

MATH 5301 Elementary Analysis - Homework 3

Jonas Wagner

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Problem 1

Let X denote the universal set. Two subsets A and B are said to have the same cardinality if there is a bijection $f : A \rightarrow B$. Notation: $|A| = |B|$.

- 1a) **Prove that $|A| = |B|$ is an equivalence relation on the power set of X**
- 1b) **Is it true that if $|A_1| = |B_1|$ and $|A_2| = |B_2|$ then $|A_1 \cup A_2| = |B_1 \cup B_2|$?**

Problem 2

Finish the proof of the Cantor-Bernstein theorem: For the sets A and B , such that $|A| \leq |B|$ and $|B| \leq |A|$ define A_∞ as the set of all elements of A having infinite order, A_0 as the set of all elements of A having even order, and A_1 the set of all elements of A having odd order. Similarly for B .

Let

$$A, B : (|A| \leq |B|) \wedge (|B| \leq |A|)$$

Define

$$A_\infty = \{a \in A : \mathcal{O}(a) = \infty\}$$

$$A_0 = \{a \in A : \mathcal{O}(a) = 0\}$$

$$A_1 = \{a \in A : \mathcal{O}(a) = 1\}$$

$$B_\infty = \{b \in B : \mathcal{O}(b) = \infty\}$$

$$B_0 = \{b \in B : \mathcal{O}(b) = 0\}$$

$$B_1 = \{b \in B : \mathcal{O}(b) = 1\}$$

- 2a) Show that $|A_\infty| = |B_\infty|$.
- 2b) Construct an injective mapping $A_1 \rightarrow B_0$.
- 2c) Show that this mapping is also surjective.

Problem 3

Set A is called countable if $|A| \leq |\mathbb{N}|$. Prove that the following sets are countable.

- 3a) Set \mathbb{Z}_+ of all non-negative integer numbers
- 3b) Set $2\mathbb{N}$ of all even numbers
- 3c) Set \mathbb{Z}^2 of all ordered pairs of integer numbers
- 3d) Set \mathbb{Q} of all rational numbers
- 3e) Set \mathbb{Q}^2 of all ordered pairs of rational numbers

Problem 4

Prove that the following sets are countable.

4a) Set $P_5(\mathbb{Z})$ of all polynomials of degree 4 with integer coefficients

4b) Any collection of non-intersecting discs on a plane

4c) Any collection of non-intersecting T-shapes on a plane

Note: T-shape consists of two perpendicular line segments such that one of the segments is attached by one of its endpoints to the center of the other segment. The lengths of these segments can be arbitrary. The orientation of the T-shape can be arbitrary.

4d) Set \mathbb{P} of all prime numbers

4e) Set \mathbb{A} of all algebraic numbers

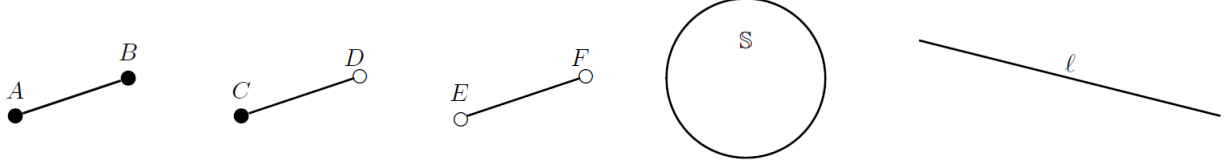
Note: Algebraic numbers are numbers which are roots of some polynomials with integer coefficients.

Problem 5

Prove that for any infinite set A there exists $B \subset A$, so that $|B| = |\mathbb{N}|$.

Problem 6

Prove that the following sets have the same cardinality.



These sets can all be represented as a set of real number ordered pairs.

These are constructed with arbitrary constants: $x_a, x_b, x_c, x_d, x_e, x_f, y_a, y_b, y_c, y_d, y_e, y_f, x_s, y_s, r_s, m_l, b_l$

$$\begin{aligned}
 AB &= \{(x, y) \in \mathbb{R}^2, t \in \mathbb{R} : (x = (1-t)x_a + tx_b) \wedge (y = (1-t)y_a + ty_b) \wedge (0 \leq t \leq 1)\} \\
 CD &= \{(x, y) \in \mathbb{R}^2, t \in \mathbb{R} : (x = (1-t)x_c + tx_d) \wedge (y = (1-t)y_c + ty_d) \wedge (0 \leq t < 1)\} \\
 EF &= \{(x, y) \in \mathbb{R}^2, t \in \mathbb{R} : (x = (1-t)x_e + tx_f) \wedge (y = (1-t)y_e + ty_f) \wedge (0 < t < 1)\} \\
 \mathbb{S} &= \{(x, y) \in \mathbb{R}^2, t \in \mathbb{R} : (x = r_s \cos(2\pi t) + x_s) \wedge (y = \sin(2\pi t) + y_s) \wedge (0 \leq t < 1)\} \\
 l &= \{(x, y) \in \mathbb{R}^2, t \in \mathbb{R} : (x = t) \wedge (y = m_l t + b_l)\}
 \end{aligned}$$

It is clear that each set is defined parametrically with bijective equations mapping the parameter t into a 2-D coordinates (x, y) , thus proving that the cardinality of each set parameter is sufficient to showing the each set has the same cardinality.

Let $T_i = \{t \in A_i\}$ for each of the sets A_i , then (from the reasoning above) the following can be said:

$$\begin{array}{ll}
 T_{AB} = \{t \in \mathbb{R} : 0 \leq t \leq 1\}, & |T_{AB}| = |AB| \\
 T_{CD} = \{t \in \mathbb{R} : 0 \leq t < 1\}, & |T_{CD}| = |CD| \\
 T_{EF} = \{t \in \mathbb{R} : 0 < t < 1\}, & |T_{EF}| = |EF| \\
 T_{\mathbb{S}} = \{t \in \mathbb{R} : 0 \leq t < 1\}, & |T_{\mathbb{S}}| = |\mathbb{S}| \\
 T_l = \{t \in \mathbb{R}\}, & |T_l| = |l|
 \end{array}$$

Clearly, the equivalent definition of T_{CD} and $T_{\mathbb{S}}$ indicates

$$|CD| = |T_{CD}| = |T_{\mathbb{S}}| = |\mathbb{S}|$$

The equivalence of the other sets is more difficult then via definition.

First, the baseline cardinality can be shown to be \aleph_1 as (by definition of T_l)

$$|T_l| = |\mathbb{R}| = \aleph_1$$

Next, the equivalence of T_{EF} and T_l can be shown with the biforjective mapping

$$f_1 : \mathbb{R} \rightarrow T_{EF} = \frac{2\pi \tan^{-1}(x) + 1}{2}$$

Therefore,

$$|EF| = |T_{EF}| = |T_l| = |\mathbb{R}| = \aleph_1$$

Next, due to the nature of infinite sets, the addition of $t = 0$ from T_{EF} to T_{CD} does not affect the overall cardinality of T_{CD} , thus

$$|CD| = |T_{CD}| = |T_{EF}| = |\mathbb{R}| = \aleph_1$$

Similarly, the addition of $t = 1$ from T_{CD} to T_{AB} will stil result in

$$|AB| = |T_{AB}| = |T_{CD}| = |\mathbb{R}| = \aleph_1$$

Ultimently this means that

$$|AB| = |CD| = |EF| = |\mathbb{S}| = |l| = |\mathbb{R}| = \aleph_1$$