

MATH 5301 Elementary Analysis - Homework 2

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Problem 1

For a function $f : A \rightarrow B$, show the following for any $X \subset A, Y, Z \subset B$

1a) $X \subset f^{-1}(f(X))$

1b) $f(f^{-1}(Y)) \subset Y$

1c) $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$

1d) $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$

Problem 2

Show that:

$$\mathbf{2a)} \quad A \cap \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A)$$

Let $\Lambda := \{1, 2, \dots, n\}$,

$$\begin{aligned} A \cap \bigcup_{\lambda \in \Lambda} A_\lambda &= A \cap (A_1 \cup A_2 \cup \dots \cup A_n) \\ &= (A \cap A_1) \cup (A \cap A_2) \cup \dots \cup (A \cap A_n) \\ &= \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A) \end{aligned}$$

Therefore,

$$\boxed{A \cap \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A)}$$

$$\mathbf{2b)} \quad \left(\bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda \right) \subseteq \bigcap_{\lambda \in \Lambda} (A_\lambda \cup B_\lambda)$$

Let $\Lambda_A := \{1, 2, \dots, n\}$ and $\Lambda_B := \{1, 2, \dots, m\}$,

$$\begin{aligned} \bigcap_{\lambda \in \Lambda} (A_\lambda \cup B_\lambda) &= (A_1 \cup B_1) \cap (A_1 \cup B_2) \cap \dots \cap (A_1 \cup B_m) \cap (A_2 \cup B_1) \cap \dots \cap (A_n \cup B_m) \\ &= (A_1 \cup (B_1 \cap B_2 \cap \dots \cap B_m)) \cap \dots \cap (A_n \cup (B_1 \cap B_2 \cap \dots \cap B_m)) \\ &= (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup \dots \cup (A_2 \cap A_3) \dots \cup (A_{n-1} \cap A_n) \\ &\quad \cup (A_1 \cap B_1) \cup \dots \cup (A_1 \cap B_m) \cup \dots \cup (A_n \cap B_m) \\ &\quad \cup (B_1 \cap B_2) \cup \dots \cup (B_1 \cap B_m) \cup \dots \cup (B_{m-1} \cap B_m) \\ &= \left(\bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda \right) \cup (A_1 \cap B_1) \cup \dots \cup (A_1 \cap B_m) \cup \dots \cup (A_n \cap B_m) \end{aligned}$$

Therefore,

$$\boxed{\left(\bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda \right) \subseteq \left(\bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda \right) \cup (A_1 \cap B_1) \cup \dots \cup (A_1 \cap B_m) \cup \dots \cup (A_n \cap B_m)}$$

Problem 3

Problem: Which of these are equivalence relations?

Solution: a, c, & d

(The following explain why)

3a)

for $a, b \in \mathbb{R}$, let $a\mathcal{R}b$ if $a - b \in \mathbb{Q}$

i) Reflective:

$$x\mathcal{R}x = x - x = 0 \in \mathbb{Q}$$

ii) Symetric:

$$\begin{aligned} x\mathcal{R}y &\implies y\mathcal{R}x \\ x\mathcal{R}y = x - y \in \mathbb{Q} &\implies y - x \in \mathbb{Q} \end{aligned}$$

Since $x - y = -(y - x)$, $(x - y)$ and $(y - x)$ will both be either rational or not rational, this is true.

iii) Transitive:

$$\begin{aligned} x\mathcal{R}y \wedge y\mathcal{R}z &\implies x\mathcal{R}z \\ (x - y \in \mathbb{Q}) \wedge (y - z \in \mathbb{Q}) &\implies (x - z \in \mathbb{Q}) \\ \left(\frac{x_a}{x_b} - \frac{y_a}{y_b} \in \mathbb{Q}\right) \wedge \left(\frac{y_a}{y_b} - \frac{z_a}{z_b} \in \mathbb{Q}\right) &\implies \left(\frac{x_a}{x_b} - \frac{z_a}{z_b} \in \mathbb{Q}\right) \end{aligned}$$

This also means that:

$$\begin{aligned} (x_a y_b - x_b y_a \in \mathbb{N}) \wedge (x_b y_b \neq 0 \in \mathbb{N}) \wedge (y_a z_b - y_b z_a \in \mathbb{N}) \wedge (y_b z_b \neq 0 \in \mathbb{N}) \\ \implies (x_a z_b - x_b z_a \in \mathbb{N}) \wedge (x_b z_b \neq 0 \in \mathbb{N}) \end{aligned}$$

Since this statments indicates that $x_b, y_b, z_b \neq 0$ and that $x_a y_b - x_b y_a, y_a z_b - y_b z_a \in \mathbb{N}$, the following will always be true as well: $x_a z_b - x_b z_a$. Therefore, the relation is transitive.

3b)

for $a, b \in \mathbb{R}$, let $a\mathcal{R}b$ if $a - b \notin \mathbb{Q}$

i) Reflective:

The relationship is NOT reflective:

$$\begin{aligned} a\mathcal{R}b &= a - b \notin \mathbb{Q} \\ x\mathcal{R}x &= x - x = 0 \in \mathbb{Q} \end{aligned}$$

3c)

for $a, b \in \mathbb{R}$, let $a\mathcal{R}b$ if $a - b$ is a square root of a rational number.
i.e.

$$a\mathcal{R}b = (a - b)^2 \in \mathbb{Q}$$

i) Reflective:

$$\begin{aligned} a\mathcal{R}b &= (a - b)^2 \in \mathbb{Q} \\ x\mathcal{R}x &= (x - x)^2 = 0^2 = 0 \in \mathbb{Q} \end{aligned}$$

ii) Symetric:

$$\begin{aligned} x\mathcal{R}y &\implies y\mathcal{R}x \\ (x - y)^2 \in \mathbb{Q} &\implies (y - x)^2 \in \mathbb{Q} \\ (x - y)^2 &= (y - x)^2 \therefore x\mathcal{R}y \implies y\mathcal{R}x \end{aligned}$$

iii) Transative:

$$\begin{aligned} x\mathcal{R}y \wedge y\mathcal{R}z &\implies x\mathcal{R}z \\ ((x - y)^2 \in \mathbb{Q}) \wedge ((y - z)^2 \in \mathbb{Q}) &\implies ((x - z)^2 \in \mathbb{Q}) \\ (x^2 - 2xy + y^2 \in \mathbb{Q}) \wedge (y^2 - 2yz + z^2 \in \mathbb{Q}) &\implies (x^2 - 2xz + z^2 \in \mathbb{Q}) \\ (x^2 \in \mathbb{Q}) \wedge (y^2 \in \mathbb{Q}) \wedge (z^2 \in \mathbb{Q}) \wedge (-2xy \in \mathbb{Q}) \wedge (-2yz \in \mathbb{Q}) &\implies (x^2 \in \mathbb{Q}) \wedge (z^2 \in \mathbb{Q}) \wedge (-2xz \in \mathbb{Q}) \\ (xy \in \mathbb{Q}) \wedge (yz \in \mathbb{Q}) &\implies (xz \in \mathbb{Q}) \end{aligned}$$

Which is clearly transitive, so $a\mathcal{R}b = (a - b)^2 \in \mathbb{Q}$ is transitive.

3d)

Let $X = \mathbb{Z} \times \mathbb{N}$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in \mathcal{R} if $x_1y_2 = x_2y_1$.

Problem 4