1. Cohomology and Homological Algebra

1.1. **Cohomology in terms of Homological Algebra.** Recall the Universal Coefficient Theorem for Cohomology:

Theorem 1.1 (Universal Coefficient Theorme for Cohomology)**.** *Let R be a ring and A an $R-$ module. Let C_* be a complex of projective R-modules such that the subcomplex of boundaries B_* is also a complex of projective modules.*

(1) *For all n, there is a SES*
$$0 \to \mathrm{Ext}^1_R\left(H_{n-1}(C_*), A\right) \stackrel{\lambda_n}{\to} H^n\left(\mathrm{Hom}_R\left(C_*, A\right)\right) \stackrel{\mu_n}{\to} \mathrm{Hom}_R\left(H_n\left(C_*\right), A\right) \to 0$$
where both λ_n and μ_n are natural in C_* and A.

(2) *If R is a PID, then the SES in (1) is split, but it is not always naturally split.*

Also recall the basic properties:

Lemma 1.2. *For a finitely generated H, we have*

- $\mathrm{Ext}\left(H \oplus H', G\right) \cong \mathrm{Ext}\left(H, G\right) \oplus \mathrm{Ext}(H', G)$
- $\mathrm{Ext}(H, G) = 0$ *if H is free.*
- $\mathrm{Ext}(\mathbb{Z}/n, G) \cong G/nG$.

Corollary 1.3. *If the homology groups H_n and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n$ and $T_{n-1} \subset H_{n-1}$, then $H^n(\mathrm{Hom}_{\mathbb{Z}}(C_*, \mathbb{Z})) \cong (H_n/T_n) \oplus T_{n-1}$.*

Proof. By the Universal Coefficient theorem for cohomology, we have that
$$H^n\left(\mathrm{Hom}_{\mathbb{Z}}\left(C_*, \mathbb{Z}\right)\right) \cong \mathrm{Ext}^1_{\mathbb{Z}}\left(H_{n-1}(C_*), \mathbb{Z}\right) \oplus \mathrm{Hom}_{\mathbb{Z}}\left(H_n\left(C_*\right), \mathbb{Z}\right)$$
Now, $\mathrm{Ext}^1_{\mathbb{Z}}\left(H_{n-1}(C_*), \mathbb{Z}\right) \cong T_{n-1}$ and $\mathrm{Hom}_{\mathbb{Z}}\left(H_n\left(C_*\right), \mathbb{Z}\right) \cong H_n/T_n$. □

Proposition 1.4. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G.*

Proof. Suppose $\alpha\colon C_* \to C'_*$ is the chain map such that $\alpha_*\colon H_n(C_*) \to H_n(C'_*)$ is an isomorphism for all n. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}\left(H_{n-1}(C), G\right) \arrow[r] & H^n(C; G) \arrow[r, "h"] & \mathrm{Hom}(H_n(C), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(C'), G) \arrow[r] \arrow[u, "(\alpha_*)^*"', "\cong"] & H^n(C'; G) \arrow[r, "h"] \arrow[u, "\alpha^*"'] & \mathrm{Hom}(H_n(C'), G) \arrow[r] \arrow[u, "(\alpha_*)^*"', "\cong"] & 0
\end{tikzcd}$$

which follows form naturality of the Universal Coefficient theorem. Then by the 5-lemma, we obtain that α^* is an isomorphism also.

□

1.2. **Cohomology of Spaces.** Define $S^{-n}(X; A) := \mathrm{Hom}_{\mathbb{Z}}\left(S_n(X), A\right)$, so $S^*(X; A)$ is a chain complex. We define $H^n(X; A) := H_{-n}\left(S^*(X; A)\right)$, called *singular cohomology of X with coefficients in A.*

Thus an n-cochain $\varphi \in S^{-n}(X; G)$ assigns to each n-simplex $\sigma\colon \Delta^n \to X$ a value $\varphi\left(\sigma\right) \in G$. Since the n-simplicies form a basis for $S_n(X)$, these values can be chosen

arbitrarily, hence n-cochains are exactly equivalent to functions from singular n-simplices to G.
The *coboundary map* $\delta\colon S^{-n}(X;G) \to S^{-(n+1)}(X;G)$ is the dual ∂^*, so for a cochain $\varphi \in S^{-n}(X;G)$, its coboundary $\delta\varphi$ is the composition $\delta\varphi = \partial^*\varphi = \varphi \circ \partial$, i.e., the composition $C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\varphi} G$.
Hence for a singular $(n+1)$-simplex $\sigma\colon \Delta^{n+1} \to X$, we have

$$\delta\varphi(\sigma) = \sum_i (-1)^i \varphi\left(\sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_{n+1}]}\right).$$

Since δ^2 is the dual of $\partial^2 = 0$, we have $\delta^2 = 0$ also, so $H^n(X;G)$ can be defined as above.

Note. For a cochain $\varphi \in S^{-n}(X;G)$ to be a cocyle means that $\delta\varphi = \varphi\partial = 0$, i.e., it means that φ vanishes on boundaries.

1.2.1. H^0 *:* When $n = 0$, there is no Ext term and so $H^0(X;G) \cong \mathrm{Hom}(H_0(X), G)$. This can also be sen from definitions: singular 0-simplices are just points of X, so a cochain in $S^0(X;G)$ is an arbitrary function $\varphi\colon X \to G$, not necessarily continuous. For this to be a cocycle, we must have $0 = \delta\varphi = \varphi \circ \partial$. Evaluating this at some 1-simplex $[v_0, v_1]$, we find that $\varphi(v_1) - \varphi(v_0) = 0$, so φ is simply constant on each path component. Thus $H^0(X;G)$ is just the set of all functions from the path components of X into G, which is the same as the set of all group homomorphisms from $S_0(X)$ into G, i.e., $H^0(X;G) \cong \mathrm{Hom}\,(H_0(X), G)$.

1.2.2. H^1 *:* Likewise, $\mathrm{Ext}\,(H_0(X), G) = 0$ since $H_0(X)$ is free, so by the Universal Coefficient theorem, $H^1(X;G)$ is isomorphic to $\mathrm{Hom}\,(H_1(X), G)$ which, when X is path-connected, is isomorphic to $\mathrm{Hom}\,(\pi_1(X), G)$ since G is abelian.

1.2.3. *Reduced Cohomology Groups.* Reduced cohomology groups $\tilde{H}^n(X;G)$ can be defined by dualizing the augmented chain complex $\ldots \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, then taking $\ker/\mathrm{im}$. This gives $\tilde{H}^n(X;G) = H^n(X;G)$ for $n > 0$, and the Universal Coefficient theorem gives $\tilde{H}^0(X;G) = \mathrm{Hom}\left(\tilde{H}_0(X), G\right)$.

We can also describe $\tilde{H}^0(X;G) \cong \mathrm{Hom}\left(\tilde{H}_0(X), G\right)$ more explicitly by using the above interpretation of $H^0(X;G)$ as functions $X \to G$ which are constant on path-components. Recall that $\varepsilon\colon C_0(X) \to \mathbb{Z}$ sends each singular 0-simplex σ to 1, so ε^* sends a homomorphism $\varphi\colon \mathbb{Z} \to G$ to $C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\varphi} G$ which sends $\sigma \mapsto \varphi(1)$ for all 0-simplices. That is, $\varepsilon^*\varphi$ is a constant function $X \to G$, and since $\varphi(1)$ can be any element of G, the image of ε^* consists of precisely the constant functions. Thus $\tilde{H}^0(X;G)$ is all functions $X \to G$ that are constant on path-components modulo the functions that are constant on all of X.

1.2.4. *Relative Groups and the LES of a Pair.* To define relative groups $H^n(X, A; G)$ for a pair (X, A), we first dualize the SES

$$0 \to S_n(A) \xrightarrow{i} S_n(X) \xrightarrow{j} S_n(X, A) \to 0$$

by applying $\mathrm{Hom}(-, G)$ to get

$$0 \leftarrow S^n(A;G) \xleftarrow{i^*} S^n(X;G) \xleftarrow{j^*} S^n(X, A; G) \leftarrow 0 \tag{Ω}$$

where $S^n(X,A;G) := \mathrm{Hom}\,(C_n(X,A),G)$.
To see exactness of the dual sequence, we note the following: i^* restricts cochains on X to cochains on A, so for a function from singular n-simplices in X to G, the image of this function under i^* is obtained by restricting the domain of the function to singular n-simplices in A. Every function from singular n-simplices in A to G can be extended to all singular n-simplices in X, for example, by assigning the value 0 to all singular n-simplices not in A, so i^* is surjective. The kernel consists of cochains taking the value 0 on all singular n-simplices in A. Such cochains are the same as homomorphisms $C_n(X,A) = C_n(X)/C_n(A) \to G$, so the kernel of i^* is exactly $C^n(X,A;G) = \mathrm{Hom}\,(C_n(X,A),G)$, giving the desired exactness.

Note. Note that we can view $C^n(X,A;G)$ as the functions from signular n-simplices in X to G that vanish on simplices in A, since the basis for $C_n(X)$ consisting of singular n-simplices in X is the disjoint union of the simplices with image contained in A and the simplices with image no contained in A.

Relative coboundary maps $\delta\colon C^n(X,A;G) \to C^{n+1}(X,A;G)$ are obtained as restrictions of the absolute δ's.

Since the maps i^* and j^* commute with δ (since i and j commute with ∂), the maps i^* and j^* induce chain maps $i^*\colon S^*(X;G) \to S^*(A;G)$ and $j^*\colon S^*(X,A;G) \to S^*(X;G)$, and in particular, since $i^*j^* = 0$, this is a SES of chain complexes, hence induces a LES of cohomology groups (Thm. 6.5.5, AlgTop1):

$$\ldots \to H_n\left(S^*(X,A;G)\right) \xrightarrow{j^*} H_n\left(S^*(X;G)\right) \xrightarrow{i^*} H_n(S^*(A;G)) \xrightarrow{\delta} H^{n+1}(X,A;G) \to \ldots$$

By similar reasoning, one obtains a LES of reduced cohomology groups for a pair (X,A) with A nonempty. In particular, if A is a point, we find $\tilde{H}^n(X;G) \cong H^n(X,x_0;G)$.

More generally, there is a LES for a triple (X,A,B) coming from the SES

$$0 \leftarrow C^n(A,B;G) \xleftarrow{i^*} C^n(X,B;G) \xleftarrow{j^*} C^n(X,A;G) \leftarrow 0.$$

1.2.5. *Duality between Connecting Homomorphisms.* There is a duality between $\delta\colon H^n(A;G) \to H^{n+1}(X,A;G)$ and $\partial\colon H_{n+1}(X,A) \to H_n(A)$ as depicted in the following diagram:

$$\begin{tikzcd}
H^n(A;G) \arrow[r, "\delta"] \arrow[d, "h"] & H^{n+1}(X,A;G) \arrow[d, "h"] \\
\mathrm{Hom}\,(H_n(A),G) \arrow[r, "\partial^*"] & \mathrm{Hom}\,(H_{n+1}(X,A),G)
\end{tikzcd}$$

Proof. Recall that the connecting homomorphisms were defined by the diagrams

$$\begin{tikzcd}
 & S^{n+1}(X;G) & S^{n+1}(X,A;G) \arrow[l] \\
S^n(A;G) \arrow[urr, dashed] & S^n(X;G) \arrow[l] \arrow[u] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& S_{n+1}(X) \arrow[r] & S_{n+1}(X,A) \arrow[dll, dashed] \\
S_n(A) \arrow[r] & S^n(X;G) \arrow[u] &
\end{tikzcd}$$

where the dashed arrows are only there when the chain and cochain groups are replaced by homology and cohomology groups.

To see that $h\delta = \partial^* h$, let $\alpha \in H^n(A;G)$ be represented by a cocycle $\varphi \in S^n(A;G)$. To compute $\delta(\alpha)$, first extend φ to $\overline{\varphi} \in S^n(X;G)$ by letting $\overline{\varphi}$ be 0 on all cochains not contained in A. Then composing $\overline{\varphi}$ with $\partial\colon S_{n+1}(X) \to S_n(X)$ to get a cochain $\delta\overline{\varphi} = \overline{\varphi}\partial \in S^{n+1}(X;G)$, and since $\varphi \in S^n(A;G)$ was a cocycle, we have that $\delta\varphi = 0$, so $\overline{\varphi} \circ \partial$ is actually in $S^{n+1}(X,A;G)$ and represents $\delta(\alpha) \in H^{n+1}(X,A;G)$. Now applying h to $\overline{\varphi}\partial$ simply restricts the domain of $\overline{\varphi}\partial$ to relative cycles in $S_{n+1}(X,A)$, i.e., $(n+1)$-chains in X whose boundary lies in A. On such chains $\overline{\varphi}\partial = \varphi\partial$ since the extension of φ to $\overline{\varphi}$ has no effect here. Thus $h\delta(\alpha)$ is represented by $\varphi\partial$.

One the other hand, let us consider $\partial^* h(\alpha)$. Recall that $S^n(X;A) = \operatorname{Hom}_{\mathbb{Z}}(S_n(X), A)$, so φ is a homomorphism $S_n(X) \to A$. Applying h to φ then restricts the domain to n-cycles in A. Then applying ∂^* composes with the map which sends a relative $(n+1)$-cycle in X to its boundary in A. Thus $\partial^* h(\alpha)$ is represented by $\varphi\partial$ just as $h\delta(\alpha)$ was, hence the square commutes.

$\square$

1.2.6. *Induced Homomorphisms.* If we have a chain map $f_\#\colon S_*(X) \to S_*(Y)$ induced by $f\colon X \to Y$, the we also have dual cochain maps $f^\#\colon S^*(Y;G) \to S^n(X;G)$. The relation $f_\#\partial = \partial f_\#$ dualizes to $\delta f^\# = f^\#\delta$, so $f^\#$ is a cochain chain map and hence induced homomorphisms $f^*\colon H^n(Y;G) \to H^n(X;G)$.

In the relative case, a map $f\colon (X,A) \to (Y,B)$ induces $f^*\colon H^n(Y,B;G) \to H^n(X,A;G)$ by the same reasoning.

Since f induces a map between short exact sequences of cochain complexes, it induces a map between long exact sequences of cohomology groups with commuting squares.

The properties $(fg)^\# = g^\# f^\#$ and $\mathrm{id}^\# = \mathrm{id}$ imply $(fg)^* = g^* f^*$ and $\mathrm{id}^* = \mathrm{id}$, so $X \mapsto H^n(X;G)$ and $(X,A) \mapsto H^n(X,A;G)$ are contravariant functors.

The algebraic Universal Coefficient theorem applies also to the relative cohomology since the relative groups $C_n(X,A)$ are free, and there is a naturality statement also: a map $f\colon (X,A) \to (Y,B)$ induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(X,A),G) \arrow[r] & H^n(X,A;G) \arrow[r, "h"] & \operatorname{Hom}(H_n(X,A),G) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(Y,B),G) \arrow[r] \arrow[u, "(f_*)^*"] & H^n(Y,B;G) \arrow[r, "h"] \arrow[u, "f^*"] & \operatorname{Hom}(H_n(Y,B),G) \arrow[r] \arrow[u, "(f_*)^*"] & 0
\end{tikzcd}$$

1.2.7. *Axioms for Cohomology.* These are exactly dual to the axioms for homology. Restricting attention to CW complexes, a (reduced) *cohomology theory* is a sequence of contravariant functors $\tilde{h}^n$ from CW complexes to abelian groups, together with natural coboundary homomorphisms $\delta\colon \tilde{h}^n(A) \to \tilde{h}^{n+1}(X/A)$ for CW pairs (X,A), satisfying the following axioms:

(1) If $f \simeq g\colon X \to Y$, then $f^* = g^*\colon \tilde{h}^n(Y) \to \tilde{h}^n(X)$.
(2) For each CW pair (X, A), there is a LES

$$\dots \xrightarrow{\delta} \tilde{h}^n(X/A) \xrightarrow{q^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \xrightarrow{q^*} \dots$$

(3) For a wedge sum $X = \bigvee_\alpha X_\alpha$ with inclusions $i_\alpha\colon X_\alpha \hookrightarrow X$, the product map $\prod_\alpha i_\alpha^*\colon \tilde{h}^n(X) \to \prod_\alpha \tilde{h}^n(X_\alpha)$ is an isomorphism for each n.

1.2.8. *Simplicial Cohomology.* If X is a Δ-complex and $A \subset X$ is a subcomplex, then the simplciail chain groups $\Delta_n(X, A)$ dualize to simplicial cochain groups $\Delta^n(X, A; G) := \operatorname{Hom}(\Delta_n(X, A), G)$, and the resulting cohomology groups are by definition the simplicial cohomology groups $H^n_\Delta(X, A; G)$. Since the inclusions $\Delta_n(X, A) \hookrightarrow C_n(X, A)$ induce isomorphisms $H_n^\Delta \cong H_n(X, A)$, then Corollary 1.3 implies that the dual maps $S^n(X, A; G) \to \Delta^n(X, A; G)$ also induce isomorphisms $H^n(X, A; G) \cong H^n_\Delta(X, A; G)$.

Exercise 1.5. (1) Directly from the definitions, compute the simplicial cohomology groups of $S^1 \times S^1$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients, using the product Δ-complex structure.
(2) Do the same for $\mathbb{RP}^2$ and the Klein bottle.

Solution. We are asked to do it directly from definitions, so let us track through each step: we can give $S^1 \times S^1 =: T$ the Δ-complex structure indicated Figure 1.

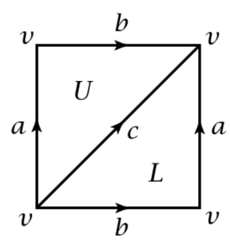

FIGURE 1.

Then $\Delta_n(S^1 \times S^1) \cong \begin{cases} \mathbb{Z}^2, & n = 2 \\ \mathbb{Z}^3, & n = 1 \\ \mathbb{Z}, & n = 0 \\ 0, & \text{else} \end{cases}$ And in particular, we get a chain complex

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0$$

so dualizing, we get that since $\operatorname{Hom}(\mathbb{Z}^k, \mathbb{Z}) \cong \mathbb{Z}^k$, then the dual complex becomes

$$0 \leftarrow \mathbb{Z}^2 \xleftarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}^*} \mathbb{Z}^3 \xleftarrow{0} \mathbb{Z} \leftarrow 0.$$

By linear algebra, we know that the dual of a matrix is given by its transpose, so this chain complex becomes

$$0 \leftarrow \mathbb{Z}^2 \xleftarrow{\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xleftarrow{0} \mathbb{Z} \leftarrow 0.$$

From this, we can read off the homology groups of the chain complex. In degree 2, $\ker \delta = \mathbb{Z}^2$, so

$$H^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}^2 / (U = L, U = -L) \cong 0.$$

For H^1, the kernel of the transposed matrix is $\ker \delta = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle$ which is thus isomorphic to $\mathbb{Z}$. Since the boundary map is 0, we find that $H^1 \left(S^1 \times S^1, \mathbb{Z} \right) \cong \mathbb{Z}$. Lastly, $H^0 \left(S^1 \times S^1, \mathbb{Z} \right) \cong \mathbb{Z}$ and $H^k = 0$ for all other k.
Note that this agrees with what Poincaré duality tells us.

If instead, we wanted to calculate $H^* \left(S^1 \times S^1; \mathbb{Z}_2 \right)$, then since $\mathrm{Hom} \left(\mathbb{Z}^k, \mathbb{Z}_2 \right) \cong \mathbb{Z}_2^k$, we find that the dual chain complex takes the form

$$0 \leftarrow \mathbb{Z}_2^2 \xleftarrow{\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}} \mathbb{Z}_2^3 \xleftarrow{0} \mathbb{Z}_2 \leftarrow 0$$

In this case, the image in degree three is $\langle (1,1) \rangle$, so $H^2 \left(\mathbb{RP}^2, \mathbb{Z}_2 \right) \cong \mathbb{Z}_2^2 / \left\{ (1,1) = (0,0) \right\} \cong \mathbb{Z}_2$. The kernel in degree 1 also now becomes $\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}_2^2$, so $H^1 \left(\mathbb{RP}^2, \mathbb{Z}_2 \right) \cong \mathbb{Z}_2^2$, and clearly, $H^0 \left(\mathbb{RP}^2, \mathbb{Z}_2 \right) \cong \mathbb{Z}_2$.
(2) We use the Δ-complexes for $\mathbb{RP}^2$ and the Klein bottles shown in Figure 2.

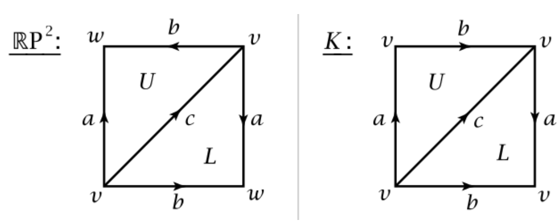

FIGURE 2.

In this case, the Δ-chain complex for $\mathbb{RP}^2$ becomes

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}} \mathbb{Z}^2 \to 0$$

The dual complex becomes

$$0 \leftarrow \mathbb{Z}^2 \xleftarrow{\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xleftarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}} \mathbb{Z}^2 \leftarrow 0$$

We thus find that $H^2(\mathbb{RP}^2;\mathbb{Z}) = 0$, $H^1\left(\mathbb{RP}^2;\mathbb{Z}\right) \cong \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle / \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle \cong \mathbb{Z}$, and $H^0\left(\mathbb{RP}^2;\mathbb{Z}\right) \cong \left\langle \begin{pmatrix}1\\-1\\0\end{pmatrix} \right\rangle \cong \mathbb{Z}$.

2. A Homotopy Construction of Cohomology

Theorem 2.1. *There are natural bijections $T\colon [X, K(G,n)]_* \to H^n(X;G)$ for all CW complexes X and all $n > 0$, with G any abelian group. Such a T has the form $T([f]) = f^*(\alpha)$ for a certain distinguished class $\alpha \in H^n(K(G,n);G)$.*

Definition 2.2 (Fundamental Class)**.** A class $\alpha \in H^n(K(G,n);G)$ with the property stated in Theorem 2.1 is called a *fundamental class*.

Note. The theorem also holds with $[X, K(G,n)]_*$ replaced by $[X, K(G,n)]$, the non-basepointed homotopy classes.
When $n > 1$, every map $X \to K(G,n)$ can be homotoped to take basepoint to basepoint and every homotopy between basepoint-preserving maps can be homotoped to be basepoint-preserving since $K(G,n)$ is simply-connected.
When $n = 1$, $[X, K(G,n)]_* = [X, K(G,n)]$ for abelian G according to an exercise for section 4.A in Hatcher.
For $n = 0$, $H^0(X;G) = [X, K(G,0)]$ and $\tilde{H}^0(X;G) = [X, K(G,0)]_*$.

The main two steps in the proof, will be the following two assertions:

(1) The functors $h^n(X) = [X, K(G,n)]_*$ define a reduced cohomology theory on the category of based CW complexes.
(2) If a reduced cohomology theory h^* defined on CW complexes has coefficient group $h^n(S^0)$ which are zero for $n \neq 0$, then there are natural isomorphisms $h^n(X) \cong \tilde{H}^n(X; h^0(S^0))$ for all CW complexes X and all n.

Towards proving (1), we will study a more general question: When does a sequence of spaces K_n define a cohomology theory by setting $h^n(X) = [X, K_n]_*$? Note that since $[X, K_n]_*$ is trivial when X is a point, this will be a reduced cohomology theory.

The first part to address is putting a group structure on the set $[X, K]_*$. When $X = S^n$, we have $[S^n, K]_* = \pi_n(K)$, which has a group structure when $n > 0$. The definition of this

2.1. Fibrations.

By convention, a fibration will be a map $p\colon E \to B$ having the homotopy lifting property with respect to all spaces. I.e., fibrations will mean Hurewicz fibrations.

Proposition 2.3. *For a fibration $p\colon E \to B$, the fibers $F_b = p^{-1}(b)$ over each path component of B are all homotopy equivalent.*

Proof. From a path $\gamma\colon I \to B$, we can construct a homotopy $g\colon F_{\gamma(0)} \times I \to B$ by $g(x,t) = \gamma(t)$. The inclusion $F_{\gamma(0)} \hookrightarrow E$ provides a lift $\tilde{g}_0\colon F_{\gamma(0)} \times \{0\} \to E$, so we have the following diagram:

$$\begin{tikzcd} F_{\gamma(0)} \times \{0\} \arrow[r, "\tilde{g}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\ F_{\gamma(0)} \times I \arrow[r, "g"] \arrow[ur, dashed, "\tilde{g}"] & B \end{tikzcd}$$

so since p is a fibration, there exists a lift $\tilde{g}\colon F_{\gamma(0)} \times I \to E$ making the diagram commute. Hence $p \circ \tilde{g} = g$ maps $F_{\gamma(0)}$ to $\gamma(t)$ for all t, hence $\tilde{g}\left(F_{\gamma(0)} \times \{t\}\right) \subset F_{\gamma(t)}$ for all $t \in I$. In particular, let L_γ be the composition $F_{\gamma(0)} \hookrightarrow F_{\gamma(0)} \times \{1\} \xrightarrow{\tilde{g}} F_{\gamma(1)}$. The association $\gamma \mapsto L_\gamma$ has the following basic properties:

(1) If $\gamma \simeq \gamma' \operatorname{rel} \partial I$, then $L_\gamma \simeq L_{\gamma'}$. In particular, the homotopy class of L_γ is independent of the choice of the lifting $\tilde{g}_t$ of g_t.
(2) For a composition of paths $\gamma\gamma'$, $L_{\gamma\gamma'}$ is homotopic to the composition $L_{\gamma'}L_\gamma$.

Note. From these statement, it follows that L_γ is a homotopy equivalence with homotopy inverse $L_{\overline{\gamma}}$.
Note also that it is true in general that a fibration has the homotopy lifting property for pairs $(X \times I, X \times \partial I)$. This is because $(I \times I, I \times \{0\} \cup \partial I \times I) \cong (I \times I, I \times \{0\})$ which naturally has the property that any map on $I \times \{0\}$ can be extended to $I \times I$, and hence $X \times (I \times I, I \times \{0\} \cup \partial I \times I) = (X \times I \times I, X \times I \times \{0\} \cup X \times \partial I \times I)$ also has this same property which is equivalent to the pair $(X \times I, X \times \partial I)$ having the homotopy extension property.

Now, to prove (a), let $\gamma\colon I \times I \to B$ be a homotopy from γ to γ'. This determines a family $g\colon F_{\gamma(0)} \times I \times I \to B$ with $g\left(s, t, F_{\gamma(0)}\right) = \gamma(s,t)$. Now we want to define a map $G\colon F_{\gamma(0)} \times I \times I \to E$. We start by letting $G|_{F_{\gamma(0)} \times \{0\} \times \{t\}}$ be equal to the composition $F_{\gamma(0)} \hookrightarrow F_{\gamma(0)} \times \{1\} \xrightarrow{\tilde{g}} F_{\gamma(1)}$, and similarly $G|_{F_{\gamma(0)} \times \{1\} \times \{t\}}$ but with γ'. Denote $G|_{F_{\gamma(0)} \times \{s\} \times \{t\}}$ by $G_{s,t}$. Then $G_{0,t} = L_\gamma$ and $G_{1,t} = L_{\gamma'}$. Let $G|_{F_{\gamma(0)} \times \{s\} \times \{0\}}$ be given by the inclusion $F_{\gamma(0)} \hookrightarrow E$ for all s. Then using the homotopy lifting property for the pair $\left(F_{\gamma(0)} \times I, F_{\gamma(0)} \times \partial I\right)$, we can extend G to a map $F_{\gamma(0)} \times I \times I \to B$. Letting now $t = 1$, we get a homotopy $F_{\gamma(0)} \times I \to B$ from $G_{F_{\gamma(0)} \times \{0\} \times \{1\}} = G_{0,1} = L_\gamma$ to $G_{1,1} = L_{\gamma'}$.
For (b), if $\tilde{g}_t$ and $\tilde{g}'_t$ define L_γ and $L_{\gamma'}$, respectively, then we obtain a lift defining $L_{\gamma\gamma'}$ by taking $\tilde{g}_{2t}$ for $0 \le t \le \frac{1}{2}$ and $\tilde{g}'_{2t-1}L_\gamma$ for $\frac{1}{2} \le t \le 1$.

$\square$

Definition 2.4 (Fiber-preserving)**.** Given fibrations $p_i\colon E_i \to B$, $i = 1, 2$, a map $f\colon E_1 \to E_2$ is called *fiber-preserving* if $p_1 = p_2 f$.

Definition 2.5 (Fiber homotopy equivalence)**.** A fiber-preserving map $f\colon E_1 \to E_2$ is a *fiber homotopy equivalence* if there is a fiber-preserving map $g\colon E_2 \to E_1$ such that both compositions fg and gf are homotopic to the identity through fiber-preserving maps.

Proposition 2.6. *Given a fibration $p\colon E \to B$ and a homotopy $f_t\colon A \to B$, the pullback fibrations $f_0^*(E) \to A$ and $f_1^*(E) \to A$ are fiber homotopy equivalent.*

Remark. This is meant in the sense that there exists a fiber homotopy equivalence $f_0^*(E) \to f_1^*(E)$ with respect to these fibrations.

Corollary 2.7. *A fibration $E \to B$ over a contractible base B is fiber homotopy equivalent to a product fibration $B \times F \to B$.*

2.1.1. *Pathspace Constructions.*

Definition 2.8 (Construction of)**.** Given a map $f\colon A \to B$, let

$$E_f := \{(a,\gamma) \mid a \in A, \gamma\colon\ (I,\{0\}) \to (B, f(a))\}.$$

We topologize E_f as a subspace of $A \times B^I$ where B^I has the compact-open topology.

Proposition 2.9. *The map $p\colon E_f \to B$ with $p(a,\gamma) = \gamma(1)$ is a fibration.*

Note. We have a natural embedding of A as the set of pairs $(a,\gamma) \in E_f$ with γ being the constant path at $f(a)$. Then E_f deformation retracts onto this subspace by restricting all the paths γ to shorter and shorter initial segments. The map $p\colon E_f \to B$ restricts to f on the subspace A, so we have factored a map $f\colon A \to B$ as the composition $A \hookrightarrow E_f \to B$ of a homotopy equivalence and a fibration.

Definition 2.10 (Homotopy fiber)**.** The fiber F_f of $E_f \to B$ is called the *homotopy fiber* of f. It consists of all pairs (a,γ) with $a \in A$ and γ a path in B from $f(a)$ to a basepoint $b_0 \in B$.

Remark. If $f\colon A \to B$ is the inclusion of a subspace, then E_f is the space of paths in B starting at points of A. In this case, a map $f\colon\ \left(I^{i+1}, \partial I^{i+1}, J^i\right) \to (B, A, x_0)$ can be regarded as a map $\left(I^i, \partial I^i\right) \to (F_f, \gamma_0)$ where γ_0 is the constant path at x_0 and F_f is the fiber of E_f over x_0.
Therefore, $\pi_{i+1}(B, A, x_0)$ can be identified with $\pi_i(F_f, \gamma_0)$, so the LES of homotopy groups of the pairs (B, A) and of the fibration $E_f \to B$ can be identified (since also $E_f \simeq A$).

Definition 2.11. An important special case of the above construction is when f is the inclusion of the basepoint $\{b_0\} \hookrightarrow B$. Then E_f is the space PB of paths in B starting at b_0, and $p\colon PB \to B$ sends each path to its endpoint. The fiber $p^{-1}(b_0)$ is the loopspace ΩB consisting of all loops in B based at b_0.
Since PB is contractible by progressively truncating paths, the LES of homotopy groups for the path fibration $PB \to B$ yields another proof that $\pi_n(X, x_0) \cong \pi_{n-1}(\Omega X, x_0)$ for all n.

Theorem 2.12 (Milnor 1959, [?])**.** *The loopspace of a CW complex is homotopy equivalent to a CW complex.*

Proposition 2.13. *If $p\colon E \to B$ is a fibration, then the inclusion $E \hookrightarrow E_p$ is a fiber homotopy equivalence. In particular, the homotopy fibers of p are homotopy equivalent to the actual fibers.*

Proof. Using that p is a fibration, we apply the HLP to the homotopy $g\colon E_p \times I \to B$ given by $g\left((e,\gamma), t\right) = \gamma(t)$, with the initial lift $\tilde{g}_0\colon E_p \times \{0\} \to E$ given by $\tilde{g}_0\left((e,\gamma), 0\right) = e$. This is indeed a lift on $E_p \times \{0\}$ since $g((e,\gamma),0) = \gamma(0) = p(e) = p \circ \tilde{g}_0\left((e,\gamma), 0\right)$. Given the homotopy $\tilde{g}\colon E_p \times I \to E$, we can now construct a homotopy $h\colon E_p \times I \to E_p$ by $h\left((e,\gamma),t\right) = \left(\tilde{g}\left((e,\gamma),t\right), \gamma \circ \varphi_{[t,1]}\right)$, where $\varphi_{[t,1]}\colon\ [0,1] \to [t,1]$ reparametrizes $[0,1]$ to $[t,1]$. Since $p \circ \tilde{g} = g$, we have that $p \circ \tilde{g}\left((e,\gamma),t\right) = g\left((e,\gamma),t\right) = \gamma(t) = \gamma \circ \varphi_{[t,1]}(0)$, so h is indeed a map $E_p \times I \to E_p$. Also, since the endpoints of the γ are unchaged, $h|_{E_p \times \{t\}}$ is fiber-preserving: $p\left(e,\gamma\right) = \gamma(1) = \gamma \circ \varphi_{[t,1]}(1) = p \circ h|_{E_p \times \{t\}}\left(e,\gamma\right)$.

Next note that $h_0(e,\gamma) = (\tilde{g}((e,\gamma),0),\gamma) = (e,\gamma)$, so $h_0 = \mathrm{id}$. Also, $h_1(e,\gamma) = (\tilde{g}((e,\gamma),1),\gamma(1))$ which belongs to the elements of E_p for which the path is constant - this is precisely what we identified E with, so $h_1(E_p) \subset E$. Also $h\left((e,c_{p(e)}),t\right) = \left(\tilde{g}\left((e,c_{p(e)}),t\right), c_{p(e)} \circ \varphi_{[t,1]}\right)$ which likewise belongs to E, so $h(E \times I) \subset E$. Let i denote the inclusion $E \hookrightarrow E_p$. Then $i \circ h_1 \simeq \mathrm{id}$ via h and $h_1 i \simeq \mathrm{id}$ via h, so i is a fiber homotopy equivalence.

Now, recall that a map $f\colon E_1 \to E_2$ is fiber preserving if $p_1 = p_2 f$, or, in other words, if $f\left(p_1^{-1}(b)\right) \subset p_2^{-1}(b)$. Let $q\colon E_p \to B$ be one fibration and $p\colon E \to B$ the other. Then let $F_{p,b}$ be the fiber above b for p and $F_{q,b}$ the fiber above b for q.

Since then since i and h_1 are fiber-preserving, they restrict to maps $F_{p,b} \to F_{q,b}$ and $F_{q,b} \to F_{p,q}$, respectively, whose compositions are homotopic to the identity, so $F_{p,q} \simeq F_{q,b}$ for each $b \in B$, showing that the fibers and homotopy fibers of p are homotopy equivalent. □

We have seen that loopspaces occur as fibers of fibrations $PB \to B$ with contractible total space PB. Here is something of a converse:

Proposition 2.14. *If $F \to E \to B$ is a fibration or fiber bundle with E contractible, then there is a weak homotopy equivalence $F \to \Omega B$.*

Proof. Composing a contraction of E with a projection $p\colon E \to B$, we obtain for each point $x \in E$ a path γ_x in B from $p(x)$ to a basepoint $b_0 = p(x_0)$, where x_0 is the point to which E contracts. This yields a map $E \to PB, x \mapsto \overline{\gamma_x}$ whose composition with the fibration $PB \to B$ sending sending a path in B starting at b_0 to its endpoint, is p. By restriction, this then gives a map $F \to \Omega B$, where $F = p^{-1}(b_0)$, and the LES of homotopy groups for $F \to E \to B$ maps to the LES for $\Omega B \to PB \to B$ (by section 6.6 in the AlgTop1 notes) Since E and PB are contractible, the five-lemma implies that $F \to \Omega B$ is a weak homotopy equivalence. □

2.1.2. *Puppe Sequence/Fibration Sequence.* Given a fibration $p\colon E \to B$ with fiber $F = p^{-1}(b_0)$, we know that the inclusion of F into the homotopy fiber F_p is a homotopy equivalence. Recall that F_p consists of pairs (e,γ) with $e \in E$ and γ a path in B from $p(e)$ to b_0. The inclusion $F \hookrightarrow E$ extends to a map $i\colon F_p \to E, i(e,\gamma) = e$ which is obviously a fibration. In fact, it is the pullback via p of the path fibration $PB \to B$ which sends each path to its endpoint.

This allows us to iterate, taking the homotopy fiber F_i with its map to F_p, and so on, as in the first row of the following diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & F_j \arrow[r] \arrow[d, "\simeq"] & F_i \arrow[r, "j"] \arrow[d, "\simeq"] & F_p \arrow[r, "i"] \arrow[d, "\simeq"] & E \arrow[r, "p"] \arrow[d, equal] & B \arrow[d, equal] \\
\ldots \arrow[r] & \Omega E \arrow[r, "\Omega p"] & \Omega B \arrow[r] & F \arrow[r] & E \arrow[r, "p"] & B
\end{tikzcd}$$

3. Cohomology in terms of geometry

3.1. **Terminology.** Recall the following definitions:

Definition 3.1 (k-forms)**.** Let

$$L^k(V) := \left\{ \text{multilinear forms } \underbrace{V \times \ldots \times V}_{k \text{ times}} \to F \right\}$$

which equals $\mathrm{Bil}(V)$ when $k=2$. The elements of $L^k(V)$ are called *k-forms*.

Note. We define a 0-form on V to simply be a linear map $F \to F$. Given such a linear map f, we can thus just identify f with $f(1)$, so $L^0(V) = F$.

Definition 3.2 (Alternating k-forms)**.** A k-form $w \in L^k(V)$ is called *alternating* if $w\left(v_1, \ldots, v_k\right) = 0$ for every k-tuple of vectors with two vectors equal. The vector space of alternating k-forms is denoted $A^k(V)$.

Note. Again $A^0(V) = F$.

Definition 3.3. We define an action of S_k on $L^k(V)$ as follows. For $\sigma \in S_k$ and $w \in L^k(V)$, define $\sigma \cdot w \in L^k(V)$ by

$$\sigma \cdot w\left(v_1, \ldots, v_k\right) = w\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right).$$

Definition 3.4 (Skew forms)**.** $w \in L^k(V)$ is called *skew* if $\sigma \cdot w = \mathrm{sgn}(\sigma) w$ for all $\sigma \in S_k$.

Lemma 3.5. *Every alternating k-form is skew, and if* $\operatorname{char} F \neq 2$*, then every skew k-form is alternating.*

Definition 3.6. The determinant $\det(A)$ of A is the scalar that satisfies

$$w\left(Av_1, \ldots, Av_n\right) = \det\left(A\right) w\left(v_1, \ldots, v_n\right)$$

for all $w \in A^n(V)$ and all $v_1, \ldots, v_n$.

Definition 3.7. Let $I \subset \{1, \ldots, n\}$ with elements $1 \leq i_1 < \ldots < i_k \leq n$. We define k-forms $y_I = y_{i_1} \cdots y_{i_k}$ and $\widehat{y_I} = \Sigma_{\sigma \in S_k} \mathrm{sgn}(\sigma) \sigma \cdot y_I$ in $L^k(V)$.

Theorem 3.8. *Let $0 \leq k \leq n$. The dimension of $A^k(V)$ is $\begin{pmatrix} n \\ k \end{pmatrix}$ and the set of all $\widehat{y_I}$, where $I \subset \{1, \ldots, n\}$ has k elements, is a basis.*

3.2. **Wedge product/exterior product.** Let us define first a bilinear map $A^k(V) \times A^l(V) \to A^{k+l}(V)$ for all integers $k, l \geq 0$. Let $m = k + l$.
Consider the product $S_k \times S_l$ as a subgroup of S_m by letting $(\sigma, \tau) \in S_k \times S_l$ act on $\{1, \ldots, m\}$ through the permutation π given by

$$\pi(i) = \sigma(i),\ \pi(k+j) = k + \tau(j), \quad 1 \leq i \leq k, 1 \leq j \leq l,$$

which leaves invariant the two sets $\{1, \ldots, k\}$ and $\{k+1, \ldots, m\}$. Denote by $[\pi]$ the coset in $S_m / \left(S_k \times S_l\right)$ corresponding to a permutation $\pi \in S_m$.

Lemma 3.9. *Let $w_1 \in L^k(V)$ and $w_2 \in L^l(V)$. Then*

$$V^m \ni \left(v_1, \ldots, v_m\right) \mapsto w_1\left(v_1, \ldots, v_k\right) w_2\left(v_{k+1}, \ldots, v_m\right) \in F$$

defines an m-form $w_1 \cdot w_2 \in L^m(V)$. If $w_1 \in A^k(V)$ and $w_2 \in A^l(V)$, then

$$w_1 \wedge w_2 = \Sigma_{[\pi] \in S_m/(S_k \times S_l)} \mathrm{sgn}(\pi) \pi \cdot \left(w_1 \cdot w_2\right)$$

defines an alternating m-form $w_1 \wedge w_2 \in A^m(V)$.

Definition 3.10 (Wedge product/exterior product)**.** The alternating m-form $w_1 \wedge w_2$ is called the *wedge product* of w_1 and w_2. By linear extension, it gives a binary operation on

$$A(V) := \bigoplus_{k=1}^{\infty} A^k(V) = \bigoplus_{k=1}^{n} A^k(V)$$

which becomes an algebra, called the *alternating algebra* of V.

The operation is associative.

Lemma 3.11. *If $w \in A^p(V)$ and $\eta \in A^q(V)$, then*

$$w \wedge \eta = (-1)^{pq} \eta \wedge w.$$

So the wedge product makes $A(V)$ into a graded anticommutative ring.

3.3. **The cotagent space.** Let M be an m-dimensional smooth manifold, and let $p \in M$.

Definition 3.12. The dual space $(T_pM)^*$ of T_pM is denoted T_p^*M and called the *cotangent space* of M at p. Its elements are called cotangent vectors or covectors.

Example 3.13. For $f \in C^\infty(M)$, $df_p \in T_p^*M = \mathrm{Hom}\,(T_pM, \mathbb{R})$.
Hence, in particular for f the coordinate maps, we get that $dx_i(p) = d(x_i)_p \in T_p^*M$ for all p in the domain of the chart for (x_i).

Lemma 3.14. *For $\big(U, (x^i)\big)$ a chart on M, let $p \in U$. Then the dual basis of $\left(\frac{\partial}{\partial x^i}|_p\right)$ is $(dx_i(p))$.*

Proof.

$$dx_i(p)\left(\frac{\partial}{\partial x^j}|_p\right) = \frac{\partial x^i}{\partial x^j}(\widehat{p})\frac{d}{dt}|_{x_i(p)} = \delta_{i,j}\frac{d}{dt}|_{x_i(p)}$$

where we used that

$$df_p\left(\frac{\partial}{\partial x^i}|_p\right) = \frac{\partial \widehat{f}^j}{\partial x^i}(\widehat{p})\frac{\partial}{\partial y^j}|_{f(p)} \qquad \text{(Lee (3.10))}$$

□

3.4. **Covector fields.**

Definition 3.15 (Covector field)**.** A *covector field* ξ on M is an assignment of covectors

$$\xi(p) \in T_p^*(M)$$

to each $p \in M$. It is called *smooth* if, for each chart $\sigma\colon U \to M$ in a given atlas of M, there exist smooth functions $a_1, \dots, a_m \in C^\infty(\sigma(U))$ such that

$$\xi(p) = \sum_i a_i(p) dx_i(p)$$

for all $p \in \sigma(U)$. The space of smooth covector fields is denoted $\mathfrak{X}^*(M)$.

Note. The space of smooth *vector* fields on M is denoted $\mathfrak{X}(M)$.

Note. Note that a covector $\xi(p)$ can be applied to the tangent vectors in T_pM, so if $U \subset M$ is open, for example, and Y is a vector field on U, then we can define $\xi(Y)\colon U \to \mathbb{R}$ by

$$\xi(Y)(p) = \xi(p)\left(Y(p)\right).$$

Lemma 3.16. *Let ξ be a covector field on M. Then ξ is smooth if and only if for each open set $U \subset M$ and each smooth vector field $Y \in \mathfrak{X}(U)$, the function $\xi(Y)$ belongs to $C^\infty(U)$.*

Proof. Suppose ξ is smooth. Let U be open and (x^i) a chart. For each $Y \in \mathfrak{X}(U)$, we write $Y(p) = \Sigma b_j(p)\frac{\partial}{\partial x^i}|_p$, where the $b_j(p)$ are smooth as functions of p by assumption on Y being smooth. Now

$$\xi(Y)(p) = \xi(p)\left(Y(p)\right) = \Sigma_i a_i(p) dx_i(p)\left(\Sigma_j b_j(p)\frac{\partial}{\partial x^i}|_p\right) = \Sigma_i a_i(p) b_i(p)$$

since $dx_i(p)\left(\frac{\partial}{\partial x^j}|_p\right) = \delta_{i,j}$. Thus $\xi(Y)$ is smooth.

Conversely, suppose $\xi(Y)$ is smooth. Let $Y = \frac{\partial}{\partial x^i}$, and $a_i(p) = \xi(p)\left(\frac{\partial}{\partial x^i}|_p\right)$. Then $\xi(Y) = \xi(p)\left(Y(p)\right) = \xi(p)\left(\frac{\partial}{\partial x^i}|_p\right) = a_i(p)$ is smooth by assumption. Now, since $(dx_i(p))$ is the dual basis to $\left(\frac{\partial}{\partial x^i}|_p\right)$, we get that because $\xi(p)\left(\frac{\partial}{\partial x^i}|_p\right) = a_i(p)$, we can thus write

$$\xi(p) = \sum_i a_i(p) dx_i(p).$$

Hence ξ is a smooth covector field as a_i is smooth. □

Note. If ξ is a smooth covector field on M and $\varphi \in C^\infty(M)$, then $\varphi\xi$ is again a smooth covector field on M defined by $(\varphi\xi)(p) = \varphi(p)\xi(p)$.

Lemma 3.17. *If $f \in C^\infty(M)$, then df is a smooth covector field on M. For each chart σ on M, the expression for df by means of the coordinate basis is*

$$df = \sum_{i=1}^{m} \frac{\partial(f \circ \sigma)}{\partial u_i} dx_i \tag{A}$$

The differentials satisfy the rule $d(fg) = gdf + fdg$ for all $f, g \in C^\infty(M)$.

Proof. Smoothness and the rule for $d(fg)$ follow from (A). To show (A), we apply df at an arbitrary point p in a chart on M first to obtain a covector and then apply this covector to the standard covector basis. So suppose $\left(U, (x^i)\right)$ is a chart and $p \in U$. Then

$$df_p\left(\frac{\partial}{\partial x^i}|_p\right) = \frac{\partial \widehat{f}}{\partial x^i}(p)\frac{d}{dt}|_{\widehat{p}_i}.$$

So writing df_p in terms of $dx_i(p)$, we get

$$df_p = \sum_{i=1}^{m} \frac{\partial \widehat{f}}{\partial x^i}(p) dx_i(p)$$

□

Example 3.18. Let $M = \mathbb{R}^2$ with coordinates (x_1, x_2) and $f \in C^\infty(\mathbb{R}^2)$. Then

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2.$$

Definition 3.19 (Exact covector field)**.** A smooth covector field $\xi = a_1 dx_1 + a_2 x_2$ with $a_1, a_2 \in C^\infty(\mathbb{R}^2)$ is said to be *exact* if it has the form df for some function f.

3.5. **Differential Forms.**

Definition 3.20 (k-form)**.** A k-form w on M is an assignment of an element

$$w(p) \in A^k(T_pM)$$

for each $p \in M$.

Note. Given a chart $\big(U, (x^i)\big)$ on M, the elements $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$ where $1 \leq i_1 < \ldots < i_k \leq m$ are k-forms on the open subset $\sigma(U)$ of M. For each $p \in U$, a basis for $A^k\left(T_pM\right)$ is obtained from these elements, so every k-form w on M has a unique expression on U given by

$$w = \sum_{I=\{i_1,\ldots,i_k\}} a_I dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

where $a_I \colon U \to \mathbb{R}$.

Definition 3.21 (Smooth/differential k-form)**.** We call w *smooth* or *differential* if all the functions a_I are smooth, for each chart σ in an atlas of M. The space of differential k-forms on M is denoted $A^k(M)$.

Note that since $A^0\left(T_pM\right) = \mathbb{R}$, we get simply that $A^0(M) = C^\infty(M)$. Likewise, $A^1(M) = \mathfrak{X}^*(M)$.

Lemma 3.22. *A k-form w is smooth if and only if $w\left(X_1, \ldots, X_k\right) \in C^\infty(\Omega)$ for all open sets Ω and all $X_1, \ldots, X_k \in \mathfrak{X}(\Omega)$.*

Lemma 3.23. *Let w be a k-form on M and φ a real function on M, and define the product φw pointwise by*

$$\varphi w(p) = \varphi(p)w(p).$$

Then φw is a smooth k-form if φ and w are smooth.
Let θ be an l-form on M, and define the wedge product $\theta \wedge w$ pointwise by

$$\left(\theta \wedge w\right)(p) = \theta(p) \wedge w(p)$$

for each p. Then $\theta \wedge w$ is a smooth $k + l$-form if w and θ are smooth.

Proof. The first part is easy. Now, the wedge product of basis elements of the vector space of alternating k-forms and l-forms is again a basis of the vector space of alternating $(k + l)$-forms. $\square$

3.6. **Pull back.**

Definition 3.24 (Pull back of k-form)**.** Let $f \colon M \to N$ be a smooth map of manifolds, and let $w \in A^k(N)$. We define $f^*(w) \in A^k(M)$, called the *pull back* of w, by

$$f^*w(p)(v_1, \ldots, v_k) = w\left(f(p)\right)\left(df_p(v_1), \ldots, df_p(v_k)\right)$$

for all $v_1, \ldots, v_k \in T_pM$ and $p \in M$.

Clearly, the operator $f^*w(p)$ is multiplinear and alternating for each p, so $f^*w(p) \in A^k(T_pM)$.

Question 3.25. Is f^*w smooth?

To answer this, note first that f^*w depends linearly on w, and now let

$$w = \sum_{I=\{i_1,\ldots,i_k\}} a_I dy_{i_1} \wedge \ldots \wedge dy_{i_k}$$

be the expression for w by means of the coordinates of a chart $\tau\colon N \to V$.

Lemma 3.26. *With the notation above,*

$$f^*w = \sum_I (a_I \circ f)\, d\,(y_{i_1} \circ f) \wedge \ldots \wedge d\,(y_{i_k} \circ f)$$

on $f^{-1}(\tau^{-1}(V))$. In particular, f^*w is smooth.

To prove the lemma, we need the following rules:

$$\begin{aligned} f^*(\varphi w) &= (\varphi \circ f)\, f^*w \\ f^*\,(\theta \wedge w) &= f^*(\theta) \wedge f^*(w) \\ f^*(d\varphi) &= d\,(\varphi \circ f) \end{aligned}$$

We verify them as follows:

$$\begin{aligned} f^*(\varphi w)(p)(v_1,\ldots,v_k) &= (\varphi w)(f(p))\,(df_p(v_1),\ldots,df_p(v_k)) \\ &= \varphi(f(p))w(f(p))\,(df_p(v_1),\ldots,df_p(v_k)) \\ &= \varphi(f(p))f^*w(p)\,(v_1,\ldots,v_k) \\ &= ((\varphi \circ f)\, f^*w)\,(p)(v_1,\ldots,v_k) \end{aligned}$$

giving $f^*\,(\varphi w) = (\varphi \circ f)\, f^*w$.

$$\begin{aligned} f^*\,(\theta \wedge w)\,(p)\,(v_1,\ldots,v_{k+l}) &= (\theta \wedge w)\,(f(p))\,(df_p(v_1),\ldots,df_p(v_{k+l})) \\ &= \sum_{[\pi]} \mathrm{sgn}(\pi)\theta(f(p))(df_p(v_{\pi(1)}),\ldots,df_p(v_{\pi(k)}))w(f(p))\,\big(df_p(v_{\pi(k+1)}),\ldots,df_p(v_{\pi(k+l)})\big) \\ &= \sum_{[\pi]} \mathrm{sgn}(\pi)f^*(\theta)(p)(v_{\pi(1)},\ldots,v_{\pi(k)})f^*(w)(p)\,\big(v_{\pi(k+1)},\ldots,v_{\pi(k+l)}\big) \\ &= (f^*(\theta)(p) \wedge f^*(w)(p))\,(v_1,\ldots,v_{k+l}) \\ &= (f^*(\theta) \wedge f^*(w))\,(p)\,(v_1,\ldots,v_{k+l})\,. \end{aligned}$$

giving $f^*(\theta \wedge w) = f^*(\theta) \wedge f^*(w)$.
Now, note that

$$\begin{aligned} d\,(\varphi \circ f)\,(p)\left(\frac{\partial}{\partial x^k}\right) &= \sum_{i=1}^m \frac{\partial(\varphi \circ f)}{\partial x^i}(p)dx_i(p)\left(\frac{\partial}{\partial x^k}|_p\right) \\ &= \frac{\partial \varphi}{\partial x^j}(f(p))\frac{\partial f^j}{\partial x^k}(p)\frac{d}{dt}|_{\varphi\circ f(p)} \end{aligned}$$

while

$$\begin{aligned} f^*(d\varphi)(p)\left(\frac{\partial}{\partial x^k}|_p\right) &= d\varphi(f(p))\frac{\partial f^j}{\partial x^k}(p)\frac{\partial}{\partial y^j}|_{f(p)} \\ &= \frac{\partial f^j}{\partial x^k}(p)\frac{\partial \varphi}{\partial y^j}(f(p))\frac{d}{dt}|_{\varphi\circ f(p)} \end{aligned}$$

giving $f^*(d\varphi) = d\,(\varphi \circ f)$.

Proof of Lemma 3.26. Let

$$w = \sum_{I=\{i_1,\dots,i_k\}} a_I dy_{i_1} \wedge \dots \wedge dy_{i_k}.$$

Then

$$\begin{aligned} f^*w &= \sum_{I=\{i_1,\dots,i_k\}} (a_I \circ f)\, f^*dy_{i_1} \wedge \dots \wedge f^*dy_{i_k} \\ &= \sum_{I=\{i_1,\dots,i_k\}} (a_I \circ f)\, d\,(y_{i_1} \circ f) \wedge \dots \wedge d\,(y_{i_k} \circ f)\,. \end{aligned}$$

□

Definition 3.27 (Restriction of a k-form to a submanifold)**.** Let M be a submanifold of N and $i\colon M \to N$ the inclusion map. Then we define $w|_M = i^*w$, given at $p \in M$ by

$$w|_M(p)(v_1,\dots,v_k) = w(p)(v_1,\dots,v_k)$$

for $v_1,\dots,v_k \in T_pM \subset T_pN$.

Lemma 3.28. *Let $f\colon M \to N$ be a smooth map between two manifolds of the same dimension k. Let $x_1,\dots,x_k$ denote the coordinate functions of a chart σ on M, and let $y_1,\dots,y_k$ denote the coordinate functions of a chart τ on N. Then*

$$f^*\left(dy_1 \wedge \dots \wedge dy_k\right) = \det(df) dx_1 \wedge \dots \wedge dx_k$$

where $\det(df)$ in p is the determinant of the matrix for df_p with respect to the standard bases for T_pM and $T_{f(p)}N$ given by the charts.

3.7. **Exterior Differentiation.** Notice that the usual differentiation operator sending a function f to its derivative at a point is an operator

$$A^0(M) = C^\infty(M) \to \mathfrak{X}^*(M) = A^1(M)$$

sending $f \mapsto df$.

We now define the exterior derivative which is a generalization of this, defined as a map $A(M) \to A(M)$ sending k-forms to $(k+1)$-forms.

Definition 3.29 (Exterior differentiation)**.** Fix a chart $\left(U,(x^i)\right)$ on a manifold M and let $w \in A^k(M)$. We write

$$w = \sum_I a_I dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

where $I = (i_1,\dots,i_k)$ with $1 \le i_1 < \dots < i_k \le m = \dim M$, and where $a_I \in C^\infty(U)$ for each I. On U, we define

$$dw = \sum_I da_I \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \tag{Ω}$$

where da_I is the differential of a_I. Then $dw \in A^{k+1}(U)$. The operator d is called *exterior differentiation*.

Can we extend this operator to all of M and is it independent of the chart?

Theorem 3.30. *There exists for each $k \in \mathbb{N}$ a unique map $d\colon A^k(M) \to A^{k+1}(M)$ such that (Ω) holds on $\sigma(U)$ for each chart σ in a given atlas for M. This map d is independent of the choice of the atlas. Furthermore, these maps have the following properties:*

(1) *If* $k = 0$*, then* d *agrees with the differential on functions.*
(2) $d\colon A^k(M) \to A^{k+1}(M)$ *is linear.*
(3) $d\,(\varphi w) = d\varphi \wedge w + \varphi dw$ *for* $\varphi \in C^\infty(M)$ *and* $w \in A^k(M)$*.*
(4) $d\,(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^k w_1 \wedge dw_2$ *for* $w_1 \in A^k(M)$ *and* $w_2 \in A^l(M)$*.*
(5) $d\,(df_1 \wedge \ldots \wedge df_k) = 0$ *for all* $f_1, \ldots, f_k \in C^\infty(M)$*.*
(6) $d\,(dw) = 0$ *for all* $w \in A^k(M)$*.*

Definition 3.31 (Differential forms)**.** The elements of $A(M)$ are called differential forms. Thus a differential form is a map which associates to each point $p \in M$ a member of the exterior algebra $A\,(T_pM)$ in a smooth manner. The operators of Theorem 3.30 can then be combined into a single linear operator, also denoted $d\colon A(M) \to A(M)$.