MATH 5301 Elementary Analysis - Homework 10

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Problem 1

Prove that the closure and the interior of a convex set $A \subset \mathbb{R}^n$ are also convex.

Definition 1. *The set A is called convex if*

$$\forall_{x,y \in A} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in A$$

Definition 2. *For a given set $A \subseteq (S, d)$,*

a. the interior of A is defined as

$$int(A) = \{x \in A \ : \ \exists_{\epsilon > 0} B_\epsilon(x) \subset A\}$$

b. the closure of A is defined as

$$\bar{A} = \{x \in S \ : \ \forall_{\epsilon > 0} B_\epsilon \cap A \neq \emptyset\}$$

Problem 2

Prove that the intersection of an arbitrary collection of convex sets $\cap_{i \in 1} C_i$ is also convex.

Problem 3

Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of nested convex sets in $\mathbb{R}^n$, i.e. $C_i \subset C_{i+1}$. Prove that $\cup_{i=1}^{\infty} C_i$ is also convex.

Problem 4

Definition 3. *A* <u>convex hull</u> *is defined as*

$$Hull =$$

a)

Show that the convex hull of any open sets in $\mathcal{R}^n$ is open.

b)

Provide an example of a closed set $A \subset \mathcal{R}^n$, such that its convex hull is not closed.

Problem 5

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $A \subset \mathbb{R}^n$ be a bounded set. Prove that $f(A)$ is bounded in $\mathbb{R}$.

Problem 6

Show that the convex hull of a compact set $A \subset \mathbb{R}^n$ is compact. (*Hint:* Caratheodory theorem)