MATH 5301 Elementary Analysis - Homework 4

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Problem 1

Use the axioms of the ordered field, prove the following:

a) $(a > c) \wedge (b > d) \implies a + b > c + d$

$$(a > c) \wedge (b > d) \implies (a + b) > (c + d)$$

From (O3):

$$(a > c) \implies ((a + b) \geq (b + c)) \wedge ((a + d) \geq (c + d))$$
$$(b > d) \implies ((a + b) \geq (a + d)) \wedge ((b + c) \geq (c + d))$$

From (02):

$$((a + b) \geq (b + c)) \wedge ((b + c) \geq (c + d)) \implies (a + b) > (c + d)$$

b) $(a > c > 0) \wedge (b > d > 0) \implies ab > cd > 0$

$$(a > c > 0) \wedge (b > d > 0) \implies ab > cd > 0$$

From (O4):

$$(a > c > 0) \wedge (b > 0) \implies ab > bc > 0$$
$$(b > d > 0) \wedge (c > 0) \implies bc > cd > 0$$

From (O2):

$$(ab > bc > 0) \wedge (bc > cd > 0) \implies ab > cd > 0$$

c) $a > b > 0 \implies \frac{1}{a} < \frac{1}{b}$

$$a > b > 0 \implies \frac{1}{b} < \frac{1}{b}$$

From

$$a > 0 \implies a^{-1} > 0$$
$$b > 0 \implies b^{-1} > 0$$

From (O4):

$$(a > b > 0) \wedge (a^{-1} > 0) \implies aa^{-1} = 1 > ba^{-1} = \frac{b}{a} > 0$$
$$(a > b > 0) \wedge (b^{-1} > 0) \implies ab^{-1} = \frac{a}{b} > bb^{-1} = 1 > 0$$
$$(\frac{a}{b} > 1 > 0) \wedge (a^{-1} > 0) \implies \frac{a}{b}a^{-1} = \frac{1}{b} > (1)(a^{-1}) = \frac{1}{a} > 0$$

Therefore,

$$\frac{1}{a} < \frac{1}{b}$$

d) Let,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & X < 0 \end{cases}$$

prove,

$$\forall a, b \implies |a - b| \geq ||a| - |b||$$

$$\forall a, b \implies |a - b| \geq ||a| - |b||$$

When $a > b > 0$ (or $b > a > 0$),

$$\begin{aligned} |a - b| &= a - b \\ |a| &= a \\ |b| &= b \\ ||a| - |b|| &= a - b \\ |a - b| &= a - b = ||a| - |b|| \end{aligned}$$

The same is true for $0 < a < b$ and $0 < b < a$ by similar arguments.
For $a > 0 > b$,

$$\begin{aligned} |a| &= a \\ |b| &= -b \\ |a - b| &= |a| + |b| \\ |a| - |b| &= a - (-b) = a + b \\ ||a| - |b|| &= \begin{cases} |a| - |b| & |a| > |b| \\ |b| - |a| & |a| < |b| \end{cases} \end{aligned}$$

From (03):

$$\begin{aligned} |a - b| &= |a| + |b| \geq |a| - |b| \\ |a - b| &= |a| + |b| \geq |b| - |a| \\ \therefore\ |a - b| &\geq ||a| - |b|| \end{aligned}$$

Therefore $\forall a, b$,

$$|a - b| \geq ||a| - |b||$$

Problem 2

Determine which of the axioms satisfied by the set of real numbers are not satisfied by the following set:

a) Set $\mathbb{Q}$ of all rational numbers.

Set $\mathbb{Q}$ of rational numbers can be an ordered field, $\langle \mathbb{Q}, +, 0, \cdots, 1 \rangle$, but lacks (C) completness:

$$\forall A \subset \mathbb{Q} \ \nexists c \in \mathbb{Q} : c = \sup A$$

b) Set $\mathbb{Q}(\sqrt{2})$ of all numbers of form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$

Set $\mathbb{Q} := \left\{ a + b\sqrt{2} : a, b \in \mathbb{Q} \right\}$ can be an ordered field, $\langle \mathbb{Q}(\sqrt{2}), +, 0, \cdots, 1 \rangle$, but lacks completness (C):

$$\forall A \subset \mathbb{Q}(\sqrt{2}) \ \nexists c \in \mathbb{Q} : c = \sup A$$

c) Set $\mathbb{C}$ of all pairs of real numbers (a, b) with addition $(a, b) + (c, d) = (a + c, b + d)$, multiplication $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$, and ordered relation $(a, b) < (c, d) \iff (b \leq d) \wedge ((b = d \vee a < c))$.

Set $\mathbb{C} := \{(a, b) : a, b \in \mathbb{R}\}$ can satisfie the field conditions, $\langle \mathbb{C}, +, 0, \cdots, 1 \rangle$, but it is not ordered becouse it does not satisfy (O1).

Problem 3

Using the method of mathematical induction, prove the following statments: ($n \in \mathbb{N}$)

a) Bernoulli inequality: $\forall n \in \mathbb{N},\ \forall x > -1$,

$$(1+x)^n \geq 1 + nx$$

Theorem 1. $\forall n \in \mathbb{N},\ \forall x > -1$,

$$(1+x)^n \geq 1 + nx$$

Proof. For $n = 1$,

$$\begin{aligned}(1+x)^n &\geq 1 + nx \\ (1+x)^1 &\geq 1 + (1)x \\ 1 + x &\geq 1 + x\end{aligned}$$

For $n > 1$,

$$\begin{aligned}(1+x)^n \geq 1 + nx &\implies (1+x)^{n+1} \geq 1 + (n+1)x \\ &\implies (1+x)^n(1+x) \geq 1 + (n+1)x\end{aligned}$$

$\square$

b) For $n \in \mathbb{N}$,

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

c) For $n, q \in \mathbb{N}$,

$$(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) = \frac{1 - q^{2^{n+1}}}{1-q}$$

d) For $n \in \mathbb{N}$,

$$1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2 + 4n + 1)$$

e) For $n, k \in \mathbb{N}$,

$$\sum k = 0n(-1)^k \frac{n!}{k!(n-k)!} = 0, \sum k = 0n \frac{n!}{k!(n-k)!} = 2^n$$

Problem 4

Show that $\forall n \in \mathbb{N}, n \geq 2,$

a)

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

b)

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$

c)

$$\left(\frac{n+1}{2}\right)^n > n!$$

d)

$$2^{2^n} - 6 \vdots 10$$

Problem 5

a) Show that $\sqrt{2} \notin \mathbb{Q}$

Definition 1. $\sqrt{2} := x > 0 : x^2 = 2$

Theorem 2. $\sqrt{2} \notin \mathbb{Q}$

Proof. Assume $\sqrt{2} \in \mathbb{Q}$,

$$\sqrt{2} \in \mathbb{Q} \implies \exists m, n \in \mathbb{N} : \frac{m}{n} = \sqrt{2}$$

Also assume that m, n are coprime. (i.e) $\gcd(m, n) = 1$
Let $m = \sqrt{2}n$,

$$m = \sqrt{2}n \implies m^2 = 2n^2 \implies m^2 \vdots 2 \implies m \vdots 2$$

$$m \vdots 2 \implies \exists k \in \mathbb{N} : m = 2k \implies m^2 = (2k)^2 = 4k^2$$

$$4k^2 = 2n^2 \implies 2k^2 = n^2 \implies n^2 \vdots 2 \implies n \vdots 2$$

This is false becouse with $\gcd(m, n) = 1$, m and n cannot both be even. $\square$

b) Show that $\forall a, b \in \mathbb{Q}, a < b \implies \exists x \in \mathbb{R} \backslash \mathbb{Q} : a < x < b$

Theorem 3. $\forall a, b \in \mathbb{Q}, a < b \implies \exists x \in \mathbb{R} \backslash \mathbb{Q} : a < x < b$

Proof. idk $\square$

c) Show that $\forall a, b \in \mathbb{R} \backslash \mathbb{Q}, a < b \implies \exists x \in Q : a < x < b$

Theorem 4. $\forall a, b \in \mathbb{R} \backslash \mathbb{Q}, a < b \implies \exists x \in Q : a < x < b$

Proof. idk $\square$

Problem 6

a)

look at original doc....