MATH 5301 Elementary Analysis - Homework 8

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Problem 1

Show that the norms $\|\cdot\|_1$, $\|\cdot\|_p$ for $p > 1$, and $\|\cdot\|_\infty$ are equivalent.

Definition 1. *For $\|\cdot\|_a, \|\cdot\|_b$ on S, $\|\cdot\|_a$ is said to be* stronger *then $\|\cdot\|_b$ if*

$$\forall \{x_n\} \subset S \ : \ x_n \underset{d_a}{\longrightarrow} x \implies x_n \underset{d_b}{\longrightarrow} x$$

Definition 2. *$\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be* equivalent, *$\|\cdot\|_a \sim \|\cdot\|_b$, if $\|\cdot\|_a$ is stronger then $\|\cdot\|_b$ and $\|\cdot\|_b$ is stronger then $\|\cdot\|_a$. This means that*

$$\|\cdot\|_a \sim \|\cdot\|_b \iff \exists \alpha, \beta \in \mathbb{R}_{>0} : \forall_{x \in S} \alpha \|\cdot\|_b \leq \|\cdot\|_a \leq \beta \|x\|_b$$

Definition 3. *The following norms are defined as*

a. $\|\cdot\|_1 := \|x\|_1 = \sum_{i=1}^n |x_i| = |x_1| + |x_2| + \cdots + |x_n|$

b. $\|\cdot\|_2 := \|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} = \left(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2\right)^{1/2}$

c. $\|\cdot\|_p := \|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p} = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}, \ p > 1$

d. $\|\cdot\|_\infty := \|x\|_\infty = \max_{i=1}^n |x_i| = \max(|x_1|, |x_2|, \ldots, |x_n|)$

Theorem 1. *The norms $\|\cdot\|_1, \|\cdot\|_p$, and $\|\cdot\|_\infty$ are equivalent.*

Proof.

Lemma 1. $\|\cdot\|_1 \sim \|\cdot\|_p$

Proof. $\|\cdot\|_1 \sim \|\cdot\|_p$ is true iff

$\forall_x \exists_{\alpha,\beta\in\mathbb{R}_+}$:

$$\alpha\|x\|_p \leq \|x\|_1 \leq \|x\|_p$$

$$\alpha\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} \leq \sum_{i=1}^{n}|x_i| \leq \beta\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}$$

From the Holder's inequality we have

$$\begin{aligned}\|x\|_1 = \sum_{i=1}^{n}|x_i| &= \sum_{i=1}^{n}|x_i|(1)\\ &\leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|1|^{(1-p)}\right)^{1/(1-p)}\\ &\leq n^{1/(1-p)}\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\end{aligned}$$

So for $0 < \alpha \leq n^{1/(1-p)}$ and $\beta \geq n^{1/(1-p)}$,

$$\begin{aligned}\alpha\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} &\leq \sum_{i=1}^{n}|x_i| &&\leq \beta\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\\ \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} &\leq n^{1/(1-p)}\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} &&\leq n^{1/(1-p)}\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\end{aligned}$$

Therefore,

$$\|x\|_p \leq \|x\|_1 \leq n^{\frac{1}{1-p}}\|x\|_p$$

which proves $\|\cdot\|_1 \sim \|\cdot\|_p$. ☐

Lemma 2. $\|\cdot\|_1 \sim \|\cdot\|_\infty$

Proof. $\|\cdot\|_1 \sim \|\cdot\|_\infty$ is true iff

$\forall_x \exists_{\alpha,\beta\in\mathbb{R}_+}$:

$$\alpha\|x\|_\infty \leq \|x\|_1 \leq \beta\|x\|_\infty$$

$$\alpha \max_{i=1}^{n}|x_i| \leq \sum_{i=1}^{n}|x_i| \leq \beta \max_{i=1}^{n}|x_i|$$

Clearly, this is true for when $\alpha \in (0,1]$. Similarly, when $\beta \geq n$ then $\sum_{i=1}^{n}\max_{i=1}^{n}|x_i|$ and then clearly greater then the $\|x\|_1$; therefore $\|\cdot\|_1 \sim \|\cdot\|_\infty$. ☐

From, Lemma 1 and Lemma 2, it is clear that $\forall_{p>1}$:

$$\|x\|_\infty \leq \|x\|_p \leq \|x\|_1 \leq n^{1/1-p}\|x\|_p \leq n\|x\|_\infty$$

Therefore, $\|\cdot\|_1 \sim \|\cdot\|_p \sim \|\cdot\|_\infty$ $(\forall_{p>1})$. ☐

Problem 2

Let $(S, \|\cdot\|)$ and $(S', \|\cdot\|')$ to be two normed spaces. Show that the following norms on $S \times S'$ are equivalent.

a. $\|(x,y)\|_1 = \|x\| + \|y\|'$

b. $\|(x,y)\|_2 = \sqrt{\|x\|^2 + (\|y\|')^2}$

c. $\|(x,y)\|_p = \left(\|x\|^p + (\|y\|')^p\right)^{1/p}$

d. $\|(x,y)\|_\infty = \max\left\{\|x\| + \|y\|'\right\}$

Theorem 2. *The norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_p$, and $\|\cdot\|_\infty$ are all equivalent on $S \times S'$.*

Proof.

Lemma 3. $\|\cdot\|_1 \sim \|\cdot\|_2$

Proof. $\|\cdot\|_1 \sim \|\cdot\|_2$ is true iff

$\forall_{(x,y)\in S\times S'} \exists_{\alpha,\beta\in\mathbb{R}_+}$:

$$\alpha\|(x,y)\|_2 \leq \|(x,y)\|_1 \leq \beta\|(x,y)\|_2$$
$$\alpha(\|x\|^2 + (\|y\|')^2)^{1/2} \leq \|x\| + \|y\|' \leq \beta(\|x\|^2 + (\|y\|')^2)^{1/2}$$

First, the following demonstrates that $\|(x,y)\|_2 \leq \|(x,y)\|_1$

$$\begin{aligned}
\|(x,y)\|_1^2 &= (\|x\| + \|y\|')^2 \\
&= \|x\|^2 + (\|y\|')^2 + \|x\|\|y\|' \\
&\leq \|x\|^2 + (\|y\|')^2 + (\|x\|)^2 + (\|y\|')^2 \\
&= 2\|x\|^2 + 2(\|y\|')^2 \\
&= 2\|(x,y)\|_2^2 \\
\frac{1}{2}\|(x,y)\|_1^2 &\leq \|(x,y)\|_2^2
\end{aligned}$$

Therefore,

$$\frac{1}{\sqrt{2}}\|(x,y)\|_2 \leq \|(x,y)\|_1$$

and this is also true for any $p > 2$ as well using an arbitrary number of power expansions.
Next, from the Cauchy Schwartz's inequality we have

$$\begin{aligned}
\|(x,y)\|_1 &= \|x\| + \|y\|' \\
&= \left(\|x\|(1) + \|y\|'(1)\right) \\
&\leq \left(\|x\|^2 + (\|y\|')^2\right)^{\frac{1}{2}} \left(1^2 + 1^2\right)^{1-\frac{1}{2}} \\
&= (2)^{\frac{1}{2}} \left(\|x\|^2 + (\|y\|')^2\right)^{\frac{1}{2}} \\
&= \sqrt{2}\sqrt{\|x\|^2 + (\|y\|')^2} \\
\|(x,y)\|_1 &\leq \sqrt{2}\|(x,y)\|_2
\end{aligned}$$

Therefore,

$$\frac{1}{\sqrt{2}}\|x\|_2 \leq \|x\|_1 \leq \sqrt{2}\|x\|_2$$

which proves $\|\cdot\|_1 \sim \|\cdot\|_2$. ∎

Lemma 4. $\|\cdot\|_1 \sim \|\cdot\|_p$

Proof. $\|\cdot\|_1 \sim \|\cdot\|_p$ is true iff

$$\forall_{(x,y)\in S\times S'} \exists_{\alpha,\beta\in\mathbb{R}_+} :$$
$$\alpha\|(x,y)\|_p \leq \|(x,y)\|_1 \leq \beta\|(x,y)\|_p$$
$$\alpha\big(\|x\|^p + (\|y\|')^p\big)^{1/p} \leq \|x\| + \|y\|' \leq \beta\big(\|x\|^p + (\|y\|')^p\big)^{1/p}$$

From the Holder's inequality we have

$$\begin{aligned}
\|x\|_1 = \|x\| + \|y\|' & \\
&= \|x\|(1) + \|y\|'(1) \\
&\leq \big(\|x\|^p + (\|y\|')^p\big)^{1/p} \left(\sum_{i=1}^{2} |1|^{(1-p)}\right)^{\frac{1}{1-p}} \\
&= n^{\frac{1}{1-p}} \big(\|x\|^p + (\|y\|')^p\big)^{1/p} \\
&= n^{\frac{1}{1-p}} \|(x,y)\|_p
\end{aligned}$$

Therefore,

$$\|(x,y)\|_1 \leq n^{\frac{1}{1-p}} \|(x,y)\|_p$$

and, since $p > 1$, then the remainder of the arguments from Lemma 3 can be applied here to any arbitrary $p > 1$ to prove the norm equivalence with 1, 2, and any p norms. □

Lemma 5. $\|\cdot\|_1 \sim \|\cdot\|_\infty$

Proof. $\|\cdot\|_1 \sim \|\cdot\|_\infty$ is true iff

$$\forall_{(x,y)\in S\times S'} \exists_{\alpha,\beta\in\mathbb{R}_+} :$$
$$\alpha\|(x,y)\|_\infty \leq \|(x,y)\|_1 \leq \beta\|(x,y)\|_\infty$$
$$\alpha \max\big\{\|x\| + \|y\|'\big\} \leq \|x\| + \|y\|' \leq \beta \max\big\{\|x\| + \|y\|'\big\}$$

Clearly, this is true for when $\alpha \in (0, 1]$. Similarly, when $\beta \geq 2$ then $\max \|x\|, \|y\|'$ and is clearly greater then the $\|(x,y)\|_1$; therefore $\|\cdot\|_1 \sim \|\cdot\|_\infty$. □

□

Problem 3

Let X be a vector space and V be a normed space. The function $f : X \to V$ is called bounded if $\exists M : \forall_{x \in X} \implies \|f(x)\| < M$. Consider the set $\mathcal{B}(X, V)$ of all bounded functions from $X \to V$.

a)

Show that $\mathcal{B}(X, V)$ is a vector space.

Definition 4. *A* Vector space *over a field is the set V along with two operations (vector addition and vector multiplication) satisfying the basic vector properties.*

a. *Associativity of vector addition*
$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

b. *Commutativity of vector addition*
$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

c. *Identity element of vector addition (zero vector)*
$$\forall_{\mathbf{v} \in V} \exists_{\mathbf{0} \in V} : \mathbf{v} + \mathbf{0} = v$$

d. *Inverse elements of vector addition (additive inverse)*
$$\forall_{\mathbf{v} \in V} \exists_{-v \in V} : \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

e. *Compatibility of scalar and field multiplication*
$$a(b\mathbf{v}) = (ab)\mathbf{v}$$

f. *Identity element of scalar multiplication (multiplicative identity)*
$$\exists_{1 \in F} \mathbf{1v} = v$$

g. *Distributivity of scalar multiplication with vector addition*
$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

h. *Distributivity of scalar multiplication with field addition*
$$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$$

Definition 5. *$\mathcal{B}(X, V)$ is the set of all functions $f : X \to V$ that are bounded under the definition:*
$$\exists_{M \in V} : \forall_{x \in X} \implies \|f(x)\| < M$$

Theorem 3. *$\mathcal{B}(X, V)$ is a vector space.*

Proof. It is known that X is a vector space and V is a normed vector space. For all functions between X and V the normed space result implies many of the required vector space properties directly. For instance, assuming standard function addition and multiplication methods, the mapped results of the new superimposed function will satisfy Associativity, Commutativity, Identity and inverse for addition, Compatibility and identity of multiplication. The Distributivity properties require more justification as they do not clearly result from the results of a single function. Fortunately, the boundedness of $\mathcal{B}(X, V)$ provides that an upper bound exists for the output and so the complicated parts of accounting for weirder functions allows for a proof of distributivity based on the output and the induced addition and multiplication operations will satisfy all of the superposition properties. $\square$

b)

Show that the function $\|\cdot\|_\infty : \mathcal{B}(X,V) \to \mathbb{R}_+$:

$$\|f\|_\infty := \sup_{x\in X} \|f(x)\|$$

defines a norm on $\mathcal{B}(X,V)$.

Definition 6. *A* norm *is a function* $\|\cdot\| : V \to \mathbb{R}_+$ *satisfying*

a. Non-negativity

$$\forall_{x\in V} \|x\| \geq 0 \implies \|x\| = 0 \iff x = 0$$

b. Homogeneity

$$\|\lambda \cdot x\| = |\lambda| \|x\|$$

c. Triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|$$

Theorem 4. $\|\cdot\|_\infty$ *is a norm on* $\mathcal{B}(X,V)$*.*

Proof. Since $\mathcal{B}(X,V)$ is a vector space, the important properties of a field and simple vector operations can be assumed.
First, by definition, the non-negativity is satisfied by the mapped results are within $\mathbb{R}_+$.
Second, the original norm properties from the normed vector space V can be applied to each of the $\|f(x)\|$ within the $\sup_{x\in X}$, resulting in the homogeneity required for a norm.
Third, The triangle inequality can also be easily seen with the following:

$$\|x + y\| \leq \|x\| + \|y\|$$
$$\sup_{(x+y)\in X} \|f(x+y)\| \leq \sup x \in X \|f(x)\| + \sup_{y\in X} \|f(y)\|$$
$$\sup_{x,y\in X \;:\; x+y\in X} \|f(x+y)\| \leq \sup_{x,y\in X} \|x\| + \|y\|$$

which is clearly true considering the definition of the original normed space V. $\square$

Problem 4

Let A be a dense set in metric space (S,d), let (V,d_1) be a complete metric space, and $f : A \to Y$ be a uniformly continuous function.
Note: assuming that there was a typo and that Y is also dense within V.

a) Show that if $\{x_n\}$ is a Cauchy sequence in A then $\{f(x_n)\}$ is a Cauchy sequence in Y.

Definition 7. *A set A is* dense *within metric space (S,d) if and only if $A = X$.*

Definition 8. *Metric space (S,d) is called a* complete metric space *if every cauchy sequence $\{a_n\} \subset S$ converges in S.*

$$\forall_{\{a_n\}\subset S \,:\, \{a_n\} \text{ cauchy } \implies \exists_{a\in S} \,:\, \lim_{n\to\infty} a_n = a}$$

Definition 9. *$\{a_n\}$ is said to be a* cauchy *sequence if*

$$\{a_n\} : \forall_{\epsilon>0}\exists_N : \forall_{n,m>N} \implies d(a_n, a_m) < \epsilon$$

Definition 10. *A function $f : X \to Y$ is said to be* uniformly continuous *if and only if*

$$\forall_{\epsilon>0}\exists_{\delta>0}\forall_{x,x'\in X} d_X(x,x') < \delta \implies d_Y(f(x), f(x')) < \epsilon$$

Theorem 5. *If $\{x_n\}$ is a Cauchy sequence in A then $\{f(x_n)\}$ is a Cauchy sequence in Y.*

Proof.

$$\{x_n\} \text{ cauchy} \implies \{f(x_n)\} \text{ cauchy}$$
$$\forall_{\epsilon,\delta>0}\exists_{N(\epsilon,\delta)\in\mathbb{N}} : \forall_{n,m>N} \implies d(x_n, x_m) < \epsilon) \implies d_1(f(x_{n_1}), f(x_{n_1})) < \epsilon_1)$$

By definition of the complete metric space Y, every cauchy sequence within Y will converge to within Y, therefore every cauchy sequence that gets mapped from the dense A to Y via the uniformly continuos function f will be guaranteed to be cauchy as well. (this could also be written out in a more complicated way using quantifiers, but the words just explained it better) $\square$

b) Show that there is only one continuous function $g : X \to Y$ so that $g(x) = f(x)$ forall $x \in A$.

Theorem 6. *There is only one continuous $g : X \to Y$ so that $g(x) = f(x)$ forall $x \in A$.*

Proof. This means that each continuous function from X or A to Y only has one complimentary function that produces the same image as it in Y. For f, the definition of uniformly continuous is

$$\forall_{\epsilon>0}\exists_{\delta>0}\forall_{x,x'\in A} d(x,x') < \delta \implies d_1(f(x), f(x')) < \epsilon$$

Similarly, g being continuous is defined by

$$\forall_{x\in X}\forall_{\epsilon>0}\exists_{\delta>0}\forall_{x,x'\in X} d(x,x') < \delta \implies d_1(g(x), g(x')) < \epsilon$$

There is only one possible mapping for each complete relation between each of the element $x \in A$ to $f(x) \in Y$, so since A is dense in X, each element $x \in X$ can be mapped from X to the image in Y. Although this is simple and the result clearly follows, the quantifiers can be used to demonstrate this as follows:

$$(\forall_{\epsilon>0}\exists_{\delta>0}\forall_{x,x'\in A} d(x,x') < \delta \implies d_1(f(x), f(x')) < \epsilon) \wedge$$
$$\wedge\, (\forall_{x\in X}\forall_{\epsilon>0}\exists_{\delta>0}\forall_{x,x'\in X} d(x,x') < \delta \implies d_1(g(x), g(x')) < \epsilon)$$
$$\forall_{\epsilon>0}\exists_{\delta>0}\forall_{x,x'\in A} d(x,x') < \delta \implies d_1(f(x), f(x')) < \epsilon \wedge \exists_{\epsilon_1(\epsilon,x,x')} d_1(g(x), g(x'))) < \epsilon_1$$

And since we want $g(x) = f(x)$, it is clear that the exact same restrictions (for each element across all of A and X) will be in place, which implies that only one function will be able to satisfy it. $\square$

Problem 5

Let $(L, \|\cdot\|)$ be a Banach space. Let L_0 be a closed subspace of L. Define the factor-space L/L_0 as : $L_1 := L/L_0 = \{x + y \ : \ x \in L, y \in L_0\}$. In other works, L_1 consists of all subsets of L obtained from L_0 by shifting all its elements by some element x.

a) Show that L_1 is a vector space.

Theorem 7. *L_1 is a vector space.*

Proof. From the definition of a vector space, Definition 4, it is necessary for all vector spaces to have additive and multiplicative operations that satisfy the multiple properties of superposition. Directly by definition of the of L_1, being composed of a Banach space and a closed subspace within it, it may be possible to directly claim that it is also Banach. Regardless, the definition of each element within L_1 as a summation of two elements within a Banach (and therefore vector) space, already demonstrates the additive properties. Similarly, the composition of L_1 as a linear combination of elements from L (and L_0) that individually satisfy all of the vector space properties will imply that an induced (yet technically undefined in the problem statement) set of additive and multiplicative operations that obey superposition. □

b)

Define the function $\|\cdot\| : L_1 \to \mathbb{R}_+$ as $\|x\|_1 = \inf_{x-y \in L_0} \|y\|$. Show that this function defines a norm on the space L_1.

Theorem 8. *$\|x\|_1$ is a norm on L_1.*

Proof. Since L_1 is a vector space, the important properties of a field and simple vector operations can be assumed.
First, by definition, the non-negativity is satisfied by the mapped results are within $\mathbb{R}_+$.
Second, the original norm properties from L apply to each $\|y\|$ within the $\inf_{x-y\in L_0}$, resulting in the homogeneity required for a norm.
Third, The triangle inequality can also be easily seen with the following:

$$\begin{aligned} \|(a,b) + (x,y)\| &\leq \|(a,b)\| + \|(x,y)\| \\ \inf_{(a+x)-(b+y)\in L_0} \|b + y\| &\leq \inf_{a-b\in L_0} \|b\| + \inf_{x-y\in L_0} \|y\| \\ &\leq \inf_{(a-b),(x-y)\in L_0} \|b + y\| \end{aligned}$$

which is clearly true considering the definition of the original banach space L. □

c) Show that L_1 is a Banach space.

Definition 11. *A complete, normed, space is called a* Banach space.

Theorem 9. *L_1 is a Banach space.*

Proof. From the first two parts of this question, we know that L_1 is a vector space, and $(L_1, \|x\|_1)$ is a normed space. The only remaining requirement is that of completeness, which by Definition 8, means

$$\forall \{l_n\} \subset L_1 \ : \ \{l_n\} \text{ cauchy } \implies \exists_{l\in L_1} \ : \ \lim_{n\to\infty} l_n = l$$

Since L_1 is composed of the complete collection of projections of the closed L_0 shifted by arbitrary values within L, both of which are complete (since they are Banach), so any possible cauchy sequences within L_1 will tend towards another element within L_1. □

Problem 6

Let $C([-1,1])$ be the space of all continuous real-valued functions $f(x)$ with $x \in [-1,1]$. Let $\|f\|_\infty := \sup_{x\in[-1,1]} |f(x)|$. Find the distance from point $p = x^{2021}$ to the space P_{2020} of all polynomials of degree less than or equal to 2020.

Definition 12. *The* distance *between a point and a set is defined by*

$$dist(x, A) := \inf_{a\in A} d(x,a) = \inf_{a\in A} \|x - a\|$$

The distance between point $p = x^{2021}$ and P_{2020} is calculated as follows:

$$\begin{aligned}
\text{dist}(p, P) &= \inf_{y\in P_{2020}} d(p,y) = \inf_{y\in P_{2020}} \|p - y\| \\
&= \inf_{y\in P_{2020}} \sup_{x\in[-1,1]} |p(x) - y(x)| \\
&= \inf_{y\in P_{2020}} \sup_{x\in[-1,1]} \left| x^{2021} - \sum_{i=0}^{2020} a_i^{(y)} x^i \right|
\end{aligned}$$

Since the supremum has to bound the maximum value, it can be assumed that the suppremmum of an absolute value will only occur when $a_i^{(y)} < 0$ and $x = \max x \in [-1,1] = 1$.

$$\begin{aligned}
&= \inf_{y\in P_{2020}} (1)^{2021} - \sum_{i=0}^{2020} -\left|a_i^{(y)}\right| (1)^i \\
&= \inf_{y\in P_{2020}} 1 + \sum_{i=0}^{2020} \left|a_i^{(y)}\right|
\end{aligned}$$

We can then see that the largest lower bound would be when $a_i^{(y)} = 0,\ \forall_{i\in[0,2020]}$.
Therefore,

$$\text{dist}(p = x^{2021}, P_{2020}) = 1$$