

MATH 5301 Elementary Analysis - Homework 8

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Problem 1

Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two equivalent norms on \mathbb{R}^n .

Definition 1. For $\|\cdot\|_a, \|\cdot\|_b$ on S , $\|\cdot\|_a$ is said to be stronger than $\|\cdot\|_b$ if

$$\forall \{x_n\} \subset S : x_n \xrightarrow{d_a} x \implies x_n \xrightarrow{d_b} x$$

Definition 2. $\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be equivalent, $\|\cdot\|_a \sim \|\cdot\|_b$, if $\|\cdot\|_a$ is stronger than $\|\cdot\|_b$ and $\|\cdot\|_b$ is stronger than $\|\cdot\|_a$. This means that

$$\|\cdot\|_a \sim \|\cdot\|_b \iff \exists \alpha, \beta \in \mathbb{R}_{>0} : \forall x \in S \alpha \|\cdot\|_b \leq \|\cdot\|_a \leq \beta \|\cdot\|_b$$

a) Prove that if the set A is closed in the a -norm, then it is closed in b -norm.

Definition 3. The set $A \subset V$ is called open if

$$\forall x \in A \exists \epsilon > 0 : B_\epsilon(x) \subset A$$

or equivalently,

$$\forall x \in A \exists \epsilon > 0 : \forall y \in V \|x - y\| < \epsilon \implies y \in A$$

Definition 4. The set $A \subset V$ is called closed if A^c is open.

Theorem 1. If the set A is closed in the a -norm, then it is closed in b -norm.

Proof. Set A being closed in a -norm implies A^c is open in a -norm.

$$\forall x \in A^c \exists \epsilon_a > 0 : \forall y \in S \|x - y\|_a < \epsilon_a \implies y \in A^c$$

Additionally, since $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$ means that

$$\exists \alpha, \beta > 0 : \forall x \in S \alpha \|\cdot\|_b \leq \|\cdot\|_a \leq \beta \|\cdot\|_b$$

Therefore, $\|x - y\|_a \leq \beta \|x - y\|_b$ and then

$$\begin{aligned} \forall x \in A^c \exists \epsilon_a > 0 : \forall y \in S \|x - y\|_a \leq \beta \|x - y\|_b < \epsilon_a &\implies y \in A^c \\ \forall x \in A^c \exists \epsilon_b > 0 : \forall y \in S \|x - y\|_b < \epsilon_b &\implies y \in A^c \end{aligned}$$

where $\epsilon_b \geq \frac{\epsilon_a}{\beta}$

□

b) **Prove that if the set A is compact in the a -norm then it is compact in the b -norm.**

Definition 5. Let (S, d) be a metric space with $A \subset S$,

a. For $\{U_\alpha\}_{\alpha \in A}$, $U_\alpha \subset S$, is a **cover** of the set A if

$$A \subset \bigcup_{\alpha \in A} U_\alpha$$

b. A cover $\{U_\alpha\}_{\alpha \in A}$ of A is an **open cover** if $\forall_{\alpha \in A} U_\alpha$ is an open set.

c. $\{V_\beta\}_{\beta \in B}$ is called a **subcover** of $\{U_\alpha\}_{\alpha \in A}$ if

(a) $\{V_\beta\}_{\beta \in B}$ is a cover of A

(b) $\forall_{\beta \in B} \exists_{\alpha \in A} V_\beta = U_\alpha$

d. A cover with a finite number of sets is called a **finite cover**.

Definition 6. For $A \subset (S, d)$, A is **compact** if for every open cover of A there exists a finite sub cover. Which is equivalent to saying all sequences within A converge to a set point in A . (i.e)

$$\forall_{a_k, k \in \mathbb{N}} \exists_{a_{n_k}} : a_{n_k} \rightarrow a \in A$$

Definition 7. A sequence $\{x_n\}$ is called **Cauchy** if

$$\forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{l_1, l_2 \geq N} \|x_{n_{l_1}} - x_{n_{l_2}}\| < \epsilon$$

Theorem 2. If the set A is compact in the a -norm, then it is compact in the b -norm.

Proof. Set A being compact in a -norm means that every sequence in A satisfies the Cauchy sequence property:

$$\forall_{\epsilon > 0} \exists_{N \in \mathbb{N}} \forall_{l_1, l_2 \geq N} \|x_{n_{l_1}} - x_{n_{l_2}}\|_a < \epsilon$$

Additionally, since $\|\cdot\|_a$ is equivalent to $\|\cdot\|_b$ means that

$$\exists_{\alpha, \beta > 0} : \forall_{x \in S} \alpha \|\cdot\|_b \leq \|\cdot\|_a \leq \beta \|\cdot\|_b$$

Therefore, $\|x - y\|_a \leq \beta \|x - y\|_b$ and then

$$\forall_{\epsilon_a > 0} \exists_{N \in \mathbb{N}} \forall_{l_1, l_2 \geq N} \|x_{n_{l_1}} - x_{n_{l_2}}\|_a \leq \beta \|x_{n_{l_1}} - x_{n_{l_2}}\|_b < \epsilon_a$$

$$\forall_{\epsilon_b > 0} \exists_{N \in \mathbb{N}} \forall_{l_1, l_2 \geq N} \|x_{n_{l_1}} - x_{n_{l_2}}\|_b < \epsilon_b$$

where $\epsilon_b \geq \frac{\epsilon_a}{\beta}$

□

Problem 2

Consider the set l^∞ of all real-valued sequences, endowed with the sup-norm: $\|l\|_\infty = \sup_{n \in \mathbb{N}} |l_n|$.

a) Prove that l^∞ is complete.

Definition 8. A metric space is (S, d) is a complete metric space if every Cauchy sequence in S converges.

Definition 9. The set A in norm space is $(S, \|\cdot\|)$ is a complete set if every Cauchy sequence in A converges to a limit in A .

Definition 10. Let the set l^∞ be the set of real-valued sequences:

$$l^\infty := \{\{l_n\}_{n \in \mathbb{N}} : l_n \in \mathbb{R}\}$$

Definition 11. Let the norm space be defined as $(l^\infty, \|l\|_\infty)$ where

$$\|l\|_\infty = \sup_{n \in \mathbb{N}} |l_n|$$

Theorem 3. The set l^∞ is complete.

Proof. asdf

□

b) Prove that l^∞ is not compact.

Problem 3

Consider the set $\mathbb{B}([0, 1], \mathbb{R})$ of all bounded real-valued functions on the unit interval endowed with the sup-norm: $\|f\|_\infty = \sup_{x \in [0, 1]} |f(x)|$. Denote the closed unit ball as $B_1 := \{f \in \mathbb{B}([0, 1], \mathbb{R}) : \|f\|_\infty \leq 1\}$.

a) **Prove B_1 is closed.**

Theorem 4. B_1 is the closed.

Proof. Set B_1 being closed implies B_1^c is open.

$$\begin{aligned} \forall f \in B_1^c \exists \epsilon > 0 : \forall g \in \mathbb{B} \quad \|f - g\|_\infty < \epsilon &\implies g \in B_1^c \\ \forall f \in \mathbb{B} \quad \|f\|_\infty > 1 \exists \epsilon > 0 : \forall g \in \mathbb{B} \quad \|f - g\|_\infty < \epsilon &\implies g \in \mathbb{B} : \|g\|_\infty > 1 \\ \forall f, g \in \mathbb{B} \exists \epsilon_1 > 0 : \|f\|_\infty > 1 \wedge \|f - g\|_\infty < \epsilon_1 &\implies \|g\|_\infty > 1 \end{aligned}$$

Additionally, since $\|f\|_\infty > 1$ and $\|f - g\|_\infty$ is bounded, the only way these are both true this must also be true: $\|g\|_\infty > 1$.

Alternatively, you can just recognize that and $f \in \mathbb{B} : \|\cdot\|_\infty > 1 \implies f \notin B_1 \implies f \in B_1^c$ which demonstrates B_1^c is open, and therefore, B_1 is closed. \square

b) **Prove that B_1 is bounded.**

Theorem 5. B_1 is bounded, i.e.

$$\exists_N : \forall f \in B_1 \quad \|f\|_\infty < N$$

Proof. Since, by definition, $\|f\| \leq 1$, B_1 is clearly bounded for any $N > 1$. \square

c) **Prove that B_1 is not compact.**

Theorem 6. B_1 is not compact.

Proof. B_1 not being compact is equivalent to saying

$$\neg \left(\forall_{f_k \in \mathbb{B}} f_k \in \mathbb{B} : \exists_{f_{n_k}} : f_{n_k} \rightarrow f \in B_1 \right) \\ \exists_{f_k \in \mathbb{B}} f_k \in \mathbb{B} : \forall_{f_{n_k}} : f_{n_k} \rightarrow f \notin B_1$$

\square

Problem 4

Let $\{V, \|\cdot\|\}$ be a normed space. Show that the function $f(x) = \|x\| : V \rightarrow \mathbb{R}$ is continuous on V .

Definition 12. A function $f : (S_1, d_1) \rightarrow (S_2, d_2)$ is continuous on S_1 if

$$\forall x \in S_1 \forall \epsilon > 0 \exists \delta(x, \epsilon) > 0 \forall y \in S_1 d_1(x, y) < \epsilon \implies d_2(f(x), f(y)) < \delta$$

Theorem 7. The function $f(x)$ is continuous on V , i.e.

$$\forall x \in V \forall \epsilon > 0 \exists \delta(x, \epsilon) > 0 \forall y \in V \|x - y\| < \epsilon \implies |f(x) - f(y)| < \delta$$

Proof.

$$\begin{aligned} \forall x \in V \forall \epsilon > 0 \exists \delta(x, \epsilon) > 0 \forall y \in V \|x - y\| < \epsilon &\implies \left| \|x\| - \|y\| \right| < \delta \\ \forall x \in V \forall \epsilon_1 > 0 \exists \delta_2(x, \epsilon_1) > 0 \forall y \in V \|x - y\| \leq \|x\| + \|y\| < \epsilon_1 &\implies \left| \|x\| - \|y\| \right| \leq \|x\| + \|y\| < \delta_2 \\ \forall x \in V \forall \epsilon_1 > 0 \exists \delta_2(x, \epsilon_1) > 0 \forall y \in V \|x\| + \|y\| < \epsilon_1 &\implies \|x\| + \|y\| < \delta_2 \end{aligned}$$

which is clearly true, therefore $f(x) = \|x\|$ is continuous on V . □

Problem 5

Let (X, d_1) and (Y, d_2) are two metric spaces. Assume also that Y is a vector space. Construct and example of two continuous functions $f, g : X \rightarrow Y$ such that $f + g$ is discontinuous.

Definition 13. Let $f : X \rightarrow Y$ be defined by

$$f(x) :=$$

Definition 14. Let $g : X \rightarrow Y$ be defined by

$$g(x) :=$$

Theorem 8. Functions f and g are continuous, but $f + g$ is discontinuous.

Proof. **a)**

Lemma 1. f is a continuous function.

Proof.

$$\begin{aligned} \forall_{x \in X} \forall_{\epsilon > 0} \exists_{\delta(x, \epsilon)} \forall_{y \in X} : d_1(x, y) < \epsilon &\implies d_2(f(x), f(y)) < \delta \\ &\implies d_2() < \delta \end{aligned}$$

□

□

Problem 6

Construct an example of a sequence $\{f_n\}$ of nowhere continuous functions $[0, 1] \rightarrow \mathbb{R}$ such that f_n converge in the sup-norm to continuous functions.