1. Let $X = V(xw - yz) \subset \mathbb{A}^4$ and $f = \frac{\overline{x}}{\overline{y}} \in k(X)$.
We assume k is algebraically closed throughout.
(a) Let

$$J_f = \{g \in \Gamma(X) \colon gf \in \Gamma(X)\}.$$

By lecture note 11 page 7, $V(J_f)$ is the pole set of f.

We thus claim $V(y,w) = V(J_f)$.

Proof: Claim $(\overline{y}, \overline{w}) \subset J_f$.
We have that $\overline{y}f = \overline{y}\frac{\overline{x}}{\overline{y}} = \overline{x} \in \Gamma(X)$, since $\overline{yx} = \overline{xy}$ and hence the last equality follows in $k(X)$ by definition.
Now, since $\overline{xw} - \overline{yz} = 0$ in $\Gamma(X)$, we have $\frac{\overline{x}}{\overline{y}} = \frac{\overline{z}}{\overline{w}}$ in $k(X)$. Therefore $f = \frac{\overline{z}}{\overline{w}}$ in $k(X)$, so $\overline{w}f = \overline{w}\frac{\overline{z}}{\overline{w}} = \overline{z} \in \Gamma(X)$ since $\overline{wz} = \overline{zw}$, and thus the last equality follows by definition in $k(X)$. Therefore $\overline{y}, \overline{w} \in J_f$.

Claim: $J_f \subset (\overline{y}, \overline{w})$. Let $I_X(W)$ denote the image of an ideal W in $\Gamma(X)$.
The result follows if we can show $V(J_f) \supset V(\overline{y}, \overline{w}) = V(y,w) \cap X = \{(x,0,z,0) \colon x, z \in \mathbb{A}\} = A$. Let $g \in J_f$. Then $gf = l \in \Gamma(X)$, so $g\overline{x} = l\overline{y} \in \Gamma(X)$. Letting $P \in A$, we find that $g(P)\overline{x}(P) = l(P)\,\overline{y}(P) = 0$, so letting P range over $\{(x,0,z,0) : x, z \in \mathbb{A}, x \neq 0\}$, we get $g \in I_X(\{(x,0,z,0) : x,z \in \mathbb{A}, x \neq 0\})$.
Similarly, using the expression $f = \frac{\overline{z}}{\overline{w}}$, we get $g \in I_X(\{(x,0,z,0) : w \neq 0\})$.
Thus g is zero on the x,z plane except possibly at $0 = (0,0,0,0)$, but as that is a dense open set in the x,z plane, g is zero one all of the x,z plane. Alternatively, we can note that $g(x,0,0,0)$ is zero for every nonzero x, and as k is algebraically closed it is infinite by problem 1.6 in Fulton, hence $g(x,0,0,0)$ has infinitely many roots so by problem 1.8 in Fulton, it is the zero polynomial. Hence $g(0,0,0,0) = 0$ as well.
Thus we get $g \in I_X(A)$ and hence $A \subset V(I_X(A)) \subset V(g)$, so $A \subset \bigcap_{g \in J_f} V(g) = V\left(\bigcup_{g \in J_f}\{g\}\right) = V(J_f)$.

Thus $A \subset V(J_f)$, and hence $J_f \subset \sqrt{J_f} \subset I_X(V(J_f)) \subset I_X(A) = (\overline{y}, \overline{w})$, where we used Nullstellensatz as k is algebraically closed.

(b) Assume it were possible to write $f = \frac{a}{b}$ for $a, b \in \Gamma(X)$ where $b(P) \neq 0$ for every P where f is defined. Then $V(b)$ is precisely the set of poles of f: $V(b) = V(J_f) = V(\overline{y}, \overline{w})$, and so $\sqrt{(b)} = \sqrt{(\overline{y}, \overline{w})} \supset (\overline{y}, \overline{w})$. Thus $b \mid \overline{y}^k$ and $b \mid \overline{w}^l$. Let $b' \in k[x,y,z,w]$ be such that b is the image of b' in $\Gamma(X)$. Then there exists $h, j \in k[x,y,z,w]$ such that $b'h - y^k = (xw - yz)j$. As the y^k term on the right hand side has coefficient 0, we have that $b'h$ contains a y^k term with $k \geq 1$, i.e. $b' \mid y^k$. Similarly we get $b' \mid w^l$. But as $k[x,y,z,w]$ is UFD and $\gcd(y^k, w^l) = 1$, we have that b' is constant. Hence b is constant, but then $V(b) = V(b') \cap X = X$, however, f has poles at $V(y,w) \subset X$, contradicting $V(J_f) = V(b) = X$ being the pole set of f.
Alternatively, assume such b existed. Then we claim that there exists a point in $V(xw - yz) - V(y,w)$ such that b vanishes at the point. Now, $b(0,0,0,0) = 0$ as shown, so $b(0,y,0,w)$ is either constant 0 or has infinitely many zeros as a function $b(0,y,0,w) \in k[y,w]$ by problem 1.14 in Fulton. If it were non constant, there thus exists a nonzero point in $V(xw - yz) - V(y,w)$ such that b vanishes at the point. Hence b cannot be defined such $f = \frac{a}{b}$ everywhere where f is defined.

2.
(a) We must show that $\mathcal{O}_P(X)$ is nonempty, closed under subtraction and under multiplication.
Firstly, the function $1 = \frac{1}{1} \in k(X)$ is defined everywhere, so $1 \in \mathcal{O}_P(X)$, hence $\mathcal{O}_P(X)$ is nonempty.
Now assume $\frac{a}{b}, \frac{c}{d} \in \mathcal{O}_P(X)$. Then $b(P) \neq 0 \neq d(P)$. Now

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$$

by definition of subtraction in rings of fractions, and $(bd)(P) = b(P)d(P) \neq 0$ as both $b(P)$ and $d(P)$ are nonzero at P and k is a field and thus especially an integral domain, so there are no zero divisors. Thus $\frac{a}{b} - \frac{c}{d} \in \mathcal{O}_P(X)$.
For multiplication, we have

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

by definition of multiplication in rings of fractions. Now $(bd)(P) = b(P)d(P) \neq 0$ with the same argument as above.
Thus $\frac{a}{b} \cdot \frac{c}{d} \in \mathcal{O}_P(X)$.

(b) We have $R = \mathcal{O}_0(V(0))$, so by the previous exercise, R is a subring of $k(V(0))$.
Let $I \subset R$ be all elements of R that are non-units.
Assume $\frac{a}{b} \in I$. Since $\frac{a}{b} \in I \subset R$, we in particular have that $b(0) \neq 0$. Now, if $a(0) \neq 0$, then $\frac{b}{a} \in R$ and hence $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = \frac{1}{1} = 1$, so $\frac{a}{b}$ would be a unit. Thus by contraposition, we must have that if a is not a unit then $a(0) = 0$.
Conversely, suppose $a(0) = 0$. Then if $\frac{ab}{cd} = \frac{a}{b} \cdot \frac{c}{d} = \frac{1}{1} = 1$, we would have

$$0 = \frac{0}{\underbrace{c(0)d(0)}_{\neq 0}} = \frac{a(0)b(0)}{c(0)d(0)} = \frac{(ab)(0)}{(cd)(0)} = \frac{1(0)}{1(0)} = \frac{1}{1} = 1$$

which is a contradiction as R contains $1 \in k$ and is thus not the zero ring. Hence $\frac{a}{b}$ is not a unit.
Thus we see that $\frac{a}{b} \in R$ is a non-unit if and only if $a(0) = 0$, i.e.

$$I = \left\{ \frac{a}{b} \in k(x) \mid a, b \in k[x], a(0) = 0, b(0) \neq 0 \right\}$$

We must show that I is a subring and closed under left and right multiplication by elements of R.
Firstly, I is nonempty since $0 = \frac{0}{1} \in I$ by the above arguments.
Now, if $\frac{a}{b}, \frac{c}{d} \in I$ then

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

$(ad - bc)(0) = \underbrace{a(0)}_{=0} d(0) - b(0) \underbrace{c(0)}_{=0} = 0 - 0 = 0$ and $(bd)(0) = b(0)d(0) \neq 0$ since R $k(x)$ is an integral domain, we have $\frac{a}{b} - \frac{c}{d} \in I$.
Similarly, we have

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

and $(ac)(0) = a(0)c(0) = 0$ as both are zero, and $(bd)(0) = b(0)d(0) \neq 0$ as both are nonzero and $k(x)$ is an integral domain. Hence also $\frac{a}{b} \cdot \frac{c}{d} \in I$.
Thus I is a subring of R.

Now let $\frac{a}{b} \in I$ and $\frac{r}{s} \in R$. Then $\frac{a}{b} \cdot \frac{r}{s} = \frac{ar}{bs}$ and as $ar(0) = \underbrace{a(0)}_{=0} r(0) = 0$ and $(bs)(0) = \underbrace{b(0)}_{\neq 0} \underbrace{s(0)}_{\neq 0} \neq 0$ by definition of R, we have $\frac{a}{b} \cdot \frac{r}{s} \in I$.
Noting that $k(x)$ is commutative, we also get $\frac{r}{s} \cdot \frac{a}{b} \in I$, so I is indeed an ideal. By the definition/lemma in section 2.4 in Fulton, R is thus a local ring.

3: Assume I is prime in $\mathcal{O}_P(X)$. We claim that $I \cap \Gamma(X)$ is prime in $\Gamma(X)$.
If $a, b \in \Gamma(X)$ with $ab \in I \cap \Gamma(X)$, then $ab \in I$ in particular, and as I is prime in $\mathcal{O}_P(X)$ and $a, b \in \mathcal{O}_P(X)$, we have $a \in I$ or $b \in I$, so either $a \in I \cap \Gamma(X)$ or $b \in I \cap \Gamma(X)$. Thus $I \cap \Gamma(X)$ is prime in $\Gamma(X)$.

We now show that $I \cap \Gamma(X)$ generates I.
By problem 1.22, $\Gamma(X)$ is Noetherian, so choose generators $f_1, \ldots, f_r$ for the ideal $I \cap \Gamma(X)$ of $\Gamma(X)$. For any $f \in I \subset \mathcal{O}_P(X)$, there is $b \in \Gamma(X)$ with $b(P) \neq 0$ and $bf \in \Gamma(V)$, so $bf \in \Gamma(V) \cap I$, so $bf = \sum a_i f_i, a_i \in \Gamma(V)$ and so $f = \sum \left(\frac{a_i}{b}\right) f_i$.

Now, by problem 5 on homework 5, we have that there is a bijection between the algebraic subsets of X and radical ideals in $\Gamma(X)$. We claim that this induces a bijection between subvarieties of X and prime ideals in $\Gamma(X)$.
We do this two-step: firstly, by the solution to problem 5 homework 5, the bijection between radical ideals containing $I(X)$ and algebraic subsets of X is given by sending an algebraic subset $Z \subset X$ to $I(Z) \supset I(X)$ which is radical, and sending a radical ideal $J \supset I(X)$ to $V(J) \subset X$ which is algebraic.
Now by problem 1 homework 3, there is a bijection between radical ideals in $k[x_1, \ldots, x_n]$ containing

$I(X)$ and radical ideals in $\Gamma(X)$ given by the canonical homomorphism $\pi\colon k[x_1,\dots,x_n] \to \Gamma(X)$.
We thus find that if P is a prime ideal in $\Gamma(X)$ then $\pi^{-1}(P)$ is a radical ideal in $k[x_1,\dots,x_n]$ containing $I(X)$. We further claim it is prime.
Let $ab \in \pi^{-1}(P)$, then $\pi(a)\pi(b) \in P$ which is prime and hence $\pi(a) \in P$ or $\pi(b) \in P$, i.e. $a \in \pi^{-1}(P)$ or $b \in \pi^{-1}(P)$.
Similarly, if P' is a prime ideal in $k[x_1,\dots,x_n]$ containing $I(X)$ then we claim $P = \pi(P')$ is a prime ideal.
Suppose $ab \in P$, then since $a,b \in \Gamma(X)$, we can find $a',b' \in k[x_1,\dots,x_n]$ such that $\pi(a') = a$ and $\pi(b') = b$, so $\pi(a'b') \in P$, so there exists $p \in P'$ such that $\pi(a'b') = \pi(p)$ so $\pi(a'b' - p) = 0$ and thus $a'b' - p \in I(X) \subset P'$ so since $p \in P'$, we have $a'b' \in P'$, and since P' is prime, we have $a' \in P'$ or $b' \in P'$, so $a \in \pi(P') = P$ or $b \in \pi(P') = P$, so P is a prime ideals.

Now, consider a prime ideal J in $k[x_1,\dots,x_n]$ containing $I(X)$, then by the first bijection, this maps to $V(J) \subset V(I(X)) = X$, and $V(J)$ is irreducible since J is prime - by the proposition on lecture note 3, page 2.
Conversely, if $W \subset X$ is a subvariety of X then the first bijection maps this to the prime ideal $I(W) \supset I(X)$ - again using the aforementioned proposition.

Let I be a prime ideal in $\mathcal{O}_P(X)$. This corresponds to a prime ideal $I \cap \Gamma(X)$ in $\Gamma(X)$. The corresponding subvariety is $V\left(\pi^{-1}\left(I \cap \Gamma(X)\right)\right)$ which contains P if and only if $\pi^{-1}\left(I \cap \Gamma(X)\right)$ is contained in $I(P)$ if and only if $\pi\left(I(P)\right)$ contains $I \cap \Gamma(X)$. Now, I is a proper ideal by definition of being prime, so it contains no units, and hence it indeed only consists of fractions $\frac{a}{b}$ such that $a(P) = 0$.

If conversely, W is a subvariety of X that passes through P, then W corresponds to the prime ideal $I_X(W) = \pi\left(I(W)\right)$ in $\Gamma(X)$ whose elements vanish at P since if $\overline{f} \in \pi\left(I(W)\right)$ then $\overline{f}(P) = f(P) = 0$ as $f \in I(W)$ and $P \in W$. Now we claim that the ideal $\pi\left(I(W)\right)$ generates in $\mathcal{O}_P(X)$ is prime:

Lemma: Let R be a commutative ring with 1. Prime ideals in $D^{-1}R$ for a multiplicatively closed subset $D \subset R$ are precisely the subsets $D^{-1}P$ with P are prime ideal of R and $P \cap D = \varnothing$.

Proof: Suppose J is prime in $D^{-1}R$. Let $P = J \cap R$. P is prime in R since if $a,b \in R$ with $ab \in P = J \cap R$ then in particular, $ab \in J$ so as J is prime, $a \in J$ or $b \in J$ and hence $a \in J \cap R = P$ or $b \in J \cap R = P$, so P is prime.
If $d \in P \cap D$ then $\frac{d}{1} \in J$ and $\frac{1}{d} \in D^{-1}R$, so $\frac{1}{d}\frac{d}{1} = 1 \in J$, hence $J = D^{-1}R$, but J is prime and hence proper, contradiction. Thus $P \cap D = \varnothing$.

Now let $j \in J$, so $j = \frac{r}{d}$ for $r \in R$ and $d \in D$. Then $\frac{r}{1} = \frac{r}{d} \cdot \frac{d}{1} \in J$, so $r \in J \cap R = P$, so $\frac{r}{d} \in D^{-1}P$, so $J \subset D^{-1}P$.

Conversely, if $r \in D^{-1}P$ then there exist $p \in P = J \cap R$ and $d \in D$ such that $r = \frac{p}{d}$, hence in particular $\frac{p}{1} \in J$ and since J is an ideal, $\frac{p}{d} = \frac{1}{d}\frac{p}{1} \in J$, so $D^{-1}P \subset J$.

Therefore $D^{-1}P = J$, so all prime ideals in $D^{-1}R$ are of this form.

If P now is a prime in R with $P \cap D = \varnothing$, then we claim $D^{-1}P$ is prime in $D^{-1}R$.
Let $\frac{a}{b}, \frac{c}{d} \in D^{-1}R$ with $\frac{ac}{bd} \in D^{-1}P$. There exists p, d' with $d' \in D, p \in P$ such that $acd' = bdp \in P$, so either $ac \in P$ or $d' \in P$, however, by assumption, $P \cap D = \varnothing$, so $ac \in P$ and thus either $a \in P$ or $c \in P$. Therefore $\frac{a}{b} \in D^{-1}P$ or $\frac{c}{d} \in D^{-1}P$.

Since $\pi\left(I\left(W\right)\right)$ consists purely of non-units, and $\mathcal{O}_P(X)$ is the ring of fractions $R = \Gamma(X)$ with D, the multiplicatively closed set, being the set of functions not vanishing at P, we have $\pi\left(I\left(W\right)\right) \cap D = \varnothing$, so the ideal it generates in $\mathcal{O}_P(X)$ is prime.

We thus get the complete bijective correspondence between prime ideals in $\mathcal{O}_P(X)$ and prime ideals in $\Gamma(X)$ of functions that vanish on P. By composing with the other bijection, this gives the complete bijective correspondence between prime ideals in $\mathcal{O}_P(X)$ and subvarieties of X that pass through P.

4:

(a) Let $f = y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy$.
Then $f_y = 3y^2 - 2y + 6xy + 3x^2 + 2x$ and $f_x = 3x^2 - 2x + 6yx + 3y^2 + 2y$. Now if $f_y(P) = 0 = f_x(P)$, then $-2y + 2x = -2x + 2y$, so $x = y$. Thus f_y becomes $3y^2 + 6y^2 + 3y^2 + 2y - 2y = 12y^2$ and f_x becomes $12x^2$. These are both zero if and only if $y = x = 0$. Thus $V(f_x, f_y) = \{(0,0)\}$, so $(0,0)$ is the only singular point of $V(f)$.
Since $f_2 = -y^2 - x^2 + 2xy \neq 0$, f_2 is the lowest term of f, so 2 is the multiplicity of $(0,0) \in V(f)$.
The tangent cone is $V(f_2) = V\left(y^2 + x^2 - 2xy\right) = V((y - x)(y - x)) = V(y - x)$ which is a line.

(b) Let $g = x^4 + y^4 - x^2y^2$. Then $g_y = 4y^3 - 2x^2y$ and $g_x = 4x^3 - 2y^2x$. If $g_x(P) = 0 = g_y(P)$ for $P = (x, y)$ then $y(2y^2 - x^2) = 0 = x(2x^2 - y^2)$. If $2y^2 - x^2 = 0$, then $x = \pm\sqrt{2}y$ and if $2x^2 - y^2 = 0$ then $y = \pm\sqrt{2}x \in \left\{\pm 2y^2\right\}$, so $y = 0$ and so $x = 0$. Similarly, if $y = 0$ then $x(2x^2) = 0$ implies $x = 0$ and likewise if $x = 0$ we get $y = 0$.
Thus the only singular point of g is $(0,0)$.

Now the lowest term of g is g_4 since $g = g_4 = x^4 + y^4 - x^2y^2$, so the multiplicity of $(0,0) \in V(g)$ is 4, and letting $\omega = e^{i\frac{2\pi}{3}}$, we have $x^4 + y^4 - x^2y^2 = (y - i\omega x)(y + i\omega x)(y - i\omega^2 x)(y + i\omega^2 x)$, so

$$V(g) = V(y - i\omega x) \cup V\left(y + i\omega x\right) \cup V\left(y - i\omega^2 x\right) \cup V\left(y + i\omega^2 x\right).$$

each of which is a line.

(c) Let $h = x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1$. Then $h_x = 3x^2 - 6x + 3y$ and $h_y = 3y^2 - 6y + 3x$. If $h_x(x, y) = 0 = h_y(x, y)$ then $-3(y^2 - x^2) - 9x + 9y = 3(y - x)(-(y + x) + 3) = 0$, so either $y = x$ or $y + x = -3$. Now if $y = -3 - x$ then $0 = h_x = 3x^2 - 6x + 3(-3 - x)$ and $0 = h_y = 3y^2 - 6y + 3(-3 - y)$ implies $x, y \in \left\{\frac{3}{2} - \frac{\sqrt{21}}{2}, \frac{3}{2} + \frac{\sqrt{21}}{2}\right\}$, but then $x + y \neq -3$, so we find that $y = x$ is the only possibility. In this case $0 = h_x = 3x^2 - 3x = 3x(x - 1)$, so $y = x = 0$ or $y = x = 1$. And this also satisfies $h_y = 0$. Thus the two singular points are $(0,0)$ and $(1,1)$. Now, at $(0,0)$, h is 1, so $(0,0) \notin V(h)$ and thus it does not have a multiplicity or tangent cone defined.
However, $h(1,1) = 1 + 1 - 3 - 3 + 3 + 1 = 0$, so $(1,1) \in V(h)$. Now the multiplicity of $(1,1) \in V(h)$ is the multiplicity of $(0,0) \in V(\varphi^* h)$ with φ being the translation $(x, y) \to (x + 1, y + 1)$. Thus $V(\varphi^* h) = V(h(x + 1, y + 1))$. Now, $h(x+1, y+1) = (x+1)^3 + (y+1)^3 - 3(x+1)^2 - 3(y+1)^2 + 3(x+1)(y+1) + 1 = x^3 + 3xy + y^3$. This has lowest degree term $3xy$ which is degree 2, so the multiplicity of $(0,0)$ in $V(\varphi^* h)$ which is the degree of $(1,1)$ in $V(h)$ is 2. Now, the tangent cone to $V(h)$ at $(1,1)$ is the image under φ of the tangent cone of $V(\varphi^* h)$ at $(0,0)$. Now Now, the tangent cone of $V(\varphi^* h)$ at $(0,0)$ is $V(3xy) = V(x) \cup V(y)$. So the tangent cone of $V(h)$ at $(1,1)$ is $\varphi(V(x) \cup V(y)) = \varphi(V(xy)) = V(\varphi^*(xy)) = V(\varphi^* x \varphi^* y) = V((x + 1)(y + 1)) = V(x + 1) \cup V(y + 1) = \{(x, y) \mid x, y \in \mathbb{C}, x = -1 \vee y = -1\}$. So the tangent cone of $V(h)$ at $(1,1)$ is the union of the lines $V(x + 1)$ and $V(y + 1)$.

5:

(a) Since $(\varphi^* f)(P) = f(\varphi(P)) = f(Q) =$ as $Q \in V(f)$, we have $P \in V(\varphi^* f)$.

(b) Let T be the translation sending $(x, y) \to (x, y) + P$. The multiplicity of $V(\varphi^* f)$ at P is the multiplicity of $V(T^* \varphi^* f) = V\left((\varphi \circ T)^* f\right)$ at 0. Now, $\varphi \circ T$ is the map sending $0 \to P \to Q$, so this is precisely the multiplicity of $V(f)$ at Q since the composition of translation maps is a translation map (so $\varphi \circ T$ is a translation sending $(x, y) \to \varphi((x, y) + P) = (x, y) + P + (Q - P) = (x, y) + Q$.)

(c) First, assume $P, Q = (0,0)$. Since φ is a polynomial map, we can write $\varphi = (\varphi_1, \varphi_2)$ with $\varphi_1, \varphi_2 \in k[x, y]$ such that $\varphi(R) = (\varphi_1(R), \varphi_2(R))$ for all points $R \in \mathbb{A}^2$. Since $\varphi(0,0) = (\varphi_1(0,0), \varphi_2(0,0)) = (0,0)$, we have that $\varphi_1(0,0) = 0$ and $\varphi_2(0,0) = 0$, so both φ_1 and φ_2 have zero constant term. Then composing with φ does not decrease the lowest degree term of f since for any homogenous polynomial f_i, $f_i \circ \varphi$ will have terms of degree $\geq i$ only. So the multiplicity of $V(\varphi^* f)$ at $(0,0)$ is greater than or equal to the multiplicity of $V(f)$ at $(0,0)$.

Now assume either $Q \neq (0,0)$ or $P \neq (0,0)$. Let T be the translation sending $(0,0) \to Q$ and S the translation sending $(0,0) \to P$. Then by (b), we have that the multiplicity of $V(T^* \varphi^* f)$ at $(0,0)$ is the same as the multiplicity of $V(\varphi^* f)$ at Q, and the multiplicity of $V(S^* f)$ at $(0,0)$ is the same as the multiplicity of $V(f)$ at P. Now, as S is a translation, it is invertible and its inverse is a translation sending $P \to (0,0)$. Thus $R = S^{-1} \circ \varphi \circ T$ is a polynomial map sending $(0,0) \to Q \to P \to (0,0)$,

so letting $\psi = (S)^* f = f \circ S$ which is a polynomial map as the composition of polynomial maps is a polynomial map by a previous homework exercise. Now by by the case $P, Q = (0,0)$ before, we have that the multiplicity of $V(R^*\psi) = V\left(\left(S^{-1} \circ \varphi \circ T\right)^* \psi\right) = V\left(T^* \varphi^* \left(S^{-1}\right)^* \psi\right)$ is greater than or equal to the multiplicity of $V(S^* f) = V(\psi)$ at $(0,0)$ which by (b) is equal to the multiplicity of $V(f)$ at Q. Now $V\left(T^* \varphi^* \left(S^{-1}\right)^* \psi\right) = V\left(T^* \varphi^* \left(f \circ S \circ S^{-1}\right)\right) = V\left(T^* \varphi^* f\right)$, and by an application of (b) again, we find that the multiplicity of $V(R^*\psi)$ at $(0,0)$ is equal to the multiplicity of $V(\varphi^* f)$ at P.
Thus we find that the multiplicity of $V(\varphi^* f)$ at P is greater than or equal to the multiplicity of $V(f)$ at Q.