MATH 5301 Elementary Analysis - Homework 9

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2021, November 5$^{\text{th}}$

Problem 1

Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two equivalent norms on $\mathbb{R}^n$.

Definition 1. *For $\|\cdot\|_a, \|\cdot\|_b$ on S, $\|\cdot\|_a$ is said to be stronger then $\|\cdot\|_b$ if*

$$\forall \{x_n\} \subset S \ : \ x_n \xrightarrow[d_a]{} x \implies x_n \xrightarrow[d_b]{} x$$

Definition 2. *$\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be equivalent, $\|\cdot\|_a \sim \|\cdot\|_b$, if $\|\cdot\|_a$ is stronger then $\|\cdot\|_b$ and $\|\cdot\|_b$ is stronger then $\|\cdot\|_a$. This means that*

$$\|\cdot\|_a \sim \|\cdot\|_b \iff \exists \alpha, \beta \in \mathbb{R}_{>0} : \forall_{x \in S} \alpha \|\cdot\|_b \leq \|\cdot\|_a \leq \beta \|x\|_b$$

a) Prove that if the set A is closed in the a-norm, then it is closed in b-norm.

Definition 3. *The set $A \subset V$ is called open if*

$$\forall_{x \in A} \exists_{\epsilon > 0} : B_\epsilon(x) \subset A$$

or equivalently,

$$\forall_{x \in A} \exists_{\epsilon > 0} : \forall_{y \in V} \|x - y\| < \epsilon \implies y \in A$$

Definition 4. *The set $A \subset V$ is called closed if A^c is open.*

Theorem 1. *If the set A is closed in the a-norm, then it is closed in b-norm.*

Proof. Set A being closed in a-norm implies A^c is open in a-norm.

$$\forall_{x \in A^c} \exists_{\epsilon_a > 0} : \forall_{y \in S} \ \|x - y\|_a < \epsilon_a \implies y \in A^c$$

Additionally, since $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$ means thats

$$\exists_{\alpha, \beta > 0} : \forall_{x \in S} \ \alpha \|\cdot\|_b \leq \|\cdot\|_a \leq \beta \|\cdot\|_b$$

Therefore, $\|x - y\|_a \leq \beta \|x - y\|_b$ and then

$$\begin{aligned}
\forall_{x \in A^c} \exists_{\epsilon_a > 0} \ &: \ \forall_{y \in S} \ \|x - y\|_a \leq \beta \|x - y\|_b < \epsilon_a \implies y \in A^c \\
\forall_{x \in A^c} \exists_{\epsilon_b > 0} \ &: \ \forall_{y \in S} \ \|x - y\|_b < \epsilon_b \implies y \in A^c
\end{aligned}$$

where $\epsilon_b \geq \dfrac{\epsilon_a}{\beta}$ $\square$

b) Prove that if the set A is compact in the $a-norm$ then it is compact in the b-norm.

Definition 5. *Let (S, d) be a metric space with $A \subset S$,*

a. *For $\{U_\alpha\}_{\alpha\in A}$, $U_\alpha \subset S$, is a **cover** of the set A if*

$$A \subset \bigcup_{\alpha\in A} U_\alpha$$

b. *A cover $\{U_\alpha\}_{\alpha\in A}$ of A is an **open cover** if $\forall_{\alpha\in A}\ U_\alpha$ is an open set.*

c. *$\{V_\beta\}_{\beta\in B}$ is called a **subcover** of $\{U_\alpha\}_{\alpha\in A}$ if*

   (a) *$\{V_\beta\}_{\beta\in B}$ is a cover of A*

   (b) *$\forall_{\beta\in B}\exists_{\alpha\in A} V_\beta = U_\alpha$*

d. *A cover with a finite number of sets is called a **finite cover**.*

Definition 6. *For $A \subset (S, d)$, A is **compact** if for every open cover of A there exists a finite sub cover. Which is equivalent to saying all sequences within A converge to a set point in A. (i.e)*

$$\forall_{a_{k\,k\in\mathbb{N}}}\exists_{a_{n_k}} : a_{n_k} \to a \in A$$

Definition 7. *A sequence $\{x_n\}$is called Cauchy if*

$$\forall_{\epsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{l_1,l_2\geq N}\left\|x_{n_{l_1}} - x_{n_{l_2}}\right\| < \epsilon$$

Theorem 2. *If the set A is compact in the a-norm, then it is compact in the b-norm.*

Proof. Set A being compact in a-norm means that every sequence in A satisfies the Cauchy sequence property:

$$\forall_{\epsilon>0}\ \exists_{N\in\mathbb{N}}\ \forall_{l_1,l_2\geq N}\left\|x_{n_{l_1}} - x_{n_{l_2}}\right\|_a < \epsilon$$

Additionally, since $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$ means thats

$$\exists_{\alpha,\beta>0} : \forall_{x\in S}\ \alpha\|\cdot\|_b \leq \|\cdot\|_a \leq \beta\|\cdot\|_b$$

Therefore, $\|x-y\|_a \leq \beta\|x-y\|_b$ and then

$$\forall_{\epsilon_a>0}\ \exists_{N\in\mathbb{N}}\ \forall_{l_1,l_2\geq N}\left\|x_{n_{l_1}} - x_{n_{l_2}}\right\|_a \leq \beta\left\|x_{n_{l_1}} - x_{n_{l_2}}\right\|_b < \epsilon_a$$
$$\forall_{\epsilon_b>0}\ \exists_{N\in\mathbb{N}}\ \forall_{l_1,l_2\geq N}\left\|x_{n_{l_1}} - x_{n_{l_2}}\right\|_b < \epsilon_b$$

where $\epsilon_b \geq \dfrac{\epsilon_a}{\beta}$ $\square$

Problem 2

Consider the set l^∞ of all real-valued sequences, endowed with the sup-norm: $\|l\|_\infty = \sup_{n\in\mathbb{N}} |l_n|$.

a) Prove that l^∞ is complete.

Definition 8. *A metric space is (S, d) is a complete metric space if every Cauchy sequence in S converges.*

Definition 9. *The set A in norm space is $(S, \|\cdot\|)$ is a complete set if every Cauchy sequence in A converges to a limit in A.*

Definition 10. *Let the set l^∞ be the set of real-valued sequences:*

$$l^\infty := \{\{l_n\}_{n\in\mathbb{N}} \ : \ l_n \in \mathbb{R}\}$$

Definition 11. *Let the norm space be defined as $(l^\infty, \|l\|_\infty)$ where*

$$\|l\|_\infty = \sup_{n\in\mathbb{N}} |l_n|$$

Theorem 3. *The set l^∞ is complete.*

Proof. Let $\{x_m\}$ denote any cauchy sequence in l^∞, which is in l^∞ by definition. For all $m \geq 1$, define

$$l_m = \{x_1^{(m)}, x_2^{(m)}, \cdots\} \in l^\infty$$

Clearly, $\forall_{j\in\mathbb{R}_{>0}}$ the sequence $\{x_j^{(m)}\}$ is a Cauchy sequence (in $\mathbb{R}$) therefore it converges to $x_j \in \mathbb{R}$. Since $\mathbb{R}$ is complete and $\forall l_m \in l^\infty \implies \lim_{m\to\infty} l_m = l$ where $l \in l^infty$, the set l^∞ is complete (because all Cauchy sequences in l^∞ converge within l^∞). $\square$

b) Prove that l^∞ is not compact.

Theorem 4. *The set l^∞ is not compact.*

Proof. Set l^∞ not being compact means that there exists an open cover $\{U_\alpha\}_{\alpha\in l^\infty}$ without a finite subcover. This can be proven by constructing an open cover that consists of an infinite set of subcovers. Let $l_m \in l^\infty$ be constructed with m elements, i.e

$$l_m = \{x_1^{(m)}, x_2^{(m)}, \cdots, x_m^{(m)}\} \in l^\infty$$

An open cover

$$\{U_\alpha\}_{\alpha\in A}$$

can then be constructed by an infinite number of sets

$$V_\beta = \left\{l_m^{(\beta)}\right\}_{m\in\{1,2,\ldots,\beta\}}$$

Which can be constructed with each additional iteration with β will increase the size of the $\{V_\beta\}_{\beta\in B}$ subcover, and will cover l^∞ when taken to infinity, but is not finite. Therefore, the open cover $\{U_\alpha\}_{\alpha\in A}$ will not have an associated finite subcover; implying l^∞ is not compact. $\square$

Problem 3

Consider the set $\mathbb{B}([0,1],\mathbb{R})$ of all bounded real-valued functions on the unit interval endowed with the sup-norm: $\|f\|_\infty = \sup_{x\in[0,1]}|f(x)|$. Denote the closed unit ball as $B_1 := \{f \in \mathbb{B}([0,1],\mathbb{R}) \,:\, \|f\|_\infty \leq 1\}$.

a) Prove B_1 is closed.

Theorem 5. *B_1 is the closed.*

Proof. Set B_1 being closed implies B_1^c is open.

$$\forall_{f\in B_1^c}\exists_{\epsilon>0} \,:\, \forall_{g\in\mathbb{B}}\ \|f-g\|_\infty < \epsilon \implies g \in B_1^c$$
$$\forall_{f\in\mathbb{B}}\|f\|_\infty > 1\exists_{\epsilon>0} \,:\, \forall_{g\in\mathbb{B}}\ \|f-g\|_\infty < \epsilon \implies g \in \mathbb{B} \,:\, \|g\|_\infty > 1$$

Additionally, since $\|f\|_\infty > 1$ and $\|f-g\|_\infty$ is bounded, the only way these are both true this must also be true: $\|g\|_\infty > 1$.
Alternatively, you can just recognize that and $f \in \mathbb{B} \,:\, \|\cdot\|_\infty > 1 \implies f \notin B_1 \implies f \in B_1^c$ which demonstrates B_1^c is open, and therefore, B_1 is closed. □

b) Prove that B_1 is bounded.

Theorem 6. *B_1 is bounded, i.e.*

$$\exists_N \,:\, \forall_{f\in B_1}\ \|f\|_\infty < N$$

Proof. Since, by definition, $\|f\| \leq 1$, B_1 is clearly bounded for any $N > 1$. □

c) Prove that B_1 is not compact.

Theorem 7. *B_1 is not compact.*

Proof. B_1 not being compact is equivalent to saying

$$\neg\Big(\forall_{f_{k\,k\in\mathbb{N}}} f_k \in \mathbb{B} \,:\, \exists_{f_{n_k}} : f_{n_k} \to f \in B_1\Big)$$
$$\exists_{f_{k\,k\in\mathbb{N}}} f_k \in \mathbb{B} \,:\, \forall_{f_{n_k}} : f_{n_k} \to f \notin B_1$$

□

Problem 4

Let $\{V, \|\cdot\|\}$ be a normed space. Show that the function $f(x) = \|x\| \ : \ V \to \mathbb{R}$ is continuous on V.

Definition 12. *A function* $f : (S_1, d_1) \to (S_2, d_2)$ *is continuous on* S_1 *if*

$$\forall_{x \in S_1} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in S_1} d_1(x, y) < \epsilon \implies d_2(f(x), f(y)) < \delta$$

Theorem 8. *The function* $f(x)$ *is continuous on* V*, i.e.*

$$\forall_{x \in V} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in V} \|x - y\| < \epsilon \implies |f(x) - f(y)| < \delta$$

Proof.

$$\begin{gathered}
\forall_{x \in V} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in V} \|x - y\| < \epsilon \implies |\|x\| - \|y\|| < \delta \\
\forall_{x \in V} \forall_{\epsilon_1 > 0} \exists_{\delta_2(x,\epsilon_1) > 0} \forall_{y \in V} \|x - y\| \leq \|x\| + \|y\| < \epsilon_1 \implies |\|x\| - \|y\|| \leq \|x\| + \|y\| < \delta_2 \\
\forall_{x \in V} \forall_{\epsilon_1 > 0} \exists_{\delta_2(x,\epsilon_1) > 0} \forall_{y \in V} \|x\| + \|y\| < \epsilon_1 \implies \|x\| + \|y\| < \delta_2
\end{gathered}$$

which is clearly true, therefore $f(x) = \|x\|$ is continuous on V. $\square$

Problem 5

(X, d_1) and (Y, d_2) are two metric spaces. Assume also that Y is a vector space. Construct and example of two continuous functions $f, g : X \to Y$ such that $f + g$ is discontinuous.

Definition 13. *Let $f : X \to Y$ be defined by*

$$f(x) := \begin{cases} -x & x < 0 \\ x & x > 0 \\ 0 & x = 0 \end{cases}$$

Definition 14. *Let $g : X \to Y$ be defined by*

$$g(x) := \begin{cases} x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Theorem 9. *Functions f and g are continuous, but $f + g$ is discontinuous.*

Proof. **a)**

Lemma 1. *f is a continuous function.*

Proof.

$$\begin{aligned} \forall_{x\in X}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)}\forall_{y\in X} \ : \ d_1(x,y) < \epsilon &\implies d_2(f(x), f(y)) < \delta \\ &\implies \begin{cases} d_2(-x,-y) < \delta & x < 0, y < 0 \\ d_2(x,-y) < \delta & x > 0, y < 0 \\ d_2(-x,y) < \delta & x < 0, y > 0 \\ d_2(0,-y) < \delta & x = 0, y < 0 \\ d_2(0,y) < \delta & x = 0, y > 0 \\ d_2(-x,0) < \delta & x < 0, y = 0 \\ d_2(x,0) < \delta & x > 0, y = 0 \\ d_2(0,0) < \delta & x = 0, y = 0 \end{cases} \end{aligned}$$

All of which are clearly true for d_2 within a vector space. □

b)

Lemma 2. *g is a continuous function.*

Proof.

$$\begin{aligned} \forall_{x\in X}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)}\forall_{y\in X} \ : \ d_1(x,y) < \epsilon &\implies d_2(g(x), g(y)) < \delta \\ &\implies \begin{cases} d_2(x^2,y^2) < \delta & x \neq 0, y \neq 0 \\ d_2(0,y^2) < \delta & x = 0, y \neq 0 \\ d_2(x^2,0) < \delta & x \neq 0, y = 0 \\ d_2(0,0) < \delta & x = 0, y = 0 \end{cases} \end{aligned}$$

All of which are clearly true for d_2 within a vector space. □

c)

Lemma 3. *$f + g$ is a discontinuous function.*

Proof. Proof by contradiction, assume $f + g$ is continuous:

$$\forall_{x\in X}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)}\forall_{y\in X} \ : \ d_1(x,y) < \epsilon \implies d_2(f(x)+g(x), f(y)+g(y)) < \delta$$
$$\implies \begin{cases} d_2(-x+x^2, -y+y^2) < \delta & x < 0, y < 0 \\ d_2(x+x^2, -y+y^2) < \delta & x > 0, y < 0 \\ d_2(-x+x^2, y+y^2) < \delta & x < 0, y > 0 \\ d_2(0, -y+y^2) < \delta & x = 0, y < 0 \\ d_2(0, y+y^2) < \delta & x = 0, y > 0 \\ d_2(-x+x^2, 0) < \delta & x < 0, y = 0 \\ d_2(x+x^2, 0) < \delta & x > 0, y = 0 \\ d_2(0,0) < \delta & x = 0, y = 0 \end{cases}$$

however, in the cases when x or y are zero and the other is not, the statement of continuity is not always true due to a discontinuity immediately surrounding $x = 0$. □

□

Problem 6

Construct an example of a sequence $\{f_n\}$ of nowhere continuous functions $[0,1] \to \mathbb{R}$ such that f_n converge in the sup-norm to continuous functions.

Definition 15. *Let* $\mathbb{NC}$ *be defined as the sequence of all nowhere continuous functions from* $[0,1] \to \mathbb{R}$*, i.e*

$$\mathbb{NC} := \left\{ f : [0,1] \to \mathbb{R} \;:\; \forall_{x \in [0,1]} \exists_{\epsilon > 0} \forall_{\delta(x,\epsilon)} \exists_{y \in [0,1]} \;:\; 0 < \|x - y\| < \delta \wedge \|f(x) - f(y)\| \geq \epsilon \right\}$$

Definition 16. *Let* $1_{\mathbb{Q}} : [0,1] \to \mathbb{R}$ *be defined as the* <u>*Dirichlet function*</u> *as the indicator for the set of rational numbers* $\mathbb{Q}$*, i.e*

$$1_{\mathbb{Q}}(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

which is a nowhere continuous function with the binary output of 1 *or* 0*.*

Definition 17. *Let the sequence* $\{f_n\} \in \mathbb{NC}$ *be defined by*

$$\{f_n\}_{n \in \mathbb{N}} := \left\{ f_n \;:\; f_n(x) = \left(\frac{2}{n} \right) (1_{\mathbb{Q}}(x) - 0.5) \right\}$$

Theorem 10. *Within the sup-norm,* $\|l\|_\infty = \sup_{n \in \mathbb{N}} |f_n|$*, the sequence of functions* $\{f_n\}$ *is continuous.*

Proof. The definition of continuity for the sequence under the sup-norm is given as:

$$\begin{aligned} \forall_{x \in X} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon)} \forall_{y \in X} \;:\; \|x - y\| < \epsilon &\implies \|f_n(x) - f_n(y)\|_\infty < \delta \\ &\implies \|f_n(x) - f_n(y)\|_\infty \leq \|f_n(x)\|_\infty + \|f_n(y)\|_\infty < \delta_2 \\ &\implies \sup_{n \in \mathbb{N}} |f_n(x)| + \sup_{n \in \mathbb{N}} |f_n(y)| < \delta \end{aligned}$$

Since $|f_n(x)|$ is clearly bounded from above by $frac1n$, $\sup_{n \in \mathbb{N}} |f_n(x)| \leq \sup_{n \in \mathbb{N}} \frac{1}{n} = 1$, therefore, taking $\delta = 2$ results in the definition of continuity being satisfied for f_n under the sup-norm. $\square$