

MATH 5301 Elementary Analysis - Final Exam

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Problem 1

For each $n \in \mathbb{N}$ define the set

$$Q_n := \left\{ \frac{1}{pq} : 0 < p < q \leq n; p + q > n; \gcd(p, q) = 1 \right\}$$

Let $f(n)$ be the sum of all elements of Q_n .
Find $\inf_n f(n)$.

Definition 1. Let the set Q_n be defined for all $n \in \mathbb{N}$ as

$$Q_n := \left\{ \frac{1}{pq} : 0 < p < q \leq n; p + q > n; \gcd(p, q) = 1 \right\}$$

Definition 2. Let $f(n)$ be the sum of all elements within Q_n .

Definition 3. A lower bound of subset A in the partially ordered set (S, \leq) is defined by

$$a \in S : a \leq x \forall x \in A$$

A lower bound of a is called an infimum of set $A \in (S, \leq)$, denoted as $a = \inf A$, is the greatest lower bound.
i.e.

$$\forall y \in S : a \leq x \forall x \in A \implies y \leq a$$

Definition 4. The Greatest Common Divisor of two nonzero integers $a, b \in \mathbb{Z} \neq 0$, $\gcd(a, b)$, is defined as the largest positive integer, $d \in \mathbb{Z}_+$, so that d is a divisor of both a and b . i.e:

$$\gcd(a, b) := d \in \mathbb{Z}_+ : (a : d) \wedge (b : d) \wedge (\forall x \in \mathbb{Z}_+ : a, b : x \implies d \geq x)$$

Additionally, a and b are considered coprime if $\gcd(a, b) = 1$.

Assumption 1. For this problem it is assumed that \gcd is only defined within \mathbb{Z}_+ , although I believe this can also be expanded to other less-strict ordered sets in the same way.

Assumption 2. It is assumed that the sum of all elements in the empty set is 0, i.e. $\sum_i \emptyset = 0$.

Theorem 1.

$$\inf_{n \in \mathbb{N}} f(n) = 0$$

Proof. Proof by induction.

For $n = 1$, $\neg \exists_{p,q \in \mathbb{Z}} : 0 < p < q \leq 1$ meaning that $Q_1 = \emptyset$.

This implies that $f(1) = \sum_i \emptyset = 0$ and that $f(1) \geq 0$.

For $n = 2$,

$$(p, q) \in \{(p, q) : 0 < p < q \leq 2; p + q > n; \gcd(p, q) = 1\} = \{(1, 2)\}$$

The set Q_2 is then defined as

$$Q_2 = \left\{ \frac{1}{pq} : (p, q) \in \{(1, 2)\} \right\} = \left\{ \frac{1}{(1)(2)} \right\} = \left\{ \frac{1}{2} \right\}$$

Therefore,

$$f(2) = \sum_i \left\{ \frac{1}{2} \right\} = \frac{1}{2}$$

It is clear that $f(2) = \frac{1}{2} \geq 0$.

For $n = 3$,

$$(p, q) \in \{(p, q) : 0 < p < q \leq 3; p + q > n; \gcd(p, q) = 1\} = \{(1, 3), (2, 3)\}$$

The set Q_3 is then defined as

$$Q_3 = \left\{ \frac{1}{pq} : (p, q) \in \{(1, 3), (2, 3)\} \right\} = \left\{ \frac{1}{(1)(3)}, \frac{1}{(2)(3)} \right\} = \left\{ \frac{1}{3}, \frac{1}{6} \right\}$$

Therefore,

$$f(3) = \sum_i \left\{ \frac{1}{3}, \frac{1}{6} \right\} = \frac{1}{3} + \frac{1}{6} = \frac{2+1}{6} = \frac{3}{6} = \frac{1}{2}$$

It is clear that $f(3) = \frac{1}{2} \geq 0$.

For $n = 4$,

$$(p, q) \in \{(p, q) : 0 < p < q \leq 4; p + q > n; \gcd(p, q) = 1\} = \{(1, 4), (2, 3), (3, 4)\}$$

The set Q_4 is then defined as

$$Q_4 = \left\{ \frac{1}{pq} : (p, q) \in \{(2, 3), (3, 4)\} \right\} = \left\{ \frac{1}{(1)(4)}, \frac{1}{(2)(3)}, \frac{1}{(3)(4)} \right\} = \left\{ \frac{1}{4}, \frac{1}{6}, \frac{1}{12} \right\}$$

Therefore,

$$f(4) = \sum_i \left\{ \frac{1}{6}, \frac{1}{12} \right\} = \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = \frac{3+2+1}{12} = \frac{6}{12} = \frac{1}{2}$$

It is clear that $f(4) = \frac{1}{2} \geq 0$.

For and arbitrary $n \in \mathbb{N}$,

$$(p, q) \in \{(p, q) : 0 < p < q \leq n; p + q > n; \gcd(p, q) = 1\} \\ = \{(1, n), (2, n - \star), (3, n - \star) \dots, (n - 2, n - 1), (n - 1, n)\}$$

$$Q_n = \left\{ \frac{1}{pq} : (p, q) \in \{(1, n), (2, n - \star), \dots, (n - 2, n - 1), (n - 1, n)\} \right\} \\ = \left\{ \frac{1}{(1)(n)}, \frac{1}{(2)(n - 1)}, \dots, \frac{1}{(n - 2)(n - 1)}, \frac{1}{(n - 1)(n)} \right\} \\ = \left\{ \frac{1}{n}, \frac{1}{2(n - \star)}, \dots, \frac{1}{(n - 2)(n - 1)}, \frac{1}{n(n - 1)} \right\}$$

where \star is dependent for on divisibility properties between n and 2, 3, 4, etc. It is important to note that each increase of n will cause every term to decrease in magnitude individually but additional elements are added that result to adding up to $\frac{1}{2}$ again.

However, eventually this will reach a point where a lack of prime numbers in a region makes it so that the only coprime numbers satisfying the conditions are adjacent to one another, which leads to the following:

$$\begin{aligned}
f(n) &= \sum_i Q_n = \frac{1}{n} + \cdots + \frac{1}{(\frac{n}{2})(\frac{n}{2} + 1)} + \cdots + \frac{1}{n(n-1)} \\
f(n+1) &= \left(\sum_i Q_n \right) \left(\frac{n!}{(n+1)!} \right) + \frac{1}{(n+1)} \\
&= \frac{1}{n} \frac{n!}{(n+1)!} + \cdots + \frac{1}{(\frac{n}{2})(\frac{n}{2} + 1)} \frac{n!}{(n+1)!} + \cdots + \frac{1}{n(n-1)} \frac{n!}{(n+1)!} + \frac{1}{n+1} \\
&= \frac{n!}{n(n+1)n!} + \cdots + \frac{n!}{\frac{n}{2}(\frac{n}{2} - 1)(n+1)n!} + \cdots + \frac{n!}{n(n-1)(n+1)n!} + \frac{1}{n+1} \\
&= \sum_i Q_{n+1} = \frac{1}{n+1} + \cdots + \frac{1}{(\frac{n+1}{2})(\frac{n+1}{2} + 1)} + \cdots + \frac{1}{n(n+1)}
\end{aligned}$$

essentially every (p, q) becomes $(q, q+1)$ and the new $\frac{1}{(n+1)}$ is added.

Anyway, the point is that $\forall_{n \in \mathbb{N}} : n > 1, f(n) \geq \frac{1}{2}$; however, because $f(n)$ is included, $\frac{1}{2} \leq f(n) \forall_{n \in \mathbb{N}}$ since $Q_1 = \emptyset \implies f(1) = 0$.

Therefore,

$$\inf_n f(n) = 0$$

□

Problem 2

Let (X, d) be a metric space. Let $B_r(a)$ denote the open ball of radius r centered at a . i.e. Can it happen that $B_{r_1}(a) \subset B_{r_2}(a)$ but $r_1 > r_2$?

Definition 5. Within the metric space (X, d) , the open ball of radius $r \in X$ centered at $a \in X$, denoted as $B_r(a)$, is defined as:

$$B_r(a) := \{x \in X : d(a, x) < r\}$$

Assumption 3. First it will be assumed that (X, d) is a normed vector space in which the triangle inequality holds. i.e.

$$\forall_{x, y, z \in X} d(x, z) \leq d(x, y) + d(y, z)$$

This can also be denoted as $(X, \|\cdot\|)$ to distinguish between them.

Theorem 2. For $r_1 > r_2$ then it is not possible for $B_{r_1}(a) \subset B_{r_2}(b)$ within $(X, \|\cdot\|)$:

Proof. Proof by contradiction.

Let

$$B_{r_1}(a), B_{r_2}(b) \subset X$$

with $0 < r_2 < r_1$ and $a \in B_{r_2}(b)$.

To minimize the amount of the set existing outside of the set, we need to set $a = b$. Next, let c be a point within the punctured open ball $B_{r_2}(b)$. i.e.

$$c \in B_{r_2}(b) \setminus \{b\}$$

c can then be used to construct a point that is contained in $B_{r_2}(b)$ but not in $B_{r_1}(a)$:

$$p + \frac{r_1 + r_2}{2} \frac{ac}{\|ac\|} \in B_{r_1}(a) \setminus B_{r_2}(b)$$

□

Assumption 4. Now relax the metric so that (X, d) is not restricted by the triangle inequality.