p. 109:
25: Show that the punctured torus deformation-retracts onto the one-point union of two circles.

Solution: Let $X = I^2 - (\frac{1}{2}, \frac{1}{2})$. Then we can give X the usual partition for the torus (i.e. we paste opposite sides of the square) inducing an identification space X^* for the punctured torus where we can assume without loss of generality that the usual image in the torus of $(\frac{1}{2}, \frac{1}{2})$ is the removed point. Let $\pi\colon X \to X^*$ denote the identification map.
Now, define a map $F\colon X \times I \to X$ given by

$$F((x,y),t) = \left(\frac{t}{2\max\left\{\left|x-\frac{1}{2}\right|, \left|y-\frac{1}{2}\right|\right\}} + 1 - t\right)\left(x - \frac{1}{2}, y - \frac{1}{2}\right) + \left(\frac{1}{2}, \frac{1}{2}\right).$$

This is continuous. Furthermore, $F\left((x,y),1\right) \in \partial I^2$, and for $(x,y) \in \partial I^2$, $\max\left\{\left|x-\frac{1}{2}\right|, \left|y-\frac{1}{2}\right|\right\} = \frac{1}{2}$, so

$$F((x,y),1) = (x,y).$$

Hence F is a deformation retraction of X onto ∂I^2. Thus $\pi \circ F$ is a deformation retraction of X onto $\pi\left(\partial I^2\right)$.

We wish to show that $\pi \circ F$ factors through X^*.

For any $x', y' \in X$ with $x' \neq y'$ such that $\pi(x') = \pi(y')$, we have that $x', y' \in \partial I^2$ and are symmetric about either the line $x = \frac{1}{2}$ or the line $y = \frac{1}{2}$. But $\pi \circ F$ maps such points to the same point, so we can define a map $\tilde{f}\colon X^* \times I \to \pi\left(\partial I^2\right)$ such that $\pi \circ F = \tilde{f} \circ (\pi, \mathbb{1})$ and by theorem 4.1, $\tilde{f}$ is continuous - $(\pi, \mathbb{1})$ is an identification map as a consequence of exercise 11, section 29 in Munkres, and since $I = [0,1]$ is a locally compact Hausdorff space.

Thus $\tilde{f}(\pi(x), 0) = \pi(x)$, $\tilde{f}\left(\pi(x), 1\right) = \pi\left(F(x,1)\right) \in \pi\left(\partial I^2\right)$, and for $\pi(x) \in \pi\left(\partial I^2\right)$, we have $\tilde{f}\left(\pi(x), 1\right) = \pi\left(F(x,1)\right) = \pi(x)$, so $\tilde{f}$ is a deformation retraction of X^* onto $\pi\left(\partial I^2\right)$.

26: Consider the following examples of a circle C embedded in a surface S :
(a) $S =$ Möbius strip, $C =$ boundary circle;
(b) $S =$ torus, $C =$ diagonal circle $= \left\{(x,y) \in S^1 \times S^1 \mid x = y\right\}$;
(c) $S =$ cylinder, $C =$ one of boundary circles.

In each case, choose a base point in C, describe generators for the fundamental groups of C and S, and write down in terms of these generators the homomorphisms of fundamental groups induced by the inclusion of C in S.

Solution:
(a) Consider the Möbius strip as the identification space of $X = I^2$ with $(0,t)$ and $(1, 1-t)$ identified for all t. Suppose we choose as a base point for C the point $p = (0,0)$ and let $\pi\colon X \to X^* = S$ be the identification map.

Then letting $\gamma\colon I \to X$ be given by

$$\gamma(t) = \begin{cases} (2t, 0), & t \in \left[0, \frac{1}{2}\right] \\ (2t-1, 1), & t \in \left[\frac{1}{2}, 1\right] \end{cases},$$

$\pi \circ \gamma\colon I \to X^*$ is the boundary circle.

Now, consider the path $\alpha\colon I \to X$ given by

$$\alpha(t) = (t,t).$$

Since $(0,0)$ and $(1,1)$ are identified, $g := \pi \circ \alpha$ is a loop in S.

Define the map $F\colon X \times I \to X$ by

$$F((x,y),t) = (x, y + t(x-y)).$$

F is a deformation retraction of X onto the image of α. Now, since

$$\begin{aligned}\pi \circ F((0,s),t) &= \pi\left(0, s(1-t)\right) \\ &= \pi\left(1, 1-(1-t)s\right) \\ &= \pi\left(1, 1-s+ts\right) \\ &= \pi\left(1, (1-s)+t(1-(1-s))\right) \\ &= \pi \circ F\left((1,1-s),t\right)\end{aligned}$$

we have that $\pi \circ F$ factors through $X^* \times I$, so there exists a map $\tilde{f}\colon X^* \times I \to X^*$ such that

$$\pi \circ F = \tilde{f} \circ (\pi, \mathbb{1}).$$

Again, we have that $\tilde{f}$ is continuous as in problem 25, so $\tilde{f}$ is a deformation retraction of X^* onto the image of $\pi \circ \alpha$ which is a circle.
Thus $\pi_1\left(S, p(0,0)\right) \cong \mathbb{Z}$ and furthermore, since $\tilde{f}(-,1)_*$ is an isomorphism according to theorem 5.18, we have that $\tilde{f}\left(-,1\right)_*\left(\pi \circ \alpha\right) = \pi \circ \alpha$ which is an generator of the image of $\tilde{f}\left(-,1\right)$ which is $\pi \circ \alpha$. Hence $\pi \circ \alpha$ is also a generator for $X^* = S$.
Let $i\colon C \to S$ be the inclusion map.
Then $i \circ \pi \circ \gamma$ is a path in S and

$$\tilde{f}\left(i \circ \pi \circ \gamma(s), t\right) = \pi \circ F\left(\gamma(s), t\right) = \pi \circ \begin{cases} (2s, 2ts), & t \in \left[0, \frac{1}{2}\right] \\ (2s-1, 1+t\left(2s-2\right)), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

is a homotopy of $\pi \circ \gamma$ to $\pi \circ \alpha.\alpha$. Thus $\langle \pi \circ \gamma\rangle = \langle \pi \circ \alpha.\alpha\rangle$, so the generator $\pi \circ \gamma$ in C is equivalent to winding around the Möbius strip twice. I.e. the inclusion induces the map $i_*\colon \mathbb{Z} \cong \pi_1(C, p(0,0)) \to \pi_1\left(S, p(0,0)\right) \cong \mathbb{Z}$ given by $i_*(1) = 2$.

(b) We have $\pi_1(S) = \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1\left(S^1\right) \cong \mathbb{Z} \times \mathbb{Z}$ and the fundamental group of C is $\mathbb{Z}$ as it is a circle.
Let $X = \mathbb{R}^2$ and identify $(a, b+t) \sim (c, d+t)$ and $(a+s, b) \sim (c+s, d)$ for $a, b, c, d \in Z$ and $s, t \in I$, giving the identification space of the torus. Let $\pi\colon X \to X^* = S$ be the identification map.
Let $\pi(0,0)$ be the chosen base point.
Then the diagonal circle C is given by $\pi \circ \alpha$ with $\alpha\colon I \to X$ given by $\alpha(t) = (t,t)$, and $\langle \pi \circ \alpha\rangle$ is a generator for $\pi_1\left(C, p(0,0)\right)$

Now, we can view the above in terms of orbit space by letting $\mathbb{Z} \times \mathbb{Z}$ act on the plane as a group of homeomorphisms. As in the proof of theorem 5.13, we then find that defining $\varphi\colon G \to \pi_1\left(X/G, \pi(x_0)\right)$ by $\varphi(g) = \langle \pi \circ \gamma\rangle$ where $x_0 \in X$ and γ is a path joining x_0 to $g(x_0)$ given some $g \in G$ gives an isomorphism of G and $\pi_1\left(X/G, \pi(x_0)\right)$. Now, $(1,0)$ and $(0,1)$ generate $\mathbb{Z} \times \mathbb{Z}$, so their images generate $\pi_1\left(X/G, \pi(x_0)\right) = \pi_1\left(S, \pi(0,0)\right)$.
Since $(1,0).(0,0) = (1,0)$ where the left element is a group element of $\mathbb{Z} \times \mathbb{Z}$ acting on the right element $(0,0)$ which is our base point in $\mathbb{R}^2$, and in the same way $(0,1).(0,0) = (0,1)$, we have that letting $\beta\colon I \to X$ be the straight line connecting $(0,0)$ and $(0,1)$ and $\delta\colon I \to X$ be the straight line connecting $(0,0)$ and $(1,0)$, $\varphi(0,1) = \langle \pi \circ \beta\rangle$ and $\varphi(1,0) = \langle \pi \circ \delta\rangle$ generate $\pi_1(S)$ which correspond to the meridean and longitudinal circle on the torus. Furthermore, commutativity in $\mathbb{Z} \times \mathbb{Z}$ of $(0,1)$ and $(1,0)$ implies commutativity $\langle \pi \circ \beta\rangle$ and $\langle \pi \circ \delta\rangle$ in $\pi_1(S)$.
Let $i\colon C \to S$ denote the inclusion. We claim that $\langle i \circ \pi \circ \alpha\rangle = \langle \pi \circ \delta\rangle\langle \pi \circ \beta\rangle$.
We can define a map $F((x,y),t) = (x,y) + t\left(\left(\frac{x+y}{2}, \frac{x+y}{2}\right) - (x,y)\right)$. This is a homotopy rel $\{0,1\}$ of α and $\delta.\beta_1$ where $\beta_1(t)$ is given by $\beta(t) + (1,0)$. Now, $\pi\left(\beta\right) = \pi\left(\beta_1\right)$, so $\langle i \circ \pi \circ \alpha\rangle = \langle \pi \circ \delta.\beta_1\rangle = \langle \pi \circ \delta.\pi \circ \beta_1\rangle = \langle \pi \circ \delta.\pi \circ \beta\rangle = \langle \pi \circ \delta\rangle\langle \pi \circ \beta\rangle$.

Hence $i_*\colon \mathbb{Z} \cong \pi_1\left(C, \pi(0,0)\right) \to \pi_1\left(S, \pi(0,0)\right) \cong \mathbb{Z} \times \mathbb{Z}$ is given by $i_*(1) = (1,1)$, so $i_*(n) = (n,n)$.

(c) We consider the cylinder as embedded as $S^1 \times I$, so $S = S^1 \times I$. Let $C = S^1 \times \{0\}$, and let $(1,0)$ be the base point. Then $\gamma\colon I \to S^1 \times I$ by $\gamma(t) = \left(e^{2\pi i t}, 0\right)$ defines a loop giving the boundary circle C.

Define $F\colon S \times I \to S$ by $F\left(\left(e^{2i\pi\theta}, s\right), t\right) = \left(e^{2i\pi\theta}, s(1-t)\right)$. This is a deformation retraction of S

onto $S^1 \times \{0\} \subset S$ (the boundary circle, C). Thus $\pi_1(S,(1,0)) \cong \pi_1(C,(1,0)) \cong \mathbb{Z}$. Now, by example 4 on page 104, S and C have the same homotopy type, so by theorem 5.18 and its proof, we have that $(F(-,1))_*$ is an isomorphism of $\pi_1(S,(1,0))$ and $\pi_1(C,(1,0))$. Now, $(F(-,1))(\langle\gamma\rangle) = \langle F(-,1)\circ\gamma\rangle = \langle\gamma\rangle$ which is a generator of $\pi_1(C,(1,0))$ and thus $\langle\gamma\rangle$ is also a generator of $\pi_1(S,(1,0))$ as $(F(-,1))_*$ is an isomorphism. Thus, with these generators, $i_* \colon \mathbb{Z} \to \mathbb{Z}$ is given by $i_*(1) = 1$, and hence $i_*(n) = n$.

p. 111:
33: Which of the following spaces have the fixed-point property?
(a) The 2-sphere
(b) the torus
(c) the interior of the unit disc
(d) the one-point union of two circles.

Solution:
(a) Let $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$. Define a map $f\colon S^2 \to S^2$ by $f(x,y,z) = (-x,-y,-z)$. This is continuous as each coordinate function is continuous, and has no fixed point since if $f(x,y,z) = (x,y,z)$ then $-x = x, -y = y$ and $-z = z$ imply $(x,y,z) = (0,0,0) \notin S^2$. Thus the 2-sphere does not have the fixed-point property.

(b) The torus does not have the fixed-point property either. For example, consider the map $f\colon S^1 \times S^1 \to S^1 \times S^1$ by $f(e^{i\theta}, e^{i\alpha}) = \left(e^{i(\theta+\frac{\pi}{2})}, e^{i\alpha}\right)$. This map is continuous as its coordinate maps are continuous, and has no fixed point since $e^{i\theta} = e^{i\left(\theta+\frac{\pi}{2}\right) = ie^{i\theta}} \implies 1 = i$ since $\left|e^{i\theta}\right| = 1$ and thus dividing is possible. We thus derive a contradiction, so the map has no fixed point.

(c) The interior of the unit disc does not have the fixed point property:
Consider $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq\leq 1\}$. Define the map $f\colon \left(D^2\right)^\circ \to \left(D^2\right)^\circ$ by $f(x,y) = \left(\frac{x^2+y^2+1}{2}, 0\right)$. This is continuous as the coordinate functions are contiuous. However, $(x,y) = f(x,y) = \left(\frac{x^2+y^2+1}{2}, 0\right)$ implies $y = 0$ so $\frac{x^2+1}{2} = x$ which implies $x = 1$, however $(1,0) \notin \left(D^2\right)^\circ$.

(d) Suppose we identify the two copies of S^1 at 1. Define a map $f\colon S^1 \vee S^1 \to S^1 \vee S^1$ by sending $x \mapsto -x$ for x in the first copy of S^1 (i.e. each point is mapped to its antipodal point in the first copy) and $x \mapsto -1$ for x in the second copy of S^1. This map is continuous on each of the circles separately, and agree on 1, so f is continuous and has no fixed point.

Lemma: If $p\colon X \to Y$ is an identification map and if Z is a locally compact Hausdorff space, then the map

$$\pi = p \times \mathbb{1}_Z \colon X \times Z \to Y \times Z$$

is a quotient map.

Proof: Firstly, π is continuous. Let $A \subset Y \times Z$ and $\pi^{-1}(A)$ be open and suppose it contains $x \times y$ which we will write for (x, y). Now, choose open sets U_1 and V with $\overline{V}$ compact such that $x \times y \in U_1 \times V$ and $U_1 \times \overline{V} \subset \pi^{-1}(A)$ (we can find such a V because Z is locally compact).
Now, given