MATH 5301 Elementary Analysis - Homework 6

Jonas Wagner

2021, October 17^{th}

Problem 1 Provide examples of the sets $A, B \subset \mathbb{R}^2$ so that:

Definition 1. *Set A is* connected *if it is not disconnected.*

Definition 2. *Set A is* disconnected *if $\forall U, V$ - open:*

a. $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$

b. $(U \cap V) \cap A \neq \emptyset$ *Note: this implies $U \neq \emptyset$ and $V \neq \emptyset$*

c. $A \subset U \cup V \neq \emptyset$

a) A and B are connected but $A \cup B$ is not.

Sets A and B are individually connected but that just aren't near each other. An example are these open balls:

$$A = \mathcal{B}_r((0,0)) := \left\{(x,y) \in \mathbb{R} \ : \ x^2 + y^2 < r^2\right\}$$
$$B = \mathcal{B}_r((-5r, 5r)) := \left\{(x,y) \in \mathbb{R} \ : \ (x+5r)^2 + (y-5r)^2 < r^2\right\}$$

b) A and B are connected but $A \cap B$ is not.

Sets A and B are individually connected, but they intersect in two areas resulting in a disjoint intersection... Such as this donut and square sets:

$$A = \mathcal{B}_R^{(2)}((0,0)) \backslash \mathcal{B}_r^{(2)}((0,0)) = \left\{(x,y) \in \mathbb{R} \ : \ r^2 < x^2 + y^2 < R^2\right\}$$
$$B = \mathcal{B}_r^{(\infty)}((0,0)) = \{(x,y) \in \mathbb{R} \ : \ x < r \wedge y < r\}$$

c) A and B are not connected but $A \cup B$ is connected.

A and B individually are disjoint shapes but each individual disjointed part intersects with one from the other set. An example would be a couple of disjoint (then connected when a union) regions.

$$A := \{(x,y) \in \mathbb{R} \ : \ x \in [-1,0] \cup [1,2] \wedge y \in [-2,2]\}$$
$$B := \{(x,y) \in \mathbb{R} \ : \ x \in [-2,1.25] \cup [1.75,4] \wedge y \in [-1,3]\}$$

d) A and B are not connected but $A \cap B$ is connected.

A and B individually are disjoint shapes, there is then only one connected region where they intersect. An example is two overlapping regions, each with random other disjoint parts.

$$A := \{(x,y) \in \mathbb{R} \ : \ x \in [0,1] \cup [15,30], y \in [0,1]\}$$
$$B := \{(x,y) \in \mathbb{R} \ : \ x \in [-1,1] \cup [-15,-20], y \in [0,1]\}$$

e) A and B are not connected but $A \setminus B$ is connected.

A and B individually are disjoint shapes, with a part of B covering all but one disjoint region of A. An example is a two overlapping regions with disjoint parts and A having one (only one) disjoint part not fully covered by B.

$$A := \{(x, y) \in \mathbb{R} \ : \ x \in [0, 1] \cup [15, 30], y \in [0, 1]\}$$
$$B := \{(x, y) \in \mathbb{R} \ : \ x \in [-1, 0.5] \cup [15, 30], y \in [0, 1]\}$$

Problem 2

a) Prove that every monotone bounded sequence in $\mathbb{R}$ converge.

Definition 3. *The function $f : \mathbb{R} \to \mathbb{R}$ is called* monotone *if and only if it is either entirely non-increasing or non-decreasing.*

a. f *is* non-decreasing *if* $\forall_{x,y\in\mathbb{R}} x \leq y \implies f(x) \leq f(y)$.

b. f *is* non-increasing *if* $\forall_{x,y\in\mathbb{R}} x \leq y \implies f(x) \geq f(y)$.

Definition 4. *The function $f : \mathbb{R} \to \mathbb{R}$ is called* bounded *if*

$$\exists_{N\in\mathbb{R}} : \forall_{x\in\mathbb{R}} \ |f(x)| \leq N$$

Definition 5. *A sequence $\{a_n\}$ is said to* converge *to limit a if*

$$\forall_{\epsilon>0}\exists_{N(\epsilon)} : \forall_{n>N}|a_n - a| < \epsilon$$

Theorem 1. *Every monotone bounded sequence in $\mathbb{R}$ converges.*

Proof. Define sequence $\{a_n\}$ bounded and monotone. (i.e)

$$((\forall_{n,m\in\mathbb{N}:m>n} a_n \geq a_m) \vee (\forall_{n,m\in\mathbb{N}:m>n} a_n \leq a_m)) \wedge (\exists_{N\in\mathbb{R}}\forall_{n\in\mathbb{N}}|a_n| \leq N)$$

Taking only the positive case (non-decreasing and bounded from above), a proof can be made without loss of generality (as thats how $|\cdot|$ works...).

$$\begin{aligned}
(\forall_{n,m\in\mathbb{N}:n>m} a_n \leq a_m) \wedge (\exists_{a\in\mathbb{R}}\forall_{n\in\mathbb{N}}|a_n| \leq a) &\implies \forall_{\epsilon>0}\exists_{N(\epsilon)} : \forall_{n>N}|a_n - a| < \epsilon \\
\exists_{a\in\mathbb{R}}\forall_{n,m\in\mathbb{N},n>m} a_n \leq a_m \leq a &\implies \forall_{\epsilon>0}\exists_{N(\epsilon)}\forall_{n>N} a - a_n < \epsilon \\
\forall_{\epsilon>0}\exists_{N(\epsilon)}\exists_{a\in\mathbb{R}}\forall_{n,m\in\mathbb{N},n>m>N} a_n \leq a_m \leq a &\implies a - a_n < \epsilon
\end{aligned}$$

Clearly since the sequence is non-decreasing and bounded from above it will converge to the upper boundary. This can then be applied again for below to fully prove this. □

b) Provide an example of the set $A \in \mathbb{R}$ having exactly four limit points.

Definition 6. *For a set $A \subset S$ in metric space (S, d), $x \in S$ is called a* limit point *of A if*

$$\forall_{\epsilon>0}\exists_{y\in A} : 0 < d(x, y) < \epsilon$$

Define A as a simple union of simple single limit point sets:

$$A := \{\frac{1}{n}, n \in \mathbb{N}\} \cup \{\sqrt{2} + \frac{1}{n}, n \in \mathbb{N}\} \cup \{\pi + \frac{1}{n}, n \in \mathbb{N}\} \cup \{e + \frac{1}{n}, n \in \mathbb{N}\}$$

In this case the limit points are: 0, $\sqrt{2}$, π, and e.

c) Provide an example of a sequence $\{a_n\}$, so that every point in the interval $[2019, 2021]$ is a limit point of it.

Because sinusoidal functions are fun, lets use:

$$\{a_n\} := \{a_n = 2000 + \sin(n), \ n = 1, 2, \dots\}$$

Problem 3

a)

Provide an example of a sequence $\{a_n\}$ so that a_n diverges, but $\lim_{n\to\infty}(a_n - a_{2n}) = 0$.
This is perhaps going too fun with it, but how about:

$$\{a_n\} := \{a_n = \begin{cases} a_{n/2} + \frac{1}{n} & n \vdots 2 \\ n^2 & \text{else} \end{cases}\}$$

b)

Provide an example of two sequences $\{a_n\}$ and $\{b_n\}$ so that

$$\left(\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n\right) < \liminf_{n\to\infty}(a_n+b_n) < \left(\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n\right) < \limsup_{n\to\infty}(a_n+b_n) < \left(\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n\right)$$

Definition 7. *A* Limit Superior*, denoted as* $\limsup_{n\to\infty}$*, is the limit of an superior function on the extremes of a sequence. This is defined by*

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right)$$

Definition 8. *A* Limit Inferior*, denoted as* $\liminf_{n\to\infty}$*, is the limit of an inferior function on the extremes of a sequence. This is defined by*

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right)$$

Defining two sinusoidal functions with the same frequency but with a phase shift to incur the strictness requirements.

$$\{a_n\} := \{a_n = \cos(\omega_1 n + \theta_1)\}_{n\in\mathbb{N}}$$
$$\{b_n\} := \{b_n = \cos(\omega_2 n + \theta_2)\}_{n\in\mathbb{N}}$$

with $\omega_1 = \omega_2 = \frac{\pi}{25}$, $\theta_1 = 0$, and $\theta_2 = \frac{\pi}{8}$.

Problem 4

Show the equivalence of the norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_p, p > 1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$.

Definition 9. *A* norm *is a function* $\|\cdot\| : S \to \mathbb{R}_+$ *satisfying:*

a. $\|\vec{\mathbf{x}}\| > 0,\ \vec{\mathbf{x}} \neq 0$

b. $\|\lambda\vec{\mathbf{x}}\| = |\lambda|\|\vec{\mathbf{x}}\|$

c. $\|\vec{\mathbf{x}} + \vec{\mathbf{y}}\| \leq \|\vec{\mathbf{x}}\| + \|\vec{\mathbf{y}}\|$

Definition 10. *For* $\|\cdot\|_a, \|\cdot\|_b$ *on* S*,* $\|\cdot\|_a$ *is said to be* stronger *then* $\|\cdot\|_b$ *if*

$$\forall\{x_n\} \subset S\ :\ x_n \underset{d_a}{\longrightarrow} x \implies x_n \underset{d_b}{\longrightarrow} x$$

Which is equivalent to

Definition 11. $\|\cdot\|_a$ *and* $\|\cdot\|_b$ *are said to be* equivalent*,* $\|\cdot\|_a \sim \|\cdot\|_b$*, if* $\|\cdot\|_a$ *is stronger then* $\|\cdot\|_b$ *and* $\|\cdot\|_b$ *is stronger then* $\|\cdot\|_a$*. Additionally, the following is also true:*

$$\|\cdot\|_a \sim \|\cdot\|_b \implies \exists\alpha, \beta \in \mathbb{R}_{>0} : \forall_{x\in S}\alpha\|\cdot\|_b \leq \|\cdot\|_a \leq \beta\|x\|_b$$

Definition 12. *The standard norms are defined as*

a. $\|\cdot\|_1 := \|x\|_1 = \sum_{i=1}^n |x_i| = |x_1| + |x_2| + \cdots + |x_n|$

b. $\|\cdot\|_2 := \|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} = \left(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2\right)^{1/2}$

c. $\|\cdot\|_p := \|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p} = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p},\ p > 1$

d. $\|\cdot\|_\infty := \|x\|_\infty = \max_{i=1}^n |x_i| = \max(|x_1|, |x_2|, \ldots, |x_n|)$

Theorem 2. *The norms* $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_p$*, and* $\|\cdot\|_\infty$ *are all equivalent on* $\mathbb{R}^n$*.*

Proof.

Lemma 1. $\|\cdot\|_1$ *is equivalent to* $\|\cdot\|_2$

Proof.

$$\|\cdot\|_1 \sim \|\cdot\|_2 \implies \exists\alpha, \beta \in \mathbb{R}_{>0}\ :\ \forall_{x\in\mathbb{R}}\alpha\|x\|_2 \leq \|x\|_1 \leq \beta\|x\|_2$$

$$\alpha\left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \leq \sum_{i=1}^n |x_i| \leq \beta\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$$

$$\left(\alpha\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right)^2 \leq \left(\sum_{i=1}^n |x_i|\right)^2 \leq \left(\beta\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\right)^2$$

$$\alpha^2\sum_{i=1}^n |x_i|^2 \leq \left(\sum_{i=1}^n |x_i|\right)\left(\sum_{j=1}^n |x_j|\right) \leq \beta^2\sum_{i=1}^n |x_i|^2$$

From the Cauchy-Schwartz inequality $(\sum_{i=1}^n u_i v_i)^2 \leq (\sum_{i=1}^n u_i^2)(\sum_{i=1}^n v_i^2)$

$$\alpha^2\sum_{i=1}^n |x_i|^2 \leq \sum_{i=1}^n |x_i x_i| \leq \sum_{i=1}^n |x_i|^2 \leq \left(\sum_{i=1}^n |x_i|\right)\left(\sum_{j=1}^n |x_j|\right) \leq \left(\sum_{i=1}^n |x_i|\right)^2 \leq \beta^2\sum_{i=1}^n |x_i|^2$$

Clearly, this is true for when $\alpha \in (0, 1]$ and $\beta \in [1, \infty)$; therefore $\|\cdot\|_1 \sim \|\cdot\|_2$. □

□

Problem 5

Are there any open sets A and B in $\mathbb{R}^4$ so that $dist(A,B)=0$ but $A\cap B=\emptyset$?

Definition 13. *For sets $A,B\subset S$ in metric space (S,d),*

$$dist(A,B) := \inf\{d(x,y)\ :\ x\in A, y\in B\}$$

Proposition 1. *There exists open sets A and B in $\mathbb{R}^2$ so that $d(A,B)=0$ but $A\cap B=\emptyset$.*

Proof. $A,B\in\mathbb{R}^4$ open means

$$\forall_{x\in A}\exists_{\epsilon>0}\forall_{y\in\mathbb{R}^4} d(x,y)<\epsilon \implies y\in A$$

and

$$\forall_{x\in B}\exists_{\epsilon>0}\forall_{y\in\mathbb{R}^4} d(x,y)<\epsilon \implies y\in B$$

$dist(A,B)=0$ means that

$$dist(A,B)=\inf\{d(x,y)\ :\ x\in A, y\in B\}=0$$

Additionally, $A\cap B=\emptyset$ implies that $\forall_{x\in A}\forall_{y\in B} x\neq y$.
In order for $dist(A,B)=0$ and $A\cap B=\emptyset$,

$$\exists_{x\in A} : \inf_{y\in B}\{d(x,y)\}=0\wedge x\notin B$$

One solution to this is that

$$\exists_{x\in\partial A, y\in\partial B}\ :\ d(x,y)=0$$

Therefore, it is possible for open sets A and B in $\mathbb{R}^4$ so that $dist(A,B)=0$ and $A\cap B=\emptyset$.
Essentially, $\partial A\cap\partial B\neq\emptyset$.
An example would be:

$$A := \{(x,y)\in\mathbb{R}^2 : x^2+y^2<r_1^2\}$$

and

$$B := \{(x,y)\in\mathbb{R}^2 : (x+2r_2)^2+y^2<r_2^2\}$$

with $r_1=r_2=r\in\mathbb{R}$. $\square$

Problem 6

Let $\mathcal{B}([0,1])$ denote the set of all bounded functions from $[0,1]$ to $\mathbb{R}$. Define the metric on $\mathcal{B}([0,1])$ as $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$.

a) Show that this is indeed a metric.

Theorem 3. *The metric* $d : \mathcal{B}([0,1]) \to \mathbb{R}$

$$d(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|$$

is a metric on $\mathcal{B}([0,1])$*.*

Proof. A metric must satisfy (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

i) Non-negativy

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| \geq 0$$

Since $|f(x) - g(x)| \geq 0$, $\sup_{x\in[0,1]} |f(x) - g(x)| \geq 0$ and therefore $d(f,g) \geq 0$.

ii) Symmetry

$$d(f,g) = d(g,f)$$

$$\begin{aligned} d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| &= d(g,f) = \sup_{x\in[0,1]} |g(x) - f(x)| \\ \sup_{x\in[0,1]} |f(x) - g(x)| &= \sup_{x\in[0,1]} |(-1)(f(x) - g(x))| \\ \sup_{x\in[0,1]} |f(x) - g(x)| &= \sup_{x\in[0,1]} |(f(x) - g(x))| \end{aligned}$$

Therefore it is symmetric.

iii) Triangle Inequality

$$d(f,h) \leq d(f,g) + d(g,h)$$

Denoting $N_f(M_f)$, $N_g(M_g)$, and $N_h(M_h)$ as the upper (and lower) bounds within $x \in [0,1]$ of $f(x)$,$g(x)$, and $h(x)$ respectively. <u>Note:</u> this is not a very efficient proof...

$$\begin{aligned} \sup_{x\in[0,1]} |f(x) - g(x)| &\leq |N_f - M_g| + |N_g - M_f| \\ \sup_{x\in[0,1]} |g(x) - h(x)| &\leq |N_g - M_h| + |N_h - M_g| \\ \sup_{x\in[0,1]} |f(x) - h(x)| &\leq |N_f - M_h| + |N_h - M_f| \end{aligned}$$

$$d(f,h) = \sup_{x\in[0,1]} |f(x) - h(x)| \leq d(f,g) + d(g,h) = \sup_{x\in[0,1]} |f(x) - g(x)| + \sup_{x\in[0,1]} |g(x) - h(x)|$$
$$|N_f - M_g| + |N_g - M_f| \leq |N_g - M_h| + |N_h - M_g| + |N_f - M_h| + |N_h - M_f|$$

Which demonstrates something.... I'll have to look at this again because it is problematic $\square$

b) Prove that the space $(\mathcal{B}([0,1]), d)$ is a complete metric space.

Definition 14. *Metric space (S, d) is called a* complete metric space *if every cauchy sequence $\{a_n\} \subset S$ converges in S.*

$$\forall_{\{a_n\} \subset S} : \{a_n\} \text{ cauchy } \implies \exists_{a \in S} : \lim_{n\to\infty} a_n = a$$

Theorem 4. *$(\mathcal{B}([0,1]), d)$ is a complete metric space.*

Proof. Let $\{a_n\}$ be a cauchy sequence (i.e.)

$$\{a_n\} : \forall_{\epsilon > 0} \exists_N : \forall_{n,m>N} \implies d(a_n, a_m) < \epsilon$$

Consider set $D_N = \{x \in S \ : \ \forall_{n<N} a_n < x\}$

a. D is bounded (i.e $x \in D < a_N + \epsilon$)

b. D is nonempty (i.e $a_N - \epsilon \in D$)

For a fixed ϵ,

$$\exists_N \ : \ \forall_{n>N+1} \implies d(a_n, a_{N+1}) < \epsilon$$

We then take $a = \sup\{D\}$.
Claim $a = \lim_{n\to\infty} a_n$,

$$\forall_{\epsilon>0} \exists_N \ : \ \forall_{n>N} \implies d(a, a_n) < \epsilon$$

Claim $y < a \implies y \in D$,

$$y \in D \wedge z < y \implies z \in D$$

$$\forall_{\epsilon>0} \exists_{N(\epsilon)} \ : \ \forall_{n>N} a_n > a - \frac{\epsilon}{2}$$

However, it is also true by definition of cauchy sequence:

$$d(a_m, a_n) < \frac{\epsilon}{2} \ \forall_{m,n>\hat{N}}$$

Also,

$$n > \max[N, \hat{N}] \ : \ d(a_n, a) \leq d(a_n, a_m) + d(a_m, a) \leq \epsilon$$

Since $\epsilon > 0$ is arbitrarily selected and $d(a_n, a) < \epsilon$ whenever $n \geq N$, this implies $\{a_n\}$ converges to a. $\square$

c) Is the unit ball $\mathcal{B}_1(0) = \{f(x) \ : \ d(f,0) \leq 1\}$ compact?

Definition 15. *Let (S,d) be a metric space with $A \subset S$,*

a. *For $\{U_\alpha\}_{\alpha\in A}$, $U_\alpha \subset S$, is a* ***cover*** *of the set A if*

$$A \subset \bigcup_{\alpha\in A} U_\alpha$$

b. *A cover $\{U_\alpha\}_{\alpha\in A}$ of A is an* ***open cover*** *if $\forall_{\alpha\in A}\ U_\alpha$ is an open set.*

c. *$\{V_\beta\}_{\beta\in B}$ is called a* ***subcover*** *of $\{U_\alpha\}_{\alpha\in A}$ if*

   (a) $\{V_\beta\}_{\beta\in B}$ is a cover of A

   (b) $\forall_{\beta\in B}\exists_{\alpha\in A}V_\beta = U_\alpha$

d. *A cover with a finite number of sets is called a* ***finite cover****.*

Definition 16. *For $A \subset (S,d)$, A is* ***compact*** *if for every open cover of A there exists a finite sub cover.*

Proposition 2. *$\mathcal{B}_1(0) = \{f(x) \ : \ d(f,0) \leq 1\}$ is compact.*

Proof.

$$\begin{aligned}
\mathcal{B}_1(0) &= \{f(x) \ : \ d(f,0) \leq 1\} \\
&= \left\{ f(x) \ : \ \sup_{x\in[0,1]} |f(x) - 0| \leq 1 \right\} \\
&= \left\{ f(x) \ : \ \sup_{x\in[0,1]} |f(x)| \leq 1 \right\} \\
&= \big\{ f(x) \ : \ \forall_{x\in[0,1]} |f(x)| \leq 1 \big\} \\
&= \big\{ f(x) \ : \ \forall_{x\in[0,1]} -1 \leq f(x) \leq 1 \big\}
\end{aligned}$$

In other words, the ball is totally bounded. Since it is also known that this bounded subset of $\mathcal{B}([0,1])$ is complete, it is therefore also compact. □