MATH5301 Elementary Analysis. Homework 4. Due: 10/01/2021, 11:59 pm

First Name: **Last Name:**

4.1

Let (S_1, d_1) and (S_2, d_2) be two metric spaces. Show that each of the following determines the metric on $S_1 \times S_2$ (here $x_j \in S_1$, $y_j \in S_2$):

(a) $d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$

(b) $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$

(c) $d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2}$

4.2

(a) A set A in the metric space (S, d) is called bounded, if $\exists R > 0 \wedge \exists x \in S : A \subset B_R(x)$. Prove that if A is unbounded then there exists a sequence $\{x_n\} \subset A$ such that $\forall m, n \in \mathbb{N} \Rightarrow d(x_n, x_m) > 1$.

(b) Show that in the normed space $(V, |\cdot|)$ the open unit ball $B_r = \{x \in V : |x| < 1\}$ is a convex set, i.e.

$$\forall x, y \in B_r,\ \forall t \in [0, 1] \Rightarrow tx + (1 - t)y \in B_r.$$

4.3

For $\mathbb{R}^2$ equipped with the usual Euclidean metric

(a) Show that $D = \{(x, y) : x^2 + y^2 \leqslant 1\}$ is a closed set.

(b) Find the infinite collection of open sets $\{A_n\}$ such that $\bigcap_n A_n = \overline{B_1(0)}$

4.4

Let $S\mathbb{R}^2$. Which of those sets are open or close

$$A = \{(x, y) : x^2 + y^2 < 1\}$$

$$B = \{(x, y) : x = 0 \wedge -1 \leqslant y \leqslant 1\}$$

$$C = \{(x, y) : 1 < x < 2 \wedge -1 \leqslant y \leqslant 1\}$$

$$D = \{(x, y) : |x| + |y| < 2\}$$

$$E = \{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\}$$

(a) In the Euclidean metric

(b) in the Manhattan metric

(c) In the highway metric: $d_h((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2 \\ |y_1| + |y_2| + |x_1 - x_2|, & \text{if } x_1 \neq x_2 \end{cases}$

4.5

Let (S, d) be a metric space.

(a) Show that for all $A \subset B \subset S$ one has $int(A) \subseteq int(B)$, $\bar{A} \subseteq \bar{B}$. Provide an example, showing that these relations cannot be made strict.

(b) Is the following true: $int(A \cup B) = int(A) \cup int(B)$?

(c) Is the following true: $\overline{A \cap B} = \bar{A} \cap \bar{B}$?

4.6

Give a topological proof of the infinitude of the set of prime numbers. (H. Furstenberg, 1955).

Denote $N_{a,b} := \{a + nb \mid b \in \mathbb{Z}\} \subset \mathbb{Z}$. Define the topology on $\mathbb{Z}$ as follows: The set U will be called open if for any $a \in U$ there exists $b \in \mathbb{Z}$ such that $N_{a,b} \subset U$. Note that every open set is infinite.

(a) Show that it is indeed a topology, i.e. any union of open sets is open and any finite intersection of open sets is open.

(b) Show that $N_{a,b}$ is closed.

(c) Show that $\mathbb{Z} \setminus \{-1, 1\}$ is open

(d) Prove that the set $\mathbb{P}$ of prime numbers cannot be finite. Hint: $\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in \mathbb{P}} N_{0,p}$.