MATH 5301 Elementary Analysis - Homework 6

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Problem 1

a) Show that $\forall_{x>0}\ x \in \mathbb{R} \implies \lim_{n\to\infty} x^{1/n} = 1$

Definition 1. *for* $f : (S_1, d_1) \to (S_2, d_2)$,
if $a \in \bar{S}_1$ *then we say*

$$\lim_{x\to a} f(x) = b$$

if

$$\forall_{\epsilon>0}\exists_{\delta(\epsilon)>0}\ :\ \forall_{x\in\dot{B}_\delta(a)\subset S_1} \implies f(x) \in B_\epsilon(b) \subset S_2$$

Theorem 1.

$$\forall_{x>0}\ x \in \mathbb{R} \implies \lim_{n\to\infty} x^{1/n} = 1$$

Proof.

$$\begin{aligned}
\forall_{x>0}\ x \in \mathbb{R} &\implies \lim_{n\to\infty} f(x) = x^{1/n} = 1 \\
&\implies \forall_{\epsilon>0}\exists_{\delta(\epsilon)>0}\ :\ \forall_{x\in\dot{B}_\delta(a)} \implies f(x) \in B_\epsilon(b) \\
&\implies \forall_{\epsilon>0}\exists_{\delta(\epsilon)>0}\ :\ \forall_{x\in\mathbb{R}} : d_1(a, x) < \delta(\epsilon) \implies x^{1/n} \in \mathbb{R} : d_2(b, x^{1/n}) < \epsilon
\end{aligned}$$

$\square$

Problem 2 True or False?

Definition 2. *for $f : (S_1, d_1) \to (S_2, d_2)$, $f(x)$ is a continuous function iff*

$$\forall_{x \in S_1} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in S_1} : d_1(x, y) < \epsilon \implies d_2(f(x), f(y)) < \epsilon$$

a) If $f : (S_1, d_1) \to (S_2, d_2)$ is continuous and $U \subset S_1$ is open then $f(U) \subset S_2$ is also open

Theorem 2. *If $f : (S_1, d_1) \to (S_2, d_2)$ is continuous and $U \subset S_1$ is open then $f(U) \subset S_2$ is also open.*

Proof.

$$\begin{aligned} \forall_{x \in S_1} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in S_1} : d_1(x, y) < \epsilon &\implies d_2(f(x), f(y)) < \delta \wedge \\ &\wedge \forall_{x \in U} \exists_{\epsilon > 0} \ : \ B_\epsilon(x) \subset U \implies \\ &\implies \forall_{f(x) \in F(U)} \exists_{\epsilon > 0} \ : \ B_\epsilon(f(x)) \subset U \\ \forall_{x \in S_1} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in S_1} : d_1(x, y) < \epsilon &\implies d_2(f(x), f(y)) < \delta \wedge \\ &\wedge \forall_{x \in U} \exists_{\epsilon_1 > 0} \ : \ \forall_{y \in S_1} d(x, y) < \epsilon_1 \subset U \implies \\ &\implies \forall_{f(x) \in F(U)} \exists_{\epsilon_2 > 0} \ : \ \forall_{f(x) \in f(U)} : \forall_{f(y) \in S_2} : d_2(f(x), f(y)) < \epsilon_2 \subset U \end{aligned}$$

which is clearly true. □

b) $f : (S_1, d_1) \to (S_2, d_2)$ is continuous $\iff \forall_{C \subset S_2}$ closed, the set $f^{-1}(C) \subset S_1$ is also closed.

Theorem 3. *$f : (S_1, d_1) \to (S_2, d_2)$ is continuous $\iff \forall_{C \subset S_2}$ closed, the set $f^{-1}(C) \subset S_1$ is also closed.*

Proof.

$$\begin{aligned} \forall_{x \in S_1} \forall_{\epsilon > 0} \exists_{\delta(x,\epsilon) > 0} \forall_{y \in S_1} : d_1(x, y) < \epsilon &\implies d_2(f(x), f(y)) < \delta \iff \\ &\iff \forall_{C \subset S_2} : \forall_{x \in C^{\complement}} \exists_{\epsilon > 0} : B_\epsilon(x) \in C^{\complement} \implies \\ &\implies \forall_{x \in f^{-1}(C)^{\complement}} \exists_{\epsilon > 0} : B_\epsilon(x) \in f^{-1}(C)^{\complement} \end{aligned}$$

□

Problem 3 Prove the following properties of continuous functions:

a) $\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2}$ **continuous** $\implies (af+bg)(x) := af(x)+bg(x) : S_1 \to S_2$ **continuous.**

Theorem 4. $\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2}$ *continuous* $\implies (af+bg)(x) := af(x)+bg(x) : S_1 \to S_2$ *continuous.*

Proof.

$$\forall_{a,b\in\mathbb{R}}\forall_{f,g:S_1\to S_2}$$

□