) = \{0\}$ and thus separating. Therefore we get a disconnected space with as many vertices as edges whose Euler characteristic must add to $2 = \chi\left(S^2\right)$, so it must precisely have 1 face each, i.e., they are disks) which will descend to give the desired isotopy in $S_{0,2}$.
So assume no intersection. Let φ be an orientation preserving homeomorphism fixing the marked points. Then $\varphi(\alpha)$ is isotopic to α, so φ is isotopic to a homeomorphism which fixes α pointwise, call it ψ. This induces a homeomorphism on $S^2 - \alpha$ which is a disk that is the identity on the boundary, and hence isotopic to the identity homeomorphism on the disk since $\operatorname{Mod}\left(D^2\right) \approx \{0\}$. This isotopy gives an isotopy of ψ to the identity. The composition of all these isotopies gives an isotopy of φ with the identity. Hence the map is injective.

Theorem 4.9. *The homomorphism*

$$\sigma\colon \operatorname{Mod}\left(T^2\right) \to \mathrm{SL}(2, \mathbb{Z})$$

given by the action on $H_1(T; \mathbb{Z}) \approx \mathbb{Z}^2$ is an isomorphism.

Proof. Additional notes on the proof: why can we for any element $f \in \operatorname{Mod}\left(T^2\right)$ choose a representative φ that fixes a basepoint for T^2? □

Corollary 4.10. *Since $H_1(S_{1,1}; \mathbb{Z}) \approx \mathbb{Z}^2$, there is a homomorphism $\sigma\colon \operatorname{Mod}(S_{1,1}) \to SL(2, \mathbb{Z})$ which is determined which isomorphism the homomorphism induces in homology. This map is an isomorphism.*

Exercise 4.11. Prove this explicitly.

The mapping class group of $S_{0,4}$. Consider the torus T^2 as $I^2/\sim$ under the usual identification. Then consider the linear map $\iota\colon \mathbb{R} \to \mathbb{R}$ by $\iota = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \in SL(2, \mathbb{Z})$ which rotates about the origin by π radians.
The map is equivarient with respect to the quotient map so it induces a map $I^2/\sim \to I^2/\sim$ and we wish to take the quotient space that identifies fibers of this map. This is equivalent to taking the quotient space of $\mathbb{R}^2$ induced by the following actions: for $(a, b) \in \mathbb{R}^2$,

(1) sending (a, b) to $(a + 2k, b)$ for $k \in \mathbb{Z}$,
(2) sending (a, b) to $(a, b + 2t)$ for $t \in \mathbb{Z}$,
(3) or sending (a, b) to $(-a, -b)$.

We claim the quotient of $[0, 2] \times I$ under this action is a fundamental domain for the action. Clearly, the action is transitive. Now if $(a, b), (c, d) \in (0, 2) \times I$ are in the same orbit, then

$$\begin{aligned} a &= (-1)^{\alpha} c + 2k \\ b &= (-1)^{\alpha} d + 2t \end{aligned}$$

for some $k, t \in \mathbb{Z}$. But then if $b + d = 2t$, we get $b + d \in 2\mathbb{Z} \cap (0, 2) = \varnothing$, so α must be even, and $b = d$. But then $a - c \in 2\mathbb{Z} \cap (-2, 2) = \{0\}$, so $a = c$ and $b = d$. The identifications on the boundary become as in figure 6 which becomes S^2.

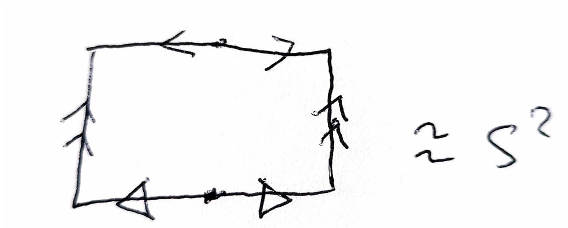


FIGURE 6.

We identify the quotient by $S_{0,4}$ where the 4 marked points are the 4 fixed points under the involution, namely, the images of the center of I^2, the midpoints of the edges and corner vertices. This is clearly also a 2-fold cover of the sphere.
Now, since for any $A \in \mathrm{SL}(2,\mathbb{Z})$, $A(-I) = (-I)A$, each element of $\mathrm{Mod}\left(T^2\right)$ induces an element of $\mathrm{Mod}\left(S_{0,4}\right)$ by descending to the quotient.

Proposition 4.12. *The hyperelliptic involution induces a bijection between the set of homotopy classes of essential simple closed curves in T^2 and the set of homotopy classes of essential simple closed curves in $S_{0,4}$.*

Proof. Notes:

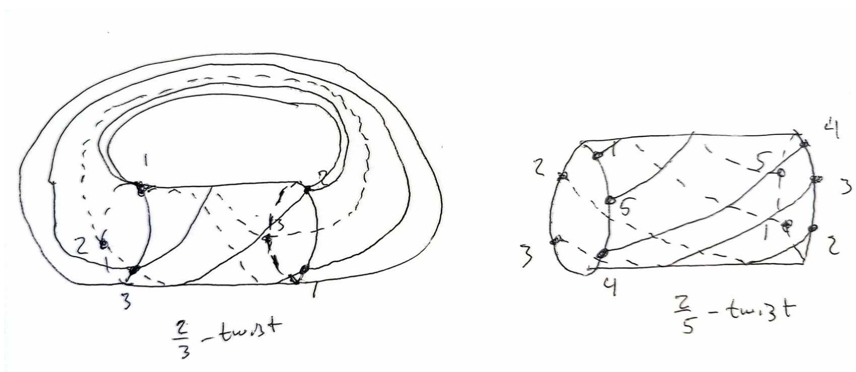


FIGURE 7. Twists along the meridian circle on the torus

Why is the preimage of a (p,q)-curve in $S_{0,4}$ in T^2 a $(2p,2q)$-curve? $\square$

Proposition 4.13. $\mathrm{Mod}\left(S_{0,4}\right) \approx \mathrm{PSL}\left(2,\mathbb{Z}\right) \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$.

Lemma 4.14. *If the short exact sequence of groups*

$$1 \to N \to G \to H \to 1$$

has a right inverse for $G \to H$, then G is naturally isomorphic to $N \ltimes H$.

Proof. Let $f\colon N \to G, g\colon G \to H$ and $h\colon H \to G$ be the inverse. Then f and g are injective. Suppose $z \in f(N) \cap h(H)$. Then there exists a $v \in N$ and $u \in H$ such that $f(v) = z = h(u)$, so $u = g\left(h(u)\right) = g(z) = g(f(v)) = 0$, so $z = 0$. Since $f(N)$ is the kernel of g, it is normal in G, so $f(N)h(H)$ forms a subgroup of G. Now suppose $p \in G - f(N)h(H)$. Then since $g(p - h(g(p))) = 0$, there exists $n \in N$ such that $p = f(n) + h(g(p)) \in f(N)h(H)$, contradiction. So $G = f(N)h(H)$, giving $G = f(N) \ltimes h(H) \approx N \ltimes H$. ☐

Proof. To show 4.13, it thus suffices to find a homomorphism $\mathrm{Mod}\,(S_{0,4}) \to \mathrm{PSL}\,(2, \mathbb{Z})$ with a right inverse, and show that the kernel is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
Notes on the proof: for the involutions ι_1, ι_2, we can lift them to homeomorphisms of T^2 by the lifting theorem [Bredon, Thm 4.1]. But why would these necessarily have to be homeomorphisms that rotate one of the factors of $T^2 \approx S^1 \times S^1$ by π? ☐

4.1.1. *The Alexander method.*

Proposition 4.15 (Alexander method)**.** *Let S be a compact surface, possibly with marked points, and let $\varphi \in \mathrm{Homeo}^+(S, \partial S)$. Let $\gamma_1, \ldots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in S with the following properties.*

(1) The γ_i are pairwise in minimal position.
(2) The γ_i are pairwise nonisotopic.
(3) For distinct i, j, k, at least one of $\gamma_i \cap \gamma_j, \gamma_i \cap \gamma_k$, or $\gamma_j \cap \gamma_k$ is empty.
(i) If there is a permutation σ of $\{1, \ldots, n\}$ so that $\varphi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to ∂S for each i, then $\varphi(\cup\gamma_i)$ is isotopic to $\cup\gamma_i$ relative to ∂S.
If we regard $\cup\gamma_i$ as a (possibily disconnected) graph Γ in S, with vertices at the intersection points and at the endpoints of arcs, then the composition of φ with this isotopy gives an automorphism φ_* of Γ.
(ii) Suppose now that $\{\gamma_i\}$ fills S. If φ_* fixes each vertex and each edge of Γ with orientations, then φ is isotopic to the identity. Otherwise, φ has a nontrivial power that is isotopic to the identity.

Lemma 4.16. *Let S be a compact surface, possibly with marked points, and let $\gamma_1, \ldots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in S that satisfy the three properties from proposition 4.15. If $\gamma_1', \ldots, \gamma_n'$ is another such collection so that γ_i' is isotopic to γ_i relative to ∂S for each i, then there is an isotopy of S relative to ∂S that takes γ_i' to γ_i for all i simultaneously and hence takes $\cup\gamma_i$ to $\cup\gamma_i'$.*

Remark. I put lemma 4.16 here since it's a slight generalization of (i) above and is used in the proof of Alexander's method too.

5. Dehn Twists

Let S be an oriented surface and let α be a simple closed curve in S. Let N be a tubular neighborhood of α and choose an orientation preserving homeomorphsim $\varphi\colon A \to N$. We then obtain a homeomorphism $T_\alpha\colon S \to S$, called a *Dehn twist about* α, as follows:

$$T_\alpha(x) = \begin{cases} \varphi \circ T \circ \varphi^{-1} & \text{if } x \in N \\ x & \text{if } x \in S - N \end{cases}.$$

"By the uniqueness of regular neighborhoods, the isotopy class of T_α does not depend on the choice of N or the choice of homeomorphism φ. Nor does T_α depend on the choice of simple closed curve α within its isotopy class." Huh, why???

Dehn twists on the torus. Via the isomorphism $\mathrm{Mod}\left(T^2\right) \to \mathrm{SL}\left(2, \mathbb{Z}\right)$ from 4.9, the Dehn twists about the $(1,0)$-curve and the $(0,1)$-curve in $\mathrm{Mod}\left(T^2\right)$ correspond to the matrices

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

5.0.1. *Dehn twist facts.*

Proposition 5.1. *Let a be the isotopy class of a simple closed curve α in a surface S. If α is not homotopic to a point or a puncture of S, then the Dehn twist T_a is a nontrivial element of* $\mathrm{Mod}(S)$.

Proposition 5.2. *Let a and b be arbitrary isotopy classes of essential simple closed curves in a surface and let k be an arbitrary integer. We have*

$$i\left(T_a^k(b), b\right) = |k|\, i\left(a, b\right)^2 .$$

Proposition 5.3. *Let α and β be simple closed curves in a surface. Suppose that α and β are in minimal position. Given a third simple closed curve γ, there exists a simple closed curve γ' that is homotopic to γ and that is in minimal position with respect to both α and β.*

Proposition 5.4. *Let $a_1, \ldots, a_n$ be a collection of pairwise disjoint isotopy classes of simple closed curves in a surface S and let $M = \Pi_{i=1}^n T_{a_i}^{e_i}$. Suppose that $e_i > 0$ for all i or $e_i < 0$ for all i. If b and c are arbitrary isotopy classes of simple closed curves in S, then*

$$\left| i\left(M(b), c\right) - \sum_{i=1}^{n} |e_i|\, i(a_i, b) i(a_i, c) \right| \leqslant i(b, c).$$

Definition 5.5 (Pairs of filling curves)**.** We say a pair of isotopy classes $\{a, b\}$ of simple closed curves in a surface S *fill* if any pair of minimal position representatives fill (i.e., the complement of the representatives in the surface is a collection of disks and once-punctured disks). This is equivalent to saying that for every isotopy class c of essential simple closed curves in the surface, either $i\left(a, c\right) > 0$ or $i(b, c) > 0$.

Proposition 5.6. *Let $g, n \geqslant 0$ and assume $\chi\left(S_{g,n}\right) < 0$. Then there exists a pair of simple closed curves in $S_{g,n}$ that fill $S_{g,n}$.*

The equivalence is intuitive since otherwise, it would be contained in one of the disks and hence not be essential, and the converse is similarly seen.

5.0.2. *Basic facts about Dehn twists.* Throughout this section, a and b denote arbitrary (unoriented) isotopy classes of simple closed curves.

Lemma 5.7. $T_a = T_b \iff a = b$.

Lemma 5.8. *For any $f \in \mathrm{Mod}(S)$ and any isotopy class a of simple closed curves in S, we have*

$$T_{f(a)} = f T_a f^{-1}.$$

Corollary 5.9. *For any $f \in \mathrm{Mod}(S)$ and any isotopy class a of simple closed curves in S, we have*

$$f \text{ commutes with } T_a \iff f(a) = a.$$

Corollary 5.10. *If a and b are nonseparating simple closed curves in S, then T_a and T_b are conjugate in* $\mathrm{Mod}(S)$.

Lemma 5.11. *For any two isotopy classes a and b of simple closed curves in a surface S, we have*

$$i(a,b)=0 \iff T_a(b)=b \iff T_aT_b=T_bT_a.$$

Proposition 5.12. *The above results have the following analogues for powers of Dehn twists: for $f \in \mathrm{Mod}(S)$, we have*

$$fT_a^j f^{-1} = T_{f(a)}^j$$

and so f commutes with T_a^j if and only if $f(a)=a$. Also, for nontrivial Dehn twists T_a, T_b and nonzero integers j,k, we have

$$T_a^j = T_b^k \iff a=b \text{ and } j=k$$
$$T_a^jT_b^k = T_b^kT_a^j \iff i(a,b)=0.$$

5.0.3. *The center of the mapping class group.*

Theorem 5.13. *For $g \geqslant 3$, $Z(G)$, for any finite index subgroup $G \leqslant \mathrm{Mod}(S_g)$, is trivial. In particular, $Z(\mathrm{Mod}(S_g))$ is trivial for $g \geqslant 3$.*

5.0.4. *Relations between two Dehn twists.*

Proposition 5.14 (Braid relation)**.** *If a and b are isotopy classes of simple closed curves with $i(a,b)=1$, then*

$$T_aT_bT_a = T_bT_aT_b.$$

Equivalently, this read $T_aT_b(a)=b$.

Question 5.15. Does the converse also work? I.e., if two Dehn twists satisfy the braid relation algebraically, do the corresponding curves necessarily have intersection number equal to 1?

Proposition 5.16. *If a and b are distinct isotopy classes of simple closed curves and the Dehn twists T_a and T_b satisfy $T_aT_bT_a=T_bT_aT_b$, then $i(a,b)=1$.*

Groups generated by two Dehn twists.

Theorem 5.17. *Let a and b be two isotopy classes of simple closed curves in a surface S. If $i(a,b)\geqslant 2$, then $\langle T_a, T_b\rangle \approx F_2$, where F_2 is the free group of rank 2.*

Proof. The slick proof makes use of this basic but nice lemma from geometric group theory:

Lemma 5.18 (Ping pong lemma)**.** *Let G be a group acting on a set X. Let $g_1,\dots,g_n$ be elements of G. Suppose that there are nonempty, disjoint subsets $X_1,\dots,X_n$ of X with the property that, for each i and each $j\neq i$, we have $g_i^k(X_j)\subset X_i$ for every nonzero integer k. Then the group generated by the g_i is a free group of rank n.*

Exercise 5.19. Complete the proof.

∎

5.1. **Cutting, capping and including.**

5.1.1. *Including.* When S is a closed subsurface of a surface S', we can define a natural homomorphism $\eta\colon \mathrm{Mod}(S) \to \mathrm{Mod}(S')$ as follows. For $f \in \mathrm{Mod}(S)$, we represent it by some $\varphi \in \mathrm{Homeo}^+(S, \partial S)$. Then, if $\hat{\varphi} \in \mathrm{Homeo}^+(S', \partial S')$ denotes the element that agrees with φ on S and is the identity outside of S, we define $\eta(f)$ to be the class of $\hat{\varphi}$. The map η is well defined because any homotopy between two elements of $\varphi \in \mathrm{Homeo}^+(S, \partial S)$ gives a homotopy between the corresponding elements of $\mathrm{Homeo}^+(S', \partial S')$ (the homotopy is simply relative to $S' - S$).

The goal is to find $\ker \eta$.

Lemma 5.20. *Let $\alpha_1, \ldots, \alpha_n$ be a collection of homotopically distinct simple closed curves in a surface S, each not homotopic to a point in S. Let β and β' be simple closed curves in S that are both disjoint from $\cup\alpha_i$ and are homotopically distinct from each α_i. If β and β' are isotopic in S, then they are isotopic in $S - \cup\alpha_i$.*

Theorem 5.21 (The kernel of the inclusion homomorphism)**.** *Let S be a closed subsurface of a surface S'. Assume that S is not homeomorphic to a closed annulus and that no component of $S' - S$ is an open disk. Let $\eta\colon \mathrm{Mod}(S) \to \mathrm{Mod}(S')$ be the induced map. Let $\alpha_1, \ldots, \alpha_m$ denote the boundary components of S that bound once-punctured disks in $S' - S$ and let $\{\beta_1, \gamma_1\}, \ldots, \{\beta_n, \gamma_n\}$ denote the pairs of boundary components of S that bound annuli in $S' - S$. Then the kernel of η is the free abelian group*

$$\ker \eta = \left\langle T_{\alpha_1}, \ldots, T_{\alpha_m}, \ldots, T_{\beta_1} T_{\gamma_1}^{-1}, \ldots, T_{\beta_n} T_{\gamma_n}^{-1} \right\rangle.$$

In particular, i no connected component of $S' - S$ is an open annulus, an open disk, or an open once-marked disk, then η is injective.

5.1.2. *The capping homomorphism.* One useful special case of theorem 5.21 is the case where $S' - S$ is a once-punctured disk. We say that S' is the surface obtained from S by *capping* one boundary component. In this case, we have

Proposition 5.22 (The capping homomorphism)**.** *Let S' be the surface obtained from a surface S by capping the boundary component β with a once-marked disk; call the marked point in this disk p_0. Denote by $\mathrm{Mod}(S, \{p_1, \ldots, p_k\})$ the subgroup of $\mathrm{Mod}(S)$ consisting of elements that fix the punctures $p_1, \ldots, p_k$, where $k \geqslant 0$. Let $\mathrm{Mod}(S', \{p_0, \ldots, p_k\})$ denote the subgroup of $\mathrm{Mod}(S')$ consisting of elements that fix the marked points $p_0, \ldots, p_k$ and then let $\mathcal{C}ap\colon \mathrm{Mod}(S, \{p_1, \ldots, p_k\}) \to \mathrm{Mod}(S', \{p_0, \ldots, p_k\})$ be the induced homomorphism. Then the following sequence is exact:*

$$1 \to \langle T_\beta \rangle \to \mathrm{Mod}(S, \{p_1, \ldots, p_k\}) \xrightarrow{\mathcal{C}ap} \mathrm{Mod}\left(S', \{p_0, \ldots, p_k\}\right) \to 1$$

Remark. In the case where S' is capped by an unmarked disk, the kernel is isomorphic to $\pi_1(TS')$, i.e., the fundamental group of the tangent bundle of S'.

6. Braid Groups

Definition 6.1 (Braids)**.** Let $p_1, \ldots, p_n$ be distinguished points in $\mathbb{C}$. A *braid* is a collection of n paths $f_i\colon [0,1] \to \mathbb{C} \times [0,1], 1 \leqslant i \leqslant n$, called *strands*, and a permutation $\overline{f} \in \Sigma_n$ such that the following hold:

- the strands $f_i([0,1])$ are disjoint
- $f_i(0) = p_i$

- $f_i(1) = p_{\overline{f}(i)}$
- $f_i(t) \in \mathbb{C} \times \{t\}$.

Usually we picture this by its *braid diagram* which is the projection of the images of the strands to the plane $\mathbb{R} \times [0,1]$ (with indications as to which strands pass over and under which others).

Definition 6.2. The *braid group on n strands*, denoted B_n, is the group of isotopy classes of braids.

Remark. Here an isotopy of the braid is a collection of isotopies $(h_1(x,t), \ldots, h_n(x,t))$ where h_i is an isotopy of f_i and such that $(h_1(-,t), \ldots, h_n(-,t))$ is a braid for each $t \in [0,1]$.

Definition 6.3. The product of the braid $(f_i(t))$ and the braid $(g_i(t))$ is the braid $(h_i(t))$, where

$$h_i(t) = \begin{cases} f_i(2t), & t \in \left[0, \frac{1}{2}\right] \\ g_{\overline{f}(i)}(2t-1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

For $1 \leqslant i \leqslant n-1$, let $\sigma_i \in B_n$ denote the braid whose only crossing is the $(i+1)$ st strand passing in front of the i th strand.
We claim that the group B_n is generated by the elements $\sigma_1, \ldots, \sigma_{n-1}$. This claim follows from the fact that any braid β can be isotoped so that its finitely many corssings occur at different horizontal levels.

6.1. **Fundamental groups of configuration spaces.**

Definition 6.4. Let S be a surface and let $C^{ord}(S,n)$ denote the configuration space of n distinct, ordered points in S, given by $C^{ord}(S,n) = \bigtimes_n S - \mathrm{BigDiag}(\bigtimes_n S)$ where $\mathrm{BigDiag}\left(\bigtimes_n S\right) = \{(x_1, \ldots, x_n) \in \bigtimes_n S \mid \exists 1 \leqslant i < j \leqslant n \colon x_i = x_j\}$.

Now, the symmetric group Σ_n acts on $\bigtimes_n S$ by permuting the coordinates. This action preserves $\mathrm{BigDiag}\left(\bigtimes_n S\right)$ and thus induces an action of Σ_n by homeomorphisms on $C^{ord}(S,n)$. Since the action of Σ_n permutes the n coordinates and since these coordinates are always distinct for points in $C^{ord}(S,n)$, we see that this action is free. The quotient space

$$C(S,n) = C^{ord}(S,n)/\Sigma_n$$

is just the configuration space of n distinct, *unordered* points in S.
Now we note the following lemma [LeeTM, Cor 12.27] and proposition [LeeTM, Prop 12.22]

Lemma 6.5. *Let M be a connected n-manifold on which a discrete group Γ acts continuously, freely, and properly. Then M/Γ is an n-manifold.*

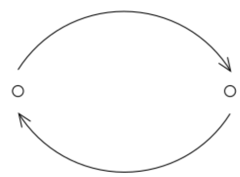
Proposition 6.6. *Every continuous action of a compact topological group on a Hausdorff space is proper*

Since $C(S,n)$ is the quotient of a manifold by a continuous free action of a finite group (hence compact), it follows that $C(S,n)$ is a manifold.
Since each strand of a braid is a map $f_i \colon I \to \mathbb{C} \times I$ with $f_i(t) \in \mathbb{C} \times \{t\}$, we can think of each f_i as a map $I \to \mathbb{C}$, so that a braid essetially becomes a path $I \to \mathbb{C}^n$ with equal end points, i.e., a loop, where each t is mapped to the point whose ith coordinate is $I \xrightarrow{f_i} \mathbb{C} \times \{t\} \xrightarrow{\mathrm{proj}} \mathbb{C}$. By assumption, the strands are disjoint, so each

t is mapped to a point in $C^{\text{ord}}(\mathbb{C}, n)$, and essentially, forgetting the strand index, we can regard this as a map $I \to C(\mathbb{C}, n)$. Said in a different way, if we consider a slice $\mathbb{C} \times \{t\}$ and intersect it with any braid, then we get a point in $C(\mathbb{C}, n)$, so the whole braid, which can be seen as a path in $C(\mathbb{C}, n)$ between these intersections, gives an element of $\pi_1(C(\mathbb{C}, n))$, and this gives the isomorphism

$$B_n \approx \pi_1(C(\mathbb{C}, n))$$

In this case, the generator σ_i of B_n corresponds to the element of $\pi_1(C(\mathbb{C}, n))$ given by the loop of n-point configurations in $\mathbb{C}$ where the i th and $(i+1)$ st points switch places by moving in a clockwise fashion, and the other $n-2$ points remain fixed.



6.2. **Mapping Class Group of a punctured disk.** Let D_n be a closed disk D^2 with n marked points. Then

$$B_n \approx \operatorname{Mod}(D_n) = \pi_0\left(\operatorname{Homeo}^+(D_n, \partial D_n)\right).$$

To see this, let φ be a representative of some element in $\operatorname{Mod}(D_n)$ which leaves the set of marked points invariant under φ - i.e., if $\{x_0, \ldots, x_n\} \subset D_n$ are the n marked points, then $\varphi(\{x_0, \ldots, x_n\}) = \{x_0, \ldots, x_n\}$. Note that we do not regard these marked points as punctures because we will not want to consider isotopies which "move" these marked points around which is not allowed when the marked points represent punctures. We see that φ is simply a homeomorphism of D^2 fixing ∂D^2 pointwise, so by the Alexander lemma, φ is isotopic to the identity. Now, throughout such an isotopy, the marked points must again be send to themselves, albeit they might move around through in the interior of D^2 (which we identify with $\mathbb{C}$) throughout the isotopy. Thus this isotopy produces a loop of these marked points, i.e., it produces a loop in $C(\mathbb{C}, n)$. So we have produced a braid.

Question 6.7. Does this association give a well-defined homomorphism $B_n \to \operatorname{Mod}(D_n)$ and would this be an isomorphism?

The answer is yes, and since σ_i generate B_n, the images generate $\operatorname{Mod}(D_n)$. Now, the image of σ_i will be the homotopy class of a homeomorphism of D_n that has support a twice-punctured disk and is described on this support by figure 8.

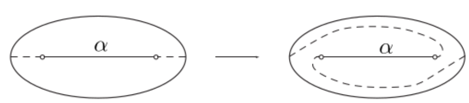


FIGURE 8. A half-twist

We denote such a half-twist as H_α, and we can think of α as either a simple closed curve with two punctures in its interior or a simple proper arc connecting two punctures.

6.3. **The Birman Exact Sequence.** Let S be any surface, possibly with punctures (but no marked points) and let (S, x) denote the surface obtained from S by marking a point x in the interior of S. There is a natural homomorphism

$$\mathcal{F}orget\colon \mathrm{Mod}\,(S, x) \to \mathrm{Mod}(S)$$

called the forgetful map which is realized by forgetting that the point x is marked. This map is surjective as any homeomorphism of S can be modified by isotopy to fix x (why?). The group $\mathrm{Mod}(S, x)$ is isomorphic to the subgroup G of $\mathrm{Mod}\,(S - x)$ consisting of homeomorphisms preserving the puncture coming from x.
The forgetful map can then be interpreted as the map $G \to \mathrm{Mod}(S)$ obtained by "filling in" the puncture x. I.e., $\mathcal{F}orget$ is the map induced by the inclusion $S - x \hookrightarrow S$.

6.3.1. *Analyzing the kernel of $\mathcal{F}orget$.* Let $f \in \mathrm{Mod}(S, x)$ be an element of the kernel of $\mathcal{F}orget$ and let φ be a homeomorphism representing f. We can think of φ as a homeomorphism $\overline{\varphi}$ of S.
Since $\mathcal{F}orget(f) = 1$, there is an isotopy from $\overline{\varphi}$ to $\mathbb{1}_S$. During this isotopy, the image of the point x traces out a loop α in S based at x. Now we will introduce the $\mathcal{P}ush$ map: given a loop α in S based at x, we can consider $\alpha\colon [0, 1] \to S$ as an isotopy $h\colon \{x\} \times I \to S$ with $h(x, 0) = h(x, 1) = x$ and extend this to the whole surface using 3.26 (here, $V = \{x\}$), denote this by $h\colon S \times I \to S$ also. Let $\varphi(x) = h(x, 1)$ be the homeomorphism of S obtained at the end of the isotopy. Taking its isotopy class in $\mathrm{Mod}\,(S, x)$, we get $[\varphi] =\colon \mathcal{P}ush(\alpha) \in \mathrm{Mod}\,(S, x)$. Think of $\mathcal{P}ush(\alpha)$ as placing your finger on x and pushing x along α, draggin the rest of the surface along as you go (indeed, locally, this is what must happen).
The question of whether this mapping class is independent of the choice of isotopy extension as well as the choice of α within its homotopy class. I.e., whether $\mathcal{P}ush$ defines a well-defined map

$$\mathcal{P}ush\colon \pi_1\,(S, x) \to \mathrm{Mod}\,(S, x)\,.$$

For that, we show look at the Birman exact sequence.

Theorem 6.8 (Birman exact sequence)**.** *Let S be a surface with $\chi(S) < 0$, possibily with punctures and/or boundary. Let (S, x) be the surface obtained from S by marking a point x in the interior of S. Then the following sequence is exact:*

$$1 \to \pi_1\,(S, x) \overset{\mathcal{P}ush}{\to} \mathrm{Mod}\,(S, x) \overset{\mathcal{F}orget}{\to} \mathrm{Mod}(S) \to 1.$$

Lemma 6.9. *$\mathcal{P}ush\colon \pi_1\,(S, x) \to \mathrm{Mod}(S, x)$ is injective when $\chi(S) < 0$.*

Proof. Let φ represent $\mathcal{P}ush\,(\alpha) \in \mathrm{Mod}\,(S, x)$. Then φ is a map $(S, x) \to (S, x)$, so in particular, we have an induced homomorphism $\varphi_*\colon \pi_1\,(S, x) \to \pi_1\,(S, x)$. This homomorphism is the inner automorphism $I_\alpha(\gamma) = \alpha\gamma\alpha^{-1}$ (why?). Now, for $\chi(X) < 0$, the center of $\pi_1(S)$ is trivial, so I_α is nontrivial whenever $\alpha \neq c_x$. If φ had been homotopic to the identity, their induced homomorphisms would be equal, so we conclude that $\mathcal{P}ush\,(\alpha)$ is nontrivial whenever α is nontrivial. □

6.3.2. *Push maps along loops in terms of Dehn twists.* Let $\alpha \in \pi_1\,(S, x)$ be simple. Let $S^1 \times [0, 2]$ be an annulus about α with $S^1 \times \{1\}$ identified with α with $(0, 1)$ identified with the marked point x. We orient $S^1 \times [0, 2]$ with the standard orientations on S^1 and $[0, 2]$.

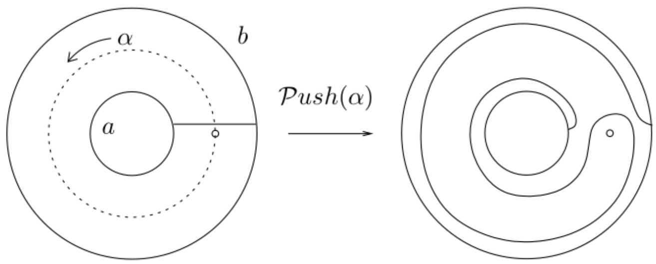
Let $F\colon A \times I \to A$ be the isotopy given by

$$F\left(\left(\theta, r\right), t\right) = \begin{cases} \left(\theta + 2\pi rt, r\right), & 0 \leqslant r \leqslant 1 \\ \left(\theta + 2\pi(2-r)t, r\right), & 1 \leqslant r \leqslant 2. \end{cases}$$

By 3.26, we can extend F to an ambient isotopy of S. Restricting F to $\{x\} \times [0,1]$, we get

$$F\left((0,1), t\right) = \left(2\pi t, 1\right),$$

so F pushes x around the core of the annulus. Now, the homeomorphism (representing $\mathcal{P}ush\left(\alpha\right)$) φ of (S,x) given by $F\left((\theta,r),1\right)$ is a product of two Dehn twists, $T_a T_b^{-1}$ where a is the closed curve identified with $S^1 \times \{0\}$ and b is the closed curve identified with $S^1 \times \{2\}$.



Remark (Naturality). For any $h \in \mathrm{Mod}(S,x)$ and any $\alpha \in \pi_1\left(S,x\right)$, we have

$$\mathcal{P}ush\left(h_*\left(\alpha\right)\right) = h\mathcal{P}ush\left(\alpha\right)h^{-1}.$$

Proof. Write a proof of the Birman exact sequence. □

6.4. **Generalized Birman Exact Sequence.** We now return to Question 6.7. We wish to show that $\pi_1\left(C\left(\mathbb{C}, n\right)\right) \approx \mathrm{Mod}\left(D_n\right)$.

Definition 6.10 (n-stranded surface braid group of S)**.** For an arbitrary surface S, we call $\pi_1\left(C\left(S,n\right)\right)$ the n-stranded surface braid group of S.

Let S be a compact finite-type surface without marked points. Let $(S, \{x_1, \ldots, x_n\})$ denote S with n marked points $x_1, \ldots, x_n$ in the interior. Equivalently to the construction in the proof of the Birman exact sequence, there is a fiber bundle

$$\mathrm{Homeo}^+\left(\left(S, \{x_1, \ldots, x_n\}\right), \partial S\right) \to \mathrm{Homeo}^+\left(S, \partial S\right) \to C\left(S^\circ, n\right)$$

where S° is the interior of S and $\mathrm{Homeo}^+\left(\left(S, \{x_1, \ldots, x_n\}\right), \partial S\right)$ is the group of orientation-preserving homeomorphisms of S that preserve the set $\{x_1, \ldots, x_n\}$ (allowing permutations, however) and fix the boundary of S pointwise.

Theorem 6.11 (Birman exact sequence generalized)**.** *Let S be a surface without marked points and with $\pi_1\left(\mathrm{Homeo}^+\left(S, \partial S\right)\right) = 1$. The following sequence is exact:*

$$1 \to \pi_1\left(C\left(S,n\right)\right) \overset{\mathcal{P}ush}{\to} \mathrm{Mod}\left(S, \{x_1, \ldots, x_n\}\right) \overset{\mathcal{F}orget}{\to} \mathrm{Mod}(S) \to 1.$$

Corollary 6.12. *When $S = D^2$, this gives the exact sequence*

$$1 \to \underbrace{\pi_1\left(C\left(D^2, n\right)\right)}_{B_n} \to \mathrm{Mod}\left(D_n\right) \to \underbrace{\mathrm{Mod}\left(D^2\right)}_{1} \to 1.$$

Hence $B_n \approx \mathrm{Mod}\left(D_n\right)$.

6.5. **Algebraic Structure of the Braid Group.**

Theorem 6.13.

$$\begin{aligned} B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \forall i \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, \quad |i-j| > 1 \end{aligned}$$

7. Exercises

Problem 7.1. Give an example of a surface S of finite type and self-diffeomorphism φ of S which is homotopic to id_S but not isotopic to id_S.

Solution. Let S be the closed unit disk D^1, and let $\varphi\colon D^1 \to D^1$ by the map $(x, y) \to (x, -y)$. This is a self-diffeomorphism which is homotopic to the identity map by $F\colon D^1 \times I \to D^1$, $F(x, t) = \varphi(x)t + (1 - t)x$. However, if there were an isotopy from id_S to φ, then since this would have to be a topological embedding, it would have to map the boundary to the boundary at all times during the isotopy. But then restricting to the boundary, the isotopy would descend to an isotopy of $\mathrm{id}_{\partial D^1}$ to $\varphi|_{\partial D^1}$. However, these maps have degree 1 and -1, respectively, and since degree is a homotopy invariant, we conclude that no such isotopy can exist.

8. Glossary

Definition 8.1 (Equivariant maps)**.** Suppose a group G acts on spaces X and Y, and let $f\colon X \to Y$ be a map. Then f is said to be equivariant if $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$ and all $g \in G$.

Definition 8.2 (Closed surface)**.** A *closed surface* is a surface that is compact and without boundary.

Definition 8.3 (Isotopy)**.** A topological isotopy is a homotopy $F\colon X \times I \to Y$ such that for each $t_0 \in I$, $F(x, t_0)\colon X \to Y$ is a topological embedding (homeomorphism onto some subspace of Y).
Two embeddings $f, g\colon X \to Y$ are said to be isotopic if there exists an isotopy $F\colon X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

Definition 8.4 (Orientation)**.** A closed n-manifold M is called orientable if $H_n(M; \mathbb{Z}) = \mathbb{Z}$. The choice of generator $[M]$ in $\mathbb{Z}$ is called an orientation, and the generator is called the fundamental class of M. A manifold together with a choice of orientation is called oriented. A compact n-manifold M with boundary is called orientable if $H_n(M, \partial M; \mathbb{Z}) = \mathbb{Z}$. The choice of generator $[M, \partial M]$ in $\mathbb{Z}$ is called an orientation, and $[M, \partial M]$ is referred to as the fundamental class of M.
A smooth manifold M is orientable if and only if the restriction of its tangent bundle to every smooth curve is trivial.

Remark. This makes sense since T. Radó showed that every surface is triangulable and it is clear then that the 2-cycles form a cyclic group. A choice of generator corresponds to choosing an orientation of each 2-simplex in the triangulation (compatibly).

Definition 8.5 (Inner and outer automorphisms)**.** Let G be any group and $\gamma \in G$. A conjugate automorphism

$$I_\gamma\colon g \mapsto \gamma g \gamma^{-1}$$

is called an *inner automorphism* of G. The group of inner automorphisms is denoted by $\mathrm{Inn}(G)$. It is isomorphic to $G/Z(G)$, and is a normal subgroup of $\mathrm{Aut}(G)$. The quotient

$$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$$

is called the *outer automorphism group* of G.

9. Appendix

9.1. **Fiber Bundles.**

9.1.1. *Bredon.*

Definition 9.1. Let X, B and F be Hausdorff spaces and $p\colon X \to B$ a map. Then p is called a bundle projection with fiber F, if each point of B has a neighborhood U such that there is a homeomorphism $\varphi\colon U \times F \to p^{-1}(U)$ such that $p(\varphi\langle b, y\rangle) = b$ for all $b \in U$ and $y \in F$. That is, on $p^{-1}(U)$, p corresponds to projection $U \times F \to U$. Such a map φ is called a *trivialization* of the bundle over U.

Definition 9.2. An action of a group G on a space X is said to be *effective* if

$$(\forall x \in X\colon gx = x) \implies g = e.$$

Definition 9.3. Let K be a topological group acting effectively on the Hausdorff space F as a group of homeomorphisms. Let X and B be Hausdorff spaces. By a *fiber bundle* over the base space B with total space X, fiber F, and structure group K, we mean a bundle projection $p\colon X \to B$ together with a collection Φ of trivializations $\varphi\colon U \times F \to p^{-1}(U)$, of p over U, called *charts* over U, such that

(1) each point of B has a neighborhood over which there is a chart in Φ ;
(2) if $\varphi\colon U \times F \to p^{-1}(U)$ is in Φ and $V \subset U$ then the restriction of φ to $V \times F$ is in Φ;
(3) if $\varphi, \psi \in \Phi$ are charts over U, then there is a map $\theta\colon U \to K$ such that $\psi \langle u, y\rangle = \varphi \langle u, \theta(u)(y)\rangle$;
(4) the set Φ is maximal among collections satisfying (1), (2) and (3).

The bundle is called *smooth* if all these spaces are manifolds and all maps involved are smooth.

Definition 9.4. A *vector bundle* is a fiber bundle in which the fiber is a Euclidean space and the structure group is the general linear group if this Euclidean space or some subgroup of that group.

9.1.2. *Hatcher.*

Definition 9.5 (Homotopy lifting property)**.** A map $p\colon E \to B$ is said to have the homotopy lifting property with respect to a space X if, given a homotopy $g\colon X \times I \to B$ and a map $\tilde{g}_0\colon X \times \{0\} \to E$ lifting $g(x,0)$, so $p\tilde{g}_0(t,0) = g(t,0)$, then there exists a homotopy $\tilde{g}\colon X \times I \to E$ lifting g_t.

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{g}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{g}"'] & B \end{tikzcd}$$

This is a special case of the *lift extension property for a pair* (Z, A) (See [Munkres])

Definition 9.6 (Fibration)**.** A fibration is a map $p\colon E \to B$ having the homotopy lifting property with respect to all spaces X. For example, a projection $B \times F \to B$ is a fibration since we can choose lifts of the form $\tilde{g}(x,t) = (g(x,t), h(x))$ where $\tilde{g}(x,0) = (g(x,0), h(x))$.

Definition 9.7 (Homotopy lifting property for a pair (X, A))**.** The map $p\colon E \to B$ is said to have the *homotopy lifting property for a pair* (X, A) if each homotopy $f\colon X \times I \to B$ lifts to a homotopy $\tilde{g}\colon X \times I \to E$ starting with a given lift $\tilde{g}_0\colon X \times \{0\} \to E$ and extending a given lift $\tilde{g}\colon A \times I \to E$.

Theorem 9.8. *Suppose $p\colon E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geqslant 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_*\colon \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geqslant 1$. Hence if B is path-connected, there is a long exact sequence*

$$\ldots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \ldots \to \pi_0(E, x_0) \to 0$$

Definition 9.9 (Fiber bundle)**.** A fiber bundle structure on a space E, with fiber F, consists of a projection map $p\colon E \to B$ such that each point of B has a neighborhood

U for which there is a homeomorphism $h\colon p^{-1}(U) \to U \times F$ making the following diagram commute

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "proj"] \\
& U &
\end{tikzcd}$$

Thus $h\left(p^{-1}(b)\right) = \{b\} \times F$ by the projection map $p\colon E \to B$, but to indicate what the fiber is we sometimes write a fiber bundle as $F \to E \to B$, a "short exact sequence of spaces". The space B is called the *base space* of the bundle, and E is the *total space*.

Remark. This is just the definition of a covering map without the restriction of F having the discrete topology.

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