MATH 5301 Elementary Analysis - Homework 4

Jonas Wagner

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Problem 1

Use the axioms of the ordered field, prove the following:

a) $(a > c) \wedge (b > d) \implies a + b > c + d$

$$(a > c) \wedge (b > d) \implies (a + b) > (c + d)$$

From (O3):

$$\begin{aligned} (a > c) &\implies ((a + b) \geq (b + c)) \wedge ((a + d) \geq (c + d)) \\ (b > d) &\implies ((a + b) \geq (a + d)) \wedge ((b + c) \geq (c + d)) \end{aligned}$$

From (02):

$$((a + b) \geq (b + c)) \wedge ((b + c) \geq (c + d)) \implies (a + b) > (c + d)$$

b) $(a > c > 0) \wedge (b > d > 0) \implies ab > cd > 0$

$$(a > c > 0) \wedge (b > d > 0) \implies ab > cd > 0$$

From (O4):

$$\begin{aligned} (a > c > 0) \wedge (b > 0) &\implies ab > bc > 0 \\ (b > d > 0) \wedge (c > 0) &\implies bc > cd > 0 \end{aligned}$$

From (O2):

$$(ab > bc > 0) \wedge (bc > cd > 0) \implies ab > cd > 0$$

c) $a > b > 0 \implies \frac{1}{a} < \frac{1}{b}$

$$a > b > 0 \implies \frac{1}{b} < \frac{1}{b}$$

From

$$a > 0 \implies a^{-1} > 0$$
$$b > 0 \implies b^{-1} > 0$$

From (O4):

$$(a > b > 0) \wedge (a^{-1} > 0) \implies aa^{-1} = 1 > ba^{-1} = \frac{b}{a} > 0$$
$$(a > b > 0) \wedge (b^{-1} > 0) \implies ab^{-1} = \frac{a}{b} > bb^{-1} = 1 > 0$$
$$(\frac{a}{b} > 1 > 0) \wedge (a^{-1} > 0) \implies \frac{a}{b}a^{-1} = \frac{1}{b} > (1)(a^{-1}) = \frac{1}{a} > 0$$

Therefore,

$$\frac{1}{a} < \frac{1}{b}$$

d) Let,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & X < 0 \end{cases}$$

prove,

$$\forall a, b \implies |a - b| \geq ||a| - |b||$$

$$\forall a, b \implies |a - b| \geq ||a| - |b||$$

When $a > b > 0$ (or $b > a > 0$),

$$\begin{aligned} |a - b| &= a - b \\ |a| &= a \\ |b| &= b \\ ||a| - |b|| &= a - b \\ |a - b| &= a - b = ||a| - |b|| \end{aligned}$$

The same is true for $0 < a < b$ and $0 < b < a$ by similar arguments.
For $a > 0 > b$,

$$\begin{aligned} |a| &= a \\ |b| &= -b \\ |a - b| &= |a| + |b| \\ |a| - |b| &= a - (-b) = a + b \\ ||a| - |b|| &= \begin{cases} |a| - |b| & |a| > |b| \\ |b| - |a| & |a| < |b| \end{cases} \end{aligned}$$

From (03):

$$\begin{aligned} |a - b| &= |a| + |b| \geq |a| - |b| \\ |a - b| &= |a| + |b| \geq |b| - |a| \\ \therefore\ |a - b| &\geq ||a| - |b|| \end{aligned}$$

Therefore $\forall a, b$,

$$|a - b| \geq ||a| - |b||$$

Problem 2

Determine which of the axioms satisfied by the set of real numbers are not satisfied by the following set:

a) Set $\mathbb{Q}$ of all rational numbers.

Set $\mathbb{Q}$ of rational numbers can be an ordered field, $\langle \mathbb{Q}, +, 0, \cdots, 1 \rangle$, but lacks (C) completeness:

$$\forall A \subset \mathbb{Q} \ \nexists c \in \mathbb{Q} : c = \sup A$$

b) Set $\mathbb{Q}(\sqrt{2})$ of all numbers of form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$

Set $\mathbb{Q} := \left\{ a + b\sqrt{2} : a, b \in \mathbb{Q} \right\}$ can be an ordered field, $\langle \mathbb{Q}(\sqrt{2}), +, 0, \cdots, 1 \rangle$, but lacks completeness (C):

$$\forall A \subset \mathbb{Q}(\sqrt{2}) \ \nexists c \in \mathbb{Q} : c = \sup A$$

c) Set $\mathbb{C}$ of all pairs of real numbers (a, b) with addition $(a, b) + (c, d) = (a + c, b + d)$, multiplication $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$, and ordered relation $(a, b) < (c, d) \iff (b \leq d) \wedge ((b = d \vee a < c))$.

Set $\mathbb{C} := \{(a, b) : a, b \in \mathbb{R}\}$ can satisfy the field conditions, $\langle \mathbb{C}, +, 0, \cdots, 1 \rangle$, but it is not ordered because it does not satisfy (O1).

Problem 3

Using the method of mathematical induction, prove the following statements: ($n \in \mathbb{N}$)

a) Bernoulli inequality: $\forall n \in \mathbb{N},\ \forall x > -1,\ (1+x)^n \geq 1 + nx$

Theorem 1. $\forall n \in \mathbb{N},\ \forall x > -1,$

$$(1+x)^n \geq 1 + nx$$

Proof. Proof by induction:
For $n = 1$,

$$\begin{aligned} (1+x)^n &\geq 1 + nx \\ (1+x)^1 &\geq 1 + (1)x \\ 1 + x &\geq 1 + x \end{aligned}$$

For $n > 1$,

$$\begin{aligned} (1+x)^n &\geq 1 + nx \\ (1+x)^n(1+x) &\geq (1+nx)(1+x) \\ (1+x)^{n+1} &\geq (1 + x + nx + nx^2) \\ &\geq 1 + (n+1)x + nx^2 \end{aligned}$$

Since $n \geq 2 \implies nx^2 > 0$

$$1 + (n+1)x + nx^2 \geq 1 + (n+1)x$$

From (O2):

$$(1+x)^{n+1} \geq 1 + (n+1)x$$

Therefore $\forall n > 1$,

$$(1+x)^n \geq 1 + nx \implies (1+x)^{n+1} \geq 1 + (n+1)x$$

Therefore $\forall n \in \mathbb{N},\ \forall x > -1,$

$$(1+x)^n \geq 1 + nx$$

$\square$

b) For $n \in \mathbb{N}$, $\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$

Theorem 2. *For $n \in \mathbb{N}$,*

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

Proof. Proof by induction: For $n = 1$,

$$\begin{aligned} \frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} &= 2 - \frac{n+2}{2^n} \\ \frac{1}{2} = 2 - \frac{1+2}{2^1} &= 2 - \frac{3}{2} \\ \frac{1}{2} &= \frac{1}{2} \end{aligned}$$

For $n > 1$,

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n} \implies \frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^{n+1}} = 2 - \frac{n+2}{2^{n+1}}$$

$$\begin{aligned} \frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} &= 2 - \frac{n+2}{2^n} \\ \frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}} &= 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} \\ &= 2 - \frac{n+2}{2^n}\frac{2}{2} + \frac{n+1}{2^{n+1}} \\ &= 2 - \frac{2(n+2)}{2^{n+1}} + \frac{n+1}{2^{n+1}} \\ &= 2 + \frac{n+1-2(n+2)}{2^{n+1}} \\ &= 2 + \frac{n+1-2n-2}{2^{n+1}} \\ &= 2 + \frac{-n-1}{2^{n+1}} \\ &= 2 + \frac{-(n+1)-2}{2^{n+1}} \\ &= 2 - \frac{(n+1)+2}{2^{n+1}} \end{aligned}$$

Therefore $\forall n > 1$,

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n} \implies \frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}} = 2 - \frac{(n+1)+2}{2^{n+1}}$$

Therefore For $n \in \mathbb{N}$,

$$\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

□

c) For $q \in \mathbb{R}, n \in \mathbb{N}$, $(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) = \frac{1-q^{2^{n+1}}}{1-q}$

Theorem 3. *For $q \in \mathbb{R}, n \in \mathbb{N}$,*

$$(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) = \frac{1-q^{2^{n+1}}}{1-q}$$

Proof. Proof by induction:
For $n = 1$,

$$\begin{aligned}
(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) &= \frac{1-q^{2^{n+1}}}{1-q} \\
(1+q)(1+q^{2^1}) &= \frac{1-q^{2^{1+1}}}{1-q} \\
(1+q^2+q+q^3) &= \frac{1-q^4}{1-q} \\
(1+q+q^2+q^3)(1-q) &= \frac{1-q^4}{1-q}(1-q) \\
1+q+q^2+q^3-q-q^2-q^3-q^4 &= 1-q^4 \\
1-q^4 &= 1-q^4
\end{aligned}$$

For $n > 1$

$$\begin{aligned}
(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) &= \frac{1-q^{2^{n+1}}}{1-q} \\
(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n})(1+q^{2^{n+1}}) &= \frac{1-q^{2^{n+1}}}{1-q}(1+q^{2^{n+1}}) \\
&= \frac{\left(1-q^{2^{n+1}}\right)\left(1+q^{2^{n+1}}\right)}{1-q} \\
&= \frac{1-q^{2^{n+1}}+q^{2^{n+1}}+\left(-q^{2^{n+1}}\right)\left(q^{2^{n+1}}\right)}{1-q} \\
&= \frac{1-q^{2^{n+1}+2^{n+1}})}{1-q} \\
&= \frac{1-q^{2\left(2^{n+1}\right)}}{1-q} \\
(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n})(1+q^{2^{n+1}}) &= \frac{1-q^{2^{n+2}}}{1-q}
\end{aligned}$$

Therefore $\forall n > 1$,

$$(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) = \frac{1-q^{2^{n+1}}}{1-q} \implies (1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n})(1+q^{2^{n+1}}) = \frac{1-q^{2^{n+2}}}{1-q}$$

Therefore $\forall n \in \mathbb{N}$,

$$(1+q)(1+q^2)(1+q^4)\cdots(1+q^{2^n}) = \frac{1-q^{2^{n+1}}}{1-q}$$

$\square$

d) For $n \in \mathbb{N}$, $1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1)$

Theorem 4. *For $n \in \mathbb{N}$,*

$$1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1)$$

Proof. Proof by induction:
For $n = 1$,

$$\begin{aligned}
1^3 + 3^3 + \cdots + (2n+1)^3 &= (n+1)^2(2n^2+4n+1) \\
1^3 + (2(1)+1)^3 &= ((1)+1)^2(2(1)^2+4(1)+1) \\
1^3 + 3^3 &= (2)^2(2(1)+4+1) \\
1 + 27 &= (4)(2+4+1) \\
28 &= (4)(7) \\
28 &= (4)(7) \\
28 &= 28
\end{aligned}$$

For $n > 1$,

$$\begin{aligned}
1^3 + 3^3 + \cdots + (2n+1)^3 &= (n+1)^2(2n^2+4n+1) \\
1^3 + 3^3 + \cdots + (2n+1)^3 + (2(n+1)+1)^3 &= (n+1)^2(2n^2+4n+1) + (2(n+1)+1)^3 \\
&= (n+1)^2(2n^2+4n+1) + (2n+3)^3
\end{aligned}$$

$$\begin{aligned}
&= (n+1)(n+1)(2n^2+4n+1) + (2n+3)(2n+3)(2n+3) \\
&= (n^2+2n+1)(2n^2+4n+1) + 27 + 54n + 36n^2 + 8n^3 \\
&= 2n^4 + 8n^3 + 11n^2 + 6n + 1 + 8n^3 + 36n^2 + 54n + 27 \\
&= 2n^4 + 16n^3 + 47n^2 + 60n + 28 \\
&= (n+2)^2(2n^2+8n+7) \\
&= ((n+1)+1)^2(2(n+1)^2 - 4n - 2 + 4(n+1) + 4n - 4 + 7) \\
&= ((n+1)+1)^2(2(n+1)^2 + 4(n+1) + 1)
\end{aligned}$$

Therefore $\forall n > 1$,

$$1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1) \implies$$
$$\implies 1^3 + 3^3 + \cdots + (2n+1)^3 + (2(n+1)+1)^3 = ((n+1)+1)^2(2(n+1)^2 + 4(n+1) + 1)$$

Therefore $\forall n \in \mathbb{N}$,

$$1^3 + 3^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1)$$

$\square$

e) For $n, k \in \mathbb{N}$**,** $\sum_{k=0}^{n} (-1)^k \dfrac{n!}{k!(n-k)!} = 0, \sum_{k=0}^{n} \dfrac{n!}{k!(n-k)!} = 2^n$

Definition 1. *The factorial of a number, $n!$, is defined as*

$$n! := (1)(2)(3)\cdots(n-1)(n)$$

Definition 2. *The combination of two numbers, $\binom{n}{k}$, is defined as*

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

i) $\sum_{k=0}^{n} (-1)^k \dfrac{n!}{k!(n-k)!} = 0$

Theorem 5. *For $n, k \in \mathbb{N}$,*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof. For $n = 1$,

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k \binom{n}{k} &= 0 \\
\sum_{k=0}^{1} (-1)^k \binom{1}{k} &= (-1)^0 \binom{1}{0} + (-1)^1 \binom{1}{1} \\
&= (1)(1) + (-1)(1) \\
&= 0
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{1} (-1)^k \binom{1}{k} = 0$$

For $n > 1$,

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k \binom{n}{k} &= 0 \\
\sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} &= 0 \\
(-1)^0 \frac{n!}{0!(n-0)!} + (-1)^1 \frac{n!}{1!(n-1)!} + \cdots + (-1)^n \frac{n!}{n!(n-n)!} &= 0
\end{aligned}$$

$$(1)\frac{n!}{0!n!} + (-1)\frac{n!}{1!(n-1)!} + (1)\frac{n!}{(2!(n-2)!} + \cdots + (-1)^{n-2}\frac{n!}{(n-2)!2!} + (-1)^{n-1}\frac{n!}{(n-1)!1!} + (-1)^n \frac{n!}{n!0!}$$

$$(1)\frac{n!}{n!} + (-1)\frac{n!}{(n-1)!} + (1)\frac{n!}{(n-2)!2} + \cdots + (-1)^{n-2}\frac{n!}{(n-2)!2} + (-1)^{n-1}\frac{n!}{(n-1)!} + (-1)^n \frac{n!}{n!} = 0$$

multiply by $\frac{(n+1)}{(n+1)}$ and add another add/subtract set... □

ii) $\sum_{k=0}^{n} \dfrac{n!}{k!(n-k)!} = 2^n$

Theorem 6. *For $n, k \in \mathbb{N}$,*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof. By induction:
For $n = 1$,

$$test$$

$\square$

Problem 4

Show that $\forall n \in \mathbb{N}, n \geq 2$,

a) $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

Theorem 7. *For $n \in \mathbb{N}$ and $n \geq 2$,* $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$

Proof. By Induction:
For $n = 2$,

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} &> \sqrt{n} \\
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} &> \sqrt{2} \\
1 + \frac{1}{2} &> \sqrt{2} \\
\frac{3}{2} &> \sqrt{2} \\
\left(\frac{3}{2}\right)^2 &> \left(\sqrt{2}\right)^2 \\
\frac{9}{4} &> 2 \\
9 &> (4)(2) \\
9 &> 8
\end{aligned}$$

Therefore, for $n = 2$,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

For $n > 2$,

$$\begin{aligned}
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} &> \sqrt{n} \\
\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} &> \sqrt{n} + \frac{1}{\sqrt{n+1}} \\
\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}\right)\sqrt{n+1} &> \left(\sqrt{n} + \frac{1}{\sqrt{n+1}}\right)\sqrt{n+1} \\
\frac{\sqrt{n+1}}{\sqrt{1}} + \frac{\sqrt{n+1}}{\sqrt{2}} + \cdots + \frac{\sqrt{n+1}}{\sqrt{n}} + \frac{\sqrt{n+1}}{\sqrt{n+1}} &> \sqrt{n}\sqrt{n+1} + 1
\end{aligned}$$

$\square$

b) $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$

Theorem 8. $\forall n \in \mathbb{N}, n \geq 2$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$

Proof. For $n = 2$,

$$\begin{aligned}
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} &> 1 \\
\frac{1}{(2)+1} + \frac{1}{(2)+2} + \cdots + \frac{1}{3(2)+1} &> 1 \\
\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{7} &> 1 \\
\sum_{k=3}^{7} \frac{1}{k} &> 1 \\
\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} &> 1 \\
\frac{140}{420} + \frac{105}{240} + \frac{84}{240} + \frac{70}{240} + \frac{60}{420} &> 1 \\
\frac{459}{420} &> 1 \\
459 &> 420
\end{aligned}$$

Therefore for $n = 2$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$

For $n > 2$,

$$\begin{aligned}
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} &> 1 \\
\sum_{k=n+1}^{3n+1} \frac{1}{k} &> 1 \\
\sum_{k=n+1}^{3n+1} \frac{1}{k} + \sum_{k=3n+2}^{3(n+1)+1} \frac{1}{k} &> 1 + \sum_{k=3n+2}^{3(n+1)+1} \frac{1}{k} \\
\frac{1}{n+1} + \sum_{k=(n+1)+1}^{3(n+1)+1} \frac{1}{k} &> 1 + \sum_{k=3n+2}^{3(n+1)+1} \frac{1}{k}
\end{aligned}$$

Since
$\frac{1}{n+1} > \sum_{k=3n+2}^{3(n+1)+1} \frac{1}{k}$,

□

c)

$$\left(\frac{n+1}{2}\right)^n > n!$$

d)

$$2^{2^n} - 6 \vdots 10$$

Problem 5

a) Show that $\sqrt{2} \notin \mathbb{Q}$

Definition 3. $\sqrt{2} := x > 0 : x^2 = 2$

Theorem 9. $\sqrt{2} \notin \mathbb{Q}$

Proof. Assume $\sqrt{2} \in \mathbb{Q}$,

$$\sqrt{2} \in \mathbb{Q} \implies \exists m, n \in \mathbb{N} : \frac{m}{n} = \sqrt{2}$$

Also assume that m, n are coprime. (i.e) $\gcd(m, n) = 1$
Let $m = \sqrt{2}n$,

$$m = \sqrt{2}n \implies m^2 = 2n^2 \implies m^2 \vdots 2 \implies m \vdots 2$$

$$m \vdots 2 \implies \exists k \in \mathbb{N} : m = 2k \implies m^2 = (2k)^2 = 4k^2$$

$$4k^2 = 2n^2 \implies 2k^2 = n^2 \implies n^2 \vdots 2 \implies n \vdots 2$$

This is false becouse with $\gcd(m, n) = 1$, m and n cannot both be even. ☐

b) Show that $\forall a, b \in \mathbb{Q}, a < b \implies \exists x \in \mathbb{R} \backslash \mathbb{Q} : a < x < b$

Theorem 10. $\forall a, b \in \mathbb{Q}, a < b \implies \exists x \in \mathbb{R} \backslash \mathbb{Q} : a < x < b$

Proof. Since $\mathbb{Q}$ lacks completeness, some irrational element $x \in \mathbb{R} \backslash \mathbb{Q}$ exists between all two rational numbers $a, b \in \mathbb{Q} : a \neq b$. Therefore, $a < b \implies \exists x : a < x < b$. ☐

c) Show that $\forall a, b \in \mathbb{R} \backslash \mathbb{Q}, a < b \implies \exists x \in Q : a < x < b$

Theorem 11. $\forall a, b \in \mathbb{R} \backslash \mathbb{Q}, a < b \implies \exists x \in Q : a < x < b$

Proof. Since the set of all irrational numbers, $\mathbb{R} \backslash \mathbb{Q}$, lacks completeness, some rational elements $x \in \mathbb{Q}$ must exists between each irrational numbers $a, b \in \mathbb{R} \backslash \mathbb{Q}$. Therefore $a < b \implies \exists x : a < x < b$. ☐

Problem 6

a)

look at original doc....