MATH 5301 Elementary Analysis - Homework 4

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Problem 1

Let (S_1, d_1) and (S_2, d_2) be two metric spaces. Show that each of the following determines the metric on $S_1 \times S_2$.
Let $x_j \in S_1, y_j \in S_2$:

a) $d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$

Theorem 1. *The metric*

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$$

is a metric on $S_1 \times S_2$.

Proof. A metric $d : S_1 \times S_2 \to \mathbb{R}$ must satisfy (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

i) Non-negativy

$$d((x_1, y_1), (x_2, y_2)) \geq 0$$

Since $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$,

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \geq 0$$

ii) Symmetry

$$d((x_1, y_1), (x_2, y_2)) = d((x_1, y_1), (x_2, y_2))$$

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} = \max\{d_2(y_1, y_2), d_1(x_1, x_2)\} = d((x_2, y_2), (x_1, x_2)$$

iii) Triangle Inequality

$$d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$$

$$\begin{aligned}
d((x_1, y_1), (x_3, y_3)) &= \max\{d_1(x_1, x_3), d_2(y_1, y_3)\} \\
d((x_1, y_1), (x_2, y_2)) &= \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \\
d((x_2, y_2), (x_3, y_3)) &= \max\{d_1(x_2, x_3), d_2(y_2, y_3)\} \\
\max\{d_1(x_1, x_3), d_2(y_1, y_3)\} &\leq \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} + \max\{d_1(x_2, x_3), d_2(y_2, y_3)\} \\
d((x_1, y_1), (x_3, y_3)) &\leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))
\end{aligned}$$

□

b) $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$

Theorem 2. *The metric*

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$$

is a metric on $S_1 \times S_2$.

Proof. A metric $d : S_1 \times S_2 \to \mathbb{R}$ must satisfy (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

i) Non-negativy

$$d((x_1, y_1), (x_2, y_2)) \geq 0$$

Since $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$,

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2) \geq 0$$

ii) Symmetry

$$d((x_1, y_1), (x_2, y_2)) = d((x_1, y_1), (x_2, y_2))$$

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2) = d_1(x_2, x_1) + d_2(y_2, y_1) = d((x_2, y_2), (x_1, x_2)$$

iii) Triangle Inequality

$$d((x_1, y_1), (x_3, y_3)) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))$$

$$\begin{aligned}
d((x_1, y_1), (x_3, y_3)) &= d_1(x_1, x_3) + d_2(y_1, y_3) \\
d((x_1, y_1), (x_2, y_2)) &= d_1(x_1, x_2) + d_2(y_1, y_2) \\
d((x_2, y_2), (x_3, y_3)) &= d_1(x_2, x_3) + d_2(y_2, y_3) \\
d_1(x_1, x_3) + d_2(y_1, y_3) &\leq d_1(x_1, x_2) + d_2(y_1, y_2) + d_1(x_2, x_3) + d_2(y_2, y_3) \\
d((x_1, y_1), (x_3, y_3)) &\leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))
\end{aligned}$$

$\square$

c) $d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2}$

Theorem 3. *The metric*

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2}$$

is a metric on $S_1 \times S_2$.

Proof. A metric $d : S_1 \times S_2 \to \mathbb{R}$ must satisfy (i) non-negativity, (ii) Symmetry, and (iii) Triangle Inequality.

i) Non-negativy

$$d((x_1, y_1), (x_2, y_2)) \geq 0$$

Since $d_1(x_1, x_2) \geq 0$ and $d_2(y_1, y_2) \geq 0$,

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} \geq 0$$

ii) Symmetry

$$d((x_1, y_1), (x_2, y_2)) = d((x_1, y_1), (x_2, y_2))$$

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} = \sqrt{(d_1(x_2, x_1))^2 + (d_2(y_2, y_1))^2} = d((x_2, y_2), (x_1, x_2)$$

iii) Triangle Inequality

$$d((x_1, y_1), (x_3, y_3)) \leq \sqrt{(d_1(x_1, x_3))^2 + (d_2(y_1, y_3))^2}$$

$$\begin{aligned}
d((x_1, y_1), (x_3, y_3)) &= \sqrt{(d_1(x_1, x_3))^2 + (d_2(y_1, y_3))^2} \\
d((x_1, y_1), (x_2, y_2)) &= \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} \\
d((x_2, y_2), (x_3, y_3)) &= \sqrt{(d_1(x_2, x_3))^2 + (d_2(y_2, y_3))^2} \\
\sqrt{(d_1(x_1, x_3))^2 + (d_2(y_1, y_3))^2} &\leq \sqrt{(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2} + \sqrt{(d_1(x_2, x_3))^2 + (d_2(y_2, y_3))^2} \\
d((x_1, y_1), (x_3, y_3)) &\leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3))
\end{aligned}$$

$\square$

Problem 2

a) A set A in the metric space (S, d) is called bounded, if $\exists_{R>0} \wedge \exists x \in S : A \subset B_R(x)$. Prove that if A is unbounded then there exists a sequence $\{x_n\} \subset A$ such that $\forall_{m,n\in\mathbb{N}} \implies d(x_n, x_m) > 1$.

Assumption 1. $m \neq n$, $m > n$

Definition 1. *The open ball set $B_r(x)$ over metric space (S, d) is defined as*

$$B_r(x) := \{y \in S : d(x, y) < r\}$$

Definition 2. *A set A in the metric space (S, d) is called* <u>*bounded*</u>*, if*

$$\exists_{R>0} \wedge \exists_{x\in S} : A \subset B_R(x)$$

Definition 3. *A set A in the metric space (S, d) is called* <u>*unbounded*</u>*, if it is not bounded, (i.e.)*

$$\forall_{R>0} \wedge \forall_{x\in S} : A \not\subset B_R(x)$$

Theorem 4. *If $A \in (S, d)$ is unbounded, then*

$$\exists \{x_n\} \subset A : \forall_{m,n\in\mathbb{N}} \implies d(x_n, x_m) > 1$$

Proof. $A \in (S, d)$ being unbounded means that

$$\forall_{R>0} \wedge \forall x \in S : A \not\subset B_R(x)$$

Since $\forall_{R>0} \wedge \forall_{x\in S} : A \not\subset B_R(x)$ and $B_R(x) := \{y \in S : d(x, y) < R\}$,

$$\forall_{x\in A} \exists_{y\in A} : d(x, y) \geq R$$

Since it is true for any $R > 0$ a sequence $\{x_n\}$ can be constructed with subsequent x_{n+1} so that $d(x_n, x_{n+1}) > R = 1$. $\square$

b) Show that in the normed space $(V, |\cdot|)$ the open unit ball $B_r = \{x \in V : |x| < 1\}$ is a convex set. (i.e) $\forall_{x,y\in B_r}, \forall_{t\in[0,1]} \implies tx + (1 - t)y \in B_r$

Definition 4. *The* <u>*open unit ball*</u> *is defined as:*

$$B_r := \{x \in V : |x| < 1\}$$

Definition 5. *The set A is convex if and only if*

$$\forall_{x,y\in A}, \forall_{t\in[0,1]} \implies tx + (1 - t)y \in A$$

Theorem 5. *The open unit ball set B_r is convex in $(V, |\cdot|)$.*

Proof.

$$\begin{aligned} \forall_{x,y\in B_r}, \forall_{t\in[0,1]} &\implies tx + (1 - t)y \in B_r \\ \forall_{x,y\in V} : (|x| < 1) \wedge (|y| < 1), \forall_{t\in[0,1]} &\implies tx + (1 - t)y \in V : |tx + (1 - t)y| \end{aligned}$$

Since $\forall_{x,y\in V}, \forall_{t\in[0,1]} \implies tx + (1 - t)y \in V$

$$(|x| < 1 \wedge |y| < 1 \implies |tx + (1 - t)y| < 1) \iff tx + (1 - t)y \in B_r$$

Clearly,

$$\forall_{x,y\in V} |x|, |y| < 1, \forall_{t\in[0,1]} tx + (1 - t)y < 1$$

Therefore, the open unit ball set B_r is convex in $(V, |\cdot|)$. $\square$

Problem 3

For $(\mathbb{R}^2 = (x, y), d = \sqrt{x^2 + y^2})$,

a) Show that $D = \{(x, y) : x^2 + y^2 \leq 1\}$ is a closed set.

Definition 6. *The set $A \subset V$ is called open if*

$$\forall_{x \in A} \exists_{\epsilon > 0} : B_\epsilon(x) \subset A$$

Definition 7. *The set $A \subset V$ is called closed if A^c is open.*

Theorem 6. *The set $D = \big\{(x, y) : x^2 + y^2 \leq 1\big\}$ is closed.*

Proof. By definition, D is closed iff D^c is open.
D^c is defined by

$$D^c = \big\{(x, y) : x^2 + y^2 > 1\big\}$$

By definition, D^c is open if

$$\forall_{(x,y) \in D^c} \exists_{\epsilon > 0} : B_\epsilon((x, y)) \subset D^c$$

This means that every element in D^c must have an associated open ball set centered at that element with a positive radius that is fully contained by D^c.
Let $(x_b, y_b) \in B_\epsilon((x, y))$ for $\epsilon > 0$. This means

$$d((x, y), (x_b, y_b)) = \sqrt{x_b^2 + y_b^2} < \epsilon$$

By definition,

$$(x, y) \in D^c \implies x^2 + y^2 > 1$$

and therefore,

$$\sqrt{x^2 + y^2} = d((0, 0), (x, y)) > 1$$

From the triangle inequality, we have

$$\begin{aligned} d((x_b, y_b), (0, 0)) &\leq d((0, 0), (x, y)) + d((x, y), (x_b, y_b)) \\ d((x, y), (x_b, y_b)) &\geq d((x_b, y_b), (0, 0)) - d((0, 0), (x, y)) \\ d((x, y), (x_b, y_b)) = \epsilon &> d((x_b, y_b), (0, 0)) - 1 > 0 \end{aligned}$$

Therefore D^c is open and therefore D is closed. $\square$

b) Find the infinite collection of open sets $\{A_n\}$ so that

$$\left\{A_n : \bigcap_n A_n = \overline{B_1(0)}\right\}$$

This means that the intersection of all sets in $\{A_n\}$ is the closure of the unit ball set.

Definition 8. *The* <u>*interier*</u> *of set A in (S, d) is the union of all open sets contained within A. (i.e.)*

$$int(A) = \{x \in A : \exists_{\epsilon>0} B_\epsilon(x) \subset A\}$$

Definition 9. *The* <u>*closure*</u> *of set A in (S, d) is the intersection of all closed sets containing A, (i.e)*

$$\overline{A} = \{x \in S : \forall_{\epsilon>0} B_\epsilon(x) \cap A \neq \emptyset\}$$

Note: The interior and closures are complementary sets. (i.e.) $\overline{A} = (\text{int}(A))^c$
$B_1(0)$ is defined as

$$B_1(0) := \big\{(x, y) \in \mathbb{R}^2 : d((0, 0), (x, y)) < 1\big\} = \left\{(x, y) : \sqrt{x^2 + y^2} < 1\right\}$$

The closure of $B_1(0)$, $\overline{B_1(0)}$ is defined by

$$\begin{aligned}
\overline{B_1(0)} &:= \big\{(x, y) \in \mathbb{R}^2 : \forall_{\epsilon>0} B_\epsilon((x, y)) \cap B_1(0) \neq \emptyset\big\} \\
&= \big\{(x, y) \in \mathbb{R}^2 : \forall_{\epsilon>0} \exists_{(x_b, y_b)\in\mathbb{R}^2} (d((x, y), (x_b, y_b)) < \epsilon) \wedge (d((0, 0), (x, y)) < 1)\big\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_n &:= \big\{A_n \subset \mathbb{R}^2 : \forall_{(x,y)\in\mathbb{R}^2} \forall_{\epsilon>0} B_\epsilon((x, y)) \cap B_1(0) \neq \emptyset \implies (x, y) \in A\big\} \\
&= \big\{A \subset \mathbb{R}^2 : \big(\forall_{(x,y)\in\mathbb{R}^2} \exists_{(x_b,y_b)\in\mathbb{R}^2} d((x, y), (x_b, y_b)) < \epsilon \implies d((0, 0), (x, y)) < 1\big)\big\} \\
&= \left\{A \subset \mathbb{R}^2 : \left(\forall_{(x,y)\in\mathbb{R}^2} \exists_{(x_b,y_b)\in\mathbb{R}^2} \sqrt{(x - x_b)^2 + (y - y_b)^2} < \epsilon \implies \sqrt{x^2 + y^2} < 1\right)\right\}
\end{aligned}$$

Problem 4

Let $S = \mathbb{R}^2$. Are the following sets open or closed within the metrics below?

$$\begin{aligned}
A &= \left\{(x, y) : x^2 + y^2 < 1\right\} \\
B &= \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\} \\
C &= \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\} \\
D &= \{(x, y) : |x| + |y| < 2\} \\
E &= \left\{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\right\}
\end{aligned}$$

a) Euclidean Metric: $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

i) $A = \left\{(x, y) : x^2 + y^2 < 1\right\}$

Open

ii) $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$

Closed

iii) $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$

Neither

iv) $D = \{(x, y) : |x| + |y| < 2\}$

Open

v) $E = \left\{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\right\}$

Open

b) Manhattan Metric:

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - y_2| + |y_1 - y_2|$$

i) $A = \left\{(x, y) : x^2 + y^2 < 1\right\}$

Open

ii) $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$

Closed

iii) $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$

Neither

iv) $D = \{(x, y) : |x| + |y| < 2\}$

Open

v) $E = \left\{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\right\}$

Open

c) Highway Metric:

Definition 10. *The highway metric is defined as*

$$d_h((x_1, y_1), (x_2, y_2)) := \begin{cases} |y_1 - y_2|, & x_1 = x_2 \\ |y_1| + |y_2| + |x_1 - x_2|, & x_1 \neq x_2 \end{cases}$$

i) $A = \{(x, y) : x^2 + y^2 < 1\}$

Neither

ii) $B = \{(x, y) : x = 0 \wedge -1 \leq y \leq 1\}$

Neither

iii) $C = \{(x, y) : 1 < x < 2 \wedge -1 \leq y \leq 1\}$

Neither

iv) $D = \{(x, y) : |x| + |y| < 2\}$

Open

v) $E = \{(x, y) : x^2 - y^2 < 1 \wedge |x| + |y| < 4\}$

Neither

Problem 5

Let (S, d) be a metric space.

a) Show that for all $A \subset B \subset S$ one has $\mathbf{int}(A) \subseteq \mathbf{int}(B)$ and $\overline{A} \subseteq \overline{B}$. Also provide an example of non-strictness.

Theorem 7. *For the metric space (S, d), and $\forall A \subset B \subset S$ the following are true:*

i) $\mathbf{int}(A) \subseteq \mathbf{int}(B)$

Proof.

$$\begin{aligned}
\text{int}(A) &= \{x \in A : \exists_{\epsilon>0} B_\epsilon(x) \subset A\} \\
&= \{x \in A : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in A\} \\
\text{int}(B) &= \{x \in B : \exists_{\epsilon>0} B_\epsilon(x) \subset B\} \\
&= \{x \in B : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in B\}
\end{aligned}$$

Since $(\text{int}(A) \subset A) \wedge (\text{int}(B) \subset B) \wedge (A \subset B)$,

$$\{x \in A \subset B : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in A \subset B\}$$
$$\forall x \in \{x \in A : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in A\} \implies x \in \{x \in B : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in B\}$$

Therefore,

$$\text{int}(A) \subseteq \text{int}(B)$$

This cannot be a strict inequality becouse it would be equal when $A = B$. □

ii) $\overline{A} \subseteq \overline{B}$

Proof.

$$\begin{aligned}
\overline{A} &= \{x \in S : \forall_{\epsilon>0} B_\epsilon(x) \cap A \neq \emptyset\} \\
&= \{x \in S : \forall_{\epsilon>0} \exists_{x_b \in S} : d(x, x_b) < \epsilon \wedge x_b \in A\} \\
\overline{B} &= \{x \in S : \forall_{\epsilon>0} B_\epsilon(x) \cap B \neq \emptyset\} \\
&= \{x \in S : \forall_{\epsilon>0} \exists_{x_b \in S} : d(x, x_b) < \epsilon \wedge x_b \in B\}
\end{aligned}$$

Since $(\overline{A} \subset A) \wedge (\overline{A} \subset B) \wedge (A \subset B)$,

$$\{x \in S : \forall_{\epsilon>0} \exists_{x_b \in S} : d(x, x_b) < \epsilon \wedge x_b \in A \subset B\}$$
$$\forall x \in \{x \in S : \forall_{\epsilon>0} \exists_{x_b \in S} : d(x, x_b) < \epsilon \wedge x_b \in A\} \implies x \in \{x \in S : \forall_{\epsilon>0} \exists_{x_b \in S} : d(x, x_b) < \epsilon \wedge x_b \in B\}$$

Therefore,

$$\overline{A} \subseteq \overline{B}$$

This cannot be a strict inequality becouse it would be equal if $A = B$. □

b) Is the following true: $\textbf{int}(A \cup B) = \textbf{int}(A) \cup \textbf{int}(B)$?

$$
\begin{aligned}
\text{int}(A \cup B) &= \{x \in A : \exists_{\epsilon>0} B_\epsilon(x) \subset A \cup B\} \\
&= \{x \in A \cup B : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge (x_b \in A \vee x_b \in B)\} \\
&= \{x \in A \cup B : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in A\} \\
&\cup \{x \in A \cup B : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in B\} \\
&= \{x \in A : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in A\} \\
&\cup \{x \in A : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in B\} \\
&\cup \{x \in B : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in A\} \\
&\cup \{x \in B : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in B\} \\
&= \text{int}(A) \cup \text{int}(B) \\
&\cup \{x \in A : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in B\} \\
&\cup \{x \in B : \exists_{\epsilon>0} \forall_{x_b \in S} d(x, x_b) < \epsilon \wedge x_b \in B\}
\end{aligned}
$$

Therefore, $\text{int}(A \cup B) \subseteq \text{int}(A) \cup \text{int}(B)$ and $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$ is not true.

c) Is the following true: $\overline{A \cap B} = \overline{A} \cap \overline{B}$?

$$
\begin{aligned}
\overline{A \cap B} &= \{x \in S : \forall_{\epsilon>0} B_\epsilon(x) \cap (A \cap B) \neq \emptyset\} \\
&= \{x \in S : \forall_{\epsilon>0} \exists_{x_b \in S} : d(x, x_b) < \epsilon \wedge x_b \in A \cap B\} \\
&= \{x \in S : \forall_{\epsilon>0} \exists_{x_b \in S} : d(x, x_b) < \epsilon \wedge x_b \in A \wedge x_b \in B\} \\
&= \{x \in S : \forall_{\epsilon>0} \exists_{x_b \in S} : d(x, x_b) < \epsilon \wedge x_b \in A\} \\
&\cap \{x \in S : \forall_{\epsilon>0} \exists_{x_b \in S} : d(x, x_b) < \epsilon \wedge x_b \in B\} \\
&= \overline{A} \cap \overline{B}
\end{aligned}
$$

Therefore, $\overline{A \cap B} = \overline{A} \cap \overline{B}$ is true.

Problem 6

Give a topological proof of the infinitude of the set of prime numbers. (H. Furstenberg, 1955)
Denote $N_{a,b} := \{a + nb : b \in \mathbb{Z}\} \subset \mathbb{Z}$. Define the topology on $\mathbb{Z}$ as follows: The set U will be called open if for any $a \in U$ there exists $b \in \mathbb{Z}$ so that $N_{a,b} \in U$. Note that every open set is infinite.

Definition 11.

$$N_{a,b} := \{a + nb : b \in \mathbb{Z}\} \subset \mathbb{Z}$$

Definition 12. *The set U will be called open if*

$$\forall_{a\in U} \exists_{b\in\mathbb{Z}} : N_{a,b} \subset U$$

a) Show that it is indeed a topology.

(i.e): any union of open sets is open and any finite intersection of open sets is open.

i) $\emptyset$ and $\mathbb{Z}$ are open sets.

ii) Any union of open sets is an open set

$$\forall_{a\in U} \exists_{b\in\mathbb{Z}} : N_{a,b} \in U$$
$$\forall_{a\in U} \exists_{b\in\mathbb{Z}} : \{a + nb : b \in \mathbb{Z}\} \subset Z \subset U$$

Trivially, it can be seen that $\{U_i\}_{i\in I}$ open $\implies \bigcup_{i\in I} U_i$ open.

iii) Finite intersections is open. (i.e.) U_1, U_2 open $\implies U_1 \cap U_2$ open.

$$x \in U_1 \cap U2 \implies \exists_{a_1,a_2\in S} N_{a_1,x} \subset U_1 \wedge N_{a_2,x} \subset U_2$$

Let $a = \text{lcm}\{a_1, a_2\}$,

$$(N_{a,x} \subseteq N_{a_1,x}) \wedge (N_{a,x} \subseteq N_{a_2,x})$$

Therefore,

$$x \in S_{a,x} \subseteq U_1 \cap U_2$$

meaning that any finite interesection of open sets is open.

b) Show that $N_{a,b}$ is closed.

$$N_{a,b} = \{a + nb : b \in \mathbb{Z}\} \subset \mathbb{Z}$$
$$N^c_{a,b} = \mathbb{Z} \backslash \{a + nb : b \in \mathbb{Z}\}$$
$$N_{a,b} = \mathbb{Z} \backslash (N_{a,b+1} \cup N_{a,b+2} \cup \cdots \cup N_{a,b_a-1})$$

Since $N^c_{a,b}$ is open, $N_{a,b}$ is closed.

c) Show that $\mathbb{Z}\backslash\{-1, 1\}$ is open

$$\forall_{x\in\mathbb{Z}\backslash\{-1,1\}} \exists_{\epsilon>0} : B_\epsilon(x) \subset \mathbb{Z}\backslash\{-1, 1\}$$
$$\forall_{x\in\mathbb{Z}\backslash\{-1,1\}} \exists_{\epsilon>0} : \forall_{x_b\in Z} d(x, x_b) < \epsilon \implies x_b \in \mathbb{Z}\backslash\{-1, 1\}$$

Which is clearly open since the ball sets are always contained within the set itself.

d) Prove that the set $\mathbb{P}$ of prime numbers cannot be finite.

Hint: $\mathbb{Z}\backslash\{-1, 1\} = \bigcup_{p\in\mathbb{P}} N_{0,p}$

Assume $\mathbb{P}$ is finite. Since $\forall_{p\in\text{ı}} N_{0,p}$ closed, the union over $p \in \mathbb{P}$ would also be closed.

However, since $\mathbb{Z}\backslash\{-1, 1\}$ is open, this can't be true, so $\mathbb{P}$ must be infinite.