Notes Topics in Algebraic Geometry

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Contents

1 Toric Geometry

1.1 Monoids

Definition 1.1.1. A commutative monoid is a triple $(M, +, 0)$ with $+ : M^2 \to M$ and $0 \in M$ such that

- $+$ is associative,
- $+$ is commutative,
- 0 is an identity element.

Example 1.1.2. The following are examples of monoids:

- $(\mathbb{Z}_{>0}, \cdot, 1)$
- $(\mathbb{Z}_{\geq 0}, +, 0)$

Lemma 1.1.3. *Let R be a ring. Then the functor*

$$\mathbf{Alg}_R \to \mathbf{Mon} : R \mapsto (R, \cdot, 1)$$

has a left adjoint

$$R[-] : \mathbf{Alg}_R \leftarrow \mathbf{Mon} : M \mapsto R[M]$$

Proof. We construct $R[M]$ as a ring:

As an additive group it is the linearization $R[M]$ with multiplication given by

$$\sum_i x_i m_i \cdot \sum_j y_i n_i = \sum_{i,j} x_i y_j (m_i + y_j)$$

which is well-defined by finite support of exact sequences.

Example 1.1.4. For a ring R, we have natural isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Alg}_R}(R[\mathbb{N}], A) &\cong \mathrm{Hom}_{\mathbf{Mon}}(\mathbb{N}, (A, \cdot)) \\ &\cong A \\ &\cong \mathrm{Hom}_{\mathbf{Alg}_R}(R[x], A). \end{aligned}$$

Therefore, we conclude by the Yoneda lemma that $R[\mathbb{N}] \cong R[x]$.

1.2 Rational polyhedral cones

Definition 1.2.1. Let M be a free finitely generated abelian group with $v_1, \ldots, v_s \in M$. The cone generated by the v_i is

$$\left\{ \sum_i r_i v_i : r_i \in \mathbb{R}_{\geq 0} \right\} \subseteq M \otimes_{\mathbb{Z}} \mathbb{R} =: M_{\mathbb{R}}.$$

We identity M with the inclusion

$$M \hookrightarrow M_{\mathbb{R}} : m \mapsto m \otimes 1.$$

A subset $\sigma \subseteq M_{\mathbb{R}}$ is a rational polyhedron (RP) cone if $\exists s \in \mathbb{N}$ and $v_1, \ldots, v_s \in M$ such that σ is of this form.

Proposition 1.2.2 (Gordon's lemma)**.** *If $\sigma \in M_{\mathbb{R}}$ is an RP cone then $\sigma \cap M$ is a finitely generated monoid.*

The generators of σ don't necessarily generate $\sigma \cap M$!

1.2.1 Duality and faces

Definition 1.2.3. For any monoid we can define

$$M^\vee = \mathrm{Hom}(M, \mathbb{Z})$$

and

$$M_\mathbb{R}^\vee = \mathrm{Hom}(M_\mathbb{R}, \mathbb{R})$$

Given a cone $\sigma \subseteq M_\mathbb{R}$ we define

$$\sigma^\vee = \{n \in M_\mathbb{R}^\vee : \forall m \in \sigma, m \cdot n = n(m) \geq 0\}$$

Lemma 1.2.4. *The dual cone $\sigma^\vee$ is an RP cone.*

$(\sigma^\vee)^\vee = \sigma.$

Definition 1.2.5. We write $\langle v_1, \ldots, v_n \rangle \subseteq M_\mathbb{R}$ to be the cone generated by the v_i, concretely:

$$\langle v_1, \ldots, v_n \rangle = \left\{ \sum_i r_i v_i : \forall i, r_i \geq 0 \right\}.$$

A face of σ is a cone of the form $\sigma \cap \langle -\tau \rangle^\vee$ for any $\tau \in \sigma^\vee$.

Lemma 1.2.6. *A face of an RP cone is an RP cone.*

Lemma 1.2.7. *Let σ be an RP cone and $(\sigma') = \sigma \cap \langle -\tau \rangle^\vee$ a face. Then $\sigma^\vee \subseteq (\sigma')^\vee$ and the map*

$$R[\sigma^\vee \cap N] \to R[(\sigma')^\vee \cap N]$$

is the localization at the multiplicative subset

$$S = \langle \tau, -\tau \rangle \cap N \subseteq M^\vee \cap N.$$

Remark 1.2.8. There is a nice trick for computing duals of cones. Let $M = \mathbb{Z}^n$ and $\sigma \subseteq M_\mathbb{R} \cong \mathbb{R}^n$ a cone generated by the elements $v_1, \ldots, v_n \in M$.

We have an isomorphism $M_\mathbb{R} \cong M_\mathbb{R}^\vee$ given by $v \mapsto \langle v, - \rangle$. Take any $v \in M$, there is then a line through the origin perpendicular to v. The dual $\langle v \rangle^\vee$ corresponds to the half plane H_v corresponding to that line in which v lies.

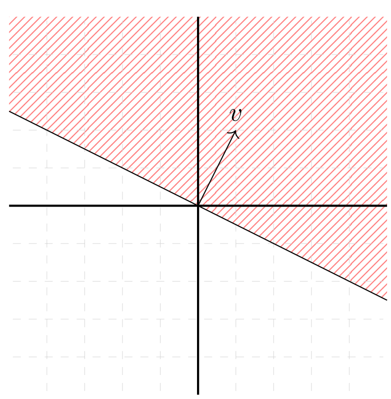

Figure 1: The dual of a cone $\langle v \rangle$.

For a general cone $\sigma = \langle v_1, \ldots, v_n \rangle$ one can then compute the dual as the intersection of the half planes:

$$\sigma^\vee = \bigcap_i H_{v_i}.$$

The right to left inclusion is easy to see: if $n \in \bigcap_i H_{v_i}$ then for all $(r_i) \in \mathbb{R}^n$

$$\begin{aligned} n \cdot \left(\sum_i r_i v_i \right) &= \sum_i r_i (n \cdot v_i) \\ &\geq 0 \end{aligned}$$

because $n \in H_{v_i}$ so $n \cdot v_i \geq 0$.

Conversely, suppose $n \in \sigma^\vee$. Then $n \cdot v_i \geq 0$ by assumption so $n \in \langle v_i \rangle^\vee = H_{v_i}$ for all i so n is also in the intersection.

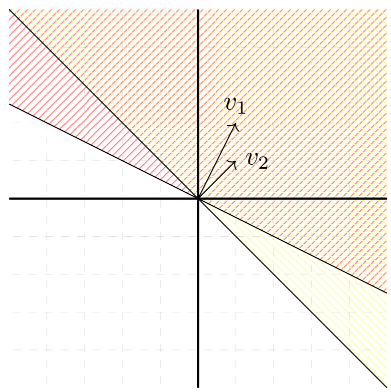

Figure 2: The dual of the cone $\langle v_1, v_2 \rangle$ is the doubly shaded region.