MATH 5301 Elementary Analysis - Homework 8

Jonas Wagner

2021, October 29$^{\text{th}}$

Problem 1

Show that the norms $\|\cdot\|_1$, $\|\cdot\|_p$ for $p > 1$, and $\|\cdot\|_\infty$ are equivalent.

Definition 1. *For $\|\cdot\|_a, \|\cdot\|_b$ on S, $\|\cdot\|_a$ is said to be* stronger *then $\|\cdot\|_b$ if*

$$\forall \{x_n\} \subset S \ : \ x_n \underset{d_a}{\longrightarrow} x \implies x_n \underset{d_b}{\longrightarrow} x$$

Definition 2. *$\|\cdot\|_a$ and $\|\cdot\|_b$ are said to be* equivalent, *$\|\cdot\|_a \sim \|\cdot\|_b$, if $\|\cdot\|_a$ is stronger then $\|\cdot\|_b$ and $\|\cdot\|_b$ is stronger then $\|\cdot\|_a$. This means that*

$$\|\cdot\|_a \sim \|\cdot\|_b \iff \exists \alpha, \beta \in \mathbb{R}_{>0} : \forall_{x \in S} \alpha \|\cdot\|_b \leq \|\cdot\|_a \leq \beta \|x\|_b$$

Definition 3. *The following norms are defined as*

a. $\|\cdot\|_1 := \|x\|_1 = \sum_{i=1}^n |x_i| = |x_1| + |x_2| + \cdots + |x_n|$

b. $\|\cdot\|_2 := \|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} = \left(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2\right)^{1/2}$

c. $\|\cdot\|_p := \|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}, \ p > 1$

d. $\|\cdot\|_\infty := \|x\|_\infty = \max_{i=1}^n |x_i| = \max(|x_1|, |x_2|, \ldots, |x_n|)$

Theorem 1. *The norms* $\|\cdot\|_1, \|\cdot\|_p$, *and* $\|\cdot\|_\infty$ *are equivalent.*

Proof.

Lemma 1. $\|\cdot\|_1 \sim \|\cdot\|_p$

Proof. $\|\cdot\|_1 \sim \|\cdot\|_p$ is true iff

$\forall_x \exists_{\alpha,\beta\in\mathbb{R}_+}$:

$$\alpha\|x\|_p \leq \|x\|_1 \leq \|x\|_p$$
$$\alpha\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} \leq \sum_{i=1}^{n}|x_i| \leq \beta\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}$$

From the Holder's inequality we have

$$\begin{aligned}\|x\|_1 = \sum_{i=1}^{n}|x_i| &= \sum_{i=1}^{n}|x_i|(1)\\ &\leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\left(\sum_{i=1}^{n}|1|^{(1-p)}\right)^{1/(1-p)}\\ &\leq n^{1/(1-p)}\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\end{aligned}$$

So for $0 < \alpha \leq n^{1/(1-p)}$ and $\beta \geq n^{1/(1-p)}$,

$$\begin{aligned}\alpha\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} &\leq \sum_{i=1}^{n}|x_i| &&\leq \beta\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\\ \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} &\leq n^{1/(1-p)}\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} &&\leq n^{1/(1-p)}\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}\end{aligned}$$

Therefore,

$$\|x\|_p \leq \|x\|_1 \leq n^{\frac{1}{1-p}}\|x\|_p$$

which proves $\|\cdot\|_1 \sim \|\cdot\|_p$. □

Lemma 2. $\|\cdot\|_1 \sim \|\cdot\|_\infty$

Proof. $\|\cdot\|_1 \sim \|\cdot\|_\infty$ is true iff

$\forall_x \exists_{\alpha,\beta\in\mathbb{R}_+}$:

$$\alpha\|x\|_\infty \leq \|x\|_1 \leq \beta\|x\|_\infty$$
$$\alpha\max_{i=1}^{n}|x_i| \leq \sum_{i=1}^{n}|x_i| \leq \beta\max_{i=1}^{n}|x_i|$$

Clearly, this is true for when $\alpha \in (0, 1]$. Similarly, when $\beta \geq n$ then $\sum_{i=1}^{n}\max_{i=1}^{n}|x_i|$ and then clearly greater then the $\|x\|_1$; therefore $\|\cdot\|_1 \sim \|\cdot\|_\infty$. □

From, Lemma **??**, Lemma 1, and Lemma 2, it is clear that $\forall_{p>1}$:

$$\|x\|_\infty \leq \|x\|_p \leq \|x\|_1 \leq n^{1/1-p}\|x\|_p \leq n\|x\|_\infty$$

Therefore, $\|\cdot\|_1 \sim \|\cdot\|_p \sim \|\cdot\|_\infty$ $(\forall_{p>1})$. □

Problem 2

Let $(S, \|\cdot\|)$ and $(S', \|\cdot\|')$ to be two normed spaces. Show that the following norms on $S \times S'$ are equivalent.

a. $\|(x,y)\|_1 = \|x\| + \|y\|'$

b. $\|(x,y)\|_2 = \sqrt{\|x\|^2 + (\|y\|')^2}$

c. $\|(x,y)\|_p = \left(\|x\|^p + (\|y\|')^p\right)^{1/p}$

d. $\|(x,y)\|_\infty = \max\left\{\|x\| + \|y\|'\right\}$

Theorem 2. *The norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_p$, and $\|\cdot\|_\infty$ are all equivalent on $S \times S'$.*

Proof.

Lemma 3. $\|\cdot\|_1 \sim \|\cdot\|_2$

Proof. $\|\cdot\|_1 \sim \|\cdot\|_2$ is true iff

$$\forall_{(x,y)\in S\times S'} \exists_{\alpha,\beta\in\mathbb{R}_+} :$$

$$\alpha\|(x,y)\|_2 \le \|(x,y)\|_1 \le \|(x,y)\|_2$$
$$\alpha(\|x\|^2 + (\|y\|')^2)^{1/2} \le \|x\| + \|y\|' \le \beta(\|x\|^2 + (\|y\|')^2)^{1/2}$$

From the Holder's inequality we have

$$\begin{aligned}
\|(x,y)\|_1 &= \|x\| + \|y\|' \\
&= \left(\|x\|(1) + \|y\|'(1)\right) \\
&\le \left(\|x\|^2 + (\|y\|')^2\right)^{\frac{1}{2}} \left(1^2 + 1^2\right)^{1-\frac{1}{2}} \\
&= (2)^{\frac{1}{2}} \left(\|x\|^2 + (\|y\|')^2\right)^{\frac{1}{2}} \\
&= \sqrt{2}\sqrt{\|x\|^2 + (\|y\|')^2} \\
\|(x,y)\|_1 &\le \sqrt{2}\|(x,y)\|_2
\end{aligned}$$

Similarily,

$$\begin{aligned}
\|(x,y)\|_2 &= \sqrt{\|x\|^2 + (\|y\|')^2} \\
&= \left(\|x\|\|x\| + \|y\|'\|y\|'\right)^{\frac{1}{2}} \\
&\le \left(\left(\|x\|^2 + (\|y\|')^2\right)^{\frac{1}{2}}\right)^2 \left(\left(\|x\|^2 + (\|y\|')^2\right)^{\frac{1}{2}}\right)^2 \\
&= 2\left(\|x\|^2 + (\|y\|')^2\right)
\end{aligned}$$

So.... this isn't complete... need to do it still... figure out why you can just take 1/epsilon for it to work... □

□

Problem 3

Let X be a vector space and V be a normed space. The function $f : X \to V$ is called bounded if $\exists M \; : \; \forall_{x \in X} \implies \|f(x)\| < M$. Consider the set $\mathcal{B}(X, V)$ of all bounded functions from $X \to V$.

a)

Show that $\mathcal{B}(X, V)$ is a vector space.
do the thing...

b)

Show that the function $\mathcal{B}(X, V) \to \mathbb{R}_+$:

$$\|f\|_\infty := \sup_{x \in X} \|f(x)\|$$

defines a norm on $\mathcal{B}(X, V)$.
Show the things that the norm needs... (mainly triangular inequality)

Problem 4

Let A be a dense set in metric space (S, d), let (V, d_1) be a complete metric space, and $f : A \to Y$ be a uniformly continuous function.

a) **Show that if $\{x_n\}$ is a Cauchy sequence in A then $\{f(x_n)\}$ is a Cauchy sequence in Y.**

b) **Show that there is only one continuous function $g : X \to Y$ so that $g(x) = f(x)$ forall $x \in$.**

Problem 5

Let $(L, \|\cdot\|)$ be a Banach space. Let L_0 be a closed subspace of L. Define the factor-space L/L_0 as $l_1 := L/L_0 = \{x + y \ : \ x \in L, y \in L_0\}$. In other works, L_1 consists of all subsets of L obtained from L_0 by shifting all its elements by some element x.

a) Show that L_1 is a vector space.

b)

Define the function $\|\cdot\| : L_1 \to \mathbb{R}_+$ as $\|x\|_1 = \inf_{x-y \in L_0} \|y\|$. Show that this function defines a norm on the space L_1.

c) Show that L_1 is a Banach space.

Problem 6

Let $C([-1,1])$ be the space of all continuous real-valued functions $f(x)$ with $x \in [-1,1]$. Let $\|f\|_\infty := \sup_{x\in[-1,1]} |f(x)|$. Find the distance from point $p = x^{2021}$ to the space P_{2020} of all polynomials of degree less than or equal to 2020.