Definition 0.1. For $z \in \mathbb{C} - \{0\}$, we let

$$\arg z = \left\{\theta \in \mathbb{R} \mid |z|e^{i\theta} = z\right\}$$

Definition 0.2 (Principal argument)**.** We define the principal argument of $z \in \mathbb{C} - \{0\}$ as the unique element

$$\operatorname{Arg} z \in \arg z \cap (-\pi, \pi]$$

Definition 0.3 (Argument function)**.** An argument function for a subset $A \subset \mathbb{C} - \{0\}$ is a function $\theta \colon A \to \mathbb{R}$ such that $\theta(z) \in \arg z$ for all $z \in A$.

Lemma 0.4. Arg *is continuous on* $\mathbb{C}_\pi = \left\{re^{i\pi} \mid r \geq 0\right\}$.

Proof. We have that $\operatorname{Arg} z$ maps $\mathbb{C}_\pi$ onto $(-\pi, \pi)$, and

$$\begin{aligned}
\operatorname{Arg} z &= \operatorname{Arccos} \frac{x}{|z|}, \quad z = x + iy, y > 0 \\
\operatorname{Arg} z &= \operatorname{Arctan} \frac{x}{y}, \quad z = x + iy, x > 0 \\
\operatorname{Arg} z &= \operatorname{Arcsin} \frac{y}{|z|}, \quad z = x + iy, y < 0
\end{aligned}$$

We have that $\operatorname{Arccos} \frac{x}{|z|}$ and $\operatorname{Arctan} \frac{x}{y}$ agree on $\{x + iy \in \mathbb{C} \mid x, y > 0\}$ and that $\operatorname{Arctan} \frac{x}{y}$ and $\operatorname{Arcsin} \frac{y}{|z|}$ agree on $\{x + iy \in \mathbb{C} \mid x > 0, y < 0\}$. All these are C^∞ functions, so in particular continuous, hence they define a continuous function on $\mathbb{C}_\pi$. $\square$

Definition 0.5 (Argument function for $\mathbb{C}_\alpha$)**.** Likewise, if $\alpha \in \mathbb{R}$, we can define

$$\mathbb{C}_\alpha = \mathbb{C} - \left\{re^{i\alpha} \mid r \geq 0\right\}.$$

Then we can define

$$\operatorname{Arg}_\alpha \colon \mathbb{C}_\alpha \to \mathbb{R}$$

by

$$\operatorname{Arg}_\alpha(z) = \operatorname{Arg}\left(e^{i(\pi - \alpha)} z\right) + \alpha - \pi.$$

As a composition of continuous maps, $\operatorname{Arg}_\alpha$ is continuous on $\mathbb{C}_\pi$.

Proposition 0.6. *There exist a continuous argument function on* $A \subset S^1 \subset \mathbb{C}$ *if and only if* $A \neq S^1$.

Proof. We first show that there does not exist a continuous argument function on S^1. Suppose there exists a continuous argument function $\theta \colon S^1 \to \mathbb{R}$. Since S^1 is compact and path-connected, the image of S^1 under θ must be a closed interval, $[a, b]$. As θ is bijective too, it is a homeomorphism. But removing any point of S^1 leaves it path-connected, while removing a point in the interior of $[a, b]$ leaves it separated, and thus not connected. Since connectedness and path-connectedness are topological properties, S^1 is not homeomorphic to $[a, b]$, so no such argument function exists.
Now, conversely, if $A \neq S^1$, then we can pick a point $e^{i\alpha} \in S^1 - A$. Then $\operatorname{Arg}_\alpha$ is a continuous argument function for A. $\square$

Proposition 0.7. *Let* $\theta \colon A \to \mathbb{R}$ *be a continuous argument function where* $A \subset \mathbb{C} - \{0\}$ *is path-connected. Then for any* $p \in \mathbb{Z}$, $\theta + 2\pi p$ *is also a continuous argument function for* A*, and all argument functions for* A *are of this form.*

Proof. Any such function is clearly a continuous argument function.
Now suppose $\gamma \colon A \to \mathbb{R}$ is another continuous argument function for A. Then $\theta - \gamma$ takes values in $2\pi\mathbb{Z}$ for all $z \in A$, and since this set is discrete and $\theta - \gamma$ is continuous, we have that $\theta - \gamma$ is constant, so $\gamma = \theta + 2\pi p$ for some $p \in \mathbb{Z}$. $\square$