MATH 5301 Elementary Analysis - Final Exam

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2021, December 7^{th}

Problem 1

For each $n \in \mathbb{N}$ define the set

$$Q_n := \left\{ \frac{1}{pq} \ : \ 0 < p < q \leq n; \ p + q > n; \ \gcd(p,q) = 1 \right\}$$

Let $f(n)$ be the sum of all elements of Q_n.
Find $\inf_n f(n)$.

Definition 1. *Let the set Q_n be defined for all $n \in \mathbb{N}$ as*

$$Q_n := \left\{ \frac{1}{pq} \ : \ 0 < p < q \leq n; \ p + q > n; \ \gcd(p,q) = 1 \right\}$$

Definition 2. *Let $f(n)$ be the sum of all elements within Q_n.*

Definition 3. *A lower bound of subset A in the partially ordered set $(S, \leq)$ is defined by*

$$a \in S \ : \ a \leq x \forall_{x \in A}$$

A lower bound of a is called an <u>infimum</u> *of set $A \in (S, \leq)$, denoted as $a = \inf A$, is the greatest lower bound. i.e.*

$$\forall_{y \in S : a \leq x \forall_{x \in A}} y \leq a$$

Definition 4. *The* <u>Greatest Common Divisor</u> *of two nonzero integers $a, b \in \mathbb{Z} \neq 0$, $\gcd(a,b)$, is defined as the largest positive integer, $d \in \mathbb{Z}_+$, so that d is a divisor of both a and b. i.e:*

$$\gcd(a,b) := d \in \mathbb{Z}_+ \ : \ (a \vdots d) \wedge (b \vdots d) \wedge (\forall_{x \in \mathbb{Z}_+ \ : \ a,b \vdots x} d \geq x)$$

Additionally, a and b are considered <u>coprime</u> *if $\gcd(a,b) = 1$.*

Assumption 1. *For this problem it is assumed that $\gcd$ is only defined within $\mathbb{Z}_+$, although I believe this can also be expanded to other less-strict ordered sets in the same way.*

Assumption 2. *It is assumed that the sum of all elements in the empty set is 0, i.e. $\sum_i \emptyset = 0$.*

Theorem 1.

$$\inf_{n\in\mathbb{N}} f(n) = 0$$

Proof. Proof by induction.
For $n = 1$, $\neg\exists_{p,q\in\mathbb{Z}\ :\ 0<p<q\leq 1}$ meaning that $Q_1 = \emptyset$.
This implies that $f(1) = \sum_i \emptyset = 0$ and that $f(1) \geq 0$.
For $n = 2$,

$$(p,q) \in \{(p,q)\ :\ 0 < p < q \leq 2;\ p+q > n;\ \gcd(p,q) = 1\} = \{(1,2)\}$$

The set Q_2 is then defined as

$$Q_2 = \left\{\frac{1}{pq}\ :\ (p,q) \in \{(1,2)\}\right\} = \left\{\frac{1}{(1)(2)}\right\} = \left\{\frac{1}{2}\right\}$$

Therefore,

$$f(2) = \sum_i \left\{\frac{1}{2}\right\} = \frac{1}{2}$$

It is clear that $f(2) = \frac{1}{2} \geq 0$.
For $n = 3$,

$$(p,q) \in \{(p,q)\ :\ 0 < p < q \leq 3;\ p+q > n;\ \gcd(p,q) = 1\} = \{(1,3),(2,3)\}$$

The set Q_3 is then defined as

$$Q_3 = \left\{\frac{1}{pq}\ :\ (p,q) \in \{(1,3),(2,3)\}\right\} = \left\{\frac{1}{(1)(3)}, \frac{1}{(2)(3)}\right\} = \left\{\frac{1}{3}, \frac{1}{6}\right\}$$

Therefore,

$$f(3) = \sum_i \left\{\frac{1}{3}, \frac{1}{6}\right\} = \frac{1}{3} + \frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

It is clear that $f(3) = \frac{1}{2} \geq 0$.
For $n = 4$,

$$(p,q) \in \{(p,q)\ :\ 0 < p < q \leq 4;\ p+q > n;\ \gcd(p,q) = 1\} = \{(1,4),(2,3),(3,4)\}$$

The set Q_4 is then defined as

$$Q_4 = \left\{\frac{1}{pq}\ :\ (p,q) \in \{(2,3),(3,4)\}\right\} = \left\{\frac{1}{(1)(4)}, \frac{1}{(2)(3)}, \frac{1}{(3)(4)}\right\} = \left\{\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\right\}$$

Therefore,

$$f(4) = \sum_i \left\{\frac{1}{6}, \frac{1}{12}\right\} = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{3+2+1}{12} = \frac{6}{12} = \frac{1}{2}$$

It is clear that $f(4) = \frac{1}{2} \geq 0$.
For and arbritrary $n \in \mathbb{N}$,

$$\begin{aligned}(p,q) \in \{(p,q)\ :\ 0 < p < q \leq n;\ p+q > n;\ \gcd(p,q) = 1\}&\\ = \{(1,n),(2,n-\star),(3,n-\star)\ldots,(n-2,n-1),(n-1,n)\}&\end{aligned}$$

$$\begin{aligned}Q_n &= \left\{\frac{1}{pq}\ :\ (p,q) \in \{(1,n),(2,n-\star),\ldots,(n-2,n-1),(n-1,n)\}\right\}\\ &= \left\{\frac{1}{(1)(n)}, \frac{1}{(2)(n-1)}, \ldots, \frac{1}{(n-2)(n-1)}, \frac{1}{(n-1)(n)}\right\}\\ &= \left\{\frac{1}{n}, \frac{1}{2(n-\star)}, \ldots, \frac{1}{(n-2)(n-1)}, \frac{1}{n(n-1)}\right\}\end{aligned}$$

where $\star$ is dependent for on divisibility properties between n and 2, 3, 4, etc. It is important to note that each increase of n will cause every term to decrease in magnitude individually but additional elements are added that result to adding up to $\frac{1}{2}$ again.
However, eventually this will reach a point where a lack of prime numbers in a region makes it so that the only coprime numbers satisfying the conditions are adjacent to one another, which leads to the following:

$$\begin{aligned}
f(n) = \sum_i Q_n &= \frac{1}{n} + \cdots + \frac{1}{(\frac{n}{2})(\frac{n}{2}+1)} + \cdots + \frac{1}{n(n-1)} \\
f(n+1) &= \left(\sum_i Q_n\right)\left(\frac{n!}{(n+1)!}\right) + \frac{1}{(n+1)} \\
&= \frac{1}{n}\frac{n!}{(n+1)!} + \cdots + \frac{1}{(\frac{n}{2})(\frac{n}{2}+1)}\frac{n!}{(n+1)!} + \cdots + \frac{1}{n(n-1)}\frac{n!}{(n+1)!} + \frac{1}{n+1} \\
&= \frac{n!}{n(n+1)n!} + \cdots + \frac{n!}{\frac{n}{2}(\frac{n}{2}-1)(n+1)n!} + \cdots + \frac{n!}{n(n-1)(n+1)n!} + \frac{1}{n+1} \\
&= \sum_i Q_{n+1} = \frac{1}{n+1} + \cdots + \frac{1}{(\frac{n+1}{2})(\frac{n+1}{2}+1)} + \cdots + \frac{1}{n(n+1)}
\end{aligned}$$

essentially every (p, q) becomes $(q, q+1)$ and the new $\frac{1}{(n+1)}$ is added.
Anyway, the point is that $\forall_{n\in\mathbb{N}\,:\,n>1} f(n) \geq \frac{1}{2}$; however, because $f(n)$ is included, $\frac{1}{2} \leq f(n) \forall_{n\in N}$ since $Q_1 = \emptyset \implies f(1) = 0$.
Therefore,

$$\inf_n f(n) = 0$$

$\square$

Problem 2

Let