1. H-Spaces, H-Groups and H-Cogroups

An H-space or H-group is a space with a product that satisfies some of the laws of a group *but only up to homotopy.* An H-cogroup is a dual notion. The "H" stands for "Hopf" or for "Homotopy".

Definition 1.1 (H-Space, homotopy associativity, homotopy inverse)**.** An *H-space* is a pointed space $X \in \mathrm{Top}_*$ with base point e, together with a map

$$\cdot\colon X \times X \to X$$

sending $(x, y) \mapsto x \cdot y$ such that $e \cdot e = e$, and the maps $X \to X$ taking $x \mapsto x \cdot e$ and $x \mapsto e \cdot x$ are each homotopic rel $\{e\}$ to the identity.

It is said to be *homotopy associative* if the maps $X \times X \times X \to X$ taking (x, y, z) to $(x \cdot y) \cdot z$ and to $x \cdot (y \cdot z)$ are homotopic rel $\{(e, e, e)\}$.

It is said to have a *homotopy inverse* $\hat{-}\colon X \to X$ if $\hat{e} = e$ and the maps $X \to X$ taking x to $x \cdot \hat{x}$ and to $\hat{x} \cdot x$ are each homotopic rel $\{e\}$ to the constant map to $\{e\}$.

Definition 1.2 (H-group)**.** An *H-group* is a homotopy associative H-space with a given homotopy inverse.

There are two main examples: the first is the class of topological groups, the second is the class of "loop spaces".

Definition 1.3 (Loop space)**.** The loop space on a space X is the space

$$\Omega X = (X, *)^{(S^1, *)},$$

i.e., X^{S^1} in the pointed category. The product is concatenation of loops, and the homotopy inverse is loop reversal. ΩX is a pointed space with base point being the constant loop at $*$.

Definition 1.4 (Operations on maps)**.** If $f\colon X \to Z$ and $g\colon Y \to W$ are maps, let $f \vee g\colon X \vee Y \to Z \vee W$ be the induced map on the one-point union.
Let $\nabla\colon Z \vee Z \to Z$ be the codiagonal, i.e., the identity on both factors.
We also define $f \veebar g\colon X \vee Y \to Z$ as the composition $f \veebar g = \nabla \circ (f \vee g)$; i.e., the map which is f on X and g on Y.

Definition 1.5 (H-cogroup)**.** An *H-cogroup* is a pointed space Y and a map $\gamma\colon Y \to Y \vee Y$ such that the following three conditions are satisfied:

(1) The constant map $*\colon Y \to Y$