MATH5301 Elementary Analysis. Homework 8. Due: 10/29/2021, 11:59 pm

First Name: **Last Name:**

8.1

Show that the norms $\|\cdot\|_1$, $\|\cdot\|_p$, for $p > 1$ and $\|\cdot\|_\infty$ are equivalent.

8.2

Let $(S, \|\cdot\|)$ and $(S', \|\cdot\|')$ be two normed spaces. Show that the following norms on $S \times S'$ are equivalent.

(a) $\|(x,y)\|_1 = \|x\| + \|y\|'$

(b) $\|(x,y)\|_2 = \sqrt{\|x\|^2 + (\|y\|')^2}$

(c) $\|(x,y)\|_p = (\|x\|^p + (\|y\|')^p)^{1/p}$

(d) $\|(x,y)\|_\infty = \max\{\|x\|, \|y\|'\}$

8.3

Let X be a vector space and V be a normed space. The function $f : X \to V$ is called bounded if $\exists M : \forall x \in X \Rightarrow \|f(x)\| < M$. Consider the set $\mathcal{B}(X, V)$ of all bounded functions from X to V.

(a) Show that $\mathcal{B}(X, V)$ is a vector space.

(b) Show that the function $\mathcal{B}(X, V) \to \mathbb{R}_+$:

$$\|f\|_\infty := \sup_{x \in X} \|f(x)\|$$

defines a norm on $\mathcal{B}(X, V)$.

8.4

Let A be a dense set in a metric space (S, d), Let (V, d_1) be a complete metric space and $f : A \to Y$ be an uniformly continuous function. Show that

(a) if $\{x_n\}$ is a Cauchy sequence in A then $\{f(x_n)\}$ is a Cauchy sequence in Y.

(b) There is only one continuous function $g : X \to Y$ such that $g(x) = f(x)$ for all $x \in A$.

8.5

Let $(L, \|\|)$ be a Banach space. Let L_0 be a closed subspace of L. Define the factor-space L/L_0 as $l_1 := L/L_0 = \{x + y \mid x \in L,\, y \in L_0\}$. In other words L_1 consists of all subsets of L, obtained from L_0 by shifting all its elements by some element x.

(a) Show that L_1 is a vector space

(b) Define the function $\| \cdot \|_{:} L_1 \to \mathbb{R}_+$ as $\|x\|_1 = \inf\limits_{x-y \in L_0} \|y\|$. Show that this function defines a norm on the space L_1.

(c) Show that L_1 is a Banach space.

8.6

Let $C([-1,1])$ be the space of all continuous real-valued functions $f(x)$ with $x \in [-1,1]$. Let $\|f\|_\infty := \sup_{x\in[-1,1]} |f(x)|$. Find the distance from the point $p = x^2021$ to the space P_{2020} of all polynomials of degree less than or equal to 2020.