STRING TOPOLOGY VIA GEOMETRIC INTERSECTION

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Contents

1. Orientations

We begin by trying to develop the notion of orientation and some of the connected theorems.

For this section, we will closely follow [1], [4] and [5], and the section is essentially a collection of different parts of the different books with added details in some places.

Definition 1.1 (Local Homology Group)**.** For $h_*(-)$ a homology theory and an n-manifold M, groups of the form $h_k(M, M-\{x\})$ are called *local homology groups.*

For a chart $\varphi\colon U \to \mathbb{R}^n$ on M centered at x, we get by excision that

$$h_k(M, M-\{x\}) \cong h_k\left(U, U-\{x\}\right) \xrightarrow{\varphi_*} h_k\left(\mathbb{R}^n, \mathbb{R}^n - \{0\}\right).$$

Hence for singular homology, we obtain $H_n\left(M, M-\{x\}\,; G\right) \cong G$.

Definition 1.2 (Local R-orientation)**.** Let R be a commutative ring. A generator of $H_n\left(M, M-\{x\}\,; R\right) \cong R$ is called a *local R-orientation* of M about x.

Let $K \subset L \subset M$. The homomorphism $r^L_K\colon h_k(M, M-L) \to h_k(M, M-K)$ induced by inclusion is called restriction. We write r^L_x when $K = \{x\}$.

Proposition 1.3. *When A is a compact, convex set contained in some chart $\mathbb{R}^n \subset M$, then r^A_x is an isomorphism for each $x \in A$ and the groups are isomorphic to the coefficient group G.*

Proof. A is contained in the interior of some closed n-disk $D \subset \mathbb{R}^n \subset M$. Thus there is a commutative diagram

$$\begin{tikzcd}
h_n(M, M-A) \arrow[r] & h_n(M, M-\{x\}) \\
h_n(\mathbb{R}^n, \mathbb{R}^n - A) \arrow[r] \arrow[u, "\cong"] & h_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \arrow[u, "\cong"'] \\
h_n(D, \partial D) \arrow[r, equal] \arrow[u, "\cong"] & h_n(D, \partial D) \arrow[u, "\cong"']
\end{tikzcd}$$

□

1.0.1. *The covering map perspective.*

Definition 1.4 (Orientation bundle - the covering map perspective)**.** We construct a covering $\omega\colon h_k(M, M-\bullet) \to M$. Define

$$h_k(M, M-\bullet) = \bigsqcup_{x \in M} h_k(M, M-\{x\})$$

where $h_k(M, M-\{x\})$ is the fiber over x and is given the discrete topology.
Let U be an open neighborhood of x such that r^U_y is an isomorphism for each $y \in U$. Define bundle charts

$$\varphi_{x,U}\colon U \times G \to \omega^{-1}(U), \quad (y, a) \mapsto r^U_y \left(r^U_x\right)^{-1}(a).$$

We then give $h_k(M, M-\bullet)$ the topology that makes $\varphi_{x,U}$ in a homeomorphism onto an open subset. In particular, since $h_k(M, M-x)$ is given the discrete topology, this is equivalent to the map $\varphi_{x,U}(-, \alpha)$ being a homeomorphism onto an open subset for each $\alpha \in h_k(M, M - x)$. It then remains to show that the transition maps

$$\varphi_{y,V}^{-1}\varphi_{x,U} \colon (U \cap V) \times h_k(M, M - \{x\}) \to (U \cap V) \times h_k(M, M - \{y\})$$

are continuous.
Let $z \in U \cap V$, and choose W such that $z \in W \subset U \cap V$ and r_w^W is an isomorphism for each $w \in W$.
Consider the diagram

$$\begin{tikzcd}
h_k(M, M - x) & h_k(M, M - U) \arrow[l, "r_x^U"'] \arrow[r, "r_w^U"] \arrow[d, "r_W^U"] & h_k(M, M - w) \\
 & h_k(M, M - W) \arrow[ur, "r_w^W"] & h_k(M, M - V) \arrow[l, "r_W^V"] \arrow[u, "r_w^V"'] \arrow[d, "r_y^V"] \\
 & & h_k(M, M - y)
\end{tikzcd}$$

Let $\varphi_{x,U,p} \colon h_k(M, M - x) \to \omega^{-1}(p)$ be defined by

$$\varphi_{x,U,p}(y) = \varphi_{x,U}(p, y).$$

Then for $w \in U \cap V$, we have

$$\varphi_{x,U,w}^{-1}\varphi_{y,V,w} = r_y^V (r_W^V)^{-1}(r_w^W)^{-1} r_w^W r_W^U (r_x^U)^{-1} = r_y^V (r_W^V)^{-1} r_W^U r_x^U$$

Firstly, this coincides with the operation of an element of the coefficient group G since it is an isomorphism $G \to G$, and secondly, note that this does not depend on w, so the map

$$g_{x,U,y,V} \colon U \cap V \to G$$

defined by $g_{x,U,y,V}(p) = \varphi_{x,U,p}^{-1}\varphi_{y,V,p}$ is constant, hence continuous.

Thus ω is indeed a covering map.
But even moreso, the fibers are groups, so for $A \subset M$, denote by $\Gamma(A)$ the set of continuous sections over A of ω. If s and t are section, we can define $(s + t)(a) = s(a) + t(a)$. Then $s + t$ is again continuous, hence $\Gamma(A)$ is an abelian group.
Denote by $\Gamma_c(A) \subset \Gamma(A)$ the subgroup of sections with compact support, i.e., the sections which have values 0 in the fiber away from a compact set.

Proposition 1.5. *Let $z \in h_k(M, M - U)$. Then $y \mapsto r_y^U z \in h_k(M, M - y) \subset h_k(M, M - \bullet)$ is a continuous section of ω over U.*

Proof. The map $U \to U \times G$ by $y \mapsto (y, r_x^U z)$ is constant in the second coordinate, hence clearly continuous. Now composing with $\varphi_{x,U}$ gives us the section in question. □

1.0.2. *Local Homology in close-by points.*

Lemma 1.6. *Let z, z' be cycles in $\Delta_n(M, M - x; G)$. Then a neighborhood V of x exists such that z, z' are cycles in $\Delta_n(M, M - p; G)$ for all $p \in V$. If the homology classes of z, z' agree at x, i.e., $[z]_x = [z']_x \in H_n(M, M - x; G)$, then they agree at all points q in a neighborhood $V' \subset V$.*

Remark. Hence $H_n(M, M-x; G) \cong= \lim_{\to} H_n(M, M-U; G)$ taken over neighborhoods of U containing x with maps the inclusions (We will introduce what this means later).

Proof. The chains $\partial z, \partial z'$ are finite linear combinations of simplices σ with $\operatorname{im}\sigma \subset M - x$. Since $\operatorname{im}\sigma$ is compact, there is a neighborhood V_σ of P such that $\operatorname{im}\sigma \subset M - V_\sigma$. Using finiteness, $V = \bigcap_\sigma V_\sigma$ is then an open neighborhood of x such that $\partial z, \partial z' \in \Delta_n(M, M-V; G)$. If $[z]_x = [z']_x$, then there exists a chain ∂c such that $z - z' - \partial c \in \Delta_n(M - x; G)$. So as we just saw, there exists a neighborhood V' of x such that $z - z' - \partial c \in \Delta_n(M - V'; G)$, and we may take $V' \subset V$. Thus for all $q \in V'$, $z - z' - \partial c \in \Delta_n(M - q; G)$. $\square$

1.0.3. *The topological situation - another perspective.*

Proposition 1.7. *We take the same orientation bundle as before:*

$$\omega \colon H_n(M, M - \bullet; G) \to M.$$

For the topology on $H_n(M, M-\bullet; G)$, we consider pairs (V, z) where V is an open subset of M and $z \in Z_n(M, M-V; G)$ is a cycle. Then define

$$V_z = \left\{ [z]_x \in H_n(M, M-x; G) \mid x \in V \right\}.$$

The collections of all such V_z is then the base of the topology on $H_n(M, M-\bullet; G)$. With respect to this topology, the map ω is a covering map. Furthermore, the maps $(u, v) \mapsto u \pm v$ of $D = \{(u, v) \in H_n(M, M-\bullet; G) \times H_n(M, M-\bullet; G) \mid \omega u = \omega v\}$ into $H_n(M, M-\bullet; G)$ is continuous. Furthermore, the map

$$\beta \colon H_n(M, M-\bullet; \mathbb{Z}) \to \mathbb{Z}, \quad \beta(u) = \|u\|$$

is continuous, i.e., locally constant. Hence

$$\tilde{M} = \tilde{M}(0) \oplus \tilde{M}(1) \oplus \tilde{M}(2) \oplus \dots$$

where $\tilde{M} = H_n(M, M-\bullet)$ and $\tilde{M}(n) = \beta^{-1}(n)$. The restricted maps $\omega|_{\tilde{M}(n)} \colon \tilde{M}(n) \to M$ are covering maps.

Proof. Let $u \in H_n(M, M-\bullet; G)$ lies in some V_z as defined earlier: if $z \in Z_n(M, M-x; G)$ represents u, then by Lemma 1.6, there exists an open neighborhood V of x, such that $z \in Z_n(M, M-V; G)$, hence $u = [z]_x \in V_z$ by definition.
Now if $u \in V'_{z'} \cap V''_{z''}$, then by definition, $u = [z']_x = [z'']_x$, so by Lemma 1.6, there exists some neighborhood $V \subset V' \cap V''$ such that $[z']_p = [z'']_p$ for all $p \in V$, so for a representative z (for example, either z' or z''), we have $[z]_p = [z']_p = [z'']_p$ for all $p \in V$, so $u \in V_z \subset V'_{z'} \cap V''_{z''}$. Thus the set of all V_z is a basis.

Next, we show that ω is a local homeomorphism. First, it maps V_z onto V bijectively by construction, hence ω is open and locally bijective. It remains to show continuity. If W is an open neighborhood of $x = \omega(u)$, then u lies in some V_z, and hence $(V \cap W)_z$ is a neighborhood of u which maps into W under ω. Hence ω is continuous.

Next, the map $(u, u') \mapsto u \pm u'$ takes $D \cap (V_z \times V_{z'})$ homeomorphically onto $V_{z \pm z'}$. We just saw that $D \cap (V_z \times V_{z'}) \cong V_z \cong V \cong V_{z \pm z'}$, which takes $(u, u') \mapsto u \pm u'$. Thus it is, in particular, continuous.

Lastly, we must show that β is locally constant and that ω is a covering map.

Firstly, given $x \in M$, let V be the interior of some closed ball around x. Since r_x^V is an isomorphism, we have that, for $z \in Z_n(M, M - V; \mathbb{Z})$, $[z]_p = r_p^V[z]$ for each $p \in V$, so $\beta\left([z]_p\right) = \|[z]_p\| = \|r_p^V[z]\| = \|[z]\|$, hence β is independent of p in V, thus constant on V_z.

Note. Importantly, the norm is in a sense, the best we can do in the general situation because we only know that r_p^V is an isomorphism, *not* the identity. It could swap signs, in particular. We will see that this failure is at the heart of the notion of orientability.

Lastly, if $[z]$ is a generator of $H_n(M, M - V; \mathbb{Z}) \cong \mathbb{Z}$, then $\omega^{-1}(V) = \bigcup_{g \in G} V_{z \otimes g}$ is a decomposition into disjoint open sets $V_{z\otimes g}$, each of which map homeomorphically onto V, so ω and each $\omega|_{\tilde{M}(n)}$ is a covering map. $\square$

1.1. **Homological Orientation.** If we specify to singular homology with coefficient group R, and again let M be an n-manifold and $A \subset M$, then we can define an orientation along A as follows

Definition 1.8 (R-orientation of M along A)**.** An *R-orientation of M along A* is a section $s \in \Gamma(A; R)$ of $\omega\colon H_n(M, M - \bullet; R) \to M$ such that $s(a) \in H_n(M, M - a; R) \cong R$ is a generator for each $a \in A$.
I.e., it is a section s such that $\beta \circ s(x) = 1$ for all $x \in A$.
Thus s glues together the local orientations in a continuous manner.
When $A = M$, we call s an *R-orientation of M* .

Note. If s is a nowhere vanishing section, then $a \mapsto \frac{s(a)}{\|s(a)\|}$ is an orientation.

Definition 1.9 (Orientation covering)**.** Let $\mathrm{Ori}(M) \subset H_n(M, M - \bullet; \mathbb{Z})$ be the subset of all generators of all fibers. Then the restriction $\mathrm{Ori}(M) \to M$ of ω gives a 2-fold covering of M, called the *orientation covering* of M.

Proposition 1.10. *The following are equivalent:*

1. *M is orientable*
2. *M is orientable along compact subsets.*
3. *The orientation covering is a trivial 2-fold covering map.*
4. *The covering $\omega\colon H_n(M, M - \bullet; \mathbb{Z}) \to M$ is a trivial covering map.*

Proof. (1) $\implies$ (2) is a subcase.
(2) $\implies$ (3). The orientation covering is trivial if and only if the covering over each component is trivial, so we may assume that M is connected. Now, if a 2-fold covering $\tilde{M} \to M$ is trivial, then $\tilde{M}$ splits as $M \times \{p, q\}$, and so $\tilde{M}$ cannot be connected. Conversely, if $\tilde{M}$ is not connected, then the covering restricted to each component must be a covering map, so the covering splits as a trivial covering.
Suppose then that $\mathrm{Ori}(M) \to M$ is non-trivial. Since $\mathrm{Ori}(M)$ is then connected, we can choose a path γ in $\mathrm{Ori}(M)$ between two points of a given fiber. The image S of such a path is compact and connected, and the covering is non-trivial over S, so by assumption (2), the orientation covering has a section s over S, but then $\gamma(0) = s\left(\omega(\gamma(0))\right) = s\left(\omega\left(\gamma(1)\right)\right) = \gamma(1)$, which gives a contradiction.
(3) $\implies$ (4).
Let $s\colon M \to \mathrm{Ori}(M) \cong M \times \{-1, 1\}$ be the section $m \mapsto (m, 1)$.

Now define a map $\varphi\colon M \times \mathbb{Z} \to H_n(M, M - \bullet; \mathbb{Z})$ by $\varphi(m, k) = ks(m)$. This is a bijective map by assumption on s being a section. It is furthermore continuous since s is continuous and since fiber-wise operations in $H_n(M, M - \bullet; \mathbb{Z})$ is continuous. Furthermore, it is also a morphism between coverings since it commutes with the projections: $\pi_M = \omega \circ \varphi$.
Lastly, one must show that it also has a continuous inverse. For this, we may take an open basis set in $M \times \mathbb{Z}$ - say $U \times \{k\}$, where $\bar{U}$ is a convex subset of $\mathbb{R}^n \subset M$. Since φ is bijective, we obtain that $\varphi(U \times \{k\}) = ks(U) = U_\alpha$ if we choose α to be the element in $H_n(M, M - U) \cong \mathbb{Z}$ which maps to k under $r_{x,U}$ for $x \in U$. And by assumption, U_α is a basis open set for the topology on $H_n(M, M - \bullet; \mathbb{Z})$.
Hence φ is a homeomorphism, and even an isomorphism of covering spaces in the sense that $\pi_M = \omega \circ \varphi$.

Note. We could also say that it is trivial since every point is in the image of some section.

(4) $\implies$ (1) : If ω is trivial, then it has a section with constant value in the set of generators.

□

1.2. **Homology in the Dimension of the Manifold.** Let M be an n-manifold and $A \subset M$ a closed subset. We will in this section use singular homology with coefficients in an abelian group G.

Proposition 1.11. *For each $\alpha \in H_n(M, M - A; G)$, the section*

$$J^A(\alpha)\colon A \to H_n(M, M - \bullet; G), \quad x \mapsto r_x^A(\alpha)$$

of ω over A is continuous and has compact support.

Proof. Choose a representative $c \in \Delta_n(M; G)$ representing α. There exists a compact set K such that c is contained in K. Suppose $A - K$ is nonempty, and let $x \in A - K$. Then the image of c under

$$\Delta_n(K; G) \to \Delta_n(M; G) \to \Delta_n(M, K; G) \to \Delta_n(M, M - x; G)$$

is zero since $K \subset M - x$. Since this image represents r_x^A, the support of $J^A(\alpha)$ is contained in $A \cap K$ which is compact.
If $A - K$ is empty, K contains A, and then the support of $J^A(\alpha)$ is a closed subset of a compact space, hence compact.

The continuity follows from the more general case of Proposition 1.5. □

Thus we obtain a homomorphism

$$J^A\colon H_n(M, M - A; G) \to \Gamma_c(A; G), \quad \alpha \mapsto \big(x \mapsto r_x^A(\alpha)\big)$$

which is also natural with respect to inclusions: i.e., for $A \subset A' \subset M$,

$$\begin{tikzcd} H_n(M, M - A'; G) \arrow[r, "J^{A'}"] \arrow[d, "i_*"] & \Gamma_c(A'; G) \arrow[d, "\rho"] \\ H_n(M, M - A; G) \arrow[r, "J^A"] & \Gamma_c(A; G) \end{tikzcd}$$

commutes.

1.3. Direct Limits.

Definition 1.12 (Directed set)**.** A *directed set* D is a partially ordered set such that, for any two elements α and β of D, there is a $\tau \in D$ with $\tau \geq \alpha$ and $\tau \geq \beta$.

Definition 1.13. Let D be a directed set and G_α an abelian group defined for each $\alpha \in D$. Suppose we are given homomorphisms $f_{\beta,\alpha} \colon G_\alpha \to G_\beta$ for each $\beta > \alpha$ in D. Assume that for all $\gamma > \beta > \alpha$ in D, we have $f_{\gamma,\beta} f_{\beta,\alpha} = f_{\gamma,\alpha}$. Such a system is called a *direct system* of abelian groups. Then $G = \lim_{\rightarrow} G_\alpha$ is defined to be the quotient group of the direct sum $G = \bigoplus G_\alpha$ modulo the relations $f_{\beta,\alpha}(g) \sim g$ for all $g \in G_\alpha$ and all $\beta > \alpha$.

Note. Hence the direct limit is just the colimit of the direct system.

The inclusions $G_\alpha \hookrightarrow \bigoplus G_\alpha$ induce homomorphisms $i_\alpha \colon G_\alpha \to \lim_{\rightarrow} G_\alpha$ and $i_\beta \circ f_{\beta,\alpha} = i_\alpha$. Moreover, for any $g \in G$, there is a $g_\alpha \in G_\alpha$ for some α such that $g = i_\alpha(g_\alpha)$. Also, for any index $\alpha \in D$, and element $g_\alpha \in G_\alpha$, we have $i_\alpha(g_\alpha) = 0$ if and only if there exists a $\beta \geq \alpha$ such that $f_{\beta,\alpha}(g_\alpha) = 0$. These properties characterize the direct limit:

Proposition 1.14. *Suppose we are given an abelian group A with homomorphisms $h_\alpha \colon G_\alpha \to A$ such that the cocone commutes. Since $\lim_{\rightarrow} G_\alpha$ is the colimit, we have a unique induced homomorphism $h \colon \lim_{\rightarrow} G_\alpha \to A$. Then*

(1) $\operatorname{im} h = \{a \in A \mid a = h_\alpha(g)$ *for some g and α*$\} = \bigcup \operatorname{im} h_\alpha$.
(2) $\ker h = \{g \in \lim_{\rightarrow} G_\alpha \mid \exists \alpha$ *and* $g_\alpha \in G_\alpha \colon g = i_\alpha(g_\alpha)$ *and* $h_\alpha(g_\alpha) = 0\} = \bigcup i_\alpha(\ker h_\alpha)$.

Proof. Define $h(g_\alpha) = h_\alpha(g_\alpha)$. Then if $f_{\beta,\alpha}(g_\alpha) \sim g_\alpha$, we have $h(g_\alpha) = h_\alpha(g_\alpha) = h_\beta \circ f_{\beta,\alpha}(g_\alpha) = h(f_{\beta,\alpha}(g_\alpha))$, so h respects the equivalence relations, thus it is well-defined.
Now property (1) is clear by the way we defined h.
As for (2), note that if g represents the equivalence class of g_α and $h(g) = 0$, then $h_\alpha(g_\alpha) = 0$ which is what (2) is saying. □

Corollary 1.15. *In the situation of Proposition 1.14, $h \colon \lim_{\rightarrow} G_\alpha \to A$ is an isomorphism if and only if the following two statements hold true:*

(1) $\forall a \in A, \exists \alpha \in D$ *and* $g_\alpha \in G_\alpha \colon h_\alpha(g_\alpha) = a$, *and*
(2) *if* $h_\alpha(g_\alpha) = 0$ *then* $\exists \beta > \alpha \colon f_{\beta,\alpha}(g_\alpha) = 0$.

Theorem 1.16. *The direct limit is an exact functor. So if we have direct systems $\{A'_\alpha\}, \{A_\alpha\}$ and $\{A''_\alpha\}$ based on the same directed set, and if we have an exact sequence $A'_\alpha \to A_\alpha \to A''_\alpha$ for each α, where the maps commute with the ones defining the direct systems, then the induced sequence*

$$\lim_{\rightarrow} A'_\alpha \to \lim_{\rightarrow} A_\alpha \to \lim_{\rightarrow} A''_\alpha$$

is exact.

Proof. We have the following diagram, where all maps commute.

$$\begin{array}{ccccc} A'_\beta & \longrightarrow & A_\beta & \longrightarrow & A''_\beta \\ \downarrow & & \downarrow & & \downarrow \\ \lim_{\rightarrow} A'_\alpha & \longrightarrow & \lim_{\rightarrow} A_\alpha & \longrightarrow & \lim_{\rightarrow} A''_\alpha \end{array}$$

Suppose $a \in \lim_{\to} A_*$ is mapped to zero in $\lim_{\to} A''_*$. Then there exists $g \in \lim_{\to} A_\alpha$ such that there exists β and $g_\beta \in A_\beta$ such that $g = i_\beta(g_\beta)$ and $h_\beta(g_\beta) = 0$.
Recall here that h_β is a homomorphism $A_\beta \to \lim_{\to} A''_*$ and i_β is the inclusion $G_\beta \to \lim_{\to} G_\alpha$.
By commutativity of the diagram, there then exists $k_\beta \in A'_\beta$ such that $i_\beta\left(d_\beta(k_\beta)\right) = d_{\lim_{\to}} i'_\beta(k_\beta)$. Hence the kernel is contained in the image.
Now suppose let $\tilde{k} = d_{\lim_{\to}}(k) \in \lim_{\to} A_*$.
Then $\tilde{k} = i_\beta\left(d(\overline{k})\right) = d_{\lim_{\to}} i'_\beta\left(\overline{k}\right)$ for some $\overline{k} \in A'_\beta$.
But now

$$d_{\lim_{\to}}(\tilde{k}) = d_{\lim_{\to}} i_\beta\left(d\left(\overline{k}\right)\right) = i''_\beta d\left(d\left(\overline{k}\right)\right) = i''_\beta(0) = 0.$$

□

Theorem 1.17. *Suppose we are given two directed sets D and E. Define an order on $D \times E$ by $(\alpha, \beta) \geq (\alpha', \beta')$ if and only if $\alpha \geq \alpha'$ and $\beta \geq \beta'$. Suppose $G_{\alpha,\beta}$ is a direct system based on $D \times E$. Then the maps $G_{\alpha,\beta} \to \lim_{\to,\beta} G_{\alpha,\beta} \to \lim_{\to,\alpha}\left(\lim_{\to,\beta} G_{\alpha,\beta}\right)$ induce an isomorphism*

$$\lim_{\to,\alpha,\beta} G_{\alpha,\beta} \xrightarrow{\cong} \lim_{\to,\alpha}\left(\lim_{\to,\beta} G_{\alpha,\beta}\right).$$

Proof. Left out for lack of time. See [1, Theorem D.5] □

Proposition 1.18. (1) *For $A \supset B$ both closed, the following diagram commutes:*

$$\begin{array}{ccc} H_n(M, M-A; G) & \longrightarrow & H_n(M, M-B; G) \\ \big\downarrow J^A & & \big\downarrow J^B \\ \Gamma_c(A, H_n(M, M-\bullet; G)) & \longrightarrow & \Gamma_c\left(B, H_n(M, M-\bullet; G)\right) \end{array}$$

(2) *For $A, B \subset M$ both closed, the sequence*

$$0 \to \Gamma_c(A \cup B, H_n(M, M-\bullet; G)) \xrightarrow{h} \Gamma_c\left(A, H_n(M, M-\bullet; G)\right) \oplus \Gamma_c\left(B, H_n(M, M-\bullet; G)\right)$$
$$\xrightarrow{k} \Gamma_c\left(A \cap B, H_n\left(M, M-\bullet; G\right)\right)$$

is exact, where h is the sum of restrictions and k is the difference of restrictions.

(3) *If $A_1 \supset A_2 \supset A_3 \supset \dots$ are all compact and $A \bigcap A_i$, then the restriction homomorphisms $\Gamma\left(A_i, H_n\left(M, M-\bullet; G\right)\right) \to \Gamma\left(A, H_n\left(M, M-\bullet; G\right)\right)$ induce an isomorphism*

$$\lim_{\to} \Gamma\left(A_i, H_n\left(M, M-\bullet; G\right)\right) \xrightarrow{\cong} \Gamma\left(A, H_n(M, M-\bullet; G)\right)$$

Proof. (1) Let $\alpha \in H_n(M, M-A; G)$, and denote by ι the inclusion $(M, M-A) \hookrightarrow (M, M-B)$. Then $\iota_* = r^A_B$, so $J^B\left(r^A_B(\alpha)\right)(x) = r^B_x\left(r^A_B(\alpha)\right)$. On the other hand, $J^A(\alpha)|_B(x) = J^A(\alpha)(x) = r^A_x(\alpha)$. Now, from the composition

$$(M, M-A) \hookrightarrow (M, M-B) \hookrightarrow (M, M-x)$$

we obtain by taking homology, that $r^A_x = r^B_x r^A_B$, which gives the result.

(2) Firstly, a section that is zero on both A and B is then also zero on $A \cup B$, which gives the injective part of h. Now, suppose $s - t$ is the zero section over

$A \cap B$ for s a section over A and t a section over B. Then s and t agree on $A \cap B$, meaning that $s \cup t$ is well-defined and continuous, where $s \cup t$ is s on A and t on B, and $h(s \cup t) = (s, t)$. Likewise, if g is a section over $A \cup B$, then $k \circ h(g) = (g|_A)\,|_{A\cap B} - (g|_B)\,|_{A\cap B} = g|_{A\cap B} - g|_{A\cap B}$ is the zero section.

(3) Left out for lack of time. See [1].

□

Theorem 1.19. *Let $A \subset M$ be closed. Then*

(1) $H_i\,(M, M - A; G) = 0$ *for* $i > n$.
(2) $J^A\colon H_n(M, M - A, G) \to \Gamma_c(A, H_n\,(M, M - \bullet; G))$ *is an isomorphism.*

Lemma 1.20 (The Bootstrap Lemma)**.** *Let $P_M(A)$ be a statement about compact sets A in a given n-manifold M^n. If $(i), (ii), (iii)$ hold, then $P_M(A)$ is true for all compact A in M^n.*
If M^n is separable metric, and $P_M(A)$ is defined for all closed sets A, and if $(i), (ii), (iii), (iv)$ hold, then $P_M(A)$ is true for all closed sets A in M.
For general M^n, if $P_M(A)$ is defined for all closed sets A in M, for all M^n, and if all five statement $(i) - (v)$ hold for all M^n, then $P_M(A)$ is true for all closed $A \subset M$ and all M^n.

Now note that for a given abelian group G and $g \in G$, the following maps are natural in $A \subset M$ (closed):

$$H_n(M, M - A) \cong H_n\,(M, M - A) \otimes \mathbb{Z} \to H_n(M, M - A) \otimes G \to H_n(M, M - A; G)$$

where the middle map is induced by the homomorphism $\mathbb{Z} \to G$ taking 1 to g.
In particular, this induces a map

$$H_n(M, M - \bullet) \to H_n(M, M - \bullet; G)$$

Lemma 1.21. *[4] The sections $\Gamma(A; G)$ of ω over A correspond bijectively to continuous maps $\lambda\colon \operatorname{Ori}(M)\,|_A \to G$ with the property $\lambda \circ t = -\lambda$, where t acts on G as multiplication by -1.*

Proof. We may assume A is connected.
Let $s \in \Gamma\,(A; G)$ be a section of ω over A. That is, $w \circ s = \mathrm{id}_A$, and s is a map $A \to H_n(M, M - \bullet; G)$. We can define an associated map $\lambda_s\colon \operatorname{Ori}(M)|_A \to G$ by sending a generator in the fiber $x \in A$ to $s(x) \in H_n\,(M, M - \{x\}\,; G) \cong G$. If one chose the other generator, one would get the negative of the above map, so we have the relation $\lambda_s \circ t = -\lambda_s$. Subject to this relation, we obtain a well-defined map $\Gamma\,(A; G) \to S \subset \operatorname{Hom}(\operatorname{Ori}(M)\,|_A, G)$, where S is the subset for which $\lambda \circ t = -\lambda$ holds. This map is injective, since the image tells us precisely the value of s at any point in A.
It is furthermore surjective, since if $\operatorname{Ori}(M)|_A$ is connected, then S can only consist of the zero section, and if it is not connected, it consists of a map on two components on which it is constant, and the relation $\lambda \circ t = -\lambda$ then determines that is must the required values to constitute the induced map of a section. □

Theorem 1.22. *Suppose $A \subset M$ is a closed connected subset. Then*

(1) $H_n(M, M - A; G) = 0$ *if A is not compact.*
(2) $H_n(M, M - A; G) \cong G$ *if M is R-orientable along A and A is compact. Moreover, $H_n(M, M - A; G) \to H_n(M, M - x; G)$ is an isomorphism for each $x \in A$.*

(3) $H_n(M, M-A; G) \cong {}_2G = \{g \in G \mid 2g = 0\}$ *if* M *is not orientable along* A *and* A *is compact.*

Proof. (1) By Lemma 7.1, a section in $\Gamma(A; G)$ is determined by its value at a single point. By the existence of the zero section, if a section is non-zero at any point, then it is non-zero at every point. Therefore, there do not exist non-zero sections with compact support over a non-compact A, so by Theorem 1.19, $H_n(M, M-A; G) \cong \Gamma_c(A; G) \cong 0$.

(2) Since A is compact, $H_n(M, M-A; G) \cong \Gamma_c(A; G) = \Gamma(A; G)$. A section is again determined by a single point. Recall now the commutative diagram

$$\begin{tikzcd} H_n(M, M-A; G) \arrow[r, "\cong"] \arrow[d, "r_x^A"] & \Gamma(A; G) \arrow[d, "b"] \\ H_n(M, M-x; G) \arrow[r, "\cong"] & \Gamma(\{x\}; G) \end{tikzcd}$$

from Proposition 1.18, the horizontal isomorphisms following from Theorem 1.19. If M is orientable along A, there by definition exists in $\Gamma(A; G)$ an element such that its value at x is a generator. Hence b is an isomorphism, and therefore also r_x^A is an isomorphism.

(3) By Lemma 1.21, a section in $\Gamma(A; G)$ corresponds to a continuous map $\lambda\colon \operatorname{Ori}(M)|_A \to G$ with $\lambda t = -\lambda$. If M is not orientable along A, then $\operatorname{Ori}(M)|_A$ is connected and therefore λ is constant as G has the discrete topology. The relation $\lambda t = -\lambda$ shows that λ is in ${}_2G$. Now by the commutative diagram from part (2), note that since λ must be constant, firstly $\Gamma(A; G) \cong {}_2G$, and secondly,b becomes injective, so $r_x^A\colon H_n(M, M-A; G) \to H_n(M, M-x; G) \cong G$ is injective and has image ${}_2G$, so the Hom term vanishes. ☐

Proposition 1.23. *Let* M *be an* n*-manifold and* $A \subset M$ *be a closed connected subset. Then the torsion subgroup of* $H_{n-1}(M, M-A; \mathbb{Z})$ *is of order* 2 *if* A *is compact and* M *non-orientable along* A*, and is* 0 *otherwise.*

Proof. By UCT for homology,

$$\begin{aligned} \mathbb{Z}/2 \cong {}_2\mathbb{Z}/2 \cong H_n(M, M-A; \mathbb{Z}/2) &\cong H_n(M, M-A) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}_1(H_{n-1}(M, M-A), \mathbb{Z}/2) \\ &\cong \operatorname{Tor}_1(H_{n-1}(M, M-A), \mathbb{Z}/2) \\ &\cong \{g \in H_{n-1}(M, M-A) \mid 2g = 0\}. \end{aligned}$$

where $H_n(M, M-A) \cong {}_2\mathbb{Z} = 0$, and $H_n(M, M-A; \mathbb{Z}/2) \cong {}_2\mathbb{Z}/2 \cong \mathbb{Z}/2$ both follow from Theorem 1.22.
To see that this is the whole torsions subgroup, note that for odd k,

$$\operatorname{Tor}_1(H_{n-1}(M, M-A), \mathbb{Z}/k) \cong H_n(M, M-A; \mathbb{Z}/k) \cong {}_2\mathbb{Z}/k \cong 0$$

When M is orientable along A and A is compact, we simply obtain

$$0 \to H_n(M, M-A) \otimes \mathbb{Z}/n \to H_n(M, M-A; \mathbb{Z}/n) \to \operatorname{Tor}_1(H_{n-1}(M, M-A), \mathbb{Z}/n) \to 0$$

and since $H_n(M, M-A) \cong \mathbb{Z}$ and $H_n(M, M-A; \mathbb{Z}/n) \cong \mathbb{Z}/n$ by Theorem 1.22, we find that Tor_1 vanishes for all n.
If A is non-compact, then Theorem 1.22 gives that Tor_1 trivially vanishes for all terms.

□

1.4. **Fundamental Class.**

Theorem 1.24. *Let M be a compact connected n-manifold. Then one of the following assertions holds:*

1. *M is orientable, $H_n(M) \cong \mathbb{Z}$, and for each $x \in M$, the restriction $H_n(M) \to H_n(M, M-x)$ is an isomorphism.*
2. *M is non-orientable and $H_n(M) = 0$.*

Proof. Special case of Theorem 1.22. □

Under the hypothesis of Theorem 1.24, the orientations of M correspond to the generators of $H_n(M)$. Such a generator will be called a *fundamental class* or *homological class/orientation* of the orientable manifold.

Definition 1.25 (Degree)**.** Let M and N be compact oriented n-manifolds. Let N be connected and suppose M has components $M_1, \ldots, M_r$. Then we have fundamental classes $z(M_j)$ for each M_j and $z(M) \in H_n(M) \cong \bigoplus_j H_n(M_j)$ is the sum of the $z(M_j)$. Now, since $H_n(N) \cong \langle z(N) \rangle \cong \mathbb{Z}$, we obtain that there exists a *degree* $d(f) \in \mathbb{Z}$ such that $f_* z(M) = d(f) z(N)$.

Lemma 1.26 (Properties)**.**
1. *The degree is a homotopy invariant.*
2. *$d(f \circ g) = d(f)d(g)$.*
3. *A homotopy equivalence has degree ± 1.*
4. *If $M = M_1 \bigsqcup M_2$, then $d(f) = d\left(f|_{M_1}\right) + d\left(f|_{M_2}\right)$.*
5. *If we pass in M or N to the opposite orientation, then the degree changes the sign.*

1.4.1. *Computations of degrees.* As usual, we can compute degrees in terms of local data of a map.
Let M and N be connected and set $K = f^{-1}(p)$. Let U be an open neighborhood of K in M. Then in particular $M - U = \overline{M-U} \subset \operatorname{int}(M - A) = M - A$, so excision gives the bottom left isomorphism in the following diagram, and the top right isomorphism follows from Theorem 1.24:

$$\begin{tikzcd}
z(M) \in \arrow[dd, mapsto] & H_n(M) \arrow[r, "f_*"] \arrow[d] & H_n(M) \arrow[d, "\cong"] & \ni z(N) \arrow[dd, mapsto] \\
& H_n(M, M-K) \arrow[r, "f_*"] & H_n(N, N-p) & \\
z(U,K) \in & H_n(U, U-K) \arrow[r, "f_*^U"] \arrow[u, "\cong"', "i_*"] & H_n(N, N-p) \arrow[u, "="'] & \ni z(N,p)
\end{tikzcd}$$

From the outer rectangle, we get $f_*^U z(U,K) = d(f) z(N,p)$, where $z(N,p)$ and $z(U,K)$ are the images of $z(N)$ and $z(M)$ under the indicated maps.
We want to show additivity of degree as in the case for spheres.
So suppose K if finite, and choose $U = \bigcup_{x \in K} U_x$ where the U_x are pair-wise disjoint open neighborhoods of x. Then

$$\bigoplus_{x \in K} H_n(U_x, U_x - x) \cong H_n(U, U-K), \quad H_n(U_x, U_x - x) \cong \mathbb{Z}.$$

The image $z(U_x, x)$ of $z(M)$ is a generator: it is the image under the following isomorphisms

$$H_n(M) \xrightarrow{\cong} H_n(M, M-x) \xrightarrow{\cong} H_n(U_x, U_x - x)$$

where the first follows from Theorem 1.24 and the second from excision. The local degree $d(f,x)$ is determined by $f_* z(U_x, x) = d(f,x) z(N,p)$, and and by additivity above, we have $d(f) = \sum_{x \in K} d(f,x)$.

Proposition 1.27. *Let M be a connected, oriented, closed n-manifold. Then there exists for each $k \in \mathbb{Z}$ a map $f\colon M^n \to S^n$ of degree k.*

Remark. If f is C^1 in a neighborhood of x, then $d(f,x)$ is the sign of the determinant of $Dg(0)$ when $Dg(x)$ is regular, where g is f in local coordinates that preserve local orientations. By this we mean that for $\varphi\colon U_x \to \mathbb{R}^n$ centered at x, $\varphi_*\colon H_n(U_x, U_x - x) \to H_n(\mathbb{R}^n, \mathbb{R}^n - 0)$ sends $z(U_x, x)$ to the standard generator. Such charts are called *positive* with respect to the given orientations.

Proof. If $f\colon M \to S^n$ has degree a and $g\colon S^n \to S^n$ degree b, then gf has degree ab. Since the proposition is true for $M = S^n$, it suffices to find f having degree ± 1. Let $\varphi\colon D^n \to M$ be an embedding. Then we have a map $f\colon M \to D^n/S^{n-1}$ which is the inverse of φ on $U = \varphi(\operatorname{int} D^n)$ and sends $M - U$ to the basepoint. This map has degree ± 1 as can be seen by choosing any neighborhood of x in the interior of U and looking at the determinant of the differential locally. □

1.5. **Manifolds with Boundary.**

Definition 1.28. For M an n-dimensional manifold with boundary, we call $z \in H_n(M, \partial M)$ a *fundamental class* if for each $x \in M - \partial M$, the restriction of z is a generator in $H_n(M, M-x)$.

Theorem 1.29. *Let M be a compact connected n-manifold with non-empty boundary. Then one of the following assertions hold:*

(1) *$H_n(M, \partial M) \cong \mathbb{Z}$, and a generator of this group is a fundamental class. The image of a fundamental class under $\partial\colon H_n(M, \partial M) \to H_{n-1}(\partial M)$ is a fundamental class. The interior $M - \partial M$ is orientable.*
(2) *$H_n(M, \partial M) = 0$, and $M - \partial M$ is not orientable.*

Proof. See [4, Thm 16.5.1]. We will follow that proof and only add a few extra words.
Let $\kappa\colon [0, \infty[\times \partial M \to U$ be a collar of M, i.e., a homeomorphism onto an open neighborhood U of ∂M such that $\kappa(0,x) = x$ for $x \in \partial M$ (See Milnor's h-cobordism book for existence). For simplicity of notation, we identify U with $[0,\infty) \times \partial M$ via κ ; similarly, for subsets of U. In this sense, $\partial M = 0 \times \partial M$. For $A = M - ([0,1) \times \partial M) \subset M - \partial M$, we have isomorphisms

$$H_n(M, \partial M) \cong H_n(M, [0,1) \times \partial M) \cong H_n(M - \partial M, (0,1) \times \partial M) \cong \Gamma(A).$$

The first one by homotopy equivalence, the second by excision, and the third using Theorem 1.19.
Since A is connected, $\Gamma(A) \cong \mathbb{Z}$ or $\Gamma(A) \cong 0$. If $\Gamma(A) \cong \mathbb{Z}$, then $M - \partial M$ is orientable along A.
Let now $A_\varepsilon \cong A$ be the complement of $[0, \varepsilon) \times \partial M$. Since each compact subset of $M - \partial M$ is contained in some such M_ε for small enough ε, we see that $M - \partial M$ is orientable along all compact subsets, hence orientable by Proposition 1.10.

The isomorphism $H_n(M - \partial M, (0,1) \times \partial M) \cong \Gamma(A)$ from above says that there exists some $z \in H_n(M - \partial M, (0,1) \times \partial M)$ which restricts to a generator of $H_n(M - \partial M, M - \partial M - x)$ for each $x \in A$. For the corresponding element $z \in H_n(M, \partial M) \cong \mathbb{Z}$, the same assertion holds for any $x \in M - \partial M$ (simply shrink the collar to not contain x).
Lastly, we must show that ∂z is a fundamental class. Let $x \in (0,1) \times \partial M$. Consider the diagram:

$$\begin{tikzcd}
H_{n-1}(\partial M) \arrow[r, "\cong"] & H_{n-1}(\partial M \cup A, A) & H_{n-1}(\partial I \times \partial M, 1 \times \partial M) \arrow[l, "\cong"'] \\
H_n(M, \partial M) \arrow[u, "\partial"] \arrow[r] \arrow[dr] & H_n(M, \partial M \cup A) \arrow[u, "\cong"', "\partial"] \arrow[d] & H_n(I \times \partial M, \partial I \times \partial M) \arrow[l, "\cong"'] \arrow[u, "\cong"', "\partial"] \\
& H_n(M, M - x) & H_n(I \times \partial M, I \times \partial M - x) \arrow[l, "\cong"']
\end{tikzcd}$$

Commutativity of the bottom left triangle tells us that the image of z under $H_n(M, \partial M) \to H_n(M, \partial M \cup A)$ gives an element whose restriction gives a generator in $H_n(M, M - x)$, but then by commutativity of the bottom right square, we get that the restriction of z transferred over by the isomorphism to $H_n(I \times \partial M, \partial I \times \partial M)$ is a generator of $H_n(I \times \partial M, I \times \partial M - x)$ at each point in $(0,1) \times \partial M$. Hence z yields a fundamental class in $H_n(I \times \partial M, \partial I \times \partial M)$.
But since z is a generator in $H_n(I \times \partial M, \partial I \times \partial M)$, the upper part shows that z is a generator in $H_{n-1}(\partial M)$, thus a fundamental class of ∂M since this characterizes fundamental classes.

□

Example 1.30. Suppose that $B\colon M \to \varnothing$ is a cobordism. We have the fundamental classes $z(B) \in H_{n+1}(B, \partial B)$ and $z(M) = \partial z_B \in H_n(M)$ (here we crucially made use of our result in Theorem 1.29). This is already a lot of information. Indeed, suppose $f\colon M \to N$ is a map which has an extension to B : $F\colon B \to N$. Then the degree of f (if defined) is zero, $d(f) = 0$, for we have $f_* z(M) = f_* \partial z(B) = F_* i_* \partial z(B) = 0$, since $i_* \partial = 0$ by the exactness of the homology sequence for the pair (B, M).
We call maps $f_\nu\colon M_\nu \to N$ *orientable bordant* if there exists a compact oriented cobordism $B\colon M_1 \to M_2$ with orientable boundary $\partial B = M_1 - M_2$ (meaning $\partial z(B) = z(M_1) - z(M_2)$) and an extension $F\colon B \to N$ of $f_1 \sqcup f_2\colon M_1 \coprod M_2 \to N$. Under these assumptions, we have $d(f_1) = d(f_2)$. This fact is called the *bordism invariance* of the degree; it generalizes the homotopy invariance.

Exercise 1.31. If M^m and N^n are manifolds, then show that $M \times N$ is orientable if and only if M and N are orientable.

Proof. We may assume that M and N are connected.
Define a map $\mu\colon \tilde{M} \times \tilde{N} \to \widetilde{M \times N}$ by

$$\mu(u, v) = u \times v \in H_{m+n}((M, M - x) \times (N, N - y)) = H_{m+n}(M \times N, M \times N - (x, y))$$

where $\times$ is the cross product.
Note that $\gamma^{M \times N} \circ \mu = \gamma^M \times \gamma^N$. We want to use this map to transfer sections: if $(s, t)\colon M \times N \to \tilde{M} \times \tilde{N}$ are sections, then we want $\mu \circ (s, t)$ to be a section. This requires us to show that μ is continuous.

Let U, V be open neighborhoods of x and y, respectively, such that $z \in Z_m(M, M - U)$ and $w \in Z_n(N, N - V)$ represent u, v, respectively.
Recall the relative Eilenberg-Zilber chain homotopy equivalence for open sets:

$$\begin{aligned} \times\colon \Delta_*(X,U) \otimes \Delta_*(Y,V) \to \Delta_*\left((X,U)\times(Y,V)\right) &= \Delta_*\left(X \times Y, X \times V \cup U \times Y\right) \\ &= \Delta_*\left(X \times Y, X \times Y - U \times V\right) \end{aligned}$$

which takes $z \otimes w$ to $z \times w$ which represents $u \times v$. Let $W_{z\times w}$ be a neighborhood of $u \times v$. We can shrink U and V such that $U \times V \subset W$. Then μ takes $U_z \times V_w$ into $W_{z\times w}$: for $(p,q) \in U \times V$,

$$\mu\left(\left([z]_p, [w]_q\right)\right) = [z]_p \times [w]_q = [z \times w]_{(p,q)}.$$

Thus μ is continuous. This gives a bilinear map

$$\Gamma A \times \Gamma B \to \Gamma\left(A \times B\right)$$

for $A \subset M, B \subset N$. Furthermore, the cross product takes a product of generators to a generator by the Künneth theorem, hence it maps a pair of orientations to an orientation which was what we wanted.

For the converse direction, we note that μ also satisfies $\beta^{M\times N}\mu\left(u,v\right) = \beta^M(v)\beta^N(v)$ since it is bilinear. Thus μ restricts to a map

$$\mu|\colon \tilde{M}(1) \times \tilde{N}(1) \to \widetilde{M \times N}(1),$$

and for each point $w \in \widetilde{M \times N}(1)$, the preimage under $\mu|$ has two points. Furthermore, since $\gamma^{M\times N} \circ \mu = \gamma^M \times \gamma^N$ and γ is a local homeomorphism, we find that μ restricts to a local homeomorphism, hence μ restricts to a two-sheeted covering map.
If either M or N is orientable, then $\tilde{M}(1)$ or $\tilde{N}(1)$ splits as $M \times \{-1,1\}$ or $N \times \{-1,1\}$, respectively, and so the LHS becomes disconnected, and the restriction to each component is a covering map, hence a homeomorphism, so the covering map is trivial.
Conversely, if the covering map is trivial, then there is a map $q\colon \tilde{M}(1) \times \tilde{N}(1) \to \{-1,1\}$ such that $\varphi\colon \tilde{M}(1)\times\tilde{N}(1) \to \widetilde{M \times N}(1)\times\{-1,1\}$ by $\varphi(u,v) = (\mu|(u,v), q(u,v))$ is a homeomorphism. Hence the LHS is not connected, but if $\tilde{M}(1)$ and $\tilde{N}(1)$ are both non-orientable, then they are connected, and so is their product, so we obtain that at least one of them must be orientable.

□

Exercise 1.32. Every manifold has a unique $\mathbb{Z}/2$-orientation.

Proof. The sections $\Gamma(M;\mathbb{Z}/2)$ of ω correspond bijectively to continuous maps $\lambda\colon \mathrm{Ori}(M) \to \mathbb{Z}/2$ such that $\lambda \circ t = -\lambda$. Hence we only have the 0 section and the constant section corresponding to the constant nontrivial map $\mathrm{Ori}(M) \to \mathbb{Z}/2$. The latter is in particular an orientation. □

Exercise 1.33. If M is a connected manifold such that $\pi_1(M)$ has no subgroups of index 2, then show that M is orientable.

Proof. If M is not orientable, then $\mathrm{Ori}\,(M)$ is a connected covering space. Manifolds are locally-path connected and semi-locally simply connected, so by the classification of covering spaces (using connectedness of M), $\mathrm{Ori}(M)$ corresponds to a subgroup of index 2 of $\pi_1(M)$ [1, Corollary 6.9]. □

Exercise 1.34. Let $p\colon M \to N$ be a covering map of n-manifolds. Then the pullback of $\mathrm{Ori}(N) \to N$ along p is $\mathrm{Ori}(M) \to M$.

Proof. The pullback of a covering map is a covering map. We have the following diagram

$$\begin{tikzcd}
Ori(M) \arrow[dr, dashed] \arrow[ddr, bend right, "q"'] & & \\
& Ori(N) \times_N M \arrow[r] \arrow[d, "\tilde{\pi}"] & Ori(N) \arrow[d, "\pi"] \\
& M \arrow[r, "p"] & N
\end{tikzcd}$$

Since q and $\tilde{\pi}$ are local homeomorphisms and $\tilde{\pi}^{-1} \circ q$ is bijective, we obtain that $\tilde{\pi}^{-1} \circ q$ is a bundle isomorphism. Thus the pullback is the orientation bundle. $\square$

Exercise 1.35. Show that the manifold $\mathrm{Ori}(M)$ is always orientable.

Proof. Recall that the orientation bundle of $\mathrm{Ori}(M)$ is

$$\omega_{\mathrm{Ori}(M)}\colon H_n(\mathrm{Ori}(M), \mathrm{Ori}(M) - \bullet) \to \mathrm{Ori}(M)$$

But ω is a local homeomorphism, so

$$\begin{aligned}
\omega_*\colon H_n(\mathrm{Ori}(M), \mathrm{Ori}(M) - u) &\cong H_n\left(\mathrm{Ori}(M) - (\mathrm{Ori}(M) - V_z), (\mathrm{Ori}(M) - u) - (\mathrm{Ori}(M) - V_z)\right) \\
&\cong H_n(V_z, V_z - u) \\
&\cong H_n\left(V, V - \omega u\right) \\
&\cong H_n\left(M, M - \omega u\right) \ni u
\end{aligned}$$

So at the point $u \in \mathrm{Ori}(M)$, we can define $s(u)$ to be the image of u under this line of isomorphisms.
Note that no $\tilde{M}(n)$ is distinguished from $\tilde{M}(1)$. Indeed, $\tilde{M}(n)$ is homeomorphic to $\tilde{M}(1)$ under the isomorphism $u \mapsto n \cdot u$ from $\tilde{M}(1)$ to $\tilde{M}(n)$ which commutes with ω_* because ω is a local homeomorphism.

$$\begin{tikzcd}
H_n\left(\tilde{M}(1), \tilde{M}(1) - u\right) \arrow[r, "\omega_*"] \arrow[d, "\cdot n"] & H_n(M, M - \omega u) \arrow[d, "\cdot n"] \\
H_n\left(\tilde{M}(n), \tilde{M}(n) - nu\right) \arrow[r, "\omega_*"] & H_n(M, M - \omega n u)
\end{tikzcd}$$

$\square$

2. Duality

Let M^n be orientable and $\vartheta_M \in \Gamma\left(M, H_n(M, M - \bullet)\right)$ an orientation.
For $K \subset M$ compact, ϑ_M restricts to to $\vartheta_K \in \Gamma\left(K, H_n(M, M - \bullet)\right) = \Gamma_c(K, H_n(M, M - \bullet)) \cong H_n(M, M - K)$, so we can regard ϑ_K as lying in $H_n(M, M - K)$. Let $\vartheta = \{\vartheta_K\}$ be the collection of all these, and we then call ϑ an orientation.

Definition 2.1. For sets $L \subset K \subset M$, we define

$$\check{H}^p(K, L; G) = \lim_{\rightarrow} \{H^p(U, V; G) \mid (U, V) \supset (K, L), U, V \text{ open}\}.$$

This is a directed system since if (U, V) and (U', V') both contain (K, L), the $(U \cap U', V \cap V')$ also contains (K, L), and the maps induced by inclusions of nested open sets satisfy the required relation to be a directed system.

This group is naturally isomorphic to that of Cech cohomology. If K and L are spaces such as ENRs (e.g., CW-complexes or topological manifolds), then this is also naturally isomorphic to singular cohomology.

2.1. **Construction of the duality map.** Suppose $(K, L) \subset (U, V)$ as above (so $L \subset K$). Then note that since $K \subset U$, we have $M - U = \overline{M - U} \subset \int(M - K) = M - K$, so by excision, $H_{n-p}(U - L, U - K) \cong H_{n-p}(M - L, M - K)$. Also, $\{V, U - L\}$ is an open cover of U, hence $H_* \left(\frac{\Delta_*(V) + \Delta_*(U-L)}{\Delta_*(U-K)}\right) \cong H_*(U, U - K) \cong H_*(M, M - K)$ where the first isomorphism follows from Theorem 7.2.2 in Algtop1, and the latter follows from excision.
Now, we have a well-defined cap product

$$\Delta^p(U, V; G) \otimes \left[\frac{\Delta_n(V) + \Delta_n(U - L)}{\Delta_n(U - K)}\right] \xrightarrow{\cap} \Delta_{n-p}(U - L, U - K; G)$$

given by $f \cap (b + c) = f \cap b + f \cap c = f \cap c$, which by the above, induces a cap product

$$H^p(U, V; G) \otimes H_n(M, M - K) \to H_{n-p}(M - L, M - K; G)$$

which is natural in (K, L).
Using the same theorem from Algtop1, we find that for $\gamma \in H_n(M, M - A)$, we can represent γ by a chain $b + c + d \in \Delta_n(V) + \Delta_n(U - L) + \Delta_n(M - K)$, so for $f \in \Delta^p(U, V; G)$, we get that

$$[f] \cap \gamma = [f \cap (b + c + d)] = [f \cap c] \in H_{n-p}(M - L, M - K; G)$$

since $f \cap b = 0$ as f vanishes on V and $f \cap d$ is a chain in $M - K$.
Thus by capping with ϑ_A for A large enough to contain K, we obtain a homomorphism

$$\cap\vartheta\colon H^p(U, V; G) \to H_{n-p}(M - L, M - K; G).$$

Now, recall that the direct limit used for $\check{H}$ has the universal property that it is the colimit under the direct system. So if the above map $\cap\vartheta$ is compatible with the direct system, then we will obtain an induced homomorphism from $\check{H}$.
Suppose $i\colon (U', V') \hookrightarrow (U, V)$ is the inclusion. Let $[f] \in H^p(U, V; G)$ and represent ϑ as $b + c + d \in \Delta_n(V') + \Delta_n(U' - L) + \Delta_n(M - A)$. Then

$$[f] \cap \vartheta = [f \cap (b + c + d)] = [f \cap c] = [f \circ i \cap c] = i^* [f] \cap \vartheta$$

so the homomorphism is compatible with the morphisms in the directed system. Furthermore, choosing a different A, say A', we similarly can write ϑ as $b + c + d \in \Delta_n(V') + \Delta_n(U' - L) + \Delta_n(M - A')$, and again $f \cap d$ becomes zero in homology since it lands inside $M - A \subset M - K$. Hence the homomorphism $\cap\vartheta$ is compatible with changing A and also with the directed system maps. Thus in the direct limit, we get a natural map

$$\cap\vartheta\colon \check{H}^p(K, L; G) \to H_{n-p}(M - L, M - K; G).$$

2.2. **Properties of the duality map.** Now that we have constructed the homomorphism we are interested in, we should prove some properties that might be useful for us. In order to do this, we give its (co)chain description.
Let $\alpha \in \check{H}^p(K, L; G)$. Then by the direct limit, there exists some open neighborhood (U, V) of (K, L) such that α is represented by a p-cocycle f of (U, V). Thus $f = 0$ on V and $\delta f = 0$ on U. Then extend f trivially to a cochain on all of M. We represent the orientation ϑ by a chain $a = b + c + d \in \Delta_n(V) + \Delta_n(U - L) + \Delta_n(M - K)$. Then $f \cap (b + c + d) = f \cap b + f \cap c + f \cap d$. But $f \cap b = 0$ and $f \cap d$ is a chain in $M - K$, so $\alpha \cap \vartheta$ is represented by $f \cap c$. In the special case for $L = \varnothing$, we take $V = \varnothing$, so b does not come into the consideration then.

Using the above description, we can now show the following lemma

Lemma 2.2. *The following diagram with arbitrary coefficients has exact rows and commutes:*

$$\begin{tikzcd}
\dots \arrow[r] & \check{H}^p(K,L) \arrow[r] \arrow[d] & \check{H}^p(K) \arrow[r] \arrow[d] & \check{H}^p(L) \arrow[r] \arrow[d] & \check{H}^{p+1}(K,L) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_{n-p}(M-L, M-K) \arrow[r] & H_{n-p}(M, M-K) \arrow[r] & H_{n-p}(M, M-L) \arrow[r] & H_{n-p-1}(M-L, M-K) \arrow[r] & \dots
\end{tikzcd}$$

where all vertical maps are the cap products with the orientation class ϑ.

Proof. The exactness of the top row follows from Theorem 1.16 together with the LES of a pair in singular cohomology.

Let us first check the first two squares. In the first one, if $[f] \in \check{H}^p(K, L)$, then going down, we get $[f] \cap \vartheta$ which includes into $H_{n-p}(M, M - K)$. In particular, this is represented by $f \cap c$. But in the same way, including first $[f]$ into $\check{H}^p(K)$ and the capping with ϑ is still represented by $f \cap c$ (as we discussed above), so we obtain commutativity of the first square. For the second square, suppose let $f \in \Delta^p(M; G)$ be such that $f|_U \in \Delta^p(U; G)$ represents $\alpha \in H^p(U; G)$ mapping to the class in question in $\check{H}^p(K; G)$. Let ϑ be represented by $a = b + c + d \in \Delta_n(V) + \Delta_n(U - L) + \Delta_n(M - K)$. Then $f \cap a$ is represented by $f \cap b + f \cap c$ in $H_{n-p}(M, M - K)$. But since we have the decomposition $a = 0 + b + (c + d) \in \Delta_n(\varnothing) + \Delta_n(V - \varnothing) + \Delta_n(M - L)$, we also have that in $H_n(M, M - L)$, $f \cap a$ becomes $f \cap b$.
On the other hand, we have f restricting to class which is the image of a class represented by $f|_V$. But then this simply becomes $f|_V \cap b = f \cap b$ in $H_{n-p}(M, M - L)$.
For the last square, we must check commutativity.
Let $f \in \Delta^p(M; G)$ such that $f|_V \in \Delta^p(V; G)$ represents $\alpha \in H^p(V; G)$ mapping to the class in $\check{H}^p(L; G)$ that we want to chase through the square (we can find such an f by the properties section above the lemma). Thus $\delta f = 0$ on V.
Now represent ϑ by $a = b + c + d \in \Delta_n(V) + \Delta_n(U - L) + \Delta_n(M - K)$. This is the decomposition of a appropriate to the (K, L) pair, but we can also decompose a as $a = 0 + b + (c + d) \in \Delta_n(\varnothing) + \Delta_n(V - \varnothing) + \Delta_n(M - L)$ which is the decomposition with respect to the pair $(L, \varnothing)$, showing that a can be used in the definition of both the cap product for $\check{H}^p(L)$ and for $\check{H}^{p+1}(K, L)$ in question. Since ϑ is a class of $(M, M - K)$, ∂a is a chain in $M - K$. Now, starting with f representing α, going to the right gives δf and then capping with a when going down gives $\delta f \cap a$. On the other hand, going down with f first gives $f \cap a$ and going right then gives $\partial (f \cap a)$.

Now recall that $\partial(f \cap a) = (\delta f) \cap a \pm f \cap \partial a$, but $f \cap \partial a$ is a chain in $M - K$, so it vanishes on passage to homology, whereby we find that the square commutes. $\square$

The following Lemma and subsequent theorem require a bit of work to prove, and we will skip this because of time constraints.

Lemma 2.3. *[1, Lemma 8.2] Let K and L be compact subsets of the n-manifold M with the orientation ϑ. Then the diagram with arbitrary coefficients*

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(K \cup L) \arrow[r] \arrow[d] & \check{H}^p(K) \oplus \check{H}^p(L) \arrow[r] \arrow[d] & \check{H}^p(K \cap L) \arrow[r, "\delta^*"] \arrow[d] & \check{H}^{p+1}(K \cup L) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{n-p}(M, M-(K \cup L)) \arrow[r] & H_{n-p}(M, M-K) \oplus H_{n-p}(M, M-L) \arrow[r] & H_{n-p}(M, M-(K \cap L) \arrow[r, "\delta_*"] & H_{n-p-1}(M, M-(K \cup L)) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical maps are the cap products with ϑ, commutes and has exact rows.

Theorem 2.4 (Poincaré-Alexander-Lefschetz Duality)**.** *[1, Theorem 8.3] Let M^n be an n-manifold oriented by ϑ, and let $K \supset L$ be compact subsets of M. Then the cap product*

$$\cap\vartheta \colon \check{H}^p(K, L; G) \to H_{n-p}(M - L, M - K; G)$$

Corollary 2.5 (Poincaré-Lefschetz Duality)**.** *[1, Corollary 8.4] If M^n is a compact orientable n-manifold and $L \subset M^n$ is closed, then we have the following diagram with exact rows and all vertical (cap products with the orientation class) being isomorphisms:*

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(M, L) \arrow[r] \arrow[d, "\cong"] & \check{H}^p(M) \arrow[r] \arrow[d, "\cong"] & \check{H}^p(L) \arrow[r] \arrow[d, "\cong"] & \check{H}^{p+1}(M, L) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & H_{n-p}(M - L) \arrow[r] & H_{n-p}(M) \arrow[r] & H_{n-p}(M, M - L) \arrow[r] & H_{n-p-1}(M - L) \arrow[r] & \cdots
\end{tikzcd}$$

The isomorphism involving M alone is called Poincaré duality. This holds with arbitrary coefficients, and M^n need not be orientable for $\mathbb{Z}/2$ as the base ring.

3. Intersection Theory

3.1. The Thom class of a disk bundle.

Definition 3.1 (k-disk bundle)**.** A k-disk bundle is the restriction of a vector bundle so that its coordinate transformations are linear maps which map the unit disk into itself and such that the local trivializations have the form $\pi^{-1}(U) \cong U \times D^k$. See [1] for a more detailed definition.

Let N^n be a connected, oriented, closed n-manifold, and W^{k+n} an $(n+k)$-manifold with boundary ∂W a $(k-1)$-sphere bundle over N^n, and let $\pi \colon W^{n+k} \to N^n$ be a k-disk bundle over N.
Let us assume that W is also oriented.

Definition 3.2. In the above situation, the *Thom class* of the disk bundle π is the class $\tau \in H^k(W, \partial W)$ given by

$$\tau = D_W(i_*[N])$$

where $D_W \colon H_{n-k}(W) \to H^k(W, \partial W)$ is the inverse of the Poincaré duality isomorphism. That is,

$$D(a) \cap [M] = a.$$

Thus

$$\tau \cap [W] = i_*[N].$$

We can deformation retract the punctured disk to its boundary, giving $H^k(W, W - N) \cong H^k(W, \partial W)$, so we will sometimes regard τ as being in $H^k(W, W - N)$.

3.2. **Transfer maps.** *Transfer* maps, also called *shriek* or *umkehr* maps, are a particular class of maps that exhibit some nice properties.
We give a definition that suffices for our purposes, but this class of maps can be treated more generally (See e.g. [5]).

Definition 3.3 (Transfer map)**.** If $f\colon N^n \to M^m$ is a map from a compact oriented n-manifold N to a compact, oriented m-manifold M, taking ∂N into ∂M, then

$$f^!\colon H^{n-p}(N) \to H^{m-p}(M) \quad \text{and} \quad f^!\colon H^{n-p}(N, \partial N) \to H^{m-p}(M, \partial M)$$

are defined by

$$f^! = D_M f_* D_N^{-1}.$$

Similarly,

$$f_!\colon H_{m-p}(M) \to H_{n-p}(N) \quad \text{and} \quad f_!\colon H_{m-p}(M, \partial M) \to H_{n-p}(N, \partial N),$$

are both defined by

$$f_! = D_N^{-1} f^* D_M.$$

Our particular interest will lie in the transfer map $i_!$ when i is an embedding of a closed oriented manifold.

This transfer map and the Thom class of a k-disk bundle are intimately connected as described in the Thom isomorphism theorem, a version of which we now state.

Theorem 3.4 (Thom Isomorphism Theorem, first version)**.** *If $\pi\colon W \to N$ is a k-disk bundle over a connected oriented closed n-manifold N^n, then there is the "Thom Isomorphism"*

$$H^p(N) \xrightarrow{\pi^*} H^p(W) \xrightarrow{\cup\tau} H^{p+k}(W, \partial W)$$

which coincides with $i^!$. Similarly, $i_! = \pm\pi_* \circ (\tau \cap -) : H_{p+k}(W, \partial W) \to H_p(N)$ is an isomorphism.

Proof. Fairly straightforward computation from the definitions. See [1]. □

We can extend this isomorphism a bit to disk bundles over base spaces which are not necessarily manifolds, but which are embedded in manifolds.

Lemma 3.5. *In the above setup, suppose $A \subset N$ is closed. Let $\tilde{A} = \pi^{-1}(A) \subset W$ and $\partial\tilde{A} = \tilde{A} \cap \partial W$. Then $\check{H}^i\left(\tilde{A}, \partial\tilde{A}\right) = 0$ for $0 < i < k$.*

Proof. Suppose first that A is compact convex subset of a Euclidean neighborhood in N. It also suffices consider the case where A is connected, so $A \cong D^n$. Consider the pullback bundle of A :

$$\begin{tikzcd} i^*(A) \arrow[r] \arrow[d] & W \arrow[d, "\pi"] \\ A \arrow[r, hook, "i"] & N \end{tikzcd}$$

Then $i^*(A) = A \times_N W \cong \pi^{-1}(A)$, so since any vector bundle over a contractible paracompact base space is trivial, we conclude that the bundle $\tilde{A} \to A$ is trivializable as $\tilde{A} \cong A \times D^k$ and $\partial\tilde{A} \cong A \times S^{k-1}$. Now the steps are as follows: calculate the

homology of $A \times D^k$ and $A \times S^{k-1}$, then use UCT to obtain the cohomology, and then use the LES to find the cohomology of $\left(A \times D^k, A \times S^{k-1}\right)$.
From here on, one can complete the proof using the Künneth theorem and the Bootstrap lemma. We will skip this for lack of time, but see [1, Lemma 11.4]. $\square$

Lemma 3.6. *The restriction $\tau_x \in \check{H}^k(\tilde{A}, \partial\tilde{A})$ of τ, when $A = \{x\}$, is a generator.*

Proof. Note that $\left(\tilde{A}, \partial\tilde{A}\right) \cong \left(D^k, S^{k-1}\right)$.
Suppose first that $\tau_x = 0$ for some x. Let $i\colon \tilde{A} \hookrightarrow W$ be the inclusion, then we have

$$0 = i_*(0) = i_*\left(\tau_x \cap \beta\right) = \tau \cap i_*\left(\beta\right),$$

for all $\beta \in H_*\left(\tilde{A}, \partial\tilde{A}\right)$.
Let U be a neighborhood around x which is evenly covered, so $\pi^{-1}(U) \cong U \times D^k$, and $\tilde{A} = \pi^{-1}(x) \cong \{x\} \times D^k$. Let $i_x\colon \{x\} \to \{x\} \times D^k$ be the zero section, and similarly for $i_y\colon \{y\} \to \{y\} \times D^k$. Let $\gamma\colon I \to U$ be a path from x to y. Then we can define a path $F\colon I \to U \times D^k$ by $F(t) = (\gamma(t), 0)$. Then $F(t) = i_{\gamma(t)}(\gamma(t))$.
We have a pullback square as follows:

$$\begin{array}{ccc} (X, X') & \longrightarrow & (W, W') \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle p} \\ I & \longrightarrow & B \end{array}$$

Then we have an isomorphism induced by inclusions:

$$w_\#\colon H_n\left(F_c, F_c'\right) \overset{\cong}{\rightarrow} H_n(X, X') \overset{\cong}{\leftarrow} H_n(F_b, F_b')$$

In particular, $i_{c*} = i_{b*} w_\#$, so if $i_{c*}\left[x\right] = 0$, then $i_{b*}\left[y\right] = i_{b*} w_\#\left[x\right] = \pm i_{c*}\left[x\right] = 0$.
Thus $\tau_y = 0$ for all y near x. Since N is connected, this implies that $\tau_y = 0$ for all $y \in N$.
By τ restricting to zero at every point does not necessarily imply a priori that τ is zero. But we can use it for a Bootstrap argument.

For closed sets $A \subset N$, let $P_N(A)$ be the statement that $\tau_A = 0$, where $\tau_A = \tau|_{(\tilde{A}, \partial\tilde{A})}$.
Suppose now that A is a convex set in some euclidean open set in N.
Note that

$$\check{H}^k(\tilde{A}, \partial\tilde{A}) \cong \check{H}^k(W - (W - \tilde{A}), (W - \tilde{A})) \cong \check{H}^k\left(W, W - \tilde{A}\right)$$

so we can regard $\tau|_A \in \check{H}^k(\tilde{A}, \partial\tilde{A})$ as living in $\check{H}^k(W, \tilde{A})$.
We have that the restriction defines an isomorphism

$$H^n\left(W, W - \tilde{A}\right) \to H^n\left(W, W - x\right)$$

if $\tilde{A}$ is a compact convex set contained in a Euclidean neighborhood. Since $\tilde{A}$ is locally of the form $C \times D^k$ for C convex, we see that for every $x \in \tilde{A}$, there is a ball at x such that its intersection with $\tilde{A}$ is convex. I.e., $\tilde{A}$ is locally convex. Also, it is closed, so each component is convex by the Tietze-Nakajima Theorem. Since the local structure of $\tilde{A}$ is as $U \times D^k$, we can for any two points in $\tilde{A}$ choose a path between the points in the image of π below, and run through this path in the

zero section above and then run the rest in D^k to connect the points. Hence $\tilde{A}$ is also connected, hence convex. Now Proposition 1.3 gives that $\check{H}^k\left(W, W-\tilde{A}\right) \cong \check{H}^k\left(W, W-\tilde{x}\right)$ for $\tilde{x} \in \tilde{A}$.
Also, we have an isomorphism

$$H_n\left(W, W-\tilde{A}\right) \cong H_n(N, N-A) \xrightarrow{\cong} H_n(N, N-x) \cong H_n(W, W-\pi^{-1}(x))$$

Let $X=\pi^{-1}(x)$, and $\partial X = X \cap \partial W$. Let U be a neighborhood of x such that $\pi^{-1}(U) \cong U \times D^k$.

$$\begin{aligned}\check{H}^k(W, W-X) &\cong \check{H}^k(W-(W-\pi^{-1}(U)), (W-X)-(W-\pi^{-1}(U)))\\ &\cong \check{H}^k(\pi^{-1}(U), \pi^{-1}(U)-X)\\ &\cong \check{H}^k(X, \partial X)\end{aligned}$$

Note that $\tau_x = i^*\tau_A$ where $i\colon \pi^{-1}(x) \hookrightarrow \tilde{A}$ is the inclusion, so if $i^*\colon H^k(\tilde{A}, \partial\tilde{A}) \to H^k\left(\pi^{-1}(x), \pi^{-1}(x) \cap \partial W\right)$ is injective, then since $\tau_x = 0$, we obtain $\tau_A = 0$.
Now, because A is a convex set in some Euclidean space, we can shrink A to be small enough so that, when considered in N, we have $\pi^{-1}(A) \cong A \times D^k$, and this deformation retract onto $\{x\} \times D^k$, so indeed i must be a homotopy equivalence (in fact, $\pi^{-1}(A) \cong A \times D^k$ directly since A is contractible and vector bundles over contractible spaces are trivial).
Next, if $P_N(A)$ and $P_N(B)$ hold, then $P_N(A \cup B)$ follows from the diagram

$$\begin{array}{ccccc}\underbrace{H^{k-1}(W\cap W, \partial(W\cap W))}_{=0} & \longrightarrow & H^k(W\cup W, \partial(W\cup W)) & \longrightarrow & H^k(W,\partial W)\oplus H^k(W,\partial W)\\ & & \big\downarrow & & \big\downarrow\\ \underbrace{\check{H}^{k-1}(\tilde{A}\cap\tilde{B}, \partial\left(\tilde{A}\cup\tilde{B}\right))}_{=0} & \longrightarrow & \check{H}^k(\tilde{A}\cup\tilde{B}, \partial\left(\tilde{A}\cup\tilde{B}\right)) & \longrightarrow & \check{H}^k(\tilde{A},\partial\tilde{A})\oplus\check{H}^k(\tilde{B},\partial\tilde{B})\end{array}$$

Here we used Theorem 1.16 to take $\check{H}$ on the bottom row. In this case, the rightmost map is $i^*_{(\tilde{A},\partial\tilde{A})} \oplus i^*_{(\tilde{B},\partial\tilde{B})}$, which maps (τ,τ) to (τ_A,τ_B), and the left map is the restriction $i^*_{\left(\tilde{A}\cup\tilde{B},\partial\left(\tilde{A}\cup\tilde{B}\right)\right)}$.
The top map is $j^* \oplus -j^*$ where j is the inclusion of W into W. Hence the map along the top and right arrow takes τ to $(\tau_A, -\tau_B) = (0,0)$, and the map along the left takes τ to $\tau_{A\cup B}$ which is mapped to $(0,0)$ along the bottom injectively. Hence $\tau_{A\cup B} = 0$.

Next suppose $P_N(A_i)$ holds for each set of decreasing sequences of closed sets $A_1 \supset A_2 \supset \ldots$. Then $\check{H}^k(A) = \lim_{\rightarrow} \check{H}^k(A_i) = 0$ since Cech cohomology commutes with direct limits. By assumption, N is closed, so by the Bootstrap lemma, $P_N(N)$ holds. That is, $\tau = 0$, however, this is impossible, since τ is the image of $[N]$ under isomorphisms, and $[N]$ is nontrivial by assumption.
We thus can conclude that $\tau_x \neq 0$ for all x.

Suppose next that τ_x is simply not a generator for some x, so say $\tau_x \in H^k(D^k, S^{k-1}) \cong \mathbb{Z}$ corresponds to $\pm m$. Let p be a prime dividing m. Then in passing to $\mathbb{Z}_p$

coefficients, τ_x becomes trivial, but all of the above goes through the same in this setting, hence we conclude that τ_x must correspond to ± 1, i.e., a generator. □

Corollary 3.7. *If N is connected, then the Thom class $\tau \in H^k(W, \partial W)$ is (up to sign) the unique class whose restriction to the fiber over each point is a generator.*

Proof. The Thom isomorphism gives $\mathbb{Z} \cong H^0(N) \cong H^0(W) \to H^k(W, \partial W)$, so if some class $\alpha \in H^k(W, \partial W)$ restricts under $\mathbb{Z} \cong H^k(W, \partial W) \to H^k(\pi^{-1}(x), \pi^{-1}(x) \cap \partial W) \cong H^k(D^k, S^{k-1}) \cong \mathbb{Z}$ to a generator, then it must be a generator in $H^k(W, \partial W)$ also. By Lemma 3.6 above, τ does restrict to a generator of each fiber, so α equals τ up to sign. □

Theorem 3.8 (Thom Isomorphism Theorem, second version)**.** *In the above situation, for any compact $A \subset N$, the map*

$$\pi(-) \cup \tau_A \colon \check{H}^i(A) \to \check{H}^{i+k}(\tilde{A}, \partial \tilde{A})$$

is an isomorphism.

Proof. An application of the Bootstrap lemma. Left out for lack of time. (Cf. e.g. [5] or [1]) □

3.3. **The Thom class arising from the normal bundle of submanifolds.** Suppose now that $i_N^W \colon N^n \to W^w$ is a smooth embedding of a smooth manifold, possibly with boundary, and assume that N meets ∂W transversally in ∂N.

Definition 3.9. In the above situation, denote $i_{N*}^W[N] \in H_n(W, \partial W)$ by $[N]_W$, and define $\tau_N^W = D_W([N]_W) \in H^{w-n}(W)$, which we will also call the Thom class of N in W. Here $D_W \colon H_n(W, \partial W) \xrightarrow{\cong} H^{w-n}(W)$.

We will show that the Thom class τ_N^W is the *image* of the Thom class of the normal $(w-n)$-disk bundle ν_N^W of N in W via

$$\varphi \colon H^{w-n}(\Xi, \Xi - N) \cong H^{w-n}(W, W - N) \to H^{w-n}(W)$$

where the first map is the inverse of the excision isomorphism.
Let $u_N = D((i_N^{\Xi})_*[N]) \in H^{n-w}(\Xi, \Xi - N)$ denote the Thom class of the normal disk bundle. Let $u_N^{W,W-N}$ denote the inverse of u_N under the excision isomorphism, and let u_N^W denote the image of $u_N^{W,W-N}$ under the restriction homomorphism. Then

Lemma 3.10. $\tau_N^W = u_N^W \in H^{w-n}(W)$.

Proof. By Poincaré duality, it suffices to show that

$$[N]_W = u_N^W \cap [W] \in H_n(W)$$

Consider the commutative diagram

$$\begin{tikzcd}
H_n(\Xi) \arrow[r, "i^W_{\Xi *}"] & H_n(W) & H^{w-n}(W) \otimes H_n(W) \arrow[l, "\cap"'] \\
H^{w-n}(\Xi, \Xi - N) \otimes H_w(\Xi, \Xi - N) \arrow[u, "\cap"] & H^{w-n}(W, W-N) \otimes H_w(W, W-N) \arrow[l] \arrow[u, "\cap"] & H^{w-n}(W, W-N) \otimes H_n(W) \arrow[l] \arrow[u]
\end{tikzcd}$$

The middle map takes $u_N^{W,W-N} \otimes [W]$ to $u_N^{W,W-N} \cap [W]$, and going around the

left square instead, it takes $u_N^{W,W-N} \otimes [W] \mapsto u_n \otimes [\Xi] \mapsto u_n \cap [\Xi] = \left(i_N^{\Xi}\right)_* [N] \mapsto \left(i_{\Xi}^W\right)_* \left(i_N^{\Xi}\right)_* [N] = \left(i_N^W\right)_* [N] = [N]_W$
As for the right square, if we start with $u_N^{W,W-N} \otimes [W]$ in the lower right, then this maps to $[N]_W$ by construction along the bottom right map and middle map as we just saw, while going the other way around gives $u_N^{W,W-N} \otimes [W] \mapsto u_N^W \otimes [W] \mapsto u_N^W \cap [W]$, so by commutativity, $u_N^W \cap [W] = [N]_W$, from which the result follows. □

3.4. **The Intersection Product.** First, suppose M^n is a compact, oriented and connected manifold, possibly with boundary. Let $D\colon H_i(M, \partial M) \to H^{n-i}(M)$ or $D\colon H_i(M) \to H^{n-i}(M, \partial M)$ be the inverse of the Poincaré duality isomorphism, so

$$D(a) \cap [M] = a.$$

Definition 3.11 (Intersection Product)**.** We define the intersection product the composite

$$\bullet\colon H_p(M)\otimes H_q(M) \overset{D\otimes D}{\to} H^{n-p}(M)\otimes H^{n-q}(M) \overset{\times}{\to} H^{2n-p-q}(M^2) \overset{\Delta^*}{\to} H^{2n-p-q}(M) \overset{D^{-1}}{\to} H_{p+q-n}(M)$$

or $H_p(M, \partial M)\otimes H_j(M) \to H_{i+j-n}$ or $H_i(M, \partial M)\otimes H_j(M, \partial M) \to H_{i+j-n}(M, \partial M)$.

Thus the intersection product of a and b is

$$a\bullet b = D^{-1}\left(D(b) \cup D(a)\right) = (D(b) \cup D(a))\cap[M] = D(b)\cap(D(a) \cap [M]) = D(b)\cap a.$$

In particular, note the change in order of a and b.

Exercise 3.12. Show that for homology classes a, b of M^n, $a\bullet b = (-1)^{n(n-\deg a)}\Delta_!(b\times a)$, where $\Delta\colon M \to M \times M$ is the diagonal map.

Proof. By definition,

$$\begin{aligned} \Delta_!(b \times a) &= D^{-1}\Delta^* D_{(N,\partial N)}\,(b \times a) \\ &= \Delta^*(z) \cap [M] \end{aligned}$$

where z is the dual of $b \times a$, given by $z \cap ([M] \times [M]) = b \times a$. Let then x, y be the duals of b and a, respectively, so $x \cap [M] = b$ and $y \cap [M] = a$. Note that $\deg x = n - \deg b, \deg y = n - \deg a$. Then $b \times a = (x \cap [M]) \times (y \cap [M]) = (-1)^{n(n-\deg a)}\,(x \times y)\cap([M] \times [M])$. Hence $z = (-1)^{n(n-\deg a)}x\times y$, so since $\Delta^*(x\times y) = x \cup y$ by definition, we obtain

$$\begin{aligned} (-1)^{n(n-\deg a)}\Delta_!(b \times a) &= \Delta^*(x \times y) \cap [M] \\ &= (x \cup y) \cap [M] \\ &= x \cap [y \cap [M]] \\ &= D(b) \cap a \\ &= a \bullet b. \end{aligned}$$

□

This lets us define the intersection product as follows instead:

Definition 3.13. Let M be an oriented manifold, and $\Delta\colon M \to M\times M$ be the diagonal embedding. For arbitrary open pairs $(V,S),(W,T)$ in M, consider the maps

$$\begin{aligned} H_i(V,S)\otimes H_j(W,T) &\xrightarrow{\times} H_{i+j}(V\times W, S\times W\cup V\times T) \\ &\to H_{i+j}(V\times W\cup(M\times M-\Delta(M)), S\times W\cup V\times T\cup(M\times M-\Delta(M))) \\ &\xrightarrow{\Delta_!} H_{i+j-n}\left(V\cap W,(S\cap W)\cup(V\cap T)\right) \end{aligned}$$

The composite is the intersection product.

3.5. **The Intersection Product as a Thom-Pontrjagin Construction.** Let M be a manifold. We claim that $TM\cong N\Delta$, where $N\Delta$ is the normal bundle of the diagonal as embedded in $M\times M$.
To see this, note first that $T(M\times M)\cong TM\oplus TM$. The tangent bundle of Δ now sits inside this as the diagonal: $TM\cong T\Delta\hookrightarrow TM\oplus TM$ induced by the diagonal map. Now, $N\Delta=(TM\oplus TM)/T\Delta$ by definition, and for each point $(p,v)\in TM$, note that $((p,v)\oplus TM)\cap T\Delta=((p,v),(p,v))$, meaning that each class in $N\Delta$ has a unique representative with second coordiante being 0, hence $(TM\oplus TM)/T\Delta\cong TM$.

Next, identify TM with TM_ε, the subbundle of small vectors, i.e., length at most $\varepsilon\ll\rho$ for ρ the injectivity radius of M [cf. Section 6]. With this identification, the map

$$\nu_M\colon TM\hookrightarrow M\times M,\quad \nu_M(x,v)=(x,\exp_x(v))$$

is a tubular neighborhood of Δ whose image is the ε neighborhood of the diagonal

$$\operatorname{im}\nu_M=U_M=\{(x,y)\in M\times M\mid |x-y|<\varepsilon\}.$$

Here the notion of distance makes sense because we assume M is a Riemannian manifold, and under this identification of TM, the bundle projection becomes the retraction $r\colon U_M\to M$ by $r(x,y)=x$.
From $\nu_M\colon TM\hookrightarrow M\times M$, we obtain a map

$$\nu_M^*\colon \Delta^n\left(M\times M, M\times M-\Delta(M)\right)\xrightarrow{\simeq}\Delta^n\left(TM,TM-M\right)$$

which is a quasi-isomorphism [cf. Section 6] as it induces an excision isomorphism. In $C^n(TM,TM-M)$, we have a representative of the Thom class for the tangent disk bundle, and we let τ_M denote the image of this representative under $(\nu_M^*)^{-1}$, i.e., $\tau_M=(v_M^*)^{-1}\left(D\left(i_{M*}^{TM}m\right)\right)$ where m is a representative for $[M]$.
We want to show a couple of things for τ_M. We want τ_M to have the property that

$$\tau_M\cap[M\times M]=\Delta_*[M]\in H_n(M\times M).$$

We can choose v_{M*} to map $[TM]$ to $[M\times M]\cong[M]\times[M]$, in which case,

$$v_M^*(\tau_M)\cap[TM]=i_{M*}^{TM}[M],$$

so applying v_{M*}, we get

$$\tau_M\cap[M\times M]=\tau_M\cap v_{M*}[TM]=v_{M*}i_{M*}^{TM}[M]=\Delta_*[M].$$

Secondly, we claim that the following map of chain complexes induces the intersection product on $H_*(M)$ up to the sign $(-1)^{n(n-q)}$:

$$\bullet_{Th}\colon\Delta_p(M)\otimes\Delta_q(M)\xrightarrow{\times}\Delta_{p+q}(M\times M)\xrightarrow{[\tau_M\cap]}\Delta_{p+q-n}(U_M)\xrightarrow{r_*}\Delta_{p+q-n}(M)$$

where the middle map is the following composition:

$$[\tau_M\cap] : \Delta_*(M\times M) \to \Delta_*\left(M \times M, M \times M - \Delta(M)\right) \overset{\simeq}{\to} \Delta_*\left(U_M, U_M - \Delta(M)\right) \overset{\tau_M\cap}{\to} \Delta_{*-n}(U_M)$$

with the middle map being a chain homotopy inverse to excision.

To see this, we follow [6]: consider the following diagram

$$\begin{tikzcd}
\Delta_p(M)\otimes\Delta_q(M) \arrow[r, "\times"] & \Delta_{p+q}(M\times M) \arrow[r, "{[\tau_M\cap]}"] & \Delta_{p+q-n}(U_M) \arrow[r, bend left, "r_*"] & \Delta_{p+q-n}(M) \arrow[l, "\Delta_*"'] \\
\Delta^{n-p}(M)\otimes\Delta^{n-q}(M) \arrow[u, "\cap m\otimes\cap m"] \arrow[r, "\times"] & \Delta^{2n-p-q}(M\times M) \arrow[u, "\cap m\times m"] \arrow[rr, "\Delta^*"] & & \Delta^{2n-p-q}(M) \arrow[u, "\cap m"']
\end{tikzcd}$$

where m represents $[M]$.
The vertical maps have quasi-inverses as the chain complexes are lower bounded, so we can talk about the top composition, which is $\bullet_{Th}$ compared to the other composition which induces $\bullet$.
The left hand square commutes up to chain homotopy and sign $(-1)^{n(n-q)}$, and the right square commutes directly. Also, $r_*\Delta_* = (r\Delta)_* = \mathrm{id}_*$, so the two compositions commute up to chain homotopy, hence induce the same product on homology.
Thus

$$a \bullet b = (-1)^{n(n-\deg a)}\Delta_!(b\times a) = (-1)^{n(n-\deg a)} a \bullet_{Th} b$$

In particular, this gives some intuition that we can think of $[\tau_M\cap]$ as intersecting with the diagonal under the assumption of transversality.

3.6. **Intersection Product Geometrically.** Below, manifolds will be assumed to be smooth, compact and oriented, and possibly with boundary.
Let $i_N^W : N^n \to W^w$ be a smooth embedding of smooth manifolds with boundary, and assume that N meets ∂W transversally in ∂N.

Definition 3.14 (Thom class)**.** In the above situation, denote $i_{N*}^W[N] \in H_n(W, \partial W)$ by $[N]_W$. Also define $\tau_N^W = D_W([N]_W) \in H^{w-n}(W)$, the *Thom class.* Here $D_W : H_n(W, \partial W) \xrightarrow{\cong} H^{w-n}(W)$. Thus

$$\tau_N^W \cap [W] = [N]_W$$

Suppose that K^k and N^n are two such submanifolds of W^w. Then

$$[K]_W \bullet [N]_W = \left(\tau_K^W \cap [W]\right) \bullet \left(\tau_N^W \cap [W]\right) = \left(\tau_N^W \cup \tau_K^W\right) \cap [W].$$

Assume now that $K \pitchfork N$ in W. Then $\nu_{K\cap N}^N = \nu_K^W|_{K\cap N}$ as a canonical isomorphism of vector bundles. (cf. [5, VIII, §11] for a homological interpretation as well).
This implies that the Thom class of K in W restricts to the Thom class of $K \cap N$ in N, since its restriction to any point gives the generator in the cohomology of the fiber modulo its boundary, and the Thom class is characterized by this.
Hence $\tau_{K\cap N}^W = (i_N^W)^* \left(\tau_K^W\right)$.

Theorem 3.15. *With the assumptions above, including $K \pitchfork N$ in W, we have*

$$\tau_{K\cap N}^W = \tau_K^W \cup \tau_N^W$$

and, equivalently,

$$[K \cap N]_W = [N]_W \bullet [K]_W .$$

Proof.

$$\begin{aligned}
[K \cap N]_W &= i_{(K\cap N)*}^W [K \cap N] \\
&= i_{N*}^W i_{(K\cap N)*}^N [K \cap N] \\
&= i_{N*}^W \left(\tau_{K\cap N}^N \cap [N]\right) \\
&= i_{N*}^W \left(i_N^{W*}(\tau_N^W) \cap [N]\right) \\
&= \tau_K^W \cap i_{N*}^W [N] \\
&= \tau_K^W \cap \left(\tau_N^W \cap [W]\right) \\
&= \left(\tau_K^W \cup \tau_N^W\right) \cap [W] \\
&= [N]_W \bullet [K]_W
\end{aligned}$$

□

Corollary 3.16. *Note that from proof, we obtain the following line:*

$$[K \cap N]_W = \tau_K^W \cap i_{N*}^W [N] = \tau_K^W \cap [N]_W$$

We interpret this result as saying that for transverse intersections, capping with the Thom class gives the fundamental class of the intersection in W, which gives another interesting property for the Thom class.

To use the above proof, we do seem to require that the chain is some form of nice space, like a closed manifold, and it is not immediately clear that this interpretation works on chains.

Theorem 3.17. *[1, VI, Theorem 11.10] Let M^n be an orientable closed manifold, and let $A, B \subset M$. Let $\alpha \in H_p(A)$ and $\beta \in H_q(B)$ and let α_M and β_M denote their images in $H_*(M)$. If $\alpha_M \bullet \beta_M \neq 0$, then $A \cap B \neq \varnothing$.*

Example 3.18. Take the transverse submanifolds $K = S^n \times \{y_0\}$ and $N = \{x_0\} \times S^m$ in $S^n \times S^m$. Then the fundamental classes of $[K]$ and $[N]$ in $W = S^n \times S^m$ generate $H_n(W)$ and $H_m(W)$, and the cup product of their duals generate $H^{n+m}(S^n \times S^m)$. This can be seen since $K \cap N = (x_0, y_0)$, so

$$[N]_W \bullet [K]_W = [K \cap N]_W = \pm [(x_0, y_0)]_W$$

which generates $H_0(S^n \times S^m)$. Hence $\tau_K^W \cup \tau_N^W$ is a generator of $H^{n+m}(S^n \times S^m)$. Likewise, if K' is a shifted copy of K, then $[K] \bullet [K] = [K] \bullet [K'] = [\varnothing] = 0$, so $\left(\tau_K^W\right)^2 = 0$.

Example 3.19. Consider $W = \mathbb{CP}^2$ and consider the copies of $\mathbb{CP}^1$ by $N = \{[z_0 : z_1 : 0]\}$ and $K = \{[0 : z_1 : z_2]\}$. Their intersection is a single point, so since they have transverse intersection, $[N]$ and $[K]$ are generators of $H_2\left(\mathbb{CP}^2\right)$, so $[N] = \pm [K]$. Thus $\left(\tau_N^W\right)^2 = \tau_N^W \cup \tau_K^W = \tau_{N \cap K}^W$ is a generator of $H^4\left(\mathbb{CP}^2\right)$.

Example 3.20. Let N and K be submanifolds of W and suppose they have an odd number of transverse intersection points. Then $[K] \in H_*\left(W; \mathbb{Z}/2\right)$ is nonzero since $[K] \bullet [N] = [K \cap N] = \left[\sum_{i=0}^{2k+1} x_i\right] =$

4. Geometric Homology Theory

For this section, we will closely follow [3] and recount some of the material developed there.

We have seen that the intersection product of two homology classes admits a nice geometric description when the homology classes can be represented by "nice enough" chains.
It is not obvious that this should be possible for other operations. Certain homology theories like bordism homology (See e.g. [4]) seem like promising theories to look at, but this is not in gemenral equivalent to singular homology classes. However, M. Jakob showed that if we add some extra cohomological information, then we obtain a certain geometric homology theory that is equivalent to singular homology and makes it possible to approach many operations in a fairly geometric way.
Let us start with constructing this homology theory.

4.0.1. *Geometric cycles.* Let X be a topological space. Then a geometric cycle is a triple (M, a, f) with $f\colon M \to X$ a continuous map, M a smooth, compact, connected and oriented manifold without boundary, and $a \in H^*(M;\mathbb{Z})$.
If M is n-dimensional and $a \in H^m(M;\mathbb{Z})$, then (M, a, f) is assigned degree $n - m$.
Let $\Delta'_k(X)$ be the free abelian group generated by all geometric cycles of degree k.
We impose the following relation:

$$(M, \lambda a + \mu b, f) = \lambda\,(M, a, f) + \mu\,(M, b, f)\,.$$

4.0.2. *Relations.*

(1) *Bordism relation.* Given a map $h\colon W \to X$ where W is an oriented bordism between $f(M)$ and $g(N)$, let $i_1\colon M \hookrightarrow X$ and $i_2\colon N \hookrightarrow X$ be the canonical inclusions. Then for any $c \in H^*(W;\mathbb{Z})$, we impose

$$(M, i_1^*(c), f) = (N, i_2^*(c), g)\,.$$

This essentially says that we consider cycles in X to be geometrically homologous if they are connected by a bordism.

(2) *Vector bundle modification.* Let (M, a, f) be a geometric cycle and consider a smooth oriented vector bundle $E \xrightarrow{\pi} M$ equipped with a Riemannian metric. Then we can take the unit sphere bundle $S\,(E \oplus 1) = \{v \in E \oplus 1 \mid \|v\| = 1\}$ of the Whitney sum of E with the trivial bundle over M. Consider the zero section $\sigma_0\colon M \to E$ and any arbitrary nowhere zero section $\sigma_1\colon M \to \mathbb{R}^k$, for example $\sigma_1(m) = (1, 0, \ldots, 0)$. Then $\sigma' = \sigma_0 \oplus \sigma_1$ is a nonzero section, hence $\sigma := \frac{\sigma'}{\|\sigma'\|}\colon M \to S\,(E \oplus 1)$ is a section. Let $\sigma^!$ be the transfer associated to this section. We then impose

$$(M, a, f) = \left(S\,(E \oplus 1)\,, \sigma^!(a), f\pi\right).$$

We denote the equivalence class of the geometric cycle (M, a, f) by $[M, a, f]$, and $H_q^{'}(X)$ the abelian group of geometric classes of degree q.

Theorem 4.1. *[7, Corollary 2.3.6] The morphism*

$$\begin{aligned} compar\colon H_q^{'}(X) &\to H_q(X;\mathbb{Z}) \\ [M, a, f] &\mapsto f_*\,(a \cap [M]) \end{aligned}$$

is a group isomorphism.

Theorem 4.2 (Cap Product and Poincaré duality)**.** *[3, §3.2, cites M. Jakob] The cap product between $H^*(X;\mathbb{Z})$ and $H'_*(X)$ is given by the following formula:*

$$\begin{aligned} \cap\colon H^p(X;\mathbb{Z}) \otimes H'_q(X) &\to H'_{q-p}(X) \\ u \cap [M,a,f] &= [M, f^*(u) \cup a, f]\,. \end{aligned}$$

Let M be a n n-dimensional smooth, compact, orientable manifold without boundary. Then the morphism

$$\begin{aligned} H^p(M;\mathbb{Z}) &\to H'_{n-p}(M) \\ x &\mapsto [M, x, \mathrm{id}_M] \end{aligned}$$

is an isomorphism.

4.0.3. *Transfer maps.* We want to carry over transfer maps to geometric homology. Recall that for a map $f\colon N^n \to M^m$, we defined $f_!\colon H_{m-p}(M) \to H_{n-p}(N)$ as the composite $f_! = D_N^{-1} f^* D_M$.
In the framework of geometric homology, we can give a very geometric interpretation of transfer maps.
Following [3, §3.3], start with $i\colon X \to Y$ an oriented morphism of Hilbert manifolds and suppose that the virtual bundle $V(i) = \ker di - \operatorname{coker} di$ has rank $-d$. Define

$$i_!\colon H_p^{'}(Y) \to H'_{p-d}(X)$$

as follows: let $[M,a,f] \in H'_p(Y)$ and choose a representing cycle (M,a,f). If f is not smooth, we can smoothly homotope it to a smooth one [3, §2.1.6] and moreover, we can choose it to be transverse to i [3, Theorem 2.2.4]. So suppose $F\colon M \times I \to Y$ is this homotopy between f and f'. There are retractions $r_L\colon M \times I \to M_L$ and $r_R\colon M \times I \to M_R$, where M_L, M_R are the left and right boundaries, respectively. It is the clear that $\tilde{a} = (r_L)^*(a) = (r_R)^*(a)$, hence $a = i_j^*(\tilde{a})$ for $j = 1, 2$, so by the bordism relation, $[M,a,f] = [M,a,f']$.
Then we can form the pullback:

$$\begin{tikzcd} M \times_Y X \arrow[r, "f^*i"] \arrow[d, "\varphi"] & M \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Theorem 4.3. *[3, Theorem 3.3.1] Let $i\colon X \to Y$ be an oriented morphism of Hilbert manifolds of codimension d, and $[M,a,f]$ a geometric cycle of $H'_{p-a}(Y)$ such that f and i are transverse. Then set*

$$i_!\left([M,a,f]\right) = (-1)^{d|a|}\left[M \times_Y X, \left(f^*i\right)^*(a), \varphi\right].$$

This gives a well-defined morphism

$$i_!\colon H'_*(Y) \to H'_{*-d}(X)$$

which satisfies

(1) *If we have a pullback of Hilbert manifolds X, Y :*

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "\psi"] & Y' \arrow[d, "\varphi"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

then $i_! \varphi_* = \psi_* j_!$,

(2) *Let i and j be two composable oriented morphisms of Hilbert manifolds. Then* $i_! j_! = (ji)_!$

(3) *Let $i_!^{PT}$ be the composition of the Pontryagin-Thom collapse κ and the Thom isomorphism th, then the following diagram commutes:*

$$\begin{tikzcd} H'_*(Y) \arrow[r, "i_!"] \arrow[d, "compar"'] & H'_{*-d}(X) \arrow[d, "compar"] \\ H_*(Y;\mathbb{Z}) \arrow[r, "i_!^{PT}"'] & H_{*-d}(X;\mathbb{Z}) \end{tikzcd}$$

4.0.4. *The intersection product.* Likewise, we can redefine the intersection product. First, the cross product in geometric homology is given by

$$\times\colon H'_p(X) \otimes H'_q(Y) \to H'_{p+q}(X \times Y)$$
$$[P, a, f] \times [Q, b, g] = (-1)^{\dim(P)|b|} [P \times Q, a \times b, f \times g],$$

such that if $\tau\colon X \times Y \to Y \times X$ is the map interchanging the coordinates, then we have

$$\tau_* (\alpha \times \beta) = (-1)^{|\alpha||\beta|} \beta \times \alpha.$$

Suppose now that M is an oriented, compact, d-dimensional manifold. Let $[P, x, f] \in H'_{n_1}(M)$ and $[Q, y, g] \in H'_{n_2}(M)$, and suppose $f \pitchfork g$. Then we form the pullback

$$\begin{tikzcd} P \times_M Q \arrow[r, hook, "j"] \arrow[d, "\varphi"] & P \times Q \arrow[d, "f\times g"] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd}$$

and define

$$\bullet\colon H'_{n_1}(M) \otimes H'_{n_2}(M) \xrightarrow{\times} H'_{n_1+n_2}(M \times M) \xrightarrow{\Delta^!} H'_{n_1+n_2-d}(M).$$

Hence,

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d(|a|+|b|)+\dim(P)|b|} [P \times_M Q, j^* (a \times b), \varphi].$$

For $l\colon P \times_M Q \to P$ and $r\colon P \times_M Q \to Q$ the canonical inclusions, we have

$$[P, a, f] \bullet [Q, b, g] = (-1)^{d(|a|+|b|)+\dim(P)|b|} [P \times_M Q, l^*(a) \cup r^*(b), \varphi].$$

This makes $H'_{*+d}(M)$ into a graded commutative algebra.

Note in particular that these definitions work on a chain level, so this definition of the intersection product gives a very geometric intuition as to what the intersection product does. In the transversal case, we take the intersection of the chains. Cf. [3] for further details.

5. String Topology

5.1. **Product and Coproduct.** Let LM denote the free loop space of M, i.e.,

$$LM = \mathrm{Map}\left(S^1, M\right).$$

Let $\mathrm{ev}_0\colon LM \to M$ denote the evaluation at 0.
We now introduce a product on the homology of the loop space of M, called the Chas-Sullivan product:

5.1.1. *A Geometric Homology Approach.* Consider the graded geometric homology group $\mathbb{H}_*(LM) := H'_{*+d}(LM;\mathbb{Z})$.
This group has some interesting structural properties, and string topology studies these underlying structures.

5.1.2. *The string pullback.* The evaluation map $\mathrm{ev}_0\colon LM \to M$ is, in fact, a smooth fiber bundle. To show this, one needs to first specify what the manifold structure of LM is. We will skip this part, but it is discussed in [3, §2.3.2].
Because ev_0 is a smooth fiber bundle, it can locally be represented as a submersion, hence is transverse to any other map, so $\mathrm{ev}_0 \times \mathrm{ev}_0$ is transverse to the diagonal map, and we can form the pullback

$$\begin{tikzcd}
LM \times_M LM \arrow[r, "\tilde{\Delta}"] \arrow[d, "\mathrm{ev}_0"] & LM \times LM \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_0"] \\
M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

By transversality, the pullback is again a Hilbert manifold. Also $LM \times_M LM \cong \mathrm{Map}(8, M)$, and $\tilde{\Delta}$ is a codimension d embedding. As the normal bundle $\nu_{\tilde{\Delta}}$ is the pullback of ν_Δ (cf. Lemma 7.2), and since this last one is isomorphic to TM. But then ν_Δ admits a structure group reduction to $\mathrm{GL}_n^+(\mathbb{R})$ of its frame bundle, and so pulling back along f^* the coordinate charts gives a structure group reduction of the frame bundle for $\nu_{\tilde{\Delta}}$. We deduce that $\tilde{\Delta}$ is an oriented morphism.

Definition 5.1. A family of closed string in M is a smooth map

$$f\colon P \to LM$$

from a compact oriented manifold P.

Proposition 5.2. *[3, Prop 2.3.7] The family $P \times Q \overset{f\times g}{\to} LM \times LM$ is transverse to $\tilde{\Delta}$ if and only if $\mathrm{ev}_0 f$ and $\mathrm{ev}_0 g$ are transverse in M.*

Now let (P, f) and (Q, g) be two families of dimensions p and q, respectively, such that $f \times g$ is transverse to $\tilde{\Delta}$. Denote by $P * Q$ the pullback

$$\begin{tikzcd}
P * Q \arrow[r] \arrow[d, "\psi"] & P \times Q \arrow[d, "f\times g"] \\
LM \times_M LM \arrow[r, "\tilde{\Delta}"] & LM \times LM
\end{tikzcd} \tag{Ω}$$

So we think of $P * Q$ as the points in $P \times Q$ that map under $f \times g$ to loops with a common basepoint. The map ψ is the map taking a point in $P * Q$ to this tuple of loops.

5.1.3. *Intersection of families of closed strings.* Define a map

$$\star\colon LM \times_M LM \to LM$$

by concatinationg loops. That is

$$\star(\alpha,\beta) = \alpha \star \beta(t) = \begin{cases} \alpha(2t), & t \in \left[0, \frac{1}{2}\right] \\ \beta(2t-1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

This is well-defined because we assume only that the loops are piecewise smooth. So for the two families of closed strings (P, f) and (Q, g) above, we can define a new family of closed strings by taking $P * Q$ and composing ψ from (Ω) with $\star$: $(P * Q, \star\psi)$.
So we essentially take a point in $P * Q$ and send it to the concatenation of its associated tuple of loops.
We can now for $[P, a, f] \in H'_{n_1+d}(LM)$ and $[Q, b, g] \in H'_{n_2+d}(LM)$, define a loop product $\wedge$ as follows: first smooth f and g and make them transverse to $\tilde{\Delta}$. Then we form the pullback $P * Q$.

Definition 5.3. [3, Definition 4.2.1] Let $j\colon P * Q \to P \times Q$ be the canonical maps. Then we have the pairing

$$\wedge\colon \mathbb{H}'_{n_1}(LM) \otimes \mathbb{H}'_{n_2}(LM) \to \mathbb{H}'_{n_1+n_2}(LM)$$
$$[P, a, f] \wedge [Q, b, g] = (-1)^{d(|a|+|b|)+\dim(P)|b|} \left[P * Q, j^*(a \times b), \star\psi\right]$$

which is called the *Chas-Sullivan loop product*.

Note in particular, that if we disregard the cohomology element, the data consists of $P * Q$ and the associated loops which makes it easy to think about loop products.

Proposition 5.4. *The loop product is associative and commutative.*

Proof. Associativity follows from associativity of the intersection product, the cup product and also, up to homotopy, of $\star$.
Now consider the map

$$Ev_1\colon I \times LM \times LM \to M \times M$$

given by $Ev_1(t, \gamma_1, \gamma_2) = (\gamma_1(0), \gamma_2(t))$. We can take the pullback

$$\begin{tikzcd} \mathrm{Map}\,(8_t, M) \arrow[r, "\tilde{\Delta}_t"] \arrow[d] & I \times LM \times LM \arrow[d, "Ev_1"] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd}$$

where $\mathrm{Map}\,(8_t, M) = \{(t, \gamma_1, \gamma_2) \mid \gamma_1(0) = \gamma_2(t)\}$. We also have a map

$$comp_1\colon \mathrm{Map}\,(8_t, M) \to LM$$

given by

$$comp_1(t, \gamma_1, \gamma_2) = \begin{cases} \gamma_2(2\theta), & \theta \in [0, \frac{t}{2}) \\ \gamma_1(2\theta - t), & \theta \in [\frac{t}{2}, \frac{t+1}{2}) \\ \gamma_2(2\theta - 1), & \theta \in [\frac{t+1}{2}, 1] \end{cases}$$

I.e, we essentially insert γ_1 into γ_2 at the point corresponding to t.
Let also τ denote the interchanging map, $(\alpha, \beta) \mapsto (\beta, \alpha)$.

$$\tau\colon LM \times_M LM \to LM \times_M LM.$$

We want now to take $I \times P * Q$ and on $\{0\} \times P * Q$ put $\star\psi$ and on $\{0\} \times P * Q$ put $\star\tau\psi$.

$$\begin{tikzcd}
\mathrm{Map}(8_t, M) \times_{I \times LM \times LM} I \times P \times Q \arrow[r] \arrow[d, "\tilde{\psi}"] & I \times P \times Q \arrow[d, "\mathrm{id} \times f \times g"] \\
\mathrm{Map}(8_t, M) \arrow[r, "\tilde{\Delta}_t"] & I \times LM \times LM
\end{tikzcd}$$

This pullback consists of pairs $((t, \gamma_1, \gamma_2), (t, p, q))$ such that $f(p) = \gamma_1$ and $g(q) = \gamma_2$ and $\gamma_1(0) = \gamma_2(t)$, which is thus the same as tuples (t, p, q) such that $f(p)(0) = g(q)(t)$.
If $(\gamma_1, \gamma_2) \in LM \times_M LM$, then $(0, p, q)$ and $(1, p, q)$ are in the pullback, corresponding under $comp_1 \circ \tilde{\psi}$ to $\gamma_1 \star \gamma_2$ and $\gamma_2 \star \gamma_1$, respectively. So this gives a cobordism between

$$[P * Q, j^*(a \times b), \star\psi]$$

and

$$\begin{aligned}
[P * Q, \tau^*(j^*(a \times b)), \star\tau\psi] &= (1)^{(\dim(P) - d - |a|)(\dim(Q) - d - |b|)} [P * Q, j^*(b \times a), \star\tau\psi] \\
&= (-1)^{(\dim(P) - d - |a|)(\dim(Q) - d - |b|)} [Q * P, j^*(b \times a), \star\psi]
\end{aligned}$$

□

5.1.4. *A Chain-level approach.* We now turn to an approach to the Chas-Sullivan product which lives on the chain level. We base the remaining parts of these notes on [8] and [6].

The idea to think about for the Chas-Sullivan product is to start with a p-chain $x \in \Delta_p(LM)$ and a q-chain $y \in \Delta_q(LM)$. Then taking the intersection of the transverse chains $\mathrm{ev}_0\, x$ and $\mathrm{ev}_0\, y$, which form a p- and q-chain in M, gives a $(p + q - n)$-chain z in M. Along this chain z, we can compose the loops of x and y, and we define $x \wedge y$ to be this element of $\Delta_{p+q-n}(LM)$.

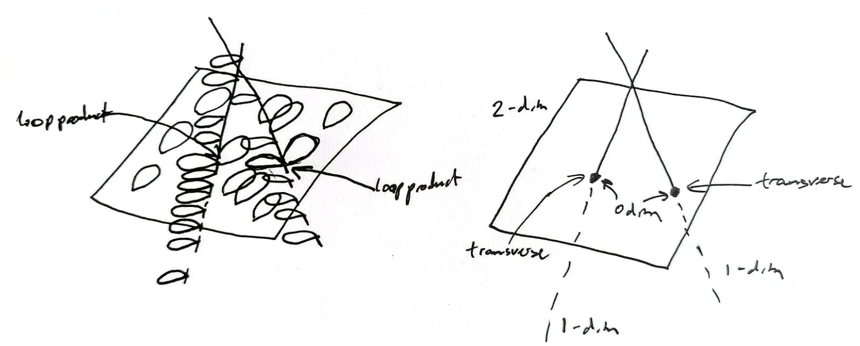

Figure 1. The transverse intersection of two 1-chains and a 2-chain in a 3-manifold giving a 0-chain as a loop product. [2, From Figure 2]

Definition 5.5 (Chas-Sullivan product $\wedge$)**.** The *Chas-Sullivan product* $\wedge$ is a lift of the intersection product making the following diagram commute:

$$\begin{tikzcd} H_p(LM) \otimes H_q(LM) \arrow[r, "\wedge"] \arrow[d, "\mathrm{ev}_0 \otimes \mathrm{ev}_0"] & H_{p+q-n}(LM) \arrow[d, "\mathrm{ev}_0"] \\ H_p(M) \otimes H_q(M) \arrow[r, "\bullet"] & H_{p+q-n}(M) \end{tikzcd}$$

We now explain the construction.
Recall the ε-neighborhood U_M of the diagonal in $M \times M$ and now just pull this back along $\mathrm{ev}_0 \times \mathrm{ev}_0$ to $LM \times LM$:

$$U_{CS} = (\mathrm{ev}_0 \times \mathrm{ev}_0)^{-1} U_M = \{(\gamma, \lambda) \in LM \times LM \mid |\gamma(0) - \lambda(0)| < \varepsilon\}.$$

Defining a retraction

$$R_{CS} \colon U_{CS} \to LM \times_M LM = \{(\gamma, \lambda) \in LM \times LM \mid \gamma(0) = \lambda(0)\} = (\mathrm{ev}_0 \times \mathrm{ev}_0)^{-1} (\Delta_M \subset M \times M)$$

by concatenating with a "geodesic stick" to connect the loops so that they form a "figure 8":

$$R_{CS}(\gamma, \lambda) = (\gamma, \lambda') \quad \text{with} \quad \lambda' = \overline{\gamma(0)\lambda(0)} \star \lambda \star \overline{\lambda(0)\gamma(0)}$$

where for $x, y \in M$ with $|x - y| < \rho$, $\overline{xy}$ denotes the unique minimal geodesic path $[0, 1] \to M$ from x to y. Figure 2 describes this intuitively.

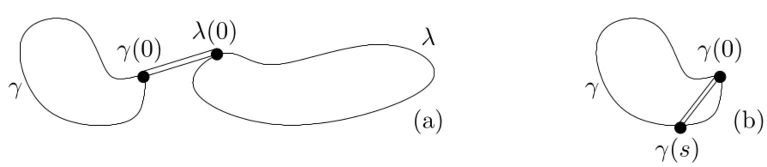

FIGURE 2. Figures/HGFHEIA.png

This is a lift of the retraction $r \colon U_M \to M$ in the sense that

$$\begin{tikzcd} U_{CS} \arrow[r, "R_{CS}"] \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_0"'] & LM \times_M LM \arrow[d, "\mathrm{ev}_0"] \\ U_M \arrow[r, "r"] & M \end{tikzcd}$$

commutes.
Pulling back τ_M along $\mathrm{ev}_0 \times \mathrm{ev}_0$ gives a cochain

$$\tau_{CS} := (\mathrm{ev}_0 \times \mathrm{ev}_0)^* \tau_M \in C^* (LM \times LM, LM \times LM - LM \times_M LM)$$

Finally, we define the Chas-Sullivan loop product on chains as follows:

Definition 5.6 (Chas-Sullivan loop Product)**.** The following sequence of chain maps induces the Chas-Sullivan product on homology:

$$\wedge \colon C_p(LM) \otimes C_q(LM) \overset{\times}{\to} C_{p+q}(LM \times LM) \overset{[\tau_{CS} \cap]}{\to} C_{p+q-n}(U_{CS})$$
$$\overset{R_{CS}}{\to} C_{p+q-n}(LM \times_M LM) \overset{\text{concat}}{\to} C_{p+q-n}(LM).$$

While this definition coincides with the Chas-Sullivan product in homology as described by Cohen-Jones, it is not necessarily clear how it coincides with the original geometric picture. However, Proposition 5.8 guarantees tells us that it does.

5.1.5. *Coproduct.* Define $e_I\colon LM \times I \to M \times M$ by $e_I(\gamma, s) = (\gamma(0), \gamma(s))$. Set

$$U_{GH} = e_I^{-1} U_M = \{(\gamma, s) \in LM \times I \mid |\gamma(0) - \gamma(s)| < \varepsilon\},$$

and we have a commutative diagram

$$\begin{tikzcd} U_{GH} \arrow[r, "R_{GH}"] \arrow[d, "e_I"] & \mathcal{F} = \{(\gamma, s) \in LM \times I \mid \gamma(0) = \gamma(s)\} \arrow[r, hook] \arrow[d, "e_I"] & LM \times I \arrow[d] \\ U_M \arrow[r, "r"] & \Delta M \arrow[r, hook] & M \times M \end{tikzcd}$$

defining a retraction map R_{GH} by concatenating with a geodesic stick to force self-intersection:

$$R_{GH}(\gamma, s) = (\gamma', s) \quad \text{with} \quad \gamma' = \gamma\,[0,s] \star \overline{\gamma(s)\gamma(0)} \star_s \overline{\gamma(0)\gamma(s)} \star [s, 0]$$

where the parametrization is chosen so that the concatenated loop passes through $\gamma(0)$ at time s; if $s = 0$ or 1 as in the case $\gamma(0) = \gamma(s)$ to begin with, the geodesic sticks are of length 0.

Definition 5.7 (Goresky-Hingston-Sullivan coproduct)**.** The following sequence of chain maps is a chain model for the Goresky-Hingston-Sullivan coproduct:

$$\vee\colon C_p(LM, M) \xrightarrow{\times I} C_{p+1}(LM \times I, LM \times \partial I \cup M \times I) \xrightarrow{[\tau_{GH} \cap]} C_{p+1-n}(U_{GH}, LM \times \partial I \cup M \times I)$$
$$\xrightarrow{R_{GH}} C_{p+1-n}(\mathcal{F}, LM \times \partial I \cup M \times I) \xrightarrow{cut} C_{p+q-n}(LM \times LM, M \times LM \cup LM \times M)$$

This induces a well-defined degree $1 - n$ map

$$\vee\colon H_p(LM, M) \to H_{p+1-n}(LM \times LM, M \times LM \cup LM \times M).$$

5.2. **Computation via geometric intersection.**

Proposition 5.8. *[6, Propositions 3.1 and 3.7]*

(1) *If $Z_1\colon \Sigma_1 \to LM$ and $Z_2\colon \Sigma_2 \to LM$ are smooth cycles with the property that the maps $\mathrm{ev}_0 \circ Z_1\colon \Sigma_1 \to M$ and $\mathrm{ev}_0 \circ Z_2\colon \Sigma_2 \to M$ are transverse, then the loop product*

$$Z_1 \wedge Z_2 = (Z_1 \star Z_2)\,|_{\Sigma_1 \times_{ev_0} \Sigma_2} \in H_*(LM)$$

is the concatenation of the loops of Z_1 and Z_2 along the locus of the basepoint-intersections $\Sigma_1 \times_{\mathrm{ev}_0} \Sigma_2 \subset \Sigma_1 \times \Sigma_2$, oriented as stated in [6]

(2) *If $Z\colon (\Sigma, \Sigma_0) \to (LM, M)$ is a smooth relative cycle with the property that the restriction of $e_I \circ (Z \times I)\colon \Sigma \times I \to M \times M$ to $(\Sigma - \Sigma_0) \times (0,1)$ is transverse to the diagonal, then*

$$\vee Z = cut \circ (Z \times I)\,|_{\overline{\Sigma_\Delta}} \in H_*(LM \times LM, M \times LM \cup LM \times M)$$

for $\overline{\Sigma_\Delta}$ the closure in $\Sigma \times I$ of the locus of basepoint self-intersecting loops $\Sigma_\Delta \subset (\Sigma - \Sigma_0) \times (0,1)$, oriented as stated in [6].

Note. Here $\Sigma_1 \times_{\mathrm{ev}_0} \Sigma_2$ is the pullback

$$\begin{tikzcd}
\Sigma_1 \times_{\mathrm{ev}_0} \Sigma_2 \arrow[r] \arrow[d] & \Sigma_2 \arrow[d, "\mathrm{ev}_0 \circ Z_2"] \\
\Sigma_1 \arrow[r, "\mathrm{ev}_0 \circ Z_1"'] & M
\end{tikzcd}$$

5.2.1. *Lens spaces.* Consider $S^3 \subset \mathbb{C}^2$ as $(r, \theta) = ((r_1, \theta_1), (r_2, \theta_2))$ with $\theta_i \in \mathbb{R}/\mathbb{Z}$ and $r_i \geq 0$, satisfying $r_1^2 + r_2^2 = 1$. Here we think of S^3 as $\left\{r_1 e^{i\theta_1}, r_2 e^{i\theta_2}\right\}$ with $r_1^2 + r_2^2 = 1$.
In particular, when r_i is zero, $(r_i, \theta_i) = 0$.
The lens space $\mathcal{L}_{p,q}$, for p, q coprime, is the quotient space $S^3/\mathbb{Z}/p$ with the $\mathbb{Z}/p$ action defined by

$$((r_1, \theta_1), (r_2, \theta_2)) \mapsto \left(\left(r_1, \theta_1 + \frac{1}{p}\right), \left(r_2, \theta_2 + \frac{q}{p}\right)\right).$$

There is a residual torus action on the lens space by

$$\begin{aligned}
\alpha \colon \left(S^1 \times S^1\right) \times \mathcal{L}_{p,q} &\to \mathcal{L}_{p,q}, \\
((s, t), (r, \theta)) &\mapsto \left(\left(r_1, \theta_1 + \frac{s}{p}\right), \left(r_2, \theta_2 + \frac{sq}{p} + t\right)\right)
\end{aligned}$$

Using this residual torus action, we define cycles $Z_{l,m}$ for pairs of integers (l, m) as follows: let

$$\delta_{l,m} \colon S^1 \to S^1 \times S^1$$

be the torus knot $t \mapsto (lt, mt)$ of slope $\frac{l}{m}$. We can combine this loop with the action α of the torus on $\mathcal{L}_{p,q}$ to get a family

$$\begin{aligned}
Z_{l,m} \colon \mathcal{L}_{p,q} &\to L\mathcal{L}_{p,q} \\
(r, \theta) &\mapsto \left[\gamma_{r,\theta}^{l,m} \colon t \mapsto \alpha\left(\delta_{l,m}(t), (r, \theta)\right)\right]
\end{aligned}$$

associated to each point (r, θ) in the lens space, the loop $\gamma_{r,\theta}^{l,m}$ is based at this point and follows the image of $\delta_{l,m}$ along the torus action.
That is, the loop is defined by

$$\gamma_{r,\theta}^{l,m}(t) = \left(\left(r_1, \theta_1 + \frac{lt}{p}\right), \left(r_2, \theta_2 + \frac{ltq}{p} + mt\right)\right).$$

Note now that Lens spaces are manifolds, and in particular, $\mathcal{L}_{p,q}$ are compact oriented 3-manifolds, so we can pick a fundamental class $[\mathcal{L}_{p,q}]$.
Denote by

$$Z_{l,m} = (Z_{l,m})_* [\mathcal{L}_{p,q}] \in H_3\left(L\mathcal{L}_{p,q}\right)$$

the associated homology class. Suppose we now postcompose with ev_0, i.e., we look at the class

$$(\mathrm{ev}_0)_* Z_{l,m} \in H_3\left(\mathcal{L}_{p,q}\right)$$

we note that the composition $\mathrm{ev}_0 \circ Z_{l,m}$ is just the identity, so this just sends $[\mathcal{L}_{p,q}]$ to itself. Thus also $Z_{l,m} \in H_3\left(L\mathcal{L}_{p,q}\right)$ is non-trivial.

5.2.2. *The loop product of the classes $Z_{l,m}$.* Let $Z_1 = Z_{l_1,m_1}$ and $Z_2 = Z_{l_2,m_2}$. Since $\mathrm{ev}_0 \circ Z_i$ is the identity for both i, $\mathrm{ev}_0 \circ Z_1$ is transverse to $\mathrm{ev}_0 \circ Z_2$. Furthermore, the locus of basepoint-intersections is the pullback

$$\begin{tikzcd} \mathcal{L}_{p,q} \times_{\mathrm{ev}_0} \mathcal{L}_{p,q} \arrow[r] \arrow[d] & \mathcal{L}_{p,q} \arrow[d, "\mathrm{ev}_0 \circ Z_2"] \\ \mathcal{L}_{p,q} \arrow[r, "\mathrm{ev}_0 \circ Z_1"'] & \mathcal{L}_{p,q} \end{tikzcd}$$

so

$$\begin{aligned} \mathcal{L}_{p,q} \times_{\mathrm{ev}_0} \mathcal{L}_{p,q} &= \{((r_1,\theta_1),(r_2,\theta_2)) \in \mathcal{L}_{p,q} \times \mathcal{L}_{p,q} \mid (r_1,\theta_1) = \mathrm{ev}_0 Z_1(r_1,\theta_1) = \mathrm{ev}_0 Z_2(r_2,\theta_2) = (r_2,\theta_2)\} \\ &= \Delta\mathcal{L}_{p,q} \end{aligned}$$

Hence using Proposition 5.8, we obtain that $Z_{l_1,m_1} \wedge Z_{l_2,m_2}$ sends each point (r,θ) of $\mathcal{L}_{p,q} \cong \Delta\mathcal{L}_{p,q}$ to the concatenated loop $Z_{l_1,m_1} \star Z_{l_2,m_2}$ based at (r,θ), where $\star$ is the concatenation of the loops in the image at their common basepoint.

We may as whether we can determined the loop product and coproduct of these classes. Computations of both are carried out in **??** and we recount some of it here in shorter form, following the source material closely.

Firstly, we turn to the product.

Proposition 5.9. *The Chas-Sullivan loop product of the classes $Z_{l,m} \in H_3(L\mathcal{L}_{p,q})$ defined above is given by summing the indices:*

$$Z_{l_1,m_1} \wedge Z_{l_2,m_2} = Z_{l_1+l_2,m_1+m_2}.$$

Proof. By the above considerations and Proposition 5.8, have

$$Z_{l_1,m_1} \wedge Z_{l_2,m_2} = (Z_{l_1,m_1} \star Z_{l_2,m_2})\,|_{\Delta\mathcal{L}_{p,q}} \colon \mathcal{L}_{p,q} \cong \Delta\mathcal{L}_{p,q} \to L\mathcal{L}_{p,q}$$

where $\star$ is the concatenation of the loops in the image at their common basepoint, i.e., it is the map $\mathcal{L}_{p,q} \to L\mathcal{L}_{p,q}$ sending $(r,\theta) \mapsto \gamma_{r,\theta}^{l_1,m_1} * \gamma_{r,\theta}^{l_2,m_2}$ which is the image under the torus action of the concatenated loops (l_1,m_1) and (l_2,m_2) in the torus. Concatenation corresponds to addition in the fundamental group, so the concatenation becomes $(l_1,m_1) * (l_2,m_2) = (l_1+l_2, m_1+m_2)$, so $\gamma_{r,\theta}^{l_1,m_1} * \gamma_{r,\theta}^{l_2,m_2} = \gamma_{r,\theta}^{l_1+l_2,m_1+m_2}$. This homotopy descends to the lens space, so up to homotopy in the codomain, $Z_{l_1,m_1} \wedge Z_{l_2,m_2} = Z_{l_1+l_2,m_1+m_2}$, and thus the classes agree in $H_3(L\mathcal{L}_{p,q})$. □

Now that we have determined the loop product of the classes, we turn to coproducts.

5.2.3. *B-classes.* Recall that the coproduct is a degree $1-3=-2$ map on $H_*(L\mathcal{L}_{p,q}, \mathcal{L}_{p,q})$:

$$\vee \colon H_3(L\mathcal{L}_{p,q}, \mathcal{L}_{p,q}) \to H_1(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}, \mathcal{L}_{p,q} \times L\mathcal{L}_{p,q} \cup L\mathcal{L}_{p,q} \times \mathcal{L}_{p,q})$$

The coproduct of the classes $Z_{l,m}$ will be given in terms of B-classes in the target. Let $\lambda \colon S^1 \to \mathcal{L}_{p,q}$ be the loop defined by $\lambda(t) = \left((1,\frac{t}{p}),0\right)$ tracing the points $(r,\theta) \in \mathcal{L}_{p,q}$ with $(r_2,\theta_2) = 0$. Then λ is a generator of $\pi_1\mathcal{L}_{p,q} \cong \mathbb{Z}/p$.
Note that

$$Z_{1,0}((1,0),0) = \gamma_{((1,0),0)}^{1,0} = \left[t \mapsto \left((1,\frac{t}{p}),0\right)\right] = \lambda$$

Projecting a path from $((1,0),0)$ to $(0,(1,0))$ under $Z_{1,0}$ gives a free homotopy from $\gamma^{1,0}_{((1,0),0)}$ to $\gamma^{1,0}_{(0,(1,0))}$. Note that

$$\gamma^{1,0}_{(0,(1,0))} = \left[t \mapsto \left(0,(1,\frac{qt}{p})\right)\right] = (\lambda')^{\star q}$$

for $\lambda' \colon S^1 \to \mathcal{L}_{p,q}$ defined by $\lambda'(t) = \left(0,(1,\frac{t}{p})\right)$.

Definition 5.10. Define the 1-cycles $B_{k,k'}$ and $B'_{k,k'}$ in $\mathcal{L}_{p,q}$ as follows: let $\lambda_s \colon S^1 \to \mathcal{L}_{p,q}$ be the rotation of λ, based at $\lambda(s)$,:

$$\lambda_s(t) = \lambda(s+t).$$

Similarly for λ'. Define

$$\begin{aligned} B_{k,k'} \colon S^1 &\to L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q} \subset L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}, \\ s &\mapsto \left((\lambda_s)^{\star k}, (\lambda_s)^{\star k'}\right) \end{aligned}$$

We also denote by $B_{k,k'} \in H_1\left(L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q}\right)$ or $H_1\left(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}\right)$ the associated homology class $(B_{k,k'})_* \left[S^1\right]$. Then the evaluation $(\mathrm{ev}_0)_* : H_1\left(L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q}\right) \to H_1\left(\mathcal{L}_{p,q}\right)$ takes

$$B_{k,k'} = (B_{k,k'})_* \left[S^1\right] \mapsto (ev_0 \circ B_{k,k'})_* \left[S^1\right] = \lambda_* \left[S^1\right] = [\lambda]$$

Define $B'_{k,k'}$ in the same way by replacing λ by λ' above.

Lemma 5.11. *[8, Lemma 2.6] Let $B'_{k,k'} \colon S^1 \to L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q}$ be the family of figure eights based at the points of λ' defined by $B'_{k,k'}(s) = \left((\lambda'_s)^{\star k}, (\lambda'_s)^{\star k'}\right)$. Then*

$$B_{k,k'} = qB'_{qk,qk'} \in H_1\left(L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} L\mathcal{L}_{p,q}\right)$$

is the sum of q copies of the class $B'_{qk,qk'}$.

Lemma 5.12. *[8, Lemma 2.7] We have that*

(1) $B_{k,k'} = B_{h,h'} \in H_1\left(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}\right)$ *if and only if* $k = h \mod p$ *and* $k' = h' \mod p$.

(2) *The relative classes*

$$\{B_{k,k'}\}_{\substack{0<k<p \\ 0<k'<p}} \in H_1\left(L\mathcal{L}_{p,q} \times L\mathcal{L}_{p,q}, \mathcal{L}_{p,q} \times L\mathcal{L}_{p,q} \cup L\mathcal{L}_{p,q} \times \mathcal{L}_{p,q}\right)$$

are linearly independent over $\mathbb{Z}/p$.

Proposition 5.13. *[8, Proposition 2.8] The coproduct of the classes $Z_{l,m} \in H_3\left(L\mathcal{L}_{p,q}, \mathcal{L}_{p,q}\right)$ with $l,m \geq 0$ is given by the formula*

$$\vee Z_{l,m} = \sum_{\substack{0<k<l \\ k,(l-k)\neq 0 \mod p}} B_{k,l-k} + q' \sum_{\substack{0<k<ql+pm \\ k,(l-kq')\neq 0 \mod p}} B_{kq',l-kq'}$$

where q' is the multiplicative inverse of $q \mod p$.

Proof. We will only give the part of the proof that is necessary for us to understand the next section.

We want to make use of (2) in Proposition 5.8.

To do this, we need $e_I \circ (Z \times I) : \mathcal{L}_{p,q} \times I \to \mathcal{L}_{p,q} \times \mathcal{L}_{p,q}$ restricted to $(L\mathcal{L}_{p,q} - \mathcal{L}_{p,q}) \times (0,1)$ to be transverse to the diagonal, where e_I evaluates the loops at 0 and $s \in (0,1) \subset I$ (here $\mathcal{L}_{p,q} \subset L\mathcal{L}_{p,q}$ is the locus of constant loops).

Recall that $Z_{l,m}$ takes (r,θ) to $\left[\gamma^{l,m}_{r,\theta} : t \mapsto \alpha\left(\delta_{l,m}(t),(r,\theta)\right)\right]$. If $(l,m) = (0,0)$, then $\delta_{0,0}$ is constant at $(0,0)$ and so α is constant at (r,θ), so $Z_{0,0}$ maps (r,θ) to the constant loop at (r,θ). Thus $Z_{0,0} = 0 \in H_3\left(L\mathcal{L}_{p,q}, \mathcal{L}_{p,q}\right)$.

Conversely, if $Z_{l,m}(r,\theta)$ is constant then $\left[\gamma^{l,m}_{r,\theta} : t \mapsto \left(\left(r_1, \theta_1 + \frac{lt}{p}\right), \left(r_2, \theta_2 + \frac{ltq}{p} + mt\right)\right)\right]$ is constant at (r,θ), so l and m must be 0, hence if $(l,m) \neq (0,0)$, $Z_{l,m}$ has no constant cycles in its image.
We can assume that $(l,m) \neq (0,0)$ and work with $(\Sigma, \Sigma_0) = (\mathcal{L}_{p,q}, \varnothing)$.
Transversality: represent the homology class of $Z_{l,m}$ by the homotopic family $\tilde{Z}_{l,m} \colon \mathcal{L}_{p,q} \to L\mathcal{L}_{p,q}$ defined by $\tilde{Z}_{l,m}(r,\theta) = \tilde{\gamma}^{l,m}_{r,\theta}$ for $\tilde{\gamma}^{l,m}_{r,\theta} \colon S^1 \to \mathcal{L}_{p,q}$ the loop based at (r,θ) given by

$$\tilde{\gamma}^{l,m}_{r,\theta}(t) = \left(\left(\tilde{r}_1(t), \theta_1 + \frac{lt}{p}\right), \left(\tilde{r}_2(t), \theta_2 + \frac{ql+pm}{p}t\right)\right),$$

where $(\tilde{r}_1(t), \tilde{r}_2(t))$ is a deformation of (r_1, r_2) with $(\tilde{r}_1(t), \tilde{r}_2(t)) = (r_1, r_2)$ only for $r_1 = 0$ or $r_2 = 0$, or when $t = 0$ or 1. For example, we can let

$$\tilde{r}_1(t) = \begin{cases} (1-2t)r_1 + 2tr_1^2, & t \in \left[0, \frac{1}{2}\right] \\ r_1^2(2-2t) + r_1(2t-1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and $\tilde{r}_2(t) = \sqrt{1 - \tilde{r}_1(t)^2}$.

We can visualize this as follows: foliate $S^4 \cong \mathbb{R}^3 \cup \{\infty\}$ by tori. Then for fixed radii, the angle coordinates trace out a torus. Then $\mathbb{Z}/p$ still acts on this torus, and it is on this quotient torus, that $\gamma^{l,m}_{r,\theta}$ lives. So understanding self-intersections amounts to understanding self-intersections on this quotient torus of the torus knot one is interested in.
For example, if $m = 0$, then for $l \geq 1$, $Z_{l,m}\,(r,\theta)$ has $l-1$ self-intersections.
But for large l and m, these loops will have many self-intersections. What $\tilde{Z}_{l,m}$ does is that it makes it so that, as we vary t, we also vary the radius such that only the torus knots corresponding to the endpoints live on the original torus knot corresponding to $Z_{l,m}$. See Figure 3
Then the map $e_I \circ \left(\tilde{Z}_{l,m} \times \mathrm{id}\right)|_{\mathcal{L}_{p,q} \times (0,1)}$ intersects the diagonal when $\tilde{\gamma}^{l,m}_{r,\theta}$ has a self-intersection $\tilde{\gamma}^{l,m}_{r,\theta}(0) = \tilde{\gamma}^{l,m}_{r,\theta}(t)$ for some $t \in (0,1)$.
Since the r coordinates must match, such a self-intersection can only happen for $r_1 = 0$ or $r_2 = 0$ since otherwise $\tilde{r}_i(t) \neq \tilde{r}_i(0) = r_i$.
For $r_2 = 0$, we have that $\tilde{r}_1(t) = r_1 = 1$ and $\tilde{r}_2(t) = r_2 = 0$, and

$$\tilde{\gamma}^{l,m}_{r,\theta}(t) = \left(\left(1, \theta_1 + \frac{lt}{p}\right), 0\right)$$

The first angle coincides with θ_1 precisely when $\frac{lt}{p} =\in \frac{1}{p}\mathbb{Z}$, i.e., when $t \in \frac{1}{l}\mathbb{Z} \cap (0,1)$.

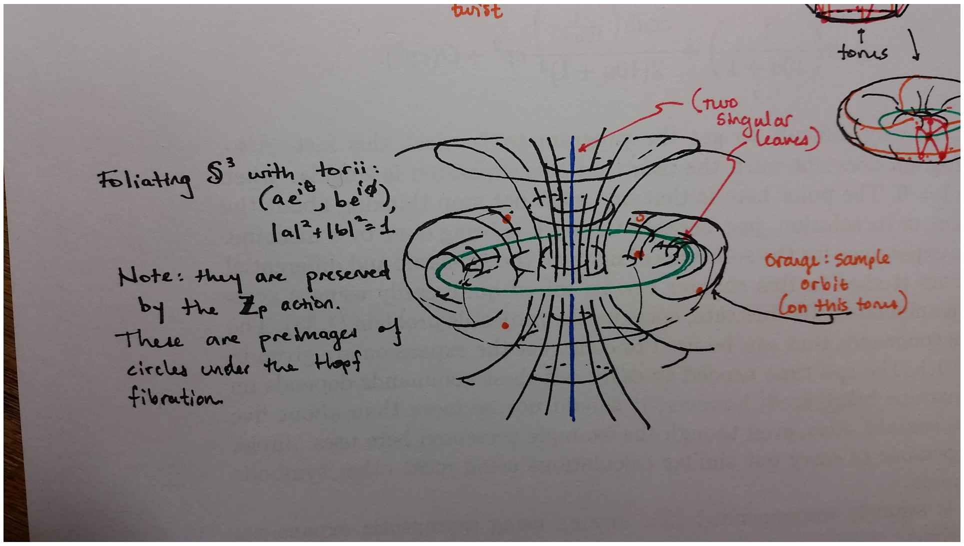

FIGURE 3. Figures/foliating-tori.jpg

When $r_1 = 0$, we get

$$\tilde{\gamma}^{l,m}_{r,\theta}(t) = \left(0, \left(1, \theta_2 + \frac{ql+pm}{p}t\right)\right)$$

Since $\gcd(p,q) = 1$, Bezout gives that $\frac{k}{p} = 0$ in the second coordinate, so the second angle coincides with θ_2 whenever $\frac{ql+pm}{p}t \in \frac{1}{p}\mathbb{Z}$.
This yields the following parameters:

$$\begin{cases} 0 < t = \frac{a}{l} < 1, \text{ for some } a \in \mathbb{N}, & r_2 = 0 \\ 0 < t = \frac{b}{ql+pm} < 1 \text{ for some } b \in \mathbb{N}, & r_1 = 0. \end{cases}$$

So the locus of self-intersections of $\tilde{Z}_{l,m} \times \mathrm{id}|_{\mathcal{L}_{p,q}\times(0,1)}$ is

$$\Sigma_\Delta = (\lambda \times I_1) \cup (\lambda' \times I_2) \subset \mathcal{L}_{p,q} \times (0,1)$$

where $I_1 = \left\{\frac{1}{l}, \ldots, \frac{l-1}{l}\right\}$, $I_2 = \left\{\frac{1}{ql+pm}, \ldots, \frac{ql+pm-1}{ql+pm}\right\}$, and λ, λ' are the loops parametrizing the points with $r_2 = 0$ and $r_1 = 0$, respectively.
Note that λ and λ' correspond to the two degenerate cases of tori in the foliation.

The rest of the proof is a check of transversality at these points. We will skip this part and refer the reader to [8].

$\square$

5.3. **Other potential operations.** Consider the 3-simplex $\Delta^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \le x_1 \le x_2 \le x_3 \le 1\}$, and consider the homology group
Since $(\Delta^3, \partial\Delta^3) \cong (D^3, S^2)$, so

$$H_3\left(\Delta^3/\partial\Delta^3\right) \cong H_3\left(D^3, S^2\right) \cong \mathbb{Z}.$$

We want to see if we can construct an interesting homomorphism

$$H_3\left(L\mathcal{L}_{p,q}\right) \to H_0\left(L\mathcal{L}_{p,q}\right) \cong \mathbb{Z}^p$$

By the above, we can consider "products" of the form

$$H_3\left(L\mathcal{L}_{p,q}\right) \otimes H_3\left(\Delta^3/\partial\Delta^3\right) \to H_0\left(L\mathcal{L}_{p,q}\right)$$

or more generally,

$$H_p\left(LM\right) \otimes H_q\left(\Delta^3/\partial\Delta^3\right) \to H_{p+q-2n}\left(LM\right)$$

which is only really interesting for $q = 0, 3$.

Let $E\colon LM \times \Delta^3/\partial\Delta^3 \to M^4$ be defined by $E\left(\gamma, (x_1, x_2, x_3)\right) = (\gamma(0), \gamma(x_2), \gamma(x_1), \gamma(x_3))$. Define a map $\Delta\times\Delta\colon M^2 \hookrightarrow M^4$ by $\Delta\times\Delta\,(m, n) = (m, m, n, n)$. Form the pullback

$$\begin{tikzcd}
LM \times_{M^4} \Delta^3/\partial\Delta^3 \arrow[r] \arrow[d] & LM \times \Delta^3/\partial\Delta^3 \arrow[d, "E"] \\
M^2 \arrow[r, "\Delta\times\Delta"] & M^4
\end{tikzcd}$$

This pullback consists of loops γ for which there exist some $0 < x_1 < x_2 < x_3 < 1$ such that $\gamma(0) = \gamma(x_2)$ and $\gamma(x_1) = \gamma(x_3)$.
On this space, we define an operation
$\Upsilon\colon LM \times_{M^4} \Delta^3/\partial\Delta^3 \to LM$ given by concatinating the loop as shown in the figure

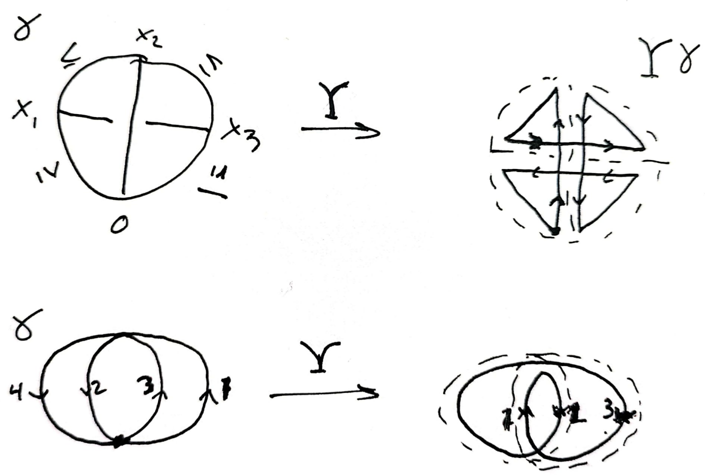

FIGURE 4. Υ of a loop γ

(or Υ could potentially be some other operation).

Now similarly to the construction of the coproduct, we can define a map $LM \times \Delta^3/\partial\Delta^3 \to LM \times_{M^4} \Delta^3/\partial\Delta^3$ by, for an element $(\gamma, (x_1, x_2, x_3))$, inserting a geodesic stick between $\gamma(0)$ and $\gamma(x_2)$ as well as another geodesic stick between $\gamma(x_1)$ and $\gamma(x_3)$, and then forming the desired loop by concatinating the different parts.
To make this precise, let us start by considering the diagonal embedding $\Delta \times \Delta\colon M \times M \hookrightarrow M^2 \times M^2$. Its normal bundle is isomorphic to the tangent bundle $T(M \times M) \cong TM \oplus TM$. Again identify $T(M \times M)$ with $T(M \times M)_\varepsilon$, $\varepsilon \ll \rho$, ρ the injectivity radius of $M \times M$.
Then the map

$$\nu_{M\times M}\colon T(M \times M) \hookrightarrow M^4 \quad \nu_{M\times M}((x,v),(y,u)) = \left(x, \exp_x v, y, \exp_y u\right)$$

gives a tubular neighborhood of $\Delta \times \Delta$, with image the ε-image of the diagonal

$$\nu_{M\times M}\colon T(M \times M) \xrightarrow{\cong} U_1 = \{(x, x', y, y') \mid |x - x'|, |y - y'| < \varepsilon\}$$

and under this identification, the bundle projection map $T(M \times M) \to M \times M$ becomes the retraction $r\colon U_1 \to M \times M$ by $r(x, x', y, y') = (x, y)$.
Let $\tau_{M\times M}$ be the image of a cochain representative for the Thom class for $T(M \times M)$ under the quasi-isomorphism

$$C^{2n}(T(M \times M), T(M \times M) - M \times M) \xrightarrow{\simeq} C^{2n}\left(M^4, M^4 - \Delta \times \Delta(M^2)\right)$$

Now let

$$U_2 = E^{-1}(U_1) = \{(\gamma, (x_1, x_2, x_3)) \mid |\gamma(0) - \gamma(x_2)|, |\gamma(x_1) - \gamma(x_3)| < \varepsilon\}$$

and set $\tau_{\Delta^3} = E^*(\tau_{M\times M})$.
On this neighborhood, we can concatenate with the desired geodesic sticks, so we obtain a retraction $R\colon U_2 \to LM \times_{M^4} \Delta^3/\partial\Delta^3$ sending

$$R(\gamma, (x_1, x_2, x_3)) = (\gamma', (x_1, x_2, x_3))$$

where γ' is the desired loop such that the following diagram commutes:

$$\begin{tikzcd}
U_2 \arrow[r, "R"] \arrow[d, "E"] & LM \times_{M^4} \Delta^3/\partial\Delta^3 \arrow[r, hook] \arrow[d, "E"] & LM \times \Delta^3/\partial\Delta^3 \arrow[d, "E"] \\
U_1 \arrow[r, "r"] & \Delta(M \times M) \arrow[r, hook] & M^4
\end{tikzcd}$$

Now define the following operation composite:

$$\begin{aligned}
\wedge_{\Delta^3}\colon C_p(LM) \otimes C_q(\Delta^3/\partial\Delta^3) \xrightarrow{\times} C_{p+q}\left(LM \times \Delta^3/\partial\Delta^3\right) &\xrightarrow{[\tau_{\Delta^3}\cap]} C_{p+q-2n}(U_2) \\
&\xrightarrow{R} C_{p+q-2n}\left(LM \times_M \Delta^3/\partial\Delta^3\right) \xrightarrow{\Upsilon} C_{p+q-2n}(LM).
\end{aligned}$$

where the middle map $[\tau_{\Delta^3}\cap]$ is defined as the composite

$$\begin{aligned}
[\tau_{\Delta^3}\cap]\colon C_*\left(LM \times \Delta^3/\partial\Delta^3\right) &\to C_*\left(LM \times \Delta^3/\partial\Delta^3, LM \times \Delta^3/\partial\Delta^3 - LM \times_M \Delta^3/\partial\Delta^3\right) \\
&\xrightarrow{\sim} C_*\left(U_2, U_2 - LM \times_M \Delta^3/\partial\Delta^3\right) \xrightarrow{\tau_{\Delta^3}\cap} C_{*-2n}(U_2)
\end{aligned}$$

Note that for $A \in C_*\left(LM \times \Delta^3/\partial\Delta^3\right)$, $[\tau_{\Delta^3}\cap](A) = \tau_{\Delta^3}\cap\rho(A)$ where $\rho\colon C_*\left(LM \times \Delta^3/\partial\Delta^3\right) \to C_*\left(LM \times \Delta^3/\partial\Delta^3\right)$ is a chain map which subdivides simplices in such a way that $\rho(A)$ is homologous to A and every simplex σ in $\rho(A)$ has support in either U_2 or

$LM \times \Delta^3/\partial\Delta^3 - U_{2,\varepsilon_0}$, and hence $\tau_{\Delta^3} \cap \rho(A)$ has support in U_2. Capping with the Thom class then kills all the simplices with support in $LM \times \Delta^3/\partial\Delta^3 - U_{2,\varepsilon_0}$ (Cf. [6]).

Hence, for example, we obtain a product

$$\wedge_{\Delta^3}: H_3\left(L\mathcal{L}_{p,q}\right) \otimes H_3\left(\Delta^3/\partial\Delta^3\right) \cong H_3\left(L\mathcal{L}_{p,q}\right) \to H_0\left(L\mathcal{L}_{p,q}\right) \cong \mathbb{Z}^p$$

Question 5.14. Consider now the classes $Z_{l,m}$ from before. One could ask what the operation Υ does to $Z_{l,m}$ on points where it satisfies the desired condition.

Now, $Z_{l,m}$ maps (r,θ) to

$$\gamma^{l,m}_{r,\theta}(t) = \left(\left(r_1, \theta_1 + \frac{lt}{p}\right), \left(r_2, \theta_2 + \frac{ltq}{p} + mt\right)\right).$$

Suppose we choose the points x_1, x_2, x_3.
Since $Z_{l,m}(r,\theta)$ is contained in a torus, our loops $\Upsilon Z_{l,m}(r,\theta)$ will be $Z_{l,m}[x_2,x_1] \star Z_{l,m}[x_3,x_2] \star Z_{l,m}[0,x_3] \star Z_{l,m}[x_1,0]$.
Now, let $S^1 \times S^1/\mathbb{Z}/p$ denote the torus quotiented out by the action of $\mathbb{Z}/p$ on it. Then we find that

$$\pi_1\left(S^1 \times S^1/\mathbb{Z}/p\right) \cong \langle a, b, c \mid ab^q = c^p\rangle^{ab}$$

So $Z_{l,m}(r,\theta)$ corresponds to $c^l b^m$.
Now, in $S^1 \times S^1/\mathbb{Z}/p$, we can take a path from $Z_{l,m}\left(r,\theta\right)\left(x_1\right) = Z_{l,m}\left(r,\theta\right)\left(x_3\right)$ to $Z_{l,m}(r,\theta)(0)$ and we can homotope our loop $Z_{l,m}\left(r,\theta\right)$ such that $Z_{l,m}\left(r,\theta\right)(0) = Z_{l,m}(r,\theta)(x_i)$ for $i = 1,2,3$. See figure 5

FIGURE 5.

But in this loop, each of the segments $Z_{l,m}[x_i,x_j]$ is a loop of its own, hence lies in $\pi_1\left(S^1 \times S^1/\mathbb{Z}/p\right)$ which is abelian, so in particular,

$$\begin{aligned}
\wedge_{\Delta^3} Z_{l,m}(r,\theta) &= Z_{l,m}[x_2,x_1] \star Z_{l,m}[x_3,x_2] \star Z_{l,m}[0,x_3] \star Z_{l,m}[x_1,0] \\
&\simeq Z_{l,m}[0,x_3] \star Z_{l,m}[x_3,x_2] \star Z_{l,m}[x_2,x_1] \star Z_{l,m}[x_1,0] \\
&= \overline{Z_{l,m}(r,\theta)}
\end{aligned}$$

where the bar denotes the inverse loop.
The question is whether this would also work for other $\tilde{Z}_{l,m}$ for example. Does it simplify to a simpler form?
We could likewise define other operations. For example, consider the following operation $\tilde{\Upsilon}$:
Using the same approach as before, this becomes $\gamma \mapsto \gamma[x_2,0] \star \gamma[x_2,1]$, i.e, first half up to x_2 is γ reversed up to x_2, and the second part is γ normally.

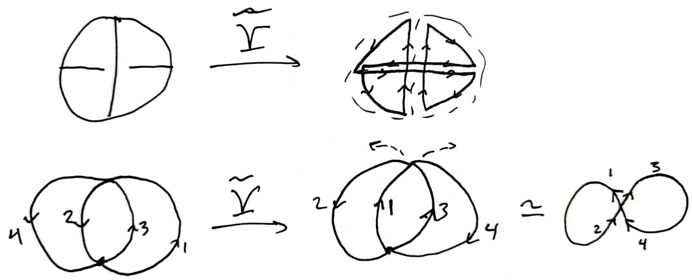

FIGURE 6. $\tilde{\Upsilon}$

What does this do to a loop?

We now turn to a different question.

Question 5.15. We have the following composition:

$$C_3(\mathcal{L}_{p,q}) \overset{Z_{l,m}}{\to} C_3\left(L\mathcal{L}_{p,q}\right) \overset{\times\Delta^3/\partial\Delta^3}{\to} C_6\left(L\mathcal{L}_{p,q} \times \Delta^3/\partial\Delta^3\right) \overset{R\circ[\tau_{\Delta^3}\cap]}{\to} C_0\left(L\mathcal{L}_{p,q} \times_M \Delta^3/\partial\Delta^3\right),$$

and we are interested in what the image of this composition is.

By definition, it will be a finite sum of loops with marked points, $(\gamma, (x_1, x_2, x_3))$, in $L\mathcal{L}_{p,q} \times_M \Delta^3/\partial\Delta^3$ with coefficients in $\mathbb{Z}$.
Let us consider what this composition is doing. For some chain σ in $\mathcal{L}_{p,q}$, we apply $Z_{l,m}$ to give a chain in $L\mathcal{L}_{p,q}$. Then we take the cross product with the quotient 3-chain $\Delta^3 \to \Delta^3/\partial\Delta^3$, which gives a 6-chain in $L\mathcal{L}_{p,q} \times \Delta^3/\partial\Delta^3$. Then $[\tau_{\Delta^3}\cap]$ subdivides this chain until simplices are either in U_2 or its complement, but still remaining homologous to the original chain, and then we apply τ_{Δ^3}, which has support in U_2, essentially only leaves the chains contained in U_2. In U_2, we can apply R, and the resulting 0-chain is the image. But furthermore, if the subdivided chain is transverse to $L\mathcal{L}_{p,q} \times_M \Delta^3/\partial\Delta^3$, the by Corollary 3.16, capping with the Thom class will just give us the intersection of the chain with $L\mathcal{L}_{p,q} \times_M \Delta^3/\partial\Delta^3$. So the problem reduces to figuring out how to make $Z_{l,m} \times \Delta^3/\partial\Delta^3$ traverse to $L\mathcal{L}_{p,q} \times_M \Delta^3/\partial\Delta^3$. But since we know that we must end in C_0 of the latter, we know that we must reduce to finite intersections.
So let us start with the $\tilde{Z}_{l,m}$ from [8].
Recall that its locus of self-intersection was

$$\Sigma_\Delta = (\lambda \times I_1) \cup (\lambda' \times I_2) \subset \mathcal{L}_{p,q} \times (0,1)$$

where $I_1 = \left\{\frac{1}{l}, \ldots, \frac{l-1}{l}\right\}$ and $I_2 = \left\{\frac{1}{pl+pm}, \ldots, \frac{ql+pm-1}{ql+pm}\right\}$, and λ, λ' are the loops parametrizing the degenerate tori (circles) at the endpoints $r_2 = 0$ and $r_1 = 0$, respectively.
For $0 < x_1 < x_2 < x_3 < 1$, if $\lambda(x_1) = \lambda(x_3)$, then $x_3 - x_1 \in \frac{1}{l}\mathbb{Z}$. If $l > 1$, there are infinitely many such pairs.

For this, we must essentially break the periodicity in the angle coordinate.
Let $g(t) = \sin(\alpha(l,m)t) - t\sin(\alpha(l,m))$. Consider the functions $\tilde{\theta}_1(t) = \theta_1 + \frac{lt}{p} + g(t)$ and $\tilde{\theta}_2(t) = \theta_2 + \frac{lqt}{p} + g(t)$, and define $Z'_{l,m}(r,\theta) = \gamma = \left[t \mapsto \left(\left(\tilde{r}_1(t), \tilde{\theta}_1(t)\right), \left(\tilde{r}_2(t), \tilde{\theta}_2(t)\right)\right)\right]$.
For $Z'_{l,m}(r,\theta)$ to be in $L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} \Delta^3/\partial\Delta^3$, we would need for some $0 < x_1 < x_2 < x_3 < 1$ that $\gamma(x_1) = \gamma(x_3)$ and $\gamma(0) = \gamma(x_2)$. Just as before, we deduce that $r_1 = 0$ or $r_2 = 0$. Next if $\gamma(x_1) = \gamma(x_3)$, we must have $g(x_1) - g(x_3) = \frac{l}{p}(x_3 - x_1)$ or $(\frac{lq}{p} + m)(x_3 - x_1)$.
If we simply choose $\alpha(l,m)$ to be small enough, we can guarantee that this never happens - for example looking at the functions' graphs. So $Z'_{l,m}$ can be made to not intersect $L\mathcal{L}_{p,q} \times_{\mathcal{L}_{p,q}} \Delta^3/\partial\Delta^3$.

6. Terminology

Definition 6.1 (Neighborhood retract)**.** If $A \subset X$ and A has a neighborhood in X of which it is a retract, then A is called a *neighborhood retract* (in X).

Note. Saying that $A \hookrightarrow X$ is a cofibration is stronger than saying that A is a neighborhood retract.

Definition 6.2 (Exponential Map)**.** Let M be a smooth manifold and $p \in M$. One can define the notion of a straight line through the point p.
Let $v \in T_pM$ be a tangent vector to the manifold at p. Then there is a unique geodesic $\gamma_v\colon [0,1] \to M$ satisfying $\gamma_v(0) = p$ with initial tangent vector $\gamma_v'(0) = v$. The corresponding *exponential map* is defined by $\exp_p(v) = \gamma_v(1)$.

Definition 6.3 (Injectivity Radius)**.** The injectivity radius of a Riemannian manifold at a point p is the supremum of all radii for which the exponential map at p is a diffeomorphism onto its image.

7. Lemmas

Lemma 7.1. *Let $\pi\colon W \to N$ be a covering map and M a connected space. Suppose $f, g\colon M \to W$ are maps such that $\pi \circ f = \pi \circ g$ and that $f(x) = g(x)$ for some $x \in M$. Then $f = g$.*

Proof. Show that the set
$$Z = \{z \in M \mid f(z) = g(z)\}$$
is closed and open. □

Lemma 7.2. *Suppose $A \subset Y$, and $f\colon X \to Y$ is transverse to A. Then we claim that $f^*(N_AY) \cong N_{f^{-1}A}(X)$. That is, the pullback of the normal bundle of A in Y is the normal bundle of the preimage of A in X.*

Proof. We have an induced map pointwise
$$\begin{tikzcd}
T_xX \arrow[r, "df"] \arrow[d] & T_{f(x)}Y \arrow[d] \\
(N_{f^{-1}(A)}X)_x = T_xX/T_xf^{-1}(A) \arrow[r, dashed, "\overline{df}"] & T_{f(x)}Y/T_{f(x)}A = (N_AY)_{f(x)}
\end{tikzcd}$$
defining $\overline{df}\colon N_{f^{-1}(A)}X \to N_AY$. We also have the bundle map $\pi\colon N_{f^{-1}}(A)X \to X$, thus inducing a map $\varphi\colon N_{f^{-1}}X \to f^*N_AY$ by $\varphi(x,[v]) = \big(x, \overline{df}\,[v]\big)$. Now, $\overline{df}$ can be seen to be injective directly, and surjectivity of $\overline{df}$ follows from surjectivity of df which follows directly from our assumption of transversality. □

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