Exercise 0.1 (1)**.** *Proof.* (i) We claim that $\left(x^2+1\right)$ is a radical, prime and maximal ideal in $\mathbb{R}[x]$. This can be seen by noting that $\mathbb{R}[x]/\left(x^2+1\right) \cong \mathbb{C}$ which is a field. Hence $\left(x^2+1\right)$ is maximal. Suppose $f^n \in \left(x^2+1\right)$. Since x^2+1 is irreducible and $x^2+1 \mid f^n$, we must have $x^2+1 \mid f$, hence $f \in \left(x^2+1\right)$, so $\sqrt{(x^2+1)} = \left(x^2+1\right)$. Over $\mathbb{C}[x]$, we claim that $\left(x^2+1\right)$ is neither prime nor maximal, but still radical. It is not prime as $x^2+1 = (x+i)(x-i)$ and hence also not maximal since $\left(x^2+1\right) \subset (x+i) \neq \mathbb{C}[x]$, where inequality follows from $(x+i)$ only having polynomials of degree ≥ 1.
Now suppose $f^n \in \left(x^2+1\right)$. Then $x+i, x-i \mid f^n$, hence both must divide f as they are irreducible, so $x^2+1 \mid f$. Thus $f \in \left(x^2+1\right)$, so $\sqrt{(x^2+1)} = \left(x^2+1\right)$ over $\mathbb{C}[x]$ as well.

(ii) Since $\left(x^2+1\right)$ is a prime ideal in $\mathbb{R}[x]$ by the previous exercise, we find by Eisenstein's criterion that y^2+x^2+1 is irreducible in $\mathbb{R}[x][y] =: \mathbb{R}[x,y]$.

(iii)
Let $C = \left\{f \in C\left(\mathbb{R}^2, \mathbb{R}\right) \mid \forall x \in \mathbb{R} \colon f(x,0) = 0\right\} \subset C\left(\mathbb{R}^2, \mathbb{R}\right)$. We claim that C is radical, but neither prime nor maximal.
To see that it is radical, suppose $g^n \in C$, so $g(x,0) \cdot \ldots \cdot g(x,0) = g^n(x,0) = 0$. Since $\mathbb{R}$ is an integral domain, this forces $g(x,0) = 0$, so $g \in C$. Thus $\sqrt{C} = C$.
Now let $h(x) = \mathbb{1}_{\geq 0}(x)x$ and $k(x) = \mathbb{1}_{\leq 0}(x)x$. Then $h, k \notin C$, but $hk \in C$. Therefore, C is not prime. Since $C\left(\mathbb{R}^2, \mathbb{R}\right)$ is a commutative ring and maximal ideals are prime over a commutative ring, we thus also conclude that C is not maximal.

(iv) The ideal (5) in $\mathbb{Z}[i]$ is not prime, hence not maximal as $\mathbb{Z}[i]$ is commutative. It is not prime because $5 = (2+i)(2-i)$.
For the radical part, if $a+bi \in \sqrt{(5)}$, then $(2+i)(2-i) \mid (a+bi)^n$, so $2+i \mid a+bi$ and $2-i \mid a+bi$ since each is irreducible, hence $5 \mid a+bi$, so $\sqrt{(5)} = (5)$.

(v) We claim that $(n) \subset \mathbb{Z}$ is prime and maximal whenever n is a prime and not otherwise. If n is not prime, then writing $n = ab$ for $a, b > 1$, we have $(n) = (a)(b)$, so (n) is not prime, hence not maximal as $\mathbb{Z}$ is commutative so all maximal ideals are prime ideals. If instead n is a prime, $n = p$, then (p) is both maximal and prime since $\mathbb{Z}/p$ is a field.
Suppose now $m \in \sqrt{(n)}$. Then $m^k \in (n)$, so $m^k = nq$ for some $q \in \mathbb{Z}$. Suppose n is squarefree. Let $p \mid n$. Then $p \mid m^k$ and thus $p \mid m$, so $n \mid m$, hence $m \in (n)$. Conversely, if n is not squarefree, then letting $n = p^k q$ for some $k > 1$, we have $p^{k-1}q \in \sqrt{(n)}$ while $p^{k-1}q \notin (n)$, so (n) is not radical. ☐

Exercise 0.2 (3)**.** Suppose $ab \in (\mathfrak{p}+\mathfrak{q}) = \mathfrak{p} \cup \mathfrak{q}$. Write $a = \sum \alpha_i x_i$ and $b = \sum \beta_i x_i$. Then $ab = \sum_{i,j} \alpha_i \beta_j x_i x_j$
We have

$$K[x_1, \ldots, x_{n+m}]/(\mathfrak{p}+\mathfrak{q}) \cong K[x_1, \ldots, x_n]/(\mathfrak{p}) \oplus K[x_{n+1}, \ldots, x_{n+m}]/(\mathfrak{q})$$

as can be seen by defining the map

$$\varphi\colon K\left[x_1,\dots,x_{n+m}\right]\to K\left[x_1,\dots,x_n\right]/\left(\mathfrak{p}\right)\oplus K\left[x_{n+1},\dots,x_{n+m}\right]/\left(\mathfrak{q}\right)$$

by

$$\sum \alpha_i x_i \mapsto \left(\sum_{i=1}^{n}\alpha_i\overline{x}_i,\sum_{i=n+1}^{n+m}\alpha_i\overline{x}_i\right).$$

This is clearly a homomorphism with kernel precisely $(\mathfrak{p}+\mathfrak{q})$ since this ideal consists precisely of the sums which have a factor that consists of a part in $(\mathfrak{p})$ and another part in $(\mathfrak{q})$. As $(\mathfrak{p})$ and $(\mathfrak{q})$ are prime, the right hand side of the isomorphism is a direct sum of integral domains which is again an integral domain.