MATH 5301 Elementary Analysis - Homework 4

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Problem 1

Use the axioms of the ordered field, prove the following:

a) $(a > c) \wedge (b > d) \implies a + b > c + d$

$$(a > c) \wedge (b > d) \implies (a + b) > (c + d)$$

From (O3):

$$(a > c) \implies ((a + b) \geq (b + c)) \wedge ((a + d) \geq (c + d))$$
$$(b > d) \implies ((a + b) \geq (a + d)) \wedge ((b + c) \geq (c + d))$$

From (02):

$$((a + b) \geq (b + c)) \wedge ((b + c) \geq (c + d)) \implies (a + b) > (c + d)$$

b) $(a > c > 0) \wedge (b > d > 0) \implies ab > cd > 0$

$$(a > c > 0) \wedge (b > d > 0) \implies ab > cd > 0$$

From (O4):

$$(a > c > 0) \wedge (b > 0) \implies ab > bc > 0$$
$$(b > d > 0) \wedge (c > 0) \implies bc > cd > 0$$

From (O2):

$$(ab > bc > 0) \wedge (bc > cd > 0) \implies ab > cd > 0$$

c) $a > b > 0 \implies \frac{1}{a} < \frac{1}{b}$

$$a > b > 0 \implies \frac{1}{b} < \frac{1}{b}$$

From

$$a > 0 \implies a^{-1} > 0$$
$$b > 0 \implies b^{-1} > 0$$

From (O4):

$$(a > b > 0) \wedge (a^{-1} > 0) \implies aa^{-1} = 1 > ba^{-1} = \frac{b}{a} > 0$$
$$(a > b > 0) \wedge (b^{-1} > 0) \implies ab^{-1} = \frac{a}{b} > bb^{-1} = 1 > 0$$
$$(\frac{a}{b} > 1 > 0) \wedge (a^{-1} > 0) \implies \frac{a}{b}a^{-1} = \frac{1}{b} > (1)(a^{-1}) = \frac{1}{a} > 0$$

Therefore,

$$\frac{1}{a} < \frac{1}{b}$$

d) Let,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & X < 0 \end{cases}$$

prove,

$$\forall a, b \implies |a - b| \geq ||a| - |b||$$

$$\forall a, b \implies |a - b| \geq ||a| - |b||$$

When $a > b > 0$ (or $b > a > 0$),

$$\begin{aligned} |a - b| &= a - b \\ |a| &= a \\ |b| &= b \\ ||a| - |b|| &= a - b \\ |a - b| &= a - b = ||a| - |b|| \end{aligned}$$

The same is true for $0 < a < b$ and $0 < b < a$ by similar arguments.
For $a > 0 > b$,

$$\begin{aligned} |a| &= a \\ |b| &= -b \\ |a - b| &= |a| + |b| \\ |a| - |b| &= a - (-b) = a + b \\ ||a| - |b|| &= \begin{cases} |a| - |b| & |a| > |b| \\ |b| - |a| & |a| < |b| \end{cases} \end{aligned}$$

From (03):

$$\begin{aligned} |a - b| &= |a| + |b| \geq |a| - |b| \\ |a - b| &= |a| + |b| \geq |b| - |a| \\ \therefore\ |a - b| &\geq ||a| - |b|| \end{aligned}$$

Therefore $\forall a, b$,

$$|a - b| \geq ||a| - |b||$$

Problem 2

Determine which of the axioms satisfied by the set of real numbers are not satisfied by the following set:

a) Set $\mathbb{Q}$ of all rational numbers.

Set $\mathbb{Q}$ of rational numbers can be an ordered field, $\langle \mathbb{Q}, +, 0, \cdots, 1 \rangle$, but lacks (C) completeness:

$$\forall A \subset \mathbb{Q} \ \nexists c \in \mathbb{Q} : c = \sup A$$

b) Set $\mathbb{Q}(\sqrt{2})$ of all numbers of form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$

Set $\mathbb{Q} := \left\{ a + b\sqrt{2} : a, b \in \mathbb{Q} \right\}$ can be an ordered field, $\langle \mathbb{Q}(\sqrt{2}), +, 0, \cdots, 1 \rangle$, but lacks completeness (C):

$$\forall A \subset \mathbb{Q}(\sqrt{2}) \ \nexists c \in \mathbb{Q} : c = \sup A$$

c) Set $\mathbb{C}$ of all pairs of real numbers (a, b) with addition $(a, b) + (c, d) = (a + c, b + d)$, multiplication $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$, and ordered relation $(a, b) < (c, d) \iff (b \leq d) \wedge ((b = d \vee a < c))$.

Set $\mathbb{C} := \{(a, b) : a, b \in \mathbb{R}\}$ can satisfy the field conditions, $\langle \mathbb{C}, +, 0, \cdots, 1 \rangle$, but it is not ordered because it does not satisfy (O1).

Problem 3

Using the method of mathematical induction, prove the following statements: ($n \in \mathbb{N}$)

a) Bernoulli inequality: $\forall n \in \mathbb{N},\ \forall x > -1,\ (1+x)^n \geq 1 + nx$

Theorem 1. $\forall n \in \mathbb{N},\ \forall x > -1,$

$$(1+x)^n \geq 1 + nx$$

Proof. Proof by induction:
For $n = 1$,

$$\begin{aligned}(1+x)^n &\geq 1 + nx \\ (1+x)^1 &\geq 1 + (1)x \\ 1 + x &\geq 1 + x\end{aligned}$$

For $n > 1$,

$$\begin{aligned}(1+x)^n &\geq 1 + nx \\ (1+x)^n(1+x) &\geq (1+nx)(1+x) \\ (1+x)^{n+1} &\geq (1 + x + nx + nx^2) \\ &\geq 1 + (n+1)x + nx^2\end{aligned}$$

Since $n \geq 2 \implies nx^2 > 0$

$$1 + (n+1)x + nx^2 \geq 1 + (n+1)x$$

From (O2):

$$(1+x)^{n+1} \geq 1 + (n+1)x$$

Therefore $\forall n > 1$,

$$(1+x)^n \geq 1 + nx \implies (1+x)^{n+1} \geq 1 + (n+1)x$$

Therefore $\forall n \in \mathbb{N},\ \forall x > -1,$

$$(1+x)^n \geq 1 + nx$$

$\square$

b) For $n \in \mathbb{N}$, $\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

Theorem 2. *For $n \in \mathbb{N}$,*

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

Proof. Proof by induction: For $n = 1$,

$$\begin{aligned}
\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} &= 2 - \frac{n+2}{2^n} \\
\frac{1}{2} = 2 - \frac{1+2}{2^1} &= 2 - \frac{3}{2} \\
\frac{1}{2} &= \frac{1}{2}
\end{aligned}$$

For $n > 1$,

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n} \implies \frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^{n+1}} = 2 - \frac{n+2}{2^{n+1}}$$

$$\begin{aligned}
\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} &= 2 - \frac{n+2}{2^n} \\
\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}} &= 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} \\
&= 2 - \frac{n+2}{2^n}\frac{2}{2} + \frac{n+1}{2^{n+1}} \\
&= 2 - \frac{2(n+2)}{2^{n+1}} + \frac{n+1}{2^{n+1}} \\
&= 2 + \frac{n+1-2(n+2)}{2^{n+1}} \\
&= 2 + \frac{n+1-2n-2}{2^{n+1}} \\
&= 2 + \frac{-n-1}{2^{n+1}} \\
&= 2 + \frac{-(n+1)-2}{2^{n+1}} \\
&= 2 - \frac{(n+1)+2}{2^{n+1}}
\end{aligned}$$

Therefore $\forall n > 1$,

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n} \implies \frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}} = 2 - \frac{(n+1)+2}{2^{n+1}}$$

Therefore For $n \in \mathbb{N}$,

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

□

c) For $q \in \mathbb{R}, n \in \mathbb{N}$, $(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) = \frac{1-q^{2^{n+1}}}{1-q}$

Theorem 3. *For $q \in \mathbb{R}, n \in \mathbb{N}$,*

$$(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) = \frac{1-q^{2^{n+1}}}{1-q}$$

Proof. Proof by induction:
For $n = 1$,

$$\begin{aligned}
(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) &= \frac{1-q^{2^{n+1}}}{1-q} \\
(1+q)(1+q^{2^1}) &= \frac{1-q^{2^{1+1}}}{1-q} \\
(1+q^2+q+q^3) &= \frac{1-q^4}{1-q} \\
(1+q+q^2+q^3)(1-q) &= \frac{1-q^4}{1-q}(1-q) \\
1+q+q^2+q^3-q-q^2-q^3-q^4 &= 1-q^4 \\
1-q^4 &= 1-q^4
\end{aligned}$$

For $n > 1$

$$\begin{aligned}
(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) &= \frac{1-q^{2^{n+1}}}{1-q} \\
(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n})(1+q^{2^{n+1}}) &= \frac{1-q^{2^{n+1}}}{1-q}(1+q^{2^{n+1}}) \\
&= \frac{\left(1-q^{2^{n+1}}\right)\left(1+q^{2^{n+1}}\right)}{1-q} \\
&= \frac{1-q^{2^{n+1}}+q^{2^{n+1}}+\left(-q^{2^{n+1}}\right)\left(q^{2^{n+1}}\right)}{1-q} \\
&= \frac{1-q^{2^{n+1}+2^{n+1}})}{1-q} \\
&= \frac{1-q^{2\left(2^{n+1}\right)}}{1-q} \\
(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n})(1+q^{2^{n+1}}) &= \frac{1-q^{2^{n+2}}}{1-q}
\end{aligned}$$

Therefore $\forall n > 1$,

$$(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) = \frac{1-q^{2^{n+1}}}{1-q} \implies (1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n})(1+q^{2^{n+1}}) = \frac{1-q^{2^{n+2}}}{1-q}$$

Therefore $\forall n \in \mathbb{N}$,

$$(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) = \frac{1-q^{2^{n+1}}}{1-q}$$

$\square$

d) For $n \in \mathbb{N}$, $1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1)$

Theorem 4. *For $n \in \mathbb{N}$,*

$$1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1)$$

Proof. Proof by induction:
For $n = 1$,

$$\begin{aligned}
1^3 + 3^3 + \cdots + (2n+1)^3 &= (n+1)^2(2n^2+4n+1) \\
1^3 + (2(1)+1)^3 &= ((1)+1)^2(2(1)^2+4(1)+1) \\
1^3 + 3^3 &= (2)^2(2(1)+4+1) \\
1 + 27 &= (4)(2+4+1) \\
28 &= (4)(7) \\
28 &= (4)(7) \\
28 &= 28
\end{aligned}$$

For $n > 1$,

$$\begin{aligned}
1^3 + 3^3 + \cdots + (2n+1)^3 &= (n+1)^2(2n^2+4n+1) \\
1^3 + 3^3 + \cdots + (2n+1)^3 + (2(n+1)+1)^3 &= (n+1)^2(2n^2+4n+1) + (2(n+1)+1)^3 \\
&= (n+1)^2(2n^2+4n+1) + (2n+3)^3
\end{aligned}$$

$$\begin{aligned}
&= (n+1)(n+1)(2n^2+4n+1) + (2n+3)(2n+3)(2n+3) \\
&= (n^2+2n+1)(2n^2+4n+1) + 27 + 54n + 36n^2 + 8n^3 \\
&= 2n^4 + 8n^3 + 11n^2 + 6n + 1 + 8n^3 + 36n^2 + 54n + 27 \\
&= 2n^4 + 16n^3 + 47n^2 + 60n + 28 \\
&= (n+2)^2(2n^2+8n+7) \\
&= ((n+1)+1)^2(2(n+1)^2 - 4n - 2 + 4(n+1) + 4n - 4 + 7) \\
&= ((n+1)+1)^2(2(n+1)^2 + 4(n+1) + 1)
\end{aligned}$$

Therefore $\forall n > 1$,

$$1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1) \implies$$

$$\implies 1^3 + 3^3 + \cdots + (2n+1)^3 + (2(n+1)+1)^3 = ((n+1)+1)^2(2(n+1)^2 + 4(n+1) + 1)$$

Therefore $\forall n \in \mathbb{N}$,

$$1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1)$$

$\square$

e) For $n, k \in \mathbb{N}$, $\sum_{k=0}^{n} (-1)^k \dfrac{n!}{k!(n-k)!} = 0, \sum_{k=0}^{n} \dfrac{n!}{k!(n-k)!} = 2^n$

Definition 1. *The factorial of a number, $n!$, is defined as*

$$n! := (1)(2)(3)\cdots(n-1)(n)$$

Definition 2. *The combination of two numbers, $\binom{n}{k}$, is defined as*

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

i) $\sum_{k=0}^{n} (-1)^k \dfrac{n!}{k!(n-k)!} = 0$

Theorem 5. *For $n, k \in \mathbb{N}$,*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof. For $n = 1$,

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k \binom{n}{k} &= 0 \\
\sum_{k=0}^{1} (-1)^k \binom{1}{k} &= (-1)^0 \binom{1}{0} + (-1)^1 \binom{1}{1} \\
&= (1)(1) + (-1)(1) \\
&= 0
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{1} (-1)^k \binom{1}{k} = 0$$

For $n > 2$,

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k \binom{n}{k} &= 0 \\
\sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} &= 0 \\
(-1)^0 \frac{n!}{0!(n-0)!} + (-1)^1 \frac{n!}{1!(n-1)!} + \cdots + (-1)^n \frac{n!}{n!(n-n)!} &= 0 \\
(1)\frac{n!}{0!n!} + (-1)\frac{n!}{1!(n-1)!} + \cdots + (-1)^{n-2} \frac{n!}{(n-2)!2!} + (-1)^{n-1} \frac{n!}{(n-1)!1!} + (-1)^n \frac{n!}{n!0!} &= 0 \\
(1)\frac{n!}{n!} + (-1)\frac{n!}{(n-1)!} + \cdots + (-1)^{n-2} \frac{n!}{(n-2)!2} + (-1)^{n-1} \frac{n!}{(n-1)!} + (-1)^n \frac{n!}{n!} &= 0 \\
(1)\frac{n!}{n!} + (-1)\frac{n!}{(n-1)!} + \cdots + (-1)^{n-1} \frac{n!}{(n-1)!} + (-1)^n \frac{n!}{n!} + (-1)^{n+1} \frac{n+1!}{n+1!} &= (-1)^{n+1} \frac{n+1!}{n+1!}
\end{aligned}$$

Therefore $\forall n \in \mathbb{N}, n \geq 2$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

□

ii) $\sum_{k=0}^{n} \dfrac{n!}{k!(n-k)!} = 2^n$

Theorem 6. *For $n, k \in \mathbb{N}$,*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof. By induction:
For $n = 2$,

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} &= 2^n \\
\sum_{k=0}^{2} \binom{2}{k} &= 2^2 \\
\binom{2}{0} + \binom{2}{1} + \binom{2}{2} &= 4 \\
1 + 2 + 1 &= 4 \\
4 &= 4
\end{aligned}$$

For $n > 2$,

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} &= 2^n \\
\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} &= 2^n \\
\frac{n!}{0!(n-0)!} + \frac{n!}{1!(n-1)!} + \cdots + \frac{n!}{n!(n-n)!} &= 2^n \\
\frac{n!}{0!n!} + \frac{n!}{1!(n-1)!} + \cdots + \frac{n!}{(n-2)!2!} + \frac{n!}{(n-1)!1!} + \frac{n!}{n!0!} &= 2^n \\
\left(\frac{n!}{0!n!} + \frac{n!}{1!(n-1)!} + \cdots + \frac{n!}{(n-2)!2!} + \frac{n!}{(n-1)!1!} + \frac{n!}{n!0!}\right)\frac{2(n+1)}{n+1} &= 2^n 2 \\
\frac{(n+1)!}{0!n!(n+1)} + \frac{(n+1)!}{1!(n-1)!(n+1)} + \cdots + \frac{(n+1)!}{(n-2)!2!(n+1)} + \frac{(n+1)!}{(n-1)!1!} + \frac{(n+1)!}{n!0!} &= 2^{n+1} \\
\frac{(n+1)!}{0!n!(n+1)} + \frac{(n+1)!}{1!(n-1)!(n+1)} + \cdots + \frac{(n+1)!}{(n-2)!2!(n+1)} + \frac{(n+1)!}{(n-1)!1!} + \frac{(n+1)!}{n!0!} &= 2^{n+1} \\
\sum_{k=0}^{n+1} \binom{n}{k} &= 2^{n+1}
\end{aligned}$$

$\square$

Problem 4

Show that $\forall n \in \mathbb{N}, n \geq 2$,

a) $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

Theorem 7. *For $n \in \mathbb{N}$ and $n \geq 2$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$*

Proof. By Induction:
For $n = 2$,

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} &> \sqrt{n} \\
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} &> \sqrt{2} \\
1 + \frac{1}{2} &> \sqrt{2} \\
\frac{3}{2} &> \sqrt{2} \\
\left(\frac{3}{2}\right)^2 &> \left(\sqrt{2}\right)^2 \\
\frac{9}{4} &> 2 \\
9 &> (4)(2) \\
9 &> 8
\end{aligned}$$

Therefore, for $n = 2$,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

For $n > 2$,

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} &> \sqrt{n} \\
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} &> \sqrt{n} + \frac{1}{\sqrt{n+1}} \\
\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}\right)\sqrt{n+1} &> \left(\sqrt{n} + \frac{1}{\sqrt{n+1}}\right)\sqrt{n+1} \\
\frac{\sqrt{n+1}}{\sqrt{1}} + \frac{\sqrt{n+1}}{\sqrt{2}} + \cdots + \frac{\sqrt{n+1}}{\sqrt{n}} + \frac{\sqrt{n+1}}{\sqrt{n+1}} &> \sqrt{n}\sqrt{n+1} + 1
\end{aligned}$$

$\square$

b) $\dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$

Theorem 8. $\forall n \in \mathbb{N}, n \geq 2,$

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$

Proof. For $n = 2$,

$$\begin{aligned}
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} &> 1 \\
\frac{1}{(2)+1} + \frac{1}{(2)+2} + \cdots + \frac{1}{3(2)+1} &> 1 \\
\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{7} &> 1 \\
\sum_{k=3}^{7} \frac{1}{k} &> 1 \\
\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} &> 1 \\
\frac{140}{420} + \frac{105}{240} + \frac{84}{240} + \frac{70}{240} + \frac{60}{420} &> 1 \\
\frac{459}{420} &> 1 \\
459 &> 420
\end{aligned}$$

Therefore for $n = 2$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$

For $n > 2$,

$$\begin{aligned}
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} &> 1 \\
\sum_{k=n+1}^{3n+1} \frac{1}{k} &> 1 \\
\sum_{k=n+1}^{3n+1} \frac{1}{k} + \sum_{k=3n+2}^{3(n+1)+1} \frac{1}{k} &> 1 + \sum_{k=3n+2}^{3(n+1)+1} \frac{1}{k} \\
\frac{1}{n+1} + \sum_{k=(n+1)+1}^{3(n+1)+1} \frac{1}{k} &> 1 + \sum_{k=3n+2}^{3(n+1)+1} \frac{1}{k}
\end{aligned}$$

Since $\frac{1}{n+1} > \sum_{k=3n+2}^{3(n+1)+1} \frac{1}{k}$, by (O3) this implies that

$$\sum_{k=(n+1)+1}^{3(n+1)+1} \frac{1}{k} > 1$$

Therefore for $n \geq 2$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$

$\square$

c) $\left(\frac{n+1}{2}\right)^n > n!$

Theorem 9. *For $n \in \mathbb{N}, n \geq 2$,*

$$\left(\frac{n+1}{2}\right)^n > n!$$

Proof. For $n = 2$,

$$\begin{aligned}
\left(\frac{n+1}{2}\right)^n &> n! \\
\left(\frac{2+1}{2}\right)^2 &> 2! \\
\left(\frac{3}{2}\right)^2 &> (2)(1) \\
\frac{9}{4} &> 2 \\
9 &> 8
\end{aligned}$$

For $n > 2$,

$$\begin{aligned}
\left(\frac{n+1}{2}\right)^n &> n! \\
\left(\frac{n+1}{2}\right)^n (n+1) &> n!(n+1) \\
\left(\frac{n+1}{2}\right)^n (n+1) &> (n+1)! \\
\frac{(n+1)^{n+1}}{2^n} &> (n+1)! \\
\frac{\frac{1}{2}((n+1)+1)^{n+1}}{2^n} &> (n+1)! \\
\frac{((n+1)+1)^{n+1}}{2^{n+1}} &> (n+1)! \\
\left(\frac{(n+1)+1}{2}\right)^{n+1} &> (n+1)!
\end{aligned}$$

Therefore for $n \geq 2$,

$$\left(\frac{n+1}{2}\right)^n > n!$$

$\square$

d) $(2^{2^n} - 6) \vdots 10$

Theorem 10. *For $n \in \mathbb{N}, n \geq 2$,*

$$(2^{2^n} - 6) \vdots 10$$

Proof. By induction,
For $n = 2$,

$$(2^{2^n} - 6) \vdots 10$$

$$(2^{2^{(2)}} - 6) \vdots 10$$

$$2^4 - 6 \vdots 10$$

$$16 - 6 \vdots 10$$

$$10 \vdots 10$$

For $n > 2$,

$$\begin{aligned}
(2^{2^n} - 6) &\vdots 10 \\
(2^{2^n} - 6) &\mod 10 = 0 \\
2^{2^n} &\mod 10 = 6 \\
(2^{2^n})^2 &\mod 10 = 6^2 = 36 \\
(2^{2^n 2}) &\mod 10 = 6 \\
(2^{2^{n+1}} - 6) &\mod 10 = 0 \\
(2^{2^{n+1}} - 6) &\vdots 10
\end{aligned}$$

Therefore $\forall n \in \mathbb{N}, n \geq 2$,

$$(2^{2^n} - 6) \vdots 10$$

$\square$

Problem 5

a) Show that $\sqrt{2} \notin \mathbb{Q}$

Definition 3. $\sqrt{2} := x > 0 : x^2 = 2$

Theorem 11. $\sqrt{2} \notin \mathbb{Q}$

Proof. Assume $\sqrt{2} \in \mathbb{Q}$,

$$\sqrt{2} \in \mathbb{Q} \implies \exists m, n \in \mathbb{N} : \frac{m}{n} = \sqrt{2}$$

Also assume that m, n are coprime. (i.e) $\gcd(m, n) = 1$
Let $m = \sqrt{2}n$,

$$m = \sqrt{2}n \implies m^2 = 2n^2 \implies m^2 \vdots 2 \implies m \vdots 2$$

$$m \vdots 2 \implies \exists k \in \mathbb{N} : m = 2k \implies m^2 = (2k)^2 = 4k^2$$

$$4k^2 = 2n^2 \implies 2k^2 = n^2 \implies n^2 \vdots 2 \implies n \vdots 2$$

This is false becouse with $\gcd(m, n) = 1$, m and n cannot both be even. $\square$

b) Show that $\forall a, b \in \mathbb{Q}, a < b \implies \exists x \in \mathbb{R} \backslash \mathbb{Q} : a < x < b$

Theorem 12. $\forall a, b \in \mathbb{Q}, a < b \implies \exists x \in \mathbb{R} \backslash \mathbb{Q} : a < x < b$

Proof. Since $\mathbb{Q}$ lacks completeness, some irrational element $x \in \mathbb{R} \backslash \mathbb{Q}$ exists between all two rational numbers $a, b \in \mathbb{Q} : a \neq b$. Therefore, $a < b \implies \exists x : a < x < b$. $\square$

c) Show that $\forall a, b \in \mathbb{R} \backslash \mathbb{Q}, a < b \implies \exists x \in Q : a < x < b$

Theorem 13. $\forall a, b \in \mathbb{R} \backslash \mathbb{Q}, a < b \implies \exists x \in Q : a < x < b$

Proof. Since the set of all irrational numbers, $\mathbb{R} \backslash \mathbb{Q}$, lacks completeness, some rational elements $x \in \mathbb{Q}$ must exists between each irrational numbers $a, b \in \mathbb{R} \backslash \mathbb{Q}$. Therefore $a < b \implies \exists x : a < x < b$. $\square$

Problem 6

Prove that for any n:

a)

Theorem 14. *For any configurations of n straight lines on a plane, one could color the plane in two colors so that every two parts with the same boundary would have different colors.*

Proof. Proof by induction:
For $n = 1$, When one line splits up a plane there are two sections that can be painted into seperate colors. Therefore Theorem 14 is true. For $n > 1$, When n lines split up a plane and each section is painted a different color, the when an additional line is added, all sections that it goes throug will be slit in two and then the sections can still be pained alternativly. Therefore Theroem 14(n) $\implies$ Theorem 14 ($n + 1$). Therefore, $\forall n \in \mathbb{N}$, Theorem 14 is true. $\square$

b)

Theorem 15. *For any set of n squares, one can partitian them into a finite amount of pieces and which will constitute one square.*

Proof. First, when $n = 1$, the square is already a square.
For every n squares capable of becoming a square, a new square (of size x^2) can then be partitioned into a square and two rectangles making an L around the n square to be the $n + 1$ square. $\square$

c) What's wrong with this theorem?

Theorem 1. All the numbers are equal. In other words, the statement P_n is true for all $n \in \mathbb{N}$, where P_n is "if $\{a_1, a_2, \dots, a_n\}$ is a collection of n numbers, then $a_1 = a_2 = \cdots = a_n$".

Proof. P_1 is true. Let $\{a_1, a_2, \dots, a_{n+1}\}$ be a collection of $n + 1$ numbers. Consider the collection $\{a_1, a_2, \dots, a_n\}$. Thanks to P_n it follows that $a_1 = \cdots = a_n$ denote this number by b. Now, consider the collection $\{a_2, a_4, \dots, a_{n+1}\}$. It contains n numbers, annd hence by P_n we get $a_2 = \cdots = a_n = a_{n+1}$. But $a_2 = \cdots = a_n = b$, therefore $a_{n+1} = a_n = b$. So $a_1 = \cdots = a_{n+1} = b$ and P_{n+1} follows.

A lot of issues exist, but one obvious one is that P_n saying a collection is equal to b does not also indicate a different collection is b. The "numbers" also doesn't indicate what numbers a_i are selected from.