Exercise 0.1. Let $[A] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$. Determine an explicit Bruhat decomposition $[A] = [S]\,[T]\,[U]$. Show that T is unique, but not $[S]$ and $[U]$.

Solution. We have $[A]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $[A]\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$. The inverse of $[A]$ is

$$[A]^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Let (x_1, x_2) be the basis $x_1 = e_1$ and x_2 whichever one of $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$ it is linearly independent with. Suppose it is $\begin{pmatrix} a \\ c \end{pmatrix}$. In this case $\sigma = (1\,2)$. Thus $S = \begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$R = \begin{pmatrix} 1 & \frac{d}{ad-bc} \\ 0 & \frac{-c}{ad-bc} \end{pmatrix}.$$

Suppose

$$A = STU = S'T'U'.$$

Then $T = S^{-1}S'T'U'U^{-1}$. So this reduces to showing that if $T = KT'L$ with K and L upper triangular, then $T = T'$. Now, T, T are one of I, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Suppose $T = I$ and $T' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $K = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $L = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$. Then

$$I = KT'L = \begin{pmatrix} bd & af + be \\ cd & ce \end{pmatrix}$$

so $cd = 0$ hence either $bd = 0$ or $ce = 0$, contradiction. So $T = T'$. If $T' = I$ and $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} ad & ae + bf \\ 0 & cf \end{pmatrix},$$

contradiction. So indeed $T = T'$.

Now, the only requirements in the proof of theorem 8.20 for x_1 and x_2 are that $x_1 \in \operatorname{span}(e_1)$ and $x_{\sigma(1)} \in A\left(\operatorname{span}(e_1)\right) = \operatorname{span}\left(\begin{pmatrix} a \\ c \end{pmatrix}\right)$ while $\operatorname{span}(x_1, x_2) = \mathbb{R}^2$. So indeed any $x_1 = (\lambda, 0)$ and $x_2 = \mu\begin{pmatrix} a \\ c \end{pmatrix}$ give a valid flag. In this case, S will for example be $S = \begin{pmatrix} \lambda & \mu a \\ 0 & \mu c \end{pmatrix}$. Thus we see that S and U are not unique.