1: Let $F = x^3 + y^3 - 2xyz$, and consider $\mathbb{V}(F) \subset \mathbb{P}^2_{\mathbb{C}}$.
(a) The tangent line to $\mathbb{V}(F)$ at $[1:1:1]$ is the projective closure of the tangent line to $V(f)$ at $(1,1)$ where $f = x^3 + y^3 - 2xy$. The tangent line to $V(f)$ at $(1,1)$ is $f_x(1,1)(x-1) + f_y(1,1)(y-1)$. Now $f_x = 3x^2 - 2y$ and $f_y = 3y^2 - 2x$, so $f_x(1,1) = 1 = f_y(1,1)$, so $T_{(1,1)}V(f) = V(x+y-2)$. Hence the tangent line to $\mathbb{V}(F)$ becomes $\mathbb{V}(x+y-2z)$.

(b) P is a singular point of $\mathbb{V}(F)$ if and only if $F(P) = F_x(P) = F_y(P) = F_z(P) = 0$.
Now, $F_x = 3x^2 - 2yz, F_y = 3y^2 - 2xz$ and $F_z = -2xy$. If $x = 0$ then $y = 0$, so $z \neq 0$. If $y = 0$ then $x = 0$ so $z \neq 0$. Since F_z must be zero at a singular point, either no singular point is in U_3^c.
So the only singular point is $[0:0:1]$.

(c) The multiplicity of $[0:0:1]$ is the multiplicity of $(0,0)$ for $f = x^3 + y^3 - 2xy$ which is 2 and the tangent cone is $V(xy) = V(x) \cup V(y)$.

2: Let $f = y^4 + y^3 - x^2$, and consider $V(f) \subset \mathbb{A}^2_{\mathbb{C}}$.
(a) The homogenization of f is $F = y^4 + y^3z - x^2z^2$, so the projective closure of $V(f)$ is $\mathbb{V}(y^4 + y^3z - x^2z^2)$.

(b) F is Eisenstein in x^2, and hence irreducible. Now $F_x = -2xz^2, F_y = 4y^3 + 3y^2z$ and $F_z = y^3 - 2zx^2$. If $x = 0$ then $y = 0$, so $z \neq 0$. If $z = 0$ then $y = 0$, so $x \neq 0$. So the only singular points are $[0:0:1]$ and $[1:0:0]$.

(c) The multiplicity of $[0:0:1]$ is the multiplicity of $(0,0)$ of f which is 2 and the tangent cone is $V(x^2) = V(x)$ so the tangent cone for $[0:0:1]$ is $\mathbb{V}(x)$.
The mult of $[1:0:0]$ is the mult of $(0,0)$ of $g = y^4 + y^3z - z^2$ which is 2 and the tangent cone of $[1:0:0]$ is $\mathbb{V}(z)$.

3:
(a)

$$\begin{aligned} I_P(xy + y^3, x^2 + 2xy + y^3) &= I_P\left(y(x+y^2), x(x+y)\right) \\ &= I_P(y,x) + I_P(y, x+y) + I_P(x+y^2, x) + I_P(x+y^2, x+y) \\ &= 1 + 1 + 2 + I_P(y(y-1), x+y) \\ &= 4 + \underbrace{I_P(y, x+y)}_{=1} + I_P(y-1, x+y) \\ &= 5. \end{aligned}$$

If we instead follow the algorithm:
First reduce to $I_P(y, x^2 + 2xy + y^3) + I_P(x + y^2, x^2 + 2xy + y^3)$ by (6).
Now, the first one reduces to $I_P(y, x^2) = 2$.
The second is $I_P(x+y^2, 2xy) = I_P(x+y^2, x) + I_P(x+y^2, y) = 2 + 1 = 3$.
So we recover $I_P\left(xy + y^3, x^2 + 2xy + y^3\right) = 5$.

(b) We have that the tangent cone of $y^2 - x^3$ is $V(y)$ and the tangent cone of $xy + x^4 + y^4$ is $V(x) \cup V(y)$.
Let $f = y^2 - x^3$ and $g = xy + x^4 + y^4$. Then $f(x,0) = -x^3$ and $g(x,0) = x^4$.
We have case 2 in (6), with $\deg f \leq \deg g$, so write $h_2 = g + xf = xy + y^4 + xy^2$. Then $I_P(f,g) = I_P(f, h_2)$.
Now back to step (2).
$h_2 = y(x + y^3 + xy)$, so h_2 and f have no common factor.
(3) $(0,0) \in V(f) \cap V(h_2)$.
(4) Now the tangent cone of h_2 is $V(xy) = V(x) \cup V(y)$, so $V(y)$ is again in common. Now $h_2(x,0) = 0$, so $\deg h_2(x,0) \leq \deg f(x,0)$. Write $h_2 = yh$, so $h = x + y^3 + xy$. Now, by (6), we have $I_P(f, h_2) = I_P(y,f) + I_P(h,f)$, and $I_P(y,f) = 3$, and $I_P(h,f) = I_P(x + y^3 + xy, y^2 - x^3)$. Back to (2).
Tangent cone of $x + y^3 + xy$ is $V(x)$ and tangent cone of $y^2 - x^3$ is $V(y)$. No lines in common, so $I_P(x + y^3 + xy, y^2 - x^3) = \mathrm{mult}_{(0,0)}(x + y^3 + xy)\mathrm{mult}_{(0,0)}(y^2 - x^3) = 2$. Thus $I_P(f,g) = 3 + 2 = 5$.

(c) Let $f = x^2 + y^2 + x^4y^4$ and $g = x^3 - y^3 + 3xy^5$.
We have that the tangent cone of f at P is $V(x^2 + y^2)$ and the tangent cone of g at P is $V(x^3 - y^3) = V((x-y)(x^2 + xy + y^2)) = V(x-y) \cup V(x^2 + xy + y^2)$. Since they have no lines in common, we have $I_P(f,g) = \mathrm{mult}_P(f)\mathrm{mult}_P(g) = 2 \cdot 3 = 6$.

4: Let $\alpha = \frac{x^2+xy}{3y^2} \in k(\mathbb{P}^1)$. What is the value of α at $[1:3] \in \mathbb{P}^1$? Where is α defined?

Solution: Since α is a fraction of homogeneous polynomials and $3y^2$ is nonzero at $[1:3]$, we have that α is defined at $[1:3]$ and is given by $\frac{1^2+1\cdot 3}{3\cdot 3^2} = \frac{4}{27}$. Now α is defined whenever $y \neq 0$, so α is defined on U_2.

5: It is by definition of closure an algebraic set.
Suppose $\overline{\varphi(X)} = U \cup V$ with U and V algebraic sets which are both proper subsets of $\overline{\varphi(X)}$. We have $\varphi^{-1}(U \cup V) = \varphi^{-1}(U) \cup \varphi^{-1}(V) \supset X$, hence $X = U' \cup V'$ where $U' = \varphi^{-1}(U) \cap X$ and $V' = \varphi^{-1}(V) \cap X$ which are both open. Hence $U' = X$ without loss of generality. So $\varphi^{-1}(U) = X$, but then $\varphi(X) = \varphi\left(\varphi^{-1}(U)\right) \subset U$, so $\overline{\varphi(X)} \subset \overline{U}$. Now We know U is open, and since $U \cup V = \overline{\varphi(X)}$, we have that U is closed in $\overline{\varphi(X)}$, so since $\overline{\varphi(X)}$ is closed in $\mathbb{A}^m$, we have that U is closed in $\mathbb{A}^m$ too. So U is both closed and open. Hence $\overline{U} = U$, so $\overline{\varphi(X)} = U$, contradiction.

6: Let $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^3$ be the morphism given by $t \mapsto \left(t, t^2, t^3\right)$. Let x, y, z be the coordinates on $\mathbb{A}^3$.
(a) We have that $\Gamma(\mathbb{A}^3) = k\left[x,y,z\right]/I\left(\mathbb{A}^3\right) = k\left[x,y,z\right]/(0) = k\left[x,y,z\right]$, and similarly, $\Gamma(\mathbb{A}^1) = k\left[t\right]$.
Now $\varphi^*(x)(t) = x\left(\varphi(t)\right) = x(t,t^2,t^3) = t$, $\varphi^*(y)(t) = y\left(t,t^2,t^3\right) = t^2$ and similarly, $\varphi^*(z)(t) = t^3$.

(b) The image of φ is indeed closed as it is $V\left(y - x^2, z - x^3\right)$. Namely, for any $t \in \mathbb{A}$, we have $\varphi(t) = \left(t, t^2, t^3\right)$ which clearly lies in $V(y - x^2, z - x^3)$. Conversely, if $(x,y,z) \in V(y - x^2, z - x^3)$ then $y = x^2$ and $z = x^3$, so $(x,y,z) = (x, x^2, x^3) = \varphi(x) \in \operatorname{Im}\varphi$.
In fact, this also follows directly from the fact that φ^* maps $x \mapsto t$ and is thus surjective. By a proposition, we then have that $\operatorname{Im}\varphi$ is an algebraic set and even that φ is an isomorphism of the domain onto its image.

(c) This follows from the last comment of (b).
Alternatively, the map $\psi\colon \mathbb{A}^3 \to \mathbb{A}^1$ by $(x,y,z) \mapsto x$ is a polynomial map given by $P \mapsto (T(P))$ where $T(x,y,z) = x$. Furthermore, suppose $(x,y,z) \in \operatorname{Im}\varphi$. Then $\varphi \circ \psi(x,y,z) = \varphi(x) = (x, x^2, x^3) = (x,y,z)$, and $\psi \circ \varphi(x) = \psi(x, x^2, x^3) = x$, so indeed φ is an isomorphism.

(d) We have
$$\varphi^{-1}\left(V(yz - x^5)\right) = V\left(\varphi^*(yz - x^5)\right) = V\left(t^2t^3 - t^5\right) = V(0) = \mathbb{A}^1.$$

7: Let $X = V\left(x^2z, x^2 + xz + yz + y^2\right) \subset \mathbb{A}^3_{\mathbb{C}}$. If $x^2z = 0$ then either $x = 0$ or $z = 0$. Suppose $x = 0$, then $yz + y^2 = y(z+y) = 0$, so either $y = 0$ or $z = -y$.
If instead $z = 0$ then $x^2 + y^2 = (x+iy)(x-iy) = 0$, so either $x = -iy$ or $x = iy$. Thus
$$X = V(x, z+y) \cup V(x+iy) \cup V(x-iy)$$
where $V(x,y) \subset V(x+iy)$.
These are irreducible since $\Gamma(V(x, z+y)) \cong k\left[x\right]$ which is an integral domain, so $I\left(V\left(x, z+y\right)\right)$ is prime and hence $V(x, z+y)$ is irreducible. Now $V(x+iy)$ and $V(x-iy)$ are irreducible since $\Gamma(V(x+iy)) \cong k\left[t,s\right] \cong \Gamma(v(x-iy))$ which is also an integral domain.

8: Let $J = \left(y^2 - x^2, y^2 + x^2\right) \subset \mathbb{C}\left[x,y\right]$.

(a) If $(x,y) \in V(J)$ then $y^2 - x^2 = 0 = y^2 + x^2$ so $x = 0$ and hence $y = 0$. So $V(J) = \{(0,0)\}$.

(b)
$$\dim_{\mathbb{C}} \mathbb{C}\left[x,y\right]/J = \dim_{\mathbb{C}} \mathbb{C}\left[x,y\right]/\left(y^2 - x^2, y^2 + x^2\right)$$
In $\mathbb{C}\left[x,y\right]/J$, we have that $y^n = y^2y^{n-2} = 0$ for $n \geq 2$ since $\frac{1}{2}\left[y^2 - x^2 + y^2 + x^2\right] = y^2 \in J$ and similarly $x^2 = 0$. Hence $\mathbb{C}\left[x,y\right]/J$ is generated by $\{1, x, y, xy\}$, so $\dim_{\mathbb{C}} \mathbb{C}\left[x,y\right]/J = 4$.

(c)
$$I_{(0,0)}\left(y^2 - x^2, y^2 + x^2\right) = I_{(0,0)}\left(y^2 - x^2, 2y^2\right) = I_{(0,0)}\left(-x^2, 2y^2\right) = I_{(0,0)}(x^2, y^2) = 4I_{(0,0)}(x,y) = 4.$$

(d) We have that $\mathbb{V}(J)$ is the projectivization of $V(J) = \{(0,0)\}$, so $\mathbb{V}(J) = \varnothing$.

9: Find the projective closure of $V(x + y^3 + z) \subset \mathbb{A}^3$ in $\mathbb{P}^3$. Is it smooth?

Solution: The projective closure is the vanishing of the homogenization:

$$\mathbb{V}\left(xw^2 + y^3 + zw^2\right) \subset \mathbb{P}^3$$

This is smooth if it has no singular points. Now $xw^2 + y^3 + zw^2$ is Eisenstein in w^2, and hence has no repeated factors. So a point $P \in \mathbb{P}^3$ is singular if and only if for $F = xw^2 + y^3 + zw^2$, we have $F(P) = F_x(P) = F_y(P) = F_z(P) = 0$. Now $F_x = w^2, F_y = 3y^2, F_z = w^2, F_w = 2xw + 2zw$. So if $P = [x, y, z, w]$ then $w = y = 0$. So any points of the form $\{a : 0 : b : 0\}$ for either $a \neq 0$ or $b \neq 0$ is singular. Hence the closure is not smooth. This can be verified since e.g. $[1 : 0 : 0 : 0] \in U_1$ and hence letting $f = w^2+y^3+zw^2$, we have that $f_y = 3y^2, f_z = w^2, f_w = 2w+2zw$ all of which disappear at $(0, 0, 0)$, so $(0, 0, 0)$ is a singular point of $V(f)$ and hence $[1 : 0 : 0 : 0]$ is a singular point of $\mathbb{V}\left(xw^2 + y^3 + zw^2\right)$.

10:

(1) Suppose L is a finite extension of k. This means that $[L : k]$ is finite, i.e., that L as a k-vector space has finite dimension, which means that there exist $l_1, \ldots, l_n$ such that $L = \sum kl_i$, i.e., L is finitely generated as a k-module. Now, L is an algebraic extension over k if for any $l \in L$, there exists some polynomial $f \in k[x]$ such that $f(l) = 0$. Now, $1, l, l^2, \ldots, l^n$ are linearly dependent as the dimension of the k-vector space L is n, so there exist $k_0, \ldots, k_n$ such that $k_0 + k_1 l + \ldots + k_n l^n = 0$ and hence the polynomial $f = k_0 + k_1 x + \ldots + k_n x^n \in k[x]$ is a polynomial such that $f(l) = 0$, so l is algebraic over k. As l was arbitrary, we have that L is an algebraic extension of k.

(2) False. Not all algebraic extensions are finite.
Consider $k = \mathbb{Q}$ and $L = \mathbb{Q}\left[\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \ldots\right]$. Then L is clearly algebraic over k; however, L is not a finite extension as a basis is the infinte set $\left\{\sqrt[n]{2}\right\}_{n\geq 2}$.

(3) True. Suppose $\varphi \colon X \to Y$ is an isomorphism with inverse $\psi \colon Y \to X$. Let first $y \in Y$. Then $\varphi(\psi(y)) = y$, so φ is surjective. Now If $x, x' \in X$ with $\varphi(x) = \varphi(x')$ then $x = \psi(\varphi(x)) = \psi(\varphi(x')) = x'$. So φ is injective.

(4) Consider the map $\varphi \colon \mathbb{A}^1 \to V\left(y^3 - x^2\right)$ by $t \mapsto (t^2, t^3)$. This is a bijection on points, however, φ is not an isomorphism, since

$$\Gamma(A^1) = k[x] \ncong k[x, y]/(y^3 - x^2)$$

since the first is UFD, however, in $k[x, y]/(y^3 - x^2)$, we have $y^3 = x^2$, so since y and x are irreducible, we have that y is associate to x, which is a contradiction.

(5) False. E.g. I is closed in the classical topology on $\mathbb{A}^1$, however not in the Zariski topology since an infinite solution set implies that the function is identically zero.

(6) True. The Zariski topology is coarser than the classical topology. This follows as polynomial maps are continuous.

(7) False. $(xy) \subset (x)$ but $(1, 0) \in V((xy))$ while $(1, 0) \notin V(x)$, so $V((xy)) \not\subset V(x)$.

(8) True.
(9) False. $[0 : 1] \in Y = \{[1 : 0], [0 : 1]\}$, however $C(Y) = V(x) \cup V(y) \not\subset V(y)$ since $(0, 1) \in C(Y)$ but $(0, 1) \notin V(y) = C(X)$.

(10) False. x vanishes on $\mathbb{V}(x)$ and y vanishes on $\mathbb{V}(y)$ but $x + y \in \mathbb{I}(\mathbb{V}(x)) + \mathbb{I}(\mathbb{V}(y)) = (x) + (y)$ does not vanish one $\mathbb{V}(x) \cup \mathbb{V}(y)$ since $[1 : 0] \in \mathbb{V}(x) \cup \mathbb{V}(y)$ but $(x + y)([1 : 0]) = 1 \neq 0$.

(11) True.

(12) False. Let $J = (x^2y^2)$. Then $[1 : 0] \in \mathbb{V}(J)$ and so $x \in \mathbb{I}(\mathbb{V}(J))$, however, $x \notin \left(x^2y^2\right)$ since we are dealing with an integral domain, so the degrees add.

(13) It suffices to show it in the affine variety, since the projective tangent space of a projective alg set $X \subset \mathbb{P}^n$ at $P \in X$ with $P \in U_i$ is the projective closure of $T_P(X \cap U_i) \subset U_i$ and the projective

tangent cone of X at P is the projective closure of $TC_P(X \cap U_i)$.
In the affine case, if we write $f = f_1 + \ldots f_m$, then if $f_1 \neq 0$, we have that $T_P(X \cap U_i) = TC_P(X \cap U_i)$. If $f_1 = 0$ then $T_P(X \cap U_1) = \mathbb{A}^{n+1}$ and $TC_P(X \cap U_i) \subset \mathbb{A}^{n+1}$.

(14) Not true. E.g. for $x^2 \in k[x,y]$, the tangent space is $\mathbb{A}^2$ but the tangent cone is $V(x)$ which does not contain all of $\mathbb{A}^2$.

(15) True.
(16) Not true. For example, (xy) is a radical ideal, however, it is not prime since $xy \in (xy)$ while neither x nor y are in (xy).
It is radical since if $f^n \in (xy)$ then $f^n = xy\alpha$ so $x, y \mid f^n$ and as x and y are irreducible, we have $x, y \mid f$ so $f \in (xy)$.

11: Let $C = V\left((x-x_0)^2 + (y-y_0)^2 - r^2\right) \subset \mathbb{A}^2_{\mathbb{C}}$ be a circle with center (x_0, y_0) and radius r.

(a) Let $\overline{C} \subset \mathbb{P}^2_{\mathbb{C}}$ be the projective closure of C, given by $\mathbb{V}\left((x-zx_0)^2 + (y-zy_0)^2 - (zr)^2\right)$. Then, the tangent line to $\overline{C}$ at $[1:i:0]$ is the tangent line to $V\left((1-zx_0)^2 + (y-zy_0)^2 - (zr)^2\right)$ at $(i,0)$. Let $\varphi \colon \mathbb{A}^2 \to \mathbb{A}^2$ be the translation $(0,0) \mapsto (i,0)$. Then the tangent line $T_{(i,0)}V\left((1-zx_0)^2 + (y-zy_0)^2 - (zr)^2\right)$ is

$$\varphi\left(T_{(0,0)}V\left((1-zx_0)^2 + (y+i-zy_0)^2 - (zr)^2\right)\right)$$

Inserting $(0,0)$, we get 0, and the degree one polynomial part is $-2zx_0 + 2iy - 2iy_0z = 2\left(-z\left(x_0+iy_0\right) + iy\right)$. So the tangent line is $V\left(-z(x_0+iy_0) + iy\right)$ which maps under φ to $V\left(-z(x_0+iy_0) + i(y-i)\right)$ whose projective closure is

$$\mathbb{V}\left(-z(x_0+iy_0) + i(y-xi)\right)$$

Similarly, for $[1:-i:0]$, we get

$$\varphi\left(T_{(0,0)}V\left((1-zx_0)^2 + (y-i-zy_0)^2 - (zr)^2\right)\right)$$

so the internal polynomial becomes $2x_0z - 2iy_0z + 2iy$ which is mapped to

$$V\left(2x_0z - 2iy_0z + 2i(y+i)\right)$$

whose projective closure becomes

$$\mathbb{V}\left(2x_0z - 2iy_0z + 2i(y+ix)\right) = \mathbb{V}\left(z(x_0-iy_0) + i(y+ix)\right)$$

(b) In (a) we found that the projective tangent line is not dependent on the radius r, so $\overline{C_1}$ and $\overline{C_2}$ have the same tangent line at $[1:i:0]$ and $[1:-i:0]$ and are thus tangent at these points.

(c) Let f be the equation for the circle C_1 and g the equation for the circle C_2. Let F and G be the homogenizations. The intersection of C_1 and C_2 in $\mathbb{A}^2_{\mathbb{C}}$ is empty if

$$\sum_{P \in \mathbb{A}^2_{\mathbb{C}}} I_P(f,g) = 0$$

by (2) in intersection multiplicities. But

$$\sum_{P \in \mathbb{A}^2_{\mathbb{C}}} I_P(f,g) = \sum_{P \in \mathbb{P}^2} I_P(F,G)$$

Now, Bezout's theorem gives that $\sum_{P\in\mathbb{P}^2} I_P(F,G) = 4$ if $\mathbb{V}(F,G)$ is finite, i.e. if F and G share no components. But if $\alpha \mid F, G$ then $\alpha \mid G - F = (zr_F)^2 - (zr_G)^2 = z^2(r_F + r_G)(r_F - r_G)$, so assume wlog. that $\alpha = z$. However, $z \nmid F, G$ since $z \nmid 1 + y^2$.
Thus $\sum_{P\in\mathbb{P}^2} I_P(F,G) = 4$. Now, we want to calculate $I_{[1:i:0]}(F,G)$. We will assume without loss of generality that $P = (0,0)$. Thus F becomes $x^2 + y^2 - (zr_F)^2$ and G becomes $x^2 + y^2 - (zr_G)^2$. Now, we

want to calculate

$$\begin{aligned}
I_{(i,0)}(1+y^2-(zr_F)^2, 1+y^2-(zr_G)^2) &= I_{(0,0)}\left(\varphi^*\left(1+y^2-(zr_F)^2\right), \varphi^*\left(1+y^2-(zr_G)^2\right)\right)\\
&= I_{(0,0)}\left(1+(y-i)^2-(zr_F)^2, 1+(y-i)^2-(zr_G)^2\right)\\
&= I_{(0,0)}(1+(y-i)^2-(zr_F)^2, z^2(r_F-r_G)(r_F+r_G))\\
&= I_{(0,0)}\left(y^2-2iy, z^2(r_F-r_G)(r_F+r_G)\right)\\
&= I_{(0,0)}\left(y, z^2\right) + \underbrace{I_{(0,0)}\left(y-2i, z^2\right)}_{=0\text{, by (2) since it does not vanish at (0,0)}}\\
&= 2
\end{aligned}$$

Hence the intersection multiplicities are each 2, so in particular, by (2), no other points is a common vanishing of both curves.

12: Let

$$F = a\left(x^2+y^2\right) + cxz + eyz + fz^2 \in k\left[x,y,z\right].$$

Let $x' = x+\alpha z$ and $y' = y+\beta z$ and $z' = rz$ for constants α, β, r. Write

$$F = a'\left(x'^2+y'^2\right) + c'x'z' + e'y'z' + f'z'^2$$

for coefficients a', c', e', f' (which are themselves polynomials in a, c, e, f). Prove that the map $[a:c:e:f] \mapsto [a':c':e':f']$ is a projective change of coordinates.

Solution: We must show that the map $(a,c,e,f) \mapsto (a',c',e',f')$ is a linear change of coordinates, i.e., an invertible linear transformation.
We have

$$\begin{aligned}
F &= a'\left((x+\alpha z)^2+(y+\beta z)^2\right) + c'(x+\alpha z)rz + e'(y+\beta z)rz + f'(rz)^2\\
&= a'\left(x^2+y^2\right) + \left(2a\alpha + c'r\right)xz + \left(2a\beta+e'r\right)yz + \left(a\left(\alpha^2+\beta^2\right) + c'\alpha r + e'\beta r + f'r^2\right)z^2
\end{aligned}$$

Comparing coefficients, we have $a' = a$, so considering the map $T\colon \mathbb{A}^4 \to \mathbb{A}^4$, we have $T_1(a,c,e,f) = a$.
Assume $r \neq 0$.
We have $c = 2a\alpha + c'r$, so $c' = \frac{c}{r} - \frac{2\alpha}{r}a$, so $T_2\left(a,c,e,f\right) = \frac{1}{r}c - \frac{2\alpha}{r}a$.
We have $e = 2a\beta + e'r$, so $e' = \frac{e-2a\beta}{r} = \frac{1}{r}e - \frac{2\beta}{r}a = T_3(a,c,e,f)$.
We have $f = a\left(\alpha^2+\beta^2\right) + \left(\frac{1}{r}c - \frac{2\alpha}{r}a\right)\alpha r + \left(\frac{1}{r}e - \frac{2\beta}{r}a\right)\beta r + f'r^2$, and again we can isolate f' to be a linear polynomial in a, c, e and f, namely

$$f' = \frac{1}{r^2}\left[f - a\left(\alpha^2+\beta^2\right) - \left(\frac{1}{r}c - \frac{2\alpha}{r}a\right)\alpha r - \left(\frac{1}{r}e - \frac{2\beta}{r}a\right)\beta r\right]$$

We have $T_4\left(1,0,0,0\right) = -\frac{\alpha^2+\beta^2}{r^2} + \frac{2\alpha^2}{r^2} + \frac{2\beta^2}{r^2} = \frac{\alpha^2+\beta^2}{r^2}$,
$T_4\left(0,1,0,0\right) = -\frac{1}{r^2}$,
$T_4\left(0,0,1,0\right) = -\frac{\beta}{r^2}$ and $T_4\left(0,0,0,1\right) = \frac{1}{r^2}$, so the matrix (T_{ij}) then becomes

$$T = \begin{pmatrix} 1 & 0 & 0 & 0\\ -\frac{2\alpha}{r} & \frac{1}{r} & 0 & 0\\ -\frac{2\beta}{r} & 0 & \frac{1}{r} & 0\\ \frac{\alpha^2+\beta^2}{r^2} & \frac{-1}{r^w} & \frac{-\beta}{r^2} & \frac{1}{r^2}\end{pmatrix}$$

which has determinant $\frac{1}{r^4} \neq 0$.

If $r = 0$, we find

$$F = a'\left(x'^2+y'^2\right) = a\left((x+\alpha z)^2+(y+\beta z)^2\right)$$

so we find that T does not become invertible. . . So it is a projective change of coordinates if $r \neq 0$.