Notes Topics in Algebraic Geometry

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1 Toric Geometry

1.1 Monoids

Definition 1.1.1. A commutative monoid is a triple $(M, +, 0)$ with $+ : M^2 \to M$ and $0 \in M$ such that

- $+$ is associative,
- $+$ is commutative,
- 0 is an identity element.

Example 1.1.2. The following are examples of monoids:

- $(\mathbb{Z}_{>0}, \cdot, 1)$
- $(\mathbb{Z}_{\geq 0}, +, 0)$

Lemma 1.1.3. *Let R be a ring. Then the functor*

$$\mathbf{Alg}_R \to \mathbf{Mon} : R \mapsto (R, \cdot, 1)$$

has a left adjoint

$$R[-] : \mathbf{Alg}_R \leftarrow \mathbf{Mon} : M \mapsto R[M]$$

Proof. We construct $R[M]$ as a ring:

As an additive group it is the linearization $R[M]$ with multiplication given by

$$\sum_i x_i m_i \cdot \sum_j y_i n_i = \sum_{i,j} x_i y_j (m_i + y_j)$$

which is well-defined by finite support of exact sequences.

Example 1.1.4. For a ring R, we have natural isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Alg}_R}(R[\mathbb{N}], A) &\cong \mathrm{Hom}_{\mathbf{Mon}}(\mathbb{N}, (A, \cdot)) \\ &\cong A \\ &\cong \mathrm{Hom}_{\mathbf{Alg}_R}(R[x], A). \end{aligned}$$

Therefore, we conclude by the Yoneda lemma that $R[\mathbb{N}] \cong R[x]$.

1.2 Rational polyhedral cones

Definition 1.2.1. Let M be a free finitely generated abelian group with $v_1, \ldots, v_s \in M$. The cone generated by the v_i is

$$\left\{ \sum_i r_i v_i : r_i \in \mathbb{R}_{\geq 0} \right\} \subseteq M \otimes_{\mathbb{Z}} \mathbb{R} =: M_{\mathbb{R}}.$$

We identity M with the inclusion

$$M \hookrightarrow M_{\mathbb{R}} : m \mapsto m \otimes 1.$$

A subset $\sigma \subseteq M_{\mathbb{R}}$ is a rational polyhedron (RP) cone if $\exists s \in \mathbb{N}$ and $v_1, \ldots, v_s \in M$ such that σ is of this form.

Proposition 1.2.2 (Gordon's lemma)**.** *If $\sigma \in M_{\mathbb{R}}$ is an RP cone then $\sigma \cap M$ is a finitely generated monoid.*

The generators of σ don't necessarily generate $\sigma \cap M$!

1.2.1 Duality and faces

Definition 1.2.3. For any monoid we can define

$$M^\vee = \mathrm{Hom}(M, \mathbb{Z})$$

and

$$M_\mathbb{R}^\vee = \mathrm{Hom}(M_\mathbb{R}, \mathbb{R})$$

Given a cone $\sigma \subseteq M_\mathbb{R}$ we define

$$\sigma^\vee = \{n \in M_\mathbb{R}^\vee : \forall m \in \sigma, m \cdot n = n(m) \geq 0\}$$

Lemma 1.2.4. *The dual cone $\sigma^\vee$ is an RP cone.*

$(\sigma^\vee)^\vee = \sigma$.

Definition 1.2.5. We write $\langle v_1, \ldots, v_n \rangle \subseteq M_\mathbb{R}$ to be the cone generated by the v_i, concretely:

$$\langle v_1, \ldots, v_n \rangle = \left\{ \sum_i r_i v_i : \forall i, r_i \geq 0 \right\}.$$

A face of σ is a cone of the form $\sigma \cap \langle -\tau \rangle^\vee$ for any $\tau \in \sigma^\vee$.

Lemma 1.2.6. *A face of an RP cone is an RP cone.*

Remark 1.2.7. There is a nice trick for computing duals of cones. Let $M = \mathbb{Z}^n$ and $\sigma \subseteq M_\mathbb{R} \cong \mathbb{R}^n$ a cone generated by the elements $v_1, \ldots, v_n \in M$.

We have an isomorphism $M_\mathbb{R} \cong M_\mathbb{R}^\vee$ given by $v \mapsto \langle v, - \rangle$. Take any $v \in M$, there is then a line through the origin perpendicular to v. The dual $\langle v \rangle^\vee$ corresponds to the half plane H_v corresponding to that line in which v lies.

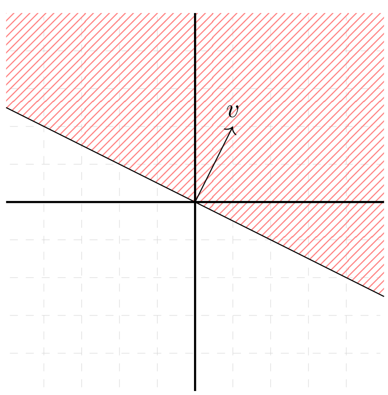

Figure 1: The dual of a cone $\langle v \rangle$.

For a general cone $\sigma = \langle v_1, \ldots, v_n \rangle$ one can then compute the dual as the intersection of the half planes:

$$\sigma^\vee = \bigcap_i H_{v_i}.$$

The right to left inclusion is easy to see: if $n \in \bigcap_i H_{v_i}$ then for all $(r_i) \in \mathbb{R}^n$

$$\begin{aligned} n \cdot \left(\sum_i r_i v_i \right) &= \sum_i r_i (n \cdot v_i) \\ &\geq 0 \end{aligned}$$

because $n \in H_{v_i}$ so $n \cdot v_i \geq 0$.

Conversely, suppose $n \in \sigma^\vee$. Then $n \cdot v_i \geq 0$ by assumption so $n \in \langle v_i \rangle^\vee = H_{v_i}$ for all i so n is also in the intersection.

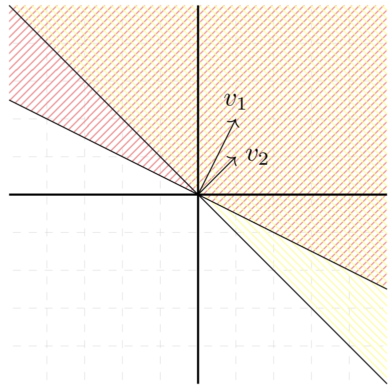

Figure 2: The dual of the cone $\langle v_1, v_2 \rangle$ is the doubly shaded region.

Definition 1.2.8. An RP cone $\sigma \subseteq M_\mathbb{R}$ is strictly convex (SRP cone) if

$$\sigma \cap -\sigma = 0$$

or equivalently σ does not contain a line through the origin.

1.2.2 Fans

Definition 1.2.9. A fan F in $M_\mathbb{R}$ is a set of SRP cones in $M_\mathbb{R}$ satisfying the following properties:

i. $\{0\} \subseteq F$,

ii. for all $\sigma \in F$ and faces τ of σ we have $\tau \in F$,

iii. for all $\sigma, \sigma' \in F$ we have that $\sigma \cap \sigma'$ is a face of σ.

If σ is an SRP cone then the set $\{\text{faces of } \sigma\}$ is a fan.

Example 1.2.10. Let $M = \mathbb{Z}^2$. Then the following is a fan:

$$F = \{\langle (1,0), (0,1) \rangle, \langle 1, 0 \rangle, \langle 0, 1 \rangle, \{0\}\}.$$

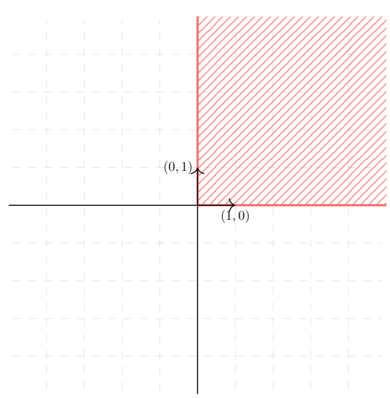

Figure 3: The fan F.

Example 1.2.11. Let $M = \mathbb{Z}$. Then

$$F = \{\sigma_2 = \langle 1 \rangle, \sigma_1 = \langle -1 \rangle, \sigma_0 = \{0\}\}$$

is a fan.

1.3 Toric schemes

We fix some ring $R \in \mathbf{CRing}$.

Definition 1.3.1. Let σ be an SRP cone. The affine toric scheme associated to σ is given by

$$X_\sigma = \mathrm{Spec}(R[\sigma^\vee \cap M^\vee]).$$

We often write $S_\sigma = \sigma^\vee \cap M^\vee$.

Remark 1.3.2. If $\sigma' \subseteq \sigma$ then $\sigma^\vee \subseteq (\sigma')^\vee$ inducing a map $S_\sigma \to S_{\sigma'}$ and therefore also a map $X_{\sigma'} \to X_\sigma$.

Example 1.3.3. If $M = \mathbb{Z}^m$ we can take

$$\sigma = R^a_{\geq 0} = \mathbb{R}^a_{\geq 0} \times \{0\}^{n-1} \subseteq M_\mathbb{R} = \mathbb{R}^n.$$

Then

$$X_\sigma = \mathbb{A}^a_R \times_R \mathbb{G}^{n-a}_{m,R}.$$

This is because $\sigma^\vee = \mathbb{R}^a_{\geq 0} \times \mathbb{R}^{n-a}$ so $S_\sigma = \mathbb{N}^a \times \mathbb{Z}^{n-a}$. We know that $R[\mathbb{N}] \cong R[x]$ and $R[\mathbb{Z}] \cong R[x, x^{-1}]$.

Exercise 1.3.4. The following are equivalent:

i. A cone σ is strictly convex,
ii. the linear span of $\sigma^\vee$ is $M^\vee_\mathbb{R}$,
iii. the map $X_{\{0\}} \to \mathrm{Spec}\, R[\sigma^\vee \cap M^\vee]$ is an open immersion[1].

Therefore, the affine schemes we get this way are those with a torus as a dense open subset.

Remark 1.3.5. The scheme $\mathrm{Spec}\, R[t, t^{-1}]$ is often called the torus.

Lemma 1.3.6 (Gluing schemes)**.** *Let I be an index set, X_i a scheme for all $i \in I$, for all i, j an open $U_{ij} \subseteq X_i$ such that $U_{ii} = X_i$ and $\varphi_{ij} : U_{ij} \cong U_{ji}$ and for all i, j, k we have $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$ and the diagram below commutes:*

$$\begin{tikzcd}
U_{ij} \cap U_{ik} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & U_{ki} \cap U_{kj} \\
& U_{ji} \cap U_{jk} \arrow[ur, "\varphi_{jk}"'] &
\end{tikzcd}$$

Then there is a scheme X and opens U_i such that $\varphi_i : X_i \cong U_i$ and

i $X = \bigcup_i U_i$,
ii $\varphi_i(U_{ij}) = U_i \cap U_j$,
iii $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

[1] We write $R[S_\sigma]$ because we only defined X_s if σ is indeed strictly convex.

Lemma 1.3.7. *Let σ be an RP cone and $(\sigma') = \sigma \cap \langle \ell \rangle^\vee$ a face. Then $\sigma^\vee \subseteq (\sigma')^\vee$ and the map*

$$R[S_\sigma] \to R[S_{\sigma'}]$$

is the localization at the multiplicative subset.

$$T = \langle \ell \rangle \cap M^\vee.$$

Proof. We prove this is the localization using the universal property:

$$\begin{aligned} \mathrm{Hom}_R(R[S_{\sigma'}], A) &\cong \mathrm{Hom}_{\mathbf{Mon}}(S_{\sigma'}, (A, \times)) \\ &\cong \left\{ f \in \mathrm{Hom}(S_\sigma, A) : f(\ell) \in A^\times \right\} \\ &\cong \left\{ f \in \mathrm{Hom}_R(R[S_\sigma, A]) : f(\ell) \in A^\times \right\} \\ &\cong \mathrm{Hom}_R(T^{-1} R[S_\sigma], A). \end{aligned}$$

Lemma 1.3.8. *Given a fan F we can define gluing data. We let the index set be F and define $X_\sigma = X_\sigma$ for all $\sigma \in F$. Given $\sigma, \sigma' \in F$ let $\tau = \sigma \cap \sigma'$ then define $U_{\sigma\sigma'} = X_\tau$ with isomorphism $\varphi_{\sigma\sigma'} = \mathrm{Id} : X_\tau \to X_\tau$.*

Proof. We need to verify $X_\tau \to X_\sigma$ is an open immersion and the triple overlap condition.

The overlap condition boils down to verifying the following: if τ, τ' are faces of σ then $\tau \cap \tau'$ is a face of σ and $X_{\tau \cap \tau'} = X_{\tau \cap \tau'} \subseteq X_\sigma$.

Let σ' be a face of σ. We show that $X_{\sigma'} \to X_\sigma$ is an open immersion. Let $\sigma' = \langle -\ell \rangle^\vee$ for some $\ell \in M^\vee$. Then $(\sigma')^\vee = \sigma^\vee + \langle -\ell \rangle$. The statement now follows from Lemma 1.3.7.

We leave the overlap condition to the reader.

Definition 1.3.9. Given a fan F we define X_F to be the result of the gluing data of Lemma 1.3.8. This can also be described as a colimit of affine schemes $F \to \mathbf{Sch}$: $\sigma \mapsto X_\sigma$ with the natural open immersions as morphisms.

Remark 1.3.10. Suppose F is a fan given by the faces of a single cone σ. Then $X_F = X_\sigma$.

Proof. In this case there is a natural inclusion $\tau \subseteq \sigma$ for all $\tau \in F$. Therefore, X_σ is a terminal object in the cocone of schemes given by $F \to \mathbf{Sch}$ which is then also the colimit of the diagram by general abstract nonsense.

Example 1.3.11. Consider the fan F from Example 1.2.11. Then the corresponding scheme is given by $\mathbb{P}^1_R$. Note that we have

$$\begin{aligned} X_{\sigma_0} &= \operatorname{Spec} R[\mathbb{Z}] = \operatorname{Spec} R[t, t^{-1}], \\ X_{\sigma_1} &= \operatorname{Spec} R[\mathbb{Z}_{\geq 0}] = \operatorname{Spec} R[t], \\ X_{\sigma_2} &= \operatorname{Spec} R[\mathbb{Z}_{\leq 0}] = \operatorname{Spec} R[t^{-1}]. \end{aligned}$$

There is a pushout diagram of schemes:

$$\begin{tikzcd} \operatorname{Spec} R[t, t^{-1}] \arrow[r] \arrow[d] & \operatorname{Spec} R[t] \arrow[d] \\ \operatorname{Spec} R[t^{-1}] \arrow[r] & \mathbb{P}^1_R \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

which gives that the gluing gives $\mathbb{P}^1_R$.

Lemma 1.3.12. *Suppose we have a finite number of finitely generated groups $(M_i)_{i\leq n}$ and SRP cones $\sigma_i \subseteq (M_i)_R$. Then the product $\prod_i \sigma_i \subseteq \prod_i M_i$ is corresponds to the product of schemes $X_{\sigma_1} \times X_{\sigma_2}$.*

Proof. We note that

$$\left(\prod_i \sigma_i\right)^\vee = \prod_i \sigma_i^\vee$$

so $S_{\prod_i \sigma_i} = \prod_i S_{\sigma_i}$.

We also have

$$\prod_i S_\sigma = \bigoplus_i S_\sigma$$

because we have finitely many M_i.

We show using Yoneda that $R[\prod_i S_{\sigma_i}] \cong \bigotimes_i R[S_{\sigma_i}]$:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Alg}_R}\left(R\left[\prod_i S_{\sigma_i}\right], A\right) &= \mathrm{Hom}_{\mathbf{Mon}}\left(\prod_i S_{\sigma_i}, (A, \times)\right) \\
&= \mathrm{Hom}_{\mathbf{Mon}}\left(\bigoplus_i S_{\sigma_i}, (A, \times)\right) \\
&= \prod_i \mathrm{Hom}_{\mathbf{Mon}}(S_{\sigma_i}, (A, \times)) \\
&= \prod_i \mathrm{Hom}_{\mathbf{Alg}_R}\left(R\left[\sigma_i\right], A\right) \\
&= \mathrm{Hom}_{\mathbf{Alg}_R}\left(\bigotimes_i R[\sigma_i], A\right).
\end{aligned}$$

This is because tensor products are coproducts in the category of commutative R-algebras. The right adjoint functor Spec : $\mathbf{CRing}^{op} \to \mathbf{Sch}$ preserves limits, so it sends the tensor product to the product of schemes.