Class 2

Goal of topology: define invariants of top. spaces. Find out when 2 spaces are equivalent. Very rarely a complete classification.

Need a source of spaces to consider. Need methods to tell if two spaces are equivalent.

Note: Metric graphs and outer space.

Chapter 1

Paths and homotopy

Setup: A path in a space X is a continuous map $f\colon I \to X$.
A path $f\colon I \to X$ such that $f(0) = f(1) = x_0 \in X$ is called a loop, and the common starting and ending point x_0 is referred to as the basepoint. The set of all homotopy classes $[f]$ of loops $f\colon I \to X$ at the basepoint x_0 is denoted $\pi_1(X, x_0)$.

Proposition 1.3: $\pi_1(X, x_0)$ is a group with respect to the product $[f]\,[g] = [f \cdot g]$ where

$$f \cdot g(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Let a reparametrization of a path f be a composition $f\varphi$ where $\varphi\colon I \to I$ is any continuous map such that $\varphi(0) = 0$ and $\varphi(1) = 1$.

Then given paths f, g, h with $f(1) = g(0), g(1) = h(0)$, we want a reparametrization $\varphi\colon I \to I$ such that $f \cdot (g \cdot h)$ is a reparametrization of $(f \cdot g) \cdot h$; i.e. we want $f\left(4\varphi(\frac{1}{2}\right) = f(1)$, so $\varphi(\frac{1}{2}) = \frac{1}{4}$. Then $g\left(4\varphi(\frac{3}{4}) - 1\right) = g(1)$ so $\varphi\left(\frac{3}{4}\right) = \frac{1}{4}$ and $h\left(2\varphi(1) - 1\right) = h(1)$ so $\varphi(1) = 1$. Then

$$\varphi(t) = \begin{cases} \frac{1}{2}t, & t \in \left[0, \frac{1}{2}\right] \\ t - \frac{1}{4}, & t \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ 2t - 1, & t \in \left[\frac{3}{4}, 1\right] \end{cases}$$

gives $f \cdot (g \cdot h)\,\varphi = (f \cdot g) \cdot h$. Hence $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$. Thus the product in $\pi_1(X, x_0)$ is associative. To make $\pi_1(X, x_0)$ into a group, we msut also show identity and inverse:
given $f\colon I \to X$ a path, let c be the constant path at $f(1)$, so $c(s) = f(1)$ for $s \in I$. Then $f \cdot c = f\varphi$ with $\varphi = \begin{cases} 2t, & t \in \left[0, \frac{1}{2}\right] \\ 1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$ Similarly, for $c(s) = f(0)$, we have $c \cdot f = f\varphi$ with $\varphi(t) = \begin{cases} 0, & t \in \left[0, \frac{1}{2}\right] \\ 2t - 1, & t \in \left[\frac{1}{2}, 1\right] \end{cases}$.
Thus $c \cdot f \simeq f \simeq f \cdot c$, so the homotopy class of the constant path at x_0 is a two-sided identity for $\pi_1\left(X, x_0\right)$.

Now, letting $\overline{f}(s) = f(1-s)$, we have $f \cdot \overline{f}$ is homotopic to the constant path by the homotopy $f_t \cdot g_t$ with $f_t(s) = \begin{cases} f(s), & s \in [0, 1-t] \\ f(1-t), & s \in [1-t, 1] \end{cases}$ and $g_t(s) = \begin{cases} f(1-t), & s \in [0, 1-t] \\ f(s), & s \in [1-t, 1] \end{cases}$. Then $f_t(s) \cdot g_t(s)$ is a loop and $f_0 \cdot g_0 = f \cdot \overline{f}$ while $f_1 \cdot g_1 = f(0) \cdot f(0)$. Taking f to be a loop at the basepoint x_0, we deduce that $\left[\overline{f}\right]$ is a two-sided inverse for $[f]$ in $\pi_1\left(X, x_0\right)$.

Def: change-of-basepoint map: is there a relationship between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ if x_0 and x_1 lie in the same path-component? Let $h\colon I \to X$ be a path from x_0 to x_1. Then we define a **change-of-basepoint** map $\beta_h\colon \pi_1(X, x_1) \to \pi_1\left(X, x_0\right)$ by $\beta_h\,[f] = \left[h \cdot f \cdot \overline{h}\right]$.

Proposition 1.5: The map $\beta_h\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ is an isomorphism.

Proof: Let $f, g \in \pi_1\left(X, x_1\right)$. Then $\beta_h(f \cdot g) = h\left(f \cdot g\right)\overline{h} = (hf\overline{h}) \cdot (hg\overline{h}) = \beta_h(f) \cdot \beta_h(g)$. So β_h is a homomorphism. It is an isomorphism since $\beta_h\beta_h^{-1}(f) = \beta_h\left(\overline{h}fh\right) = h\overline{h}fh\overline{h} = f$ and similarly for $\beta_h^{-1}\beta_h$.

So the group $\pi_1(X, x_0)$ is invariant up to isomorphism on any path-connected space. In this case $\pi_1(X, x_0)$ is often abbreviated $\pi_1(X)$ or just $\pi_1 X$.

A space is called **simply-connected** if it is path-connected and has trivial fundamental group.
Proposition 1.6: A space is simply-connected iff there is a unique homotopy class of paths connecting any two points in X.

The fundamental group of the circle

Goal: Showing $\pi_1(S^1) \approx \mathbb{Z}$.

Theorem 1.7: $\pi_1\left(S^1\right)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ based at $(1, 0)$.

Elements and ideas from proof: Note $[\omega]^n = [\omega_n]$ where $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ for $n \in \mathbb{Z}$.
Thus the theorem says that every loop in S^1 based at $(1, 0)$ is homotopic to ω_n for a unique $n \in \mathbb{Z}$. Consider the map $p\colon R \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$. This map can be visualized geometrically by embedding $\mathbb{R}$ in $\mathbb{R}^3$ as the helix parametrized by $s \to (\cos 2\pi s, \sin 2\pi s, s)$, and then p is the restriction to the helix of the projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$, $(x, y, z) \to (x, y)$. Then ω_n is the composition $p\tilde{\omega}_n$ where $\tilde{\omega}\colon I \to R$ is the path $\tilde{\omega_n}(s) = ns$, starting at 0 and ending at n, winding around the helix $|n|$ times, upward i $n > 0$ and downward if $n < 0$. The relation $\omega_n = p\tilde{\omega_n}$ is expressed by saying that $\tilde{\omega_n}$ is a **lift** of ω_n.

In category theory, given a morphism $f\colon X \to Y$ and a morphism $g\colon Z \to Y$, a **lift** or **lifting** of f to Z is a morphism $h\colon X \to Z$ such that $f = g \circ h$. We say that f factors through h.

Given a space X, a **covering space** of X consists of a space $\tilde{X}$ and a map $p\colon \tilde{X} \to X$ satisfying the following condition:

For each point $x \in X$ there is an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p.

Such a U will be called **evenly covered**.

More precisely: a **covering** of a topological space X is a continuous map $p\colon E \to X$ suc that there exists a discrete space D and for every $x \in X$ and open neighborhood $U \subset X$, such that $p^{-1}(U) = \bigsqcup_{d \in D} V_d$ and $p|_{V_d}\colon V_d \to U$ is a homeomorphism for every $d \in D$.

So e.g. for the previously defined map $p\colon R \to S^1$, any open arc in S^1 is evenly covered.

For theorem 1.7, we will need the following two facts about covering spaces $p\colon \tilde{X} \to X$: **Lemma:** for each path $f\colon I \to X$ starting at a point $x_0 \in X$ and each $\tilde{x_0} \in p^{-1}(x_0)$ there is a unique lift $\tilde{f}\colon I \to \tilde{X}$ starting at $\tilde{x_0}$.

Lemma: For each homotopy $f_t\colon I \to X$ of paths starting at x_0 and each $\tilde{x_0} \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde{f_t}\colon I \to \tilde{X}$ of paths starting at $\tilde{x_0}$.

Both of these lemmas can be deduced from a more general assertion about covering spaces $p\colon \tilde{X} \to X$:

Lemma: Given a map $F\colon Y \times I \to X$ and a map $\tilde{F}\colon Y \times \{0\} \to \tilde{X}$ lifting $F|Y \times \{0\}$, then there is a unique map $\tilde{F}\colon Y \times I \to \tilde{X}$ lifting F and restricting to the given $\tilde{F}$ on $Y \times \{0\}$.

Exercise: Prove the first 2 lemmas from this lemma.

Theorem 1.9 (Brouwer fixed point theorem in dimension 2): Every continuous map $h\colon D^2 \to D^2$ has a fixed point, that is, a point $x \in D^2$ with $h(x) = x$.

Trivia: This theorem was first proved by Brouwer around 1910, quite early in the history of topology. Brouwer in fact proved the corresponding result for D^n, and we shall obtain this generalization in Corollary 2.15 using homology groups in place of π_1.

The techniques used to calculate $\pi_1(S^1)$ can be applied to prove the Borsuk-Ulam theorem in dimension two:

Theorem 1.10 (Borsuk Ulam theorem in dimension two): For every continuous map $f\colon S^2 \to \mathbb{R}^2$ there exist a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

The theorem holds more generally for maps $S^n \to \mathbb{R}^n$, as we will show in corollary 2B.7.
The theorem says in particular that there is no one-to-one continuous map from S^2 to R^2, so S^2 is not homeomorphic to a subspace of $\mathbb{R}^2$.

Corollary 1.11: Whenever S^2 is expressed as the union of three closed sets A_1, A_2 and A_3, then at least one of these sets must contain a pair of antipodal points $\{x, -x\}$.

The number 3 in this result is best possible: consider a sphere inscribed in a tetrahedron. Projecting the four faces of the tetrahedron radially onto the sphere, we obtain a cover of S^2 by four closed sets, none of which contains a pair of antipodal points.
Assuming the higher-dimensional version of the Borsuk-Ulam theorem, the same arguments show that S^n cannot be covered by $n+1$ closed sets without antipodal pairs of points, though it can be covered by $n+2$ such sets, as the higher dimensional analog of a tetrahedron shows.
For $n = 1$, this says that if the circle is covered by two closed sets, one of them must contain a pair of antipodal points.

Proposition 1.12: $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$ if X and Y are path-connected.

Example 1.13: The Torus: We have $\pi_1(S_1 \times S_1) \approx \mathbb{Z} \times \mathbb{Z}$. Here $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ corresponds to a loop that winds p times around one S_1 factor of the torus and q times around the other S^1 factor, for example the loop $\omega_{pq}(s) = (\omega_p(s), \omega_q(s))$. Check later

Induced Homomorphisms

Let $\varphi\colon X \to Y$ be a map taking the basepoint $x_0 \in X$ to the basepoint $y_0 \in Y$; writing $\varphi\colon (X, x_0) \to (Y, y_0)$. Then φ induces a homomorphism $\varphi_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$, defined by composing loops $f\colon I \to X$ based at x_0 with φ, that is, $\varphi_*[f] = [\varphi f]$.
This is well-defined since a homotopy f_t of loops based at x_0 yields a composed homotopy φf_t of loops based at y_0, so $\varphi_*[f_0] = [\varphi f_0] = [\varphi f_1] = \varphi_*[f_1]$. Furthermore, φ_* is a homomorphism since $\varphi(f \cdot g) = (\varphi f) \cdot (\varphi g)$, both functions having the value $\varphi f(2s)$ for $0 \le s \le \frac{1}{2}$ and the value $\varphi g(2s-1)$ for $\frac{1}{2} \le s \le 1$.

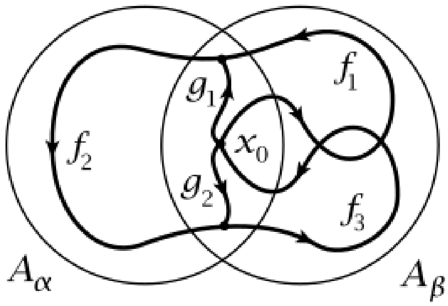
Two basic properties of induced homomorphisms are:

- $(\varphi\psi)_* = \varphi_*\psi_*$ for a composition $(X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$
- $\mathbb{1}_* = \mathbb{1}$, which is a concise way of saying that the identity map $\mathbb{1}\colon X \to X$ induces the identity map $\mathbb{1}\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$.

The first follows since map composition is associative: $(\varphi\psi) f = \varphi(\psi f)$; the second is obvious. These make the fundamental group a functor.

As an application: if φ is homeo with inverse φ, then φ_* is an iso with inverse φ_* since $\varphi_*\psi_* = (\varphi\psi)_* = \mathbb{1}_* = \mathbb{1}$ and similarly $\psi_*\varphi_* = \mathbb{1}$. Check this later
We have two categories: $\mathcal{A}$ whose objects are topological spaces and whose morphisms are basepoint preserving maps between spaces.
Another category will be the fundamental groups and the morphisms will be the induces homomorphisms.
Proposition 1.14: $\pi_1(S^n) = 0$ if $n \ge 2$.

Lemma 1.15: If a space X is the union of a collection of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.



Corollary 1.16: $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.

The more general statement that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ if $m \neq n$ can be proved in the same way using either the higher homotopy groups or homology groups. In fact, nonempty open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ can be homeomorphic only if $m = n$, as we will see in theorem 2.26 using homology.

Proposition 1.17: If a space X retracts onto a subspace A, then the homomorphism $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i\colon A \to X$ is injective. If A is a deformation retract of X, then i_* is an isomorphism.

Proof: If $r\colon X \to A$ is a retraction, then $ri = \mathbb{1}$, hence $r_* i_* = \mathbb{1}$, which implies that i_* is injective. If $r_t\colon X \to X$ is a deformation retraction of X onto A, so $r_0 = \mathbb{1}, r_t|_A = \mathbb{1}$, and $r_1(X) \subset A$, then for any loop $f\colon I \to X$ based at $x_0 \in A$ the composition $r_t f$ gives a homotopy of f to a loop in A, so i_* is also surjective.

This gives another way of seeing that S^1 is not a retract of D^2 : the inclusion-induced map $\pi_1(S^1) \to \pi_1\left(D^2\right)$ is a homomorphism $\mathbb{Z} \to 0$ which cannot be injective.

If $\varphi_t\colon X \to Y$ is a homotopy that takes a subspace $A \subset X$ to a subspace $B \subset Y$ for all t, then we speak of a homotopy of maps of pairs, $\varphi_t\colon (X, A) \to (Y, B)$. In particular, a **basepoint-preserving homotopy** $\varphi_t\colon (X, x_0) \to (Y, y_0)$ is the case that $\varphi_t(x_0) = y_0$ for all t. We have:

- If $\varphi_t\colon (X, x_0) \to (Y, y_0)$ is a basepoint-preserving homotopy, then $\varphi_{0*} = \varphi_{1*}$,

since $\varphi_{0*}[f] = [\varphi_0 f] = [\varphi_1 f] = \varphi_{1*}[f]$.

There is a notion of homotopy equivalence for spaces with basepoints. One says $(X, x_0) \simeq (Y, y_0)$ if there are maps $\varphi\colon (X, x_0) \to (Y, y_0)$ and $\psi\colon (Y, y_0) \to (X, x_0)$ with homotopies $\varphi\psi \simeq \mathbb{1}$ and $\psi\varphi \simeq \mathbb{1}$ through maps fixing the basepoints.
Then the induced maps on π_1 satisfy $\varphi_*\psi_* = (\varphi\psi)_* = \mathbb{1}_* = \mathbb{1}$ and likewise $\psi_*\varphi_* = \mathbb{1}$, so φ_* and ψ_* are inverse isomorphisms $\pi_1(X, x_0) \approx \pi_1(Y, y_0)$.

This gives another proof that a deformation retraction induces an isomorphism on fundamental groups, since if X deformation retracts onto A then $(X, x_0) \simeq (A, x_0)$ for any choice of basepoint $x_0 \in A$.

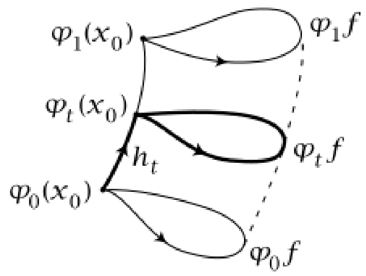
Having to pay so much attention to basepoints when dealing with the fundamental group is something of a nuisance. For homotopy equivalence one does not have to be quite so careful:

Proposition 1.18: If $\varphi\colon X \to Y$ is a homotopy equivalence, then the induced homomorphism $\varphi_*\colon \pi_1(X, x_0) \to \pi(Y, \varphi(x_0))$ is an isomorphism for all $x_0 \in X$.

Lemma 1.19: If $\varphi_t\colon X \to Y$ is a homotopy and h is the path $\varphi_t(x_0)$ formed by the images of a basepoint $x_0 \in X$, then the three maps in the diagram satisfy $\varphi_{0*} = \beta_h \varphi_{1*}$.

$$\begin{tikzcd}
 & \pi_1(Y,\varphi_1(x_0)) \arrow[d, "\beta_h"] \\
\pi_1(X,x_0) \arrow[ur, "\varphi_{1*}"] \arrow[r, "\varphi_{0*}"'] & \pi_1(Y,\varphi_0(x_0))
\end{tikzcd}$$

Proof: Let h_t be the restriction of h to the interval $[0,t]$, with a reparametrization so that the domain of h_t is still $[0,1]$. Explicitly, we can take $h_t(s) = h(ts)$. Then if f is a loop in X at the basepoint x_0, the product $h_t \cdot (\varphi_t f) \cdot \overline{h_t}$ gives a homotopy of loops at $\varphi_0(x_0)$ (see figure). Restricting this homotopy to $t = 0$ and $t = 1$, we see that $\varphi_{0*}([f]) = \beta_h(\varphi_{1*}([f]))$.



Lecture 4 - Van Kampen's Theorem

Methods to find presentations of $\pi(X, x_0)$. Group presentations: like $\langle r, s \mid \text{conditions}\rangle$.

Free products groups: Given a collection of groups $\{G_\alpha\}_{\alpha\in A}$, we can form the free product $*_\alpha G_\alpha$ as follows: as a set, the free product $*_\alpha G_\alpha$ consists of all words $g_1 g_2 \dots g_m$ of arbitrary finite length $m \geq 0$, where each letter g_i belonds to a group G_{α_i} and is not the identity element of G_{α_i}, and adjacent letters g_i and g_{i+1} belong to different groups G_α, that is, $\alpha_i \neq \alpha_{i+1}$. Words satisfying these conditions are called *reduced*. The empty word is allowed, and will be the identity element of $*_\alpha G_\alpha$. The group operation in $*_\alpha G_\alpha$ is juxaposition, $(g_1 \dots g_m)(h_1 \dots h_n) = g_1 \dots g_m h_1 \dots h_n$.

In particular, we have the free product $\mathbb{Z} * \mathbb{Z}$ as described on p. 40. The elements of a free group are uniquely representable as reduced words in powers of generators for various copies of $\mathbb{Z}$, with one generator for each $\mathbb{Z}$. These generators are called a *basis* for the free group, and the number of basis elements is the *rank* of the free group. The abelianization of a free group is a free abelian group with basis the same set of generators, so since the rank of a free abelian group is well-defined, independent of the choice of basis, the same is true for the rank of a free group.

A basic property of the free product $*_\alpha G_\alpha$ is that any collection of homomorphisms $\varphi_\alpha \colon G_\alpha \to H$ extends uniquely to a homomorphism $\varphi\colon *_\alpha G_\alpha \to H$. Namely, the value of φ on $g_1 \dots g_n$ with $g_i \in G_{\alpha_i}$ must be $\varphi_{\alpha_1}(g_1)\dots\varphi_{\alpha_n}(g_n)$. For example, for the free product $G * H$, the inclusions $G \to G \times H$ and $H \to G \times H$ induce a surjective homomorphism $G * H \to G \times H$.

The Van Kampen Theorem

Suppose a space X decomposes as the union of a collection of path-connected open subsets A_α, each of which contains the basepoint $x_0 \in X$. Then the homomorphisms $j_\alpha \colon \pi_1(A_\alpha) \to \pi_1(X)$ induced by the inclusions $A_\alpha \to X$ extend to a homomorphism $\Phi\colon *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$. The van Kampen theorem will say that Φ is very often surjective, but we can expect Φ to have a nontrivial kernel in general. For if $i_{\alpha\beta}\colon \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the homomorphism induced by the inclusion $A_\alpha \cap A_\beta \to A_\alpha$ then $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$, both these compositions being induced by the inclusion $A_\alpha \cap A_\beta \to X$, so the kernel of Φ contains all the elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$. Van Kampen's theorem asserts that under fairly broad hypotheses this gives a full description of Φ :

Theorem 1.20: If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi\colon *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is

the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and hence Φ induces an isomorphism $\pi_1(X) \approx *_\alpha \pi_1(A_\alpha)/N$.
Fischer-Zastrow

If Y is a spanning tree of X, each edge in $X - Y$ gives a loop going through x_0.

Can 1A.4 be used to show every subgroup of a free abelian group is free abelian?

Reidemeister Schreier algorithm: find gen. set. for finite-index subgroups of a free group.

Sphere theorem of Papakyriokopoulos.