

# MATH 5301 Elementary Analysis - Homework 2

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## Problem 1

For a function  $f : A \rightarrow B$ , show the following for any  $X \subset A, Y, Z \subset B$

**1a)**  $X \subset f^{-1}(f(X))$

$$\begin{aligned} f(X) &= \{y \in B : \exists x \in X : y = f(x)\} \\ f^{-1}(Y) &= \{x \in A : \exists y \in Y : y = f(x)\} \\ f(f^{-1}(X)) &= \{\tilde{x} \in \tilde{X} : \exists x \in A : \tilde{x} = f(f^{-1}(x))\} \\ &= \{\tilde{x} \in \tilde{X} : \exists y \in Y : y = f(\tilde{x}) : \exists x \in X : y = f(x)\} \implies \\ &\implies \tilde{X} \subset X \\ &\therefore X \subset f^{-1}(f(X)) \end{aligned}$$

**1b)**  $f(f^{-1}(Y)) \subset Y$

$$\begin{aligned} f^{-1}(Y) &= \{x \in A : \exists y \in Y : y = f(x)\} \\ f(X) &= \{y \in B : \exists x \in X : y = f(x)\} \\ f^{-1}(f(Y)) &= \{\tilde{x} \in \tilde{X} : \exists x \in A : \tilde{x} = f^{-1}(f(x))\} \\ &= \{\tilde{x} \in \tilde{X} : \exists y \in Y, \exists x \in X : (y = f(\tilde{x})) \wedge (y = f(x))\} \implies \\ &\implies \tilde{X} \subset X \\ &\therefore X \subset f^{-1}(f(X)) \end{aligned}$$

$$\mathbf{1c)} \quad f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z)$$

$$\begin{aligned} f^{-1}(Y) &= \{x \in A : \exists y \in Y : y = f(x)\} \\ f^{-1}(Z) &= \{x \in A : \exists z \in Z : z = f(x)\} \\ f^{-1}(Y \cup Z) &= \{x \in A : (\exists y \in Y y = f(x)) \vee (\exists z \in Z : z = f(x))\} \\ &= \{x \in A : (\exists y \in Y y = f(x))\} \cup \{x \in A : \vee (\exists z \in Z : z = f(x))\} \\ &= f^{-1}(Y) \cup f^{-1}(Z) \\ \therefore f^{-1}(Y \cup Z) &= f^{-1}(Y) \cup f^{-1}(Z) \end{aligned}$$

$$\mathbf{1d)} \quad f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$$

$$\begin{aligned} f^{-1}(Y) &= \{x \in A : \exists y \in Y : y = f(x)\} \\ f^{-1}(Z) &= \{x \in A : \exists z \in Z : z = f(x)\} \\ f^{-1}(Y \cap Z) &= \{x \in A : (\exists y \in Y y = f(x)) \wedge (\exists z \in Z : z = f(x))\} \\ &= \{x \in A : (\exists y \in Y y = f(x))\} \cap \{x \in A : \vee (\exists z \in Z : z = f(x))\} \\ &= f^{-1}(Y) \cap f^{-1}(Z) \\ \therefore f^{-1}(Y \cap Z) &= f^{-1}(Y) \cap f^{-1}(Z) \end{aligned}$$

## Problem 2

Show that:

$$\mathbf{2a)} \quad A \cap \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A)$$

Let  $\Lambda := \{1, 2, \dots, n\}$ ,

$$\begin{aligned} A \cap \bigcup_{\lambda \in \Lambda} A_\lambda &= A \cap (A_1 \cup A_2 \cup \dots \cup A_n) \\ &= (A \cap A_1) \cup (A \cap A_2) \cup \dots \cup (A \cap A_n) \\ &= \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A) \end{aligned}$$

Therefore,

$$\boxed{A \cap \bigcup_{\lambda \in \Lambda} A_\lambda = \bigcup_{\lambda \in \Lambda} (A_\lambda \cap A)}$$

$$\mathbf{2b)} \quad \left( \bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left( \bigcap_{\lambda \in \Lambda} B_\lambda \right) \subseteq \bigcap_{\lambda \in \Lambda} (A_\lambda \cup B_\lambda)$$

Let  $\Lambda_A := \{1, 2, \dots, n\}$  and  $\Lambda_B := \{1, 2, \dots, m\}$ ,

$$\begin{aligned} \bigcap_{\lambda \in \Lambda} (A_\lambda \cup B_\lambda) &= (A_1 \cup B_1) \cap (A_1 \cup B_2) \cap \dots \cap (A_1 \cup B_m) \cap (A_2 \cup B_1) \cap \dots \cap (A_n \cup B_m) \\ &= (A_1 \cup (B_1 \cap B_2 \cap \dots \cap B_m)) \cap \dots \cap (A_n \cup (B_1 \cap B_2 \cap \dots \cap B_m)) \\ &= (A_1 \cap A_2) \cup (A_1 \cap A_3) \cup \dots \cup (A_2 \cap A_3) \dots \cup (A_{n-1} \cap A_n) \\ &\quad \cup (A_1 \cap B_1) \cup \dots \cup (A_1 \cap B_m) \cup \dots \cup (A_n \cap B_m) \\ &\quad \cup (B_1 \cap B_2) \cup \dots \cup (B_1 \cap B_m) \cup \dots \cup (B_{m-1} \cap B_m) \\ &= \left( \bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left( \bigcap_{\lambda \in \Lambda} B_\lambda \right) \cup (A_1 \cap B_1) \cup \dots \cup (A_1 \cap B_m) \cup \dots \cup (A_n \cap B_m) \end{aligned}$$

Therefore,

$$\boxed{\left( \bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left( \bigcap_{\lambda \in \Lambda} B_\lambda \right) \subseteq \left( \bigcap_{\lambda \in \Lambda} A_\lambda \right) \cup \left( \bigcap_{\lambda \in \Lambda} B_\lambda \right) \cup (A_1 \cap B_1) \cup \dots \cup (A_1 \cap B_m) \cup \dots \cup (A_n \cap B_m)}$$

### Problem 3

**Problem:** Which of these are equivalence relations?

**Solution:** a, c, & d

(The following explain why)

**3a)**

for  $a, b \in \mathbb{R}$ , let  $a\mathcal{R}b$  if  $a - b \in \mathbb{Q}$

**i) Reflective:**

$$x\mathcal{R}x = x - x = 0 \in \mathbb{Q}$$

**ii) Symetric:**

$$\begin{aligned} x\mathcal{R}y &\implies y\mathcal{R}x \\ x\mathcal{R}y = x - y \in \mathbb{Q} &\implies y - x \in \mathbb{Q} \end{aligned}$$

Since  $x - y = -(y - x)$ ,  $(x - y)$  and  $(y - x)$  will both be either rational or not rational, this is true.

**iii) Transitive:**

$$\begin{aligned} x\mathcal{R}y \wedge y\mathcal{R}z &\implies x\mathcal{R}z \\ (x - y \in \mathbb{Q}) \wedge (y - z \in \mathbb{Q}) &\implies (x - z \in \mathbb{Q}) \\ \left(\frac{x_a}{x_b} - \frac{y_a}{y_b} \in \mathbb{Q}\right) \wedge \left(\frac{y_a}{y_b} - \frac{z_a}{z_b} \in \mathbb{Q}\right) &\implies \left(\frac{x_a}{x_b} - \frac{z_a}{z_b} \in \mathbb{Q}\right) \end{aligned}$$

This also means that:

$$\begin{aligned} (x_a y_b - x_b y_a \in \mathbb{N}) \wedge (x_b y_b \neq 0 \in \mathbb{N}) \wedge (y_a z_b - y_b z_a \in \mathbb{N}) \wedge (y_b z_b \neq 0 \in \mathbb{N}) \\ \implies (x_a z_b - x_b z_a \in \mathbb{N}) \wedge (x_b z_b \neq 0 \in \mathbb{N}) \end{aligned}$$

Since this statments indicates that  $x_b, y_b, z_b \neq 0$  and that  $x_a y_b - x_b y_a, y_a z_b - y_b z_a \in \mathbb{N}$ , the following will always be true as well:  $x_a z_b - x_b z_a$ . Therefore, the relation is transitive.

**3b)**

for  $a, b \in \mathbb{R}$ , let  $a\mathcal{R}b$  if  $a - b \notin \mathbb{Q}$

**i) Reflective:**

The relationship is NOT reflective:

$$\begin{aligned} a\mathcal{R}b &= a - b \notin \mathbb{Q} \\ x\mathcal{R}x &= x - x = 0 \in \mathbb{Q} \end{aligned}$$

**3c)**

for  $a, b \in \mathbb{R}$ , let  $a\mathcal{R}b$  if  $a - b$  is a square root of a rational number.  
i.e.

$$a\mathcal{R}b = (a - b)^2 \in \mathbb{Q}$$

**i) Reflective:**

$$\begin{aligned} a\mathcal{R}b &= (a - b)^2 \in \mathbb{Q} \\ x\mathcal{R}x &= (x - x)^2 = 0^2 = 0 \in \mathbb{Q} \end{aligned}$$

**ii) Symetric:**

$$\begin{aligned} x\mathcal{R}y &\implies y\mathcal{R}x \\ (x - y)^2 \in \mathbb{Q} &\implies (y - x)^2 \in \mathbb{Q} \\ (x - y)^2 &= (y - x)^2 \therefore x\mathcal{R}y \implies y\mathcal{R}x \end{aligned}$$

**iii) Transative:**

$$\begin{aligned} x\mathcal{R}y \wedge y\mathcal{R}z &\implies x\mathcal{R}z \\ ((x - y)^2 \in \mathbb{Q}) \wedge ((y - z)^2 \in \mathbb{Q}) &\implies ((x - z)^2 \in \mathbb{Q}) \\ (x^2 - 2xy + y^2 \in \mathbb{Q}) \wedge (y^2 - 2yz + z^2 \in \mathbb{Q}) &\implies (x^2 - 2xz + z^2 \in \mathbb{Q}) \\ (x^2 \in \mathbb{Q}) \wedge (y^2 \in \mathbb{Q}) \wedge (z^2 \in \mathbb{Q}) \wedge (-2xy \in \mathbb{Q}) \wedge (-2yz \in \mathbb{Q}) &\implies (x^2 \in \mathbb{Q}) \wedge (z^2 \in \mathbb{Q}) \wedge (-2xz \in \mathbb{Q}) \\ (xy \in \mathbb{Q}) \wedge (yz \in \mathbb{Q}) &\implies (xz \in \mathbb{Q}) \end{aligned}$$

Which is clearly transitive, so  $a\mathcal{R}b = (a - b)^2 \in \mathbb{Q}$  is transitive.

**3d)**

Let  $X = \mathbb{Z} \times \mathbb{N}$ ,  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  are in  $\mathcal{R}$  if  $x_1y_2 = x_2y_1$ .  
i.e.

$$\begin{aligned} a_1, b_1 &\in \mathbb{Z}, a_2, b_2 \in \mathbb{N}, \\ a_1b_2 &= a_2b_1 \implies (a_1, a_2)\mathcal{R}(b_1, b_2) \end{aligned}$$

**i) Reflective:**

$$\begin{aligned} a_1b_2 &= a_2b_1 \implies (a_1, a_2)\mathcal{R}(b_1, b_2) \\ x_1x_2 &= x_2x_1 \implies (x_1, x_2)\mathcal{R}(x_1, x_2) \end{aligned}$$

**ii) Symetric:**

$$\begin{aligned} x\mathcal{R}y &\implies y\mathcal{R}x \\ (x_1y_2 = x_2y_1 \implies (x_1, x_2)\mathcal{R}(y_1, y_2)) &\implies (y_1x_2 = y_2x_1 \implies (y_1, y_2)\mathcal{R}(x_1, x_2)) \end{aligned}$$

**iii) Transative:**

$$\begin{aligned} x\mathcal{R}y \wedge y\mathcal{R}z &\implies x\mathcal{R}z \\ (x_1y_2 = x_2y_1 \implies (x_1, x_2)\mathcal{R}(y_1, y_2)) \wedge (y_1z_2 = y_2z_1 \implies (y_1, y_2)\mathcal{R}(z_1, z_2)) &\implies (x_1z_2 = x_2z_1 \implies (x_1, x_2)\mathcal{R}(z_1, z_2)) \\ (x_1y_2 = x_2y_1) \wedge (y_1z_2 = y_2z_1) &\implies (x_1z_2 = x_2z_1) \end{aligned}$$

Which is clearly transitive, so  $(a_1, a_2)\mathcal{R}(b_1, b_2)$  is transitive, and therefore an equivalence relation.

## Problem 4

For the relation  $(x, y) \succeq (a, b)$  if  $(x \geq a)$  and  $(y \geq b)$  on the set of ordered pairs of  $\{1, 2, 3\} \times \{1, 2, 3\}$ . i.e.  $x, a \in \{1, 2, 3\}$  and  $y, b \in \{1, 2, 3\}$ ,

$$(x \geq a) \wedge (y \geq b) \implies (x, y) \succeq (a, b)$$

### 4a) Show that the above relation is an order relation.

An ordered relation requires (i) reflexivity, (ii) anti-symmetry, and (iii) transitivity.

#### i) Reflective:

$$\begin{aligned} (x \geq a) \wedge (y \geq b) &\implies (x, y) \succeq (a, b) \\ (x \geq x) \wedge (y \geq y) &\implies (x, y) \succeq (x, y) \end{aligned}$$

#### ii) Anti-Symmetry:

$$\begin{aligned} ((x, y) \succeq (a, b)) \wedge ((a, b) \succeq (x, y)) &\implies (x, y) = (a, b) \\ ((x \geq a) \wedge (y \geq b)) \wedge ((a \geq x) \wedge (b \geq y)) &\implies (x, y) = (a, b) \\ ((x \geq a) \wedge (a \geq x)) \wedge ((b \geq y) \wedge (y \geq b)) &\implies (x, y) = (a, b) \end{aligned}$$

#### iii) Transitivity

$$\begin{aligned} (x, y) \mathcal{R}(a, b) \wedge (a, b) \mathcal{R}(\alpha, \beta) &\implies (x, y) \mathcal{R}(\alpha, \beta) \\ ((x \geq a) \wedge (y \geq b)) \wedge ((a \geq \alpha) \wedge (b \geq \beta)) &\implies (x \geq \alpha) \wedge (y \geq \beta) \\ ((x \geq a) \wedge (a \geq \alpha)) \wedge ((y \geq b) \wedge (b \geq \beta)) &\implies (x \geq \alpha) \wedge (y \geq \beta) \\ (x \geq a \geq \alpha) \wedge (y \geq b \geq \beta) &\implies (x \geq \alpha) \wedge (y \geq \beta) \end{aligned}$$

Therefore, the relation  $(x, y) \succeq (a, b)$  is an order relation.

### 4b) Can you make it the total order?

#### i) Totality:

$$\begin{aligned} \forall (x, y), (a, b) \in \{1, 2, 3\} \times \{1, 2, 3\} &\implies ((x, y) \succeq (a, b)) \vee ((a, b) \succeq (x, y)) \\ \forall x, a \in \{1, 2, 3\}, \forall y, b \in \{1, 2, 3\} &\implies ((x \geq a) \wedge (y \geq b)) \vee ((a \geq x) \wedge (y \geq b)) \end{aligned}$$

#### 4c) How many different total orderings can be constructed?

Multiple total orderings of relation on subsets can satisfy this. A network constructed from the following relations would demonstrate the multiple paths.

$$\begin{aligned}(3, 3) &\succeq (x, y) \forall (x, y) \in \{1, 2, 3\} \times \{1, 2, 3\} \\(3, 2) &\succeq (x, y) \forall (x, y) \in \{1, 2, 3\} \times \{1, 2\} \\(3, 1) &\succeq (x, y) \forall (x, y) \in \{1, 2, 3\} \times \{1\} \\(2, 3) &\succeq (x, y) \forall (x, y) \in \{1, 2\} \times \{1, 2, 3\} \\(2, 2) &\succeq (x, y) \forall (x, y) \in \{1, 2\} \times \{1, 2\} \\(2, 1) &\succeq (x, y) \forall (x, y) \in \{1, 2\} \times \{1\} \\(1, 3) &\succeq (x, y) \forall (x, y) \in \{(1, 1), (1, 2), (1, 3)\} \\(1, 2) &\succeq (x, y) \forall (x, y) \in \{(1, 1), (1, 2)\} \\(1, 1) &\succeq (1, 1)\end{aligned}$$

If you only count the total orders of length 2 (the number of actual network edges with self-edges) this would be 36. If drawn as a tree, these 36 pairs would be created as directed edges and the total number of unique tree paths can also be counted using various algorithms.

## Problem 5

Provide an example of  $f : \mathbb{Z} \rightarrow \mathbb{N}$  such that

5a)  $f$  is surjective, but not injective

$$y = f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

5b)  $f$  is injective, but not surjective

$$y = f(x) = \begin{cases} x^2 & x \geq 0 \\ x^2 + 1 & x < 0 \end{cases}$$

5c)  $f$  is surjective and injective (bijective)

$$y = f(x) = \begin{cases} 2x & x \geq 0 \\ -2x - 1 & x < 0 \end{cases}$$

5d)  $f$  is neither surjective nor injective

$$y = f(x) = 0$$



## Problem 6

**Problem:** Is the following statement correct?

**Theorem 1.** *If the relation  $\mathbb{R}$  on  $A$  is symmetric and transitive, then it is reflexive.*

*Proof:* For any  $a \in A$  let  $b \in A$  be such that  $a\mathbb{R}b$ . Then by symmetry  $b\mathbb{R}a$ . Then by symmetry  $a\mathbb{R}a$ .

**Solution:** No. Specifically, the final statement of the proof states to use symmetry to conclude that it is reflexive, but it actually requires the transitivity property to make that conclusion.

The following is a proposed corrected statement:

**Theorem 1.** *If the relation  $\mathcal{R}$  is symmetric and transitive on  $A$ , then  $\mathcal{R}$  is also reflexive on  $A$ .*

*Proof:* Let  $a \in A$  and  $b \in A$  be selected so that  $a\mathcal{R}b$ . Since  $\mathcal{R}$  is symmetric:

$$a\mathcal{R}b \implies b\mathcal{R}a$$

Since  $\mathcal{R}$  is transitive:

$$(a\mathcal{R}b) \wedge (b\mathcal{R}a) \implies a\mathcal{R}a$$

Therefore,  $\mathcal{R}$  is also reflexive.