

**Theorem 0.1** (Linear function  $\mathbb{R}^m \rightarrow \mathbb{R}$ ). *A linear function  $f: \mathbb{R}^m \rightarrow \mathbb{R}$  is uniformly continuous.*

*Proof.* Let  $e_1, \dots, e_m$  be the usual basis for  $\mathbb{R}^m$ . By linearity,

$$\begin{aligned} |f(x)|^2 &= |x_1 f(e_1) + \dots + x_m f(e_m)|^2 \\ &\leq (x_1^2 + \dots + x_m^2) (f(e_1)^2 + \dots + f(e_m)^2) \end{aligned} \quad (\text{Cauchy-Schwarz})$$

□

So for all  $x \in \mathbb{R}$ , we have

$$|f(x)| \leq M \|x\|, \quad M = (f(e_1)^2 + \dots + f(e_m)^2)^{\frac{1}{2}}.$$

It follows that

$$|f(x) - f(x_0)| = |f(x - x_0)| \leq M \|x - x_0\|.$$