MATH 5301 Elementary Analysis - Homework 3

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Problem 1

Let X denote the universal set. Two subsets A and B are said to have the same cardinality if there is a bijection $f : A \to B$. Notation: $|A| = |B|$.

1a) Prove that $|A| = |B|$ is an equivalence relation on the power set of X

The relation $\mathcal{R} = \text{``}|A| = |B|''$ [1] is defined as:

$$\mathcal{R} = \text{``}|A| = |B|'' = \{(A, B) : \exists(f : A \xrightarrow{\text{B}} B)\}$$

Equivilence can be demonstrated by proven by demonstrating: (i) Reflexivity, (ii) Symetry, and (iii) Transivity

i) Reflective:

$$A\mathcal{R}A = \{(A, A) : \exists(f : A \xrightarrow{\text{B}} A)\}$$

Since $f : A \xrightarrow{\text{B}} A = a \in A$ is true $\forall A \in 2^X$, $\mathbb{R}$ is Reflective.

ii) Symetric:

$$A\mathcal{R}B \implies B\mathcal{R}A$$
$$\{(A, B) : \exists(f : A \xrightarrow{\text{B}} B)\} \implies \{(B, A) : \exists(f : B \xrightarrow{\text{B}} A)\}$$

Since $f : A \xrightarrow{\text{B}} B \implies g : B \xrightarrow{\text{B}} A = f^{-1}$ is true $\forall A, B \in 2^X$, $\mathbb{R}$ is Symetric.
(Essentially if f is bijective one way, f^{-1} is bijective for the other way)

iii) Transative:

$$(A\mathcal{R}B) \wedge (B\mathcal{R}C) \implies A\mathcal{R}C$$
$$(\{(A, B) : \exists(f : A \xrightarrow{\text{B}} B)\}) \wedge \{(B, C) : \exists(g : B \xrightarrow{\text{B}} C)\} \implies \{(A, C) : \exists(h : A \xrightarrow{\text{B}} C)\}$$

Since $(f : A \xrightarrow{\text{B}} B) \wedge (g : B \xrightarrow{\text{B}} C) \implies (h : A \xrightarrow{\text{B}} C = A \xrightarrow{f} B \xrightarrow{g} C)$ is true $\forall A, B, C \in 2^X$, $\mathcal{R}$ is Transative.

Therefore $\mathcal{R} = \text{``}|A| = |B|''$ is an equivalence relation over 2^X.

[1] using $\mathcal{R}$ for simplicyity/reusability

1b) Is it true that if $|A_1| = |B_1|$ and $|A_2| = |B_2|$ then $|A_1 \cup A_2| = |B_1 \cup B_2|$?

$$\begin{aligned}
(|A_1| = |B_1|) \wedge (|A_2| = |B_2|) &\implies |A_1 \cup A_2| = |B_1 \cup B_2| \\
(A_1 \mathcal{R} B_1) \wedge (A_2 \mathcal{R} B_2) &\implies (A_1 \cup A_2) \mathcal{R} (B_1 \cup B_2) \\
\{(A_1, B_1) : \exists (f_1 : A_1 \xrightarrow{\mathsf{B}} B_1)\} \wedge \{(A_2, B_2) : \exists (f_2 : A_2 \xrightarrow{\mathsf{B}} B_2)\} &\implies \{((A_1 \cup A_2), (B_1 \cup B_2)) : \exists (f : (A_1 \cup A_2) \xrightarrow{\mathsf{B}} (B_1 \cup B_2))\}
\end{aligned}$$

This itself is false, as in the case when

$$(A_1 \cap A_2 \neq \emptyset) \wedge (B_1 \cap B_2 = \emptyset) \implies (f : (A_1 \cup A_2) \xrightarrow{\mathsf{I}} (B_1 \cup B_2)) \wedge (f^{-1} : (B_1 \cup B_2) \xrightarrow{\mathsf{S}} (B_1 \cup B_2))$$

But, f is not surjective and f^{-1} is not injective, so f cannot be bijective.

Problem 2

Finish the proof of the Cantor-Bernstein theorem: For the sets A and B, such that $|A| \leq |B|$ and $|B| \leq |A|$ define A_∞ as the set of all elements of A having infinite order, A_0 as the set of all elements of A having even order, and A_1 the set of all elements of A having odd order. Similarly for B.
Let

$$A, B : (|A| \leq |B|) \wedge (|B| \leq |B|)$$

Define

$$A_\infty = \{a \in A : \mathcal{O}(a) = \infty\}$$

$$A_0 = \{a \in A : \mathcal{O}(a) \vdots 2\}$$

$$A_1 = \{a \in A : (\mathcal{O}(a) + 1) \vdots 2\}$$

$$B_\infty = \{b \in B : \mathcal{O}(b) = \infty\}$$

$$B_0 = \{b \in B : \mathcal{O}(b) \vdots 2\}$$

$$B_1 = \{b \in B : (\mathcal{O}(b) + 1) \vdots 2\}$$

2a) Show that $|A_\infty| = |B_\infty|$.

Since,

$$|A| \leq |B| \implies \exists(f_{AB} : A \xrightarrow{\text{I}} B) \therefore f_{AB}(A_\infty) = B_\infty$$

Similarly,

$$|B| \leq |A| \implies \exists(f_{BA} : B \xrightarrow{\text{I}} A) \therefore f_{BA}(B_\infty) = A_\infty$$

Since,

$$(\exists f_{AB} : A \xrightarrow{\text{I}} B) \wedge (\exists f_{BA} : B \xrightarrow{\text{I}} A) \implies (\exists f : A \xrightarrow{\text{B}} B)$$

Therefore, since a bijective mapping exists,

$$|A_\infty| = |B_\infty|$$

2b) Construct an injective mapping $A_1 \to B_0$.

From the definition of A_1, it is known that A_1 contains all elements whose ancestors (for f and g) are of odd order.
Similarly, from the definition of B_0 it is known that B_0 contains all elements whose ancestors (for f and g) are of even order.
Since all elements of B_0 are of even order, then that implies that acording to the mappings f and g,

$$a \in \{a \in A : \exists b \in B_0 : f(a) = b\} \implies a \in A_1$$

therefore, f is a direct mapping from A_1 to B_0 (which is technically bijective so it is also injective)

2c) Show that this mapping is also surjective.

Since, from the previous argument, f is bijective so it is also surjective.

Problem 3

Set A is called countable if $|A| \le |\mathbb{N}|$. Prove that the following sets are countable.

3a) Set $\mathbb{Z}_+$ of all non-negative integer numbers

Define $f_0 : \mathbb{N} \to \mathbb{Z}_+$,

$$f_0(x) := x - 1$$

Claim $f_0 : \mathbb{N} \xrightarrow{s} \mathbb{Z}_+$ (surjective). This is true becouse $\forall x \in \mathbb{N}\ \exists f_0(x) \in \mathbb{Z}_+$.
When (i) $x \in \mathbb{N} = 1$, $f_0(x) = x - 1 = 0$ which is in $\mathbb{Z}_+$.
Similarly, when (ii) $x \in \mathbb{N} > 1$, $f_0(x) = x - 1$ which is in $\mathbb{Z}_+$.

Since f_0 is in fact surjective, $\mathbb{Z}_+$ is countable. (i.e)

$$f_0 : \mathbb{N} \xrightarrow{s} \mathbb{Z}_+ \implies |\mathbb{Z}_+| \le |\mathbb{N}|$$

3b) Set $2\mathbb{N}$ of all even numbers

Define $f : \mathbb{N} to 2\mathbb{N}$,

$$f(x) := 2x$$

Claim $f : \mathbb{N} \xrightarrow{s} 2\mathbb{N}$ (surjective). This is true becouse $\forall x \in \mathbb{N} \exists f(x) \in 2\mathbb{N}$.
When $x \in \mathbb{N}$, $f(x) = 2x$ which is in $2\mathbb{N}$.

Since f is surjective, $2\mathbb{N}$ is countable. (i.e)

$$f : \mathbb{N} \xrightarrow{s} 2\mathbb{N} \implies |2\mathbb{N}| \le |\mathbb{N}|$$

3c) Set $\mathbb{Z}^2$ of all ordered pairs of integer numbers

Define $g : \mathbb{N} \to \mathbb{Z}^2$,

$$g(x) := \begin{cases} (0,0) & x = 1 \\ (1,0) & x = 2 \\ (1,1) & x = 3 \\ (0,1) & x = 4 \\ (-1,1) & x = 5 \\ (-1,0) & x = 6 \\ (-1,-1) & x = 7 \\ (0,-1) & x = 8 \\ (1,-1) & x = 9 \\ (2,-1) & x = 10 \\ (2,0) & x = 11 \\ \vdots & \vdots \end{cases}$$

Claim $g : \mathbb{N} \xrightarrow{s} \mathbb{Z}^2$ (surjective). This is true becouse $\forall x \in \mathbb{N} \exists g(x) \in \mathbb{Z}^2$ as is clearly evident in the spiraling mapping discribed by g mapping to all $\mathbb{Z}^2$.

Since g is surjective, $\mathbb{Z}^2$ is countable. (i.e)

$$g : \mathbb{N} \xrightarrow{s} \mathbb{Z}^2 \implies \left|\mathbb{Z}^2\right| \le |\mathbb{N}|$$

3d) Set $\mathbb{Q}$ of all rational numbers

Define $h : \mathbb{N} \to \mathbb{Q}$,

$$h(x) := \begin{cases} 0 & x = 1 \\ 1 & x = 2 \\ -1 & x = 3 \\ 2 & x = 4 \\ \frac{3}{2} & x = 5 \\ \frac{1}{2} & x = 6 \\ \frac{-1}{2} & x = 7 \\ \frac{-3}{2} & x = 8 \\ -2 & x = 9 \\ 3 & x = 10 \\ \frac{8}{3} & x = 11 \\ \frac{5}{2} & x = 12 \\ \frac{7}{3} & x = 13 \\ \frac{5}{3} & x = 14 \\ \frac{4}{3} & x = 15 \\ \frac{2}{3} & x = 16 \\ \frac{1}{3} & x = 17 \\ \frac{-1}{3} & x = 18 \\ \vdots & \vdots \end{cases}$$

Claim $h : \mathbb{N} \xrightarrow{s} \mathbb{Q}$ (surjective). This is true becouse $\forall x \in \mathbb{N} \exists h(x) \in \mathbb{Q}$ as is clearly evident in the spiraling mapping discribed by h mapping to all $\mathbb{Q}$.

Since h is surjective, Q is countable. (i.e)

$$g : \mathbb{N} \xrightarrow{s} \mathbb{Q} \implies |Q| \leq |\mathbb{N}|$$

3e) Set $\mathbb{Q}^2$ of all ordered pairs of rational numbers

Define $m : \mathbb{N} \to \mathbb{Q}^2$,

$$m(x) := \begin{cases} (0,0) & x = 1 \\ (0,1) & x = 2 \\ (1,0) & x = 3 \\ (1,1) & x = 4 \\ (0,-1) & x = 5 \\ (1,-1) & x = 6 \\ (-1,0) & x = 7 \\ (-1,1) & x = 8 \\ (-1,-1) & x = 9 \\ (0,2) & x = 10 \\ \vdots & \vdots \\ (2,2) & \vdots \\ (0,\frac{3}{2}) & \vdots \\ \vdots & \vdots \\ (\frac{3}{2},\frac{3}{2}) & \vdots \\ (0,\frac{1}{2}) & \vdots \\ \vdots & \vdots \\ (\frac{1}{2},\frac{1}{2}) & \vdots \\ (0,\frac{-1}{2}) & \vdots \\ \vdots & \vdots \\ (\frac{-1}{2},\frac{-1}{2}) & \vdots \\ (0,\frac{-3}{2}) & \vdots \\ \vdots & \vdots \\ (\frac{-3}{2},\frac{-3}{2}) & \vdots \\ (0,3) & \vdots \\ \vdots & \vdots \\ (3,3) & \vdots \\ (0,\frac{8}{3}) & \vdots \\ \vdots & \vdots \end{cases}$$

Note: $m(x)$ continues to spiral folowing the pattern in a way that ultimently combines the mappings of f_0, g, and h.

Claim $m : \mathbb{N} \xrightarrow{s} \mathbb{Q}^2$ (surjective). This is true becouse $\forall x \in \mathbb{N} \exists m(x) \in \mathbb{Q}^2$ as is clearly evident in the spiraling mapping discribed by m mapping to all $\mathbb{Q}^2$.

Since m is surjective, Q^2 is countable. (i.e)

$$g : \mathbb{N} \xrightarrow{s} \mathbb{Q}^2 \implies \left|\mathbb{Q}^2\right| \leq |\mathbb{N}|$$

Problem 4

Prove that the following sets are countable.

4a) Set $\mathbf{P}_5(\mathbb{Z})$ of all polynomials of degree 4 with integer coefficients

Set $\mathbf{P}_5(\mathbb{Z})$ can be defined as:

$$\mathbf{P}_5(\mathbb{Z}) := \{ax^4 + bx^3 + cx^2 + dx + e : a, b, c, e, d \in \mathbb{Z}\}$$

A direct mapping $f_0 : \mathbb{Z}^5 \to \mathbf{P}_5(\mathbb{Z})$ can be created by either redefining or mapping into:

$$\mathbf{P}_5(\mathbb{Z}^5) := \{ax^4 + bx^3 + cx^2 + dx + e : (a, b, c, e, d) =\in \mathbb{Z}^5\}$$

Clearly $|\mathbf{P}_5(\mathbb{Z})| = \left|\mathbf{P}_5(\mathbb{Z}^5)\right| = \left|\mathbb{Z}^5\right|$ so if now if $\mathbb{Z}^5$ can be shown to be countable, then $\mathbf{P}_5(\mathbb{Z})$ would also be countable.

Define $f_1 : \mathbb{N} \to \mathbb{Z}^5$,

$$f_1 := \begin{cases} (0,0,0,0,0) & x = 1 \\ (0,0,0,0,1) & x = 2 \\ (0,0,0,1,0) & x = 3 \\ (0,0,0,1,1) & x = 4 \\ \quad\vdots & \quad\vdots \\ (1,1,1,1,1) & x = 33 \\ (0,0,0,0,-1) & x = 34 \\ (0,0,0,1,-1) & x = 35 \\ (0,0,0,-1,0) & x = 35 \\ (0,0,0,-1,1) & x = 36 \\ (0,0,0,-1,-1) & x = 37 \\ (0,0,-1,0,0) & x = 38 \\ \quad\vdots & \quad\vdots \\ (-1,-1,-1,-1,-1) & \quad\vdots \\ (0,0,0,0,2) & \quad\vdots \\ \quad\vdots & \quad\vdots \end{cases}$$

Claim $f_1 : \mathbb{N} \xrightarrow{\mathsf{S}} \mathbb{Z}^5$ (surjective). Clearly,

$$\forall x \in \mathbb{N} \exists f_1(x) \in \mathbb{Z}^5$$

Therefore, f_1 is surjective which implies $|\mathbb{Z}^5| \leq |\mathbb{N}|$.

Since f_1 is surjective and f_0 is bijective (and therefore surjective) the set $\mathbf{P}_5(\mathbb{Z})$ is countable:

$$(f_1 : \mathbb{N} \xrightarrow{\mathsf{S}} \mathbb{Z}^5) \wedge (f_0 : \mathbb{Z}^5 \xrightarrow{\mathsf{B}} \mathbf{P}_5(\mathbb{Z})) \implies |\mathbf{P}_5(\mathbb{Z})| = |\mathbb{Z}^5| \leq |\mathbb{N}| = \aleph_0$$

4b) Any collection of non-intersecting discs on a plane

Let,

$$A := \Big\{(x, y, r) \in \mathbb{R}^3 : \forall (x_i, y_i, r_i) \in A : (x - x_i)^2 + (y - y_i)^2 \geq (r + r_i)^2\Big\}$$

Clearly, $A \subseteq \{(x, y, r) \in \mathbb{R}^3\}$, therefore $|A| \leq |\mathbb{R}^3| = |\mathbb{R}| = \aleph_1$. This itself does not actually say anything about if it is countable.

Alternativly, the restction of r_0 being arbritrarily selected would redefine the set as

$$A := \Big\{(x, y) \in \mathbb{R}^2 : \forall (x_i, y_i) \in A : (x - x_i)^2 + (y - y_i)^2 \geq (2r_0)^2\Big\}$$

Although this could also be looked at as just $A \subseteq \{(x, y) \in \mathbb{R}^2\} \implies |A| \leq |\mathbb{R}^2| = \aleph_1$ this could also be seen as the subset of the optimaly organized set A_0^*,

$$A_0^* := \{(x, y, r_0) : (x, y) \in 2r_0\mathbb{N}^2\}$$

where $2r_0\mathbb{N}^2$ is defined as

$$2r_0\mathbb{N}^2 := \{(2r_0x, 2r_0y) : (x, y) \in \mathbb{N}^2\}$$

Clearly, $|A_0^*| \leq \left|2r_0\mathbb{N}^2\right|$. Using a simple 1-1 mapping between $2r_0\mathbb{N}^2$ and $\mathbb{N}$, $2r_0\mathbb{N}^2$ is countable. Additionally, this implies that A_0^* is also countable.

$$(f : \mathbb{N} \xrightarrow{\text{S}} 2r_0\mathbb{N}^2) \wedge (g : 2r_0\mathbb{N}^2 \xrightarrow{\text{B}} A_0^*) \implies |A_0^*| \leq \left|2\mathbb{N}^2\right| = |\mathbb{N}| = \aleph_0$$

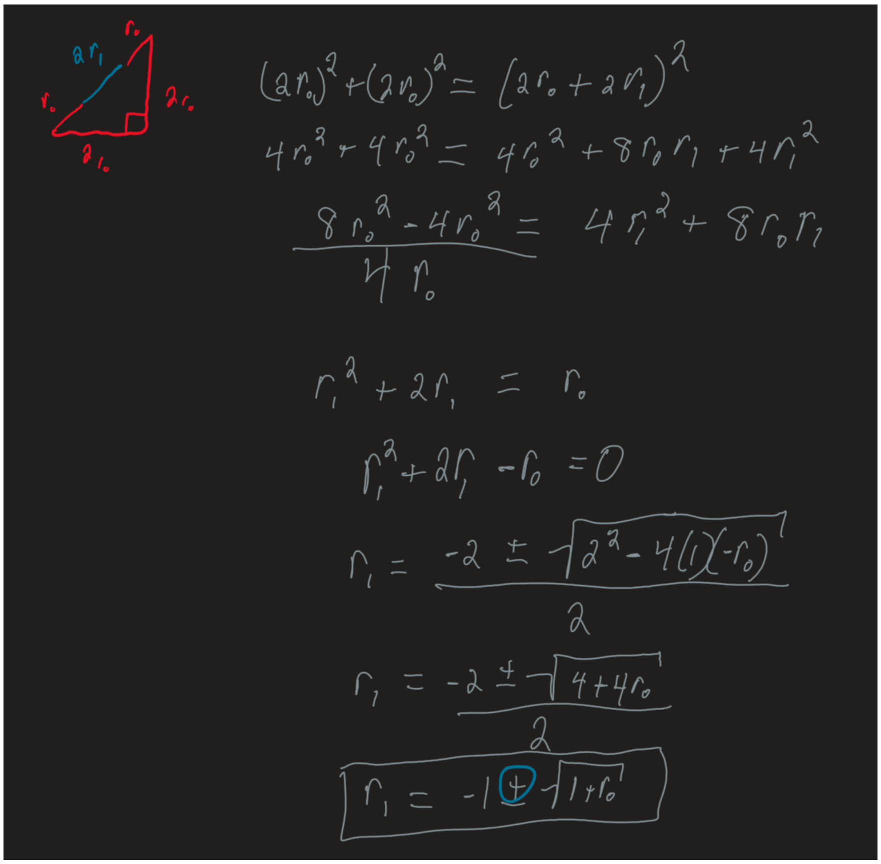
This can all be used then to claim (and subsequently prove) that A is countable.

$$A \subseteq A_0^* \implies |A| \leq |A_0^*|$$
$$(|A| \leq |A_0^*|) \wedge (|A_0^*| \leq 2\left|2r_0\mathbb{N}^2\right| = |\mathbb{N}| = \aleph_0) \implies |A| \leq \aleph_0$$

This result can then be expanded to include an additional set of non-intersecting discs optimaly placed in the unconstrained space. This next set of dics A_1^* can be defined as

$$A_1^* := \left\{(x + r_0, y + r_0, r_1) : (x, y) \in 2r_0\mathbb{N}^2\right\}$$

with a selected r_1 such that the two sets of discs do not intersect.
Such r_1 can be calculated so that $(2r_0)^2 + (2r_0)^2 \leq (2r_0 + 2r_1)$: thus $r_1 \leq -1 + \sqrt{1 + r_0}$.



This is then proven as countable by the same logic as A_0^*.
Additionally, this process can be repeated for more collections of disks of decreasing size until r_i is infinitey small.
At first glance the union of all these sets may not actually be proven to be coundable, but each additional set will no longer consist of all potential optimal pairs so it may be possible to limit this to $\aleph_0$

4c) Any collection of non-intersecting T-shapes on a plane

Note: T-shape consists of two perpendicular line segments such that one of the segments is attached by one of its endpoints to the center of the other segment. The lengths of these segments can be arbitrary. The orientation of the T-shape can be arbitrary.

Define set A,

$$A := \Bigg\{ ((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in (\mathbb{R}^2)^3 : \\ \forall \Big((x_1^{(i)}, y_1^{(i)}), (x_2^{(i)}, y_2^{(i)}), (x_3^{(i)}, y_3^{(i)}) \Big) \in (\mathbb{R}^2)^3 : \\ ((x_1, y_1)\mathcal{R}_1(x_2, y_2) \cap ((x_1^{(i)}, y_1^{(i)})\mathcal{R}_1(x_2^{(i)}, y_2^{(i)}) = \emptyset) \\ \wedge \left((\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{c})\mathcal{R}(x_3, y_3) \right) \cap \left(\frac{x_1^{(i)} + x_2^{(i)}}{2}, \frac{y_1^{(i)} + y_2^{(i)}}{c} \right) = \emptyset) \Bigg\}$$

with $\mathcal{R}_1$ defined as

$$\mathcal{R}_1 := \Bigg\{ (((x_1, y_1), (x_2, y_2), (x_3, y_3)), ((x_1, y_1), (x_2, y_2), (x_3, y_3))) \in (\mathbb{R}^2)^3 \times (\mathbb{R}^2)^3, \forall t \in \mathbb{R}, 0 \leq 1 : \\ ((x_1, y_1)\mathcal{R}_0(x_2, y_2)) \wedge \left(\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \mathcal{R}_0(x_3, y_3) \right) \Bigg\}$$

and $\mathcal{R}_0$ defined as

$$\mathcal{R}_0 := \{(x, y) \in x \times y,\ t \in \mathbb{R} : (x = (1 - t)x_a + tx_b) \wedge (y = (1 - t)y_a + ty_b) \wedge (0 \leq t \leq 1)\}$$

From this definition, the sum of an infinite collection of non-intersecting T-shapes on a plane can be bounded by $\aleph_1$:

$$\left(A \subset (\mathbb{R}^2)^3\right) \wedge \left(\exists f : \mathbb{R} \xrightarrow{\text{B}} (\mathbb{R}^2)^3\right) \implies |A| \leq |(\mathbb{R}^2)^3| = |\mathbb{R}| = \aleph_1$$

This doesn't imply countability, but it is an important bound for every possible combinination of $T-shapes$.

When restricting to an individual set of arbritray parameters l_1, l_2 and orientation θ set A^* can be defined as:

$$A^* := \Bigg\{ ((x_1, y_1), (x_2, y_2), (x_3, y_4)) \in A : \\ \wedge ((x_2 - x_1) = l_1 * \cos(\theta)) \wedge \left((x_2^{(i)} - x_1^{(i)}) = l_2 * \cos(\theta) \right) \\ \wedge ((y_2 - y_1) = l_1 * \sin(\theta)) \wedge \left((y_2^{(i)} - y_1^{(i)}) = l_2 * \sin(\theta) \right) \Bigg\}$$

This can then be bounded follwoing a similar procedure as shown in the previous proof.

4d) Set $\mathbb{P}$ of all prime numbers

By definition, all prime numbers are natural numbers, thus

$$\mathbb{P} \subseteq \mathbb{N}$$

This then implies that $|\mathbb{P}| \leq |\mathbb{N}|$, therefore $\mathbb{P}$ is countable.

$$\mathbb{P} \subseteq \mathbb{N} \implies |\mathbb{P}| \leq |\mathbb{N}|$$

4e) Set $\mathbb{A}$ of all algebraic numbers

Note: Algebraic numbers are numbers which are roots of some polynomials with integer coefficients.

Define $\mathbb{A}$ as,

$$\mathbb{A} := \{x \in \Re : \forall i \in \mathbb{N} \exists p \in \mathbf{P}_i(\mathbb{Z}) : p(x) = 0\}$$

Since $\forall i \in \mathbb{N} : |\mathbf{P}_i(\mathbb{Z})| \leq |\mathbb{N}| = \aleph_0$, and $\exists f : \mathbf{P}_i(\mathbb{Z}) \xrightarrow{\mathsf{S}} \mathbb{A}$. $\mathbb{A}$ is countable.

$$(\forall i \in \mathbb{N} : |\mathbf{P}_i(\mathbb{Z})| \leq |\mathbb{N}| = \aleph_0) \wedge (\exists f : \mathbf{P}_i(\mathbb{Z}) \xrightarrow{\mathsf{S}} \mathbb{A}) \implies |\mathbb{A}| \leq |\mathbf{P}_i(\mathbb{Z})| \leq |\mathbb{N}| = \aleph_0$$

Problem 5

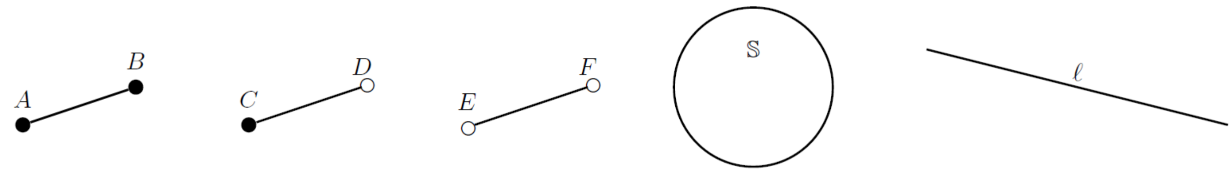
Prove that for any infinite set A there exists $B \subseteq A$, so that $|B| = |\mathbb{N}|$.

By definition an infinite set, $|A| \geq |\mathbb{N}| = \aleph_0$. A very, very simple proof of the fact $\exists B \subseteq A : |B| = |\mathbb{N}| = \aleph_0$ is as follows:
Define $f : \mathbb{N} \xrightarrow{\mathsf{B}} B$ (bijective) to directly map $\mathbb{N}$ to an element of subset B. Since f is bijective, $|B| = |\mathbb{N}| = \aleph_0$.
Next, define $g : B \xrightarrow{\mathsf{I}} A$ (injective). Becouse A is an infinite set, and injective function g can in fact map the infinite set B into A, which then implies $B \subseteq A$.

$$(f : \mathbb{N} \xrightarrow{\mathsf{B}} B) \wedge (g : B \xrightarrow{\mathsf{I}} A) \implies |A| \geq |B| = |\mathbb{N}| = \aleph_0$$

Problem 6

Prove that the following sets have the same cardinality.



These sets can all be represented as a set of real number ordered pairs.
These are constructed with arbitrary constants: $x_a, x_b, x_c, x_d, x_e, x_f, y_a, y_b, y_c, y_d, y_e, y_f, x_s, y_s, r_s, m_l, b_l$

$$
\begin{aligned}
AB &= \{(x,y) \in \mathbb{R}^2,\ t \in \mathbb{R} : (x = (1-t)x_a + tx_b) \wedge (y = (1-t)y_a + ty_b) \wedge (0 \leq t \leq 1)\} \\
CD &= \{(x,y) \in \mathbb{R}^2,\ t \in \mathbb{R} : (x = (1-t)x_c + tx_d) \wedge (y = (1-t)y_c + ty_d) \wedge (0 \leq t < 1)\} \\
EF &= \{(x,y) \in \mathbb{R}^2,\ t \in \mathbb{R} : (x = (1-t)x_e + tx_f) \wedge (y = (1-t)y_e + ty_f) \wedge (0 < t < 1)\} \\
\mathbb{S} &= \{(x,y) \in \mathbb{R}^2,\ t \in \mathbb{R} : (x = r_s cos(2\pi t) + x_s) \wedge (y = sin(2\pi t) + y_s) \wedge (0 \leq t < 1)\} \\
l &= \{(x,y) \in \mathbb{R}^2,\ t \in \mathbb{R} : (x = t) \wedge (y = m_l t + b_l)\}
\end{aligned}
$$

It is clear that each set is defined parmetrically with bijective equations maping the parameter t into a 2-D corndates (x, y), thus proving that the cardinality of each sets parameter is sufficent to showing the each set has the same cardinality.

Let $T_i = \{t \in A_i\}$ for each of the sets A_i, then (from the reasoning above) the following can be said:

$$
\begin{aligned}
T_{AB} &= \{t \in \mathbb{R} : 0 \leq t \leq 1\}, & |T_{AB}| &= |AB| \\
T_{CD} &= \{t \in \mathbb{R} : 0 \leq t < 1\}, & |T_{CD}| &= |CD| \\
T_{EF} &= \{t \in \mathbb{R} : 0 < t < 1\}, & |T_{EF}| &= |EF| \\
T_{\mathbb{S}} &= \{t \in \mathbb{R} : 0 \leq t < 1\}, & |T_{\mathbb{S}}| &= |\mathbb{S}| \\
T_l &= \{t \in \mathbb{R}\}, & |T_l| &= |l|
\end{aligned}
$$

Clearly, the equivalent definition of T_{CD} and $T_{\mathbb{S}}$ indicates

$$|CD| = |T_C D| = |T_{\mathbb{S}}| = |\mathbb{S}|$$

The equivalence of the other sets is more difficult then via definition.
First, the baseline cardinality can be shown to be $\aleph_1$ as (by definition of T_l))

$$|T_l| = |\mathbb{R}| = \aleph_1$$

Next, the equivalence of T_{EF} and T_l can be shown with the biforjective mapping

$$f_1 : \mathbb{R} \to T_{EF} = \frac{2\pi \tan^{-1}(x) + 1}{2}$$

Therefore,

$$|EF| = |T_{EF}| = |T_l| = |\mathbb{R}| = \aleph_1$$

Next, due to the nature of infinite sets, the addition of $t = 0$ from T_{EF} to T_{CD} does not affect the overall cardinality of T_{CD}, thus

$$|CD| = |T_{CD}| = |T_{EF}| = |\mathbb{R}| = \aleph_1$$

Similarly, the addition of $t = 1$ from T_{CD} to T_{AB} will stil result in

$$|AB| = |T_{AB}| = |T_{CD}| = |\mathbb{R}| = \aleph_1$$

Ultimently this means that

$$|AB| = |CD| = |EF| = |\mathbb{S}| = |l| = |\mathbb{R}| = \aleph_1$$