Notes Topics in Algebraic Geometry

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1 Toric Geometry

1.1 Monoids

Definition 1.1.1. A commutative monoid is a triple $(M, +, 0)$ with $+ : M^2 \to M$ and $0 \in M$ such that

- $+$ is associative,
- $+$ is commutative,
- 0 is an identity element.

Example 1.1.2. The following are examples of monoids:

- $(\mathbb{Z}_{>0}, \cdot, 1)$
- $(\mathbb{Z}_{\geq 0}, +, 0)$

Lemma 1.1.3. *Let R be a ring. Then the functor*

$$\mathbf{Alg}_R \to \mathbf{Mon} : R \mapsto (R, \cdot, 1)$$

has a left adjoint

$$R[-] : \mathbf{Alg}_R \leftarrow \mathbf{Mon} : M \mapsto R[M]$$

Proof. We construct $R[M]$ as a ring:

As an additive group it is the linearization $R[M]$ with multiplication given by

$$\sum_i x_i m_i \cdot \sum_j y_i n_i = \sum_{i,j} x_i y_j (m_i + y_j)$$

which is well-defined by finite support of exact sequences.

Example 1.1.4. For a ring R, we have natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Alg}_R}(R[\mathbb{N}], A) &\cong \mathrm{Hom}_{\mathbf{Mon}}(\mathbb{N}, (A, \cdot)) \\
&\cong A \\
&\cong \mathrm{Hom}_{\mathbf{Alg}_R}(R[x], A).
\end{aligned}$$

Therefore, we conclude by the Yoneda lemma that $R[\mathbb{N}] \cong R[x]$.

1.2 Rational polyhedral cones

Definition 1.2.1. Let M be a free finitely generated Abelian group with $v_1, \dots, v_s \in M$. The cone generated by the v_i is

$$\left\{ \sum_i r_i v_i : r_i \in \mathbb{R}_{\geq 0} \right\} \subseteq M \otimes_{\mathbb{Z}} \mathbb{R} =: M_{\mathbb{R}}.$$

We identify M with the inclusion

$$M \hookrightarrow M_{\mathbb{R}} : m \mapsto m \otimes 1.$$

A subset $\sigma \subseteq M_{\mathbb{R}}$ is a rational polyhedron (RP) cone if $\exists s \in \mathbb{N}$ and $v_1, \dots, v_s \in M$ such that σ is of this form.

Proposition 1.2.2 (Gordon's lemma)**.** *If $\sigma \in M_{\mathbb{R}}$ is an RP cone then $\sigma \cap M$ is a finitely generated monoid.*

The generators of σ don't necessarily generate $\sigma \cap M$!

1.2.1 Duality and faces

Definition 1.2.3. For any monoid we can define

$$M^\vee = \mathrm{Hom}(M, \mathbb{Z})$$

and

$$M_\mathbb{R}^\vee = \mathrm{Hom}(M_\mathbb{R}, \mathbb{R})$$

Given a cone $\sigma \subseteq M_\mathbb{R}$ we define

$$\sigma^\vee = \{n \in M_\mathbb{R}^\vee : \forall m \in \sigma, m \cdot n = n(m) \geq 0\}$$

Lemma 1.2.4. *The dual cone $\sigma^\vee$ is an RP cone.*

$(\sigma^\vee)^\vee = \sigma$.

Definition 1.2.5. We write $\langle v_1, \ldots, v_n \rangle \subseteq M_\mathbb{R}$ to be the cone generated by the v_i, concretely:

$$\langle v_1, \ldots, v_n \rangle = \left\{ \sum_i r_i v_i : \forall i, r_i \geq 0 \right\}.$$

A face of σ is a cone of the form $\sigma \cap \langle -\tau \rangle^\vee$ for any $\tau \in \sigma^\vee$.

Lemma 1.2.6. *A face of an RP cone is an RP cone.*

Remark 1.2.7. There is a nice trick for computing duals of cones. Let $M = \mathbb{Z}^n$ and $\sigma \subseteq M_\mathbb{R} \cong \mathbb{R}^n$ a cone generated by the elements $v_1, \ldots, v_n \in M$.

We have an isomorphism $M_\mathbb{R} \cong M_\mathbb{R}^\vee$ given by $v \mapsto \langle v, - \rangle$. Take any $v \in M$, there is then a line through the origin perpendicular to v. The dual $\langle v \rangle^\vee$ corresponds to the half plane H_v corresponding to that line in which v lies.

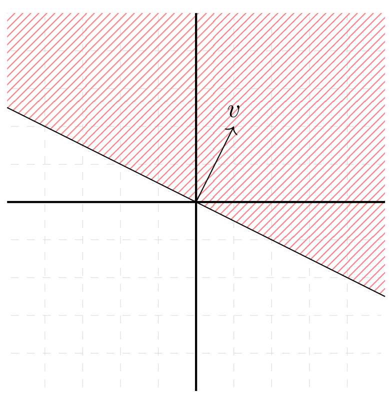

Figure 1: The dual of a cone $\langle v \rangle$.

For a general cone $\sigma = \langle v_1, \ldots, v_n \rangle$ one can then compute the dual as the intersection of the half planes:

$$\sigma^\vee = \bigcap_i H_{v_i}.$$

The right to left inclusion is easy to see: if $n \in \bigcap_i H_{v_i}$ then for all $(r_i) \in \mathbb{R}^n$

$$\begin{aligned} n \cdot \left(\sum_i r_i v_i \right) &= \sum_i r_i (n \cdot v_i) \\ &\geq 0 \end{aligned}$$

because $n \in H_{v_i}$ so $n \cdot v_i \geq 0$.

Conversely, suppose $n \in \sigma^\vee$. Then $n \cdot v_i \geq 0$ by assumption so $n \in \langle v_i \rangle^\vee = H_{v_i}$ for all i so n is also in the intersection.

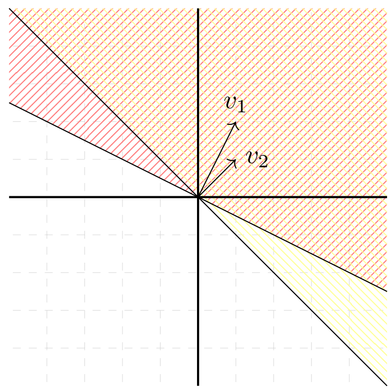

Figure 2: The dual of the cone $\langle v_1, v_2 \rangle$ is the doubly shaded region.

Definition 1.2.8. An RP cone $\sigma \subseteq M_{\mathbb{R}}$ is strictly convex (SRP cone) if

$$\sigma \cap -\sigma = 0$$

or equivalently σ does not contain a line through the origin.

1.2.2 Fans

Definition 1.2.9. A fan F in $M_{\mathbb{R}}$ is a set of SRP cones in $M_{\mathbb{R}}$ satisfying the following properties:

i. $\{0\} \subseteq F$,

ii. for all $\sigma \in F$ and faces τ of σ we have $\tau \in F$,

iii. for all $\sigma, \sigma' \in F$ we have that $\sigma \cap \sigma'$ is a face of σ.

If σ is an SRP cone then the set $\{\text{faces of } \sigma\}$ is a fan.

Example 1.2.10. Let $M = \mathbb{Z}^2$. Then the following is a fan:

$$F = \{\langle (1,0), (0,1) \rangle, \langle 1, 0 \rangle, \langle 0, 1 \rangle, \{0\}\}.$$

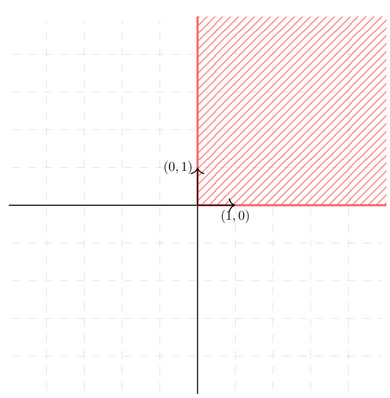

Figure 3: The fan F.

Example 1.2.11. Let $M = \mathbb{Z}$. Then

$$F = \{\sigma_2 = \langle 1 \rangle, \sigma_1 = \langle -1 \rangle, \sigma_0 = \{0\}\}$$

is a fan.

1.3 Toric schemes

We fix some ring $R \in \mathbf{CRing}$.

Definition 1.3.1. Let σ be an SRP cone. The affine toric scheme associated to σ is given by

$$X_\sigma = \operatorname{Spec}(R[\sigma^\vee \cap M^\vee]).$$

We often write $S_\sigma = \sigma^\vee \cap M^\vee$.

Remark 1.3.2. If $\sigma' \subseteq \sigma$ then $\sigma^\vee \subseteq (\sigma')^\vee$ inducing a map $S_\sigma \to S_{\sigma'}$ and therefore also a map $X_{\sigma'} \to X_\sigma$.

Example 1.3.3. If $M = \mathbb{Z}^m$ we can take

$$\sigma = R^a_{\geq 0} = \mathbb{R}^a_{\geq 0} \times \{0\}^{n-1} \subseteq M_\mathbb{R} = \mathbb{R}^n.$$

Then

$$X_\sigma = \mathbb{A}^a_R \times_R \mathbb{G}^{n-a}_{m,R}.$$

This is because $\sigma^\vee = \mathbb{R}^a_{\geq 0} \times \mathbb{R}^{n-a}$ so $S_\sigma = \mathbb{N}^a \times \mathbb{Z}^{n-a}$. We know that $R[\mathbb{N}] \cong R[x]$ and $R[\mathbb{Z}] \cong R[x, x^{-1}]$.

Exercise 1.3.4. The following are equivalent:

i. A cone σ is strictly convex,

ii. the linear span of $\sigma^\vee$ is $M^\vee_\mathbb{R}$,

iii. the map $X_{\{0\}} \to \operatorname{Spec} R[\sigma^\vee \cap M^\vee]$ is an open immersion[1].

Therefore, the affine schemes we get this way are those with a torus as a dense open subset.

Remark 1.3.5. The scheme $\operatorname{Spec} R[t, t^{-1}]$ is often called the torus.

Lemma 1.3.6 (Gluing schemes)**.** *Let I be an index set, X_i a scheme for all $i \in I$, for all i, j an open $U_{ij} \subseteq X_i$ such that $U_{ii} = X_i$ and $\varphi_{ij} : U_{ij} \cong U_{ji}$ and for all i, j, k we have $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$ and the diagram below commutes:*

$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & U_{ki} \cap U_{kj} \\ & U_{ji} \cap U_{jk} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

Then there is a scheme X and opens U_i such that $\varphi_i : X_i \cong U_i$ and

i $X = \bigcup_i U_i$,

ii $\varphi_i(U_{ij}) = U_i \cap U_j$,

iii $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

[1] We write $R[S_\sigma]$ because we only defined X_s if σ is indeed strictly convex.

Lemma 1.3.7. *Let σ be an RP cone and $(\sigma') = \sigma \cap \langle \ell \rangle^\vee$ a face. Then $\sigma^\vee \subseteq (\sigma')^\vee$ and the map*

$$R[S_\sigma] \to R[S_{\sigma'}]$$

is the localization at the multiplicative subset.

$$T = \langle \ell \rangle \cap M^\vee.$$

Proof. We prove this is the localization using the universal property:

$$\begin{aligned}
\operatorname{Hom}_R(R[S_{\sigma'}], A) &\cong \operatorname{Hom}_{\mathbf{Mon}}(S_{\sigma'}, (A, \times)) \\
&\cong \left\{ f \in \operatorname{Hom}(S_\sigma, A) : f(\ell) \in A^\times \right\} \\
&\cong \left\{ f \in \operatorname{Hom}_R(R[S_\sigma, A]) : f(\ell) \in A^\times \right\} \\
&\cong \operatorname{Hom}_R(T^{-1}R[S_\sigma], A).
\end{aligned}$$

Lemma 1.3.8. *Given a fan F we can define gluing data. We let the index set be F and define $X_\sigma = X_\sigma$ for all $\sigma \in F$. Given $\sigma, \sigma' \in F$ let $\tau = \sigma \cap \sigma'$ then define $U_{\sigma\sigma'} = X_\tau$ with isomorphism $\varphi_{\sigma\sigma'} = \mathrm{Id} : X_\tau \to X_\tau$.*

Proof. We need to verify $X_\tau \to X_\sigma$ is an open immersion and the triple overlap condition.

The overlap condition boils down to verifying the following: if τ, τ' are faces of σ then $\tau \cap \tau'$ is a face of σ and $X_{\tau\cap\tau'} = X_{\tau\cap\tau'} \subseteq X_\sigma$.

Let σ' be a face of σ. We show that $X_{\sigma'} \to X_\sigma$ is an open immersion. Let $\sigma' = \langle -\ell \rangle^\vee$ for some $\ell \in M^\vee$. Then $(\sigma')^\vee = \sigma^\vee + \langle -\ell \rangle$. The statement now follows from Lemma 1.3.7.

We leave the overlap condition to the reader.

Definition 1.3.9. Given a fan F we define X_F to be the result of the gluing data of Lemma 1.3.8. This can also be described as a colimit of affine schemes $F \to \mathbf{Sch}$: $\sigma \mapsto X_\sigma$ with the natural open immersions as morphisms.

Remark 1.3.10. Suppose F is a fan given by the faces of a single cone σ. Then $X_F = X_\sigma$.

Proof. In this case there is a natural inclusion $\tau \subseteq \sigma$ for all $\tau \in F$. Therefore, X_σ is a terminal object in the cocone of schemes given by $F \to \mathbf{Sch}$ which is then also the colimit of the diagram by general abstract nonsense.

Example 1.3.11. Consider the fan F from Example 1.2.11. Then the corresponding scheme is given by $\mathbb{P}^1_R$.

We first compute S_{σ_i} for $i = 0, 1, 2$.

For $\sigma_0 = \{0\}$ the dual cone is all of $M^\vee_\mathbb{R}$. Therefore, $S_{\sigma_0} = M^\vee_\mathbb{R} \cap M^\vee = M^\vee$.

For $\sigma_1 = \langle -1 \rangle$

Note that we have

$$\begin{aligned}
X_{\sigma_0} &= \operatorname{Spec} R[\mathbb{Z}] = \operatorname{Spec} R[t, t^{-1}], \\
X_{\sigma_1} &= \operatorname{Spec} R[\mathbb{Z}_{\geq 0}] = \operatorname{Spec} R[t], \\
X_{\sigma_2} &= \operatorname{Spec} R[\mathbb{Z}_{\leq 0}] = \operatorname{Spec} R[t^{-1}].
\end{aligned}$$

There is a pushout diagram of schemes:

$$\begin{tikzcd}
\operatorname{Spec} R[t, t^{-1}] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} R[t] \arrow[d] \\
\operatorname{Spec} R[t^{-1}] \arrow[r] & \mathbb{P}^1_R
\end{tikzcd}$$

which gives that the gluing gives $\mathbb{P}^1_R$.

Lemma 1.3.12. *Suppose we have a finite number of finitely generated groups $(M_i)_{i\leq n}$ and SRP cones $\sigma_i \subseteq (M_i)_R$. Then the product $\prod_i \sigma_i \subseteq \prod_i M_i$ is corresponds to the product of schemes $X_{\sigma_1} \times X_{\sigma_2}$.*

Proof. We note that

$$\left(\prod_i \sigma_i\right)^\vee = \prod_i \sigma_i^\vee$$

so $S_{\prod_i \sigma_i} = \prod_i S_{\sigma_i}$.

We also have

$$\prod_i S_\sigma = \bigoplus_i S_\sigma$$

because we have finitely many M_i.

We show using Yoneda that $R[\prod_i S_{\sigma_i}] \cong \bigotimes_i R[S_{\sigma_i}]$:

$$\begin{aligned}
\operatorname{Hom}_{\mathbf{Alg}_R}\left(R\left[\prod_i S_{\sigma_i}\right], A\right) &= \operatorname{Hom}_{\mathbf{Mon}}\left(\prod_i S_{\sigma_i}, (A, \times)\right) \\
&= \operatorname{Hom}_{\mathbf{Mon}}\left(\bigoplus_i S_{\sigma_i}, (A, \times)\right) \\
&= \prod_i \operatorname{Hom}_{\mathbf{Mon}}(S_{\sigma_i}, (A, \times)) \\
&= \prod_i \operatorname{Hom}_{\mathbf{Alg}_R}\left(R\left[\sigma_i\right], A\right) \\
&= \operatorname{Hom}_{\mathbf{Alg}_R}\left(\bigotimes_i R[\sigma_i], A\right).
\end{aligned}$$

This is because tensor products are coproducts in the category of commutative R-algebras. The right adjoint functor Spec : $\mathbf{CRing}^{op} \to \mathbf{Sch}$ preserves limits, so it sends the tensor product to the product of schemes.

1.4 Toric morphisms

Now we define morphisms of toric schemes to turn the function from fans to schemes into a functor.

Definition 1.4.1. Let $f : M \to M'$ be a morphism of monoids, $\sigma \subseteq M_{\mathbb{R}}, \sigma' \subseteq M'_{\mathbb{R}}$ and f^* the pullback map $\operatorname{Hom}(M'^{\vee}_{\mathbb{R}}, \mathbb{R}) \to \operatorname{Hom}(M^{\vee}_{\mathbb{R}}, \mathbb{R})$. Then if $f[\sigma] \subseteq \sigma'$ we get a map of monoids $S_{\sigma'} \to S_\sigma$ inducing a morphism of rings $R[S_{\sigma'}] \to R[S_\sigma]$ or equivalently a scheme morphism $X_\sigma \to X_{\sigma'}$.

Remark 1.4.2. The functor $R[-]$ is not full so not every morphism between these rings is induced by monoid morphisms. Consider a map $\mathbb{Z}[\mathbb{N}] = R[x] \to \mathbb{Z}[\mathbb{N}] = R[x]$ sending $x \mapsto x^2$. This is not induced by a monoid map $\mathbb{N} \to \mathbb{N}$.

Definition 1.4.3. Let F, F' be fans on M, M' respectively and $f : M \to M'$ be a monoid morphism such that for each $\sigma \in F$ there is a $\sigma' \in F'$ such that $f[\sigma] \subseteq \sigma'$. Then f is called compatible with F and F'.

Theorem 1.4.4. *Let $f : M \to M'$ be a morphism compatible with the fans F, F'. Then it induces a morphism $X_F \to X_{F'}$. Such a map is called a toric morphism.*

Proof. First we note that for any σ there is a smallest σ' which contains it. To see this note that F^{op} is directed: if $\tau, \tau' \in F'$ then $\tau \cap \tau' \in F'$. Now because F is finite this process terminates eventually. We write σ' for the smallest cone in F' containing $\sigma \in F$.

Given any $\sigma \in F$ we know there is a map $X_\sigma \to X_{\sigma'} \to X_{F'}$. If we show that this collection of maps forms a cocone then it induces a map $X_F \to X_{F'}$ by the universal property. To prove this, by functoriality it suffices to show that the diagram

$$\begin{tikzcd} S_\tau \arrow[r] \arrow[d] & S_{\tau'} \arrow[d] \\ S_\sigma \arrow[r] & S_{\sigma'} \end{tikzcd}$$

is a commutative diagram of monoids for any $\sigma \subseteq \tau \in F$. This is immediate because the vertical maps are inclusions and the horizontal map is the restriction of the pullback f^* on the ambient space.

Proposition 1.4.5. *The map $F \mapsto X_F$ extends to a functor from fans to schemes.*

Proof. We define the functor on morphisms of fans using the morphisms defined in Theorem 1.4.4.

This functor preserves the identity because for any fan map $F \to F$ the induced map $S_\sigma \to S_\sigma$ is the identity for all $\sigma \in F$ because pullbacks preserve identity.

Similarly, this is multiplicative because the pullback preserves composition so the pullback given by a composition $F \to F' \to F''$ is the composition of the individual pullbacks.

Example 1.4.6. Let $\pi : \mathbb{Z}^n \to \mathbb{Z}$ be the projection onto the first coordinate and F a fan with support contained in $\mathbb{R}^{\geq 0} \times \mathbb{R}^{n-1}$. Let F' be the fan $F' = \left\{\{0\}, \mathbb{R}^{\geq 0}\right\}$. Then π is compatible with the fans. We therefore get a map

$$X_F \to X_{F'} \cong \operatorname{Spec} R[x] = \mathbb{A}^1_{\mathbb{R}}.$$

In particular if we take F from Example 1.2.10 then we get a morphism of rings

$$R[x] \to R[x, y] : x \mapsto R[x, y]$$

which corresponds to the natural projection $\mathbb{A}^2_R \twoheadrightarrow \mathbb{A}^1_R$.

Example 1.4.7. Let $M = \mathbb{Z}^n$ and F a collection of cones. Then $f(x) = mx$ is compatible with F for fixed $m \in \mathbb{N}$. Any cone $\sigma \in F$ is closed under scaling with a non-negative number so $f[\sigma] \subseteq \sigma \in F$. This gives an endomorphism $X_F \to X_F$.

1.5 The torus action

Definition 1.5.1. Let M be a finitely generated Abelian group of rank n and F a fan on M. Writing $T = \operatorname{Spec} R[M^\vee]$ for the torus there is a map

$$X_F \times_R T \to X_F$$

locally given for each $\sigma \in F$ by a map $R[S_\sigma] \to R[S_\sigma] \otimes_R T$ sending $x \mapsto x \otimes x$ where we embed $R[S_\sigma] \subseteq T$ by the natural inclusion $S_\sigma \to M^\vee$.

Proof. We show that this map exists. Notice that $(X_\sigma \times_R T)_{\sigma \in F}$ is an open cover of the fibre product. Therefore, if the maps are compatible on this affine cover it defines a scheme morphism.

Let $\tau \subseteq \sigma$ be cones in F. To show compatibility of the cone map means showing commutativity of the following diagram:

$$\begin{tikzcd} R[S_\sigma] \arrow[r] \arrow[d] & R[S_\sigma] \otimes T \arrow[d] \\ R[S_\tau] \arrow[r] & R[S_\tau] \otimes T \end{tikzcd}$$

This is trivial because the map into the tensor product is a natural transformation between functors and the vertical maps are both the natural localization map.

Theorem 1.5.2. *Let $M \to M'$ be a map of monoids such that F, F' are compatible with it. Then the torus action is compatible with the induced $X_F \to X_{F'}$. Concretely the diagram*

$$\begin{tikzcd} X_F \times T_M \arrow[r] \arrow[d] & X_{F'} \times T_{M'} \arrow[d] \\ X_F \arrow[r] & X_{F'} \end{tikzcd}$$

commutes.

Proof. It is sufficient the diagram commutes locally. Concretely we must have that the following commutes:

$$\begin{tikzcd} R[S_\sigma] \arrow[r] \arrow[d] & R[S_{\sigma'}] \arrow[d] \\ R[S_\sigma] \otimes R[M] \arrow[r] & R[S_{\sigma'}] \otimes R[M'] \end{tikzcd}$$

This is clearly true.

Remark 1.5.3. Given a toric scheme X_F over R. The toric action gives an action $X_F(R) \times T(R) \to X_F(R)$ of the torus rational points on the rational points of X_F which is compatible with toric morphisms.

1.6 Flat morphisms

Definition 1.6.1. Let A, B be two commutative rings and $f : A \to B$ a ring morphism. Then f is called flat if B is a flat A-module by this map.

Example 1.6.2. The localization map $R \to S^{-1}R$ is flat for any ring R and multiplicatively closed subset $S \subseteq R$.

1.7 Proper morphisms

Definition 1.7.1. We quickly define a couple properties of morphisms. Let $f : X \to Y$ be a morphism of schemes

i. f is called separated if the diagonal map $X \mapsto X \times_Y X$ is a closed immersion,

ii. f is called of finite type if for all opens $V \subseteq Y$ the map $\mathcal{O}_Y(V) \to f_*\mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}[V])$ turns $f_*\mathcal{O}_X(V)$ into a finitely generated $\mathcal{O}_Y(V)$ algebra,

iii. f is called universally closed when any pullback along $Z \to Y$ has that $X \times_Y Z \to X$ is a closed embedding of topological spaces,

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

iv. A morphism is proper if it has all three of the above properties.

Example 1.7.2. Most "natural" morphisms of schemes are separated and of finite type. However, not all nice maps are universally closed. The standard example is the map $\mathbb{A}^1 \to \mathbb{Z}$. Pulling back along $\mathbb{A}^1 \to \mathbb{Z}$ gives the projection map $\mathbb{A}^2 \to \mathbb{A}^1$ which is not closed because the closed $Z(xy-1) \subseteq \mathbb{A}^2$ is mapped onto the open $D(x) \subseteq \mathbb{A}^1$.

Definition 1.7.3. We write $|F| = \bigcup_{\sigma \in F} \sigma \subseteq M_{\mathbb{R}}$ and call it the support of a fan.

We are going to work towards the following theorem:

Theorem 1.7.4. *Let M, M' be finitely generated Abelian groups, F, F' fans in the respective lattices and $f : M \to M'$ a morphism compatible with the fans. Then the induced $f : X_F \to X_{F'}$ of toric schemes is proper if and only if $f^{-1}[|F'|] = F$.*

Example 1.7.5. Consider the fan from Example 1.2.10 and the following fan

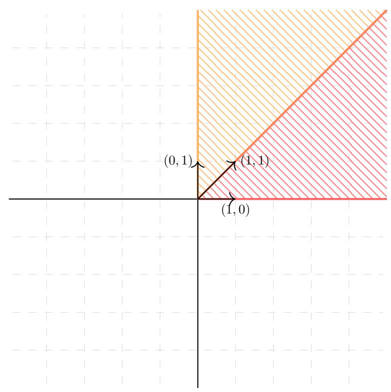

Figure 4: A "refinement" F'.

The identity on $\mathbb{Z}^2$ is compatible with $F' \to F$. The only points of $M_{\mathbb{R}}$ mapped into $|F|$ are those already in $|F'|$. Therefore, this defines a proper map of toric schemes (later we will see this corresponds to a blow-up).

Example 1.7.6. A non-example we have already seen is sending the fan $\{\langle 1 \rangle\} \subseteq \mathbb{R}$ to the zero fan in 0. It corresponds to the structure map $\mathbb{A}^1_R \to R$ which is not proper because $-1 \notin \pi^{-1}[0]$.

Definition 1.7.7. Let R be a domain. A discrete valuation is a map $v : R \to \mathbb{N} \cup \{\infty\}$ with the following properties:

i $v(x) = \infty$ if and only if $x = 0$,

ii $v(xy) = v(x) + v(y)$,

iii $v(x+y) \geq \min(v(x), v(y))$.

It extends to the fraction field K with a function $K \to \mathbb{Z} \cup \{\infty\}$ by $v({}^{x}\!/_{y}) = v(x) - v(y)$.

We call R a discrete valuation ring (DVR) if we can recover $R = \{x \in K : v(x) \geq 0\}$.

Example 1.7.8. Let k be a field. Then the ring $k[x]_{(x)}$ is a valuation ring with valuation

$$v(f) = \max\left\{i \in \mathbb{N} : x^i \mid f\right\}.$$

It has fraction field $k(x)$ the fraction field of $k[x]$.

Theorem 1.7.9 (Valuative criterion of properness)**.** *Let $X \to Y$ be a morphism of Noetherian schemes*[2] *Then it is proper if and only if for all discrete valuation rings R the following commutative square has a unique lift:*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

Now we can start the proof of Theorem 1.7.4. We will prove both implications in separate lemmas.

Lemma 1.7.10. *Let M, M' be finitely generated Abelian groups, F, F' fans in the respective lattices and $f : M \to M'$ a morphism compatible with the fans. If the induced $f : X_F \to X_{F'}$ of toric schemes over R is proper then $f^{-1}[|F'|] = F$.*

Proof. Suppose $X_F \to X_{F'}$ is proper, but there is a $w \in M \setminus |F|$ that has $f(w) \in |F'|$. We want to arrive at a contradiction: find some square such that Theorem 1.7.9 has no lift. To do this we fix some $\sigma' \in F'$ with $f(w) \in \sigma'$.

We want to find some field k and a map $R \to k$. This is purely to construct a DVR for the valuative criterion, the choice does not matter. We fix a maximal ideal $\mathfrak{m} \triangleleft R$ and set $k = R/\mathfrak{m}$.

We construct a commutative square by giving maps $\operatorname{Spec} k(x) \to X_F$ and $\operatorname{Spec} k[x]_{(x)} \to X_{F'}$:

$$\begin{aligned} \lambda_w : \operatorname{Spec} k(x) &\to \operatorname{Spec} R[M^\vee] \subseteq X_F \\ k(x) &\leftarrow R[M^\vee] \\ x^{u(w)} &\leftarrow\!\shortmid u \in M^\vee, \\ \lambda_{f(w)} \operatorname{Spec} k[x]_{(x)} &\to \operatorname{Spec} X_\sigma \subseteq X_{F'} \\ k[x]_{(x)} &\leftarrow R[\sigma^\vee \cap M^\vee] \\ x^{u(f(w))} &\leftarrow\!\shortmid u \in \sigma^\vee \cap M^\vee. \end{aligned}$$

This gives a commutative square with lift by assumption of properness. We track in which affine open X_σ the point $(x) \triangleleft \mathbb{Q}[x]_{(x)}$ lands, and obtain a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k(x) \arrow[r, "\lambda_w"] \arrow[d] & X_\sigma \arrow[d] \\ \operatorname{Spec} k[x]_{(x)} \arrow[r, "\lambda_{f(w)}"] \arrow[ur, dashed, "\exists!"] & X_{F'} \end{tikzcd}$$

Note that $w \notin \sigma$. This means that there is some $u \in \sigma^\vee \cap M^\vee$ with $u(w) < 0$. We obtain a commutative triangle of rings

$$\begin{tikzcd} k(x) & R[\sigma^\vee \cap M^\vee] \arrow[l] \arrow[dl] \\ k[x]_{(x)} \arrow[u] & \end{tikzcd}$$

but u is mapped to $x^{u(w)} \in k(x)$ which has $v(x^{u(w)}) = u(w) < 0$ so u cannot be sent into $k[x]_{(x)}$. This is a contradiction so no such w can exist, proving the lemma.

[2]There is a more general version of this for non-Noetherian schemes with all valuation rings. We need DVR's because we can construct maps to $\mathbb{Z}$ from them which live in dual spaces.

Now we prove the opposite implication, first over $\mathbb{Z}$ and extend our results using results about base changes of proper maps. We do this to work with Noetherian schemes for which the weakened valuative criterion is sufficient.

Lemma 1.7.11. *Let M, M' be finitely generated Abelian groups, F, F' fans in the respective lattices and $f : M \to M'$ a morphism compatible with the fans. If $f^{-1}[|F'|] = F$ then the induced $f : X_F \to X_{F'}$ of toric schemes over $\mathbb{Z}$ is proper.*

Proof. We show that the induced map $X_F \to X_{F'}$ is proper using the valuative criterion of Noetherian schemes. Toric schemes over $\mathbb{Z}$ are Noetherian because affine toric schemes correspond to finitely generated $\mathbb{Z}$-algebras and therefore any toric scheme over $\mathbb{Z}$ can be covered by finitely many affine Noetherian schemes.

Let (R, v) be a DVR with fraction field K such that we have a commutative square. We want to find a lift and show its uniqueness.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X_F \arrow[d, "f_*"] \\ \operatorname{Spec} R \arrow[r] & X_{F'} \end{tikzcd}$$

We can assume that the top map $\operatorname{Spec} K \to X_F$ factors through the torus because X_F is irreducible[3,4].

Consider any $\sigma' \in F'$ be such that $\operatorname{Spec} R \to X_F$ factors through $X_{\sigma'}$. Then we get a commutative square of rings

$$\begin{tikzcd} K & \mathbb{Z}[M^\vee] \arrow[l, "\alpha"'] \\ R \arrow[u] & \mathbb{Z}[\sigma'^\vee \cap M'^\vee] \arrow[l] \arrow[u, "f^*"'] \end{tikzcd}$$

Because $M^\vee$ is a group, the map $M^\vee \to K$ takes values in the units of K. This means that we obtain a map $v \circ \alpha \circ f^* : \sigma'^\vee \cap M'^\vee \to \mathbb{Z}$, it factors through R where all elements have non-negative valuation: we have a map $v \circ \alpha \circ f^* : \sigma'^\vee \cap M'^\vee \to \mathbb{N}$. This corresponds to some point in $\sigma' \cap M'$ which we will denote $v \circ \alpha \circ f^*$. It is in the image of f: it is given by $f(v \circ \alpha)$. By the assumption on f this means that there is a cone $\sigma \in F$ with $v \circ \alpha \in \sigma$. Now we fix a particular cone $\sigma' \in F'$ such that $f[\sigma] \subseteq \sigma'$ to obtain a commutative diagram

$$\begin{tikzcd} K & \mathbb{Z}[M^\vee] \arrow[l, "\alpha"'] & \mathbb{Z}[\sigma^\vee \cap M^\vee] \arrow[l] \\ R \arrow[u] & \mathbb{Z}[\sigma'^\vee \cap M'^\vee] \arrow[l] \arrow[u, "f^*"'] \arrow[ur] & \end{tikzcd}$$

Because $v \circ \alpha$ is a point in σ the map $v \circ \alpha : \sigma^\vee \cap M^\vee \to \mathbb{Z}$ takes values in $\mathbb{N}$ and therefore gifts a lift of the diagram. This shows the existence part of the valuative criterion.

Now we show uniqueness. If we show that $X_F \to X_{F'}$ is separated, we get the uniqueness of such lifts for free by the valuative criterion of separatedness. Any toric scheme over $\mathbb{Z}$ is separated[5]. We now make use of the following properties of separated morphisms: if $X \to Y \to Z$ is separated then $X \to Y$ is separated. This gives immediately that the map is separated and thus uniqueness of the lift.

We conclude that $X_F \to X_{F'}$ satisfies the valuative criterion for properness and is therefore proper.

Now we can prove the remaining implication Theorem 1.7.4 for arbitrary rings using base change.

[3]The toric scheme X_F is irreducible because $\mathbb{Z}[M^\vee]$ is a dense irreducible open.

[4]This is not an easy thing to see, and the proof involves some complicated scheme theory. We refer to `https://mathoverflow.net/questions/68648/valuation-criterion-of-properness-irreducible-varieties`.

[5]This is true for a toric scheme over any ring. We will not show this but refer to any standard text e.g. Theorem 3.1.5 of Toric Varieties by Cox, Little, and Schenck.

Corollary 1.7.12. *Let M, M' be finitely generated Abelian groups, F, F' fans in the respective lattices and $f : M \to M'$ a morphism compatible with the fans. If $f^{-1}[|F'|] = F$ then the induced $f : X_F \to X_{F'}$ of toric schemes over any ring R is proper.*

Proof. Let R be any ring. We will write X_F for the toric scheme over $\mathbb{Z}$ and X_F^R for the scheme over R.

Suppose we satisfy all hypotheses, then $X_F \to X_{F'}$ is a proper map. We show that $X_F^R \to X_{F'}^R$ is a proper map by exhibiting it as a base change. Proper maps are preserved by base change, so the resulting map is also proper.

We can give X_F^R is given by a pullback diagram

$$\begin{tikzcd}
X_F^R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d] \\
X_F \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

Now we can construct the morphism we are interested in as a pullback:

$$\begin{tikzcd}
X_F^R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{F'}^R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d] \\
X_F \arrow[r] & X_{F'} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

This concludes the proof.