Exercise 0.1 (9.16)**.** Let $[A]$ be a real $n \times n$ matrix. Show for its complex Jordan normal form that every Jordan block with a nonreal eigenvalue can be paired with a Jordan block of the same size and the conjugated eigenvalue.

Solution. Suppose $[A|_{M_i}]$ is a Jordan block occuring in some normal form of A over $\mathbb{C}$. Let λ_i be the eigenvalue. Then by assumption, letting M_i denote the generalized eigenspace for the eigenvalue λ_i, we get $M_i = \left\{ x \in V \mid \exists k > 0\colon\ (A - \lambda)^k \, x = 0 \right\}$. Now, taking the complex conjugate over this equation (considered as matrices such that $[A]$ is real), we get

$$\left([A] - \overline{\lambda} I\right)^k \overline{x} = 0$$

so $\overline{x} \in M_{\overline{\lambda_i}}$, hence $\overline{\lambda_i}$ must be some λ_j. And clearly the above argument gives $\dim M_i = \dim M_j$, the bijective correspondence being $x \mapsto \overline{x}$.