MATH5301 Elementary Analysis. Homework 7. Due: 10/22/2021, 11:59 pm

First Name: **Last Name:**

7.1

Provide an examples of the sets $A, B \subset \mathbb{R}^2$ such that

(a) A and B are connected, but $A \cup B$ is not.

(b) A and B are connected, but $A \cap B$ is not.

(c) A and B are not connected, but $A \cup B$ is connected.

(d) A and B are not connected, but $A \cap B$ is connected.

(e) A and B are not connected, but $A \setminus B$ is connected.

7.2

(a) Prove that every monotone bounded sequence in $\mathbb{R}$ converge.

(b) Provide an example of the set $A \in \mathbb{R}$ having exactly four limit points.

(c) Provide an example of a sequence $\{a_n\}$, such that every point of the interval $[2019, 2021]$ is a limit point of it.

7.3

(a) Provide an example of a sequence $\{a_n\}$ such that a_n diverges, but $\lim_{n\to\infty}(a_n - a_{2n}) = 0$

(b) Provide an example of two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$(\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n) < \liminf_{n\to\infty}(a_n + b_n) < (\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n) < \limsup_{n\to\infty}(a_n + b_n) < (\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n)$$

7.4

Show the equivalence of the norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_p$, $p > 1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$

7.5

Are there any open sets A and $B4$ in $\mathbb{R}^2$ such that $d(A, B) = 0$ but $A \cap B = \emptyset$?

7.6

Let $\mathcal{B}([0,1])$ denote the set of all bounded functions from $[0,1]$ to $\mathbb{R}$. Define the metric on $\mathcal{B}[0,1]$ as $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$.

(a) Show that this is indeed a metric.

(b) Prove that the space $(\mathcal{B}([0,1]), d)$ is complete metric space.

(c) Is the unit ball $B_1(0) = \{f(x) \mid d(f,0) \leqslant 1\}$ compact?