MATH 5301 Elementary Analysis - Midterm Exam

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Problem 1

Problem: 100 soldiers stayed in a rank in front of the corporal. The corporal ordered all of them to turn left, but the soldiers were newbies, so they were not certain was it left from their perspective or from the corporal's point of view. So, some of them turned left and some turned right. After that at every second if two neighboring soldiers find themselves facing each other, they rotate by 180 degrees. Show that this process will not last forever.

Problem Formulation

Problem 2

Let $\mathbb{R}^\infty$ be the set of all sequences of real numbers

$$\mathbb{R}^\infty = \{a_1, a_2, \cdots \mid a_j \in \mathbb{R}\}$$

Define the relation $\mathcal{R}$ on $\mathbb{R}^\infty$ as follows: $a\mathcal{R}b$ if for some $j \in \mathbb{N} : a_j > b_j$ and for all $k < j \implies a_k = b_k$.
Does it mean this?

$$\exists_{j\in\mathbb{N}} : ((a_j > b_j) \wedge (\forall_{k<j} \implies a_k = b_k))$$

or this?

$$(\exists_{j\in\mathbb{N}} : (a_j > b_j \wedge \forall_{k<j})) \implies a_k = b_k$$

Prove that such a relation is an order relation on R^∞. Is it a total order?

$$\exists_{j\in\mathbb{N}} : (\forall_{k<j}\ a_k = b_k) \wedge (a_j > b_j)$$

Problem 3

Alice wrote some finite sequence of zeros and ones on the paper (e.g. 010010). Bob is allowed to replace any pair "10" by "00···01" with any (but finite) amount of zeros in front of 1. Bob can repeat this procedure as many times as he wants (if he will find "10" in the resulting sequence). Prove that Bob can perform such operation only finitely many times.

Problem 4

Present two essentially different total orderings of the field $\mathbb{Q}(\sqrt{2}) := \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$.

Problem Formulation

Definition 1. *The field $\mathbb{F} = \langle \mathbb{Q}(\sqrt{2}), +, 0, \cdot, 1 \rangle$ is defined with the set*

$$\mathbb{Q}(\sqrt{2}) := \left\{ a + b\sqrt{2} \;\middle|\; a, b \in \mathbb{Q} \right\}$$

and the operators

$$\begin{aligned} + &: \mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2}) := (a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2) \\ \cdot &: \mathbb{Q}(\sqrt{2}) \times \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2}) := (a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 + 2a_2 b_2, a_1 b_2 + a_2 b_1) \end{aligned}$$

It is also assumed that the standard field properties all apply.

This is not good... need to fix it....:

Problem 5

Infinitely many wizards $W_1, W_2, \ldots$ stay in the line. Each wizard wears a hat of one the three colors: Red, Yellow or Green. Every wizard W_n can see the hats of all the next wizards in line (i.e. Wn+1, Wn+2, etc.) Starting with the wizard W_1 every one has to guess the color of his own hat. If the wizard guesses correctly, he can go free. Otherwise he got dematerialized. Wizards discussed their strategy before this event. Show that if the wizards were smart enough, then only finitely many of them will disappear.