2.1.1: Let $\{U_i\}_{i\in I}$ be an open covering of $X \times Y$ equipped with the product topology. Fix an $x \in X$. Then for each $y \in Y$, there exists some U_y containing (x, y), and since this is open, and

$$\{V_i \times W_i \mid V_i \text{ open in } X \text{ and } W_i \text{ open in } Y\}$$

forms a basis for the product topology, we can find $V_y \times W_y$ with V_y open in X and W_y open in Y such that $(x, y) \in V_y \times W_y \subset U_y$. Then since $\bigcup_{y\in Y} W_y = Y$, and Y is compact, we can find a finite subcollection $y_1, \dots, y_n$ such that $Y = W_{y_1} \cup \dots \cup W_{y_n}$. Then $V_x = V_{y_1} \cap \dots V_{y_n}$ is nonempty and open as $x \in V_x$ and it is the intersection of a finite collection of open sets. We then get

$$V_x \times Y = V_x \times W_{y_1} \cup \dots \cup V_x \times W_{y_n} \subset U_{y_1} \cup \dots \cup U_{y_n}.$$

What this means is that for any $x \in X$, there exists a finite subcollection $\mathcal{A}_x \subset \{U_i\}_{i\in I}$, such that $V_x \times Y \subset \bigcup_{A\in\mathcal{A}_x} A$.
Now since $\bigcup_{x\in X} V_x = X$ and X is compact, there exist some finite subcollection $V_{x_1}, \dots, V_{x_n}$ such that $V_{x_1} \cup \dots \cup V_{x_n} = X$. Hence

$$X \times Y = \bigcup_{i=1}^{n} V_x \times Y \subset \bigcup_{i}^{n} \bigcup_{A\in\mathcal{A}_{x_i}} A \subset X \times Y,$$

so

$$X \times Y = \bigcup_{i}^{n} \bigcup_{A\in\mathcal{A}_{x_i}} A$$

which is a finite subcover of $\{U_i\}_{i\in I}$. Thus $X \times Y$ is compact.

2.1.2: Let $(x_n) \subset I$ be any sequence in I. If (x_n) converges in I, we are done, so we may assume (x_n) does not converge. In particular, (x_n) thus consists of infinitely many distinct points. Therefore, either $\left[0, \frac{1}{2}\right]$ or $\left[\frac{1}{2}, 1\right]$ must contain infinitely many distinct points of the sequence. Let I_1 denote a half of I_0 satisfying this.
Now define for a general $n \in \mathbb{Z}_+$, I_n to be a half of I_{n-1} which contains infinitely many distinct points - such an I_n must exist, as otherwise I_{n-1} contains only finitely many points in contradiciton with its construction.
We have a countable collection $\{I_n\}_{n\in\mathbb{N}_0}$ with $I_0 \supset I_1 \supset I_2 \supset \dots$. By the axiom of countable choice, we can choose a sequence $(x_{n_k})_{k\in\mathbb{N}_0}$ such that $x_{n_k} \in I_k$. We can assume this to be a subsequence of (x_n), i.e. $n_i < n_k$ if $i < k$, since otherwise if $i < k$ with $n_k \le n_i$, we can replace x_{n_k} by x_{n_i} since $I_{n_i} \subset I_{n_k}$.

Now let $\varepsilon > 0$ be arbitrary fixed. By the Archimedean property, there exists $N \in Z_+$ such that $\frac{1}{\varepsilon} < N$ and hence $0 < \frac{1}{2^N} < \frac{1}{N} < \varepsilon$. For any $r, s \ge N$, assuming without loss of generality that $r \le s$, we have $x_{n_r}, x_{n_s} \in I_r$ and since the diameter of I_r is $\frac{1}{2^r}$, we have

$$|x_{n_r} - x_{n_s}| \le \frac{1}{2^r} \le \frac{1}{2^N} < \varepsilon.$$

So $(x_{n_k})_{k\in\mathbb{N}_0}$ is a cauchy sequence, and since $\mathbb{R}$ is complete, it follows that (x_{n_k}) converges to a point in I (since I is closed and the sequence is in I).

2.1.3: An equatorial cut would produce a mobius strip which is essentially twice the length and has twice the twists of the original strip. An equatorial cut along each of the resulting pieces would create two strips of the same length and same number of twists - i.e. two twists.