

1.2.4: Firstly project all points in $\mathbb{R}^3 - X$ radially onto S^2 by the map $(x, t) \rightarrow x(1 - t) + t \frac{x}{\|x\|}$. This is a deformation retraction, so the fundamental group is isomorphic to $\pi_1(\mathbb{R}^3 - X)$. Our resulting space is the 2-sphere with $2n$ points removed, where n is the number of lines in X . Choose any removed point and stereographically project the sphere with $2n$ holes onto \mathbb{R}^2 with $2n - 1$ holes removed - this is a homeomorphism, so the fundamental group is invariant.

Solution 1: Let X' denote this space. We can now define a path $\gamma: I \rightarrow X'$ which in the interval $\left[\frac{k}{2n-1}, \frac{k+1}{2n-1}\right]$ completes a loop around the k th hole where we enumerate the holes in some arbitrary way. The resulting image is $\vee_{2n-1} S^1$. Now, we define a deformation retraction of \mathbb{R}^2 minus the holes onto $\gamma(I)$. For points not inside one of the circles, we send it to the closest point on $\gamma(I)$. For points inside $\gamma(I)$, we project from the hole radially out onto $\gamma(I)$, and on $\gamma(I)$ we let the deformation retraction be the identity. Then this function deformation retracts our space X' onto $\vee_{2n-1} S^1$, so in particular, the fundamental groups are isomorphic, and by van Kampen, we have $\pi_1(X') \cong \pi_1(\vee_{2n-1} S^1) \cong *_{2n-1} \mathbb{Z}$ by example 1.21.

I hope it is okay if I include a second solution which I originally wrote that I'm not completely sure whether works.

Solution 2: Now let d denote the minimal distance between the $2n - 1$ points. Then for each hole, x_i , we define $(x, t) \rightarrow \begin{cases} (t - 1)x + t(\frac{x - x_0}{\|x - x_0\|} \frac{d}{3} + x_0), & \|x - x_i\| \leq \frac{d}{3} \\ x, & \|x - x_i\| > \frac{d}{3} \end{cases}$ for all x in the plane with $2n - 1$ holes, which is a deformation retraction that expands the holes into removed disks in the plane.

Now, the closed disks are compact, so there exists $R > 0$ such that all the removed disks are within $B(0, R)$ in our space. Now deformation retract the space by $(x, t) \rightarrow x$ for all t if $x \in B(0, R)$ and $(x, t) \rightarrow x(1 - t) + R \frac{x}{\|x\|} t$ if $x \in B(0, R)^c$.

Let Y denote the space $D(0, R)$ with $2n - 1$ disks removed. Y is path-connected, so choose a basepoint $x_0 \in Y$ and connect it to one circle which we then connect to the next circle by a path and so on with a path from the last circle to x_0 , forming a loop. We wish to build Y as a CW-complex, so to attach a 2-cell, we attach another edge with a point on its free end to any point on the 1-skeleton. To the free point, we attach an edge whose endpoints are both the free point, thereby creating a loop.

Thus Y can be built as a CW-complex with $2n + 1$ 0-cells (one for each circle that is the boundary of the removed disk, and one for the chosen basepoint and one for the free point), $4n$ 1-cells connecting, $2n - 1$ used to construct circles and $2n - 1$ used to connect the circle to the basepoint and 2 used for the attaching map to attach to. To this 1-skeleton, we attach a 2-cell to get a disk with 4 open disks removed where we can view the disk as a square, letting one edge wrap around the original 1-skeleton, 2 edges fusing onto the last added edge and the fourth edge fusing onto the last added loop.

Since X^1 is path-connected, we then find by proposition 1.26 that $X^1 \rightarrow X^2$ induces a surjection $\pi_1(X^1) \rightarrow \pi_1(X^2)$ whose kernel is generated by the attaching loop which can be decomposed as $a = a_1 a_2 \dots a_{2n}$ where each a_i is a generating loop for the corresponding circle's fundamental group. Thus the fundamental group of the space is isomorphic to the fundamental group of the 1-skeleton modulo a . Now, as each edge in the 1-skeleton is contractible, we get by page 11 that the quotient space obtained by collapsing this to a point is homotopy equivalent to the one skeleton X^1 ; and by proposition 1.18, $\pi_1(X^1)$ is isomorphic to the fundamental group of the quotient space which is $\pi_1(\vee_{2n} S^1) = *_{2n} \mathbb{Z}$. Therefore the fundamental group of the space is

$$\pi_1(X^2) \cong *_{2n} \mathbb{Z} / (a_1 a_2 \dots a_{2n}) \cong *_{2n-1} \mathbb{Z}.$$

1.2.22: Since X is a deformation retract of $\mathbb{R}^3 - K$ by assumption, we wish to calculate $\pi_1(X)$. Firstly, X is path connected, so pick any $p \in X$ on the plane. Next deformation retract the plane onto p by $(x, t) \rightarrow (1 - t)x + tp$. This collapses the edges of the R_i to p as well. Now, each R_i is a rectangular piece, and we can deformation retract this to its edge - i.e. we deformation retract it along its length. Since the edges connect to the plane T are identified, we end up with a circle for each R_i , wrapping around K once. For the relation, choose a concrete point, say on the plane in the image where R_i intersects R_k , i.e. the leftmost corner in the diagram from Hatcher. Then taking the path $x_i x_j x_i^{-1}$ gives a loop that is

homotopy equivalent to the path x_k by sliding the loop across the crossing using S_l . Thus $x_i x_j x_i^{-1} = x_k$ for each square S_l where the indices are as in the figure in Hatcher.

(b) We have

$$\pi_1(\mathbb{R}^3 - K) \cong \pi_1(X) \cong \langle x_1, x_2, \dots, x_n \mid x_i x_j x_i^{-1} x_k^{-1}, \text{ for all } S_l \rangle$$

Abelianization of this results from letting the basis be the basis of a free abelian group. In this case, all relations for S_l become

$$x_i x_j x_i^{-1} x_k^{-1} = x_i x_i^{-1} x_j x_k^{-1} = x_j x_k^{-1}$$

So for any bridge, identify loops on each side. Since all rectangles are separated by some amount of rectangles and bridges, this makes all loops identify, so we simply get $\pi_1(\mathbb{R}^3 - K) \cong \langle x \rangle = \mathbb{Z}$ under abelianization.