MATH 5301 Elementary Analysis - Homework 2

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Problem 1

For a function $f : A \to B$, show the follwoing for any $X \subset A, Y, Z \subset B$

1a) $X \subset f^{-1}(f(X))$

$$\begin{aligned}
f(X) &= \{y \in B : \exists x \in X : y = f(x)\} \\
f^{-1}(Y) &= \{x \in A : \exists y \in Y : y = f(x)\} \\
f(f^{-1}(X)) &= \{\tilde{x} \in \tilde{X} : \exists x \in A : \tilde{x} = f(f^{-1}(x))\} \\
&= \{\tilde{x} \in \tilde{X} : \exists y \in Y : y = f(\tilde{x}) : \exists x \in X : y = f(x)\} \implies \\
&\implies \tilde{X} \subset X \\
&\therefore X \subset f^{-1}(f(X))
\end{aligned}$$

1b) $f(f^{-1}(Y)) \subset Y$

$$\begin{aligned}
f^{-1}(Y) &= \{x \in A : \exists y \in Y : y = f(x)\} \\
f(X) &= \{y \in B : \exists x \in X : y = f(x)\} \\
f^{-1}(f(Y)) &= \{\tilde{x} \in \tilde{X} : \exists x \in A : \tilde{x} = f^{-1}(f(x))\} \\
&= \{\tilde{x} \in \tilde{X} : \exists y \in Y, \exists x \in X : (y = f(\tilde{x})) \wedge (y = f(x)\} \implies \\
&\implies \tilde{X} \subset X \\
&\therefore X \subset f^{-1}(f(X))
\end{aligned}$$

1c) $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$

$$
\begin{aligned}
f^{-1}(Y) &= \{x \in A : \exists y \in Y : y = f(x)\} \\
f^{-1}(Z) &= \{x \in A : \exists z \in Z : z = f(x)\} \\
f^{-1}(Y \cup Z) &= \{x \in A : (\exists y \in Y y = f(x)) \vee (\exists z \in Z : z = f(x))\} \\
&= \{x \in A : (\exists y \in Y y = f(x))\} \cup \{x \in A : \vee(\exists z \in Z : z = f(x))\} \\
&= f^{-1}(Y) \cup f^{-1}(Z) \\
&\therefore f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)
\end{aligned}
$$

1d) $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$

$$
\begin{aligned}
f^{-1}(Y) &= \{x \in A : \exists y \in Y : y = f(x)\} \\
f^{-1}(Z) &= \{x \in A : \exists z \in Z : z = f(x)\} \\
f^{-1}(Y \cap Z) &= \{x \in A : (\exists y \in Y y = f(x)) \wedge (\exists z \in Z : z = f(x))\} \\
&= \{x \in A : (\exists y \in Y y = f(x))\} \cap \{x \in A : \vee(\exists z \in Z : z = f(x))\} \\
&= f^{-1}(Y) \cap f^{-1}(Z) \\
&\therefore f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)
\end{aligned}
$$

Problem 2

Show that:

2a) $A \cap \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A)$

Let $\Lambda := \{1, 2, \cdots, n\}$,

$$\begin{aligned} A \cap \bigcup_{\lambda \in \Lambda} A_\lambda &= A \cap (A_1 \cup A_2 \cup \cdots \cup A_n) \\ &= (A \cap A_1) \cup (A \cap A_2) \cup \cdots \cup (A \cap A_n) \\ &= \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A) \end{aligned}$$

Therefore,

$$\boxed{A \cap \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A)}$$

2b) $\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda\right) \subseteq \bigcap_{\lambda \in \Lambda} (A_\lambda \cup B_\lambda)$

Let $\Lambda_A := \{1, 2, \cdots, n\}$ and $\Lambda_B := \{1, 2, \cdots, m\}$,

$$\begin{aligned} \bigcap_{\lambda \in \Lambda} (A_\lambda \cup B_\lambda) &= (A_1 \cup B_1) \cap (A_1 \cup B_2) \cap \cdots \cap (A_1 \cup B_m) \cap (A_2 \cup B_1) \cap \cdots \cap (A_n \cup B_m) \\ &= (A_1 \cup (B_1 \cap B_2 \cap \cdots \cap B_m)) \cap \cdots \cap (A_n \cup (B_1 \cap B_2 \cap \cdots \cap B_m)) \\ &= (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup \cdots (A_2 \cap A_3) \cdots \cup (A_{n-1} \cap A_n) \\ &\quad \cup (A_1 \cap B_1) \cup \cdots \cup (A_1 \cap B_m) \cup \cdots \cup (A_n \cap B_m) \\ &\quad \cup (B_1 \cap B_2) \cup \cdots \cup (B_1 \cap B_m) \cup \cdots \cup (B_{m-1} \cap B_m) \\ &= \left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda\right) \cup (A_1 \cap B_1) \cup \cdots \cup (A_1 \cap B_m) \cup \cdots \cup (A_n \cap B_m) \end{aligned}$$

Therefore,

$$\boxed{\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda\right) \subseteq \left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda\right) \cup (A_1 \cap B_1) \cup \cdots \cup (A_1 \cap B_m) \cup \cdots \cup (A_n \cap B_m)}$$

Problem 3

Problem: Which of these are equivalence relations?

Solution: a, c, & d
(The following explain why)

3a)

for $a, b \in \mathbb{R}$, let $a\mathcal{R}b$ if $a - b \in \mathbb{Q}$

i) Reflective:

$$x\mathcal{R}x = x - x = 0 \in \mathbb{Q}$$

ii) Symetric:

$$\begin{aligned} x\mathcal{R}y &\implies y\mathcal{R}x \\ x\mathcal{R}y = x - y \in \mathbb{Q} &\implies y - x \in \mathbb{Q} \end{aligned}$$

Since $x - y = -(y - x)$, $(x - y)$ and $(y - x)$ will both be either rational or not rational, this is true.

iii) Transitive:

$$\begin{aligned} x\mathcal{R}y \wedge y\mathcal{R}z &\implies x\mathcal{R}z \\ (x - y \in \mathbb{Q}) \wedge (y - z \in \mathbb{Q}) &\implies (x - z \in \mathbb{Q}) \\ \left(\frac{x_a}{x_b} - \frac{y_a}{y_b} \in \mathbb{Q}\right) \wedge \left(\frac{y_a}{y_b} - \frac{z_a}{z_b} \in \mathbb{Q}\right) &\implies \left(\frac{x_a}{x_b} - \frac{z_a}{z_b} \in \mathbb{Q}\right) \end{aligned}$$

This also means that:

$$\begin{aligned} &(x_a y_b - x_b y_a \in \mathbb{N}) \wedge (x_b y_b \neq 0 \in N) \wedge (y_a z_b - y_b z_a \in \mathbb{N}) \wedge (y_b z_b \neq 0 \in N) \\ &\qquad\qquad\qquad\qquad \implies (x_a z_b - x_b z_a \in \mathbb{N}) \wedge (x_b z_b \neq 0 \in N) \end{aligned}$$

Since this statments indicates that $x_b, y_b, z_b \neq 0$ and that $x_a y_b - x_b y_a, y_a z_b - y_b z_b \in \mathbb{N}$, the following will always be true as well: $x_a z_b - x_b z_a$. Therefore, the relation is transitive.

3b)

for $a, b \in \mathbb{R}$, let $a\mathcal{R}b$ if $a - b \notin \mathbb{Q}$

i) Reflective:

The relationship is NOT reflective:

$$\begin{aligned} a\mathcal{R}b &= a - b \notin \mathbb{Q} \\ x\mathcal{R}x &= x - x = 0 \in \mathbb{Q} \end{aligned}$$

3c)

for $a, b \in \mathbb{R}$, let $a\mathcal{R}b$ if $a-b$ is a square root of a rational number.
i.e.

$$a\mathcal{R}b = (a-b)^2 \in \mathbb{Q}$$

i) Reflective:

$$\begin{aligned} a\mathcal{R}b &= (a-b)^2 \in \mathbb{Q} \\ x\mathcal{R}x &= (x-x)^2 = 0^2 = 0 \in \mathbb{Q} \end{aligned}$$

ii) Symetric:

$$\begin{aligned} x\mathcal{R}y &\implies y\mathcal{R}x \\ (x-y)^2 \in \mathbb{Q} &\implies (y-x)^2 \in \mathbb{Q} \\ (x-y)^2 = (y-x)^2 \therefore\ x\mathcal{R}y &\implies y\mathcal{R}x \end{aligned}$$

iii) Transative:

$$\begin{aligned} x\mathcal{R}y \wedge y\mathcal{R}z &\implies x\mathcal{R}z \\ \big((x-y)^2 \in \mathbb{Q}\big) \wedge \big((y-z)^2 \in \mathbb{Q}\big) &\implies \big((x-z)^2 \in \mathbb{Q}\big) \\ \big(x^2 - 2xy + y^2 \in \mathbb{Q}\big) \wedge \big(y^2 - 2yz + z^2 \in \mathbb{Q}\big) &\implies \big(x^2 - 2xz + z^2 \in \mathbb{Q}\big) \\ (x^2 \in \mathbb{Q}) \wedge (y^2 \in \mathbb{Q}) \wedge (z^2 \in \mathbb{Q}) \wedge (-2xy \in \mathbb{Q}) \wedge (-2yz \in \mathbb{Q}) &\implies (x^2 \in \mathbb{Q}) \wedge (z^2 \in \mathbb{Q}) \wedge (-2xz \in \mathbb{Q}) \\ (xy \in \mathbb{Q}) \wedge (yz \in \mathbb{Q}) &\implies (xz \in \mathbb{Q}) \end{aligned}$$

Which is clearly transitive, so $a\mathcal{R}b = (a-b)^2 \in \mathbb{Q}$ is transitive.

3d)

Let $X = \mathbb{Z} \times \mathbb{N}$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $\mathcal{R}$ if $x_1 y_2 = x_2 y_1$.
i.e.

$$\begin{aligned} a_1, b_1 \in \mathbb{Z}&, a_2, b_2 \in \mathbb{N}, \\ a_1 b_2 = a_2 b_1 &\implies (a_1, a_2)\mathcal{R}(b_1, b_2) \end{aligned}$$

i) Reflective:

$$\begin{aligned} a_1 b_2 = a_2 b_1 &\implies (a_1, a_2)\mathcal{R}(b_1, b_2) \\ x_1 x_2 = x_2 x_1 &\implies (x_1, x_2)\mathcal{R}(x_1, x_2) \end{aligned}$$

ii) Symetric:

$$\begin{aligned} x\mathcal{R}y &\implies y\mathcal{R}x \\ (x_1 y_2 = x_2 y_1 \implies (x_1, x_2)\mathcal{R}(y_1, y_2)) &\implies (y_1 x_2 = y_2 x_1 \implies (y_1, y_2)\mathcal{R}(x_1, x_2)) \end{aligned}$$

iii) Transative:

$$\begin{aligned} x\mathcal{R}y \wedge y\mathcal{R}z &\implies x\mathcal{R}z \\ (x_1 y_2 = x_2 y_1 \implies (x_1, x_2)\mathcal{R}(y_1, y_2)) \wedge (y_1 z_2 = y_2 z_1 \implies (y_1, y_2)\mathcal{R}(z_1, z_2)) &\implies (x_1 z_2 = x_2 z_1 \implies (x_1, x_2)\mathcal{R}(z_1, z_2)) \\ (x_1 y_2 = x_2 y_1) \wedge (y_1 z_2 = y_2 z_1) &\implies (x_1 z_2 = x_2 z_1) \end{aligned}$$

Which is clearly transitive, so $(a_1, a_2)\mathcal{R}(b_1, b_2)$ is transitive, and therefore an equivalence relation.

Problem 4

For the relation $(x,y) \succeq (a,b)$ if $(x \geq a)$ and $(y \geq b)$ on the set of ordered pairs of $\{1,2,3\} \times \{1,2,3\}$. i.e. $x,a \in \{1,2,3\}$ and $y,b \in \{1,2,3\}$,

$$(x \geq a) \land (y \geq b) \implies (x,y) \succeq (a,b)$$

4a) Show that the above relation is an order relation.

An ordered relation requires (i) reflexitivity, (ii) anti-symmetry, and (iii) transivity.

i) Reflective:

$$\begin{aligned}(x \geq a) \land (y \geq b) &\implies (x,y) \succeq (a,b)\\ (x \geq x) \land (y \geq y) &\implies (x,y) \succeq (x,y)\end{aligned}$$

ii) Anti-Symmetry:

$$\begin{aligned}((x,y) \succeq (a,b)) \land ((a,b) \succeq (x,y)) &\implies (x,y) = (a,b)\\ ((x \geq a) \land (y \geq b)) \land ((a \geq x) \land (b \geq y)) &\implies (x,y) = (a,b)\\ ((x \geq a) \land (a \geq x)) \land ((b \geq y) \land (y \geq b)) &\implies (x,y) = (a,b)\end{aligned}$$

iii) Transivity

$$\begin{aligned}(x,y)\mathcal{R}(a,b) \land (a,b)\mathcal{R}(\alpha,\beta) &\implies (x,y)\mathcal{R}(\alpha,\beta)\\ ((x \geq a) \land (y \geq b)) \land ((a \geq \alpha) \land (b \geq \beta)) &\implies (x \geq \alpha) \land (y \geq \beta)\\ ((x \geq a) \land (a \geq \alpha)) \land ((y \geq b) \land (b \geq \beta)) &\implies (x \geq \alpha) \land (y \geq \beta)\\ (x \geq a \geq \alpha) \land (y \geq b \geq \beta) &\implies (x \geq \alpha) \land (y \geq \beta)\end{aligned}$$

Therefore, the relation $(x,y) \succeq (a,b)$ is an order relation.

4b) Can you make it the total order?

i) Totality:

$$\begin{aligned}\forall (x,y),(a,b) \in \{1,2,3\} \times \{1,2,3\} &\implies ((x,y) \succeq (a,b)) \lor ((a,b) \succeq (x,y))\\ \forall x,a \in \{1,2,3\}, \forall y,b \in \{1,2,3\} &\implies ((x \geq a) \land (y \geq b)) \lor ((a \geq x) \land (y \geq b))\end{aligned}$$

4c) How many different total orderings can be constructed?

Multiple total orderings of relation on subsets can satisfy this. A network constructed from the following relations would demonstrate the multiple paths.

$$
\begin{aligned}
&(3,3) \succeq (x,y) \forall (x,y) \in \{1,2,3\} \times \{1,2,3\} \\
&(3,2) \succeq (x,y) \forall (x,y) \in \{1,2,3\} \times \{1,2\} \\
&(3,1) \succeq (x,y) \forall (x,y) \in \{1,2,3\} \times \{1\} \\
&(2,3) \succeq (x,y) \forall (x,y) \in \{1,2\} \times \{1,2,3\} \\
&(2,2) \succeq (x,y) \forall (x,y) \in \{1,2\} \times \{1,2\} \\
&(2,1) \succeq (x,y) \forall (x,y) \in \{1,2\} \times \{1\} \\
&(1,3) \succeq (x,y) \forall (x,y) \in \{(1,1),(1,2),(1,3)\} \\
&(1,2) \succeq (x,y) \forall (x,y) \in \{(1,1),(1,2)\} \\
&(1,1) \succeq (1,1)
\end{aligned}
$$

If you only count the total orders of length 2 (the number of actual network edges with self-edges) this would be 36. If drawns as a tree, these 36 pairs would be created as directed edges and the total number of unique tree paths can also be counted using various algorithms.

Problem 5

Provide and example of $f : \mathbb{Z} \to \mathbb{N}$ such that

5a) f is surjective, but not injective

$$y = f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

5b) f is injective, but not surjective

$$y = f(x) = \begin{cases} x^2 & x \geq 0 \\ x^2 + 1 & x < 0 \end{cases}$$

5c) f is surjective and injective (bijective)

$$y = f(x) = \begin{cases} 2x & x \geq 0 \\ -2x - 1 & x < 0 \end{cases}$$

5d) f is niether surjective nor injective

$$y = f(x) = 0$$

Problem 6

Problem: Is the following statement correct?

Theorem 1. *If the relation $\mathbb{R}$ on A is symetric and transistive, then it is reflexive.*
Proof: For any $a \in A$ let $b \in A$ is such that $a\mathcal{R}b$. Then by symmetry $b\mathcal{R}a$. Then by symmetry $a\mathcal{R}a$.

Solution: No. Specifically, the final statement of the proof states to use symmetry to conclude that it is reflexive, but it actually requires the transivity property to make that conclusion.

The following is a proposed corrected statement:

Theorem 1. *If the relation $\mathcal{R}$ is symetric and transistive on A, then Rel is also reflexive on A.*
Proof: Let $a \in A$ and $b \in A$ be selected so that $a\mathcal{R}b$ Since $\mathcal{R}$ is symetric:

$$a\mathcal{R}b \implies b\mathcal{R}a$$

Since $\mathcal{R}$ is transitive:

$$(a\mathcal{R}b) \wedge (b\mathcal{R}a) \implies a\mathcal{R}a$$

Therefore, $\mathcal{R}$ is also reflexive.