ASSIGNMENT 5

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Exercise 0.1 (1)**.** (2)
We claim that $\mathbb{Z}[x_1, x_2, x_3]$ is a finite extension. Now, clearly, we can express $1, x_1, x_1^2, x_1^3$ as linear combinations over $1, x_1, x_1^2, x_1^3$. Suppose we can express x_1^n as a linear combination of $1, x_1, x_1^2, x_1^3$ for $n = 1, \ldots, N-1$ for some $N \geq 4$. Since

$$\begin{aligned}
x^n &= x_1^{n-1}(x_1 + x_2 + x_3) - x_1^{n-1}x_2 - x_1^{n-1}x_3 \\
&= x_1^{n-1}\sigma_1 - x_1^{n-2}\sigma_2 + x_1^{n-2}x_2x_3 \\
&= x_1^{n-1}\sigma_1 - x_1^{n-2}\sigma_2 + x_1^{n-3}\sigma_3
\end{aligned}$$

we find that for $N \geq 4$, x_1^N can be written as a linear combination over x_1^{N-1}, x_1^{N-2} and x_1^{N-3} which by the inductive assumption can be written as linear combinations of $1, x_1, x_1^2, x_1^3$. Hence $\mathbb{Z}[x_1, x_2, x_3]$ is a finite extension over $\mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]$ with

$$\left\{1, x_1, x_1^2, x_1^3, x_2, x_2^2, x_2^3, x_3, x_3^2, x_3^3\right\}$$

as a finite generating set. By proposition 10.11, this also implies that the extension is integral.

(3) We claim that $\mathbb{Z}[x, y]$ is not a finite extension of $\mathbb{Z}[x, xy]$. Suppose $\{g_1, \ldots, g_n\} \in \mathbb{Z}[x, y]$ is a generating set as a module. Let N be the maximal degree of y over $g_1, \ldots, g_n$. Then $y^{N+1} \in \operatorname{span}(g_1, \ldots, g_n)$, so let $y^{N+1} = f_1(x, xy)g_1(x, y) + \ldots + f_n(x, xy)g_n(x, y)$.
Writing each $f_i(x, xy) = \sum \alpha_{i,k,l} x^k (xy)^l$, we see that

$$\begin{aligned}
y^{N+1} &= \sum \alpha_{1,k,l} x^k (xy)^l g_1(x, y) + \ldots + \sum \alpha_{n,k,l} x^k (xy)^l g_n(x, y) \\
&= \sum \left(\alpha_{1,k,l} g_1(x, y) + \ldots + \alpha_{n,k,l} g_n(x, y)\right) x^k (xy)^l.
\end{aligned}$$

So in particular, we must have that for $(k, l) \neq (0, 0)$,

$$\alpha_{1,k,l} g_1(x, y) + \ldots + \alpha_{n,k,l} g_n(x, y) = 0.$$

But then we get

$$y^{N+1} = \alpha_{1,0,0} g_1(x, y) + \ldots + \alpha_{n,0,0} g_n(x, y)$$

which has maximal y degree N, giving a contradiction. Thus the extension is not finite. However, clearly, it is a finite-type extension, since y together with $\mathbb{Z}[x, xy]$ precisely generate all of $\mathbb{Z}[x, y]$. By proposition 10.11, we then find that $\mathbb{Z}[x, y]$ is not an integral extension.

(7) If the map $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x, y, z] / \left(z^2 - xy\right)$ were integral, proposition 10.6 gives that $\mathbb{C}[x, y] \subset \mathbb{C}[x, y, z] / \left(z^2 - xy\right)$ would be finitely generated as a $\mathbb{C}[x]$-module. Suppose $f_1, \ldots, f_n$ generated $\mathbb{C}[x, y]$ as a $\mathbb{C}[x]$ module in $\mathbb{C}[x, y, z] / \left(z^2 - xy\right)$. If y^N is the maximal degree of y among $f_1, \ldots, f_n$, then $y^{N+1} = \sum g_i f_i$ for $g_i \in \mathbb{C}[x]$.

However, there is clearly no way to obtain y^{N+1} in such a way. So the extension is not integral. Since it is clearly finite type, it is also not finite by proposition 10.11.

(9) If $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x,y,z] / \left(z^2 - xy, x^3 - yz\right)$ were integral, z would be integral over $\mathbb{C}[x]$, so there would be some linear combination

$$f_n(x)z^n + \ldots + f_0(x) = 0$$

However, there is not relation converting xz to something different, so if x^{k_n} is the highest term of x in $f_n(x)$, then $x^{k_n}z^n$ is a term that cannot cancel in the above linear combination. So the extension cannot be integral. Hence it can also not be finite.

(10) The extension $\mathbb{C} \hookrightarrow \mathbb{C}[x_1, x_2, x_3, \ldots] / \left(x_1^2, x_2^2, x_3^2, \ldots\right)$ is not finite: suppose it were generated by $f_1, \ldots, f_n \in \mathbb{C}[x_1, x_2, \ldots] / \left(x_1^2, x_2^2, \ldots\right)$, and let m be maximal such that one of the f_i has a term with x_m. Then $x_{m+1} = c_1 f_1 + \ldots + c_n f_n$ with $c_i \in \mathbb{C}$. However, then multiplying both sides by x_{m+1}, we see that each non-zero term in $c_1 f_1 + \ldots + c_n f_n$ must have a x_{m+1}, contradicting maximality of m.

The extension is integral, however, since for any $b \in \mathbb{C}[x_1, x_2, \ldots] / \left(x_1^2, x_2^2, \ldots\right)$, let $x_{i_1}, \ldots, x_{i_k}$ be the x_i which appear in b. Then $b\left(x_{i_1} \cdots x_{i_k}\right) = 0 \subset \left(x_{i_1} \cdots x_{i_k}\right)$, and $\left(x_{i_1} \cdots x_{i_k}\right)$ is clearly finitely generated, so by proposition 10.6, b is integral over $\mathbb{C}$.