MATH 5301 Elementary Analysis - Final Exam

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Problem 1

For each $n \in \mathbb{N}$ define the set

$$Q_n := \left\{ \frac{1}{pq} \ : \ 0 < p < q \leq n; \ p + q > n; \ \gcd(p,q) = 1 \right\}$$

Let $f(n)$ be the sum of all elements of Q_n.
Find $\inf_n f(n)$.

Definition 1. *Let the set Q_n be defined for all $n \in \mathbb{N}$ as*

$$Q_n := \left\{ \frac{1}{pq} \ : \ 0 < p < q \leq n; \ p + q > n; \ \gcd(p,q) = 1 \right\}$$

Definition 2. *Let $f(n)$ be the sum of all elements within Q_n.*

Definition 3. *A lower bound of subset A in the partially ordered set $(S, \leq)$ is defined by*

$$a \in S \ : \ a \leq x \forall_{x \in A}$$

A lower bound of a is called an infimum of set $A \in (S, \leq)$, denoted as $a = \inf A$, is the greatest lower bound. i.e.

$$\forall_{y \in S : a \leq x \forall_{x \in A}} y \leq a$$

Definition 4. *The Greatest Common Divisor of two nonzero integers $a, b \in \mathbb{Z} \neq 0$, $\gcd(a,b)$, is defined as the largest positive integer, $d \in \mathbb{Z}_+$, so that d is a divisor of both a and b. i.e:*

$$\gcd(a,b) := d \in \mathbb{Z}_+ \ : \ (a \vdots d) \wedge (b \vdots d) \wedge (\forall_{x \in \mathbb{Z}_+ \ : \ a,b \vdots x} d \geq x)$$

Additionally, a and b are considered coprime if $\gcd(a,b) = 1$.

Assumption 1. *For this problem it is assumed that $\gcd$ is only defined within $\mathbb{Z}_+$, although I believe this can also be expanded to other less-strict ordered sets in the same way.*

Assumption 2. *It is assumed that the sum of all elements in the empty set is 0, i.e. $\sum_i \emptyset = 0$.*

Theorem 1.

$$\inf_{n\in\mathbb{N}} f(n) = 0$$

Proof. Proof by induction.
For $n = 1$, $\neg\exists_{p,q\in\mathbb{Z}\,:\,0<p<q\leq 1}$ meaning that $Q_1 = \emptyset$.
This implies that $f(1) = \sum_i \emptyset = 0$ and that $f(1) \geq 0$.
For $n = 2$,

$$(p, q) \in \{(p, q) \;:\; 0 < p < q \leq 2;\; p + q > n;\; \gcd(p, q) = 1\} = \{(1, 2)\}$$

The set Q_2 is then defined as

$$Q_2 = \left\{\frac{1}{pq} \;:\; (p, q) \in \{(1, 2)\}\right\} = \left\{\frac{1}{(1)(2)}\right\} = \left\{\frac{1}{2}\right\}$$

Therefore,

$$f(2) = \sum_i \left\{\frac{1}{2}\right\} = \frac{1}{2}$$

It is clear that $f(2) = \frac{1}{2} \geq 0$.
For $n = 3$,

$$(p, q) \in \{(p, q) \;:\; 0 < p < q \leq 3;\; p + q > n;\; \gcd(p, q) = 1\} = \{(1, 3), (2, 3)\}$$

The set Q_3 is then defined as

$$Q_3 = \left\{\frac{1}{pq} \;:\; (p, q) \in \{(1, 3), (2, 3)\}\right\} = \left\{\frac{1}{(1)(3)}, \frac{1}{(2)(3)}\right\} = \left\{\frac{1}{3}, \frac{1}{6}\right\}$$

Therefore,

$$f(3) = \sum_i \left\{\frac{1}{3}, \frac{1}{6}\right\} = \frac{1}{3} + \frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

It is clear that $f(3) = \frac{1}{2} \geq 0$.
For $n = 4$,

$$(p, q) \in \{(p, q) \;:\; 0 < p < q \leq 4;\; p + q > n;\; \gcd(p, q) = 1\} = \{(1, 4), (2, 3), (3, 4)\}$$

The set Q_4 is then defined as

$$Q_4 = \left\{\frac{1}{pq} \;:\; (p, q) \in \{(2, 3), (3, 4)\}\right\} = \left\{\frac{1}{(1)(4)}, \frac{1}{(2)(3)}, \frac{1}{(3)(4)}\right\} = \left\{\frac{1}{4}, \frac{1}{6}, \frac{1}{12}\right\}$$

Therefore,

$$f(4) = \sum_i \left\{\frac{1}{6}, \frac{1}{12}\right\} = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{3+2+1}{12} = \frac{6}{12} = \frac{1}{2}$$

It is clear that $f(4) = \frac{1}{2} \geq 0$.
For an arbitrary $n \in \mathbb{N}$,

$$\begin{aligned}(p, q) \in \{(p, q) \;:\; 0 < p < q \leq n;\; p + q > n;\; \gcd(p, q) = 1\} = \\ = \{(1, n), (2, n - \star), (3, n - \star) \ldots, (n - 2, n - 1), (n - 1, n)\}\end{aligned}$$

$$\begin{aligned}Q_n &= \left\{\frac{1}{pq} \;:\; (p, q) \in \{(1, n), (2, n - \star), \ldots, (n - 2, n - 1), (n - 1, n)\}\right\} \\ &= \left\{\frac{1}{(1)(n)}, \frac{1}{(2)(n-1)}, \ldots, \frac{1}{(n-2)(n-1)}, \frac{1}{(n-1)(n)}\right\} \\ &= \left\{\frac{1}{n}, \frac{1}{2(n-\star)}, \ldots, \frac{1}{(n-2)(n-1)}, \frac{1}{n(n-1)}\right\}\end{aligned}$$

where $\star$ is dependent for on divisibility properties between n and 2, 3, 4, etc. It is important to note that each increase of n will cause every term to decrease in magnitude individually but additional elements are added that result to adding up to $\frac{1}{2}$ again.
However, eventually this will reach a point where a lack of prime numbers in a region makes it so that the only coprime numbers satisfying the conditions are adjacent to one another, which leads to the following:

$$
\begin{aligned}
f(n) = \sum_i Q_n &= \frac{1}{n} + \cdots + \frac{1}{(\frac{n}{2})(\frac{n}{2}+1)} + \cdots + \frac{1}{n(n-1)} \\
f(n+1) &= \left(\sum_i Q_n\right)\left(\frac{n!}{(n+1)!}\right) + \frac{1}{(n+1)} \\
&= \frac{1}{n}\frac{n!}{(n+1)!} + \cdots + \frac{1}{(\frac{n}{2})(\frac{n}{2}+1)}\frac{n!}{(n+1)!} + \cdots + \frac{1}{n(n-1)}\frac{n!}{(n+1)!} + \frac{1}{n+1} \\
&= \frac{n!}{n(n+1)n!} + \cdots + \frac{n!}{\frac{n}{2}(\frac{n}{2}-1)(n+1)n!} + \cdots + \frac{n!}{n(n-1)(n+1)n!} + \frac{1}{n+1} \\
&= \sum_i Q_{n+1} = \frac{1}{n+1} + \cdots + \frac{1}{(\frac{n+1}{2})(\frac{n+1}{2}+1)} + \cdots + \frac{1}{n(n+1)}
\end{aligned}
$$

essentially every (p, q) becomes $(q, q+1)$ and the new $\frac{1}{(n+1)}$ is added.
Anyway, the point is that $\forall_{n\in\mathbb{N}\,:\,n>1} f(n) \geq \frac{1}{2}$; however, because $f(n)$ is included, $\frac{1}{2} \leq f(n)\forall_{n\in N}$ since $Q_1 = \emptyset \implies f(1) = 0$.
Therefore,

$$\inf_n f(n) = 0$$

$\square$

Problem 2

Let (X, d) be a metric space. Let $B_r(a)$ denote the open ball of radius r centered at a. i.e. Can it happen that $B_{r_1}(a) \subset B_{r_2}(a)$ but $r_1 > r_2$?

Definition 5. *Within the metric space (X, d), the open ball of radius $r \in X$ centered at $a \in X$, denoted as $B_r(a)$, is defined as:*

$$B_r(a) := \{x \in X \ : \ d(a, x) < r\}$$

Assumption 3. *First it will be assumed that (X, d) is a normed vector space in which the triangle inequality holds. i.e.*

$$\forall_{x,y,z \in X} d(x, z) \leq d(x, y) + d(y, z)$$

This can also be denoted as $(X, \|\cdot\|)$ to distinguish between them. It is also assumed that X is complete.

Theorem 2. *For $r_1 > r_2$ then it is not possible for $B_{r_1}(a) \subset B_{r_2}(b)$ within $(X, \|\cdot\|)$:*

Proof. Proof by contradiction.
Let

$$B_{r_1}(a), B_{r_2}(b) \subset X$$

with $0 < r_2 < r_1$ and $a \in B_{r_2}(b)$.
To minimize the amount of the set existing outside of the set, we need to set $a = b$. Next, let c be a point within the punctured open ball $B_{r_2}(b)$. i.e.

$$c \in B_{r_2}(b) \backslash \{b\}$$

c can then be used to construct a point that is contained in $B_{r_2}(b)$ but not in $B_{r_1}(a)$:

$$p + \frac{r_1 + r_2}{2} \frac{ac}{\|ac\|} \in B_{r_1}(a) \backslash B_{r_2}(b)$$

Meaning that there is no possible way for an open ball of greater radius (within a normed metric space). $\square$

Assumption 4. *The previous assumption, Assumption 3, is now relax the metric so that d is not restricted by completeness or*

Theorem 3. *It is possible for $B_{r_1}(a) \subset B_{r_2}(b)$ within (X, d) when $r_1 > r_2$:*

Proof. Proof by example:
Let metric space (X, d) be defined by

$$X := 0 \cup [5, \infty)$$

$$d(x, y) := |x - y|$$

For $r_1 = 4$, $r_2 = 3$,
Let $B_4(0)$ be defined as

$$B_4(0) := \{4x \in X \ : \ d(0, x) < 4\} = \{0\} \cup [2, 4)$$

Let $B_3(2)$ be defined as

$$B_3(2) := \{x \in X \ : \ d(2, x) < 3\} = \{0\} \cup [2, 5)$$

Clearly, $B_3(2) \subset B_4(0)$. Since $r_1 = 4 > r_2 = 3$, this exists as an example that satisfies the conditions. $\square$

Problem 3

Let M be the set of all bounded sequences

$$M = \left\{ \{a_j\}_{j=1}^{\infty} \ : \ |a_j| < \infty \right\}$$

Define $\rho(\{a_n\}, \{b_n\}) = \max_{n \in \mathbb{N}} |a_n - b_n|$

a) Show that (M, ρ) is a metric space.

b) Show that M does not contain a dense