MATH 5301. Final exam.

Full credit is given for 100 points.

Date: 12/08/2021

First name:	
Last name:	
Score:	

1	2	3	4	5	6	$\sum$

Problem 1 (20 pts)**.** For each $n \in \mathbb{N}$ define the set

$$Q_n := \left\{ \frac{1}{pq} \mid 0 < p < q \leqslant n;\ p + q > n;\ \gcd(p, q) = 1 \right\}$$

Let $f(n)$ be the sum of all elements of Q_n.
Find $\inf\limits_{n} f(n)$.

Problem 2 (20 pts)**.** Let (X, d) be a metric space. Let $B_r(a)$ denote the open ball of radius r centered at a. In other words

$$B_r(a) := \{x \in X \mid d(a, x) < r\}$$

Can it happen that $B_{r_1}(a) \subset B_{r_2}(b)$ but $r_1 > r_2$?

Problem 3 (20 pts)**.** Let M be the set of all bounded sequences

$$M = \{\{a_j\}_{j=1}^{\infty} \mid |a_j| < \infty\}$$

Define $\rho(\{a_n\}, \{b_n\}) = \max_{n \in \mathbb{N}} |a_n - b_n|$.

1. Show that (M, ρ) is a metric space.
2. Show that M does not contain dense countable subset.

   (*Hint:* Recall the very first example of uncountable set.)

Problem 4 (20 pts)**.** Does there exist a metric space, containing a sequence of nested bounded closed sets $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ such that

$$\bigcap_{n \in \mathbb{N}} F_n = \emptyset?$$

(*Hint:* If $d(x, y)$ is a usual Euclidean metric on $\mathbb{R}$ one can show that $\dfrac{d(x, y)}{1 + d(x, y)}$ is also a metric. Such metric is often called *bounded* metric.)

Problem 5 (20 pts)**.** Show that there exists unique continuous function $f(x)$ on the interval $[0,1]$, satisfying the equation

$$f(x) = \int_0^1 \sin\left(x^2 + y^2\right) f(y)dy$$

Problem 6 (20 pts)**.** Let V be a complete metric space without isolated points. Show that V is uncountable ($|V| > |\mathbb{N}|$)