1. Double and Total Complexes

Definition 1.1 (Double complex)**.** A *double complex* (or *bicomplex*) in an abelian category $\mathcal{A}$ is a family $\{C_{p,q}\}$ of objects of $\mathcal{A}$, together with maps

$$d^h \colon C_{p,q} \to C_{p-1,q} \quad \text{and} \quad d^v \colon C_{p,q} \to C_{p,q-1}$$

such that $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$.

It is useful to picture the double complex as a lattice in which the maps d^h go horizontally, the maps d^v go vertically, and each square anticommutes.

Each row C_{*q} and each columns C_{p*} is a chain complex.
We say that the double complex C is *bounded* if C has only finitely many nonzero terms along each diagonal line $p+q=n$. For example, if C is concentrated in the first quadrant of the plane (a *first quadrant double complex*).

1.0.1. *Sign Trick.* Are the maps d^v and d^h maps in Ch?
Because of anticommutativity, the chain map conditions fail, but we can construct chain maps f_{*q} from $C_{*,q}$ to $C_{*,q-1}$ by introducing signs:

$$f_{p,q} = (-1)^p d^v_{p,q} \colon C_{p,q} \to C_{p,q-1}.$$

Using this sign trick, we can identify the category of double complexes with the category $\mathrm{Ch}\,(\mathrm{Ch})$.

1.0.2. *Total Complexes.* To see why the anticommutativity condition $d^v d^h + d^h d^v = 0$ is useful, we define the *total complexes* $\mathrm{Tot}(C) = Tot^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ as follows:

Definition 1.2 (Total complexes)**.** We define

$$\mathrm{Tot}^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q} \quad \text{and} \quad \mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

The formula $d = d^h + d^v$ define maps

$$d\colon \mathrm{Tot}^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q} \quad \text{and} \quad d\colon \mathrm{Tot}^{\oplus}(C)_n \to \mathrm{Tot}^{\oplus}(C)_{n-1}$$

such that $d \circ d = 0$, making $\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ into chain complexes.

Exercise 1.3. Check that $d = d^h + d^v$ define maps as claimed.

Solution. Let $(\alpha_{p,q}) \in \mathrm{Tot}^{\Pi}(C)_n$, so $p+q=n$. Then $d\left((\alpha_{p,q})\right) = d^h\left((\alpha_{p,q})\right) + d^v\left((\alpha_{p,q})\right) = (\alpha_{p-1,q}) + (\alpha_{p,q-1}) \in \prod_{p+q=n-1} C_{p,q}$. Clearly, this also works for direct products since the number of non-zero terms under d just multiplies by 2, hence is still finite. We also want to show that $d \circ d = 0$. For this, note that

$$\begin{aligned} d \circ d\,(\alpha) = d\left(d^h(\alpha) + d^v(\alpha)\right) &= d^h\left(d^h(\alpha) + d^v(\alpha)\right) + d^v\left(d^h(\alpha) + d^v(\alpha)\right) \\ &= d^h d^h(\alpha) + d^h d^v(\alpha) + d^v d^h(\alpha) + d^v d^v(\alpha) \\ &= 0. \end{aligned}$$

2. Introduction to Spectral Sequences

Consider the problem of computing the homology of the total chain complex $T_* = \mathrm{Tot}(E_{**})$ where E_{**} is a first quadrant double complex.
Firstly, it is convenient to forget the horizontal differentials and add a superscript zero, retaining only the vertical differentials d^v along the columns E^0_{p*}.
Let E^1_{pq} be the vertical homology $H_q\left(E^0_{p*}\right)$ at the (p,q) spot.

3. Filtrations

Definition 3.1 (Filtered R-module)**.** A *filtered R-module* is an R-module A with an increasing sequence of submodules $\{F_p\}_{p\in\mathbb{Z}}$ such that $F_pA \subset F_{p+1}A$ for all p and such that $\bigcup_p F_pA = A$ and $\bigcap_p F_pA = \{0\}$.
A filtration is said to be *bounded* if $F_pA = \{0\}$ for p sufficiently small and $F_pA = A$ for p sufficiently larger.

Definition 3.2 (Associated graded module)**.** The *associated graded module* is defined by $G_pA = F_pA/F_{p-1}A$.

Definition 3.3 (Filtered chain complex)**.** A *filtered chain complex* is a chain complex (C_*, ∂) together with a filtration $\{F_pC_i\}_{p\in\mathbb{Z}}$ of each C_i such that the differential preserves the filtration, i.e., s.t. $\partial\left(F_pC_i\right) \subset F_pC_{i-1}$.

Note that we, in particular, obtain an induced differential $\partial\colon G_pC_i \to G_pC_{i-1}$ by the universal property of cokernels

$$\begin{tikzcd}
F_pC_i \arrow[r, "\partial"] \arrow[d] & F_pC_{i-1} \arrow[d] \\
F_{p-1}C_i \arrow[r, "\partial"] \arrow[d, "\mathrm{coker}"] & F_{p-1}C_{i-1} \arrow[d, "\mathrm{coker}"] \\
G_pC_i \arrow[r, dashed, "\partial"] & G_pC_{i-1}
\end{tikzcd}$$

so we obtain an associated graded chain complex G_pC_*.

The filtration on C_* also induces a filtration on the homology of C_* by

$$F_pH_i(C_*) = \{\alpha \in H_i(C_*) \mid (\exists x \in F_pC_i) : \alpha = [x]\}\,.$$

This filtration has associated graded pieces $G_pH_i(C_*)$ which, in favorable cases, determine $H_i(C_*)$.

3.1. **Example.** Suppose we have a chain complex C_* and a filtration consisting of a single F_0C_*, so $F_nC_* = 0$ if $n < 0$ and $F_nC_* = F_0C_*$ if $n \geq 0$.
Then $G_nC_* = 0$ for $n \neq 0$ and $G_0C_* = F_0C_*$ and