Game Theory 2019-2020: Lecture Notes

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1 Introduction & Perfect information

Game theory is about strategically interdependent decision-making. In such situations the result of a decision also depends on the decisions of others. When making a decision people have to think about what others will do, who in turn are thinking about what others do and so on. Game theory offers several concepts and insights for understanding and analysing such situations and for making better strategic decisions

Definition 1.1 (Game Theory)**.** Game Theory studies mathematical models of strategic interaction (actions) among rational decision-makers (players).

Definition 1.2 (Normative Game Theory)**.** Normative Game Theory is about improving real-world results of games.

Definition 1.3 (Postive Game Theory)**.** Positive Game Theory is describing how people behave in real-world situations.

Definition 1.4 ((Non-)cooperative Game Theory)**.** A game is cooperative if the players can form binding commitments externally enforced. Cooperative game theory focuses on predicting which coalitions will form, the actions that groups take and the resulting payoffs.

Non-cooperative game theory , also know as competitive game theory, studies games with competition between individual players.

Definition 1.5 (Action)**.** A set A consisting of all the actions a that, under some circumstances, are available to the decision-maker.

Definition 1.6 (Rational choice)**.** An action chosen by a decision-maker is at least as good, according to her preferences, like every other available action.

Remark 1.7. Within game Theory, we assume people can rank any two actions. However, this is not always the case. We are not always 100% rational.

Definition 1.8 (Payoff function)**.** The payoff function $u : A \to \mathbb{R}$ represents a decision-maker's preferences if, for any actions $a, b \in A$

$$u(a) > u(b) \iff \text{the decision-maker prefers a to b}$$

Definition 1.9 (Strategic game)**.** A strategic game - or normal form game - consists of three things:

- a set of players N,
- for each player a set of actions A,
- for each player, (ordinal) preferences over the set of action profiles $u_i : A \to \mathbb{R}$ for all $i \in N$.

Example 1.10 (Prisoner's Dilemma)**.** This game has two players. Each player has two actions: Fink or Quiet. The prisoners' dilemma has the action profiles with corresponding payoff:

- If Player 1 and Player 2 each betray the other, each of them serves two years in prison.
- If Player 1 betrays Player 2 but Player 2 remains silent, Player 1 will be set free and Player 2 will serve three years in prison (and vice versa)
- If Player 1 and Player 2 both remain silent, both of them will serve only one year in prison.

This can be visualized in the following payoff matrix:

	Quiet	Fink
Quiet	$2, 2$	$0, 3$
Fink	$3, 0$	$1, 1$

Table 1: Payoff matrix corresponding to the Prisoner's Dilemma.

Theorem 1.11. *Any function applied to the utility which does not change the oridinality*[1]*, will not change the game.*

Example 1.12 (Battle of the sexes)**.** This game has two players: a wife and a husband. Each player has two actions: Bach or Stravinsky. The husband would prefer to go to Bach. The wife would rather go to Stravinsky. Both would prefer to go to the same place rather than different ones. This can be visualized in the following payoff matrix:

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

Table 2: Payoff matrix corresponding to the Battle of the sexes.

Example 1.13 (Stag hunt)**.** This game has two players: two identical hunters. Each hunter can individually choose to hunt a stag or hunt a hare. If an individual hunts a stag, they must have the cooperation of their partner to succeed. An individual can get a hare by himself, but a hare is worth less than a stag. This can be visualized in the following payoff matrix:

	Stag	Hare
Stag	$2, 2$	$0, 1$
Hare	$1, 0$	$1, 1$

Table 3: Payoff matrix corresponding to Stag Hunt.

Definition 1.14 (Nash equilibrium)**.** A Nash equilibrium is an action profile with the property that no player can do better given that other players adhere to their actions. *Formally:* the action profile a^* in a strategic game with ordinal preferences is a Nash equilibrium if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i, a^*_{-i}) in which i chooses a_i while every other player j chooses a^*_j.

Or, for every player i

$$u_i(a^*) > u_i(a_i, a^*_{-i})$$

for every action a_i of player i where u_i represents player i's preferences.

[1] if $x > y$, then $f(x) > f(y)$

Definition 1.15 (Best Response)**.** The Best Response for player i is the set of player i's best actions (denoted by $\mathcal{B}_i(a_{-i})$) when the list of the other players' actions is a_{-i} *Formally*:

$$\mathcal{B}_i(a_{-i}) = \{a_i \mid a_i \in A_i : u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \forall a_i' \in A_i\}$$

Corollary 1.16. *In a Nash equilibrium every player plays best response to the other players' actions. Formally:*

$$a^* \textit{ is Nash equilibrium} \iff a_i^* \in \mathcal{B}_i(a_{-i}^*) \forall i$$

Definition 1.17 (Strictly dominating actions)**.** We say a player i's action a_i'' strictly dominates action a_i' if

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$$

for every list of other players' actions a_{-i}.

This means that a player's action strictly dominates another action if it is superior, no matter what the other players do.

Definition 1.18 (Strictly dominated actions)**.** If an action strictly dominates the action a_i, we say that a_i is strictly dominated.

Corollary 1.19. A strictly dominated action is not used in any Nash equilibrium.

Definition 1.20 (Weakly Dominated actions)**.** We say a player i's action a_i'' strictly dominates action a_i' if

$$u_i(a_i'', a_{-i}) \geq u_i(a_i', a_{-i})$$

for every list of other players' actions a_{-i}, and

$$u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$$

for some list a_{-i} of other players' actions.

Remark 1.21. A Weakly dominated action may be used in a Nash equilibrium.

Example 1.22. Consider the following payoff matrix:

	Left	Right
Left	$2, 2$	$0, 3$
Right	$3, 0$	$1, 1$

Table 4: Payoff matrix corresponding to the Prisoner's Dilemma.

Row Player's action *Bottom* is weakly dominated by *Top* Column Player's action *Right* is weakly dominated by *Left* Nevertheless, there are two Nash equilibria: (*Top*, *Left*) and (*Bottom*, *Right*).

Illustration 1.23 (Contributing to a public good)**.** When building some public good (like street lights), there is a cost to that (c_i) and a value function ($v_i(c)$). Furthermore, each player has a maximum (w_i) they can contribute. This gives the following game:

- Players: two people.
- Actions: $0 < c_i < w_i$.

- Preferences: $u_i(c_1, c_2) = v_i(c_1 + c_2) - c_i$ for $i = 1, 2$.

Consider the perspective of player 1. We assume $0 < \mathcal{B}_1(0) < w_1$ where $\mathcal{B}_1(0)$ is the best response when the other player contributes 0. Suppose that $u_1(c_1, 0)$ has an interior maximum thus $b_1(0)$ is player 1's best response to $c_2 = 0$; Now consider $\mathcal{B}_1(k)$, 1's best response to $c_2 = k$:

$$u_1(c_1 + k, 0) = v_1(c_1 + k) - c_1 - k;$$
$$u_1(c_1, k) = v_1(c_1 + k) - c_1;$$

Now it follows, by adding k to both sides:

$$u_1(c_1 + k, 0) + k = v_1(c_1 + k) - c_1;$$
$$u_1(c_1, k) = v_1(c_1 + k) - c_1;$$

Therefore,

$$u_1(c_1, k) = u_1(c_1 + k, 0) + k;$$

This means: for me to be indifferent to spend c_1 euros and player 2 spending k euros or spending c_1 euros, player 2 spending 0 and receiving k euros.

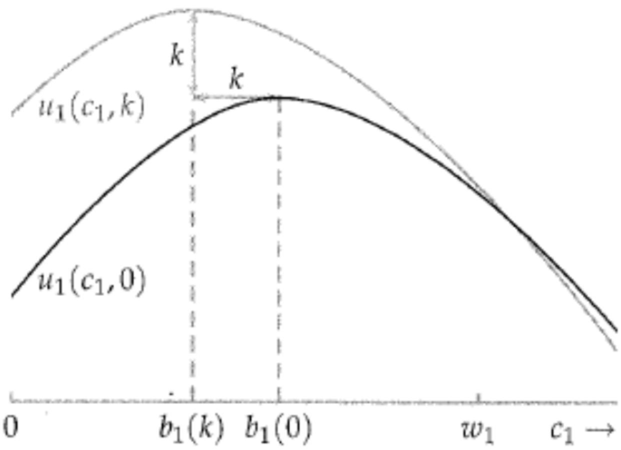


Figure 43.1 The relation between player 1's best responses $b_1(0)$ and $b_1(k)$ to $c_2 = 0$ and $c_2 = k$ in the game of contributing to a public good.

We now can say: if $k < b_1(0)$ then $b_1(k) = b_1(0) - k$; if $k > b_1(0)$ then $b_1(k) = 0$; When doing a similar analysis applies to player 2: assume $b_1(0) > b_2(0)$. Now we can say: if $k < b_2(0)$ then $b_2(k) = b_2(0) - k$; if $k > b_2(0)$ then $b_2(k) = 0$;

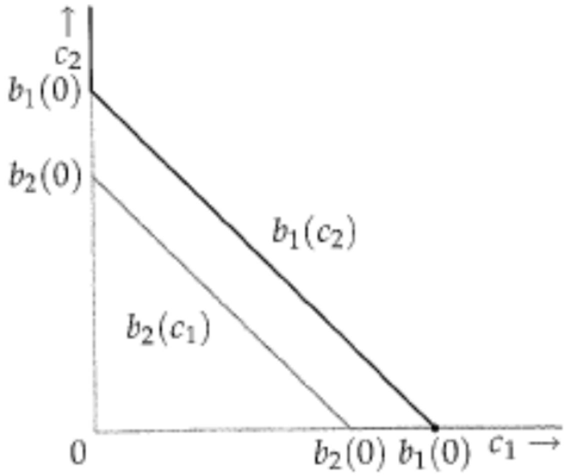


Figure 44.1 The best response functions for the game of contributing to a public good in a case in which $b_1(0) > b_2(0)$. The best response function of player 1 is the black line; that of player 2 is the gray line.

Since each player chooses best response, from the Graph it follows: $(b_1(0), 0)$ is the Nash equilibrium if $b_1(0) > b_2(0)$. When $b_1(0) = b_2(0)$, the Nash equilibrium are all $(c_1, c_2) : c_1 + c_2 = b_1(0) = b_2(0), c_1, c_2 \geq 0$. Lastly, when $b_1(0) < b_2(0)$,the Nash equilibrium is $(0, b_2(0))$.

Definition 1.24 (Symmetric games)**.** A strategic game with ordinal preferences is a **symmetric game** if the players' sets of actions are the same and the players' preferences are represented by u_i with $u_i(ai, a_{-i}) = u_j(a_j, a_{-j})$ for all i, j.

Definition 1.25 (Symmetric Nash equilibrium)**.** An action profile a^* in a symmetric game is a **symmetric Nash equilibrium** if it is a Nash equilibrium and a_i^* is the same for each player i.

2 Mixed strategy equilibrium

Definition 2.1 (Mixed strategy Nash equilibrium)**.** The mixed strategy profile α^* is a **mixed strategy Nash equilibrium** of a strategic game with vNM preferences if for each player i and every mixed strategy α_i

$$U_i(\alpha^*) \geq U_i\left(\alpha^*_{-i}, \alpha_i\right)$$

holds.

Example 2.2 (Matching pennies)**.** In the matching pennies game, each of two players flip a penny to heads or tails. The players then reveal their choices simultaneously. We have the following table corresponding to the payoff. Assume player 1 plays Head with

	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

Table 5: Payoff matrix for the Matching Pennis game

probability p (thus tail with probability $1-p$) and player 2 plays Head with probability q (thus tail with probability $1-q$). Then:

$$\mathbb{E}_1[\pi(\text{head})] = q + (1-q)(-1) = 2q - 1$$

$$\mathbb{E}_1[\pi(\text{tail})] = q(-1) + (1-q) = 1 - 2q$$

Now, we can easily find the best response function for player one if player 2 is allowed to mix.

$$\mathcal{B}_1(q) = \begin{cases} \{0\} & \text{if } q < \frac{1}{2} \\ \{p : 0 \leq p \leq 1\} & \text{if } q = \frac{1}{2} \\ \{1\} & \text{if } q > \frac{1}{2} \end{cases}$$

$$\mathcal{B}_2(p) = \begin{cases} \{1\} \text{ if } p < \dfrac{1}{2} \\ \{q : 0 \leq q \leq 1\} \text{ if } p = \dfrac{1}{2} \\ \{0\} \text{ if } p > \dfrac{1}{2} \end{cases}$$

When drawing these two functions, there is an intersection. This intersection is the Nash equilibrium. We can say:

$$a^* : p = \frac{1}{2}; q = \frac{1}{2}$$

Proposition 2.3. *A mixed strategy profile α^* in a strategic game with vNM preferences in which each player has finitely many actions is a mixed strategy Nash equilibrium*
$\iff$
for each player i

- *the expected payoff, given α^*_{-i}, to every action to which α^*_i assigns positive probability is the same;*
- *the expected payoff, given α^*_{-i}, to every action to which α^*_i assigns zero probability is smaller than or equal to the expected payoff to any action to which α^*_i assigns positive probability.*

Remark 2.4. *In simple terms, this means you are indifferent about the actions you are using and the actions that you are not using cannot give you a higher expected utility. This means you are choosing your best response. If every player chooses their best response, it is a Nash equilibrium.*

Corollary 2.5. *Each player's expected payoff is in an equilibrium is her expected payoff to any action that she uses with positive probability.*

Proof. Given a set of actions with positive probability for a player. If any action deviates from the other, either

- this action is a better option, and she should drop the other actions in this set.
- this action is a worse option, and she should drop this action from the set.

By induction, we can conclude that all expected payoffs are equal. □

Proposition 2.6. *Every strategic game with vNM preferences in which each player has finitely many actions has a mixed strategy equilibrium.*

Remark 2.7. *Note the use of 'finite' here. There might be, but might not always be, a mixed strategy equilibrium in the case for infinite games. For example, the game where you choose the highest number. Here, there are infinite action with always a better option (choosing a higher number).*

Definition 2.8 (Dominated actions in strategic games with vNM preferences)**.** Player i's mixed strategy α_i strictly dominates action α if

$$u_i(\alpha_i'', \alpha_{-i}) > u_i(\alpha_i', \alpha_{-i})$$

for every list a of other players' actions.

Example 2.9. Consider the following game;: T_1 does not look like a good option. Claim:

	T_1	T_2	T_3
S_1	$2,2$	$0,3$	$1,2$
S_2	$3,1$	$1,0$	$0,2$

for player 2 the action profile $\alpha_2 = (0, q, 1-1)$ is strictly better than than T_1. For this, there need to be

- For S_1: $3 \cdot q + 2(1-q) > 2 \iff q > 0$
- For S_2: $0 \cdot q + 2(1-q) > 1 \iff q < \frac{1}{2}$

Now we can conclude our claim is true for $0 < q < \frac{1}{2}$. Thus T_1 is stricly dominated by α_2 for some action profile (for example $(0, \frac{1}{3}, \frac{2}{3})$), and therefor not used in any mixed strategy Nash equilibrium. We can see in general when player 1 has profile $\alpha_1 = (0, \lambda, 1-\lambda)$, that

$$U(T_1) = 2 \cdot \lambda + 1 - \lambda = \lambda + 1 < \frac{7}{3}\lambda + \frac{4}{3}(1-\lambda) = \lambda + \frac{4}{3}\lambda = U(\alpha_2)s$$

Illustration 2.10 (Volunteers dilema)**.** We have the following game:

- Players: n people witnessing a crime
- Each player i chooses from {call, don't call};
- Preferences: vNM preferences
  - $U[\text{nobody calls}] = 0$
  - $U[i \text{ calls}] = v - c$
  - $U[i \text{ doesn't call, at least one other does}] = v$

This means calling costs c and solving the crime gives v.
Claim 1: this game has Nash equilibrium. One person calling is enough. Say one person calls, that person doesn't want to not call ($U[\text{nobody calls}] < U[i \text{ calls}]$). For the other persons, they don't want to call ($U[i \text{ calls}] < U[i \text{ doesn't call, at least one other does}]$). This means that the action profile where one player calls is a Nash equilibrium. This also means there are n Nash equilibria: every player i can call.

Claim 2: there also is a symmetric Nash equilibrium (a Nash equilibrium where every player uses the same mixed strategy). This means every player is calls with the same probability p. We must be indifferent between these two actions (see Corollary 2.3).

$$\begin{aligned}
EU[\text{call}] &= EU[\text{not calling}] \\
v - c &= 0 \cdot P[\text{no one else calls}] + v \cdot P[\text{at least one player calls}] \\
v - c &= v \cdot P[\text{at least one player calls}] \\
v - c &= v(1 - P[\text{no one player calls}]) \\
v - c &= v - v \cdot P[\text{no one else calls}]) \\
-c &= -v \cdot P[\text{no one else calls}]
\end{aligned}$$

$$\begin{aligned}
P[\text{no one else calls}] &= \frac{c}{v} \\
(1-p)^{n-1} &= \frac{c}{v} \\
(1-p) &= (\frac{c}{v})^{\frac{1}{n-1}} \\
p &= 1 - (\frac{c}{v})^{\frac{1}{n-1}}
\end{aligned}$$

Claim 3: when n increases, p decreases

3 Extensive games with perfect information

Definition 3.1. Definition of extensive game with perfect information

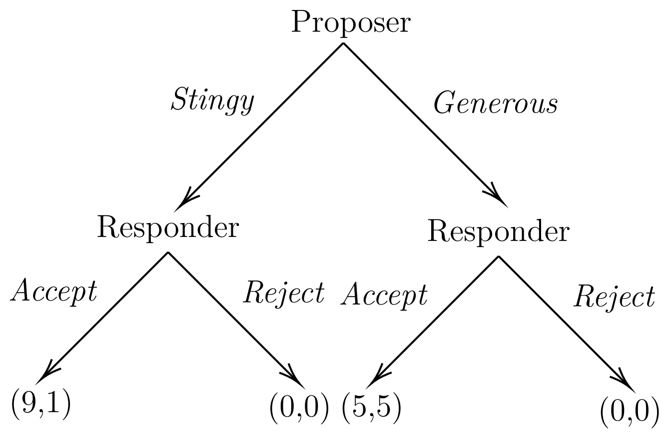
- set of players
- set of terminal histories
- player function
- for each player, preferences terminal histories

Example 3.2 (Mini-ultimatum game Proposer)**.** We have two players: proposer and responder. Proposer stats with a choice: *String* or *Generous*. Then, the responder has the choice *Accept* or *Reject*. More formally, the specification of mini-ultimatum game is as follows:

- Players: proposer and responder
- Terminal histories: (Stingy, Accept); (Stingy, Reject) (Generous, Accept); (Generous, Reject)
- Player function:
  - $P(\varnothing)$ = Proposer (Start of the game)
  - $P(\text{Stingy})$ = Responder
  - $P(\text{Generous})$ = Responder
- Preferences:

$$
\begin{aligned}
U_2(\text{Stingy}, \text{Accept}) &= 1\\
U_1(\text{Stingy}, \text{Accept}) &= 9\\
U_2(\text{Stingy}, \text{Reject}) &= 0\\
U_1(\text{Stingy}, \text{Reject}) &= 0\\
U_2(\text{Generous}, \text{Accept}) &= 5\\
U_1(\text{Generous}, \text{Accept}) &= 5\\
U_2(\text{Generous}, \text{Reject}) &= 0\\
U_1(\text{Generous}, \text{Reject}) &= 0
\end{aligned}
$$

This equal to the following tree:



Drawing a graph might be more intuitive, but not every game is discrete. For continuos games, working with formulas is easier (or sometimes the only possibility).

Definition 3.3 (Strategy)**.** A **strategy** of a player in an extensive game is a complete action plan. That is, it specifies an action for every situation where it is her turn to make a move.

Example 3.4 (Strategies in the mini-ultimatum game)**.** In the mini-ultimatum game, the responder has 4 strategies

1. A A = Accept after Stingy, Accept after Generous
2. A R = Accept after Stingy, Reject after Generous
3. R A = Reject after Stingy, Accept after Generous
4. R R = Reject after Stingy, Reject after Generous

This extensive form game can be written in strategic form: pure strategy Nash equi-

	A A	A R	R A	R R
Stingy	(9,1)	(9,1)	(0,0)	(0,0)
Generous	(5,5)	(0,0)	(5,5)	(0,0)

libria:

1. (Stingy, A A)
2. (Stingy, A R)
3. (Generous, R A)

Definition 3.5 (Nash equilibrium in an extensive game)**.** The strategy profile s^* in an extensive game with perfect information is a Nash equilibrium if, for every player i and every strategy r_i of player i:

$$U_i(s^*) > U_i(r_i, s^*_{-i})$$

Definition 3.6 (Subgame)**.** Let Γ be an extensive game with perfect information. For any non-terminal history h of Γ, the subgame $\Gamma(h)$ following the history h is the following extensive game.

- Players: players in Γ
- Terminal histories: set of sequences h' such that (h, h') is a terminal history of Γ
- Player function: $P(h, h')$ is assigned to each subhistory h' of a terminal history
- Preferences: each player prefers h' to h'' if and only if she prefers (h, h') to (h, h'')

Example 3.7 (Subgames of the mini-ultimatum game Proposer)**.** The mini-ultimatum game has three subgames:

1. The whole game
2. The game after Stingy
3. The game after Generous

Definition 3.8 (Subgame prefect equilibrium)**.** A subgame perfect equilibrium is a strategy profile s^* with the property that in no subgame can any player i do better by choosing a strategy different from s_i^*, given that every other player j adheres to strategy s_j^*.

Corollary 3.9. *From this definition, it follows:*

- *every subgame perfect equilibrium is a Nash equilibrium*
- *a subgame perfect equilibrium induces a Nash equilibrium in every subgame*
- *not every Nash equilibrium is subgame perfect*

Example 3.10 (Subgame prefect equilibrium of the mini-ultimatum game)**.** The mini-ultimatum game has three subgames:

1. (Stingy, A A) is a N.E. and subgame perfect
2. (Stingy, A R) is a N.E. but not subgame perfect
3. (Generous, R A) is a N.E. but not subgame perfect, this N.E. uses *incredible threat*[2]

Example 3.11 (Twist to the mini-ultimatum game)**.** Let us define the utility function by $U_i(x, y) = x - \frac{1}{4}|x - y|$. Now, our table is

	A A	A R	R A	R R
Stingy	(7,-1)	(7,-1)	(0,0)	(0,0)
Generous	(5,5)	(0,0)	(5,5)	(0,0)

Now it is the case that unique subgame perfect equilibrium [Generous, Reject Accept].

[2]The threat of choosing *reject* when choosing *stingy*, the *proposer* wants to choose *generous*. However, if the *proposer* still chooses *stingy*, the *responder* will still *accept*

Illustration 3.12 (Stackelberg's model of duopoly)**.** Cournot setting where firms move sequentially

$$P_d(Q) = \begin{cases} (\alpha - Q) & \text{if } Q \leq \alpha \\ 0 & \text{if } Q > \alpha \end{cases}$$

$$C_i(q_i) = \mathrm{cq}_i$$

- Players: 2 firms
- Terminal histories: set of all quantity pairs $(q_1, q_2), q_1 \geq 0, q_2 \geq 0$ ·
- Player function: $P(\emptyset) = 1$ and $P(q_1) = 2$ for all q_1 ·
- Preferences: $\pi_i = q_i P_d(q_1 + q_2) - c_i(q_i) = \begin{cases} q_i(\alpha - c - q_1 - q_2) & \text{if } Q \leq \alpha \\ -cq_i & \text{if } Q > \alpha \end{cases}$

To find the Subgame perfect equilibrium, we use backward induction:
First Order Condition: if $Q \leq \alpha$, set $\frac{d\pi_2}{dq_2} = 0$ (+ check Second Order Condition)
This implies $\frac{d\pi_2}{dq_2} q_2(\alpha - c - q_1 - q_2) = \alpha - c - q_1 - q_2 - q_2 = 0$, and thus $q_2 = \frac{1}{2}(\alpha - c - q_1)$.
Thus

$$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c - q_1) & \text{if } q_1 \leq \alpha - c \\ 0 & \text{if } q_1 > \alpha - c \end{cases}$$

This was the smallest subgame. Now, solve a bigger subgame. This is the whole game. Of course, player 1 anticipates player 2's response and maximizes

$$\pi_1 = q_1(\alpha - c - q_1 - (\alpha - c - q_1)/2) = q_1(\alpha - c - q_1)/2$$

Again, set $\frac{d\pi_1}{dq_1} = 0$ and calculate the derivative:

$$0 = \frac{d\pi_1}{dq_1} q_1(\alpha - c - q_1 - q_2) = (\alpha - c - q_1)/2 - q_1/2$$

Subgame perfect outcome	Payoff	Subgame perfect equilibrium
$q_1^* = \frac{1}{2}(\alpha - c)$	$\pi_1^* = \frac{1}{8}(\alpha - c)^2$	$q_1^* = \frac{1}{2}(\alpha - c)$
$q_2^* = \frac{1}{4}(\alpha - c)$	$\pi_2^* = \frac{1}{16}(\alpha - c)^2$	$b_2(q_1) = \begin{cases} \frac{1}{2}(\alpha - c - q_1) & \text{if } q_1 \leq \alpha - c \\ 0 & \text{if } q_1 > \alpha - c \end{cases}$

Now, firm 1 is more aggressive and earns higher profit than in Cournot duopoly $[\frac{1}{9}(\alpha - c)^2]$

4 Bayesian games

Definition 4.1 (Bayseian Game)**.** A **Bayesian game** is a game in which players have incomplete information about the other players. A player may not know the exact payoff functions of the other players, but instead have beliefs about these payoff functions. These beliefs are represented by a probability distribution over the possible payoff functions.

Example 4.2 (Battle of Sexes)**.** Let us start with an example using the Battle of the Sexes game. We assume:

- player 1 does not know player 2's preference
- player 2 knows player 1's preference

This means there are two cases: *player 2 wants to meet* or *player 2 does not want to meet.* The case of *player 2 wants to meet* is the standard Battle of the Sexes game. Player 2 will have payoff 0 if they meet, and keep her preference for S over B.

	B	S
B	(2,1)	(0,0)
S	(0,0)	(1,2)

Table 6: Situation where player 2 wants to meet

	B	S
B	(2,0)	(0,2)
S	(0,1)	(1,0)

Table 7: Situation where player 2 does not want to meet

Example 4.3 (Nash Equiliberum in Battle of Sexes)**.** Let us give both situations a chance: $p_{m(eet)} = p_{a(void)} = 0.5$. Only player 2 can choose, since she knows player 1's preference. However, player 1 does not know about player 2's preference. We can still make table for the player 1's expected payoff. Here (A, B) is choice A when not meeting and B when meeting.

	(B,B)	(B,S)	(S,B)	(S,S)
B	2	1	1	0
S	0	0.5	0.5	1

Table 8: Expected payoffs for player 1.

From this table, we can read that the best response for player 1 is B. When player 1 chooses B, player two would want to pick B (payoff 1) with $p_m = 0.5$ and S (payoff 2) with $p_a = 0.5$. This means $\mathcal{B}_2(B) = (B, S)$. This means $(B, (B, S)$ is a Nash Equilibrium of this game.

When player 1 chooses S, player two would want to pick S (payoff 2) with $p_m = 0.5$ and B (payoff 1) with $p_a = 0.5$. However, player 1 would prefer B. This means $(S, (S, B)$ is not a Nash Equilibrium of this game. In formulae: $\mathcal{B}_2(S) = (S, B)$ but $\mathcal{B}_1(S, B) = B \implies$ player 1 will not choose S.

Definition 4.4 (Formal Bayesian Game)**.** A **Bayesian Game** has

- players: a set N

- states: a set ω
- actions: a set A
- signals: a function $\tau : \omega \to S$ where S is a belief of states of the world.
- Bernoulli payoff function over (a, ω)

Example 4.5 (Battle of Sexes as a Bayesian Game)**.** We can give the Battle of Sexes as a Bayesian Game as follows:

- players $N = \{\text{player 1, player 2}\}$
- states $\omega = \{$ meet, avoid $\}$
- actions $a = \{B, S\}$ for each player
- signals
  - (uninformed) player 1: $\tau(\text{meet}) = \tau(\text{avoid}) = z$
  - (informed) player 2: $\tau(\text{meet}) = m; \tau(\text{avoid}) = v$
- given signal, belief about states
  - player 1 assigns prob. 1/2 to either state after z
  - player 2 assigns prob. 1 to meet after m
  - player 2 assigns prob. 1 to avoid after v
- Bernoulli payoff function over (a, ω), best viewed in Table 8

Example 4.6 (Battle of Sexes with both players uncertain)**.** This game is like the example above, but now player 1 is also uncertain. Let y be 'wants to meet' and n be 'does not want to meet'. Let ab be player 1 wants action a and player 2 wants action b. Players only know their own state. We give y_i and y_i to players $i \in \{1, 2\}$

- player 1's belief: player 2 wants to meet with probability 1/2
- player 2's belief: player 1 wants to meet with probability 2/3
- player 1 receives signals y_1 in yy or yn, n_1 in ny or nn
- player 2 receives signals y_2 in yy or ny, n_2 in yn or nn

This gives:

- Type y_1 believes $pr[yy] = pr[yn] = 1/2$
- Type y_2 believes $pr[yy] = 2/3$; $pr[ny] = 1/3$

Claim: ((B,B), (B,S)) N.E.; ((S,B), (S,S)) N.E.

Proof. We first proof two lemmas.

Lemma 1. $\mathcal{B}_1(B, S) = (B, B)$
We have two cases y_1 and n_1, which both have two cases B and S.

(i) type y_1 of player 1

- $\pi_{y_1}(B,(B,S)) = (1/2)\cdot 2 + (1/2)\cdot 0 = 1$
- $\pi_{y_1}(S,(B,S)) = (1/2)\cdot 0 + (1/2)\cdot 1 = 1/2$

(ii) type n_1 of player 1

- $\pi_{n_1}(B,(B,S)) = (1/2)\cdot 0 + (1/2)\cdot 2 = 1$
- $\pi_{n_1}(S,(B,S)) = (1/2)\cdot 1 + (1/2)\cdot 0 = 1/2$

Lemma 2. $\mathcal{B}_2(B,B) = (B,S)$
We have two cases y_2 and n_2, which both have two cases B and S.

(i) type y_2 of player 2

- $\pi_{y_2}((B,B),B) = (2/3)\cdot 1 + (1/3)\cdot 1 = 1$
- $\pi_{y_2}((B,B),S) = (2/3)\cdot 0 + (1/3)\cdot 0 = 0$

(ii) type n_2 of player 2

- $\pi_{n_2}((B,B),B) = (2/3)\cdot 0 + (1/3)\cdot 0 = 0$
- $\pi_{n_2}((B,B),S) = (2/3)\cdot 2 + (1/3)\cdot 2 = 2$

By Lemma 1, 2 we can conclude $((B,B),(B,S))$ is a Nash equilibrium. The proof for $((S,B),(S,S))$ is similar and left as an exercise for the reader. $\square$

Example 4.7. We can give the Battle of Sexes with two uncertain players as a Bayesian Game as follows:

- players $N = \{$ player 1, player 2$\}$
- states $\omega = \{$ yy,yn,ny,nn$\}$
- actions $a = \{B, S\}$ for each player
- signals
  - player 1:
    - $\tau_1(yy) = \tau_1(yn) = y_1$
    - $\tau_1(ny) = \tau_1(nn) = n_1$
  - player 2:
    - $\tau_2(nn) = \tau_2(yn) = n_2$
    - $\tau_2(ny) = \tau_2(yy) = y_2$
- given signal, belief about states
  - player 1 assigns prob. 1/2 to each state yy and yn after $y1$
  - player 1 assigns prob. 1/2 to each state ny and nn after $n1$
  - player 2 assigns prob. 2/3 to yy and 1/3 to ny after $y2$
  - player 2 assigns prob. 2/3 to yn and 1/3 to nn after $n2$

,

- Bernoulli payoff function over (a, ω), best viewed in a table from the (old) lecture notes.

Definition 4.8 (Nash equilibrium of a Bayesian game)**.** A **Nash equilibrium of a Bayesian game** is a Nash equilibrium of the strategic game (with vNM preferences) defined as follows:

- *Players*: set of all pairs (i, t_i) in which i is the player and t_i is one of the signals that i may receive
- *Actions*: set of actions of each player (i, t_i) is the set of actions of i in Bayesian game
- *Preferences*: a Bernoulli payoff function of (i, t_i) such that

$$\pi_i[a_i, t_i] = \sum_{\omega \varepsilon \Omega} \Pr(\omega \mid t_i)\, u_i \left[\left\{a_i, a^+_{-i}(\omega)\right\}, \omega\right]$$

where

$$a^+(\omega) = a_j(\tau_j(\omega))$$

Example 4.9 (Public good II)**.** Public good is provided if and only if at least one player contributes (at known cost c) Each individual is privately informed about v_i , draw from $F(v)$, $0 < v_min < c < v_max$

Example 4.10 (Bayesian game for Public good II)**.** The Public good II can be modelled as:

- Players: n individuals
- States: set of all profiles $(v_1, \ldots, v_n)$
- Actions: each player chooses 0 or c
- Signals: $_i(v_1, \ldots, v_n) = v_i$
- Beliefs: player i assigns probability $F(v_1)F(v_2)\ldots F(v_{i-1})F(v_{i+1})\ldots F(v_n)$ to the event that the valuation of every other player j is at most v_j
- Preferences of player i:
  - $\pi_i = 0$ if no one contributes
  - $\pi_i = v_i$ if i does not contribute and at least one other does
  - $\pi_i = v_i - c$ if i contributes

Example 4.11 (Symmetric Nash Equilibrium of Public good II)**.** An example of a symmetric action profile is: 0 for all i. This is symmetric, but not a Nash Equilibrium. I would rather spend c to receive $v_i - c$. Another such option is c for all i. But then, deviating to 0 will increase my payoff from $v_i - c$ to v_i. Thus, this is also not a Nash Equilibrium. A candidate for such a Nash Equilibrium is: c if $c \geq v^*$.

If a player i draws v^*, she is indifferent about playing c or 0. To prove this, we look at the expected utility. This is

$$\mathbb{E}_i(\{c\} \mid v_i, s_{-1}) = v_i - c$$

and

$$\begin{aligned}\mathbb{E}_i(\{0\} \mid v_i, s_{-1}) &= 0 \cdot \mathbb{P}(\text{others do not contribute}) + v_i \cdot \mathbb{P}(\text{at least one other contributes}) \\ &= 0 + v_i(1 - \mathbb{P}(\text{others do not contribute})) \qquad (\text{Note: } F(v^*) = \mathbb{P}(v < v^*) \\ &= v_i(1 - F(v^*)^{n-1})\end{aligned}$$

Player i which $v_i = v^*$ will be indifferent if $\mathbb{E}_i(\{c\} \mid v_i, s_{-1}) = \mathbb{E}_i(\{0\} \mid v_i, s_{-1})$. This gives

$$\begin{aligned}v^* - c &= v^*(1 - F(v^*)^{n-1}) \\ v^* - c &= v^* - v^*F(v^*)^{n-1}) \\ c &= v^*F(v^*)^{n-1})\end{aligned}$$

If $v_i < v^*$, then $v_iF(v^*)^{n-1} < v^*F(v^*)^{n-1} = c$. Then $-v_iF(v^*)^{n-1} > -c$. Now $v_i - v_iF(v^*)^{n-1} > v_i - c$ which is equal to $\mathbb{E}_i(\{0\} \mid v_i, s_{-1}) > \mathbb{E}_i(\{c\} \mid v_i, s_{-1})$. Thus, if $v_i < v^*$ then i would want to not contribute.

If $v_i > v^*$, then $v_iF(v^*)^{n-1} > v^*F(v^*)^{n-1} = c$. Then $-v_iF(v^*)^{n-1} < -c$. Now $v_i - v_iF(v^*)^{n-1} < v_i - c$ which is equal to $\mathbb{E}_i(\{0\} \mid v_i, s_{-1}) < \mathbb{E}_i(\{c\} \mid v_i, s_{-1})$. Thus, if $v_i > v^*$ then i would want to contribute.

Now, we have identified the strategy 'c if $v_i > v^*$' as an equilibrium.

Remark 4.1. *In this example, the equilibrium is not in the best interest of the public good.*

5 Extensive games with imperfect information

Definition 5.1 (Inforamtion Set)**.** An **information set** is a collection of histories with the the property that the player who is going to move, doesn't know which history has been reached given that one of these histories was reached.

Definition 5.2 (information Partition)**.** An **information partition** is a collection of information sets.

Example 5.3. Assume that after player i moves $H_i = \{C, D, E\}$

P1: $\{C\}, \{D, E\}$: player knows whether C occurred or not

P2: $\{C\}, \{D\}, \{E\}$: player knows whether C, D or E occurred

P3: $\{C, D, E\}$: player does not know whether C, D or E occurred

Remark 5.1. *Any strategic game can be presented as an extensive game with imperfect information.*

Example 5.4 (Battle of Sexes)**.** In the Battle of Sexes, player 2 does not know wether she is on the left or right side of the tree. This is denoted by the red line with '2'. In any information set, it must be the case that the player is choosing between the same actions. If this is not the case, the property (unknown history) from an information set is not longer true.

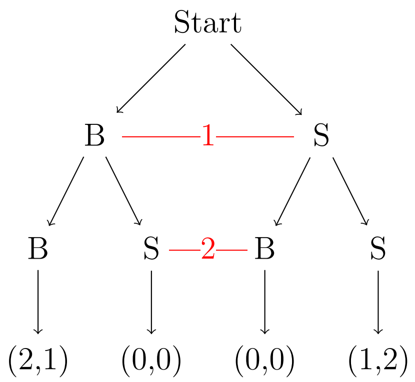


Illustration 5.5 (Definition extensive game/application to BoS)**.** Formally, a extensive game has the following form:

- Players
  - Player 1
  - Player 2
- Terminal histories
  - (B,B)
  - (B,S)
  - (S,B)
  - (S,S)
- Player function
  - $P(\varnothing) = 1$
  - $P(B) = 2$

  - $P(S) = 2$
- Chance moves: none
- Information partitions:
  - Player 1: $\varnothing$
  - Player 2: $\{B, S\}$
- Preferences: as given in the graph above.

Illustration 5.6 (Card game)**.** Our Card game is played as follows:

- Player 1 and player 2 put $1 in the pot.
- A card is drawn.
- Player 1 is informed whether card is 'high' or 'low', player 2 is not informed.
- Player 1 can choose between ('see' and 'raise')
  - If Player 1 chooses 'see', then the game is finished. The card determines who wins dollar. 'high' means player 1 gets the money, 'low' means player 2 gets the money.
  - If player 1 chooses 'raise', then she adds $1 to the pot. Then Player 2 can choose ('pass' or 'meet')
    * If player 2 chooses 'pass', player 1 takes the money.
    * If player 2 chooses 'meet', she adds $1 to pot.

Illustration 5.7 (Nash Equilibrium of Card game)**.** To find the Nash Equilibrium of the Card Game, write down the game in strategic form. In this table, (Raise, See) denotes player 1's strategy to Raise after High and See after Low.

	Pass	Meet
Raise, Raise	(1,-1)	(0,0)
Raise, See	(0,0)	$(\frac{1}{2}, -\frac{1}{2})$
See, Raise	(1,-1)	$(-\frac{1}{2}, \frac{1}{2})$
Raise, Raise	(0,0)	(0,0)

Table 9: Strategic form of Card game

We can conclude that there are no pure strategies Nash equilibrium. Furthermore, (See, See) is strictly dominated by ($\frac{1}{2}$, Raise Raise; $\frac{1}{2}$, Raise See). This means (See, See) is not played in any Nash equilibrium. We also see that (See, Raise) is weakly dominated by (Raise, Raise). Given that $q = 1$ (probability 1 to pass) cannot be part of Nash equilibrium, (See, Raise) is not used in any Nash equilibrium. We can now reduce our table.

There are mixed strategies Nash equilibria in the remaining game. We have row-player indifferent if $q = (1/2)(1 - q)$, thus $q = 1/3$. We also have column-player indifferent if: $-p = -(1/2)(1 - p)$, thus $p = 1/3$.

So, unique Nash equilibrium: player 1 assigns 1/3 to (Raise, Raise); 2/3 to (Raise, See) and player 2 assigns 1/3 to pass; 2/3 to meet

	Pass	Meet	
Raise, See	(0,0)	$(\frac{1}{2},-\frac{1}{2})$	p
See, Raise	(1,-1)	$(-\frac{1}{2},\frac{1}{2})$	1 -p
	q	1-q	

Table 10: Reduced Strategic form of Card game with probabilities

In words, this means player 1 will always chooses 'raise' after 'high' and bluffs after 'low' with probability 1/3

This means that Card Game is favorable to player 1. You should refuse the role of player 2 if you can.

Illustration 5.8 (Repated games: prisoner's dilemma)**.** The main idea of this game is to sustain cooperation in equilibrium by threatening to switch to punishment if the other does not cooperate.

	C	D
C	(2,2)	(0,3)
D	(3,0)	(1,1)

Table 11: Payoff matrix for the prisoner's dilemma

There are two strategies:

Grim trigger strategy

- start with C
- continue with C as long as other player chooses C
- if in any period other chooses D, then choose D in every subsequent period

Tit-for-tat

- start with C
- do whatever the other player did in previous period

We have some notation. Player i uses discount factor δ for which $0 < \delta < 1$

$$u_i(a^1, a^2, \dots, a^T) = u_i(a^1) + \delta u_i(a^2) + \delta^2 u_i(a^3) + \dots + \delta^{T-1} u_i(a^T) = \sum_{t=1}^{T} \delta^{t-1} u_i(a^t)$$

There is a possibility that $T \to \infty$. Or that there is a p: a probability the game is finished at a given period.

Example 5.9. Let c be the value that makes player indifferent between payoffs $c, c, c, ...$ and payoffs $w^1, w^2, w^3, ...$ Let V denote the value of discounted sum $w^1, w^2, w^3, ...$ We can find this c as follows:

$$\begin{aligned} V &= c + c\delta + c\delta^2 + c\delta^3 + ... \\ V\delta &= c\delta + c\delta^2 + c\delta^3 + ... \\ V - \delta V &= c && \text{(subtract } \delta\text{V from both sides)} \\ (1-\delta)V &= c \end{aligned}$$

So discounted average c equals $(1-\delta)V$

Remark 5.2. *A repeated game is an extensive game with perfect information and simultaneous moves. This mean we can use concept of subgame perfection. If the game is finitely repeated n times, we can apply backwards induction (starting at the nth period).*

Example 5.10 (Finitely repeated Prisoner's dilemma)**.** Unique subgame perfect equilibrium: each player's strategy chooses D in every period (proof by backward induction). In any finitely repeated game every Nash equilibrium generates outcome (D,D) in every period. To support cooperation in equilibrium, it is required that the game is infinitely repeated.

Illustration 5.11. Some Nash equilibria of infinitely repeated PD

1. Every player always plays D whatever happened

2. Grim trigger strategy is N.E. if δ is sufficiently large. Assume other player plays grim trigger and that layer i considers to deviate to D in period k. The discounted average deviation from period k equals $(1-\delta)(3+\delta+\delta 2+\delta 3+...) = (1-\delta)(3+\delta/(1-\delta)) = 3(1-\delta)+\delta$ discounted average if i sticks to grim trigger $= 2$ In conclusion, Grim trigger is a best response if

$$2 > 3(1-\delta)+\delta \iff \delta > 1/2$$

3. Tit-for-tat Assume other player plays tit-for-tat and that player i considers to deviate to D in period k

   (a) Player i reverts to C in period $k+1$. If this is part of her best response, she should continue alternating (situation is stationary). The discounted average payoff alternating from period k:

   $$(1-\delta)(3+0\delta+3\delta 2+0\delta 3+3\delta 4+...) = 3(1-\delta)/(1-\delta 2) = 3/(1+\delta)$$

   Alternating does not beat tit-for-tat if

   $$3/(1+\delta) < 2 \iff \delta > 1/2$$

   (b) player i sticks to D after k. Then she earns

   $$(1-\delta)(3+\delta+\delta 2+\delta 3+... = (1-\delta)(3+\delta/(1-\delta)) = 3(1-\delta)+\delta \leq 2 \iff \delta \geq 1/2$$

But many more outcomes can be supported in equilibrium of the infinitely repeated prisoner's dilemma.

Theorem 5.12 (Folk Theorem)**.**
- *For any discount factor δ with $0 < \delta < 1$, the discounted average payoff of any player i in any Nash equilibrium is at least $u_i(D, D)$.*
- *Let (x_1, x_2) be a feasible pair of payoffs such that $x_i > u_i(D, D)$ for each i. There exists a $\delta^* \leq 1$ such that if $\delta > \delta^*$, then the game has a N.E. in which the discounted payoff of each player i is xi .*
- *For any δ the game has a N.E. in which the average discounted payoff of player i equals $u_i(D, D)$*

Proof. □

Illustration 5.13 (Nash equilibria of infinitely repeated PD with mistakes)**.** What about environments where players may make mistakes? grim trigger does not do so well, tit-for-tat works better. But, the winner is Pavlov strategy ("win-stay lose-change"):

- Start with C
- Choose C if previous outcome was (C,C) or (D,D)
- Choose D after any other history

6 Weak sequential equilibrium

Definition 6.1 (Stategy)**.** A strategy in an extensive game tells you what a player does for every information set where she has the move. It it gives one action for every information set.

Definition 6.2 (Weak sequential equilibrium)**.** We want to want to determine which equilibrium is better. We force each player whose turn it is to move forms a belief about the histories in her information set. A belief system is a collection of beliefs, one for each information set We are working with behavioral strategy in extensive game. This is a probability function that assigns to each information set a probability distribution over the possible actions in that set. Behavioral strategies are equivalent to mixed strategies, but easier to work with

Example 6.3 (card game)**. Behavioral strategy:** With behavioral strategy, you give a probability distribution for $\{Raise, See\}$ after High card other probability distribution for $\{Raise, See\}$ after Low card
Mixed strategy: Using mixed strategy, we make one probability distribution for the set $\{(Raise, Raise), (Raise, See), (See, Raise), (See, See)\}$

Definition 6.4 (Weak sequential equilibrium)**.** (Kreps and Wilson, 1982) A Weak sequential equilibrium, also know as a Perfect Bayesian Equilibrium is an assessment, that is a behavioral strategy profile β and a belief system μ, that satisfies

(i) **Sequential rationality:** Each player's strategy is optimal in the part of the game that follows each of her information sets, given the strategy profile and her belief about the history in the information set that has occurred.

(ii) **Weak consistency of beliefs with strategies:** For every Information Set reached with positive probability given the strategy profile β, the probability assigned by the belief system to each history h^* in the information set I_i is given by Bayes' rule. That is:

$$\mathbb{P}[h^* \mid I_i] = \frac{\mathbb{P}[h^* \text{ according to } \beta]}{\sum_{h \in I_i} \mathbb{P}[h \text{ according to } \beta]}$$

If an information set is not reached with positive probability, any belief is allowed.

Definition 6.5 (Signaling games)**.** (Spence, 1973)

- Why do gazelles engage in stotting when approached by cheetah?
- Why do men give expensive flowers to women?
- Why do students follow a costly but (sometimes) useless education?

To signal their quality.

- In signaling game, first mover has private information that is relevant to second mover
- First mover wants to convince second mover to choose particular beneficial action
- "Good" types may send costly signals to distinguish themselves from "bad" types

  - Signaling games have **separating equilibria**, in which different types choose different signals and second movers infer types from signals
  - Signaling games have **pooling equilibria**, in which different types choose the same signal and second movers do not update their beliefs

Part II
Appendix

7 List of symbols

7.1 General Probability

- $\mathbb{E}(X)$: Expected value of X.
- $\mathbb{P}(X)$: Probability of X.
- $\mathbb{P}(X \mid Y)$: Probability of X given Y.

7.2 Game Theory

- A: Set of actions.
- a: a action in A; an action profile.
- α: an action profile.
- $\mathcal{B}_i$: Best response for player i.
- $\mathcal{B}_i(a)$: Best response for player i for the action profile a.
- N: a set of players.
- $u_i(a)$: Utility (payoff) for player i with action profile a.
- $U_i(a)$: Expected payoff for player i with mixed action profile a.
- $P(A) = 1$: Player function. When the history is A, it is 1's turn.

8 Remarks

8.1 Some remarks on the exam

- The final exam will be a three-hour written examination with three open questions. One question will be relatively simple, one will be relatively difficult and the other is in between. Simple questions have a rating of 1 and difficult questions a rating of 3.
- The sum of the ratings of the questions at the exam will approximately be 6.0.
- The probability that you get (exactly) one of the questions of Osborne is 0. Still, it will help you a lot if you practice those questions. Try to actively make the questions yourself. My judgment is that if you feel comfortable with the questions dealt with at the tutorials and the material presented on the slides, you will do well on the exam. It is not necessary to practice other questions, but you can of course always do this. Then you are best advised to practice questions with a black circle, so that you can compare your answers with the ones Osborne provides on his website.

- If your answers are in the "same spirit" as the ones that you received during the tutorials or the ones that you can find on the website of Osborne, you are doing well. However, in game theory questions often have multiple solutions, so usually there is more than one way to earn the maximum number of points. Some people prefer to avoid notation as much as possible and to use words instead. If you are one of those people, try to be as precise as possible in words.

8.2 Additional Material

- Nash Equilibrium (from A Beautiful Mind)