MATH 5301 Elementary Analysis - Final Exam

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Problem 1

For each $n \in \mathbb{N}$ define the set

$$Q_n := \left\{ \frac{1}{pq} \;:\; 0 < p < q \leq n;\; p+q > n;\; \gcd(p,q) = 1 \right\}$$

Let $f(n)$ be the sum of all elements of Q_n.
Find $\inf_n f(n)$.

Definition 1. *Let the set Q_n be defined for all $n \in \mathbb{N}$ as*

$$Q_n := \left\{ \frac{1}{pq} \;:\; 0 < p < q \leq n;\; p+q > n;\; \gcd(p,q) = 1 \right\}$$

Definition 2. *Let $f(n)$ be the sum of all elements within Q_n.*

Definition 3. *A lower bound of subset A in the partially ordered set $(S, \leq)$ is defined by*

$$a \in S \;:\; a \leq x \forall_{x \in A}$$

A lower bound of a is called an infimum *of set $A \in (S, \leq)$, denoted as $a = \inf A$, is the greatest lower bound. i.e.*

$$\forall_{y \in S : a \leq x \forall_{x \in A}} y \leq a$$

Definition 4. *The* Greatest Common Divisor *of two nonzero integers $a, b \in \mathbb{Z} \neq 0$, $\gcd(a,b)$, is defined as the largest positive integer, $d \in \mathbb{Z}_+$, so that d is a divisor of both a and b. i.e:*

$$\gcd(a,b) := d \in \mathbb{Z}_+ \;:\; (a \vdots d) \wedge (b \vdots d) \wedge (\forall_{x \in \mathbb{Z}_+ \;:\; a,b \,\vdots\, x}\; d \geq x)$$

Additionally, a and b are considered coprime *if $\gcd(a,b) = 1$.*

Assumption 1. *For this problem it is assumed that $\gcd$ is only defined within $\mathbb{Z}_+$, although I believe this can also be expanded to other less-strict ordered sets in the same way.*

Assumption 2. *It is assumed that the sum of all elements in the empty set is 0, i.e. $\sum_i \emptyset = 0$.*

Theorem 1.

$$\inf_{n\in\mathbb{N}} f(n) = 0$$

Proof. Proof by induction.
For $n = 1$, $\neg\exists_{p,q\in\mathbb{Z}\ :\ 0<p<q\leq 1}$ meaning that $Q_1 = \emptyset$.
This implies that $f(1) = \sum_i \emptyset = 0$ and that $f(1) \geq 0$.
For $n = 2$,

$$(p,q) \in \{(p,q)\ :\ 0 < p < q \leq 2;\ p+q > n;\ \gcd(p,q) = 1\} = \{(1,2)\}$$

The set Q_2 is then defined as

$$Q_2 = \left\{\frac{1}{pq}\ :\ (p,q) \in \{(1,2)\}\right\} = \left\{\frac{1}{(1)(2)}\right\} = \left\{\frac{1}{2}\right\}$$

Therefore,

$$f(2) = \sum_i \left\{\frac{1}{2}\right\} = \frac{1}{2}$$

It is clear that $f(2) = \frac{1}{2} \geq 0$.
For $n = 3$,

$$(p,q) \in \{(p,q)\ :\ 0 < p < q \leq 3;\ p+q > n;\ \gcd(p,q) = 1\} = \{(1,3),(2,3)\}$$

The set Q_3 is then defined as

$$Q_3 = \left\{\frac{1}{pq}\ :\ (p,q) \in \{(1,3),(2,3)\}\right\} = \left\{\frac{1}{(1)(3)}, \frac{1}{(2)(3)}\right\} = \left\{\frac{1}{3}, \frac{1}{6}\right\}$$

Therefore,

$$f(3) = \sum_i \left\{\frac{1}{3}, \frac{1}{6}\right\} = \frac{1}{3} + \frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

It is clear that $f(3) = \frac{1}{2} \geq 0$.
For $n = 4$,

$$(p,q) \in \{(p,q)\ :\ 0 < p < q \leq 4;\ p+q > n;\ \gcd(p,q) = 1\} = \{(1,4),(2,3),(3,4)\}$$

The set Q_4 is then defined as

$$Q_4 = \left\{\frac{1}{pq}\ :\ (p,q) \in \{(2,3),(3,4)\}\right\} = \left\{\frac{1}{(1)(4)}, \frac{1}{(2)(3)}, \frac{1}{(3)(4)}\right\} = \left\{\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\right\}$$

Therefore,

$$f(4) = \sum_i \left\{\frac{1}{6}, \frac{1}{12}\right\} = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{3+2+1}{12} = \frac{6}{12} = \frac{1}{2}$$

It is clear that $f(4) = \frac{1}{2} \geq 0$.
For an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}(p,q) \in \{(p,q)\ :\ 0 < p < q \leq n;\ p+q > n;\ \gcd(p,q) = 1\} = \\ = \{(1,n),(2,n-\star),(3,n-\star)\ldots,(n-2,n-1),(n-1,n)\}\end{aligned}$$

$$\begin{aligned}Q_n &= \left\{\frac{1}{pq}\ :\ (p,q) \in \{(1,n),(2,n-\star),\ldots,(n-2,n-1),(n-1,n)\}\right\} \\ &= \left\{\frac{1}{(1)(n)}, \frac{1}{(2)(n-1)}, \ldots, \frac{1}{(n-2)(n-1)}, \frac{1}{(n-1)(n)}\right\} \\ &= \left\{\frac{1}{n}, \frac{1}{2(n-\star)}, \ldots, \frac{1}{(n-2)(n-1)}, \frac{1}{n(n-1)}\right\}\end{aligned}$$

where $\star$ is dependent for on divisibility properties between n and 2, 3, 4, etc. It is important to note that each increase of n will cause every term to decrease in magnitude individually but additional elements are added that result to adding up to $\frac{1}{2}$ again.
However, eventually this will reach a point where a lack of prime numbers in a region makes it so that the only coprime numbers satisfying the conditions are adjacent to one another, which leads to the following:

$$
\begin{aligned}
f(n) = \sum_i Q_n &= \frac{1}{n} + \cdots + \frac{1}{(\frac{n}{2})(\frac{n}{2}+1)} + \cdots + \frac{1}{n(n-1)} \\
f(n+1) &= \left(\sum_i Q_n\right)\left(\frac{n!}{(n+1)!}\right) + \frac{1}{(n+1)} \\
&= \frac{1}{n}\frac{n!}{(n+1)!} + \cdots + \frac{1}{(\frac{n}{2})(\frac{n}{2}+1)}\frac{n!}{(n+1)!} + \cdots + \frac{1}{n(n-1)}\frac{n!}{(n+1)!} + \frac{1}{n+1} \\
&= \frac{n!}{n(n+1)n!} + \cdots + \frac{n!}{\frac{n}{2}(\frac{n}{2}-1)(n+1)n!} + \cdots + \frac{n!}{n(n-1)(n+1)n!} + \frac{1}{n+1} \\
&= \sum_i Q_{n+1} = \frac{1}{n+1} + \cdots + \frac{1}{(\frac{n+1}{2})(\frac{n+1}{2}+1)} + \cdots + \frac{1}{n(n+1)}
\end{aligned}
$$

essentially every (p, q) becomes $(q, q+1)$ and the new $\frac{1}{(n+1)}$ is added.
Anyway, the point is that $\forall_{n \in \mathbb{N}\,:\,n>1} f(n) \geq \frac{1}{2}$; however, because $f(n)$ is included, $\frac{1}{2} \leq f(n) \forall_{n \in N}$ since $Q_1 = \emptyset \implies f(1) = 0$.
Therefore,

$$\inf_n f(n) = 0$$

□

Problem 2

Let (X, d) be a metric space. Let $B_r(a)$ denote the open ball of radius r centered at a. i.e. Can it happen that $B_{r_1}(a) \subset B_{r_2}(a)$ but $r_1 > r_2$?

Definition 5. *Within the metric space (X, d), the open ball of radius $r \in X$ centered at $a \in X$, denoted as $B_r(a)$, is defined as:*

$$B_r(a) := \{x \in X \ : \ d(a, x) < r\}$$

Assumption 3. *First it will be assumed that (X, d) is a normed vector space. This restricts the metric and metric space into a normed space. This can also be denoted as $(X, \|\cdot\|)$ to distinguish between them. It is also assumed that X is complete.*

Theorem 2. *For $r_1 > r_2$ then it is not possible for $B_{r_1}(a) \subset B_{r_2}(b)$ within $(X, \|\cdot\|)$:*

Proof. Proof by contradiction.
Let

$$B_{r_1}(a), B_{r_2}(b) \subset X$$

with $0 < r_2 < r_1$ and $a \in B_{r_2}(b)$.
To minimize the amount of the set existing outside of the set, we need to set $a = b$. Next, let c be a point within the punctured open ball $B_{r_2}(b)$. i.e.

$$c \in B_{r_2}(b) \backslash \{b\}$$

c can then be used to construct a point that is contained in $B_{r_2}(b)$ but not in $B_{r_1}(a)$:

$$p + \frac{r_1 + r_2}{2} \frac{ac}{\|ac\|} \in B_{r_1}(a) \backslash B_{r_2}(b)$$

Meaning that there is no possible way for an open ball of greater radius (within a normed metric space). $\square$

Assumption 4. *The previous assumption, Assumption 3, is now relax the metric so that d is not restricted to be a norm (i.e. may not be linear).*

Theorem 3. *It is possible for $B_{r_1}(a) \subset B_{r_2}(b)$ within (X, d) when $r_1 > r_2$:*

Proof. Proof by example:
Let metric space (X, d) be defined by

$$X := 0 \cup [5, \infty)$$

$$d(x, y) := |x - y|$$

For $r_1 = 4$, $r_2 = 3$,
Let $B_4(0)$ be defined as

$$B_4(0) := \{4x \in X \ : \ d(0, x) < 4\} = \{0\} \cup [2, 4)$$

Let $B_3(2)$ be defined as

$$B_3(2) := \{x \in X \ : \ d(2, x) < 3\} = \{0\} \cup [2, 5)$$

Clearly, $B_3(2) \subset B_4(0)$. Since $r_1 = 4 > r_2 = 3$, this exists as an example that satisfies the conditions. $\square$

Problem 3

Let M be the set of all bounded sequences

$$M = \left\{\{a_j\}_{j=1}^{\infty} \ : \ |a_j| < \infty\right\}$$

Define $\rho(\{a_n\}, \{b_n\}) = \max_{n\in\mathbb{N}} |a_n - b_n|$

Definition 6. *Function $d : X \times X \to \mathbb{R}$ is considered a* metric *if it satisfies all of the following:*

a. Non-negativity:

$$d(a, b) \geq 0$$

b. Symmetry:

$$d(a, b) = d(b, a)$$

c. Triangle Inequality:

$$d(a, c) \leq d(a, b) + d(b, c)$$

a) Show that (M, ρ) is a metric space.

Theorem 4. *Let M be defined as the set of all bounded sequences:*

$$M = \left\{\{a_j\}_{j=1}^{\infty} \ : \ |a_j| < \infty\right\}$$

Let the metric ρ be defined on M such that

$$\rho(\{a_n\}, \{b_n\}) = \max_{n\in\mathbb{N}} |a_n - b_n|$$

The metric space (M, ρ) is in fact a metric space.

Proof. From Definition 6, ρ is a metric if $\forall_{\{a_n\},\{b_n\},\{c_n\}} \in M$ these three conditions are all satisfied: (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

a. Non-negativity:

$$d(a, b) \geq 0$$
$$\rho(\{a_n\}, \{b_n\}) = \max_{n\in\mathbb{N}} |a_n - b_n| \geq 0$$

b. Symmetry:

$$d(a, b) = d(b, a)$$
$$\rho(\{a_n\}, \{b_n\}) = \max_{n\in\mathbb{N}} |a_n - b_n| = \max_{n\in\mathbb{N}} |b_n - a_n| = \rho(\{a_n\}, \{b_n\})$$

c. Triangle Inequality:

$$d(a, c) \leq d(a, b) + d(b, c)$$
$$\rho(\{a_n\}, \{c_n\}) \leq \rho(\{a_n\}, \{b_n\}) + \rho(\{b_n\}, \{c_n\})$$
$$\max_{n\in\mathbb{N}} |a_n - c_n| \leq \max n \in \mathbb{N}|a_n - b_n| + |b_n - c_n| \leq \max_{n\in\mathbb{N}} |a_n - b_n| + \max_{n\in\mathbb{N}} |b_n - c_n|$$

$\square$

b) Show that M does not contain a dense countable subset.

Definition 7. *The* <u>Closure</u>*,* $\overline{A}$*, of* $A \subset X$ *is defined as*

$$\overline{A} = A \cup \left\{ \lim_{n\to\infty} a_n \ : \ a_n \in A \forall_{n\in\mathbb{N}} \right\}$$

Definition 8. *A set* $A \subset X$ *is considered* <u>dense</u> *in* X *if* $\overline{A} = X$*.*

Theorem 5. *For the power set,* $\mathcal{P}(A)$*, defined as the collections of all sets constructed from the elements of* A*, then the cardinality of* $\mathcal{P}(A)$ *will always be strictly greater then that of* A*. i.e.*

$$\left|2^A\right| > |A|$$

- *This is also applicable to infinite sets with whether it is countable or not. i.e*

$$\left|2^{\mathbb{N}}\right| = \aleph_1 > |\mathbb{N}| = \aleph_0$$

- *The theorem itself is that any mapping from* A *to* $\mathcal{P}(A)$ *is not surjective which is then proven false. It then follows that* $f : A \xrightarrow{!} \mathcal{P}(A)$ *is injective, which is equivalent to saying that* $|A| < |\mathcal{P}(A)|$*.*

Theorem 6. M *does not contain any dense countable subsets.*

Proof. Proof by contradiction inspired by Cantor's Theorem (5).
Let $A_N \subset M$ be defined as

$$A_N := \{\{a_j\}_{j=1}^{\infty} \ : \ |a_j| < N\}$$

Similarly to Cantor's theorem, even when restricting a_j from a finely sized set, the only mapping that exists from a countable set into A_n are strictly injective.
Next, taking $A = \lim_{N\to\infty} A_N$, we will prove that in order for A to be dense, A would no longer be countable. From Definition 7 and Definition 8, it is known that in order for A to be dense within M, $\overline{A} = M$. Since M itself is an infinite set, even if for sequences of a finite set of numbers, A would become infinite and ultimately uncountable with $|A| \leq |M|$. □

Problem 4

Does there exist a metric space, containing a sequence of nested bounded closed sets $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ such that

$$\bigcap_{n \in \mathbb{N}} F_n = \emptyset$$

Hint: If $d(x, y)$ is a usual Euclidean metric on $\mathbb{R}$, one can shown that $\frac{d(x,y)}{1+d(x,y)}$ is also a metric. Such metric is often called a bounded metric...

Definition 9. *The set A in metric space (X, d) is considered* <u>open</u> *if*

$$\forall_{x \in A} \exists_{\epsilon > 0} \;:\; \forall_{y \in X} d(x, y) < \epsilon$$

Definition 10. *The set A in metric space (X, d) is considered* <u>closed</u> *if the set A^c is open.*

Definition 11. *The set A in metric space (X, d) is called* <u>bounded</u> *if*

$$\forall_{x \in A} \exists_{R > 0} \;:\; \forall_{y \in A} d(x, y) < R$$

Theorem 7. *There does exist a metric space (X, d) containing the sequence of nested bounded closed sets $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ such that*

$$\bigcap_{n \in \mathbb{N}} F_n = \emptyset$$

Proof. Let the metric space (X, d) be defined with

$$X := \mathbb{R} \backslash \{0\}$$

and endowed with the Euclidean metric $d : \mathbb{R} \times \mathbb{R}$ defined by:

$$d(x, y) := \sqrt{(x - y)^2}$$

Let $F_1 \subset X$ be defined for:

$$F_1 := B_{r_1}(0) = \{x \in X \;:\; d(0, x) \leq r_1\}$$

where r_1 is initialized arbitrarily large.
For $n = 2, 3, \ldots,$ $F_n \supset \cdots \supset F_2 \subset F_1 \subset X$ is defined by:

$$F_n := B_{r_n}(0) = \{x \in X \;:\; d(0, x) \leq r_n\}$$

where $r_{n+1} = \frac{r_n}{1+r_n}$.
Finally, the solution is very obvious that the origin is the only limit point of the intersection.

$$\lim_{N \to \infty} \cap_{n < N} F_n = \{0\}$$

However, within this particular metric space, where $X = \mathbb{R} \backslash \{0\}$, this limit is not within the sets themselves. Therefore,

$$\cap_{n \in \mathbb{N}} F_n = 0$$

□

Problem 5

Show that there exists a unique continuous function, $f(x)$ on the interval $[0,1]$, satisfying the equation

$$f(x) = \int_0^1 \sin\left(x^2 + y^2\right) f(y)\mathrm{d}y$$

Theorem 8. *There exists a unique continuous function $f : [0,1] \to \mathbb{R}$ that satisfies the following equation*

$$f(x) = \int_0^1 \sin\left(x^2 + y^2\right) f(y)\mathrm{d}y \tag{1}$$

Proof. I'll prove that the is only a single solution by essentially treating the integral statement as a set of systems of equations (but with an operator) and then demonstrate that the solution function is unique as a contradiction would occur otherwise.
Let T be a functional mapping of the $f(y)$ to $f(x)$, $T : \mathbb{C}([0,1]) \to \mathbb{C}([0,1])$, defined as:

$$T(f(y)) := \int_0^1 \sin\left(x^2 + y^2\right) f(y)\,\mathrm{d}y$$

Now we take $T(f(x)) = f(x)$ and make the claim $f(x)$ is not unique. This would mean that $T(g(x)) = g(x)$ is another solution. If there are multiple solutions, then the following would be true:

$$\begin{aligned} T(f(x)) - f(x) &= T(g(x)) - g(x) \\ \int_0^1 \sin\left(x^2 + y^2\right) f(y)\,\mathrm{d}y - f(x) &= \int_0^1 \sin\left(x^2 + y^2\right) g(y)\,\mathrm{d}y - g(x) \\ f(x) - g(x) &= \int_0^1 \sin\left(x^2 + y^2\right) f(y)\,\mathrm{d}y - \int_0^1 \sin\left(x^2 + y^2\right) g(y)\,\mathrm{d}y \\ f(x) - g(x) &= \int_0^1 \sin\left(x^2 + y^2\right) (f(y) - g(y))\,\mathrm{d}y \end{aligned}$$

Since $|\sin x| \leq 1 \implies \int_0^1 \sin(x)\,\mathrm{d}x < 1$,

$$< \int_0^1 (f(y) - g(y))\,\mathrm{d}y \leq$$

Since we can look at the integral over the region as less then the maximum value times the width:

$$\begin{aligned} &\leq (1-0) \sup_{y\in[0,1]} (f(y) - g(y)) \\ f(x) - g(x) &< \sup_{y\in[0,1]} (f(y) - g(y)) \end{aligned}$$

Which is not possible, leading to the claim that two solutions exist to be false. $\square$

Problem 6

Let V be a complete metric space without isolated points. Show that V is uncountable ($|V| > |\mathbb{N}|$).

Definition 12. *A metric space (X, d) is considered* Complete *if every Cauchy sequence of points in X has a limit within X.*

- *A sequence $x_1, x_2, \ldots$ in metric space (X, d) is considered* Cauchy *if*

$$\forall_{r>0} \exists_N \ : \ \forall_{m,n>N} d(x_m, x_n) < r$$

- *x is the* limit *of sequence (x_n), $\lim_{n\to\infty} x_n$, if*

$$\forall_{\epsilon>0} \exists_{N\in\mathbb{N}} \ : \ \forall_{n\geq N} |x_n - x| < \epsilon$$

Definition 13. *A point within metric space (X, d) is considered an* isolated point *of set $A \subset X$ in which no other points are within the neighborhood of x. i.e.*

$$\exists_{\epsilon>0} \ : \ \forall_{y\in X} \ : \ d(x, y) < \epsilon y \implies \notin A$$

- *A complete set A that contains no isolated points is called* dense-in-itself.

Definition 14. *A* one to one correspondance *is also known as a bijective function that maps $\mathbb{N} \to X$.*

- *A function $f : \mathbb{N} \to A$ is said to be* surjective *if*

$$\exists_{f:\mathbb{N}\xrightarrow{S}A} \iff \exists_{f:\mathbb{N}\to A} \ : \ \forall_{x\in\mathbb{N}} \exists f(x) \in A$$

- *A function $f : \mathbb{N} \to A$ is said to be* injective *if*

$$\exists_{f:\mathbb{N}\xrightarrow{I}A} \iff \exists_{f:\mathbb{N}\to A} \ : \ \forall_{f(x)\in A} \exists_{x\in\mathbb{N}}$$

- *A function $f : A \to B$ is said to be* bijective *if f is both surjective and injective. i.e.*

$$\exists_{f:A\xrightarrow{B}B} \iff \exists_{f:A\to B} \ : \ \big(\forall x \in A \exists_{f(x)\in B}\big) \wedge \big(\forall y \in B \exists_{f^{-1}(y)\in A}\big)$$

Definition 15. *The* Cardinality *of set A, denoted as $|A|$, is the number of unique elements contained within A.*

- *A set A is considered* Countable *if $|A| \leq |\mathbb{N}|$. This is also said to be true if a surjective function exists mapping $\mathbb{N}$ to A.*
- *Set A and B within metric space (X, d) are said be of the same cardinality, $|A| = |B|$, if there exists a bijective mapping between A and B, $f : A \xrightarrow{B} B$.*
- *If A is an infinite set, then A is* Countably Infinite, *$|A| = \aleph_0 = |\mathbb{N}|$, if there exists a one to one correspondence from $\mathbb{N}$ to A.*
- *For A A set A is considered* uncountable *if it is not countable. i.e. $|A| > |\mathbb{N}|$. This is also said to be true if an injective function exists mapping $\mathbb{N}$ to A, but that no surjective mappings exist.*

Theorem 9. *A complete metric space, (V, d), that contains no isolated point is uncountable.*

Proof. From Definition 12, we have that all cauchy sequences in the complete metric space (X, d) must have a limit in X.
From Definition 15, it is known that within all countable sets there exists a one-to-one correspondence between $\mathbb{N}$ and the set A.
For A to be uncountable, an injective function mapping $\mathbb{N}$ to A, there exists, $f : \mathbb{N} \xrightarrow{I} A$.
From Definition 14, this means that $\forall_{f(x)\in A} \exists_{x\in\mathbb{N}}$, however, since $\mathbb{N}$ is not a complete set, it is not possible for a one-to-one correspondence to exist. Therefore, the set is not countable and therefore uncountable. □