# MATH 5301 Elementary Analysis - Homework 2

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## Problem 1

For a function $f : A \to B$, show the follwoing for any $X \subset A, Y, Z \supset B$

**1a)** $X \subset f^{-1}(f(X))$

**1b)** $f(f^{-1}(Y)) \subset Y$

**1c)** $f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$

**1d)** $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$

# Problem 2

Show that:

**2a)** $\quad A \cap \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A)$

Let $\Lambda := \{1, 2, \cdots, n\}$,

$$A \cap \bigcup_{\lambda \in \Lambda} A_\lambda = A \cap (A_1 \cup A_2 \cup \cdots \cup A_n)$$
$$= (A \cap A_1) \cup (A \cap A_2) \cup \cdots \cup (A \cap A_n)$$
$$= \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A)$$

Therefore,

$$\boxed{A \cap \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A)}$$

**2b)** $\quad \left( \bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left( \bigcap_{\lambda \in \Lambda} B_\lambda \right) \subseteq \bigcap_{\lambda \in \Lambda} (A_\lambda \cup B_\lambda)$

Let $\Lambda_A := \{1, 2, \cdots, n\}$ and $\Lambda_B := \{1, 2, \cdots, m\}$,

$$\bigcap_{\lambda \in \Lambda} (A_\lambda \cup B_\lambda) = (A_1 \cup B_1) \cap (A_1 \cup B_2) \cap \cdots \cap (A_1 \cup B_m) \cap (A_2 \cup B_1) \cap \cdots \cap (A_n \cup B_m)$$
$$= (A_1 \cup (B_1 \cap B_2 \cap \cdots \cap B_m)) \cap \cdots \cap (A_n \cup (B_1 \cap B_2 \cap \cdots \cap B_m))$$
$$= (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup \cdots (A_2 \cap A_3) \cdots \cup (A_{n-1} \cap A_n)$$
$$\cup (A_1 \cap B_1) \cup \cdots \cup (A_1 \cap B_m) \cup \cdots \cup (A_n \cap B_m)$$
$$\cup (B_1 \cap B_2) \cup \cdots \cup (B_1 \cap B_m) \cup \cdots \cup (B_{m-1} \cap B_m)$$
$$= \left( \bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left( \bigcap_{\lambda \in \Lambda} B_\lambda \right) \cup (A_1 \cap B_1) \cup \cdots \cup (A_1 \cap B_m) \cup \cdots \cup (A_n \cap B_m)$$

Therefore,

$$\boxed{\left( \bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left( \bigcap_{\lambda \in \Lambda} B_\lambda \right) \subseteq \left( \bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left( \bigcap_{\lambda \in \Lambda} B_\lambda \right) \cup (A_1 \cap B_1) \cup \cdots \cup (A_1 \cap B_m) \cup \cdots \cup (A_n \cap B_m)}$$

# Problem 3

**Problem:** Which of these are equivalence relations?

**Solution:** a, c, & d
(The following explain why)

## 3a)

for $a, b \in \mathbb{R}$, let $a\mathcal{R}b$ if $a - b \in \mathbb{Q}$

    **i) Reflective:**

$$x\mathcal{R}x = x - x = 0 \in \mathbb{Q}$$

    **ii) Symetric:**

$$x\mathcal{R}y \implies y\mathcal{R}x$$
$$x\mathcal{R}y = x - y \in \mathbb{Q} \implies y - x \in \mathbb{Q}$$

Since $x - y = -(y - x)$, $(x - y)$ and $(y - x)$ will both be either rational or not rational, this is true.

    **iii) Transitive:**

$$x\mathcal{R}y \wedge y\mathcal{R}z \implies x\mathcal{R}z$$
$$(x - y \in \mathbb{Q}) \wedge (y - z \in \mathbb{Q}) \implies (x - z \in \mathbb{Q})$$
$$\left(\frac{x_a}{x_b} - \frac{y_a}{y_b} \in \mathbb{Q}\right) \wedge \left(\frac{y_a}{y_b} - \frac{z_a}{z_b} \in \mathbb{Q}\right) \implies \left(\frac{x_a}{x_b} - \frac{z_a}{z_b} \in \mathbb{Q}\right)$$

This also means that:

$$(x_a y_b - x_b y_a \in \mathbb{N}) \wedge (x_b y_b \neq 0 \in N) \wedge (y_a z_b - y_b z_a \in \mathbb{N}) \wedge (y_b z_b \neq 0 \in N)$$
$$\implies (x_a z_b - x_b z_a \in \mathbb{N}) \wedge (x_b z_b \neq 0 \in N)$$

Since this statments indicates that $x_b, y_b, z_b \neq 0$ and that $x_a y_b - x_b y_a, y_a z_b - y_b z_b \in \mathbb{N}$, the following will always be true as well: $x_a z_b - x_b z_a$. Therefore, the relation is transitive.

## 3b)

for $a, b \in \mathbb{R}$, let $a\mathcal{R}b$ if $a - b \notin \mathbb{Q}$

    **i) Reflective:**

The relationship is NOT reflective:

$$a\mathcal{R}b = a - b \notin \mathbb{Q}$$
$$x\mathcal{R}x = x - x = 0 \in \mathbb{Q}$$

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## 3c)

for $a, b \in \mathbb{R}$, let $a\mathcal{R}b$ if $a - b$ is a square root of a rational number.
i.e.
$$a\mathcal{R}b = (a - b)^2 \in \mathbb{Q}$$

### i) Reflective:

$$a\mathcal{R}b = (a - b)^2 \in \mathbb{Q}$$
$$x\mathcal{R}x = (x - x)^2 = 0^2 = 0 \in \mathbb{Q}$$

### ii) Symetric:

$$x\mathcal{R}y \implies y\mathcal{R}x$$
$$(x - y)^2 \in \mathbb{Q} \implies (y - x)^2 \in \mathbb{Q}$$
$$(x - y)^2 = (y - x)^2 \therefore \ x\mathcal{R}y \implies y\mathcal{R}x$$

### iii) Transative:

$$x\mathcal{R}y \wedge y\mathcal{R}z \implies x\mathcal{R}z$$
$$\left((x - y)^2 \in \mathbb{Q}\right) \wedge \left((y - z)^2 \in \mathbb{Q}\right) \implies \left((x - z)^2 \in \mathbb{Q}\right)$$
$$\left(x^2 - 2xy + y^2 \in \mathbb{Q}\right) \wedge \left(y^2 - 2yz + z^2 \in \mathbb{Q}\right) \implies \left(x^2 - 2xz + z^2 \in \mathbb{Q}\right)$$
$$(x^2 \in \mathbb{Q}) \wedge (y^2 \in \mathbb{Q}) \wedge (z^2 \in \mathbb{Q}) \wedge (-2xy \in \mathbb{Q}) \wedge (-2yz \in \mathbb{Q}) \implies (x^2 \in \mathbb{Q}) \wedge (z^2 \in \mathbb{Q}) \wedge (-2xz \in \mathbb{Q})$$
$$(xy \in \mathbb{Q}) \wedge (yz \in \mathbb{Q}) \implies (xz \in \mathbb{Q})$$

Which is clearly transitive, so $a\mathcal{R}b = (a - b)^2 \in \mathbb{Q}$ is transitive.

## 3d)

Let $X = \mathbb{Z} \times \mathbb{N}$, $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $\mathcal{R}$ if $x_1 y_2 = x_2 y_1$.
i.e.
$$a_1, b_1 \in \mathbb{Z}, a_2, b_2 \in \mathbb{N},$$
$$a_1 b_2 = a_2 b_1 \implies (a_1, a_2)\mathcal{R}(b_1, b_2)$$

### i) Reflective:

$$a_1 b_2 = a_2 b_1 \implies (a_1, a_2)\mathcal{R}(b_1, b_2)$$
$$x_1 x_2 = x_2 x_1 \implies (x_1, x_2)\mathcal{R}(x_1, x_2)$$

### ii) Symetric:

$$x\mathcal{R}y \implies y\mathcal{R}x$$
$$(x_1 y_2 = x_2 y_1 \implies (x_1, x_2)\mathcal{R}(y_1, y_2)) \implies (y_1 x_2 = y_2 x_1 \implies (y_1, y_2)\mathcal{R}(x_1, x_2))$$

### iii) Transative:

$$x\mathcal{R}y \wedge y\mathcal{R}z \implies x\mathcal{R}z$$
$$(x_1 y_2 = x_2 y_1 \implies (x_1, x_2)\mathcal{R}(y_1, y_2)) \wedge (y_1 z_2 = y_2 z_1 \implies (y_1, y_2)\mathcal{R}(z_1, z_2)) \implies (x_1 z_2 = x_2 z_1 \implies (x_1, x_2)\mathcal{R}(z_1, z_2))$$
$$(x_1 y_2 = x_2 y_1) \wedge (y_1 z_2 = y_2 z_1) \implies (x_1 z_2 = x_2 z_1)$$

Which is clearly transitive, so $(a_1, a_2)\mathcal{R}(b_1, b_2)$ is transitive, and therefore an equivalence relation.

# Problem 4

# Problem 5

# Problem 6

**Problem:** Is the following statement correct?

**Theorem 1.** *If the relation $\mathbb{R}$ on A is symetric and transistive, then it is reflexive.*
*Proof:* For any $a \in A$ let $b \in A$ is such that $a\mathcal{R}b$. Then by symmetry $b\mathcal{R}a$. Then by symmetry $a\mathcal{R}a$.

**Solution:** No. Specifically, the final statement of the proof states to use symmetry to conclude that it is reflexive, but it actually requires the transivity property to make that conclusion.

The following is a proposed corrected statement:

**Theorem 1.** *If the relation $\mathcal{R}$ is symetric and transistive on A, then Rel is also reflexive on A.*
*Proof:* Let $a \in A$ and $b \in A$ be selected so that $a\mathcal{R}b$ Since $\mathcal{R}$ is symetric:

$$a\mathcal{R}b \implies b\mathcal{R}a$$

Since $\mathcal{R}$ is transitive:

$$(a\mathcal{R}b) \wedge (b\mathcal{R}a) \implies a\mathcal{R}a$$

Therefore, $\mathcal{R}$ is also reflexive.