ASSIGNMENT 4

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Exercise 0.1. Let $\dim V = n < \infty$ and assume $\operatorname{char} F \neq 2$. Let $B \in \operatorname{Bil}(V)$ be represented by a matrix $[B]$ with respect to some basis. Prove that B is symmetric, respectively skew-symmetric if and only if $[B]$ has this property. Then determine the dimensions of $\operatorname{Sym}(V), \operatorname{Skew}(V), \operatorname{Alt}(V)$ and $\operatorname{Quad}(V)$.

Solution. Let $\{x_1, \dots, x_n\}$ be a basis for V. If B is symmetric then $B(u,v) = B(v,u)$ for all $u, v \in V$, so in particular, $B(x_i, x_j) = B(x_j, x_i)$. Hence

$$[B]_{ij} = B(x_i, x_j) = B(x_j, x_i) = [B]_{ji} = [B]^t_{ij}$$

showing that the matrix $[B]$ is symmetric. If $[B]$ is symmetric, then

$$B(x_i, x_j) = [B]_{ij} = [B]_{ji} = B(x_j, x_j)$$

for all i, j, so B is symmetric.
Similarly, if B is skew-symmetric, then

$$[B]_{ij} = B(x_i, x_j) = -B(x_j, x_i) = -[B]_{ji} = -[B]^t_{ij}$$

so $[B]$ is skew-symmetric, and conversely, if $[B]$ is skew-symmetric, then

$$B(x_i, x_j) = [B]_{ij} = -[B]_{ji} = -B(x_j, x_i)$$

for all i, j.

Let $\{y_1, \dots, y_n\}$ be the dual basis for $\{x_1, \dots, x_n\}$. Let $w_{ij} = y_i y_j \in \operatorname{Bil}(V)$. Then by theorem 4.3, any symmetric bilinear form B can be written

$$B = \sum_{i<j} B(x_i, x_j)(w_{ij} + w_{ji}) + \sum_i B(x_i, x_i) w_{ii}$$

and similarly, skew symmetric matrices B can be written

$$\sum_{i<j} B(x_i, x_j)(w_{ij} - w_{ji})$$

(no w_{ii} since skew-symmetric matrices are alternating for $\operatorname{char} F \neq 2$). Hence $\mathcal{B} = \{w_{ij} + w_{ji}\}_{i<j} \cup \{w_{ii}\}_i$ spans $\operatorname{Sym}(V)$ and $\mathcal{B}' = \{w_{ij} - w_{ji}\}_{i<j}$ spans $\operatorname{Skew}(V)$. Since $\{w_{ij}\}$ forms a basis for $\operatorname{Bil}(V)$, they are linearly independent, so in particular $\mathcal{B}$ and $\mathcal{B}'$ are linearly independent since each w_{ij} only occurs once in each set, and hence form bases for $\operatorname{Sym}(V)$ and $\operatorname{Skew}(V)$, respectively. Therefore, $\dim \operatorname{Sym}(V) = 1 + 2 + \ldots + (n-1) + n = \frac{(n+1)n}{2}$ while $\dim \operatorname{Skew}(V) = 1 + 2 + \ldots + (n-1) = \frac{(n-1)n}{2}$.

Similarly, we see that $\{w_{ij}\}_{i \neq j}$ is a basis for $\operatorname{Alt}(V)$, so $\dim \operatorname{Alt}(V) = n^2 - n$.

Lastly, since $\operatorname{char} F \neq 2$, lemma 4.19 gives an isomorphism $\operatorname{Sym}(V) \cong \operatorname{Quad}(V)$, so in particular, $\dim \operatorname{Quad}(V) = \dim \operatorname{Sym}(V) = \frac{(n+1)n}{2}$.