p. 47, 3: Use the Heine-Borel theorem to show that an infinite subset of a closed interval must have a limit point.

Solution: Assume for contradiction that I is an closed interval and an infinite subset $A \subset I$ has no limit point.
Now, a point $x \in I$ is a limit point of A if and only if for any neighborhood N of x in I, there exists a point $a \in A \cap (N - \{x\})$, so negating each side we find that $x \in I$ is not a limit point of A if and only if there exists a neighborhood N of x in I such that $A \cap (N - \{x\}) = \varnothing$. Now, since A has no limit point in I, we have that for all $x \in I$, we can find a neighborhood N_x of x in I such that $A \cap (N_x - \{x\}) = \varnothing$; hence $A \cap N_x \subset \{x\}$. Now, $I = \bigcup_{x \in I} \{x\} \subset \bigcup_{x \in I} N_x \subset I$, so $I = \bigcup_{x \in I} N_x$. Now, I is compact by Heine-Borell, so there exists a finite subcover $N_{x_1} \cup \ldots \cup N_{x_n} = I$. Then by construction

$$A = A \cap I = A \cap (N_{x_1} \cup \ldots \cup N_{x_n}) = (A \cap N_{x_1}) \cup \ldots \cup (A \cap N_{x_n}) \subset \{x_1, \ldots, x_n\},$$

contradicting A being infinite.

p. 50., 14: Let $f \colon X \to Y$ be an injective continuous map. If we restrict it to a function $f \colon X \to f(X)$ then f is injective and surjective. We have that X is Hausdorff; now we claim $f(Y)$ is Hausdorff with the induced subspace topology.
Let $x, y \in f(X)$. Then $x, y \in Y$, so there exist open sets U, V in Y such that $x \in U, y \in V$ and $U \cap V = \varnothing$. Then the sets $U' = U \cap f(X)$ and $V' = V \cap f(X)$ are open in the subspace topology by definition, and as $x, y \in f(X)$, also $x \in U'$ and $y \in V'$, and $U' \cap V' \subset U \cap V = \varnothing$, so $U' \cap V' = \varnothing$. Hence $f(X)$ is Hausdorff. Now by theorem 3.7, $f \colon X \to f(X)$ is a homeomorphism, so by definition, f is an embedding of X in Y.

p. 55. 21: If A and B are compact, and if W is a neighborhood of $A \times B$ in $X \times Y$, find a neighborhood U of A in X and a neighborhood V of B in Y such that $U \times V \subset W$.

Solution: Fix an $a \in A$. Then for each $b \in B$, since W is a neighborhood of (a, b), we can find a basis element $(a, b) \in U_b \times V_b \subset W$ with U_b and V_b neighborhoods of respectively a and b in respectively X and Y. Now, $\bigcup_{b \in B} V_b$ covers B, so as B is compact, there exists a finite subcover $V_{b_1} \cup \ldots \cup V_{b_n}$. Now let $V_a = V_{b_1} \cup \ldots \cup V_{b_n}$. $N_a = U_{b_1} \cap \ldots \cap U_{b_n}$. Then we claim $\{a\} \times B \subset N_a \times V_a \subset W$. Let $(a, b) \in \{a\} \times B$. Since V_a covers B, we can find a V_{b_i} containing b, so $(a, b) \in N_a \times V_{b_i} \subset N_a \times V_a$ giving the first inclusion.
Now if $(a', b') \in N_a \times V_a$, then there exists b_i such that $(a', b') \in N_a \times V_{b_i}$ and hence $(a', b') \in U_{b_i} \times V_{b_i} \subset W$, giving the other inclusion.

Now $\bigcup_{a \in A} N_a$ covers A with open sets as N_a is a finite intersection of open sets, hence as A is compact, there exists a finite subcover $N_{a_1} \cup \ldots \cup N_{a_m}$. Let $V = V_{a_1} \cap \ldots \cap V_{a_m}$ and $U = N_{a_1} \cup \ldots \cup N_{a_m}$. Both U and V are open as the union of open sets and the finite intersection of open sets, and we claim $A \times B \subset U \times V \subset W$.
For the first inclusion, let $(a, b) \in A \times B$. Then as U covers A, there exists a_i such that $a \in N_{a_i}$, and as each V_{a_j} contains B, $b \in V_{a_j}$ as well, so $(a, b) \in (N_{a_i} \times V_{a_1}) \cap \ldots \cap (N_{a_i} \times V_{a_m}) = N_{a_i} \times V \subset U \times V$.

Now, if $(u, v) \in U \times V$, then there exists N_{a_i} such that $u \in N_{a_i}$, so $(u, v) \in N_{a_i} \times V \subset N_{a_i} \times V_{a_i} \subset W$, giving the other inclusion.