Problem 0.1. Let F be the homotopy fibre of the map $S^n \to S^n$ of degree k, for $n \geq 2$.

(1) Show that $H^i(F) = 0$ for $0 < i < n$.
(2) Using the Serre spectral sequence, compute that

$$H^i(F) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/k, & i = 1 + m(n-1),\ m > 0 \\ 0, & \text{otherwise} \end{cases}.$$

(3) Show that for $x, y \in H^*(F)$, if $\deg(x), \deg(y) > 0$, then $x \smile y = 0$.

Proof. (1) Since $\pi_1 S^n = 0$, the Serre spectral sequence to the homotopy fiber sequence

$$F \to S^n \to S^n$$

gives the following double complex:

$$\begin{tikzcd}
H^{n-1}(F) \arrow[d, dash] \arrow[ddddrr] & & \\
\vdots \arrow[d, dash] & & \\
H^2(F) \arrow[d, dash] & & \\
H^1(F) \arrow[d, dash] & & \\
\mathbb{Z} \arrow[rr, dash] & & \mathbb{Z} \arrow[r, dash] & \cdots \\
& & n &
\end{tikzcd}$$

We apply the LSSS for cohomology and find that $H^i(S^n) = F_0^n$, and since $H^i(F)$ is the only nontrivial entry on the antidiagonal in degree i, and since there are no maps to kill off $H^i(F)$ for $0 < i < n-1$, we obtain that $H^i(F) = H^i(S^n) = 0$ for $0 < i < n-1$.
All that's missing is $i = n-1$. For this, note that by the LES for the fibration, we get the following exact sequence:

$$\underbrace{\mathbb{Z}}_{\pi_n(S^n)} \xrightarrow{\cdot k} \underbrace{\mathbb{Z}}_{\pi_n(S^n)} \to \pi_{n-1}(F) \to \underbrace{0}_{\pi_{n-1}(S^n)}$$

hence $\pi_{n-1}(F) \cong \operatorname{coker}\left(\mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z}\right) \cong \mathbb{Z}/k$, and by the Hurewicz theorem, we get $H_{n-1}(F) \cong \pi_{n-1}(F) \cong \mathbb{Z}/k$. Now using the UCT, we obtain

$$0 \to \underbrace{\operatorname{Ext}\left(H_{n-2}(F), \mathbb{Z}\right)}_{=0} \to H^{n-1}(F) \to \underbrace{\operatorname{Hom}\left(\underbrace{H_{n-1}(F)}_{=\mathbb{Z}/k}, \mathbb{Z}\right)}_{=0} \to 0$$

so $H^{n-1}(F) = 0$ as we wanted.
(2)
By the LSSS, the E^∞ page has the form $E^\infty_{0,0} = E^\infty_{n,0} \cong \mathbb{Z}$, so in particular, on the E^k page, we get the following double complex:

$$\begin{tikzcd}
\vdots \arrow[d, dash] & \\
H^{3(n-1)}(F) \arrow[d, dash] \arrow[dr] & H^{3(n-1)}(F) \\
H^{2(n-1)}(F) \arrow[d, dash] \arrow[dr] & H^{2(n-1)}(F) \\
\mathbb{Z} \cong H^{n-1}(F) \arrow[d, dash] \arrow[dr] & \mathbb{Z} \cong H^{n-1}(F) \\
\mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

This is the only page on which the horizontal maps can be nontrivial, so given the E^∞ page, we conclude that the maps must be isomorphisms (including the trivial ones by just inductively shifting down by $n-1$ enough times). Hence we get periodicity, so

$$H^i(F) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/k, & i = 1 + m(n-1),\ m > 0 \\ 0, & \text{otherwise} \end{cases},$$

which was what we wanted to show.

(3) Suppose $\deg(x) + \deg(y) = 2$ so both are of degree 1, then since $H^1(F) = 0$, we have $x = 0 = y$ so $x \smile y = 0$. Suppose we have shown it for $\deg(x) + \deg(y) \leq N - 1$ now. If $\deg x + \deg y = N$, then firstly we can assume $x, y \neq 0$ since otherwise $x \smile y = 0$. Hence $x \in H^{1+m(n-1)}(F)$ and $y \in H^{1+m'(n-1)}$, so $x \smile y \in H^{2+(m+m')(n-1)}(F) = 0$, so directly, $x \smile y = 0$.

$\square$