

MATH 5301 Elementary Analysis - Homework 4

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Problem 1

Let (S_1, d_1) and (S_2, d_2) be two metric spaces. Show that each of the following determines the metric on $S_1 \times S_2$.

Let $x_j \in S_1, y_j \in S_2$:

a) $d((x_1, y_1), (x_2, y_2)) = \max \{d_1(x_1, x_2), d_2(y_1, y_2)\}$

Theorem 1. *The metric*

$$d((x_1, y_1), (x_2, y_2)) = \max \{d_1(x_1, x_2), d_2(y_1, y_2)\}$$

is a metric on $S_1 \times S_2$.

Proof. A metric $d : S_1 \times S_2 \rightarrow \mathbb{R}$ must satisfy (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

i) Non-negativity

$$d((x_1, y_1), (x_2, y_2)) \geq 0$$

Since $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$,

$$d((x_1, y_1), (x_2, y_2)) = \max \{d_1(x_1, x_2), d_2(y_1, y_2)\} \geq 0$$

ii) Symmetry

$$d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$$

$$d((x_1, y_1), (x_2, y_2)) = \max \{d_1(x_1, x_2), d_2(y_1, y_2)\} = \max \{d_2(y_1, y_2), d_1(x_1, x_2)\} = d((x_2, y_2), (x_1, y_1))$$

iii) Triangle Inequality

$$d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$$

$$d((x_1, y_1), (x_3, y_3)) = \max \{d_1(x_1, x_3), d_2(y_1, y_3)\}$$

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$$d((x_2, y_2), (x_3, y_3)) = \max \{d_1(x_2, x_3), d_2(y_2, y_3)\}$$

$$\max \{d_1(x_1, x_3), d_2(y_1, y_3)\} \leq \max \{d_1(x_1, x_2), d_2(y_1, y_2)\} + \max \{d_1(x_2, x_3), d_2(y_2, y_3)\}$$

$$d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$$

□

b) $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$

Theorem 2. *The metric*

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$$

is a metric on $S_1 \times S_2$.

Proof. A metric $d : S_1 \times S_2 \rightarrow \mathbb{R}$ must satisfy (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

i) Non-negativity

$$d((x_1, y_1), (x_2, y_2)) \geq 0$$

Since $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$,

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2) \geq 0$$

ii) Symmetry

$$d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$$

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2) = d_1(x_2, x_1) + d_2(y_2, y_1) = d((x_2, y_2), (x_1, y_1))$$

iii) Triangle Inequality

$$d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$$

$$d((x_1, y_1), (x_3, y_3)) = d_1(x_1, x_3) + d_2(y_1, y_3)$$

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$$

$$d((x_2, y_2), (x_3, y_3)) = d_1(x_2, x_3) + d_2(y_2, y_3)$$

$$d_1(x_1, x_3) + d_2(y_1, y_3) \leq d_1(x_1, x_2) + d_2(y_1, y_2) + d_1(x_2, x_3) + d_2(y_2, y_3)$$

$$d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$$

□

c) $d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2}$

Theorem 3. *The metric*

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2}$$

is a metric on $S_1 \times S_2$.

Proof. A metric $d : S_1 \times S_2 \rightarrow \mathbb{R}$ must satisfy (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

i) Non-negativity

$$d((x_1, y_1), (x_2, y_2)) \geq 0$$

Since $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$,

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} \geq 0$$

ii) Symmetry

$$d((x_1, y_1), (x_2, y_2)) = d((x_2, y_2), (x_1, y_1))$$

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} = \sqrt{(d_1(x_2, x_1))^2 + (d_2(y_2, y_1))^2} = d((x_2, y_2), (x_1, y_1))$$

iii) Triangle Inequality

$$d((x_1, y_1), (x_3, y_3)) \leq \sqrt{(d_1(x_1, x_3))^2 + (d_2(y_1, y_3))^2}$$

$$d((x_1, y_1), (x_3, y_3)) = \sqrt{(d_1(x_1, x_3))^2 + (d_2(y_1, y_3))^2}$$

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2}$$

$$d((x_2, y_2), (x_3, y_3)) = \sqrt{(d_1(x_2, x_3))^2 + (d_2(y_2, y_3))^2}$$

$$\sqrt{(d_1(x_1, x_3))^2 + (d_2(y_1, y_3))^2} \leq \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} + \sqrt{(d_1(x_2, x_3))^2 + (d_2(y_2, y_3))^2}$$

$$d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$$

□

Problem 2

- a) A set A in the metric space (S, d) is called **bounded**, if $\exists_{R>0} \wedge \exists x \in S : A \subset B_R(x)$.
 Prove that if A is unbounded then there exists a sequence $\{x_n\} \subset A$ such that

$$\forall_{m,n \in \mathbb{N}} \implies d(x_n, x_m) > 1.$$

Assumption 1. $m \neq n, m > n$

Definition 1. The open ball set $B_r(x)$ over metric space (S, d) is defined as

$$B_r(x) := \{y \in S : d(x, y) < r\}$$

Definition 2. A set A in the metric space (S, d) is called bounded, if

$$\exists_{R>0} \wedge \exists_{x \in S} : A \subset B_R(x)$$

Definition 3. A set A in the metric space (S, d) is called unbounded, if it is not bounded, (i.e.)

$$\forall_{R>0} \wedge \forall_{x \in S} : A \not\subset B_R(x)$$

Theorem 4. If $A \in (S, d)$ is unbounded, then

$$\exists \{x_n\} \subset A : \forall_{m,n \in \mathbb{N}} \implies d(x_n, x_m) > 1$$

Proof. $A \in (S, d)$ being unbounded means that

$$\forall_{R>0} \wedge \forall x \in S : A \not\subset B_R(x)$$

Since $\forall_{R>0} \wedge \forall_{x \in S} : A \not\subset B_R(x)$ and $B_R(x) := \{y \in S : d(x, y) < R\}$,

$$\forall_{x \in A} \exists_{y \in A} : d(x, y) \geq R$$

Since it is true for any $R > 0$ a sequence $\{x_n\}$ can be constructed with subsequent x_{n+1} so that $d(x_n, x_{n+1}) > R = 1$. \square

- b) Show that in the normed space $(V, |\cdot|)$ the open unit ball $B_r = \{x \in V : |x| < 1\}$ is a convex set. (i.e) $\forall_{x,y \in B_r}, \forall_{t \in [0,1]} \implies tx + (1-t)y \in B_r$

Definition 4. The open unit ball is defined as:

$$B_r := \{x \in V : |x| < 1\}$$

Definition 5. The set A is convex if and only if

$$\forall_{x,y \in A}, \forall_{t \in [0,1]} \implies tx + (1-t)y \in A$$

Theorem 5. The open unit ball set B_r is convex in $(V, |\cdot|)$.

Proof.

$$\begin{aligned} \forall_{x,y \in B_r}, \forall_{t \in [0,1]} &\implies tx + (1-t)y \in B_r \\ \forall_{x,y \in V : (|x| < 1) \wedge (|y| < 1)}, \forall_{t \in [0,1]} &\implies tx + (1-t)y \in V : |tx + (1-t)y| \end{aligned}$$

Since $\forall_{x,y \in V}, \forall_{t \in [0,1]} \implies tx + (1-t)y \in V$

$$(|x| < 1 \wedge |y| < 1 \implies |tx + (1-t)y| < 1) \iff tx + (1-t)y \in B_r$$

Clearly,

$$\forall_{x,y \in V} |x|, |y| < 1, \forall_{t \in [0,1]} tx + (1-t)y < 1$$

Therefore, the open unit ball set B_r is convex in $(V, |\cdot|)$. \square

Problem 3

For $(\mathbb{R}^2 = (x, y), d = \sqrt{x^2 + y^2})$,

a) Show that $D = \{(x, y) : x^2 + y^2 \leq 1\}$ is a closed set.

Definition 6. The set $A \subset V$ is called open if

$$\forall x \in A \exists \epsilon > 0 : B_\epsilon(x) \subset A$$

Definition 7. The set $A \subset V$ is called closed if A^c is open.

Theorem 6. The set $D = \{(x, y) : x^2 + y^2 \leq 1\}$ is closed.

Proof. By definition, D is closed iff D^c is open.

D^c is defined by

$$D^c = \{(x, y) : x^2 + y^2 > 1\}$$

By definition, D^c is open if

$$\forall (x, y) \in D^c \exists \epsilon > 0 : B_\epsilon((x, y)) \subset D^c$$

This means that every element in D^c must have an associated open ball set centered at that element with a positive radius that is fully contained by D^c .

Let $(x_b, y_b) \in B_\epsilon((x, y))$ for $\epsilon > 0$. This means

$$d((x, y), (x_b, y_b)) = \sqrt{x_b^2 + y_b^2} < \epsilon$$

By definition,

$$(x, y) \in D^c \implies x^2 + y^2 > 1$$

and therefore,

$$\sqrt{x^2 + y^2} = d((0, 0), (x, y)) > 1$$

From the triangle inequality, we have

$$\begin{aligned} d((x_b, y_b), (0, 0)) &\leq d((0, 0), (x, y)) + d((x, y), (x_b, y_b)) \\ d((x, y), (x_b, y_b)) &\geq d((x_b, y_b), (0, 0)) - d((0, 0), (x, y)) \\ d((x, y), (x_b, y_b)) &= \epsilon > d((x_b, y_b), (0, 0)) - 1 > 0 \end{aligned}$$

Therefore D^c is open and therefore D is closed. □

b) Find the infinite collection of open sets $\{A_n\}$ so that $\bigcap_n A_n = \overline{B_1(0)}$

$$\{A_n : \bigcap_n A_n = \overline{B_1(0)}\}$$

This means that the intersection of all sets in $\{A_n\}$ is the boundary of the unit ball set.

Problem 4

Let $S = \mathbb{R}^2$. Are the following sets open or closed within the metrics below?

$$A = \{(x, y) : x^2 + y^2 < 1\}$$

$$B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$$

$$C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$$

$$D = \{(x, y) : |x| + |y| < 2\}$$

$$E = \{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\}$$

a) Euclidean Metric: $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \|(x_1, y_1) - (x_2, y_2)\|_2$

i) $A = \{(x, y) : x^2 + y^2 < 1\}$

Open

ii) $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$

Closed

iii) $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$

Neither

iv) $D = \{(x, y) : |x| + |y| < 2\}$

Open??? *test it later...

v) $E = \{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\}$

Open??? *test it later...

b) Manhattan Metric:

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2| = \|(x_1, y_1) - (x_2, y_2)\|_1$$

i) $A = \{(x, y) : x^2 + y^2 < 1\}$

Open

ii) $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$

Closed

iii) $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$

Neither

iv) $D = \{(x, y) : |x| + |y| < 2\}$

Open

v) $E = \{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\}$

Open??? *test it later...

c) Highway Metric:

Definition 8. *The highway metric is defined as*

$$d_h((x_1, y_1), (x_2, y_2)) := \begin{cases} |y_1 - y_2|, & x_1 = x_2 \\ |y_1| + |y_2| + |x_1 - x_2|, & x_1 \neq x_2 \end{cases}$$

i) $A = \{(x, y) : x^2 + y^2 < 1\}$

Neither?? check again...

ii) $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$

Neither?? check again...

iii) $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$

Neither?? check again...

iv) $D = \{(x, y) : |x| + |y| < 2\}$

Open

v) $E = \{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\}$

Neither?? check again...

Problem 5

Let (S, d) be a metric space.

- a) Show that for all $A \subset B \subset S$ one has $\int(A) \subseteq \int(B)$ and $\overline{A} \subseteq \overline{B}$. Also provide an example of non-strictness.
- b) Is the following true: $\int(A \cup B) = \int(A) \cup \int(B)$?
- c) Is the following true: $\overline{A \cap B} = \overline{A} \cap \overline{B}$?

Problem 6

Give a topological proof of the infinitude of the set of prime numbers. (H. Furstenberg, 1955)

Denote $N_{a,b} := \{a + nb : b \in \mathbb{Z}\} \subset \mathbb{Z}$. Define the topology on \mathbb{Z} as follows: The set U will be called open if for any $a \in U$ there exists $b \in \mathbb{Z}$ so that $N_{a,b} \subset U$. Note that every open set is infinite.

a) Show that it is indeed a topology.

(i.e): any union of open sets is open and any finite intersection of open sets is open.

b) Show that $N_{a,b}$ is closed.

c) Show that $\mathbb{Z} \setminus \{-1, 1\}$ is open

d) Prove that the set \mathbb{P} of prime numbers cannot be finite.

Hint: $\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \in \mathbb{P}} N_{0,p}$