GEOMETRIC TOPOLOGY

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Contents

1. Continuous maps

Definition 1.1. For a continuous map $f\colon M \to N$ between topological manifolds,

- f is called an immersion if locally at each point of M, it is of the form $\mathbb{R}^m \to \mathbb{R}^n$ sending $x \mapsto (x, 0)$.
- f is an embedding if it is an immersion, injective and induces a homeomorphism with its image.
- f is a submersion if it is locally of the form to $(x, y) \mapsto x$.

Definition 1.2 (Bundle as defined by Robert (is this supposed to be a fiber bundle?))**.** If $f\colon M \to N$ is a continuous map between topological manifolds, then f is called a bundle if it is locally on N of the form $X \times V \xrightarrow{\pi_2} V$. That is, there exist charts, in which f takes the form of a projection.

2. Bundles

2.1. **Fibre Bundle Theory.** I will define things slightly differently.

Definition 2.1 (Bundle)**.** A bundle is simply a triple (E, p, B) where $p\colon E \to B$ is a map.
The pullback

$$\begin{tikzcd} x^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ x \arrow[r] & B \end{tikzcd}$$

is called the fiber over x.

Definition 2.2 (Fiber bundle)**.** A fiber bundle over B with standard fibre F is a bundle over B such that, given any $x\colon 1 \to B$, the pullback of E along x is isomorphic to F: $x^*E \cong F$.

Definition 2.3 (Locally trivial fibre bundle)**.** If C is a site (???), then a locally trivial fibre bundle over B with typical fibre F is a bundle over B with a cover $(j_\alpha\colon U_\alpha \to B)_\alpha$ such that, for each index α, the pullback E_α of E along j_α is isomorphic in the slice category C/U_α to the trivial bundle $U_\alpha \times F$.

Definition 2.4 (Morphisms of bundles)**.** Let (E, p, B) and (E', p', B') be two bundles. A bundle morphism $(u, f) : (E, p, B) \to (E', p', B')$ is a pair of maps $u \colon E \to E'$ and $f \colon B \to B'$ such that

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

commutes.

Lemma 2.5. *Bundles together with bundle morphisms form a category, which we denote* Bun

Proof. Composition of two morphisms (u, f) and (u', f') is simply done component-wise: $(u', f') \circ (u, f) = (u' \circ u, f' \circ f)$. Now, clearly for a bundle (E, p, B), we have that $(\mathrm{id}_E, \mathrm{id}_B)$ forms an identity morphism, and associativity is inherited from associativity of morphism composition of the ambient category.

□

Definition 2.6 (Slice category)**.** For a category C and an object $c \in C$, we form the category c/C whose objects are morphisms $f \colon c \to x$ with domain c and in which a morphism from $f \colon c \to x$ to $g \colon c \to y$ is a map $h \colon x \to y$ such that

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

commutes. Likewise, there is a category C/c whose objects are morphisms $f \colon x \to c$ with codomain c, and where a morphism from $f \colon x \to c$ to $g \colon y \to c$ is a map $h \colon x \to y$ such that

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes.
The categories c/C and C/c are called the **slice categories** of C **under** and **over** c, respectively.

Proposition 2.7 ([1])**.** *If C is complete and cocomplete, then so are the slice categories c/C and C/c for any $c \in C$.*

So in particular, we have that since Top is complete and cocomplete, so is Top/X for any $X \in \mathrm{Top}$. So the product $E \times_X E'$ exists in Top/X for any $[E \to X], [E' \to X] \in \mathrm{Top}/X$.

Definition 2.8 ($\mathrm{Bun}(N)$)**.** For an object N in the category C, we let $\mathrm{Bun}(N)$ be the slice category C/N.

Definition 2.9 (Topological and smooth fiber bundles with structure group)**.** Let K be a topological group acting on a Hausdorff space F as a group of homeomorphisms. Let X and B be Hausdorff spaces. By a *fiber bundle* over a base space B with total space X, fiber F and structure group K, we mean a bundle map $p \colon X \to B$ together with a maximal chart atlas Φ over B. Explicitly, Φ is a collection of trivializations $\varphi \colon U \times F \to p^{-1}(U)$ such that

(1) each point of B has a neighborhood over which there is a chart in Φ
(2) if $\varphi\colon U \times F \to p^{-1}(U)$ is in Φ and $V \subset U$, then the restriction $\varphi|_{V\times F}$ is also in Φ.
(3) If $\varphi, \psi \in \Phi$ are charts over U then there exists a map $\theta\colon U \to K$ such that $\psi(u, y) = \varphi(u, \theta(u)(y))$
(4) the set Φ is maximal among the collections satisfying the (1),(2) and (3)

The fiber bundle is called smooth if all the spaces are smooth manifolds and all maps involved are smooth.

Example 2.10. The product bundle If we have a space $B = X \times Y$ and let $p\colon B \to X$ be the projection $p(x, y) = x$, then seeing as $p^{-1}(X) = X \times Y$, we automatically obtain an trivialization $\varphi\colon p^{-1}(X) \cong X \times Y$. The sections (aka cross sections) of B, i.e., continuous maps $X \to X \times Y$ is then just simply equivalent to graphs of maps $X \to Y$. The fibres are all homeomorphic. Since a single trivialization works for all of X, this exhibits $X \times Y$ as a fiber bundle over X with trivial structure group.

Example 2.11 (Möbius band)**.** Take the base space $X = S^1$ obtained from I by identifying ends. Let $Y = I$ be the fibre. We can obtain the Möbius bundle from $I \times I$ by matching the ends by a twist. This descends to a projection $p\colon B \to S^1 = X$ where B is the Möbius band. There are many cross-sections: any curve $I \to I \times I$ by $t \mapsto (t, \gamma(t))$ for some $\gamma\colon I \to I$ such that $\gamma(0) = 1 - \gamma(1)$ works. In particular, any two cross-sections agree on at least one point (see picture [2, p. 4]. The structure group is $\mathbb{Z}/2$.

Example 2.12 (Klein Bottle)**.** The Klein bottle can be obtained similarly, choosing I as the fibre but S^1 as the base space and then quotienting the ends of $S^1 \times I$. Again, see [2, p. 4].

Example 2.13 (Covering Spaces)**.** A covering space B of a space X is another example of a bundle. The projection $p\colon B \to X$ is the covering map. In particular, a covering space is a locally trivial fibre bundle where the fibre is a discrete space.

2.1.1. *Coordinate bundles and fibre bundles.*

Definition 2.14 (Transformation groups)**.** Recall that if G is a topological group and Y is a topological space, we say that G is a topological transformation group of Y relative to a map $\eta\colon G \times Y \to Y$ if

(1) η is continuous
(2) $\eta(e, -) = \mathrm{id}$
(3) $\eta(g_1 g_2, y) = \eta(g_1, \eta(g_2, y))$.

We shall often implicitly assume η as given and abbreviate $\eta(g, y)$ by $g \cdot y$, so that the above become that $\cdot$ is continuous, $e \cdot y = y$ for all y and $(g_1 g_2) \cdot y = g_1 \cdot (g_2 \cdot y)$.

Definition 2.15 (Effective action)**.** We say that G is effective if $g \cdot y = y$ for all y implies that $g = e$.

Definition 2.16 (Coordinate Bundle)**.** A coordinate bundle $\mathcal{B}$ is a collection as follows:

(1) A bundle space B
(2) a base space X

(3) a projection $p\colon B \to X$
(4) a space Y called the fibre
(5) an effective topological transformation group G acting on Y, called the (structure) group of the bundle
(6) A family $\{V_\alpha\}$ of open sets covering X called coordinate neighborhoods
(7) trivializations φ_α giving homeomorphisms

$$\varphi_a\colon V_a \times Y \to p^{-1}(V_a)$$

called coordinate functions.

restricted to the following requirements

(1)

$$\begin{tikzcd} V_\alpha \times Y \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi_1"'] & & p^{-1}(V_\alpha) \arrow[dl, "p"] \\ & V_\alpha & \end{tikzcd}$$

commutes.

(2) letting the map $\varphi_{j,x}\colon Y \to p^{-1}(x)$ be defined by

$$\varphi_{j,x}(y) = \varphi_j(x, y)$$

then for each $x \in V_\alpha \cap V_\beta$, $\varphi_{j,x}^{-1}\varphi_{i,x}(-)\colon Y \to Y$ is the same as $g\cdot(-)\colon Y \to Y$ for some $g \in G$.

(3) the map $g_{\alpha\beta}\colon V_\alpha \cap V_\beta \to G$ by $g_{\alpha\beta}(x) = \varphi_{\alpha,x}^{-1}\varphi_{\beta,x}$ is continuous.

And immediate consequence of the definition is that

$$g_{\gamma\beta}(x)g_{\beta\alpha}(x) = g_{\gamma\alpha}(x), \quad x \in V_\alpha \cap V_\beta \cap V_\gamma.$$

It is also convenient to introduce the map $p_\alpha\colon p^{-1}(V_\alpha) \to Y$ given by $p_\alpha(b) = \varphi_{\alpha,p(b)}^{-1}(b)$.
We obtain the identities

$$\begin{aligned} p_\alpha\varphi_\alpha(x, y) &= y \\ \varphi_\alpha\left(p(b), p_\alpha(b)\right) &= b \\ g_{\alpha\beta}\left(p(b)\right) \cdot p_\beta(b) &= p_\alpha(b) \end{aligned}$$

Definition 2.17 (Fibre bundle in terms of coordinate bundles)**.** Two coordinate bundles $\mathcal{B}$ and $\mathcal{B}'$ are equivalent in the strict sense i they have the same bundle space, base space, projection, fibre and structure group and their coordinate functions satisfy that

$$\overline{g}_{kj} = \varphi_{k,x}'^{-1}\varphi_{j,x}$$

coincide with the operation of an element of G and the map $\overline{g}_{kj}\colon V_j \cap V_k' \to G$ is continuous.
Then a fibre bundle is a maximal coordinate bundle with respect to this equivalence relation.

Definition 2.18 (Mappings of fibre bundles)**.** Let $\mathcal{B}$ and $\mathcal{B}'$ be two coordinate bundles having the same fibre and structure group. A map $h\colon \mathcal{B} \to \mathcal{B}'$ is a tuple

$\left(h, \overline{h}\right)$ with $h\colon B \to B'$ and $\overline{h}\colon X \to X'$ such that

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\overline{h}} & X' \end{array}$$

commutes and

$$\overline{g}_{\alpha\beta}(x) = \varphi'^{-1}_{\alpha,x'} h_x \varphi_{\beta,x} = p'_k h_x \varphi_{\beta,x}$$

coincides with the operation of some $g \in G$ on Y. Here $h_x\colon Y_x \to Y_{x'}$ is the map $h(x, -)$, where $x' = \overline{h}(x)$. Furthermore, the map

$$\overline{g}_{\alpha\beta}\colon V_\beta \cap \overline{h}^{-1}\left(V'_\alpha\right) \to G$$

is assumed to be continuous.

In particular, since $\overline{g}_{\alpha\beta}(x)$ acts by some $g \in G$ on Y which is through homeomorphisms, we obtain that since $\varphi'^{-1}_{\alpha,x'}$ and $\varphi_{\beta,x}$ are also homeomorphisms, h_x is a homeomorphism of the fibres.
The mapping transformations $\overline{g}_{\alpha\beta}$ satisfy

$$\begin{aligned} \overline{g}_{\alpha\beta}(x) g_{\beta\gamma}(x) &= \overline{g}_{\alpha\gamma}(x) \qquad (\Omega) \\ g'_{\alpha\beta}\left(\overline{h}(x)\right)\overline{g}_{\beta\gamma}(x) &= \overline{g}_{\alpha\gamma}(x). \end{aligned}$$

Lemma 2.19. *Let $\mathcal{B}, \mathcal{B}'$ be coordinate bundles having the same fibre Y and group G, and let $\overline{h}\colon X \to X'$ be a map of one base space into the other. Let $\overline{g}_{kj}\colon V_j \cap \overline{h}^{-1}(V'_k) \to G$ be a set of continuous maps satisfying (Ω). Then there exists a unique fibre bundle map $h\colon \mathcal{B} \to \mathcal{B}'$ inducing $\overline{h}$ and having $\left\{\overline{g}_{jk}\right\}$ as its mapping transformations.*

Lemma 2.20. *Let $\mathcal{B}, \mathcal{B}'$ be coordinate bundles having the same fibre and group, and let $h\colon \mathcal{B} \to \mathcal{B}'$ be a bundle map such that the induced map $\overline{h}\colon X \to X'$ is a homeomorphism. Then h has a continuous inverse $h^{-1}\colon B' \to B$, and h^{-1} is a bundle map $\mathcal{B}' \to \mathcal{B}$.*

Definition 2.21. Two coordinate bundles $\mathcal{B}$ and $\mathcal{B}'$ with the same base space, fibre and group are said to be equivalent if there exists a fibre bundle map $\mathcal{B} \to \mathcal{B}'$ which induces the identity of the common base space.
Two fibre bundles having the same base space, fibre and group are said to be equivalent if they have representative coordinate bundles which are equivalent.

Lemma 2.22. *Let $\mathcal{B}, \mathcal{B}'$ be coordinate bundles having the space base space, fibre and group. Then they are equivalent if and only if there exist continuous maps*

$$\overline{g}_{kj}\colon V_j \cap V'_k \to G$$

such that

$$\begin{aligned} \overline{g}_{ki}(x) &= \overline{g}_{kj}(x) g_{ji}(x) \\ \overline{g}_{lj}(x) &= g'_{lk}(x)\overline{g}_{kj}(x) \end{aligned}$$

.

Lemma 2.23. *Let $\mathcal{B}, \mathcal{B}'$ be two coordinate bundles with the same base space, fibre, group and coordinate neighborhoods. Let g_{ji}, g'_{ji} denote their coordinate transformations. Then $\mathcal{B}, \mathcal{B}'$ are equivalent if and only if there exist continuous functions $\lambda_j \colon V_j \to G$ such that*

$$g'_{ji}(x) = \lambda_j(x)^{-1} g_{ji}(x) \lambda_i(x).$$

Lemma 2.24. *Let $\mathcal{B}, \mathcal{B}'$ be coordinate bundles having the same fibre and group, and let $h \colon \mathcal{B} \to \mathcal{B}'$ be a fibre bundle map. Corresponding to each section $f' \colon X' \to B'$, there exists a unique section $f \colon X \to B$ such that*

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ X & \xrightarrow[\bar{h}]{} & X' \end{array}$$

commutes. The section f is said to be induced by h and f' and will be denoted $h^* f'$.

2.1.2. *Construction of a bundle from coordinate transformations.*

Definition 2.25. Let G be a topological group and X a space. By a *system of coordinate transformations in X with values in G* is meant an indexed covering $\{V_j\}$ of X by open sets and a collection of continuous maps

$$g_{ji} \colon V_i \cap V_j \to G$$

such that

$$g_{kj}(x) g_{ji}(x) = g_{ki}(x).$$

Remark. We have so far seen that any bundle over X with group G determines such a set of coordinate transformations. We now state a converse.

Theorem 2.26 (Existence)**.** *If G is a topological transformation group of Y, and $\{V_j\}, \{g_{ij}\}$ is a system of coordinate transformations in the space X, then there exists a bundle $\mathcal{B}$ with base space X, fibre Y, group G and coordinate transformations $\{g_{ij}\}$. Furthermore, any such bundles are equivalent.*

2.1.3. *Factor/Quotient/Coset Spaces of Groups.*

Definition 2.27 (Local section of G)**.** Let G be a closed subgroup of B. Then G is a point $x_0 \in B/G$. A *local section* of G in B is a function f mapping a neighborhood V of x_0 continuously into B and such that $pf(x) = x$ for each $x \in V$.

2.1.4. *Enlarging the group of a bundle.* Let H be a closed subgroup of the topological group G. If $\mathcal{B}$ is a bundle with group H, the same coordinate neighborhoods, and the same coordinate transformations, altered only by regarding their values as belong to G, define a new bundle called the *G-image of $\mathcal{B}$*.

Note. If H operates on the fibre Y, it may or may not occur that G operates on Y or even that such operations can be defined.

Definition 2.28 (G-equivalence)**.** Let $H, K \leq G$ be closed subgroups, and let $\mathcal{B}, \mathcal{B}'$ be bundles having the same base space and structure groups H, K, respectively. We say that $\mathcal{B}, \mathcal{B}'$ are *equivalent in G* or *G-equivalent* if the G-images of $\mathcal{B}$ and $\mathcal{B}'$ are equivalent.

2.1.5. *The Principal Bundle and the Principal Map.*

Definition 2.29 (Principal G-bundle)**.** A bundle $\mathcal{B} = \{B, p, X, Y, G\}$ is called a principal bundle if $Y = G$ and G operates on Y by left translations.

Definition 2.30 (Associated prinicipal bundle)**.** Let $\mathcal{B} = \{B, p, X, Y, G\}$ be an arbitrary bundle. The *associated principal bundle* $\tilde{B}$ of $\mathcal{B}$ is the bundle given by the construction/existence theorem using the same base space, the same $\{V_j\}$, the same $\{g_{ji}\}$ and the same group G as for $\mathcal{B}$, but replacing Y by G and allowing G to operate on itself by left translations.

Theorem 2.31 (Equivalence theorem)**.** *Two bundles having the same base space, fibre and group are equivalent if and only if their associated principal bundles are equivalent.*

Proof. By Lemma 2.22, equivalence of bundles is purely a property of the coordinate transformations. □

Definition 2.32 (Manifold bundle)**.** Let M be a smooth manifold. A manifold bundle over M with structure group G is a fiber bundle $W \to E \to M$ with structure group G such that E is a manifold and $E \to M$ is continuous.
We say a manifold bundle over M is a smooth manifold bundle if it is a smooth fiber bundle as well as a manifold bundle and G acts by diffeomorphisms on M.

Definition 2.33 (Associated bundles)**.** Let M be a smooth manifold, and fix a manifold bundle $E \xrightarrow{\xi} M$ with fibre a smooth manifold W and structure group $G \leq \mathrm{Homeo}(W)$. Given another smooth manifold W' such that there exists an injective group homomorphism $\iota\colon G \hookrightarrow \mathrm{Homeo}\,(W')$, the associated W'-manifold bundle of ξ is defined as follows. Let $\{U_\alpha, \varphi_\alpha\}_\alpha$ be a cover of M by open neighborhoods together with trivializations φ_α of ξ. Transition maps $\varphi_\alpha\varphi_\beta^{-1}$ give rise to transition function $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G \leq \mathrm{Homeo}(W)$ satisfying the cocycle condition. We define the associated W'-manifold by gluing trivializations $U_\alpha \times W'$ along transition maps

$$\iota \circ g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G \xrightarrow{\iota} \mathrm{Homeo}\,(W')\,.$$

Definition 2.34 (Structure group reduction)**.** Fix a manifold bundle $\xi\colon E \to M$ over a smooth manifold M, with fibre a smooth manifold W and structure group G. Given a subgroup $H \leq G$, ξ is said to admit a structure group reduction to H if it is isomorphic to a bundle so that all transition maps $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ take values in H.

Problem 2.35 (Change of fibres of bundles)**.** Let W_0 and W_1 be two smooth manifolds, and let G be a group which we assume as a simultaneous subgroup of both $\mathrm{Homeo}(W_0)$ and $\mathrm{Homeo}(W_1)$, i.e., we have injective group homomorphisms $\iota_0\colon G \hookrightarrow \mathrm{Homeo}(W_0)$ and $\iota_1\colon G \hookrightarrow (W_1)$. Given a fixed smooth manifold M, construct a bijection $\mathrm{Bun}_G^{W_0}(M) \to \mathrm{Bun}_G^{W_1}(M)$, where $\mathrm{Bun}_G^{W_i}(M)$ denotes the set of isomorphism classes of manifold bundles with fibre W_i and structure group G.

Proof. Let $\mathcal{B} = \{B, p, X, W_0, G\} \in \mathrm{Bun}_G^{W_0}$. By Theorem 2.31, the bundle $\mathcal{B}$ is equivalent to its associated principal bundle $\tilde{\mathcal{B}} = \{B, p, X, G, G\}$. But by assumption, G embeds into $\mathrm{Homeo}(W_1)$, so by Theorem 2.26, also $\tilde{B}$ is equivalent to $\{B, p, X, W_1, G\} =: \mathcal{B}'$ which has the same coordinate transformations. Thus $\tilde{B}$ and $\tilde{B}'$ are equivalent. Now, seeing as equivalence of bundles is purely determined by their base space, fibre, structure group and coordinate transformations, this gives an injective map $\mathrm{Bun}_G^{W_0} \to \mathrm{Bun}_G^{W_1}$. Seeing as we can do the exact same thing to obtain an injective map $\mathrm{Bun}_G^{W_1} \to \mathrm{Bun}_G^{W_0}$, we obtain a bijection by Schröder-Bernstein. □

2.1.6. *Associated bundles and relative bundles.*

Definition 2.36. Two bundles, having the same base space X and the same group G, are said to be *associated* if their associated principal bundles are equivalent.

Exercise 2.37. Check that the relation of being associated is reflexive, symmetric and transitive.

Definition 2.38 (Relative bundle)**.** Let $\mathcal{B} = \{B, p, X, Y, G\}$ be a bundle. Let $A \subset X$ be a closed subspace and $H \leq G$ a closed subgroup. If, for every i, j and every $x \in V_i \cap V_j \cap A$, the coordinate transformation $g_{ji}(x)$ is an element of H, then the portion of the bundle over A may be regarded as a bundle with group H. One simply restricts the coordinate neighborhoods and functions to A. Whenever this occurs, we say that $\mathcal{B}$ is a *relative* (G, H)*-bundle over the base space* (X, A).

Definition 2.39 (((G,H)-equivalence)**.** Let $\mathcal{B}$ be a (G,H)-bundle over (X,A) and let $\mathcal{B}'$ be an (H,H)-bundle over (X,A). A (G,H)-equivalence of $\mathcal{B}$ and $\mathcal{B}'$ is a map $h\colon \mathcal{B} \to \mathcal{B}'$ which is, first, a G-equivalence of the two absolute bundles over X, and, second, an H-equivalence when restricted to the portions of $\mathcal{B}, \mathcal{B}'$ lying over A.

Slogan. The smaller the group of a bundle, the simpler the bundle.

2.1.7. *The canonical section of a relative bundle.* Let $\mathcal{B}$ be a (G,H)-bundle over (X,A). Let $\mathcal{B}'$ denote the associated bundle over X having G/H as fibre and G acting on the fibre by left translations. Let e_0 denote the coset of H treated as an element of G/H. We define a section over A of the bundle $\mathcal{B}'$ by

$$f_0(x) = \varphi_j'(x, e_0), \quad x \in V_j \cap A.$$

If $x \in V_i \cap V_j \cap A$, then

$$\varphi_j'(x, e_0) = \varphi_i'\left(x, g_{ij}(x) \cdot e_0\right) = \varphi_i'(x, e_0)$$

since $g_{ij}(x) \in H$. Thus f_0 defines a section over A. We call f_0 the *canonical section of the* (G,H)*-bundle.*

2.1.8. *Structure Group Reduction.*

Definition 2.40. For a bundle where the fibres are of the form G/H, if G operates effectively on G/H, we obtain an associated bundle; otherwise, a weakly associated bundle.

Theorem 2.41. *Let $H \leq G$ be a closed subgroup which has a local section. A (G,H)-bundle over (X,A) is (G,H)-equivalent to an (H,H)-bundle over (X,A) if and only if the canonical section (defined only over A) can be extended to a full section of the weakly associated bundle with fibre G/H.*

Corollary 2.42. *If H has a local section in G, then a G-bundle over X is G-equivalent to an H-bundle if and only if the weakly associated bundle with fibre G/H has a section.*

Tomorrow, check out the link https://math.stackexchange.com/questions/2015174/structure-group-of-tangent-bundle-of-riemannian-manifold

2.1.9. *Associated frame bundles and structure group reductions.*

Problem 2.43. For a rank d vector bundle $\xi\colon E \to M$ over a smooth manifold, we define the associated frame bundle $\mathrm{Fr}(\xi)$ as the associated $\mathrm{GL}_d(\mathbb{R})$-bundle.

(1) For M a smooth d-dimensional manifold, we define its frame bundle $\mathrm{Fr}(M)$ as the associated frame bundle of its tangent bundle TM. Show that $\mathrm{Fr}(M) \to M$ is a principal $\mathrm{GL}_d(\mathbb{R})$-bundle.
(2) Show that a manifold is orientable if and only if its frame bundle $\mathrm{Fr}(M)$ admits a $\mathrm{GL}_d^+(\mathbb{R})$ reduction of its structure group, where $\mathrm{GL}_d^+(\mathbb{R})$ is the subgroup of the general linear group consisting of invertible matrices with positive determinant.
(3) Show that a structure bundle reduction of the frame bundle $\mathrm{Fr}(M)$ to the orthogonal group $O(n) \leq \mathrm{GL}_d(\mathbb{R})$ corresponds to a choice of a bundle metric on the tangent bundle TM of M.

2.1.10. *The Induced Bundle.*

Definition 2.44 (First definition of the induced bundle)**.** Suppose we have a bundle $\mathcal{B}'$ over a base space X', fibre Y and group G which is uniquely determined up to isomorphism by a system of coordinate transformations $\{V'_\alpha\}$ and $\left\{g'_{\alpha\beta}\right\}$. Suppose now we have a map $\eta\colon X \to X'$. The *induced bundle* $\eta^*\mathcal{B}'$ having base space X, fibre Y and group G is defined by pulling back the system of coordinate transformations by letting $\{V_\alpha\}$ with $V_\alpha = \eta^{-1}(V'_\alpha)$ and $\{g_{\alpha\beta}\}$ with $g_{\alpha\beta}(x) = g'_{\alpha\beta} \circ \eta(x)$ be the system of coordinate transformations of $\eta^*\mathcal{B}'$ and then constructing a bundle using the Existence theorem (Theorem 2.26). We define a map $h\colon \eta^*\mathcal{B}' \to \mathcal{B}'$ (which, recall, is a map $B \to B'$) by

$$h(b) = \varphi'_j\left(\eta p(b), p_j(b)\right), \qquad p(b) \in V_j$$

Recall that $p_j\colon p^{-1}(V_j) \to Y$ is given by $p_j(b) = \varphi^{-1}_{j,p(b)}(b) \in Y$. Indeed then $\eta p(b) \in X'$, so $(\eta p(b), p_j(b)) \in X' \times Y$, and φ'_j is defined on some open subset of this space. To show that h is well-defined, we must show that it agrees on overlaps. If $p(b) \in V_i \cap V_j$, then

$$\begin{aligned}\varphi'_j\left(\eta(p(b)), p_j(b)\right) &= \varphi'_i\left(\eta(p(b)), g'_{ij}\left(\eta\left(p(b)\right) \cdot p_j(b)\right)\right)\\ &= \varphi'_i\left(\eta(p(b)), g_{ij}(x) \cdot p_j(b)\right) = \varphi'_i\left(\eta(p(b)), p_i(b)\right)\end{aligned}$$

Furthermore, all the maps in the definition of h are continuous, so h is continuous.

In particular, $p'h(b) = \eta(p(b))$, so indeed h induces η on $X \to X'$. I.e.,

$$\begin{array}{ccc} B & \xrightarrow{h} & B' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\eta} & X' \end{array}$$

commutes. Lastly, we want to show that h is a bundle map.
This means that we must show that for $x \in V_j \cap \eta^{-1}(V'_k)$, the map

$$\overline{g}_{kj}(x) = \varphi'^{-1}_{k,x'} h_x \varphi_{j,x} = p'_k h_x \varphi_{j,x}\colon Y \to Y$$

coincides with the operation of some $g \in G$ on Y. That is, that $\overline{g}_{kj}\colon V_j \cap V_k \to G$ is continuous for any k, j. But indeed

$$\begin{aligned}\overline{g}_{kj}(x) \cdot y &= \varphi'^{-1}_{k,x'} h_x \varphi_{j,x}(y)\\ &= \varphi'^{-1}_{k,x'} \varphi'_j\left(x', p_j\left(\varphi_{j,x}(y)\right)\right)\\ &= \varphi'^{-1}_{k,x'} \varphi'_j\left(x', y\right)\\ &= \varphi'^{-1}_{k,x'} \varphi'_{j,x'}(y)\\ &= g'_{k,j}(x') \cdot y\end{aligned}$$

so $\overline{g}_{kj} = g'_{kj} \circ \eta = g_{kj}$, and it is a continuous map of $V_k \cap V_j$ into G.

Definition 2.45 (Second definition of the induced bundle)**.** Suppose $\mathcal{B}', X$ and η are as before. Form the product space $X \times B'$ and let $p\colon X \times B' \to X, h\colon X \times B' \to B'$ be the natural projections. Define $B = X \times_{X'} B' := \{(x, b') \in X \times B' \mid \eta(x) = p'(b')\}$ to be the fibered product.

We want to give $[p\colon B \to X]$ a fibre bundle structure (by giving it a coordinate bundle structure). Define $V_j = \eta^{-1}(V_j')$ and set

$$\varphi_j(x,y) = \left(x, \varphi_j'\left(\eta(x), y\right)\right).$$

Let's give these maps some motivation. For these to be trivializations, we want φ_j to be homeomorphisms $p^{-1}(V_j) \cap B = p|_B^{-1}(V_j) \cong V_j \times Y$. Now, φ_j simply maps x to x in the first coordinate, but φ_j' by assumption maps $V_j' \times Y$ homeomorphically onto $p'^{-1}(V_j')$. Hence in particular, $\varphi_j'\left(\eta(x), y\right) \in p'^{-1}(V_j') \subset B'$. So $\left(x, \varphi_j'\left(\eta(x), y\right)\right) \in B$ if and only if $\eta(x) = p'\left(\varphi_j'\left(\eta(x), y\right)\right)$, but this is true by assumption. Furthermore, $\left(x, \varphi_j'\left(\eta(x), y\right)\right) \in X \times B'$, so applying p, we get $p\left(x, \varphi_j'\left(\eta(x), y\right)\right) = x$ which is in V_j when $x \in V_j$. Hence putting things together, φ_j maps $V_j \times Y$ to $p^{-1}(V_j) \cap B$. We, in fact, want to show that φ_j is a homeomorphism of these spaces. For this, simply note that the map $(u,v) \mapsto \left(u, \pi_2 \circ \varphi_j'^{-1}(v)\right)$ is an inverse.
Lastly, let for $x \in V_i \cap V_j$, $g_{ij}(x) = \varphi_{i,x}^{-1}\varphi_{j,x} = p_i\varphi_{j,x}$
Note then that

$$\begin{aligned} g_{ij}(x)y &= p_i\varphi_{j,x}(y) \\ &= p_i\left(x, \varphi_j'\left(\eta(x), y\right)\right) \\ &= p_i'\varphi_j'\left(\eta(x), y\right) \\ &= g_{ij}'\left(\eta(x)\right)y \end{aligned}$$

So the clutching functions are simply $g_{ij}' \circ \eta$ which are indeed continuous.

Theorem 2.46 (Equivalence Theorem/pullbacks of fibre bundles with the same fibre and group exist)**.** *Let $\mathcal{B}, \mathcal{B}'$ be two bundles having the same fibre and group and $h\colon \mathcal{B} \to \mathcal{B}'$ a bundle map. Let $\eta\colon X \to X'$ be the induced map of base spaces. Then the induced bundle $\eta^*\mathcal{B}'$ is equivalent to $\mathcal{B}$, and there is an equivalence $h_0\colon \mathcal{B} \to \eta^*\mathcal{B}'$ such that h is the composite $h = h^* \circ h_0$ where $h^*\colon \eta^*\mathcal{B}' \to \mathcal{B}'$ is the induced map:*

$$\begin{tikzcd}
B \arrow[dr, dashed, "\exists"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_{X'} B' \cong \eta^* B' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B' \arrow[d] \\
& X \arrow[r, "\eta"] & X'
\end{tikzcd}$$

Definition 2.47 (Orientability)**.** A smooth manifold M is called *orientable* if for all smooth maps $S^1 \to M$, f^*TM is trivializable. That is, $[f^*TM \to S]$ is a trivial bundle.

2.2. **A Bundle Theory.**

Note. A "Bundle Theory" is also called a Cartesian Fibration over Sm.

Definition 2.48 (Essential fibers)**.** For a functor $F\colon \mathcal{B} \to \mathcal{C}$ and an object $M \in \mathcal{C}$, the (essentail) fiber above M is the fibered category $\mathcal{B} \times_{\mathcal{C}} \mathbb{1}$ making

$$\begin{tikzcd} \mathcal{B} \times_{\mathcal{C}} \mathbb{1} \arrow[r] \arrow[d] & \mathcal{B} \arrow[d, "F"] \\ * \arrow[r, "* \mapsto M"] & \mathcal{C} \end{tikzcd}$$

commute.

Definition 2.49 (Bundle Theory)**.** A bundle theory is a functor from some arbitrary category $\mathcal{B}$ to Sm subject to the following conditions.
Given a map $f\colon M \to N$ between smooth manifolds in Sm, there exists a map $f^*\colon \mathcal{B}(N) \to \mathcal{B}(M)$.
The solid arrows in the diagram below, the dashed lifts are in bijection and the diagram commutes.

$$\begin{tikzcd} B' \arrow[r, dashed, "\psi"] \arrow[d] \arrow[rr, bend left] & f^*B \arrow[r] \arrow[d] \arrow[dr, phantom, "\exists"] & B \arrow[d] \\ N \arrow[r, dashed, "\varphi"] \arrow[rr, bend right] & P \arrow[r, "f"] & M \end{tikzcd}$$

In the sense that given φ, there exists a ψ, everything commutes and composite map above is mapped under the functor to the composite map below.
Furthermore, it is required to satisfy gluing (the cocycle condition): given $U_{ijk} \hookrightarrow U_{ij} \hookrightarrow U_i \hookrightarrow M$ and a bundle $B \in \mathcal{B}(M)$, we can consider the restricted bundles $B|_{U_i} = B_{U_i} = B_i \in \mathcal{B}(U_i)$ for each i, and likewise for B_{ij} and B_{ijk} for all combinations of i, j and k. For these, we have transition
A bundle $B \to M$ is called locally trivial if for each point $x \in M$, there exists a neighborhood $x \in U \overset{i}{\hookrightarrow} M$ and there exists a bundle $B' \to *$ and a pullback along $\pi\colon U \to *$ for B' such that there exists an isomorphism $i^*B \cong \pi^*B'$.

2.3. **Principal G-bundles.** Let G be a discrete group. Consider the category Sm^G where objects are smooth manifolds equipped with a free, fixed point free action by G which is properly discontinuous: the exists a cover $\{U_\alpha\}_{\alpha \in A}$ of M so that $\{g \cdot U_\alpha\}$ are pairwise disjoint for all $\alpha \in A$ and $g \in G$. Furthermore, morphisms are smooth maps which are G-equivariant: $f\colon M \to N$ is such that $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in M$.

Problem 2.50. (1) Show that for $M \in \mathrm{Sm}^G$, the quotient M/G admits a structure of a smooth manifold so that the map $M \to M/G$ is a local diffeomorphism.
(2) Check that the association $M \mapsto M/G$ defines a functor $\mathrm{Sm}^G \to \mathrm{Sm}$, and show that this defines a locally trivial bundle theory on smooth manifolds.

Proof. (1) (I will assume that G acts by homeomorphisms on M) Using the covering space quotient theorem (theorem 12.14 in Lee's book on Topological Manifolds), we find that $M \to M/G$ is a covering space. To construct a smooth structure on M/G, let $p \in M/G$ and U an evenly covered open neighborhood of p. Then U splits into homeomorphic copies $\sqcup U_\alpha$ in M with $\pi|_{U_\alpha}\colon U_\alpha \cong U$ homeomorphisms. For $\tilde{p} \in U_\alpha$, choose a smooth chart $(V_{\tilde{p}}, \varphi_{\tilde{p}})$ contained in U_α. Since $\tilde{p} = g \cdot p$ for some g, we may as well denote these charts as $(V_{g,p},\psi_{g,p})$. Now consider the charts $\left(\pi|_g(V_{g,p}),\psi_{g,p}\circ(\pi|_g)^{-1}\right)$. On an overlap the transition functions have the form

$$\psi_{g,p}\circ(\pi|_g)^{-1}\left(\psi_{g',p'}\circ(\pi|_{g'})^{-1}\right)^{-1}=\psi_{g,p}\circ(\pi|_g)^{-1}\pi|_{g'}\circ\psi_{g',p'}^{-1}=\psi_{g,p}\circ\psi_{g',p'}^{-1}$$

on the overlap, which is smooth by assumption. Hence we indeed obtain a smooth structure on M/G. In particular, the map $\pi\colon M\to M/G$ has coordinate form

$$\left(\psi_{g,p}\circ\pi|_g^{-1}\right)\pi\circ\psi_{g,p}^{-1}=\mathrm{id}$$

which is a diffeomorphism. So π is a local diffeomorphism when we equip M/G with this smooth structure.

(2) Define the functor $F\colon \mathrm{Sm}^G\to\mathrm{Sm}$ sending $M\mapsto M/G$ with the smooth structure defined in the first part of the exercise. Here, since maps $f\colon M\to N$ in Sm^G are G-equivariant, they, in particular, descend to smooth maps $\overline{f}\colon M/G\to N/G$, and we let $F(f)=\overline{f}$. Then indeed $F(\mathrm{id}_M)=\overline{\mathrm{id}_M}=\mathrm{id}_{M/G}$ and if $f\colon M\to N$ and $g\colon N\to P$, then $F(g\circ f)=\overline{g\circ f}$. But by pasting the two squares

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[r, "g"] \arrow[d] & P \arrow[d] \\ M/G \arrow[r, "\overline{f}"] & N/G \arrow[r, "\overline{g}"] & P/G \end{tikzcd}$$

we find that $\overline{g\circ f}=\overline{g}\circ\overline{f}$. So $F(g\circ f)=F(g)\circ F(f)$.
This shows that F is indeed a functor.
We want to show that this defines a bundle theory on Sm. So suppose we have some $N\in\mathrm{Sm}^G$ and $f\colon M\to N/G$ in Sm. Now, the quotient map $N\to N/G$ is a submersion (show this), so the pullback along f exists in Sm, giving

$$\begin{tikzcd} f^*N \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & N \arrow[d] \\ M \arrow[r] & N/G \end{tikzcd}$$

Lastly, we must then show that f^*N is in Sm^G. For this, note that the induced bundle f^*N is precisely the pullback which is equivalent as a fibre bundle to $M\times_{N/G}N$. But this inherits a natural action of G given by $g\cdot(m,n)=(m,g\cdot n)$. Choosing the same cover $\{U_\alpha\}$ for N as given in the condition of it being in Sm^G, i.e., $\{g\cdot U_\alpha\}$ being disjoint for all g and α, the neighborhoods $M\times U_\alpha\cap f^*N$ then satisfy the same conditions under this action of G. Lastly, the map $f^*N\cong M\times_{N/G}N\to N$ given by the projection to the N component which is the top map in the pullback diagram is naturally G-equivariant. This shows that the above diagram indeed can be made.

Now suppose we have some $P \in \mathrm{Sm}^G$ and a bundle map $P \to N$ giving the solid part of the diagram

$$\begin{tikzcd}
P \arrow[r, dashed] \arrow[rr, bend left] \arrow[d] & M \times_{N/G} N \arrow[r] \arrow[d] & N \arrow[d] \\
P/G \arrow[r] & M \arrow[r] & N/G
\end{tikzcd}$$

where the map $P \to N$ descends to the composite map $P/G \to M \to N/G$ on the bottom.

We then want to show that the dashed map exists. Let $p\colon P \to P/G$ and $q\colon f^*N \cong M \times_{N/G} N \to M$ be the projection. Let $k\colon P \to N$ be the map on the top. Let $f\colon P/G \to M$ be the map on the bottom. Define a map $h\colon P \to M \times_{N/G} N$ by $h(x) = (f(p(x)), k(p))$. Then if $l\colon M \to N/G$ denotes the map on the bottom, $l \circ f\,(p(x)) = \pi\,(k(p))$ where $\pi\colon N \to N/G$. By definition then $h(x) \in M \times_{N/G} N$. Furthermore,

$$h\,(g \cdot x) = (f\,(p\,(g \cdot x))\,, k\,(g \cdot x)) = (f\,(p\,(x))\,, g \cdot k(x)) = g{\cdot}(f\,(p(x))\,, k(x)) = g{\cdot}h(x),$$

so h is G-equivariant.

Next we must check that the bundle theory is locally trivial. That is, we must check that for any $M \in \mathrm{Sm}^G$ and any point $x \in M/G$, there exists an open neighborhood U about x such that if we let $\pi\colon U \to *$ be the unique map and $i\colon U \to M/G$ the open embedding, there exists a manifold $N \in \mathrm{Sm}^G$ such that $N/G \cong *$, and such that the pullbacks are isomorphic: $i^*M \cong \pi^*N$.
Note that these pullbacks are really

$$\begin{tikzcd}
U \times_{M/G} M \cong i^*M \arrow[r] \arrow[d] & M \arrow[d, "p"] \\
U \arrow[r] & M/G
\end{tikzcd}$$

But clearly if $(u, m) \in U \times_{M/G} M$, then essentially $\overline{m} = u$, so $U \times_{M/G} M \cong p^{-1}(U)$, and

$$\begin{tikzcd}
U \times N \cong U \times_* N \arrow[r] \arrow[d] & N \arrow[d] \\
U \arrow[r] & *
\end{tikzcd}$$

So we find that the condition is indeed equivalent to the usual one: the existence of a neighborhood U about x and a homeomorphism $p^{-1}(U) \cong U \times N$. In this case, suppose $x \in M/G$ and simply choose one of the U_α such that $x \in p\,(U_\alpha)$. Note that this is open in M/G since the $g \cdot U_\alpha$ are pairwise disjoint and g acts by homeomorphisms (G is discrete and each g has g^{-1} as inverse). Choosing $U = p\,(U_\alpha)$, we get $p^{-1}(U) = \sqcup_{g \in G} U_\alpha \cong U_\alpha \times G \cong U \times G$ where $G \in \mathrm{Sm}^G$ is precisely G considered as a smooth manifold with the trivial charts $g \mapsto *$, at each $g \in G$. Indeed then $G/G \cong *$, so this satisfies the condition above. I.e., the functor $\mathrm{Sm}^G \to \mathrm{Sm}$ is locally trivial.

Lastly, we must check gluing. Namely that for $M \in \mathrm{Sm}^G$ and some open coordinate neighborhoods $U_i, U_j, U_k \subset M/G$, with coordinate maps $g_{ij}\colon U_i \cap U_j \to G$, $g_{jk}\colon U_j \cap U_k \to G$ and $g_{ki}\colon U_k \cap U_i \to G$, the maps satisfy $g_{ik}(x) = g_{ij}(x)g_{jk}(x)$ for $x \in U_i \cap U_j \cap U_k$. As we saw above, $p^{-1}(U_i) = U_i \times G$, and we shall call this coordinate function $\varphi_i\colon U_i \times G \to p^{-1}(U_i)$. Let $g_{ij}(x) = \varphi_{i,x}^{-1}\varphi_{j,x}$ where $\varphi_{i,x}(y) = \varphi_i(x, y)$ is the function considered only as a function of y. But then the condition $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$ follows trivially.

This completes the proof that the functor we constructed $\mathrm{Sm}^G \to \mathrm{Sm}$ is indeed a bundle theory over Sm. $\square$

Note. The bundles constructed by sending an object $M \in \mathrm{Sm}^G$ to M/G above exhibits $M \to M/G$ as a principal G-manifold bundle.

Lemma 2.51. *For any locally trivial bundle theory $\mathcal{B} \to \mathrm{Sm}$, every $B \in \mathcal{B}(\mathbb{R})$ is trivial. (here $\mathcal{B}(\mathbb{R})$ denotes the fiber of $\mathbb{R}$ under the functor)*

2.4. **Vector Bundles.**

Definition 2.52 (Vector Bundle)**.** A vector bundle over $X \in \mathrm{Top}$ consists of the following data:

- An object $\left[E \xrightarrow{\pi} X\right]$ in Top/X.
- An $\mathbb{R}$-vector space structure internal to Top/X :
  (1) a morphism $+\colon E \times_X E \to E$
  (2) a morphism $\cdot\colon \mathbb{R} \times E \to E$
  which satisfy the vector space axioms.

which are required to satisfy

- (local triviality) there exists an open cover $\{U_\alpha\}$ of X where if $U := \sqcup_{\alpha \in I} U_\alpha$ $[U \to X] \in \mathrm{Top}/X$ is such that there exists an isomorphism of vector space objects in Top/U
$$U \times_I \mathbb{R}^n \cong U \times_X E$$
  where $n\colon I \to \mathbb{N}$. Here $\mathbb{R}^n = \sqcup_{i \in I} \mathbb{R}_i^{n(i)}$

Definition 2.53 ($\mathrm{Vect}(X)$)**.** Topological vector bundles over X and bundle morphisms between them constitute a category denoted $\mathrm{Vect}(X)$.

Viewed in top, the last condition implies that there is a diagram of the form

$$\begin{tikzcd}
U \times k^n \arrow[r, "\cong"] \arrow[dr] & U \times_X E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "\pi"] \\
& U \arrow[r] & X
\end{tikzcd}$$

where the homeomorphism in the top left is fiber-wise linear.

All of this is fine so far, but we want to look at smooth manifolds, so we now reformulate our definitions a bit.

Remark. From now on, Bun will denote that subcategory consisting of topological manifolds. Then $\mathrm{Bun}(N)$ will denote Sm/N.

We would like to define vector bundles the same as before but replacing Top by Sm. However, the category Sm is not complete, so what is $+\colon E \times_X E \to E$ supposed to be?

Lemma 2.54. *If $f\colon M \to N$ and $g\colon P \to N$ are morphisms in* Sm *and f is a submersion, then the pullback exists:*

$$\begin{tikzcd}
\exists X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & M \arrow[d, "f"] \\
P \arrow[r, "g"] & N
\end{tikzcd}$$

Now, since $\pi\colon E \to X$ is a bundle, it is a submersion, so the pullback $E \times_X E$ exists. Then we can define $+\colon E \times_X E \to E$ in the same way as before.

Definition 2.55 (Vect)**.** Topological vector bundles form a category Vect whose morphisms are bundle maps

$$\begin{tikzcd} E \arrow[r] \arrow[d] & E' \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}$$

such that

$$\begin{tikzcd} E \arrow[r] \arrow[dr] & E' \times_{X'} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E' \arrow[d, "\pi"] \\ & X \arrow[r] & X' \end{tikzcd}$$

commutes.

2.5. **Gluing vector bundles.**

Proposition 2.56 (Topological vector bundles reconstructed from transition functions (see ncatlab))**.** *Let $[\pi\colon E \to X]$ be a topological vector bundle, $\{U_i \subset X\}_{i\in I}$ an open cover of the X and $\left\{U_i \times k^n \overset{\varphi_i}{\to} E|_{U_i}\right\}_{i\in I}$ be local trivializations.*
Write

$$\left\{g_{ij} := \varphi_j^{-1} \circ \varphi_i \colon U_i \cap U_j \to \mathrm{GL}(n,k)\right\}_{i,j\in I}$$

for the corresponding transition functions. Then there is an isomorphism of vector bundles over X :

$$((\sqcup_{i\in I} U_i) \times k^n) \,/\, \left(\{g_{ij}\}_{i,j\in I}\right) \overset{(\varphi_i)_{i\in I}}{\to} E$$

from the vector bundle glued from the transition functions to the original bundle E.

2.6. **Examples of Vector Bundles.**

Lemma 2.57. *The bundle $\Lambda^k E$ of k-fold exterior powers is a vector bundle when (E, π, X) is an n-dimensional vector bundle with bundle atlas $\mathcal{U}$. We construct this by forming a pre-vector bundle. Define $\Lambda^k E := \bigsqcup_{x\in X} \Lambda^k E_x$. Each $\Lambda^k E_x$ has dimension $\binom{n}{k}$, hence $\varphi_x \colon \Lambda^k E_x \cong \mathbb{R}^{\frac{n!}{k!(n-k)!}}$. Let the projection be the canonical one. Let the atlas be given by $\{(f_\alpha, U_\alpha)\}$ where $f_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{\frac{n!}{k!(n-k)!}}$ is given by $\pi(-) \times \varphi_-$ and $(f, U_\alpha) \in \mathcal{U}$.*

Lemma 2.58 (Orientation cover)**.** *Using the exerior power bundle, we can construct the 1-dimensional bundle $\Lambda^n E$ for (E, π, X) an n-dimensional vector bundle. Define the equivalence relation in $\Lambda^n E - \{$zero section $\}$ by $x \sim y \iff y = \lambda x$ for some $\lambda > 0$. Give the equivalence classes $\tilde{X}(E)$ the quotient topology. Then we obtain a two sheeted cover of X by the canonical projection*

$$\tilde{X}(E) \overset{\tilde{\pi}}{\to} X$$

which is called the orientation cover of E.

3. Morse Theory

3.1. **The Cobordism Category.**

Definition 3.1 (Smooth manifold triad)**.** $(W; V_0, V_1)$ is a smooth manifold triad if W is a compact smooth n-manifold and ∂W is the disjoint union of two open and closed submanifolds V_0 and V_1.

Definition 3.2. If $(W; V_0, V_1)$ and $(W'; V_1', V_2')$ are two smooth manifold triads and $h\colon V_1 \to V_1'$ is a diffeomorphism, then we can form a third triad $(W \cup_h W'; V_0, V_2')$ where $W \cup_h W'$ is the space formed from W and W' by identifying points of V_1 and V_1' under h according to the following theorem.

Theorem 3.3. *There exists a smooth structure which is unique up to diffeomorphism fixing $V_0, h(V_1) = V_1'$ and V_2' on $W \cup_h W'$ such that the inclusion maps $W \hookrightarrow W \cup_h W', W' \hookrightarrow W \cup_h W'$ are diffeomorphisms onto their images.*

Definition 3.4 (Cobordism)**.** Given two closed smooth n-manifolds M_0 and M_1 (so M_0, M_1 compact and $\partial M_0 = \partial M_1 = \varnothing$), a *cobordism* from M_0 to M_1 is a 5-tuple $(W; V_0, V_1; h_0, h_1)$ where $(W; V_0, V_1)$ is a smooth manifold triad and $h_i\colon V_i \to M_i$ is a diffeomorphism for $i = 0, 1$.

Definition 3.5 (Equivalence)**.** Two cobordisms $(W; V_0, V_1; h_0, h_1)$ and $(W'; V_0', V_1'; h_0', h_1')$ from M_0 to M_1 are said to be *equivalent* if there exists a diffeomorphism $g\colon W \to W'$ carrying V_0 to V_0' and V_1 to V_1', such that for $i = 0, 1$, the following triangle commutes:

$$\begin{tikzcd} V_i \arrow[rr, "g|_{V_i}"] \arrow[dr, "h_i"'] & & V_i' \arrow[dl, "h_i'"] \\ & M_i & \end{tikzcd}$$

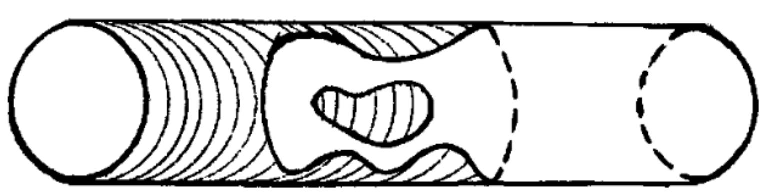
Definition 3.6 (Composition of cobordisms)**.** Given a cobordism equivalence class c from M_0 to M_1 and c' from M_1 to M_2, there is a well-defined class cc' from M_0 to M_2 formed using Theorem 3.3 as follows: let $(W; V_0, V_1; h_0, h_1)$ be the cobordism from M_0 to M_1 and $(W'; V_0', V_1'; h_0', h_1')$ from M_1 to M_2. Then the cobordism formed by $(W \cup_{\mathrm{id}} W'; V_0, V_1'; h_0, h_1')$ is a cobordism from M_0 to M_2, and furthermore, the inclusions $j_h\colon W \to W \cup_{\mathrm{id}} W'$ and $j_{h'}\colon W' \to W \cup_{\mathrm{id}} W'$ are diffeomorphisms onto their images.

This composition is associative.

Definition 3.7 (Identity cobordism)**.** For every closed manifold M, the identity cobordism class ι_M is the equivalence class of $(M \times I; M \times 0, M \times 1; p_0, p_1)$ where $p_i(x, i) = x$, for $x \in M$ and $i = 0, 1$. Hence $\iota_{M_1} c = c = c\iota_{M_2}$ when c is a cobordism class from M_1 to M_2.

Definition 3.8 (Trivial cobordism)**.** A cobordism $c = (W; V_0, V_1; h_0, h_1)$ is called a trivial cobordism if it is equivalent to an identity cobordism.

Note. Note also that there are non-trivial inverses: In particular, the manifolds in



a cobordism are **not** assumed to be connected.

Definition 3.9. Consider cobordism classes from M to itself. These form a monoid H_M. The invertible coboridisms in H_M form a group G_M.

Definition 3.10 (c_h)**.** Given a diffeomorphism $h\colon M \to M'$, define c_h as the class of $(M \times I; M \times 0, M \times 1; j, h_1)$ where $j(x, 0) = x$ and $h_1(x, 1) = h(x)$ for $x \in M$.

So a diffeomorphism $M \to M'$ gives a cobordism c_h from M to M'.

Theorem 3.11. *$c_h c'_h = c_{h'h}$ for any two diffeomorphisms $h\colon M \to M'$ and $h'\colon M' \to M''$.*

Proof. Let $W = M \times I \cup_h M' \times I$. Let $c_h = (M \times I, M \times 0, M \times I, ; j_0, j_1)$ and $c_{h'} = (M' \times I, M' \times 0, M' \times 1, j'_0, j'_1)$. So recall that this is formed by taking a tube on M and a tube on M' and then gluing an end of the tube of M to an end of the tube of M' through a twist by the diffeomorphism h. Then W is still a smooth manifold. The resulting cobordism is $(W, M \times 0, M' \times 1, j_0, j'_1)$. We must show that this is the same, or more precisely, that this cobordism is *equivalent* to the cobordism $(M \times I, M \times 0, M \times 1, j, h_1)$ where $j(x, 0) = x$ and $h_1(x, 1) = h'h(x)$. So we must define a diffeomorphism $g\colon M \times I \to W$ carrying $M \times 0$ to $M \times 0$ and $M \times 1$ to $M' \times I$, such that for $i = 0, 1$, the following triangle commutes

$$\begin{tikzcd} M \times 1 \arrow[rr, "g|_{M\times 1}"] \arrow[dr, "h_1"'] & & M' \times 1 \arrow[dl, "j'_1"] \\ & M'' & \end{tikzcd}$$

and

$$\begin{tikzcd} M \times 0 \arrow[rr, "g|_{M\times 0}"] \arrow[dr, "j"] & & M \times 0 \arrow[dl, "j_0"] \\ & M & \end{tikzcd}$$

Define $g\colon M \times I \to W$ by

$$g(x,t) = \begin{cases} j_h(x, 2t), & t \in \left[0, \frac{1}{2}\right] \\ j_{h'}\left(h(x), 2t-1\right), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

where $j_h\colon M \times I \to W$ is the inclusion and $j_{h'}\colon M' \times I \to W$ is the other inclusion given in the construction of $c_h c_{h'}$. Then indeed $g|_{M\times 0}$ maps into $M \times 0$ and $g|_{M\times 1}$ maps into $M' \times 1$. Furthermore, $j_0 \circ g(x, 0) = j_0 \circ j_h(x, 0) = x$ and $j'_1 \circ g(x, 1) = j'_1 \circ j_{h'}(h(x), 1) = j'_1(h(x), 1) = h'h(x) = h_1(x, 1)$, so $j'_1 \circ g = h_1$. □

3.1.1. *Isotopies and Pseudo-Isotopies.*

Definition 3.12. Two diffeomorphisms $h_0, h_1\colon M \to M'$ are (smoothly) isotopic if there exists a smooth map $f\colon M \times I \to M'$ such that $f_t = f(-, t)\colon M \to M'$ is a diffeomorphism for every t and $f_0 = h_0$ and $f_1 = h_1$.
Two diffeomorphisms $h_0, h_1\colon M \to M'$ are *pseudo-isotopic* if there exists a diffeomorphism $g\colon M \times I \to M' \times I$ such that $g(x, 0) = (h_0(x), 0)$ and $g(x, 1) = (h_1(x), 1)$.

Lemma 3.13. *Isotopy and pseudo-isotopiy are equivalence relations.*

Theorem 3.14. *$c_{h_0} = c_{h_1}$ if and only if h_0 is pseudo-isotopic to h_1.*

Proof. Let $g\colon M \times I \to M' \times I$ be a pseudo-isotopy between h_0 and h_1. Define $h_0^{-1} \times \mathrm{id}\colon M' \times I \to M \times I$ by

$$\left(h_0^{-1} \times \mathrm{id}\right)(x,t) = \left(h_0^{-1}(x), t\right).$$

We claim that $\left(h_0^{-1} \times \mathrm{id}\right) \circ g$ is an equivalence between c_{h_1} and c_{h_0}. Firstly, $\left(h_0^{-1} \times \mathrm{id}\right) \circ g$ is indeed a map $M \times I \to M \times I$. If we write $c_{h_0} = (M \times I; M \times 0, M \times 1; j_0, k_0)$ and $c_{h_1} = (M \times I; M \times 0, M \times 1; j_0', k_0')$ where $j_0(x,0) = x$, $j_0'(x,0) = x$ and $k_0(x,1) = h_0(x)$ and $k_0'(x,1) = h_1(x)$, then firstly, $\left(h_0^{-1} \times \mathrm{id}\right) \circ g(x,0) = \left(h_0^{-1} \times \mathrm{id}\right)(h_0(x),0) = (x,0)$ and $\left(h_0^{-1} \times \mathrm{id}\right) \circ g(x,1) = \left(h_0^{-1} \times \mathrm{id}\right)(h_1(x),1) = \left(h_0^{-1} \circ h_1(x), 1\right) \in M \times 1$, and lastly,

$$k_0 \circ \left(h_0^{-1} \times \mathrm{id}\right) \circ g(x,1) = k_0\left(h_0^{-1} \circ h_1(x), 1\right) = h_1(x) = k_0'(x,1)$$

and

$$j_0 \circ \left(h_0^{-1} \times \mathrm{id}\right) \circ g(x,0) = j_0(x,0) = x = j_0'(x,0)$$

so $\left(h_0^{-1} \times \mathrm{id}\right) \circ g$ defines an equivalence from c_{h_1} to c_{h_0}. □

3.1.2. *Interlude.* A different way to define a cobordism is as follows:

Definition 3.15. A smooth compact n-dimensional manifold is said to be a cobordism between two $(n-1)$-dimensional smooth manifolds M_L and M_R if there exist open embeddings $M_L \times \mathbb{R} \hookrightarrow M$ and $M_R \times \mathbb{R} \hookrightarrow M$ such that the images of $M_R \times [0,\infty)$ and $M_L \times (-\infty, 0]$ are closed. We denote this by $M_L \rightsquigarrow M_R$

Definition 3.16 (Gluing cobordisms/composition of cobordisms)**.** Given cobordisms $M_L \rightsquigarrow M_R = N_L \rightsquigarrow N_R$, we can form the composite cobordism $M \circ N$ as the pullback

$$\begin{tikzcd}
M_L \times \mathbb{R} \arrow[dr, hook] & & M_R \times \mathbb{R} = N_L \times R \arrow[dl, hook] \arrow[dr, hook] & & N_R \times \mathbb{R} \arrow[dl, hook] \\
& M \arrow[dr, hook] & & N \arrow[dl, hook] & \\
& & M \circ N \arrow[u, phantom, "\wedge"] & &
\end{tikzcd}$$

Definition 3.17 (Isomorphism/Equivalence of Cobordisms)**.** In this definition, two cobordisms $M_1\colon M_L \rightsquigarrow M_R$ and $M_2\colon M_L \rightsquigarrow M_R$ are isomorphic/equivalent when there exist maps making the following diagram commute:

$$\begin{tikzcd}
M_L \times R \arrow[r, hook] \arrow[d, dash, "\mathrm{id}"'] & M_1 \arrow[d, "f"', "\cong"] & M_R \times \mathbb{R} \arrow[l, hook'] \arrow[d, dash, "\mathrm{id}"] \\
M_L \times \mathbb{R} \arrow[r, hook] & M_2 & M_R \times \mathbb{R} \arrow[l, hook']
\end{tikzcd}$$

Definition 3.18 (Identity cobordism)**.** For a smooth compact manifold M, the identity cobordism of M is the cobordism from M to M given by $M \times \mathbb{R}$ where we embed $M \times \mathbb{R}_{<0} \hookrightarrow M \times \mathbb{R}$ and $M \times \mathbb{R}_{>0} \hookrightarrow M \times \mathbb{R}$ by the inclusions.

Definition 3.19 (Trivial cobordism)**.** A cobordism is trivial if it is equivalent to an identity cobordism.

3.2. **Elementary Cobordisms.**

Definition 3.20 (Gradient-like vector fields for Morse functions)**.** Let f be a Morse function for the triad $(W^n; V, V')$. A vector field ξ on W^n is a *gradient-like vector field for* f if

(1) $\xi(f) > 0$ throughout the complement of the set of critical points of f
(2) Given any critical point p of f, there are coordinates $(x, y) = (x_1, \dots, x_\lambda, x_{\lambda+1}, \dots, x_n)$ in a neighborhood U of p such that $f = f(p) - |x|^2 + |y|^2$ and ξ has coordinates $(-x_1, \dots, -x_\lambda, x_{\lambda+1}, \dots, x_n)$ throughout U.

Remark. We identify the triad $(W; V_0, V_1)$ with the cobordism $(W; V_0, V_1; i_0, i_1)$ where $i_0 \colon V_0 \to V_0$ and $i_1 \colon V_1 \to V_1$ are the identity maps.

Definition 3.21 (Product cobordism)**.** A triad $(W; V_0, V_1)$ is said to be a *product cobordism* if it is diffeomorphic to the trivial cobordism $(V_0 \times [0,1]; V_0 \times 0, V_0 \times 1)$.

Theorem 3.22 (Identifying product/trivial cobordisms)**.** *If the Morse number* μ *of a triad* $(W; V_0, V_1)$ *is zero, then* $(W; V_0, V_1)$ *is a product cobordism.*

Theorem 3.23 (Collar Neighborhood Theorem)**.** *Let* W *be a compact smooth manifold with boundary. There exists a neighborhood of* ∂W *(called a collar neighborhood) diffeomorphic to* $\partial W \times [0, 1)$.

Definition 3.24 (Two-sided)**.** A connected, closed submanifold $M^{n-1} \subset W^n - \partial W^n$ is said to be *two-sided* if some neighborhood of M^{n-1} on W^n is cut into two components when M^{n-1} is deleted.

Theorem 3.25 (The Bicollaring Theorem)**.** *Suppose that every component of a smooth submanifold* M *of* W *is compact and two-sided. Then there exists a "bicollar" neighborhood of* M *in* W *diffeomorphic to* $M \times (-1, 1)$ *in such a way that* M *corresponds to* $M \times 0$.

3.2.1. *Problems.*

Problem 3.26 (Invertible cobordisms and boundaries of compact manifolds)**.** Let $W_0 \colon M_0 \rightsquigarrow \varnothing$ and $W_1 \colon M_1 \rightsquigarrow \varnothing$ be two compact d-dimensional smooth cobordisms from compact $(d-1)$-dimensional smooth manifolds M_0 and M_1 to the empty manifold, viewed as a $(d-1)$-manifold. In other words, we have a smooth embedding $M_i \times \mathbb{R} \hookrightarrow W_i$ satisfying that $M_i \times (-\infty, 0]$ is closed, and such that their complement $W_i - (M_i \times \mathbb{R})$ is compact. We define $\operatorname{int}(W_i)$ to be the complement of the image of $M_i \times (-\infty, t]$ for some $t \in \mathbb{R}$ (and hence any $t \in \mathbb{R}$), and observe that $\operatorname{int}(W_i)$ is again a smooth manifold, being an open subset of W_i.

(1) Assume that in the situation of the above, $\operatorname{int}(W_0)$ is diffeomorphic to $\operatorname{int}(W_1)$. Show that M_0 and M_1 are invertibly cobordant, i.e., there exists a cobordism $M_0 \rightsquigarrow M_1$ which is invertible in the category Cob_d.
(2) Let W be a smooth, open (i.e., non-compact) d-manifold. We define a compact closure of W to be a compact cobordism $W' \colon M \rightsquigarrow \varnothing$ such that W is diffeomorphic to $\operatorname{int}(W')$. Assume that W admits a comapct closure $W' \colon M \rightsquigarrow \varnothing$. Show that the set of compact closures of W up to isomorphism of their interiors is in bijection with the set of invertible cobordisms over M.

Proof. (1)
Saying that $M_0 \rightsquigarrow M_1$ is invertible in Cob_d is precisely saying that there exists a cobordism $M_1 \rightsquigarrow M_0$ such that the composite cobordism $M_0 \rightsquigarrow M_1 \rightsquigarrow M_0$ is equivalent to the trivial cobordism $M_0 \rightsquigarrow M_0$. We will do this using the usual definition of cobordisms with boundaries. Then the problem is equivalently to show that we can find coborisms $M_0 \rightsquigarrow M_1 \rightsquigarrow M_0$ such that the composite is a product cobordism - i.e., has Morse number 0. In this case, we are dealing with closed compact manifolds W_0, W_1 such that $\partial W_0 \cong M_0$ and $\partial W_1 \cong M_1$. Furthermore, the boundaries have closed collar neighborhoods $\partial W_i \times I$, and removing some open/usual collar neighborhoods of these boundaries $\partial W_i \times [0, 1)$ leaves us with compact spaces which are, by assumption, diffeomorphic. Now, take the cobordism W_0 and choose a collar neighborhood of ∂W_0: $M_0 \times [0, 1]$, where M_0 is identified with $M_0 \times 0$ in W_0. By assumption, there is a diffeomorphism $W_0 - (M_0 \times [0, 1]) \cong W_1 - (M_1 \times [0, 1])$. Now, the diffeomorphism extends to the closure of the interiors which is also M_i since the collar is a cylinder, so we obtain a diffeomorphism $h \colon M_0 \times 1 \cong M_1 \times 1$. Without loss of generality, we can reparametrize, to get the diffeomorphism $h \colon M_0 \times 1 \to M_1 \times 0$ since the boundaries of the interiors must map to each other. Now we can glue the collars by gluing the cobordisms they represent using theorem 1.4 in Milnor's book on h-cobordisms to get a cobordism c_h which is the manifold $M_0 \times [0, 1] \cup_h M_1 \times [0, 1]$. This indeed now gives a cobordism $M_0 \rightsquigarrow M_1$. We can likewise obtain the cobordism $M_1 \rightsquigarrow M_0$ which is also obtained by gluing $M_1 \times [0, 1]$ with $M_0 \times [0, 1]$ along $M_1 \times 1$ and $M_0 \times 0$. Denote this cobordism by $c_{h'}$. We claim that $c_h c_{h'} = \mathrm{id}_{M_0}$. That is, that $c_h c_{h'}$ is a product cobordism/trivial cobordism of M_0. One way to see this is by using theorem 1.6 in Milnor's book on h-cobordisms which says that $c_h c_{h'} = c_{h'h} = c_{\mathrm{id}_{M_0}}$ which indeed is the trivial cobordism. Alternatively, each collar neighborhood has no critical values, so c_h and $c_{h'}$ both have Morse number 0, and then corollary 3.8 in Milnor's book on h-cobordisms gives that $c_h c_{h'}$ also has Morse number 0, hence is trivial by theorem 3.4 in the same book. $\square$

3.3. **Morse Functions.** The goal is to be able to factor cobordisms into compositions of simpler cobordisms.

Definition 3.27 (Critical points and non-degenerate critical points)**.** Let W be a smooth manifold and $f\colon W \to \mathbb{R}$ a smooth function. A point $p \in W$ is a critical point of f if, in some coordinate system,

$$\frac{\partial f}{\partial x^1}|_p = \frac{\partial f}{\partial x^2}|_p = \ldots = \frac{\partial f}{\partial x^n}|_p = 0.$$

Such a point is called a non-degenerate critical point if $\det\left(H(f)_p\right) = \det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}|_p\right) \neq 0$

Lemma 3.28 (Morse Lemma)**.** *If p is a non-degenerate critical point of f, then in some coordiante system about p,*

$$f\left(x_1, \ldots, x_n\right) = c - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2$$

for λ between 0 and n and c some constant.

Definition 3.29 (Index of a critical point)**.** The λ from the Morse Lemma (Lemma 3.28) is called the index of the critical point p.

Definition 3.30 (Morse Function)**.** A *Morse function* on a smooth manifold triad $(W; V_0, V_1)$ is a smooth function $f\colon W \to [a, b]$ such that

(1) $f^{-1}(a) = V_0$ and $f^{-1}(b) = V_1$
(2) All the critical points of f are interior (lie in $W - \partial W$) and are non-degenerate.

Corollary 3.31. *A Morse function has only finitely many zeros.*

Proof. Suppose we have a Morse function $f\colon W \to [a, b]$ and suppose that p is a critical point. By definition, it is non-degenerate since f is a Morse function, so by the Morse Lemma, in some neighborhood of p, f takes the form

$$f\left(x_1, \ldots, x_n\right) = c - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2$$

so in particular, $\frac{\partial f}{\partial x^i}\left(x_1, \ldots, x_n\right) = -2x_i$ in this neighborhood for all i. Hence $(x_1, \ldots, x_n) = (0, \ldots, 0)$ in this neighborhood is the only critical point (in particular, in local coordinates, $p = (0, \ldots, 0)$). This shows that critical points of a Morse function are isolated. Since the manifold of a smooth manifold triad is, in particular, compact, there are only finitely many critical points since a collection of isolated points in a compact space is finite. □

Definition 3.32 (Morse number μ)**.** The *Morse number* μ of $(W; V_0, V_1)$ is the minimum over all Morse functions f on $(W; V_0, V_1)$ of the number of critical points of f.

Theorem 3.33 (Existence of Morse functions)**.** *Every smooth manifold triad $(W; V_0, V_1)$ possesses a Morse function.*

To prove the existence theorem of Morse functions, we need the following lemmas:

Lemma 3.34. *There exists a smooth function $f\colon W \to [0, 1]$ with $f^{-1}(0) = V_0, f^{-1}(1) = V_1$, such that f has no critical points in a neighborhood of the boundary of W.*

Lemma 3.35 (M. Morse)**.** *If f is a C^2 mapping of an open subset $U \subset R^n$ to the real line, then, for almost all linear mappings $L\colon R^n \to R$, the function $f + L$ has only nondegenerate critical points.*

Proof. The idea of the proof is to consider the manifold $U \times \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$ the its submanifold $M = \{(x, L) \mid d(f(x) + L(x)) = 0\}$. Then $x \mapsto (x, -df(x))$ is a diffeomorphism $U \cong M$. Composing with a projection $\pi\colon M \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ sending $(x, L) \mapsto L$, one sees that π is critical at $(x, L) \in M \cong U$ if and only if $d\pi = -\frac{\partial^2 f}{\partial x_i \partial x_j}$ is singular. That is, $f + L$ has a degenerate critical point □

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