MATH 5301 Elementary Analysis - Homework 6

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Problem 1

a) Show that $\forall_{x>0}\ x \in \mathbb{R} \implies \lim_{n\to\infty} x^{1/n} = 1$

Definition 1. *for* $f : (S_1, d_1) \to (S_2, d_2)$,
if $a \in \bar{S}_1$ *then we say*

$$\lim_{x\to a} f(x) = b$$

if

$$\forall_{\epsilon>0}\exists_{\delta>0}\forall_{x\in S_1} : 0 < d_1(a,x) < \delta \implies d_2(f(x), b) < \epsilon$$

or written in Ball form:

$$\forall_{\epsilon>0}\exists_{\delta(\epsilon)>0}\ :\ \forall_{x\in \dot{B}_\delta(a)\subset S_1} \implies f(x) \in B_\epsilon(b) \subset S_2$$

Theorem 1.

$$\forall_{x>0}\ x \in \mathbb{R} \implies \lim_{n\to\infty} x^{1/n} = 1$$

Proof.

$$\forall_{x>0}\ x \in \mathbb{R} = \forall_{x\in\mathbb{R}_+} \implies \forall_{\epsilon>0}\exists_{\delta>0}\forall_{x\in S_1} : 0 < d_1(a,x) < \delta \implies d_2(f(x), b) < \epsilon$$

□

b) Show that for any bounded sequence $\{a_n\}$ and any sequence $\{b_n\}$, converging to zero, the sequence $\{a_n b_n\}$ converges to zero.

Theorem 2. $\forall\{a_n\}$ *bounded* $\land\ \forall\{b_n\}$ *converging to zero* $\implies$ $\{a_n b_n\}$ *converges to zero.*

Proof.

$$(\exists_{N_1>0}\ :\ \forall_{n\in\mathbb{N}} a_n < N_1) \land \left(\forall_{\epsilon>0}\exists_{N_2(\epsilon)>0}\forall_{n>N_2} b_n < \epsilon\right) \implies \forall_{\epsilon>0}\exists_{N_3(\epsilon)>0}\forall_{n>N_3} a_n b_n < \epsilon$$
$$\forall_{\epsilon_{2,3}>0}\exists_{N_1,N_2,N_3>0}\ :\ \forall n > \max(N_2, N_3)((a_n < N_1) \land (b_n < \epsilon_2) \implies a_n b_n < \epsilon_3)$$

Which clearly implies $\{a_n b_n\}$ converges to zero. □

c) Find the limit $\lim_{n\to\infty} a_n$. Prove the convergence.

$$a_n = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}$$

This is a pretty simple that it is just 2. This true for any of a similar type of problem.

d) Find the limit $\lim_{n\to\infty} a_n$ where $a_n = 2 + \cfrac{1}{2 + \cfrac{1}{\ddots + \frac{1}{2}}}$. Prove the convergence.

This one just converges to around 2.4... not that that is the answer, I just ran out of time to do correctly...

Problem 2 True or False?

Definition 2. *for $f : (S_1, d_1) \to (S_2, d_2)$, $f(x)$ is a continuous function iff*

$$\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x, y) < \delta \implies d_2(f(x), f(y)) < \epsilon$$

a) If $f : (S_1, d_1) \to (S_2, d_2)$ is continuous and $U \subset S_1$ is open then $f(U) \subset S_2$ is also open

Theorem 3. *If $f : (S_1, d_1) \to (S_2, d_2)$ is continuous and $U \subset S_1$ is open then $f(U) \subset S_2$ is also open.*

Proof.

$$\begin{aligned}
\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x, y) < \delta &\implies d_2(f(x), f(y)) < \epsilon \wedge \\
&\wedge \forall_{x\in U}\exists_{\epsilon>0} \ : \ B_\epsilon(x) \subset U \implies \\
&\implies \forall_{f(x)\in F(U)}\exists_{\epsilon>0} \ : \ B_\epsilon(f(x)) \subset U \\
\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x, y) < \delta &\implies d_2(f(x), f(y)) < \delta\wedge \\
&\wedge \forall_{x\in U}\exists_{\epsilon_1>0} \ : \ \forall_{y\in S_1} d(x, y) < \epsilon_1 \subset U \implies \\
&\implies \forall_{f(x)\in F(U)}\exists_{\epsilon_2>0} \ : \ \forall_{f(x)\in f(U)} : \forall_{f(y)\in S_2} : d_2(f(x), f(y)) < \epsilon_2 \subset U
\end{aligned}$$

which is clearly true. □

b) $f : (S_1, d_1) \to (S_2, d_2)$ is continuous $\iff$ $\forall_{C\subset S_2}$ closed, the set $f^{-1}(C) \subset S_1$ is also closed.

Theorem 4. *$f : (S_1, d_1) \to (S_2, d_2)$ is continuous $\iff$ $\forall_{C\subset S_2}$ closed, the set $f^{-1}(C) \subset S_1$ is also closed.*

Proof.

$$\begin{aligned}
\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x, y) < \delta &\implies d_2(f(x), f(y)) < \epsilon \iff \\
&\iff \forall_{C\subset S_2} : \neg\text{open} \implies f^{-1}(C)\neg\text{open} \quad \iff \forall_{c\in C\subset S_2}\exists_{!x\in f^{-1}(C)\subset S_1} : f(x) = c
\end{aligned}$$

By definition of a function, $\forall_{y\in Y}\exists_{!x\in f^{-1}(Y)} : f(x) = y$ so there exists a one-to-one mapping and since f is continuous it therefore can maintain closeness. So true. □

c) If $f, g : (S, d) \to \mathbb{R}$ continuous $\implies$ $m(x) := \max(f(x), g(x))$ and $n(x) := \min(f(x), g(x))$ are both continuous.

Theorem 5. *If $f, g : (S, d) \to \mathbb{R}$ continuous $\implies$ $m(x) := \max(f(x), g(x))$ and $n(x) := \min(f(x), g(x))$ are both continuous.*

Proof.

$$\begin{aligned}
\forall_{x,y\in S}\forall_{\epsilon_{1,2}>0}\exists_{\delta(x,\epsilon)>0} : d(x, y) < \delta &\implies d(f(x), f(y)) < \epsilon_1 \wedge d(g(x), g(y)) < \epsilon_2 \implies \\
&\implies \forall_{x,y\in S}\forall_{\epsilon_{3,4}>0}\exists_{\delta(x,\epsilon)>0} : d(x, y) < \delta \implies \\
&\implies d(\max(f(x), g(x)), \max(f(y), g(y))) < \epsilon_3\wedge \\
&\quad \wedge d(\min(f(x), g(x)), \min(f(y), g(y))) < \epsilon_4
\end{aligned}$$

Upon inspection, $\epsilon_3, \epsilon_4 \leq \epsilon_1 + \epsilon_2$ at the worst case (at least thats what I learned when doing uncertainty calculations in the past...), therefore the implication is obvious. □

d) If $f : (S, d) \to \mathbb{R}$ continuous the for any Cauchy sequence $\{x_n\} \in S$, the sequence $f(x_n)$ is a Cauchy sequence in $\mathbb{R}$

Definition 3. *For the metric space (S, d), a sequence $\{x_n\}$ is considered a **Cauchy Sequence** if*

$$\forall_{\epsilon>0}\exists_{N\in\mathbb{N}} \,:\, \forall_{m,n>N} d(x_m, x_n) < \epsilon$$

Theorem 6. *If $f : (S, d) \to \mathbb{R}$ continuous the for any Cauchy sequence $\{x_n\} \in S$, the sequence $f(x_n)$ is a Cauchy sequence in $\mathbb{R}$.*

Proof.

$$\begin{aligned} \forall_{x,y\in S}\forall_{\epsilon_1>0}\exists_{\delta(x,\epsilon_1)>0} : d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon_1 \implies \\ \implies (\forall_{\epsilon_2>0}\exists_{N\in\mathbb{N}} \,:\, \forall_{m,n>N} d(x_m, x_n) < \epsilon_2) \implies \\ \implies (\forall_{\epsilon_3>0}\exists_{N\in\mathbb{N}} \,:\, \forall_{m,n>N} d(f(x_n), f(x_m)) < \epsilon_3) \end{aligned}$$

From the continuous implication that: $\delta \to 0 \implies d(f(x), f(y)) \to 0$, the fact that $\{x_n\}$ converges mean that $\epsilon_2 \to 0 \implies d(f(x_n), f(x_m)) \to 0$. Therefore the sequence $\{f(x_n)\}$ would be considered a Cauchy sequence. $\square$

Problem 3 Prove the following properties of continuous functions:

a) $\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2}$ continuous $\implies (af+bg)(x) := af(x)+bg(x) : S_1 \to S_2$ continuous.

Theorem 7. *$\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2}$ continuous $\implies (af+bg)(x) := af(x)+bg(x) : S_1 \to S_2$ continuous.*

Proof.

$$\begin{aligned}
&\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2} :\\
&\qquad\qquad \left(\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(f(x),f(y)) < \epsilon\right)\wedge\\
&\qquad\qquad \wedge\left(\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(g(x),g(y)) < \epsilon\right) \implies\\
&\qquad\qquad \implies \left(\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(af(x)+bg(x),af(y)+bg(y)) < \epsilon\right)\\
&\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2} :\\
&\qquad\qquad \forall_{x\in S_1}\forall_{\epsilon_{1,2,3}>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies\\
&\qquad\qquad \implies (d_2(f(x),f(y) < \epsilon_1) \wedge (d_2(g(x),g(y)) < \epsilon_2)) \implies\\
&\qquad\qquad\qquad \implies d_2(af(x)+bg(x),af(y)+bg(y)) < \epsilon_3
\end{aligned}$$

Which by the additive and multiplicative properties of a metric space, along with the triangular inequality, this is clearly true:

$$d_2(af(x)+bg(x),af(y)+bg(y)) \le ad_2(f(x),f(y)) + bd_2(g(x),g(y))$$

Therefore this statement is true. □

b) $\forall_{f:S_1\to S_2}$ continuous and $\forall_{h:S_1\to\mathbb{R}}$ continuous $\implies (hf)(x) := h(x)\cdot f(x) : S_1 \to S_2$ continuous.

Theorem 8. *$\forall_{f:S_1\to S_2}$ continuous and $\forall_{h:S_1\to\mathbb{R}}$ continuous $\implies (hf)(x) := h(x)\cdot f(x) : S_1 \to S_2$ continuous.*

Proof.

$$\begin{aligned}
&\forall_{f:S_1\to S_2} : \forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(f(x),f(y)) < \epsilon)\wedge\\
&\qquad \wedge\forall_{h:S_1\to\mathbb{R}} : \forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(h(x),h(y)) < \epsilon) \implies\\
&\qquad\qquad \implies \left(\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(h(x)\cdot f(x),h(y)\cdot f(y)) < \epsilon\right)\\
&\forall_{f:S_1\to S_2}\wedge\forall_{h:S_1\to\mathbb{R}} : \forall_{x\in S_1}\forall_{\epsilon_{1,2,3}>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies\\
&\qquad \implies (d_2(f(x),f(y)) < \epsilon_1 \wedge d_2(h(x),h(y)) < \epsilon_2 \implies d_2(h(x)\cdot f(x),h(y)\cdot f(y)) < \epsilon_3)
\end{aligned}$$

Which by the multiplicative properties of a metric space this is clearly true:

$$d_2(h(x)\cdot f(x),h(y)\cdot f(y)) \le d_2(h(x),h(y))\cdot d_2(f(x),f(y))$$

Therefore this statement is true. □

c) $h(x) \neq 0 \forall_{x \in S_1} \implies \frac{1}{h(x)}$ is continuous.

Note: assuming $h : S_1 \to \mathbb{R} \backslash \{0\}$ and continuous.

Theorem 9. *$\forall_{h:S_1 \to \mathbb{R}\backslash\{0\}}$ continuous $\implies (h(x))^{-1} := \frac{1}{h(x)} : S_1 \to \mathbb{R}$ continuous.*

Proof.

$$\begin{aligned} \forall_{h:S_1\to\mathbb{R}\backslash\{0\}} :& \forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(h(x),h(y)) < \epsilon) \implies \\ &\implies \left(\forall_{x\in S_1}\forall_{\epsilon>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2\left(\frac{1}{h(x)}, \frac{1}{h(y)}\right) < \epsilon\right) \\ \forall_{h:S_1\to\mathbb{R}\backslash\{0\}} :& \forall_{x\in S_1}\forall_{\epsilon_{1,2}>0}\exists_{\delta(x,\epsilon)>0}\forall_{y\in S_1} : d_1(x,y) < \delta \implies d_2(h(x),h(y)) < \epsilon) \implies \\ &\implies d_2(h(x),h(y)) < \epsilon_1 \wedge d_2\left(\frac{1}{h(x)}, \frac{1}{h(y)}\right) < \epsilon_2 \end{aligned}$$

Within $(\mathbb{R}\backslash\{0\}, d_2)$,

$$d_2(a,b) < \epsilon_1 \implies \exists_{\epsilon_2>0} : d_2\left(\frac{1}{a}, \frac{1}{b}\right) < \epsilon_2$$

Therefore it's true. □

Problem 4 Prove the following statement:

If A and B are two closed nonempty disjoint sets in the metric space (S,d) then there exists a continuous function $\mathcal{X}(x)$ such that $\mathcal{X}(x) = 0$ for all $x \in A$ and $\mathcal{X} = 1$ for all $x \in B$.

Definition 4. *A function $f : (S_1, d_1) \to (S_2, d_2)$ is considered **Lipschitz Continous Function** if*

$$\exists_{L>0} : \forall_{x,y \in S_1} d_2(f(x), f(y)) \leq L d_1(x, y)$$

Theorem 10.

$$A, B \neq \emptyset \in (S, d) : A \cap B = \emptyset \implies \exists_{\mathcal{X}:S \to \mathbb{R}} \text{continuous} : (\forall_{x \in A} \mathcal{X} = 0) \wedge (\forall_{x \in B} \mathcal{X} = 1)$$

Proof. Define the distance from the point x to the set A as

$$\rho_A(x) := \inf_{y \in A} d(x, y)$$

Lemma 1.

$$\rho_A(x) = 0 \iff x \in \bar{A}$$

Proof. The lower bound of $d(x, y)$ for $y \in A$ will never be zero unless $x \in \bar{A}$, so clearly the inf will be zero if and only if $x \in \bar{A}$. □

Lemma 2. $\rho_A(x)$ *is lipshits with* $L = 1$

Proof.

$$\exists_{L>0} : \forall_{x,y \in S} d(\rho_A(x), \rho_A(y)) \leq L d(x, y)$$

Let $L = 1$,

$$\forall_{x,y \in S} d(\rho_A(x), \rho_A(y)) \leq d(x, y)$$

By definition of $\rho_A(x)$,

$$\rho_A(x) = \inf_{y \in A} d(x, y) \leq d(x, y) \forall y \in A$$

Therefore, $d(\rho_A(x), \rho_A(y)) \leq d(x, y)$ and $\rho_A(x)$ is lipshits with $L = 1$. □

Consider $\mathcal{X}(x) = \dfrac{\rho_B(x)}{\rho_A(x) + \rho_B(x)}$, Although I believe it has merit I will actually be using the similar version: $\mathcal{X}(x) = \dfrac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$ From lemma 2 it is clear that $\rho_A(x)$ and $\rho_B(x)$ are Lipschitz and therefore this rational combination of these will still be continuous within its defined regions. From Lemma 1, it is clear that $\dfrac{\rho_A(x)}{\rho_A(x) + \rho_B(x)} = 0 \forall_{x \in \bar{A}} and \dfrac{\rho_A(x)}{\rho_A(x) + \rho_B(x)} = 1 \forall x \in \bar{B}$. □

Problem 5 Which of following sets in $\mathbb{R}^2$ are compact?

Definition 5. *Let (S, d) be a metric space with $A \subset S$,*

*a. For $\{U_\alpha\}_{\alpha\in A}$, $U_\alpha \subset S$, is a **cover** of the set A if*

$$A \subset \bigcup_{\alpha\in A} U_\alpha$$

*b. A cover $\{U_\alpha\}_{\alpha\in A}$ of A is an **open cover** if $\forall_{\alpha\in A}\ U_\alpha$ is an open set.*

*c. $\{V_\beta\}_{\beta\in B}$ is called a **subcover** of $\{U_\alpha\}_{\alpha\in A}$ if*

(a) $\{V_\beta\}_{\beta\in B}$ is a cover of A

(b) $\forall_{\beta\in B}\exists_{\alpha\in A}V_\beta = U_\alpha$

*d. A cover with a finite number of sets is called a **finite cover**.*

Definition 6. *For $A \subset (S, d)$, A is **compact** if for every open cover of A there exists a finite sub cover.*

a) $A = \{(x, y)\ :\ x^2 - y^2 \leq 1\}$

This set (a cone I believe) is not bounded and therefore no, through contradiction of the necessary (but insufficient) condition of boundedness, it is not compact.

b) $B = \{(x, y)\ :\ 0 < x^2 + y^2 \leq 1\}$

This set is a disk, but because it excludes the origin it is no longer closed which is a necessary (but insufficient) condition of boundedness, therefore it is not compact.

c) $C = \{(x, y)\ :\ x^2 + y^4 \leq 1\}$

This set is compact. Although it is technically not a disk, it follows the same principle because it contains its boundary (meaning it is closed), but generally it is both complete and totally bounded, which is equivalent to being compact.

d) $D = \left\{\left(1, \frac{1}{n}\right)\ :\ n \in \mathbb{N}\right\} \cup (1, 0)$

For the one single dimension $(1/n)$ it is clear that it is closed and bounded (which are necessary but not sufficient conditions). Similarly, the dimension where $1 = 1$ is also both closed and bounded. To prove compactness, one could use the equivalency of compactness with: $\forall_{{a_k}_{k\in\mathbb{N}}}\exists_{a_{n_k}} : a_{n_k} \to a \in A$.

Problem 6 Let $A \subset S$ be a compact set. Show the following:

Note: it is assumed that $B, C \subset S$ as well as I believe that was the intent.

a) ∂A is compact.

When A is compact, it will be closed meaning $\partial A \subset A$. Additionally, every sequence within A will have a converging subsequence and since ∂A will maintain both its completeness and its totally boundedness. From the original definition of a compact set, it is known that a finite subcover exists around the entirety of the set A, and from $\partial A \subset A$ it can be concluded that a finite subcover will exist for this boundary itself; therefore ∂A is compact.

b) For any closed B, $A \cap B$ is compact.

When A is compact, it will also be closed and so since B is closed, $A \cap B$ is also closed. Additionally, by definition of an intersection of sets, $A \cap B \subset A$. Since A is compact, it is both complete and totally bounded which I believe means that closed subsets of A will remain complete and totally bounded. Since $A \cap B$ is closed and $A \cap B \subset A$, $A \cap B$ is complete and totally bounded $\iff$ $A \cap B$ is compact.

c) For any compact C, $A \cup C$ is compact.

From the original definition of a compact set, it is known that a finite subcover exists around the entirety of sets A and C. $A \cup C$ will be closed (because it is a finite union of closed sets) and I believe the completeness and boundedness from each compact set will be maintained. Since both A and C individually were covered by finite subcovers, $A \cup C$ could then also be covered by a finite number of subcovers. Therefore $A \cup C$ satisfies the criteria to be a compact set.

d) Union of infinitely many compact sets may be not compact.

One of the necessary (but not sufficient) conditions of a compact set is that it is bounded. If an infinitely many compact sets are structured so that they become unbounded this could violate that necessary condition.