MATH 5301 Elementary Analysis - Homework 10

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2021, November 12$^{\text{th}}$

Problem 1

Compute the derivatives of the following functions:

Definition 1. *Let $f : (a, b) \to \mathbb{R}$ be a function.*

a. *The* <u>derivative of the function at point x_0</u> *is defined as*

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

b. *If the derivative is defined at x_0, then it is* <u>differentiable at x_0</u>.

c. *If the derivative is defined for all $x_0 \in (a, b)$, then the function f is said to be* <u>differentialble</u>.

d. *When f is differentiable, the* <u>derivative</u> *of $f(x)$ is defined as:*

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

a) $x^2 \sin \frac{1}{x}$

Let

$$f(x) = x^2 \sin \frac{1}{x}$$

Then, by definition, the derivative of $f(x)$ is calculated as

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 \sin\left(\frac{1}{x+h}\right) - x^2 \sin\left(\frac{1}{x}\right)}{h} \\
&= \lim_{h \to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\left((x+h)^2 \sin\left(\frac{1}{x+h}\right) - x^2 \sin\left(\frac{1}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}h} h} \\
&= \lim_{h \to 0} \frac{2(x+h) \sin\left(\frac{1}{x+h}\right) + (x+h)^2 \left(\frac{-1}{(x+h)^2}\right) \cos\left(\frac{1}{x+h}\right)}{1} \\
&= \lim_{h \to 0} 2(x+h) \sin\left(\frac{1}{x+h}\right) - \cos\left(\frac{1}{x+h}\right)
\end{aligned}$$

$$\boxed{f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)}$$

b) $\dfrac{e^x + e^{-x}}{2}$

Let

$$f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

Then, by definition, the derivative of $f(x)$ is calculated as

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{e^{(x+h)} + e^{-(x+h))}}{2} + \dfrac{e^x + e^{-x}}{2}}{h} \\
&= \lim_{h\to 0} \frac{e^{x+h} + e^{-x-h} + e^x + e^{-x}}{2h} \\
&= \lim_{h\to 0} \frac{e^x e^h + e^{-x} e^{-h} + e^x + e^{-x}}{2h} \\
&= \lim_{h\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\left(e^x e^h + e^{-x} e^{-h} + e^x + e^{-x}\right)}{\frac{\mathrm{d}}{\mathrm{d}h} 2h} \\
&= \lim_{h\to 0} \frac{e^x e^h - e^{-x} e^{-h}}{2} \\
&= \lim_{h\to 0} \frac{e^{x+h} - e^{-x-h}}{2}
\end{aligned}$$

$$\boxed{f'(x) = \frac{e^x - e^{-x}}{2} = \sinh(x)}$$

c) $\dfrac{e^x - e^{-x}}{2}$

Let

$$f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}$$

Then, by definition, the derivative of $f(x)$ is calculated as

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{e^{(x+h)} - e^{-(x+h))}}{2} + \dfrac{e^x - e^{-x}}{2}}{h} \\
&= \lim_{h\to 0} \frac{e^{x+h} - e^{-x-h} + e^x - e^{-x}}{2h} \\
&= \lim_{h\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\left(e^{x+h} - e^{-x-h} + e^x - e^{-x}\right)}{\frac{\mathrm{d}}{\mathrm{d}h} 2h} \\
&= \lim_{h\to 0} \frac{e^{x+h} + e^{-x-h}}{2} \\
&= \lim_{h\to 0} \frac{e^{x+h} + e^{-x-h}}{2}
\end{aligned}$$

$$\boxed{f'(x) = \frac{e^x + e^{-x}}{2} = \cosh(x)}$$

d) $\quad e^x + e^{e^x} + e^{e^{e^x}}$

Let

$$f(x) = e^x + e^{e^x} + e^{e^{e^x}}$$

Then, by definition, the derivative of $f(x)$ is calculated as

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{e^{x+h} + e^{e^{x+h}} + e^{e^{e^{x+h}}} - \left(e^x + e^{e^x} + e^{e^{e^x}}\right)}{h} \\
&= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} + \lim_{h\to 0} \frac{e^{e^{x+h}} - e^{e^x}}{h} + \lim_{h\to 0} \frac{e^{e^{e^{x+h}}} - e^{e^{e^x}}}{h} \\
&= \lim_{h\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\left(e^{x+h} - e^x\right)}{\frac{\mathrm{d}}{\mathrm{d}h}h} + \lim_{h\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\left(e^{e^{x+h}} - e^{e^x}\right)}{h} + \lim_{h\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\left(e^{e^{e^{x+h}}} - e^{e^{e^x}}\right)}{\frac{\mathrm{d}}{\mathrm{d}h}h} \\
&= \lim_{h\to 0} \frac{(1)e^{x+h}}{1} + \lim_{h\to 0} \frac{(1)(e^{x+h})(e^{e^{x+h}})}{1} + \lim_{h\to 0} \frac{(1)(e^{x+h})(e^{e^{x+h}})(e^{e^{e^{x+h}}})}{1} \\
&= \lim_{h\to 0} e^{x+h} + \lim_{h\to 0} e^{x+h}e^{e^{x+h}} + \lim_{h\to 0} e^{x+h}e^{e^{x+h}}e^{e^{e^{x+h}}}
\end{aligned}$$

$$\boxed{f'(x) = e^x + e^x e^{e^x} + e^x e^{e^x} e^{e^{e^x}} = e^x + e^{x+e^x} + e^{x+e^x+e^{e^x}}}$$

e) $x^{x^{x^x}}$

Let

$$f(x) = x^{x^{x^x}}$$

Then, by definition, the derivative of $f(x)$ is calculated as

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^{(x+h)^{(x+h)^{(x+h)}}} - x^{x^{x^x}}}{h} \\
&= \lim_{h\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\left((x+h)^{(x+h)^{(x+h)^{(x+h)}}} - x^{x^{x^x}}\right)}{\frac{\mathrm{d}}{\mathrm{d}h} h} \\
&= \lim_{h\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}h}\left((x+h)^{(x+h)^{(x+h)^{(x+h)}}}\right)}{1}
\end{aligned}$$

Exponent Rule: $a^b = e^{b \ln a}$

$$= \lim_{h\to 0} \frac{\mathrm{d}}{\mathrm{d}h}\left(e^{\left((x+h)^{(x+h)^{(x+h)}} \ln(x+h)\right)}\right)$$

Chain Rule: $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$

$$= \lim_{h\to 0} \frac{\mathrm{d}e^u}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}h}$$

with $u = (x+h)^{(x+h)^{(x+h)}} \ln(x+h)$

$$\begin{aligned}
&= \lim_{h\to 0} e^u \frac{\mathrm{d}}{\mathrm{d}h}\left((x+h)^{(x+h)^{(x+h)}} \ln(x+h)\right) \\
&= \lim_{h\to 0} e^u \Bigg[\left((x+h)^{(x+h)^{(x+h)}}\right)\frac{1}{x+h} \\
&\quad + \ln(x+h)\frac{\mathrm{d}}{\mathrm{d}h} e^{(x+h)^{(x+h)} \ln(x+h)}\Bigg] \\
&\vdots \\
&= \lim_{h\to 0} (x+h)^{(x+h)^{(x+h)^{(x+h)}}} \\
&\qquad \Bigg[(x+h)^{(x+h)^{(x+h)}-1} \\
&\qquad\quad + \Bigg((x+h)^{(x+h)^{(x+h)}} \ln(x+h)\Big((x+h)^{h+x-1} \\
&\qquad\qquad + (x+h)^{x+h} \ln(x+h)(1 + \ln(x+h))\Big)\Bigg)\Bigg]
\end{aligned}$$

Then

$$\boxed{f'(x) = x^{x^{x^x}}\left(x^{x^x} \ln(x)\left(x^x \ln(x)(\ln(x) + 1) + x^{x^{x-1}}\right) + x^{x^x - 1}\right)}$$

Problem 2

Definition 2. *A* <u>differentiable function</u> *over* (a, b) *is a function* f *so that* $f'(x)$ *exists for all* $x \in (a, b)$.

Definition 3. *A* <u>bounded function</u> *is a function so that*

$$\exists_{N \in \mathbb{R}} \ : \ \forall_{x \in (a,b)} |f(x)| < N$$

Definition 4. *An* <u>unbounded function</u> *is a function that is not bounded.*

Definition 5. *A function* $f : X \to Y$ *is* <u>Uniformly Continous</u> *if*

$$\forall_{\epsilon > 0} \exists_{\delta(\epsilon) > 0} \ : \ \forall_{x,y \in X} \|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$$

Note: this is stricter then continuity itself with the only difference that $\exists_\delta$ *is constant* $\forall_{x,y \in X}$*; whereas a* <u>Continous</u> *function only requires that this is true for some* δ *dependent on* ϵ*,* x*, and* y*:*

$$\forall_{\epsilon > 0} \forall_{x,y \in X} \exists_{\delta(\epsilon,x,y) > 0} \ : \ \|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$$

a) Prove the following:

Theorem 1. *If* $f \ : \ (-1, 1) \to \mathbb{R}$ *is a differentiable unbounded function, then* f' *is also unbounded on* $[-1, 1]$.

Proof. When f is unbounded,

$$\forall_{N \in \mathbb{R}} \exists_{x \in (-1,1)} \ : \ |f(x)| \geq N$$

then f' is defined by

$$f'(x) := \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Since the derivative is an instantaneous 'rise' over 'run', in order for no bound to exists within a bounded interval $(-1, 1)$, the differentiable (and therefore continuous) function must have an asymptote at the boundaries. This implies that within the closed interval, the derivative must also be unbounded to obtain the the asymptote on the boundary. □

b) Provide an example of bounded differentiable function on $[-1, 1]$ with an unbounded derivative.

Theorem 2. $f(x) = \sqrt{x + 1}$ *is a bounded differentiable function on* $[-1, 1]$*, but has an unbounded derivative.*

Proof. Clearly, $f(x)$ is fully defined and bounded on $[-1, 1]$. The domain of $\sqrt{x + 1}$ is $\{x \in \mathbb{R} \ : \ x \geq -1\}$ while the range for $x \in [-1, 1]$ is $\{y \in [0, \sqrt{(3)}]\}$.
The derivative of f is defined as

$$f'(x) = \frac{1}{2\sqrt{x + 1}}$$

which is actually bounded on $x \in (-1, 1]$, but an asymptote exists at $x = -1$ in which

$$\lim_{x \to -1^+} \frac{1}{2\sqrt{x + 1}} = +\infty$$

and is clearly unbounded. □

c) Prove the following:

Theorem 3. *If $f : (-1,1) \to \mathbb{R}$ is a differentiable function, such that f' is bounded on $[-1,1]$, then f is uniformly continuous.*

Proof. f differentiable on $x \in (-1,1)$ means

$$\exists f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

f' bounded on $x \in [-1,1]$ means $f'(x)$ is bounded on $x \in (-1,1)$ and that the following is bounded:

$$\lim_{h\to 0^{+/-}} \frac{f(x+h)-f(x)}{h}$$

which is equivalent to saying that these two limits exists and are bounded.

$$\lim_{x\to(+/-)1^{+/-}} f(x)$$

Since it is already known that f is differentiable on $x \in (-1,1)$, this implies that f is continuous, i.e:

$$\forall_{\epsilon>0}\forall_{x,y\in(-1,1)}\exists_{\delta(\epsilon,x,y)>0} \ : \ \|x-y\| < \delta \implies \|f(x)-f(y)\| < \epsilon$$

and since f' is bounded on the entire closed domain, a constant finite bound δ will exist to bound the 'run' ($\|x-y\|$) for an arbitrary 'rise' ($\|f(x)-f(y)\|$). Additionally, the lack of asymptotes at the boundaries also implies that continuity of the open domain will extend to the closed domain. Therefore f is uniformly continuous, i.e:

$$\forall_{\epsilon>0}\exists_{\delta(\epsilon)>0} \ : \ \forall_{x,y\in[-1,1]}\|x-y\| < \delta \implies \|f(x)-f(y)\| < \epsilon$$

□

Problem 3

Find $f^{(n)}(0)$ for the following functions:

Definition 6. *Let $f : (a,b) \to \mathbb{R}$ be a differentiable function.*

a. *If $f' : (a,b) \to \mathbb{R}$ is also differentiable, f is called* twice differentiable.

b. *The derivative of f' is then denoted as f'' and is called the* Secound Derivative.

c. *The* n-th Derivative, *denoted by $f^{(n)} : (a,b) \to \mathbb{R}, n \in \mathbb{N}$, is defined by repeating differentiation on the next derivative:*

   (a) $n = 0,\ f^{(0)} = f$

   (b) *If $f^{(n)}$ is defined, for $n \geq 0$, then $f^{(n+1)} = \frac{\mathrm{d}}{\mathrm{d}x}\big(f^{(n)}\big)$.*

d. *If f has derivatives up until the order $n, n \geq 1$, then f is* n-times differentiable.

e. *If $f^{(n)}$ is continuous, the f is* n-times continously differentiable*; which is also known as f is of class C^n.*

Theorem 4. ***Leibnitz Formula:*** *Let $f, g : (a,b) \to \mathbb{R}$ be n-differentiable functions. Then for $1 \leq m \leq n$, the m-th derivative of $f(x)g(x)$ is given by:*

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}(f(x)g(x)) = \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)}(x)g^{(k)}(x)$$

Proof. Proof provided in the lecture notes: Theorem 6.20 □

a) $\sin(ax)\cos(bx)$

It is known, and can be proven using Euler's identity, that the derivatives of sinusoidal functions follow the following progression:

a. $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$

b. $\frac{\mathrm{d}}{\mathrm{d}x} - \sin(x) = -\cos(x)$

c. $\frac{\mathrm{d}}{\mathrm{d}x} - \cos(x) = \sin(x)$

d. $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$

Additionally, from the chain rule, the m-th derivative of a domain scaled function, $f(ax)$ will be given by

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} f(ax) = a^m f^{(m)}(u),\ u = ax$$

Let $f(x) = \sin(ax)$ and $g(x) = \cos(bx)$, by Euler's formula,

$$f(x) = \sin(ax) = \frac{e^{jax} - e^{-jax}}{2j}$$

and

$$g(x) = \cos(bx) = \frac{e^{jbx} + e^{-jbx}}{2}$$

The m-th derivatives can be defined in many ways, but the following begins by recognizing that $g(x) = a^{-1}f'(x) = a^{-1}f^{(1)}(x)$; and additionally, $g^{(m)}(x) = a^{-m-1}b^m f^{(m+1)}(\frac{b}{a}x)$.
The m-th derivatives of f and g are then clearly given as:

$$f^{(m)}(x) = a^m(j)^m \frac{e^{jax} - (-1)^m e^{-jax}}{j2} = a^m(j)^{m-1}\frac{e^{jax} + (-1)^{m-1}e^{-jax}}{2}$$

and

$$g^{(m)}(x) = b^m (j)^m \frac{e^{jbx} + (-1)^m e^{-jbx}}{2}$$

By the Leibnitz Formula, the m-th derivative of $f(x)g(x)$ can be calculated as follows:

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}(f(x)g(x)) = \sum_{k=0}^{m} \binom{m}{k} f^{(m-k)}(x) g^{(k)}(x)$$

$$\begin{aligned}
\frac{\mathrm{d}^m}{\mathrm{d}x^m}(\sin(ax)\cos(bx)) &= \sum_{k=0}^{m} \binom{m}{k} \left(a^{m-k} (j)^{m-k-1} \frac{e^{jax} + (-1)^{m-k-1} e^{-jax}}{2} \right) \left(b^k (j)^k \frac{e^{jbx} + (-1)^k e^{-jbx}}{2} \right) \\
&= \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k (j)^{m-k-1+k} \frac{e^{jax} + (-1)^{m-k-1} e^{-jax} + e^{jbx} + (-1)^k e^{-jbx}}{2}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}^m}{\mathrm{d}x^m}(\sin(x)\cos(bx)) = \frac{1}{2} \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k (j)^{m-k-1+k} \left[e^{jax} + (-1)^{m-k-1} e^{-jax} + e^{jbx} + (-1)^k e^{-jbx} \right]}$$

Which, using Euler's formula again, can be converted back into many different configurations of sinusoidal forms. Then by evaluating for $f^{(n)}(0)$,

$$f^{(n)}(0) = \frac{1}{2} \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k (j)^{m-k-1+k} \left[e^{jax} + (-1)^{m-k-1} e^{-jax} + e^{jbx} + (-1)^k e^{-jbx} \right]$$

b) $x^k \sin \frac{1}{x}$

c) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

Problem 4

Construct an example of infinitely many times differentiable function $f(x)$ such that $f(x) = 0$ for $x \leq 0$, $f(x) = 1$ for $x \geq 1$ and $f(x)$ is strictly monotone on the interval $(0, 1)$.
Using such function you could construct, for example, a monotone function $g(x)$ such that $\lim_{x\to+\infty} g(x) = 0$ but $\lim_{x\to+\infty} g'(x) \neq 0$. (How?)

Problem 5

Find the following limits

a) $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

b) $\lim_{x\to 0} \frac{\arctan(\arcsin x) - \arcsin(\arctan x)}{\sin x - \tan x}$

c) $\lim_{x\to +\infty} \frac{x^{\ln x}}{(\ln x)^x}$

Problem 6

Find the example of a function $f(x)$ which is continuous at every point of the interval $(0, 1)$, but is not differentiable at every point $(0, 1)$.
Read about the construction of the function, which is differentiable at every point of $(0, 1)$ but whose derivative is continuous at every point of $(0, 1)$.