MATH 5301 Elementary Analysis - Homework 10

Jonas Wagner

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Problem 1

Prove that the closure and the interior of a convex set $A \subset \mathbb{R}^n$ are also convex.

Definition 1. *The set A is called* <u>convex</u> *if*

$$\forall_{x,y \in A} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in A$$

Definition 2. *For a given set $A \subseteq (S,d)$,*

a. the <u>interior</u> *of A is defined as*

$$int(A) = \{x \in A \;:\; \exists_{\epsilon>0} B_\epsilon(x) \subset A\}$$

b. the <u>closure</u> *of A is defined as*

$$\overline{A} = \{x \in S \;:\; \forall_{\epsilon>0} B_\epsilon(x) \cap A \neq \emptyset\}$$

Theorem 1. *If $A \subset \mathbb{R}^n$ is a convex set, then the closure of A, $\overline{A}$, is also convex.*

Proof. A being convex means that

$$\forall_{x,y \in A} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in A$$

$\overline{A}$ is defined by

$$\overline{A} = \{x \in \mathbb{R}^n \;:\; \forall_{\epsilon>0} B_\epsilon(x) \cap A \neq \emptyset\}$$

For $\overline{A}$ to be convex, the following would be true:

$$\forall_{x,y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A}$$

Additionally, since $\overline{A} = A \cup \partial A$, $\overline{A}$ is convex if

$$\left(\forall_{x \in A} \forall_{y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A}\right) \wedge \left(\forall_{x \in \partial A} \forall_{y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A}\right)$$

Since $A \subset \overline{A}$, by definition the first statement is true,

$$\forall_{x \in A} \forall_{y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A}$$

Additionally, since the boundary of A, ∂A, is the collection of limit points of A and the limit points all exist within the neighborhood of elements in A,

$$\forall_{x \in \partial A} \forall_{y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A}$$

Therefore,

$$\forall_{x,y \in \overline{A}} \forall_{t \in [0,1]} ((t)x + (1-t)y) \in \overline{A}$$

□

Problem 2

Prove that the intersection of an arbitrary collection of convex sets $\cap_{i \in I} C_i$ is also convex.

Theorem 2. *If each of the sets within the collection $C_i \subset (S, d)$ are convex, then the intersection of the collection, $\cap_{i \in I}$ is also convex.*

Proof. For $\cap_{i \in I}$ to be convex, the following must be true:

$$\forall_{x,y \in \cap_{i \in I} C_i} \forall_{t \in [0,1]} (t)x + (1-t)y \in \cap_{i \in I} C_i$$

Which is the same as:

$$\forall_{x,y \in S} \; : \; \forall_{i \in I} x, y \in C_i \implies \forall_{t \in [0,1]} \forall_{i \in I} (t)x + (1-t)y \in C_i$$

Since all the sets C_i are convex, by definition:

$$\forall_{x,y \in C_i} \forall_{t \in [0,1]} (t)x + (1-t)y \in C_i$$

Therefore this is true $\forall_{i \in I}$:

$$\wedge_{i \in I} \forall_{x,y \in C_i} \implies \forall_{t \in [0,1]} (t)x + (1-t)y \in C_i$$

Which is equivalent to:

$$\forall_{x,y \in \cap_{i \in I} C_i} \forall_{t \in [0,1]} (t)x + (1-t)y \in \cap_{i \in I} C_i$$

□

Problem 3

Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of nested convex sets in $\mathbb{R}^n$, i.e. $C_i \subset C_{i+1}$. Prove that $\cup_{i=1}^{\infty} C_i$ is also convex.

Theorem 3. *For the sequence of nested convex sets in $\mathbb{R}^n$, $\{C_i\}_{i\in\mathbb{N}}$, a union of all the elements, $\cup_{i=1}^{\infty} C_i$, is also convex.*

Proof. Proof by induction.
For $n = 1$, the set $\cup_{i=1}^{n} C_i = C_1$ is convex.
For $n = 2$, the set $\cup_{i=1}^{n} C_i = C_1 \cup C_2$ is convex.

Proof. Since $C_1 \subset C_2$, $C_1 \cup C_2 = C_2$ and C_2 is convex. □

Assuming for $n = k$, $\cup_{i=1}^{k} C_i = C_k$ is convex, then for $n = k+1$, $\cup_{i=1}^{k+1} C_i = C_{k+1}$ is convex.

Proof. Since $C_k \subset C_{k+1}$,

$$\cup_{i=1}^{k+1} C_i = \cup_{i=1}^{k} C_i \cup C_{k+1} = C_{k+1}$$

which is convex. □

Therefore, by induction,

$$\forall_{n\in\mathbb{N}} \cup_{i=i}^{n} C_i$$

is convex. This implies $\cup_{i=1}^{\infty} C_i$. □

Problem 4

Definition 3. *The* <u>convex hull</u> *for set* $A \in (S, d)$ *is defined as*

$$\operatorname{conv}(A) = \cap_{C \supseteq A \,:\, C \text{ convex}}$$

Additionally, for $A \subset \mathbb{R}^n$,

$$\operatorname{cov}(A) = \cup_{m=1}^{\infty} C_m$$

$$C_m = \left\{ x \in \mathbb{R}^n \;:\; x = \alpha_1 a_1 + \cdot + \alpha_m a_m,\ a_1, \ldots, a_m \in A,\ \alpha_i \geq 0,\ \sum_i \alpha_i = 1 \right\}$$

Definition 4. *The set* $A \subset V$ *is called* <u>open</u> *if*

$$\forall_{x \in A} \exists_{\epsilon > 0} : B_\epsilon(x) \subset A$$

or equivalently,

$$\forall_{x \in A} \exists_{\epsilon > 0} : \forall_{y \in V} \|x - y\| < \epsilon \implies y \in A$$

Definition 5. *The set* $A \subset V$ *is called* <u>closed</u> *if* A^c *is open.*

a)

Show that the convex hull of any open sets in $\mathcal{R}^n$ is open.

Theorem 4. *For any open set* $A \subset \mathbb{R}^n$*, then the convex hull,* $\operatorname{conv}(A)$*, is open.*

Proof. By definition, A will satisfy the open condition:

$$\forall_{x \in A} \exists_{\epsilon > 0} : \forall_{y \in \mathbb{R}^n} \|x - y\| < \epsilon \implies y \in A$$

The convex hull of A, $\operatorname{conv}(A)$, is the intersection of all convex sets that contain A. This means that the smallest convex set containing A will share a portion of the boundary of A. Then the open condition will be true along the portion of the boundary in which A and $\operatorname{conv}(A)$ share will be open. The smallest convex set that shares this boundary will then also be open since the open $\operatorname{int}(C)$ is contained by any closed set with the same boundary. Therefore, the convex hull would be open. □

b)

Provide an example of a closed set $A \subset \mathcal{R}^n$, such that its convex hull is not closed.

Example 1. *Let closed set* $A \subset \mathbb{R}^2$ *be defined as*

$$A := \left\{ (x, y) \in \mathbb{R}^2 \;:\; y \geq e^{-x^2} \right\}$$

Clearly, A *is closed, however the convex hull of* A *is actually open due to the asymptote that occurs on the* x*-axis:*

$$\operatorname{conv}(A) := \left\{ (x, y) \in \mathbb{R}^2 \;:\; y > 0 \right\}$$

Problem 5

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $A \subset \mathbb{R}^n$ be a bounded set. Prove that $f(A)$ is bounded in $\mathbb{R}$.

Definition 6. *Function $f : [a, b] \to \mathbb{R}$ is considered a* <u>convex function</u> *if*

$$\forall_{x_1,x_2 \in [a,b]} \forall_{t \in [0,1]} f((t)x_1 + (1-t)x_2) \leq (t)f(x_1) + (1-t)f(x_2)$$

Theorem 5. *If the function $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, then for a bounded set $A \subset \mathbb{R}^n$, the image $f(A)$ is bounded in $\mathbb{R}$.*

Proof. f being convex means that

$$\forall_{\vec{\mathbf{x}},\vec{\mathbf{y}} \in \mathbb{R}^n} \forall_{t \in [0,1]} f((t)\vec{\mathbf{x}} + (1-t)\vec{\mathbf{y}}) \leq (t)f(\vec{\mathbf{x}}) + (1-t)f(\vec{\mathbf{y}})$$

A being bounded means

$$\exists_N \ : \ \forall_{\vec{\mathbf{x}} \in A} \|\vec{\mathbf{x}}\| \leq N$$

It is known that the a convex function with a non-constant output cannot obtain its maximum within $\text{int}(A)$; therefore, $\arg\max_{\vec{\mathbf{x}} \in A} f(x) \in \partial A$. Since A is bounded,

$$\exists_N \ : \ \forall_{x \in \partial A} \|x\| < N$$

Therefore,

$$\exists_N \ : \ \forall_{\vec{\mathbf{x}} \in A} f(x) < \max_{\vec{\mathbf{x}} \in \partial A} f(x) < N$$

Which means $f(A)$ is bounded. $\square$

Problem 6

Show that the convex hull of a compact set $A \subset \mathbb{R}^n$ is compact. (*Hint:* Caratheodory theorem)