MATH5301 Elementary Analysis. Homework 3. Due: 09/17/2021, 11:59 pm

First Name: **Last Name:**

3.1

Let X denote the universal set. Two subsets A and B are said to have the same cardinality if there is a bijection $f : A \to B$. Notation: $|A| = |B|$.

(a) Prove that $|A| = |B|$ is an equivalence relation on the set of all subsets of X.

(b) Is it true that if $|A_1| = |B_1|$ and $|A_2| = |B_2|$ then $|A_1 \cup A_2| = |B_1 \cup B_2|$?

3.2

Finish the prove of the Cantor-Bernstein theorem: For the sets A and B, such that $|A| \leqslant |B|$ and $|B| \leqslant |A|$ define A_∞ as the set of all elements of A having infinite order, A_0 as the set of all elements of A having even order and A_1 the set of all elements of A having odd order. Similarly for B.

(a) Show that $|A_\infty| = |B_\infty|$

(b) Construct an injective mapping $A_1 \to B_0$

(c) Show that this mapping is surjective

3.3

Set A is called countable if $|A| \leqslant |\mathbb{N}|$. Prove that the following sets are countable

(a) Set $\mathbb{Z}_+$ of all non-negative integer numbers

(b) Set $2\mathbb{N}$ of all even numbers

(c) Set $\mathbb{Z}^2$ of all ordered pairs of integer numbers

(d) Set $\mathbb{Q}$ of all rational numbers

(e) Set $\mathbb{Q}^2$ of all ordered pairs of rational numbers

3.4

Prove that the following sets are countable

(a) Set $\mathbf{P}_5(\mathbb{Z})$ of all polynomials of degree 5 with integer coefficients

(b) Any collection of non-intersecting discs on a plane

(c) Any collection of non-intersecting T-shapes on a plane. T-shape consists of two perpendicular line segments such that one of the segments is attached by one of its endpoints to the center of the other segment. The lengths of these segments can be arbitrary. The orientation of the T-shape can be arbitrary.

(d) Set $\mathbb{P}$ of prime numbers.

(e) Set $\mathbb{A}$ of all algebraic numbers, i.e. the numbers which are roots of some polynomials with integer coefficients.

3.5

Prove that for any infinite set A there exists $B \subseteq A$, so that $|B| = |\mathbb{N}|$.

3.6

Prove that the following sets have the same cardinality (avoid using decimal representation of real numbers)