ASSIGNMENT 8

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Recall that the cohomology with integral coefficients of $K(\mathbb{Z}/3,2)$ up to degree 6 is:

$$H^*\left(K(\mathbb{Z}/3,2);\mathbb{Z}\right) \cong \begin{cases} \mathbb{Z}, & * = 0, \\ \mathbb{Z}/3, & * = 3,5, \\ 0, & * = 1,2,4,6. \end{cases}$$

With $\mathbb{F}_3$-coefficients we have

$$H^*\left(K\left(\mathbb{Z}/3,2\right);\mathbb{F}_3\right) \cong \mathbb{F}_3\left\langle 1,a,b,a^2,ab\right\rangle \quad \text{for } * \leq 5$$

where $|a| = 2$ and $|b| = 3$.

Problem 0.1. The goal is to classify up to homotopy equivalence all CW complexes X with the following properties:

(1) X has a single cell in each dimension $0,3,4,6$, and no cells in other dimensions.
(2) $H_3(X) \cong \mathbb{Z}/3$.

Let $Y = X^{(4)}$ be the 4-skeleton of X.

(1) Show that Y is uniquely determined up to homotopy equivalence.
(2) Compute the E_4-page of the cohomology spectral sequence (up to degree 6) for the fiber sequence

$$K\left(\mathbb{Z}/3,2\right) \to \tau_{>3}Y \to Y.$$

Assuming the differential $d_3\colon E_3^{0,5} \to E_3^{3,3}$ is non-trivial, show that

$$\pi_k(Y) \cong \begin{cases} 0, & * = 0,1,2,4,5 \\ \mathbb{Z}/3, & * = 3 \end{cases}$$

and that $\pi_6(Y)$ has at least three elements.
(3) Redo step 3 with $\mathbb{F}_3$-coefficients. Deduce that the d_3-differential in step 2 must indeed have been non-trivial.
(4) Show that X is unique up to homotopy.

Solution. (1) The Δ-chain complex for Y has the form

$$\ldots \to 0 \to \mathbb{Z} \xrightarrow{a} \mathbb{Z} \to 0 \to 0 \to \mathbb{Z} \to 0 \to \ldots$$

where $\operatorname{coker} a = \mathbb{Z}/3$, hence a must be multiplication by 3. But then taking homology, we obtain

$$\ldots \to 0 \to 0 \to \mathbb{Z}/3 \to 0 \to 0 \to \mathbb{Z} \to 0 \to \ldots$$

I.e.,

$$\tilde{H}_k(Y) \cong \begin{cases} \mathbb{Z}/3, & k = 3 \\ 0, & k \neq 3 \end{cases}$$

so Y is a Moore space $M(\mathbb{Z}/3,3)$. We have seen, (Hatcher, example 4.34), that Moore spaces are unique up to homotopy equivalent from which the claim follows.

(2) Firstly, since X and hence Y only have cells in dimension 0 and then > 1, we obtain by the cellular approximation theorem that $\pi_1(Y) \cong \pi_1(X) \cong 0$, so that π_1 acts trivially on homology. Hence we can use the LSSS. By the UCT,

$$\tilde{H}^k(Y) \cong \begin{cases} \mathbb{Z}/3, & k = 4 \\ 0, & k \neq 4 \end{cases}$$

so we obtain a double complex as follows:

$$\begin{tikzcd}
 & \vdots \arrow[d, no head] & & & \\
5 & \mathbb{Z}/3 \arrow[d, no head] \arrow[dr, dashed] & \mathbb{Z}/3 & \mathbb{Z}/3 & \\
3 & \mathbb{Z}/3 \arrow[d, no head] \arrow[drr, dashed] & \mathbb{Z}/3 & \mathbb{Z}/3 & \\
 & \mathbb{Z} \arrow[rr, no head] & & \mathbb{Z}/3 \arrow[r, no head] & \cdots \\
 & & 3 & 4 &
\end{tikzcd}$$

where $H^3\left(Y; H^k\left(K\left(\mathbb{Z}/3,2\right);\mathbb{Z}\right)\right) \cong \operatorname{Hom}\left(H_3(Y), H^k\left(K\left(\mathbb{Z}/3,2\right)\right)\right) \cong \operatorname{Hom}\left(\mathbb{Z}/3,\mathbb{Z}/3\right) \cong$ $\mathbb{Z}/3$ for $k = 3,5$ from the UCT and $H^4\left(Y; H^k\left(K\left(\mathbb{Z}/3,2\right);\mathbb{Z}\right)\right) \cong \operatorname{Ext}\left(H_3(Y), H^k\left(K\left(\mathbb{Z}/3,2\right);\mathbb{Z}\right)\right) \cong$ $\operatorname{Ext}\left(\mathbb{Z}/3,\mathbb{Z}/3\right) \cong \mathbb{Z}/3$ again from the UCT.

Since there can only be trivial maps in E_2, the same double complex forms E_3 where we also obtain the topmost indicated dashed map as the only possible nontrivial map.
We assumed that this map is nontrivial, hence must be an isomorphism, so the E_4 page will be as follows:

$$\begin{tikzcd}
 & \vdots & & & \\
5 & & \mathbb{Z}/3 & \mathbb{Z}/3 & \\
3 & \mathbb{Z}/3 \arrow[rrd, dashed] & & \mathbb{Z}/3 & \\
 & \mathbb{Z} & & \mathbb{Z}/3 & \cdots \\
 & & 3 & 4 &
\end{tikzcd}$$

Since $H^3\left(\tau_{>3}Y;\mathbb{Z}\right) \cong \operatorname{Hom}\left(H_3(\tau_{>3}Y),\mathbb{Z}\right) \cong 0$ by UCT, we must have that the indicated map is injective, hence an isomorphism. Thus E^4 will look as follows:

$$\begin{tikzcd}
 & \vdots & & & \\
5 & & \mathbb{Z}/3 & \mathbb{Z}/3 & \\
3 & & & \mathbb{Z}/3 & \\
 & \mathbb{Z} & & & \cdots \\
 & & 3 & 4 &
\end{tikzcd}$$

From this, we can read off that $H^7\left(\tau_{>3}Y;\mathbb{Z}\right) \cong \mathbb{Z}/3$. Now, recall that $\tau_{>3}Y$ is a CW-complex, hence its homology and cohomology groups are finitely generated abelian, so we can use the structure theorem for finitely generated abelian groups and analyse its torsion part and torsion free part, into which it separates by a direct product. Now

$$0 \to \operatorname{Ext}\left(H_6(\tau_{>3}Y),\mathbb{Z}\right) \to \underbrace{H^7\left(\tau_{>3}Y;\mathbb{Z}\right)}_{\cong\mathbb{Z}/3} \to \operatorname{Hom}\left(H_7\left(\tau_{>3}Y\right),\mathbb{Z}\right) \to 0$$

and $\operatorname{Hom}\left(H_7\left(\tau_{>3}Y\right),\mathbb{Z}\right)$ is the torsion-free part of $H_7\left(\tau_{>3}Y\right)$, so since $\mathbb{Z}/3$ surjects onto this part, it must be 0. Thus $\operatorname{Ext}\left(H_6\left(\tau_{>3}Y\right),\mathbb{Z}\right) \cong H^7\left(\tau_{>3}Y;\mathbb{Z}\right) \cong \mathbb{Z}/3$, but $\operatorname{Ext}\left(H_6\left(\tau_{>3}Y\right),\mathbb{Z}\right)$ is the torsion part of $H_6\left(\tau_{>3}Y\right)$, so $H_6\left(\tau_{>3}Y\right) \cong \mathbb{Z}/3 \bigoplus \text{(free part)}$, where the free part could be trivial. In any case, $H_6(\tau_{>3}Y)$ is nontrivial, while all $H_k(\tau_{>3}Y) \cong 0$ for $1 \le k \le 5$ by the UCT. Hence by Hurewicz,

$$\pi_k(\tau_{>3}Y) \cong \begin{cases} \mathbb{Z}/3 \bigoplus (\text{free part}), & k = 6 \\ 0, & 1 \le k \le 5 \end{cases}$$

Combining this with $\pi_k(Y)$ for $k \le 3$ which by Hurewicz and the previous problem is

$$\pi_k(Y) \cong \begin{cases} \mathbb{Z}/3, & k = 3 \\ 0, & 0 \le k \le 2 \end{cases}$$

we obtain

$$\pi_k(Y) \cong \begin{cases} \mathbb{Z}/3 \bigoplus (\text{free part}), & k = 6 \\ \mathbb{Z}/3, & k = 3 \\ 0, & k = 0, 1, 2, 4, 5 \end{cases}$$

(3) By UCT,

$$0 \to \operatorname{Ext}(H_{n-1}(Y), \mathbb{Z}/3) \to H^n(Y; \mathbb{Z}/3) \to \operatorname{Hom}(H_n(Y), \mathbb{Z}/3) \to 0$$

so when $n = 3$, Ext vanishes, so $H^3(Y;\mathbb{Z}/3) \cong \operatorname{Hom}(\mathbb{Z}/3, \mathbb{Z}/3) \cong \mathbb{Z}/3$ and when $n = 4$, Hom vanishes, so $H^4(Y;\mathbb{Z}/3) \cong \operatorname{Ext}(H_3(Y), \mathbb{Z}/3) \cong \mathbb{Z}/3$. Furthermore, $H^0(Y;\mathbb{Z}/3) \cong \mathbb{Z}/3$. In all other columns, H^p vanishes.
So consider the double complex

$$\begin{tikzcd}
 & \vdots \arrow[d, dash] & & & \\
5 & \mathbb{Z}/3ab \arrow[d, dash] \arrow[ddr, "d_3"] \arrow[dddrr, "d_4"] & \mathbb{Z}/3abx & \mathbb{Z}/3aby & \\
4 & \mathbb{Z}/3a^2 \arrow[d, dash] \arrow[ddr] & \mathbb{Z}/3a^2x & \mathbb{Z}/3a^2y & \\
3 & \mathbb{Z}/3b \arrow[d, dash] \arrow[ddrr, "\cong"] & \mathbb{Z}/3bx & \mathbb{Z}/3by & \\
2 & \mathbb{Z}/3a \arrow[d, dash] \arrow[dr, "\cong"] & \mathbb{Z}/3ax & \mathbb{Z}/3ay & \\
 & \mathbb{Z}/3 \arrow[r, dash] & \mathbb{Z}/3x \arrow[r, dash] & \mathbb{Z}/3y \arrow[r, dash] & \cdots \\
 & & 3 & 4 &
\end{tikzcd}$$

Since $H^p(\tau_{>3}Y;\mathbb{Z}/3) \cong 0$ for $p = 2, 3$, we must have that the maps eminating from $\mathbb{Z}/3a$ and $\mathbb{Z}/3b$ are injective, hence they must be isomorphisms since any nontrivial group homomorphism $\mathbb{Z}/3 \to \mathbb{Z}/3$ is an isomorphism. Thus $d_3(a) = x$ and $d_4(b) = y$. These are both forced since all other maps eminating from these groups or terminating at them are 0-maps. Now, using the multiplicative structure, we can calculate d evaluated at the other generators. Using the Leibniz rule, we get

$d_3(a^2) = d(a)a + (-1)^{|a|}ad(a) = 2ax \in \mathbb{Z}/3ax$ which still generates $\mathbb{Z}/3ax$, hence $d_3\colon \mathbb{Z}/3a^2 \to \mathbb{Z}/3ax$ is an isomorphism. Likewise

$$d_3(ab) = xb + ad_3(b) = xb \in \mathbb{Z}/3xb$$

so since $d_3\colon \mathbb{Z}/3ab \to \mathbb{Z}/3xb$ maps generators to generators, it is an isomorphism. Thus, the page E_4 looks as follows:

$$\begin{array}{c|ccc}
 & \vdots & & \\
5 & & \mathbb{Z}/3abx & \mathbb{Z}/3aby \\
4 & & \mathbb{Z}/3a^2x & \mathbb{Z}/3a^2y \\
3 & & & \mathbb{Z}/3by \\
2 & & & \mathbb{Z}/3ay \\
\hline
 & \mathbb{Z}/3 & & \cdots \\
 & & 3 & 4
\end{array}$$

Clearly, on this and all subsequent pages, all maps are trivial, so $E_4 = E_\infty$, and we get that $H^p(\tau_{>3}Y; \mathbb{Z}/3) \cong \begin{cases} \mathbb{Z}/3, & p = 0, 6 \\ 0, & 1 \le p \le 5 \end{cases}$

Now, suppose that the map $\mathbb{Z}/3 \to \mathbb{Z}/3$ in the integral case from part (2) of the problem were not an isomorphism - i.e., suppose it were trivial. Then since that group was the only group of total degree 5 and since all maps from it or terminating at it on subsequent pages would be 0-maps, we would get that $H^5(\tau_{>3}Y; \mathbb{Z}) \cong \mathbb{Z}/3$. But then by UCT, and the above part of this problem, $0 = H^5(\tau_{>3}Y; \mathbb{Z}/3) \cong H^5(\tau_{>3}Y; \mathbb{Z}) \otimes \mathbb{Z}/3 \cong \mathbb{Z}/3 \otimes \mathbb{Z}/3 \neq \mathbb{Z}/3 \neq 0$ gives a contradiction. Hence the map must have been nontrivial, which was what we wanted to show.

Now proceeding as in step (2), we can conclude all the same things.

(4)
We will make use of the following proposition a few times:

Proposition 0.2 (Hatcher, Proposition 0.18)**.** *If (X_1, A) is a CW pair and we have attaching maps $f, g\colon A \to X_0$ that are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \operatorname{rel} X_0$.*

Now, a CW complex can be constructed by successively adding cells of higher and higher dimension to the previously constructed skeleton (as described in Hatcher

p. 5), so suppose X and X' are spaces which have a single $0, 3, 4$ and 6 cell and such that $H_3(X) \cong H_3(X') \cong \mathbb{Z}/3$.
Since X and X' are CW complexes, cellular homology tells us from the cellular chain complex that the attaching maps for each CW complex of the 4-cell is a degree 3 map.

Now, to construct X and X', start with a single 0-cell. Then we attach the 3-cell via the unique attachig map $S^2 \to *$ where $*$ is the 0-cell.
This space is necessarily a 3-sphere.
The attaching map of the 4-cell is now a map $S^3 \to S^3$ which is of degree 3. We use the following lemma:

Lemma 0.3. *Two maps $f, g\colon S^n \to S^n$ have the same degree if and only if they are homotopic.*

Proof. If they are homotopic, they have the same degree as degree is a construction on homology.
Conversely, suppose $\deg f = \deg g$. Recal that the Hurewicz isomorphism is natural with the commutative diagram

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{\varphi_*} & \pi_n(Y, y_0) \\ \downarrow h & & \downarrow h \\ H_n(X) & \xrightarrow{\varphi_*} & H_n(Y) \end{array}$$

where $\varphi\colon X \to Y$ is a map.
Now, since f and g might not be basepoint preserving maps (once we choose a basepoint), we need to fix this issue. One way to do this would be for example to compose f and g with a rotation of the sphere such that it maps the basepoint to itself - noting that rotations are homotopic to the identity, hence also do not change the degrees. Thus we can without issue assume that f and g Using this, we obtain a commutative diagram as follows:

$$\begin{array}{ccc} \pi_n(S^n) & \xrightarrow{f_*} & \pi_n(S^n) \\ \downarrow \cong & & \downarrow \cong \\ H_n(S^n) & \xrightarrow{f_* = g_*} & H_n(S^n) \\ \cong \uparrow & & \cong \uparrow \\ \pi_n(S^n) & \xrightarrow{g_*} & \pi_n(S^n) \end{array}$$

Thus we find that f and g induce the same maps on π_n, so in particular, $[f] = f_*[1] = g_*[1] = [g]$, hence f and g are homotopic.

For $n = 1$, where we cannot use the Hurewicz map, we can use that the abelianization map is an isomorphism and then note that $\pi_1(S^1) \cong \mathbb{Z}$ is an isomorphism.

$\square$

Using this lemma, we can conclude that the attaching map for the 4-cell on X' is homotopic to the degree 3-map for the attaching map on X, hence using the Proposition, we conclude that $X^{(4)} \simeq (X')^{(4)} (\operatorname{rel} X^{(3)} = X'^{(3)})$. Next, we attach

a 6-cell to each complex. But note that we showed in part (3) that $\pi_5(X) = 0 = \pi_5(X')$, so suppose φ is the attaching map $S^5 \to X^{(4)}$ and φ' the attaching map $S^5 \to X'^{(4)}$. Then both a homotopic to a constant map, say $\varphi \simeq c_p$ and $\varphi' \simeq c_{p'}$. But also π_0 of either space is trivial, so they are path-connected, hence $c_p \simeq c_*$ where $*$ is the 0-cell which, in particular, is contained in the 3-skeleton of both X and X'. Thus also $c_{p'} \simeq c_*$. Let $G\colon X^{(4)} \to (X')^{(4)}$ be the homotopy equivalence rel $X^{(3)} = X'^{(3)}$ with homotopy inverse H, so both G and H restrict to the identity on the 3-skeleton. Then G and H extend to maps $\tilde{G}\colon X^{(4)} \cup_{c_*} D^6 \to (X')^{(4)} \cup_{c_*} D^6$ and $\tilde{H}\colon (X')^{(4)} \cup_{c_*} D^6 \to X^{(4)} \cup_{c_*} D^6$, respectively, by being the identity on D^6. Since the homotopies $GH \simeq \mathrm{id}$ and $HG \simeq \mathrm{id}$ are also the identity on the 3-skeleton at all times, we can extends these homotopies to $\tilde{G}\tilde{H} \simeq \mathrm{id}$ and $\tilde{H}\tilde{G} \simeq \mathrm{id}$.
And now we are done because

$$X = X^{(4)} \cup_\varphi D^6 \simeq X^{(4)} \cup_{c_*} D^6 \simeq (X')^{(4)} \cup_{c_*} D^6 \simeq (X')^{(4)} \cup_{\varphi'} D^6 = X'.$$