MATH5301 Elementary Analysis. Homework 4. Due: 09/24/2021, 11:59 pm

First Name: **Last Name:**

4.1

Using axioms of the ordered field, prove

(a) $a > c \wedge b > d \Rightarrow a + b > c + d$

(b) $a > c > 0 \wedge b > d > 0 \Rightarrow ab > cd > 0$

(c) $a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b}$

(d) Denote $|x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{else} \end{cases}$. Then for any a, b it follows $|a - b| \geqslant ||a| - |b||$.

4.2

Determine which of the axioms satisfied by the set of real numbers are not satisfied by the following set:

(a) Set $\mathbb{Q}$ of all rational numbers.

(b) Set $\mathbb{Q}(\sqrt{2})$ of all numbers, having the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$

(c) Set $\mathbb{C}$ of all pairs of real numbers (a, b) with the addition defined as: $(a, b) + (c, d) = (a + c, b + d)$, multiplication defined as $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$ and the order relation $(a, b) < (c, d)$ if and only if $(b \leqslant d) \vee (b = d \wedge a < c)$.

4.3

Using the method of mathematical induction, prove the following statements:

(a) Bernoulli inequality: for all $n \in \mathbb{N}$ and for all $x > -1$ it follows that

$$(1+x)^n \geqslant 1 + nx$$

(b) For all $n \in \mathbb{N}$ it follows

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

(c)

$$(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) = \frac{1-q^{2^{n+1}}}{1-q}$$

(d)

$$1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1)$$

(e)

$$\sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!} = 0, \qquad \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} = 2^n,$$

4.4

Show that for all $n \in \mathbb{N}$, $n \geqslant 2$

(a)

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

(b)

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$

(c)

$$\left(\frac{n+1}{2}\right)^n > n!$$

(d) The number $2^{2^n} - 6$ is divisible by 10.

4.5

(a) Show that $\sqrt{2} \notin \mathbb{Q}$

(b) Show that for any $a, b \in \mathbb{Q}$, $a < b$ there exists $x \in \mathbb{R} \setminus \mathbb{Q} :\ a < x < b$

(c) Show that for any $a, b \in \mathbb{R} \setminus \mathbb{Q}$, $a < b$ there exists $x \in \mathbb{Q} :\ a < x < b$

4.6

Prove that for any n

(a) For the any configuration of n straight lines on a plane one could color the plane in two colors, so that every two parts, having common boundary would have different colors.

(b) For any set of n squares, one can partition them on finite amount of pieces and which will constitute one square.

(c) What's wrong with this "theorem"?

Theorem 1. *All the numbers are equal. In other words, the statement P_n is true for all $n \in \mathbb{N}$, where P_n is "if $\{a_1, a_2, \ldots, a_n\}$ is a collection of n numbers, then $a_1 = a_2 = \cdots = a_n$".*

Proof. P_1 is definitely true: if $\{a_1\}$ is a collection of 1 number, then $a_1 = a_1$.

Assume that P_n is true. Let $\{a_1, \ldots, a_{n+1}\}$ be a collection of $n+1$ number. Consider the collection $\{a_1, \ldots, a_n\}$. Thanks to P_n it follows that $a_1 = \cdots = a_n$ denote this number by b. Now, consider the collection $\{a_2, \ldots, a_n, a_{n+1}\}$. It contains n numbers, and hence by P_n we get $a_2 = \cdots = a_n = a_{n+1}$. But $a_2 = \cdots = a_n = b$, therefore $a_{n+1} = a_n = b$. So $a_1 = \cdots = a_{n+1} = b$ and P_{n+1} follows. $\square$