HOMOTOPY THEORY

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For these notes, we will follow [2], [1] and [3].

1. Cofibrations

For this section, we will follow chapter VII.1 in [1].

One of the fundamental questions in topology is the "extension problem". Namely, given a map $g\colon A \to Y$ defined on a subspace A of X, when can we extend this map to all of X.
This cannot always be done - for example, as is the case with $A = Y = S^n$ and $X = D^{n+1}$ choosing the map to be any degree -1 map.

Question 1.1. Is the extension problem a *homotopy-theoretic* problem? That is, does the answer depend only on the homotopy class of g?

The answer is: generally not. For example, we can take $X = [0,1]\,, A = \{0\} \cup \left\{\frac{1}{n} \mid n = 1, 2, \ldots\right\}$ and $Y = CA$, the cone on A. Choosing g to be the inclusion of A into Y, this cannot be extended to X as the extension would be discontinuous at $\{0\}$. However, $g \simeq g'$ with g' being the constant map of A to the vertex of the cone, and g' easily extends to X by the constant map.

It turns out, however, that under some very mild conditions on the spaces, the problem becomes homotopy theoretic. We will now discuss this.

Definition 1.2 (Homotopy extension property)**.** Let (X, A) and Y be given spaces. Then (X, A) is said to have the *homotopy extension property* with respect to Y if the following diagram can always be completed to be commutative.

$$\begin{tikzcd}
A \times I \cup X \times \{0\} \arrow[r] \arrow[d, hook] & Y \\
X \times I \arrow[ur, dashed] &
\end{tikzcd}$$

One can also depict this by the following diagram:

$$\begin{tikzcd}
A \times \{0\} \arrow[rr, hook] \arrow[dd, hook] & & A \times I \arrow[dl] \arrow[dd, hook] \\
& Y & \\
X \times \{0\} \arrow[ur] \arrow[rr, hook] & & X \times I \arrow[ul, dashed]
\end{tikzcd}$$

If (X, A) has the homotopy extension property with respect to Y, then the extensibility of maps $g\colon A \to Y$ depends only on the homotopy class of g. For suppose $H\colon g \simeq g'$ and g' can be extended to $\tilde{g'}\colon X \to Y$, then define the map $A \times I \cup X \times \{0\}$ by $\tilde{g'} \times \{0\}$ on $X \times \{0\}$ and H on $A \times I$. The homotopy extension property for the pair (X, A) then guarantees the existence of a map $G\colon X \times I \to Y$ which equals g on $A \times \{1\}$, so $H(-, 1) : X \to Y$ extends g.

Definition 1.3 (Cofibration)**.** Let $f\colon A \to X$ be a map. Then f is called a *cofibration* if one can always fill in the following commutative diagram given the solid arrows:

$$\begin{tikzcd}
A \times \{0\} \arrow[rr] \arrow[dd, "f\times\mathrm{id}"] & & A \times I \arrow[dl] \arrow[dd, "f\times\mathrm{id}"] \\
& Y & \\
X \times \{0\} \arrow[ur] \arrow[rr] & & X \times I \arrow[ul, dashed]
\end{tikzcd}$$

for *any* space Y.

Note. If f is an inclusion, the this is the same as the homotopy extension property for all Y. That attribute is sometimes referred to as the *absolute homotopy extension property.*

Theorem 1.4. *For an inclusion $A \subset X$, the following are equivalent:*

(1) *The inclusion map $A \hookrightarrow X$ is a cofibration.*
(2) *$A \times I \cup X \times \{0\}$ is a retract of $X \times I$.*

Proof. If the inclusion is a cofibration, then choosing $Y = A \times I \cup X \times \{0\}$ with all arrows being inclusions in the diagram of a cofibration, we obtain a map $X \times I \to A \times I \cup X \times \{0\}$ which is the identity on $A \times I \cup X \times \{0\}$.
Conversely, if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$, then we can always complete the diagram by mapping $X \times I \to A \times I \cup X \times \{0\} \to Y$ where the second map takes the maps $A \times I \to Y$ and $X \times \{0\} \to Y$ from the diagram. □

Corollary 1.5. *If A is a subcomplex of a CW-complex X, then the inclusion $A \hookrightarrow X$ is a cofibration.*

Proof. We want to construct a retraction $X \times I \to A \times I \cup X \times \{0\}$. We will do so by constructing a retraction $\left(\left(A \cup X^{(r)}\right) \times I\right) \cup (X \times \{0\}) \to (A \times I) \cup (X \times \{0\})$ by induction on r. If it has been defined on the $(r-1)$-skeleton, then extending it over an r-cell is simply a matter of extending a map on $S^{r-1} \times I \cup D^r \times \{0\}$ over $D^r \times I$ which can be done since the pair $\left(D^r \times I, S^{r-1} \times I \cup D^r \times \{0\}\right)$ is homeomorphic to $(D^r \times I, D^r \times \{0\})$. See Figure 1
These maps for each cell fit together to give a map on the r-skeleton because of the weak topology on $X \times I$. The union of these maps for all r gives a map on $X \times I$, again because of the weak topology on $X \times I$. □

Theorem 1.6. *Assume that $A \subset X$ is closed and that there exists a neighborhood U of A and a map $\varphi\colon X \to I$ such that*

(1) *$A = \varphi^{-1}(0)$.*
(2) *$\varphi(X - U) = \{1\}$.*

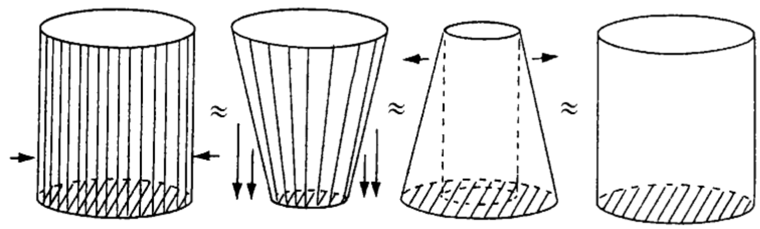

FIGURE 1. A homeomorphism of pairs.

(3) *U deforms to A through X with A fixed. That is, there is a map $H\colon U\times I \to X$ such that $H(a,t)=a$ for all $a\in A, H(u,0)=0$, and $H(u,1)\in A$ for all $u\in U$.*

Then the inclusion $A\hookrightarrow X$ is a cofibration. The converse also holds.

Proof. We may assume that $\varphi=1$ on a neighborhood of $X-U$ by replacing φ with $\min(2\varphi,1)$. It suffices to show that there exists a retract $\Phi\colon U\times I\to X\times\{0\}\cup A\times I$ since then the map

$$r(x,t)=\begin{cases}\Phi\left(x,t\left(1-\varphi(x)\right)\right), & x\in U\\ (x,0), & x\notin U\end{cases}$$

gives a retraction $X\times I\to A\times I\cup X\times\{0\}$.
We define Φ by

$$\Phi(u,t)=\begin{cases}H\left(u,\frac{t}{\varphi(u)}\right)\times\{0\}, & \varphi(u)>t\\ H\left(u,1\right)\times\{t-\varphi(u)\}, & \varphi(u)\le t.\end{cases}$$

The only thing that needs checking here is that Φ is continuous at points $(u,0)$ such that $\varphi(u)=0$, i.e., points $(a,0)$ for $a\in A$ - indeed here the expression for $\varphi(u)>t$ is not defined.
Recall that a map $f\colon X\to Y$ is continuous if for every point $x\in X$ and any neighborhood U of $f(x)$, there exists a neighborhood V of x such that $f(V)\subset U$. So let W be a neighborhood of $a=H(a,t)$. Then there exists a neighborhood $V\subset W$ containing a such that $H\left(V\times I\right)\subset W$, by assumption of H being continuous. □

2. Homotopy Groups

2.1. **Homotopy.** We follow chapter 14 of [1] for this subsection.

To start of, we recall the basic definitions of homotopies.

Definition 2.1 (Homotopy)**.** Two maps $f_0, f_1 \colon X \to Y$ are said to be *homotopic* if there exists a homotopy $F\colon X \times I \to Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$.

Definition 2.2 (Homotopy equivalence)**.** A map $f\colon X \to Y$ is said to be a *homotopy equivalence* if it is an isomorphism in hTop.

Lemma 2.3 (Reparametrization Lemma)**.** *Let φ_1, φ_2 be maps $(I, \partial I) \to (I, \partial I)$ which are equal on ∂I. Let $F\colon X \times I \to Y$ be a homotopy and let $G_i(x,t) = F(x, \varphi_i(t))$ for $i = 1, 2$. Then $G_1 \simeq G_2 \operatorname{rel} X \times \partial I$.*

We shall use c to denote the constant homotopy.

Proposition 2.4. *$F * c \simeq F \operatorname{rel} X \times \partial I$ and $c * F \simeq F \operatorname{rel} X \times \partial I$.*

Definition 2.5. If $F\colon X \times I \to Y$ is a homotopy, then we define $F^{-1}\colon X \times I \to Y$ by $F^{-1}(x,t) = F(x, 1-t)$.

Note that F^{-1} is precisely the inverse to F in hTop.

Proposition 2.6. *For any homotopies F, G, H for which the concatenations are defined, we have*
$$(F * G) * H \simeq F * (G * H) \operatorname{rel} X \times \partial I.$$

Proposition 2.7. *For homotopies F_1, F_2, G_1, G_2, if $F_1 \simeq F_2 \operatorname{rel} X \times \partial I$ and $G_1 \simeq G_2 \operatorname{rel} X \times \partial I$, then $F_1 * G_1 \simeq F_2 * G_2 \operatorname{rel} X \times \partial I$.*

Note that all of the discussion of concatenation of homotopies goes through with no difficulties for the cases in which all homotopies are relative to some subspace $A \subset X$ or are homotopies of pairs $(X, A) \to (Y, B)$.
It follows that homotopy between maps of pairs $(X, A) \to (Y, B)$ is an equivalence relation. The set of homotopy classes of these maps is commonly denoted by $[X, A; Y, B]$ or just $[X; Y]$ if $A = \varnothing$.

Theorem 2.8. *If $f_0 \simeq f_1 \colon X \to Y$ then $M_{f_0} \simeq M_{f_1} \operatorname{rel} X + Y$ and $C_{f_0} \simeq C_{f_1} \operatorname{rel} Y +$ vertex.*

To show this, one needs the following basic topological proposition:

Proposition 2.9. *If $f\colon X \to Y$ is a quotient map and K is locally compact Hausdorff, then $f \times 1\colon X \times K \to Y \times K$ is a quotient map.*

Proof of Theorem 2.8. First, let $F\colon X \times I \to Y$ be the homotopy between f_0 and f_1. Now define $h\colon M_{f_0} \to M_{f_1}$ by $h(y) = y$ for $y \in Y$ and
$$h(x,t) = \begin{cases} F(x, 2t), & t \leq \frac{1}{2} \\ (x, 2t-1), & \frac{1}{2} \leq t. \end{cases}$$
Define $k\colon M_{f_1} \to M_{f_0}$ likewise by the identity on Y nad
$$k(x,t) = \begin{cases} F^{-1}(x, 2t), & t \leq \frac{1}{2} \\ (x, 2t-1), & \frac{1}{2} \leq t \end{cases}.$$

Then the composition $kh\colon M_{f_0} \to M_{f_1}$ is the identity on Y and $F * \left(F^{-1} * E\right)$ on the cylinder portion, where $E\colon X \times I \to M_{f_0}$ is induced by the identity on $X \times I \to X \times I$. This is homotopic to the identity $\operatorname{rel} X \times \{1\} + Y$. Similarly for hk. In now remains to check the continuity of this homotopy. We have a homotopy $M_{f_0} \times I \to M_{f_0}$. We now claim that $M_{f_0} \times I \cong M_{f_0 \times I}$. Indeed then, using that $M_{f_0 \times I} = \frac{X \times I \times I \sqcup Y \times I}{((x,0,k) \sim (f_0(x),k)}$, it suffices to show continuity of the composition $X \times I \times I \sqcup Y \times I \to M_{f_0} \times I \to M_{f_0}$. For on $Y \times I$, it is the constant homotopy and on $X \times I \times I$ it is $F * \left(F^{-1} * E\right) \simeq E \operatorname{rel} X \times \partial I$. Now, that $M_{f_0} \times I \cong M_{f_0 \times I}$ follows from Proposition 2.9.

□

Let $f\colon X \to Y$. If $\varphi\colon Y \to Y'$ is a map, then there is the induced map $F\colon M_f \to M_{\varphi \circ f}$ induced from φ on Y and the identity on $X \times I$.

Theorem 2.10. *If $\varphi\colon Y \to Y'$ is a homotopy equivalence then so is $F\colon (M_f, X) \to (M_{\varphi\circ f}, X)$ and hence so is $F\colon C_f \to C_{\varphi\circ f}$.*

Proof. Let $\psi\colon Y' \to Y$ be a homotopy inverse of φ and let $G\colon M_{\varphi\circ f} \to M_{\psi\circ\varphi\circ f}$ be the map induced by ψ on Y' and the identity on $X \times I$. The composition $GF\colon M_f \to M_{\psi\circ\varphi\circ f}$ is induced from $\psi \circ \varphi\colon Y \to Y$ and the identity on $X \times I$. Let $H\colon Y \times I \to Y$ be a homotopy from id to $\psi \circ \varphi$; i.e., $H(y, 0) = y$ and $H(y, 1) = \psi\varphi(y)$. By the proof of Theorem 2.8, there is a homotopy equivalence $h\colon M_f \to M_{\psi\circ\varphi\circ f} \operatorname{rel} X$ given by $h(y) = y$ and

$$h(x,t) = \begin{cases} H\left(f(x), 2t\right), & t \le \frac{1}{2} \\ (x, 2t-1), & t \ge \frac{1}{2} \end{cases}.$$

We claim that $h \simeq GF \operatorname{rel} X$. Indeed, the homotopy H can be extended to $M_f \times I \to M_{\psi\circ\varphi\circ f}$ by putting

$$H\left((x,s),t\right) = \begin{cases} H\left(f(x), 2s+t\right), & 2s+t \le 1 \\ \left(x, \frac{2s+t-1}{t+1}\right), & 2s+t \ge 1 \end{cases}.$$

Then $H(-,0) = h$ and $H(-,1) = GF$, so since GF is a homotopy equvalence, so is h. Define $F'\colon M_{\psi\circ\varphi\circ f} \to M_{\varphi\circ\psi\circ\varphi\circ f}$ as the induced map on mapping cones with φ on Y and the identity on $X \times I$. Then similarly, $F'G$ is a homotopy equivalence. If k is a homotopy inverse of GF then $GFk \simeq \mathrm{id}$. If k' is a homotopy inverse of $F'G$ then $k'F'G \simeq \mathrm{id}$. Thus G has a right and left homotopy inverse: $R = Fk$ and $L = k'F'$. Then $R = \mathrm{id} \circ R \simeq (LG)\,R = L\,(GR) \simeq L \circ \mathrm{id} = L$, so $R \simeq L$. That is, G has a homotopy inverse. Therefore, G is a homotopy equivalence. Since G and GF are homotopy equivalences, so is F. □

Problem 2.11. [1, Ex 14.1] Let $S^2 \cup A$ denote the union of the unit 2-sphere and the line segment joining the north and south poles. Show that $S^2 \vee S^1 \simeq S^2 \cup A$.

Proof. Define two maps $f_0, f_1\colon \{0,1\} \to S^2$ where $f_0(t) = (\cos(2\pi t), \sin(2\pi t), 0)$ and f_1 is the constant map at $(1,0,0)$. Then $f_0 \simeq f_1$, so $C_{f_0} \simeq C_{f_1}$. Now, $C_{f_0} = S^2 \cup A$ while $C_{f_1} = S^2 \vee S^1$. □

Problem 2.12. [1, Ex 14.2] Show that the union of a 2-sphere and a flat unit 2-cell through the origin is homotopically equivalent to the one-point union of two 2-spheres.

Proof. A 2-cell is contractible, an a 2-sphere with a 2-cell inside it is precisely the cone of the map $S^1 \sqcup S^1 \to S^1$ with the identity on both. By [1, Thm 14.19], this is homotopy equivalent to the cone on $S^1 \sqcup S^1 \to \{*\}$ which is $S^2 \vee S^2$. □

Problem 2.13. Show that the union of a standard 2-torus with two disks, one spanning a latitudinal circle and the other spanning a longitudinal circle of the torus, is homotopically equivalent to a 2-sphere.

Proof. Using the identification of the torus as the quotient space of I^2 in the usual way, we can choose on spanning circle to be a 2-cell attached along $\{0\} \times I$ and the other to be a 2-cell attached along $I \times \{0\}$. These are contractible, and the quotient space becomes a 2-sphere. □

2.2. **Homotopy Groups.** Recall that $[X, A; Y, B]$ denotes the set of homotopy classes of maps $X \to Y$ carrying A into B such that A goes into B during the entire homotopy.
To make a group then, we can select a point $y_0 \in Y$ and consider the set

$$[X \times I, X \times \partial I; Y, \{y_0\}]$$

In this case, the operation of concatenation of homotopies makes this set into a group. It is technically also better to choose a basepoint $x_0 \in X$ and consider

$$[X \times I, \{x_0\} \times I \cup X \times \partial I; Y, \{y_0\}] .$$

For the moment, let us set $A = \{x_0\} \times I \cup X \times \partial I$. Then maps $X \times I \to Y$ which carry A into $\{y_0\}$ are in bijective correspondence with maps $(X \times I)/A \to Y$ which take the point $\{A\}$ into $\{y_0\}$.

Definition 2.14 (Reduced Suspension)**.** We define the *reduced suspension* of X to be

$$SX = (X \times I)/A = (X \times I) / (\{x_0\} \times I \cup X \times \partial I)$$

The set of homotopy classes of pointed maps of a pointed space X to a pointed space Y with homotopies preserving the base points will be denoted by $[X; Y]_*$.
Thus $[SX; Y]_*$ is in canonical bijective correspondence with $[X \times I, A; Y, \{y_0\}]$.
Now, suppose we have pointed maps $f, g \colon SX \to Y$. Then they induce homotopies $f', g' \colon X \times I \to Y$ by precomposing with the quotient map $X \times I \to SX$. We can then define $f' * g' \colon X \times I \to Y$ as usual. The resulting pointed map $SX \to Y$ will be denoted $f * g$. Geometrically, $f * g$ is obtained by putting f on the bottom and g on the top of the one-point union $SX \vee SX$ and composing the resulting map $SX \vee SX \to Y$ with the map $SX \to SX \vee SX$ obtained by collapsing the middle parameter value $\frac{1}{2}$ copy of X in SX to the base point.
For a map $f \colon (SX, \{A\}) \to (Y, \{y_0\})$, we denote its homotopy class in $[SX; Y]_*$ by $[f]$, and we define

$$[f]\,[g] = [f * g]$$

Under this operation, the set $[SX; Y]_*$ becomes a group.

Proposition 2.15. *The reduced suspension gives $SS^{n-1} \cong S^n$.*

Thus, we can define S^n as the n-fold reduced suspension of S^0. As a special case, the set $[S^n; Y]_*$ then becomes a group for $n > 0$.

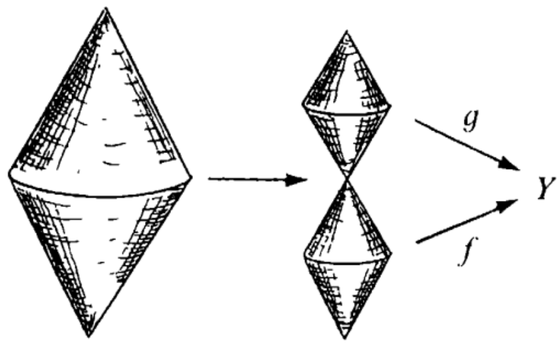

FIGURE 2. The product of two map classes $SX \to Y$.

Definition 2.16 (n th homotopy group)**.** We define

$$\pi_n(Y, y_0) = [S^n; Y]_*$$

with this operation.

2.2.1. *A different way of defining* $\pi_n(Y, y_0)$. Note that reduced suspension supplies a parameter in $[0, 1]$ and the space S^n as constructed is the quotient space of I^n obtained by collapsing the boundary of the cube to a point. Pointed maps $S^n \to Y$ are in bijective correspondence with maps $I^n \to Y$ taking ∂I^n to the base point of Y. This is a more traditional way of defining $\pi_n(Y)$. This becomes the group of homotopy classes of maps $(I^n, \partial I^n) \to (Y, \{y_0\})$ with the operation being

$$f * g(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n), & t_1 \in \left[0, \frac{1}{2}\right] \\ g(2t_1 - 1, t_2, \ldots, t_n), & t_1 \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Proposition 2.17. *For $n \geq 2$, $\pi_n(X, x_0)$ is abelian.*

Proof. Consider the homotopy in Figure 3. We begin by shrinking the domains of f and g to smaller subcubes of I^n, where the region outside is mapped to the basepoint. This allows us to move the boxes around in a continuous manner. The rest is clear. □

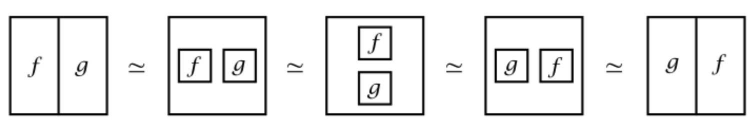

FIGURE 3. The homotopy in question

Next, we want to show that following:

Proposition 2.18. *If X is path-connected, then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for any two $x_0, x_1 \in X$.*

For this, we introduce an action of π_1 on π_n.

Definition 2.19 (The action of π_1 on π_n)**.** Given a path $\gamma\colon I \to X$ from x_0 to x_1, we associate to a map $f\colon (I^n, \partial I^n) \to (X, x_1)$ the map $\gamma f\colon (I^n, \partial I^n) \to (X, x_0)$ by shrinking the domain of f to a smaller concentric cube in I^n, then inserting the path γ on each radial segment in the shell between this smaller cube and ∂I^n. See Figure 4

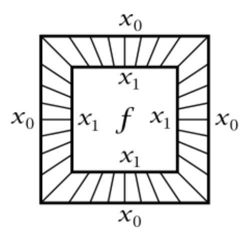

FIGURE 4. Depiction of γf.

Note. We have the following properties

(1) $\gamma\,(f+g) \simeq \gamma f + \gamma g$.
(2) $(\gamma\eta)\, f \simeq \gamma\,(\eta f)$.
(3) $\mathrm{id} f \simeq f$, where id denotes the constant path.

To see (1), first deform f and g to be constant on the right and left halves of I^n, respectively, producing maps which we may call $f+0$ and $0+g$, then we can excise a progressively wider symmetric middle slab of $\gamma(f+0)+\gamma(0+g)$ (which can be seen on the left in Figure 5) until it becomes $\gamma\,(f+g)$ (shown on the right).

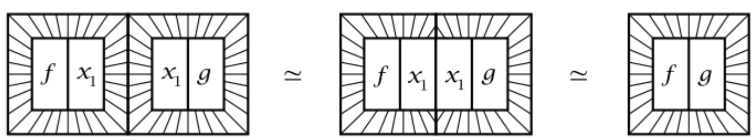

FIGURE 5.

Now if $\beta_\gamma\colon \pi_n(X, x_1) \to \pi_n(X, x_0)$ is the change-of-basepoint transformation, $\beta_\gamma\,[f] = [\gamma f]$, then the above note shows that β_γ is a group isomorphism. This proves Proposition 2.18. If we restrict attention to loops γ at x_0, then since $\beta_{\gamma\eta} = \beta_\gamma \beta_\eta$, the map $[\gamma] \mapsto \beta_\gamma$ defines a homomorphism from $\pi_1\,(X, x_0)$ to $\mathrm{Aut}\,(\pi_n\,(X, x_0))$ called the *action of* π_1 *on* π_n .

Note. For $n > 1$, this action makes $\pi_n(X, x_0)$ into a module over the group ring $\mathbb{Z}\,[\pi_1\,(X, x_0)]$.

Definition 2.20 (Simple/abelian spaces)**.** A space with trivial π_1 action on π_n is called 'n-simple', and 'simple' means ' n-simple for all n '. We call a space *abelian* if it has trivial action of π_1 on all homotopy groups π_n.

Proposition 2.21 (π_n is a functor)**.** *A map $\varphi\colon (X, x_0) \to (Y, y_0)$ induces a map $\varphi_*\colon \pi_n\,(X, x_0) \to \pi_n\,(Y, y_0)$ defined by $\varphi_*\,[f] = [\varphi f]$. It is immediate from the definitions that φ_* is well-defined and a homomorphism for $n \geq 1$. The functorial properties are also clear.*

Corollary 2.22. *Homotopy equivalent spaces have isomorphic homotopy groups.*

Proposition 2.23. *A covering space projection* $p\colon \left(\tilde{X}, \tilde{x}_0\right) \to (X, x_0)$ *induces isomorphisms* $p_*\colon \pi_n\left(\tilde{X}, \tilde{x}_0\right) \to \pi_n(X, x_0)$ *for all* $n \geq 2$.

Proof. Since S^n is path-connected and locally path-connected, and simply connected for $n \geq 2$, we find that any map $(S^n, s_0) \to (X, x_0)$ lifts to a map $(S^n, s_0) \to \left(\tilde{X}, \tilde{x}_0\right)$ when $n \geq 2$. This gives surjectivity of p_*. For injectivity, suppose $p_*[f] = [0]$ where $f\colon (S^n, s_0) \to \left(\tilde{X}, \tilde{x}_0\right)$. Let $c_{\tilde{x}_0}$ be the constant map at $\tilde{x}_0$. Then $p_*[\tilde{x}_0] = [0]$, so by uniqueness of the lifting theorem, $[f] = [c_{\tilde{x}_0}] = [0]$. □

Definition 2.24 (Aspherical)**.** Spaces with $\pi_n = 0$ for all $n \geq 2$ are called *aspherical.*

Corollary 2.25. S^1, T^n *and* K *are aspherical since they have contractible covering spaces.*

Proposition 2.26.

$$\pi_n\left(\prod_\alpha X_\alpha\right) \cong \prod_\alpha \pi_n(X_\alpha)$$

Next we define relative homotopy groups.

Definition 2.27 (Relative homotopy groups)**.** Regard I^{n-1} as a face of I^n with the last coordinate $s_n = 0$ and let J^{n-1} be the closure of $\partial I^n - I^{n-1}$. Then we define

$$\pi_n(X, A, x_0) := \left[I^n, \partial I^n, J^{n-1}; X, A, x_0\right]$$

We shall leave $\pi_0(X, A, x_0)$ undefined for now.

We can define a sum operation on $\pi_n(X, A, x_0)$ in the same way as for $\pi_n(X, x_0)$, except now the coordinate s_n now must remain free, so we must use one of the other coordinates. Thus we must have at least one other coordinate to define the same operation. So $\pi_n(X, A, x_0)$ is a group for $n \geq 2$, and it is abelian for $n \geq 3$. For $n = 1$, we have $I^1 = [0,1], I^0 = \{0\}$ and $J^0 = \{1\}$, so $\pi_1(X, A, x_0) = [I, \{0\}, \{1\}; X, A, x_0]$ is the set of homotopy classes of paths in X from a varying point in A to the fixed basepoint $x_0 \in A$. In general, this is not a group in any natural way.

Now, we saw before that $\pi_n(X, x_0)$ can be regarded as homotopy classes of maps $(S^n, x_0) \to (X, x_0)$. Similarly, collapsing J^{n-1} to a point, converts $\left(I^n, \partial I^n, J^{n-1}\right)$ to $\left(D^n, S^{n-1}, s_0\right)$. In this case, addition is done by the map $c\colon D^n \to D^n \vee D^n$ collapsing $D^{n-1} \subset D^n$ to a point.

2.3. **Problem set 1.**

2.3.1. *Exercises.*

Exercise 2.28 (The action of the fundamental gorup, part 2)**.** Let X be a path-connected, semi-locally simply-connected space with basepoint x and $p\colon \tilde{X} \to X$ its universal cover. Show that for $n \geq 2$ and $\tilde{x} \in X$ with $p(\tilde{x}) = x$, the isomorphism $p_* = \pi_n(p)\colon \pi_n\left(\tilde{X}, \tilde{x}\right) \cong \pi_n(X, x)$ allows us to identify the action of $\pi_1(X, x)$ on $\pi_n(X, x)$ with the action of $\pi_1(X, x)$ on $\pi_n\left(\tilde{X}, \tilde{x}\right)$ induced by the group of deck transformations, i.e., the natural action of $\pi_1(X, x)$ on $\tilde{X}$. In particular, make the statement precise.

Proof. We want to show that for $[\gamma] \in \pi_1(X, x)$ and $[f] \in \pi_n(X, x)$, if $\tilde{g}$ is the lift for γ starting at $\tilde{x}_0$, and $\tilde{f}\colon\ (S^n, s_0) \to \left(\tilde{X}, \tilde{x}_0\right)$ is the lift of f, then $p_*\left(\tilde{\gamma}\tilde{f}\right) = \gamma f$. But this follows directly from how $\tilde{\gamma}\tilde{f}$ and γf we constructed. Namely, applying p to the square used in the definition, we see that we obtain γf from $\tilde{\gamma}\tilde{f}$ since $p \circ \tilde{\gamma} = \gamma$ and $p \circ \tilde{f} = f$.

□

Exercise 2.29. Let X and Y be pointed spaces and $n \geq 2$. Show that the inclusion $X \vee Y \hookrightarrow X \times Y$ induces a surjection $\pi_n(X \vee Y) \to \pi_n(X \times Y)$ for all n. Furthermore, this exhibits $\pi_n(X \times Y)$ as a retract of $\pi_n(X \vee Y)$ for all n. (Is this also true for $n = 1$?)

Proof. a □

2.3.2. *Problems.*

Problem 2.30. Fix an isomorphism $H_n(S^n) \cong \mathbb{Z}$. We define the degree $\deg f$ of a map $f\colon S^n \to S^n$ to be the integer such that $f_*\colon H_n(S^n) \to H_n(S^n)$ sends 1 to $\deg f \in \mathbb{Z}$.

(1) Show that taking the degree of a map $S^n \to S^n$ induces a well-defined map
$$\deg\colon \pi_n(S^n) \to \mathbb{Z}$$
(2) Show that deg is a group homomorphism.
(3) Show that the map deg is surjective.
(4) Suppose that $n \geq 2$. Show that $\pi_n(S^n) \cong \mathbb{Z} \times A$ for some abelian group A.

Proof.
(1) Let $[f] \in \pi_n(S^n)$ and suppose f, f' are two representatives of this class. Then f and f' are homotopic by definition, so $f_* = (f')_* : \mathbb{Z} = H_n(S^n) \to H_n(S^n) = \mathbb{Z}$ are equal. In particular, $\deg f = f_*(1) = (f')_*(1) = \deg f'$. So the map is well-defined.
(2) To show that degree is a group homomorphism, we must show that $\deg(f+g) = \deg f + \deg g$.

For this, we will show a couple of results.

Proposition 2.31. *Let $X = S^n_1 \vee \ldots \vee S^n_k$ for $n > 0$. Then the homomorphism $H_n(S^n_1) \oplus \ldots \oplus H_n(S^n_k) \to H_n(X)$ induced by the inclusion maps is an isomorphism whose inverse is induced by the projections $X \to S^n_i$.*

To prove this proposition, we must show the following lemma.

Lemma 2.32. *Let X be a Hausdorff space and let $x_0 \in X$ be a point having a closed neighborhood N in X of which $\{x_0\}$ is a strong deformation retract. Let Y be a Hausdorff space and let $y_0 \in Y$. Define $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y$. Then the inclusion maps induce isomorphisms $\tilde{H}_i(X) \oplus \tilde{H}_i(Y) \cong \tilde{H}_i(X \vee Y)$ whose inverse is induced by the projections of $X \vee Y$ to X and Y.*

Proof of lemma. Consider $A = X$ and $U = X - N$ which is open, and $\overline{U} \subset A$. Then by excision, $H_*(X \vee Y, X) \cong H_*(N \cup Y, N) \cong \tilde{H}_*(Y)$

Consider the LES of the triple $(X \vee Y, \{x_0\} \times Y, \{x_0\} \times \{y_0\})$. We obtain

$$\ldots \to H_p(\{x_0\} \times Y, (x_0, y_0)) \xrightarrow{i_*} H_p(X \vee Y, (x_0, y_0)) \xrightarrow{j_*} H_p(X \vee Y, \{x_0\} \times Y) \to \ldots$$

Since $\pi_Y \circ i = \mathrm{id}_{\{x_0\} \times Y}$, i_* is injective.

Furthermore, we have

$$H_p(X \vee Y, (x_0, y_0)) \xrightarrow{(\pi_X)_*} H_p(\{x_0\} \times Y, (x_0, y_0)) \cong H_p(X \vee Y, \{x_0\} \times Y)$$

so $j_* = (\pi_X)_*$ under these identifications, so, in particular, j_* is surjective. Therefore, our exact sequence is a SES:

$$0 \to H_p(Y, pt) \xrightarrow{i_*} H_p(X \vee Y, pt) \xrightarrow{j_*} \underbrace{H_p(X \vee Y, Y)}_{\cong H_p(X, pt)} \to 0$$

It remains to show that this SES is split, but since $\pi_X \circ \iota_X = \mathrm{id}_{\{x_0\} \times X}$, we have that ι_{X*} provides a section.

□

Proof of proposition. This follows by induction on the lemma. □

Next, suppose that $E_1, \ldots, E_k$ are disjoint open subsets of S^n, each homeomorphic to $\mathbb{R}^n$ for $n > 0$. Let $f \colon S^n \to Y$ be a map which takes $S^n - \bigcup E_i$ to y_0. Then f factors through the quotient space $S^n / (S^n - \bigcup E_i) \cong S_1^n \vee \ldots \vee S_k^n$ where $S_i^n = S^n / (S^n - E_i)$:

$$f \colon S^n \xrightarrow{g} S_1^n \vee \ldots \vee S_k^n \xrightarrow{h} Y$$

Let $\iota_j \colon S_j^n \hookrightarrow S_1^n \vee \ldots \vee S_k^n$ be the j th inclusion and let $p_j \colon S_1^n \vee \ldots \vee S_k^n \to S_j^n$ be the j th projection. Then by the proposition, $\sum_j \iota_{j*} p_{j*} = \mathrm{id}_* \colon H_n(S_1^n \vee \ldots \vee S_k^n) \to H_n(S_1^n \vee \ldots \vee S_k^n)$. Let $g_j = p_j \circ g \colon S^n \to S_j^n$ and $h_j = h \circ \iota_j \colon S_j^n \to Y$ and let $f_j = h_j \circ g_j \colon S^n \to Y$. That is, f_j is the map which is f on E_j and maps the complement of E_j to the basepoint y_0.

Theorem 2.33. *In the above situation, $f_* = \sum_{j=1}^k f_{j*} \colon H_n(S^n) \to H_n(Y)$.*

Proof of theorem. We have $f_* = h_* \circ g_* = \sum_j h_* i_{j*} p_{j*} g_* = \sum_j h_{j*} g_{j*} = \sum_j f_{j*}$. □

Now we get back to showing that $\deg(f + g) = \deg f + \deg g$.

Note that by way of defining $f + g$, this essentially maps I^n by f on the left half and g on the right half with the boundary mapping to the base point x_0. In particular, this factors through the quotient $I^n \to I^n / \partial I^n \cong S^n$, where now the two halves can be interpreted as, say, the upper and lower hemispheres. In particular, the equator is by assumption also mapped to x_0, so we can quotient further by $S^n \to S^n \vee S^n$ by "pinching" the equator to a point. This is essentially what the proposition above describes. In particular, $f+g$ can be covered by the two open hemispheres and maps the equator to x_0, so by the theorem, we have $(f+g)_* = f_* + g_*$, i.e., $\deg(f+g) = (f+g)_*(1) = f_*(1) + g_*(1) = \deg f + \deg g$, as we wanted to show.

(3) Next we show that deg is surjective. First note that $\deg \mathrm{id} = \mathrm{id}_*(1) = 1$ by functoriality since $\mathrm{id}_* = \mathrm{id}_{H_n(S^n)}$. By functoriality, we thus hit all of $\mathbb{Z}$. More precisely, $\deg(*_n \mathrm{id}) = n$ for $n \in \mathbb{N}$ as deg is a homomorphism. Also $\deg(*_n(-\mathrm{id})) = -n$ for $n \in \mathbb{N}$ and $\deg(c_{x_0}) = 0$, so deg is surjective.

(4) Let $n \geq 2$. We have a SES

$$0 \to \ker \deg \to \pi_n(S^n) \overset{\deg}{\to} \mathbb{Z} \to 0.$$

Since $\mathbb{Z}$ is projective, this splits, so $\pi_n(S^n) \cong \mathbb{Z} \oplus \ker \deg$. But $\ker \deg$ is a subgroup of $\pi_n(S^n)$ which is abelian, hence is itself abelian.

$\square$

Problem 2.34. Fix $n \geq 1$. We say that a space X is *n-connected* if it is non-empty, path-connected, and $\pi_k(X,x) = 0$ for all $1 \leq k \leq n$ and $x \in X$. For (X, x_0) a pointed, path-connected space, show that the following are equivalent:

(1) X is n-connected.
(2) $\pi_k(X, x_0) = 0$ for all $1 \leq k \leq n$.
(3) Every map $S^k \to X$ can be extended to a map $D^{k+1} \to X$ for all $k \leq n$.
(4) Every map $S^k \to X$ is homotopic to a constant map for all $k \leq n$.

Proof. ($1 \implies 2$): this follows since X being n-connected means that $\pi_k(X,x) = 0$ for all $x \in X$ and all $1 \leq k \leq n$, hence in particular for x_0.
($2 \implies 3$) : Let $f \colon S^k \to X$ be a map. Then f represents some homotopy class $[f] \in \pi_k(X, x_0)$. But since $\pi_k(X, x_0) = 0$, f is homotopic to the constant map at $x_0 \operatorname{rel} s_0$. Let $H \colon S^k \times I \to X$ be this homotopy. Define $\tilde{f} \colon D^{k+1} \to X$ by $\tilde{f}(x) = H(x, \|x\|)$. Then $\tilde{f}$ is continuous as a composite of continuous maps and $\tilde{f}|_{S^k}(-) = H(-,1) = f(-)$, so $\tilde{f}$ indeed extends f.
($3 \implies 4$) : Let $f \colon S^k \to X$ be a map. Extends f to a map $\tilde{f} \colon D^{k+1} \to X$. Define now a homotopy $H \colon S^k \times I \to X$ by $H(x,t) = \tilde{f}(xt)$. This is continuous and $H(x,1) = \tilde{f}(x) = f(x)$ while $H(x,0) = \tilde{f}(0) \in X$ is constant. Hence this gives a homotopy between f and $c_{\tilde{f}(0)}$.

($4 \implies 3$) : Let $f \colon S^k \to X$ be a given map. By assumption, there exists a homotopy $H \colon S^k \times I \to X$ such that $H(-,1) = f(-)$ and $H(-,0) = c$ where c is some constant map at a point in X. But then H factors through the quotient

$$\begin{tikzcd} S^k \times I \arrow[d] \arrow[dr, "H"] & \\ D^{k+1} \arrow[r, "\tilde{H}"'] & X \end{tikzcd}$$

where we identify $S^k \times \{0\}$ to a point. But then $\tilde{H}|_{S^k}(-) = H(-,1) = f(-)$, so $\tilde{H}$ extends f.

($3 \implies 2$) : Let $[f] \in \pi_k(X, x_0)$ and f a representative. We want to show that f is homotopic to the constant map at x_0 relative ∂I^k. Extend f to a map $\tilde{f}\colon D^{k+1} \to X$, and let $H\colon S^k \times I \to X$ be given by $H(x,t) = \tilde{f}(ts_0 + (1-t)x)$. This gives a homotopy between f and the constant map at x_0.

$(2 \implies 1)$: the only thing that requires showing is that given that $\pi_k(X, x_0) = 0$ for all k, we then have $\pi_k(X, x) = 0$ for all k and all $x \in X$. But this is precisely what the given hint says we are allowed to assume since X is path connected. So we are done. $\square$

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