

# MATH 5301 Elementary Analysis - Final Exam

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2021, December 7<sup>th</sup>

## Problem 1

For each  $n \in \mathbb{N}$  define the set

$$Q_n := \left\{ \frac{1}{pq} : 0 < p < q \leq n; p + q > n; \gcd(p, q) = 1 \right\}$$

Let  $f(n)$  be the sum of all elements of  $Q_n$ .  
Find  $\inf_n f(n)$ .

**Definition 1.** Let the set  $Q_n$  be defined for all  $n \in \mathbb{N}$  as

$$Q_n := \left\{ \frac{1}{pq} : 0 < p < q \leq n; p + q > n; \gcd(p, q) = 1 \right\}$$

**Definition 2.** Let  $f(n)$  be the sum of all elements within  $Q_n$ .

**Definition 3.** A lower bound of subset  $A$  in the partially ordered set  $(S, \leq)$  is defined by

$$a \in S : a \leq x \forall x \in A$$

A lower bound of  $a$  is called an infimum of set  $A \in (S, \leq)$ , denoted as  $a = \inf A$ , is the greatest lower bound.  
i.e.

$$\forall y \in S : a \leq x \forall x \in A \implies y \leq a$$

**Definition 4.** The Greatest Common Divisor of two nonzero integers  $a, b \in \mathbb{Z} \neq 0$ ,  $\gcd(a, b)$ , is defined as the largest positive integer,  $d \in \mathbb{Z}_+$ , so that  $d$  is a divisor of both  $a$  and  $b$ . i.e:

$$\gcd(a, b) := d \in \mathbb{Z}_+ : (a : d) \wedge (b : d) \wedge (\forall x \in \mathbb{Z}_+ : a, b : x \implies d \geq x)$$

Additionally,  $a$  and  $b$  are considered coprime if  $\gcd(a, b) = 1$ .

**Assumption 1.** For this problem it is assumed that  $\gcd$  is only defined within  $\mathbb{Z}_+$ , although I believe this can also be expanded to other less-strict ordered sets in the same way.

**Assumption 2.** It is assumed that the sum of all elements in the empty set is 0, i.e.  $\sum_i \emptyset = 0$ .

**Theorem 1.**

$$\inf_{n \in \mathbb{N}} f(n) = 0$$

*Proof.* Proof by induction.

For  $n = 1$ ,  $\neg \exists_{p,q \in \mathbb{Z}} : 0 < p < q \leq 1$  meaning that  $Q_1 = \emptyset$ .

This implies that  $f(1) = \sum_i \emptyset = 0$  and that  $f(1) \geq 0$ .

For  $n = 2$ ,

$$(p, q) \in \{(p, q) : 0 < p < q \leq 2; p + q > n; \gcd(p, q) = 1\} = \{(1, 2)\}$$

The set  $Q_2$  is then defined as

$$Q_2 = \left\{ \frac{1}{pq} : (p, q) \in \{(1, 2)\} \right\} = \left\{ \frac{1}{(1)(2)} \right\} = \left\{ \frac{1}{2} \right\}$$

Therefore,

$$f(2) = \sum_i \left\{ \frac{1}{2} \right\} = \frac{1}{2}$$

It is clear that  $f(2) = \frac{1}{2} \geq 0$ .

For  $n = 3$ ,

$$(p, q) \in \{(p, q) : 0 < p < q \leq 3; p + q > n; \gcd(p, q) = 1\} = \{(1, 3), (2, 3)\}$$

The set  $Q_3$  is then defined as

$$Q_3 = \left\{ \frac{1}{pq} : (p, q) \in \{(1, 3), (2, 3)\} \right\} = \left\{ \frac{1}{(1)(3)}, \frac{1}{(2)(3)} \right\} = \left\{ \frac{1}{3}, \frac{1}{6} \right\}$$

Therefore,

$$f(3) = \sum_i \left\{ \frac{1}{3}, \frac{1}{6} \right\} = \frac{1}{3} + \frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

It is clear that  $f(3) = \frac{1}{2} \geq 0$ .

For  $n = 4$ ,

$$(p, q) \in \{(p, q) : 0 < p < q \leq 4; p + q > n; \gcd(p, q) = 1\} = \{(1, 4), (2, 3), (3, 4)\}$$

The set  $Q_4$  is then defined as

$$Q_4 = \left\{ \frac{1}{pq} : (p, q) \in \{(2, 3), (3, 4)\} \right\} = \left\{ \frac{1}{(1)(4)}, \frac{1}{(2)(3)}, \frac{1}{(3)(4)} \right\} = \left\{ \frac{1}{4}, \frac{1}{6}, \frac{1}{12} \right\}$$

Therefore,

$$f(4) = \sum_i \left\{ \frac{1}{6}, \frac{1}{12} \right\} = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{3+2+1}{12} = \frac{6}{12} = \frac{1}{2}$$

It is clear that  $f(4) = \frac{1}{2} \geq 0$ .

For an arbitrary  $n \in \mathbb{N}$ ,

$$\begin{aligned} (p, q) \in \{(p, q) : 0 < p < q \leq n; p + q > n; \gcd(p, q) = 1\} = \\ = \{(1, n), (2, n - \star), (3, n - \star), \dots, (n - 2, n - 1), (n - 1, n)\} \end{aligned}$$

$$\begin{aligned} Q_n &= \left\{ \frac{1}{pq} : (p, q) \in \{(1, n), (2, n - \star), \dots, (n - 2, n - 1), (n - 1, n)\} \right\} \\ &= \left\{ \frac{1}{(1)(n)}, \frac{1}{(2)(n - 1)}, \dots, \frac{1}{(n - 2)(n - 1)}, \frac{1}{(n - 1)(n)} \right\} \\ &= \left\{ \frac{1}{n}, \frac{1}{2(n - \star)}, \dots, \frac{1}{(n - 2)(n - 1)}, \frac{1}{n(n - 1)} \right\} \end{aligned}$$

where  $\star$  is dependent for on divisibility properties between  $n$  and 2, 3, 4, etc. It is important to note that each increase of  $n$  will cause every term to decrease in magnitude individually but additional elements are added that result to adding up to  $\frac{1}{2}$  again.

However, eventually this will reach a point where a lack of prime numbers in a region makes it so that the only coprime numbers satisfying the conditions are adjacent to one another, which leads to the following:

$$\begin{aligned}
f(n) &= \sum_i Q_n = \frac{1}{n} + \cdots + \frac{1}{(\frac{n}{2})(\frac{n}{2} + 1)} + \cdots + \frac{1}{n(n-1)} \\
f(n+1) &= \left( \sum_i Q_n \right) \left( \frac{n!}{(n+1)!} \right) + \frac{1}{(n+1)} \\
&= \frac{1}{n} \frac{n!}{(n+1)!} + \cdots + \frac{1}{(\frac{n}{2})(\frac{n}{2} + 1)} \frac{n!}{(n+1)!} + \cdots + \frac{1}{n(n-1)} \frac{n!}{(n+1)!} + \frac{1}{n+1} \\
&= \frac{n!}{n(n+1)n!} + \cdots + \frac{n!}{\frac{n}{2}(\frac{n}{2} - 1)(n+1)n!} + \cdots + \frac{n!}{n(n-1)(n+1)n!} + \frac{1}{n+1} \\
&= \sum_i Q_{n+1} = \frac{1}{n+1} + \cdots + \frac{1}{(\frac{n+1}{2})(\frac{n+1}{2} + 1)} + \cdots + \frac{1}{n(n+1)}
\end{aligned}$$

essentially every  $(p, q)$  becomes  $(q, q+1)$  and the new  $\frac{1}{(n+1)}$  is added.

Anyway, the point is that  $\forall_{n \in \mathbb{N}} : n > 1, f(n) \geq \frac{1}{2}$ ; however, because  $f(n)$  is included,  $\frac{1}{2} \leq f(n) \forall_{n \in \mathbb{N}}$  since  $Q_1 = \emptyset \implies f(1) = 0$ .

Therefore,

$$\inf_n f(n) = 0$$

□

## Problem 2

Let  $(X, d)$  be a metric space. Let  $B_r(a)$  denote the open ball of radius  $r$  centered at  $a$ . i.e. Can it happen that  $B_{r_1}(a) \subset B_{r_2}(a)$  but  $r_1 > r_2$ ?

**Definition 5.** Within the metric space  $(X, d)$ , the open ball of radius  $r \in X$  centered at  $a \in X$ , denoted as  $B_r(a)$ , is defined as:

$$B_r(a) := \{x \in X : d(a, x) < r\}$$

**Assumption 3.** First it will be assumed that  $(X, d)$  is a normed vector space in which the triangle inequality holds. i.e.

$$\forall_{x, y, z \in X} d(x, z) \leq d(x, y) + d(y, z)$$

This can also be denoted as  $(X, \|\cdot\|)$  to distinguish between them. It is also assumed that  $X$  is complete.

**Theorem 2.** For  $r_1 > r_2$  then it is not possible for  $B_{r_1}(a) \subset B_{r_2}(b)$  within  $(X, \|\cdot\|)$ :

*Proof.* Proof by contradiction.

Let

$$B_{r_1}(a), B_{r_2}(b) \subset X$$

with  $0 < r_2 < r_1$  and  $a \in B_{r_2}(b)$ .

To minimize the amount of the set existing outside of the set, we need to set  $a = b$ . Next, let  $c$  be a point within the punctured open ball  $B_{r_2}(b)$ . i.e.

$$c \in B_{r_2}(b) \setminus \{b\}$$

$c$  can then be used to construct a point that is contained in  $B_{r_2}(b)$  but not in  $B_{r_1}(a)$ :

$$p + \frac{r_1 + r_2}{2} \frac{ac}{\|ac\|} \in B_{r_1}(a) \setminus B_{r_2}(b)$$

Meaning that there is no possible way for an open ball of greater radius (within a normed metric space).  $\square$

**Assumption 4.** The previous assumption, Assumption 3, is now relax the metric so that  $d$  is not restricted by completeness or

**Theorem 3.** It is possible for  $B_{r_1}(a) \subset B_{r_2}(b)$  within  $(X, d)$  when  $r_1 > r_2$ :

*Proof.* Proof by example:

Let metric space  $(X, d)$  be defined by

$$X := 0 \cup [5, \infty)$$

$$d(x, y) := |x - y|$$

For  $r_1 = 4$ ,  $r_2 = 3$ ,

Let  $B_4(0)$  be defined as

$$B_4(0) := \{4x \in X : d(0, x) < 4\} = \{0\} \cup [2, 4)$$

Let  $B_3(2)$  be defined as

$$B_3(2) := \{x \in X : d(2, x) < 3\} = \{0\} \cup [2, 5)$$

Clearly,  $B_3(2) \subset B_4(0)$ . Since  $r_1 = 4 > r_2 = 3$ , this exists as an example that satisfies the conditions.  $\square$

### Problem 3

Let  $M$  be the set of all bounded sequences

$$M = \left\{ \{a_j\}_{j=1}^{\infty} : |a_j| < \infty \right\}$$

Define  $\rho(\{a_n\}, \{b_n\}) = \max_{n \in \mathbb{N}} |a_n - b_n|$

- a) **Show that  $(M, \rho)$  is a metric space.**
- b) **Show that  $M$  does not contain a dense**

## Problem 4

Does there exist a metric space, containing a sequence of nested bounded closed sets  $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$  such that

$$\bigcap_{n \in \mathbb{N}} F_n = \emptyset$$

Hint: If  $d(x, y)$  is a usual Euclidean metric on  $\mathbb{R}$ , one can show that  $\frac{d(x, y)}{1 + d(x, y)}$  is also a metric. Such metric is often called a bounded metric...

**Definition 6.** The set  $A$  in metric space  $(X, d)$  is considered open if

$$\forall x \in A \exists \epsilon > 0 : \forall y \in X d(x, y) < \epsilon$$

**Definition 7.** The set  $A$  in metric space  $(X, d)$  is considered closed if the set  $A^c$  is open.

**Definition 8.** The set  $A$  in metric space  $(X, d)$  is called bounded if

$$\forall x \in A \exists R > 0 : \forall y \in A d(x, y) < R$$

**Theorem 4.** There does not exist a metric space  $(X, d)$  containing the sequence of nested bounded sets  $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$  such that

$$\bigcap_{n \in \mathbb{N}} F_n = \emptyset$$

*Proof.* Let the metric space  $(X, d)$  be defined with field  $X$  ...

□

## Problem 5

Show that there exists a unique continuous function,  $f(x)$  on the interval  $[0, 1]$ , satisfying the equation

$$f(x) = \int_0^1 \sin(x^2 + y^2) f(y) dy$$

**Definition 9.** continuous function

**Definition 10.** Linear operator (integration is a linear operator... also multiplication by a value at a single point... sin and squared obviously isn't though)

**Theorem 5.** *There exists a unique continuous function  $f : [0, 1] \rightarrow [0, 1]$  that satisfies the following equation*

$$f(x) = \int_0^1 \sin(x^2 + y^2) f(y) dy \tag{1}$$

*Proof.* A unique function that satisfies the

□

## Problem 6

Let  $V$  be a complete metric space without isolated points. Show that  $V$  is uncountable ( $|V| > |\mathbb{N}|$ ).

**Definition 11.** A metric space  $(X, d)$  is considered complete if:

**Definition 12.** A point within metric space  $(X, d)$  is considered isolated if

**Definition 13.** • The Cardinality of set  $A$ , denoted as  $|A|$ , is the number of unique elements contained within  $A$ .

- A function  $f : \mathbb{N} \rightarrow A$  is said to be surjective if

$$\exists_{f:\mathbb{N} \rightarrow A} \iff \exists_{f:\mathbb{N} \rightarrow A} : \forall_{x \in \mathbb{N}} \exists f(x) \in A$$

- A function  $f : \mathbb{N} \rightarrow A$  is said to be injective if

$$\exists_{f:\mathbb{N} \rightarrow A} \iff \exists_{f:\mathbb{N} \rightarrow A} : \forall_{f(x) \in A} \exists x \in \mathbb{N}$$

- A function  $f : \mathbb{N} \rightarrow A$  is said to be bijective if  $f$  is both surjective and injective. i.e.

$$\exists_{f:\mathbb{N} \rightarrow A} \iff \exists_{f:\mathbb{N} \rightarrow A} : (\forall x \in \mathbb{N} \exists_{f(x) \in A}) \wedge (\forall y \in A \exists_{f^{-1}(y) \in \mathbb{N}})$$

A bijective function is also known as a one to one correspondance.

- A set  $A$  is considered Countable if  $|A| \leq |\mathbb{N}|$ . This is also said to be true if a surjective function exists mapping  $\mathbb{N}$  to  $A$ .
- If  $A$  is an infinite set, then  $A$  is countably infinite,  $|A| = \aleph = |\mathbb{N}|$ , if there exists a one to one correspondence from  $\mathbb{N}$  to  $A$ .

**Theorem 6.** A complete