MATH 5301 Elementary Analysis - Homework 6

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Problem 1

a) Show that $\forall_{x>0}\ x \in \mathbb{R} \implies \lim_{n\to\infty} x^{1/n} = 1$

Definition 1. *for* $f : (S_1, d_1) \to (S_2, d_2)$,
if $a \in \bar{S}_1$ *then we say*

$$\lim_{x\to a} f(x) = b$$

if

$$\forall_{\epsilon>0}\exists_{\delta(\epsilon)>0}\ :\ \forall_{x\in \dot{B}_\delta(a)\subset S_1} \implies f(x) \in B_\epsilon(b) \subset S_2$$

Theorem 1.

$$\forall_{x>0}\ x \in \mathbb{R} \implies \lim_{n\to\infty} x^{1/n} = 1$$

Proof.

$$\begin{aligned}
\forall_{x>0}\ x \in \mathbb{R} \implies & \lim_{n\to\infty} f(x) = x^{1/n} = 1 \\
& \implies \forall_{\epsilon>0}\exists_{\delta(\epsilon)>0}\ :\ \forall_{x\in \dot{B}_\delta(a)} \implies f(x) \in B_\epsilon(b) \\
& \implies \forall_{\epsilon>0}\exists_{\delta(\epsilon)>0}\ :\ \forall_{x\in\mathbb{R}} : d_1(a, x) < \delta(\epsilon) \implies x^{1/n} \in \mathbb{R} : d_2(b, x^{1/n}) < \epsilon
\end{aligned}$$

□

Problem 2 True or False?

Definition 2. *for $f : (S_1, d_1) \to (S_2, d_2)$, $f(x)$ is a continuous function iff*

$$\forall_{x \in S_1} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in S_1} : d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \epsilon$$

a) If $f : (S_1, d_1) \to (S_2, d_2)$ is continuous and $U \subset S_1$ is open then $f(U) \subset S_2$ is also open

Theorem 2. *If $f : (S_1, d_1) \to (S_2, d_2)$ is continuous and $U \subset S_1$ is open then $f(U) \subset S_2$ is also open.*

Proof.

$$\begin{aligned}
\forall_{x \in S_1} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in S_1} : d_1(x, y) < \delta \implies & \; d_2(f(x), f(y)) < \epsilon \wedge \\
& \wedge \forall_{x \in U} \exists_{\epsilon > 0} \; : \; B_\epsilon(x) \subset U \implies \\
& \implies \forall_{f(x) \in F(U)} \exists_{\epsilon > 0} \; : \; B_\epsilon(f(x)) \subset U \\
\forall_{x \in S_1} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in S_1} : d_1(x, y) < \delta \implies & \; d_2(f(x), f(y)) < \delta \wedge \\
& \wedge \forall_{x \in U} \exists_{\epsilon_1 > 0} \; : \; \forall_{y \in S_1} d(x, y) < \epsilon_1 \subset U \implies \\
& \implies \forall_{f(x) \in F(U)} \exists_{\epsilon_2 > 0} \; : \; \forall_{f(x) \in f(U)} : \forall_{f(y) \in S_2} : d_2(f(x), f(y)) < \epsilon_2 \subset U
\end{aligned}$$

which is clearly true. □

b) $f : (S_1, d_1) \to (S_2, d_2)$ is continuous $\iff$ $\forall_{C \subset S_2}$ closed, the set $f^{-1}(C) \subset S_1$ is also closed.

Theorem 3. *$f : (S_1, d_1) \to (S_2, d_2)$ is continuous $\iff$ $\forall_{C \subset S_2}$ closed, the set $f^{-1}(C) \subset S_1$ is also closed.*

Proof.

$$\begin{aligned}
\forall_{x \in S_1} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in S_1} : d_1(x, y) < \delta \implies & \; d_2(f(x), f(y)) < \epsilon \iff \\
& \iff \forall_{C \subset S_2} : \forall_{x \in C^{\complement}} \exists_{\epsilon > 0} : B_\epsilon(x) \in C^{\complement} \implies \\
& \implies \forall_{x \in f^{-1}(C)^{\complement}} \exists_{\epsilon > 0} : B_\epsilon(x) \in f^{-1}(C)^{\complement}
\end{aligned}$$

□

Problem 3 Prove the following properties of continuous functions:

a) $\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2}$ continuous $\implies (af+bg)(x) := af(x)+bg(x) : S_1 \to S_2$ continuous.

Theorem 4. $\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2}$ *continuous* $\implies (af+bg)(x) := af(x)+bg(x) : S_1 \to S_2$ *continuous.*

Proof.

$$\begin{aligned}
&\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2}: \\
&\qquad \left(\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1}: d_1(x,y)<\delta \implies d_2(f(x),f(y))<\epsilon\right)\wedge \\
&\qquad \wedge \left(\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1}: d_1(x,y)<\delta \implies d_2(g(x),g(y))<\epsilon\right) \implies \\
&\qquad \implies \left(\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1}: d_1(x,y)<\delta \implies d_2(af(x)+bg(x),af(y)+bg(y))<\epsilon\right) \\
&\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2}: \\
&\qquad \forall_{x\in S_1}\forall_{\epsilon_{1,2,3}>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1}: d_1(x,y)<\delta \implies \\
&\qquad \implies (d_2(f(x),f(y)<\epsilon_1)\wedge(d_2(g(x),g(y))<\epsilon_2)) \implies \\
&\qquad\qquad \implies d_2(af(x)+bg(x),af(y)+bg(y))<\epsilon_3
\end{aligned}$$

Which by the additive and multiplicative properties of a metric space, along with the triangular inequality, this is clearly true:

$$d_2(af(x)+bg(x),af(y)+bg(y)) \leq ad_2(f(x),f(y)) + bd_2(g(x),g(y))$$

Therefore this statement is true. □

b) $\forall_{f:S_1\to S_2}$ continuous and $\forall_{h:S_1\to\mathbb{R}}$ continuous $\implies (hf)(x) := h(x)\cdot f(x) : S_1 \to S_2$ continuous.

Theorem 5. $\forall_{f:S_1\to S_2}$ *continuous and* $\forall_{h:S_1\to\mathbb{R}}$ *continuous* $\implies (hf)(x) := h(x)\cdot f(x) : S_1 \to S_2$ *continuous.*

Proof.

$$\begin{aligned}
&\forall_{f:S_1\to S_2}: \forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1}: d_1(x,y)<\delta \implies d_2(f(x),f(y))<\epsilon)\wedge \\
&\qquad \wedge \forall_{h:S_1\to\mathbb{R}}: \forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1}: d_1(x,y)<\delta \implies d_2(h(x),h(y))<\epsilon) \implies \\
&\qquad\qquad \implies \left(\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1}: d_1(x,y)<\delta \implies d_2(h(x)\cdot f(x),h(y)\cdot f(y))<\epsilon\right) \\
&\forall_{f:S_1\to S_2}\wedge\forall_{h:S_1\to\mathbb{R}}: \forall_{x\in S_1}\forall_{\epsilon_{1,2,3}>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1}: d_1(x,y)<\delta \implies \\
&\qquad \implies (d_2(f(x),f(y))<\epsilon_1 \wedge d_2(h(x),h(y))<\epsilon_2 \implies d_2(h(x)\cdot f(x),h(y)\cdot f(y))<\epsilon_3)
\end{aligned}$$

Which by the multiplicative properties of a metric space this is clearly true:

$$d_2(h(x)\cdot f(x),h(y)\cdot f(y)) \leq d_2(h(x),h(y))\cdot d_2(f(x),f(y))$$

Therefore this statement is true. □

c) $h(x) \neq 0 \forall_{x \in S_1} \implies \frac{1}{h(x)}$ is continuous.

Note: assuming $h : S_1 \to \mathbb{R} \backslash \{0\}$ and continuous.

Theorem 6. *$\forall_{h:S_1 \to \mathbb{R}\backslash\{0\}}$ continuous $\implies (h(x))^{-1} := \frac{1}{h(x)} : S_1 \to \mathbb{R}$ continuous.*

Proof.

$$\begin{aligned}
\forall_{h:S_1\to\mathbb{R}\backslash\{0\}} :& \forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(h(x),h(y)) < \epsilon) \implies \\
&\implies \left(\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2\left(\frac{1}{h(x)}, \frac{1}{h(y)}\right) < \epsilon\right) \\
\forall_{h:S_1\to\mathbb{R}\backslash\{0\}} :& \forall_{x\in S_1}\forall_{\epsilon_{1,2}>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(h(x),h(y)) < \epsilon) \implies \\
&\implies d_2(h(x),h(y)) < \epsilon_1 \wedge d_2\left(\frac{1}{h(x)}, \frac{1}{h(y)}\right) < \epsilon_2
\end{aligned}$$

Within $(\mathbb{R}\backslash\{0\}, d_2)$,

$$d_2(a,b) < \epsilon_1 \implies \exists_{\epsilon_2>0} : d_2\left(\frac{1}{a}, \frac{1}{b}\right) < \epsilon_2$$

Therefore it's true. $\square$

Problem 4 Prove the following statement:

If A and B are two closed nonempty disjoint sets in the metric space (S,d) then there exists a continuous function $\mathcal{X}(x)$ such that $\mathcal{X}(x) = 0$ for all $x \in A$ and $\mathcal{X} = 1$ for all $x \in B$.

Theorem 7.

$$A, B \neq \emptyset \in (S, d) : A \cap B = \emptyset \implies \exists_{\mathcal{X}:S \to \{0,1\}} \text{continuous} : (\forall_{x \in A} \mathcal{X} = 0) \wedge (\forall_{x \in B} \mathcal{X} = 1)$$

Proof. Define the distance from the point x to the set A as

$$\rho_A(x) := \inf_{y \in A} d(x, y)$$

Lemma 1.

$$\rho_A(x) = 0 \iff x \in \bar{A}$$

Proof. □

□

Problem 5 Which of following sets in $\mathbb{R}^2$ are compact?

Definition 3. *Let (S, d) be a metric space with $A \subset S$,*

a. *For $\{U_\alpha\}_{\alpha \in A}$, $U_\alpha \subset S$, is a **cover** of the set A if*

$$A \subset \bigcup_{\alpha \in A} U_\alpha$$

b. *A cover $\{U_\alpha\}_{\alpha \in A}$ of A is an **open cover** if $\forall_{\alpha \in A}\ U_\alpha$ is an open set.*

c. *$\{V_\beta\}_{\beta \in B}$ is called a **subcover** of $\{U_\alpha\}_{\alpha \in A}$ if*

   (a) *$\{V_\beta\}_{\beta \in B}$ is a cover of A*

   (b) *$\forall_{\beta \in B} \exists_{\alpha \in A} V_\beta = U_\alpha$*

d. *A cover with a finite number of sets is called a **finite cover**.*

Definition 4. *For $A \subset (S, d)$, A is **compact** if for every open cover of A there exists a finite sub cover.*

a) $A = \{(x, y)\ :\ x^2 - y^2 \leq 1\}$

This set (a cone I believe) is not bounded and therefore no, through contradiction of the necessary (but insufficient) condition of boundedness, it is not compact.

b) $B = \{(x, y)\ :\ 0 < x^2 + y^2 \leq 1\}$

This set is a disk, but because it excludes the origin it is no longer closed which is a necessary (but insufficient) condition of boundedness, therefore it is not compact.

c) $C = \{(x, y)\ :\ x^2 + y^4 \leq 1\}$

This set is compact. Although it is technically not a disk, it follows the same principle because it contains its boundary (meaning it is closed), but generally it is both complete and totally bounded, which is equivalent to being compact.

d) $D = \left\{\left(1, \frac{1}{n}\right)\ :\ n \in \mathbb{N}\right\} \cup (1, 0)$

For the one single dimension $(1/n)$ it is clear that it is closed and bounded (which are necessary but not sufficient conditions). Similarly, the dimension where $1 = 1$ is also both closed and bounded. To prove compactness, one could use the equivalency of compactness with: $\forall_{{a_k}_{k \in \mathbb{N}}} \exists_{a_{n_k}} : a_{n_k} \to a \in A$.

Problem 6 Let $A \subset S$ be a compact set. Show the following:

Note: it is assumed that $B, C \subset S$ as well as I believe that was the intent.

a) ∂A is compact.

When A is compact, it will be closed meaning $\partial A \subset A$. From the original definition of a compact set, it is known that a finite subcover exists around the entirety of the set A, and from $\partial A \subset A$ it can be concluded that a finite subcover will exist for this boundary itself; therefore ∂A is compact.

b) For any closed B, $A \cap B$ is compact.

c) For any compact C, $A \cup C$ is compact.

d) Union of infinitely many compact sets may be not compact.

One of the necessary (but not sufficient) conditions of a compact set is that it is bounded. If an infinitely many compact sets are structured so that they become unbounded this could violate that necessary condition.