# MATH 5301 Elementary Analysis - Homework 4

Jonas Wagner

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## Problem 1

Let $(S_1, d_1)$ and $(S_2, d_2)$ be two metric spaces. Show that each of the following determines the metric on $S_1 \times S_2$.
Let $x_j \in S_1, y_j \in S_2$:

### a) $d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$

**Theorem 1.** *The metric*

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$$

*is a metric on $S_1 \times S_2$.*

*Proof.* A metric $d : S_1 \times S_2 \to \mathbb{R}$ must satisfy (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

#### i) Non-negativy

$$d((x_1, y_1), (x_2, y_2)) \geq 0$$

Since $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$,

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \geq 0$$

#### ii) Symmetry

$$d((x_1, y_1), (x_2, y_2)) = d((x_1, y_1), (x_2, y_2))$$

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} = \max\{d_2(y_1, y_2), d_1(x_1, x_2)\} = d((x_2, y_2), (x_1, x_2)$$

#### iii) Triangle Inequality

$$d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$$

$$\begin{aligned}
d((x_1, y_1), (x_3, y_3)) &= \max\{d_1(x_1, x_3), d_2(y_1, y_3)\} \\
d((x_1, y_1), (x_2, y_2)) &= \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \\
d((x_2, y_2), (x_3, y_3)) &= \max\{d_1(x_2, x_3), d_2(y_2, y_3)\} \\
\max\{d_1(x_1, x_3), d_2(y_1, y_3)\} &\leq \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} + \max\{d_1(x_2, x_3), d_2(y_2, y_3)\} \\
d((x_1, y_1), (x_3, y_3)) &\leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))
\end{aligned}$$

$\square$

## b) $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$

**Theorem 2.** *The metric*

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$$

*is a metric on* $S_1 \times S_2$.

*Proof.* A metric $d : S_1 \times S_2 \to \mathbb{R}$ must satisfy (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

### i) Non-negativy

$$d((x_1, y_1), (x_2, y_2)) \geq 0$$

Since $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$,

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2) \geq 0$$

### ii) Symmetry

$$d((x_1, y_1), (x_2, y_2)) = d((x_1, y_1), (x_2, y_2))$$

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2) = d_1(x_2, x_1) + d_2(y_2, y_1) = d((x_2, y_2), (x_1, x_2)$$

### iii) Triangle Inequality

$$d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$$

$$\begin{aligned}
d((x_1, y_1), (x_3, y_3)) &= d_1(x_1, x_3) + d_2(y_1, y_3) \\
d((x_1, y_1), (x_2, y_2)) &= d_1(x_1, x_2) + d_2(y_1, y_2) \\
d((x_2, y_2), (x_3, y_3)) &= d_1(x_2, x_3) + d_2(y_2, y_3) \\
d_1(x_1, x_3) + d_2(y_1, y_3) &\leq d_1(x_1, x_2) + d_2(y_1, y_2) + d_1(x_2, x_3) + d_2(y_2, y_3) \\
d((x_1, y_1), (x_3, y_3)) &\leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))
\end{aligned}$$

□

**c)** $d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2}$

**Theorem 3.** *The metric*

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2}$$

*is a metric on* $S_1 \times S_2$.

*Proof.* A metric $d : S_1 \times S_2 \to \mathbb{R}$ must satisfy (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

**i) Non-negativy**

$$d((x_1, y_1), (x_2, y_2)) \geq 0$$

Since $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$,

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} \geq 0$$

**ii) Symmetry**

$$d((x_1, y_1), (x_2, y_2)) = d((x_1, y_1), (x_2, y_2))$$

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} = \sqrt{(d_1(x_2, x_1))^2 + (d_2(y_2, y_1))^2} = d((x_2, y_2), (x_1, x_2)$$

**iii) Triangle Inequality**

$$d((x_1, y_1), (x_3, y_3)) \leq \sqrt{(d_1(x_1, x_3))^2 + (d_2(y_1, y_3))^2}$$

$$\begin{aligned}
d((x_1, y_1), (x_3, y_3)) &= \sqrt{(d_1(x_1, x_3))^2 + (d_2(y_1, y_3))^2} \\
d((x_1, y_1), (x_2, y_2)) &= \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} \\
d((x_2, y_2), (x_3, y_3)) &= \sqrt{(d_1(x_2, x_3))^2 + (d_2(y_2, y_3))^2} \\
\sqrt{(d_1(x_1, x_3))^2 + (d_2(y_1, y_3))^2} &\leq \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} + \sqrt{(d_1(x_2, x_3))^2 + (d_2(y_2, y_3))^2} \\
d((x_1, y_1), (x_3, y_3)) &\leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))
\end{aligned}$$

□

## Problem 2

### a) A set $A$ in the metric space $(S, d)$ is called bounded, if $\exists_{R>0} \wedge \exists x \in S : A \subset B_R(x)$. Prove that if $A$ is unbounded then there exists a sequence $\{x_n\} \subset A$ such that $\forall_{m,n\in\mathbb{N}} \implies d(x_n, x_m) > 1$.

**Assumption 1.** *$m \neq n$, $m > n$*

**Definition 1.** *The open ball set $B_r(x)$ over metric space $(S, d)$ is defined as*

$$B_r(x) := \{y \in S : d(x, y) < r\}$$

**Definition 2.** *A set $A$ in the metric space $(S, d)$ is called <u>bounded</u>, if*

$$\exists_{R>0} \wedge \exists_{x\in S} : A \subset B_R(x)$$

**Definition 3.** *A set $A$ in the metric space $(S, d)$ is called <u>unbounded</u>, if it is not bounded, (i.e.)*

$$\forall_{R>0} \wedge \forall_{x\in S} : A \not\subset B_R(x)$$

**Theorem 4.** *If $A \in (S, d)$ is unbounded, then*

$$\exists\{x_n\} \subset A : \forall_{m,n\in\mathbb{N}} \implies d(x_n, x_m) > 1$$

*Proof.* $A \in (S, d)$ being unbounded means that

$$\forall_{R>0} \wedge \forall x \in S : A \not\subset B_R(x)$$

Since $\forall_{R>0} \wedge \forall_{x\in S} : A \not\subset B_R(x)$ and $B_R(x) := \{y \in S : d(x, y) < R\}$,

$$\forall_{x\in A}\exists_{y\in A} : d(x, y) \geq R$$

Since it is true for any $R > 0$ a sequence $\{x_n\}$ can be constructed with subsequent $x_{n+1}$ so that $d(x_n, x_{n+1}) > R = 1$. □

### b) Show that in the normed space $(V, |\cdot|)$ the open unit ball $B_r = \{x \in V : |x| < 1\}$ is a convex set. (i.e) $\forall_{x,y\in B_r}, \forall_{t\in[0,1]} \implies tx + (1-t)y \in B_r$

**Definition 4.** *The <u>open unit ball</u> is defined as:*

$$B_r := \{x \in V : |x| < 1\}$$

**Definition 5.** *The set $A$ is convex if and only if*

$$\forall_{x,y\in A}, \forall_{t\in[0,1]} \implies tx + (1-t)y \in A$$

**Theorem 5.** *The open unit ball set $B_r$ is convex in $(V, |\cdot|)$.*

*Proof.*

$$\forall_{x,y\in B_r}, \forall_{t\in[0,1]} \implies tx + (1-t)y \in B_r$$
$$\forall_{x,y\in V} : (|x| < 1) \wedge (|y| < 1), \forall_{t\in[0,1]} \implies tx + (1-t)y \in V : |tx + (1-t)y|$$

Since $\forall_{x,y\in V}, \forall_{t\in[0,1]} \implies tx + (1-t)y \in V$

$$(|x| < 1 \wedge |y| < 1 \implies |tx + (1-t)y| < 1) \iff tx + (1-t)y \in B_r$$

Clearly,

$$\forall_{x,y\in V}|x|, |y| < 1, \forall_{t\in[0,1]}tx + (1-t)y < 1$$

Therefore, the open unit ball set $B_r$ is convex in $(V, |\cdot|)$. □

## Problem 3

For $(\mathbb{R}^2 = (x,y), d = \sqrt{x^2+y^2})$,

### a) Show that $D = \{(x,y) : x^2 + y^2 \leq 1\}$ is a closed set.

**Definition 6.** *The set $A \subset V$ is called open if*

$$\forall_{x\in A}\exists_{\epsilon>0} : B_\epsilon(x) \subset A$$

**Definition 7.** *The set $A \subset V$ is called closed if $A^c$ is open.*

**Theorem 6.** *The set $D = \left\{(x,y) : x^2 + y^2 \leq 1\right\}$ is closed.*

*Proof.* By definition, $D$ is closed iff $D^c$ is open.
$D^c$ is defined by

$$D^c = \left\{(x,y) : x^2 + y^2 > 1\right\}$$

By definition, $D^c$ is open if

$$\forall_{(x,y)\in D^c}\exists_{\epsilon>0} : B_\epsilon((x,y)) \subset D^c$$

This means that every element in $D^c$ must have an associated open ball set centered at that element with a positive radius that is fully contained by $D^c$.
Let $(x_b, y_b) \in B_\epsilon((x,y))$ for $\epsilon > 0$. This means

$$d((x,y),(x_b,y_b)) = \sqrt{x_b^2 + y_b^2} < \epsilon$$

By definition,

$$(x,y) \in D^c \implies x^2 + y^2 > 1$$

and therefore,

$$\sqrt{x^2+y^2} = d((0,0),(x,y)) > 1$$

From the triangle inequality, we have

$$\begin{aligned} d((x_b,y_b),(0,0)) &\leq d((0,0),(x,y)) + d((x,y),(x_b,y_b)) \\ d((x,y),(x_b,y_b)) &\geq d((x_b,y_b),(0,0)) - d((0,0),(x,y)) \\ d((x,y),(x_b,y_b)) = \epsilon &> d((x_b,y_b),(0,0)) - 1 > 0 \end{aligned}$$

Therefore $D^c$ is open and therefore $D$ is closed. $\square$

**b) Find the infinite collection of open sets $\{A_n\}$ so that $\bigcap_n A_n = \overline{B_1(0)}$**

$$\{A_n : \bigcap_n A_n = \overline{B_1(0)}\}$$

This means that the intersection of all sets in $\{A_n\}$ is the boundary of the unit ball set.

## Problem 4

Let $S = \mathbb{R}^2$. Are the following sets open or closed within the metrics below?

$$\begin{aligned}
A &= \left\{(x, y) : x^2 + y^2 < 1\right\} \\
B &= \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\} \\
C &= \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\} \\
D &= \{(x, y) : |x| + |y| < 2\} \\
E &= \left\{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\right\}
\end{aligned}$$

**a) Euclidean Metric:** $d((x, y)) = \sqrt{(x)^2 + (y)^2} = \|(x, y)\|_2$

- **i)** $A = \left\{(x, y) : x^2 + y^2 < 1\right\}$
- **ii)** $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$
- **iii)** $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$
- **iv)** $D = \{(x, y) : |x| + |y| < 2\}$
- **v)** $E = \left\{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\right\}$

**b) Manhattan Metric:** $d((x, y)) = |x| + |y| = \|(x, y)\|_1$

- **i)** $A = \left\{(x, y) : x^2 + y^2 < 1\right\}$
- **ii)** $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$
- **iii)** $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$
- **iv)** $D = \{(x, y) : |x| + |y| < 2\}$
- **v)** $E = \left\{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\right\}$
- **vi)** $A = \left\{(x, y) : x^2 + y^2 < 1\right\}$
- **vii)** $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$
- **viii)** $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$
- **ix)** $D = \{(x, y) : |x| + |y| < 2\}$
- **x)** $E = \left\{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\right\}$
- **xi)** $A = \left\{(x, y) : x^2 + y^2 < 1\right\}$
- **xii)** $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$
- **xiii)** $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$
- **xiv)** $D = \{(x, y) : |x| + |y| < 2\}$
- **xv)** $E = \left\{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\right\}$
- **xvi)** $A = \left\{(x, y) : x^2 + y^2 < 1\right\}$
- **xvii)** $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$
- **xviii)** $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$
- **xix)** $D = \{(x, y) : |x| + |y| < 2\}$
- **xx)** $E = \left\{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\right\}$