MATH5301 Elementary Analysis. Homework 6. Due: 10/17/2021, 11:59 pm

First Name: **Last Name:**

6.1

(a) Show that for any $x > 0, x \in \mathbb{R}$, $\lim_{n\to\infty} x^{1/n} = 1$.

(b) Show that for any bounded sequence $\{a_n\}$ and any sequence $\{b_n\}$, converging to zero, the sequence $\{a_n b_n\}$ converges to zero.

(c) Find the limit $\lim_{n\to\infty} a_n$ where $a_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n}$. Prove the convergence.

(d) Find the limit $\lim_{n\to\infty} a_n$ where $a_n = 2 + \cfrac{1}{2+\cfrac{1}{\ddots + \cfrac{1}{2}}}$. Prove the convergence.

6.2 True or False?

(a) If f is continuous function $(S_1, d_1) \mapsto (S_2, d_2)$ and $U \subset S_1$ is open then $f(U) \subset S_2$ is also open?

(b) If f is continuous function $(S_1, d_1) \mapsto (S_2, d_2)$ if and only if for any closed set $C \subset S_2$ the set $f^{-1}(C) \subset S_1$ is also closed.

(c) If f and g are continuous functions $(S, d) \mapsto \mathbb{R}$, then $m(x) := \max(f(x), g(x))$ and $n(x) = \min(f(x), g(x))$ are also continuous?

(d) If f is continuous function $(S, d) \mapsto \mathbb{R}$ then for any Cauchy sequence $\{x_n\}$ in S, the sequence $f(x_n)$ is Cauchy sequence in $\mathbb{R}$.

6.3

Prove the following properties of continuous functions:

(a) For any $a, b \in \mathbb{R}$ and for any two continuous functions $f, g : S_1 \mapsto S_2$, it follows

$$(af + bg)(x) := af(x) + bg(x) : S_1 \mapsto S_2$$

is also continuous.

(b) For any continuous $f : S_1 \mapsto S_2$ and for any continuous $h : S_1 \mapsto \mathbb{R}$ the function

$$(hf)(x) := h(x) \cdot f(x) : S_1 \mapsto S_2$$

is also continuous.

(c) If $h(x) \neq 0$ for any $x \in S_1$ then $\dfrac{1}{h(x)}$ is also continuous function from S_1 to $\mathbb{R}$.

6.4

Prove the following statement: If A and B are two closed nonempty disjoint sets in the metric space (S, d) then there exists a continuous function $\chi(x)$ such that $\chi(x) = 0$ for all $x \in A$ and $\chi(x) = 1$ for all $x \in B$.

Hint:

(a) Define the distance from the point x to the set A as

$$\rho_A(x) := \inf_{y \in A} d(x, y)$$

(b) Show that $\rho_A(x) = 0 \Leftrightarrow x \in \bar{A}$

(c) Show that $\rho_A(x)$ is Lipshits with constant 1.

(d) Consider $\chi(x) = \dfrac{\rho_B(x)}{\rho_A(x) + \rho_B(x)}$.

6.5

Which of the following sets in $\mathbb{R}^2$ are compact?

(a) $A = \{(x, y) \mid x^2 - y^2 \leqslant 1\}$

(b) $B = \{(x, y) \mid 0 < x^2 + y^2 \leqslant 1\}$

(c) $C = \{(x, y) \mid x^2 + y^4 \leqslant 1\}$

(d) $D = \{(1, \frac{1}{n}) \mid n \in \mathbb{N}\} \cup (1, 0)$

6.6

Let $A \subset S$ be compact set. Show that

(a) ∂A is compact.

(b) For any closed B, $A \cap B$ is compact.

(c) For any compact C, $A \cup C$ is compact.

(d) Union of infinitely many compact sets may be not compact.