MATH5301 Elementary Analysis. Homework 10. Due: 11/12/2021, 11:59 pm

First Name: **Last Name:**

10.1

Compute the derivatives of the following functions

(a) $x^2 \sin \dfrac{1}{x}$

(b) $\dfrac{e^x + e^{-x}}{2}$

(c) $\dfrac{e^x - e^{-x}}{2}$

(d) $e^x + e^{e^x} + e^{e^{e^x}}$

(e) $x^{x^{x^x}}$

10.2

(a) Prove the following

Theorem 1. *If $f : (-1, 1) \to \mathbb{R}$ is differentiable unbounded function, then f' is also unbounded on $[-1, 1]$.*

(b) Provide an example of bounded differentiable function on $[-1, 1]$ with unbounded derivative.

(c) Prove the following

Theorem 2. *If $f : (-1, 1) \to \mathbb{R}$ is differentiable function, such that f' is bounded on $[-1, 1]$, then f is uniformly continuous.*

10.3

Find $f^{(n)}(0)$ for the functions

(a) $\sin(ax)\cos(bx)$

(b) $x^k \sin\dfrac{1}{x}$

(c) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$

10.4

Construct an example of infinitely many times differentiable function $f(x)$ such that $f(x) = 0$ for $x \leqslant 0$, $f(x) = 1$ for $x \geqslant 1$ and $f(x)$ is strictly monotone on the interval $(0, 1)$.

Using such function you could construct for example a monotone function $g(x)$ such that $\lim\limits_{x \to +\infty} g(x) = 0$ but $\lim\limits_{x \to +\infty} g'(x) \neq 0$. (How?)

10.5

Find the limit

(a) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^3}$

(b) $\lim\limits_{x\to 0} \dfrac{\arctan(\arcsin x) - \arcsin(\arctan x)}{\sin x - \tan x}$

(c) $\lim\limits_{x\to +\infty} \dfrac{x^{\ln x}}{(\ln x)^x}$

10.6

Find the example of a function $f(x)$ which is continuous at every point of the interval $(0, 1)$, but is not differentiable at every point of $(0, 1)$.

Read about the construction of the function, which is differentiable at every point of $(0, 1)$ but whose derivative is discontinuous at every point of $(0, 1)$.