MATH5301 Elementary Analysis. Homework 9. Due: 11/05/2021, 11:59 pm

First Name: **Last Name:**

9.1

Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two equivalent norms on $\mathbb{R}^n$.

(a) Prove that if the set A is closed in the a-norm, then it is closed in b-norm.

(b) Prove that if the set A is compact in the a-norm, then it is compact in b-norm.

9.2

Consider the set ℓ^∞ of all real-valued sequences, endowed with the sup-norm: $\|l\|_\infty = \sup\limits_{n\in\mathbb{N}} |l_n|$.

(a) Prove that ℓ^∞ is complete.

(b) Prove that ℓ^∞ is not compact.

9.3

Consider the set $\mathbb{B}([0,1],\mathbb{R})$ of all bounded real-valued functions on the unit interval, endowed with the sup-norm: $\|f\|_\infty = \sup\limits_{x\in[0,1]} |f(x)|$. Denote by $B_1 := \{f \in \mathbb{B} : \|f\|_\infty \leqslant 1\}$ be close unit ball.

(a) Prove B_1 is closed.

(b) Prove that B_1 is bounded.

(c) Prove that B_1 is not compact.

9.4

Let $\{V, \|\cdot\|\}$ be a normed space. Show that the function $f(x) = \|x\| : V \to \mathbb{R}$ is continuous on V.

9.5

Let (X, d_1) and (Y, d_2) are two metric spaces. Assume also that Y is a vector space. Construct an example of two continuous functions $f, g : X \to Y$ such that $f + g$ is discontinuous.

9.6

Construct an example of a sequence $\{f_n\}$ of nowhere continuous functions $[0,1] \to \mathbb{R}$ such that f_n converge in sup-norm to continuous function.