MATH 5301 Elementary Analysis - Homework 10

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Problem 1

Prove that the closure and the interior of a convex set $A \subset \mathbb{R}^n$ are also convex.

Definition 1. *The set A is called* <u>convex</u> *if*

$$\forall_{x,y \in A} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in A$$

Definition 2. *For a given set $A \subseteq (S, d)$,*

a. the <u>interior</u> *of A is defined as*

$$int(A) = \{x \in A \ : \ \exists_{\epsilon > 0} B_\epsilon(x) \subset A\}$$

b. the <u>closure</u> *of A is defined as*

$$\overline{A} = \{x \in S \ : \ \forall_{\epsilon > 0} B_\epsilon(x) \cap A \neq \emptyset\}$$

Theorem 1. *If $A \subset \mathbb{R}^n$ is a convex set, then the closure of A, $\overline{A}$, is also convex.*

Proof. A being convex means that

$$\forall_{x,y \in A} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in A$$

$\overline{A}$ is defined by

$$\overline{A} = \{x \in \mathbb{R}^n \ : \ \forall_{\epsilon > 0} B_\epsilon(x) \cap A \neq \emptyset\}$$

For $\overline{A}$ to be convex, the following would be true:

$$\forall_{x,y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A}$$

Additionally, since $\overline{A} = A \cup \partial A$, $\overline{A}$ is convex if

$$\left(\forall_{x \in A} \forall_{y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A} \right) \wedge \left(\forall_{x \in \partial A} \forall_{y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A} \right)$$

Since $A \subset \overline{A}$, by definition the first statement is true,

$$\forall_{x \in A} \forall_{y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A}$$

Additionally, since the boundary of A, ∂A, is the collection of limit points of A and the limit points all exist within the neighborhood of elements in A,

$$\forall_{x \in \partial A} \forall_{y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A}$$

Therefore,

$$\forall_{x,y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A}$$

$\square$

Problem 2

Prove that the intersection of an arbitrary collection of convex sets $\cap_{i \in I} C_i$ is also convex.

Theorem 2. *If each of the sets within the collection $C_i \subset (S, d)$ are convex, then the intersection of the collection, $\cap_{i \in I}$ is also convex.*

Proof. For $\cap_{i \in I}$ to be convex, the following must be true:

$$\forall_{x,y \in \cap_{i \in I} C_i} \forall_{t \in [0,1]} (t)x + (1-t)y \in \cap_{i \in I} C_i$$

Which is the same as:

$$\forall_{x,y \in S} \ : \ \forall_{i \in I} x, y \in C_i \implies \forall_{t \in [0,1]} \forall_{i \in I} (t)x + (1-t)y \in C_i$$

Since all the sets C_i are convex, by definition:

$$\forall_{x,y \in C_i} \forall_{t \in [0,1]} (t)x + (1-t)y \in C_i$$

Therefore this is true $\forall_{i \in I}$:

$$\wedge_{i \in I} \forall_{x,y \in C_i} \implies \forall_{t \in [0,1]} (t)x + (1-t)y \in C_i$$

Which is equivalent to:

$$\forall_{x,y \in \cap_{i \in I} C_i} \forall_{t \in [0,1]} (t)x + (1-t)y \in \cap_{i \in I} C_i$$

□

Problem 3

Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of nested convex sets in $\mathbb{R}^n$, i.e. $C_i \subset C_{i+1}$. Prove that $\cup_{i=1}^{\infty} C_i$ is also convex.

Theorem 3. *For the sequence of nested convex sets in $\mathbb{R}^n$, $\{C_i\}_{i\in\mathbb{N}}$, a union of all the elements, $\cup_{i=1}^{\infty} C_i$, is also convex.*

Proof. Proof by induction.
For $n = 1$, the set $\cup_{i=1}^{n} C_i = C_1$ is convex.
For $n = 2$, the set $\cup_{i=1}^{n} C_i = C_1 \cup C_2$ is convex.

Proof. Since $C_1 \subset C_2$, $C_1 \cup C_2 = C_2$ and C_2 is convex. □

Assuming for $n = k$, $\cup_{i=1}^{k} C_i = C_k$ is convex, then for $n = k+1$, $\cup_{i=1}^{k+1} C_i = C_{k+1}$ is convex.

Proof. Since $C_k \subset C_{k+1}$,

$$\cup_{i=1}^{k+1} C_i = \cup_{i=1}^{k} C_i \cup C_{k+1} = C_{k+1}$$

which is convex. □

Therefore, by induction,

$$\forall_{n\in\mathbb{N}} \cup_{i=i}^{n} C_i$$

is convex. This implies $\cup_{i=1}^{\infty} C_i$. □

Problem 4

Definition 3. *A* <u>convex hull</u> *is defined as*

$$Hull =$$

a)

Show that the convex hull of any open sets in $\mathcal{R}^n$ is open.

b)

Provide an example of a closed set $A \subset \mathcal{R}^n$, such that its convex hull is not closed.

Problem 5

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $A \subset \mathbb{R}^n$ be a bounded set. Prove that $f(A)$ is bounded in $\mathbb{R}$.

Problem 6

Show that the convex hull of a compact set $A \subset \mathbb{R}^n$ is compact. (*Hint:* Caratheodory theorem)