MATH 5301 Elementary Analysis - Homework 1

Jonas Wagner

2021, September 01

1 Problem 1

Problem: Prove the following tautologies by writing the true-false table. Also translate each of these statements into human language.

a. $A \vee \sim A$

b. $(A \vee B) \Rightarrow A$

c. $(A \wedge B) \Rightarrow A$

d. $(A \Rightarrow B) \iff (\sim B \Rightarrow \sim A)$

e. $\sim (A \vee B) \iff (\sim A \wedge \sim B)$

f. $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$

g. $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$

Solution:

language.

		a)	b)	c)	d)	e)	f)	g) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$	
A	B	$A + A'$	$A + B \Rightarrow A$	$AB \Rightarrow A$	$(A \Rightarrow B) \Leftrightarrow (B' \Rightarrow A')$	$(A+B)' \Leftrightarrow (A'B')$	$((A \Rightarrow B) \Rightarrow A) \Rightarrow A$	$C = 0$	$C = 1$
1	1	1	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1	1
0	1	1	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1	1

a. A or not A

b. A or B implies A

c. A and B implies A

d. A implying B implies not B implying not A

e. Neither A nor B occurs if and only if not A and not B

f. A implying B implies A which implies A

g. A implying B implies C implies that A implying B implies A implying C

2 Problem 2

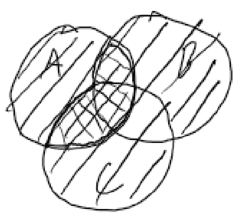
Problem: Prove the following identities for the set operations.

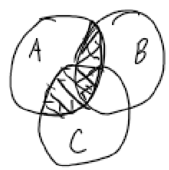
a. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b. $(A \backslash B) \cup C = ((A \cup C) \backslash B) \cup (B \cap C)$

Solution:

a) $\underline{A}(\underline{B+C}) = \underline{AB} + \underline{AC}$ (Distributive)





K-map

AB \ C	A(B+C) 1	A(B+C) 0	AB + AC 1	AB + AC 0
1 1	1	1	1	1
1 0	1	0	1	0
0 1	0	0	0	0
0 0	0	0	0	0

A

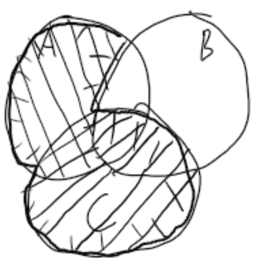
A(B+C)

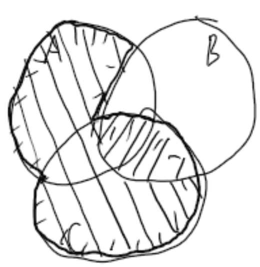
B+C

AC AB

AC + AB

b) $\underline{AB'} + \underline{C} = ((\underline{A+C)B'}) + \underline{BC}$





AB \ C	AB' + C 1	AB' + C 0	(A+C)B' + BC 1	(A+C)B' + BC 0
1 1	1	0	1	0
1 0	1	1	1	1
0 1	1	0	1	0
0 0	1	0	1	0

AB'

AB' + C

C

(A+C)B' BC

(A+C)B' + BC

3 Problem 3

Problem: Write the following statements using quantifiers.

a. Even elements of the sequence $\{a_n\}$ may be arbitrarly large.

b. The sequence $\{a_n\}$ contains arbitary large even elements.

c. The sequence $\{a_n\}$ contains infinitely many even elements.

Solution:

a. $\forall n \in \mathbf{N} : a_n \vdots 2 \Rightarrow \exists a \in \mathbf{N} : a_n \geq a$

b. $\exists n \in \mathbf{N} : a_n \vdots 2 \Rightarrow \forall a \in \mathbf{N} : a_n > a$

c. $\forall n \in \mathbf{N} : a_n \vdots 2 : \exists m \in \mathbf{N} : m > n : a_m \vdots 2$

4 Problem 4

Problem: Show that:

a. $\exists_x : (p(x) \vee q(x)) \iff (\exists_x : p(x)) \vee (\exists_x : q(x))$

b. $(\forall_x p(x) \vee \forall_x q(x)) \Rightarrow \forall_x (p(x) \vee q(x))$

c. Why is there no left arror implication on the previous line?

Solution:
Part a)

$\exists_x p(x)$	$\exists_x q(x)$	$\exists_x (p(x) \vee q(x))$	$(\exists_x p(x)) \vee (\exists_x q(x))$	$\exists_x : (p(x) \vee q(x)) \iff (\exists_x : p(x)) \vee (\exists_x : q(x))$
1	1	1	1	1
1	0	1	1	1
0	1	1	1	1
0	0	0	0	1

Part b)

$\forall_x p(x)$	$\forall_x q(x)$	$\forall_x (p(x) \vee q(x))$	$(\forall_x p(x)) \vee (\forall_x q(x))$	$(\forall_x p(x) \vee \forall_x q(x)) \Rightarrow \forall_x (p(x) \vee q(x))$	$(\forall_x p(x) \vee \forall_x q(x)) \Leftarrow \forall_x (p(x) \vee q(x))$
1	1	1	1	1	1
1	0	1	1	1	1
0	1	1	1	1	1
0	0	0	0	1	1
0	0	1	0	1	0

Part c)
This is becouse there are cases when $p(x)$ and $q(x)$ themselves are not satisfied $\forall x$, but together at least one of them are true $\forall x$.

5 Problem 5

Problem: Show that one needs only one logic operation to construct all the 16 binary operations on statements A and B.

A	B	$A \star B$
0	0	1
0	1	0
1	0	0
1	1	0

Define $A \star B$ via the following table:
Show that one can construct $\sim A, A \vee B$,and$A \wedge B$ using only $\star$. Then show that any other binary operation can be obtained from $\{\sim, \vee, \wedge\}$.

Solution:

$$\sim A := A * A$$

$$A \vee B := (A * B) * (A * B)$$

$$A \wedge B := (A * A) * (B * B)$$

. **and** B.

A	B	$A \star B$	$A * A$	$A * A * B$	$(A*A)*(B*B)$	$A*B*A$	$(A*B)*(A*B)$
0	0	1	1	0	0	0	0
0	1	0	1	0	0	1	1
1	0	0	0	1	0	0	1
1	1	0	0	0	1	0	1

(using the notation learned in digital circuits)

A	B	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	0	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

$0 := (A + A')'$

$1 := (A+B)'$

$2 := (A+B')'$

$3 := A'$

$4 := (A'+B)'$

$5 := B'$

$6 := (AB + A'B')'$

$7 := (AB)'$

$8 := AB$

$9 := AB + A'B$

$10 := B$

$11 := A'+B$

$12 := A$

$13 := A+B'$

$14 := A+B$

$15 := A+A'$

6 Problem 6

Problem: How many subsets does the set $A = \{a, p, p, l, e\}$ have?
Solution: On the surface this problem can be solved simply by considering the selection of each letter to be a part of a subset as a binary diget of T/F for inclusion, resulting in $2^n = 2^5 = 32$. However, due to the repeaterd element, p, this is not as simple but a similar combinatorics proccess can be used to account for this.

Let the following be defined:
$n :=$ number of elements within A
$m :=$ number of unique elements within A
$a_i :=$ sequence of the unique elements within A ordered from the most occuring to the least occuring
$n_i :=$ sequence of the number of unique elements, a_i, within A.
$(a_i, n_i) :=$ set of ordered pairs of unique elements of a unique element, a_i, within A paired with the number of a_i elements contained within A, n_i.

The calculation can split into the sum of all the possible subsets of lengths, l, from 0 to n.
For $l = 0$ the only possible subset is $\emptyset$, so

$$N_0 = \binom{n}{0} = 1.$$

Similarily, for $l = n$ the only possible subset of A is A, so

$$N_n = \binom{n}{n} = 1.$$

For $l = 1$ there exists m unique sets consisting of the elements in $\{a_i\}$. Alternativly, this can be calculated as

$$N_1 = \binom{n}{1} - \sum_{i=1}^{m}(n_i - 1)$$

.
For $l = 2$ (and all $l > 1$) the computation becomes more complicated. One solution is to define the ordered pairs $\{(b_i, n_i - 1)\} : b_i = a_i \forall i : n_i > 1$. The new sequence, $\{b_i\}$ can then be used to add the nessicary extra repeated element subsets. Additionally, let the number of elements within sequence $\{b_i\}$ be defined as m_2. The collection of subsets with $l = 2$ would be all possible combinations of a_i, $\left(N_{l=2}^{(all)} = \binom{n}{2}\right)$, minus the number of repeated subsets constructed with $b_i, a_j \forall i \in \{1, 2, \cdots, m_2\}, j \in \{1, 2, \cdots, m_2\}$, $\left(N_{l=2}^{(duplicates)} = m_2 * m\right)$. Therefore,

$$N_2 = \binom{m}{2} - m_2.$$

Next, this proccess can be repeated for $2 < l < n$, such that a new set of ordered pairs is defined, $\{c_i, n + (l - 1)\} : c_i = a_i \forall n > (l - 1)$, and then the number of subsets with l elements is calculated as the sum of all possible combinations of the elements of A of length l, $\left(N_l^{(all)} = \binom{n}{2}\right)$, minus the number of repeated subsets constructed with $c_i, \cdots, b_j, a_k \forall i \in \{1, 2, \cdots, m_l\}, \cdots, j \in \{1, 2, \cdots, m_2\}, k \in \{1, 2, \cdots, m\}$, $(N_l)^{(duplicates)} = m_l * \cdots * m_2 * m$. Therefore,

$$N_l = \binom{n}{l} - m_l * \cdots * m_2 * m$$

.
The total number of possible subsets can then be calculated as the sum

$$N = \sum_{l=1}^{n} N_l = \sum_{l=1}^{n} \binom{n}{l} - \prod_{1}^{l} m_l$$